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Preface and Introduction

Preface: CERME14 – finally in Bozen-Bolzano!

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The 14th Congresss of the European Society for Research in Mathematics Education (CERME14) marks a genuine return to normality for the congress, in two ways. First, the congresses have been biannual and held in February since CERME2, held 2002 in Prague. Second, we originally planned to have CERME12 in Bolzano in 2021. As we all know, the COVID pandemic disturbed everything from early 2020, resulting in a postponement of CERME12 to 2022, and the decision in December 2021 to have the congress online "from Bolzano" but, indeed, not "in Bolzano". Subsequently, CERME13 was held in Budapest in July 2023, as the first on-site CERME since 2019. Having finally been able to realise CERME14 in February 2025, and in Bolzano, thus marks a restoration of the regular pattern, and of the plan to realise CERME in the capital of the South Tyrol region.

The community of research which is drawn to CERME owes much to the team of Free University of Bozen-Bolzano (UniBz), having organised both CERME12 and CERME14. There was nothing given in the fact that this team, led by Giorgio Bolondi and Camilla Spagnolo, would propose to organise CERME14, after the many challenges and disappointments that had been caused by the pandemic. Indeed, CERME12 became a "real" CERME thanks to them, and despite the impossibility, at the time, to convene on-site. But the dream to have the community come to the beautiful city of Bolzano survived, and was realised as an indisputable success in 2025. The community is rightfully grateful for the heroic efforts of the team over seven years – indeed, the planning for CERME12 had started in 2018 – culminating with the congress that the present proceedings report on. The Local Organising Committee of CERME14 mobilised not only colleagues from Bolzano but also from neighbouring regions of Italy, and was supported by the wonderful administrative staff of UniBz. To all of you dear colleagues, the warmest thoughts and thanks are due, from the entire ERME community.

The organisation of the scientific programme of CERME14 was led by Marianna Bosch (Barcelona) and Susana Careira (Portugal), as well as by the entire International Programme Committee and the leader teams of no less than 29 Thematic Working Group (TWG). The community is equally grateful for your tireless and generous efforts to make the congress a rich and fruitful site of scientific communication, collaboration and cooperation. CERME is not a congress like any other – its success relies on the careful preparation of all participants, whether authors, reviewers, group leaders or committees, to enable the congress to focus on discussion of the contributions rather than mere presentations and decorative questioning. The present proceedings present the outcome of this collective work, not simply the individual contributions which were submitted back in September of 2024. Each TWG represents a specialized subcommunity of ERME, and the interaction of its members makes both the event and its products much more than the sum of what was initially submitted. It is this experience of international scientific community which makes CERME such a productive event and which makes it worthwhile for so many to work for its continued development and success. At CERME14, 848 participants from 50 countries contributed to this work (Table 1).

Table 1: Participants at CERME14 by country

Algeria : 2	Cyprus : 2	Iceland : 2	Mexico : 8	Romania : 1
Australia : 8	Czech Rep.: 15	Ireland : 7	Morocco : 1	Slovakia : 9
Austria : 25	Denmark : 25	Israel : 44	Netherlands: 30	South Africa : 3
Belgium : 3	Estonia : 4	Italy : 94	New Zealand: 1	Spain : 42
Brazil : 7	Faroe Islands : 1	Japan : 7	Nigeria : 1	Sweden : 51
Canada : 10	Finland : 2	Kuwait : 1	Norway : 64	Switzerland: 14
Chile : 5	France : 10	Latvia : 3	Poland : 3	Thailand : 1
China : 3	Germany : 168	Lithuania : 1	Portugal : 18	Turkey : 31
Costa Rica:1	Greece : 20	Luxembourg : 1	Puerto Rico : 1	UK : 39
Croatia : 5	Hungary : 12	Malta : 1	Quatar : 1	US : 40

As the table illustrates, the ERME community is not limited to the geographical region of Europe. The contribution of colleagues from around the world obviously adds to the importance and quality of the CERME concept. Mathematics education research is, indeed, a global endeavour with far-reaching stakes both in terms of the intellectual and societal importance of its subject. Research evidently benefits hugely from considering the variations, difficulties and advances that exist within each national and cultural context. The original purpose of ERME – to share results and ideas across countries in Europe – has clearly caught up with globalization in this sense.

In my preface for the proceedings of CERME13 (Winsløw, 2013), I briefly touched upon the concrete numerical challenges, which the success of CERME poses to organisers. The ERME board has worked closely with both the LOC and the IPC to implement a model which unites inclusivity and fairness in a situation where there is an increasing number of proposals submitted, in fact, more than the capacity of academic venues can currently accommodate. This was already the case for CERME14 and is likely to be so at CERME15 in Bratislava. The solution, to be further developed by the entire ERME community, involves not only a necessary "acceptance rate" (fixed at 75% for CERME14) but also a development of further opportunities for specialized subcommunities to convene to communicate, collaborate and cooperate on research, through still more ERME topic conferences. It also involves an increased effort to support first-time contributors to CERME, a refocused effort to extend financial support to emergent research communities in Europe, and new online events to prepare young researchers for CERME and other activities organised by our society.

Finally, on a personal note, my four years as president of ERME ended with CERME14. I wish to extend my heartfelt thanks to all the board members and other colleagues whose support and kindness have made these years some of the most professionally enriching end enjoyable of my life.

References

Winsløw, C. (2023). Preface: CERME13 in Budapest, and beyond. In: Drijvers, P., Csapodi, C., Palmér, H., Gosztonyi, K., & Kónya, E. (Eds.). (2023). *Proceedings of the Thirteenth Congress of the European Society for Research in Mathematics Education (CERME13)*, pp. 1-2. Alfréd Rényi Institute of Mathematics and ERME.

Introduction to the proceedings of the fourteenth Congress of the European Society for Research in Mathematics Education (CERME14)

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About CERME14

The fourteenth Congress of the European Society for Research in Mathematics Education (CERME14) took place in Bolzano, Italy, from 4 to 8 February 2025. It was hosted by the Free University of Bozen-Bolzano, with the patronage of Associazione Italiana di Ricerca in Didattica della Matematica (AIRDM), of Unione Matematica Italiana (UMI), of Stadtgemeinde Bozen-Comune di Bolzano, of M.C. Escher Foundation, in collaboration with the Tourist Office of the town of Bolzano, and with support of Stiftung Südtiroler Sparkasse (Fondazione Cassa di Risparmio di Bolzano). The congress brought together 924 among researchers, teachers, and PhD students from 53 countries across all continents, who share an interest in advancing the field of mathematics education research through dialogue, collaboration, and exchange.

Marianna Bosch and Susana Carreira were chair and co-chair of the International Programme Committee, which included Giorgio Bolondi, Paul Drijvers, Csaba Csapodi, Alf Coles, Sibel Kazak, Bozena Maj-Tatsis, Janka Medova, Hanna Palmer, Sigal Rotem, Daniel Thurm, Floriane Wozniak (from July 2023). Giorgio Bolondi chaired the Local Organizing Committee, which was co-chaired by Michael Gaidoschik and Camilla Spagnolo.

As in previous editions, the scientific heart of the congress was the work carried out within the Thematic Working Groups (TWGs). CERME14 hosted 29 TWGs, listed in Table 1. TWG7 on *Mathematics for Work, Society and Personal Development: Lifelong Learning*, which was inactive at CERME13, has been reactivated for CERME14. TWG11, previously under the title *Algorithmics*, has re-emerged as an active group on *Teaching and Learning of Discrete and Computational Mathematics*. TWG25 on *Inclusive Mathematics Education – Challenges for Students with Special Needs* was discontinued and replaced by a new TWG25 on *Teaching and Learning of Calculus*. Titles of several TWGs were updated to better reflect evolving research focuses and current terminology. TWG2 changed from *Arithmetic and Number Systems* to *Arithmetic, Numbers and Operations* TWG9 was expanded from *Mathematics and Language* to *Interaction, Communication and Language in Mathematics Education*. Finally, TWG24 was extended from *Representations in Mathematics Teaching and Learning* to *Representations in Mathematics Teaching, Learning and Research*. In total, 159 TWG leaders and co-leaders contributed to shaping the discussions and fostering constructive and critical exchanges. Their efforts, along with those of hundreds of participants who

presented and discussed research papers and posters, lie at the core of CERME's identity: a conference grounded in community, dialogue, and shared commitment to improving mathematics education.

Prior to the congress, the YERME Day offered a dedicated space for early career researchers to present their work, receive feedback, and connect with others. This tradition continues to play a vital role in sustaining and renewing the ERME community, attracting 125 participants.

Table 1: Overview of the TWG leadership teams

Thematic Working Group	Leader	Co-leaders
TWG01: Argumentation and Proof	Andreas Moutsios-Rentzos	Nadia Azrou, Orly Buchbinder, Viviane Durand-Guerrier, David Alexander Reid, Fiene Bredow, Dimitrios Deslis, Mei Yang
TWG02: Arithmetic, Numbers and Operations	Ioannis Papadopoulos	Elisabeth Rathgeb-Schnierer, Renata Carvalho, Robert Gunnarsson, Eszter Kónya, Linda Devi Fitriana
TWG03: Algebraic Thinking	Maria Chimoni	Luis Radford, Jorunn Reinhardtsen, Stefan Korntreff
TWG04: Geometry Teaching and Learning	Alik Palatnik	Lina Brunheira, Taro Fujita, Petra Surynková, Chrissa Papadaki
TWG05: Probability and Statistics Education	Aisling Leavy	Daniel Frischemeier, Orlando Gonzalez, Sibel Kazak, María Magdalena Gea, Luca Lamanna
TWG06: Applications and Modelling	Jonas Bergman Ärlebäck	Britta Eyrich Jessen, Gilbert Greefrath, Katrin Vorhölter, Serife Sevinç, Yana Lacek
TWG07: Mathematics for Work, Society and Personal Development: Lifelong Learning	Pauline Vos	Beth Kelly, Peter Frejd, Michele Giuliano Fiorentino, Javier Díez-Palomar, Oda Heidi Bolstad
TWG08: Affect and the Teaching and Learning of Mathematics	Chiara Andrà	Domenico Brunetto, Robin Göller, Inés María Gómez-Chacón, Hanna Viitala, Janina Krawitz
TWG09: Interaction, Communication and Language in Mathematics Education	Jenni Ingram	Kirstin Erath, Alexander Schüler-Meyer, Máire Ní Ríordáin, Ingólfur Gíslason
TWG10: Diversity and Mathematics Education: Social, Cultural and Political Challenge	Timo Dexel	Laura Black, Juuso Nieminen, Sabrina Bobsin Salazar, Mariam Makramalla, Daniela Steflitsch
TWG11: Teaching and Learning of Discrete and Computational Mathematics	Simon Modeste	Janka Medova, Ulrich Kortenkamp, Sylvia van Borkulo, David Zenkl

TWG12: History in Mathematics Education	Antonio M. Oller-Marcén	Bjørn Smestad, Wendy Goemans, Tanja Hamann, Marianne Thomsen
TWG13: Early Years Mathematics	Andrea Maffia	Camilla Björklund, Dorota Lembrér, Eva Nováková, Chronoula Voutsina, Ergi Acar Bayraktar
TWG14: University Mathematics Education	Irene Biza	Olov Viirman, Ignasi Florensa, Ghislaine Gueudet, Athina Thoma, Mathilde Hitier
TWG15: Teaching Mathematics with Technology and Other Resources	Bärbel Barzel	Ornella Robutti, Eirini Geraniou, Nicola Bretscher, Gülay Bozkurt, Andreas Tamborg
TWG16: Learning Mathematics with Technology and Other Resources	Osama Swidan	Jana Trgalova, Florian Schacht, Eleonora Faggiano, Rogier Bos, Sara Bagossi
TWG17: Theoretical Perspectives and Approaches in Mathematics Education	Heather Johnson	Felix Lensing, Pedro Nicolás, Amalie Sødal
TWG18: Mathematics Teacher Education and Professional Development	Marita Friesen	Raffaele Casi, Andreas Ebbelind, Janne Fauskanger, Tracy Helliwell, Pedro Ivars, Libuše Samková, Gil Schwarts, Fanni Biró
TWG19: Mathematics Teaching and Teacher Practice(s)	Reidar Mosvold	Mark Hoover, Helena Grundén, Nada Vondrova, Qëndresa Morina, Derek Maher
TWG20: Mathematics Teacher Knowledge, Beliefs and Identity	Miguel Montes	Veronika Hubenaková, Federica Ferretti, Thorsten Scheiner, Felix Woltron
TWG21: Assessment in Mathematics Education	Francesca Morselli	Christina Drüke-Noe, Michal Ayalon, Johanna Rämö, Gözde Kaplan Can
TWG22: Curricular Resources and Task Design in Mathematics Education	Annalisa Cusi	Dubravka Glasnović Gracin, Shai Olsher, Hendrik Van Steenbrugge, Patrick Ebers
TWG23: Implementation of Research Findings in Mathematics Education	Linda Marie Ahl	Mario Sánchez Aguilar, Boris Koichu, Oda Helenius, Julie Vangsøe Færch
TWG24: Representations in Mathematics Teaching, Learning and Research	Michal Tabach	Elisa Miragliotta, Anna Baccaglini-Frank, Kate O'Brien
TWG25: Teaching and Learning of Calculus	Laura Branchetti	Anatoli Kouropatov, Vahid Borji, Farzad Radmehr, Jana Peters
TWG26: Mathematics in the Context of STEM Education	Michelle Stephan	Clelia Cascella, M. Sencer Corlu, Nelleke Den Braber
TWG27: The Professional Practices, Preparation and Support of Mathematics Teacher Educators	Ronnie Karsenty	Stefan Zehetmeier, Bettina Rösken-Winter, Hilda Borko, Francisco Javier Rojas, Birte Pöhler-Friedrich

TWG28: Collaborative Settings in Mathematics Teacher Education	Stéphane Clivaz	Pernilla Mårtensson, Aoibhinn Ní Shúilleabháin, Geoff Wake, Paula Gomes
TWG29: Embodied and Material Studies of Mathematical Behaviour	Anna Shvarts	Elizabeth de Freitas, Ricardo Nemirovsky, Nathalie Sinclair, Christina Krause, Giulia Ferrari

Editorial information

These proceedings are available as a complete volume on the ERME website. Each individual contribution is also available through the HAL open archive, where it can be found through keywords, title, or author name. This has been the practice since CERME9, to increase the visibility of the huge work done in CERME conferences.

After the preface by the president of ERME, Carl Winsløw, and this brief introduction by the proceedings' editors, this volume continues with texts of three plenary activities of CERME14. The three plenary sessions address a question that has not yet been tackled directly at CERME. It concerns the evolution of the mathematical curriculum in light of the new society demands, particularly with regard to critical issues such as sustainability and the rapid development of data science. The plenary lecture by Ferdinando Arzarello was entitled "New landscapes for mathematics curricula: reflections from an Italian experience". Drawing on a few recent Italian curricular stories of recent years, the lecture provides an original overview of the challenging issues surrounding the mathematics curriculum that are often overlooked in mathematics education research. It offers productive theoretical frameworks with which to address these issues. The second plenary was a dialogue between Kjellrun Hiis Hauge and Alf Coles titled "Climate change and mathematics education: entanglements, values and hopes". The dialogue explores the entanglements of our living world and how mathematics education could respond to them, building on different intricate "gestures" that can be employed to avoid overlooking them. The contrasts between the English and Norwegian curricula provide insights into possible development paths, highlighting both their urgency and the critical constraints that could hinder the involvement of mathematics in addressing the pressing issues of our world. The third plenary was a dialogue between Michelle Wilkerson and Rolf Biehler titled "The new role of data in society: Exploring challenges and opportunities for mathematics education in the age of data science and AI". The authors provide a comprehensive and helpful overview of research in data science and AI, emphasising the essential role they play in our societies and highlighting the need for curricula to be more closely aligned with them. They also point to critical difficulties associated with the field's inherent interdisciplinarity, while offering concrete and highly stimulating examples based on their own research experiences that open up the prospect of a possible future.

The biggest part of these proceedings consists of the sections corresponding to the TWGs. These sections start with an introduction by the TWG leadership team. According to the team's choice, this is a 4-page introduction presenting the contributions, or an extended 8-page one, which proposes an additional analysis of the current research on the theme of the TWG, and perspectives for the future. Next, the section consists of the paper and poster contributions—in alphabetical order by first author's name.

Acknowledgment

These CERME14 proceedings represents a collaborative effort involving the CERME14 IPC and LOC, the TWG leaders and co-leaders, and the editorial team. We express our sincere gratitude to all those who contributed to the reviewing, editing, and organisational processes that made this publication possible. Particular thanks are due to Domenico Brunetto, Sofia Cavalletti, Chiara Giberti, and Veronica Manzoni for their editorial work. We warmly thank all participants for their contributions, and we hope that these proceedings will continue to nurture the collaborative spirit that defines ERME, inspiring further research and dialogue in mathematics education across Europe and beyond.

Plenary Lectures

Old and new landscapes for mathematics curricula: reflections from an Italian experience

Ferdinando Arzarello

Mathematics Department, University of Torino, Italy

The international research underlines the elusive and multifaceted aspects of what is meant by a mathematics curriculum across countries. Two major driving forces have been identified as significant, even if not always convergent: that of cultural values and that of political movements. Teachers' involvement, ownership and commitment to a curriculum reform must be regarded as a must for its successful implementation: for having a suitable didactic transposition, concrete bottom-up and top-down strategies are needed, developed with a careful theoretical analysis. With new areas and links to other disciplines being introduced in mathematics curriculum reforms, foundational questions about what constitutes mathematics in the school mathematics curriculum are raised, and these are linked to the current changes in professional mathematics and to the rapid increase of technology. The discussion of these issues is mainly developed through examples based on the experiences of the author.

Keywords. Mathematical curriculum, didactic transposition, citizenship education, artificial intelligence.

Looking backward for going forward

The mathematics curriculum: an elusive notion

In his contribution to the *ICMI Study 24* (Ruiz et al., 2022), Mogen Niss recalls the different meanings that the word "curriculum" carries and depicts a (mathematics) curriculum with respect to a given educational setting as a vector with six components, which correspond to its main *facets*: Goals; Content; Materials; Forms of Teaching; Student Activities; Assessment. Moreover, other scholars (e.g. Kulm & Li, 2009) point out that curricula can be analysed at different *levels*: from micro to macro.

At the micro-level, the specific facets of curriculum practices and changes are the focus, e.g. according to the Niss version: these, on their turn, embrace the TIMSS classification (Mullis & Martin, 2017) in *Intended* (mainly Goals and Content), *Implemented* (mainly Materials, Forms of Teaching, Student Activities), and *Attained* (mainly Assessment) curriculum.

At the macro-level, the socio-cultural and historical specificities of curriculum practices and changes are emphasised. For example, the ideas of inquiry, problem solving, and student-centred instruction in the development of China's new standards were built on a strong foundation of learning basic skills, using a centuries-old philosophy of Confucianism to provide support for more modern ideas. In Italy, the same words are used with a very different background, e.g., as we will show in the next section, Italian curricula in the first decades of this century have been inspired to the approach of US New Standards, as well as to the Italian tradition represented for example by the reflections of F. Enriques (Giacardi, 2012) and E. Castelnuovo (Arzarello et al., 2013), hence with an integration of Italian and abroad theoretical frames.

This framework puts forward a structural approach to curriculum analysis with a suitable combination of micro- and macro- features, but without losing their specificity. In fact, it is the same general world globalisation that pushes towards such an integration:

> In recent years the internationalisation and globalisation of the economy, universality of technological development and related needs for new skills and knowledge play the role of strong motivations for curriculum reforms that have brought calls for unified standards for mathematics in school. (Shimizu & Vital, 2023, p. 6)

Specifically, the stress is mainly on promoting at a micro-level specific and more or less sophisticated ways of thinking, like *critical thinking* and *high-order thinking*; *inquiry-based mathematical education*; or *quantitative literacy*.

As discussed by Bosch and O'Meara (2023), such processes can interact productively according to three main macro-level aspects:

- *Theoretical* (analysis tools to deal with the present rapidly evolving situation).
- *Cultural* (what mathematics to teach and how to do it);
- *Political* (how to reconcile the need for a cultural preparation suitable to live in a globalised world, with the links to be kept with personal and cultural reality?).

This double classification (in macro-/micro- levels and micro-facets) illustrates the main problems with curricula: the risk of a *double gap*. On the one side, there are difficulties for an actual merging between the macro- and the micro- levels; on the other side, it may also be difficult to produce an effective and productive interaction among its various micro-facets (e.g., those described by M. Niss). The main problem poses the necessity of an effective theoretical analysis of the double gap to elaborate suitable actions in order to overcome it at the concrete level of curricula implementation.

The Anthropological Theory of the Didactic by Y. Chevallard (2019) can help in solving this problem through the *didactic transposition* construct. It identifies conditions that go beyond the narrow space of the classroom and the subject that has to be studied in it, and deals with:

> the transformations an object or a body of knowledge undergoes from the moment it is produced, put into use, selected, and designed to be taught until it is actually taught in a given educational institution. (Chevallard & Bosch, 2014, p. 170)

In fact, knowledge is not taught directly, but undergoes transformations at different levels of the didactic transposition chain, and different actors participate in this transposition work: producers of knowledge, teachers, curriculum designers, etc. The didactic transposition consists of two basic steps (Figure 1): external didactic transposition carried out outside school, and internal didactic transposition inside the school (Chevallard, 1991; Winsløw, 2011). More specifically, in the external didactic transposition process, the scholarly knowledge that institutions outside school, such as universities or colleges, produce is transposed into knowledge to be taught by the noosphere, in accordance with the social and cultural needs of a society (Bosch & Gascón, 2006; Chevallard, 1991).

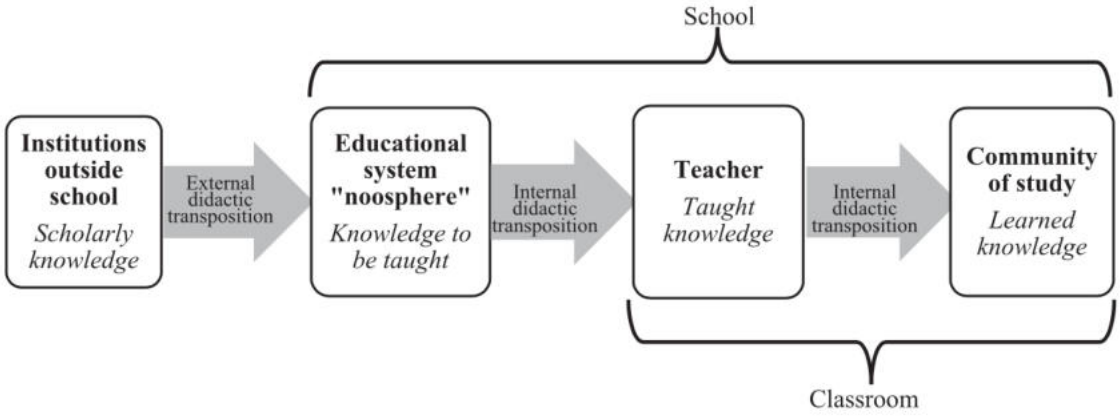

Figure 1. The didactic transposition process
(Akar, N. & Işıksal-Bostan, M., 2021, p. 631; Bosch & Gascón, 2006, p. 56; Winsløw, 2011, p. 121)

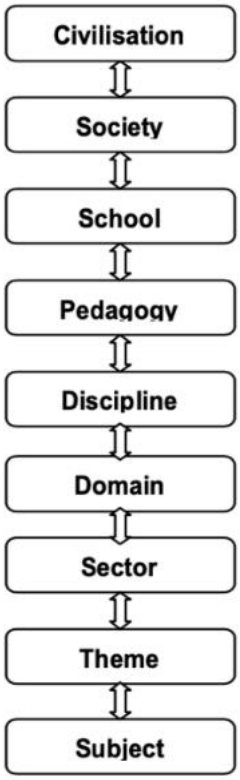

Figure 2. Scale of levels of determinacy
(Bosch & Gascón, 2006, p. 61)

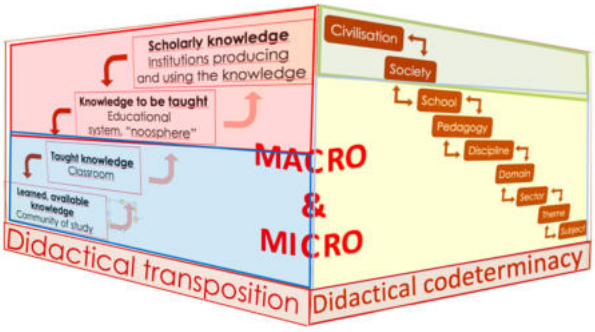

Figure 3. The two scales together

Chevallard introduces also a scale of "levels of codeterminacy" (Figure 2): it suitably frames in a unique process the macro- and micro- levels of a curriculum together with its different facets and can

help researchers to "go beyond the narrow space of the classroom and the subject that has to be studied in it" (Bosch & Gascón, 2006, p. 61).

Figure 3 illustrates the two scales within a double parallel frame: it goes from micro- (white background) to macro- (reddish background) features of the didactic transposition scale, contrasted with the corresponding codeterminacy levels. It must be observed that what we have called the curriculum facets involve more the micro-levels of didactic transposition/codeterminacy, while the macro-levels concern issues that appeared later on the scene of curricula but the variety and speed with which they are appearing in these last years is progressively increasing. For example, they concern globalisation, computational thinking, unified standards, etc., topics that will be discussed in the second part of the paper.

In the next section, I will illustrate how some Italian curricular projects progressively and, so to say "unconsciously", had to concretely face the double gap: their development represents a productive interaction between their micro-facets and a concrete progress from micro- to macro- levels. I will point out how this could happen because of an interesting didactical phenomenon, the meta-didactical transposition.

Mathematics Education: what is already there

I will start my discussion from some important Italian projects and experiences of mathematics teaching, which also deeply influenced national curricula during this first quarter of the century (see Figure 4):

a) *Matematica per il Cittadino* (2000-2005);
b) M@t.abel (2006-2012);
c) *Liceo Matematico* (2015-currently).

I will briefly give some information about the rationale of each project basing mainly on two papers: Arzarello et al. (2021), and Bernardi et al. (2024). I will so be able to elaborate some more general issues that will lead to the reflections of the successive section.

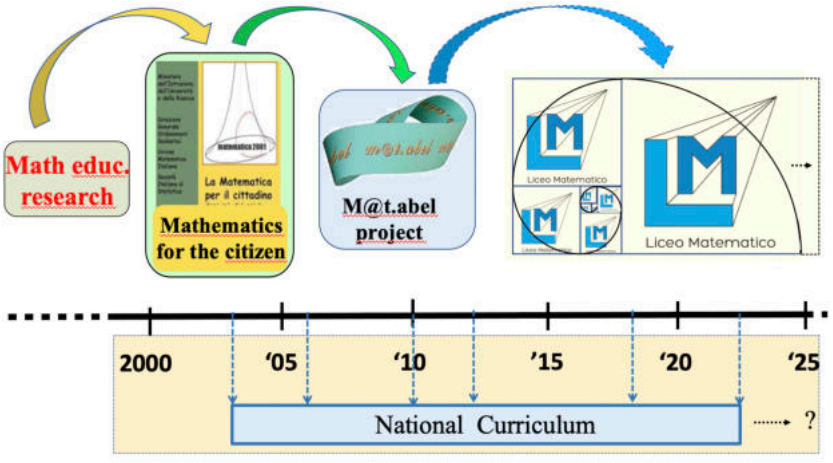

Figure 4: The three main Italian projects for Math teaching in the first quarter of this century

Mathematics for the citizen

At the beginning of the new century, a working group, on behalf of the Italian Ministry of Education, of UMI (Italian Mathematical Union) and SIS (Italian Society for Statistics), was charged to elaborate an updated and compact curriculum of mathematics for primary and secondary schools. It was made of about 30 experts (including myself as chair): they were academics in mathematics and mathematics education, policy makers, and mathematics teachers. All the people involved in the group had a strong expertise in mathematics education, and most of them had been part of research groups promoted since the 1970s by the Italian National Council of Research (CNR) in order to improve the teaching of mathematics in the schools (Arzarello & Bartolini Bussi, 1998). As a result of their work, which lasted four years, the *Matematica per il Cittadino* (MpC: Mathematics for the Citizen) was produced. MpC was inspired by the NCTM Principles and Standards for School Mathematics[1] and by other curricular projects that appeared in those years, by the results of the international mathematics education research and, of course, it took also into consideration the Italian tradition and practices in mathematics teaching. The proposed curriculum was based on the idea of mathematics laboratory, not so much as a place but as a methodology based on activities in which the students can learn by doing, seeing, imitating, communicating with each other, under the guidance of the teacher—that is, "practicing", as in an Italian Renaissance workshop (Arzarello et al., 2003, 2004; Chapman & Robutti, 2008; UMI-CIIM, 2001). The proposed teaching practices fostered a close interaction between novices (students) and expert (teacher), in the frame of cognitive apprenticeship, namely it "stressed the learning-through-guided-experiences based on cognitive and meta-cognitive, rather than on physical, skills and processes" (Collins et al., 1989, p. 455). Another important issue of the project was the particular attention to processes (and not only to contents), such as the use of different communication strategies, conjectures, argumentation and proof, problem-posing and problem-solving, measuring, and modelling. In consonance with the NCTM documents, MpC considered four main areas of mathematics contents (Numbers and Algorithms; Space and Figures; Relations and Functions; Data and Forecasts) and three main transversal areas (Arguing, Conjecturing and Proving; Measuring; Problem Posing and Problem Solving), essentially the same for all the grades of pre-university schools. The curriculum, which distinguished between skills and knowledge, was accompanied by about 150 examples of teaching activities, which illustrated the meaning of the curriculum itself: for each of the areas, the examples made explicit the skills and the knowledge, at which the activity aimed. Each activity was also accompanied by the relative assessment tests. All this huge work was published in three volumes (AAVV, 2001, 2003, 2004), which were sent as books to all the Italian schools, and later made freely available in the web as a PDF publication. The new proposal was spread through the country with many conferences, where its main points were showed to the schools and its content was used in many programs for teachers' professional development; moreover, the Ministry of Education used it to design – in the school reform – the new Mathematics Curriculum (in 2003, 2006, 2010, and 2012, respectively).

[1] https://www.nctm.org/Standards-and-Positions/Principles-and-Standards/

The MpC project can be classified according to the two following indicators: (i) the materials were books, available on the website; (ii) the interactions happened through the traditional format programs for teachers' professional development: meetings promoted by the Ministry of Education and by the schools, where the proposal was illustrated by experts, who had previously shared experiences in mathematical education through the CNR research groups mentioned above.

M@t.abel

Although the new Italian National Curriculum inspired by the MpC project mirrored the influence of this project, school reality was however quite far from being broadly influenced by the new perspectives: innovation was bounded to isolated cases and to primary or middle schools, more than to secondary schools. The believes of teachers about the effective ways they can teach mathematics to "mean achievers" were at the origin of a *didactical gap*, widely studied in the literature (e.g. see Kulm & Li, 2009; Silver, 2009). Hence, in order to fill the gap at the secondary level, the Ministry of Education and the Italian Agency of School (INDIRE) promoted in 2006 a new project, M@t.abel (it is not a web address but an acronym: *Matematica: apprendimenti di base con e-learning* [Basic mathematics with e-learning])[2]. The aim of M@t.abel (M@) was to disseminate the activities and teaching practices of MpC in a mathematics teacher professional development program, carried out over Italy from 2006 until 2012 in blended modality. It involved 6500 teachers, 412 tutors, 56 authors and researchers. The M@ activities were adapted to a double modality—multimodal and static, as downloadable file—presentation in a platform online. PISA tests (2006-2009) certified the success of the project (Stacey & Turner, 2015, pp. 249-260).

A careful analysis of this complex process made ex-post allowed to describe it as a process of *meta-didactic transposition* (MDT: Arzarello et al., 2014; Arzarello et al., 2021; Cusi et al., 2023) between the *praxeologies* of the three subjects involved in it: researchers, tutors, teachers. The word "praxeology" is a neologism introduced by Chevallard made of praxis as the "know how", and logos as the "knowledge", namely the theoretical discourses on praxis (Chevallard, 1999). According to Chevallard, a praxeology is made of the following components: a type of tasks, a technique, that is a way of performing the tasks (*praxis*), and its theoretical justification (*logos*). While Chevallard uses this construct to analyse teachers' didactical praxeologies in their school activity, the MDT adapts it to the analysis of professional development programs for teachers, considering the different praxeologies involved in the M@ program: those of the teachers, because of their basic professional experience in the school, those of the tutors, as experienced teachers, and those of researchers as designers and coachers of the teachers' educational program, because of their competence in mathematics education and in MpC. During the development of the program, the meta-didactical praxeologies of researchers and teachers (different at the beginning) may evolve towards shared praxeologies (Arzarello et al., 2014), through the passage of some of their components from external into being internal (namely used by individuals or by a community). Evolution of the praxeologies does not mean that all the teachers (or researchers) involved in the educational program evolve in the

[2] http://www.scuolavalore.indire.it/superguida/matabel/

same way with the same transformation of components: in fact, different teachers may evolve in different ways, with respect to their histories and experiences.

The M@ program develops in two steps. The first step concerns tutors' training (made by university researchers): the meta-didactical praxeologies of the researchers are shared with those of tutors, giving rise to shared praxeologies. In the second step, the teachers were organized in communities of inquiry (Jaworski & Goodchild, 2006), composed of 15-20 teachers and supervised by tutors. The communities of teachers worked first in some face-to-face meetings with tutors: the tutors presented the activities and the spirit of the project in some meetings, asking the teachers to analyse them from a didactical point of view. Then, the teachers experimented with some chosen activities in their own classrooms and observed their students' processes, writing their notes in a logbook to be uploaded on the platform: in this way, teachers' praxeologies started to change. Finally, the tutor coordinated the group of teachers from remote, through synchronous meetings and asynchronous discussions.

MDT had its strength in the use of a digital *platform* for synchronous and asynchronous activities among teachers. The platform was the environment that gave new techniques to teachers, influencing and supporting them in changing their praxeologies: specifically, if a teacher had worked for many years in a traditional and isolated way, now she was forced to discuss new methodological issues through the platform: this one, together with the brokering function of the tutors, aimed to build a community of teachers with *shared praxeologies*. For instance, the Dynamic Geometry Software productions could be used as concrete boundary objects between the community of tutors and that of teachers. The platform worked as a *communicational-representational infrastructure* (Hegedus & Moreno-Armella, 2009), fostering the use of a shared desktop where the teachers could work together on-line on the same topic or mathematical object. In the MDT model, the platform constitutes an example of an *external component* that becomes an *internal* one over the course of the project.

Through MDT as a process, a complete intertwining of the micro-level facets was realized and important connections between them and the macro-level institutional and cultural components was realized: particularly, the shared platform played a crucial role as a communicational and representational infrastructure in promoting them.

The M@ project ended in 2012 and influenced further the new versions of the official national curriculum until and after that date (Figure 1). However, as it is well known, from that time new characters were already treading the scene of mathematics and of mathematics teaching. In fact, since some decades, mathematics was dramatically changing both for endogenous and exogenous reasons (Borovic et al., 2022; Lolli, 2022). As a long-lasting example of the former, see the Langlands Program (Frenkel, 2013). For the second, one can advocate the development of interconnectivity and the growth of planet-wide crises, for example the effects of COVID-19 on the former and the climate changes on the latter. As a result of such human and social problems, new questions and goals were beginning to emerge. For example:

- What mathematics is necessary to understand interconnectivity?
- What mathematics is needed to understand climate?
- What mathematics is needed to understand biodiversity?
- What mathematics is needed to understand wealth distribution? (Chan et al., 2021, p. 9)

It is not by case that all the three plenary sessions in this conference address the problem of the evolution of the mathematical curriculum given the new society demands, especially concerning critical issues like sustainability and the high development of data science.

Such and similar questions require a concrete content revision of the usual didactical transposition processes. In Italy the experience of the *Liceo Matematico* represents a first partial concrete answer to such queries.

Liceo Matematico

A third initiative that involves Italian mathematics teachers in an ever-growing number consists of the *Liceo Matematico* (LM: Mathematical High School) activities.[3] The general idea of the LM is to give more space to mathematics and science, not to introduce a greater number of notions, but to reflect on foundations and ideas, broaden cultural horizons, deepen, understand better, and in particular underline connections with other disciplines, including humanities. A strongly interdisciplinary approach to the initiative is therefore highlighted. The LM-schools do not require adherence to a rigid scheme of activities but are based on the sharing of some principles that are flexibly declined in the different local realities. The activities always take place in a laboratory manner, in line with the idea of a mathematics laboratory as outlined in the MpC and M@ projects (in fact, some participants and organizers of LM come from those experiences). The design of the activities is featured by a strong collaboration between university researchers and schoolteachers, in both cases not only of mathematics. The collaboration is expressed through the organization of periodic meetings aimed at designing and discussing laboratory paths to be tested and implemented in the classes. The different experiences build a precious common heritage that is shared and compared during an annual National Seminar, where the made experiences, the used teaching methodologies and the addressed disciplinary content are critically discussed.

The training project includes, for the classes involved, an additional hour of mathematics every week. Moreover, for the training of teachers, cycles of meetings with researchers are organized during the school year.

Hence, the main features of LM can be so summarized:

- Collaboration between mathematics education researchers and teachers (MDT frame);
- Additional hours (about 30 per year);
- Mathematics laboratory (from MpC and M@ projects);
- Statistics: currently, more than 25 universities, 150 secondary schools, 400 classes, 7000 students, 300 teachers distributed throughout Italy; but the numbers are increasing.

A recent exploratory study (Bernardi et al., 2024) has investigated whether the LM students learning is on average better than the learning of students at the same schools that do not participate in the project and how the LM project has an effective impact on this learning. It has based its research on the yearly surveys of INVALSI, the Italian National Institute for the Educational Evaluation of

[3] https://www.liceomatematico.it

Instruction and Training[4], which, according to some researches (Gambini et al., 2022), have a good predictive value for students' university careers. The research consists in a longitudinal analysis over time, which compares the performance trend in learning Mathematics and Italian between students who attend the LM and students who do not attend it, within the same schools. Since for many reasons it was not possible the adoption of a probabilistic sampling, a reasoned one was adopted: hence this method prevents the generalization of the results obtained at the level of the entire population but has a value for the examined cohort of students. The research regarded the same cohort of 2588 students from 11 schools in centre-south of Italy and considered their scores in the National INVALSI survey when they attended grade 8 (2017), 10 (2019), and 13 (2022). Of the total of 2588 students, 2232 (86.2%) belong to sections that have not followed the LM, while 356 (13.8%) belong to sections that attended it.

The data from this first exploratory study seem to indicate that participation in the LM project has a positive impact on the level of learning achieved in both Mathematics and Italian, and more markedly for students who obtain low scores in both subjects at the end of the first cycle. Figure 5 gives an idea of the type of results shown by the research. It reports the means of the distributions of scores for Mathematics in the different grades considered, respectively for LM and No-LM students.

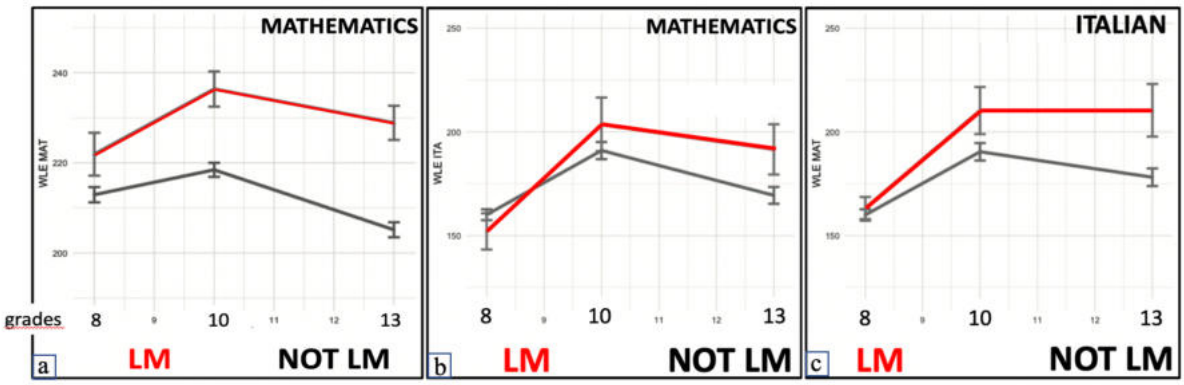

Figure 5. Data from the research on LM students (Bernardi et al., 2024, p. 80, 87)

Figure 5a corresponds to the entire cohort of students, while Figures 5b, and 5c correspond to the subgroup of students who reported at grade 8 (the last year before higher secondary school) a score lower than the first quartile of the distribution in both Mathematics and Italian. In the group under examination, a total of 325 students were identified, of which 44 were participants in the LM and the remaining 281 belonged to non-LM sections. Beyond the limitations due to the relatively small numbers, it is interesting to note from the analysis of the graphs in Figures 5b and 5c that, even for the students apparently most disadvantaged at the start, a significant improvement in the scores of both subjects was found, to which participation in the LM seems to have contributed.

[4] https://www.invalsi.it/invalsi/index.php

Mathematics Education: what is beginning to be there

In this second part of the paper, I will give some examples and develop some reflections on the changes that are beginning to feature the new landscape of mathematics and of mathematics learning.

Mathematics for world awareness

I will start with an example from the activities of LM, emblematic of these new trends. It is described in a paper written by the teacher (Giovanna Guidone) and four students of the class where the activities were developed in grades 11 and 12 (Di Tuoro et al., in press) in a LM Mathematics and Physics laboratory (the two disciplines are taught by the same teacher), significantly called *Numbers for world awareness.*

The idea of laboratory as a central didactical tool comes from the Italian tradition as illustrated by the work of E. Castelnuovo and further elaborated by the MpC and M@ experiences (see comments in the first section). The contents are outside the usual traditional ones, involve the skills for an aware citizenship through mathematics, and in this sense, they represent a concrete example of the types of new entries within the stream of the didactic transposition at the macro-level, also strongly determining and reshaping the curriculum facets at the micro-level.

Here is a short summary of the activities. In grade 11 the students work to learn how to make 'Fermi estimates' (Nityananda, 2014) with the spirit of a civic education; for example: "how much energy savings would result from reducing shower time from 15 to 5 minutes for ten million of Italians". In grade 12 they face real-life problems like the following: in their town (Fermo, in the Centre of Italy, not far from the sea at about 43° lat. N with a mean temperature of 6°-12° C in December) every year in the main square of the town (34×13 meters) an ice-skating rink is installed for the Christmas holidays for about 50 days: its ice thickness is 7 cm, therefore its mass is 10^5 kg. Consequently, the students face these related questions: "How much energy is needed for maintaining the ice rink? How big is it? What temperature is it at? What is the ambient temperature? How much energy does it absorb? By what mechanism? How long does it take to heat up? What is the average power used? How much energy? From what sources?". To answer these questions the students, with the help of the teacher, rely on an important work of I. Newton about the law of cooling[5] which provides the analytical expression of the temperature *T* as time *t* varies, starting from an initial condition (in grade 12 students learn the concept and basic machinery of derivatives and already know the exponentials from previous years). The proposed model was coherent with the curricular topics, in which they had discussed some models of exponential growth and decay. Making a concrete experiment with different equal containers, each with the same volume of water and the same shape, and measuring with a certain precision the time necessary for getting ice, they could estimate a physical parameter

[5] Newton's law of cooling (also called "of heating") claims that the rate at which the temperature $T(t)$ of a body changes, as time t varies, is proportional to the difference between the temperature $T(a)$ of the environment in which the body is located and the temperature $T(t)$ of the body itself: $T'(t) = k(T_a - T(t)), k > 0$. From this one can get $T(t) = T_a - (T_a - T_i) \cdot e^{-kt}$; knowing the value of the parameter k, it is possible to obtain how long does it take for the heating from an initial temperature T_i to a final temperature T_f.

in the Newton formula[6], so getting a law according to which they could estimate the following quantities (Figure 6):

- the heat quantity necessary to raise the ice rink temperature from -5 °C to -1 °C (~ $2 \cdot 10^8$ J);
- the time τ to pass from -5 °C to -1 °C (~ 1500 s);
- energy necessary to prevent heating (~ 50 000 kWh).

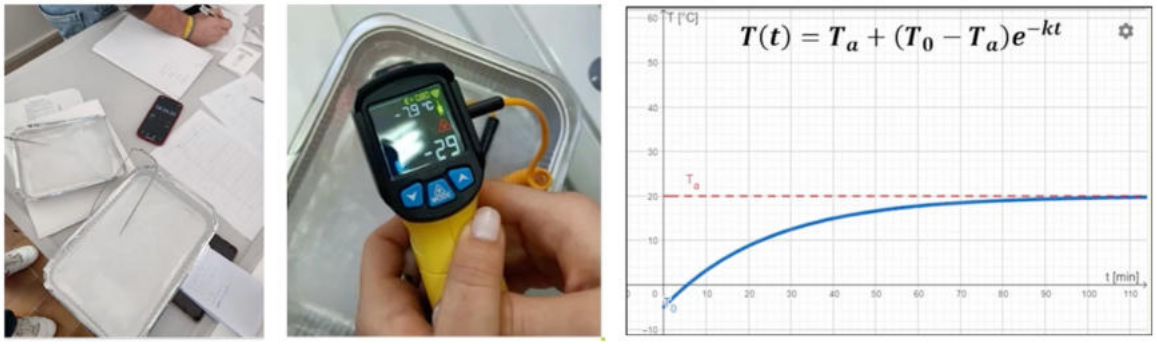

Figure 6. Students' experiment for estimating the parameter *k* in the Newton formula (Di Tuoro et al., in press)

The last estimated quantity poses a typical *citizenship question*: *is it a lot*? *what does it mean?* In order to make the result more intelligible, the students suggested to relate it to the consumption of some activities that are frequently performed in everyday life. The research for such data in the web produced interesting results: for example, 50000 kWh is approximately equivalent to the consumption of an average family's washing machines for 500 years (estimating about 300 washes per year, each of which consumes about 1 kWh); or it is equivalent to the amount of energy used to perform more than two million cell phone recharges; etc. Moreover, through some further calculation and research on the mean temperature in Fermo during winter, they could realize that the rink in their city consumes five times as much as a rink located in a place where temperatures are around 0 °C. Last, but not least, they also estimated that in Italy there is an emission of 0.5 kg of CO_2 per kW of electricity consumption. For the ice rink this is equivalent to 25 000 kg of CO_2: from further research, they found that a hectare of larches absorbs them in one year. This last observation had an interesting "political" result. The class informed the city council about their results and asked to plant 500 larches to offset CO_2 emissions: because of this, the mayor encountered the class and welcomed the request, asking the students to send the technical office some proposals for locations where the trees could be placed. And now we are at this point of the story.

Its value is educationally indisputable: it represents an effective experience of *conscious citizenship* for students. Because of this, it is also relevant from the point of view of the didactic transposition dynamics. In fact, the praxeologies (Chevallard, 1999) that the teacher developed in her class have a double inverse direction. From the one side, they come from the macro levels of nowadays changes in our society and as such have deeply marked the lower microlevels in the two Chevallard's scales (Figures 2 and 3) producing *numbers for world awareness* in students within a *conscious citizenship*

[6] The parameter *k* of Note 1: it was found equal to about $2{,}5 \cdot 10^{-4} s^{-1}$.

frame. From the other side, because of the interaction with the mayor, their research has possibly also reversed the arrow and (hopefully) influenced the higher levels of the scales.

I think that this can suggest an educational productive approach to the educational discourse in schools, where the instances of developing a conscious citizenship are more and more present in curricula. It is precisely these kinds of skills that are needed in schools today, as pointed out by Fadel et al. (2015, p. 41):

> In order to deepen and enhance the learning in these three dimensions—knowledge, skills, and character qualities—there is an important additional fourth dimension needed for a fully comprehensive twenty-first century education: meta-learning (often called learning to learn—the internal processes by which we reflect and adapt our learning). It is not enough to implicitly include this fourth dimension in all the other dimensions—its significance must be highlighted explicitly, so that we are constantly reminded to incorporate meta-learning strategies into the knowledge, skills, and character portions of our learning experiences, learning how to strive to improve no matter what goals we set for ourselves.

In this spirit, I end the section quoting the conclusion of the mentioned paper of Di Tuoro et al. (in press):

> As we mentioned at the beginning, our project was born from the need to make a sensation quantitative, namely the fact that in our latitudes the installation of an ice rink is an ecologically questionable manoeuvre. As we continued with our analyses, the need to share the results obtained made its way into us. In fact, thanks to the sensitivity acquired with our activities on estimates, we increasingly realize that the people around us have behaviours whose global impact they do not understand: it is not a question of bad will, but of misinformation. When the numbers at play are made clear, many can stop to reflect on the fact that changing some behaviours actually costs very little sacrifice, but is a much more responsible way of being in the world. This is the fundamental reason why we thought of writing this work: we would like, through teachers, to be able to spread among our peers an ecological awareness based not on suggestions (in one sense or another), but on reliable and shared scientific models.

Such pieces of activities represent exactly what "begins to be there" in the school and we all hope that they will expand more and more.

It is important that people in mathematics education start (continue) working in this direction, keeping together the two sides of the micro- and macro- levels of the didactic transposition scale. It is an important task for people currently working on curricula: forgetting some of the components in each facet could have negative effects on the school activities, particularly for mathematics.

Moreover, working with such a perspective can also produce aside results of more classical research findings in mathematics education. As an example, the work by Guidone sketched above and other similar activities (e.g. Bagossi, 2024) suggested to the authors an interesting stream of research for what they have called *grounded covariation*. It is a particular aspect of second order covariation (Swidan et al., 2022), which appears in the way Guidone's class has answered the questions about the energy necessary for the ice rink and generally in many other examples where Fermi estimates

are drawn. Grounded covariation is mathematically, physically, and cognitively relevant insofar it allows to unexpectedly grasp the meaning of the relationship between two quantities through a third one within a known environment. The *world awareness* framework allows to achieve this specific mathematical sense of covariation.

However, while from the point of view of the stream represented by the previous discussion, possible didactically and epistemologically clear and feasible paths to follow appear at the horizon of teachers' activities, things seem epistemologically less clear for what regards another issue, that of computational thinking. In this case, it seems that more elaboration is still necessary to have feasible paths through the scales of didactic transposition. This issue will be the focus of the next section.

Changes in mathematics: a new 'New Math'?

Through the LM experience some new faces of mathematics started to enter concrete classroom activities:

- Effective teaching approaches to sustainability;
- From Mathematics Laboratory to citizenship competencies.

Before entering the issue of current transformations in the mathematics thought, a methodological premise is necessary. Changes in mathematics are the rule and not an exception and any temptation to elaborate a fresh comprehensive and normative definition is doomed to precariousness and falsification:

> Falsification in the sense of certain imminent denial and falsification in the sense of a false representation of the present and the past. Such a definition presupposes, in fact, contrary to the repeated teachings of history, that mathematics has reached its true and definitive nature. But whoever proposes it cannot control the elements fermenting in society and which, not yet incorporated into the new synthesis, will inevitably cause cracks. On the other hand, a new proposal can only use the cultural tools available in the present and tested, albeit perhaps in an original way, with the addition of some innovative contribution; these tools, derived from the recent past, however, were developed to serve other problems and other objectives. By bending the tools already possessed to new needs, even unconsciously, the reading of history is distorted into a description either of failures and confusions or of anticipations that were alien to the real concerns of the protagonists. (Lolli, 2022, p. 253, translation by the author)

With this proviso in mind, we can point out that in fact, in the last decades, mathematics has faced some important changes of different types. I will mention two of them: the reduced difference between pure and applied mathematics, and the increasing complexity of the discipline in its new results and the consequent increasing use of automated (ATP) and interactive (ITP) theorem provers in its research activities.

For the first, the increased demand for mathematical application has appeared in two ways: because of multiple "infections" from the science world (social sciences, biology, etc.), or because many aspects of mathematics regarded as "impregnable pure" are now being applied, e.g. algebraic geometry in control theory; graph theory in economics; fibre-bundles in physics; algebraic invariant theory in error-correcting codes.

For the second, it is evident that the research level mathematics is becoming more and more complex, and its proofs are becoming terribly long and difficult so that ATPs and ITPs are used massively by researchers. Today's proof assistants can do more than verification. Their ability to work in interactive mode, allow mathematicians to take the next step and use them to create a new proof from scratch. Well known examples of these are: the four-color theorem (verified in 2005 with the help of the proof assistant Coq); the Kepler conjecture (proved with HOL); the Pythagorean triple problem (proved with Coq and SAT), and many others. This fresh approach is having epistemological and foundational consequences, e.g. the so-called univalent foundations of Vladimir Voevodsky (Grayson, 2018), aiming to explain how mathematics can be encoded in type theory and how this encoding reveals a potentially viable foundation for all of modern mathematics that can serve as an alternative to set theory, via the Curry-Howard isomorphism (Sørensen & Urzyczyn, 2006). Mathematics research, including that relevant to exogenous 21st century challenges (e.g. climate change, the environment, etc.) are likely to be developed from the bottom up in new, proof-assistants-friendly ways. As we will discuss below, these features have also important consequences for mathematics education.

ATPs typically use brute force methods to perform large calculations, while ITPs act as proof assistants that can verify the accuracy of an argument and check existing proofs for errors. However, these two strategies, even when combined (as in the case of newer theorem provers), do not lead to full automated reasoning. Modern proof assistants such as Isabelle/HOL provide increasingly intuitive languages for constructing and representing proofs, but significant scientific progress is still needed to achieve this ambitious goal. Finding a comprehensive solution that integrates both notions of proof can even be understood as a challenge for AI.

The switch to the use of computers as ITPs in production of proofs requires rethinking traditional approaches not only to mathematics (Bayer et al., 2024) but also to mathematics education, which, as hinted above, is struggling through a crisis of its own, socio-economic and political by its nature (Borovic et al., 2022).

For example, Fadel et al. (2015) see the situation of the first decades of this century as "the perfect learning storm", caused by four converging forces: knowledge work, thinking tools, digital lifestyles, and learning research. In fact, today's learners are growing up more and more in a world that is increasingly shaped by digital artifacts (AI technologies included), but many times they have not a social control on them (Benanti, 2022). This trend is making it critical for mathematics education to properly address these new technological realities, going deeply beyond its purely technical features. The same term "computational thinking" is one of the most used (and abused) buzzwords: politicians, ministers, media, educators attribute different meanings to it, some more oriented towards computer science, others more linked to its interdisciplinary value in STEM disciplines. As a consequence, the content of the didactic transposition and determination scales of Figure 5 is going to face a deep change, whose features at macro- and micro- levels are dramatically and rapidly evolving with new productive but previously unsuspected dynamical links between the two. And it is important that mathematics education elaborates suitable epistemological, cognitive, discursive, and instrumental tools to face the new landscape. As pointed out by P. Richard, it is necessary a complete and complex integration of digital artefacts into mathematical activity:

> [It] goes beyond merely facilitating or accelerating tasks; it transforms the nature of mathematical work itself by opening new pathways for problem-solving, exploration, and learning. The human-machine interaction resulting from this integration produces new forms of mathematical activity […] including the dynamic interplay of representations, automated calculations and reasoning, exploratory conjecture, instant feedback, and the cognitive impact of artifact interactions. (Richard, 2024, p. 2214)

This observation points out the complexity of the interactions that this integration entails, provided it must not be reduced to purely technical interventions. For example, it changes the usual "didactical triangle" of Brousseau (knowledge-teacher-students), which, through the instrumentation approach (Artigue, 2002), had been able to integrate the old digital artefacts into its frame. With AI, especially with generative AI, represented e.g. by the interactions with chatbots, things have deeply changed: the interactions happen among four subjects, of which one is not human but has some not so clear resemblance with them. What seems necessary from an educational standpoint is an integration among the different components, considering in particular the specific complexity given by the two forms of AI (symbolic and statistic), which both can be important in fostering and enhancing mathematical explorations and validations but, at the moment, seem limited in performing a full mathematical reasoning.

Some people use terms like "hybridization", "synergy", "technological counterpoint", and the like to indicate the particular new form of integration between human and technology in the new classroom: each underlines a specific feature of the interactions, highlighting this or that particular aspect. Possibly, this variety of expressions indicates that the theoretical elaboration is not yet satisfactory and needs further thoughts and deeper analysis, as it is natural in a situation of fast and radical changes, such as those we are considering. In any case, the literature is beginning to provide concrete examples of integration among the different types of AI for educational purposes (for a survey see Richard, 2024). This "integration problem" is one of the crucial current problems that mathematics education has to face and solve. Together with the issues of "citizenship awareness", introduced above with the ice rink example, they constitute the new stuff upon which new mathematics curricula will be built with fresh and deeply intertwined macro- and micro- facets of the didactic transposition/determinacy scales.

Hoping to give a small contribute to this issue, I will conclude giving an example of an ongoing research I am participating to, and which tries a specific form of the integration hinted above, with some initial consequent theoretical elaboration about the new forms of interactions in the class triggered by some current AI technologies. Of course, one must be aware that since the performance and reliability of LLMs (Large Language Models) are increasing at an astonishing rate, what one writes today may be scaled back or become obsolete tomorrow.

A teaching experiment

Such devices like ChatGPT and chatbots in general have assumed great resonance in the public debate and their use in the classroom can be motivating and productive for the students. A useful methodology may be using them in mathematics teaching/learning as an additional voice that join that of the teacher, and of the students (Conte & Cusi, 2024).

However, ChatGPT, like current generative artificial intelligence systems, is "intelligent" in a different way from human beings: in a sense it is an "alien" voice. Its answers, contrary to those provided by symbolic artificial intelligence, are grounded on a generative AI, hence do not base on a formal theory. For these reasons, as it is well known, even a highly performing system like ChatGPT risks falling and falls into mistakes. These, when detected, can give useful suggestions to those who follow carefully, with intelligence, its ability to produce also fallacious arguments.

It is possible to base on these observations to design a didactical approach to proofs in geometry (Paola, 2025). With a group of Italian researchers and teachers[7] I am now working to a pilot teaching experiment in a class of higher secondary school (10th grade). The didactical scenario involves experimenting with "open problem" activities, in a geometric context in which small groups of students are asked to carry out explorations of geometric configurations. The explorations are made first in paper and pencil environments, then with GeoGebra in such a way as to detect regularities and produce conjectures, validating them with the tools made available by GeoGebra (in particular the `Relation` command[8]), in such a way as to convince themselves of their validity. After this first part of activities, students are asked to interact with ChatGPT trying to detect why the conjectures produced do work or not. ChatGPT, however, may stumble when asked to explain why and sometimes gives answers that are poor or even contradictory; it is the students' task to properly interact with ChatGPT in order to detect, like Sherlock Holmes, when this is the case and why. We have called this an "epistemic challenge" that they have to overcome. After having interacted with ChatGPT, the different groups write in a Padlet the full path they followed for the solutions: its content is then shared in the whole class (Lemmo et al., 2024). In the end, there is a whole class mathematical discussion orchestrated by the teacher (Bartolini Bussi, 1996; Giberti et al., 2025), where their findings are discussed and critically analysed with the teacher.

The aim of our research was to check, through the development of teaching activities like the one above, whether a suitably integration of a geometry software like GeoGebra (which has graphical, numerical, and symbolic features) with ChatGPT (generative AI) can be used to facilitate students' introduction to theoretical knowledge in geometry. The proposal is based on the idea that ChatGPT can and should be used in mathematics teaching and learning as an additional *voice* alongside that of the teacher and the students. It is a particularly significant voice for at least two reasons: first, ChatGPT has gained considerable resonance in public debate, and therefore there is a good chance its use will be motivating for students; second, it is an alien voice. ChatGPT, like current generative AI systems, is intelligent in a profoundly different way than humans. Its answers, unlike those provided by symbolic artificial intelligence, do not refer to a theory: as P. Benanti writes, with

[7] We call ourselves "The skew group" (our logo is a skew quadrilateral). Its members are: F. Arzarello, S. Bagossi, S. Beltramino, P. Cazzaninga, F. Ferretti, C. Giberti, D. Molinari, D. Paola, C. Spagnolo, E. Taranto.

[8] The command previously was purely numerical (e.g. it verified numerically the validity of a property within a certain threshold, the user being warned in a message that the reached conclusion was only numerically valid), now can also activate a symbolic calculation translating the given geometric figure into a collection of polynomial equations and considering, systematically, as a thesis, the algebraic translation of the possible relations one has referred to, between the chosen geometric elements (Kovács et al., 2022, p. 26).

generative artificial intelligence systems "we are witnessing the birth of a new epistemological paradigm, moving from causality to correlation" (Benanti, 2022, p. 82, translation by the author).

Now I will sketch some first provisional results we have found with the first activity of the teaching experiment. Students were asked to investigate the properties of a quadrilateral whose vertices M, N, P, Q are the midpoints of the sides of an arbitrary given quadrilateral ABCD. They were (hopefully) supposed to find and prove that these always form a parallelogram[9]. The justifications (the groups talk about "proof") were always given with the help of GeoGebra (with measurements and with the "Relation" tool, both with numeric and symbolic checks) and only in one case a group proposed a real proof of the theorem in a particular case (when ABCD is a rectangle) using the congruence criteria of triangles. At this point students were asked to engage in a dialogue with ChatGPT to move from their (almost) empirical justifications or "zero-knowledge proofs" (in the case of numerical and symbolic checkings with GeoGebra) to overcome the epistemic challenge, that is, understanding "why" MNPQ is a parallelogram, whatever ABCD is. "Understanding why", in fact, means not only learning how to solve a problem or answer a question, but also understanding—in the deepest sense of knowing—why things are in a certain way, and not only "knowing" that MNPQ "is" indeed a parallelogram and being simply happy with that.

I now move on to some observations relating the first result of our teaching experiment, in particular considering the groups' answers in the Padlet. I limit to refer to some of the main theoretical frameworks that we have chosen for our analysis: (i) the instrumental genesis (Artigue, 2002); (ii) the dialogical perspective of knowledge (Mercer et al., 2020).

(i) According to our a priori analysis, the instrumental genesis would have involved the use of GeoGebra commands, of ChatGPT prompts, and of the communication space provided by Padlet, with recognition of their strengths and limitations. We found that the groups feel comfortable with GeoGebra and Padlet, tools they had already used in the past. The instrumentation process seems less advanced in terms of the ability to modify the prompts so that ChatGPT provides answers more suited to the interlocutor's expectations, although at the end of the activity we observed an increased attention to the posed questions and an attempt to modify and refine them to obtain the desired answers.

(ii) In the a priori analysis, we had conjectured that:

- Dialogue is the tool that fosters the exercise of critical thinking, fostering awareness of the weaknesses and strengths of one's own opinions and those of others. The search for justifications shifts from "knowing how" to "knowing why". This competence seems to have grown up in the long run: the posts in the Padlet contain some significant reflections that show the activation of critical thinking in some cases, although less frequently than the evaluation of elements of correctness. Moreover, we have found observations related to the language used by ChatGPT and considerations on the proposed proofs and on their strengths and weaknesses.

[9] It is the well-known problem of Varignon, whose proof was published in 1731 (Oliver, 2001)

- Today, there is a greater need than in the past for the skills needed to understand a text, verify data and their plausibility, and identify false or incorrect information. Here the results are more than positive: the various groups have understood that ChatGPT's responses cannot be taken as valid, but that they must always be subjected to critical scrutiny and careful verification. This is a very important educational objective from the perspective of citizenship education.

- It is through the dialogical practices implemented in working groups and in whole-class discussions that the primary educational objective should be realized: namely, that students truly understand (after having accepted it in the didactic contract) the role and function of theoretical knowledge as an environment in which the meaning of the different "whys" is clarified and one has to attempt answering them. In short, this is the dialogic practice (in the group, with ChatGPT, with the teacher). This objective seems not to have been completely achieved for the majority of the class. We have not found explicit evidence, except in one group, of this tension towards "understanding why", namely towards the dimension of theoretical knowledge, even if it is present early in at least two groups. However, this tension emerges in the final phase for some students: for reasons of space, I report only a couple of quotes from their posts in the Padlet showing that they are on the way of achieving this delicate goal:

S1: This is why, from the class project, I realized that, first of all, although we want to know the solution to the exercise, we must also ask ourselves how and why the problem was solved that way, in order to understand and use ChatGPT as a learning tool and not just as an exercise solver.
[…] Therefore, the software functions as a coach and not as an exercise solver, to better understand the topic.

S2: It is therefore up to the reader to filter the information obtained to understand whether it is correct or not, in two ways: by pointing again with Chat GPT, asking further questions about it and also explicitly asking if it was sure of what it had just said, or by looking it up in the textbook. It is during this verification step that we learn the most, because if the information provided by Chat GPT is correct, we have gained a useful understanding; otherwise, we are led both to find a correct answer and to understand what was incorrect in that given information.

In these quotes (as in others not reported here), the focus on "correctness" is evident, but a more or less explicit attention also begins to emerge towards the communicative aspects and those related to understanding.

So, in short, the analysis of students' productions and especially of their comments on the Padlet indicate that, even if not for the entire class, they have certainly achieved the goal of understanding the need for a conscious and critical approach to AI tools, whose answers must always be verified. They also understood that it is possible to modify the questions to obtain increasingly satisfactory answers. The focus is always on correctness (epistemic dimension), but some initial tendencies emerge in the final responses, primarily regarding attention to communicative aspects, and, in some cases, also to those related to understanding why, even though it seems that there is no clear commitment to what we have called "the epistemic challenge". On the other side, it is important to consider that the challenge condense a family of objectives of high educational value, which cannot be achieved with an activity alone, however targeted, structured, and well-conducted. The fact that, however, because a tendency toward these objectives appears evident, it seems that this approach to

math learning through AI can be successful, even if with some important refinements to the task, to which our group is now working.

Conclusions

Basing on the Italian experiences in the first decades of this century, the paper illustrates how the innovative aspects of mathematics curricula can become an effective shared heritage in the mathematics teaching community, provided they effectively enter into teachers practices and believes. The concrete course of action through which this process is realized is described by means of the meta-didactical transposition process. In general, from the curricular standpoint, the main analysis tool to properly focus these dynamics is the didactic transposition structure. This theoretical lens also allows to enter into the novelties that currently are modifying mathematics and mathematics teaching and learning in the school, because of cultural, technological and socio-economic reasons.

In the second part, the paper discusses how these big changes are influencing the links between the macro- and micro- facets in the didactic transposition scale and are starting to produce consequent changes in school practices and in their curricula, especially for mathematics.

Within this stream of thought, three issues have been introduced as emblematic examples of these changes and of the necessity of properly and concretely rethinking the mathematics curricular transposition scales:

- the growing use of automated and interactive theorem provers in mathematics research, which in the long run will influence (are already influencing) also mathematics teaching practices;
- the way in which computational thinking can properly enter into school with a critical approach to the new technological instances;
- the contribution of mathematics for developing a form of mathematics for world awareness in students within a wider educative aim of developing a general citizenship awareness.

The changes that these issues bring with them constitute just as many points of attack against the many barriers and constraints at the different levels of the codeterminacy scale and that can hinder possible evolutions of teaching and learning processes: these issues are the main problems that researchers, teachers and policy makers face today to properly address mathematics curricula.

The examples discussed in the paper show how “the spirit of the time” is pushing towards strong changes, of which we are only still partially aware. I think that we are in fact starting to live the wonderful experience that B. Brecht (1963, p. 6) describes for Galileo, when the scholar sensed the big changes of the scientific revolution in everyday phenomena:

> When a young man in Siena, I saw how a couple of builders, after a dispute of five minutes, replaced a thousand-year-old system for moving granite blocks by a new and more practical arrangement of the tackle. Then and there I knew: the old age is past and a new age is here.
>
> (translation by the author)

It is an exciting challenge, in which all of us are called to participate with intelligence and creativity. I think that the motto with which the article starts, *looking backward for going forward*, is more alive than ever and that we, as mathematics education researchers and teachers, must keep it well in mind

to avoid the mistakes made many decades ago with the infatuation for the so-called “new math” in teaching.

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Climate change and the mathematics curriculum: gestures and entanglements for the future

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Mathematics can be taught as a neutral and value-free enterprise, disentangled from the living world and focusing on non-complex problems with only a single solution. We consider what entanglements mathematics education could respond to, in the context of a world on fire. These include the entanglement of humans and ecosystems, the entanglement of people across the globe and across time, and the entanglement of values, hopes and climate change from a critical, democratic perspective. We discuss how these can be connected to mathematics and mathematics education, and how the entanglements may be overlapping and contradicting. The Norway and the England curriculum are analysed in terms of what values and entanglements they convey, related to climate change, and what kind of citizen is imagined. Finally, we offer examples of projects which point to a curriculum for the future that acknowledges the proposed entanglements, and which can foster hope.

Keywords: *socio-ecological; hope; democracy; values.*

Introduction

In a tragically prescient article, Latour (2004) begins "Wars. So many wars." (p. 225). Latour proposes that, as academics, we have been too slow to adapt the focus of our work and critique to the changing circumstances of the globe, lamenting: "Are we not like those mechanical toys that endlessly make the same gesture when everything else has changed around them?" (Latour, 2004, p. 225). In this article, we want to consider what "gestures" mathematics education has made (we define the term in the next section) and what alternative gestures we could make, in the context of a world on fire (Delpit, 2021), particularly in relation to curriculum. Following this introduction, our writing takes the form of a dialogue—with voices indicated by the major headings. We conclude with a final section in a joint voice. We have chosen a dialogic form of writing, in part because this writing has arisen from a plenary talk which involved the two of us in a dialogue and, in part, because a belief we share is about the importance of dialogue in relation to the content of this writing (Barwell et al., 2022).

A word that will recur in both our sets of writing is that of entanglement, and hence we say briefly what we take this to mean. Entanglement is a concept used in quantum physics (Barad, 2007) which acknowledges the deep and ineradicable interconnection across phenomena often kept apart in more classical thinking (such as culture/nature, mind/body, inner/outer, organism/environment). We use the word in this metaphorical sense. In writing about entanglement, we are asserting something about the common materiality of the binaries just listed (see de Freitas & Sinclair, 2014). We also want to draw on the way the word entanglement, in common language, can point to connection and complication, to joining and to the erecting of barriers.

Mathematics can be taught as a neutral and value-free enterprise, disentangled from the living world and focused on non-complex problems with only a single solution. We do not consider such an approach to be befitting of the world as we now experience it, nor of the needs of students, let alone possible needs of society. We have both been influenced by the insights of critical mathematics education (Skovsmose, 1994) into the way that mathematics formats the world, influencing the very realities it models and predicts. Mathematics has the power to shape worldviews. Skovsmose (1994) called this the *formatting power of mathematics* and, later, the *performative interpretations of mathematics* (Skovsmose, 2020). And, following from the insight about mathematical models and technologies formatting the world, it becomes clear that ethics and values are part of the doing of mathematics and the responsibility of mathematicians.

As well as our focus on curriculum, we consider the particular issues of climate change, climate justice and ask what they might mean for the mathematics curriculum and whether the mathematics curriculum has anything to offer in relation to them. Our broader concern is with the socio-ecological pressures facing the world, of which climate change/climate justice are examples. And, we want to suggest that mathematics has a specific responsibility in relation to climate, such that education about climate cannot just be left to the sciences or other subjects. In particular, mathematics is applied to describe, understand and communicate climate change, its consequences and possible solutions (Barwell, 2013). A range of mathematical representations are used to communicate issues of climate change: numbers, tables, graphs, charts, Sankey diagrams, colour maps of the world, fractions of areas, coloured climate stripes. As citizens, students need to be acquainted with these mathematical representations to understand what is communicated. Mathematics is a tool needed to understand the world. Students need mathematical competence to achieve insights into climate change issues, either from the media, or work with data on the internet or collect data themselves. Climate sceptics and deniers point to "flaws" and limitations of climate models, avoiding communicating uncertainty in presenting their alternative understanding. There is fake news related to climate change and mathematics is needed to discern levels of manipulation.

With these ideas in mind, in the next section, Alf outlines the notion of a gesture and the three gestures for mathematics education, proposed by Coles, Solares-Rojas and le Roux (2024). Kjellrun then responds, adding two further gestures, which she sees as missing, relating to democracy and hope. Kjellrun continues, by looking at the current Norwegian curriculum, in relation to the five gestures we outline. Alf does the same for England and considers similarities and differences across the systems. We then look to the future (with sections in each voice) and consider and exemplify possible gestures. We argue that every mathematics classroom is making a gesture (more or less explicitly) around values—hence the need for reflexivity about the meta-messages of our own practices, and our instantiations of mathematics, and the values of the institutions where we work.

Socio-ecological gestures of mathematics education (Alf)

Influenced by the pioneering work of de Freitas and Sinclair (2014), Coles et al. (2024) propose the term "socio-ecological" to point to the attempt to bring the ecological into our concerns as mathematics educators and to recognise that much (but by no means all) of our research has taken the living world as a stable and benign background to our concerns. Such assumptions, where they exist, are surely no longer tenable; we are perhaps in a situation a little like a scene from a Star Wars film

when the characters land on what they think is an asteroid, only to find they are inside an enormous creature. If we did ever think we were conducting our work on a stable surface, it is now clear that the surface is alive and has begun to move. The three "gestures" proposed, which I will elaborate briefly, are: listening for socio-ecological entanglement; attending to the scales of socio-ecological entanglements; and living entanglement (Coles et al., 2024).

The first gesture, listening for entanglement, "focuses on where we place our attention as mathematics educators […] Listening to entanglement will necessarily take our attention away from a sole focus on mathematics and on education and, we believe, can enrich both in the process" (Coles et al., 2024, p. 177). This gesture might be as simple as acknowledging how we are entangled, even if our concerns remain centrally on, say, mathematics learning. In other words, to engage in this gesture might imply becoming conscious that we are bracketing out the "ecological" in order to focus on purely "social" or "socio-political" concerns, in our mathematics education research. This gesture might also involve sitting with, and dwelling in, the potential consequences of recognising that even something as seemingly abstract as learning (say) the calculus in a classroom is a socio-ecological event, an event with ecological (as well as socio-political) implications, an event imbued with values (e.g., precision, coherence, logical reasoning).

The second gesture, scales of entanglement, involves "bringing to awareness from where we are standing, when we attend to aspects of the landscape (both literal and academic), as mathematics educators" (p. 177). This gesture might be expressed through attention to spatial and temporal scales. Spatial gesture(s) imagined here can arise from the disjuncture that is often experienced between the local and global, for example, when the historical accumulation of local actions has had a planetary impact, and yet, as individuals, it seems we are powerless to effect change. How might we work with students while acknowledging such tensions? Alternatively, considering temporal scales implies attention to the past, present and future of any place, context, question, issue. When a school sits on a polluted river, we might consider the oral histories of elders (past), the current levels of pollution (present) and potential routes back to health (future) (Solares et al., 2022). Attention to temporal scales implies the need for activism as a key element of mathematics education. Again, values are implicated in this second gesture, in the move into activism.

The third gesture (or set of gestures), living entanglement, raises ontological and epistemological issues arising from what it means to take seriously the unity of the social and the ecological. If we really believe there is no discontinuity between culture and nature, how does that change how we think about students, and about mathematics and about learning? What does it mean, for example, to acknowledge the materiality of a mathematical concept (de Freitas & Sinclair, 2014), or the culture present in plants? And there is caution needed here. Bullock (2025) warns that too quick a move to extend human concerns to the non-human risks an unquestioned racism by not acknowledging those humans who have never been offered fully human status within colonial, capitalist institutions that span international, national and local scales. Living entanglement implies an ongoing gesture of openness to noticing our own blind spots and a solidarity (Povey & Angier, 2024) with all those (human and non-human) marginalised by historical and present injustices. Yet again, within this gesture, there is a surfacing of values and a need to stand for some values and against other values.

Gestures of mathematics education relating to climate change (Kjellrun)

In this section, I continue to think with the notion of gestures, but now particularly applying them to issues of mathematics education and climate change. In doing this, I have identified two further relevant gestures, democratic entanglement and entanglements of hope—ideas which I see as potentially missing from the three gestures previously outlined.

Starting with democratic entanglement, this gesture acknowledges that mathematics students (and teachers and researchers) may have very different worldviews and thereby very different understandings related to climate change. Because climate change issues are often complex and controversial, there may be disagreement on what the problem is and what should be considered as appropriate solutions, so-called wicked problems (Rittel & Webber, 1973). Whether solutions are considered preferable or not depends on people's values and worldviews.

Much of my work is rooted in post-normal science, which is a philosophy of science that opposes the idea that science and mathematics alone can provide the best answers to complex policy problems, i.e., in cases where knowledge is uncertain, stakes are high and decisions are urgent, such as climate change (Funtowicz & Ravetz, 1994). Because what is considered a good solution also depends on opinions on what is at stake, Funtowicz and Ravetz (1994) argue that quality in science for policy also depends on how values are handled and represented in both how the problem is posed and how it is analysed, including mathematically. A certain choice of simplifying or limiting the scope of a problem through mathematics may benefit one stakeholder over another, which means that knowledge becomes value-laden.

This implies that the process of peer-reviewing expert knowledge is not sufficient for assessing the quality of science for policy. To answer this challenge, Funtowicz and Ravetz (1994) suggest what they call an extended peer-review process, where stakeholders and citizens can contribute by presenting and discussing other forms of knowledge and worldviews, supplying issues on risk and associated risk bearers, and evaluating the relevance of expert knowledge. A range of frameworks have been developed and applied to facilitate extended peer review communities, all of which are participatory and to various degrees deliberative (for an overview, see O'Connor & Douguet, 2024). Similarly with other forms of democratic participation and representation, there are benefits and challenges with extended peer review processes.

There are several connections between post-normal science and mathematics education in relation to climate change. First, both Funtowicz and Ravetz started their careers as mathematicians, and they have given substantial attention to the limitations of mathematics as a tool to cope with complex, wicked or post-normal problems, also on how numerical results could be presented to communicate uncertainty aspects where statistical measures do not represent uncertainty sufficiently (see e.g. Funtowicz & Ravetz, 1990). Second, several scholars within critical mathematics education have pointed to hidden values in mathematical models applied in society (e.g. Ernest, 2018; Skovsmose, 2020). Third, the idea of critical democratic citizenship, proposed by Skovsmose (2023) and others, where the idea is for the students to get acquainted with the benefits and limitations of mathematics in society in order to act on unfairness or what may be harmful in some sense, is quite similar to the idea of extended peer communities (Hauge & Barwell, 2017). Fourth, both post-normal science and

critical mathematics education appreciate the resources laypersons/students bring to the table/class, valuing their views and experiences (Hauge, 2024).

Taken together, the gesture of democratic entanglement can be understood as consisting of two qualities related to democratic citizenship. One is acknowledging the various worldviews of students, teachers and researchers, and nurturing democratic discussions and virtues in mathematics classrooms. The other quality is acknowledging that mathematics is a powerful tool for exploring and understanding climate change, but acknowledging that mathematics can also conceal embedded values in mathematics-based argumentation. This entanglement thus opposes any gesture that mathematics is a value-free enterprise which fosters rational reasoning, as an ideal for democracy and citizenship (for further discussion of citizenship and democracy, see e.g., Chronaki, 2024).

The second further gesture I offer, which again I relate specifically to climate change, is one I label the *entanglement of hopes*. Surveys show that many are worried about climate change. For example, 56% of the survey respondents in Norway answered that they were worried or very worried, and youngsters seem to be more worried than the rest of the population (Gregersen et al., 2023). This is not a trend particular to Norway. Hickman et al. (2021) found that 59% of the 16–25-year-old respondents from ten countries were very or extremely worried, varying from 46% in both Finland and the USA to 84% in the Philippines. Further, youngsters seem not to feel listened to and thereby become disillusioned about adults' ability to solve the climate change crisis (see e.g. Jones & Lucas, 2023). This suggests that they have lost hope of achieving a sustainable future.

How much people worry depends on their beliefs about the causes and consequences of climate change (see e.g. Gregersen et al., 2020). Snyder et al. (2001) operate with three aspects of hope when they discuss the problem of inaction. These are goals, pathways and agency. Relating this to climate change, people have different goals for the future concerning climate change, depending on their beliefs and values. Even in cases where the goal of a sustainable future is the same, people can picture different pathways to reach the goal. For example, techno-optimists will have certain suggestions for how to reach a sustainable future. Concerning agency, research finds correlations between people's environmental engagement and agency (see e.g. Ojala, 2023).

A central question is therefore what gestures mathematics educators can make, which might foster students' hope, when the problems they face are wicked and the future looks grim. A range of concerns need to be balanced, and here follow five suggestions. One is that information about climate change, and why there are reasons to worry, is necessary to awaken a wish to act, but at the same time, it cannot be too overwhelming so that students become apathetic and lose hope. Ojala (2017) uses the term "critical hope" to emphasise the need to both acknowledge the crises and to envision a different future. By introducing complexity to the issues of climate change, more realistic pathways of hope can be developed, enabling students to see the issues from different perspectives to understand others' opinions and suggested solutions. Satterfield (2000) argues that hope is most developed when one has this ability. The complexity requires mathematical competence, including understanding the strengths and limitations of mathematics, when exploring topics through mathematics-based argumentation in reports or in the media, or through applying mathematics themselves. Second, teachers need to be aware that students wish for different futures, so that fostering hope in relation to climate change might collide with other goals which students find more

important. They may not worry about a non-sustainable future. Third, even though students may have the same goal of an environmental-friendly future, they may have different ideas and beliefs on how to reach this goal. Fourth, students may have very different emotions connected to climate change, which requires a particular awareness among educators. A study by Ojala (2021) shows that teachers have different attitudes to climate change emotions in the classroom and have different strategies of dealing with them. While some teachers regarded emotions as irrelevant to teaching about climate change, others regarded all emotions as potentially constructive in mobilising an interest in climate change or willingness to make a change. However, these teachers emphasised that how to cope with emotions is highly situational. Both the first and the fourth point align with the democratic entanglement as they require a deliberative approach in the classroom, combined with critical thinking related to the topic and the applied mathematics. Yet, deliberation and democratic practices can result in opinions or actions that contradict climate change. Fifth, agency is considered key to hope, and studies suggest that people engaging in collective action have more hope (Lorimer et al., 2020).

Hope is an important topic, both for students' well-being and for encouraging environmental-friendly actions. Yet, the significance of values and worldviews must be taken into account in classrooms to strengthen students' values that align with a sustainable future, but also avoid the suppression of the students who do not think that actions need to be taken to reduce climate change.

Summarising the five gestures

We will be using the notions of the five gestures in the remainder of this paper; hence, I summarise them here, giving some examples, before we each use them to consider curricula of the present and the future. The first entanglement is about how we all, teachers and students, are entangled in the issue of climate change, both as being part of nature, e.g. experiencing weather changes, and as members of society, e.g., through politics and regulations. The first gesture is about mathematics education listening for such entanglements.

The second entanglement denotes how we all are being part of nature and connected to other people, both at present and in the future, locally and globally. The second gesture is about becoming aware of the temporal and spatial scales of our entanglement, exploring these relationships (across humans and non-humans) and how actions have consequences at these various scales. We are wary of the dangers in shifting concerns beyond the human, that there are humans who have never been accorded fully human status (Bullock, 2025). Attending to the scales of our entanglement implies a refusal to flatten the experiences of humans and a reckoning with the racism inherent in capitalism.

The third entanglement (and we change the order of presentation from above), the democratic entanglement, is about bringing awareness to how people may disagree on issues of climate change and how mathematics-based argumentation plays an important role in discussions and decision-making. The third gesture is therefore to foster democratic virtues and critical thinking. This implies that students use mathematics as a tool to understand climate change and to provide argumentation, but also explore mathematics-based argumentation in society and what values they are based on.

The fourth entanglement is about the entanglements of emotions related to climate change. Because many worry about the future, a needed gesture in mathematics education is to listen to students and

respond to their emotions regarding climate change. This requires that students have knowledge about climate change (to avoid false hope) and a sense of acting together. An act or actions could be a classroom discussion, where students' voices are heard; a letter to a politician, or an op-ed in a newspaper; or making a practical change, such as reducing the school's CO_2 emissions.

The final entanglement, the living entanglement (put last because it is the most philosophical), is about ontological and epistemological entanglements when taking the other entanglements into account in mathematics teaching and learning. A gesture this entanglement requires is a move from mathematics being a neutral enterprise. It may also include understanding nature as having agency and navigating the complex ecologies binding human and more-than-human life. Another gesture is understanding the relationship between teachers and students as more equal. The complexity of the climate issue makes it impossible for a teacher to be an expert, and the teacher may be unaware of a range of values. This means that students' voices are needed, for their knowledge, perspectives, ideas, experiences, emotions and their values. Students can contribute with something new and something other than the teacher, so that the teacher and students can explore and learn together.

Having set out these gestures and, given our focus on curriculum, in the next section, Kjellrun reflects on the Norwegian curriculum. The aim here is to look at the current curriculum in terms of the gestures (and Alf will also consider the curriculum in England in a similar way) and think about what might be missing, before we both engage in thinking about possible futures.

Climate change in the mathematics curriculum in Norway (Kjellrun)

The Norway Education Act states that "The pupils and apprentices shall learn to think critically and act ethically and with environmental awareness. They shall have joint responsibility and the right to participate." The act thus implies that teaching and learning should include sustainability issues, such as climate change. The mention of environmental awareness suggests that gestures of *listening for socio-ecological entanglement* are relevant, and *democratic entanglement* is supported through highlighting students' critical thinking and participation. It also allows for uniting the social and the ecological, which aligns with our understanding of *living entanglement*.

The statement in the act is further elaborated on under *Respect for nature and environmental awareness,* one of the six core values of the education and training in the Norway curriculum (Ministry of Education and Research, 2017). Statements include "Human beings are part of nature and are responsible for taking good care of it", which can be understood as a gesture that embraces the *socio-ecological entanglement*. The two quotes "The pupils shall develop awareness of how our lifestyles impact nature and the climate, and thus also our societies" and "Children and young people will need to deal with the today's and tomorrow's challenges, and our common future depends on the coming generations and their willingness and ability to protect our world" (Ministry of Education and Research, 2017, p. 9), both relate to *entanglements of scales*, the first regarding geographical scales and the second regarding time.

Another core value in the Norway curriculum is *Democracy and participation* (Ministry of Education and Research, 2017). Two of the associated statements are: "School shall promote democratic values and attitudes that can counteract prejudice and discrimination. Pupils shall learn in school to respect the fact that people are different and learn to solve conflicts peacefully" (p. 9). Climate change can

be argued to be a relevant issue for fostering this core value, as climate change issues are often associated with prejudice and conflicts. The core value emphasises promoting democratic values, which is in line with what we have called *democratic entanglement*, but it can also be associated with *entanglements of emotions*, as learning to solve conflicts peacefully can foster hope. The text explaining the core value also highlights the need to "develop awareness of minority and majority perspectives and ensure that there is room for collaboration, dialogue and disagreement". An aim of this value is to "prepare the pupils for becoming responsible citizens in society" (p. 10). These pieces of text can also be related to climate change, and they align with *democratic entanglement*.

The core values in the Norway curriculum show that teachers are encouraged, and partly obliged, to include activities that align with all the gestures we have proposed in this paper. The curriculum is quite radical. In a blogpost (Hauge, 2022), I reflect on a paradox it represents, as it aims at fostering students to become something different from the societal environment they grow up in. As mathematics educators, we need to start thinking about how to cope with this paradox.

While the core values reflect all five gestures, the mathematics curriculum does not reflect them distinctly. However, there are some statements and elements that are relevant for democratic entanglement and working with climate change. Examples include "Mathematics is an important subject for understanding the patterns and relationships within society and nature through the use of modelling and applications" and "Critical thinking in mathematics includes critical evaluation of reasonings and arguments and can arm the pupils to make their own decisions and take a stand on important questions in their own life and in society" (The Norwegian Directorate for Education and Training, 2019, p. 2). The mathematics curriculum specifically addresses its role regarding the topic of democracy and citizenship. It explains that it includes "giving the pupils the competence to explore and analyse findings from real datasets and data collected from nature, society, working life and everyday life" and "learning how to evaluate the validity of such findings" (p. 4). The aim of such activities is "to formulate one's own arguments and participate in public debate" (p. 4d). A competence aim worth noting is for year 9: "interpret and critically evaluate statistical representations found in media and the local community" (The Norwegian Directorate for Education and Training, 2019, p. 14). The mathematics curriculum thus explicitly includes the gesture of handling *democratic entanglement* through critical thinking on societal issues.

Climate change in the mathematics curriculum in England (Alf)

The current (in 2025) National Curriculum in England was introduced in 2014. The curriculum before then had emphasised skills and competencies. The changes made in 2014 were explicitly about moving towards a view of "knowledge" not skills as being central to education, as reflected in the aims, which are to provide: "pupils with an introduction to the essential knowledge they need to be educated citizens. It introduces pupils to the best that has been thought and said, and helps engender an appreciation of human creativity and achievement." (DfE, 2014).

There was criticism of curriculum documents for their lack of reference to climate change; such criticism was rebuffed by the Department for Education (DfE) during consultation (DfE, 2013). The result is that the first reference to climate change appears in the curriculum for 11-14 year olds, and the only reference is in the statements for Geography and Science, and without any links to wider

social justice issues relating to climate change. The word "climate" appears twice in the Science curriculum for 15-16 year olds (around the "potential" effects of greenhouse gases and "evidence and uncertainties" about human-caused climate change), and three times in the Geography curriculum at the same age. The insufficiency of this situation led to belated recognition of the importance of climate change education after 9 years. In 2023, a government strategy document was published on "Sustainability and climate change" for education (DfE, 2023). The "vision" of this document is that "the United Kingdom is the world-leading education sector in sustainability and climate change by 2030", although no changes were made to the national curriculum. As part of this strategy, a "National Education Nature Park" was launched in October 2023. This initiative is focused on students taking action "to engage with nature in their local areas" (DfE, 2024). One aim is that schools turn some of their grounds to becoming a nature park. A new government in the UK (elected in 2024) is undertaking a wide-ranging curriculum and assessment review—they have promised "evolution" not "revolution" and so it seems unlikely sustainability will feature centrally.

The curriculum in England that was introduced in 2014 represented a backwards step in relation to the extent to which climate change and questions of sustainable development were present across ages and subjects. There is, perhaps, a salutary lesson here that there is no inexorable bending of policy towards a broadly liberal or even scientific consensus.

In terms of the mathematics curriculum, as can be implied from the discussion above, there is no direct mention of climate change or sustainability. Perhaps the closest approach is the mention of "modelling" in the curriculum for 11-14 year olds ("begin to model situations mathematically and express the results using a range of formal mathematical representations") and 15-16 year olds ("model situations mathematically and express the results using a range of formal mathematical representations, reflecting on how their solutions may have been affected by any modelling assumptions") (DfE, 2014/2021 update). There is also limited mention of data literacy and data science, which, along with the study of statistics, could offer scope for considering climate change as a context for learning. There is, therefore, little in the curriculum itself which aligns with any of the gestures we have outlined. Of course, mathematics teachers in their practice may well be engaging in gestures such as *listening for entanglement*, or *democratic entanglement*, but such gestures would seem to me to be taking place in spite of, not because of, the curriculum.

A report by The Royal Society (2024), in the UK, set out a vision for how the mathematics curriculum might change. A key suggestion of this group is combining mathematics and data education, proposing far more use of technology and a closing of the gap between how mathematics is used outside school and what happens in classrooms. The word "climate" has eight mentions in this report and so, although not central, is certainly within the scope of the new plans. It is not currently known to what extent this report will influence the curriculum review.

Reflections on the curricula (Alf)

In this section, I look across the two curriculum contexts above and draw out possible implications. I consider the vexed issue of how and when teachers can take a political stance. The contrasts between England and Norway's curriculum are stark, with the Norwegian case much more closely aligned to UN initiatives such as Action for Climate Empowerment. The Norwegian curriculum includes several

topics (not present in England) with explicit references to sustainability, climate, and criticality. The contrast can also be observed in the aims of education, which for Norway include mention of environmental awareness, compared to a focus on knowledge in the case of England. Gestures relating to *listening*, *democracy*, and *hope* seem visible in Norway's curriculum, which are barely present in England. Starting from these two examples, I want to consider possible aims of a future (mathematics) curriculum, in relation to the five gestures. One of the few commonalities across England and Norway is that, for both countries, the aims of education are linked to the notion of "citizen". In England, the aim is to create "educated citizens", and in Norway, questions around sustainability are linked to the need for "democratic engagement from citizens". So, in discussing the aims of education, we might first need to consider what concept of citizen is being invoked.

Chronaki (2024) has offered a powerful and original analysis of the concept of the "citizen" through time and how this is relevant to the role of mathematics education. Chronaki (2024) traces a move through ideals of a "competent" citizen (in Antiquity) to an "insurgent" citizen (of Modernity) to a "creative" citizen of current (posthuman) times. In each case, she charts the paradoxes or antimonies associated with each image of citizenship. For example, in ancient Greece, citizens were required to develop ethics and wisdom, to partake in democracy. And this requirement for competency served as a means of exclusion of the marginalised. Chronaki traces a direct link to a mathematics classroom focused on consensus and establishing "norms" (as might be seen to be needed in gestures towards democracy), where the space for disagreement and subject agency is reduced (particularly for certain subjects). A key conclusion (Chronaki, 2024) is that the concept of the citizen is politically inflected and, without careful scrutiny, any call for mathematics education to support citizenship: "converts, easily, into a legal tool that, alongside mathematical biopower […] governs people, imposes enclosures, and justifies injustices" (p. 16). Such "biopower" can be observed in the use of mathematics in algorithms for allocating resources (Skovsmose, 2023).

Instead, Chronaki proposes bringing attention to the paradoxes inherent in the concept of citizenship (paradoxes with relevance to living and democratic entanglement), and aligning with grassroot activity that counters oppressive logics of control. Chronaki's (2024) proposals are ones "requiring active/passive participation in political activity that interrogates the institutional legitimisation of injustice always in alliance with people and their social movements" (p. 16). So, when Norwegian and English curricula call for the development of citizenship, it is important to interrogate what image of citizenship is invoked and what paradoxes or tensions are entailed.

The English curriculum advocates for "educated" citizens, educated with the knowledge of the best of what has been "thought and said". Norway calls for citizens able to partake in democracy. Particularly in the English case, I see connections to Chronaki's notion of the "competent" citizen and, following her argument, it is pertinent to ask who are the subjects deemed not to be educated, or without the "basic tenets" that allow them to partake in democracy. In explicitly aiming for a vision of a competent citizen, there is the potential for a legitimisation of the power to exclude individuals and groups who are deemed not to have reached the necessary level of competence.

Chronaki's (2024) work draws on other scholars (e.g., Gutiérrez, 2013; Valero, 2004) who raise questions of politics in relation to mathematics education. And, in considering questions of climate change, it is impossible to avoid questions of politics, both at the macro level of curriculum and the

micro level of the classroom. For example, we are conscious of research into the role of right-wing think tanks and lobbying groups, which are reported as having an enduring influence on education policy in the USA (Foster & Davis, 2004; McDonald, 2014). And that such influence is spreading (Applebaum, 2023). The 2014 curriculum in England was initiated by a right-wing government, explicitly wanting to re-dress a previous competency-based approach. One element of right-wing education policy tends to be the attempt to make teaching "neutral" or "a-political".

Of course, what it means to advocate for "a-political" teaching is an acceptance of the status quo. And while this could seem desirable in times of peace or stability, in relation to the ever-worsening climate chaos being experienced around the world, I want to suggest that, in contexts where they are expected to be a-political, there are acute pressures placed on teachers. In England, for example, DfE guidance (from 2020) banned the use of teaching materials produced by organisations which advocated an end to capitalism (such groups being considered "extremist"). It therefore becomes a potentially career-threatening decision by a teacher to raise questions such as whether there should be an end to the quest for economic growth. Anecdotally, some secondary schools in England are explicit about steering teachers away from discussion of current political issues, partly, no doubt, through fear of media reaction. Our focus on curriculum in this article means we want to consider other possibilities than what is the standard now. In the next section, I look at some research suggestions about climate change education, and then we both move into reviewing work we have each done and what this might imply for the future, continuing to link to the five gestures.

Considerations in thinking about curriculum innovation (Alf)

In a review of past research, Monroe et al. (2017) identified four themes relevant to effective climate change education: (1) deliberative discussion; (2) interactions with scientists; (3) addressing misconceptions; (4) school and community projects (p. 801). All of these four themes are potentially relevant to the mathematics classroom. In the examples of curriculum innovation that follow, each of these themes is present. In another review of past research, Rousell and Cutter-Mackenzie-Knowles (2020) distinguished approaches to climate change education which are knowledge-based from those which are interdisciplinary, affect-driven and experiential. Their review pointed to the predominance of knowledge-based approaches and called for more of the latter. Again, although they were addressing the broader field of climate change education, it seems such a provocation to think that affect and experiential learning are highly relevant to our concerns in this article.

In thinking through how the curriculum, or the teaching approaches, might need to change, it seems important to recognise that the ethics of action are ambiguous (Boylan, 2016) because we cannot know what the effects of our actions will be, and there is a need to navigate ethical dimensions with conflicting priorities. Barwell et al. (2022) proposed a dialogical ethics as a starting point for navigating this ambiguity—listening to the voice of the Other (whether human or not) and answering. And, in a deliberately circular and mutual formulation, we proposed that it is through being recognised by the Other that we gain our capacity for answering and acknowledging our unique obligations (Barwell et al., 2022). Any gesture of living entanglement implies the need for ongoing reflexive work, to continue engaging in and being mindful of how we are ethically implicated.

Ideas around a dialogical ethics relate to Chronaki's (2024) challenge around the paradoxes of citizenship and Bullock's (2025) concerns around the potential racism that lies uncovered or unnoticed in moves to think beyond the human. One potential avenue for thinking through these challenges is through a questioning of the individualism in our concept of "human". Gestures of listening, scales and living entanglement prompt towards a shift away from thinking about mathematics education as relating to individuated beings. In Coles and Sinclair (2024), we try to think through a notion of mathematical "connections", where there is not the assumption that well-defined individuals come to an encounter. If we imagine an encounter momentarily bringing an individual into being (for the duration of that encounter), how might that change our perspective on learning or curriculum? A shift away from individualism might go with attention to the ecologies of habit in which we are embedded. A moment's reflection on any culturally-typical behaviour we engage in (tea with milk in the UK, black coffee in Italy) will point to how patterned our actions are, through all that has come before us. We live most of our lives in patterns of action we do not even notice (and we could not survive long if we did not do so). It might be said that we enact the patterns of others' actions (Coles, 2017). In other words, we never act as an individual and, furthermore, our actions are part of laying down the patterns of others' actions, with all the responsibility this implies. Such thoughts might lead to exploring what a communal mathematics education and curriculum could look like and what gestures and entanglements might be involved.

A curriculum for an uncertain future (Alf)

In the final two sections (before we conclude in a joint section), we begin looking to the future of the curriculum (first, projects I have been involved with and second, Kjellrun). Having considered the gestures of mathematics education and the gestures visible in current curricula, and some broader considerations around climate change education, what possibilities might there be for the future? I draw on the ideas of gestures and ethics set out above to put forward some proposals, drawing on work in which I am currently engaged.

I was privileged to be invited to do some work for the youth group "Teach the Future" (TTF). Students in England are generally excluded from discussions about their curriculum. TTF is a lobbying organisation, run by school students, that has been successful in gaining the ear of politicians about how the curriculum should change (aiming at *listening for entanglement*). One project they ran was "re-imagining" the National Curriculum. I worked with Sarah-Jane Compton-Williams to look at the Mathematics curriculum in England, for ages 11 to 16 (see, https://www.teachthefuture.uk/tracked-changes-project). This work made me realise just how old the mathematics is, in the current offer. My guess would be that, up to age 11, there is little mathematics taught that would not have been recognizable in England 500 years ago.

Sarah-Jane and I wanted to make space for introducing topics including systems thinking and projective geometry—this latter so that students might have an experience of two systems of thought (Euclidean and projective geometry), which lead to contradictory conclusions (such as lines in the same direction meeting or not meeting at a point), allowing reflection on the role of assumptions. One thing we considered is what might we *not* teach if we want to make room for different kinds of mathematics. And a conclusion we came to, in England, is that there is far too much emphasis on number work. For example, do we really need to teach fraction notation?

In framing our alternative proposal, we drew on Roe et al. (2018)'s work on mathematics for sustainability at the undergraduate level. We proposed moving away from curriculum categories such as "Number" and "Algebra" (as they currently exist in England) and instead proposed the following five categories—each with a guiding question:

Measuring: how big (or small) can things get?

Changing/flowing: how quickly do things change/move?

Mapping: how do things occupy space and time?

Risking: when is a risk worth taking?

Deciding: how do we act ethically?

We imagined work within these areas being guided by those key questions, which could be returned to repeatedly, motivating the study of both contexts and mathematics. In my own research, I am currently trying out this curriculum arrangement and the use of the guiding questions, in outreach work I am doing in a prison, fairly local to where I live (for me, this represents an extension of the *scales of entanglement* in mathematics education to which I usually attend, just in terms of context).

Prison education is one of the most marginalised and under-funded sectors of the education system in England (Coates, 2016). The number of prisoners doing "level 3" qualifications (which allow university entrance) has dropped 90% in the last 10 years (House of Commons Education Committee, 2022). The courses on offer (at level 1 and 2, which would be achieved at age 11 and 16 in school), from what I can see, seem to be experienced (by teachers and prisoners) as little more than "tick box" exercises. My university started an access course for STEM degrees, and I wanted to see if we might be able to run this in prison. After a year or so of chasing around a range of different committees, the answer came back as a "no". There were concerns about how quality assurance might happen and about the status of the prisoners as students. This latter point is interesting in terms of the critique I offered around the "competent" citizen (Chronaki, 2024). Here is a clear example of how it seemed impossible for a university committee to imagine how a marginalised group might be given "full" status, as students, and hence as humans, we might say.

My university has been happy for me to engage in "outreach" work in prison, however, so long as the work is not accredited. I have been trying to "mirror" as far as possible the STEM access course run by my university. As it happens, there are some advantages to not offering the full course, not least that I can offer it for free, rather than costing the prisoners several thousand pounds.

If I reflect on the work in prison, moments that stand out are ones of a prisoner seeming to find joy—and these have all been moments of joy in an element of pure mathematics, in amongst the work in context. A future curriculum is surely likely to question the disciplinary boundaries of the present and be more trans/inter-disciplinary. And there is something that remains significant about disciplinary engagement in mathematics. I interpret the experience of joy (e.g., in one case contemplating how differentiating could be iterated and wondering what would happen for different functions) as a form of *living entanglement*. If part of what we are arguing for is a non-separation of nature and culture, then coming into contact with a mathematical structure (such as differentiation of functions) is as much a socio-ecological experience as, e.g., taking a walk by a river (Coles, 2025).

The final example I want to touch on is work with teachers. At the University of Bristol, for the last 10 years, we have slowly been working on introducing questions of sustainability into our teacher education course (Coles & Helliwell, 2022). As part of one Master's unit, done on the course, prospective teachers have to design a task relating to a "global challenge" (including, but not limited to, climate change), implement the task with a class they are teaching and then reflect. Some of these tasks have been written up in a professional journal (see Brown et al., 2021a, 2021b). We also support teachers to work in collaborative groups, doing action research, for some, with a climate change focus (Coles et al., 2023). In the plenary lecture at CERME, we gave participants time to talk in 3s, partway through, to share their thoughts—with the instruction to engage in active listening, not looking to add your own anecdotes, but supporting the articulation of the person speaking. That experience speaking in groups of three, and supporting each other to articulate ideas, is a small example of a practice we use regularly (established by Laurinda Brown)—in which teachers' learning becomes entwined, in a communal endeavour. I am struck by the importance of listening. So, while in an English context, work on climate change in mathematics lessons in schools has a "one-off" feel, developing skills around listening and dialogue can be developed with any curriculum.

I end this section reflecting further on the presence of the five gestures in the practices above and, also, on the questions and challenges, above, from Chronaki (2024) and Bullock (2025) around the paradoxes of the notion of the citizen, and implications arising from how we frame the human and who is allowed human status. Material and social conditions of life are entangled in prison life (as we point to in *living entanglement*), in the sense that a socially sanctioned decision (to jail someone) leads to their physical detention. The curriculum framing around large questions: "How quickly do things change?"; "How do things occupy space?" does seem to have allowed the study of mathematics to be linked to socio-ecological issues of relevance to learners (*listening for entanglement*). Lessons have included an emphasis on dialogue. Time has unusual qualities in a prison—the repetition of the routines of day—and a sense of wasted time, of time standing still, as prisoners look to their release date. Prisoners have little capacity to make changes to the organisation of their lives. Making sense of mathematics is one (small) area of their lives in which they can, perhaps, engage as equals with one another and with people outside prison—it is the quality of arguments that determines what is acceptable, not social position or status (*entanglement of democracy*). Kjellrun's *entanglement of hope* is also relevant. The men are explicit about engaging in study with a view to their possible futures. It is a different kind of hope, perhaps, to think about climate and other earth systems, but it feels significant, nonetheless.

In relation to the teacher education work I have mentioned, I have collaborated previously on writing where we have argued that our course adopts a "communal" approach to initial teacher education (Coles et al., 2023). Our teacher education curriculum is framed at a "meta-level" around supporting teachers with the skills to learn about teaching (not the details of how they might teach). There is a "relentless consistency" (Brown, 2012, p. 31) to practices of moving from "experiences to issues to actions" (Brown et al., 2021) throughout the course. The course aims to equip teachers with the skills to adapt their practice—ready for the curriculum change to come (which relates to temporal *scales of entanglement*). In focusing teacher education at a meta-level, there is a (democratic) space for teachers to bring their own contexts and histories to university sessions and be supported in acting

effectively in those contexts. The intelligent use of affect is a key to the programme, for instance, in de-brief discussions about lessons, we invite prospective teachers to bring to mind moments which felt comfortable or uncomfortable. Using affect, in reflection, helps to identify the focus for work, often in relation to changing habits and noticing implicit assumptions. Prospective teachers are entangled in each other's journeys of learning to teach through consistent three-way discussions and an emphasis on dialogue and working communally on mathematics.

Perhaps one way to respond to the challenges of Chronaki (2024) and Bullock (2025) is to contemplate what it might mean to move to a position where we no longer demand that anyone needs to prove to us their competence, their rationality, their understanding, their humanity (Wolfe, 2016). Recalling the words of Angier and Povey (2025), who draw on Bridle (2022), instead of asking, "Are you like us?", might we be curious about, "What is it like to be you?", shifting, "from empathy and identity to solidarity, recognizing and valuing other things without having to identify with them as being like us" (Angier & Povey, 2015, p. 105). It is perhaps almost unthinkable how schooling in England might not demand that students prove their competence. Current framings of education (e.g., the curriculum and school inspectorate) emphasise tracking of student progress, with an almost constant assessment of competence. And yet, perhaps looking at practices in the early years (children younger than 5) offers some examples, and inspiration even, where it seems to me quite precisely the case that young children are not required to prove anything about themselves. Inspiration from art education may also be relevant, with a focus on the process, not the product of learning (Hay, 2022).

A curriculum for an uncertain future (Kjellrun)

I now present a project which addresses some of the entanglements and which illustrates some challenges with addressing climate change, or sustainable development in general, in mathematics classrooms. A couple of years ago, four preservice teachers, all Master's students in mathematics education, were asked to plan and conduct a three-week project on sustainable development during their practicum. This task was developed by their practicum teacher and two researchers, Trude Fosse and me. The project was to be carried out in all mathematics, natural science and social science classes, and the practicum teacher gave the preservice teachers a list of relevant curriculum goals in the three school subjects. To make the project relevant for the students, who were 14-15 years old, we asked them to start the project with something close to their everyday life (for more information, see Fosse & Hauge, 2025; Hauge, 2024; Hauge et al., 2024).

Giving the task to develop a project on sustainable development originated from a concern about consumption, emissions, pollutions and the loss of nature. It can thereby be considered a gesture of listening to how we humans, and our mathematical practices, are entangled with nature. So, what gestures did the preservice teachers make?

The preservice teachers chose textiles and clothing as the sustainability topic and developed a range of tasks. They showed documentaries and TV debates on problems related to consumption and working conditions in Cambodia and China; they gave a lesson about UN Sustainable Development Goals, and the students learnt about textiles and pollution. This indicates the gestures of *listening* and of *scales of entanglement* because the focus was on the consequences of consumption, which take place in other parts of the world. Because the three-week project facilitated an awareness of a global

problem which the students, as consumers, could respond to through a change in habits, the preservice teachers made a gesture of *living entanglement*.

The students chose subtopics to work with in groups, and they participated in a debate, a role play on the responsibilities of different actors regarding the bad working conditions in factories. They were to find information on the Internet and back up their statements and argumentation with quantified information. The freedom of choice regarding subtopic and how to get hold of relevant information can be interpreted as gestures of *democratic practices* and *hope*. However, the students did not seem to be given the opportunity to voice their opinions about the issues they were working with, and perhaps only the students who were concerned about sustainable development achieved a sense of purpose with the three-week projects.

Projects on sustainable development, such as this one, are formative and call for change. This is challenging for different reasons, as I will now illustrate. The first task given to the students was to write a reflection note on their own habits and consumption of clothes. Later, the practicum teacher and the four preservice teachers discussed this activity, and one of the preservice teachers, Alex, said,

> Alex: I felt many reflected well on their own habits, and so on, but then I kind of "what is OK to ask?" […] If someone can't afford the clothes they want, it can be hurtful for some. […] It can't be like, "Yes, very good, you donate to the Salvation Army." We're not judging people.

Alex points to two ethical considerations related to the task. First, the students at this school came from families with different incomes. Because clothes are a status and identity marker, clothes could be a hurtful topic for some. Second, the task implicitly conveys a moral stance, indicating that certain consumption patterns and certain handling of clothes are better than others. Bringing attention to this may cause shaming of those who feel their practices are not good enough. This challenge will be present when working with sustainable development with the purpose of changing habits. A consequence is that mathematics educators need to think about how to be prepared for emotions in their classrooms, such as shame, anxiety, anger and hope, and how to prepare preservice teachers for students' emotions. Because how to deal with emotions is situational, there is no prescriptive method that can be applied (Ojala, 2021). Conducting projects that promote awareness and change may demand new kinds of relationships between school, family and enterprises.

Towards the end of the three weeks, the students presented their projects for the class, the preservice teachers graded their work and gave the students feedback. One student group had chosen to work with fast fashion because they were concerned about the negative impacts of this industry. They received a good grade, and part of the feedback from the preservice teachers is the following.

> Drew: […] you have worked well and efficiently throughout the whole process. You cooperated well and made a very nice presentation. […] You appear unbound by scripts, and we can hear and see that you have worked well together and are collaborating. Your academic content is sound, and you are very familiar with the content, are reflective, and have included criteria from all the school subjects in a good way. We sense that you have been very united and agreed on what to do in class and at the presentation. And cooperate well.

Then, one of the other preservice teachers, Billie, continues. She goes more into detail, and in the middle of her feedback, she makes the one and only comment related to the problems the students are raising:

> Billie: [..] And I think it was very nice that you talked about problems with fast fashion […]

This made me realise that we need to think carefully about projects on sustainable development and grading. If concerns about nature, working conditions and opinions are less valued than facts and presentation form when grading tasks, engaged students can lose hope and trust that the school supports them in their engagement. Grading and feedback can thereby be an obstacle to all entanglements and gestures: *listening, scales, democratic engagement, hope* and *living entanglements*. We, mathematics educators, need to resolve this obstacle.

Although all the gestures were present in the textile project, there are ways to make them more apparent. For example, discussing solutions and finding ways to act might both have strengthened hope and the gesture of living entanglements. In addition, discussing what is possible, what is desirable and why change is challenging could emphasise the gesture of democratic entanglements. Valuing students' opinions and practising democratic virtues would be vital, together with addressing structural obstacles to change, moving the essential responsibility from the individual.

The following project illustrates the qualities of gestures of democratic practices. Three researchers, a PhD student and a Master's student, went to a town in Lofoten to arrange a classroom discussion with 13-14-year-olds on whether to open the area to petroleum exploitation (for more information, see Breivega & Rangnes, 2019; Herheim et al., 2022). There has been a public and political debate about whether to open offshore Lofoten areas to the oil industry since the 90ies. So far, it is banned because of the important fishing grounds. The scenery in Lofoten is stunning, so tourism is also very important for the area.

The students first worked in groups to develop argumentation, followed by a plenary discussion. The following excerpt is from the last part of the plenary discussion. Previous to the excerpt, some students have argued that if the area is opened to petroleum exploitation, their hometown will grow so that it will need a shopping mall.

> Melissa: You talked about getting a shopping mall here. It's the same thing as if you would remove the Eiffel Tower in Paris and rather build a shopping mall. Then tourism there would drop at once, and the same can happen here. That if we get big oil installations and a shopping mall, and we only have industry here, then it will drop here, too.

A couple of minutes later, Roy declares,

> Roy: I think, personally, that change is the best for Vestvik right now. We need to take a few chances, because now we have lived on fish and dried fish, and tourism, for many, many years—and we are committed […] to try to become a city […] There will be a shopping mall. When you want to go for a walk, there are still mountains […] We're not going to blast [the mountains] to create a shopping mall.

While Melissa is against petroleum exploitation and argues that a shopping mall will affect the iconic value of the area, Roy favours a cultural change and supports the oil industry. This excerpt illustrates that the students clearly disagreed on what is best for their hometown and have different hopes for the future, but still, they showed respect for each other, as illustrated in the next excerpt.

The next excerpt is from earlier in the plenary discussion. Mary states that she and her group are against oil exploitation based on the following argument,

> Mary: The fish can be scared away, and tourism can decline. And it can affect the fisheries.
>
> Researcher (facilitator): Can you say something about why tourism will decline?

Mary struggles to answer this, but Roy wants to help,

> Roy: I can. Most tourists come to Vestvik because they want to see the scenery and to fish. But, if we assume [oil exploitation], then it will be full of oil platforms out there—and when we search "Vestvik" and see how it looks here, then probably the first thing to appear would be oil platforms. […] And not very many will come to look at the oil platforms. Yes.

Here, Roy helps Mary with providing arguments that support her claim, even though he disagrees and supports the oil industry. The classroom discussion can be understood as a *gesture of democratic practices*. The students were encouraged to give their own opinions and to disagree on a controversial issue related to sustainable development and climate change. The research team made efforts to show that the students' contributions were valued. The students showed each other respect as well and had important dialogical qualities. They practised democratic virtues, such as listening, responding, respecting (which Roy was an example of), exploring and including critical thinking when developing argumentation in line with *democratic entanglement* and *listening to socio-ecological entanglement,* in particular when ecological effects were part of the argumentation. The discussion was also an arena for the students to express their hopes for the future.

The other gestures were perhaps not so prominent, and mathematics was not much present in the discussion. With more time, both could be implemented. For example, the students were quite capable of mapping risks (Breivega et al., 2022), which could be explored later through mathematical tools, which again could draw attention to listening to entanglements as positive and negative impacts of an oil industry would become clearer.

Conclusions and discussion

In this paper, we have argued that mathematics education needs to change in response to the impacts of our "new" climate. Our focus has been on curriculum. We proposed five possible "gestures" for an alternative mathematics education. We do not offer these with any sense of completeness, and we also recognise there are significant overlaps. The five gestures and some prompting questions are offered in Table 1 (the questions are not exhaustive of the gesture but rather, specific examples):

Gesture	**Example prompts**

Listening to entanglements	What do we take as fixed for the purposes of our research? What are the implications of taking the environment as fixed?
Scales of entanglements	How has the past influenced the inequalities and issues we face now? What are the implications of our actions for other places on the earth? For other species?
Democratic entanglements	To what extent is the mathematics education we are involved with fostering democracy and critical thinking in relation to climate?
Entanglements of emotions	Whose voices are heard, and how might hope or courage be fostered, as well as awareness of the scale of challenges?
Living entanglements	What does it mean to act from a position of recognising our lives as emerging out of symbiotic collective relations? How do we conceptualise the "human"?

Table 1: Five gestures

We considered the national curriculum in both Norway and England, which showed up some stark contrasts in terms of aims and the presence of questions of the climate and sustainability. It seems extraordinary to us that, in 2025, it still has to be said (e.g., looking at the English National Curriculum) that there needs to be more attention paid to knowledge of climate change and an engagement in what it might mean to respond. We have attempted to show and argue that mathematics is an important tool for exploring and understanding climate change topics. We also questioned the image of the "citizen" evoked in different National curricula and what that image may occlude, and who may be marginalised.

In the section just above, we considered some projects involving curriculum innovation, which we have both been involved in, and what these might suggest about possibilities for the future of the mathematics curriculum and any gestures it may embody. Our perspective is that every mathematics classroom is making a gesture (more or less explicitly) around values—hence there is a need for reflexivity about the meta-messages of our own practices, and our instantiations of mathematics, as well as the values of the broader institutions within which we work. So, whether our specific proposals make sense, or seem relevant in your context, our broader point (which itself is a gesture of *living entanglement*) is a provocation to consider what values are being promoted through our work (e.g., are they values of democracy? do they support awareness and listening for what is ignored? is it possible to consider a current time and place in relation to other times and places?). We have tried to argue that mathematics education is not exempt from taking responsibility to engage in issues relating to climate change and climate justice.

Reflecting on our writing about curriculum projects, there are several points of learning relevant to thinking about the future. One commonality is the way current assessment structures get in the way of interdisciplinary projects and the capacity to allow space for students' ideas to be aired and taken seriously. Alf called for students not to need to prove their competence, their humanity, and Kjellrun reflected on the obstacles of grading, in relation to projects displaying the gestures we have outlined. If a schooling system without assessment seems unthinkable—can we stretch our imaginations?

A second issue arising is the complexity of bringing socio-ecological issues into the classroom. Teachers may need to engage in gestures that have not typically been required in mathematics classrooms in Norway and England. New gestures might need new ways of working, for example: skills of handling opposing and strongly held views; consideration and sensitivity to students' different backgrounds and experiences; skills of noticing and managing students' emotions; skills of allowing for the expression and discussion of controversial ideas in an environment that still feels safe for participants; skills of navigating ambiguity; addressing politics explicitly. One framing of what may be required is a move towards a communal mathematics education. And we do not underestimate the skills of teaching required and the implications for professional development and initial teacher education. This does not mean that teachers need to be experts in everything, but rather that there might need to be the development of a sense of exploring issues together with students. And while this might imply the need for teachers to take risks, there is also a sense that there may be much to be gained from a schooling focused on co-exploration of genuine issues in the world.

To pull out, from the writing, some further suggestions for the future, we want to propose (a) mathematics has a vital role in climate change education (beyond the sciences), (b) there is an urgent need for reconsidering not just how mathematics is taught, but also what mathematics is taught. In relation to (a), we noted right at the start the significance of the recognition that mathematics is not a neutral subject and developing an awareness of the ambiguity and ethics of the uses of mathematics is surely a key element of a curriculum for the future, to say nothing of the mathematical skills required in making sense of and communicating about climate issues. In relation to (b), several proposals can be interpreted through what we have offered, including: inter-disciplinary projects arising from local issues; the importance of dialogue; re-framing of the subject in terms of processes relating to sustainability; reducing an obsession (as we see it in current curriculum documents) with number work; introducing a study of systems (e.g., to develop an awareness of runaway effects, local stabilities, catastrophic change, etc.), including the climate.

We want to end by thanking the organisers of CERME14 for allowing us space to bring to the conference issues relating to climate, justice, sustainability and the future of mathematics education, particularly as these are currently relatively marginal pursuits, looking at mathematics education research in Europe as a whole. We hope that CERME continues to be a space for developing a research agenda around how the living world matters for mathematics education and how mathematics education matters for the living world. We hope others might take up, challenge, refine or rework some of the gestures outlined, and we hope we can continue to engage in conversation and collaboration around such issues at future CERME conferences.

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The new role of data in society: Challenges and opportunities for mathematics education in the age of data science and AI

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Data Science and Artificial Intelligence (AI) are reshaping society, entangling (Coles & Hiis Hauge, 2025) mathematics with people, values, and the wider world. This transformation brings new challenges and opportunities for mathematics education, while also calling for broader curricular shifts that connect with other school subjects. In this talk, we examine key skills and literacies students need to navigate a data-driven world and imagine their role in shaping it—critical data literacy, statistical, media, computational, and AI literacy. We discuss how these literacies should evolve within mathematics education and emphasize the importance of interdisciplinary collaboration across data science, the natural and social sciences, and computer science. Classroom examples from our own projects illustrate how these ideas can be brought into practice.

Keywords: Data science education, Data literacies, AI literacy.

1. Introduction

The topic of data has always fallen between the cracks—especially between mathematics and computer science:

> Computer scientists, by nature, don't respect data. They have traditionally been taught that the algorithm was the thing, and that data was just meat to be passed through a sausage grinder.
>
> So to qualify as an effective data scientist, you must first learn to think like a real scientist. Real scientists strive to understand the natural world, which is a complicated and messy place. By contrast, computer scientists tend to build their own clean and organized virtual worlds and live comfortably within them. Scientists obsess about discovering things, while computer scientists invent rather than discover. (Skiena, 2017, p. 2)

We invite readers to consider whether there might be a similar attitude toward data among mathematics educators—that the term "computer scientists" in the passage above might be replaced with the word "mathematicians". Of course, data are part of statistics education, which is taught in the mathematics classroom, and data play a certain role in mathematical modelling. In computer science education "data and information" is a standard topic. And as the quote highlights, data are also central to inquiry across the natural and social sciences.

However, standard current educational practice is far away from taking the emergent new roles of data in society into account. Classes and curricula are still typically organized according to relatively siloed and static visions of school subjects, limiting the extent to which scientific, social, mathematical, and computational explorations can be meaningfully brought together to enable students to make discoveries with data. At the same time, the confluence of all these disciplines and approaches makes planning developmentally appropriate and tractable activities with data

challenging. Exactly what should students know about data, and when should they learn it? And, how can educators be prepared to teach such a dynamic and interdisciplinary set of practices?

In this paper, we review the current state of the field of data science education and provide some examples from our own research and development work. It is based on a plenary talk we gave at CERME 14 in February 2025, where the slides contain many illustrating examples (https://bit.ly/cerme14). We begin by defining *data science* as an emerging, interdisciplinary field and examining its impacts on society, science, and education. We then present pedagogical examples of data science driven activities, drawn from our own work and others at the interface with the natural, social, and computer sciences. Our goal is to plot a course for considering the impact of data science, and related innovations such as machine learning and generative AI, *across* the curriculum—and the implications of these curricular impacts for mathematics education. We invite our readers to consider, within this context:

- What is important for students to know about data?
- What should students be able to do with data?
- What is (or should be) the role of mathematics education in contributing to the knowledge, skills, and dispositions needed for students to understand life with data?

A first answer, which one could have provided at least 20 years ago is: Students should have experience with analysing "tidy data" (Wickham, 2014) with digital tools from the perspective of statistical investigation (Wild & Pfannkuch, 1999). Tidy data are multivariate data, organized in a table in rows and columns, like a survey where each row is a person and columns record their attributes. Attributes can be numerical or categorical. Digital tools for school use have been available (Biehler et al., 2013), as well as material, data sets, and ideas for classroom use, backed by research (see e.g., Ben-Zvi et al., 2018, and the website of the International Association of Statistics Education iase-web.org). However, implementation in standard curricula is far behind the consented goals among statistics educators in many countries.

The new developments of data in science and society demand an even more substantial place of data in the curriculum, which can build on traditional approaches and integrate new aspects. We see the emerging societal pressure on schools to integrate more data literacy or AI literacy as helpful for curricular reform. The Organization for Economic Co-operation and Development (OECD), for example, recently announced an effort to develop an AI literacy framework that extends across the curriculum, so that "educators will be able to better assess students' understanding of AI, as well as broader issues like privacy concerns and digital citizenship with clear, measurable benchmarks [...] not only for academic success but also for active, well-informed participation in an AI-driven, digitally evolving society" (Schleicher, 2025).

2. Characterizing Data Science and its Evolving Meanings and Roles in Society

What counts as data has changed dramatically and rapidly. Data now includes everything from images, text, videos, and webpages to audio data (speech), sensor data, and records of transactions. We have to think also about posts to social media, location data from GPS or location-based services, and even behavioural or health data collected by wearable devices. The explosion of digital data means that conceptions of data literacy, the ability to access, interpret, critically assess, manage, handle, and ethically use data, has to be updated as well as what we understand as democratic participation in a society being transformed by datafication.

Preparing K-12 students for this data-rich reality means ensuring they understand not just how to calculate with numbers, but how to make sense of, use, and critically evaluate data in its many forms.

This changing nature and increasing availability of data have led to new and significant roles for data, data-based models, and data-related algorithms across various spheres.

Data play a new and expanding role in the sciences. In fields like statistics and mathematical modelling, we now encounter new types of data and require new workflows. For instance, the CRISP-DM workflow model (Shearer, 2000) provides new emphasis on data preparation, communication, and deployment. It differs from the traditional PPDAC cycle (Wild & Pfannkuch, 1999, discussed in more detail in Section 4.1) and mathematical modelling cycles, even if they include data as a separate component (English, 2021). Related to the new role of data, new types of models become relevant: predictive models (the dominant type of model in machine learning), algorithmic and computational models (Breiman, 2001), sometimes with millions of parameters. Computer science has seen the emergence of data-driven Artificial Intelligence (AI) and data-driven digital artifacts. Even in natural, social, and life sciences, datafication and new research methods are transforming how questions are asked and answered. For example, data science techniques are being used to understand complex biological systems.

At the same time, data science is emerging as a new discipline itself. It is an interdisciplinary field that combines statistical methods, computer science, and expertise in a particular subject area to gain insights from data. Hazzan and Mike (2023, pp.19) devote a whole chapter to the origins and different characterizations of data science.

Perhaps most visibly for young people, data has new roles in everyday life, including the lives of students. Students interact constantly with data-driven digital artifacts in their private lives through search engines, streaming services, and social media. Wearables and fitness trackers collect personal data. Digital data are woven into the fabric of our cultural, social, political, and economic worlds. Educational conceptions that focus on these aspects of data can be found under the heading of personal data literacies (Pangrazio & Selwyn, 2019) and building students' data awareness (Höper & Schulte, 2024).

Similarly, data and its visualizations have new roles in the media. We see more data and new visualizations in traditional media, as well as websites with embedded interactive visualizations and dashboards. Data journalism is an emerging field. Understanding data in the media requires a new lens, especially when dealing with spatial data or issues of spatial justice (da Silva et al., 2021; Engledowl & Weiland, 2021; Rubel et al. 2021). Media items use data and visualizations to inform the public, present opinions, justify predictions or recommendations, and validate government policies. Data visualizations are not neutral; they are social texts authored from specific points of view that narrate particular stories (Gal, 2022).

Beyond consumer applications, data-based models, and data-related algorithms have profound new roles in society at large. Algorithmic decision systems, often based on machine learning, are used in healthcare (predictive diagnosis, treatment recommendations), finances, transportation, job applications, educational administration, and predictive policing. Many applications raise ethical questions. Examples include examining racial bias in healthcare AI (Chin et al., 2023), predicting school dropouts (Bowers, 2021), or the controversial use of tools like COMPAS software in the justice system (https://en.wikipedia.org/wiki/COMPAS_(software)).

These new roles of data have led to significant political discourses on the ethical aspects of data and AI use, involving associations, initiatives, and state regulations. New scientific journals are devoted to this topic such *AI and Society* and *Big Data & Society*. Key concerns raised in the 400 pages report of Deutscher Ethikrat (2023) include the danger to individuals through statistical stratification, the protection of privacy and autonomy versus threats from surveillance, and the pervasive issues of bias

and discrimination embedded in data and algorithms. Transparency and traceability are crucial for control and responsibility. Counterbalancing these concerns are efforts focused on AI and Data Science for Social Good (https://www.datascienceforsocialgood.org). Critically evaluating data use also requires examining data quality and the strength of evidence presented, issues that are increasingly discussed in the media.

Given this landscape, preparing students for a data-driven world involves fostering data literacy that includes the ability to critically interpret and evaluate data and its sources. This is reflected in educational conceptions such as *critical data literacy* (Weiland, 2017). Data visualizations, while powerful tools for communication, are embedded within structural and ideological relationships of power and can become sites where meanings are contested. Access to learning about data visualization without a deep interrogation of identity and status, national and commercial interests, and power relations can even be counterproductive, potentially reproducing societal hierarchies in the classroom (Bilstrup et al., 2022; d'Ignazio & Klein, 2020; Hall & Dávila, 2022; Lee et al, 2022; Philip et al., 2016). Students need to understand that data can be used to both uphold and expose systems of oppression (Martínez-Castro et al., 2023). Critically reading data visualizations involves understanding how reality is mathematically formatted, how variables and relationships are framed, and what stories are narrated, recognizing that these are texts created by humans with specific viewpoints. Ignoring the social complexities of data and its representation would defeat the democratic purpose of teaching data literacy.

3. How is the Growing Role of Data in Society Impacting K-12 Policy?

Much has been written about the role of data in educational administration and decision-making (e.g., Mandinach et al., 2015; Schildkamp & Datnow, 2022). While these impacts of data are certainly important, here we are interested in a complementary issue: what must students themselves now learn, given the increasing role of data in science and society?

To address this question, we consider what Chevallard called *didactic transposition* of knowledge (1985). Didactic transposition refers to the process by which scholarly knowledge is transformed to become teachable. The challenge with data science is that the relevant scholarly knowledge is in itself interdisciplinary and unstable, and practitioners have different perspectives regarding its nature and function as a field (Tanweer & Steinhoff, 2024). This makes it difficult to identify a stable body of knowledge to be taught or locate where data science might live within the broader curriculum. Chevallard also notes that didactic transposition can introduce tensions between maintaining the legitimacy of the taught knowledge in the eyes of experts and adapting that knowledge for novices in ways that are accessible and socially relevant. This tension is already playing out as school systems grapple with whether and how to introduce data (see Boaler et al., 2024 for a recent example), and how to prepare students for its significant impacts on society outside of academic disciplines, as described in the prior section.

One approach educators have taken to address these tensions when introducing data science at the K-12 level is through integration with a variety of existing school subjects. A recent position statement authored jointly by a collection of organizations in the United States representing teachers of Mathematics, Statistics, Science, Social Studies, and Computer Science argued that "Data science bridges disciplines and thus should be introduced and taught across the curriculum in K-12 schools [...] All subjects in school should recognize the contribution of data to their discipline and take curricular approaches that integrate data with disciplinary lessons where appropriate" (NCTM et al., 2024, p. 1). Such initiatives require a developmental approach to coordinate data science integration

across the grades and subjects, leading some organizations to call for the development of comprehensive frameworks to inform curriculum and instruction (DS4E, 2025; NASEM, 2023).

Cross-curricular integration can be challenging, however, as it requires coordination across different academic departments and potentially changes to existing curricula. To address these challenges, integrative approaches typically emphasize that data science is fundamentally an *investigative* process that involves formulating a question, collecting or considering data, analysing the data, and interpreting and communicating results (Lee, Mojica, et al., 2022). Such references to this cycle of investigation are increasingly emphasized in new related standards and frameworks (e.g., Bargagliotti, 2020; IDSSP Curriculum Team, 2019). And while they may use different terminology, the core concept of inquiry—pursuing a compelling question or evaluating a claim using data—is shared across many disciplines (Manz & Beckert, 2023; Schanzer et al., 2022; Shreiner, 2018; Wild & Pfannkuch, 1999).

Another approach educators have taken to introducing data science at the K-12 level has been through its connections to citizenship education. This approach positions data as a way to empower students to understand and address real-world issues, potentially leading to pro-civic actions and understandings (Engel & Ridgway, 2022; Louie et al., 2021; Rubel et al., 2016). However, defining *what kind of citizenship* data science education aims to foster is itself a challenge. Various projects and curricula cite goals that vary from empowering individual students as (future) citizens; fostering cultural coherence and understanding across diverse societies; advancing society through supporting social mobility and equal opportunity; and boosting economic growth for students through workforce education. However, different stakeholders may prioritize these goals differently, leading to tensions in curriculum design and implementation. An emphasis on workforce preparation might focus heavily on technical skills for specific industries, while an emphasis on equal opportunity might prioritize critical data literacy for evaluating societal claims and advocating for justice (see Rosenberg & Jones, 2024). Navigating these competing priorities and ensuring that data science education effectively contributes to a balanced set of educational goals is a significant challenge for K-12 systems.

Whether data are introduced to students as a tool of inquiry across domains, or as a tool for civic and social engagement, the societal and ethical dimensions of data and its use have become increasingly important (Baumer et al., 2022; European Commission. Directorate General for Education, Youth, Sport and Culture., 2022; Utts, 2021). As a result, many of the emerging standards and frameworks described above emphasize not only data-related *knowledge and skills* but also related *dispositions* and *ethical considerations* as key learning goals for students. For example, the Dana Center's Data Science Course Framework (2024) describes design principles for courses that will "empower students to live skillfully and productively in our data-driven society" (p.4), which includes activities such as "Provid[ing] authentic experiences that empower students to make a positive difference in their communities through data science projects" (p.7).

4. How Could the Growing Role of Data in Society Impact Education and especially the K-12 Curriculum?

While administrators and policymakers are still working to understand how data might best be integrated into K-12 curriculum and instruction, many teachers, researchers, and curriculum developers have already explored new pedagogies, tools, and dispositions to introduce students to data science. A number of pedagogical frameworks concerned with data have been introduced, though typically with a focus on tertiary and professional education rather than the developmental steps needed to prepare students at primary and secondary levels for these intensive tertiary experiences. Here, we review four such adaptations and innovations related to how students are

introduced to data; what are the tools and technologies they use to analyse them; how they are encouraged to develop critical data literacies regarding how data can be used to understand and advocate for change in the world; and how is it possible to develop students' awareness of how datafication is impacting both personal life and societal structures.

4.1 New Models of the Data and Statistical Workflow

Integrating data science into K-12 education demands a fundamental rethinking of how we teach students to work with real-world quantities. For decades, reform-oriented statistics, science, and mathematics educators have emphasized the importance of moving away from solely demonstrative or obviously predetermined "cookbook" data analysis activities toward more authentic engagement in inquiry (Cobb, 1999; Hancock et al., 1992; Lehrer & and Romberg, 1996). These sorts of inquiry-based engagements with data were traditionally described as a structured cycle in which students pose their own investigative questions, collect their own data, and explore those data using appropriate tools and methods. For example, Wild and Pfannkuch's PPDAC (Problem, Plan, Data, Analysis, Conclusions; Wild & Pfannkuch, 1999) cycle is foundational to reform-oriented statistics education. However, data science, with its emphasis on new forms of data and tools, disrupts several components of the investigation cycle, requiring us to further adapt this foundation.

Perhaps the most significant shift that has been introduced to these traditional descriptions has to do with the nature and origin of the *Data* at the centre of the cycle. K-12 statistics and science lessons oriented toward student inquiry often engage students in designing experiments or conducting fieldwork in their schoolyards or neighbourhoods to collect their own small first-hand datasets. Data science, however, frequently involves working with large, complex datasets that already exist – data that are publicly "open" (Zapata-Cardona, 2025), "repurposed" (Wilkerson & Laina, 2018), and so "messy" (Rosenberg et al., 2020) that students may find themselves "awash" (Erickson, 2025).

Consider an example drawn from McDonald's *Handbook of Biological Statistics* (McDonald, 2014, pp. 204–205). Students are invited to address the question "Does species diversity decrease as you move away from the equator?" using the Audubon Society's Christmas Bird Count, a yearly participatory citizen science activity that reports the total count of species spotted in different locations around the United States. The activity presents a small subset of species counts from approximately 15 locations, along with those locations' names and latitudes. The reader is expected to run a linear regression with latitude as the independent variable and species count as the dependent variable, which yields an equation in which the species count declines slightly as latitude increases, and the result does not reach statistical significance. Inviting the student to engage in common-sense reasoning, McDonald writes, "Even if it were significant, I don't know what you'd do with the equation; I suppose you could extrapolate and use it to predict that above the 49th parallel, there would be fewer than zero bird species" (p. 205).

This activity engages students with data in meaningful and careful ways. It encourages them to consider the context and implications of mathematical methods and points out their limitations. But from a data science perspective, there is much more that can be done due to the availability and richness of data we have about the world. Consider, for example, that with the information provided (location names and latitudes), it is relatively easy to link these records with additional geographic information: with not only latitude but also longitude, daily weather patterns, elevation, geological features, and more. Merging the available bird species counts with this richer set of available information (Figure 1) reveals that the subset of bird count records selected by the author are drawn from the U. S. mid-Atlantic region, a location and scale for which bird species diversity is less likely to vary by latitude as the question might suggest, but more likely to vary by proximity to the coast.

Location	Latitude	Longitude*	Number of Species
Bombay Hook, DE	39.22	-75.48	128
Cape Henlopen, DE	38.80	-75.10	137
Middletown, DE	39.47	-75.74	108
Milford, DE	38.96	-75.46	118
Rehoboth, DE	38.60	-75.10	135
Seaford-Nanticoke, DE	38.58	-75.62	94
Wilmington, DE	39.73	-75.56	113
Crisfield, MD	38.03	-75.87	118
Denton, MD	38.90	-75.87	96
Elkton, MD	39.53	-75.86	98
Lower Kent County, MD	39.13	-76.08	121
Ocean City, MD	38.32	-75.11	152
Salisbury, MD	38.33	-75.63	108
S. Dorchester County, MD	38.37	-76.11	118
Cape Charles, VA	37.20	-76.03	157
Chincoteague, VA	37.97	-75.41	125
Wachapreague, VA	37.67	-75.69	114

* Obtained by finding locations using Google Maps.

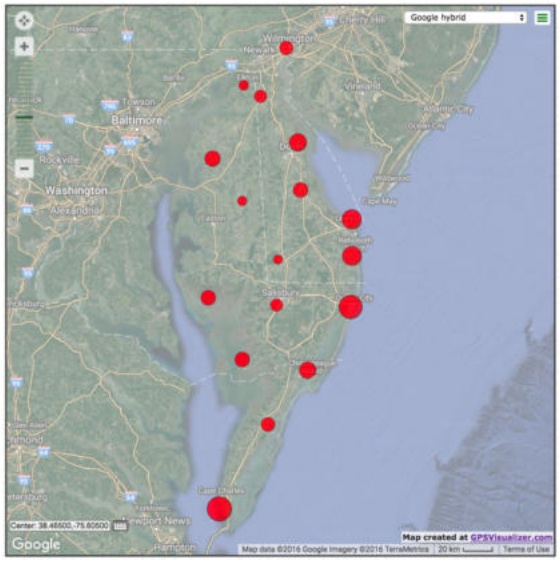


Figure 1. An expanded dataset, constructed by merging locations and latitudes with available longitude information (left) and the resulting map of bird species counts (right)

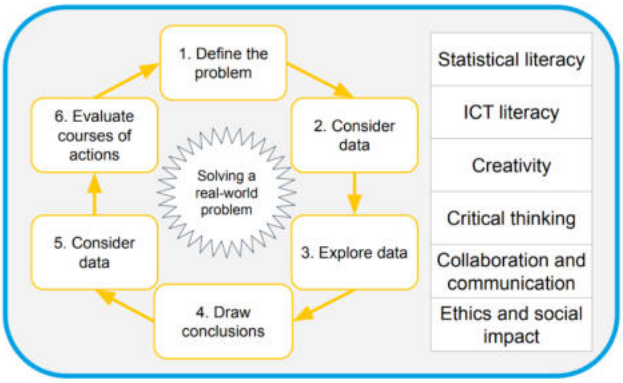
In this way, data that are collected by or provided to students can be merged, supplemented, and otherwise transformed using "found data" in ways that fundamentally change other components of the PPDAC cycle (see, e.g. Biehler & Schulte, 2018). When considering the *Planning* of an investigation, students' attention may shift from a focus on designing data collection protocols toward data preparation and exploration, emphasizing concepts of metadata, and data quality as students consider how datasets acquired from third parties likely include potential anomalies that require attention and further processing. The possibilities and constraints of an expanded, but potentially less reliable, set of data sources can in turn impact the *Problems* that an investigator is willing to take on. Instead of a linear or progressively iterative process where the problem is defined first and then data are collected, students might take a more exploratory approach with a large dataset to identify interesting patterns or generate questions.

The *Analysis* stage that results from an expanded set of available data and data types is also transformed. While traditional statistical tests are still relevant, data science more heavily leverages computational tools for exploration and visualization. As machine learning algorithms and other computational techniques become more common, investigators need to understand which data were used to train a given model and whether such training is appropriate for the new application context. To build on the bird count example, a biostatistician may wonder whether models developed from data in other geographic regions with different characteristics are appropriate to apply in this case.

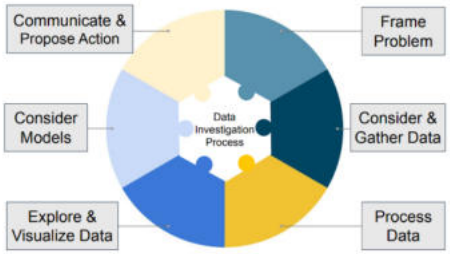
Finally, the diversity of potential data sources and analysis methods and the complex considerations each brings changes how investigators should *Communicate* the results of their analysis. They should report potential biases in data sources; shifts in paths of inquiry as a result of available or constrained datasets; evaluation of the appropriateness of the algorithms used and their training, and so on. throughout the entire process. This task has become so important that topics such as "data storytelling" have become a focus in disciplines such as the information sciences (Segel & Heer, 2010) and journalism (Matei & Hunter, 2021). Investigators are also increasingly expected to report on the ethical implications of their findings. The goal is not just to analyze data, but to use it responsibly to understand and potentially impact the world. In data science practice, communication

should take into account the principles of reproducible research (Dogucu & and Çetinkaya-Rundel, 2022). How this can be made accessible to secondary students is an open question.

In response to the perturbations to the data and statistical workflow caused by these new data sources and data types, several proposals have emerged to re-envision the data investigation cycle (Figure 2). Kazak and colleages' (2023) Data Analytics Cycle, for example, introduces two separate steps both dedicated to *considering* and then *exploring* data. The explicit inclusion of "considering" data here highlights how the increasing availability and complexity of data requires explicit attention to "getting to know" the data—including reviewing metadata, exploring the history and initial purpose for data collection, evaluating validity, and so on. Lee and colleagues' (2022) Data Investigation Cycle isolates a different dimension, data processing or the preparation, cleaning, re-coding, and otherwise transforming the data in order to permit the desired analyses, as a specific component. Wise's (2020) continuum of "data scientist perspectives" toward data further highlights data generation and storage as concerns in themselves (Figure 2).



Data Analytics Cycle & Competency Areas
Reproduced from Kazak, et al (2023)



Data Investigation Cycle
Reproduced from Lee, et al (2023)

Subject	Data Scientist					Audience
	Generation	Storage	Transformation	Interpretation	Representation	

Subject, Audience, and Data Scientist Perspectives Toward Data
Reproduced from Wise (2020)

Figure 2. New proposals for data-science-ready investigation cycles as described in Kazak et al (2023, top left), Lee et al (2023, top right), and Wise (2020, bottom)

In data practice, especially those that are influenced by computer science, even more different workflow models are used, such as the CRISP-DM cycle (Biehler & Schulte, 2018; Shearer, 2000). One major difference is that an extra stage *deployment* is added, similar to what computer science workflow models already include. Data science does not just end with a conclusion, but a system is constructed (such as a machine learning model) that is implemented in a socio-technical system, where ethical considerations have to be taken into account. Though their emphases are different, all of these proposals highlight how expanding data types and access have shifted definitions of what it means to pursue inquiry with data.

We note that while many of these frameworks are referenced by educators and researchers concerned with K-12 education, most were developed for, and/or heavily informed by, tertiary and professional levels. Even frameworks well known to the K-12 statistics and data science education community such as PPDAC (Wild & Pfannkuch, 1999) and Lee and colleagues' (Lee et al., 2022) more recent data investigations cycle were developed through interviews with professionals and statistics-oriented

students. These shed much needed light on some aspirational *goals* for modern statistics and data science education, but there is still much to be done in order to understand the *developmental processes* that might lead students to these endpoints, and to identify the available pathways through the existing curriculum that might help set the foundations for students' competence and confidence with data.

4.2 New Mathematical and Technological Tools for Working with Data

Access to new tools and technologies is also fundamentally reshaping how we should teach the data and statistical workflow in K-12 education. For decades educators have highlighted the potential for digital tools to transform students' engagement with data and modelling (Ben-Zvi, 2000; Biehler, 1997; Edelson & Gordin, 1998; Hancock et al., 1992; Lehrer & and Romberg, 1996). Topics such as probability and statistics were well situated to benefit from the introduction of computers into schools, because they allowed the easy visualization of large, multivariate datasets and the simulation of probabilistic events that form the foundations of statistical models. Ben-Zvi (2001), building from Pea (1985), argued that computers have redefined statistics as a cultural and disciplinary practice by reorganizing which statistical objects students are able to explore and manipulate, offering them fundamentally new cognitive models and new ways to connect statistical practice to the real world. Some of these same redefinitions are the ones that transformed the professional practice of data analysis and led to the emergence of data science (Biehler, 1993; Cleveland, 2001; Kaplan, 2007).

Since these beginnings, a variety of genres of educational technologies focused on data and statistical processes have become available. Several reviews and classifications of these tools exist, organized by function, interface type, or other such principles (e.g., Ben-Zvi, 2000; Biehler, 2019; Chance et al., 2007; Foley et al., 2024; Israel-Fishelson et al., 2025; A. McNamara, 2019; Pimentel et al., 2022; Pratt et al., 2011; Rubin, 2007), and we refer the reader to those for a comprehensive overview. Here it suffices to note that in general, these reviews include references to tools and technologies for collecting data from the world through automated measurements and recordings; tools for visualizing data through the easy creation of charts and graphs; tools for manipulating and calculating data; mapping tools that associate data with specific geospatial locations; and tools for generating simulated data from user-defined models. More recently, such reviews and frameworks have also expanded to consider how the design of particular tools may facilitate students' transition to professional practice (Foley et al., 2024; A. McNamara, 2019). One family of tools deserves special note here because of their impacts on statistics and data education research and because we both use them extensively in our own work. These include *Fathom* (a powerful GUI-driven educational data analysis and visualization tool; Finzer, 2001), *Tinkerplots* (a version developed for younger students, www.tinkerplots.com), and the *Common Online Data Analysis Platform* (codap.concord.org; Erickson, 2025), a more recent online data analysis tool that builds on the design principles of Fathom and Tinkerplots. It is a cost-free web application in several interface languages. Compared to Fathom and Tinkerplots it can use geographical data and build statistical maps and offers flexible hierarchical data structures. CODAP can be extended by using plug-ins, and we refer to the Arbor plug-in later, when we discuss decision trees. We refer readers to papers by Konold (2007), Biehler et al. (2013), and Finzer and Damelin (2015) for more information about the history of this family of software tools.

Despite this long history, a new focus on "data science education" has led to some new developments in these established tools and technologies that are worth noting here. First, many tools now support or directly integrate more diverse datasets, especially larger, publicly accessible datasets. Tools like CODAP and ESRI's free educational version of ArcGIS Online, a data mapping tool, allow students to easily import a variety of real-world data. New efforts have also made it easier for students to

participate in and access data from citizen science campaigns that are directly relevant to their lives or communities. For example, the Census@School project allows students to contribute to an international dataset recording demographic and behavioral data (Day, 2025), and the WeatherBlur project combined citizen science efforts and environmental probes to allow students to explore weather and wildlife patterns in their local environment (Kermish-Allen et al., 2019). Similarly, some tools, or curricula that are developed to accompany tools, highlight the use of APIs (Application Programming Interfaces) that support students to access dynamic ("live") data from the internet, from sources of data as about topics ranging from weather and sports to music (Fergusson & Wild, 2021; Weintrop & Israel-Fishelson, 2024).

Tools designed for novices are also beginning to highlight new data types and analysis techniques. Some tools and approaches emphasize network data structures and allow students to apply data reasoning to explore traditionally distant domains, such as analyzing relationships described in historical or literary texts (Craig et al., 2021; Legacy et al., 2025). Similarly, curricula and tools have emerged that support students in conducting computational text analysis (Horton et al., 2023) and image analysis. Others allow students to engage with or visualize various predictive modeling and machine learning methods such as decision trees (Biehler & Fleischer, 2021; Erickson & Engel, 2023) and digital classification (Grizioti & Kynigos, 2024). Anna Fergusson (teaching.statistics-is-awesome.org) has developed a suite of tools to help teachers visualize and experiment with simulation-based introductions to a variety of core statistics concepts (Fergusson & Pfannkuch, 2022; Fergusson, 2017).

Finally, as we will review in more detail below, more complex and socially embedded data often demands more interpretive narrative work in order to communicate results clearly and carefully. Many tools for data analysis have started to more explicitly facilitate new ways of communicating the processes and results of a data investigation. Story Builder, a plugin for the CODAP tool, allows students to save "moments" of an analysis cycle and annotate each with photos, rich text, and other explanatory materials (Wilkerson et al., 2021). Similarly, Fleischer and colleagues (Fleischer, Biehler et al., 2022; Fleischer, Hüsing et al., 2022) have begun to explore how students' work with Jupyter Notebooks can be structured to support the construction of "computational essays" that integrate code, outputs (including visualizations), and explanatory text into a coherent narrative. In this approach, half-worked examples and template structures encourage students to write narratives, in the process engaging with issues of data validity and bias, alongside their work through the technical details of an analysis process.

Modern data science practice uses tools with extended libraries that require coding such as R/RStudio or Python. Coding poses obstacles for many, especially younger students. It is an unsolved challenge, how far genuine data analysis practice can be supported at school level without the need of coding, using tools with GUI such as CODAP, and when it is reasonable and possible to move to tools requiring coding and how this transition can be bridged and supported (Fergusson, 2022; McNamara, 2015).

4.3 Critical Data Literacies: New Frameworks for Addressing Socio-scientific and Socio-ecological Problems

In recent years, we can observe two different but related developments, one in statistics and data science education and another in science and mathematical modelling education. In the context of statistics and data science education, we see new emerging frameworks for statistical literacy and data literacy, namely critical data or statistical literacy (Weiland, 2017; Louie, 2022; Martínez-Castro et al., 2023), civic statistical literacy (Ridgway, 2022) or data feminism (d'Igazio & Klein, 2020; Lee et al., 2022). They recognize that data are not neutral but is produced through social processes and

can perpetuate or exacerbate unequal power relations. A further common feature is that the context in which data science should be taught and learned should be actual problems in society and the world, encouraging the development of a critical stance. Similar educational movements include the so-called socio-scientific problem approach in science education (Sadler & Dawson, 2012), socio-ecological problems in mathematics education (Coles, 2023), and the tradition of critical mathematical modeling (Siller et al., 2024; Rosa & Orey, 2015). *Education for sustainable development* and *global citizen education* (which includes, e.g., human rights and democracy education in addition to ecological problems) have also to be mentioned in this context.

In statistics education, an important distinction is between the so-called consumer and producer perspective related to data and statistical products (Gal, 2002). Citizens are much more in the situation of a consumer, when reading newspapers or other documents with data and statistics as compared to doing their own data analysis. Though, from the perspective of education, the experiences and competences gained through conducting one's own analyses may contribute to a critical consumer perspective. The emerging new data literacies are closer to a consumer perspective in the sense that they raise the need to critically reconstruct and deconstruct data uses in science and society instead of inventing completely new ones (Rubel et al., 2021). Advocates of the new literacies sometimes refer to the critical literacy of Freire (2005), who coined the term of *reading and writing the world,* and add "with data" (Weiland, 2022). Writing the world is a metaphor of moving beyond understanding a certain social reality and arguing or working to change it for the better. This of course raises the question of how to deal with political, cultural and social value systems in the classroom.

Data science provides the opportunity to engage students in addressing a range of socio-scientific problems relevant to their lives and communities. This can include exploring environmental issues. Examples include urban planning, participatory mapping, citizen science, and climate investigations (Aridor et al., 2023; Engledowl & Weiland, 2021; Lanouette et al., 2024; Lee et al., 2022; Lim et al., 2023; Louie et al., 2023; Rubel et al., 2021; Wilkerson & Laina, 2018).

For instance, students can analyse some of the "Five Most Important Datasets of Climate Science" (Guimarães et al., 2022), such as CO_2 concentration over the years, using data measured at sites like Mauna Loa, Hawaii (since 1960). For comparison with concentrations before the industrial age, data reconstructed from ice core measurements in places like Vostok, Antarctica (since for over 400.000 years) can be used. The educational challenge is that climate change is not "obvious" from just two data graphs, but evidence relies on theories that explain why measurements taken at two specific places are representative and can be compared at all, and on scientific theories that relate CO_2 concentration to global warming and make plausible that the increase of CO_2 concentration is human-made (Biehler et al., 2025)

Economic issues like the gender pay gap are also fertile ground for critical data investigations. Students can engage in activities exploring the gender pay gap in the U.S., for example, by analysing sample microdata using the new data access tools in CODAP described in Section 4.2 to visualize graphs of wages by sex and educational attainment for a random sample of employed U.S. individuals, using boxplots and dotplots. This allows them to investigate how the gender pay gap varies with the level of education (Louie et al., 2022). On the other hand, the gender pay gap is a complex socio-economic problem, and students will need more background information, for instance what is adjusted and unadjusted pay gap than can be gained through data explorations alone. For instance, the Wikipedia article (https://en.wikipedia.org/wiki/Gender_pay_gap) provides detailed information and several complex graphs that describe historical and geographical variation of the gender pay gap, as well as its variation in different professions, that are usually not taught in statistics

classrooms. This exemplifies the tensions between independent data exploration and critical understanding of argumentative texts of popular science with embedded graphs.

Relegating data to only mathematics and statistics courses may burden such topics, calling for a contribution of the natural and social sciences. Connections between these topics and rich datasets are not hard to find. Many of the topics described above, such as climate and the environment, have a natural home in school science. Data allows students in these courses to also examine deeper intersections between the natural and social, such as through investigations of the disproportionate impacts of climate change on communities of colour and the Global South (Barton et al., in press; Levy & Patz, 2015; Sayavedra et al., under review).

There is a growing literature examining the use and teaching of data within the social sciences (Shreiner, 2018). For example, health issues, prominently highlighted during the COVID-19 pandemic, involve extensive data use and analysis (Calabrese Barton et al., 2021; Engledowl & Weiland, 2021; Rubel et al., 2021). Furthermore, data science is essential for investigating inequalities in society related to race, ethnicity, and migration. Examples include investigating racial bias or uncovering social inequality with data from sources like the U.S. Census (Kahn, 2020; Louie et al., 2022; Rubel et al., 2016).

Different theoretical traditions contribute to shaping the understanding of critical data literacy. In the United States, discussions often center on scrutinizing data for biases and potential misdirection (Engledowl & Weiland, 2021; Kahn et al., 2022), and using data to understand and challenge social oppression and work towards more equitable outcomes (Louie et al., 2022; Rubel et al., 2016). This aligns closely with the principles of teaching mathematics for social justice ("TMSJ"; Gutstein, 2012), which aims to equip students to recognize and counter social injustices using mathematics and data. Key goals include developing social and political consciousness, building agency for social change, and fostering positive identities. This approach views critical data literacy as essential for creating a more just world with data.

The European context, exemplified by initiatives like the ProCivicStat project (Ridgway, 2022), emphasizes Civic Statistics as a means to foster civic engagement through the exploration of evidence. Understanding civic statistics requires comprehending complex social phenomena, often involving dynamic, multivariate data presented through rich texts and novel visualizations. A critical interpretation is essential for informed citizenship in democratic societies, going beyond typical statistics curricula to consider the meaning and implications of data for society and policy. This approach encourages critical thinking and informed decision-making based on evidence. Examples explored in the various chapters include analyzing data related to racial bias in European football, climate change data, or engaging with official statistics and considering their social implications. The project developed a multidimensional framework that shows the multifaceted nature of civic statistics (Figure 3).

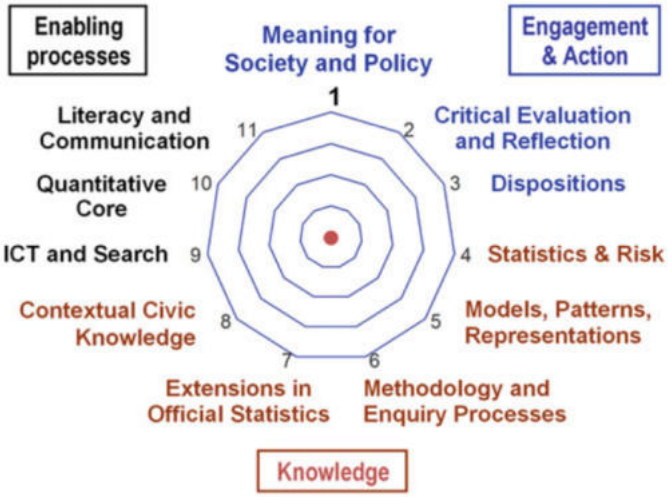


Figure 3. Civic statistical skills, attitudes, knowledge and competences. Graph provided by the authors of Gal et al. (2022).

A particularly rich tradition for understanding critical data literacies emerges from Latin America, heavily influenced by the work of Paulo Freire (2005) and the field of critical mathematics education (Skovsmose, 2014; 2023). This perspective views education as a process of liberation, where individuals develop critical consciousness to understand reality and act to transform it. This involves using statistical investigations to explore societal crises and inequalities, such as repression, conflict, misery, inequity, ecological devastation, and exploitation. The broader field of "critical data studies with Latin America" further elaborates on these ideas, incorporating frameworks of dependence, oppressions, and liberation, viewing datafication as intertwined with existing systems of inequality shaped by class, gender, and race (Campos, 2016; Martínez-Castro et al., 2023; Raffaghelli, 2022; Tygel & Kirsch, 2016; Valente & Grohmann, 2024).

Thus, critical data literacy across these traditions moves beyond technical proficiency to encompass a critical understanding of how data are produced, used, and interpreted within its social, political, and ethical context. It empowers individuals not just to consume data-based arguments critically but to use data actively to understand, challenge, and potentially transform unjust realities. While implementing this approach faces challenges, such as the need for appropriate tools, time, cost, and adequate teacher preparation, the potential benefits for fostering informed and active citizens make it a crucial area for K-12 data science education. However, a central open question is, how to empirically assess the complex goals in knowledge, attitudes, skills and competencies. The available studies do only contribute fragmentary knowledge in specific small-scale studies.

4.4 Presenting the Need for Further Data Literac(ies): From Personal Data Literacy to Datafication Literacy

The datafication of science and society has led to two quite new data-related literacies, which are distinct from what we have discussed so far: *personal data literacies* (Pancrazio & Selwyn, 2019) and *critical datafication literacy* (Sander, 2024a).

Personal data literacies aim at understanding and critically engaging with personal data. This area, sometimes referred to as data awareness, emphasizes that we live in a world where digital devices and platforms are ubiquitous and constantly collect data from us, often without our full awareness. This creates a significant "information asymmetry", where users may not understand where their data

goes, why it is being collected, or how it will be used. The goal of fostering personal data literacies is to help students overcome feelings of powerlessness and develop a sense of self-determination and agency in these interactions.

One framework designed to support this is the *data awareness framework* (Höper & Schulte, 2024; Höper et al., 2024), which provides students with an explanatory model of how data-driven digital artifacts (ddA) work and the role data plays in them. This model helps students become aware of explicit and implicit data collection, the primary and secondary purposes for data processing, and the creation of data models about themselves, often called digital doppelgangers. It also highlights their own role as data sources in these interactions. For instance, one teaching unit designed to build personal data literacy explores location data collected when using a mobile phone, including looking at authentic data like that made available by German politician Malte Spitz. Another unit focuses on recommender systems in streaming services, allowing students to work with a miniaturized engine to see how personalized recommendations are generated based on user data, such as finding "*k* nearest neighbours" to understand how their preferences are used to recommend new films. Learning this explanatory model and applying it to everyday examples seems to encourage students to reflect on these data practices and potentially think about their own behaviour, which is a step toward feeling more empowered rather than resigned (https://www.prodabi.de/en/unterricht/).

Datafication literacy (Sander 2024a; 2024b) broadens the focus from individual data to the larger societal change processes driven by data. In one sentence: The socio-scientific problem addressed by critical datafication theory is the need to critically evaluate the datafication of society and science itself. Critical datafication literacy aims to equip students with the ability to understand and critically reflect on how data are deeply intertwined with social processes and power dynamics An approach that strongly aligns with this is data feminism (d'Ignazio & Klein, 2020), which offers principles such as examining power, challenging power, considering context, and making labour visible, to understand the relationships between data and power. Datafication literacy encourages students to see the labour involved in creating data and to question who benefits and who is marginalized by data practices.

Ultimately, both personal data literacies and datafication literacy are essential for students to navigate our increasingly data-driven world responsibly. Understanding the data collected about oneself (personal data literacies) can serve as a foundational step towards critically reflecting on the broader societal implications of data use (datafication literacy). Issues surrounding the privacy of personal data, for instance, are relevant from both perspectives. However, it's crucial to empower students to look beyond individual actions and understand the need for systemic approaches and collective action to address the challenges posed by datafication. By fostering these literacies, K-12 data science education can move students from passive users to informed and empowered citizens who can critically understand the data that shapes their lives and society and potentially use data to advocate for change.

4.5 AI Literacy from the perspective of mathematics, statistics and computer science education

The data use addressed by these two literacies is also closely connected to data-driven AI systems, particularly those involving machine learning models and algorithms, while also encompassing broader applications of data use that are independent of AI. There has been extensive discussion of AI literacy (e.g. Almatrafi et al., 2024), which analyses these developments in terms of educational requirements.

The emergence of machine learning as a central topic in contemporary AI has led to a paradigm shift in computer science concerning the role of data. Training systems with data instead of classical

programming is the new principle, machines program themselves in a sense, creating algorithms, by being trained with data. This is acknowledged in computer science education, where the new role of data induced a paradigm shift in computer science education itself (Olari & Romeike, 2024; Tedre & Vartiainen, 2023). The topic of what secondary school students should—and are able to—learn about artificial intelligence is primarily explored within the computer science education community. A growing number of publications are addressing this issue, including several comprehensive literature reviews (Almatrafi et al., 2024; Casal-Otero et al., 2023; Chiu & Sanusi, 2024; Olari & Romeike, 2024).

An influential approach are the *five big ideas of artificial intelligence (AI)* of ai4k12.org. They encompass how computers *perceive* the world using sensors, how agents *represent* information and use it for *reasoning* to solve problems, how computers *learn* from data to find patterns (which is machine learning), how intelligent agents interact *naturally* with humans, and the *societal impact* of AI in both positive and negative ways. Data-based machine learning is the closest topic to statistics and mathematics.

Recent literature (Casal-Otero et al., 2023; Long & Magerko, 2020) highlights that AI/ML literacy is not a monolithic concept but spans multiple domains of understanding and engagement. These domains reflect different educational priorities and philosophical orientations toward technology. Three broad categories help to structure current discussions: learning about AI, learning for life with AI, and learning how AI works. The latter entails a conceptual and sometimes technical understanding of how AI systems are developed and deployed. This includes the overall workflow of a machine learning system, from data collection to model deployment, and the role humans play in shaping and overseeing these processes. Within this category, one of the most pressing challenges is addressing what has been termed the multiple opacity problem (Burrell, 2016). Machine learning systems often operate as "black boxes", meaning that the logic behind their outputs is not easily interpretable—even to their creators. This opacity stems from three sources: the complexity of learning algorithms, the abstract nature of the resulting models, and the hidden assumptions embedded in training data.

One approach to dealing with the opacity problem is *black box algorithm auditin*g (Morales-Navarro et al., 2022; 2024). Alternatively, scholars recommend a "white" or "grey" box approach with algorithms where this is possible (Hazzan & Mike, 2023). This means that algorithmic details can be made accessible, at least a simplified version of the professional ones, and reasons for choosing this algorithm. This is very difficult with artificial neural networks, because the required mathematical knowledge about matrices, partial differentiation, and special sigmoid functions are used as activation functions. In contrast, algorithms of methods such as decision trees, k-nearest-neighbours, and linear regression which can be re-interpreted from a machine learning perspective, are mathematically much simpler. We will provide a pedagogical example in section 5.2

4.6 Summary

The frameworks we have reviewed often each come with classroom examples that illustrate the pedagogical approaches and the data science facets that they are designed to support. Although these can be a rich source of guidance for educators, it is clear that not all examples, frameworks, or even data science facets can be taught and learned at K-12 level. Curriculum designers of the various subjects have to select what is suitable and feasible in their educational context. Very recently, the U. S. based *Data Science 4 Everyone* organization, in collaboration with teachers, youth, and experts, have started an initiative to map K-12 data science progressions across the curriculum (DataScienceLearning.org), which is a promising direction though still early in its development.

In the next section, we present pedagogical examples from our own work with colleagues, which of course can illustrate only some of the frameworks and data science facets. These examples serve to illustrate the pedagogical challenges we face already with the relatively small teaching units implemented under very favourable conditions, but also the rich potential such activities hold for conducting truly integrative data investigations even with young students.

5. Pedagogical Examples

To illustrate the many changes and emphasis described above, here we present examples of curriculum focused on data science and data-intensive AI. For each example, we include versions of our curricular approaches that were delivered in excellent conditions, as well as versions that are better adapted to the current constraints of most classrooms and therefore have high potential for immediate scale up. Our goal is to provide some concrete examples of what data science education can look like, and to encourage mathematics educators to consider how such activities might be adopted, supported by, or otherwise intersect with classroom instruction.

5.1 Integrating Data with the Natural and Social Sciences through Data Storytelling in K-12

Teaching data science at the K-12 level presents a significant opportunity to engage students with the world around them. However, moving beyond procedural instruction to foster genuine understanding requires a framework that resonates with how students make sense of complex information. Many approaches have begun to leverage *storytelling* as a potentially powerful, integrative framework for teaching K-12 data science, allowing students to see data analysis not just as a collection of steps or calculations, but as a process of constructing and interpreting meaningful narratives grounded in evidence (e.g., Amato et al., 2022; Finzer & Wilkerson, 2020; Reigh et al., 2022; Sanei et al., 2024; Stornaiuolo, 2020). As we will show, storytelling can make space for students to engage with some of the novel forms of data, workflows, and tools described in Sections 4.1 and 4.2. To the extent that they engage with issues of social and civic relevance, they can also engage students in many of the issues described in Section 4.3, regarding critical data literacies.

In contrast to a traditional statistical report which might review the key features of a dataset, detail the analyses performed, and describe the key results of an analysis and their implications, data storytelling approaches emphasize context and personal expression (Stornaiuolo, 2020), highlighting what might be investigators' motivations for examining a question, providing an accounting of the decisions and processes that underlie the reported analysis (D'Ignazio & Klein, 2020), and invoking aesthetics and emotion to engage the reader and perhaps even persuade them to pursue particular actions as a result of the findings (Reigh et al., 2022). If we consider the producer/consumer distinction introduced above to these ideas about data storytelling, these contextual and social considerations can be written into the stories that students themselves tell as they perform and communicate data analyses, and they can be examined in the stories of others as students work to "read" the stories told in visualizations and statistical reporting in science communication, civic documents, and the media.

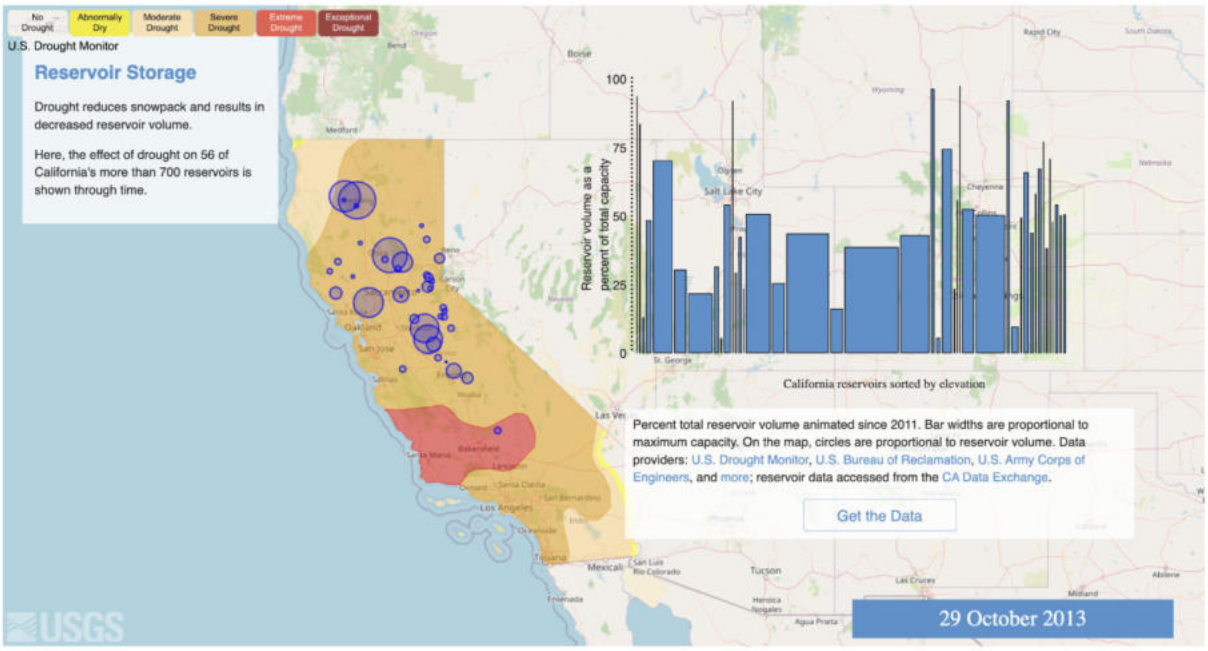
Narrative is often identified as a primary mode of human thought, distinct from but complementary to the logical or paradigmatic thinking often associated with mathematics and science (Bruner, 1991). While logical thinking is associated with decontextualized and verifiable facts, narrative is inherently temporal, contextual, and focused on the meaning and interpretation of events unfolding over time (Healy & Sinclair, 2007; Mor & Noss, 2008). Storytelling provides cognitive and cultural structures to organize thought and motivate coherence in understanding complex situations and multiple perspectives (Bal, 2009; Ochs et al., 1992; Schank & Abelson, 2013). Framing data science through

narrative can bring the analytic and interpretive dimensions of analysis that are so often separated back together (Kemble & Wilkerson, 2024; Pfannkuch et al., 2010).

Example 1: The California Drought. Given the new available tools and technologies described in Section 4.2, students and teachers now have unprecedented access to datasets which often include temporal and geographic features. This creates especially fertile ground for narrative engagement with data. In our work, we have leveraged storytelling approaches to engage science classrooms in week-long investigations of issues at a variety of scales, from personal nutritional decision-making to explorations of regional environmental conditions, to statewide and global climate trends. In these units, a major goal is for students to produce "data stories" that combine data analysis with journalistic research, interviews, personal stories, and expository and persuasive text to highlight issues of personal relevance.

Here, we focus on one investigation we did in collaboration with a 7th grade science teacher, in which students examined water reservoir levels in the U. S. State of California (where this activity was done), from 2011 through 2017. The investigation focused on the question: *What were the main factors that contributed to the California Drought?* We had obtained the dataset and associated visualization that was produced by the U. S. Geological Survey (USGS), and we begin the investigation by exploring a dynamic visualization that was quite complex. It featured a map (Figure 4) that was annotated with circles that changed size to indicate the location and volume of reservoirs and that coloured different regions to indicate regional drought declarations over time. It also featured a bar chart to indicate reservoir volume as a percentage of total capacity. All three of these representations–the reservoir point-locations, regional drought declaration colours, and bar chart–were animated over time. Despite this complexity, the visualization was recommended by the U. S. National Aeronautics and Space Administration (NASA) educational division for use by students as young as 6th grade (students of approximately 11-12 years of age).

There are many patterns, causes, effects, and experiences–some explicit, some implicit, and others only suggested—that a viewer might attribute to the drought as visualized here. When played from beginning to end, it demonstrates dramatic seasonal oscillations in reservoir levels as water is replenished each winter, but also a growing depletion of reserves year-over-year as this replenishment proves insufficient, until the drought finally ends in 2017. Regional patterns demonstrate how the northern regions collect and distribute water to the more populous, and drier, southern cities in the state. Observers with connections to the region may experience emotional reactions as they observe especially dramatic dips in water reserves in 2015, and recall associated public concerns and water use restrictions.



Credit: United States Geological Survey, CC0 1.0 Universal/Public Domain

Figure 4. A dynamic visualization developed by the United States Geological Survey to visualize California drought conditions. As a user scrolls their mouse, the date in the lower right advances and the bar chart and map overlay animate to display changes in reservoir volume across time. We encourage readers to explore the animation at https://bit.ly/cerme14drought or at https://labs.waterdata.usgs.gov/visualizations/ca_drought/index.html.

After exploring this visualization as a class, students were asked to describe what they noticed. We found that students attended to quantitative features inscribed in the visualization, but also to specific locations, time periods, and events represented therein. We also found that many students asked questions about the causes and effects of these patterns and referenced individual water wasting or conserving behaviours, with fewer students also mentioning municipal policies and the impacts of climate change (Barton et al., under review). After they noted these observations, we also provided students with age-appropriate news articles about the drought conditions and impacts. We encouraged students to look across these sources to identify a specific question to investigate related to factors that contributed to the drought, about which they would write a "data story" by combining the provided resources with further data analysis they would conduct throughout the week. Analysing these materials through a storytelling framework encourages students to look for the "plot", and connect resources that might otherwise be treated as isolated.

Based on students' more specific attention and the questions they generated, we then provided the reservoir volume dataset that was used to generate the visualization in CODAP. The dataset included each reservoir's total capacity, location, and the weekly mean volume of water stored for the entire period from Jan 2011 to November 2017. We also made other datasets that addressed students' questions about environmental factors (e.g. monthly precipitation, snowpack, and temperature for each reservoir's nearest weather station) available in CODAP for further investigation. Constructing a data story involves attributing significance and exploring relationships between variables toward a coherent explanation of all factors. As the students engaged with these and other resource materials, competing theories emerged in the class about what is most predictive of drought conditions over time—theories that can be examined, though incompletely, through the lens of data. Students asserted, supported, and analysed one and others' claims using the available datasets, constructing

over time an understanding of the interconnected natural and engineered systems (precipitation, snowpack, water use and distribution via interconnected reservoirs) that impact water access and availability.

This example is one of many from a longer design-based research project exploring students' narrative engagement with data as part of the DataSketch and Writing Data Stories projects (Finzer & Wilkerson, 2020; Kemble & Wilkerson, 2024; Reigh et al., 2022; Roberto et al., 2022; Wilkerson & Laina, 2018). We—Michelle Wilkerson with a team of collaborators including Vasikili Laina, Bill Finzer, Tim Erickson, Hollylynne Lee, Lisette Lopez, Kris Gutiérrez, Emily Reigh, and Cherise McBride—worked in close collaboration with partner grades 7-9 science teachers to identify how curriculum units they already planned to teach could be deepened by asking students to build data stories. Each unit engages students in constructing data stories using a combination of resources including specific publicly available datasets, news stories, scientific argumentation, interviews with family and community members, and personal stories to build connections between data, science, and the social world. It embraces many of the shifts around the pedagogical treatment of data highlighted in Section 4 by leveraging public datasets that incorporate contemporary formats such as GIS and images; interactive artifacts from the media that represent new tools for understanding data in the world; and highlights connections between the data analysed and students' own lives and communities. Additional data storytelling units and associated materials include other dimensions, such as datafication through students' recording and analysis of their own nutritional consumption, can be found at corelab.berkeley.edu and https://bit.ly/wds-units.

Example 2: The DataBytes Discussion Protocol. Weeks-long, open-ended data engagements such as the one described above are not always possible in every classroom. However, there are other ways for students to build deeper interpretive connections to data. One practice that has become increasingly common and can better fit into many existing day-to-day school routines are "data talks", classroom discussions about data visualizations that are intended to build student's everyday competencies as data consumers (Arnold et al., 2022; Boaler et al., 2021; Kahn et al., 2022). Importantly, such discussions are easy to integrate into any school subject, and exploring data visualizations can foreground the relevance of mathematics for understanding topics as varied as global climate, literary sentiment analysis, and consumer opinion.

We developed Data Story Bytes or "DataBytes" to bring a storytelling lens, with attention to the explanatory, social, and contextual features of data analysis and communication, to data talks. DataBytes are a collection of brief structured discussion guides we developed for use with specific visualizations, along with a more general questioning protocol that can help educators lead such discussion about any data visualization (Lee, Wilkerson, et al., 2022; Wilkerson et al., 2025). The DataBytes protocol centres around a four-part questioning framework designed to guide learners through making sense of a data visualization, considering its connections to their own lives and communities, considering the history and context of the data presented, and envisioning how they might use the data and findings in new ways.

To illustrate how the DataBytes protocol works in practice, consider an article from the *New York Times* titled, "Is Sushi 'Healthy'? What About Granola? Where Americans and Nutritionists Disagree" (see https://bit.ly/cerme-nyt; Quealy & Sanger-Katz, 2016). The article features a scatterplot with icons of various food items (e.g. pizza, tofu, coconut oil) positioned to indicate the proportion of "all Americans" (corresponding to the X axis) and the proportion of "nutritionists" (corresponding to the Y axis) that rate each food as "healthy". Already there is interesting mathematical work to be done in the interpretation of this representation, as students are asked to make sense of the position of each food, to reason through the existence and meaning of a dotted line

on the visualization representing $x = y$, and to develop sensibilities around what it means for a food or group of foods to be located "above" or "below" this line.

Moving beyond quantitative interpretation, the discussion guide then prompts students to connect the information and patterns in the data visualization to their own lives and communities. Can they find foods they eat on the graph? Where is their favourite food located, or why might it be missing? Is their own perception of the health of various foods consistent with what is reflected in the visualization? Some might begin to think about how the concept of "healthy" food relates to their own dietary needs, cultural background, or family habits. It is during this phase that we have observed teachers begin to focus more specifically on the scientific context of the data, connecting the quantities indicated in the visualization to the more complex underlying factors and processes they are meant to represent (Wilkerson et al., 2025).

Next, the DataBytes protocol encourages teachers and students to delve deeper into the history and context of the data that are presented within the visualization. This involves considering where the data came from, how it was collected, by whom, and for what purpose (these are called "worry questions" in Gal, 2002)—considerations that might take special precedence as one notices there are particularly salient connections or disconnections between the presentation of data and their relevance to personal or community interests. For example, we find educators and students in this phase begin to question how the specific 52 food items that are featured in the visualization were selected, especially if some of their own favourite foods are missing or appear to be inappropriately assessed. They wonder who was surveyed as part of the "all Americans" or "nutritionists" groups, how those people were contacted, and whether either group may exhibit selection bias that could impact the results.

The final part of the framework asks students to think critically about the data's potential applications, limitations, and future possibilities. This step encourages students to imagine different ways data could be collected or used to answer new questions or represent different perspectives. What would be needed for readers to use the information to make their own dietary decisions? How can the concept of health be indicated more transparently, or represented with more accuracy or precision (for example, through describing macronutritional facts associated with each food, to validate survey participants' responses)? Might the visualization be redesigned to be made more useful or inclusive? In semi-structured clinical interviews with teachers who were considering using the protocol in their own classrooms, we found this step prompted the deepest integration of mathematical considerations with subject matter and social concerns. This is because teachers considered what measurements, calculations, or ways of representing patterns might best reflect the complexities and human impacts of the topic (Wilkerson et al., 2025).

Adopting storytelling as an integrative framework for K-12 data science has helped us to address some of the challenges and necessary adjustments described in Section 4—around the new data types that are now available to explore, the new tools and technologies available, and around social and civic literacies and the impact of data on society. Storytelling is flexible enough to adapt to new representational forms and socially relevant issues. It also can be used across a variety of topic areas and school subjects, allowing for the kind of interdisciplinarity highlighted in the introduction to this chapter. We have found that this approach allows students to engage deeply with data as one part of a bigger story, seeing it as the basis for compelling narratives about the world. Using concrete examples like the California drought, students can explore and model quantitative patterns across time and space while also engaging in the interconnectedness of large, complex systems.

5.2 Integrating Data Science and AI Literacy into Mathematics and Computer Science Education: Decision Trees as an example

We will illustrate the conceptions we characterized in section 4.5. by an example from the ProDaBi project, which is a collaborative project of mathematics and computer science educators, led by Rolf Biehler and Carsten Schulte. Our intention was to integrate mathematical, statistical and computer science aspects into a data science and AI curriculum. We worked with teachers and classrooms in elective courses at secondary level, where no curricular and time constraints limited our experimentation with new ideas and methods, see Biehler et al. (2018) and Heinemann et al. (2018) for the initially different perspectives. One component of the curriculum was machine learning methods. In addition to artificial neural networks, we choose decision trees as relatively transparent machine learning method.

Over the past few years, we—Rolf Biehler, together with Yannik Fleischer and Susanne Podworny—have investigated how decision trees (DT) can be introduced as a machine learning method to students across various grade levels (5/6, 9/10, and 12/13) within the context of the ProDaBi project (Biehler & Fleischer, 2021; Fleischer et al., 2022, 2024; Fleischer & Biehler, 2025; Podworny et al., 2021; 2022; 2025). Classroom materials are available at (https://www.prodabi.de/en/unterricht/), Table 1 provides an overview. The use of DT also illustrates new models of data and workflows with new features (see 4.1). The age verification example we sketch below provides a contribution to critical data literacies (4.3), especially concerning hidden data processing in social media.

Grades and courses	No of lessons	Data	Tools
5/6 (computer science)	10	Nutritional values of food	Data cards (Jupyter Notebook without coding)
9/10 (compulsory elective)	10	Media use	(data cards) CODAP, ProDaBi-JupyterNotebook (without coding)
11-13 (computer science)	16	Media use, data from medicine	Data cards, CODAP, ProDaBi-Jupyter Notebook (without and with coding)
12/13 (elective)	Part of a year-long course	Media use, MNIST data, Parking lot (historical and live data), ..	CODAP, ProDaBi-Jupyter Notebook (without and with coding), coding with Python

Table 1. Decision trees in ProDaBi, different tools and datasets

Exploring decision trees in K-12 education offers a practical way to teach several of the following core concepts, contributing to students'' AI and data literacy. Decision trees are a form of machine learning, directly illustrating how computers can *learn* from data to identify patterns and make predictions. Their transparent, flowchart-like structure makes the hierarchical rule system derived from data relatively easy to grasp, showing students one way knowledge can be *represented* and used for *reasoning* to make classifications or decisions. Furthermore, applying decision tree models to real-world data, such as the media use data of teenagers, clearly demonstrates the *societal impact* of AI, revealing how data collected about individuals can be used to build profiles and influence online experiences through prediction. Discussions around the performance evaluation using metrics like the confusion matrix, and the consequences of different error types (False Positives and False

Negatives) within relatable scenarios (like age verification on platforms), encourage students to think critically about the ethical and societal implications of AI systems that make decisions based on data.

The process of building a decision tree with data cards follows a simplified heuristic of a machine learning algorithm. Students learn to perform a *data split* by selecting a *predictor variable* (like fat content) and a *threshold value* (e.g., more or less than 8 g of fat). The data cards are then divided into two subgroups based on this rule. For each subgroup, the majority label (e.g., most foods in this subgroup are "recommendable") becomes the decision for that branch.

Students evaluate the quality of their decision tree by counting the *number of misclassifications*—the food items that are incorrectly labelled by the rule based on their training data label. To build a multi-level tree, they repeat this process on the subgroups of data that still contain misclassified items, adding more rules based on other predictor variables.

The unplugged approach using data cards offers several benefits. It is a hands-on, active learning experience that helps students deeply understand that a decision tree is a system of rules derived from data. By manually sorting and splitting cards, students can "play the machine", gaining insight into the process before seeing it automated. The relatable context of food makes the topic engaging for young students. Studies on this teaching unit have shown promising results, providing evidence that teaching decision trees as an elementarized data-based model-building process is teachable to middle school students (Fleischer et al., 2024; Fleischer & Biehler, 2025; Podworny et al., 2025)

Decision Trees with CODAP/Arbor in Grades 9-10. After gaining a conceptual understanding through unplugged activities, students can transition to plugged activities using the digital tool CODAP (Common Online Data Analysis Platform, www.codap.concord.com) with the Arbor plugin (see arbor on codap.xyz) for decision trees (Erickson & Engel, 2023). This tool allows students to explore decision trees with larger datasets and witness the automation of the tree-building process. They can also delve into more advanced concepts like evaluating trees with separate test data that was not used for training, understanding overfitting (when a tree performs very well on training data but poorly on test data), and methods like pruning to optimize performance for new data. Combining unplugged and plugged activities provides a balanced approach, where the manual work builds foundational understanding and the digital tools enable exploration of automation and more complex ideas.

The teaching module for grades 9/10 centres around a relevant context for students: analysing data related to teenagers' media use. This is based on the annually conducted representative JIM (Jugend, Information, Medien [Youth, Information, Media]) study in Germany (Rathgeb & Schmid, 2020), which only publishes aggregate data. Our project collected its own microdata with a similar questionnaire, known as JIM-Paderborn (JIM-PB) or YOU-PB, from approximately 1200 teenagers, encompassing 161 questions about their media consumption habits, including digital device ownership, online platform usage (like Instagram, TikTok, YouTube), and gaming. Students are tasked with adopting the role of platform owners using this data to identify users (based on other media use variables) likely to play online games frequently ("frequently" or "rarely") for targeted advertising of a new game. This scenario resonates with students' daily encounters with personalized online experiences.

The Arbor plug-in enables students to build decision trees manually using a visual interface. Students can drag variables from the dataset, which is a reduced version of the full YOU-PB data in this phase for better manageability, into the tree structure. Arbor automatically splits the data based on the variable's values and displays the distribution of the target variable within each resulting branch. Students can assign a predicted label to each branch based on the majority class. A key feature is

Arbor's immediate calculation and display of true positives (TP), true negatives (TN), false positives (FP), and false negatives (FN) for the current tree configuration. This feedback helps students understand the direct outcomes of their chosen rules, seeing how many cases are correctly or incorrectly classified at each step. By adding more variables in layers, students can manually build and refine a multi-level tree, observing the cumulative effect of rules on classification results.

Arbor supports the comparison of different predictor variables' split in terms of misclassifications, so the students can easily improve their trees. After that, they understand the machine learning algorithms, namely how a full automation can happen if a system tests systematically all available predictor variables and continues to add layers as long as further predictor variables are available.

For our educational purposes, specially prepared menu-based Jupyter Notebooks can make powerful ML algorithms accessible to students without requiring extensive programming knowledge by hiding complex code behind interactive widgets. Yannik Fleischer programmed menu-based Jupyter Notebooks with a simplified version of professional algorithms. Using these notebooks, students can apply a decision tree algorithm to larger datasets, including the full YOU-PB data with nearly 1200 cases and 161 variables. This automated process, where a tree is built in seconds, powerfully illustrates the automation central to ML.

We use a second classification topic with these data, namely pursuing a question. To begin, we posed the question: How might TikTok determine a user's real age based on their behaviour on the platform? This was based on a real-world case from a few years ago, when TikTok sought to identify users under 13—who are not permitted on the platform—by making hidden inferences from their usage patterns. Exploring such implicit inferences is an important aspect of citizenship education in the digital age. Following this introduction, we presented a related and solvable proxy problem using the YOU-PB dataset: How can age be predicted based on other variables, such as patterns of media consumption and ownership of digital devices?" (see Figure 5).

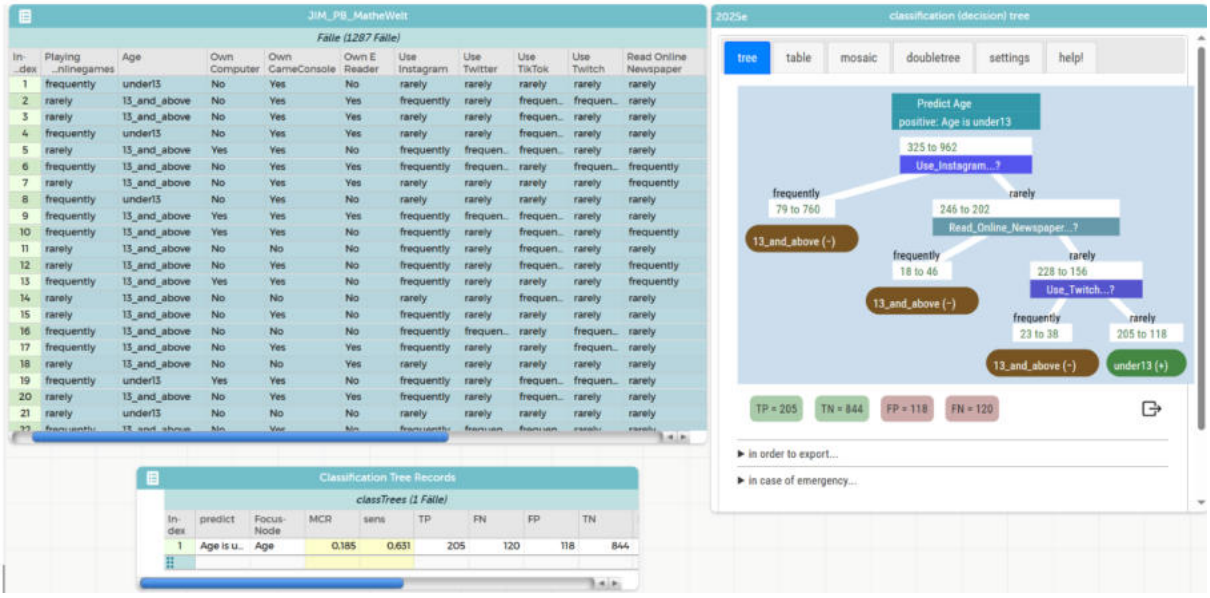


Figure 5. Screenshot from CODAP with plug-in Arbor. The decision tree predicts the variable *age* ("under_13" or "13_and_above") using the variables *Use_Instagram_?*, *Read_Online_Newspaper_?*, *Use_Twich_?* These predictor variables can have the values "rarely" or "frequently". The overall misclassification rate of this tree is 18.5%

The example allowed to discuss the different consequences of two types of misclassifications:

- Predicting that a student is 13 or above 13 years, although younger than 13
- Predicting that a student is younger than 13, although being 13 or above 13 years

The first error is more severe, because the students can use TiKTok although too young. In the second case a ban from the platform is temporary only, because they can present age proofs.

Discussing these nuances is crucial for developing students' critical understanding of how AI systems function and their real-world implications. Plugged tools like Jupyter Notebooks make it straightforward to generate and analyse the confusion matrix and other performance metrics for models built on larger datasets. They also facilitate exploring techniques like pruning, where parts of a potentially overfitted tree are removed to improve its performance on test data. Observing a tree with high training accuracy (e.g., 94.9% in one example) that performs significantly worse on test data (e.g., 75.6%) clearly illustrates overfitting and the value of evaluation and optimization methods.

For students in grades 12 and 13 who are already familiar with Python, the final phase includes hands-on programming: they either develop parts of the algorithm themselves or adapt pre-existing code within Jupyter Notebooks. These notebooks function as worked examples, aligning with the approach outlined by Atkinson et al. (2000). This progression of tools is consistent with the recommendations of Pimentel et al. (2022), commissioned paper for the workshop of the National Academies of Sciences, Engineering, and Medicine (2023), and is supported by research highlighting the educational benefits of using Jupyter Notebooks (Barba et al., 2019) and the instructional value of worked examples (Atkinson et al., 2000).

Further design research is essential to enhance instructional materials, teacher support, and pedagogical strategies. This includes gaining deeper insights into students' prior conceptions, their learning challenges and progress, as well as the specific difficulties teachers face when introducing decision trees in the classroom. While several qualitative studies in experimental classroom settings have already been conducted (Fergusson & Pfannkuch, 2024; Fleischer & Biehler, 2025; Fleischer, Biehler, et al., 2022; Fleischer et al., 2024; Podworny et al., 2025; Zieffler et al., 2021), research aimed at broader implementation and scaling of these teaching approaches remains largely unexplored.

6. Future Challenges: Addressing the multidisciplinary and transdisciplinary nature of data science and AI by staying within and moving beyond discipline-oriented school curricula

At the beginning of this paper, we highlighted Skeina's (2017) assertion that computer scientists "don't respect data" to emphasize the ways in which the domains of mathematics and computer science might be allowing some of the most important features of data—their contextual specificity and social/scientific embeddedness—to slip through the cracks of students' educational experiences. When working with data, Skeina claims, one must "[...] think like a real scientist. Real scientists strive to understand the natural world, which is a complicated and messy place" (p. 2).

It might be no surprise, then, that the examples we present do not actually come from mathematics classrooms, but rather from science and highly contextualized computer science contexts. And even in these cases, we both recognize areas in which our efforts have fallen short. While Michelle has gathered evidence that data storytelling activities engage students deeply in inquiry with data, there is not enough time, space, or preparation in most Science classrooms to address important questions related to validity and uncertainty, such as "how sure are we of this pattern or trend?" More research has begun to focus on the important role of uncertainty in K-12 science instruction (Manz & Suarez,

2018), and this may be one area where deeper data engagements can play a bigger part in the future. And Rolf at times has difficulty convincing collaborators and teachers of the equal importance of situational knowledge and data alongside algorithmic treatment as an important learning goal in Computer Science. Quite a few mathematics teachers are already reluctant to include more data exploration activities in their classroom, given their otherwise full curriculum. Teaching for understanding AI may be assigned to the computer science teacher and ethical, political, and social questions to the social science teacher. However, there is a new trend in mathematics education to explore how generative AI may support teaching and learning processes (Pepin et al., 2025). This may also open the minds for more teaching for understanding the underlying mathematical and statistical principles of AI.

Indeed, effectively integrating data science into the K-12 curriculum presents many challenges. Finzer (2013) calls this the data science education dilemma, lamenting:

> Data science is inherently interdisciplinary, yet it currently has no natural home within the existing, often fragmented landscape of school subjects. Mathematics, while quantitative, often prioritizes abstraction and may be uncomfortable with the rich, messy contexts real data provides. Physical and biological sciences, despite relying heavily on experimental data, are frequently constrained by the need to cover core concepts, pushing data analysis to a secondary, illustrative role. The social sciences, which are increasingly data-driven in practice, have struggled to equip K-12 educators with the quantitative perspective needed for meaningful data use. This lack of a clear disciplinary home means that structured, meaningful opportunities for K-12 students to engage with data analysis are rare, and many teachers may lack the necessary skills and confidence to teach data-driven lessons. (Finzer, 2013, p. 1)

Compounding these practical implementation challenges is the nature of the data science education research field itself, which is also fragmented (Rosenberg & Jones, 2024; Wilkerson, 2025). The field is currently built upon distinct, somewhat isolated clusters of foundational literature that emphasizes data science for undergraduate majors (focused on statistics and computing skills), K-12 education through learning sciences perspectives (focused on data literacy or storytelling), or computational approaches for non-majors (emphasizing specific methods like data mining or simulation). These distinct orientations highlight the need for better communication and collaboration across these scholarly divides.

These orientations are also reflected in the many different definitions and frameworks currently active in the literature, as we briefly reviewed in Section 4.1. We contend that there is nothing problematic in itself about having many different definitions, because these all serve different audiences and purposes. However, these different audiences and purposes should be clearly articulated. The field must stop trying to find a "one size fits all" solution and instead consider what definitions make sense in what contexts, and toward what ends. For example, as described above in Section 4.1, most current frameworks seem most appropriate for the tertiary level and for students who are expected to become specialists in the areas of statistics and data science. And even in our examples, we note that not *all* of the shifts and issues raised in Section 4 are addressed by a single curricular design alone. While they offer examples of what approaching *some* of these new needs and challenges may look like, we are still unsure of *which* challenges are most important to address, and *when*. If we are clearer on these more specific contexts and goals, we can understand how different definitions may build on one another or suggest progressions and paths for students and educators.

Moving forward requires addressing these challenges on multiple fronts (see e.g., Biehler et al., submitted). We need ongoing curriculum development efforts that provide clear progressions of data science concepts and skills, allowing for incremental learning and building upon prior experiences

(see DS4E, 2025 or OECD 2025 for examples). This must be supported by dedicated teacher development and training to build capacity and comfort with data-driven instruction, across subject areas and grade levels (Fergusson & Pfannkuch, 2022; Hudson et al., 2024; NASEM, 2023, Sukol, 2024a; 2024b). Building consensus on standards and core learning goals for K-12 data science is also vital to provide a shared vision and coherence. From a research perspective, fostering deeper engagement and collaboration across existing communities is crucial to build a common literature, shared vision, and convergent understanding of effective pedagogical approaches and learning trajectories.

Ultimately, the goal remains to empower students with the skills and habits of mind necessary to navigate and make sense of the data-rich world. This means incorporating data science thoughtfully across the curriculum, supported by appropriate resources and teacher training. While the field of K-12 data science education research is still coalescing, continued effort to bridge conceptual differences between the disciplines and collaborate on curriculum and pedagogy will be key to realizing the potential of data science education for all students.

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TWG 01: Argumentation and Proof

Introduction to Thematic Working Group 1: Argumentation and Proof

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Introduction

Building upon the rich tradition established in previous CERME conferences, TWG1 has consistently attracted the interest of researchers investigating the multifaceted nature of mathematical argumentation and proof across all educational levels. The group's focus encompasses both theoretical investigations into the nature of mathematical proof and empirical studies of how students and teachers engage with reasoning and argumentation in classroom settings.

At CERME14, TWG1 brought together 32 participants from 15 countries who engaged in in-depth discussions around 26 contributions: 24 papers and 2 posters. The contributions spanned a range of methodologies and educational levels, reflecting the field's diversity, depth, and international scope. The TWG1 sessions featured three whole-group and four split sessions, which fostered critical, collaborative, and respectful engagement among participants while also ensuring the coherence and unity of the TWG1 spirit in the split groups. The main points of interest from each split-group session were reported in a whole-group session for a comprehensive overview of all TWG1 contributions.

In this introductory paper, drawing upon the discussions of our group, the TWG1 papers are organised in four themes: a) Argumentation and Proof in School and Teacher Education, b) Cognitive, Affective, and Metacognitive Dimensions, c) Evaluative Practices, Contextual influences and Epistemological Perspectives, and d) Technological and Collaborative Dimensions.

Argumentation and Proof in School and Teacher Education

A broad range of studies focussed on how argumentation and proof are addressed in both school curricula and teacher education. Szűcs and Slavíčková conducted an international comparison of textbook treatments of Thales' theorem, revealing notable regional variations in reasoning opportunities and prompting discussion about curriculum design. Platz studied how children in primary education engage with proving, while Dilberoğlu, Çakıroğlu, and Haser examined a sixth grader's first proving experience, emphasizing the importance of nurturing proving practices early. At the high-school level, Niederquell and Kempen explored students' use of generic examples for

conjecturing, raising questions about how instructional approaches can help learners move from empirical to formal reasoning.

Teacher preparation and development emerged as a parallel focus. Højbjerg studied pre-service teachers' mathematical knowledge for teaching proof, highlighting the complexity of professional preparation, while Parsinia and Rott investigated how teacher training could enhance justification abilities. Knipping, Bredow, Miyakawa, Otani, and Shinno provided a cross-cultural perspective by comparing proof instruction in Germany and Japan, while Deslis, Stylianides, and Jamnik offer insights into teachers' motivational views on Lakatos-style investigation using a value-cost-expectancy framework. Hofmann and Gustavsen focused on how teachers facilitate understanding of fraction multiplication through proving, while Lestyanto and Pfeiffer explored approaches to teaching mathematical induction. Butler and Buchbinder used linguistic analysis to reveal how beginning teachers conceptualise and communicate about reasoning and proof.

The discussions focused on the importance of coherence across educational levels, the role of institutional supports, and the value of integrating research-based approaches into teacher education. Moreover, the need for more research on how teacher preparation programs can effectively integrate proof instruction across different mathematical domains was stressed. The group identified significant gaps in understanding how teachers transition from personal mathematical knowledge to pedagogical content knowledge for proof instruction. The group also discussed the role of collaborative learning in teacher education and the importance of providing teachers with experiences as mathematical reasoners before expecting them to facilitate student reasoning.

Affective Dimensions

This theme reflected the growing interest in the affective dimensions that shape students' proof-related activity. Stylianides, Stylianides, Zhang, Deslis, and Seo identified affect-related constructs—such as value, self-concept, and self-efficacy—that predict proof competence, prompting conversations on how emotional and motivational factors influence engagement with mathematical proof. Hapipi and Stylianides focused on the cognitive skill of negation and its connection to understanding proof by contraposition. Moutsios-Rentzos and Mantziou introduced the construct of proof awareness, a complex affective and meta-affective lens for gaining deeper understanding about the diverse ways that learners evaluate and reflect on proofs.

The group discussed the bidirectional relationship between cognitive and affective factors in proof learning, noting that emotional responses to mathematical arguments can both facilitate and hinder reasoning processes. Assessment practices can exacerbate proof-related anxiety, underscoring the importance of supportive learning environments, particularly during transitions to formal proof.

Evaluative Practices and Epistemological Perspectives

The evaluation of arguments and proofs and the ways these practices are shaped by broader contexts was the focus of several contributions. Saccoletto studied how students shift their argumentative strategies according to their goals and interlocutors, a topic that prompted wide discussion about the authenticity of proof tasks and communicative expectations. Fahse and Lutz proposed the RPA (rich, pseudo-factual, apodictic) model of argumentation, which distinguishes between justification and

proof; this led to reflection on how these constructs are treated across different educational cultures. Kovács and Kónya explored situational cues that influence whether students recognize a task as proof-related, shedding light on the interplay between task design and learners' framing.

Epistemological and theoretical questions were also prominent. Durand-Guerrier proposed a model-theoretic lens for studying proof didactically, while Boero discussed implicit criteria of proof validity and how they shape conjecturing processes. Matteucci, Tortoriello, and Veronesi examined how students interact with AI systems in argumentative contexts, raising questions about the epistemological status of machine-generated proofs. Fischer provided a Lakatos-inspired framework for generating refutations, while Asenova argued for integrating fallibilism with formal logic in the conceptualization of mathematical proof.

Reflections were raised about what constitutes acceptable mathematical proof and how such judgments are communicated and internalized by learners. Discussions centred on how epistemological assumptions inform instructional decisions and how proof validation practices vary across contexts. The need for educational environments that support multiple modes of reasoning while also cultivating shared norms of rigor and justification was expressed. The group also discussed the importance of making implicit evaluation criteria explicit in both research and teaching contexts.

Technological and Collaborative Dimensions

Digitally enriched learning environments were featured in several contributions. Perivolari and Psycharis examined how dynamic geometry software combined with paper folding supports proof construction, illustrating how physical manipulation can complement digital visualization. Sears and Buchbinder reported on the use of the *FullProof* digital platform to support learners' geometric reasoning through guided interactions. The two posters of TWG1 were focused on this theme, showcasing the growing interest. Ceballos, van den Bogaart, Spandaw, van Ginkel, and Drijvers discussed collaborative problem solving to develop higher-order reasoning, while Sisco and Ouvrier-Buffet presented findings on how students interact with automated theorem provers in proof-writing.

The group engaged in extensive discussions about the transformative potential of artificial intelligence in mathematics education, while emphasizing the importance of teacher agency in selecting and orchestrating technological resources, and designing tasks that leverage technology to deepen conceptual understanding and support proof writing. The participants discussed how technology can enhance collaborative learning experiences, highlighting the need for research on the long-term effects of technology integration on students' mathematical reasoning abilities and the importance of maintaining a balance between technological support and independent reasoning. These works generated lively debate about how to balance automation and human judgement in proof learning, and how to ensure that technological tools support rather than supplant critical mathematical thinking. There was also interest in how collaborative tools and digital environments could promote equitable participation and foster inclusive proof cultures in diverse classrooms.

Discussion and Ways Forward

The TWG1 sessions at CERME14 featured rich discussions that illuminated several cross-cutting concerns and future directions for the field. As reflected in the TWG1 discussions and the four

interconnected themes, three interrelated domains emerged as pivotal for advancing research and practice in argumentation and proof: theory-practice relations, interdisciplinarity and the digital, and socio-cultural perspectives.

The tension and synergy between theoretical development and classroom practice were central in TWG1 discussion: *How can theory remain productively close to the lived realities of students and teachers, while still offering robust conceptual frameworks?* There was a call to develop more holistic constructs that are relevant to the complexity of human argumentation, proof, and proving, which can also meaningfully inform instructional design and curricular decisions. This requires a bidirectional dynamic, not only translating theory into practice, but engaging in a recursive process where classroom observations and empirical work help refine and problematise theoretical assumptions.

The socio-cultural aspects of argumentation and proof were also discussed as an underexplored yet essential dimension, especially when considering that technology accessibility remains a significant equity issue: *How do socio-cultural and institutional contexts shape students' access to and participation in argumentation and proof, and what strategies promote equitable engagement across diverse learner populations?* A way forward involves the greater use of cross-cultural comparative research, longitudinal case studies, and collaborations that include researchers and educators from underrepresented contexts to examine how cultural, linguistic, and institutional factors influence students' opportunities to engage in argumentation and proof.

Interdisciplinarity is a crucial dimension in research about argumentation and proof, which was evident in last CERMEs. At CERME14, interdisciplinarity combined with and digital/multimodal tools emerged as a powerful, yet complex, area of innovation: *In what ways can interdisciplinary integration –drawing from philosophy, linguistics, and computer science– expand our understanding of mathematical argumentation and proof as educational practices?* Argumentation and proof inherently intersect with other domains (e.g., epistemology, logic, linguistics, computer science). These perspectives need to be researched, as AI and technologies such as interactive proof platforms, automated theorem provers, and multimodal environments become increasingly prominent in educational settings. The ways that technological tools affect the nature of argumentation, the epistemic roles of teachers and students, and the balance between automation and human reasoning may be among the areas for future research. The collaboration with other CERME working groups seems to be a fruitful way to gain deeper understanding about these issues. Moreover, to address this sustained and growing interest, the co-leaders of TWG1 initiated the organization of an ERME Topic Conference (ETC) entitled "Argumentation and proof in teaching and learning of mathematics and interactions with other disciplines", which will take place in June 2026 in Milano, Italy.

These areas of interest reflect the central ideas that emerged through our TWG1 deliberations, and should not be interpreted as indicating preferences for particular research directions over others. Mathematical argumentation and proof are investigated through multiple lenses and methodological approaches (including work in other thematic working groups), and TWG1 remains dedicated to celebrating and amplifying this rich diversity of perspectives.

Integrating fallibilism and formal logic: A Peircean approach to Lakatos' methodology

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Authentic learning of argumentation and proof in mathematics requires active conjecture development rather than passive reproduction of proofs. Lakatos' approach, widely used in mathematics education research, emphasizes this active process. Although Lakatos rejects formal logic as useless for knowledge development, investigating a potential logical framework for his approach could provide insights into its logical foundations and connect fallibilist and classical views on argumentation and proof in mathematics education research. This theoretical paper explores how an epistemic-logical model, mainly based on an intuitionistic version of the Peircean first-order existential graphs, can frame Lakatos' methodology, contributing to theoretical perspectives and offering a new analytic tool for researchers.

Keywords: Lakatos' approach, epistemic-logical model, intuitionistic existential graphs, fallibilism, classical logic.

Rationale

Lakatos' approach in designing and analysing meaningful mathematical activities is widely spread in mathematics education research (e.g. Deslis et al., 2022; Stylianides et al., 2022). Even though, to the best of my knowledge, investigations on the logic that could frame this approach have never been carried out in this research area. Lakatos (1976) calls his approach "the logic of discovery," (p. 3). Still, the term *logic* is intended in that context as rational scientific enterprise inspired by informal logic, i.e. "the logic of proof or of thought-experiment or of construction" (Lakatos, 1976, p. 81), rather than as deductive logic in a formal system (Lakatos, 1976, p. 3). In this sense, trying to frame Lakatos' approach by a logical system might seem nonsense. Nevertheless, discussing this aspect could be useful for at least two reasons in mathematics education research: (1) it could help to shed light on the logical bases of this approach and situate it in the panorama of logical approaches to proof in mathematics education; (2) it could act as a bridge that allows communication between fallibilist approaches to argumentation and proof in mathematics education (e.g. Stylianides et al., 2022) and approaches that suppose mathematical knowledge to be, once established, certain and unrefutable and thus framed by classical logic (e.g. Durand-Guerrier, 2005). This theoretical paper discusses how and to what extent the epistemic-logical model presented in Asenova (2023) and deepened in Asenova (2024), extended here to first-order intuitionistic logic, can frame Lakatos' approach, also keeping the door open for considering mathematical knowledge as certain, according to classical logic. It aims to contribute to the development of theoretical and methodological perspectives. and the design of an analytic tool for research in mathematics education.

Method

Hereafter, some elements of Lakatos' approach are exposed, and the epistemic-logical model (Asenova, 2023, 2024) is presented and extended from propositional to first-order logic. Then the

suitability of the model for framing Lakatos' approach from a logical viewpoint is discussed. Finally, Lakatos' lemma-incorporation method is exemplified, and the conclusions are drawn.

Lakatos' approach and the methods of discovery and verification

Lakatos (1976) challenged the traditional view that mathematical knowledge is timeless, certain, and a priori. He combined Hume's problem of induction with Popper's theory of falsification to highlight the relativity of mathematical knowledge. Lakatos (1976) provided a list of heuristically detected methods used by mathematicians in their mathematical practice for improving a conjecture and its proof through counterexamples. In this process, the type of the discovered counterexample plays a crucial role. A counterexample can be local or global; it is *local* when it affects at least one step of the proof (based on a lemma acting as an argument in the proof); it is *global* when it impacts the main conjecture and thus the conclusion itself. A counterexample can be both local and global or only one of them: if it is only local, the conclusion is not affected and only the 'guilty' argument should be changed; if it is only global, the arguments seem to work but the conclusion does not and one should search for a hidden assumption and incorporate it explicitly as a condition in the conjecture; if the counterexample is both local and global, the problematic proof step affects the conclusion and should thus be incorporated explicitly as a condition.

Lakatos' most famous example (Lakatos, 1976) is the historical refinement of the justification of Euler's formula on the relation between the number of vertices V, edges E, and faces F of a polyhedron: V-F+E=2. Lakatos presents his inquiry in a dialogic form, where a maths class discusses an informal proof of this 'naïve conjecture' provided by the teacher (it is the proof historically provided by Cauchy), introducing counterexamples (e.g. the hollow cube) that force the class to explicitly define the involved concepts and to refine the proof. The proof consists of a thought-experiment that is supposed to: remove a face of the polyhedron; stretch its surface upon a plane; section the surface following certain rules based on triangulation; remove parts of the stretched surface following rules that do not change the initial relation between V, F, and E; show that at the end the relation between V, F and E is V-F+E=2 and thus must have been so at the beginning of the procedure. Below, after introducing the epistemic-logical model, this paradigmatic example is used to exemplify the modeling of one of Lakatos' methods: lemma-incorporation.

The epistemic-logical model with quantifiers

The model, which consists of three elements —Vergnaud's (2009) concepts-in-action and theorems-in-action (Vergnaud, 2009), Duval's value of a verbalised proposition (Duval, 2007), and an adaptation of Oostra's intuitionistic propositional existential graphs (Oostra, 2022), as presented in Asenova (2024) — is extended here to predicate logic to frame the use of (counter)examples.

Concepts-in-action, theorems-in-action and the value of a verbalised proposition

Vergnaud's theory of conceptual fields (2009) aims to establish connections between the operational form of knowledge, linked to actions, and the predicative form of knowledge, linked to symbolic and linguistic expressions. In this context, two main ideas are concepts-in-action and theorems-in-action that are implicit and need to be deduced from a student's actions. Theorems-in-action are expressed as propositional sentences and are either true or false, whereas concepts-in-action are relevant or not

relevant. According to Vergnaud (2009), even if someone believes a false proposition to be true, it still qualifies as a theorem-in-action. What is important is that the individual acts as if the proposition were true. This perspective focuses on the student's knowledge, regardless of how it aligns with normative standards. In Duval's framework of the meaning space of a verbalised proposition (2007), the value-dimension links the proposition to its meaning. Here we focus only on the epistemic value (e.g., obvious, absurd, possible) and the logical value (true or false) because they allow to characterise the kind of reliability that students attribute to their knowledge: the epistemic value aligns with the idea of theorem-in-action, which are based on implicit, proposition-like knowledge, while the logical value corresponds to theorems and is supported by explicit concepts.

The system of intuitionistic existential graphs with quantifiers

Starting from Peirce's diagrammatic logic of existential graphs (EGs), divided into Alpha EGs for propositional logic and Beta EGs for predicate logic, Oostra (2022) formalised a topological system of classical Alpha and Beta EGs and intuitionistic Alpha and Beta EGs. Alpha-intuitionistic EGs (Alpha-IEGs) are composed of some or all the following elements: sheet of assertion (the plane surface), propositional letters (capital letters representing the propositions), two types of closed curves: cuts, that means closed curves that 'cut' the sheet of assertion into two regions: one inside and one outside of the closed curve (Figure 1 a) or scrolls (Figure 1 c, d). Each scroll is composed of a cut with one or more loops folded inside it (e.g., single or double scroll, Figure 1 c, d). According to Peirce, a scroll is: "a curved line without contrary flexure and returning into itself after once crossing itself" (Peirce, CP 4.564). The areas on the sheet of assertion are even or odd, according to the number of curves drawn around them (Figure 1).

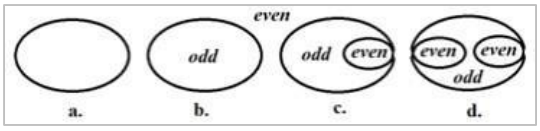

Figure 1: Elements and areas of IEGs: empty cut (a); single and double scrolls (c, d); even and odd areas (b, c, d, and the surface they are drawn upon) (Oostra, 2022, p. 136)

In a scroll, the region limited by the cut and the loops is the outer area of the scroll, and the interior part of the loop is the inner area. The scroll is a purely intuitionistic element as classical EGs work only with cuts; this is the main difference between classical and intuitionistic EGs. The interpretation of the classical and intuitionistic connectives is represented in Figure 2, where '=' stands for equivalence.

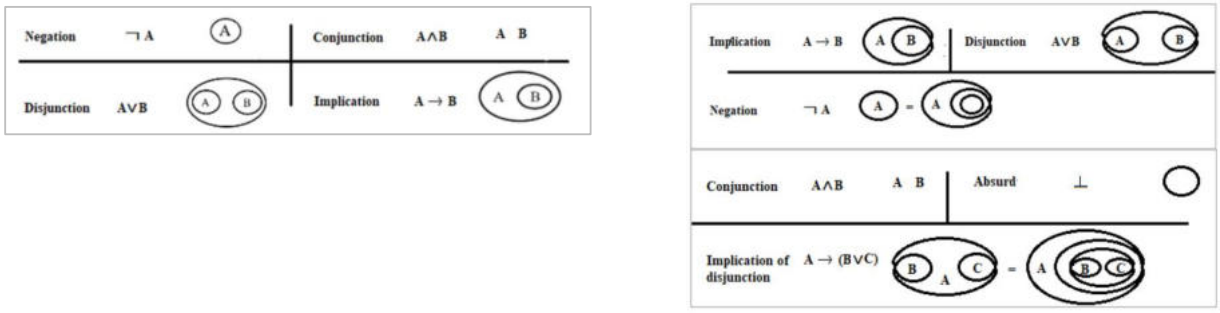

Figure 2: Basic and derived connectives for classical EGs (on the left) and intuitionistic EGs (on the right) (adapted from Oostra (2022, pp. 135–136))

According to Oostra: “we obtain the system of intuitionistic Beta graphs by adding to the Alpha system a line, which stands for a subject, element, or individual, and letters with attached lines, which stand for relatives, or predicates” (Oostra, 2022, p. 147). In this way, the system of Alpha IEGs is extended to the Beta-IEGs.

Graphically, Beta-IEGs differ from Alpha-IEGs by the use of (possibly branched) lines, called lines of identity (Figure 3a), and the use of capital letters with a positive number of (ordered) lines attached, symbolising predicates (Figure 3b); the existential quantifier corresponds to the line of identity, and the universal quantifier to the combination of the line with a scroll (Figure 3c) (Oostra, 2022).

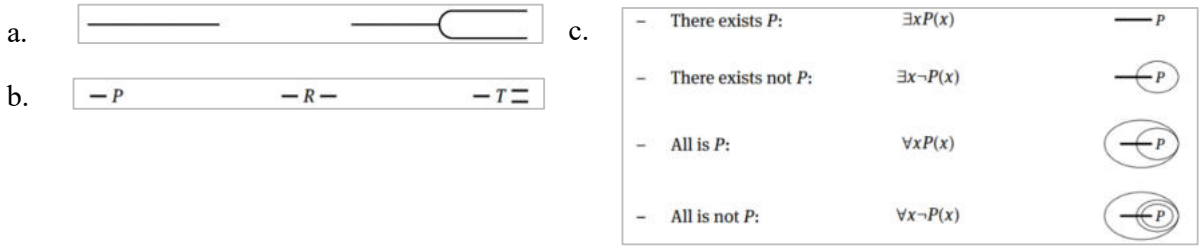

Figure 3: Existential and universal quantifier in Beta-IEGs (adapted from Oostra, 2022 (pp. 147-148))

Summing up, a Beta IEG is a diagram consisting of a finite combination of one or more of the following elements: predicate letters, cuts, scrolls, and lines of identity.

The interpretation of the intuitionistic Beta-IEGs is based on the clauses for Alpha-IEGs, with the addition of the following: drawing an identity line means asserting the existence of an individual; writing a letter with lines attached to it means that the predicate it represents holds for the involved individuals; joining two lines of identity means identifying the individuals they represent.

In Asenova (2024) the Alpha IEGs were introduced with some additional transformation rules compared to Oostra’s (2022) system, without affecting its soundness because these rules can be derived from the others. The additional rules are introduced due to the specific needs in mathematics education research which is interested in linking together aspects related to a logic concerning the epistemic value (here represented by the intuitionistic logic) and aspects related to a logical value concerning certainty (here represented by classical logic), as well as to the need to express an increasing/decreasing epistemic value. Even thought the additional transformation rules are not necessary, they are explicitly formulated as rules to render the system more user-friendly.

The lines of identity do not affect the parity of the areas. Hence, the rules of transformation that hold for Alpha IEGs hold also for Beta-IEGs, with some additions related to the line of identity. In this way, the transformation rules for Beta-IEGs can be conceived as an extension of the transformation rules for Alpha-IEGs and the latter can be conceived as an extension of the transformation rules for classical EGs. Therefore, the system of transformation rules for Beta-IEGS presented below includes the transformation rules (TRs) for classical logic (the part before the first slash in the first five rules), intuitionistic propositional logic (the part between the first end the second slash in the first five rules and rules 6 and 8), and intuitionistic predicate logic (the part after the second slash in rules from 1 to 5 and after the first slash in rules 6 and 8). TR 7 is universal and holds for all kinds of graphs. Below, ‘ITRn’ stands for ‘intuitionistic transformation rule number n.’

ITR1: *Erasure*. In an even area, any graph may be deleted/ any loop within an even area may be erased, together with its contents/ in an even area, any line of identity may be cut. ITR2. *Insertion*. In an odd area, any graph may be added/ if the odd area is limited externally by a cut, a loop containing any graph may be added to the cut/ in an odd area, two lines of identity may be joined. ITR3. *Iteration*. A graph can be repeated in its own area or in any area contained in it that is not part of the graph itself/ any loop may be iterated, together with its contents, on its own cut/ a branch with a loose end may be added to any line and any loose end of a line may be extended inwards through cuts or loops; when there are lines of identity involved in the graph to be iterated, they must correspond exactly to those of the original graph. ITR4. *Deiteration*. Any graph may be deleted if a copy of it persists in the same area or any area around it; a loop, together with its contents, may be erased if another loop with the same contents is present on its cut/ a branch with a loose end may be removed from any line; any loose end of a line may be retracted from the outside (retracted, as a reverse of iterated) through cuts or loops; when there are lines of identity involved in the graph to be deiterated, they must correspond exactly to those of the outside copy of the graph. ITR5. *Scrolling*. A scroll with an empty outer area may be drawn around or removed from any graph on any area/ The application of this rule is not prevented by the presence of lines that cross both the cut and the loop of the scroll with an empty outer area, that is, that pass from outside the scroll to the inside of the loop. ITR6. *Increasing or decreasing loop and increasing or decreasing cut*. In a scroll, the area enclosed by the loop can increase (↑) or decrease (↓) and the area of the cut can increase (↑) or decrease (↓)/ when the initial loop or scroll crosses lines, the increasing or decreasing loop or scroll must keep crossed the same lines. ITR7. *Topological equivalence*. A graph can be transformed into another equivalent by continuously transforming the curves it is composed of. ITR8. *Detachment*. A loop can be detached from the cut/ the detached loop must keep crossed the same lines as before.

The ITRs from 1 to 5 are the original rules formulated by Oostra (2022), while the ITRs 6, 7 and 8 are the rules added for the purposes in mathematics education research. With respect to Asenova (2024), ITR 6 is extended by introducing increasing/ decreasing cuts; this can be justified by ITR7.

Framing Lakatos' approach by the epistemic-logical model with quantifiers

One of the main aspects that makes the model described above useful for framing Lakatos' approach is that it is based on intuitionistic existential graphs which see classical existential graphs as a special case. Indeed, the main difference between intuitionistic and classical EGs is the meaning of the universal quantifier within the corresponding first-order logic.

The graph in Figure 4a represents the intuitionistic universal quantifier referred to the predicate P and its meaning is 'All is P;' in Figure 4b the classical universal quantifier $\forall x P(x)$ is represented; in Figure 4c the negation of 'All is P,' i.e., 'All is not P,' is represented.

Figure 4: Graphs representing the universal intuitionistic quantifier (a), the universal classical quantifier (b), the negation of the universal intuitionistic quantifier (c), and the steps to switch from 'All is not P' to 'There are P' (d)

In intuitionistic logic 'All is not P' does not imply that there are not objects satisfying P, as this is the case in classical logic, where $\forall x \neg P(x)$ implies $\neg \exists x P(x)$ ('$\neg$' represents classical negation and '$\exists$' represents the classical existential quantifier). Indeed, in intuitionistic logic the statement 'There are P' can be derived from the statement 'All is not P' (Figure 4d). This means that the judgment on the domain of a conjecture is formulated 'until proven otherwise.' In this sense, even if at a given moment it seems that a conjecture fails because it is falsified, one could still detect a safety domain where it holds. This aspect is captured by intuitionistic logic and is graphically represented by the existential graphs. Vice versa, in classical logic, the statement '$\exists x P(x)$' cannot be derived from $\forall x \neg P(x)$, as they are contradictory. Summing up, intuitionistic universal quantification keeps open the possibility of increasing or decreasing the domain of a conjecture, as it refers only to the already examined domain of the underlying concept, which could be altered if counterexamples or previously unconsidered examples are discovered.

Furthermore, the model can frame the epistemic value assigned by a subject (obvious, absurd, possible, probable, etc.) to a statement, suspending the judgment on its logical value and expressing his/her increasing or decreasing convincement (ITR6). Nevertheless, the model does not require to agree with the idea that mathematical theorems are never definitely true or false, embracing a fallibilist position on mathematical knowledge. Indeed, it keeps the door open to refutation, as described above, but also allows to shift to certainty. According to the system of IEGs presented in the previous section, switching from the graph represented in Figure 4a to the one in Figure 4b is allowed (ITR8), but not vice versa. This does not contradict Lakatos' position: even if it is generally fallibilist, it does not deny the existence of statements that are undoubtedly true (e.g., *The sum of the angles of a triangle in plane geometry is 180°*), but highlights that they belong to "mature theories," while the fallibilist approach refers to theories that are "under construction" (Lakatos, 1976, p. 42).

Mathematics taught at school (and to a large extent at university) belongs to mature theories (e.g. Euclidean Geometry or Calculus), at least from the expert's point of view. Nevertheless, from the student's point of view, this does not need to be the case: when students formulate a conjecture and try to prove it or try to identify a safety domain for it, they might be more or less convinced that it is right (according to Duval's distinction, they assign an epistemic value to the statement rather than a logical one). If (counter)examples or exceptions arise, they must be considered revising the definition of the involved concepts, the conjecture, or the just provided proof. Usually, the teacher knows the largest domain of the conjecture and the appropriate definition of the involved concept that makes the proof work. Once this domain is achieved, she can support students in shifting from the epistemic to the logical value. Summing up, from a logical point of view, the presented epistemic-logical model reconciles a perspective based on a constructive approach, interested in conjecture formulation and discovery, and based on epistemic aspects, with an approach based on truth-valued logic.

The lemma-incorporation method represents, according to Lakatos, the core of his proof-and-refutation method. It allows for the improvement of the proof itself by incorporating a local counterexample that refutes a proof step, i.e. a lemma. According to the polyhedra example proposed by Lakatos, the result of the lemma-incorporation method is that a polyhedron must be simple and with simple connected faces. This allows for the incorporation of two 'guilty' lemmas that undermine the proof. Lakatos states that this method "virtually *summed up the proof in the lemma-incorporating*

theorem" (Lakatos, 1976, p. 42, emphasis in original). The sequence of Beta-IEGs in Figure 7 shows how the basic idea of lemma-incorporation can be modelled.

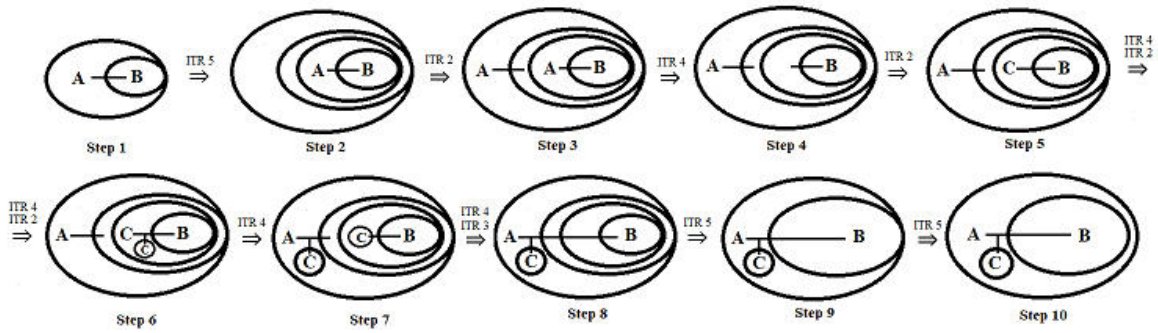

Figure 7: Modelling of the lemma-incorporation method by the epistemic-logical model

In Figure 7, Step 1 represents the statement 'All A are B,' i.e., 'All simple polyhedra are Eulerian.' Steps from 2 to 5 are technical steps needed to introduce a lemma (C) used in the proof. In this way, in step 5, we have the statement 'All A are B because of C', that means 'All simple polyhedra (A) are Eulerian (B) because, in the triangulating process, by drawing a new diagonal edge, one always increases by one the number of edges and faces (C)'. According to Lakatos (1976), this lemma becomes suspicious as it turns out that in the case of the crested cube (a cube with a smaller cube setting on top of it), one needs to increase the edges by two to increase the number of faces by one. Step 6 represents the 'appearance' of the counterexample, i.e., the crested cube, which is a simple polyhedron but for which the lemma does not work. The counterexample is represented by the negation of C: a cut with the propositional letter 'C' inside it. In step 7, the lemma is turned into the condition 'polyhedra for which any face dissected by a diagonal edge falls into two pieces,' equivalent to the 'guilty' lemma. Steps 8 and 9 are technical steps that show how the graphs evolve to the final statement in step 9 ('All simple polyhedra for which any face dissected by a diagonal edge falls into two pieces are Eulerian'), while step 10 represents the possible shift to certainty and classical logic.

The lemma-incorporation method leads to consider the interplay of concepts-in-action (e.g., the concept of simple polyhedron) and theorems-in-action (the incorporated lemma) and their evolution to explicit concepts and theorems. Indeed, step 10 represents the possible shift from a theorem-in-action to a theorem and thus from an epistemic to a logical value, showing how the first two elements of the theoretical framework are used to interpret the third one.

Conclusions

In this paper, the suitability of the presented epistemic-logical model in framing Lakatos' approach to proof and refutations (Lakatos, 1976) was discussed and exemplified. From a theoretical viewpoint, this allows us to situate Lakatos' approach in the panorama of logical perspectives on argumentation and proof in mathematics education, associating it to a constructivist logic that focuses on processes rather than on states of affairs. From a didactic point of view, this analytical tool can be used by the researchers to better frame student's reasoning when it represents aspects difficult to frame with classical logic. Indeed, the technical part of the model allows to symbolically represent the reasoning similarly as this is done by symbolic logic, but the topological form of the representation allows us accounting for continuities and discontinuities in the reasoning process and for growing and

decreasing epistemic values, as it is shown in the examples in Asenova (2023, 2024). Furthermore, it accounts for shifts between an epistemic and a logical value, and thus for shifts from implicit theorems-in-actions to theorems, as shown in the example in figure 7. As the model involves not only a fallibilist but also a truth-value perspective, it helps to bridge communication between fallibilist approaches to argumentation and proof (e.g. Stylianides et al., 2022) and approaches that suppose mathematical knowledge to be, once established, certain and unrefutable (e.g., Durand-Guerrier, 2005). Further research should be done to deepen the model's potential in bridging this gap and in using it for the design of tasks for teacher training courses.

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Presumed criteria of validity of proof drive student teachers' actions in conjecturing and proving: a case study

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The aim of this paper is to show and interpret, in the perspective of Habermas' rationality, how prospective teachers attending a course of mathematics education of the Master Degree curriculum in mathematics behave in a conjecturing and proving activity in elementary theory of numbers. What emerges from the reported study is that (for all students) only algebraic proof guarantees the necessary rigor, and that (for some students) only the exploration of the conjecturing situation by means of algebraic expressions and properties is a productive way of getting conjectures. Moreover, epistemic and strategic pitfalls affect several students' algebraic conjecturing and proving activities. The implications of the results for mathematics teacher education will be discussed.

Keywords: Habermas' construct of rationality, proving and proof, prospective teachers.

Introduction

In the Abstract of their paper (Healy & Hoyles, 2000) the Authors wrote:

> After surveying high-attaining 14- and 15-year-old students about proof in algebra, we found that students simultaneously held 2 different conceptions of proof: those about arguments they considered would receive the best mark and those about arguments they would adopt for themselves. In the former category, algebraic arguments were popular. In the latter, students preferred arguments that they could evaluate and that they found convincing and explanatory, preferences that excluded algebra. (…) The most successful students presented proofs in everyday language, not using algebra (p. 396).

Students' behaviors may depend (among other factors) on teachers' conceptions and experiences about proof and proving, and related teaching choices. Several researchers investigated teachers and prospective teachers' conceptions (see for instance Knuth, 2002, who refers to the USA, and Lesseig, Hine, Na, et al., 2019, who present a comparative study concerning Australia, USA and Korea). A smaller number of studies concern teachers' and prospective teachers' proving. Among them, Dogan and Williams-Pierce (2021) investigate the use of generic examples in teachers' proving.

The aim of this paper is to investigate how prospective mathematics teachers in the first year of their Master Degree studies in mathematics deal with a conjecturing and proving task in elementary theory of numbers. The results of this study shed light on the possible source of school students' behaviors through their assimilation of teachers' ideas and practices developed during the university studies in mathematics, and suggest teacher education choices like, for instance, those presented in Guala & Boero (2017). Results also put into evidence distortions of conjecturing and proving processes, due to the fact that the related epistemic and strategic requirements are not taken in charge by students, thus suggesting more general changes in the mathematics curriculum of prospective teachers.

Theoretical assumptions

Concerning proof, I will refer to the construct of Theorem by Mariotti (2006), which consists of a triple (statement, proof, reference theory), where reference theory provides not only the primitive elements, the axioms, the definition and the theorems that may be used in the proof, but also the criteria of validity of proof. This choice on one side allows to take in charge and compare theorems and proofs of different historical periods and different disciplines, on the other provides the teacher (and the students too!) with the opportunity of considering different kinds of acceptable proofs.

Concerning the proving process, I will refer to the distinction between the process of proving and its product (proof), and to the distinction between two phases of proving: exploration, and construction of the proof (not necessarily in this order, since difficulties in the phase of construction may bring to a new phase of exploration). In the literature we find more detailed segmenting of the proving process, but for the purpose of this study it is sufficient to distinguish between these main phases.

In recent years, the Habermas' construct of rationality (Habermas, 1998) has been used to compare different rationalities in mathematics (Guala & Boero, 2017), to inform the teachers' questioning ("rational questioning", in Zhuang & Conner, 2022), to analyze students' mathematical activities (Urhan & Bülbül, 2023). Indeed, the Habermas' elaboration allows to take in charge both the product of an activity, and the subject's 'rational behavior' during the activity. While the tension towards what should be a rational, ideal behavior may inform actions to promote students' rational behavior (like rational questioning). This variety of uses of the Habermas' construct depends on the nature of its three components in interaction among them: epistemic rationality (justifications for the claims in a given cultural context), teleological rationality (including the identification of the reasons why the chosen strategy resulted or not in the attainment of the goal), communicative rationality (the choice of the means, in a social context, to reach the interlocutor – even the subject, in the inner dialogue). Regarding the subject of this paper, I will make a distinction between the epistemic requirements that should inform the two main phases of proving, in strict connection with the teleological dimension (concerning the strategic choices that result in the evolution of the process towards the aim to be attained). In the exploration phase, a statement resulting from an induction or an abduction or an analogy is acceptable, what counts is its use to build the proof. However during the exploration epistemic and teleological criteria are not irrelevant: algebraic transformations or references to known results must be correct, and the link with the aim to attain (pertinence) must be taken into account.

Method

15 students attended a mathematics education course in the second semester of the first year of the Master Degree studies in Mathematics in an Italian university, after three years and one half of university studies in mathematics; at the beginning of the fourth year they had made the option of a MD curriculum oriented towards the profession of teaching mathematics in secondary schools. This option included about one half of credits of advanced courses of mathematics, and one half of credits for courses that were more or less strictly related to teacher education (mathematics education, history of mathematics, epistemology of mathematics, etc.). The course was based on students' work on problem solving (including proving), with problems that may be currently proposed in the last years of the high school or at the beginning of the university studies, followed by discussions on students'

individual solutions (different strategies, mistakes, obstacles met by students), possibly followed by comparisons with high school and first-year university students' solutions. This activity resulted in theoretical developments (e.g. didactic contract, theories of teaching and learning) and in implications for secondary school teaching of mathematics. The final evaluation of students was based only on their discussion and revision of the personal written productions during the course (solutions of mathematical problems and ideas on the educational problems dealt with during the course). This mode of evaluation encouraged students to write down all their thoughts and essays during problem solving activities (in the perspective of their final revision). The following individual task (to be performed in about 90') was proposed after two weeks of the course. It was aimed at opening a cycle of lessons on proof and proving in school mathematics, according to Guala & Boero (2017).

> Which is the Greatest Common Divisor (GCD) of the products of three consecutive natural numbers? Justify your conjecture in a rigorous way! Then try to generalize your conjecture, and justify such more general statement. Write down your reasonings, even those that you consider useless, according to their order.

Different kinds of proof are possible, according to different epistemic criteria for validity: a purely semantic proof (by considering where multiples of 2 and 3 are in the sequence of natural numbers), a mixed semantic-algebraic proof (by considering the products n(n+1)(n+2) and the necessary presence of one multiple of 3 and at least one multiple of 2 among the factors), and a purely algebraic proof (for instance, the algebraic proof by induction of the divisibility by 6 of the products of three consecutive numbers, based on the lemma that odd and even numbers alternate in the sequence of natural numbers). Students' individual written productions (38 A4-pages from 15 students) and audio-recordings of the subsequent 90' discussion were available. Solutions by 6 students are reported below. They were chosen as representatives of the whole set of solutions.

Analysis of students' productions

Federica starts algebraic exploration: From x(x+1)(x+2) to x^3+3x^2+2x. From (x+n)(x+n+1)(x+n+2) to the division of the resulting polynomial by x^3+3x^2+2x (one page of calculations, ending with "not this way"). Numerical essays follow: 2.3.4, 7.8.9, 10.11.12, 21.22.23, 101.102.103, each of them resulting in careful prime numbers factorization, like for instance 101.102.103=101.2.3.17.103

> It seems that 6 is the GCD because each product of three consecutive numbers contains 2 and 3. Indeed at least one number out of three is even, thus it is divisible by 2, and another will be multiple of 3, because in each triple of consecutive numbers there is 3: n(n+1)(n+2). (n-1)(n-2)(n-3). If n is even, n and n+2 are even, thus divisible by 2. If n is odd, either n+1 even or n-1, n-3 even, thus divisible by 2. If n is multiple of 3, the product is divisible by 3. If n is not multiple of 3, n+1 or n+2 divisible by 3 (indeed I find one multiple of 3 in the product of three consecutive numbers). But this only confirms that 6 is the GCD. I must prove it.

One page of calculations follows, with division of x^3+3x^2+2x by x-6 (repeated twice) and other trials with expressions like those written at the beginning. With the comment "I don't succeed". Federica engages in the use of algebraic transformations to produce the conjecture. Then numerical essays suggest the conjecture. Federica performs well in the following exploration, which in the reality

results in a mixed semantic-algebraic proof construction. In the discussion she will say: “I was not sure of the validity of the proof, only algebra ensures it”. Hence she comes back to the algebraic expressions and their unsuccessful manipulations.

No reflection on chosen unsuccessful strategies emerge from the text (a requirement for teleological rationality). Epistemic criteria privileging algebraic proving prevail on the whole activity (including conjecturing) and inform it in a blind way. Algebraic manipulations are carefully performed, but a clear connection of algebraic expressions with the aims to attain (a conjecture, then a proof) is lacking.

Amelia

> 1.2.3=6 5.6.7=210 I consider the product 1.2.3=6 This is a product of 3 natural numbers, thus the GCD of all the numbers obtained by multiplying three consecutive natural numbers will be one of the numbers 1, 2, 3, 6. I start by an even number: $2m(2m+1)(2m+2)=2[m(2m+1)(2m+2)]$. $(2m+1)(2m+2)(2m+3)= 2[(2m+1)(m+1)(2m+3)$]. Thus 2 is a common divisor of all the products of three consecutive natural numbers. I consider the list of multiples of 3. 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18,19, 20, 21, 22, 23, 24. Given how this table is made up, after 3, I find that after two numbers that are not multiple of 3, I find one number which is multiple of 3. Then I consider the case in which the first number that I choose is the first number after a multiple 3n of 3: $3n+1$ 1°, $3n+2$ 2°, $3n+3$ 3°. Their product will be: $(3n+1)(3n+2)(3n+3)=3(3n+1)(3n+2)(n+1)$. Thus a number that is divisible by 3.

The same reasoning for the second number after a multiple of 3.

> Then my conjecture is that the GCD of all the numbers that are products of three consecutive numbers is 6. In my opinion the only step to be justified in a more rigorous way is the step in which I say that after two consecutive numbers that are not divisible by 3, I find one number multiple of 3. If by absurd 3 would be not a divisor of $n(n+1)(n+2)$ then $n(n+1)(n+3)\neq 3k$ (…) $n^3+3n^2+2n\neq 3k$. If $n=3m$ we get the absurd: $(3m)^3+3n^2+2.3m\neq 3k$. By this way I think to be unable to reach something. I try to exploit the unique factorization of natural numbers.

An algebraic essay with general factorization of n follows.

> It looks useless. I reflect on the construction of the multiples of 3. 3n means 3+3+3…+3 n times. The following multiple of 3 will be 3+3+3…+3 (n times) +3, which means 3+3+3…+3 +3 (n+1 times). Then two numbers, which are not multiples of 3, are between two multiples of 3 by definition of multiplication. The generalization is that the GCD of the products of n consecutive numbers is n!

During the discussion of this proof Amelia says: “It is not a rigorous proof. I was willing to use algebra, but I was not able to do it”. Amelia looks free from the obligation of using algebraic expressions in the conjecturing phase, which in the reality contains a ‘justification’ that looks as a mixed semantic-algebraic proof. In the proving phase she does not rely upon the work done before and tries a proof by absurd. Then she comes back to her previous reasoning, and she moves towards an exploration of what happens on the number line. The reasoning stops and a general conjecture is produced. Then two different unsuccessful trials of proving (by absurd, and by using the unique factorization theorem) follow. Finally, Amelia produces a semantic proof that will not be

acknowledged by her as "rigorous" in the discussion. We may identify in Amelia's work a confusion between the epistemic requirements of conjecturing and those of proving, resulting in a confuse choice of strategies to attain the related aims, without any teleological reflection on them. For her, semantic justifications do not fit epistemic criteria of validity.

Sara gives the definition of GCD of two numbers, then she finds the GCD of 1.2.3=6 and 2.3.4=24, of 2.3.4=24 and 4.5.6=120, of 1.2.3=6 and of 5.6.7=210. For 10.11.12=1332 and 8.9.10=720, she gets the GCD of 1332 and 720 through the Euclid's algorithm and she finds 36 (which in any case is not the GCD of 10.11.12=1220 ≠1332). Then she states that:

> I must find a general rule and to prove it. I observe that all the GCD that I found till now are multiple of 3 and 2. GCD (h,z) is the greatest multiple of 3 that divides h and z. Each natural number h greater than 0 which is the product of 3 consecutive natural numbers contains one multiple of 3, thus h=3k. I re-write GCD (h,z)=GCD (3k, 3f). By definition of GCD, GCD(3k,3f) is a multiple of 3. Now I must prove that g= GCD (h,z) is the greatest multiple of 3 which divides h and z, that is: all the other multiples of 3 are not the GCD. By absurd, let us suppose that g'<g, g'=GCD (h,z) g'=3t g=3k g'<g =>t<k, false because, by definition of GCD, this must be the greatest of the common divisors.

Sara considers a hypothetical GCD suggested by numerical exploration as a multiple of 3 and 2, then she tries to prove that such GCD is the greatest multiple of 3 that divides two products of three consecutive numbers. The first Sara's (logical) mistake derives from the lack of epistemic control of the nature of the conjecture in the transition to proof; it is followed by a second mistake in the management of the proof by absurd.

Giorgio writes that to find the GCD of a(a+1)(a+2) and c(c+1)(c+2) we may represent (according to the theorem of factorization of natural numbers) the two products as products of prime numbers a_1, …a_p and c_1… c_s. "As a generalization, I consider these products: a(a+1)(a+2)… (a+n) and c(c+1)(c+2)….(c+m) where a, n, c, m are natural numbers. These numbers are products of prime numbers, so if I take each number and I write it as product of prime numbers, I can find the GCD".

Giorgio presents what for him is a general method to find the GCD in the case of the products of three consecutive numbers, without realizing that the considered couples of products might not lead to it (epistemic check related to the request of the task is lacking, like in the case of Sara), then he moves to describe a similar method for a more general situation (the GCD of products of any number of consecutive factors). The effort of using the factorization theorem of natural numbers brings Giorgio to a higher level of not well specified generalization. Does the generalization mean that we consider all the products of a whatever number of factors each? Then the GCD would be 2. Giorgio's whole production lacks a clear distinction between the phases of exploration and proof; also the epistemic check of the connection between what could be obtained from the chosen strategy (the GCD of two products of numbers) and what should be obtained (the GCD of all the products of the same number of factors) is lacking in general.

Cecilia starts by considering some examples: 5.6.7, 3.4.5, 6.7.8, 2.3.4 and she finds that all of them are divisible by 6, then she focuses on 1.2.3=6 and she makes the conjecture that a number ≤6

is the GCD of all the products of three consecutive numbers (including 6, which is a divisor of 1.2.3). "The product of three consecutive numbers is even because they may be: even.odd.even=even or odd.even.odd=even", thus Cecilia concludes that " 2≤ MCD ≤6, thus GCD may be 2, 3, 6". Then Cecilia writes the product of any three consecutive numbers in this way: "n≥ 2, (n-1)n(n+1) = n^3-n n^3-n is divisible by 6 ⇔ a natural, positive number k exists, such that n^3-n=6k". Then she draws the Cartesian graph of y=x^3-x ("for x>0 because they are natural numbers") and two parallel lines for k=0 and k=? She finds that (2,8) belongs to the graph of y=x^3-x and she writes that the straight line y=6k meets such graph in the point (2,8) when k=1: "8-2=6k 6=6k=> k=1". Her conclusion is that "n^3-n is always divisible by 6, because k exists, such that n^3-n=6k, which implies that 6 is the GCD".

Cecilia moves from the conjecture "GCD may be 2, 3, 6" (we observe that 3 as GCD contradicts the previous exploration) to a proof related to GCD=6 without any explicit reason for choosing 6. The conjecturing phase looks poor (or poorly reported in her text), even if the justification of the possible value 6 for the GCD shows that attention is paid to the epistemic requirements inherent in the definition of the GCD. The proof results from the solution of an equation, supported by graphic considerations (an algebraic-analytic method). During proof construction, Cecilia loses the contact with the situation modeled by the algebraic expression (a requirement of epistemic type): she looks unaware of the fact that she has justified the validity of the conjecture only in the case n=2.

Anna starts by writing and transforming algebraic expressions (similar to Federica's). After a few essays, she moves to a semantic exploration by using the Euclidean algorithm. She considers 5.6.7 and 18.19.20 and she finds that their GCD is 6 and she writes the conjecture: "The GCD of products of three consecutive numbers is a multiple of 3". Then she tries to prove algebraically that the GCD of n(n+1)(n+2) and (n+8)(n+9)(n+10) is divisible by 3; she transforms both products into polynomials in the x variable, then she concludes "I am not able to go further". Then she generalizes her conjecture to the GCD of the products of n numbers by writing that it should be n, and she tests this conjecture with the products 1.2.3.4 and 9.10.11.12. Then she tries a proof of the conjecture related to the products of 3 numbers by multiplying the polynomials that she had already got before and she finds that some of the coefficients of the product are divisible by 3, while others are not divisible by 3.

Anna's exploration, performed in the case of two products of three consecutive numbers, results in the general conjecture that the GCD is a multiple of 3. This conjecture is immediately considered for an algebraic proof. The failure is followed by a general conjecture (n is the GCD of the products of n numbers) that is weaker than what was conjectured in the case n=3. The conjecturing phase is poor on the teleological side (only one case of two triples of consecutive numbers is considered) and on the epistemic side (multiplication of two polynomials in the x variable). Like in the case of Federica, the initial algebraic exploration is unrelated to the search for the GCD.

Results and conclusion

By considering the students' texts, we may realize that no student considers the possibility of delivering a non-algebraic proof as a valid proof, and even those who produce a semantic or an algebraic-semantic proof (Federica, Cecilia, Amelia) look unsatisfied with it and try to get a purely algebraic proof (by algebraic transformations or by using the unique factorization theorem), or an algebraic-analytic proof, like in the case of Cecilia. For Federica, Giorgio and Anna, conjecturing is

driven by the need of using algebraic expressions and algebraic manipulations, or a theorem. The reason for this behavior was made explicit in the subsequent discussion: 'Only algebra guarantees that the conjecture is a good conjecture" (Federica), "I agree" (Giorgio and Anna). It is like if, in order to be a good conjecture, it must be derived through the same process (algebraic, or relying upon a theorem) that should bring to prove its validity. Even in the case of an exploration of semantic nature, what prevails is not the specific quality of a conjecture (strength in particular), but the immediate search for its validity.

Teleological requirements for conjecturing (to explore the situation in order to get the widest information and to notice regularities, and then to evaluate the outcomes and the adopted ways of exploration) and proving (specially, to evaluate the failures in order to identify possible, alternative strategies) are not taken in charge. What emerges in some cases is that students move from one essay to the other, the only criterium is the use of algebraic language or the unique factorization theorem or the Euclidean algorithm both in the case of conjecturing and in the case of proving.

In general, we may say that epistemic (presumed) reasons for validity of proof drive students' actions in conjecturing (Federica, Giorgio, Anna among the students of this paper, and 9 students out of 15 attending the course) and in proving (5 out of 6 students of this paper, with the only exception of Sara, as well as in the case of 12 students out of 15 attending the course).

The fact that the privileged kind of proof is the algebraic one agrees with Healy and Hoyles' results with younger students (and we may imagine that if no intervention is made on these prospective teachers, they will reproduce a similar situation in the school). Also the fact that only some conjecturing and proving processes that keep a verbal contact with semantic are successful agrees with Healy & Hoyles (2000), with a striking similarity in spite of more than seven intermediate years of studies in mathematics. The only difference concerns the fact that "students preferred arguments that they could evaluate and that they found convincing and explanatory" (Healy & Hoyles, 2000, p. 396), while the prospective teachers of the present study are not satisfied with those arguments.

In some cases (Sara, Giorgio, Cecilia), the contact is lost with the epistemic requirement of mathematical modelling, regarding the necessity of checking the relationship between the model (the algebraic expression) and the modelled situation (in the domain of natural numbers). See Morselli & Boero (2011). And logical mistakes concerning conjecturing and proving emerge: in particular, in Giorgio's work the presumption to be able to derive the GCD of all the products from comparison of factorization of couples of products (performed by using the theorem of factorization of any natural number in an unique product of prime numbers); and the conjecture concerning the GCD of all the products of m and n natural numbers (without realizing that the GCD would be 2). Also Sara tries to derive the conjecture from the Euclidean algorithm applied to couples of products, while Amelia tries to do it to get a valid proof. It is like if the only reference to theorems or theoretically based procedures, learned in the Algebra course, would guarantee the validity of the results.

The number of students in this study is low, but similar behaviors were observed in the same course in previous years on similar tasks. What emerges from the performed analyses suggests that a big gap exists between how students were prepared in the standard mathematics courses at the university, and the needs inherent in their future profession as teachers. Authentic conjecturing looks as a rather

strange activity for the majority of the students considered in this study; proving is dominated by the need of conformity with an algebraic proof. Semantic proofs or mixed algebraic-semantic proofs are excluded from the range of 'rigorous proofs'. However a program of cultural analysis of the content to be taught, like that hypothesized and exemplified in Guala & Boero (2017) and related to Mariotti' construct of theorem (Mariotti, 2006), might be only a partial answer to the needs emerging from the present study. Indeed, the present study puts into evidence more general, problematic aspects of the basic preparation in mathematics (proving in particular) of future mathematics teachers. The resulting needs concern, in particular, genuine conjecturing and proving activities by students at the high school and the university levels, and related reflections on what conjecturing and proving and proof mean. Moreover a careful checking of propositions in proving should be related to the aims of their use, according to a joint consideration of epistemic and teleological requirements of proving and proof.

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Linguistic analysis of beginning mathematics teachers' discourse about reasoning and proof

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Humans' use of language and the choice of words they use in their discourse on a particular topic, may reveal their inner conceptions and feelings about this topic. We apply this perspective to analyze three beginning secondary mathematics teachers' language about reasoning and proof (RP) – the topic that poses persistent challenges for students and teachers alike. In this longitudinal study, we examined characteristics of beginning teachers' RP language during their supervised student teaching (internship) and the first year of autonomous teaching. The data collected through observation debriefs and interviews were analyzed using a combination of innovative computational methods (R-software) and grounded theory. The results reveal that, while the three participants had similar mathematical and pedagogical backgrounds in RP, they recontextualized RP differently in their classroom practice, according to their respective contexts and views.

Keywords: Reasoning and proof, Linguistic perspective, Novice secondary mathematics teachers

Introduction

Teacher education programs worldwide aim to prepare future mathematics teachers to enact reasoning and proving (RP) in their classrooms (Conner et al., 2014; Wasserman et al., 2023). As teacher candidates transition from university to school teaching, they face additional challenges, specifically integrating RP into their teaching practices. For example, Stylianides et al. (2013) found that novice elementary teachers with strong mathematical knowledge and productive RP beliefs had to overcome students' pre-existing unproductive classroom norms and habits of mind. Given their limited knowledge of students' conceptions, these beginning teachers struggled to enact proof-related tasks and maintain high cognitive demand. Even more experienced teachers often struggle to enact RP in their classrooms (e.g., Bieda, 2010). These challenges have been attributed to teachers' insufficient mathematical knowledge of proof or unfavorable dispositions toward proof (Ko, 2010). Other researchers point to alternative external factors like rigid school culture and norms, lack of professional support, or institutional constraints (Herbst & Chazan, 2011; Stylianides et al., 2017).

To support teacher preparation specific to RP, Buchbinder and McCrone (2023) developed a special capstone course *Mathematical Reasoning and Proving for Secondary Teachers*. The course supported multiple cohorts of prospective secondary teachers developing mathematical knowledge for teaching proof, productive dispositions toward proof, and practical skills for integrating RP in classrooms. The course participants were followed for four years after graduation into their supervised internship and autonomous teaching practice to examine how their competence for teaching RP evolved.

In this paper, we focus on three beginning teachers from that project, Olive, Nancy, and Diane (pseudonyms) within two periods: a year-long internship and the first year of teaching. We collected rich data about each teacher from multiple sources (lesson plans, sample student work, classroom videos, and interviews). We conducted a linguistic analysis of their discourse by analyzing the words

they used when talking about RP and their mathematics teaching, in general, to reveal how these beginning teachers recontextualize (Ensor, 2001) RP in their teaching as they transition from university to school practice. Our research question was: *How do beginning mathematics teachers talk about RP in their teaching within more general mathematical discourse?*

Theoretical Perspectives

We draw on several theoretical perspectives. First, in considering beginning teachers' transition from university teacher preparation programs to school teaching, we draw on Herbst and Chazan's (2011) theory of practical rationality. This theory conceptualizes teaching as a social practice embedded in the institution of schooling. Teaching decisions are grounded in four professional obligations: to mathematics as a discipline, to students as individuals, to classrooms as collectives, and to schools as educational institutions. Teachers wishing to enact RP in their classrooms cannot act solely upon their beliefs but must consider these obligations (either consciously or unconsciously).

Additionally, we conceptualize teachers' learning-to-teach under Vygotsky's (1978) sociocultural perspective, holding that learning is a fundamentally social process with language as a key mediator of this process. Learning is also situated within social contexts. As teachers transition between contexts: university, internship, and schools, they adjust their practices and recontextualize discourses constructed in one setting into another (Ensor, 2001; Gomez Marchant et al., 2021). Focusing on language, we further conceptualize it within the psychological theory of "words as attention," which holds that the specific words people choose, consciously or unconsciously, in spoken language provide insight into aspects of their psychology by revealing how individuals cast their attention (Boyd & Schwartz, 2021). Dating back to the 1960s, this perspective has an extensive history in psychology, with numerous studies connecting word frequencies to various psycho-social conceptions (e.g., people with higher confidence and social status more frequently used "you" and "we" pronouns than "I" pronouns (Boyd & Schwartz, 2021)). While psychological approaches typically sort frequent words into predefined categories, we use an open approach to categorize frequent RP and non-RP-related words to understand the specific mathematical and pedagogical features of their use by our teachers.

Methods

Three beginning secondary teachers, Olive, Nancy, and Diane, volunteered to participate in the study and remained with it for the entire duration of the project, allowing for continuous and comprehensive data collection. As undergraduates, the participants excelled in both mathematical and pedagogical coursework and exhibited positive dispositions toward proof. Thus, these teachers were well-prepared and eager to integrate RP into their teaching. However, transitioning to classroom teaching brought challenges that resulted in RP-related topics being sparsely represented in their discourse. For this reason, the data from a year-long internship and the first year of autonomous teaching were combined.

The primary data sources were video-recorded interviews conducted about an observed or teacher-described lesson (five during the internship and three during autonomous teaching). All interviews were transcribed and initially coded with Herbst & Chazan's (2011) professional obligations framework. For the current analysis, we exclusively focus on obligation to the discipline of

mathematics and specifically the place of RP discourse within it. To capture and characterize the specifics of these discourses, we used *R*-computational software to analyze the most frequent substantive words in teachers' utterances. First, within general mathematical talk, we identified utterances specific to RP. Next, the *SpacyR* package was used to filter only nouns, verbs, and adjectives and to "tokenize" these words according to their underlying roots (e.g., "understand," "understood," and "understanding" are replaced with their common root "understand"). The *TidyText* package was applied to determine the most frequent words in each teacher's discourse (Silge & Robinson, 2017). A word was judged as "frequent" if it appeared at least three times in a teacher's RP and/or mathematics-specific utterances. For each teacher, the package identified 15 frequent words. These words were categorized using open coding (Corbin & Strauss, 2015) into meaningful categories based on the interview transcripts to capture different aspects of teachers' talk about RP and mathematics. For each frequent word, we examined the surrounding text to understand the meaning in which the word was used and to sort it into appropriate categories.

Results

Figure 1 summarizes the percentage of frequent RP and non-RP mathematical words by the three participants. For each teacher, the data from the internship and autonomous teaching was combined. We use the abbreviation INT for interns and NT for novice teaching periods.

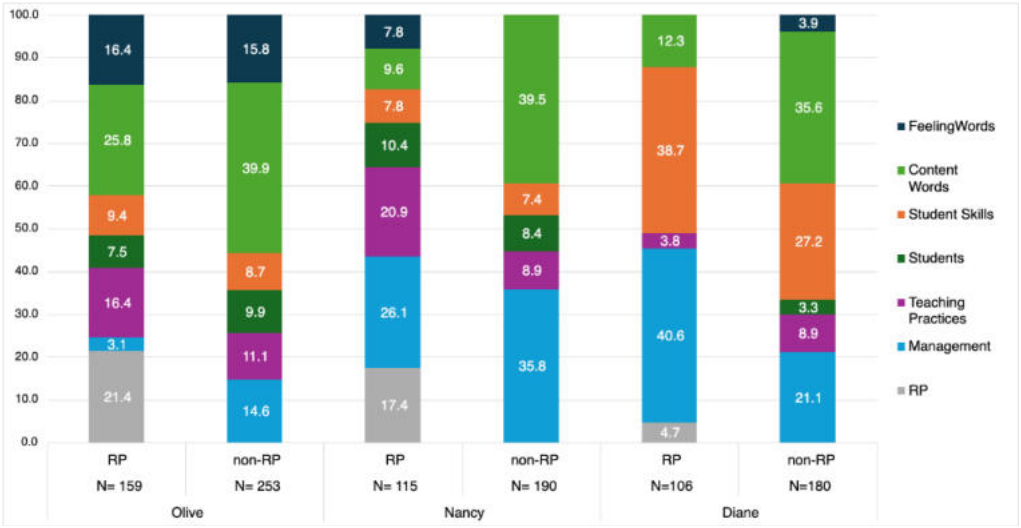

Figure 1: Teachers Frequent Words by Group, in percentages

Reasoning and Proof-Specific Terms

RP-related terms included words such as "proof," "reasoning," "true," "counterexample," and others. The three teachers used RP terms to different degrees. Olive used RP-related terms substantially: 21% of her RP-related talk included words like "proof," "counter," "true," and "reasoning." For example, when describing a lesson where students explored how the sign of inequality changes when inequality is multiplied by a negative number, Olive said: "Obviously, an example doesn't constitute proof, and we all know that. [but] I kind of scratched that idea when I said [the rule] is always true. […] it was

probably more, a more reasoning way of proving". For Olive, RP is a spectrum of justifying mathematical truths, between informal "reasoning" to formal "proof."

Nancy also frequently used RP-related words (17.4%), but her talk was skewed toward "reasoning" as a primary teaching focus in her algebra class (e.g., "We are doing a lot more reasoning than doing proving […] I ask them why and how questions […] It gets them thinking about justifying and explaining."). Diane, who, as an NT, taught a geometry course largely focused on "two-column proofs," talked about how she intended to "weave proofs into every unit." She emphasized proof as a wide-range tool for mathematical justification.

Management

Within mathematics-focused discourse, there were frequent words describing how teachers managed classroom environments to integrate RP. All teachers discussed how they plan, implement, and refine the mathematical content they teach but their discourse varied in the words and frequency of the words used. Olive mostly focused on "time," saying that she "would love to do [more RP activities], which we haven't had the time for." But beyond that, management was Olive's least frequent category in both RP and non-RP-related talk (3.1% and 14.6%, respectively). In Nancy and Diane's discourse, words related to managing the mathematics teaching were both prominent (~20-40%) and diverse (e.g., "activity," "lesson," "class," and "unit" and the "topic" or "stuff" (i.e., content)). Diane's language communicated the effort it took to reconcile her institutional obligation to uphold the school curriculum with integrating RP into her "lessons", "activities", and "units," saying things like: "I forget to sit down for every single lesson and think, oh, how could I make this more exploration-like?" and that she needed to "move forward" or "move on" with the curriculum. Alternatively, Nancy's management talk focused on making her "activities" more 'exploratory" to facilitate RP-oriented "goals" (e.g., "my goal was just for [students] to explore pi", "I'm trying to do more exploratory stuff […] where I'm trying to get them to reason and try to get them to see things on their own"). The teachers talked differently of RP and classroom management: Diane saw it as a planning burden, Nancy as helping students achieve learning outcomes, and Olive as being time-consuming.

In all three teachers' non-RP-related talk, the words related to managing mathematics learning were more prominent. Diane and Olive both took a descriptive stance, talking about the "activities" they implemented. Opposite to Diane, Nancy seemed willing and eager to modify the existing curriculum, talking about how she "changed" "activities" and "lessons" based upon whether they were too "hard" or too "easy" for students and aiming to make them "cool" or "fun" for them.

Teaching Practices

Frequent words in the teaching practice category included "answer," "wrong," "explain," "question," "rule," and others. This category comprised 16.4% of Olive's and 20.9% of Nancy's RP-related talk. For Diane, this category comprised just 3.8% of her RP-related talk and 8.9% of her non-RP-related talk. In her RP-related talk, Diane often used the word "question" to express her efforts to "throw in like little questions, so [students] can form an opinion on something and argue it, whether it's super proof-y or not, but just to hear why they made a choice or why they think something." In her non-RP-related talk, Diane frequently used "rule," expressing her desire to have students understand the

origins and proper application of mathematical rules (e.g., "I would add a row [to the example] that is $x^5 \cdot x^3$, just so that they [students] know that the rule also works with variables").

Nancy discussed how she focused on asking students "why and how questions" to engage students in reasoning. Among Nancy's frequent non-RP terms was the word "talk," capturing how she presented information to students. She positioned "talk[ing]" as "direct" instruction, contrasting it with RP-related teaching practices, which she saw as inferior to RP ways of teaching (e.g., "I'm trying to get them to reason, see things on their own, but sometimes I feel like I'm doing more direct instruction."

Olive frequently used "wrong" and "answer" when talking about her teaching but used them differently in RP vs. non-RP-related talk. In non-RP talk, the word "wrong" signified incorrect student answers or methods. In her RP talk, Olive described wanting students to explore these "wrong" "answers" to foster their justification skills and increase student confidence (e.g., "Can you see why what you're doing doesn't work? It's not that it's wrong; you're just not getting anywhere," "They are all going to start saying, why, why does that happen? And [...] we should give them an answer"). That is, Olive viewed the justification aspect of RP as essential to her teaching.

Students and Student Skills

Students were foregrounded to varying degrees and in different ways in teachers' talk. Both Olive and Nancy used terms like "kids" and "people" in addition to "students." Student-related terms comprised 10.4% of Nancy's RP-related codes and 8.4% of non-RP-related codes, indicating her strong focus on student engagement with RP, whereas "student" did not appear in Diane's RP-related codes and only in 2.4% of non-RP-related codes. However, Diane had the largest percentage of words in the students' skills category: 38.7% of RP-related codes and 27.2% of non-RP-related codes.

The word usage in the student skills category provides further insight into how teachers viewed students' roles in the mathematics discipline. Diane's RP-related activities had students "discover" mathematical "rules," emphasizing that students should "figure out" problems for themselves. But the goal of these discovery activities was to aid students "memoriz[ing]" the "rules." Diane's frequent words include "remember," "memorize," and "stick (in student memory)". Diane explained:

> These kinds of activities jump out, and when there is a hands-on piece to it, they [students] remember it. […] Just so they would have that process so that when they do get stuck on a quiz and they don't remember the rule, they know what they could do to figure it out.

Outside of the RP context, Diane's emphasis on student investigations was less prominent, although she still wanted them to "create" physical models, "measure" quantities, and "figure out" problems. In the non-RP-related talk, these words were paired with more traditional words like "practice" and "talk" in the sense of classroom discussions. Thus, while she still had elements of student-led activity in her non-RP-related talk, this was less focused on discovery than in her RP-related talk.

Olive showed the least word variation in the student skills category, but interestingly, she used the same words with different meanings. For example, in her RP-related talk, Olive used the word "understanding" to convey her goal of students obtaining "conceptual understanding" and "understanding why" mathematical rules work, while in the non-RP context, "understanding" referred to mastering content. Similarly, the word "struggle" in the RP-related context was positioned

as productive for student learning (e.g., "when they really, really understand something is when […] "struggle" alongside other students to arrive at a conclusion"). In her non-RP-related talk, Olive wanted students to explain answers rather than "struggle". Thus, her emphasis on the justification was consistent across her RP and non-RP-related talk.

For Nancy, students' RP skills manifested in students being able to "explain" concepts to one another, justify their answers, and "participate" in the class (e.g., "Reasoning was in this activity where the students were explaining to each other how they were doing things). In another lesson, she reflects that her students were able to "participate openly and just have a discussion... students were recognizing what was happening, seeing, and were able to participate and engage". Nacny's notion of "explaining" is similar to Diane's notion of "discover[y]" and to Olive's "understanding why" since all teachers wanted their students to be able to recognize where mathematical results come from.

Content Words

The teachers also frequently mentioned words related to the mathematical content they were teaching. Mathematical vocabulary was more frequently used in non-RP-related rather than RP-related ones, indicating that our teachers tended to talk about RP more generally, focusing on the affordances and challenges of RP in their classrooms. The variation across teachers and frequent terms highlights the range of content our teachers had to handle in their classrooms. For example, Olive, in her non-RP-related talk, used many verbs (e.g., "move", "flip", "set", "fold"), while Nancy and Diane tend to use a single verb ("stay" and "expand") focusing more on nouns (e.g. "equation", "angle", "line"). Olive's emphasis on mathematical actions in addition to mathematical objects suggests that she emphasized mathematical processes in her teaching, while Nancy and Diane's focus on mathematical objects suggests a greater emphasis on concepts.

Additionally, both Nancy and Diane frequently mentioned mathematical subject areas such as "geometry" and "algebra" but did so exclusively in their RP-related talk, while reflecting on how some content areas more aptly lent themselves to the incorporation of RP. For example, Nancy who taught both algebra and geometry, said: "I feel like I know a lot of more exploratory and application stuff, more critical thinking stuff for geometry. And I'm struggling with algebra, trying to get them to experience things." While these teachers were trained in integrating RP across mathematical contents (Buchbinder & McCrone, 2023), their expertise in this was still developing. As they were adjusting to the responsibilities of independent teaching, they were more apt to integrate RP into geometry than into algebra while still expressing a desire to learn how to integrate RP more widely in their teaching.

Feeling Words

Our teachers also frequently used words to convey a sentiment (e.g., "love," "nice") or hedge their thoughts (e.g., "sense," "guess" (in the meaning of supposing)). Nancy and Olive used the term "feel" to indicate their perceptions and intuitions about classroom events or teaching approaches (e.g., "That's something I feel like I've wanted to do" - Olive, "I just feel like some of the lessons were just over some students' heads" -Nancy).

The teachers varied in their expression of affect and emotion across their RP and non-RP-related talk. Olive had the greatest variation in these types of terms but used them approximately evenly through

her RP and non-RP-related talk (16.4% vs 15.8% respectively), indicating that such words may be intrinsic features of her vernacular. Nancy and Diane have less variation in the types of these words. Nancy expressed feelings more frequently in RP-related talk, while Diane did in non-RP-related talk. This variation is significant as it indicates that teachers express different levels of confidence in RP.

Discussion

The different aspects of RP identified in the teachers' talk highlight how RP can be incorporated into teaching to varied extents and in different ways. While all teachers mention that it is difficult to find time for RP activities, Nancy prioritized these types of tasks, modifying curriculum resources given to her, because of their affordance for supporting students' exploration, while Daine found such activities difficult to incorporate. Additionally, all teachers used RP teaching practices as a way to center student contributions in their classrooms. Diane and Nancy accomplished this by focusing on "questions" and Olive focusing on "answers," but all communicate the importance of RP for student thinking. Our teachers also viewed RP as a means of fostering students' deeper understandings of mathematical content (c.f., Buchbinder & McCrone, 2023; Ko, 2010) but formulated this in distinct ways, with Diane focusing on students' "remembering" and Olive focusing on students "understanding." However, this implementation varied depending on the content to be taught, with several teachers finding RP more suited to geometry courses than algebra.

While the teachers in this study had similar backgrounds in RP during their teacher training, we see that, in their first year of teaching, they emphasized different aspects of RP in their classroom enactment of RP. While individual differences are expected, our findings emphasize the flexibility of ways to integrate RP into teaching. That is, despite their differing individual personalities, teaching philosophies, and institutional contexts (Herbst & Chazan, 2011), each beginning teacher had a strong sense of the role of RP across aspects of their teaching. These teachers varied in the frequency and nature of discussion of the identified categories, but these categories provide a sense of the areas in which beginning teachers and pre-service teachers need support to integrate RP into their teaching.

Our study offers a methodological contribution, showing how computational language tools can be incorporated alongside qualitative approaches to understand the nuances with which participants speak about mathematical topics (Silge & Robinson, 2017). While we have described here how teachers' conceptualizations of RP in the classroom vary, future studies should examine whether and how these differences impact student learning.

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Examining teachers' motivational views on Lakatos-style investigation: A value-cost-expectancy approach

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Despite the affordances of Lakatos-style investigation (LSI) for mathematics instruction, research has focused on students, and left a gap in understanding teachers' perspectives. To address this gap, we explore primary teachers' views on incorporating LSI in the classroom, focusing on perceived value, cost, and expectancy for themselves and their students. Through a survey with 331 participants, we found a generally positive inclination towards LSI, though perceptions vary by demographic factors. Female teachers perceive higher personal costs but also higher personal value, while older teachers report lower personal costs and reduced expectations of students. Academic background relates to teacher views, with stronger mathematical foundations linked to higher personal expectancy and lower perceived personal costs. These findings underscore that tailoring teacher education to demographic-specific factors may be crucial for promoting teachers' openness to LSI.

Keywords: Proof and proving, Lakatos-style investigation, primary school teachers, teacher views, value-cost-expectancy framework.

Introduction

Mathematical proof has long been a cornerstone of the mathematics discipline, and its significance in school mathematics is increasingly recognised (Stylianides et al., 2022). Despite this recognition, opportunities for students to engage with proof tasks remain limited, and even more limited are authentic proof-related activities that can expose students to the nature and genuine practices of mathematical inquiry, like the interplay between conjecture formulation and proof construction (Stylianides et al., 2022). This dynamic relationship, especially the critical role of example examination, is central to Lakatos' (1976) philosophical-historical account of the growth of mathematical knowledge. A growing body of educational research suggests that elements of Lakatos' account can effectively be incorporated into school mathematics, even at the primary level (Komatsu, 2016; Reid, 2002; Zack, 1997). By engaging with an authentic mathematical activity that has enabled historical advancements, students can develop an understanding of the nature of mathematics and the underlying principles of mathematical investigation, thus helping bridge the gap between academic and school mathematics (Stylianides et al., 2022).

However, research on integrating methods of mathematical investigation akin to those described in Lakatos' account into school mathematics has predominantly focused on students and on the design of classroom instruction, leaving a gap in the field's understanding of teachers' perspectives on such instructional approaches. Teachers' ability to facilitate proof-related activities is closely tied to their own understanding of proof (Deslis et al., 2021, 2023; Lada & Forbregd, 2022). Beyond capability, however, a crucial yet underexplored factor is teachers' willingness and motivation to incorporate Lakatos-style investigation into their classrooms. This paper aims to address this gap by exploring the following research question: *What are primary school teachers' views about the perceived value,*

cost, and expectancy related to Lakatos-style investigation, and how are these views associated with primary school teachers' gender, age, academic and professional backgrounds?

Theoretical Frameworks

Framework LSI: Lakatos-style Investigation

Building on Lakatos (1976) and classroom applications of his approach (e.g., Komatsu, 2016; Reid, 2002; Zack, 1997), we developed Framework *Lakatos-style Investigation* (LSI), which offers an iterative approach to conjecture exploration through systematic exploration (Deslis et al., 2021). This model, structured in four interrelated phases, describes participants' engagement in a dynamic process of conjecture development and refinement. Initially, solvers are presented with a proving task, prompting them to formulate a conjecture (*Phase 1: Formulation*). As they begin to check various cases, they may identify examples that appear to support the conjecture, suggesting its potential truth (*Phase 2: Validation*). However, they might also encounter counterexamples that challenge the conjecture's truth (*Phase 3: Refutation*). Reflecting on the attributes of both the supportive examples and counterexamples, solvers proceed to modify their original conjecture. This modification might involve narrowing the conjecture's scope to exclude the counterexamples or broadening it so that previously identified counterexamples now support the conjecture (*Phase 4: Modification*). The process then repeats, allowing for continuous refinement and deeper investigation of the conjecture. Past findings suggest that this activity is within students' reach; for example, Komatsu (2016) reports that fifth graders effectively used examples to develop a conjecture related to the sums of two-digit integers and reflected on the evidence provided by the counterexamples they discovered to refine it.

Framework VCE: Value-Cost-Expectancy

We developed Framework *VCE* (Value-Cost-Expectancy) by synthesising insights from research in educational psychology (e.g., Eccles & Wigfeld, 2020), education (Dixon et al., 2024) and mathematics education (e.g., Gaspard et al., 2015; Rach & Schukajlow, 2024) to examine the motivational factors that influence teachers' willingness to engage with LSI. The framework identifies three primary view dimensions that shape motivation: Value, Cost, and Expectancy. The *Value* dimension refers to the perceived *Importance*, *Usefulness*, and *Interest* that teachers associate with the activity. The *Cost* dimension involves the perceived cost in *Time*, *Effort*, and *Emotional* cost required to engage in the activity. The *Expectancy* dimension pertains to the anticipated *Ease* of the task, the anticipated *Competence*, and the anticipated *Likelihood of success*.

Crucially, Framework VCE posits that teachers assess these factors not only for themselves but also consider their implications for their students. Thus, each component is analysed through two distinct strands of view components: the *Teacher strand [T]* and the *Student strand [S]*, which span these three dimensions, resulting in six view components: *Value [T], Value [S], Cost [T], Cost [S], Expectancy [T], Expectancy [S]* (see Figure 1). For instance, when evaluating the costs of introducing a new activity into the classroom, teachers may consider how much effort this endeavour would demand both from themselves and their students, and whether the required effort is manageable for both themselves and their students. This dual-dimensional approach allows for a more comprehensive understanding of how motivational factors influence both teaching and learning in the context of LSI.

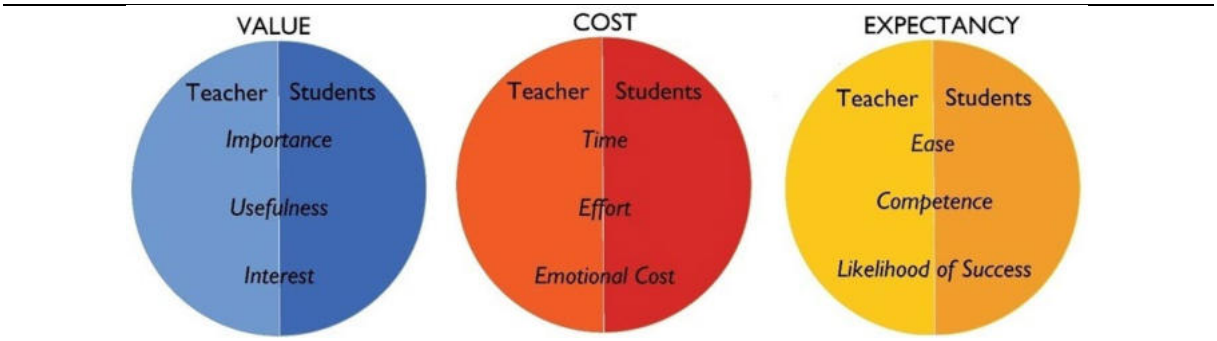

Figure 1: The three interconnected VCE dimensions: Value, Cost, Expectancy, and the six view components: Value [T], Value [S], Cost [T], Cost [S], Expectancy [T], Expectancy [S]

Methods

Data Collection

Data were collected using an online survey, which employed contextualised comic-style vignettes to illustrate classroom situations (see Figure 2). The survey was completed by a sample of 331 participants, comprising 149 in-service primary teachers, teaching at various schools across Greece, and 182 pre-service primary teachers, studying at nine different Greek universities. The vignettes depicted classroom episodes adapted from Zack (1997) and Reid (2002), featuring comic-style teacher and student characters exploring the "Count the Squares" problem (Zack, 1997): "How many squares are there in a four-by-four square grid? Try to design or imagine other similar grids of your choice. How many squares are there in each of them? Can you find a general rule that applies to grids of all sizes?" Specifically, the characters discuss the conjecture: "The number of squares in an n-by-n grid is a multiple of five" while navigating through all four LSI phases. The survey consisted of 16 short episodes, showcasing a diverse range of reasoning strategies, student understandings, and teaching practices related to the LSI phases. These episodes were developed based on insights gained from a prior interview study with ten primary teachers (Deslis et al., 2021) to probe various aspects of the participants' LSI-related mathematical knowledge (see Deslis et al., 2023 for more details).

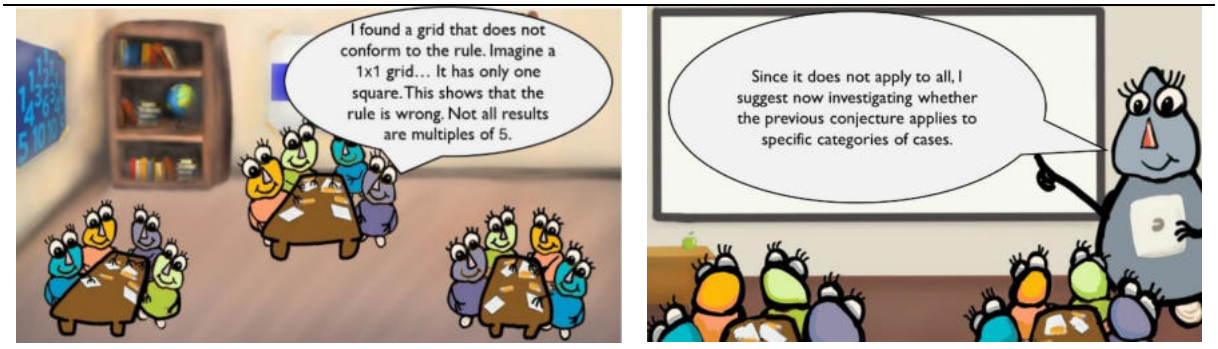

Figure 2: Two sample comic-style episodes pertaining to LSI Phases 3 (Refutation) and 4 (Modification)

Upon engaging with all 16 episodes, which offered participants a good sense of what this activity is and what it entails, they were invited to respond to 18 Likert-type items (ranging from 1: disagree to 5: agree), each corresponding to a sub-component of VCE (see Figure 1). For example, the following

item was used to address the sub-component Interest [S]: "*I think that students would find activities like this interesting*". These responses were transformed into 0–1 scores (0, 0.25, 0.50, 0.75, 1) for each view sub-component. The score for each view component was the average score of its sub-components (see Figure 1); for example, Value [T] = (Importance [T] + Usefulness [T] + Interest [T]) / 3 and Value [S] = (Importance [S] + Usefulness [S] + Interest [S]) / 3.

Data Analysis

Considering participants' VCE scores as the dependent variables and their demographics as the independent variables, we employed various statistical analysis techniques, adapting them to the nature of each independent variable: independent samples t-tests for binary variables, one-way ANOVAs for categorical variables, and linear regression analyses for continuous variables. Adopting an exploratory approach, each view score was tested against all independent variables.

Findings

Figure 3 below presents the mean scores for the six view components for the entire sample, and Table 1 gives an overview of the associations between view scores and demographics.

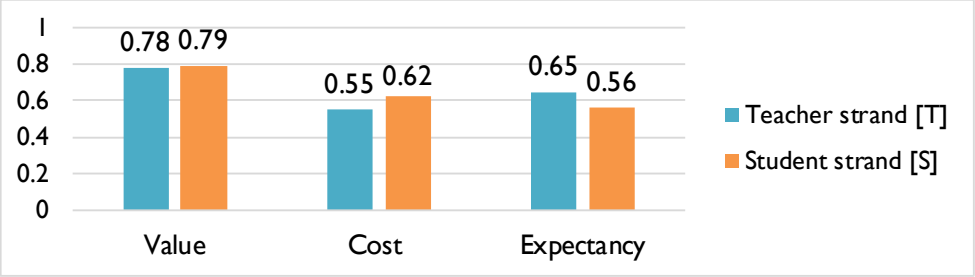

Figure 3: View scores by VCE component

Gender & Age. The t-tests revealed that female participants (n=281) perceived significantly higher costs for both students ($t(329)= 2.005$, $p < .05$) and teachers ($t(329)= 3.339$, $p < .001$), as well as a higher value for teachers ($t(329)= 2.198$, $p < .05$), compared to male participants (n=50). Specifically, the mean scores were higher for females in Cost [S] (M=.63, SD=.20 vs. M=.57, SD=.22), Cost [T] (M=.57, SD=.25 vs. M=.44, SD=.20), and Value [T] (M=.79, SD=.21 vs. M=.72, SD=.26). Gender was not found to be associated with any other view components.

Significant regression equations were identified for both Expectancy [S] and Cost [T] scores as functions of age. For Expectancy [S], the equation ($F(1,329)= 9.01$, $p < .01$, $R^2 = .027$) indicated that the predicted score is .658, decreasing by 0.003 for each additional year of age. Similarly, the regression for Cost [T] ($F(1,329)= 4.00$, $p < .05$, $R^2 = .012$) showed a predicted score of .636, also decreasing by .003 per year. No other view components were significantly associated with age.

Academic Background. The t-tests revealed a significant difference between pre- (n=182) and in-service teachers' (n=149) Expectancy [S] scores; specifically, pre-service teachers had significantly higher expectations of students (M=.61, SD=.17), than in-service teachers (M=.51, SD=.22), $t(329) = 4.685$, $p<.001$. Pre- and in-service teachers did not have any other significant differences.

In the final high school year, students select a direction of studies and take the Panhellenic Examinations, with subjects and university options varying by direction. Schools of Primary Education accept students from any direction, including those without mathematics. The t-tests showed that those who chose a direction including mathematics (n=89) perceived a lower personal cost (Cost [T], M=.48, SD=.23) than those who chose a non-mathematical direction (n=242, M =.58, SD=.24), $t(329) = 3.563$, $p < .001$. They also had higher Expectancy [T] (M=.72, SD =.17 vs. M =.62, SD =.20), $t(329) = -3.856$, $p < .001$. The differences for the other view scores were not significant.

According to t-tests, participants with a master's degree in education (n=136) had significantly lower expectations of students (Expectancy [S]; M=.50, SD=.21) compared to those without (n=195; M=.60, SD=.17), $t(329) = 4.823$, $p<.001$, but no other significant differences. Furthermore, no statistically significant differences were found in relation to PhD studies.

Participants who had taken compulsory mathematics education courses at undergraduate level (n=317) showed significantly higher Value [T] scores (M=.77, SD=.22) than those who had not (n=14; M=.64, SD=.22; $t(329)=-2.434$, $p <.05$), but non-significant differences in the other view components. Additionally, attending or planning to attend optional mathematics education courses (has attended or will attend: n=209 vs. n=122) was significantly associated with lower Cost [T] (M= .53, SD=.25 vs. M=.59, SD=.23) and higher Expectancy [T] scores (M=.67, SD=.21 vs. M=.61, SD=.18), while for the other view components no significant association was found.

Table 1: Overview of the associations between demographics and VCE scores (Note: —, non-significant)

	Value [T]	Cost [T]	Exp [T]	Value [S]	Cost [S]	Exp[S]
Gender (Female/Male)	F>M	F>M	—	—	F>M	—
Age (↑)	—	↓	—	—	—	↓
Pre-/In-service	—	—	—	—	—	Pre>In
Math direction	—	No>Yes	Yes>No	—	—	—
Master's	—	—	—	—	—	No>Yes
PhD	—	—	—	—	—	—
Compulsory math ed	Yes>No	—	—	—	—	—
Optional math ed	—	No>Yes	Yes>No	—	—	—
Teaching experience (↑)	—	—	—	—	—	↓
Upper primary/secondary	—	—	—	—	Yes>No	No>Yes
Public/private	—	—	—	—	—	—
School location	—	—	—	—	—	—

Professional Background. Single linear regression analyses were used to test if teaching experience significantly predicted any of the participants' scores. A significant regression equation was found

($F(1,291)=12.16$, $p<.001$) with an R^2 of .040. Expectancy [S] decreased .005 for each year of teaching experience. No other scores were significantly associated with teaching experience.

To investigate whether the grades participants had taught were associated with any differences, we employed ANOVAs with a categorical variable: 0 "No grade taught" (n=38); 1 "has taught, but neither upper primary, nor secondary grades" (n=104); 2 "has taught upper primary and/or secondary grades" (n=186), indicating significant differences in two components. Those taught in upper primary or secondary school had significantly lower Expectancy [S] (M=.53, SD=.20) than the other two groups (0: M=.62, SD=.14; 1: M=.60, SD=.20) ($F(2, 325) = 4.07$, $p < .05$). Additionally, this group exhibited significantly higher Cost [S] (M=.64, SD=.20) than those who had teaching experience but not in upper primary nor secondary school (M=.58, SD=.21) ($F(2, 325) = 4.03$, $p < .05$).

The results of the independent samples t-tests suggest that whether participants taught at a public or a private school at the time of the study was not associated with any significant differences. Likewise, the examination of various alternative binary or categorical variables related to participants' school location (e.g., large city vs. countryside) did not generate any significant differences.

Discussion

The findings reveal a generally positive inclination among participants towards incorporating LSI into their classrooms, as evidenced by the high perceived value for both teachers and students, the moderate perceived costs, and the cautious optimism reflected in their expectations (Eccles & Wigfield, 2002). The analysis also uncovered several interesting associations between teachers' demographic and professional characteristics and their views on LSI.

Female teachers perceived higher costs for themselves and students and greater personal value compared to males, recognising both the challenges and benefits of such activities. Gender-based differences have also been observed in other studies employing variations of the expectancy-value theory (Eccles & Wigfield, 2002). For example, in Gaspard et al.'s (2015) intervention targeting high school students' perceived task values, female students benefited more. These differences may stem from broader socio-cultural or biological factors, an interesting avenue for future research on LSI.

Age was linked to decreased personal cost, though the lack of association with teaching experience suggests broader professional development influences. Conversely, both age and years of teaching experience were associated with lower expectations of student success in engaging with LSI, reflecting conservative expectations likely shaped by classroom challenges. Shifts in motivation-related views as teacher grow older and gain teaching experience have also been reported in past studies. For instance, Dixon et al. (2024) found that the value and expectancy teachers perceive in pursuing postgraduate degrees evolved throughout their academic journey. Additionally, in this paper pre-service teachers exhibited higher expectations of students than in-service teachers, likely due to exposure to contemporary theories during training (Rach & Schukajlow, 2024), which could contrast with the pragmatic challenges faced by in-service teachers. Additionally, those who had taught upper primary or secondary grades reported lower expectations and higher costs for students, indicating that experience in higher grades might lead to a more cautious or even sceptical view of students' capacity to engage with advanced LSI. Furthermore, those with a master's degree, also had lower expectations of students, indicating that concerns about the practical challenges of implementing innovative

teaching approaches like LSI in the classroom are shaped by both classroom experience and teachers' postgraduate education (Dixon et al., 2024). Qualitative studies can illuminate why and how teachers' motivational views change with teaching and academic experience.

Academic background is also connected with teachers' views, particularly in terms of their perceived personal value, cost, and expectancy. Those who had attended compulsory mathematics education courses perceived greater value in LSI, likely due to their exposure to contemporary developments in mathematics education, which may foster a deeper appreciation for innovative teaching methods (Rach & Schukajlow, 2024). Additionally, teachers who pursued a mathematical direction in high school or took optional mathematics education courses demonstrated higher personal expectancy and lower perceived costs. This suggests that a strong mathematical foundation or a personal affinity for the subject contribute to greater openness and confidence in implementing new instructional approaches. In line with this, Rach and Schukajlow (2024) report that students strongly interested in school mathematics were more willing to put more effort in mathematics tasks, suggesting that such interest may play a reinforcing role in shaping their motivational views.

The findings have implications for the design of teacher education and professional development programs. Tailoring programs to address the specificities of different teacher demographics could be crucial. For instance, training programmes could differentiate strategies by addressing the high expectations of pre-service teachers while encouraging more mature teachers to maintain elevated expectations for student success. Furthermore, the significant influence of academic background on teachers' perceptions underscores the importance of strengthening teachers' understanding of mathematics education during teacher training (Rach & Schukajlow, 2024). Encouraging pre-service teachers to take additional mathematics education courses and providing ongoing professional development for in-service teachers could potentially enhance their confidence and reduce perceived costs. Overall, while mathematical knowledge should be a central focus in professional development programs (Lada & Forbregd, 2022), it is crucial to also address motivational views or the interactions between knowledge and motivational views (Deslis et al., 2023). Demonstrating the benefits and feasibility of implementing authentic mathematical activities like LSI (Stylianides et al., 2022) through examples and hands-on experiences, such programs could help increase teachers' motivation and willingness to incorporate these activities into their classrooms (Eccles & Wigfield, 2002). A limitation of this study is the multiple comparisons problem arising from the large number of statistical tests performed. This increases the likelihood of Type I errors, meaning some significant results could be due to chance. While this issue is common in exploratory research, the findings should be interpreted with caution, and further studies are needed to validate these results. Future research could expand on our findings by exploring similar questions in different cultural or educational settings or employing qualitative methods. Gaining a more comprehensive understanding of how various factors influence teacher views can inform professional development programmes, ultimately contributing to the more widespread adoption of innovative teaching practices like LSI.

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A sixth-grade student's first instance of mathematical proof

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This paper presents findings from an individual teaching experiment study investigating the mechanisms behind a sixth-grade student's transitioning from the empirical to the deductive proof scheme. From the set of learning goals targeted throughout the ten-week teaching experiment, here we[1] report findings related to a particular learning goal – understanding "a first instance of mathematical proof". We introduce this concept as a major "goal understanding" in one's transitioning towards the deductive proof scheme. Using the constructs "intellectual need" from the DNR-Based Instruction in Mathematics and "reflective abstraction" from the Learning Through Activity (LTA) theoretical frameworks, we illustrate designing for a sixth-grade student's –Beren's– construction of such a first instance of her own. Findings of the study provide theoretical insight (rather than practical) into individual's learning of deductive proof, regardless of age or grade level.

Keywords: Deductive proof scheme, DNR-Based Instruction in Mathematics (DNR), Learning Through Activity (LTA), individual teaching experiments.

Introduction

The empirical proof scheme, using a limited number of confirming examples to validate a mathematical generalization, is pervasive among students, and does not fade away effortlessly as individuals learn about disciplinary ways of proving (Harel, 2008). This makes transitioning from the empirical to the deductive proof scheme a two-fold learning goal: eliminating the empirical proof scheme and promoting the deductive proof scheme. The first aspect was addressed in Stylianides and Stylianides's (2009) study, where the researchers introduced an empirically tested instructional sequence. Engaging in the task sequence, the participants of the study realized the limitations of empirical arguments and developed an intellectual need for a secure method of validating mathematical generalizations. When the forthcoming instruction demanded construction of a valid proof, the participants were not able to do so without the researchers' scaffolding. The participants showed no evidence of anticipating what a valid proof would look like. We explored this second aspect of the learning goal – how to promote learners' transitioning toward the deductive proof scheme after the elimination of the empirical one – with an individual sixth grade student. The broader study explored the processes by which an individual sixth-grade student may (1) come to recognize a deductive argument as an instance of mathematical proof for the first time and then (2) practice producing other instances of proper proofs to develop the character of the deductive proof scheme. Here we focus on the former aspect only.

[1] We present this work as part of the first author's doctoral dissertation. We respectfully commemorate the contribution of her recently deceased supervisor, Prof. Dr. Erdinç Çakıroğlu.

Defining the first instance of mathematical proof

The idea of spotlighting a first instance of proof in one's transitioning toward the deductive proof scheme is grounded in Duality Principle of the DNR-based instruction in mathematics (DNR) framework (Harel, 2008). DNR is an acronym for three instructional principles: *duality, necessity,* and *repeated reasoning*. Duality is a two-part principle referred to by Harel as "Duality I" and "Duality II," addressed in the below quote as parts (a) and (b) respectively.

> Duality principle: (a) Learners come with a set of ways of thinking, some desirable and some undesirable, that inevitably affect the ways of understanding we intend to teach them. (b) Learners develop desirable ways of thinking only through repeated application of proper ways of understanding. (Harel, 2021, p. 711)

Ways of understanding are cognitive *products* of particular *mental acts* through which mathematics is practiced, such as proving, modelling, or defining. *Ways of thinking* are the *cognitive characteristics* shared by the multitude of these products. Mathematics is the discipline consisting of *all the ways of understanding* and *the ways of thinking* that the mathematics community has accepted throughout history (Harel, 2008). *Mental act-way of understanding-way of thinking* is a generalization of the triad *proving-proof-proof scheme* (Harel 2008). From this perspective, one's proof scheme is her/his way of thinking associated with the act of proving, and the particular proofs she produces are her/his ways of understanding particular acts of proving.

Duality I implies that the learners in Stylianides and Stylianides's (2009) study did not have the deductive proof scheme. Neither the empirical, since now was eliminated, nor the deductive proof scheme, which was not yet constructed, did frame the participants' subsequent acts of proving. Duality II suggests that the learners develop the deductive proof scheme by repeatedly experiencing proper ways of understanding proofs. However, how one can recognize a proper proof as such without having the deductive proof scheme yet. This suggests the students must undergo a cognitive shift at some point from not recognizing to recognizing a mathematical argument, a product of their own or someone else's activity, as an example of a valid proof for the first time. We refer to this moment as the first instance of mathematical proof.

Recognizing the first example of mathematical proof as a proper member of the category without yet possessing the deductive proof scheme is no different than a case of Learning Paradox (Pascual-Leone, 1976) —a consequence of the constructivist notion of *assimilation*. According to Piaget's theory of *equilibration*, experiences new to the individual are assimilated into one's existing concepts and operations (one's existing cognitive schemes) only if they produce expected results for the individual's actions based on the individual's prior experiences. The unfitting experience, which is contradictory to the individual's existing assimilatory structures, creates a *perturbation* (a disequilibrating experience) in the individual's cognitive structures and – not necessarily, as discussed in Simon et al. (2004) – leads to an *accommodation*. Referring to a rearrangement of one's existing schemes, accommodation is the process that re-establishes the desired state of equilibrium.

As raised in Simon et al. (2004), a disequilibrating experience may stimulate a process of accommodation – in Stylianides and Stylianides's (2009) study, the cognitive conflict mechanism

results in such stimulation –, but it does not necessarily result in constructive reorganization in the desired direction (toward the deductive proof scheme in this case). One possibility for the solution to the learning paradox was postulated by Piaget (2001) as the process of "*reflective abstraction*." Simon et al. (2004) elaborated on the construct as the *learner's reflection on the activity-effect relationship*, which later, constituted the basis for the Learning Through Activity (LTA) research program. LTA aims to explicate the mechanisms of mathematics conceptual learning by studying individual students' transition from one conceptual state to another (from not recognizing to recognizing a mathematical argument as a proof, in the case of current study). The basis for studying conceptual change here is the close observation of individual students' goal-directed activities, engendered by purposefully designed task sequences that are also evolving as the research study progresses. Simon et al. (2018) defines *concepts* to be *researcher's constructs* that are articulated at the beginning of such a research process to be able to work on and progressively refine the "goal understandings" that the study aims to promote. Concepts that are constructed through a *coordination of actions* type of reflective abstraction are defined as "*CoA concepts*." A CoA concept is "a researcher's articulation of intended or inferred student knowledge of the logical necessity involved in a particular mathematical relationship" (Simon, 2020, p. 123). Such concepts include not only the objects but also the logical necessity of the relationships among them.

Based on the view of CoA concept as a researcher's construct, this study investigated whether CoA type of reflective abstraction could frame the study of an individual student's constructing of a conceptual understanding of a particular proof (proper way of understanding associated with the act of proving) for the first time through *abstracting from* her/his activity. The intended concept of this study can be viewed as understanding the logical necessity of the relationships involved in a particular deductive proof. The initial hypothesis of the study was that, in line with the LTA instructional design approach (Simon et al., 2018), an individual student, in response to a given task, would engage in a sequential activity that would result in a flowchart proof when completed. Reflecting on the result of her/his activity, the student would have a chance to realize that she indeed created a proof for a specific mathematical generalization. The following research questions guided the study:

> How does (or may) an individual student come to understand a deductive argument as an instance of mathematical proof for the first time? Can a coordination of actions (CoA) type of reflective abstraction explain the process of such learning?

Methods

The 6th grade student, Beren (pseudonym), participated in the ten-week teaching experiment study. She was selected for the study based on her competence in four operations, ability to express mathematical ideas, and willingness to learn mathematics. As the deductive structure of mathematical proofs were not explicit focus of middle school curriculum in Turkey, she had no previous interaction with mathematical proof. The broader teaching experiment study consisted of five major phases. Understanding a first instance of mathematical proof was the focus of the third phase, after the first two phases prepared the student for the conceptions needed in constructing the intended CoA concept. The first phase prepared the student about basic number theory concepts (such as modular structure, parity, and divisibility) to be coordinated at the third phase. The representation system for arithmetic

number patterns, ($kq+r$), called "general representation," namely the idea of using identical counting items and identical cups containing equal number of these items, was introduced here. The second phase challenged the student's empirical proof scheme by adapting Stylianides and Stylianides's (2009) task sequence including the Monstrous Counterexample Illustration. This phase triggered an intellectual need (Harel, 2008) in the student for learning about secure methods of proving.

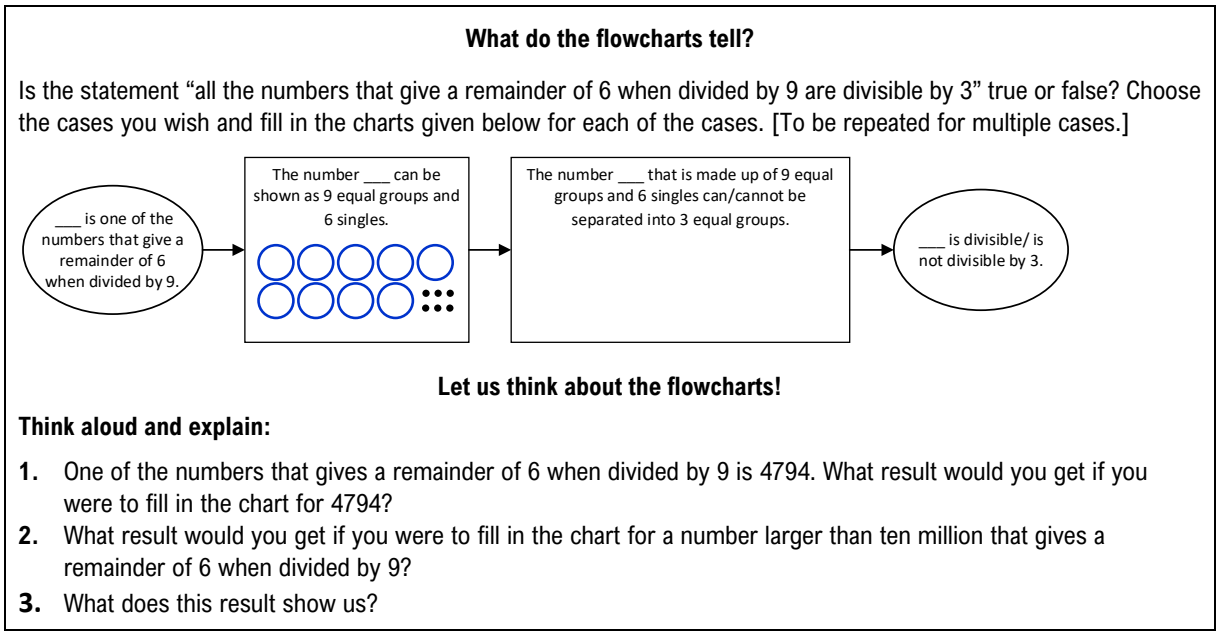

Figure 1: The modified task

The third phase consisted of two consecutive cycles, as the initial task design did not result in the intended learning. Both the initial task and the modified task (given in Figure 1) guided the student to carry out two consecutive actions sequentially to complete two distinct structures of flowcharts. In each case, reflecting upon flowchart activity, the student was expected to realize that the flowcharts indeed conveyed a new type of connected meaning beyond the distinct actions carried out, and this meaning was related to the goal of proving the conjecture being investigated.

While the activity elicited by the initial task started with constructing the general representation of a given arithmetic number pattern in a single flowchart, the modified task treated the elements of the pattern individually, in case-by-case flowcharts. The initial task failed to foster Beren's understanding of the structure *8k+4* as an *abstract structure* detached from the particular examples. Beren demonstrated a solid understanding of the one-to-one correspondence between the elements of the number pattern and the number of counting items contained in each of the cups for each of these elements. However, the intended shift of attention from seeing the representation *8k+4* as an expression of generality defining the set 12, 20, 28, 36, 44, 52, 60, 68, 76, 84, … had not been realized. Therefore, the modified task has built on the *operational* rather than *structural* origins of the concept of modular congruence (Sfard, 1991). The modified task guided the student to carry out the following two actions sequentially:

$A_{0a\ (modified)}$: Given a particular element of an arithmetic number pattern, identify its modular structure according to the given modulus n.

A_{0b}: Given a modular structure, make appropriate divisibility inferences.

As with the initial one, the modified task was expected to be the basis for the student's understanding of the logical necessity that the divisibility inference made about the particular modular structure is valid for all the numbers that share the same structure. Hence, the result of her activity ($A_{0a} \rightarrow A_{0b}$) is a proof to a particular theorem. Construction of the intended CoA concept was analysed based on the third-phase data by using the LTA theoretical account of conceptual learning (Simon et al., 2018). Any indications of understanding or lack of understanding of the deductive relations were captured through the observation of abstract structures revealed in students' thinking.

Findings

For the first case-by-case flowchart (see Figure 2) Beren selected the number 15. Thinking about the third box of the flowchart, her focus was on dividing the number 15 by 3, independently of its configuration in the general representation.

Beren: 15… Let me say this… Separable.
Researcher: How?
Beren: We separate five by five.

But then her focus shifted from the value of the number 15 to its specific form in general representation. Using the story, she had made-up previously in Phase 1, she drew 3 baskets with 3 cups and 2 singles in each, keeping an eye on the total number of cookies in every basket.

Beren: Christmas baskets again to be distributed among three persons. Inside these ones each (points to one of the cups) there is one. [The basket] consists of five. Others likewise. The number 15, one second, then, we should mark 'can be separated.' The number 15 is perfectly divisible by 3.

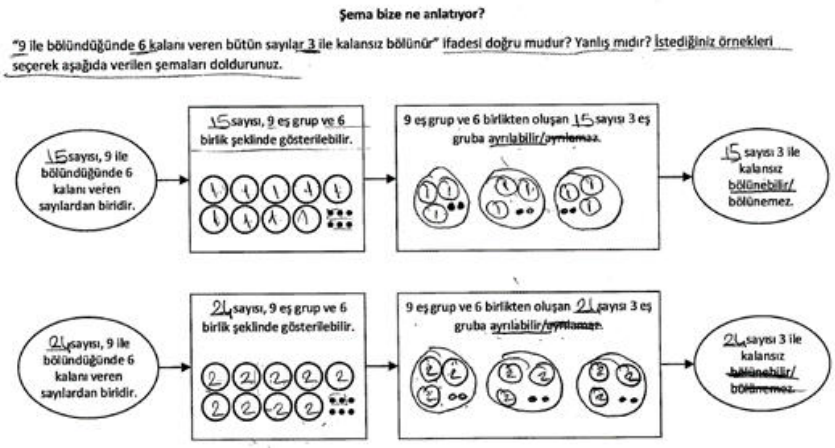

Figure 2: Beren's work on the case-based flowcharts for the numbers 15 and 24

For the second flowchart, she picked the number 24.

Beren: The number 24 that is made up of 9 equal groups and 6 singles can, cannot be separated into three equal groups. I am counting right now, 3, 6, 9, 12, 15, 18, 21, 24, it's ok (Her focus is again on dividing the number 24 by 3). (Drew three baskets.) Here (in the third step of the case 15) five (counting items) go in each (basket). How many go in here (in the case of 24)?

She first structurally mapped the third step look from the case of 15 to the case of 24. While doing this structural mapping, she demonstrated an instance of the intended abstract structure for the first time. That is, for Beren, 9k+6 instantly became a structure standing for itself, separated from the specific case of 24 being examined.

Beren: Three equal groups. Wait a minute, 3 okay. Same two of singles thing we will have. Two more will be in each of these (cups) and three of these (cups)… But it is not 24 then? (Probably thought each cup to contain a single cookie as in the previous case.)
Researcher: Why not?
Beren: No. Wait a minute. Whatever the number is. Okay I now understand (Seems to experience an a-ha moment). No matter what the number it is, my things here (points to the items in the second step) are the same. That means, again 3 cups and 2 single packs go in (each basket). Only the number inside the cup(s) changes. Okay, I got it. So, in total, this time, there will be […] 8 cookies.
Researcher: What is your decision? (When Beren came to marking the last box)
Beren: Can be separated. They all can be separated, in fact.
Researcher: What do you mean they all can be separated?
Beren: Well, think of it this way. You know, the number that is right here (Points to the number 24 shown in general representation as 9 groups of 2 singles plus 6 more singles), how can I explain it… It is not that the number 24 here when we actually look at it. I mean, you know it is (a thing) consisting of 9 equal groups and 6 singles.
Researcher: Yes.
Beren: It already. Well, let me say this way. This one is separable, in any way, it is separable into three equal parts and it is again three cups and two single packs. I can't put it into words but… is divisible by three.

Even if the intended abstract structure was observed in the second flowchart, the researcher encouraged Beren to work on further appropriate cases. She picked the numbers 33 and 42 for the next two flowcharts. Increasing the number inside the cups by one at a time, she consciously and purposefully recited the steps of the flowchart to match the same structure as in the first two ones. For each pick, she mentally did the division operation first (33:3 and 42:3) to check the result of the divisibility decision. She didn't seem as confident as she had just been when transitioning from the second to the third flowchart about the steps she was following. Only after making sure that the number selected was perfectly divisible by three, she drew three baskets in the third steps by rote, equally distributed cups and singles to each of the baskets and then marked the number of items in each of the cups to obtain the correct quotient (11 and 14). Then she considered choosing an "upper number" (üst rakam), as she put it. She selected 105 and calculated 11 items to go in each of the cups. This time she was about to make her decision without making the division operation 105:3. She attempted to reason based on the modular structure of the number 105 (refers to this structure as a "rule" in the below except). As she was not fully certain, Beren questioned whether the conjecture was a proven one or not. Unlike observed in the initial task, this time she ascertained herself by making a connection to the Monstrous Counterexample Illustration.

Beren: The number 105 […] can be separated. Why it can be separated? Because, in the end, it goes again with the same rule and all of them that go with this rule necessarily have to be. Well, it might not be proven, but I think it holds… It holds, isn't it? But wait a minute, it did hold, because that number… that day there was a number when I tried to read it, I couldn't read, starting with 30…. that in fact was a single number but I read as multiple numbers. Even if it goes up to those numbers, based on this thing (points to the step two), it already is divisible. Alright "divisible", that is it.

When she was asked to mentally run the flowchart activity for the number 4794 and a number that is greater than ten million, her reasoning was based on the same abstract structure again—the property of all the numbers in the conjecture domain.

Beren: You know this (step one) and this one (step two) already holds. This one (step three), as well. This one (step four) certainly holds as well. If, in here, one of them holds (points to the statement 'One of the numbers that give a remainder of 6 when divided by 9 is 4794.') […] again, divisible this is, because based on the digits (numbers) here (points to the configuration in box 2). Just I would have some trouble figuring out the number of cookies in each cup.

Beren: For a number greater than ten million […] I would again obtain the same result. If for example it was something around eleven million, again that number would be one of those numbers that gives a remainder of 6 when divided by 9. Again, that number can be shown as 9 equal groups and 6 singles. I just wouldn't know the number in the equal groups, well, which I would be able to find if I think and try.

She was certain of the truth of the conjecture. In the rest of the dialog, she used the expression "we prove" for the first time. Compared to her previous uses of the verb, where she attributed absolute certainty to an outside authority, here she expressed confidence in her decision.

Beren: After all, if we add 6 to one of the numbers that is a multiple of 9, it is already divisible by 3 without a remainder.

Researcher: Why?

Beren: We have already divided them into three here. And here we prove that it is divisible. And the reason is, well, for me there is not a reason for this. That just means all numbers that give a remainder of 6 when divisible by 9 can also be divided by 3 without a remainder.

Discussion

During her participation in the modified task, when Beren first detached the modular structure *9k+6* from the particular case being examined, the number 24, she also seemed to instantly have abstracted the logical necessity of the relationship whose correctness was being investigated. We interpret the two abstractions as having occurred simultaneously since the continuous data did not reveal any time lag in between. In Beren's expression, the two abstract structures appeared immediately after one another. First, she revealed an understanding of the structure *9k+6* as an invariant property of any and all of the numbers in the pattern "*15, 24, 33, 42, 51, 60, 69*, …" Then, she referred to this abstraction as the reason why the relationship between being a member of the pattern and being divisible by 3 was invariant.

Beren: No matter what the number it is, my things here (cups and counting items) are the same. (The first abstraction) That means, again 3 cups and 2 single packs go in (each basket). Only the number inside the cup(s) changes. (The second abstraction)

This marked the momentary abstraction of the intended logical necessity relationship. Beren utilized both abstractions together to explain why, in her words, "they (the elements of the pattern) all can be separated." However, this was only an instantaneous emergence of the intended structural understanding of the first instance of proof. While working on the subsequent case-based flowcharts for the numbers 33, 42, and 105, Beren did not maintain the logical necessity relationship that seemed to have been abstracted. Only after making a connection to the Monstrous Counterexample Illustration, while working on the case 105, the intended coordination of actions was completed.

This connection seems to have unexpectedly fostered a second reflection on Beren's way of thinking about the act of proving, leading her to relate the whole activity she engaged into the goal of proving. We speculate the twenty-six-digit number, from the Monstrous Counterexample Illustration, might have caused Beren to better envisage the extent of the conjecture domain to reflect on her abstraction as a *for all* argument (her way of understanding the particular relationship) and to reflect on her proof scheme (her way of thinking regarding the act of proving) to resolve the cognitive conflict she experienced previously in the second phase. Even though the intended learning took place only after introducing a modification to the initially designed task, Beren's learning from her activity provided supporting evidence for the initial hypothesis of the study, with a slight difference. Her abstraction of the intended logical necessity relationship was a result of the coordination of actions mechanism combined with her intellectual need for a secure method of proving.

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A model-theoretic approach to didactic research on proof and proving in mathematics

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Abstract – The model-theoretic approach developed by Tarski in the first half of the 20th century aimed at building a bridge between formal systems and reality, including mathematics. In this theoretical paper, we first present the main features of the model-theoretic approach, which we illustrate by presenting and discussing the axiomatic systems provided by Tarski for the foundations of arithmetic of real numbers. In the second part, we highlight and illustrate, on the basis of previous works, the value of a model-theoretic approach for didactic research on proof and proving in mathematics. This supports the claim that the model-theoretic approach developed by Tarski provides researchers with a logical framework for addressing critical questions about the dialectical relationship between syntax and semantics in proof and proving, from a didactic perspective.

Keywords: Didactics and epistemology of mathematics, model-theoretic approach to proof and proving, syntax, semantics, proof by interpretation.

Introduction

In a paper published in 2001, Hourya Sinaceur enlightened the fruitful contribution of Tarski to metamathematics:

> Tarski's specific contribution consisted in exploring the mutual relations between sentences and their mathematical models, showing that semantic analysis is furthered and not superseded by syntactic analysis and in weaving metamathematical and mathematical threads in a much tighter fashion than anyone had done before him. (Sinaceur, 2001, p.52)

This focus on the interplay between syntax and semantics and its relevance for clarifying the relationship between *truth* in an interpretation and *logical validity* was for us the main initial motivation for investigating the relevance of Tarski's perspective for didactic analysis. This relevance has been acknowledged in Durand-Guerrier (2008) for what concerns the relationship between truth and validity. It has been the logical background for much of our research since then.

In this theoretical paper, we first present the main features of the model-theoretic approach developed by Tarski in the first half of the 20th century (Tarski, 1994). We then illustrate this approach by presenting and discussing the axiomatic systems provided by Tarski for the foundations of arithmetic of real numbers. The second part of the paper is devoted to highlighting the value of a model-theoretic approach for conducting didactic research on proof and proving in mathematics.

All along the present paper, *logic* refers to *classical first-order logic* in which only variables that range over individuals are quantified, and the *excluded middle principle* holds.

Main features of a model-theoretic approach

As developed in Durand-Guerrier (2008, pp. 376–377), the initial step was done by Tarski with his definition of truth as "materially adequate" and "formally correct" (Tarski, 1983, p. 159). Tarski's

perspective was to bridge formal system and reality (the latter being either material reality, mathematical theory, or local theory): a proposition is true if it designates a state of things (Tarski, 1944). Although the term "model theory" appeared first in a publication in 1954, the model-theoretic approach was first developed in his book *Introduction to logic and to the methodology of deductive sciences,* first published in 1936[1].In the chapter VI of this book, *Methodology of deductive sciences*, Tarski introduced, by the example of the *miniature theory* of the congruence of segments, a method to associate to a given deductive theory a formal axiom system in which the statements of the initial deductive theory "become sentential functions containing as free variables those symbols by which the primitive terms have been replaced, and not containing any constant but those belonging to logic" (Tarski, 1994, p.117). This formal axiom system might then be reinterpreted using other *interpretation*s. In case the reinterpretations of all the axioms of the formal system are true, the new interpretation is a *model* of the axiom system. Of course, the initial given theory is a model of the axiom system. Then, Tarski proves an important result:

> Every theorem of a given deductive theory is satisfied by any model of the axiom system of this theory, and moreover to every theorem there corresponds a general statement which can be formulated and proved within the framework of logic, and which asserts the fact that the theorem in question is satisfied by any such model. (ibid., p.117)

This leads to what Tarski names *method of proof by exhibition of a model* or *proof by interpretation*. Let us consider the case where a sentence A in a deductive theory seems undoubtedly true, but we fail to find a proof.

> Thus, the conjecture arises that Sentence A simply cannot be proved from the axioms of the theory. In order to confirm this conjecture, we argue in the following way: if sentence A could be deduced from our axioms, every model of these axioms would satisfy the sentence; therefore if we succeed in describing a model of the axiom system that does not satisfies Sentence A, then we shall have obtained a proof that Sentence A cannot be deduced from the axioms [of the theory in question]. (ibid, p.116)

An elementary example of the construction of an axiom system and of the use of these two results is provided in the case of an item of the *Math Kangaroo* (In French *Kangourou des mathématiques*) in Durand-Guerrier (2008, pp. 377–379). This method of proof by interpretation can also be used to show that the Intermediate value property is not a theorem in the axiomatic theory of ordered fields with its natural topology (i.e., the topology generated by the open intervals). Indeed, $\mathbb{Q}$ with its standard order and the associate natural topology is a model of this theory but not a model of the property: it is true for certain pair (function, interval) and not for others. Adding an axiom of completeness provides a new axiomatic system for which $\mathbb{Q}$ with its standard order and the associate natural topology is not a model, while $\mathbb{R}$ with its standard order and the associate natural topology is.

[1] In this paper we use Tarski (1994), the fourth edition of the English version.

Tarski's axiomatic system for the foundations of arithmetic of real numbers

In chapter X of Tarski (1994), we find a presentation of an axiom system for the foundations of arithmetic of real numbers, which is based on an axiom system for groups developed and discussed in previous chapters (system A in the book). There are two ways to extend System A in this chapter. We will only present and discuss the first, which is sufficient for our purposes.

The new system A' contains four primitive terms: "$\mathfrak{R}$", "$<$", "$+$" and "1", and consists of nine axioms:

Axiom 1': If $x \neq y$, then $x < y$ or $y < x$.

Axiom 2': If $x < y$, then $y \not< x$.

Axiom 3': If $x < z$, *then there exists a number y such that* $x < y$ and $y < z$.

Axiom 4': If S and T *are any sets of numbers* (That is $S \sqsubseteq \mathfrak{R}$ and $T \sqsubseteq \mathfrak{R}$) *satisfying the condition:*

for any x *belonging to* S *and any* y *belonging to* T,

we have: $x < y$,

then there exists a number z *for which the following condition holds*:

if x is any element of S, and y any element of T, and

if $x \neq z$ and $y \neq z$, then $x < z$ and $z < y$.

Axiom 5': $x + (y + z) = (x + z) + y$

Axiom 6': *For any numbers* x *and* y *there exists a number* z *such that* $x = y + z$.

Axiom 7': If $x + z < y + t$, then $x < y$ or $z < t$.

Axioms 8': $1 \in \mathcal{R}$.

Axiom 9': $1 < 1 + 1$.

Figure 1. Tarski's first axiom system for Arithmetic of real numbers (Tarski, 1994, p. 202)

Axiom 1' to 4' concern the properties of the order relation. In chapter V (section 31, pp. 91–92) Tarski defines an order relation on a class K as a relation that is asymmetric, transitive and connected. Such a relation is also irreflexive. As an example, he mentions the relation *less than* that establishes an order in any set of numbers. This can also be expressed by *less than* is an ordering relation in any set of numbers, or by any set of numbers is ordered by the relation *less than*.

In the axiom system for the set of real numbers, Axiom 1' expresses that the relation is connected; Axiom 2' expresses that the relation is asymmetric.

It is the first occurrence in the book of axioms 3' and 4'. Axiom 3' corresponds to the law of density for the relation *less than* (a particular case of the law of density for a relation). Axiom 4' expresses the law of continuity for the relation *less than*. Tarski mentions that it is also named Dedekind's axiom. After defining a relation between two sets named "precedes", Tarski gives a very simple formulation of the axiom of continuity: "If one set of numbers precedes another, then there exists at least one number separating the first one from the second one". (p. 203)

In this form, this axiom captures our intuition of the continuum and fits the method developed by Dedekind in his essay "Creation of irrational numbers" (Dedekind, 1963).

In this axiom system (Figure 1), it is easy to prove the existence of a supremum for a bounded nonempty subset of the set of real numbers, a property of the set of real numbers which is often chosen as an axiom in French universities (another axiom often chosen is the convergence of Cauchy sequences), and then to prove that the intermediate value property holds in the ordered set of real numbers (with the standard order and the natural associated topology). Together with the bijection between the Cartesian product $\mathbb{R} \times \mathbb{R}$, and the plane (as set of points) with a cartesian coordinates system, this clarifies the back and forth between the algebraic and the geometrical register in Calculus (e.g. conjecturing the existence of solutions in some intervals in the graphical register, and then moving back to the algebraic register, using the intermediate value theorem to prove the conjecture).

While constructing this theory, Tarski uses his methodology of deductive sciences as mentioned in section VI of the book: the axioms are chosen to fit with our intuition of what should be true in the mathematical theory that we consider (here, the arithmetic of real numbers).

Then, this axiom system can be interpreted in a variety of mathematical domains, for example, in various sets of numbers. A question that could be raised is, do we really need to introduce Axiom 4', or would it be possible to derive it from the axioms 1', 2', and 3'? We know that it is not the case because of the incompleteness of the set of rational numbers that satisfies the System A", with all axioms of System A', except 4'. This is a proof by interpretation that Axiom 4' cannot be proved in the System A". The reverse question could also be raised: would it be possible to derive Axiom 3' from Axiom 4'? Here also, the answer is no; indeed, the set of natural numbers ordered by the standard order satisfies the System A"' composed of Axioms 1', 2', 4', 5' to 9' of System A', but does not satisfy Axiom 3': the standard order on the set of natural number is discrete, i.e. every element of the set of natural numbers has a successor; consequently, the following statement, which is the negation of axiom 3', is satisfied in the ordered set of natural numbers (with the standard order)

There exist at least two elements x, z such that $x < z$ and for any number y, $y \leq x \text{ or } z \leq y$.

This allows to point out the difference between dense non continuous (or incomplete) ordered set, such as the set of rational numbers or the set of algebraic numbers with the standard order *less than*, and dense and continuous (complete) ordered sets, such as the set of real numbers with the same standard order. Another interest of this axiom system is that it enlightens that discreteness, density, and continuity (completeness) are properties of order relations on a set, not of the set itself, given that different relations of order can be considered on a given set. There is research-based evidence that these two points are challenging for undergraduate students and secondary prospective teachers (Durand-Guerrier, 2016; Branchetti & Durand-Guerrier, 2023).

Value of a model theoretic approach for didactic research on proof and proving in mathematics

A fruitful theory for didactic analysis of mathematical activity

Logical matters are largely underestimated by teachers, as well in secondary school as at tertiary level, in particular concerning difference between novices and experts: an expert in a mathematical

field knows when it is dangerous to slack off the rigorous application of inference rules, while novices have to learn this at the same time as they acquire the relevant mathematical knowledge. These two aspects of mathematics cannot be learned separately. The back and forth between sentences and objects, syntax and semantics, interpretation and theory are at the core of mathematical learning (Durand-Guerrier & Arsac, 2005). From a didactic perspective, the model-theoretic approach developed by Tarski provides researchers with a logical framework for addressing questions about the dialectical relationship between syntax and semantics, which are critical in proof and proving, as emphasised by Weber and Alcock (2004). As Sinaceur (1991) states, for mathematicians, we consider that Tarski's construction of a definition of truth materially adequate and formally correct, and his project of bridging formal system and reality (including mathematical theories) provides an effective epistemology in mathematics education. This opens avenues for analysing mathematical activity and reconsidering the traditional assumption in mathematics education that there is a great distance between common sense and mathematical logic. In the following sections, we develop examples drawn from our own works to illustrate this fecundity

Fostering awareness of students of the role of theory in proof and proving in mathematics

As Mariotti (2006) points out, a theorem is a complex of a statement, a theory, and a proof of this statement in this theory. However, at secondary school and even in the first years of university, the theory is mainly left in the shadow. In other words, the logos bloc of praxeologies (Chevallard & Bosch, 2020) is most often reduced to technologies, which is likely to prevent students from being aware of the critical role of theory in proof and proving.

At the heart of the model-theoretic approach is a better understanding of the relationship between truth in an interpretation and logical validity in an axiomatic system: that is, accounting for the difference between a theorem and a statement that is considered true in an interpretation, but not provable in the corresponding axiomatic system. This was discussed in the 19th century by Bolzano (1817), who, unlike Cauchy (1821), did not accept proofs of the intermediate value theorem based on geometrical considerations, but looked for a proof in the theory of equations, where the statement was formulated. It was the first step that led to the creation of the sets of real numbers in the middle of the 19th century, bringing to the fore the issue of $\mathbb{Q}$-incompleteness versus $\mathbb{R}$-completeness. It is noticeable that this issue is almost never explicitly addressed in French curricula, since the set of real numbers is implicitly considered the reference set (i.e., the set of all numbers). Nevertheless, there is research-based evidence that students face significant difficulties when confronted with non-standard activities that require a genuine conceptualisation of the objects involved (Durand-Guerrier, 2016; Branchetti & Durand-Guerrier, 2023). Defining and proving are two intertwined practises that can be traced in the history of mathematics, and that we consider valuable for students to develop**.** In Durand-Guerrier (2022), we discuss this on the epistemological side in the case of real numbers and of infinity. Based on these studies, we argue that "emphasising the intertwining between *defining and proving* rather than the relationship between *definitions and proofs* opens new research avenues on students' practices at the secondary/tertiary transition". Durand-Guerrier and Tanguay (2018) highlight in the case of $\mathbb{R}$-completeness, "the importance of the way objects are defined towards molding the proofs, even when the approaches seem rather similar at first sight (e.g. defining real numbers as decimal expansions)" (p. 32**).** This highlights the importance of introducing undergraduates and future

teachers to activities that develop such practices, to raise their awareness of the importance of theoretical choices. In the case of completeness, beyond the issue of choosing definitions for objects, another challenge arises from the variety of possible axioms when moving to axiomatic systems of complete ordered fields (Bergé, 2010). An example of a multi-proof activity focusing on the Bolzano–Weierstrass theorem and designed to address this issue is presented in Durand-Guerrier (2024).

Dealing with incorrect proof of true statements

In addition to the well-documented reluctance of many students to engage with proofs of statements they consider obvious, there is also the difficulty of reconsidering an incorrect proof of a true statement. An example using the generalised mean value theorem is given in Durand-Guerrier (2008). More recently, we have provided an example (*Tilling by dominoes*) in a paper discussing logical issues at the interface between mathematics and computer sciences (Durand-Guerrier et al., 2019, pp.121–122). The problem is as follows: Given a rectangular grid (with integral dimensions), is it possible to tile it with dominoes (1x2 rectangles)? The answer is given by the theorem: *A rectangular grid can be tiled with dominoes if and only if its area is even.* A common (incorrect) proof of the above theorem given by students is the following: "a grid can be tiled by dominoes if and only if its area is $2k$ where k is the number of dominoes, which means that the area of the grid is even". Here, in Tarski's perspective, we can consider a kind of *miniature* theory whose domain is the sets of polyominos, a measure of area with the square as unit of measure; a domino is composed of two squares; its area is 2. An *axiom* (equi-decomposability of area): If an area can be decomposed into k subareas, then the measure of the total area is the sum of all the measures of the subareas.

Re-interpreting this miniature theory in the set of rectangular grids (a subset of the set of polyominos), the statement "*A rectangular grid can be tiled by dominoes if and only if its area is even"* is true. However, the above proof is incorrect. In fact, if the proof were correct, it should apply to any polyomino consisting of an even number of squares. It is easy to find a polyomino with an even surface that cannot be tiled with dominoes. This is a proof by interpretation, in Tarski's sense, that to prove the theorem for a rectangular grid, the miniature theory needs to be enriched.

This kind of argument can be used to convince students who have provided the above proof that it is not correct. In fact, it is an incomplete proof; it is necessary to prove that a rectangular grid (with integral dimensions) and an even surface can be tiled with dominoes, which may seem obvious to students.

Conclusion

In this paper, we intend to share with the audience how Tarski's model-theoretic approach has informed our research into proof and proving in general, and specifically in the case of the completeness/incompleteness of a set of numbers. However, this does not mean that this model-theoretic approach should be taught to undergraduate mathematics students or pre-service mathematics teachers. Rather, it is a conceptual framework to guide epistemological studies from a didactic perspective, as prolegomena to the design of didactic engineering (Artigue, 2020) aimed at

fostering proof and proving skills and, consequently, the conceptualisation of the mathematical objects at stake.

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Justification not Proof as the Scope of Research: The RPA Model of Argumentation

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What kind of argumentation is shown when students try to convince an interlocutor in the context of division by zero? A grounded analysis of written argumentation leads to a classification into three groups: the RPA model. The empirically derived categories aim to provide insight (rich R-type), to imitate a mathematical argumentation (pseudo P-type), or to establish the reliability of the source of information (apodictic A-type). Attempts to proof are rare. While the R and A types are of obvious rationality, it remains an open question whether the P type follows a successful learning strategy. The empirical results are based on approximately 1,500 written texts on a special item that allows control and categorization by AI, a survey of argumentation skills at the level of learning groups, and a study of the quasi-longitudinal development of this competence.

Keywords: Argumentation, justification, discourse analysis.

Introduction

This is an exploratory study: How do students respond when asked to *justify* an opinion they hold? Will they show proofs or at least attempts to proof? A grounded analysis by inductive qualitative content analysis (QCA, Mayring, 2000) will deviate from the too narrow alignment to proof and decern any clusters. Moreover, this study does not have any particular interest in a theory of preliminary stages of proving activities, as there is already a substantial body of literature on this subject (e.g., Boero et al., 2010; Duval, 1991; Pedemonte, 2007). Instead, it focuses on the communicative purposes of any reaction to the stimulus "to justify" that is set by the task. Therefore, the main research questions of the study treated here are:

- (RQ1) What categories will be found?
- (RQ2) In what abundance?
- (RQ3) What is the quasi-longitudinal development, that is, the dependence on grade?

Due to the stimulus "to justify" and to connect this study to the existing literature, a review of definitions of the word field of "to justify" should be the first step. Three criteria are presented for the purpose of selecting appropriate definitions that will lead to the establishment of a special system of terms. The meaning of the terms for students is of importance and thus investigated, but not presented here. One of two long-term objectives of this study is to make a contribution to the monitoring of argumentation competencies on the level of learning groups, which is of interest for school development (see below). Another objective, not finished yet, is to provide a quantitative empirical review of the categories identified by Harel and Sowder (1998) in their influential paper, extending them beyond the scope of university students.

Theoretical background

There is considerable divergence in the literature with regard to the definitions of the basic notions in the conceptual field of argumentation (see Brunner 2014; especially the overview on p. 231). Even the term "proof" is not universally defined (Weber et al., 2022). Two factors contribute to the emergence of this ambiguity of terms and, consequently, of notions: The terms "argumentation", "justification", "explanation", and even "proof" are part of the everyday language register. As a result, there is a temptation to overestimate the degree to which others share one's concept images, especially, but not only, when translation is involved. Furthermore, the particular field of research requires a differentiation of the concepts in order to align them with the specific characteristics inherent to that field. To illustrate, if one is investigating university students, proof is a much more prominent feature and notions describing preliminary forms of proof would be useful. So, like in Fiallo and Gutiérrez (2017, p. 147), it is necessary to provide clear definitions of the terms used.

The literature contains a variety of argumentation typologies. It is possible to distinguish between argumentation and proof (Duval, 1991). Similarly, Tsamir and Sheffer (2000) differentiate between concrete and formal (algebraic) arguments, even with regard to division by zero. Following Pedemonte (2007) argumentations are based on a system of conceptions and related to conjectures either as "structurant" or as "constructive" argumentation. Furthermore, an argumentation may be classified as abductive, inductive, or deductive (Pedemonte, 2007, pp. 28–31). All these characteristics could be applied to our data set; however, as our study does not focus on proof, they do not adequately describe the variety of justifications found in our study. Harel and Sowder (1998) employ the term "proof schemes", which refers to "what the person offers to convince others" (p. 275). Apart from the choice of the identifier "proof", this fits well into the model of argumentation given below, aligning closely with our definition of justification, despite the criticism of this "subjective" approach by Weber et al. (2022, p. 66) as it "serves a psychological purpose", namely to convince. The classification scheme proposed by Harel and Sowder (externally conviction, empirical, and analytical proof schemes as superordinate categories; "analytical" is changed into "deductive" in Harel, 2008) will be compared to the findings of our study elsewhere. The proof scheme "external conviction" seems to be similar to our "apodictic" type (see below), but we do not identify the two terms to be able to discuss potential differences between our grounded found type and a rather theoretically structured proof scheme (second long term objective, see above).

It can be reasonably assumed that the majority of research concerning argumentation has been oriented towards the objective of developing effective pedagogical strategies for teaching especially proof, at least in a broad perspective. However, this article does not focus on proof per se due to both practical and theoretical considerations.

Firstly, it can be observed that proving activities are marginalised within the context of German schools. The national curriculum only makes reference to formal proofs as a particular instance on a continuum of argumentative activities, which also encompasses conjectures, plausibility arguments and empirical arguments (KMK, 2012). A survey of all 505 final examination tasks, published in the 2017-2024 "pool" of the national institute IQB indicates that the verb "to prove" was not utilized at all, whereas "to justify" was used 353 times (IQB, 2024). Thirty years ago, proof was by far more

prominent in classroom practice. The change may be attributed to several factors, including a higher percentage of students attending higher secondary schools, which has resulted in increased heterogeneity; a lack of constructive alignment, as cited above; the introduction of new subjects in the curriculum, such as statistics; and a resulting decline in the prominence of geometry, where the majority of proving activities had previously taken place. Secondly (and perhaps as a consequence), the data collected showed proofs to a limited extend, only. Thirdly, historically within our scientific community, the notion of proof was step for step expanded. Processes of exploring, of conjecturing (Boero et al., 2010), even of "inductive" proofs (Pedemonte, 2007) and other "informal reasoning" (Jones & Herbst, 2012) were also incorporated. The reason was to expand the scope, deepen the analysis (cf. Boero, 2017, argumentative activities in the "exploratory phase of proving"), and ensure greater relevance for existing classroom practice (cf. e.g. Fiallo & Gutiérrez 2017). Therefore, the defining feature of our study aligns with a broader developmental trajectory, on the one hand. On the other, here, argumentation is not taken as the antithesis to proof as proposed by Duval (1991), but rather as a generic term, with proof representing only one possibility among others.

In the following, definitions of the main terms are given in accordance with the following criteria: Firstly, they should be adapted to describe the data collected. Secondly, they should be consistent with the communication model employed and form a coherent system of concepts. Thirdly, they should be compatible with the language used in official documents related to the probe. In addition to these criteria, it would be beneficial to consider connections to existing didactical literature.

Following criterium three, **argumentation** has to be the generic term, encompassing proof, justification, explanation, and elucidation (KMK, 2012). It serves to settle or expand the shared content bases and rules of communication (Ger. "Rede" in the sense of Habermas, 1984). In the didactic literature, taking "argumentation" as generic term is found e.g. in Bezold (2009).

The communication model is a straightforward one (recurring among others to Habermas, 1984): Both the sender and receiver refer to a knowledge and communication basis, which they assume to be shared. The objects of justification are statements that are perceived as having varying degrees of plausibility by the two interlocutors. These degrees of plausibility are conceptualised as an ordinal structure, which models the attributed epistemic values. It is evident that this concept of argumentation is genuinely dialogic, encompassing the possibility that the receiver can also be an internal entity within the sender, or a universal audience.

Reasoning or **justification** represents a communicative reaction of the sender to a questioning of a statement by the receiver. The objective of reasoning is to increase the degree of plausibility of the statement under discussion by relating it to the basis of knowledge and communication that is assumed to be shared (Habermas, 1984).

A **proof** is constituted by a sequence of argumentative steps that rely on an accepted basis of statements, with the objective of establishing a complete logical chain of deductive steps (Duval, 1991). A mathematical proof is merely one method of justification among others, and its efficacy is contingent upon the mathematical abilities of the sender and receiver. For example, one student's proof step $7:0 = x \mid \cdot 0 \rightarrow 7 = 0$ may not be convincing to another student if the concept of equivalence transformation is not sufficiently understood. Weber et al. (2022) demonstrate that this extends even

to mathematicians, where the complexity of the proof, rather than the ability of the author, may necessitate supplementary inductive reasoning or socially constructed justifications.

An **explanation** (of "why", not "how" or "how to do") is an addressee-oriented justification with the aim of creating an "understanding".

These three different activities are not mutually exclusive but rather serve distinct purposes, namely plausibility, logical coherence, and understanding, which can be united. For instance, justifications may attempt to explain but may also just indicate reliable sources (see RPA model below).

The Study and method of analysis

Several items were pilot tested, each of which was assumed to evoke a variety of different types of justification (e.g. binomial formula, square root, statistical mean). The item presented here, division by zero, was chosen for a broader study because it showed the greatest variety of justifications. We asked the students to give the result of the division 7:0 and to "justify [their] opinion in a way that someone who doesn't know the answer is able to understand [the result]" (translation of the original task from German). The item was part of different tests administered to students in grades 7 to 13 in two secondary schools in Germany, as well as to pre-service teachers in their first lecture and in the middle of their studies, all in convenience samples (in fact, the study is being extended to 17 schools, but this part of the data is not yet available). Division by zero is covered in year 5 or 6. Different basic ideas about division are part of primary school education. The written responses of N=1442 students were analysed. In regard to the relative abundance of argumentation types, we report data from a subgroup of N=1185 students who did not take part in interventional courses.

The initial step of the analysis involved the application of a qualitative content analysis (QCA, Mayring, 2000) with inductive category development (grounded approach) to 365 texts from grades 7, 9, 11 and 13. Consequently, the student justifications were subjected to a multi-stage analysis. Initially, the texts were classified solely on the basis of their similarities, without any recourse to theoretical frameworks. Subsequently, the items were aggregated at an increasing level of abstraction, resulting in the identification of the various types of argumentation. To ensure reliability, a second step was conducted in which a pair of university students performed a separate "deductive category application" (Mayring, 2000, Section 4.2) using a coding manual describing the found types. The inter-rater reliability for this step was κ=.967 for the initial sample of N=365 texts. Further evidence on the reliability is provided by the accuracy of an AI classification by supervised learning, which achieved an accuracy of >0.93 (N=1442, size of evaluation sample 200). The AI tool (Lutz & Fahse, 2024), designed to assess argumentation skills, is currently available in German. A methodological decision that may appear surprising at first glance is that we do not assess the correctness of the responses. Our focus is on the capacity for argumentation, which we conceptualise as a construct that is not stochastically independent but nevertheless distinct from mathematical ability.

Results of the clustering by QCA (RQ1)

The QCA gives rise to three distinct types of argumentation, more specifically justification, forming the RPA model. Argumentations of a **rich (R-type)** nature are associated with an appropriate and well-structured mathematical domain, such as that pertaining to the distribution of quantities or

algebra (reversal of operations). The warrants employ basic concepts and are potentially useful even if the result may be erroneous. For example, "7:0=0 because no one gets an apple if there is no one to get any of the seven apples": This domain and the warrant are appropriate because they could lead to the insight that there is no answer to the question "How much does everybody get?" and this question is a correct potential modelling of division. There are operations – here the distribution, or more abstract, the change of representation. The rationale of this type is *to provide insight* (or, more seldom, a logical chain of arguments to check) *for a critical instance*.

The **P-type (pseudo-factual)** provides an answer that resembles a mathematical one; however, it is more concerned with appearing to be a genuine answer from a mathematics lesson than with offering insight or providing certainty for those who are critical. It employs unapt or poorly structured domains, sometimes fanciful (e.g., "All operations with 0 give 0" as an algebraic rule), and utilises only a limited number of operations. These cannot be improved to provide a rich answer, meaning they are essentially wrong. The rationale behind (why is it a justification?) is: "We do it always like this in mathematics much alike", the value is similarity.

In contrast to the first two types, which require a mathematical or content-based justification, the **A-type (apodictic)** provides reasons for the *reliability of the source* of the statement (information). There is no mention of mathematical domains or operations used for the warrants; instead, social authorities are referenced (in the broadest sense, including reference to the calculator or to the knowledge of the sender – "That's the way it is, I know it for sure.").

Results of the statistical analysis (RQ2, RQ3)

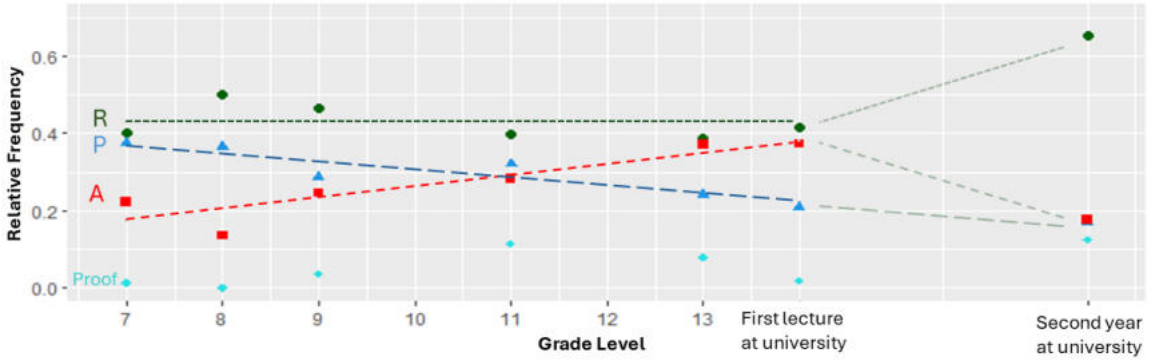
First and foremost, it is of interest to investigate the frequencies of the types in question (RQ2). In instances where more than one type is present, the cases are mapped in accordance with the rule "R before P before A". This rule, and perhaps the intention of the whole study, would be misunderstood if taken as a hierarchy of value between the three types, since the aim is to describe. On the contrary, the reason for the rule is logical: if a rich argument is presented, it achieves the objective of P, thereby rendering the attribution of P redundant. Furthermore, if R or P are identified in a text, the value of a mathematical justification is accepted, precluding the possibility of assigning A in a pure sense. It should be noted that all results are preliminary, as the study is still in progress.

With reference to Figure 1, which illustrates the quasi-longitudinal development (RQ3), it can be observed that Type R is about stagnating, P is descending, and A is demonstrating an increase. The most prevalent type is R, fluctuating about 40%. The other types are considerable, too, though they undergo changes over time. The proportion of P-type cases declines from almost 40% to about the half in opposition to the A-type, which roughly doubles from about 20% to almost 40% in the seven years. These two slopes observed for the initial 6 measurement times are significant, as indicated in Figure 1. The texts coded by "Proof" in Figure 1 (lower dots) are part of type R and comprise also texts only vaguely referring to multiplication as the inverse operation to division. Therefore, even when considering a favourable interpretation, there is a paucity of attempts to prove.

A comparison of the last two measurement times reveals a significant effect of the university, although it should be noted that only preservice teachers were included in the testing, here. They

obviously did significantly learn justifying but not proving in the same degree (the majority of them was preparing for primary school).

What is the effect of the teaching on argumentation in the various study groups? A total of 411 students belonged to 17 fixed study groups (only existing in grade levels 7 to 10, class sizes 19 to 29, mean 24.1). The prevalence of each type varies significantly with the different groups (Chi²-test in a log-linear regression with partialised grade, p<.001, D=81.1 for type R). The high value of the deviance indicates that the existing culture of argumentation within the study groups, implemented by different teachers, has a remarkable impact.



Number of students n=72, 88, 191, 78, 62, 518, 176 for grade 7 to university resp.. All cases with no answer to the division task were eliminated before. Linear model with slope significantly unequal 0 only for type P and A (t-tests, p=.009; .013 resp.).

Figure 1: Quasi-longitudinal development

Discussion

The study is limited by the use of convenience samples, which is somewhat mitigated by the sample size and the observation that the frequencies follow a satisfactory linear model. However, only the major trends are worthy of discussion in the following.

The substantial abundance of apodictic argumentation, a type of very poor justification that does not seek any understanding, could be irritating at first sight, particularly given its tendency to increase over time. It is, however, important to bear in mind that in communicative situations, the A-type perfectly fits the requirements of justification by increasing the plausibility of a statement. Thus, it is a pragmatic and sensible action, particularly well-suited in contexts such as engineering. In pedagogical practice, if the objective is to encourage an emphasis on comprehension, it would be both prudent and accurate to refrain from discrediting the A-type. Instead, it would be more beneficial to briefly address in class the various methods of enhancing plausibility, with an examination of the respective merits of each approach (plausibility independent of any mathematical ability of the interlocutors versus understanding and expanding networks of knowledge).

This recommendation relies on the communication model: The task presented to the student is to *justify* (which students do with appropriate rationality in R- and A-type), but we want them to *give* a mathematical justification instead. The latter is a request and does not serve the purpose of a justification in a real communicative situation. According to Habermas (1984), this confuses discourse (justifying) with communicative action (fulfilling a request). The teacher does not need to

be convinced as he/she is already aware of the veracity of the statement. Therefore, he/she cannot demand a proper justification other than in a simulated, purely artificial communicative situation. One implication for teaching would be to provide genuine problems (of pure mathematics or real contexts), where students attempt to convince each other and not a teacher.

What are the implications of this study? Firstly, it shows the relative frequency of the three types, indicating different ways of arguing that could, perhaps should, be discussed with students (without the abbreviations R, P, A): "Type A is appropriate when it is essential that the statement is true, for example in the context of engineering, or when the logical chain in a proof is completed by a mere reference to theorems known to be proven. Type R would be more appropriate in a different communicative setting, where the aim is to create understanding. The P type is only an imitation, but in mathematics similarity is meaningless: taking the double or the square root of 9 plus 16 seems to be similar, but the former is the double of 9 plus the double of 16, but the latter is not '3 plus 4'". The distinction between the two types A and R depends on the two interlocutors. If the proposition to which they are returning is understood and known by the receiver, no argumentation is needed at all, but only a reference. However, even in pure mathematics, the development of new theorems would be facilitated by a deeper understanding, which would lead to type R in line with our administrative regulations. In Germany, there is a consensus among didacticians and school authorities to cultivate overarching competencies such as argumentation and communication, conceptualised as a means of fostering comprehension in place of mere mechanical calculations. This suggests that formal operations should be accompanied by illustrative or explanatory activities (Ger. "Grundvorstellungen"). The extent to which a learning group has achieved this objective can be gauged in a relatively short time in the classroom using the item proposed here. The efficacy of the item is demonstrated by the markedly disparate frequencies observed at the learning group level, as well as the impact of an intervention, which is not discussed in this study.

Does the learning of mathematical argumentation through imitation (P-type) represent a futile pursuit or a valuable preliminary stage? In the field of foreign language pedagogy, the concept of inductive grammar is regarded as an effective and widely accepted approach to acquiring rules that are not too dissimilar from those encountered in mathematics. Our study indicates that the evaluation of type P in learning biographies is a relevant question for two reasons. Firstly, type P is a highly abundant type. Secondly, the instruction may play an important role, given the significant deviation observed across the different study groups. Nevertheless, the answer to this question is left to further research by longitudinal studies in the strict sense.

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A framework for refutations to recipes and algorithms

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In many areas of mathematics, a systematic approach following a recipe or algorithm can be applied. Ideally, the individual steps, in the sense of a deductive structure of mathematics, should not only be mastered but also their validity should be justified through reasoning. In this way, conceptual knowledge can also be activated for procedural tasks. For this I adapt a framework developed for proofs and refutations to outline a task-based Lakatos-style intervention aimed at promoting conceptual argumentation. I accompany it with first results from an interview study.

Keywords: Lakatos, calculus, recipes, reasoning, argumentation.

Introduction

When mathematical proofs are addressed in the school curriculum, formal rigor is often not the primary focus (Meyer & Schnell, 2020). In late secondary education, there are several reasons for this, one being that many theorems in Calculus can only be proven in a traditional (often referred to as formal) manner, which is not feasible in the school context due to the lack of necessary mathematical tools. Argumentation in secondary education has therefore been identified by experts as an area needing further development (Sommerhoff & Brunner, 2021). In Calculus, an algorithmic approach or method-based recipe is often used instead, which includes tasks such as graphical differentiation (Fischer, 2024; Fischer & Krause, 2025). A common discussion in this context is whether such tasks are conceptual or procedural, with the latter allowing for the linear execution of a recipe (Sommerlade & Eichler, 2023), characterized by the sequential application of individual steps to complete a task. In practice, topics such as curve sketching are frequently criticized for being reduced to routine procedures, a tendency that has been further exacerbated by the widespread availability of function plotters, as already observed by Tall (1993) over 30 years ago. In the context of Calculus, Koichu (2008) therefore suggests focusing on the discovery of proofs by highlighting refutations. Lakatos (1976) views the handling of proofs and refutations as essential, emphasizing their role in the development of mathematical thinking. Based on his five heuristic rules, a framework for proofs and refutations has been established (Komatsu, 2016), which I adapt to recipes and algorithms. Following Komatsu (2016) this theoretical framework plays two roles interlinking research and practice, namely understanding and prescription (Komatsu, 2016).

Literature review and theory discussion

Argumentation and proof are essential components of authentic mathematics learning (Stylianides et al., 2022), yet the concept of reasoning and its relationship to these terms have so far been insufficiently clarified in the educational discourse. We use here the theoretical framework presented by Kirsten (2021, p. 26) for reasoning activities within Calculus. In this context, argumentation and proof are considered as subcomponents of reasoning, which overlap in pre-formal proofs. From a didactic perspective, pre-formal proofs can be often recognized as valid proofs, whereas in the context of university-level mathematics, they are often classified as argumentation (Jahnke et al., 2023).

While a proof may appear as a linear sequence of deductive steps, the actual process of proving follows a more intricate, non-linear line of thought, involving different phases of progress, exploration, adjustment, and revision. Lakatos (1976) described this process through five heuristic rules which were basis for a framework for proofs and refutations (Komatsu, 2016).

Lakatos's method of proofs and refutations is often criticized for its limited applicability but proves effective in specific mathematical domains, particularly in task design involving hidden assumptions (Stylianides et al., 2022). Many procedures and methods taught in schools often contain hidden assumptions or are presented in an incomplete form. A framework describing the structural activities when specific tasks lead to refutations of a naive recipe with a prescriptive nature could serve as a basis in creating task-based instructional interventions for the classroom, building on the ideas in Stylianides et al. (2022, p. 754). While frameworks for Lakatos-style proofing have been valuable for exploring pre-formal and formal proofs and refutations in the domains of arithmetic, algebra, and geometry (Komatsu, 2016; Larsen & Zandieh, 2008; Reid, 2002), to my knowledge no adaptation to the field of recipes and naive algorithms exists.

Research questions

Accompanying this, I address the following research questions: How can Lakatos' five heuristic rules be developed into a framework for recipes and can this framework be used as a prescription causing desirable behaviors in students, namely the use of conceptual knowledge when following recipes.

We present theoretical considerations based on the literature to develop the framework and demonstrate the application of it using an excerpt from an interview study to address the second question.

Model description of Lakatos' framework for recipes

In addition to proving, another form of argumentation in the classroom is "reasoning through the application of a procedure" (Jahnke et al., 2023, S. 374, own translation), which includes recipes and (possibly naive) algorithms. Similarities and differences between recipes on the one hand and theorems and their pre-formal proofs on the other can be outlined as follows, building on first suggestions in Fischer (2024a). Assumptions in a theorem ideally specify the minimal conditions under which the statement holds, making it clear that additional assumptions can restrict the validity of a statement. At the conclusion of the proof, based on premises, lemmas, and axioms, a mathematical theorem is deductively derived (Jahnke et al., 2023). Similar, assumptions in a recipe ensure its applicability to a task or a problem. Building on this, Weber & Tanswell (2022) argue that proofs and recipes are strongly related and that so-called recipe proofs are common in mathematical practice and in undergraduate mathematics textbooks (Tanswell & Inglis, 2022; Weber & Tanswell, 2022). Related, a recipe (or naive algorithm) consists of the process of applying individual steps, provided the assumptions on the input are met, until the output as the final product is achieved. Ideally, these steps are not merely followed procedurally; conceptual knowledge allows the users of the procedure to maintain some flexibility in the steps as well as to reflect on their correctness and justification (Oldenburg, 2022). A framework for recipes based on these similarities tells us which

parts of incidents one should focus on and how to construct a Lakatos-style activity in order to derive important actions based on conceptual arguments within the respective thematic field of the recipe.

For the reader's better understanding, we restate Lakatos-style reasoning, following Yang et al. (2023, p. 1): “First, a conjecture is constructed through conscious guessing (Phase 1). The conjecture is then tested using supportive examples (Phase 2). A proof may be constructed to further validate the conjecture (Phase 3). It is also likely that counterexamples emerge that refute the conjecture or invalidate the respective proof (Phase 4), leading to the refinement of the conjecture or proof (Phase 5). Still, it is important to note that Lakatos’ philosophy is more complex than this fivephase activity, highlighting also other important aspects, such as the crucial role of the interplay between defining and proving.” Additionally, a distinction is made between so-called local counterexamples (to a specific step in the proof) and global counterexamples (to the conjecture itself). Adapting this distinguishment to recipes, the former results in a modification of a recipe-step based on argumentation, while the latter leads to changes in the overall recipe, such as adjusting assumptions or incorporating additional steps.

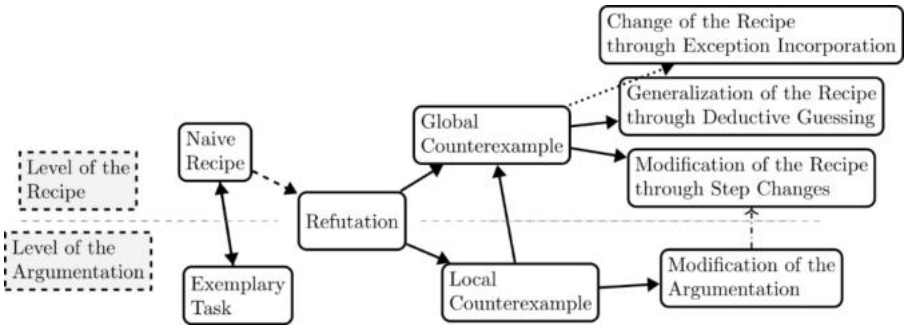

Figure 1: The framework from Lakatos, as presented in Komatsu (2016), adapted here for recipes.

Due to space limitations, it is not possible to present a complete adaptation of the framework in Figure 1. Instead, we primarily focus on Phases 4 and 5, which involve the identification of refutations, followed by a subsequent refinement of reasoning, the latter described in the original work by Lakatos (1976, p. 62) as the fourth heuristic rule: “If you have a counterexample which is local but not global, try to improve your proof-analysis by replacing the refuted lemma by an unfalsified one.”

Studies show that students often struggle to independently find counterexamples, but if they could do so, it would demonstrate a high level of conceptual understanding in the topic. Moreover, in the context of naive algorithms and recipes, students have often already received a proof by authority from a teacher or, for instance, an instructional video (Hanna, 1989), which they used exemplary multiple times. Therefore, this study design is created such that the refutations to students’ naive algorithms and recipes are presented by the interviewer respective the teacher after students developed their naive recipe (dashed line, Figure 1).

According to Weber & Tanswell (2022), the first two of the four levels of understanding a recipe involve executing a specific step of the procedure and understanding why that step can be applied. Similar to a refuted lemma in a proof, students must critically question the individual steps of the procedure when faced with a local counterexample. This occurs at an argumentative level in order to identify the step that is not properly applied or refuted. It is a central activity triggered by the local

counterexample, as this modification of argumentation may hypothetically form the basis for modifying the step, thereby leading to a modification of the recipe, adapted from Komatsu (2016). The latter occurs at the level of the recipe itself (dash-dotted arrow).

Additional Lakatos-style activities discussed in the context of math-education are “monster-barring” and “exception-barring” (Larsen & Zandieh, 2008; Reid, 2002). Adapted, the former refers to dismissing a counterexample on the grounds that it does not represent a true case application of the recipe or naive algorithm. The latter, termed “exception-barring”, refers to any response that modifies the recipe to tread the counterexample, without considering the recipe and its underlying reasoning (dotted arrow, Figure 1), it is therefore not a heuristic rule in Komatsu (2016). It may also include the addition of an extra step to handle exceptions (Larsen & Zandieh, 2008).

An explanatory episode

The data presented in the following originates from a task-based interview conducted as part of a research project focused on developing task design principles for argumentation in Calculus at the upper secondary level. Three interviews were conducted with first-semester mathematics students (BEd & BSc) using the Lakatos-style task sequence outlined below. Students were invited to bring a self-selected study partner to encourage observable communication within a group setting, as group interaction is closely linked to explanation and argumentation (Meyer & Schnell, 2020). These task-related interviews were recorded, transcribed, and examined according to the framework in Figure 1 following Mayring (2010). An important aspect of applying the framework is to analyze the students' argumentation in terms of its structural and content dimensions, as well as the processual aspects of their reasoning (Jeannotte & Kieran, 2017; Meyer & Schnell, 2020). The following excerpt serves primarily as an illustration of the proposed Lakatos-style intervention in Figure 1, rather than offering a comprehensive analysis of students' writing and cognitive processes during the task.

The initial task was to sketch the first derivative of a function represented by its graph f_1 (prompt: “Given is the graph of a function f. Draw the first derivative f'”) and then develop a recipe for this task (prompt: “Generalise the procedure just used as a general recipe”), explaining their steps with the help of the supportive example (Deslis et al., 2023). Accordingly, the interviewer then presented the students with a suitable counterexample f_2 to their recipe as illustrated in Figure 1 (dashed arrow), where solving this new task is ideally intended to lead to a revision of the recipe. Repeating this process subsequently leads to an iterative cycle of refining the recipe in a Lakatos-style activity (Deslis et al., 2023). Additionally, a pre- and post-test regarding reasoning in pre-formal proofs was conducted, results were discussed in Fischer (2024b).

The students first successfully draw the graph of the derivative. Then, referring to f_1' several times, they present their recipe which was summarised by the author as follows: M_1: Extrema of f become zeros of f', M_2: Inflection points of f become extrema of f', M_3: Determining the degree n of the polynomial f in order to deduce degree $n-1$ of its derivative f', M_4: Sign of the leading coefficient a_n of the polynomial to deduce the asymptotic behaviour of f'. Therefore, the graph of f_1' can be seen as a supporting example of their recipe, additionally the students explain M_3 and M_4 with an algebraic argumentation. The second task is again a 4th degree polynomial, but with a saddle point W, it leads to a discussion about a first sketch, traced with a finger, focussing on W.

S1: I have no idea whether this makes sense. Can this be an extremum?
S2: Yes, I just don't know. f' gives you the slope.
S1: I mean, the curvature changes. […]
S2: It [the slope] is negative, then it is zero and then it is negative again.
S1: Yes, we said that if the slope of the tangent is zero, - which is the case with the saddle point is zero, then it is also a zero, although it is not an extremum.
I: Shall we extend strategy?
S1: Yes, extrema in slope 0.
S2: When the slope is 0. [Word message almost simultaneously]

Student S1 initially does not use their recipe; the saddle point W can be viewed as a counterexample in her reasoning. Instead of relying on the recipe, she begins by making conjectures and examining other properties of the function, such as curvature, while searching for familiar similarities (Jeannotte & Kieran, 2017). Student S2 operates on the level of reasoning, classifying similarities between the saddle point and the extremum based on the slope of the tangent. Eventually, S1 agrees on an argument regarding the tangent slope, and together they sketch the derivative f_2' correctly. Despite the choice of words in the interviewer's question about extending the recipe, the students refine M_1.

Task f_3 is the graph of the normal distribution. The asymptotic behaviour should serve as a key sign here. Additionally, S1 and S2 face a problem at the beginning since f_3 is also not a polynomial.

S2: Oh man. Is it asymptotic [addressed to the interviewer]? Puh. So somehow with a fraction or, do you know what I mean? [addressed to S1]? Asymptote with fractions.
S1: Mm-hmm.
S2: I always associate them with fractions.
S1: Yes, but you don't have to.
S2: Yes, but I always associate them with fractions. [pause] But you can't see it as 2nd degree either.
S1: Hmm, but I have to say, I've never had to deal with a function that looks like this in school.
S2: Yes… [pause] So it must not take on any values that are negative, i.e. where $f < 0$. And it approaches the x-axis.
S1: But f' can still be 0.
S2: Yeahhh? [uncertain]
I: Let us just go through the steps?

The students initially focus on properties of the function that had not previously occurred, rather than applying their recipe. This only happens after prompting by the interviewer. The students later argue that M_3 and M_4 are not applicable for this task, it is solved correctly afterwards.

Discussion

The students begin by progressing through Phases 1 to 3. In the subsequent first excerpt, they recognize the saddle point as a counterexample. Their approach reveals a missing step for W in their recipe. The actions of S1 suggest that they are in Phase 4. Based on S2's response, however, the saddle point is subsequently treated as a local counterexample, leading to the identification and modification of the refuted step M_1 in their recipe. They do this based on an argumentative reasoning about the slope (Step 5). As a comparison, a second group of students S3 and S4, with a similar recipe, chose an alternative approach, commonly found in school books and online videos on the graphical differentiation task (Fischer, under review), treating saddle points as an exception and therefore a global counterexample. They added in the process an additional step. Contrary, S1 and S2 addressed

the conceptual relationship between a local extremum and a saddle point. Incorporating saddle points as exceptions removes the need for an argumentation based on the horizontal tangent slope, a fundamental concept of the derivative. In the following excerpt, the students work on the next item f_3, which presents a challenge. The horizontal asymptotes allow the students additionally to recognize that it is not a polynomial function. This leads to a metacognitive process. They do not resort to their usual recipe; this could be because they intuitively perceive f_3 as a global counterexample (Phase 4). However, rather than immediately resorting to "monster-barring", they begin with an information search and a sign analysis. They engage in a discussion about the mathematical content. Following the interviewers prompt they reason why they cannot execute steps M_3 and M_4, underlining the interpretation of f_3 as a global counterexample.

According to Tanswell and Inglis (2022), imperative forms and instructional statements are common in mathematical proof texts. This highlights both similarities and differences between proving activities and the application of algorithms and recipes, as I was able to outline based on the literature. Proofs are often structured like a recipe, relying on specific procedural instructions such as "take" and "choose" (Weber & Tanswell, 2022, p. 26). Similarly, recipes in school contexts frequently include hidden assumptions or are incomplete. However, the deductive nature of mathematics underscores that the steps of a recipe or algorithm can be understood or justified through reasoning. Building on this, reasoning about individual steps in algorithms, also an essential aspect of computational thinking, should take center stage in mathematics education and serves as another potential application of the framework (Oldenburg, 2022). Therefore recipes, when embedded in instructional tasks, provide opportunities for students to confront and resolve refutations, promoting conceptual understanding.

As outlined, the proposed framework aims to describe an intervention leading to a Lakatos-style investigation cycle, which we applied in the context of Calculus, where proving activities are often not feasible. Building on these ideas, I adapted a framework describing the structural activities when specific tasks lead to refutations of a naive recipe. It has a prescriptive nature and serves as a foundation for creating task-based instructional interventions for the classroom (Komatsu, 2016), aligning with the ideas in Stylianides et al. (2022, p. 754). This framework provides guidance on which parts of incidents to focus on and how to construct them to derive meaningful actions for conceptual arguments in the context of recipes (Komatsu, 2016). The importance of item selection must also be highlighted, as it plays a critical role in guiding students' reasoning. Refutations presented by the author were designed in such a way that students could easily identify a specific sign constituting it as a counterexample. The detailed task development process further underscores the need for additional exploration and research, particularly in teacher education. This is essential because the instructional approaches outlined in the literature can be challenging to apply in practice (Deslis et al., 2023; Stylianides et al., 2022; Yang et al., 2023).

Nonetheless, the results presented here are based on a single sequence of tasks in Calculus involving two first-year students. Yin (2009) suggests that individual case studies are justifiable when the case plays a critical role in theory development. In this regard, showcasing the process of students refining their recipe and modifying their reasoning, as described by the proposed framework, even in a single case, is significant. The results must now be further tested with additional items and in the classroom.

In this process, the students' reactions in dealing with counterexamples should be analysed to gain insights regarding prediction and generalization — two further roles in which theory acts as an intermediary between research and practice (Komatsu, 2016).

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Investigating the relationship between negating competence and understanding proof by contraposition: A study with Indonesian undergraduate mathematics education students

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Undergraduate mathematics students tend to have difficulties understanding proof by contraposition in which the concept of negation plays a significant role. Despite this significance, there is a lack of research investigating whether and how students' competence in negating statements is associated with their understanding of proof by contraposition. In this paper—which derives from a broader, ongoing study—we aimed to address this gap by analysing survey data from 87 fifth-semester undergraduate mathematics education students in Indonesia. The findings indicate a strong relationship between students' ability to negate statements–particularly conditional statements and single statements–and their understanding of proof by contraposition. These findings have significant implications for the teaching of proof by contraposition.

Keywords: Proof, negating competence, proof by contraposition, undergraduate mathematics.

Introduction

In addition to proof by contradiction (PBCd), proof by contraposition (PBCp) has been identified as another challenging indirect proof method for students (Antonini & Mariotti, 2008; Brown, 2017; Buchbinder & McCrone, 2020; Stylianides et al., 2004). Although research investigating students' understanding of PBCp is scarce compared to other proof methods, some issues regarding students' difficulties with this proof method have been identified: Students tend to lack adequate ways for validating contrapositive argumentation as a legitimate form of reasoning (Yopp, 2017, 2020). Also some students hold a "child logic" issue (Hoyles & Küchemann, 2002) considering that a conditional statement is equivalent to its converse, or that a contrapositive statement is the same as its inverse (Stylianides et al., 2004), thus failing to recognize that a conditional statement is equivalent to its contrapositive (Buchbinder & McCrone, 2020). Furthermore, students tend to find contraposition more challenging to understand in a symbolic form than in a verbal context (Stylianides et al., 2004). The inference method of justifying PBCp, known as modus tollens, is intuitively more challenging for students compared to modus ponens, which is one of the inference rules for direct proof (Antonini & Mariotti, 2008; Yopp, 2020).

Given that PBCp is based on the concept of negation, particularly as it starts by negating the conclusion of the given conditional statement, it is notable that no studies thus far seem to have investigated this relationship. In this paper, we take a step towards addressing this gap by focusing on the following research questions: *Does undergraduate mathematics education students' competence in negating statements associate with their understanding of PBCp? If so, in what ways?* This paper derives from a broader, ongoing study, and it is a continuation of our investigation of the role of students' negating competence in their understanding of indirect proof methods, which started with our investigation of this relationship in the area of PBCd (Hapipi & Stylianides, 2024).

Theoretical framework

Understanding proof

As discussed in Hapipi and Stylianides (2024), the concept of understanding in this study, rooted in the perspective of Baumberger et al. (2017), Sierpinska (2005), Skemp (2006), and Dewey (1971), denotes the profound ability to grasp meaning beyond mere recognition of facts. It entails an awareness of relationships, causes, consequences, and contextual intricacies within events or situations. To practically assess students' understanding of PBCp, this study adopts six criteria (Hapipi & Stylianides, 2024), drawing on the frameworks of Mejia-Ramos et al. (2012, 2017) and Anderson et al. (2001): (1) understanding the meaning of terms and statements, (2) evaluating the logical structure of statements and proof frameworks, (3) justifying claims, (4) summarizing using high-level concepts, (5) identifying modular structures, and (6) illustrating with examples.

Negating competence

Lin et al. (2003) explored four types of quantifying statements (all, some, only one, and none) in their study of students' understanding of PBCd, while Inglis and Simpson (2008) examined four types of conditional statements ($p \to q, \sim p \to q, p \to \sim q,$ and $\sim p \to \sim q$) in their study on conditional inference within the context of mathematical studies. Reflecting on this work, we defined negating competence as students' ability to negate a group of statements fulfilling the following three criteria, and we operationalized this definition in an instrument called the Negating Instrument (Hapipi & Stylianides, 2024). First, the statements encompass various types, as indicated by Lin et al. (2003) and Inglis and Simpson (2008). Second, the content of these statements aligns with participants' previous coursework, minimizing the likelihood that their ability to negate would be influenced by a lack of understanding of the statements. Lastly, the instrument includes the context of PBCd and PBCp when addressing conditional statements ($p \to q$), as negation can work similarly when dealing with PBCd and PBCp. Both proofs start by negating q; however, in PBCp, this is the only initial step, whereas in PBCd, it serves to simplify the negation of $p \to q$. The instrument captures this context by including statements asking students to negate $p \to q$ and separately to negate q and p.

Regarding the role of negation in the indirect proof methods of PBCd and PBCp, Hapipi and Stylianides (2024) reported that, in the context of PBCd, students' negating competence shows a significant association with their understanding of PBCd. Specifically, they found that students' competence in negating single or quantified statements significantly influences their understanding of PBCd. Interestingly, they also reported that students' ability to negate conditional statements did not show a significant link with their understanding of PBCd. Since PBCp applies exclusively to conditional statements, it would be worthwhile to investigate how students' understanding of PBCp relates to their ability to negate various types of statements.

The role of negation in PBCp

Given that PBCp is a proof method that demonstrates the truth of a statement by proving the truth of its (logically equivalent) contrapositive statement (Antonini & Mariotti, 2008; Thompson, 1996), the role that the notion of negation plays in this proof includes at least three aspects (Figure 1). First, at the beginning of the proof, PBCp starts by negating the conclusion of the conditional statement. Second, near the end of the proof, PBCp results in providing a statement that negates the premise of

the conditional statement. Lastly, at the conclusion of the PBCp, especially when distinguishing between negation, which contradicts the original statement, and the contrapositive, which is logically equivalent to the proved statement.

Steps of PBCp for a given statement $p \rightarrow q$	Logical structures of PBCp	Negation involved
Assuming the negation of the conclusion q is true	Assume not q	Negating the conclusion
This leads to producing a negation of the premise p.	leads to Not p is true	Resulting in negation p is true.
This results in producing a contrapositive statement for the given statement.	results in $\sim q \rightarrow \sim p$ is true	Contrapositive VS negation of conditional statement
As the given statement and its contrapositive are logically equivalent, this leads to the truth of the given statement.	hence $p \rightarrow q$ is true	

Figure 1: The logical structure of PBCp and the role of negation within it

Methods

This paper derives from a broader mixed-methods research study that employed survey and interview techniques to explore how undergraduate mathematics education students understand PBCd and PBCp, and their association with direct proof. Due to space limitations, in this paper we focus solely on relevant findings from the survey segment of our study. The study included 87 fifth-semester Indonesian undergraduate mathematics education students, all of whom had prior exposure to proof methods and proof-related tasks in their coursework. The participants completed two instruments in Indonesian: first, the Negating Instrument; second, the Understanding ContraProof (UCP) Instrument, which comprised six packets, three related to each of PBCp and PBCd.

UCP Instrument

The UCP Instrument, which we developed based on Mejia-Ramos et al.'s (2012, 2017) proof comprehension framework, comprised three components in each packet: a (mathematical) statement, its corresponding PBCp or PBCd, and questions that aligned with the six framework categories we mentioned earlier. Some of the contents related to PBCp were adopted from Brown (2017), while others were constructed, inspired by, or modified from Chamberlain (2017) and Mejia-Ramos et al. (2012). Four experts reviewed the instrument before it was piloted twice.

The questions within the UCP Instrument were presented in both open-ended and multiple-choice formats. Participants' responses to the open-ended questions were scored from 0 to 4. The scoring scheme was inspired by Stylianides (2019) and Stylianides and Stylianides (2009), where a score of 0 indicates a wrong answer with a wrong or no reason; 1 represents a wrong answer with a correct or partially correct reason; 2 signifies a right answer with a wrong or no reason; 3 indicates a right answer with a partially correct reason; and 4 signifies a right answer with a correct reason. Meanwhile, for the multiple-choice questions, a score of 1 indicates a correct answer, and a score of 0 is assigned for all other responses.

Negating Instrument

In the absence of an existing instrument or criteria for measuring negating competence in relation to PBCd or PBCp, this study developed one based on the three criteria mentioned earlier, under Theoretical framework. The instrument asked students to negate statements in the form of conditionals (CondS) and statements involving quantifiers (QuantS). It also required students to negate q and p within statements of the form $p \rightarrow q$ (SingleS). The content of these statements was drawn or adapted

from materials of courses that the students had previously attended, such as calculus, elementary algebra, number theory, logic and set, and real analysis.

Participants' responses were scored on a scale from 0 to 4, based on a scheme inspired by Stylianides (2019) and Stylianides and Stylianides (2009). A score of 0 denotes no response, 1 indicates an incorrect answer, 2 signifies an answer with a major mistake, 3 reflects an answer with a minor mistake, and 4 represents a correct answer. Figure 2 provides examples of coding for participants' responses negating the statement, 'If x is not a member of interval $[0,1]$, then $x^2 - x$ is positive'.

Students' responses and their translation	Scores	Notes
x bukan anggota interval [0,1] tidak x^2-x tidak positif x is not a member of interval [0,1], but $x^2 - x$ is not positive.	4	Correct answer
x bukan merupakan anggota interval [0,1] dan x^2-x negatif x is not a member of interval [0,1], but $x^2 - x$ is negative.	3	The participant wrote negative instead of not positive
Jika x tidak merupakan anggota interval [0,1] maka x^2-x negatif If x is not a member of interval [0,1], then $x^2 - x$ is negative.	2	A kind of negation but not proper and written in the wrong form, conditional form
Jika x merupakan anggota interval [0,1], maka x^2-x positif If x is a member of interval [0,1], then $x^2 - x$ is positive.	1	This is not negation. The premise is negated and expressed in wrong form
No answer	0	

Figure 2: Examples of coding for students' responses on the Negating Instrument

The instrument was piloted twice, demonstrating good reliability, as indicated by a Cronbach's alpha of 0.90 and an average interitem correlation of 0.41 (Acock, 2023). Results from confirmatory factor analysis, along with convergent and discriminant validity, also indicate the instrument met acceptability criteria (Hapipi & Stylianides, 2024).

Analysis

Participants' responses to the UCP and Negating Instruments were coded by the first author, referring to the predetermined key answers and coding scores. This process was conducted twice, a week apart, to ensure consistency. A student's individual understanding of PBCp was determined by averaging their score across six equally weighted criteria and then scaling the result by a factor of 10. Their negating competence was calculated by dividing their total score by the ideal maximum score and multiplying by 10. Correlation and regression analyses were then performed to examine the association between students' negating competence and their understanding of PBCp. The regression results do not show a causal relationship (Mehmetoglu & Jakobsen, 2017) between these two variables, but the theoretical framework in Figure 1 offers a basis on which to explain the findings.

Findings

Relationship between understanding PBCd and negating competence

Spearman's correlation was performed to explore the relationship between students' negating competence and their understanding of PBCp. Table 1 shows the summary results of the analyses.

Table 1 shows a strong correlation between students' ability to negate statements and their understanding of PBCp, with an overall correlation of $r(85) = .58, p < .001$. Moreover, students' competence in negating Conds, QuantS, and SingleS shows strong positive connection with their understanding of PBCp, with correlation coefficients of $r(85) = .41, p < .001$, $r(85) = .42, p < .001$, and $r(85) = .54, p < .001$, respectively. Additionally, students' ability to negate the conclusion and premises of statements specific to PBCp indicates a positive link with their understanding of the respective PBCp, with a correlation coefficient of $r(85) = .49, p < .001$.

Table 1: Correlation among negating competence and understanding PBCp

Categories	N	Understanding PBCp
All types of statements	87	0.58***
CondS	87	0.41***
QuantS	87	0.42***
SingleS	87	0.54***
Conclusion and premises of statements specific to PBCp	87	0.49***

* $p < 0.05$, ** $p < 0.01$, *** $p < 0.001$

A significant positive correlation means that high scores on negating competence tend to be paired with high scores on understanding PBCp, while lower scores on one align with lower scores on the other. This association indicates that as participants perform better on negating statements, they are more likely to have a better understanding of PBCp as well. To further investigate these relationships, linear and multiple regression analyses were conducted, and their results are presented below.

Influence of negating competence on understanding PBCp

Regression analyses examined how students' competence in negating statements influences their understanding of PBCp. Table 2, column 1, shows that students' competence in negating statements significantly influences their understanding of PBCp, as indicated by the coefficient of 0.80 ($t = 6.73, p < 0.001$). The regression model is also statistically significant ($F(1, 85) = 45.29, p < 0.001$) and explains approximately 35% of the variance in students' understanding of PBCp (R-squared=0.35), with a low root mean square error (RMSE = 1.49). The model meets the assumption of linearity, as well as normality and homoscedasticity of residuals.

A multiple regression analysis examined how students' competence in negating specific statement types affects their understanding of PBCp (Table 2, column 2). The results show that students' competence in negating conditional (CondS) and single statements (SingleS) significantly influences their understanding of PBCp, with coefficients of 0.43 ($t = 2.34, p < 0.05$) and 0.36 ($t = 3.61, p < 0.01$), respectively. Meanwhile, students' ability to negate statements involving quantifiers (QuantS) has no significant effect, which seems to be understandable since PBCp, as illustrated in Figure 1, only works with conditional statements. Moreover, the model is statistically significant ($F(3, 83) = 15.15, p < 0.001$), explains about 35% of the variance in students' understanding of PBCp (R-squared = 0.35), and has a low root mean square error (RMSE) of 1.50. The model also satisfies the assumptions of linearity, normality and homoscedasticity of residuals, and no multicollinearity among the independent variables.

Additionally, students' ability to negate the conclusions and premises of statements proved using PBCp significantly affects their understanding of the respective PBCp, as shown by a regression analysis 0.39 ($t = 5.11, p < 0.001$) in Table 2, column 3. The regression model is statistically significant ($F(1, 85) = 26.10, p < 0.001$), with an R-squared value of 0.24, explaining about 24% of

the variance in students' understanding of PBCp, and it meets assumptions of linearity, as well as normality and homoscedasticity of residuals.

Table 2: Regression analysis results between negating competence and understanding PBCp

	(1) Understanding PBCp	(2) Understanding PBCp	(3) Understanding PBCp
Negating Competence	0.80***		
	(6.73)		
CondS		0.41*	
		(2.34)	
QuantS		0.14	
		(1.07)	
SingleS		0.36**	
		(3.14)	
Conclusion and premises of statements specific to the given PBCp			0.39***
			(5.11)
Constant	0.10	0.06	2.42***
	(0.13)	(0.07)	(4.37)
N	87	87	87
R^2	0.35	0.35	0.24
Adj. R^2	0.34	0.33	0.22
RMSE	1.49	1.50	1.62

t statistics in parentheses, * $p < 0.05$, ** $p < 0.01$, *** $p < 0.001$

Discussion and conclusion

This paper elucidates the importance of students' ability to negate statements for understanding PBCp. Working with negation during PBCp is crucial, as having a clear understanding of the concept of negation is required at least in three different stages of PBCp, as shown in Figure 1. Our findings indicate that students' negating competence, particularly negating conditional statements (CondS) and single statements (SingleS), contributed to their understanding of PBCp. Moreover, this study found that students' ability to negate the conclusions and premises of the conditional statements proven using PBCp significantly influences their understanding of the respective PBCp. These findings seem to be reasonable for two reasons. First, as noted in Figure 1, PBCp starts by negating the conclusion of the given conditional statement, so students' ability to negate the conclusion is essential for understanding the proof. Second, the ability to negate conditional statements presumably helps students distinguish between negation and the contrapositive of the conditional statement. Moreover, this competence

might ease students to follow the flow of PBCp as PBCp works with conditional statements. Therefore, these findings shed further light on aspects, in addition to those suggested by Hoyles and Kuchemann (2002), Stylianides et al. (2004), Buchbinder and McCrone (2020), and Yopp (2017, 2020), that could shape students' understanding of PBCp.

Interestingly, the significant effect of being able to negate conditional statements only appears on students' understanding of PBCp but not PBCd as reported by Hapipi and Stylianides (2024). On the other hand, negating quantifying statements which significantly affected students' understanding of PBCd (Hapipi & Stylianides, 2024) does not affect their understanding of PBCp. These findings seem to be important in the context of indirect proofs for three reasons: they confirm the importance of negating competence in PBCp and PBCd, clarify which type of statements are relevant to each proof methods, and highlight specific negating competence to emphasize when teaching negation in relation to PBCp or PBCd.

Despite the study's contributions, certain limitations and areas for future research remain. First, the UCP Instrument included only three statements with their respective PBCp, which may not fully capture students' understanding of PBCp across broader mathematical topics. Future research could benefit from a wider range of statements. Second, the Negating Instrument and its criteria need further testing in broader contexts, as they were developed and initially tested in this study.

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Proving the rule for fraction multiplication: What kind of content knowledge is visible in preservice teachers' arguments?

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In this study, we investigate how preservice teachers construct written arguments for the rule for fraction multiplication after having worked with tasks designed in view of proof-based teaching. The aim was for the preservice teachers to explore fraction multiplication through paper folding and visual diagrams, and in this process develop content knowledge for fractions and competence in proving. We analysed written arguments of 28 preservice teachers in their fourth year of study. From our analysis we found that while most preservice teachers did not succeed in constructing a valid general argument or a proof, relevant content knowledge of fractions was visible in the arguments. We suggest that proof-based teaching can be a fruitful approach to facilitate the development of content knowledge for teaching both fraction multiplication and proving.

Keywords: Proving, proof-based teaching, fractions, teacher knowledge.

Introduction

Reasoning, argumentation and proving are central activities in the discipline of mathematics, but also in school mathematics. Proofs can have different roles, such as verification, explanation, systematization, discovery and communication (de Villiers, 1990). De Villiers (1990) emphasizes that proofs that explain are important for developing understanding for *why* a given statement is true. In Norway, there is an increased focus on reasoning and argumentation in school since the curriculum from 2020 (in short: LK20) includes "Reasoning and argumentation" as a core element (Ministry of Education and Research, 2019). Research suggests that it can be demanding for students to understand proof and proving, and that students often rely on examples in proving tasks and consequently provide empirical arguments and not valid general arguments (Aricha-Metzer & Zaslavsky, 2019). Additionally, research indicates that many students struggle with fraction tasks, which can constitute an obstacle since the learning of fractions and fraction operations is essential in further learning of mathematics (Siegler & Lortie-Forgues, 2017). Izsák (2008) suggests that "teachers' understanding of fraction multiplication (and division) is limited" (p. 100). Ball et al. (2008) discuss content knowledge that is necessary for a teacher to have. Stylianides and Ball (2008) argue more specifically that teachers' knowledge of proof is important and needs to comprise different aspects, such as "knowledge of situations for proving" (including "knowledge of different kinds of proving tasks") and knowledge of the logical and linguistic structures of proofs.

Since research points out the importance of content knowledge for teaching and difficulties that students might encounter with both fraction operations and proving, we wished to investigate preservice teachers' (PTs) knowledge within these two, more specifically, PTs' knowledge for proving the rule for fraction multiplication, and how reasoning and proving can be naturally integrated into the process of developing content knowledge. Utilising proof-based teaching (Reid & Vallejo Vargas, 2019), we engaged PTs in fraction multiplication tasks, with a goal for the PTs to further develop their reasoning skills and content knowledge for teaching the topic of fraction

multiplication (Izsák, 2008; Ball et al., 2008). In addition, as future teachers, they were supposed to reflect on how this kind of activity could be used with students in school.

The study reported in this paper is part of a larger project, where our aim is to develop proof-based teaching of fractions, both in school and in teacher education. In Hofmann and Gustavsen (2024), we analysed written work of 20 preservice teachers in the teacher education program for grades 1–7 in their first year of study. In the present paper, we extend the method of analysis in our previous short paper (Hofmann & Gustavsen, 2024) using a more detailed coding regarding aspects of fractions. Furthermore, we contrast our previous study by analysing a more advanced group of students. The PTs whose written work we analyse for the present paper, were in their fourth year of study and had chosen to specialize in mathematics education. One might expect that their skills in proving, reasoning and argumentation were developed further than the skills of PTs in their first year of study. Our research question is as follows: *How can we characterize advanced mathematics preservice teachers' arguments for the rule for fraction multiplication?*

Theoretical framework

Proof-based teaching

Proof-based teaching is a way of teaching where one seeks to integrate the learning of mathematical content knowledge into the learning of proving skills. One aim of proof-based teaching is that learners develop conceptual understanding in mathematics. One expectation in this way of teaching is that "the learners themselves conjecture and prove important theorems in mathematics" (Reid & Vallejo Vargas, 2019, p. 809). The proofs produced are expected to explain, as well as to verify the result (Reid & Vallejo Vargas, 2019). We use the definition of a 'proof' in school mathematics as in Stylianides (2007): A *proof* is a

> connected sequence of assertions for or against a mathematical claim, with the following characteristics: 1. It uses statements accepted by the classroom community (*set of accepted statements*) that are true and available without further justification; 2. It employs forms of reasoning (*modes of argumentation*) that are valid and known to, or within the conceptual reach of, the classroom community; 3. It is communicated with forms of expression (*modes of argument representation*) that are appropriate and known to, or within the conceptual reach of, the classroom community. (Stylianides, 2007, p. 291)

When preparing and implementing proof-based teaching of specific mathematical content, one needs to have in mind a 'framing theory' that "guides the development of teaching activities and their implementation in the classroom" (Reid & Vallejo Vargas, 2019, p. 809). The framing theory contains, among others, a 'tool-box' (can be compared to 'set of accepted statements' in Stylianides (2007)), and theorems that the learners are supposed to produce proofs of (target theorems) (Reid & Vallejo Vargas, 2019).

When we refer to a 'generic example', we use the concept as in Balacheff (1988), namely viewing an "object … as a characteristic representative of its class" (Balacheff, 1988, p. 219). In addition, we apply Reid and Vallejo Vargas (2019) in the sense that we require a "clarification that the example shown … is meant [to] be generic" (Reid & Vallejo Vargas, 2019, p. 811). Reid and Vallejo Vargas

(2019) express the opinion that "generic examples accompanied by explicit clarification of their intended generality are appropriate *forms of expression* at the middle school level" (p. 811), and Aricha-Metzer and Zaslavsky (2019) propose that the use of generic examples in the process of learning to prove can be productive.

Representations of fraction multiplication

Behr et al. (1983) proposed a model for the rational number construct comprising seven subconstructs, among those being part-whole, operator, measure and quotient. One way to introduce fraction multiplication is through the operator subconstruct, e.g. represented through paper folding (Wyberg et al., 2011). In this study, we focused on two representations that can be used for multiplication of two proper fractions: paper folding and drawings of grids. Paper folding can help students to visualize the process of fraction multiplication and to understand why the rule for fraction multiplication works (Wyberg et al., 2011). On the other hand, the use of manipulatives can also "interfere with their [childrens'] logical-mathematical thinking" (Behr et al., 1983, p. 108). Behr et al. (1983) also point out that the transition and interpretation from physical manipulatives (in this case paper folding) to symbolic representations (in this case grid with symbols) can be demanding for students.

Mathematical knowledge for teaching fraction multiplication

Ball et al. (2008) discuss content knowledge such as correct understanding of fractions as part of a whole. Izsák (2008) investigates mathematical knowledge for teaching fraction multiplication in depth and highlights the importance of teachers' knowledge of pedagogical use of drawings as representations in fraction tasks, of the operator subconstruct (multiplication as operator) and of change of unit in fraction multiplication. Izsák (2008) shows that, while the part-whole subconstruct of fractions is fundamental, in the operator subconstruct, when a fraction operates on another number, teachers must navigate several part-whole relationships to be able to produce drawings and explanations that can facilitate students' conceptual understanding of fraction multiplication.

Methods

The first author together with a teacher, conducted a teaching session of approximately three hours length in a master course in teacher education, in two classes separately. In this paper we focus on one class of PTs. The tasks in the session were designed in view of proof-based teaching of fraction multiplication, initially by the authors and three researchers, and then developed further in iterations by the authors and the teacher participating in the session. The PTs were in the primary and lower secondary school teacher education program (both grades 1–7 and grades 5–10, the majority from the grades 5–10 program), in their fourth year. All PTs were specializing in mathematics education in a five-year program culminating in a master thesis in either mathematics education (most of the PTs) or pedagogy. The master course the PTs attended included the discussion of theory and research literature on reasoning and proving. Our aim with the session was to introduce the PTs to representing fraction multiplication with paper folding and to initiate the process of proving the rule for multiplying two proper fractions. The PTs were 28 in total and arranged in groups of 3–4. The session started with a task where the PTs were asked to represent some fractions using paper folding. Then fraction multiplication was introduced as part of a part and exemplified with paper folding. In groups,

the PTs worked with the next two tasks, in which they were invited to fold and interpret several fraction multiplications and record the processes and the results (for that purpose we introduced tables on a whiteboard). The final task was answered individually. In this final task, the PTs were requested to give a proof of the rule for fraction multiplication. There was an emphasis on that the PTs' arguments should explain *why* the rule is true, using drawing and text. In addition, the PTs were asked to use variables to express a general argument. This task was framed in a way that it should be an argument given to a fellow PT, i.e. we view it in the context of a classroom community of PTs. In this session, we linked multiplication to the operator subconstruct, meaning that $\frac{a}{n} \cdot \frac{b}{m}$ is the same as $\frac{a}{n}$ *of* $\frac{b}{m}$ ($a \leq n,\ b \leq m$ and $n \neq 0,\ m \neq 0$). The statement (target theorem) that guided our development of the sequence of tasks and our analysis is the following: *Consider a grid with* m *equal columns and* n *equal rows. Assume that* b *of the columns are colored. In the restricted grid consisting of the* b *columns, suppose that* a *rows are colored once more. Then the part of the grid that is colored twice, is a representation of* $\frac{a}{n} \cdot \frac{b}{m}$ *and* $\frac{ab}{nm}$*, and thus these two expressions are the same.*

In the first part of the session, we used paper folding and coloring as one representation of grids and fractions. For the written task, we asked the PTs to draw a diagram, since utilising diagrams allows for a wider range of examples and a generalization, whereas using paper folding has limitations in that respect.

Analysis

We coded written arguments given by 28 preservice teachers in response to the final task described above. Since we were interested in the PTs' competence in developing a proof, we used the categories in Stylianides and Stylianides (2009) and adapted them to our setting, e.g. the "classroom community" in the definition of proof as in Stylianides (2007) is the class of preservice teachers: *proof* (according to Stylianides' (2007) definition of proof; use of generic example is included here; code M1), *valid general argument but not a proof* (deductive argument that applies "to all cases in the domain of the statement" (Stylianides & Stylianides, 2009, p. 239), but that for example is not "within the conceptual reach of the classroom community" (Stylianides, 2007, p. 291); code M2), *unsuccessful attempt for a valid general argument* (i.e. invalid or incomplete argument, but where an attempt of generalization was made evident; code M3), *empirical argument* (PT explained the truth of the statement in one or more examples, but did not make explicit an attempt to generalize; code M4) and *non-genuine argument* (non-mathematical or irrelevant argument, or the "response was potentially relevant but the relevance was not made evident" (Stylianides & Stylianides, 2009, p. 245); code M5). To supplement the codes M1–M5 in Stylianides and Stylianides (2009), we added the codes, "argument explains why the rule is true in a particular example", "uses generic example" and "uses variables to express generality". For instance, the codes "uses generic example" and "uses variables to express generality" can give us more detailed information about the arguments coded as M1, M2 or M3. In addition, we were interested in how aspects of content knowledge of fractions were reflected in the PTs' arguments, and we thus added the following codes for our analysis: "uses correct drawing", "refers to part-whole", "refers to operator" and "change of unit made explicit". The code "change of unit made explicit" was used when the PTs explicitly stated that when they take a fraction $\frac{a}{n}$ of a fraction $\frac{b}{m}$, the fraction they start with $\left(\frac{b}{m}\right)$ becomes the new whole. The code

"argument explains why the rule is true in a particular example" was added because we saw some (partial) explanations in arguments that were coded as M4 (empirical argument), while several arguments coded as M3 did not include sufficient explanations (i.e. this is an inductive code). We wish to emphasize that we do not view the categories M1–M5 as hierarchical. Both authors coded the written work of all the PTs independently, compared codes and discussed disagreements. Consensus was reached for all disagreements.

Findings and discussion

We coded the PTs' written arguments according to M1–M5 and aspects of content knowledge for teaching fraction multiplication as mentioned above. Our findings regarding the categories M1–M5 are presented in Table 1 below.

Table 1: The number of arguments in the categories M1–M5

M1	M2	M3	M4	M5
0	3	12	9	4

In all the three arguments that were coded as M2, variables to express generality were used. One of the three arguments included a generic example. The distribution of codes concerning content knowledge for teaching fraction multiplication is summarized in Table 2 below.

Table 2: The distribution of codes within content knowledge

	Correct drawing	Part-whole	Operator	Change of unit
Yes	14	8	15	6
Partially	11	12	4	5
No	3	8	9	17

These findings demonstrate that important aspects of knowledge of fractions were evident in many of the PTs' arguments, also those not coded as M2 (or M1). The three arguments that were coded as M2, all had codes "yes" or "partially" for correct drawing, part-whole and operator. Two of these mentioned explicitly a change of unit. Handling multiple units is a particular challenge in teaching fraction multiplication (Izsák, 2008).

To exemplify some of the categories, we discuss in the following the written arguments of two of the PTs, one coded as M2, and one coded as M3.

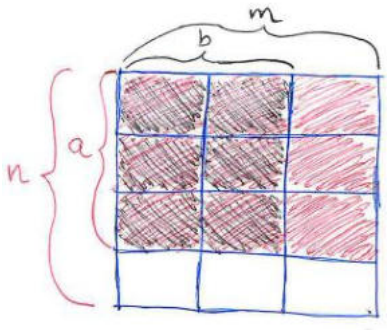

Figure 1: Reproduction of one PT's drawing, code M2

This PT started by explaining the multiplication $\frac{3}{4} \cdot \frac{2}{3}$ as taking $\frac{2}{3}$ of $\frac{3}{4}$. Since, by the definition we introduced, taking $\frac{2}{3}$ of $\frac{3}{4}$ is the same as $\frac{2}{3} \cdot \frac{3}{4}$, one could discuss commutativity of multiplication, but this was not the focus of our present study. The PT's explanations and the drawing with variables showed that this example is a generic example, as in Balacheff (1988). The PT also wrote the general rule $\frac{a}{n} \cdot \frac{b}{m} = \frac{a \cdot b}{n \cdot m}$ under the drawing. The drawing in Figure 1 is a good visualization of our target theorem. Change of unit was explicitly mentioned here, since the PT explained that one starts with four parts in the grid and then takes three parts of these four parts (representing $\frac{3}{4}$), thus "hiding the last part". The PT explained why $4 \cdot 3$ is the denominator in the product and that numerator times numerator is represented by the region that is shaded twice. The PT produced a valid general argument, but an explicit explanation of the product as part-whole is missing. We concluded that this proof attempt could not be accepted as proof in the classroom community of fellow PTs, and we coded it as M2, and not M1.

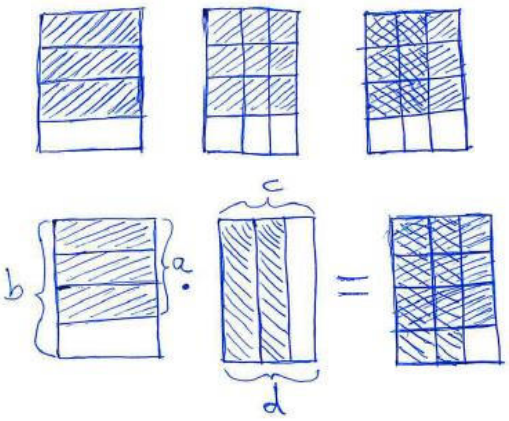

Figure 2: Reproductions of one PT's drawings, code M3

In the first sequence of drawings and explanations, the PT explained, although not in detail, why one can compute $\frac{3}{4} \cdot \frac{2}{3} = \frac{3 \cdot 2}{3 \cdot 4}$. Part-whole subconstruct was referred to in some places, operator subconstruct and change of unit remained implicit ("in the area already shaded, we wish to shade $\frac{2}{3}$"). In the next step, it becomes clear that the PT attempted to generalize, but the PT did not complete a general argument. In this context, it became evident that the example $\frac{3}{4} \cdot \frac{2}{3}$ is a generic example, although this was not made explicit. Next to the last drawing, the PT wrote the general rule $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$. In the step towards a generalization (the last sequence of drawings in Figure 2), the PT represented the "overlap" ("placed on top of each other"), i.e. the PT shaded the part $\frac{c}{d}$ of the whole grid and $\frac{a}{b}$ of the whole grid, and thus did not use the operator subconstruct. We interpret the representation of c in the drawing as an error, since the PT demonstrated good knowledge of what c represents in the example.

Our analysis revealed that many of the PTs' arguments showed a limited understanding of generality. We also noticed that many of the PTs' arguments showed content knowledge of fractions, but that the explicitness and articulation of this knowledge varied across the arguments. As pointed out by Behr et al. (1983), the transition from paper folding to drawing diagrams might have been challenging for several of the PTs.

While we analysed and discussed our codes, we experienced some challenges. One challenge was to arrive at a conclusion regarding what could be included in the "set of accepted statements" (Stylianides, 2007) in this classroom community (of PTs), e.g. what kind of statements could be "taken-as-shared" (Stylianides & Stylianides, 2009), i.e. used without explicit reference or explanation. For instance, when something was discussed in the groups or in the whole class during the paper folding activity that preceded the PTs' written work, could the PTs use this in their arguments without explicit references? When we read an argument, it can be difficult to determine what is most likely understood by the PT that produced the argument and by the classroom community (of fellow PTs). In addition, the PTs could have had in mind a classroom community of students in school.

Compared to the written arguments of preservice teachers in their first year of study, analysed in Hofmann and Gustavsen (2024), we see that when it comes to referring to part-whole or operator subconstruct, there was no big difference. In both groups, the PTs mentioned the operator subconstruct more often than the part-whole subconstruct. There might be a natural explanation for this, namely in the session, fraction multiplication was several times referred to as taking a part of a part. When it comes to expressing generality, in percentages more PTs in year 4 than in year 1 used variables to express generality, while more PTs in year 1 compared to PTs in year 4 used a generic example. The latter might be due to the fact that the PTs in year 1 (Hofmann & Gustavsen, 2024) were guided more, and the use of a generic example might be facilitated to a higher degree through the sequence of tasks used.

Conclusions and further research

Our analysis revealed that there is a great variety among the PTs' written work. Many of the written arguments show a limited understanding of generality. On the other hand, most of the written arguments show content knowledge for teaching fraction multiplication (Ball et al., 2008; Izsák, 2008), even though they could not be categorized as proofs or valid general arguments. For instance, several of the empirical arguments (coded as M4) use the part-whole or operator subconstruct. In general, the arguments coded as M4 show a fair amount of knowledge. To produce this kind of argument, PTs could not just compute one or more examples, it required some argument, involving a visual representation or some context (if not linked to paper folding). We propose that proof-based teaching can be conducive to the development of PTs' mathematical knowledge for teaching, both within fraction multiplication and proving.

As we realized that many aspects remained implicit in the written arguments, we suggest that it might be fruitful to follow up the analysis of PTs' written work with task-based interviews to get insight into the PTs' thought processes. Video recordings of PTs giving explanatory arguments to students in school might also give insights. Building upon our observations from the coding process, where we perceived a need for expanded and more nuanced codes, we suggest further research employing a more detailed coding scheme, incorporating subcodes or "graded" codes, to enhance the depth and precision of our analysis. This more detailed analysis could provide important information for the further development of a more extended teaching sequence and deeper findings about to what degree and how proving competence and fraction competence are intertwined.

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Pre-service teachers developing and using mathematical knowledge for teaching proof

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This paper explores pre-service teachers' development of mathematical knowledge for teaching proof in a practice-based setting. Proof is challenging for both students and teachers but is crucial for meaningful mathematical learning. We design an instructional sequence based on core practices, tasking pre-service teachers with planning, implementing, and analyzing proving activities. Analyzing six presentations on the process, we select two cases of maximal variation to explain and illuminate how pre-service teachers report on and utilize mathematical knowledge for teaching proof. The results suggest that a minimum level of common content knowledge for proof might be necessary for developing other forms of mathematical knowledge for teaching proof. This highlights the need for further research into the relationship between categories of mathematical knowledge for teaching proof in relation to designing instructional sequences.

Keywords: Initial teacher education, Proof, Mathematical knowledge for teaching proof.

Introduction and background

Proving is a key mathematical activity that deepens understanding, generates knowledge, and should play a central role in students' mathematical education from the primary grades onwards (Stylianides, 2007; Stylianides et al., 2017). Stylianides (2007, p. 291) offers a classroom-specific definition of proof as a mathematical argument with the following characteristics:

1. It uses statements accepted by the classroom community (*set of accepted statements*) that are true and available without further justification;
2. It employs forms of reasoning (*modes of argumentation*) that are valid and known to, or within conceptual reach of, the classroom community; and
3. It is communicated with forms of expressions (*modes of argument representation*) that are appropriate and known to, or within conceptual reach of, the classroom community.

While this definition aligns with the traditional view of proof as a valid argument with true premises, appropriately represented, it adapts the components to the understandings within a specific classroom context (Stylianides, 2007). Research shows that students struggle to understand and use the concept of proof (Stylianides et al., 2016; Stylianides et al., 2017). Similar challenges are observed for teachers and pre-service teachers, who often share the same misconceptions as students (Stylianides et al., 2016; Stylianides et al., 2017). Buchbinder and McCrone (2020) suggest that "teachers themselves often lack the mathematical knowledge for teaching proof and hence are not disposed to support students to learn proving" (p. 1). However, there is little research on these teacher-related aspects (Stylianides et al., 2017). In a Danish context, Skott et al. (2020) show that pre-service teachers struggle with the concept of proof, often mistaking empirical arguments for proofs. We thus need to know more about equipping pre-service teachers with the knowledge and skills to competently teach proof, enabling students to benefit from their expertise (Stylianides et al., 2017).

Lesseig (2016) develops a framework of mathematical knowledge for teaching proof (MKT-P), building on Ball et al.'s (2008) framework of mathematical knowledge for teaching. The MKT-P framework consists of common content knowledge for proof (CCK-P), which includes the knowledge and ability to construct a valid proof; specialized content knowledge for proof (SCK-P), which includes the components (Stylianides' definition) and functions of proof; knowledge of content and students for proof (KCS-P), which includes knowledge of students' proof schemes (Harel & Sowder, 2007) and students' (mis-)conceptions of proof; and knowledge of content and teaching for proof (KCT-P), which includes knowledge of the relationship between instruction and proof schemes, questioning and other pedagogical strategies, such as talk moves (Kazemi & Hintz, 2014). Lesseig (2016) categorizes Stylianides' definition as SCK-P, but she classifies the ability to judge whether certain statements, representations, and arguments are within students' conceptual reach as KCS-P, illustrating the fluid boundaries of these categories.

This paper reports on an instructional sequence designed for initial teacher education to develop and/or reinforce pre-service teachers' MKT-P. Pre-service teachers presented their work from the sequence, and this paper analyzes those presentations to answer the following research question:

> (RQ) How do pre-service teachers utilize and report on different elements of MKT-P in their presentations, when focusing on isolating the three components of a proof (SCK-P) and combining these to an appropriate proof (CCK-P)?

The instructional sequence

The instructional sequence is based on *core practices as a pedagogy of teacher education* (McDonald et al., 2013). By adopting a practice-based, practice-directed framework, we acknowledge that "effective mathematics teaching requires teachers not only to have good mathematical knowledge but also to use that knowledge flexibly in supporting students' learning" (Stylianides et al., 2016, p. 342). The participating pre-service teachers are second-year students in initial mathematics teacher education in the Danish educational system. To enroll in initial mathematics teacher education, the students have passing grades from high school mathematics level B, where there is some focus on the construction and evaluation of proofs (CCK-P). In their first year of initial teacher education, they have studied various aspects of proof, including Harel and Sowder's (2007) proof schemes and proof construction and evaluation (CCK-P) in the contexts of number theory and geometry.

In the first unit of the instructional sequence, pre-service teachers worked in groups on the Josephus problem, which involves n people standing in a circle, eliminating every second person until only one remains. The task allowed multiple levels of proving activity, enabling pre-service teachers to engage at a level suited to their understanding—for example, proving that the survivor is never in an even position or, for $n = 2^m$ people, that the winning seat is always 1. Although no pre-service teachers reached this point, a double induction proof showing the general formula for the surviving seat with n people was a prepared possibility. The second unit focused on the content of Stylianides' 2007 article, specifically the definition of proof relating it both to Stylianides' examples and the Josephus activity. In subsequent units, the focus remained on Stylianides' framework, with groups of pre-service teachers conducting practice teaching of designed proving activities with classmates acting as students, which were then analyzed in the following units. All practice teaching, including

the Josephus activity, was recorded on video to support subsequent analyses. A Padlet was created for each group, containing all analyses of their taught proving activity. After these units, pre-service teachers had time to teach, observe, and analyze their proving activities in real classroom settings during practicum, under the guidance of the teacher educator. In a concluding unit, pre-service teachers orally presented a) the motivation for their proving activity, and b) a proof fitting Stylianides' definition, specifying the set of accepted statements, modes of argumentation, and modes of representation. The research question reflects that pre-service teachers were tasked with presenting on at least one aspect of SCK-P, the three components of a proof, and at least one aspect of CCK-P, the construction and presentation of a proof from the students' perspective.

Methods

The experiment is a teacher education design experiment (Cobb et al., 2003), focusing on designing an instructional sequence linked to specific learning outcomes and analyzing how these outcomes are reflected in pre-service teachers' presentations. The data consists of six recordings of pre-service teachers' presentations, each lasting around 35-45 minutes, with 20-25 minutes for the presentation and 15-20 minutes for questions and discussion. The recordings were initially viewed by the author, who timestamped sections where: a) pre-service teachers presented their proving activity, b) they presented a proof fitting Stylianides' definition from the students' perspective, or c) they addressed aspects relevant to their use of MKT-P. Subsequently, relevant timestamped sections were reviewed again, with a focus on the pre-service teachers' identification of three components in Stylianides' definition, as well as written or oral formulations of the student proof as a synthesis of these components. Based on this review, two of the six presentations were selected as cases of maximal variation (Flyvbjerg, 2006). The variation criterion was the extent to which pre-service teachers used CCK-P to present a valid proof from the students' perspective. Examining the differences in MKT-P utilization between two groups with contrasting performances on this aspect could provide valuable insights.

Analyses and results

Group A. This group provided a correct sketch of two valid methods to prove their chosen statement, though neither sketch was fully completed. This is well beyond how any other group utilized their CCK-P to present a valid proof from the perspective of the students. The group designed an activity for 6th graders, asking how many triangles with an area of 6 cm² can be made, where it is clear from context that the statement they want students to prove is that there are infinitely many. In their implementation, pre-service teachers had students create a table of different heights and bases that result in an area of 6 cm². They then continue this table with the possibility of dividing the base by two and doubling the height. In the presentation they orally report that "during the whole class discussion many students realized that the base was divided by two every time and the height doubled. In that way they realized that one could go on forever." In relation to this, another pre-service teacher comments, "as long as one of the accepted statements is that when dividing by 2 you never get 0". This demonstrates a good understanding of the logical structure of the proof (CCK-P), which relies on this fact. However, it also raises the question of why they did not include it as one of the accepted

statements from the start, especially given that the statements they did choose seem to provide background knowledge about triangles rather than directly supporting the proof.

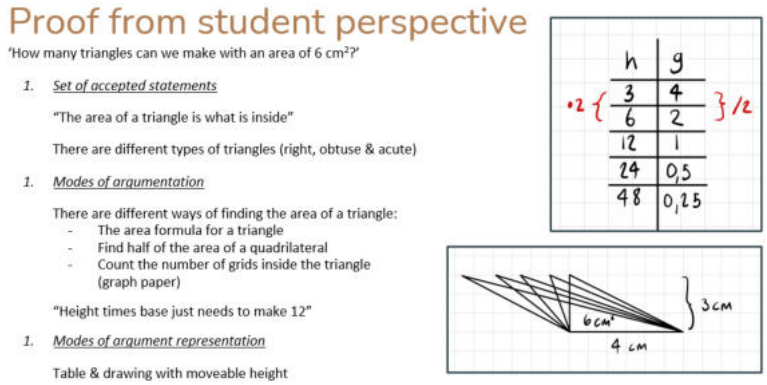

Figure 1: Slide from group A's presentation (translated from Danish)

The group effectively argues that infinitely many non-identical triangles can be generated, and that representing this through a systematic table is within students' conceptual reach. They emphasize the need to extend the table beyond the first few entries containing whole numbers, which reflects KCS-P, recognizing that students might have difficulties seeing the pattern continuing if only whole numbers are represented. They do not explain how we know that the new triangles are different from the previous ones, which is needed for the argumentation to be valid.

Group A also outlined another valid proof strategy using a triangle with a fixed base and a top point moving along an axis parallel to the base. They explain that as the top point moves, the area remains unchanged, demonstrating their grasp of the logical relationships (CCK-P) in a proof based on this fact. They did not explicitly address the fact that there are infinitely many possible positions for the top point or how this is known. Their chosen representations—a table and a drawing—align well with the envisioned proof, and they argue that both are appropriate for the classroom community.

Group A emphasized using talk moves to foster dialogue between students, but they did not specify the content of this dialogue, making it unclear how these moves supported students' engagement with proving or reflected pre-service teachers' KCS-P.

Group B. This group planned, taught, and reported on an activity for 5th graders. They never explicitly identified a statement to be proven, nor was it clear from the context what they expected students to prove. They did not present a proof, leading us to assess their use and possession of CCK-P as limited. The activity involved laying tiles around bushes, with eight tiles around the first bush and five more tiles for each additional bush. The phrase "valid forms of argumentation" (figure 2) is translated from the Danish *afslutningsformer*, a non-existent word. An introductory slide in their presentation states "The students were charged with proving a linear growth for a geometrical pattern," and the pre-service teachers orally use phrases such as "proving the activity" and "when we had to prove our proof." This may reflect confusion over terminology, but further analysis suggests that it might stem from pre-service teachers lacking a fundamental understanding of what a proof is and its role in mathematical reasoning. The pre-service teachers present the modes of argumentation as a *systematic enumeration of tiles*, relating to a form of argumentation that has been relevant

throughout the instructional sequence, as the examples from Stylianides (2007) use *systematic enumeration of all cases*. The context makes it clear that the pre-service teachers use the term without fully grasping its meaning, likely due to prior exposure. There is no systematic enumeration of tiles involved beyond simple counting, and without further explanation, it does not seem to be a relevant mode of argumentation either, as there is no given number of bushes for which we need to systematically enumerate the tiles. This suggests that the pre-service teachers have not developed sufficient CCK-P to produce or evaluate a valid proof in this context.

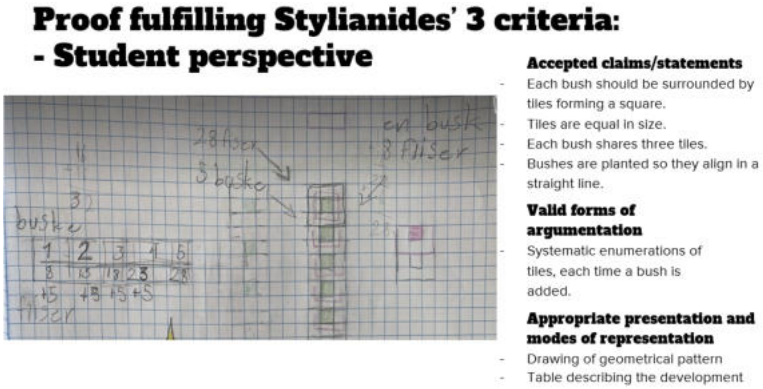

Figure 2: Slide from group B's presentation (translated from Danish)

The pre-service teachers approach the task as a problem-solving activity, focusing on guiding students to *discover* that the mathematical function $y = 5x + 3$ describes the relationship between tiles (y) and bushes (x), and we may note that in such a context, a crucial part of the process would be to count 5 tiles for each bush. In many other contexts, this would be a meaningful and rewarding mathematics activity. But in the given context, where pre-service teachers are tasked with developing proving activities, it is crucial to distinguish between finding an answer and moving beyond to prove that the answer is correct—something the pre-service teachers do not do.

Approaching the teaching activity through Stylianides' definition (SCK-P) did not help these pre-service teachers identify the lack of proving activity, likely due to their insufficient CCK-P. While the pre-service teachers identified relevant background statements and representations, it is unclear whether these elements were explicitly intended for proof construction or merely selected as relevant to the activity. Given that the pre-service teachers did not use Stylianides' definition to link these with a valid mode of argumentation to form a proof, the latter interpretation seems most plausible.

While Group B commendably emphasizes talk moves to foster communication, they provide no details on the proof-related content of the envisioned dialogue, leaving their use apparently disconnected from mathematical content and instructional goals.

General discussion and concluding remarks

Group B is more representative of the general difficulties pre-service teachers face with MKT-P. While 5 of 6 groups correctly identify a statement suitable for proving in their activities, their presentations reflect limited MKT-P, in particular CCK-P and SCK-P, as they generally do not isolate a valid mode of argumentation relative to which accepted statements and appropriate representations

can be singled out. Most pre-service teachers identify a set of accepted statements, though they tend to include background knowledge not directly relevant to the proof. And while they correctly identify the representations involved, it is not always clear whether they distinguish between representations used in the proof itself and those used in the lesson. Pre-service teachers often engaged superficially with Stylianides' components, particularly modes of argumentation, reflecting a lack of foundational understanding of valid proofs (CCK-P). When they struggle to identify the logical argumentation (CCK-P), they will naturally also face difficulties identifying accepted statements and representations used, as the logical structure of the argumentation is essentially a tying together of these. It is thus unclear if Stylianides' definition has helped the pre-service teachers. Instead, it might have led them to superficially focus on the content of the three components without engaging deeply with the nature of proof itself. This raises broader questions about the role of conceptual understanding in pre-service teachers' engagement with proof. While these challenges inevitably surface in classroom practice, they are rooted in the knowledge pre-service teachers bring to these situations.

Accordingly, this paper focuses on pre-service teachers' knowledge as distinct from their enacted classroom practices, though these are, of course, inherently interrelated. While what teachers do in classrooms is undeniably important, our focus here is solely on their development and utilization of MKT-P. This necessarily narrows our scope, as we consider pre-service teachers' classroom actions only to the extent that they reveal how they articulate and apply their MKT-P in their own reports. However, building on the example of Group B, it is reasonable to infer that if pre-service teachers' conceptual understanding of proof (CCK-P) is insufficient to the extent that students are not even presented with proving opportunities, then pedagogical strategies alone seem unlikely to compensate. Likewise, identifying students' misconceptions about proof (KCS-P) is likely to be difficult if pre-service teachers themselves hold similar misconceptions. Moreover, talk moves seem inefficient in eliciting proving (KCT-P) without a clear understanding of what needs to be proven and how.

Many groups of pre-service teachers focused on problem-solving rather than proving, which may reflect a limited understanding of the role and utility of proof in mathematics as a deductive discipline. Ball et al. (2008) define *horizon content knowledge* (HCK) as "an awareness of how mathematical topics are related over the span of mathematics included in the curriculum" (p. 403). Pre-service teachers' struggle may not only stem from the technical aspects of proving (CCK-P) but also from gaps in their own horizon content knowledge about the purpose and role of proof in mathematics. If pre-service teachers are unaware of the specific deductive nature of mathematics and the special role of proof, this could explain their focus on *problem-solving* over *proving*—they may not recognize a need for proof in the first place. While problem-solving and proving ideally occur together in teaching, there is a fundamental difference between finding an answer to a problem and moving beyond to establish with certainty that the answer is true given specific assumptions. If classroom mathematics primarily emphasizes the former, the nature and role of proof as a deductively valid argument may become unclear or even seem irrelevant. This, in turn, could likely impact the development of competencies for constructing and evaluating proofs (CCK-P), as pre-service teachers may struggle to see proof as essential to mathematical reasoning. Crucially, this issue is not confined to the pre-service teachers in this study but is embedded in their own mathematical education, often taught by teachers who might themselves lack a deep understanding of proof. This

could perpetuate a vicious cycle: when proof is not recognized as central to mathematical reasoning, it is neither emphasized in instruction nor meaningfully integrated in classroom practice. As a result, future teachers could enter the profession with the same limited appreciation of proof, reinforcing the gap between mathematics as a deductive discipline (HCK-P) and how it is taught. This is also closely linked to KCS-P and the proof schemes of pre-service teachers and students (Harel & Sowder, 2007), as a lack of appreciation for the purpose of proof—why it is needed and what it accomplishes—is likely to shape one's understanding of the concept of proof itself and what one accepts as such.

Future research should investigate whether empirical evidence supports such a hypothesized relationship between HCK-P and CCK-P. In light of this, future research should also further examine whether, as suggested by the analyses of this paper, a minimum level of CCK-P is necessary to support the development of other forms of MKT-P. If the framework of MKT-P has a hierarchical structure and certain types of MKT-P depend on foundational CCK-P, clarifying this structure can enhance our insight into how these domains interact and develop. Since all three components of Stylianides' definition—particularly *modes of argumentation*—seem to involve CCK-P, it could also be worth questioning whether categorizing the definition as SCK-P can be meaningfully upheld. If not, future research may need to adopt alternative frameworks, such as Buchbinder and McCrone's (2020), which introduces the category of *knowledge of logical aspects of proof* (KLAP), integrating core elements of both CCK-P and Stylianides' definition (Buchbinder & McCrone, 2020).

The instructional sequence reported in this paper does not appear to have significantly advanced pre-service teachers' mastery of MKT-P, likely due to their limited understanding of CCK-P—both at the outset and as developed during the sequence. This raises the crucial question of how to effectively integrate and balance proof components in initial teacher education. Given that MKT-P is directed at teaching in practice, giving up on practice-directed and practice-based instruction seems unwise. Rather than replacing practice-based instruction, a more effective approach could be to complement it with explicit proof-focused activities, such as elements from the instructional sequence in Stylianides and Stylianides (2022). This sequence fosters both an intellectual and pedagogical need for proof by engaging pre-service teachers in recognizing the limits of empirical justification and developing an operationally functional conceptualization of proof (Stylianides & Stylianides, 2022). Integrating such elements early in the instructional sequence could help pre-service teachers establish a stronger foundation in proof (CCK-P) before applying this understanding in practice-based settings. However, practice-based approaches, like the one reported here, can unintentionally divert pre-service teachers' focus. A review of Padlets containing pre-service teachers' practice teaching analyses highlights this issue: 17 of 56 points focus on talk moves, while only 9 address proof content or Stylianides' definition, with none addressing both. This suggests that while pre-service teachers are genuinely committed to fostering student collaboration and learning—an admirable and valuable emphasis—they tend to prioritize general teaching strategies over the content of proof. Talk moves are only an effective part of teachers' KCT-P when they directly support proving related dialogue—a challenge for pre-service teachers whose limited CCK-P may hinder their ability to identify and emphasize key proving concepts for students to discuss. More research is needed to explore how instructional sequences can better integrate proof-related teaching into practice-based approaches, ensuring that pedagogical strategies support rather than draw focus away from mathematical content.

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Analytic and narrative perspectives on proof and proving in mathematics classrooms - a cultural comparative approach

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Proof and proving are a complex endeavour in mathematics classrooms. Comparing such classroom ventures requires nuanced perspectives. From an analytic perspective, a structural, linguistic and functional angle permit the characterization of significant facets of the situation. To become more sensitive to cultural specifics in these situations, the tacit perspectives we hold as researchers are also important. The methodology of narratives allows the reconstruction of such tacit perspectives and the comparison of different views of the same situation. In our cultural comparative study, the comparison of different narratives of the same situations in German and Japanese classrooms inspired specifications for our theoretical framework. Analytic and narrative perspectives together enriched the cultural comparisons and the further development of the theoretical framework. In this paper, we illustrate this dialectic with regard to the evolution of our concept of structure.

Keywords: Argumentation, proof, comparative study, mathematics classrooms.

Introduction

Describing and comparing proof and proving activities in mathematics classrooms is challenging for several reasons. It is less obvious than one might believe, to agree on key aspects to look for. What constitutes a proof or proving is regularly discussed in mathematics education (e.g., Mariotti et al., 2018; Reid & Knipping, 2010). Researchers seem to not necessarily share the same epistemology (Balacheff, 2008), when talking about teaching and learning proof and proving in classrooms. Also, several studies conducted from a comparative perspective have shown the diversity of proof / proving and their role in different educational systems (e.g., Bergwall, 2021). Different cultural traditions and historically shaped conventions related to proof and proving in classrooms seem at stake. Describing these differences is not obvious (Reid, 2015). In a cultural comparative approach also language, its use and translations are an ongoing concern, which requires careful attention and reflection. In this paper the dialectic of analytic and narrative perspectives is the focus.

In our international collaborative research project on proof and proving in mathematics classrooms, several researchers are involved with research experiences in different national and cultural contexts. Aware of the above challenges, we wanted to develop together a refined cultural comparative approach for describing proof and proving in mathematics classrooms. We agreed to work together on this, both theoretically and empirically. Starting with a theoretical framework (Miyakawa & Shinno, 2021), which we describe in the next section, we focused on *structure*, *language* and *function*, to analyze similarities and differences of proof and proving activities in classroom situations at the middle school level. In the project and further collaboration, it also became obvious that we envisioned different perspectives, not only for the concept of *structure*, but also *language* and *function*. In this paper, we will focus on our concept of *structure* and how our view of it evolved towards a more multi-faceted understanding. It also occurred to us that our views of the classroom

situations were not only informed by our theoretical research backgrounds and foci, but also shaped by our cultural experiences of classrooms. Therefore, we started looking for a methodology that would allow us as researchers to capture our own different cultural views of the classroom situations we analysed. Such a reflective stance seemed critical to avoid confusing our own perspectives and prejudices with potential cultural differences in the classrooms. Our congruent and diverging views of the same classroom situations are the focus of this paper, not the classroom processes themselves.

The *methodology of narratives* (see Artigue et al., 2023), embedded in scholarly traditions of the Social Sciences (Connelly & Clandinin, 1985), seemed promising to us. Narratives, understood in this tradition as descriptive accounts of a series of related events or experiences, also called 'stories', are the starting point in this methodology. Comparing 'our distinct stories' about proof and proving in classrooms in form of narratives helped us to reveal the cultural underpinnings of our views.

> Narratives are more than just personal, individual stories, but imbricated in cultural and historical underpinnings. This means that comparing narratives allows us to reveal and understand this footing. (Artigue et al., 2023, p. 601)

Also, narratives bank on the resource of language and allow us to compare our languages about teaching and learning in mathematics classrooms. We became aware that the methodology of narratives could inspire our theoretical analyses and comparisons in the project. For these reasons, we decided to adopt the narrative methodology as complementary to our analytical approach. We sought not to conceal our tacit cultural views by an analytic perspective, but to reveal our tacit perception of processes in the classroom and to look at our tacit interpretation as complementary to our (further developing) multi-faceted theoretical framework. In this paper, we argue for such a multi-faceted cultural comparative approach, complementing analytic perspectives on proof and proving activities in mathematics classrooms by narrative accounts of these situations. We pursue the research question: How can narrative accounts of proof and proving processes in mathematics classrooms complement an analytic perspective and reconcile tacit cultural views?

Analytical perspective

Based on different theoretical perspectives of argumentation and proof in mathematics education (Mariotti et al., 2018; Reid & Knipping, 2010), Miyakawa and Shinno (2021) proposed to characterize proof and proving in a given institution by three aspects: *structure*, *language*, and *function*. The triplet allows us to understand cultural specificities of proof and proving in classrooms.

- *Structure* designates in this framework the organization of reasoning or arguments showing how different statements in a proof are connected.
- *Language* is understood as the semiotic representation (Duval, 2006) of arguments and reasoning structures, including gestures, diagrams, oral and/or written discourse.
- *Function* denotes the role which is attributed to a given proof or proving activity by the institution (e.g., verification, explanation, exploration, communication, or systematisation), and may not be restricted to these functions often mentioned in the literature.

How to analyse each of the three aspects, when characterizing proof and proving activities in classrooms, were still methodological and epistemological challenges. Characterizing the *structure*

of proof and proving, which is the focus of this paper, included in our initial theoretical framework reconstructing the *logical structure*. Particularly, it involved reconstructing the deductive steps, which led from starting statements, by the use of theorems/axioms/definitions, to conclusions in the class. Analysing the *logical structure* would include reconstructing the type of logic (e.g., propositional, predicate) which is employed, when statements and inference rules in a proof are applied (Durand-Guerrier et al., 2012). But, in many situations the presentation of proofs follows conventional styles rather than formal logical rules. So, reconstructing the *logical structure* has to acknowledge its more or less informal presentation, and include informally as well as formally presented logical structures.

Classroom practices are multi-faceted processes, where convincing others and negotiating a common argumentation basis may be important. Discourses with complex *argumentation structures* occur in classrooms, which are apparently different to the logic of deductive inference. Revealing these complex structures cannot be undertaken by means of logic alone, but requires a conceptualization of rationality related to the context of practice. Such a conceptualization has been introduced by Stephen Toulmin (1958), whose functional model of argumentation has been widely adopted in mathematics education. Reconstructing and comparing the *argumentational structures* of proving activities and their peculiar rationale in the classroom appeared to be insightful and complementary to a logical perspective. For revealing cultural differences in the organization of reasoning in class, we therefore decided to complement our analyses of the *logical structure* by analyses of *argumentation structures*.

Comparing logical and argumentation structures in different classrooms made it possible to reveal differences and similarities in proof and proving activities in classrooms. For example, an emphasis on an if-then structure in propositional logic in some of the Japanese classrooms occurred, while visual argumentations in some of the German classrooms were emphasised with less obvious *logical structures*. We further realized that our accounts and interpretations of the reasoning in the lessons diverged, beyond the structural differences we had found in these classes in terms of logic and argumentation. To better understand these disparities and our tacit diverging interpretations, we decided to capture our different points of views and to use the *methodology of narratives* (Artigue et al., 2023) for our cultural comparative approach. We characterize this methodology and our usage of it next, before we document the outcomes of this approach.

Methodology of narratives

In the Social Sciences narratives are defined as ‘distinct stories’ or descriptive accounts of a series of related events or experiences (Connelly & Clandinin, 1985). These ‘stories’ are acknowledged as a means to reconstruct tacit understandings of cultural events and experiences. Narratives of a situation are seen as more than personal stories, as these stories reveal an individual account of a situation related to a social and cultural background. Connelly and Clandinin (1985) also argue that individual’s narratives elicited by a setting expose the individual and cultural understanding and knowledge of a teaching and learning setting. Recognizing narratives as socio-cultural products and artefacts allows us to use these narratives for cultural comparative approaches (see Artigue et al., 2023). Narratives are deliberately not analytic, but foster a descriptive, eclectic and linguistically multi-faceted style. The intent is to reveal cultural underpinnings of the storylines, of the used language and linguistic styles, which are employed aside from a common research terminology or

professional language. This can help to reveal our tacit understanding of classroom teaching and learning of proof and proving, within and across cultural comparisons. This tacit understanding and revealing the cultural underpinnings of this understanding is the focus of this paper.

We adopted an approach that had been suggested by Artigue et al. (2017), in the context of the Lexicon project (Mesiti & Clarke, 2018). Stimulated by videos of Grade 8 mathematics classroom situations in Germany and in Japan, we split into different teams (German colleagues, Japanese colleagues, other colleagues) and each team produced their own narratives for each lesson. The foci of the lessons were proof and proving activities with a common topic: 'the sum of three consecutive numbers (is divisible by three)'. The narratives were team productions and besides the videos as stimuli, they also relied on transcripts of the lessons. The transcripts had first been produced in the original language of the lesson and then been translated into English; both versions were available to all teams. Different to Artigue et al.'s approach the teams only included researchers at this point, not teachers. Also, the focus was not (as in the Lexicon project) on our usage of professional language to characterize these situations, but had a focus on describing what 'proof' and 'proving' meant in the videotaped classroom situations. We therefore labelled our narratives as 'focused narratives'. The narratives included a general description of the context and progression of the lesson, followed by a description of what 'proof' and 'proving' meant and involved in the classroom situations. The intent of this approach was to capture descriptions of the situation to reconstruct our tacit interpretations.

The narratives were first written in the native language of each team, before they were translated into English, which was the language of communication in our international project. We did not want to lose specific linguistic characterisations and our reflections on them, when comparing the narratives. So, we regularly went back to the original versions of the narratives in our comparative analyses. As in the Lexicon Project, we consider language as an important resource that regulates individuals' experiences and interactions. Beyond an analytic approach, the narrative methodology also allowed us to capture and reflect our tacit use of language, when we described our experiences and observations of interactions. Comparing our descriptions in the narratives on what 'proof' and 'proving' meant in the videotaped lessons, helped us to become more sensitive for our own perspectives and how these might have been shaped by our cultural views on 'proof' and 'proving'. These intercultural comparisons and reflections also helped to distinguish our own interpretations from what teachers and students might mean by 'proof' and 'proving' in their conversations and interactions in the situation. This stimulated us to critically question, what we as researchers noticed in these classroom contexts and what we included in our theoretical perspectives.

Comparing our narratives on proof and proving

As researchers becoming aware of our tacit understandings of the classroom situations helped to see our different cultural perspectives on them. Comparing our narratives also made us aware that we 'naturally' commented on the triplet of the analytic framework. We will indicate how we described 'structures' in our narratives. This was obviously different from what we had done analytically.

The videos and transcripts stimulated us to describe the context and progression of the classroom lessons in concrete ways, characterizing the structure of proof and proving by their presentation and evolution, rather than their logical or argumentational structures. In the different narratives, we

described the intricate relationship of proof and proving, and we particularly discussed the role of the tasks and activities in class which were related with proof and proving. Cultural differences in our descriptions occurred, when it came to what we would consider as pre-activities or key activities for proof and proving in class. For example, generic-examples, figurative arguments with dot patterns or oral explanations in class, why 'the sum of three consecutive integers is three times the middle number', would be considered as proofs and proving by the German team, whereas the Japanese team would rather interpret these as a pre-proving phase, justifying a statement-in-action, from which later a symbolic proof with letters would evolve, which would prove the general statement.

Related to this, the German team's narratives described more activities as proving. Processes such as understanding, verifying, validating and justifying were all described as part of the proving activities in these narratives. In contrast, the Japanese team's narrative focused on the activities related to justifications and described the other processes as 'pre-proving activities'. Our researchers' tacit concepts of proving, as a footing of our descriptions, seemed different and leading our 'stories' on what counted as 'proving' in the classroom lessons. While the German team focused on 'proving' as a process with multiple functions, the Japanese team portrayed 'proving' rather as a process with one function, demonstrating the validity of a mathematical statement by generating a written proof, a product. Nevertheless, all teams described similar mathematical activities as characteristic for the classroom situations, specifically focusing the following: motivating the exploration of a statement, formulating the statement, looking for ideas how to justify it, formulate and present a general justification, applying or extending the statement in further activities.

Our narratives indicated that we as researchers interpreted the progression of these activities as different types of proof and proving patterns, which we saw as emerging in the classes. Comparing our different views in the narratives, we agreed to also compare our interpretations of the specific interplays of these activities, which seemed to constitute in our views different conventional styles of proof and proving in class. For example, we compared our descriptions of the Japanese lesson, when the students were engaged in pre-activities and were made familiar with a mathematical statement. We also compared how we described the teacher guiding the students through a written proof and agreed that this happened in a conventional form: 'introduction, contents, and conclusion'. We agreed across the teams that students' active engagement in proving seemed to happen later in application activities in the Japanese classrooms, where students were expected to replicate the pattern of the written proof they had seen in class. In the German classroom contexts, we saw together, despite our different views, an emphasis on the pre-activities, where students were actively engaged in justifying the statement for some examples. Together, we interpreted these activities now as establishing a common argumentative ground for generalizing the ideas of these example-based arguments, where using different representations for the arguments (figurative, generic examples, semi-paragraph) were encouraged. We agreed across the teams that no explicit conventional style for writing a proof was emphasized and found in the German classes. These comparisons made us aware that we needed to include an analysis of *didactical structures* in our theoretical framework (Shinno et al., 2024).

When comparing our characterisations of 'proof' and 'proving' in the narratives, we also realized that the foci of our interpretations were at times different and that we highlighted aspects, we had not specifically focused on in our analytical perspective.

We realized through the comparisons of our narratives and our discussions around these comparisons that our concept of structure of proof / proving referred to different 'objects'. The concept of structure refers to structures of (logical) products, when it comes to 'proof', and it refers to structures of argumentation processes, when it comes to 'proving'. These are different 'objects'. We also realized that there seem to be cultural differences in mathematics classrooms in this respect. We see a difference between the German / Japanese classrooms we analyzed: While in the German classrooms, 'proof' is not an object to teach, but a paramathematical concept (accompanied by activities), in Japanese classrooms 'proof' is an object to teach. This is clearly reflected in our narratives and comparing these made us aware of this tacit cultural underpinning.

In the German narratives, we found an emphasis on describing different mathematical approaches in the classrooms. For instance, for proving that 'The sum of three consecutive natural numbers is always divisible by 3', the narratives of the German team pointed out different mathematical approaches for the German classrooms. These narratives highlighted that first, the smallest number was represented by n and the sum was expressed and calculated as $n+(n+1)+(n+2)=3n+3=3(n+1)$. Then, a second approach was pointed out, where the middle number was represented by n and the sum was described as $(n-1)+n+(n+1)=3n$. The narrative of the Japanese team also noted these approaches, but the German narratives put more weight on this observation. These narratives highlighted that even though in both approaches the algebraic expressions were simplified and condensed, the justifications for the different approaches in class made obvious that their mathematical structures needed to be made explicit. Again, the comparisons of our different narratives pointed to the need of extending our analyses of the proof and proving activities in class. We decided to include an analysis of the mathematical content and the *mathematical structures* on which the proofs and the proving activities in class built.

Towards a cultural comparative approach

Comparing our narratives allowed us to not only become aware of our different cultural views of the classroom situations, but also that we had to come to an agreement on what we wanted to include, when analysing the structure of classroom practices of proof and proving. As argued above, our narratives made us aware that we needed to analyse the *logical* and *argumentation structures*, when analysing proof and proving as different objects in classrooms. Also, their *didactical* and *mathematical structures* needed to be analysed, closely related to proving as a process and proof as a product in the class. Acknowledging all these structures would help to characterize potentially different conventional styles of proof and proving in classrooms and their cultural contexts.

Structures of proof and proving activities in classrooms are also related to different functions that these activities have in classrooms. Defining these functions analytically is important. But, again the comparisons of our narratives indicate that the classroom situations are complex and our interpretations of activities and their role in these situations differ. While the narratives of the German team focused more on different functions of the (pre-)proving activities and proving as a process, the narratives of the Japanese team highlighted rather the progression of the lesson and focused on function and product, when looking at the written proof. Comparing and analysing this more closely,

will also help to come to a more nuanced analytical understanding of the concept of function, which has been discussed rather broadly in the mathematics education community so far.

Last, but not least, language and linguistic aspects require a closer look, analytically and pragmatically, through looking at the use of language in the classroom contexts. Looking at the semiotic representation of arguments and reasoning structures is analytically sound and well acknowledged in mathematics education as a way to analyse the register of language for proof and proving. But, what does this mean, from a pragmatic point of view? Our narratives might provide first insights related to this. As indicated above, the narratives of the German team interpreted the different representations of the general argument and their use as 'proofs' in the German classrooms. The narratives of the Japanese team on the other hand focused rather on the symbolic representation and its use in class, which was interpreted in this narrative as 'proof'. Our researchers' tacit concepts of proof not only shaped our narratives and what we considered as 'proof' in the classrooms, but also our focus on its meaning in the classroom practice. The comparison of our narratives indicates that looking at language from a pragmatic stance, and becoming aware of our cultural footing in this, might be worthwhile. In addition to a semiotic view, a focus on discourses and the pragmatic use of language in classroom contexts seemed necessary, which requires an extended theoretical and methodological approach.

Overall, we incorporated a narrative perspective to reflect our own tacit perspectives and epistemologies as researchers. This approach expanded our theoretical framework into a more multi-faceted one, as demonstrated in this article focusing on the evolution of our concept of structure. The narratives of German and Japanese researchers not only represent different views on the structure of proof and proving (e.g., logical, argumentative, didactical, and mathematical ones), but also differ in which aspects of the structure they emphasise. We propose to combine different theoretical and methodological approaches and perspectives to better understand and reconstruct cultural specifics related to proof and proving in classroom situations. As there is not enough space in this article, we will further elaborate on our networking approach of theories in future publications.

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Recognising the situation for proving is not obvious

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This paper investigates how visual perception and empirical experience influence recognising situations for proof. We examined the written works of 59 prospective math teachers on an open geometry problem based on a paper-folding task. This setting allows us to observe the cognitive process from problem identification to the formulation of conjecture and providing proof. The findings suggest that easily perceptible visual clues make it difficult to recognise the proof situation and may lead to a reliance on empirical argumentation. In contrast, situations with less perceptible clues tend to prompt the use of mathematical proofs to substantiate conjectures.

Keywords: Argumentation, mathematical proof, empirical proof scheme, paper-folding, perception of shapes.

Introduction

The knowledge teachers need to possess about mathematical proofs has been the subject of numerous studies. A. J. Stylianides and Ball (2008) have taken the initiative to complement teachers' knowledge of proofs with their understanding of the situations that require proving. The authors assert that teachers must possess the ability to recognise situations that require proof. When a mathematical problem's wording explicitly mentions proof or refutation, identifying a proof situation is simple. However, the student must first formulate mathematical relations when it comes to open tasks, such as situation-based problem-posing tasks. Students should be sure that their statement is only a conjecture that requires proof or refutation. However, various factors can influence students' identification of proof tasks. For instance, visual perception can override claims that require proof. Moreover, certain statements are straightforward to make visible through simple manipulation. The paper folding case investigated in this paper is a prime example of this phenomenon.

Paper folding is an appropriate medium for examining several questions on argumentation and proof (Mariotti et al., 2023), as well as the ability to recognise proof situations. First, paper folding frequently results in conjectures that are evident based on perception. On the other hand, manipulating geometric components can create a cognitive trap in which the learner accepts empirical reasoning as a conviction. This insight led to the research question in this paper: How do visual perception and empirical actions influence prospective mathematics teachers' ability to recognise situations requiring proof, and how do these factors shape their proof schemes?

The role of visualisation in the proving process has been examined in several studies (Biza et al., 2010). However, the literature rarely addresses the influence of visuality-driven intuition on mathematical activity. By investigating this phenomenon, our paper contributes to filling a research gap.

We draw on G. J. Stylianides (2009) conceptualisation of proof schemes to answer the research questions. This framework defines our coding system by distinguishing between empirical reasoning, rationale, and providing proof. By applying this perspective, the research links task design, based on paper-folding activities, to the broader goal of understanding how visual cues affect the recognition of proof situations and schemas.

Theoretical underpinning

Much research has been conducted on the factors that influence the argumentation process. Some of these factors depend on the circumstances under which the task is performed. For example, Dreyfus and Hadas (1996) argue that "Students' concept of proof, their appreciation for its need as well as for its roles, depends on the curriculum, on the activities students are asked to carry out, on the questions they are asked and on *how* they are asked these questions" (p. 1). In their experiment, Boero et al. (1996) reported that most seventh graders can produce theorems (conjectures and proofs) if, during the production of the conjecture, the student progressively works out their statement through intense argumentative activity. During the subsequent proving stage, they coherently link up with this process, organising some of the arguments generated during the construction of the statement into a logical chain. Intuition and perception can also influence proof-related activities. Mariotti and Pedemonte (2019) showed that when a conjecture is based on intuition or perceptual evidence, the proof process can involve a cognitive gap that learners sometimes find challenging to overcome.

The idea of *proof* in mathematics education has evolved over time; see (Hanna & Knipping, 2020) for more details. Following G. J. Stylianides (2009), in this paper, we define proof as a series of arguments, each accepted by the community, covering the key ideas needed to validate or reject a claim. As subjects in our research are prospective teachers for secondary mathematics education with strong mathematical backgrounds, the approach is also consistent with Hanna concept of "acceptable proof," i.e., "a normative concept that defines what is acceptable to qualified mathematicians" (1990, p. 6). On the contrary, the concept of *rationale* encompasses valid arguments for or against mathematical facts. However, in some respects, this is incomplete. In particular, an argument counts as a rationale rather than a proof if it does not explicitly refer to the accepted truths of a specific community; the argument is incomplete because it omits some of the necessary steps (G. J. Stylianides, 2009). *Empirical proof schemes* use physical evidence or sensory perceptions to prove, refute, or doubt hypotheses (Harel & Sowder, 1998). The paper folding process objectifies geometric shapes for instant sensory judgment. Moreover, rearranging the geometric elements by folding allows for experimental conclusions on geometric relations. However, using rigorous deductions to turn empirical argumentation into mathematical proof is sometimes challenging.

Method

Our research was based on the written homework of 59 prospective mathematics teachers (serial numbers S1-S59 in this paper). They have already completed introductory math courses, such as Euclidean geometry, and are assumed to have mathematical knowledge to mathematise the paper folding process. Data were collected in autumn 2020 and spring 2021.

The problem formulation is as follows:

From a sheet of A4 paper (ratio of sides is $1:\sqrt{2}$), cut the square with the maximum side as usual. Then, fold the two opposite vertices of the remaining rectangle to form a pentagon (Figure 1, left). Discover as many properties of the resulting pentagon as you can. It is unnecessary to list and prove properties that are true for all pentagons.

Figure 1 (right) shows the GeoGebra representation of the problem, which we will need to unify the notations in the future. For the sake of easy reference, we refer to the ADC' triangle as the "roof," MN will be the "base," D is the "top", and we refer to AMD and $C'ND$ triangles as "corner triangles." The folding line t is the perpendicular segment bisector of segment BD (diagonal of the rectangle), and C' is the reflection of C through t. The question of symmetry is whether vertices A and C' are symmetrical through the diagonal DB.

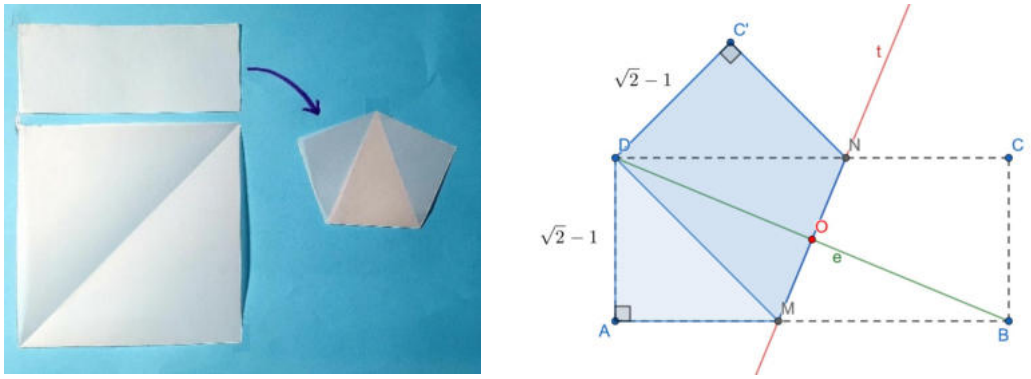

Figure 1: The problem representation: Paper folding (left); GeoGebra construction (right)

This open-ended task is a situation-based problem-posing task that simulates the complex process of mathematical activities. The geometrical problem has to be seen in a situation, mathematised, discovered and concluded with conjecture and proof or refutation.

We will only highlight two statements in this paper.

1. The pentagon is symmetrical.
2. The corner triangles are congruent isosceles right triangles, i.e., the pentagon has four equal sides.

The rationale behind the selection of these two properties is as follows:

Ad 1. We assumed that the pentagon's symmetry is an easily observable feature. Symmetry is one of the defining principles of form perception according to Gestalt psychology (Sternberg & Sternberg, 2012). In addition, brain research has suggested that humans are susceptible to symmetry. Bertamini and Makin (2014) asserted that the visual cortex processes symmetry automatically and sustainably. Therefore, we were intrigued to examine how students mathematically represent this visible characteristic, if they argue for it, and, if so, whether their argument extends beyond manipulative action.

Ad 2. Even though the isosceles triangles are symmetrical, they appear as parts of the pentagon in this situation. Therefore, the pentagon must be mentally or physically split into parts to deal with these triangles. For example, some students used backlit photographs to make this separation evident

(Figure 2, left). Although the two triangles at the corners are visible, in their photo, S58 only stated that they were congruent but did not claim that they were isosceles triangles. Thus, we assume that it is harder to perceive that the corner triangles are isosceles, but one is naturally curious about the length of these sides. We are interested in whether this researcher's attitude and interest will lead to a flawless mathematical investigation and proof.

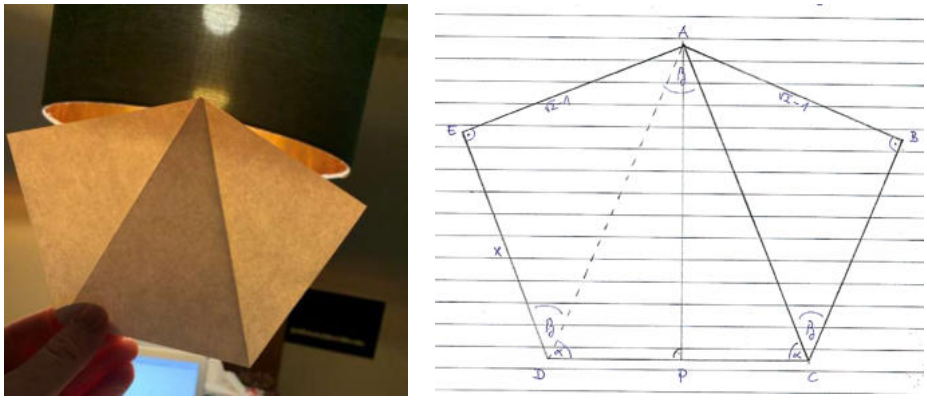

Figure 2: Problem representations: a photo included S58 in their work and a typical drawing (S41)

The two authors independently judged the proof schema for the student contributions, and we closed cases of disagreement by consensus.

Results and discussion

The high proportion of students identifying the pentagon as regular (10 of 59 students, 16.9%) was unexpected. We identified a lack of conceptual clarity regarding regular pentagons for four students. Moreover, a misinterpretation of the task also appeared, as six students did not refer to the folding situation. We excluded these students from further analysis, leaving a sample of 49.

Forming conjectures

Consistent with our hypothesis, most students (46 of 49 students, 93.8%) identified symmetry, whereas slightly fewer (39 of 49 students, 79.6%) formulated a statement for the corner triangle's property. However, the treatments for these properties differ significantly. Studying the students' figures, we can observe that the axis of symmetry is usually included in their figure, while the indication for the equal sides is usually not. A typical example is shown in Figure 2 (right). This figure captures the properties that follow from the folding: right angles, the length of the sides that make up the roof, and the corresponding alternate angles, which are marked in the drawing by β. The ED segment, a quantity to be calculated, is denoted by x. However, the axis of (assumed) symmetry is also included in the drawing. The student clarifies this in the explanation of the drawings:

S41: The pentagon is axially symmetrical. The axis of symmetry is the line connecting the midpoint of the DC side to vertex A.

This student will later calculate x using the Pythagorean theorem using the relation $AD = 1 - x$, but will no longer deal with symmetry.

Argumentation and proof schemes

Folding, i.e., applying the *empirical proof scheme,* can be used to argue for both properties. The elements of the original figure can be rearranged manually, making the claims convincing. This is obvious in the case of symmetry; see Figure 3 (left). Regarding the corner triangles, the folding shown in Figure 3 (right) is the most straightforward. From this fold, conclusions about the angles can be drawn, namely, the angles of corner triangles at the top are congruent, and they give a right angle together, so we have angles of 45°.

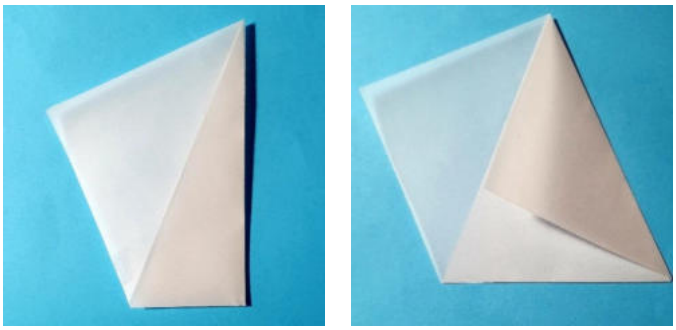

Figure 3: Demonstrating the symmetry (left); the acute angle of the corner triangle (right) with fold

To *prove* the properties of corner triangles, one can apply the Pythagorean theorem, as shown in Figure 5.

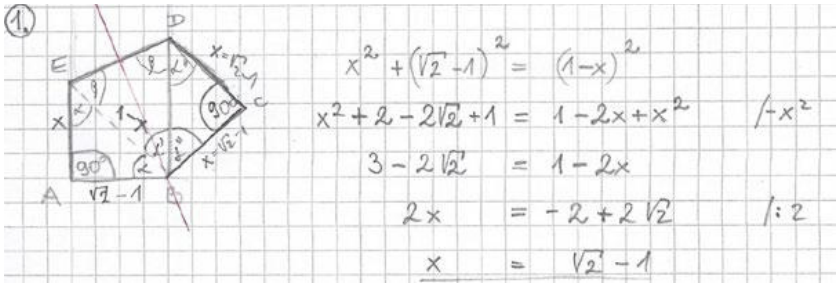

Figure 4. Drawing and calculation by S13

All of the work arguing for the pentagon's symmetry was incomplete to a greater or lesser extent, leaving it somewhere between proof and rationale. In four cases, however, the students' assumed and expected geometric knowledge easily compensated for missing arguments. We recalled these students' arguments, aligning their texts to Figure 1.

S50: AMD and $C'ND$ are congruent isosceles right-angled triangles with vertex D common and $DM = DN$. Thus, the shape is symmetric with respect to the angular bisector of the MDN angle.

Despite the student's proof's lack of details, it aligns with the group's expected geometric knowledge, as congruent triangles can be transformed into each other by the composition of axial reflections, and in the geometric situation under consideration, a single axial reflection is sufficient; see, for example, Theorem 19.9 of the textbook (Martin, 1975).

Student S33 made similar arguments. Their proofs necessitate a deeper understanding of abstract concepts of congruence and isometry in geometry.

S35: The two angles on the segment MN are equal. The two sides joined at the base are also equal. Thus, $AMNC'$ is a symmetrical trapezoid; in conclusion, the pentagon is symmetric.

The missing step in the proof is that the axis of symmetry of the trapezoid, which is the perpendicular bisector of the segment AC', has vertex D on it because it is equidistant from both A and C'.

S3: The two angles lying on the segment MN are equal. The two sides lying on the base are also equal. The roof's two sides are equal, with two opposite right angles. (These are all due to the way the folding is done.) Therefore, the resulting pentagon is axially symmetric.

S3's argument was similar to that of S35, but this student referred to the symmetric construction of the pentagon from two components. A possible necessary completion is that reflecting one side of the pentagon to the segment bisector of MN yields the other side. This reasoning is an example of the transformational argument (Simon, 1996): S3 dynamically imagines constructing the pentagon in successive steps. S3 starts at the base's endpoints and symmetrically builds the pentagon's two halves. Thus, the pentagon is not a static end state but the result of successive construction steps.

No rationale-type argument for corner triangles has been identified. Regarding symmetry, the most common example of the rationale is that the pentagon's symmetry follows the symmetry of the triangle *MND*, as shown in S16 work:

S16: I started with the isosceles triangle MND. The segment connecting the vertex D to the midpoint of the MN side divides the MND triangle into two congruent triangles. The straight line connecting points D and O is the axis of symmetry for both the triangle and the pentagon.

The argument's obvious flaw is that it does not examine whether A and C' are symmetric along the MND symmetry axis. This missing part of the proof presumably results from some mentally performed thought processes supported by visual perception.

Table 1 categorises the reasoning types students use when identifying properties of the pentagon.

Table 1. The types of reasoning used in the study

	Proof	Non-Proof argument		False argument	No argument	Sum
		Rationale	Empirical			
Corner triangle	17	0	7	2	13	39
Symmetry	4	6	5	0	30	45

After analysing the different types of argumentations, we were surprised that many students failed or missed clarifying the properties of the corner triangles. When examining the work of students who lacked an argument, we observed instances where the isosceles right triangle was held in a "typical position," i.e., with a horizontal hypotenuse (Figure 6). Some research suggests that typical positions, such as horizontal hypotenuse in our case, aid geometric shape recognition (see Chapters 3 and 8 of

Sternberg & Sternberg, 2012). Interpreted in this way, this result also supports our hypothesis that decisions based on visual perception adversely affect the recognition of the proof situation.

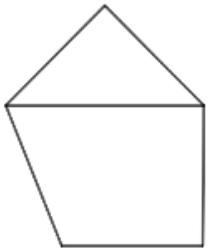

Figure 5. The isosceles right triangle in a typical position based on the photo by S8.

Conclusion

Both problems had many students who did not argue their claims. Visual perception contributed to certainty in both cases. This finding highlights the difficulty of transitioning from observable evidence to formal, deductive proof or even recognising the "need for proof" situation. Moreover, we identified a notable disparity in the treatment of corner triangle properties, where formal proofs were more prevalent. This disparity suggests that students found it easier to apply concrete, formal-symbolical mathematical methods, such as applying the Pythagorean theorem, rather than dealing with more abstract concepts, like symmetry.

Contrary to our expectations, the empirical proof scheme did not dominate any problems. Perhaps the reason may be that the teacher candidates were aware that the manual activity is not proof. As S21 notes, "By folding, an *informal deduction* can be used to prove that we have obtained an axially symmetric pentagon" [italicised by the paper's authors].

The pedagogical impact of the results presented in this article is achieved by reflecting on the experience gained during the task performance and evaluating it in the form of a meta-discussion with teacher candidates. Investigating this aspect is beyond the scope of this article and presents a topic for future research.

Acknowledgement

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A series of mathematical induction activities to promote de-ritualization processes

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In this paper, we suggest including opportunities for de-ritualization in teaching and learning activities of mathematical induction. De-ritualization is a term used in commognition theory which refers to the process of moving away from ritualized routines towards explorative routines in learning. This process plays a key-role in learning. After introducing the concept and theoretical background of de-ritualization we present a literature review on teaching and learning mathematical induction. Based on these ideas and findings, we developed a series of teaching and learning activities with the aim to advance de-ritualization and ultimately undergraduates' understanding of mathematical induction. We will outline this series and exemplify how they have potential to advance de-ritualization processes.

Keywords: De-ritualization, commognition, mathematical induction, teaching and learning activities.

Introduction

Commognition theory offers a socio-cultural view of mathematics teaching and learning, which has been emerging in recent years. In this theory, de-ritualization is a valuable process which can promote an understanding of concepts as it shifts the focus from repetitive, rule-based learning to a more dynamic, exploratory approach (Sfard, 2018). The main goal of this paper is to propose a sequence of teaching and learning activities to promote de-ritualization of mathematical induction. Teaching and learning activities, here, mean suggestions for tasks and teacher inventions.

We focus on mathematical induction (MI) because it is a powerful proving technique to improve problem-solving, persuasive argumentation, reasoning, creativity, mathematical thinking, and enabling mathematical communication (Güler, 2016). The application of mathematical induction tackles a wide range of mathematical problems, from proving fundamental identities and inequalities to establishing the correctness of algorithms and recursive structures. It is particularly valuable in developing theoretical foundations of computer science (Baker, 1996), which became very popular in recent years. Apart from its mathematical significance, Stylianides et al. (2007) argued that inclusion of proof methods is important in teacher education programs to provide them with insights into mathematical practice. They argue that in particular mathematical induction promotes conceptualization of preservice teachers' conception of proof and logical thinking. This paper presents a series of teaching and learning activities of mathematical induction promoting de-ritualization processes which we are planning to implement with a group of mathematics undergraduate students in Malang, Indonesia.

Theoretical background

De-ritualization in commognition theory

Commognition, a term that was first introduced by Anna Sfard, combining two words "communication" and "cognition", focuses on the social and discursive nature of thinking and learning. This theoretical framework views learning as a process of becoming fluent in a specific discourse, characterized by unique language or *keywords* (for instance, “three,” “triangle,” “set,” or “function”), *visual mediators* (e.g., numerals, algebraic symbols, and graphs), *routines (*repetitive patterns characteristic of the given discourse), and *endorsed narratives* (in mathematics, theorems, definitions, proof methods and computational rules, among others) (Sfard, 2008). Thinking as a form of communication is the basic assumption in this approach, and learning mathematics means participating in this special form of communication (Sfard, 2018).

Commognition theory in mathematics education examines how learners transition from everyday language to specialized mathematical discourse. Key processes include *ritualization*, where learners focus on procedures, and *de-ritualization*, where conceptual understanding and logical reasoning become the primary focus (Lavie et al., 2019; Sfard, 2007). Through this lens, thinking is conceptualized as the internalization of communicative processes (Sfard, 2018), and the classroom functions as a space where learners collaboratively develop and negotiate mathematical understandings (Sfard, 2008; Yackel & Cobb, 1996).

In commognition theory, ritualization and de-ritualization in the routines are important for meta-level learning. Ritualization is often the initial stage of meta-level learning, where learners imitate expert performances before they can fully understand the usefulness and success of their own actions, while in de-ritualization, learners start to actively participate in discourse, making persistent efforts to figure out the purpose and applicability of the discourse (Lavie et al., 2019; Sfard, 2018). These two parts of routines should be provided by teachers’ actions to achieve learning goals. Nachlieli and Tabach (2019) introduced two notions of opportunities to learn (OTLs) which identify and explain teaching goals: *ritual-enabling* OTLs, and *exploration-requiring* OTLs. The former OTLs are the teachers’ actions that primarily focus on practising established procedures and mimicking previously learned examples, and the latter OTLs are the ones that encourage students to engage in more active and independent learning (Nachlieli & Tabach, 2019). Exploration-requiring OTLs are considered crucial for developing a deeper understanding of mathematical concepts, fostering problem-solving skills, and promoting mathematical communication. However, these two OTLs are equally important as they can contribute to shifting students' focus towards exploration and problem-solving (Nachlieli & Tabach, 2019).

De-ritualization in teaching and learning of mathematical proof and the role of teachers

Several characteristics of de-ritualization have been discussed by Lavie et al (2019). These characteristics include flexibility, bondedness, applicability, agentivity, objectification and substantiability. In context of mathematical proof, according to Karavi et al. (2022), *flexibility* refers to presenting multiple approaches to the same proof, demonstrating that there are different paths to the same solution. *Bondedness* describes the links between conceptual understanding and the formal execution of steps in a proof. *Applicability* means how a specific proof strategy or concept can be applied in other mathematical contexts, showing broader relevance beyond the immediate problem. *Agentivity* refers to the extent to which students take an active role in the proving process, rather than

passively following algorithms. *Objectification* is the ability to treat mathematical concepts as objects that can be manipulated, rather than just procedures to be executed. Finally, *substantiability* is the degree to which students can provide meaningful explanations for each step in a proof, grounded in their conceptual understanding.

To obtain characteristics of de-ritualization, teachers play a crucial role in the classroom (Karavi et al., 2022; Valenta & Enge, 2022). Valenta and Enge (2022) provide a practical framework for achieving this, suggesting six practises to support students: (1) introducing new routines and guiding the interpretation of tasks, (2) promoting proving tasks that require exploration, (3) encouraging clear communication and challenging informal arguments, (4) facilitating the shift from empirical arguments to deductive reasoning, (5) emphasizing the significance of justification, pattern recognition, and the development of strategic knowledge, and (6) aiding students in recognizing structures, developing clear reasoning, and connecting various mathematical concepts.

Literature review on teaching and learning of mathematical induction

Mathematical induction is a powerful tool for proving patterns, identities, and algorithmic correctness, yet it remains a challenging topic for students (Baker, 1996; Telloni & Malara, 2021). Telloni and Malara identify four key difficulties: *logical*, *conceptual*, *technical*, and *psychological*. Students often misunderstand the sequence of logical implications that connect the base step to the induction step because they see the variable n simply as a generic fixed number, and do not reflect on the role of quantification. Many also struggle with algebraic manipulation. These challenges suggest that students frequently engage with induction as a procedural routine rather than a conceptual process. From a commognitive perspective, this reflects a ritualized mode of learning that hinders deeper understanding and flexible application.

Recent studies have proposed strategies to enhance students' conceptual understanding of mathematical induction. Telloni and Mallara (2021) identified four key activities: *exploration* and *conjecturing*, *small group work*, *proofs* and *refutations*, and a *role-playing game*. These activities encouraged students to generalize from patterns, collaboratively prove implications, analyze flawed arguments to grasp the structure of Principle Mathematical Induction (PMI), and teach peers to reflect on their understanding. Conducted with Computer Engineering sophomores in Italy, the study showed improved comprehension of PMI, especially in handling its complexity and appreciating its explanatory power. Students also developed stronger linguistic, argumentative, and reflective skills. The authors suggest this approach benefits not only students but also pre- and in-service teachers due to its focus on higher-order goals and argumentation beyond technical proficiency.

Relaford-Doyle and Núñez (2021) introduced visual proofs to investigate students' understanding of mathematical induction. Participants, both MI-trained and untrained, created tutorial videos explaining visual proof to a hypothetical audience, followed by reflective interviews. The study found that while MI-trained students had procedural knowledge, they often lacked conceptual depth, especially when applying induction across different representations. MI-untrained students struggled

with foundational concepts, such as the nature of natural numbers. The authors recommend incorporating visual proof into instructions to promote deeper conceptual understanding.

De-ritualization of mathematical induction: reframing teaching and learning activities

Based on the theoretical considerations above, we propose a sequence of teaching and learning activities consisting of three sessions, each 100 minutes long, to promote de-ritualization of mathematical induction. Table 1 illustrates our proposed sequence of teaching and learning activities. We propose two key shifts in the framing of teaching and learning activities. First, we advocate for reducing the scale of ritual-enabling OTLs that overly focus on the procedural aspects of induction proofs. Secondly, we propose incorporating exploration-requiring OTLs that foster deep engagement with the various roles of proof, such as verification, explanation, systematization, and discovery. Below we describe three of our suggested activities in detail (oral presentation and reflection from Session I, proof evaluation and peer review from Session III). We will explain how these activities have potential to support de-ritualization in mathematical induction.

Table 1: Teaching and learning activities

Session I (Introduction to MI)	Session II (Consideration of base cases)	Session III (Consideration of nature and features of MI)
(1) Students explore a visual pattern individually, discuss and make a conjecture about the sum of the first n natural numbers in groups.	(1) Students recall what they have learned in Session I.	(1) Students explain individually in writing for each of the steps in MI why they are necessary.
(2) Instructor explains the idea of MI using visual analogies (not showing an example proof).	(2) Students identify base cases for three example statements.	(2) Proof evaluation exercise (see below for detail)
(3) Students work in groups to prove their conjectures from (1).	(3) Class discussion about importance of careful consideration of base cases.	(3) Proof construction exercise.
(4) Presentation and reflection (see below for detail).	(4) Students work in groups to outline their strategies or proof ideas to prove one of the statements from (2). (5) Group work: students develop their ideas from (4) into formal proofs.	(4) Peer review activity (see below for detail)
(Closure Activity) Students discuss and summarize what they have learned in Session I.	(Closure Activity) Students summarize what they have learned in Session II.	(Closure Activity) Students summarize what they have learned in Session III.

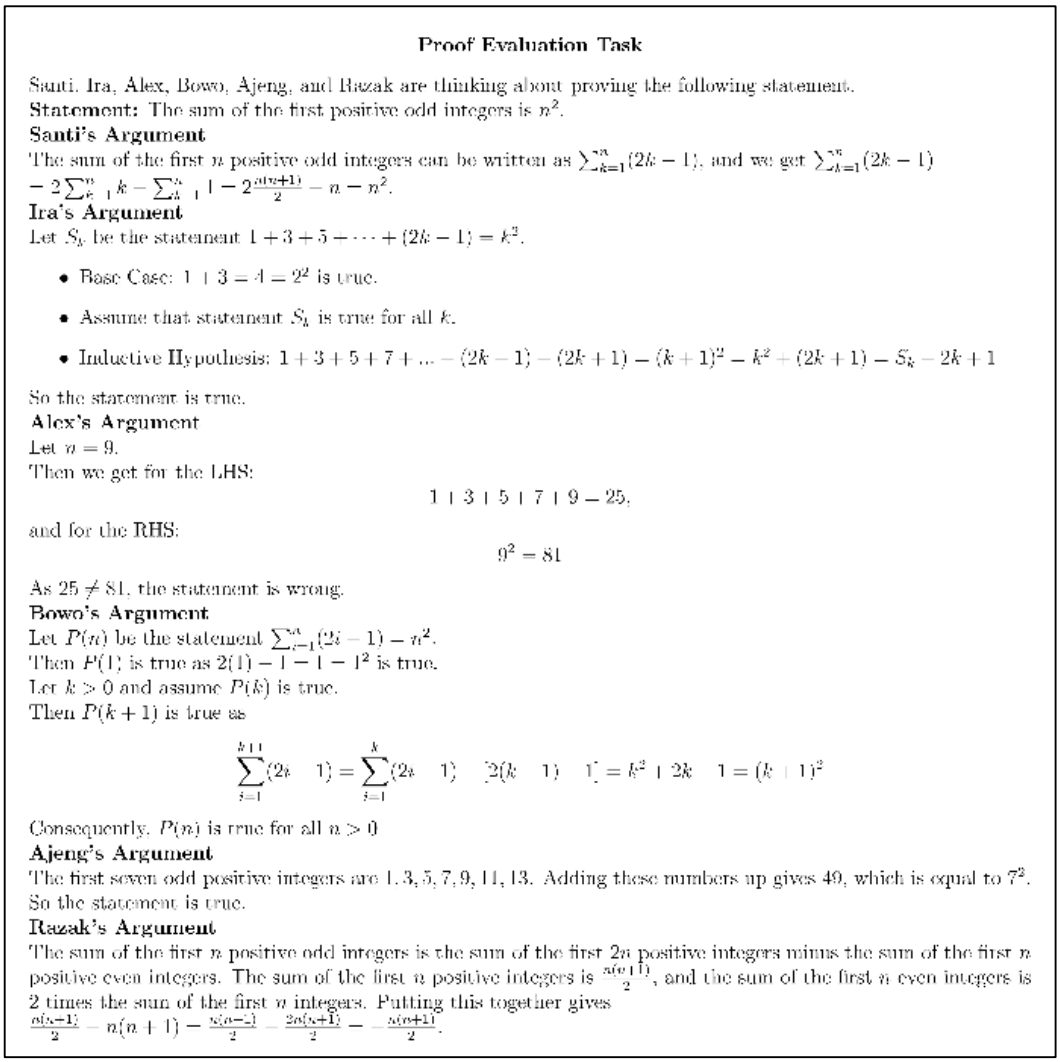
Inspired by Kontorovich & Greenwood (2023), we include oral presentation and reflection activities. After students prove their conjectures and plan their presentations in groups, some delegates present their proofs. This activity involves students writing and explaining their proofs without much interaction with the rest of the class during the presentation. Presenting their ideas allows students to validate and refine their proofs in a public setting. This activity encourages students to engage in practices similar to those of the mathematics community (Kontorovich & Greenwood, 2023). When students present their proofs, they take ownership of their reasoning process. Students move beyond ritualized execution to communicate the reasoning behind the base case, inductive assumption, and inductive step, which is the essence of de-ritualization. After presentations, students engage in a posteriori reflection to discuss the presented proof. The instructor then provides feedback, highlights any issues, and offers suggestions for improvement. These reflection activities help deepen students'

understanding of the concept of mathematical induction. By discussing their proofs, students see different perspectives and approaches, which is enriching their learning experience, and encouraging them to think critically about their work and the work of others. Reflection activities in Session I prepare students for the more advanced collaborative tasks of proof evaluation and peer reviewing in Session III.

For the proof evaluation task, students will be presented with a variety of approaches to prove a statement, of different nature with some of them being flawed, see Figure 1. Students are instructed to collaboratively: a) observe and compare the arguments, b) rank the arguments in order of their preference and include some comments to explain the ranking, c) include comments about the arguments, and d) find any errors in the arguments and advise to resolve the errors. As suggested by Pfeiffer & Quinlan (2015a), the six arguments provide a sufficient variety of different approaches to provoke learning and discussion about the nature and features of mathematical proof and about the process of proof evaluation. In our proof evaluation task, Santi's argument and Razak's argument are sufficient to prove the statement, but not using mathematical induction. Ira's argument has several errors: the base case is wrong, too much is assumed, S_{k+1} is used in an effort to prove it, and S_k is used as a value. Alex's argument interprets n incorrectly as the last odd number to be added in an effort to provide a counterexample. Bowo's argument is a valid proof using mathematical induction. Ajeng's argument is an example and does not prove the statement.

There is no "right answer" for the proof evaluation task. Incorporating a request for students to rank the proofs according to preference encourages them to evaluate the subjective qualities of each proof and recognizes the diversity of mathematical perspectives (Pfeiffer & Quinlan, 2015b). By ranking the different proposed arguments, students consider and discuss criteria that make a good proof. This activity encourages students to reflect on purposes of mathematical proofs, including how they establish truth and enhance understanding. When students discuss and compare arguments in the task, they gain insight into the structure and function of each step in mathematical induction and the importance of these steps, why these steps are missing and what happens if any part is missing, incorrect, or unclear. By critiquing flawed arguments, students may notice cases where induction is misapplied, such as incorrect base case and overextending the assumption which can make the argument invalid. By comparing different arguments, students see that there may be multiple valid ways to approach and prove statements. Proof evaluation activities address several goals of de-ritualization, to enhance students' understanding of mathematical induction, to foster their critical thinking, and to encourage a more exploratory and reflective approach to mathematical reasoning.

When implementing a proof evaluation task, the instructor has multiple roles (Pfeiffer & Quinlan, 2015b). One is to supply the students with a selection of proposed proofs that is varied and appropriately challenging. This is similar to one of practices suggested by Valenta and Enge (2022), namely aiding students in recognizing structures, developing clear reasoning, and connecting various mathematical concepts. Other roles for the instructor are to facilitate a discussion without fully directing it, to assess the students' work and to consider useful feedback to provide to the group or to individuals. Additionally, the instructor should consider other responses based on their assessment of the students' understanding and competence in mathematical proof.



Proof Evaluation Task

Santi, Ira, Alex, Bowo, Ajeng, and Razak are thinking about proving the following statement.
Statement: The sum of the first positive odd integers is n^2.

Santi's Argument

The sum of the first n positive odd integers can be written as $\sum_{k=1}^{n}(2k-1)$, and we get $\sum_{k=1}^{n}(2k-1)$ $= 2\sum_{k=1}^{n} k - \sum_{k=1}^{n} 1 = 2\frac{n(n+1)}{2} - n = n^2$.

Ira's Argument

Let S_k be the statement $1+3+5+\cdots+(2k-1)=k^2$.

- Base Case: $1+3=4=2^2$ is true.
- Assume that statement S_k is true for all k.
- Inductive Hypothesis: $1+3+5+7+\ldots-(2k-1)-(2k+1)-(k+1)^2-k^2+(2k+1)-S_k-2k+1$

So the statement is true.

Alex's Argument

Let $n=9$.
Then we get for the LHS:

$$1+3+5+7+9=25,$$

and for the RHS:

$$9^2=81$$

As $25 \neq 81$, the statement is wrong.

Bowo's Argument

Let $P(n)$ be the statement $\sum_{i=1}^{n}(2i-1)=n^2$.
Then $P(1)$ is true as $2(1)-1=1=1^2$ is true.
Let $k>0$ and assume $P(k)$ is true.
Then $P(k+1)$ is true as

$$\sum_{i=1}^{k+1}(2i-1)=\sum_{i=1}^{k}(2i-1)+2(k+1)-1=k^2+2k+1=(k+1)^2$$

Consequently, $P(n)$ is true for all $n>0$

Ajeng's Argument

The first seven odd positive integers are $1, 3, 5, 7, 9, 11, 13$. Adding these numbers up gives 49, which is equal to 7^2. So the statement is true.

Razak's Argument

The sum of the first n positive odd integers is the sum of the first $2n$ positive integers minus the sum of the first n positive even integers. The sum of the first n positive integers is $\frac{n(n+1)}{2}$, and the sum of the first n even integers is 2 times the sum of the first n integers. Putting this together gives $\frac{n(n-1)}{2}-n(n+1)-\frac{n(n-1)}{2}-\frac{2n(n+1)}{2}-\frac{n(n+1)}{2}$.

Figure 1: The proof evaluation task

The peer review exercise is scheduled after students prove the statement " For all $n \geq 1, n \in \mathbb{N}$, $1+4+7+\cdots+(3n-2)=\frac{n(3n-1)}{2}$ " collaboratively in groups. Students exchange their proofs with those from other groups for peer review. This task includes writing the other groups' proof idea, to justify whether all steps of the proof are correct, and to give feedback to the other groups. We choose peer review because it encourages students to articulate their understanding and critique their peers' work constructively. As far as we know, this approach is relatively rare in teaching mathematical induction. The activity reinforces students' ability to identify and correct errors, a skill that helps them internalize the structure of induction proofs. According to Telloni and Mallara (2021), the process of validating another student's proof forces them to think through the logic step-by-step, which justifies their own understanding. Writing constructive feedback also helps students think about proof-writing as a communication skill, which is another novel aspect for mathematical induction lessons. According to characteristics in Karavi et al. (2022), peer review tasks naturally

encourage key features that support de-ritualization such as bondedness, helping students make connections between different mathematical ideas within the proof, and agentivity, promoting the active role of students as evaluators and creators of mathematical arguments, thus enhancing their agency in the discourse. Lavie et al. (2019) emphasize that de-ritualization involves changes in students' routines, where their focus shifts from mimicking established methods to actively constructing and validating mathematical arguments. Peer review activities encourage such shifts by requiring students to justify critiques and suggestions, promoting deeper engagement with the proofs.

In the peer review activity, the instructor's role is to facilitate collaboration and open dialogue. The instructor demonstrates how to evaluate proofs constructively, focusing on logical structure, clarity, and reasoning. This practise aids students in recognizing structures, and developing clear reasoning as suggested by Valenta and Enge (2022).

Closing Remarks

Karavi et al. (2022) claimed that the scholarly exploration of instructional approaches aimed at fostering students' de-ritualization of mathematical proving practices remains underexplored. While some alternative approaches (Relaford-Doyle & Núñez, 2021; Telloni & Malara, 2021) emphasize conceptual understanding and student involvement in mathematical induction learning, these studies do not specifically explore de-ritualization in the teaching and learning of this material. In this paper, we address this gap and propose a series of teaching and learning activities to promote de-ritualization of mathematical induction. Going forward, we plan to implement these teaching and learning activities with a group of first-year mathematics undergraduate students at Universitas Negeri Malang, Indonesia, and to use commognition theory to identify de-ritualization processes and to analyze their effect on students' learning and understanding of mathematical induction.

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Debating the machine: Mediations and truth

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In this paper, we examine the relationship between disciplinary debate and mathematical discussion. The advent of generative AI has revolutionised the field of education, bringing about changes that are both profound and irreversible. A key skill that will become increasingly necessary for students in the future is the ability to evaluate AI-generated output. We have previously introduced an activity model—rooted in Engeström's Expansive Learning theory—designed to address this need. In this study, we focus on the dialogical components of the model, particularly discussion and debate, analysing their similarities and differences in light of the concept of mediation, with special attention to the role of teachers.

Keywords: Debate, artificial intelligence, mediation, discussion, argumentation.

Introduction

Pierre Rabardel begins one of his most renowned works (Rabardel, 2002) with a joke. The joke goes that in the cockpits of future airplanes, there will only be two seats: one for a human and the other for a dog; the dog's job is to prevent the human from touching the controls; the task of the human is to feed the dog. With the advent of powerful Artificial Intelligence Large Language Models (AI LLMs), there's a chance that in the aircraft of the future there will be no seats at all, and that risk will not only concern airplanes. In fact, the real risk in an AI-dominated future is not the one depicted in many apocalyptic science fiction movies featuring killing machines: the AI will not make us dead, although it may make us irredeemably lazy. But this scenario—featuring a humanity uncritically, irreversibly, and completely abdicating its decision-making role—is not inevitable.

AI is essentially a guessing machine, a piece of software that tries to give you an answer as close as possible to answers that were given to similar questions (Tao, 2024). So, how is it possible to use a tool that is very powerful yet not entirely reliable? We should devise a way to independently check its answers. Speaking of mathematics, in academic research one possibility is to pair the AI LLM with a proof assistant software (Tao, in print), but in an educational setting this might not be a viable solution. As optical illusions—such as the Müller-Lyer illusion, the Zöllner illusion, the Poggendorff illusion etc.— in challenging intuitive understandings of space and shapes, are useful in letting students understand the necessity of formal geometric reasoning and proofs, the very fact that AI-generated answers to mathematical questions—whether theorem proofs or problem-solving procedures—might look right even if factually wrong can let students understand the need for a consideration of the correctness of the solution itself. The difference between these two situations is that in the first case the subject of this pondering are properties of a mathematical object, in the latter case we are talking about the properties of an argument or a chain of arguments.

Mechanically copying patterns does not generate understanding, as Searle's Chinese Room thought experiment predicted and generative artificial intelligence tools have confirmed. This is especially evident in mathematics, aligning with the belief that mastery of mathematics does not consist merely of knowing certain definitions and possessing procedural skills (cf. Baccaglini-Frank et al., 2017). However, since Plato's era, explanatory dialogue has been regarded as a crucial tool for

preventing the pitfalls of false knowledge and highlighting the non-mechanical and non-imitative nature of knowledge (cf. Piazza, 2020), hence the idea of using dialogical activities to problematise the assessment of AI-generated answer in class. One of the difficulties in orchestrating such an assessment with students is that—unlike what happens in ordinary mathematics teaching—when dealing with AI, teachers cannot act as experts holding institutionalised knowledge: in this regard they are on the same level as their students. Nor can they make up for it with training courses, because the current rapid development of the AI sector makes it impossible to bridge this gap in a timely manner. Our research group has already begun investigating the possibility of applying dialogical methods, including debate (Matteucci & Tortoriello, 2023), to the evaluation of AI-generated answers (Matteucci et al., 2024a; Matteucci et al., 2024b). Notably, while most classroom activities in formal educational contexts that can be classified as debates consist of competitive debates (Akerman & Neale, 2011), unless otherwise stated, we will use the term 'debate' to mean disciplinary debate and metamathematical debate in particular—that is, briefly, a debate focused on the hows and whys of doing mathematics (for more details cf. Matteucci & Tortoriello, 2023). With this distinction clarified, we now provide context and details on applying debate to AI answer evaluation.

Theoretical framework and outline of the proposed model

The dystopian scenario hinted at in the introduction above can be considered the eventual limit outcome of what Rabardel (2002) called the technocentric approach: a perspective where human involvement in technological systems is seen as secondary or even undesirable, therefore characterising human activity as a mere residue. The antithetical approach to the technocentric approach, which is the anthropocentric approach, "places human activity at the heart of analysis and thus allows the reversal necessary so as to speak of things in terms of people" (Rabardel, 2002, p. 5). Of course, Rabardel writes about human activity in a highly technological environment—he is no luddite—where humans have to interact with tools, tools that—to use Rabardel's terminology—are no mere artefacts, i.e. objects considered independently of their use, but instruments—that is artefacts that have been integrated into human activity and have acquired a specific function through their use. In short, an instrument is an artefact—physical or symbolic—in use, mediating the users' interactions with their environment, helping them achieve their goals.

Rabardel's notions of artefact and instrument originate from the concept of cognitive artefact by Donald Norman. In the book where this concept is expounded, Norman (1993) also describes the difference between experiential and reflective cognition. As their names imply, the first is based on experience, immediate, intuitive, it is used in routine situations that do not require analytical thinking; the second one involves conscious thought and reasoning, it is used in planning, learning or problem-solving situations. Of the dangers that can arise in an activity that misunderstands the above difference, Norman says that the one that "poses the greatest peril today is that of experiencing when one should be reflecting" (1993, p. 27)—incidentally, this is exactly the risk we hinted at in the introduction. In building upon Norman's concept of cognitive artefact describing how an artefact can become an instrument, Rabardel (2002) also introduces the concepts of instrumentalization and instrumentation: the first refers to the process by which an artefact is modified or adapted to better fit the users' needs within an activity, while the second describes how

users develop the necessary cognitive and operational schemes to effectively use the artefact as an instrument.

Phase	Sub-phase	Description
Chatting	1. Problem Selection	The teacher selects a mathematical problem to be solved by the AI.
	2. Problem Input	The teacher inputs the problem into the GAI chatbot, requesting detailed explanations of the steps used to reach the solution.
	3. Solution Distribution	The teacher sends the problem along with the AI-generated solution to the students.
Discussing	4. Critical Review	Students critically review the solution at home, following a predetermined list of criteria: correctness, completeness, effectiveness, generalizability, clarity, understandability.
	5. Classroom Discussion	Students discuss their assessments in class under the teacher's supervision to establish consensus on the conformity of the solution to the aforementioned criteria.
	6. Motion Drafting	The teacher drafts precise debate motions based on the unresolved controversial issues from the discussion, using the metamathematical motions' taxonomy.
Debating	7. Debating	Students, divided into teams, debate the motions prepared by the teacher according to a specific protocol.
	8. Debriefing	The teacher debriefs the students on the debate results, highlighting strengths and weaknesses in reasoning and argumentation.
	9. Final Discussion	Students and teacher discuss the lessons learned from the debate and debriefing, concluding with a metacognitive reflection.

Table 1: Model of the proposed activity

Expanding on Rabardel's work about instruments and their genesis, Bartolini Bussi's theory of semiotic mediation (Bartolini Bussi & Mariotti, 2008) delves deeper into how instruments become instruments of thought in educational settings. She argues that instruments are not just objects or symbols but are embedded within a broader semiotic system where meaning is co-constructed through interaction. In this framework, the teacher plays a crucial role in guiding students' use of tools, helping them transitioning from external, concrete interactions with artefacts to internal, abstract mathematical reasoning. This process of semiotic mediation, therefore, explicitly links the development of cognitive schemes to the communicative and social dimensions of learning in the classroom, within a Vygotskian framework.

Speaking of Vygotskian frameworks, it is useful for our purposes to underline the difference between action and activity, as exemplified by Engeström and Sannino (2017) expanding on an example by Leont'ev:

> In a tribal hunt, for example, certain participants chase the game away, toward other participants who wait in ambush and kill the game. These two groups perform different actions (chasing, killing) in the collective activity of hunting. The half-life of an action is finite; an action has a definite beginning and an end. A collective activity, on the other hand, reproduces itself without a predetermined endpoint by generating seemingly similar actions over and over again. Yet there is continuous and at times dramatically discontinuous change in the activity. The very idea of expansive learning is built on this theoretically consequential *distinction between action and activity*. Expansive learning is movement from actions to activity. (p. 105)

Building on this distinction, we already proposed an activity model, built up on dialogical actions, designed to foster students' critical engagement with AI-generated solutions to mathematical questions, shifting focus from product-oriented learning to process-oriented understanding (Matteucci et al., 2024a; Matteucci et al., 2024b). In our usage, the term "dialogical" refers to any interaction between subjects that involves an exchange of responses with a back-and-forth dynamic. While such interactions are typically conceived as oral exchanges, they are not in principle limited to this form and may also occur in written formats, such as digital chat. The aforementioned model is summarised in Table 1.

In this model, the overall activity—*Tätigkeit*, to use the German word that best conveys the meaning of activity as collective interaction aimed at achieving a collective goal, instead of simply being busy with something to do (Radford, 2021)—involves the use of different "dialogical tools" that can be considered instruments in Rabardel's meaning. These instruments are both discussions, in subphases 5 and 9, and debates, in subphase 7. Debates are to be intended as metamathematical debates (Matteucci & Tortoriello, 2023; Giangrande & Matteucci, 2021), but how are they related with the discussions? Moreover, can these discussions here be considered compatible with the concept of "mathematical discussions" described by Bartolini Bussi (1996)? Because while the latter entail facilitating students' construction of object meanings through the use of signs, in this particular case, the meanings we are considering—e.g. the clarity of a proof—are not those of mathematical objects *per se*, nor is the presence of corresponding signs for them guaranteed.

Thus, the question we are posing is: how can discussions in the proposed activity model be properly understood in relation to debate? And what is the role of teachers in this context?

Protocols, dialectics, and teachers

According to Bartolini Bussi (1996), in a Vygotskian approach, the social construction of knowledge in classroom activity:

> is realised, on the one side, by the format of discussion, that gives value to joint activity in mathematical experience and, on the other side, by the systematic recourse to tools for semiotic mediations [...] hence mathematical discussion and semiotic mediation seem to be the two sides of a same coin both theoretically and experimentally (pp. 37–38)

The overall process can be conceived as an iteration of "didactical cycles" (Bartolini Bussi & Mariotti, 2008) each of which consists of: a) activities with artefacts, b) individual production of signs, c) collective production of signs; step c) is where mathematical discussion plays a crucial role. In this respect, if one examines the model in Table 1, one can see a parallel between its first phase, subphase 4, subphase 5 and steps a), b), c) above. There is one important difference, however. In the case discussed by Bartolini Bussi and Mariotti the teacher exploits the semiotic potentialities of the use of a given artefact to foster the students' move towards the learning of the properties of a given mathematical *object*, while in our model it is the properties of a mathematical *process* that teachers and students are interested in.

If one considers metamathematical debate as a tool of semiotic mediation, one can also draw an additional parallel between the aforementioned steps a), b), c) of the didactical cycle and subphases 7, 8, 9 of the model in Table 1. It is important to recognise that, also in this case, the intended mediation is different. To explain how this latter mediation is also different, consider that in the framework of Bartolini Bussi and Mariotti (2008), teachers mediate between students and mathematical knowledge using artefacts as tools of semiotic mediation. This mediation is typically intended as a bridge between the personal meanings of the students and the institutionalised meanings of mathematics as an academic discipline, and this evolution of signs, with respect to the students, towards mathematical signs is akin to the institutionalization process as found in Brousseau (2006). Conversely, in the case of the mediation during debate and debriefing actions in subphases 7, 8, 9 of the model, this mediation does not focus primarily on the abstract mathematical objects that can be found in the various branches of institutionalised mathematics. It mediates, instead, mainly between the personal experience that students have of debates and discussions (intended in a broad sense) and debate as a regulated dialectical practice. It also mediates between students' personal beliefs about how and why they engage in mathematics and the reflections on the how and why one does mathematics, according to the professionals' practice, that is the object of the so called philosophy of mathematical practice (cf. Mancosu, 2008).

The differences between these kinds of mediations and the typical mediations of mathematical discussions, and also the differences of these two kind of mediations between themselves allow us to further clarify the status of metamathematical debate—and disciplinary debate, understood in a broad sense—versus both competitive debate and mathematical discussion. First, although competitive debate can exist in isolation as a 'team game', disciplinary debate differs from it primarily because it is necessarily embedded within a broader activity—exactly as, to elaborate further along the lines of Leont'ev's example, although archery as a sport can exist in isolation, archery as a real life practice makes sense only within a broader context (e.g. hunting, war). Second, debate not only does not overlap with the concept of mathematical discussion but, in fact, within the discussion/tool dichotomy illustrated in the previous quotation by Bartolini Bussi (1996), it takes on the role of a tool. In which way can debate act as a tool or, to use a more precise term, as an instrument? Since debate, as we said above, can mediate between both the beliefs about how disputes are conducted and how and why mathematics is done, it does this with respect to its two constitutive aspects, respectively: 1) protocol, 2) dialectic.

A protocol is the set of rules that governs how a debate is conducted. It contains indications related to timing, speaking order, rules of engagement, cross-examination, moderation. Although teachers

will mainly act as moderators during the debate, their role in devising, adapting, and enforcing the protocol will in turn create the action that shall act as a mediating tool with respect to the belief about how disputes are conducted. Dialectic is a mode of argumentation theory that starts, typically, from probable premisses, instead of principles or axioms, and differs from both demonstrative reasoning—insofar as the latter is based only on deductions from certainties—and rhetoric—which does not rely exclusively on rationality but also on authority and emotion, the famous Aristotelian triad of *logos*, *ethos*, *pathos*. Keeping track of the structure of the proposed arguments during debate, and comparing their relative effectiveness, teachers will be able to prepare a debrief (subphase 8) that along with the debate proper (subphase 7) will be the tool mediating the conclusive reflection of the whole activity (subphase 9).

Speaking again of the role of teachers, we see that while in competitive debate they act mainly as moderators, here their role as mediators is paramount throughout the activity as a whole. This is due to the peculiar nature of the disciplinary debate described above. Plato, in the seventh book of The Republic, already warned about the risks of the uncritical use of dialectic by the youth, especially when done for amusement rather than in the pursuit of truth (Plato, 2013). Here, the teacher does not simply act as a referee but as a guarantor—not of the final results, which by their very nature cannot be institutionalised à la Brousseau, but of the intentionality of the entire activity in the search for truth, understood as the collective result of a process of applying mutually agreed-upon criteria of rationality to ongoing open questions.

Conclusions and perspectives

With this paper we aimed to better situate debate within the landscape of research in mathematics education, elucidating its status *vis-à-vis* other established dialogical practices—i.e. dialogical discussion. It emerged that disciplinary debate is not only non-conflatable with mathematical discussion, but also serves as an essential complement for it in contexts where *lato sensu* metamathematical issues are to be addressed.

Moreover, we contributed to clarifying the role of teachers in disciplinary debate, underlining the fact that they act not only as 'judges'—as it is customary in competitive debate—but also as mediators, and the mediations they are enacting are typically different from what usually happens in the majority of mathematical discussions as there is no single institutionalised reference to aim for. The emphasis in the previous sentence is on the word "single". The point being that, while there is a single right answer to the question about, say, what is the sum of the internal angles of a triangle in Euclidean geometry, there is no single codified right answer to questions such as: what is the most explicative proof of Gauss' formula for the sum of the first n positive integers or what is the worst error one can make in solving a quadratic inequality. Nonetheless, the fact that there is no single institutionalised answer does not mean that there are none—even institutionalised.

After having answered the questions about the relationship between discussion and debate and what is the role of teachers in this context, we are now poised to shift our focus toward briefly exploring the broader implications and future avenues of inquiry that these results envisions.

We recently completed the design of a pilot experiment (Matteucci et al., 2024b), which will be conducted in a secondary school next autumn. This pilot test aims to evaluate the feasibility and

effectiveness of our proposed model and will take place within the framework of the Liceo Matematico (Mathematical High School) research project (Rogora & Tortoriello, 2021).

Empirical testing, of course, is not the only perspective opened by the present work. There is still room for theoretical research, starting from these conclusions. Topics that deserve to be studied include, for example, how to link the concept of instrumental genesis to debate seen as a tool, and how to detail the selection and refinement of its protocol through the processes of instrumentation and instrumentalisation (Rabardel, 2008). Also, considering the specific application of the proposed model to the evaluation of AI-generated answers to mathematical questions and not only the more general application of metamathematical debate, another issue worth researching is modifying the model to include the analysis of how to modify the prompts to obtain better results and, eventually, delegate the first phase of the activity to the students themselves.

In his famous paper titled "On Proof and Progress in Mathematics," William Thurston, discussing what he considered the ultimate goal of mathematicians' work, wrote: "What we are doing is finding ways for *people* to understand and think about mathematics. The rapid advance of computers has helped dramatize this point, because computers and people are very different." The advent of generative AI has made this goal even more relevant. We maintain that the activity proposed in our model is a small step towards achieving it.

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Proof awareness and students' proof evaluations

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In this paper, we introduce the construct of proof awareness conceptualized as a complex affective system including also meta-affective dimensions about experiencing proof: self-efficacy, views, certainty, and emotions. As an exemplar, we operationalize this theoretical perspective to investigate the proof awareness of high school students (Grade 10) in Greece as they evaluate different proving statements in an algebra task. Indicative results of the empirical study are discussed.

Keywords: Proof awareness, affective system, high school, algebra.

Setting the scene

Proof appears to be the validity gatekeeper of modern mathematics. Notwithstanding the fact that school mathematics qualitatively differs from the mathematics discipline, proof holds a central role at school, even as a potentiality. Considering the complexity that mathematical proof entails and the complex mathematics classroom system, researchers have investigated aspects of the students' conceptualisations of proof and their divergences from the desired qualities. For example, considering the validation function of proof (Zaslavsky et al., 2012), the students face difficulties discerning the complex relationship between mathematical verification, which is based on a theoretical framework, and common-sense verification, which is based on empirical confirmation (Mariotti, 2006), thus being sceptical about the way that a deductive proof secures the lack of counterexamples (Chazan, 1993). The students appear to strive to realise the need for producing a proof, or even when they do, they are uncertain about the logical process to follow, the steps they need to perform, whether they have successfully completed it, or, in cases that they may construct an institutionally acceptable proof, they may remain unsure about the generalization of a proof or whether further empirical checks are needed (Herbst & Brach, 2006). Considering the fact that the learning outcomes of an educational system is strongly related to the assessment process (Boud & Falchikov, 2007), the students appear to experience a chasm between proofs they believe will receive better grades and proofs that the students themselves would adopt (Healy & Hoyles, 2000; Moutsios-Rentzos & Korda, 2018).

Raising awareness about crucial aspects of mathematics practices is essential to developing crucial qualities of mathematical thinking. Moreover, a central goal of mathematics teaching is to improve and enhance students' awareness concerning the types of arguments and proofs that are acceptable for validating various statements (Tabach et al., 2010). Following these, in this paper, we introduce the notion of proof awareness and we discuss aspects of an empirical study investigating the proof awareness of Grade 10 students when evaluating proving statements about an algebraic task.

Towards a conceptualisation of proof awareness

The importance of teachers' awareness in the teaching and learning mathematics has been stressed by various researchers (Mason, 1998; Potari, 2022). Mason (1998) notes that awareness is "a complex

concept comprising both conscious and unconscious powers and sensitivities which enable people to act freshly and creatively in the moment" (p. 243). Mason continues by drawing upon Gattegno (1987) to identify three levels of awareness:

> *awareness-in-action* (the powers of construal and of acting in the material world); awareness of awareness-in-action, or *awareness-in-discipline*, which enables articulation and formalisation of awarenesses- in-action, and is closely linked to one form of shift of attention; awareness of awareness-in-discipline or *awareness-in-counsel*, which is the self-awareness required in order to be sensitive to what others require in order to build their own awarenesses-in-action and -in-discipline. (Mason, 1998, p. 256)

Thus, the role of *attention* is crucial in one's being aware of something and qualitatively differs from one's perceptual and/or mental ability to perceive something. For example, a medical doctor may draw life-saving conclusions that a non-medically trained person cannot, although both apparently experience the same perceptual reality. Thus, awareness may be linked with the phenomenological notion of *intentionality*, which refers to "the conscious relationship we have to an object," thus implying that every "intending has its intended object" (Sokolowski, 2000 p. 8). Following these, *noema* (the phenomenologically described given object) is differentiated from *noesis* (the intentional activity). Hence, awareness may be conceptualized as an intending activity, as the experienced lifeworld is determined by the ways the intending being is related with the intended object.

Learning has also been strongly linked with the affective domain. For example, emotions have been found to be related with mathematical achievement and learning (Kleine et al., 2005). Thus, awareness is associated with the interaction between cognitive and emotional aspects, grounded in one's beliefs, abilities, and level of understanding (Hannula et al., 2004). Considering the interplay of conscious and pre-conscious affective states in learning situations, Malmivuori (2001) emphasised the role of self-regulation and self-motivation and the dynamic links between affective arousal and states and cognition within specific social mathematics learning situations. Regarding proof and proving, Selden et al. (2010) distinguish emotions from feelings ("with the former being physical and the latter being mental," p. 202) and they suggest that nonemotional feelings play an important role in the students' behavioural schemas ("habits of mind, that appear to drive many mental actions in the proving process," p. 199) when they are engaged in a proving process.

Furthermore, learning mathematics within a school environment is largely a goal-oriented activity, related with the results of the assessment process (e.g., grades). Skemp (1979) posited a theory about goal-directed thinking identifying two 'realms of consciousness': a) actuality, the "environment in which our physical actions and activities take place" (p. 21), and b) inner reality, the person's 'mental realm' of "individual experiences [...] which can also take place independently of sensory input from actuality" (p. 21). He argues that "many, perhaps most, of our actions are systematically directed toward bringing about goal states" (p. 105). Skemp explicitly discusses the role of emotions as they "give information about progress, or ability to progress, relative to goal states and anti-goal states" (Skemp, 1979, p. 18) set by an individual. Focussing on proof, Moutsios-Rentzos (2009, 2015) drew upon Skemp's theory to propose a topological organisation of the multidimensional affective and cognitive broader and task-specific processes operating in an exam-type proving question situation

to identify and give meaning to the links amongst: general affective/thinking learning dispositions (such as attitudes or cognitive styles), domain-specific affective/thinking learning dispositions (such as mathematics attitudes or mathematics learning style), and actual task-specific affective/thinking experiences (such as emotions or thinking strategies).

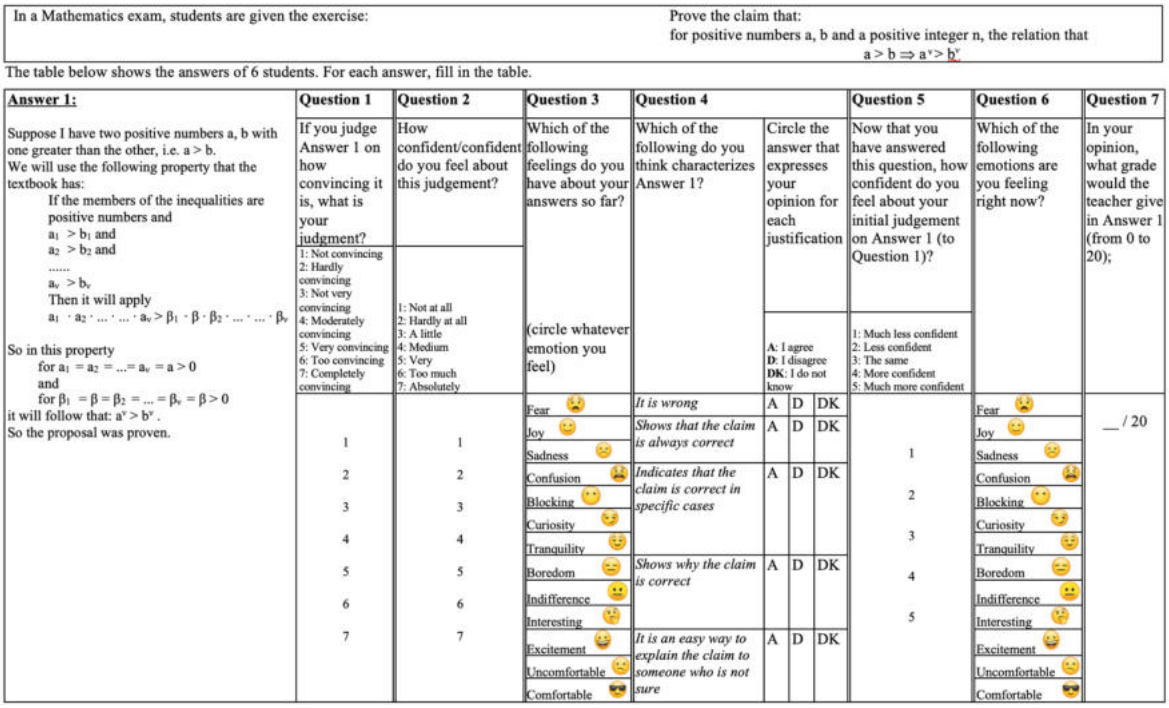
Following these, *proof awareness* is conceptualised as a partially hierarchical, complex affective system (Pepin & Roesken-Winter, 2015), with *proof self-efficacy* representing an overarching wholistic affective positioning about proof, whereas *views*, *certainty,* and *emotions* about proof are situated and given meaning in relation with this positioning. Proof awareness includes meta-affect referring "to affect about affect, affect about and within cognition about affect, and the individual's monitoring of affect through cognition (thinking about the direction of one's feelings) and/or further affect" (DeBellis & Goldin, 2006, p. 136). The meta-affective dimension of proof awareness includes, for example, views about expressed views, as well as emotions about views and emotions.

Considering proof self-efficacy, Bandura (1986) defines self-efficacy as "people's judgments of their abilities to organize and perform actions required to achieve defined forms of proof" (p. 391). In relation to mathematics, Hackett and Betz (1989) state that "mathematical self-efficacy is a casual, on a particular problem, assessment of an individual's confidence in his or her abilities to perform successfully on a particular activity or problem" (p. 262). Following these, we posit that proof self-efficacy refers to the extent that one believes that the proof influences mathematics learning and who also have confidence in their proving abilities. Furthermore, views refer to a 'softer-wired' construct than beliefs in terms of their temporal stability (Goldin, 2002; Hannula, 2012). Hence, we conceptualise views about proof to include one's views about proof, as well as his/her views about specific (claiming to be) proving statements, produced or evaluated by this person. Emotions may refer to two inter-related, complementary notions of an *affective reflex*: a state of alertness as a bodily reaction to a stimulus (Oatley & Jenkins, 1996), and/or a socially situated, and mentally processed affective reaction towards a situation (Hannula, 2012). In our perspective, proof awareness includes the affective reflex towards a proving situation (e.g., proof evaluation.). Moreover, drawing upon the notion of intentionality, in our conceptualisation of proof awareness, we explicitly include the *audience* of the proving statement (Moutsios-Rentzos, 2009; Segal, 1999) and the context within which is situated, as the *goals* of the proof-related situation may radically differ. For example, students' proof awareness may differ in a high-stakes situation (such as exams) from a non-threatening situation (such as a research interview). Finally, in line with Mason (1998), we identify three levels of the students' proof awareness: proof awareness-in action, proof awareness-in-discipline, and proof awareness-in-counsel.

Operationalizing the theoretical perspective to an empirical investigation

In order to operationalize this perspective, a study was conducted in April (towards the end of the school year) with 59 students (N=59) attending Grade 10 in a public high school in Greece. Data were collected through questionnaires and follow-up interviews (two weeks after the questionnaires), structured in line with the questionnaire with four students (two average and two above average, as identified by their teacher). The questionnaire was organised in two sections: one focusing on the students' affective relationship with mathematics, whilst the second section focused on proof

awareness in proof evaluation of specific proving statements. In this paper, we concentrate only on proof awareness and on aspects of the respective conducted analyses.



In a Mathematics exam, students are given the exercise:

Prove the claim that:
for positive numbers a, b and a positive integer n, the relation that $a > b \Rightarrow a^v > b^v$

The table below shows the answers of 6 students. For each answer, fill in the table.

Answer 1:	**Question 1**	**Question 2**	**Question 3**	**Question 4**			**Question 5**	**Question 6**	**Question 7**
Suppose I have two positive numbers a, b with one greater than the other, i.e. a > b. We will use the following property that the textbook has: If the members of the inequalities are positive numbers and $a_1 > b_1$ and $a_2 > b_2$ and ..... $a_v > b_v$ Then it will apply $a_1 \cdot a_2 \cdot \ldots \cdot \ldots \cdot a_v > \beta_1 \cdot \beta \cdot \beta_2 \cdot \ldots \cdot \ldots \cdot \beta_v$ So in this property for $a_1 = a_2 = \ldots = a_v = a > 0$ and for $\beta_1 = \beta = \beta_2 = \ldots = \beta_v = \beta > 0$ it will follow that: $a^v > b^v$. So the proposal was proven.	If you judge Answer 1 on how convincing it is, what is your judgment? 1: Not convincing 2: Hardly convincing 3: Not very convincing 4: Moderately convincing 5: Very convincing 6: Too convincing 7: Completely convincing	How confident/confident do you feel about this judgement? 1: Not at all 2: Hardly at all 3: A little 4: Medium 5: Very 6: Too much 7: Absolutely	Which of the following feelings do you have about your answers so far? (circle whatever emotion you feel)	Which of the following do you think characterizes Answer 1?	Circle the answer that expresses your opinion for each justification A: I agree D: I disagree DK: I do not know		Now that you have answered this question, how confident do you feel about your initial judgement on Answer 1 (to Question 1)? 1: Much less confident 2: Less confident 3: The same 4: More confident 5: Much more confident	Which of the following emotions are you feeling right now?	In your opinion, what grade would the teacher give in Answer 1 (from 0 to 20); __ / 20
	1 2 3 4 5 6 7	1 2 3 4 5 6 7	Fear Joy Sadness Confusion Blocking Curiosity Tranquility Boredom Indifference Interesting Excitement Uncomfortable Comfortable	*It is wrong*	A	D	DK	1 2 3 4 5	Fear Joy Sadness Confusion Blocking Curiosity Tranquility Boredom Indifference Interesting Excitement Uncomfortable Comfortable	
				Shows that the claim is always correct	A	D	DK			
				Indicates that the claim is correct in specific cases	A	D	DK			
				Shows why the claim is correct	A	D	DK			
				It is an easy way to explain the claim to someone who is not sure	A	D	DK			

Figure 1: Investigating proof awareness in proof evaluation (Answer 1)

Regarding the level of proof awareness, in this study, we mainly focus on proof awareness-in action and, in specific, on proof evaluations (and not, for example, on proof producing). Considering proof awareness and proof evaluation, first, the students completed the Proof Self-Efficacy Scale, which was constructed in line with Bandura (2006), drawing upon Hackett and Betz (1989) and May (2009). Subsequently, they completed a questionnaire investigating their views about proof (Moutsios-Rentzos & Korda, 2018). Considering proof evaluations, the students were provided six answers to an algebra task that they were asked to evaluate (see Figure 1). Drawing upon the types of proof that the students encountered during Grade 10 in Algebra, upon Balacheff (1988), upon Duval (2006) and upon Moutsios-Rentzos (2009) the following six types of answers were provided: Answer 1 (deductive, correct, expected mathematical language in Algebra; see Figure 1), Answer 2 (deductive, incorrect, expected mathematical language in Algebra), Answer 3 (deductive, correct, natural language), Answer 4 (crucial symbolic experiment, expected mathematical language in Algebra), Answer 5 (naïve empiricism, expected mathematical language), Answer 6 (geometrical, expected mathematical language in Geometry). In line with Moutsios-Rentzos and Korda (2018) and with proof awareness as conceptualized in this paper, the students were asked for each of the six answers the following (see Figure 1): to evaluate how convincing this answer is, to provide a degree of their certainty for their evaluation, to identify their emotions about this evaluation, to identify their views about whether this answer satisfies different functions of a proof (cf. Healy & Hoyles, 2000), to re-evaluate their certainty about their initial evaluation, to re-identify their emotions about this evaluation, and to provide their expectation about the grade that the teacher would assign to this

answer. At this point, it should be noted that we acknowledge the literature investigating basic emotions (e.g., Moutsios-Rentzos & Kalozoumi-Paizi, 2017), but for the purposes of that study, we considered only the students' emotions as communicated by their conscious choices. Upon completion of their evaluations about the six tasks, the students were asked to re-consider all the answers to *comparatively* re-evaluate them, to justify their choice, to state their certainty about this choice, and to identify their emotions at that stage (see Figure 2). The latter part is attempting to also explore the students' second level of proof awareness: proof awareness-in-discipline.

Of the previous answers, which would you choose in a Mathematics exam?

	Question 1	**Question 2**	**Question 3**	**Question 4**
	Mark your choice with✓.	Why did you choose this answer? Explain your view.	Now that you have answered this question, how confident do you feel about your initial choice? 1: Much less confident 2: Less confident 3: The same 4: More confident 5: Much more confident	Which of the following emotions are you feeling right now?
Answer 1			1 2 3 4 5	Fear Joy Sadness Confusion Blocking Curiosity Tranquility Boredom Indifference Interesting Excitement Uncomfortable Comfortable
Answer 2				
Answer 3				
Answer 4				
Answer 5				
Answer 6				
None of the 6				

Figure 2: Investigating proof awareness in comparative proof evaluation

Indicative results and concluding remarks

For the purposes of this paper, we consider only two of the research questions posed in the broader study: a) How is proof awareness (as conceptualized in this study) manifested in first-year high school students? b) What is the relationship between this awareness and different types of proofs and proof arguments?

Considering views and certainty, the results of the quantitative analyses revealed that the students appear to favour deductive incorrect proof (27%) and naïve empiricism (21%) as the most convincing. Moreover, they seem to be certain about their choices (more than 70% positive certainty for all the provided answers). The situation did not appear to change in the respective meta-affective dimension, when they were asked to re-consider their evaluations. It is noted that this was after the students stating that the answer: a) 'deductive-correct-expected' is correct and it explains, b) 'deductive-incorrect-expected' is correct, it holds true in some cases, and it explains, c) 'deductive-correct-natural language' is undecided whether it is correct or not, it holds true in some cases, and it explains, d) 'crucial symbolic experiment' is correct, it holds true in some cases, and it explains, e) 'naïve empiricism' is correct, it holds true in some cases, and it explains, and f) 'geometrical' is correct, it always holds true, and undecided whether it explains or not. Interestingly, the situation changed when the students considered exams: they chose 'naïve empiricism' (27%) and 'geometrical' (21%),

followed by the three deductive proofs sharing the same percentage (14%). Considering the students' emotions and meta-emotions, these did not appear to change across the different phases of the study. Moreover, they were mainly 'comfortable', 'calm' and 'curious' after their interaction with each answer. However, 'blocking' was also prevalent in students' responses for two deductive answers: 'deductive-correct-expected' and 'deductive-correct-natural language'. Consequently, the students appeared to experience a low degree of proof awareness-in action, since they did not hold appropriate views about proof—views associated with positive emotions and high degrees of certainty.

The results of the qualitative analyses partially confirmed these findings, as the students appeared to radically reposition themselves about certainty when during the interview they comparatively re-consider the answers. This also resulted in their reporting more negative emotions about their proof evaluations (e.g., uncomfortable, confusion). Hence, it appears that the comparative proof evaluations within a dynamic communicational context (such as the interview) allowed the students to question their affective positionings. It may be argued that this situation disturbed the equilibrium of the interviewees' affective system, thus allowing for the potentiality of developing more appropriate qualities of proof awareness. However, the fact that such a process entails negative emotions and lower degree of certainty requires delicate teaching decisions. Ongoing research focusses on engineering appropriate teaching designs that may identify the students' proof awareness and to constructively disturb the equilibrium towards obtaining desired proof awareness qualities.

Overall, in this paper, we concentrated on introducing the construct of proof awareness as a complex affective system. As an exemplar, we also presented a way of operationalising these ideas in an empirical study and we discussed aspects of the obtained results, with the purpose to show the potential gains of this theoretical perspective. We argue that raising the students' proof awareness may allow us to tap into important till now invisible aspects of teaching and learning mathematics.

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The use of generic examples in students' conjecturing processes

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In this paper, we focus on the use of generic examples in students' conjecturing process from the perspective of promoting mathematical reasoning in school mathematics. For this purpose, we propose the ECJ-model by dividing the conjecturing process into several subprocesses. A case study was used to gain initial insights into how learners cope with generic examples in a conjecturing process and to what extent they make use of its generic idea to justify their conjectures. Results indicate that the role of the generic example differs significantly between the different groups. The study thus considers initial approaches to the potential of generic examples as a scaffold for the transition from conjecturing to justification. By this, we highlight the relevance for further research on the effect of generic examples in mathematical reasoning.

Keywords: Conjecturing process, generic example, school mathematics, mathematical reasoning.

Introduction

In several national standards, the authors call for mathematics education to promote mathematical reasoning in everyday mathematics classes, and also the OECD highlights mathematical reasoning as an integral part of mathematical literacy (c.f., Kultusministerkonferenz, 2022; OECD, 2018). Accordingly, there is a need to offer broad fields of application for mathematical reasoning in daily school mathematics by focusing on different skills. One of these subskills is making and checking conjectures. Developing conjectures is a central component of and closely related to mathematical reasoning processes (Hsieh et al., 2012). Several international studies have shown that conjecturing has become important in teaching mathematics (Cañadas et al., 2007). However, it does not seem clear how to guide learners in their conjecturing processes so that they can subsequently verify their conjectures mathematically (ibid.). In particular, it is unclear how the transition from conjecturing to justification should be organized (Morselli, 2006). It is already known from research that generic examples can be helpful in starting a justification process and can be a fruitful element of proving processes at the school and university level (Kempen & Biehler, 2019). For the process of conjecturing and proving, the use of examples has already been highlighted in the case of university students (e.g., Morselli, 2006). However, it is still an open question how learners can benefit from generic examples during conjecturing processes.

Theoretical Background

According to Reiss and Ufer (2009), 'mathematical reasoning' includes exploring and checking activities that can be labeled as 'conjecturing'. The term conjecturing comprises activities in which "a regularity to be formulated as a (plausible) conjecture is sought or a given conjecture [...] is checked and, if necessary, adapted or corrected" (ibid., p. 157; authors' translation). Making conjectures provides a basis in that arguments or approaches can already be made for later proof production (Kirsten, 2021). Boero et al. (1996) stressed that a lack of engagement with a conjecture from the

outset can create additional obstacles in the proving process. According to the literature, a conjecture is a logical statement whose truth is not yet certain (a.o., Fischbein, 2002). “Along with this idea, conjecture is a statement concerning all the possible cases, based on the empirical facts, but with the doubtful elements.” (Sutarto et al., 2018, p. 1). However, conjectures are constructed in the course of a process. Fischbein (2002) describes the “conjecturing process [...] as the mental activity in constructing conjecture based on the possessed knowledge” (in Sutarto et al., 2018, p. 1). Connected to this, Hsieh et al. (2012, p. 288) “regard the process of proving in the classroom as a process of justifying a conjecture”. There are already several models that describe the activity of conjecturing in more detail, as Cañadas and Castro (2007) do for cases of inductive reasoning in seven stages: (1) Observing cases, (2) Organizing cases, (3) Searching for and predicting patterns, (4) Formulating a conjecture, (5) Validating the conjecture, (6) Generalizing the conjecture and (7) Justifying the generalization. To focus on the activity of conjecturing in the context of mathematical reasoning, a model was developed that represents the essential activities of exploration (E), conjecturing (C), and justification (J) as a process. The activities of the ECJ-model, shown in Figure 1, were refined by the seven stages of Cañadas and Castro (2007) so that the model allows a more focused view and classification of possible conjecturing processes of learners. Here, we focus especially on the transition from conjecturing to justification. As an additional element to support the transition to justification, a generic example has been included in the model.

This model is intended to provide a focused view and classification of a possible learners’ conjecturing process. As conjecturing is considered here as part of a wider reasoning process, the third activity justification is also modeled, but is not the focus of the empirical research presented in this paper. The generic example was meant as a scaffold to ease the transition from conjecturing to justification. As one can see in Figure 1, the generic example is placed at the transition from conjecturing to justification to use the general idea in respective examples as a starting point for such activities.

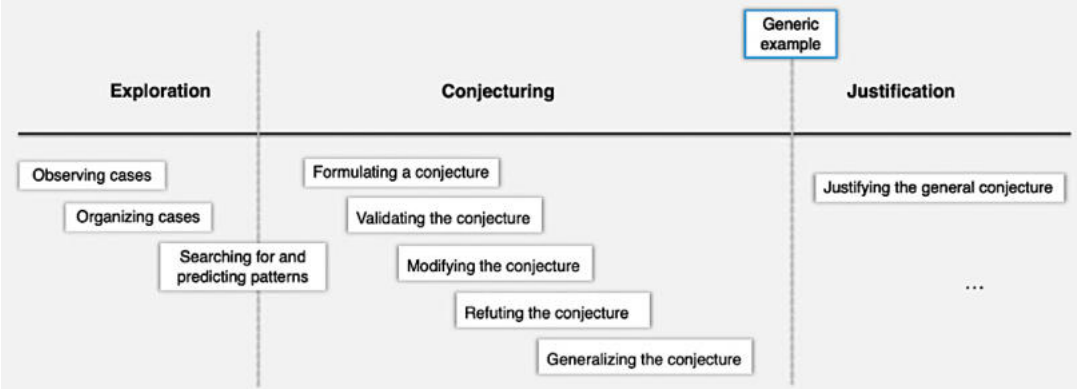

Figure 1: ECJ-model

Generic examples have been shown to be a fruitful element of proving processes at the school and university level (e.g., Kempen & Biehler, 2019). They allow learners to engage with the key ideas of a (generic) proof in an intuitive and familiar context. Potential obstacles such as the requirement for completeness of generality, formalism, and symbolism are not yet taken into account (Leron &

Zaslavsky, 2013). This can make it much easier to start an argumentation process and smooth the transition to a more formal proof structure. Our definition of a generic example is based on Mason and Pimm (1984), who define a generic example "as an actual example […] presented in such a way as to bring out its intended role as the carrier of the general" (ibid., p. 287).

Empirical Insights

Conjecturing activities are highlighted in various national mathematics standards and play an important role in mathematics education. For example, the German educational standards explicitly call for the formulation of conjectures (Kultusministerkonferenz, 2022). Similar activities around conjecturing can, for example, be found in Swedish curricula (Bergqvist, 2005). It is, therefore, surprising that an exemplary analysis of tasks on quadratic functions in German textbooks showed that although conjecturing tasks are relatively common, they are rarely embedded in reasoning processes (Niederquell & Kempen, 2024). Also, Bergqvist (2005) described that although tasks related to the wider competence mathematical reasoning are required, they are rarely found in Swedish textbooks. These findings seem problematic, as it has been shown that teachers follow the textbook very closely when designing lessons (Lepik et al., 2015). Therefore, it could be challenging for teachers to select appropriate tasks or problem situations to develop the ability to make conjectures. Furthermore, it did not seem clear how to guide learners in their conjecturing process so that they can justify their conjectures mathematically (Cañadas et al., 2007). This question (a.o.) led Morselli (2006) to investigate the use of examples in conjecturing and proving in elementary number theory. The results of her study showed that examples can help to discover patterns but are often considered insufficient for later proof productions. Students without a mathematical focus rely more on numerical examples and explain their justification verbally. The study showed that university students' mathematical background and their understanding of examples significantly influence their approach to proof. The use of generic examples as a didactic tool for learning to prove and respective research on it has increased in the last decades. Discussing or independently generating a generic example can help to initially convince oneself of the truth of a statement. Leron and Zaslavsky (2013) claimed that the benefits of generic examples are not about the explicit example itself but the process of using it. Based on the theoretical and empirical aspects described above, the question arises as to how learners make use of generic examples in conjecturing processes.

Research Questions

It has been shown above that there is a need for further research in the area of conjecturing processes, especially in school mathematics. It is still an open question to what extent generic examples can be a fruitful tool for the transition from conjecturing to justification. Therefore, our research questions focus on the use of generic examples in students' conjecturing processes: *(1) How do learners use a given generic example in a conjecturing process? (2) How does the use of a generic example influence learners' conjecturing process?*

The Learning environment

We chose the content of quadratic functions to engage students in conjecturing activities. This topic was chosen to widen the scope of activities around mathematical reasoning beyond the classical

domains such as geometry or elementary arithmetic (a.o., Furinghetti & Morselli, 2011). We used a conjecturing task on the rate of change of quadratic functions to gain initial insights into learners' process of conjecturing.

Kaya is a passenger in Tim's car and complains that he doesn't keep his distance when he speeds up. They both learned a rule of thumb at driving school to calculate braking distance, in which x is the speed (in km/h) and f(x) is the braking distance (in meters):

$$f(x) = \frac{x^2}{100}$$

Now, the two of them are discussing how the braking distance changes as the speed increases.

Figure 2: Central task in the learning environment

In the context of the braking distance of a car, the learners are guided through a complete conjecturing process. The tasks are formulated and arranged according to the model illustrated in Figure 1. The learners should first explore the pattern of quadratic change and increasingly generalise it as they progress. After the learners have generated several examples and made initial conjectures towards a generalization, they are asked to conjecture how the assumed behavior of change can be described in general. When they have completed this task, the interviewer first asks if this is their final answer. They are then given the generic example shown in Figure 3 and asked to explain whether it fits their conjecture and how far it applies to all possible cases.

$$f(40) = \frac{40^2}{100} = 16$$

$$f(2 \cdot 40) = \frac{(2 \cdot 40)^2}{100} = \frac{2^2 \cdot 40^2}{100} = 2^2 \cdot \frac{40^2}{100} = 2^2 \cdot 16$$

Figure 3: The generic example

Learners should then justify whether and why the generic example fits each of their conjectures. Finally, they are asked how confident they are that their general conjecture (which they made before the generic example came in) holds. They tick this on a six-point scale from "very uncertain" to "very certain" and are asked to state the reasons for their choice. The generic example was integrated into the learning environment to provide the initial impulse for respective justification processes. In Figure 3, the number two, marked in green, represents the general factor k, which we wanted the learners to identify. The speed 40 can also be considered a generic moment, as every number (possible speed) can be placed here. Finally, the whole equation can be regarded as generic, as respective manipulations can be performed in every case, i.e., the "two" and the "forty" could be replaced by any numbers (factor of change and speed).

Methods

A case study was conducted with three pairs of two students for the initial research. The learners, four females and two males, were fifteen years old and in the ninth grade of a grammar school. They had

already covered the topic of quadratic functions in the school year and thus should be familiar with the basic aspects of quadratic functions. Their previous performance in mathematics is considered average by their teacher. This case study aimed to gain initial insights that would enable a deeper understanding of individual, complex thinking processes. Since we still lack (empirical) insights about the use of generic examples in conjecturing processes, a degree of flexibility was required. Therefore, a deliberately small sample was used for the first laboratory study in order to capture the unique characteristics of each conjecturing process and consider them individually. The detailed observations of the three pairs were meant to generate new hypotheses about the use of generic examples in conjecturing and reasoning processes that will provide an approach for further, possibly larger, studies (Yin, 2018).

For research purposes, the learners were filmed in a laboratory setting while working in pairs in the learning environment. An interviewer observed the work and only intervened if the learners could not continue without help. The video and audio recordings were then transcribed. The transcripts were analysed using a qualitative content analysis (Mayring, 2014) with regard to the research questions above.

Results

Group 1: Tanja & Paula

Tanja and Paula started their exploration by generating different examples. As can be seen in Figure 4, they organized the three different cases (doubles, triples, quadruples) and observed patterns of quadratic change. The learners generalised each as a 'rule' to predict the pattern. This led Paula to move on to the activity of conjecturing. She formulated their first conjecture:

Paula: Our conjecture is: [...] how often you multiply the speed [...], the number is then squared.

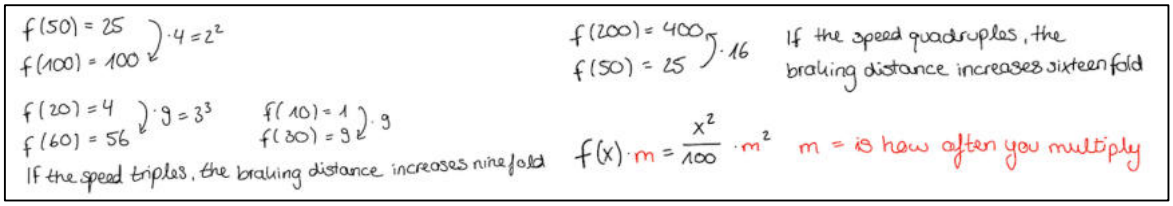

Figure 4: Conjecturing process group 1

Without being prompted, Paula and Tanja wrote their conjecture using algebra (see Figure 4, below). They validated their formula with two examples and signalled to the interviewer that they were sure that their general conjecture always held. Then, the interviewer handed out the last part of the learning environment with the generic example. Tanja and Paula immediately identified the marked “two” as the factor of change and recognised the generic character of the factor in the sense of a variable. They then discussed the structural similarities between the factor of change in the generic example and the factor they had labelled “m” themselves (Figure 4, at the bottom).

Tanja: I think we just did it like this [...] (*adds a bracket to* $f(2 \cdot (40))$ Because we wrote it on the outside [...] But then, yes, the new x would be an 80, so that works, too.

[...] In the end, it just means that our conjecture was correct. We just wrote it down differently.

As can be read in the transcript excerpt, they added a bracket around the 40. They argued that although their formula was written differently, it was just as correct because both formulae expressed the previously established 'general conjecture'. The similarities they discovered strengthened their confidence in their own conjecture. To justify their conjecture, they used the generic example and equated a term from their example with a term from the generic example. With this ‘true’ result, they confirmed that their formula must always be correct and, therefore, had no need for further investigation. The generic example thus represented a certain 'authority' for them, the validity of which was transferred to their conjecture by equating it with a concrete example.

Group 2: Maja & Gwen

Maja and Gwen began their exploration by generating different examples and searching for a pattern. They repeatedly looked at the local change between several concrete values and found some patterns. They formulated various conjectures, but when they tried to validate them, they always realised that these were not always valid. Therefore, they switched several times between the exploration and conjecturing activity. By analysing a table of several examples, they ruled out linear growth and finally formulated a supposedly general conjecture (see Figure 5).

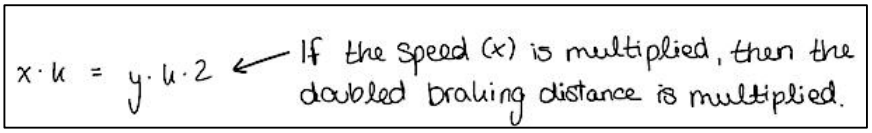

Figure 5: General conjecture group 2

Although they formulate their conjecture in general terms, they know that it does not apply in all cases. However, even when comparing it with their examples, they couldn’t identify precisely where the error was. The interviewer handed out the last part of the learning environment with the generic example. After Maja and Gwen had received the generic example, they were immediately able to explain the given equation. Finally, they recognised their mistake and the actual rate of change in relation to squaring.

Gwen: That squares itself [...] then we had it wrong somehow. [...]
Maja: Well, I mean, it still fits for the first step, because 2 squared is two times two. That's what we had. But it doesn't work after we have another point, you know?

When comparing the computations with the generic example, they recognised why their conjectures did not work. In this process, the generic example served as a stimulus for modifying the conjecture.

Maja: You have f(k times x). That's what's in front (*points to* $f(2 \cdot 40)$), because the green is always k; that's the factor. Later, Maja and Gwen went back to generalizing a conjecture and formulated the rate of change. They used the generic example to write down a general formula, which can be seen in Figure 6.

However, these students were not able to proceed to the activity of justifying the generalization and to reason why their conjecture (or the given generic example) applies to all possible cases. In the end, they were unsure whether their formula really always holds. The results of the third group are not

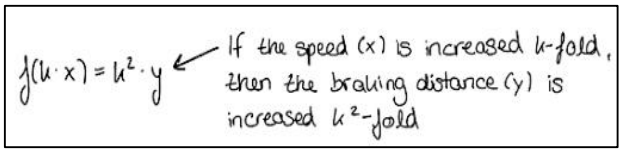

Figure 6: Revised general conjecture group 2

presented here in detail because the generic example had almost no relevance for these learners. They just noted it and then said this was just another way of solving the question. Although the generic example contradicted their conjecture, they were still sure that their conjecture was correct.

Conclusion

The results provide initial insights into how learners use a given generic example in a conjecturing process and how it influences their conjecturing process. The role of the generic example in the different conjecturing processes turned out to be very different. The analysis of Tanja's and Paula's conjecturing process shows first indications that generic examples can, in principle, serve as an *initial impulse for the transition to justification*. They actively engaged with the generic example, discussing general structures and using these structures as arguments for generalization. The situation was quite different for group 2, who had problems with the conjecturing process and could not state a general conjecture that they felt was always valid. The generic example *helped them modifying their conjecture*. As Group 3 showed, the generic example can also have *no (noticeable) effect*.

Final Remarks

The ECJ-model was helpful in describing learners' work in our learning environment and specifying the moment in the process when the generic example came in. The analysis of the results provided initial insights into the use of a generic example in a conjecturing process. It was shown that the generic example takes on different roles and thus has various influences on the further course of the process. It turns out that learners can benefit from active engagement with generic examples in the conjecturing process, for example, by encouraging discussion and the use of general structures as arguments for generalization and justification.

Overall, these initial findings on the different roles of generic examples in the conjecturing process provide a substantial starting point for further research. It seems especially interesting to investigate the extent to which a generic example can support or stimulate the transition to mathematical proof and the extent to which it influences learners conviction about the validity of conjectures.

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Enhancing pre-service mathematics teachers' justification abilities: Impact of a brief intervention on real numbers

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This study examines the development of justification abilities in the context of real numbers among pre-service teachers (PSTs). A comparative analysis was conducted between Grade 12 students (n=36) from a Cologne Gymnasium and an equal number of final-semester Masters of secondary PSTs at the University of Cologne. Initial results revealed comparable performance between groups, with PSTs showing modest advantages in proof construction but lower performance in applications. A short-term intervention with PSTs (n=20) yielded significant improvements across all areas, with overall performance increasing from 47.9% to 80.8%. Analysis employed Toulmin's model of argumentation, supported by strong internal consistency measures (pre-test $\alpha = 0.83$, post-test $\alpha = 0.87$), demonstrating that focused intervention can significantly enhance both reasoning abilities and conceptual understanding.

Keywords: Justification ability, pre-service teacher, mathematical logic.

Introduction

Mathematical justification abilities are fundamental to effective mathematics teaching, particularly in the context of real numbers where conceptual understanding intersects with formal reasoning. Recent research indicates a concerning gap between pre-service teachers' (PSTs) theoretical knowledge and their ability to construct and evaluate mathematical arguments (Stylianides & Stylianides, 2017). This gap is critical as teachers' capacity to understand and explain mathematical concepts significantly influences their ability to make mathematics reasonable and accessible to students (Ball & Bass, 2003). In German secondary schools, this challenge is particularly evident in the teaching of real numbers, where procedural fluency often takes precedence over conceptual understanding and justification skills. The Standing Conference of the Ministers of Education and Cultural Affairs (KMK) has emphasized this issue through their Mathematics Standards Agreement (KMK, 2022), calling for enhanced focus on mathematical logic and reasoning in teacher preparation programs.

The observation that PSTs' justification abilities often do not significantly exceed those of secondary school students raises fundamental questions about current teacher preparation effectiveness. While PSTs typically demonstrate competence in formal proof construction, their ability to apply mathematical reasoning in practical contexts and effectively communicate mathematical concepts remains problematic. This study addresses these challenges by examining the impact of targeted interventions on PSTs' justification abilities in the context of real numbers. By focusing on the development and evaluation of specific interventions, this research contributes to both theoretical understanding of mathematical justification development and practical improvements in teacher

education programs, particularly in bridging the gap between formal mathematical knowledge and its practical application in teaching contexts.

Literature Review

Research in mathematics education has increasingly emphasized the importance of justification abilities, particularly in the context of teacher preparation. The literature reveals several key frameworks and approaches that have been developed to understand and analyze mathematical justification processes, especially in relation to proof and proving activities in mathematics education. Stylianides and Ball (2008) highlight the importance of teachers' knowledge about proof and their ability to engage students in proving activities. Their research emphasizes that effective mathematical tasks should incorporate multiple aspects of reasoning, including pattern recognition, conjecture formation, and formal proof construction. More recent work by Stylianides et al. (2017) synthesizes research on teaching and learning of proof, suggesting that supporting teachers' development of proof competencies requires attention to both content knowledge and pedagogical approaches. Hemmi, Lepik, and Viholainen (2013, p. 374) observe that "the role of proof and proving in school mathematics is often marginal, and explicit requirements for students' proof-related competences are rare." Their comparative analysis of national mathematics curricula in Estonia, Finland, and Sweden demonstrates that proof and proving occupy a peripheral role, with explicit expectations for students' proof-related competences seldom articulated. The importance of addressing these gaps in teacher preparation is underscored by research showing that teachers' own comfort and facility with mathematical justification significantly impacts their ability to foster these skills in their students (Lo et al., 2008). Pre-service teachers often struggle with constructing and evaluating mathematical proofs, particularly when dealing with abstract concepts—a challenge compounded by limited exposure to formal proof techniques during their own secondary education. However, research indicates that carefully designed interventions can successfully bridge the gap between procedural fluency and conceptual understanding, especially when focusing on connecting intuitive understanding with rigorous mathematical reasoning.

Despite this, the existing literature reveals a critical gap in our understanding of how teacher preparation programs develop justification abilities compared to secondary education outcomes. While studies document that teachers often struggle with proof construction (Lo et al., 2008), there is limited empirical evidence comparing PSTs' actual performance with that of advanced secondary students on equivalent tasks. This comparison is crucial because it can reveal whether teacher education programs are effectively building upon students' existing mathematical reasoning foundations or merely maintaining similar competency levels.

Research Questions

The study was guided by the following research questions:

1. How do advanced secondary students and pre-service mathematics teachers compare in their ability to justify mathematical statements about real numbers?
2. To what extent does targeted intervention improve PSTs' ability to apply logical reasoning and construct valid mathematical proofs, particularly in the domain of real numbers?

These questions underscore the study's focus on the potential long-term effects of educational practices that prioritize procedural knowledge over conceptual understanding and logical reasoning, as well as the effectiveness of an intervention aimed at improving justification skills.

Methods

The study employed a two-phase research design, beginning with a comparative analysis followed by a targeted intervention specifically for pre-service teachers. In the initial phase, we conducted a comparative analysis between two distinct groups: 36 Grade 13 students from a Gymnasium in Cologne, North Rhine-Westphalia, Germany, and 36 pre-service secondary school mathematics teachers (PSTs) in their final semester of Master's studies at the University of Cologne. All participants completed a written test comprising 10 carefully designed items organized into four questions. These items were specifically designed to evaluate the participants' ability to apply their knowledge in justifying mathematical statements within the context of real numbers, requiring them to demonstrate logical reasoning, construct mathematical proofs, and use counterexamples effectively. Following this comparative analysis, the study transitioned to its second phase, focusing exclusively on the PST group. Based on the initial assessment results, we developed and implemented a targeted intervention specifically designed to address gaps identified in the PSTs' justification abilities. This intervention consisted of two intensive 90-minute workshops that focused on developing mathematical reasoning skills and proof construction abilities within the context of real numbers. The workshop design was informed by the specific areas of difficulty identified in the initial assessment, particularly focusing on enhancing PSTs' capacity for mathematical justification and proof construction. To evaluate the effectiveness of this PST-specific intervention, we administered a post-test to the intervention participants. Due to train strikes in Germany during this period, the post-test sample size was reduced to 20 PST participants, which was noted as a limitation of the study. The assessment of intervention effectiveness employed paired t-tests to compare pre-test and post-test scores within the PST group, providing a statistical measure of improvement in their justification abilities. This focused analysis allowed us to quantify the impact of the intervention specifically on the target group of pre-service teachers, while maintaining the broader comparative context established in the initial phase of the study.

Test Construction and Intervention

The assessment methodology integrates multiple theoretical frameworks to create a comprehensive evaluation system for mathematical reasoning and justification abilities. At its foundation lies Toulmin's Argumentation Model (2003), enhanced by recent developments in mathematics education research, particularly Stylianides et al.'s (2017) work on proof teaching and learning, and Knipping's (2008) adaptation for mathematics education. The evaluation employs a four-level Structural Completeness Scoring Framework (0-3 points), designed to capture increasingly sophisticated levels of mathematical argumentation. At Level 0 (No Structure), responses lack coherent mathematical argumentation, typically showing only calculations without explanation and no clear mathematical claims. Level 1 (Basic) responses demonstrate elementary argumentation with isolated mathematical claims and minimal supporting evidence, as exemplified by statements like "The sum of a rational and irrational number is always irrational because irrational numbers can't be written as fractions."

Level 2 (Developing) responses show emerging sophistication through connected mathematical claims and partial evidence, such as "When we add a rational number to an irrational number like π, the result must be irrational because π is irrational and can't be written as a fraction. Adding doesn't change this."

At Level 3 (Complete), responses demonstrate full mathematical argumentation including clear claims, comprehensive evidence, and well-structured logical warrants. A representative Level 3 response would be: "Consider a rational number r and an irrational number x. Assume their sum s is rational. Then $s = p/q$ for some integers p, q. This implies $x = s - r = p/q - r$, which can be written as a fraction, contradicting x's irrationality. Therefore, s must be irrational." The assessment comprises four distinct components: Mathematical Classification (12 points), Proof Construction (3 points), Concept Explanation (3 points), and Mathematical Applications (12 points), totaling a maximum possible score of 30 points. The scoring process involves initial evaluation using the 0-3 framework, followed by weighting based on component maximum points, enabling standardized comparison across different components while accounting for their relative complexity.

The framework's effectiveness was demonstrated through pre- and post-intervention comparisons, where pre-intervention responses typically scored at Level 1, showing basic claims without proper support, while post-intervention responses often reached Level 3, demonstrating complete argument structures with clear logical progression. Comparative analysis revealed no significant differences between PST and high school student performance, suggesting potential gaps in teacher preparation programs. However, the framework successfully captured improvements following targeted interventions, demonstrating its utility in measuring the development of mathematical reasoning abilities. This refined assessment system provides a robust tool for evaluating mathematical reasoning development, particularly valuable for teacher preparation programs and intervention assessment, while maintaining high standards of reliability and validity in measuring mathematical argumentation skills.

The intervention program was designed as a focused, two-session workshop series, with each session lasting 90 minutes. The design emphasized active learning principles and progressive skill development, allowing participants to build confidence while developing deeper mathematical understanding. The first session focused on developing robust conceptual understanding of real numbers through multiple representations and hands-on exploration. Participants began by examining the geometric interpretation of real numbers using dynamic geometry software, which allowed them to visualize and manipulate representations of both rational and irrational numbers on a number line. This visual approach helped bridge the gap between intuitive understanding and formal mathematical concepts. The session then progressed to exploring different representations of real numbers, including decimal expansions, fractional forms, and geometric constructions. Through guided discovery activities, participants investigated the relationships between rational and irrational numbers, examining properties such as density and completeness of the real number system.

Building on this foundational understanding, the second session concentrated on developing proof strategies and justification techniques. The session began with an introduction to direct proof methods, where participants learned to construct step-by-step logical arguments using definitions and

previously established theorems. This was followed by exploration of indirect proof strategies, including proof by contradiction and contraposition. Participants worked through carefully sequenced examples that demonstrated how to identify key properties needed for proofs and how to structure mathematical arguments effectively. Special attention was given to the construction and verification of counterexamples, as this skill had been identified as particularly challenging in the initial assessment. Throughout both sessions, participants alternated between individual work and small-group collaboration. Each new concept was introduced through carefully designed tasks that progressed from simple applications to more complex problems. For example, when working with irrational numbers, participants first proved the irrationality of $\sqrt{2}$ using a traditional proof by contradiction, then progressed to proving the irrationality of other square roots and finally to proving statements about operations with irrational numbers. It also incorporated regular opportunities for reflection and discussion, with participants sharing their reasoning strategies and receiving feedback from both peers and instructors. These discussions were structured using Toulmin's argumentation framework, helping participants identify the key components of effective mathematical arguments. The instructors modeled appropriate mathematical language and proof techniques throughout, emphasizing the importance of precise communication in mathematical justification.

The intervention concluded with a synthesis activity that brought together the conceptual understanding developed in the first session with the proof techniques learned in the second. Participants worked on complex problems involving real numbers that required both deep conceptual understanding and sophisticated proof techniques. This culminating activity helped reinforce the connections between different aspects of mathematical reasoning while providing participants with a model for integrating these skills in their future teaching practice.

Results

We begin with comparison of the responses of Master student versus High School students for each question on the test. The comparison revealed several interesting patterns:

Table 1: Masters students pre-test vs. high school students

Question	Masters Pre-Test	High School	Improvement	Effect Size	p-value	t-statistic
Q1	55.8%	52.5%	+3.3%	$d = 0.28$	0.183	$t = 1.35$
Q2	57.2%	44.7%	+12.5%	$d = 0.43$	0.312	$t = 1.02$
Q3	52.8%	45.6%	+7.2%	$d = 0.31$	0.428	0.428
Q4	32.5%	41.4%	−8.9%	$d = -0.38$	0.198	0.198
Overall Mean	49.6%	46.1%	+3.5%	$d = 0.25$	0.226	0.226

The table revealed nuanced patterns in mathematical reasoning abilities. For Question 1, PSTs demonstrated marginally better performance (55.8% vs 52.5%), with a small effect size and non-significant difference. The most notable difference appeared in Question 2 where PSTs achieved

57.2% compared to high school students' 44.7%, showing a moderate effect size. Similarly, Question 3 showed a small advantage for PSTs (52.8% vs 45.6%) but with substantial variability. An interesting reversal occurred in Question 4, where high school students demonstrated stronger performance (41.4% vs 32.5%), with a moderate negative effect size. This difference, while not statistically significant ($p = 0.198$), suggests potential gaps in PSTs' ability to apply mathematical concepts practically. The overall mean scores show a small advantage for PSTs (49.6% vs 46.1%), but the non-significant p-value (0.226) suggest this difference is not practically meaningful. The high Cronbach's alpha values for both groups (PSTs: 0.83, High School: 0.81) indicate good internal consistency in the assessment measures. The intervention produced substantial improvements across all assessment areas:

Table 1: Master Students pre-test vs. Master Students post-test

Question	Pre-test	Post-test	Improvement	Effect Size	p-value	t-statistic
Q1	51.7%	82.5%	30.8%	$d = 0.89$	$p < 0.001$	$t = 4.86$
Q2	55.0%	85.0%	30.0%	$d = 0.88$	$p < 0.001$	$t = 4.73$
Q3	53.3%	80.0%	26.7%	$d = 0.82$	$p < 0.001$	$t = 4.52$
Q4	31.7%	75.8%	44.1%	$d = 1.18$	$p < 0.001$	$t = 5.94$
Overall Mean	47.9%	80.8%	32.9%	$d = 0.94$	$p < 0.001$	$t = 5.12$

The intervention demonstrated substantial and statistically significant improvements across all assessment areas, with overall performance increasing from 47.9% to 80.8%. This comprehensive improvement indicates a significant enhancement in mathematical competency across all measured dimensions. The most dramatic improvement was observed in Question 4, where performance increased to 44.1%. This substantial gain was accompanied by the largest effect size, indicating that the intervention was particularly effective in developing students' ability to apply mathematical concepts in practical contexts. Question 1 showed similarly impressive gains, with overall performance increasing from 51.7% to 82.5%. This improvement in foundational mathematical reasoning abilities suggests that the intervention effectively enhanced students' basic conceptual understanding and analytical skills. The strong effect size indicates that these improvements represent meaningful practical advances in student capability. Question 2 and Question 3 demonstrated robust improvements of 30.0% and 26.7% respectively, both with strong statistical significance ($p < 0.001$) and large effect sizes. The improvement in Question 2, indicates enhanced ability in formal mathematical reasoning, while the gains in Question 3, from 53.3% to 80.0%, suggest improved capacity to articulate mathematical concepts clearly. The intervention also demonstrated strong internal consistency in post-test ($\alpha = 0.87$) which showed increase in internal consistency compare to the pre-test. This increase suggests that the intervention not only improved performance but also led to more consistent demonstration of mathematical reasoning abilities.

Discussion and Conclusion

This study investigated the development of mathematical justification abilities among pre-service teachers (PSTs) and the effectiveness of targeted intervention in enhancing these abilities. Our research addressed two key questions: comparing PSTs' justification abilities with those of advanced secondary students, and evaluating the impact of targeted intervention on PSTs' mathematical reasoning skills. Addressing our first research question, the findings reveal a concerning gap in teacher preparation. Despite their advanced mathematical training, PSTs demonstrated no significant advantage over high school students in mathematical justification tasks involving real numbers. This finding is particularly concerning when considering that PSTs represent a highly selected subset of former high school students who demonstrated strong mathematical aptitude and chose to pursue advanced mathematical studies. Given their academic trajectory, including successful completion of university-level mathematics courses and selection into competitive teacher preparation programs, one would reasonably expect PSTs to significantly outperform average high school students in mathematical reasoning tasks. The absence of such performance differences suggests that current teacher preparation programs may not be effectively building upon PSTs' existing mathematical strengths to develop the specialized reasoning and justification abilities required for effective mathematics teaching. In fact, high school students occasionally outperformed PSTs, particularly in tasks requiring logical reasoning and proof construction. This unexpected pattern suggests that current teacher preparation programs may not effectively develop these crucial mathematical competencies beyond high school level understanding. The comparable performance between groups ($p > 0.05$) raises important questions about the structure and focus of current mathematics teacher education programs. In response to our second research question, the intervention program demonstrated remarkable effectiveness in enhancing PSTs' justification abilities. The most substantial improvements were observed in proof construction (58.7% improvement) and applications of real number properties (59.9% improvement), areas that initially showed the weakest performance. Analysis through Toulmin's Argumentation Model revealed qualitative improvements in participants' ability to construct mathematical arguments, with enhanced use of warrants and backing to support their claims. The intervention particularly strengthened PSTs' abilities in classifying numbers (23.3% improvement) and explaining mathematical concepts (22.8% improvement), demonstrating growth in both procedural and conceptual understanding. These findings have significant implications for mathematics teacher education. The initial comparison results suggest that current preparation programs need to place greater emphasis on developing advanced justification abilities. However, the significant improvements observed after the intervention indicate that these skills can be effectively developed through targeted instruction. The success of our brief intervention, demonstrated by substantial gains across all measured areas ($p < 0.001$), suggests that similar approaches could be valuable additions to mathematics teacher education curricula, particularly in developing proof construction and logical reasoning abilities. This study provides compelling evidence that while current teacher preparation programs may not adequately develop mathematical justification abilities, targeted interventions can effectively address this gap. The strong effect sizes observed across all areas ($d = 0.82$ to 1.18) indicate practically significant improvements in PSTs' mathematical reasoning capabilities. As mathematics education continues to emphasize logical reasoning and proof

skills, ensuring that teachers are well-prepared to develop these abilities in their students becomes increasingly crucial.

Some limitations should be noted, including the reduced post-test sample size due to external factors and the short duration of the intervention. Future research should examine the long-term impact of such interventions, their transfer to classroom practice, and their application across different mathematical domains.

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Studying the role of synergy between paper folding and a Dynamic Geometry Environment in upper secondary students' conceptualization of conditional statement and proof

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This paper reports a research study focusing on the contribution of the synergy of a manipulative artefact (paper folding) and a Dynamic Geometry Environment (DGE) to the conceptualization of the conditional statement and proof during 10th grade students' engagement with construction-proof tasks. Under the combined use of Mathematical Working Spaces and the Theory of Semiotic Mediation the analysis highlights the particular contribution of the synergy of paper folding and the DGE in the formation and validation of conjectures, in understanding the logical dependencies between premises and the conclusion of a conditional statement, and finally in proving.

Keywords: Synergy, paper folding, dynamic geometry environment, conditional statement, proof.

Introduction

Proving and conjecturing have attracted much research interest over the years. The literature is replete with studies that focus on the role of Dynamic Geometry Environments (DGE) in students' conceptualization of these mathematical concepts. According to Mariotti (2012), with the dragging modality of a DGE the user can act upon the drawing while all the properties defined by the construction are maintained. Observing the simultaneity of invariants of a robust construction during dragging is likely to contribute to the understanding of the conditional links of a conditional statement. Specifically, the invariants referring to the single steps of the construction correspond to the premises and the invariants that appear as the consequences of the former correspond to the conclusion of the conditional statement (Mariotti, 2012). This dynamicity of the dragging combined with the measuring tool are considered to promote the transition from the exploration phase, characterized by conjectures, to the proving phase (Sinclair & Robutti, 2013). Furthermore, the investigation of different instances of the construction is claimed to support the generalization of propositions and their proof (Komatsu & Jones, 2020). Parallelly, one can find in the literature studies about the potential of manipulative artefacts, such as paper folding, to contribute to exploring geometric ideas and to discovering mathematical relations (Wiles, 2013). According to Hsieh et al. (2012), interaction with paper folding has the potential to promote the discovery of the logical steps necessary in a proof, thus supports understanding of the proving process.

In recent years, there has been intense research interest on the interplay of two artefacts of different nature. Mariotti and Montone (2020) introduced a new research field by defining the *synergy* of a duo of artefacts. Synergy is the combined effect of using different artefacts, where their interaction creates interconnected signs and meanings. This occurs when references to both artefacts form relationships between their respective signs, which can be organized to semiotic chains, i.e. evolution paths describing the move from personal situated meanings towards shared de-contextualized mathematical

meanings (Mariotti & Montone, 2020). Within this new research trend, among the few related studies, there is one (Valori et al., 2022) that examines the interplay between paper folding and GeoGebra during students' engagement with construction-proof tasks. The results confirm the value of using different artefacts to support students' coordination of operative and discursive apprehensions of the figures. However, this study focuses on the nature of students' argumentation rather than on the proof per se. This paper aims to fill the gap in the existing literature regarding the conceptualizations of conditional statement and proof arising during students' interaction with both paper folding and DGE (i.e. Geometer's Sketchpad, GSP) from the perspective of synergy. The research question is formulated as follows: What is the role of synergy between paper folding and the use of a DGE in upper secondary students' conceptualization of conditional statement and proof?

Theoretical framework

Our theoretical framework consists of the combined use of Mathematical Working Spaces (MWSs) (Kuzniak & Richard, 2014) and the Theory of Semiotic Mediation (TSM) (Bartolini Bussi & Mariotti, 2008) as a tool of design and analysis. According to MWSs, to grasp the mathematical work of individuals confronted with mathematical activities, it is necessary to take into account an epistemological plane which is related to the mathematical content and a cognitive one related to the reasoning of the individual. The evolution of mathematical work occurs progressively through the bridging of these planes, a process which takes place through three interrelated geneses; *semiotic (sem)*, *instrumental (ins)* and *discursive (dis)*. Semiotic genesis associates signs with the process of visualization and explains the dialectical relationship between syntactic and semantic perspectives on mathematical objects. Instrumental genesis reveals the connection between artefacts and the construction processes that contribute to the completion of the mathematical task. Finally, discursive genesis of proof refers to the process through which properties and results organized in the theoretical system of reference become available for mathematical reasoning and discursive validations.

On the other hand, TSM allows the study of the interaction of three categories of signs in the process of conceptualizing conditional statement and proof: artefact, pivot and mathematical. Artefact signs refer to the context of the use of the artefact and the associated meanings are personal, strictly related to the subject's experience with the artefact. Mathematical signs refer to the mathematical context and are associated with mathematical meanings. Pivot signs are bridges between the artefact and mathematical signs and can refer to the activity with the artefact and the mathematical context. The evolution of meanings can be organized into semiotic chains that highlight the process of meaning-making through the transformation of signs and the transition from meanings associated with the artefact to mathematical and de-contextualized meanings. In this study, we consider paper folding and GSP as two MWSs and we use this framework to analyze their semiotic, instrumental and discursive dimensions. The concept of semiotic chain contributes to the analysis of the connections between the MWSs through the study of the transitions from one dimension to the other and the evolution of meanings from empirical to more mathematical ones.

Methods

Participants and data

This case study involved two groups, Group 1 (S1, S2) and Group 2 (S3, S4) of students around the end of the 10th grade and the main criterion for the selection was their interest in geometry. The small number of participants aligns with the in-depth investigation of the case study method, while working in groups may foster the evolution of signs (Bartolini Bussi & Mariotti, 2008). There was a first meeting with each group in order to become familiar with GSP followed by three meetings (lasting $5\frac{1}{2}$ hours with Group 1 and $6\frac{1}{2}$ hours with Group 2) in each of which the students worked on one of the three tasks. The role of the teacher (first author) was mainly to encourage, where necessary, the emergence of pivot signs and to facilitate discussion. Data include audio, video and screen recordings, worksheets and sheets of paper with students' creases and folds.

Tasks

Task 1 first asks students to fold an A4 sheet according to the instructions given (Figure 1) and then formulate a conjecture about the final shape. The expected conjecture is that the resulting shape is an equilateral triangle, which they are asked to validate working only with the folded paper. Then, through the identification of properties and relations of the creases, students have to entirely construct the same structure step-by-step in a robust manner (preserving properties during dragging) in GSP by selecting appropriate tools. In this phase it is expected that students' mathematical interpretation of the signs of the paper folding will encourage a more deductive reasoning. The completion of the robust construction in the GSP is followed by the request to validate the conjecture also in the digital environment. Finally, a formal proof of the conjecture is requested, in which the mathematical meanings and properties emerged through the synergy of the two MWSs are expected to be exploited.

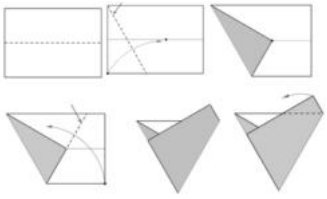

Figure 1: Folding instructions for Task 1

Tasks 2, 3 introduce some new aspects to the objectives of Task 1. Task 2 (Figure 2) involves the emergence of contradictory results on the screen and on the folded paper. While the expected conjecture in the folding paper is that the resulting shape is an isosceles triangle, the completion of the construction in GSP leads to a scalene. The contradiction arises if the premise of maintaining the ratio of the dimensions of the rectangle to those of the A4 sheet is not considered. This procedure aims at understanding the logical dependence of the premises and the conclusion of a conditional statement.

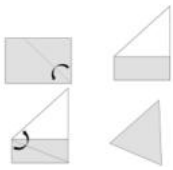

Figure 2: Folding instructions for Task 2

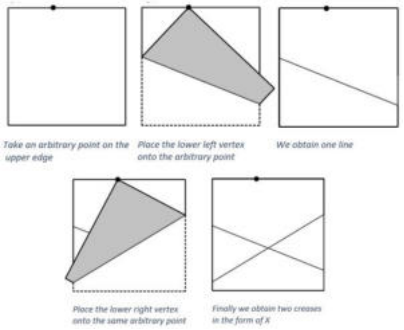

Figure 3: Folding instructions for Task 3

The particular element of Task 3 (Figure 3) is the notion of generalization. After some repetitions of the folding steps (with a different initial arbitrary point), students are asked to formulate a conjecture about the intersection points of the X-lines. They are expected to observe that these points lie on the perpendicular bisector of the upper and lower side of the square. The request to represent the construction in the GSP and to

investigate the conjecture is expected to lead to investigation procedures of invariant properties. Dragging will play a crucial role in generating multiple pairs of X-lines and in observing general properties. In addition, Task 3 invites groups to prove their conjecture in the particular case where the arbitrary point coincides with one of the square's vertices and in the general case. This process is expected to contribute to further understanding of the general-particular distinction through the need to change the proving path. In this task, too, the logical connection of the conclusion from the premises is expected to become evident on an empirical and a strictly mathematical level.

A priori analysis

We distinguish two MWSs "Paper folding" and "GSP", each of which offers different possibilities for students' work at its three dimensions. However, the focus of the research is the synergy between them. After the completion of the constructions in the paper folding, a continuous interplay between the dimensions of the two MWSs is expected to emerge and be evident in students' constructed meanings. In particular, during the transfer of the construction from the manipulative to the digital artefact, the work in the semiotic and instrumental dimension of the "Paper folding" and the empirical interpretation of the creases, is likely to be followed by the discovery of properties and invariant relations. This way the preceding semiotic and instrumental work may be mobilized in the "GSP". During the evolution of this process, the transition from personal to mathematical meanings is expected through the transformation of the students' conceptualization of signs. The cooperation of the discursive dimensions of the MWSs is expected to become apparent while exploring mathematical meanings related to the conditional statement and methods of validating conjectures and proving them. Episodes of synergy are also expected to occur in instances of contradictory results between the two MWSs and during the investigation of particular and general cases.

Method of analysis

After the transcription, the open coding method (Strauss & Corbin, 1998) was used, and codes were assigned in relation to the dimensions (semiotic, instrumental, discursive) of each of the MWSs. Then, additional codes were developed to characterize the categories of signs (artefact, pivot, mathematical). In the next step, episodes of synergy were identified and organized in semiotic chains to show the move from one MWS to the other, and the evolution of meaning generation. In these chains the transition from one sign to the adjacent stems from students' work in particular dimensions of a MWS. Finally, the episodes were categorized into four groups based on their contribution to the conceptualization of specific elements or processes related to the conditional statement.

Results

An episode from each one of the four groups is presented to illustrate the progressive conceptualization of the conditional statement and proof. Each episode brings to the fore a specific component of the conditional statement (premises, conditional links and conclusion) and the proof.

Conceptualizing the premises of the conditional statement

In all three tasks, after completing the folds and formulating a conjecture, the challenge of a robust construction in the GSP brought out episodes of synergy between the two MWSs in the students' attempt to attribute a mathematical meaning to the folds and the relationships between them.

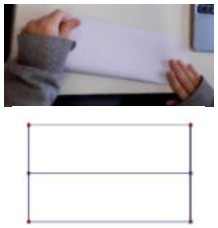

Figure 4: Student's work on paper folding and GSP

Achieving this goal led the groups to early premises and logical connections of the conditional statement. The following excerpt is from Group 1's engagement with Task 1, in which students attempt to conceptualize the first fold (Figure 4).

S1: It's the midline (a line halfway between two parallel lines).
Researcher: Why?
S2: If I fold it, they fall on top of each other.
S1: Yes that... these sides are equal.

Initially, the students working on the semiotic dimension of paper folding visualize the first fold (*artefact sign*-paper folding) and refer to it as "midline" (*mathematical sign*). When asked to justify this conjecture, they activate the instrumental dimension and folding again along the first crease, they observe that the upper and lower sides coincide (*pivot sign*-paper folding). They interpret this visual and haptic fact as symmetry (*mathematical sign*) of the sides and, finally, they justify their claim by observing the resulting equalities. This instrumental and discursive dimension of the MWS of paper folding subsequently directs their work in GSP, as they identify those properties of the sign (1st crease) that will allow it to be represented in GSP. Using appropriate commands (*pivot sign*-GSP), students construct the perpendicular bisector of the right and left sides of the rectangle, enriching the initial meaning of the midline, thus activating the instrumental and discursive dimension of the GSP. In summary, the synergy revealed premises related to the first crease, and in particular to the properties of the midline and the perpendicular bisector. However, these premises were ultimately not explicitly exploited by the students in the final proof.

Semiotic chain: 1st crease (artefact sign-paper folding) $\xrightarrow{ins\ (paper\ folding)}$ properties of 1st folding (pivot sign-paper folding) $\xrightarrow{sem\ (paper\ folding)}$ midline/symmetry (mathematical sign) $\xrightarrow{ins-dis(GSP)}$ selection of tools in GSP (pivot sign-GSP) $\xrightarrow{sem\ (GSP)}$ construction of midline from the middle points of the sides (artefact sign-GSP) $\xrightarrow{dis}$ early premises of the conditional statement (mathematical sign).

Conceptualizing the conditional links of the conditional statement

During the investigation of the contradictory results in the folding paper and GSP in Task 2, groups were led to a deductively questioning of the premises and to the conceptualization of the conditional links. The contradiction led Group 1 to generate correspondences both between the signs and the actions they were carrying out in the two MWSs. Specifically, the students first worked at the instrumental dimension of the GSP where they utilized dragging and measuring to identify instances where the triangle becomes isosceles (*pivot sign*-GSP), as they expected. This work activated the instrumental dimension of the paper folding as the students repeated the same folds on sheets of different dimensions ("manipulative counterexamples") (*pivot sign* -paper folding), discovering that only the sheets that are similar (*mathematical sign*) to A4 lead to an isosceles triangle. The comparative observation of the "manipulative counterexamples", led students to identify the necessary mathematical relationship for the desired conclusion, namely the perpendicularity of two segments (Figure 5, purple). This synergy supported the discursive genesis of meanings related to the dependence of the conclusion from the premises.

Semiotic chain: different visual feedback from the two MWSs (artefact signs) $\xrightarrow{ins\ (GSP)}$ dragging to obtain an isosceles (pivot sign-GSP) $\xrightarrow{ins\ (paper\ folding)}$ manipulative counterexamples (pivot sign-paper folding) $\xrightarrow{sem\ (paper\ folding)}$ similarity (mathematical sign) $\xrightarrow{dis\ (paper\ folding)}$ different dimensions lead to different shapes (pivot sign-folding) $\xrightarrow{dis}$ different assumptions lead to different results (mathematical sign).

Figure 5: "Manipulative counterexamples"

Developing conclusions and strengthening of the conditional links of the conditional statement

In Task 3, the students in Group 2, to test the conjecture that the intersection points of the X-lines lie on the perpendicular bisector of the two opposite sides of the square, apply two validation methods in the paper folding and two in the GSP, which are related to each other. S3, having already made 4 repetitions of the folding procedure of the task with different initial arbitrary points, in order to reject the fact that it is by chance that the intersection points are collinear, makes one more repetition (*artefact sign*-paper folding) (Figure 6). Correspondingly, this group activated in GSP the trace (*artefact sign*-GSP) of the intersection point (Figure 6). The synergy of the instrumental dimensions of the MWSs seemed to favor students' conceptualization of the locus as a set of infinite points with a common property. Another validation method was to produce a new crease (*artefact sign*-paper folding) along the line through which the intersection points seem to lie, to check whether this line (the crease) is the perpendicular bisector. Similarly, working at GSP, S4 states that to confirm that the intersection points of the X-lines are on the perpendicular bisector (*mathematical sign*) they can construct this object (*artefact sign*-GSP) and then check whether the intersection points are indeed points of this line by utilizing the dragging. The synergy in this case led to the perception of the locus as a whole, i.e. as an object or a static structure bearing certain construction properties. The chains of meaning composed due to the connections of the validation methods contribute - in addition to highlighting different aspects of a mathematical concept – to the strengthening of the constructive and ultimately logical dependencies of the premises with the conclusion.

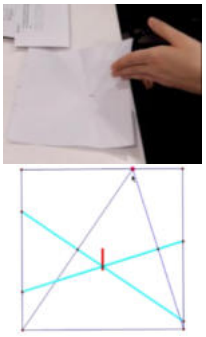

Figure 6: The intersection on paper and the trace at GSP

Semiotic chain 1: production of a new pair of X-lines (*artefact sign*-paper folding) $\xrightarrow{ins}$ trace (*artefact sign*-GSP) $\xrightarrow{dis}$ perpendicular bisector as a locus (mathematical sign). Semiotic chain 2: crease along the perpendicular bisector (artefact sing-paper folding) $\xrightarrow{ins}$ construction of the perpendicular bisector and dragging (artefact sign-GSP) $\xrightarrow{dis}$ perpendicular bisector as an object (mathematical sign). These semiotic chains express the reinforcement of the conclusion and the conditional links of the conditional statement and the enrichment of the notion of the perpendicular bisector.

Proof of the conditional statement

Students' experience stemming from the synergy of the manipulative and digital artefact revealed geometric properties and relations between signs and consequently helped them to identify the necessary components of proof. Combined with their known propositions and theorems, the revealed geometric relations structured their proofs. For example, students of Group 1, in order to prove that the triangle ABC (Figure 7) is isosceles in Task 2, utilized the geometric relationships they discovered through synergy. In a previous phase, the students proved empirically (equality as coincidence) by folding that the three triangles are equal (Figure 7), and the justification of equality was given a mathematical character when transferred to the digital environment. The equality of triangles 1, 2 was justified by exploiting a property of a diagonal of a rectangle and the equality of 1,3 was justified through symmetry which was expressed in GSP by exploiting the reflection tool. Through the logical relation "if a=b and b=c then a=c" the students justified the equality of triangles 2, 3 and finally the equality of angles ACB and CAB and recalled their well-known proposition about the equality of adjacent angles in an isosceles triangle. The semiotic chain is: equality of triangles 1,2,3 through the folding and the properties of creases (pivot sign-paper folding) $\xrightarrow{ins/dis(paper\ folding)}$ geometric relations of symmetry and properties of the rectangle (mathematical sign) $\xrightarrow{dis}$ justification of equality of the angles ACB and CAB/definition of isosceles (mathematical sign) $\xrightarrow{dis}$ proof.

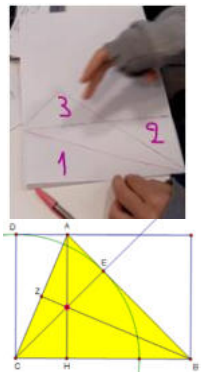

Figure 7: The Task 2 with folded paper and GSP tools

Conclusion

The analysis highlighted the contribution of the synergy of the two MWSs to the students' conceptualization of the conditional statement. The visual and haptic experience offered by paper folding activated the semiotic and instrumental dimension of this MWS, resulting in students' empirical attribution of mathematical meanings to the folds and the discovery of relations related mainly to symmetry (e.g., equality of segments and angles). The same two dimensions of GSP were activated by the request to represent the construction in GSP which led the students to search for appropriate commands and through this to develop meanings (e.g. for the midline). Moreover, in Tasks 2 and 3 the dynamic nature of GSP (dragging, trace) promoted further the exploration of contradictory results. A closer look at the emerging semiotic chains reveals that the synergy of the semiotic and instrumental dimensions of the two MWSs resulted in the interplay of their discursive dimensions and the discovery of the premises and conditional links of the conditional statement. For example, in Task 2 S1 and S2 exploited the manipulative counterexamples and dragging and provided a mathematical explanation of the source of the contradiction. In addition, the easy manipulation of the paper and the dynamicity of the GSP with their immediate feedback helped students to find methods to validate their conjectures and thus make the role of synergy more evident. For instance, in Task 3 the strengthening of conditional links and the generalization of a conjecture (i.e. perpendicular bisector as a locus) was promoted through the synergy of multiple paper foldings (different arbitrary points) and the subsequent activation of trace in GSP. Summarizing, the above episodes illustrate how the emerged meanings were utilized by the groups during the proving process. Specifically, the synergy contributed to the rich conceptualizations of the signs, the logical

dependencies between premises and conclusions and the students' use of a variety of propositions and theorems of Euclidean Geometry in structuring their proofs (e.g. the isosceles triangle in Task 2). This research contributes to the gap in the existing literature regarding the role of synergy in the conceptualization of conditional statement and proof and demonstrates that the combined use of MWS and TSM is an effective theoretical tool in revealing the aspects of the contribution of synergy in this context.

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How do children in primary school prove?

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A learning environment that allows the discovery of the assertion 'the sum of three consecutive numbers is three times the middle number' was tested with eight primary school children (grades 3 and 4) in clinical interviews. The interviews were analyzed on the epistemic actions recognizable in the proving activities using Abstraction in Context, and proof levels and indications for the further development of the learning environment were derived. The construct 'block towers' was used by a minority of children but led to proofs by generic example. The justifications of the children using the other construct, 'placeholders', remained empirical.

Keywords: Primary education, content-visual proving, reasoning, instruction support.

Introduction

By embedding proving at the center of children's mathematical experiences, they have more opportunities to engage with mathematics in depth and see it as a meaningful activity. In addition, they will not only be better prepared in secondary school to engage in proving, which they see as a natural extension of their earlier mathematical experiences but will also be more able to reason mathematically in a disciplined way (Stylianides, 2016, p. 10). In particular, content-visual proving can be implemented at the primary level (Peterßen, 2012, p. 20). A content-visual proof (Blum & Kirsch, 1991; Wittmann & Müller, 1988) is based on some concrete, visually perceptible object used to prove something general. Through trained observation, this can be understood as an object of a more general nature, whereby the more general aspect of this example must be mentally recognized to provide a content-visual proof. A verbalization must occur so that what is initially subjectively found to be universally valid can be socially shared and possibly recognized by others (Wittmann & Ziegenbalg, 2007). Proving can be realized within the framework of learning environments of the 'best quality' (Wittmann, 1998, p. 337), so-called substantial learning environments. According to Wittmann (1998, p. 339), a substantial learning environment is open in principle. Merely, the key information that the teacher gives at the beginning of each phase is determined, and further interaction with and among the pupils remains open. Concerning content-visual proving, the question arises of how such key information can look and how open the further interaction should be to stimulate a proving activity.

This paper presents the current state of development of a learning environment for content-visual proving. The learning environment is created using Design Science Research (Drechsler & Hevner, 2016) around a task that allows the discovery of the assertion 'the sum of three consecutive numbers is three times the middle number' and is tested with primary school children in clinical interviews (Wittmann, 1998) by students who have been specially prepared. This paper looks at which constructs primary school pupils use for reasoning to draw conclusions for the further development of the learning environment. Abstraction in Context (AiC) provides a model of nested epistemic actions to investigate learning processes on a micro-analytical level that lead for the learner to new constructs

(e.g., concepts, strategies, etc.). The central theoretical element of AiC is the dynamically nested epistemic action model, according to which the emergence of a new construct is described and analyzed based on three observable epistemic actions (RBC model): *(R–recognizing)* identifying situationally relevant knowledge constructs, *(B–building with)* combining known constructs to solve a local problem and *(C–constructing)* developing new constructs (Dreyfus & Kidron, 2014).

The Learning Environment

The design of the learning environment is based on the following design principles (Platz, in review) formulated in the manner of van den Acker (1999): If you want to design a learning environment for proof at the primary level, then you are best advised to

(1) awaken the need for proof right from the start by
   a) embedding the tasks in a historical context (Krauthausen, 2018) or a story (Kothe, 1979), as this can reduce the rejection of the tools used and create experiences of relevance.
   b) allowing the children to discover the assertion independently (Krumsdorf, 2015), as this increases the children's motivation.

(2) awaken the need for proof concerning generalizing by
   a) repeatedly asking 'why' questions (Krauthausen, 2001, p. 105) because we get what we accept (Higgins, 1998) – and if we settle for less, children will only give us that.
   b) the application of the concept of 'Big Numbers' (Martin & Harel, 1989), as this can help children detach themselves from empirical testing (Krumsdorf, 2015).

(3) as a learning guide, to be a reliable source of information, and to support and supplement the children's activities with suitable, not too far-reaching impulses (Wollring, 2008), namely by
   a) taking up the children's previous experiences and ideas and not directing them too strongly but instead building on their ways of thinking (Selter & Spiegel, 1997).
   b) giving the children time (Krauthausen, 2001).
   c) well preparing the implementation by identifying possible solutions/ sources of error in advance (Wollring, 2008).

The tasks within the learning environment are as follows:

Table 1: Tasks in the learning environment and corresponding design principles

Task	Design Principle(s)
Today, you will become a maths detective: a tricky case must be solved.	(1) a)
Solve the tasks. a) $2 + 3 + 4 =$ $3 \cdot 3 =$ b) $3 + 4 + 5 =$ $3 \cdot 4 =$ c) $7 + 8 + 9 =$ $3 \cdot 8 =$ What do you notice?	(1) b)
Why is this so? Give reasons!	(2) and (3)

Research Questions

In this paper, the following research questions are targeted: *RQ(1) Which epistemic actions are recognizable in the proving activities of primary school children? RQ(2) Which consequences can be drawn for the further development of the learning environment?*

Methods and procedures for collecting and analyzing data & obtained data

A multiple case study (Yin, 2018) was conducted using qualitative data collection and analysis methods. Clinical interviews (Selter & Spiegel, 1997; Wittmann, 1998) were used for data collection. Conducting a clinical interview, in which only the key questions are defined, and the requirement is to follow the children's thinking, is analogous to classroom management when implementing a substantial learning environment (Wittmann, 1998). However, as these do not coincide with the routines of maths lessons and, therefore, a teaching reference can be established more in terms of content, only limited conclusions can be drawn for implementing proving in maths lessons. Nevertheless, the data collected can provide information about the teaching and learning processes, thought processes, and learning progress. On the other hand, they help evaluate and revise the learning environments to organize teaching and learning processes even more effectively (Wittmann, 1998, p. 339). Although group interviews have certain advantages (Selter & Spiegel, 1997, p. 106), the decision was made in favor of individual interviews to avoid the disadvantages of group interviews like the dominance of individual children, loss of information, the heterogeneity of the group and mutual reinforcement, which would make it difficult to keep track of the individual children's approaches (Selter & Spiegel, 1997, p. 106).

In the seminar 'Mathematics didactic research: proving in primary school' in mathematics for the primary school teaching profession at Saarland University (Germany), students were prepared in the summer semester of 2022 for conducting the clinical interviews. They were trained to consider the design principles mentioned above. In the interviews, besides pencil and grid paper, the children were offered clamping blocks and the 'Steinchen-Applet' (Platz, 2020a). The children were free to decide which materials they wanted to use. The interviews were videotaped and transcribed. The visual and text material was analyzed using qualitative content analysis (Mayring, 2015). A category system was developed in the seminar based on theory (cf. Platz, 2020b) and is further developed inductively using the material. The analysis focussed on constructs developed independently by the children. Eight clinical interviews were conducted with 3rd and 4th graders from Saarland and Luxembourg. Balacheff's (1988) generally recognized levels of proof are used to classify the solutions: *(1) 'proof' by naive empiricism* (some arbitrarily selected concrete examples are verified), *(2) 'proof' by crucial experiment* (concrete examples are used, however, these are typical examples that allow for generalization), *(3) proof by generic example* (structural mathematical properties are stated with reference to a characteristic case that exemplifies a whole class of cases) and *(4) proof by thought experiment* (a reference is made to structural mathematical properties, but completely independent of concrete examples). Proofs on *levels 3* and *4* can be regarded as valid (Balacheff, 1988).

Results

RQ1

Firstly, ideal-typical exemplary substantive and illustrative proofs are presented (a priori analysis, Tables 2 and 4), followed by a description of the particular aspects of the constructs and possible difficulties. The ideal-typical reasoning contexts are each a content-visual proof, as the reasoning on the image works with logical arguments. Still, the warrants are not drawn from the formal system but from images, reality-related considerations, or similar (Meyer & Prediger, 2009). Therefore, the justification consists of a chain of correct conclusions based on non-formal premises (Blum & Kirsch, 1991). The reasoning can consequently be considered valid, as the operations are generally executable, regardless of the specific examples used to demonstrate them (Wittmann, 2014, p. 226). The structure of the tables is as follows: in row 1, we have the prerequisites *(C_1)*; in row two, the proof idea or the 'trick' *(C_2)*; in row three, the generalization, which makes it a valid proof *(C_3)*.

Table 2: Blocktowers (subconstructs C_1-C_3 and classification in the RBC-Modell)

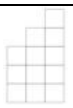
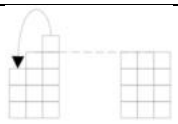
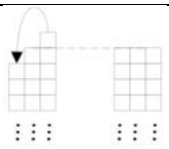
C_1: Three consecutive numbers can be represented as three towers of blocks (cardinality, *R*), where the second number has one block more than the first, and the third has one block more than the second (predecessor and successor, *B*).	
C_2: Putting the top block of the third number on the first number creates a rectangle with a side length of 3 and another side length of the middle number (constancy of the sum, opposing change, *B*). So, for this example, the sum of three consecutive numbers is three times the middle number (in this case, 4) (addition as union of sets, multiplication as continued addition, *R*).	
C_3: Extending or shortening the towers still leads to three consecutive numbers; the sum is always three times the middle number (*C*).	

Table 3: Explaining and visualizing the generality of the values

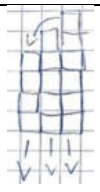
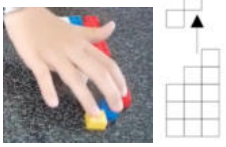
Pupil: Um (...) from 103, one goes to 101. So and so. *(draws an arrow on the top of the blocktower)*  Interviewer: That means you've already drawn 100 boxes here? Pupil: So (...) that I don't (...) have to do so much, I draw something that (...) that it still goes on. *(draws three arrows on the bottom)* […] Interviewer: And for other numbers with the same distance? Pupil: Then it works like this. Interviewer: Always? Pupil: Always!	

Table 4: Placeholder (subconstructs C_1-C_3 and classification in the RBC-Modell)

C_1: Three consecutive numbers can be represented as numerals (numeric notation, predecessor and successor, *R*).	3, 4, 5
C_2: If the summands 3 and 5 are changed in opposite directions (both to 4), the sum remains constant (constancy of the sum, opposing change, *B*). Using the definition of multiplication as continued addition leads to 3·4. So, for this example, the sum of three consecutive numbers is three times the middle number (*R*).	3+4+5 = (4-1)+4+(4+1) = (4-1+1)+4+4 = 4+4+4 = 3·4
C_3: This procedure works no matter which consecutive numbers we choose (*C*).	(□ -1)+□+(□+1) = (□-1+1)+□+□ = □+□+□ = 3·□

Among others, Krumsdorf (2015, p. 192) and Stylianides (2016, p.131) report on the 'blocktower'-construct (Table 2). In the study, two children, 3rd and 4th grade, used this construct. Both justified by verifying a statement by referring to the structural properties of mathematics using a general example (*level 3,* Balacheff, 1988). Particular in the solutions was the visualization of the generality of the values (C_3): starting from big numbers (*principle (2) b)*) the 4th-grader used arrows for visualizing the generality of the values (Table 3, left). The 3rd-grader used clamping blocks and visualized generality by adding or removing the three upper blocks arranged in the form of stairs (Table 3, right).

Among others, Krumsdorf (2015, p. 192) reports on the 'placeholder' construct (Table 4). Six children's solutions can be assigned to this construct, whereby only two children (both grade 4) used placeholders for their arguments. One child said the following *(C_2)* but did not explain why this works no matter which consecutive numbers are chosen *(C_3)*: "You must take one from the largest number and add it to the smallest, and then multiply the number in the middle by three." The others (one 3rd- and three 4th-graders) used a set visualization, which reminds of the visualization in Dilberoğlu et al. (2022, p. 162) and showed a cardinal understanding of the numbers (see Fig. 1). All children justified empirically: five with crucial experiment (*level 2*) and one by naive empiricism (*level 1*).

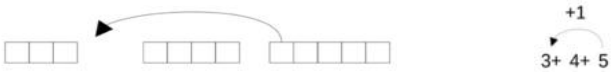


Figure 1: cardinal visualization

RQ2

As the sample is very small, no generalizable statements can be made. However, indications for the optimization of the learning environment can be found. The two children who used the construct 'blocktowers' (Table 2) successfully constructed a proof by generic example. Whereas the six children who used the construct 'placeholders' (Table 4) found the proof idea but did not express any generalization. The fact that children do not express a generalization and remain on C_2 (like in the 'placeholder'-construct) does not mean that they do not understand that their procedure works no matter which consecutive numbers are chosen. One problem could be that the pupils are required to generalize without knowledge of the algebraic language (Akinwunmi, 2012). Furthermore, the generalization may be so evident for the child that it prevents the linguistic effort (Freudenthal, 1978,

p. 262). Additionally, the transition from arithmetic to algebraic thinking (Steinweg, 2013) could cause difficulties here. To support pupils, the negotiation for a common argumentation basis is important, and that the children learn proving, e.g., by conducting a proof that follows a known pattern (Fischer & Malle, 1985; in our case, this could be proving the assertion that 'the sum of five consecutive numbers is five times the middle number' after the pattern for proving such assertions was learned). It is necessary to reflect justifications with pupils: they must learn which justification is accepted or rejected when and why, i.e., which requirements are demanded of a justification (Peterßen, 2012, p. 348). This was not stimulated in the learning environment (Table 1). Therefore, in the project's next step, authentic pupils' solutions will be reenacted and provided as short videos that can be used in the learning environment. To discuss the videos with the pupils, the interviewer can guide the pupils through stages with increasing metacognitive demands (Platz, 2022).

Conclusion

In this paper, the proving activities of eight primary school children (grades 3 and 4) were analyzed to determine which epistemic actions are recognizable in the proof activities of primary school children *(RQ1)*. Based on the results, consequences for the further development of the learning environment were drawn *(RQ2)*. The presented learning environment is rather not suitable for regular teaching, as the attention required from the teacher for the individual child (Wollring, 2008) is too extensive. Nevertheless, targeted further development of mathematical reasoning skills in primary school mathematics teaching seems to be possible in principle (Bezold, 2009; Peterßen, 2012; Vogt, 2020). In the next step, a concept for establishing a culture of justification in the classroom will be developed, including the reenacted video scenes created in the project. This way, justifications in themselves and their respective persuasiveness can be made the subject of teaching (Platz, 2022).

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Evaluating arguments: the role of goals and interlocutors

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Argumentations may serve different functions in mathematical classrooms, such as explaining, verifying or convincing. We believe that considering the function arguments serve in the classroom could influence students' conception and evaluation of arguments. This work aims to investigate students' evaluation of written arguments in two situations: to explain a problem resolution to a peer and in a class test. To this purpose, a questionnaire was administered to nineteen 11th-Grade students at the end of a teaching experiment. The analysis of the students' answers shows that students choose different arguments in the two situations investigated and take into consideration both the interlocutors and the function that the arguments serve in the two situations.

Keywords: Argumentation, explanatory function, probability, upper secondary school.

Introduction and theoretical framework

Mathematics Education researchers investigate argumentations from different perspectives (Mariotti, 2006; 2022), in which different goals, or more precisely, different *functions* of argumentation can be recognised. For example, research conducted by Boero and colleagues, within the framework of Cognitive Unity (e.g. Boero et al., 1996), emphasises the use of argumentation to promote the development of mathematical proofs, particularly focusing on supporting the validity of a claim. The research of Harel and Sowder (1998) delves into personal conviction and the persuasive power of arguments. Studies by Krummheuer (1995) and by Yackel and Cobb (1996) highlight argumentation as a tool to facilitate learning in the classroom, stressing its connection to the learning process. Finally, in Cusi, et al. (2017) argumentation is conceived both as a mean to "make the process visible", thus providing insights into students' thought processes, and as a learning objective in itself. Drawing on the research on functions of proof (e.g. de Villiers, 1990; Hanna, 2000), in Ferrari and Saccoletto (2023) we took into consideration different functions that arguments and argumentations might serve in mathematical classrooms, such as verifying, explaining, or convincing. We believe that the function of the argument plays a role in its production or evaluation, and, in classrooms, the *explanatory function* is particularly crucial. We see that the explanatory function as relational and, therefore, dependent on the interlocutor (Ferrari & Saccoletto, 2023; Stylianides et al., 2016).

To examine the explanatory function of arguments and argumentation, in Ferrari and Saccoletto (2023), we discussed the introduction of a theoretical framework based on the perspective of Mercier and Sperber (2017), related to the investigation of the social evolution of reasoning. According to Mercier and Sperber (2017), arguments and argumentation are developed primarily for social purposes. People produce reasons mainly to justify their claims, actions, or points of view to others. In doing so, they adhere and negotiate norms of what is considered reasonable, and they make their reasoning available for questioning while also asserting their right to question the reasoning of others. In their perspective, interlocutors play a crucial role. They state that "a good argument consists precisely in displaying coherence relationships that the audience can evaluate on their own" (Mercier & Sperber, 2017, p. 195). Coherence relationships should link interlocutor's knowledge and beliefs

to the speaker's claim. Although Mercier and Sperber do not explicitly characterise these coherence relationships, they emphasise the influence of context in producing and evaluating reasoning. In a mathematical classroom context, the links between the speaker's claim and the audience's knowledge and beliefs can take various forms. For example, theorems or propositions are commonly accepted to justify a mathematical result.

Interlocutors influence arguments and argumentative processes. In classroom to enhance the explanatory function of argumentation, we think peers should be the primary interlocutors. However, students also produce arguments directed at the teacher, particularly in evaluative contexts. Interlocutors and argumentative functions are interrelated. This paper investigates which written argument students prefer to justify a problem resolution in two different situations, which differ both for the interlocutors and the function of argumentation:

- explaining the problem resolution to another peer,
- during a class test.

The aim of this work is to study whether students choose different arguments and to reflect on the characteristics of the selected argument(s). Moreover, students' motivations for the selected argument are analysed to understand whether students' choice are influenced by the different interlocutors and or by the different functions.

Method

In the school year 2022-2023 we carried out a teaching experiment in an upper secondary school classroom of 19 students of a "Liceo Scientifico" (a high school with a curriculum that places particular emphasis on scientific disciplines such as mathematics). The mathematics teacher of the classroom was involved in the research group. The teaching experiment lasted from the beginning of March to the middle of April (approximately 40 hours). The activities were carefully designed with the dual aim of fostering argumentative practice and introducing students to probabilistic thinking. No probability concepts had been introduced in the classroom before the teaching experiment. Students were involved in the resolution of different problems related to the classic theory of probability and in discussion about the solution of the problems (see Ferrari and Saccoletto, 2023 for an example of an historical problem used in the activity). During the discussions students were generally asked to explain to others their solutions and to justify them. Explaining and justifying one's reasoning primarily occur in small-groups or whole-class discussion, where peers were the main interlocutors. The aim of the activities was not to convince others at all costs to agree with one's own ideas, but to foster sharing, understanding, and supporting collective learning. With this purpose, students were prompted to understand others' ideas and perspectives, and to elaborate on the rationale behind their choice. The explanatory function of argumentation was at the core of the activity, with peers as the primary interlocutors.

Among the final activities of the teaching experiment, we administered a questionnaire to students, which is at the centre of this paper. Eighteen out of nineteen students participated in the questionnaire.

In the questionnaire we presented students with a problem similar to those they had previously encountered in classroom activities. The problem, translated in English by the author, was:

According to an ISTAT[1] survey, in Italy, 34% of people (male or female) aged three or over engage in at least one sport. Among those who practise a sport, 60% are male, while among those who are sedentary, that is, those who do not engage in any physical activity, 40% are male.

a. *What is the probability that a randomly selected person aged three or over is female and engages in at least one sport?*
b. *What is the probability that a randomly selected person from the total population (aged three or over) is male?*
c. *A person aged three or over is randomly selected from the total population, and it is known that they are male. What is the probability that this person is sedentary?*

Three different solutions followed the problem: Carlotta's, Enrico's and Diletta's solutions. The solutions were created ad hoc by the author of the paper and discussed with the teacher of the classroom. All the solutions could be acceptable according to the classical probabilistic approach, but they comprised different arguments to sustain the results. Particularly, they differ in the visual representation used in the text: Carlotta's and Enrico's solutions primarily employ visual representations, namely a table and a tree diagram. Diletta's solution is strongly characterized by the use of formulas and theory recall. Tables, tree diagrams and formulas were all encountered and learned by the students in classrooms activities.

Below, we present the author's translation of the three solutions. We are aware that translation may affect the texts, however we think major differences are still recognisable.

Solution 1 (Carlotta's)

Table 1: Table of Carlotta's solution

	Engage in at least one sport	Sedentary	Engage in at least one sport or sedentary
Males	0.34•0.6=0.204	0.66 • 0.4=0.264	0.468
Females	0.34•0.4=0.136	0.66 • 0.6=0.396	0.532
Males and Females	0.34	0.66	1

a. The probability that the person is female and engages in at least one sport is 0.136, i.e. 13.6%.
b. The probability that the person is male is 0.468, i.e. 46.8%.
c. The probability that the person is sedentary, given that they are male, is $0.66 \cdot 0.4 : 0.468 = 0.564$, i.e. 56.4%

Solution 2 (Enrico's)

[1] ISTAT is the Italian National Institute of Statistics.

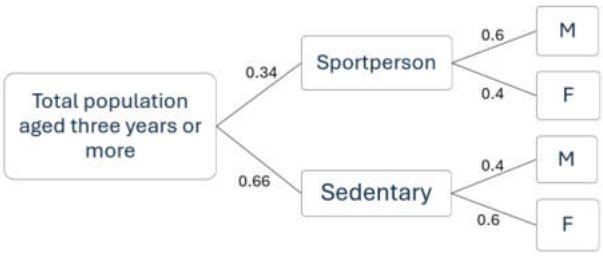

Figure 1: Tree Diagram of Enrico's solution

a. The probability that the person is female and engages in at least one sport is $0.34 \cdot 0.4 = 0.136$, i.e. 13,6%.
b. The probability that the person is male is $0.6 \cdot 0.34 + 0.4 \cdot 0.66 = 0.468$, i.e. 46.8%.
c. The probability that the person is sedentary, given that they are male, is $0.66 \cdot 0.4 : 0.468 = 0.564$, i.e. 56.4%

Solution 3 (Diletta's)

a. If among those who practice sports, 60% are male, then 40% are female. Let M denote being male, F denote being female, S denote being sedentary, and *NS* denote being non-sedentary. Then: $P(F \cap S) = P(F) \cdot P(S) = 0.4 \cdot 0.34 = 0.136$, i.e. 13.6%
b. The probability of being male is given by the sum of the probabilities of the two mutually exclusive events $M \cap S$ and $M \cap NS$. Then: $P(M) = P(M \cap S) + P(M \cap NS) = P(M) * P(S) + P(M) \cdot P(NS) = P(M) \cdot P(S) + P(M) \cdot (100\% - P(S)) = 0.6 \cdot 0.34 + 0.4\% \cdot 0.66 = 0.468$, i.e. 46.8%.
c. The probability that the person is sedentary (NS), given that they are male (M), is a conditional probability. $P(NS \mid M) = P(M \cap NS) : P(M) = (0.4 \cdot 0.66) : 0.468\% = 0.564$, i.e. 56.4%.

At the end of the questionnaire, students were asked to choose an argument to explain which solution they would employ to explain the problem resolution to other students and which one they would use in a class-test. In addition, students were required to carefully justify their choices. The two situations involve different interlocutors (peers and the teacher) and serve different functions (explaining to someone who does not know the resolution and validate one's results, showing one's reasoning to the professor in an evaluative context). Through this questionnaire, we aimed to explore whether students' choices were different in the two situations and the reasons for the various choices. Students' answers were analysed qualitatively using thematic analysis (Cohen et al., 2018) in order to identify the reasons behind students' choices.

Results

Sixteen out of eighteen students chose Enrico's solution as the most suitable for explaining the resolution of the problem to another student. One student selected Carlotta's solution, and one chose Diletta's.

From the analysis of students' justifications, it emerges that students value the graphical aspect: Enrico's solution is considered the easiest to understand, thanks to the tree diagram, which enables the steps to be visualised. Carlotta's solution is seen as less clear and intuitive, although the graphical

element is generally appreciated, while Diletta's solution is regarded as the least comprehensible. For example, two students state:

> [S6] I chose Enrico's solution because, in my opinion, it is the easiest to understand. Even visually, you immediately grasp the procedure to follow to reach the solution. I find it to be an intuitive and immediate method for understanding the problem. I excluded Carlotta's method because I find it harder to understand and less immediate than the previous one; it's easy to misread the table. In my opinion, the third method helps the least in understanding the problem because it develops the process purely through laws without explaining or providing a different visual perspective of the problem, as the other two methods do. […]

> [S18] I think Enrico's solution is the most suitable because it uses an effective graphical representation (a tree diagram), making it easier to explain the problem and helping classmates understand it (as they have a reference point to follow). While Diletta's solution is correct and justified in all steps (using proven formulas), it could be a bit harder to understand, as some might not grasp why a certain formula is being used or may not fully comprehend it. Carlotta's solution also uses a graphical representation (a table), but I find it less effective and harder to understand compared to the tree diagram.

The diagram is also positively evaluated as it allows calculations to be performed more quickly or explained more effectively.

> [S2] […] If I had to explain it to someone for the first time, drawing a tree diagram on how the exercise is structured would be extremely effective, as all the data would be easily noticeable and would allow me to explain how to calculate the percentage (40% or 60%) of one of the two initial percentages given (34% and 66%).

> [S3] Because the diagram helps to reach the solution more quickly […].

Finally, students emphasize the accessibility of Enrico's solution, which is considered easier to understand by a greater number of people and accessible without requiring specific knowledge. For instance, two students state:

> [S9] The second response explains the problem's resolution even to those who do not have an intuition for probability, thus making it easier for a wider variety of people to comprehend compared to the other two methods. The third response is a translation of the solution into mathematical language, and if one does not master it, it becomes difficult to draw the path to the solution […].

> [S11] I chose Enrico's explanation simply because it is a very simple and understandable explanation – meaning everyone will be able to grasp it. […] Personally, I am not a mathematician, but I enjoy maths. I say this because I couldn't quite understand Diletta's explanation of the problem, as it is based solely on mathematical language […].

To sum up, the focus is on comprehension, intuitiveness, ease of performing or explaining the calculations, and the accessibility of the solution.

Regarding the in-class test, twelve students indicated Diletta's solution as the most appropriate, five chose Enrico's solution, and one selected Carlotta's. Among the five students who chose Enrico's solution, two noted that although they would choose Enrico's solution, their professor would likely prefer Diletta's. Among the twelve students who indicated Diletta's solution, four stated they would probably add Enrico's tree diagram to help them solve the problem.

The analysis of students' justifications indicates that students focus on the use of formulas and symbolic language. Diletta's solution is generally considered the most formal among the three solutions:

> [S9] The first and second might be more intuitive, but they are informal. The third is academic and formal, as it uses fundamental probability laws. Moreover, it helps to generalise the resolution for these kinds of problems.

Additionally, students consider the use of formulas and symbolic language to be the most appropriate way to validate a result:

> [S8] Diletta's solution, while a bit less immediate than the others, is the only one that directly refers to the theory demonstrated in class, and therefore cannot be questioned […].

> [S6] I would choose the third because it appropriately justifies the method and the calculations used through established theorems, while the other two methods do not use theorems or laws but are simply based on diagrams that help to understand the problem. However, the calculations are not supported by formulas, as I believe they should be in a maths test, because formulas increase the validity of the response.

Finally, Diletta's solution is considered the most appreciated by the professor:

> [S11] […] it [Diletta's solution] is also the preferred method of writing for all professors, meaning our teachers want us to reach this kind of interpretation of the problem.

> [S10] I would use Diletta's response because it contains the formulas that our professor requires. However, I would also include a tree diagram to help with the calculations.

Student 10 is an example of someone who would use the tree diagram to solve the problem but still chose Diletta's solution as the most appropriate for a class test. The usefulness of the tree diagram in solving the problem is the most common reason students give for selecting Enrico's solution in an in-class test, as these answers show:

> [S12] I find it much easier to represent problems with tree diagrams, so I chose this solution. However, in a test, the professor would probably also want to understand the process, so I would likely use the final solution to describe everything.

Discussion and conclusion

The analysis shows that students' preferences differ in the two situations investigated: Enrico's solution is generally considered the best to explain the problem resolution to a peer, while Diletta's solution is the most appropriate in a class-test. In other words, there is not a "best argument" to support the result, but arguments are evaluated according to the situation which is influenced both by

the interlocutor and the function of the argument. From the analysis of students' motivation of the answers, it emerges that their choices are influenced both by the interlocutors and by the function. In the following we try to distinguish the two variables; however, we recognise a strong link between the two. First, in motivating their choices students take into consideration the interlocutor. For instance, they think about what the interlocutor might (not) know. This is in line with the perspective of Mercier and Sperber, in which the *knowledge of the interlocutor* are to be considered when developing an argument. Enrico's solution is generally considered more accessible than Diletta's, as the latter contains formulas and a symbolic language that not all might be familiar with. In this sense a tree diagram is considered a way of displaying *coherence relationships* more approachable than Diletta's discursive solution to explain to a peer. Conversely, when the teacher is the main interlocutor students prefer formulas and symbolic language, without doubting the knowledge of the teacher. Moreover, in a class-test, students think about what an interlocutor might prefer. For example, students state "It is also the preferred method of writing for all professors" or "It contains the formulas that our professor requires", highlighting the importance of adapting the texts to the interlocutors. These answers might be considered from the point of view of the pragmatic linguistic (e.g. Sperber & Wilson, 1995), in which Mercier and Sperber's (2017) perspective is rooted: students choose the text that, in their view, is the most suitable for communicating with a teacher, their intention is likely also to show they master symbolic and technical language. Moreover, in giving reasons students adhere or negotiate norms of what is reasonable (Mercier & Sperber, 2017). Thus, the norms established in the classroom, such as the *didactic contract* (Brousseau et al., 2020), likely influence these answers. Second, students take the function of the argument into account. On one hand the visual representations used in Enrico's and Carlotta's solutions are positively evaluated to explain, as they help illustrate calculus and problem resolution steps. On the other hand, formulas are considered the best way to validate a statement and to show the own knowledge of probability theory. However, some students choose Enrico's solution for the class-test, as it is the way they solve the problem they are most comfortable with.

Being able to take into account the interlocutor and the function of arguments is in line with competencies related to argumentation framed in literature, such as the competence framework of Niss (2003) and the *argumentive competencies* by Kuhn (1992).

The influence of the interlocutor(s) and of the function of the argument point out the importance of considering linguistic aspects of the argument. We are currently further investigating the characteristics of explanatory arguments drawing on the Theory of Relevance (Sperber & Wilson, 1995), which we believe will be useful in better characterising text differences and in deepening the analysis conducted.

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Using *FullProof* to develop geometrical reasoning and proof skills within a practicum course for preservice teachers

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This paper describes the technological pedagogical content knowledge (TPACK) specific to proof of three prospective secondary mathematics teachers (PSTs) who worked with FullProof - the innovative AI-style tool for writing geometry proofs. The PSTs were introduced to the tool to develop their ability to facilitate instruction, promote geometrical reasoning and proof-writing, and strengthen their TPACK. Two mathematics teacher educators (MTEs) co-taught the course. Data were collected via four video-recorded lessons. The results suggest that the use of FullProof-mediated learning experiences aided in strengthening the PSTs' TPACK knowledge and enhanced their confidence in teaching geometrical proofs. The results illustrate how FullProof can support the development of PSTs' TPACK about proof in geometry and highlight the role of MTEs in facilitating PSTs' learning.

Keywords: Future teachers, geometry, reasoning and proof, technological pedagogical content knowledge.

Introduction

There is a need to develop preservice teachers (PSTs) and in-service teachers' proof skills to support students learning how to prove. The use of technological tools during instruction can afford opportunities for individuals to explore concepts, shift the foci of what individuals attend to, impact what is learned and by whom it is learned, strengthen connections, support knowledge construction, and provide intelligent support (Hoyles, 2018). In recent times, advancements in artificial intelligence resulted in technological tools that can support students in proof writing and validating the accuracy and completeness of the proof. *FullProof* is a digital platform specifically developed to help students advance their proof-writing skills in Euclidean Geometry. Hence, we sought to examine how *FullProof* can be used to develop PST's technological pedagogical content knowledge (TPACK) about proof and proving, with the support of two mathematics teacher educators. The overarching research question for this study is: *How does the use of FullProof in a secondary mathematics education practicum course affect PSTs' TPACK knowledge about proof in geometry?*

FullProof – the innovative tool for supporting proof writing

FullProof is a digital platform developed to advance proof-writing skills in Euclidean Geometry. It holds a vast collection of proof problems at high, medium, and low difficulty levels. Each problem is accompanied by a static figure whose angles and segments become highlighted when pointed to; they can also be marked. Students write proofs in the two or three-column format by typing statements or equations and choosing supporting justifications from the menu of theorems and definitions. For

example, in Figure 1, it is given that $\overline{DE}$ is a midsegment of ΔABC. The length $BF = (1/2)FC$. The segment $\overline{AF}$ intersects $\overline{DE}$ at a point G and the segment $\overline{BE}$ at a point O. Prove that $\overline{BO} \cong \overline{EO}$. If students do not know how to proceed, the software provides hints. *FullProof* also checks the completed proof and provides line-by-line feedback indicating correct, partially correct, or incorrect steps, as well as a hint into the nature of the mistake.

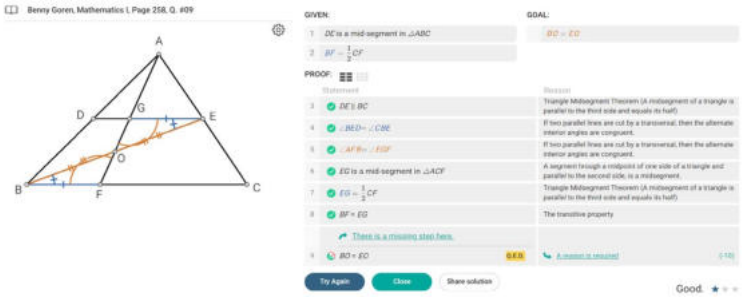

Figure 1: Sample solution in *FullProof* and the feedback provided by the software

The unique feature of *FullProof* is that it allows for multiple solution paths as long as the proof is mathematically correct. For example, the proof in Figure 1 can be completed by relying on congruent triangles or using centroid properties. Correcting their mistakes and resubmitting solutions allows students to improve their proofs, which makes *FullProof* a valuable formative assessment tool.

Theoretical Perspective

The Technological Pedagogical Content Knowledge (TPACK) framework (Mishra, 2019) is the theoretical lens used for this study. TPACK has been widely used to conceptualize the integration of technology into mathematics teaching and the various facets of teacher knowledge involved in this integration. The TPACK framework (Figure 2) comprised three knowledge domains: Technological (TK), Pedagogical (PK), and Content (CK), and their intersections. The TK domain involves teachers' familiarity with various technological tools and the ability to integrate these tools into mathematics instruction effectively. The PK domain considers teachers' ability to create effective learning environments to support student mathematical learning and engagement. Teachers' CK provides robust mathematical knowledge (Mishra, 2019). In the context of teaching geometry, CK considers the teacher's knowledge of geometric concepts, theorems, and modes of argumentation and proving, including their ability to write geometry proofs. Similarly, PK implies promoting a culture of proving in the classroom by designing learning opportunities to construct proofs, strengthen students' ability to communicate their arguments, and evaluate the arguments of others. TK involves knowledge of technological tools that can support students' proof processes.

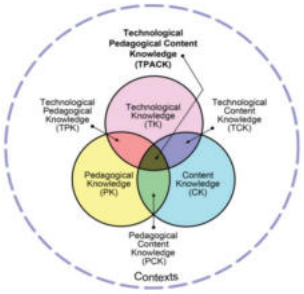

Figure 2. TPACK Framework (Mishra, 2019, p. 77)

Intersections between and among knowledge domains can occur. Teachers draw on their Technological Pedagogical Knowledge (TPK), i.e., the combination of TK and PK, when they consider how particular technological tools can support (or constrain) students' understanding of geometry concepts or developing certain geometry skills and how to structure students' experiences with technology to aid their learning. Similarly, Technological Content Knowledge (TCK) enables teachers to understand how geometry content is represented through a particular technological tool and what affordances it provides to communicate mathematics.

Method

We conducted an exploratory study in a university practicum course for PSTs enrolled in a secondary mathematics education program in the southeast region of the United States. Data were collected via a video recording of lessons and analyzed using thematic analysis. In the subsequent sections, additional information is provided about the course context, data collection, and analysis employed.

Course context

The practicum course is taken the semester before PSTs complete their final internship. The course met virtually for 2 hours and 45 minutes weekly for a 15-week semester. Three PSTs participated in the course. For the course, the PSTs were required to engage in clinical experiences for at least 60 hours. They were assigned to teach geometry lessons in which they integrated technology and strengthened students' reasoning and proof skills. The first author (MTE1) of the paper was the course instructor, the second author (MTE2) led the discussions devoted to learning how to use features of *FullProof* and participated in the sessions in which the PSTs reported on their use of the *FullProof* in their classrooms.

Participants

There were three PSTs in the practicum course who were in the 18-25 age group. The PSTs were in the fourth year of their undergraduate program, having typically completed coursework in Calculus, Discrete mathematics, Number theory, Geometry, History of mathematics, and Statistics. They also would have completed a course on technology and methods of teaching secondary mathematics. Pseudonyms are used to refer to the three PSTs (Alpha, Beta, and Gamma).

Data collection and analysis

Data were collected via recording of virtual class meetings in Fall 2023. One session was devoted to an overview of *FullProof* (August 31, 2023), and three sessions (September 7, 14, and 26, 2023) were allocated for PST to teach with *FullProof* and reflect on the enacted lessons. During the first session, MTE2 provided an overview of *FullProof*, and MTE1 provided insight into how the tool could support the teaching and learning of proof in geometry, with consideration given to evidence-based teaching practices. The preliminary training sought to aid PSTs in becoming comfortable with the tool, to increase the likelihood of using the tool to teach proof, and to support students' ability to prove. Both MTEs supported the discussion focusing on geometrical content knowledge. For the subsequent lessons, PSTs were required to engage with *FullProof* and teach their fellow classmates a 30-minute lesson using it. The mathematics teacher educators (MTEs) emphasized the domains of TPACK during their facilitation and supported PSTs in developing a conceptual understanding of the geometrical concepts, reflecting on effective pedagogical practices for helping the development of proof skills and familiarizing PSTs with varied features of *FullProof*.

The video recordings of the sessions ranged from 60 minutes to 90 minutes. They were transcribed for the analysis. The didactical discourse of the PSTs and the MTEs was analyzed using thematic analysis (Braun & Clarke, 2012), referring to the domains of the TPACK Framework, particularly referring to the focus on mathematics, pedagogy, and the role of technology. The roles of the two MTEs in facilitating learning and the nature of the didactical discourse among the PSTs and the MTEs were also examined. The data was subsequently organized by emergent themes.

Results

In the subsequent sections, we provide evidence supporting the outcomes of our analysis, which suggest that *FullProof* positively affected PSTs' TPACK knowledge regarding the teaching and learning of geometrical proofs. The MTEs played an essential role in supporting PSTs' development of TPACK knowledge and engaging in reflective practices. The MTEs modeled how to develop complete geometrical proofs and made explicit connections to the intersection of TPACK, where the tool created opportunities to reflect on pedagogical approaches such as questioning strategies. Productive struggles emerged as PSTs reflected on why their statements were incorrect and experimented to find solutions. The PSTs used *FullProof* to illustrate statements and reasons and make connections between steps. The immediate feedback provided by the *FullProof* system validated the accuracy of PSTs' progression in constructing proof and enhanced their comfort level. We organize this section to provide insight into how the domains of TPACK and intersections among them were attended to.

Technological Knowledge

Providing adequate training on *FullProof* and its features supports the likelihood that PSTs would actually use it within their practice. Thus, the first session focused on having PSTs become familiar with the tool features and elements. MTE2 noted that as PSTs explore the tool, they may notice a few instances of differences in notation since the tool was developed outside of the US. She said, "You will be able to... poke around and try it out... But you might see some things that are a little bit different

than the American notations." Nonetheless, the PSTs acknowledged the tool was generally user-friendly and helped clarify how to write proof. For instance, Alpha said: "I found it kind of simplified it for me, where I was like, oh yeah, I do remember this from high school. And it helped break certain ideas down back to me […] instead of having just definitions thrown at me, it kind of broke it down in a way of, oh, that's how I do this in a proof again!" (August 31, 2023). Similarly, Gamma indicated, "I felt like a good word to describe it [*FullProof*] is intuitive. I feel like it was pretty easy to follow along with how to use it. Um, and I think it gave good feedback and hints. […] Honestly, I wish that I had it when I was learning how to do proofs because I still struggle with proofs to this day. Because I didn't have stuff like this" (August 31, 2023).

The PSTs also shared strategies to utilize the tool with each other. For instance, the vignette below illustrates a PST sharing a shortcut in labeling angles with a peer.

Gamma: I don't know if this will help you with the type typing part, but instead of having to type out B, F, D, or whatever, you can also click on the angle if that makes it easier for you as the person typing.
Beta: I had no idea. Oh, my goodness. I love it. (October 26, 2023)

Thus, the PSTs were becoming proficient with the use of *FullProof,* such that they could help each other rather than rely solely on the MTEs for support.

Pedagogical Knowledge

As PSTs explored geometry proof, the MTEs pressed them to reflect on pedagogical considerations that can impact student learning outcomes. The importance of PCK in teaching proof was emphasized to ensure the proof task maintained cognitive rigor during enacted lessons. Throughout the lessons, MTE 1 frequently asked the PSTs to share how they knew learning occurred, saying: "What would be your evidence of [student] learning based on working through the problem? How did you know they learned anything?" (October 26, 2023). Beta responded by identifying mathematical objectives related to how interior angles and alternate interior angles can be used to structure the proof, which was among the goals for learning.

The PSTs also frequently reflected on practices that can promote reasoning and potentially address students' misconceptions. For instance, Gamma talked about how they "tend to encourage productive struggle […] try questions that probe for reasoning. […] because that's just an important part of making sure that the students are actually understanding" (September 14, 2023). The MTE1 further encouraged Gamma to engage in self-reflection on what students are "looking for" when trying to construct a proof. The PSTs' awareness of how to question and support productive struggle is critical to ensuring that students can actively engage in the proof process in a meaningful way. It also helps to ensure they can address misconceptions that may arise relative to a concept and to proving theorems. Therefore, to promote the development of pedagogical knowledge, PSTs were encouraged to reflect on their instructional decisions and the extent to which they would support learning.

The MTEs encouraged the PSTs to consider the implications of projecting their perspectives about proof onto their future students and think like teachers. MTE1 said: "Reflect on what knowledge you need to support your students. What knowledge will students need? And how do you bridge this

knowledge that you need to help your students make sense of it?" Similar sentiments were shared by Gamma, who noted, "... So I always like to start proofs with asking what do we know and what's our goal? So, we kind of have a sense of direction" (September 14, 2023).

During the lessons, consideration was also given to the learning experience. The following vignette illustrates a PST reflection on their experience. The PST indicated that their peer facilitation of the proof catalyzed productive struggle as they questioned whether the justification for the concluding statement in the proof was required.

MTE 1: Okay. Now, so you were the teacher [Beta]. Now I'm going to ask the question to the learner. …what was your learning experience that [Beta] was facilitating? Because we want to talk about the experience of the teacher, which may not necessarily be that of the learner. What was happening as [Beta] engaged in this space? What were your thoughts?

Gamma: Um, I thought that [Beta] was doing a good job at letting us have the productive struggle and giving us the space to try things when we weren't sure if uh, on the last line we had to put, reasoning. [Beta said,] Well, let's try it with and without [justification] and see if it's considered correct [by *FullProof*]. Because I personally struggle with the proper way to write certain proofs. So that is something … I don't know if that's correct, and I never really get to check both ways.

The PST response suggests that the support offered emphasized providing reasoning and multiple ways of writing the proof despite initially struggling with it.

Content Knowledge

The analysis revealed that PSTs' initial CK of proof was lacking. All three of them admitted that they had not encountered proofs extensively besides brief exposure in a geometry course in high school and at the university. The PSTs were uncomfortable with the idea that, as teachers, they would need to teach geometry proofs. For example, Beta said about the lesson plan they needed to write for the practicum: "I was scrolling through the standards […] I saw the proofs, and I [said], "I'm getting away from that. I do not want to touch that because just the idea of teaching it, and I mean, at least how I remember it in high school, is just so frustrating." The other two PSTs shared similar thoughts and feelings, saying things like "It has been a while since I've used geometric knowledge," that proofs are "scary," and "intimidating" (Alpha and Gamma, August 31, 2023).

As PSTs engaged with *FullProof* as students and then attempted to use it to teach each other various proof problems, they gained more competence in proving; consequently, their confidence in their ability to teach proof (with *FullProof*) increased. For example, Alpha appreciated that the tool allowed for multiple proving paths, saying that "forcing" teachers' way of thinking onto students will "make them [students] miserable." Instead, teachers can utilize *FullProof* to "build on their logic" (October 26, 20203).

The PSTs also sought to provide a rationale for how they constructed a proof with *FullProof*. For example, in the vignette below, Alpha explained how they constructed the proof in Figure 3.

Figure 3: Proof involving an angle bisector (Benny Goren, Geometry 1, page 241, Q.#06)

> … what I did for *FullProof*, the first thing I immediately went for was looking at the goal and breaking down what I needed to do with the goal. So, I knew I needed to find three different items to prove that these two triangles would be congruent. […] So, I looked into, […] try and use one of the various theorems, like Angle-Side-Angle and all that. Uh, the first thing I then went from there is to look into the givens. So I noticed that since we know that AC bisects the angle BAD […] I was able to go ahead and say that the angle ∡BAC is congruent to the angle ∡CAD, and ∡ACB, is congruent to the angle ∡ACD from being bisected. Uh, then from there, I looked at the triangle, and I noticed that since they share AC… So, it took me a bit to remember it was reflexivity as it says. But then, from there, I was able to get that since I knew I have two angles and a side, I could say the Angle-Side-Angle congruence theorem. I would say I definitely would want to define more stuff (Alpha, September 7, 2023)

This vignette shows the PSTs utilized theorems and definitions to justify the proof constructed related to congruent triangles and indicated PST's desire to provide additional information.

The Intersection of Various Domains of TPACK

The instructional dialogues often promoted the intersection of various TPACK knowledge domains. For instance, the overlap between technological and content knowledge became evident through didactical discourse in which the tool helped PSTs reflect on appropriate mathematical vocabulary and logical progression of proofs. *FullProof* helped PSTs remember definitions, visualize concepts, and probe for understanding. The following vignette illustrates the intersection of TK and CK. In the dialogue, the PST realized that *FullProo*f was not crediting the response of bisector, since the word should be perpendicular.

> Beta: ... I was just, I was just meaning that they were, that they were perpendicular and
> MTE 2: Perpendicular. Exactly. That's why the word bisector is not appropriate there. That's why the software is totally ... alerting.
> Beta: Okay, that makes a lot more sense. I don't know why I kept saying bisector. (September 7, 2023)

Thus, the tool supported the PST in using the correct terminology.

Discussion and Conclusion

Our study relates to the broader issue of how to prepare preservice teachers to teach proof, which has been consistently shown to be hard for teachers to understand and be able to teach. We focus on the

intersection of geometry and technology in this space. We also acknowledge the role of MTE in promoting PSTs' reflection on their didactical discourse and in supporting the development of PSTs' TPACK and proving competence. Our findings have implications for teacher preparation programs that support the development of PSTs' proof skills. Given that the PSTs acknowledged their fragile understanding of proving, the use of *FullProof* mediated the learning of geometrical content and writing proofs. The didactical discourse that the MTEs facilitated during the lessons supported PSTs in making sense of the content and encouraged them to apply their enhanced TPACK knowledge in their future classrooms. The immediate feedback offered in *FullProof* contributed to PSTs' confidence in writing proofs. This feature of the tool also contributed to rich discussions as to why the progression across proof steps was logical. It also emphasized the need to know appropriate theorems and definitions that can support claims made, thereby enriching PSTs' content knowledge. Finally, the vignettes shared illustrated questioning strategies MTEs could employ to have PSTs reflect on how to teach proof and support their future students in learning to prove. The co-teaching approach utilized by the MTEs provided opportunities for students to gain multiple insights and reflect on how they can emulate similar practices with their collaborative teacher during their clinical experiences.

In closing, this study provided insights into how MTEs within a practicum course intentionally sought to develop PSTs' proof skills. It illustrated that once PSTs become familiar with the features of *FullProof*, the tool could be used to mediate learning opportunities and support PST's development of TPACK, which is needed to support the teaching and learning of proof in geometry. The ability to adjust the level of difficulty of tasks posed could promote differentiated instruction. PSTs utilized *FullProof* to assess the accuracy of their proof and the justification for the reasoning provided. This helped the PSTs to conceptualize how to write a proof that progresses from a given to a concluding statement logically and on the suitability of mathematical definitions and theorems to be included in proofs about congruent triangles. Furthermore, the use of *FullProof* promoted scaffolding and students' engagement with geometry. Therefore, the study has implications for how technology can be used to enhance the teaching and learning of proof in geometry. Nevertheless, future studies are needed to examine further co-teaching efforts to support the development of teaching proof in teacher preparation programs.

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Exploring how affect-related constructs predict students' proof construction competence in algebra and geometry

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Research suggests that proof construction, an important proof competence, is not an isolated cognitive construct but rather may relate with certain affect-related constructs including students' productive disposition beliefs towards proof construction. Yet there is a scarcity of research on how proof construction competence may relate with such beliefs. In this paper, we take a step towards addressing this research gap. Based on preliminary analysis of survey data from 226 secondary students in England, we report an empirically-based and theoretically-meaningful model depicting how certain student beliefs concerning proof construction (affect-related constructs) predict, in isomorphic ways in algebra and geometry, students' proof construction competence (cognitive construct). Implications for research/practice and limitations are discussed in light of these findings.

Keywords: Proof construction, affect, beliefs, cognition, secondary mathematics, algebra, geometry.

Introduction

Proof construction is challenging for students, attracting considerable research attention (for a review, see Stylianides et al., 2017). Some research showed a connection between proof construction and affect-related constructs such as the *value* students place on this activity (Weber et al., 2020). Yet the field still lacks an empirically-based and theoretically-meaningful model depicting possible relationships between students' cognitive and affect-related constructs in the area of proof construction. While existing research recognizes the intertwined nature of cognitive and affect-related constructs, understanding their relationship is crucial not only within specific mathematical topics/areas (Jacobson et al., 2018), such as proof construction, but also across content domains (e.g., algebra, geometry) that may influence variably proof-related behaviour (Dawkins & Karunakaran, 2016). We recognize the potential bidirectional relationship between affect-related and cognitive constructs. Due to space constraints, we limit our exploration in this paper to the following questions: *Do certain student beliefs concerning proof construction (affect-related constructs) predict students' proof construction competence (cognitive construct)? If so, what model describes these relationships, and how does this model vary (if at all) by the content domains of algebra and geometry?*

Prior research and theoretical framework

Several theoretical frameworks have addressed the relationship between cognitive and affect-related constructs for mathematics students and teachers. Some frameworks used "knowledge" broadly to encompass both cognitive and affect-related constructs including beliefs (e.g., Ponte & Chapman, 2008). Others viewed "knowledge" as a cognitive construct separate from beliefs, acknowledging its intertwined nature with beliefs (e.g., Jacobson et al., 2018). We align with the latter perspective, though we do not associate knowledge exclusively with an individual's cognitive domain, choosing instead the notion of *(proof construction) competence* as our focal cognitive construct.

Proof construction competence (cognitive construct)

General proof competence comprises several constituent competencies including proof construction, our focus herein, and others like proof evaluation and proof comprehension (Stylianides et al., 2017). By *proof construction* we mean the activity to develop a proof to a given or a self-generated statement. The outcome of this activity is an argument for the truth or falsity of the statement that a teacher or a researcher can judge against a specified standard of proof to measure the prover's competence in proof construction. Several hierarchical frameworks have been developed to categorize students' constructed arguments for *true* statements (e.g., Harel & Sowder, 2007; Stylianides & Stylianides, 2009); the categorizations range from non-proof arguments, including *empirical arguments*, to arguments qualifying as proofs. In contrast, there is a scarcity of frameworks to categorize students' constructed arguments for *false* statements (Stylianides et al., 2017), posing methodological challenges in achieving a holistic measurement of proof construction competence.

Productive disposition beliefs towards proof construction (affect-related constructs)

Among the various perspectives on beliefs, we follow Jacobson et al.'s (2018) notion of *productive disposition beliefs* defined as topic-specific affect-related constructs with empirical links to students' mathematics learning. Productive disposition encompasses malleable beliefs, attitudes, and emotions (Jacobson & Kilpatrick, 2015) that extend beyond cognition. Thus, we distinguish between cognitive and affective constructs concerning proof construction, while acknowledging they are interrelated, interdependent, dynamic, and malleable (Jacobson et al., 2018). The literature is broad in terms of the objects of belief, or the scope of the mathematical content about which an affective evaluation is made (Jacobson et al., 2018). Some studies address beliefs related to mathematics in a general sense, others focus on specific mathematical domains, or even narrower topics such as teachers' beliefs related to teaching the topic of multidigit addition and subtraction (Jacobson et al., 2018). The latter perspective suggests a relationship between cognitive and affect-related constructs that is dependent on the mathematical topic or domain. This perspective forms the basis for our research, which investigates task-based beliefs specific to proof construction in algebra and geometry.

Diversity in the literature on beliefs also concerns their role in an individual's behavior. Some view them as "predispositions to action" (Rokeach, 1968, cited in Leatham, 2006, p. 92) while others argue that they "must be inferred from what people say, intend, and do" (Pajares, 1992, p. 207). Our research departs from dichotomous perspectives and is based on the premise that beliefs are held with varying degrees of conviction that can be task- or topic- dependent (Jacobson et al., 2018). Furthermore, we posit that beliefs about proof construction can be self-reported and related to proof construction competence in a dialectical manner (Saleh et al., 2023). Productive disposition beliefs comprise multiple affective traits that influence mathematical competence in a particular task or topic (Jacobson & Kilpatrick, 2015), in our case, proof construction. In the framework of *productive disposition beliefs towards proof construction* that we use in our study, we consider five core beliefs.

The first two, *anxiety* and *value,* derive from Jacobson et al.'s (2018) framework and describe, respectively, how students feel when engaging in proof construction ("I feel nervous when I try to answer questions like this one") and how much they value it ("It is important for me to be able to answer questions like this one"). Anxiety and value are often overlooked in proof construction

research. Yet there is evidence that anxiety can negatively affect students' performance on geometric proof construction tasks (Shimizu, 2022), while the value an individual perceives in obtaining high levels of conviction can influence their proof construction competence (Weber et al., 2020).

The previous two, alongside the third belief, *self-concept of ability*, derive from Jacobson et al.'s (2018) framework; they are all based on the expectancy value theory (Eccles & Wigfield, 2002) and were connected to proof construction by Weber et al. (2020). Self-concept of ability is similar to, but distinct from, *self-efficacy beliefs*, our fourth focal belief, also deriving from Jacobson et al. (2018). A main distinction between self-concept of ability and self-efficacy beliefs is that the former embodies a past-oriented perception while the latter denotes a future-oriented conception of one's ability to construct a proof (Bong & Skaalvik, 2003). Particularly, self-concept of ability refers to how good one believes they are engaging in proof construction ("I am good at answering questions like this one"), while self-efficacy is associated with the amount and length of effort one might expend on a task ("If I try hard, I can figure out questions like this one"), indicating an individual's confidence in their ability to overcome factors that could make proof construction difficult.

The fifth productive disposition belief, *self-evaluation beliefs*, derives from our prior research (Stylianides & Stylianides, 2009) and concerns students' beliefs about whether their arguments meet the standard of proof ("I have proved that my answer is correct"). Specifically, we found these beliefs could distinguish individuals based on their awareness of the mathematical quality of their arguments, thereby exhibiting different levels of understanding of proof. *Self-evaluation beliefs* are likely related with *self-concept of ability* and *self-efficacy beliefs*: students' subjective evaluations of their proof constructions in specific tasks (self-evaluation beliefs) may be related with their perceptions about how good they think they are at responding to proof construction tasks (self-concept of ability) and their perceptions about the role of effort in responding to such tasks (self-efficacy beliefs).

Methods

The findings are part of our ongoing analysis of data we collected from a survey study with 226 Year 10 students (14–15-year-olds) in England. The students responded to one or both of the algebra and geometry versions of the Reasoning-and-Proving Instrument (Stylianides & Stylianides, 2022b). This is a newly developed instrument that addresses several important competencies in the area of reasoning-and-proving including proof construction. Its two versions, one in algebra and the other in geometry, were designed to be isomorphic. The instrument is literature-based and has passed three rounds of review and revision, including two rounds of expert validation each with three mathematics education experts in the area of proof, and a review regarding clarity of instructions and age appropriateness by four experienced teachers.

Student participants and procedure

The instrument was administered to the students by their teachers. The students were of variable mathematics achievement, as determined by school data. We asked the teachers, if possible, to administer both the algebra and geometry tests, specifying a different order of administration per half teachers. If administering both tests was not possible, teachers gave half of their students each test, alternating the tests between students sitting next to each other.

Data

Students responded to 3 or 6 proof construction tasks if they took one or both tests, respectively, enabling the measurement of their *proof construction competence*. Additionally, they responded to 5 Likert-scale questions, ranging from 1 ("Strongly disagree") to 7 ("Strongly agree"), after each task, addressing students' *productive disposition beliefs towards proof construction* using the statements we presented earlier. Each test included two tasks with statements with undisclosed truth values (one true and one false), and one task where students had to generate (through conjecturing) the statement to be proved. We discarded the data from participants who did not respond to any of the corresponding Likert-scale questions for a specific affect-related construct in the survey, across the six proof construction tasks. In total, data from 198 students were included in the analysis.

Coding

For *true* statements, a non-zero mark was assigned only when students recognized the statement as true or generated the expected statement themselves through conjecturing. The actual mark was based on Stylianides and Stylianides's (2009) 5-level categorization of arguments for true statements, ranging from 3 (non-genuine argument) to 7 (proof). For *false* statements, a non-zero mark was assigned only when students recognized the statement as false. The actual mark was based on a new 5-level categorization that we developed, ranging from 3 (non-genuine argument) to 7 (refutation by counterexample or a valid general argument). The first 40% of the responses to proof construction tasks were coded independently by the last two authors with an inter-rater agreement (Cohen's kappa) of 0.71 to 1.00 in the different categories of the coding scheme; disagreements were discussed and resolved. The researchers then each coded independently half of the remaining data. Instances where a second opinion was needed were coded also by others within the team; disagreements were resolved through discussion. *Anxiety responses to the Likert-scale questions were reverse-scored.*

Analysis

Confirmatory Factor Analyses (CFA) and Structural Equation Models (SEM) were conducted with Mplus 8.3 using the MLM estimator (maximum likelihood estimation with robust standard errors). The Full Information Maximum Likelihood (FIML) estimation was used to handle missing data (roughly 35.6%, missing at random [MAR] caused by participants taking only one of the tests as it was one of the options we gave to the teachers).

We carried out the data analysis in four steps. First, a particular construct (e.g., proof construction, anxiety) in algebra/geometry was defined by its mean score across the three tasks in algebra/geometry. Second, descriptive and correlational statistics among variables and covariates were conducted. Following observation of the strong associations among *self-concept of ability*, *self-efficacy beliefs*, and *self-evaluation*, we ran CFA with a one-factor, a two-factor, and a three-factor model to identify an acceptable measurement model. Finally, the structural model with possible paths was specified and evaluated, exploring whether/how students' affect-related constructs predicted students' *proof construction competence*, hereafter referred to as PCC.

Findings

Table 1 presents descriptive statistics and correlations among the observed variables. PCC in algebra was significantly associated with all the *affect-related constructs* in algebra (marked in orange) but not in geometry. Similarly, PCC in geometry was significantly associated with all the *affect-related constructs* in geometry except for anxiety (marked in purple) but not with all the *affect-related constructs* in algebra. This indicated that the paths of *affect-related constructs* predicting PCC may differ by the content domain. Also, *self-concept of ability*, *self-efficacy beliefs*, and *self-evaluation* were correlated (marked in red), some of which strongly. Regarding *value* (marked in blue), we found *value* in algebra was more strongly associated with *self-concept*, *self-efficacy,* and *self-evaluation* in algebra as compared to the same constructs in geometry. A similar pattern was observed in regard with *value* in geometry. *Anxiety* was associated with certain constructs; for instance, *anxiety* in algebra was significantly related with *self-concept* in algebra and *self-evaluation* in algebra, but significantly negatively associated with *value* regardless of the content domain (marked in green).

Table 1: Descriptive statistics and correlations among observed variables

		1	2	3	4	5	6	7	8	9	10	11	12
PCC[1]	PCCa[2]												
	PCCg[2]	.153*											
SC[1]	SCa	.310***	.019										
	SCg	.242**	.351***	.220**									
SEF[1]	SEFa	.245**	.098	.547***	.302***								
	SEFg	.032	.279***	.292***	.408***	.369***							
SEV[1]	SEVa	.359**	-.109	.756***	.240**	.446***	.251***						
	SEVg	.165*	.410***	.138	.777***	.224**	.518***	.097					
VAL[1]	VALa	.171*	.090	.295***	.293***	.550***	.171*	.270***	.141*				
	VALg	.087	.161*	.135	.170*	.322***	.309***	.191**	.246***	.533***			
ANX[1]	ANXa	.174*	.066	.311***	-.046	.015	-.173*	.234**	-.123	-.205**	-.264***		
	ANXg	-.019	.020	-.060	.112	-.188**	-.072	-.090	.057	-.215**	-.267***	.230**	
Mean		1.929	1.668	3.414	3.081	4.519	4.286	3.857	3.463	4.515	4.372	4.281	3.909
SE		.129	.128	.090	.091	.095	.092	.101	.101	.109	.114	.088	.090
Min-Max		0-7	0-7	1-7	1-7	1-7	1-7	1-7	1-7	1-7	1-7	1-7	1-7

Notes: ***p<.001, **p<.01, *p<.05; [1]PCC: proof construction competence; SC: self-concept of ability; SEF: self-efficacy beliefs; SEV: self-evaluation beliefs; VAL: value; ANX: anxiety. [2]Subscripts: a (algebra) and g (geometry).

Considering the strong associations among *self-concept of ability*, *self-efficacy beliefs*, and *self-evaluation,* we used CFA to examine if these three facets of self-related constructs could be empirically explained/encapsulated as a singular, dual, or triple entities in PCC in algebra/geometry by testing a one-factor (M1), a two-factor (M2), and a three-factor model (M3) (see Table 2). The results of the three models indicated that M2 provided the best fit, suggesting that there were differences in students' *self-perceptions* about PCC across content domains (algebra and geometry).

Once an acceptable *self-perceptions* measurement model was established, we moved onto the structural part of SEM, where the *affect-related constructs* were specified and examined in predicting PCC (Table 2). Based on M2, the *self-perceptions* model predicting PCC in algebra and geometry (M4) was first specified with a good fit (*Chi-square/df* =1.676 < 3, *CFI*=0.977 > 0.95, *TLI*=0.965 > 0.95, *RMSEA*=0.058 < 0.06). This indicated that *self-perceptions*, predicted by *self-concept of ability*, *self-efficacy beliefs* and *self-evaluation*, empirically and positively predicted *proof construction*, in both algebra and geometry. Then, in M5 and M6, we successively added the constructs of *value* and *anxiety*, which served as the exogenous variables predicting *self-perception*, which, in turn, predicted PCC in algebra and geometry. We accepted M6 as the final (at this point of our ongoing analysis) structural model shown in Figure 1. Most of the model fit indices in M5 and M6 were good (Table 2). In the final structural model M6, the path in algebra showed that *value* and *anxiety* positively predicted *self-perception* ($\beta = .419, p < .001$; $\beta = .439, p < .001$, respectively), which was loaded by *self-concept of ability*, *self-efficacy beliefs*, and *self-evaluation*, and in turn positively predicted PCC ($\beta = .449, p < .001$). *Value* in algebra negatively predicted *anxiety* ($\beta = -.283, p < .001$). The path in algebra was the same as the path in geometry. Also, we found that each construct in algebra was significantly associated with the respective construct in geometry (e.g., $r_{\text{value}} = .748, p < .001; r_{\text{anxiety}} = .674, p < .001; r_{\text{self-perception}} = .569, p < .001; r_{\text{proof-construction}} = .667, p < .001$).

Table 2: Model fit statistics for the CFA and SEM models

Model	Description	χ^2	df	χ^2/df	CFI	TLI	$RMSEA$
CFA							
One-factor (M1)	SCa+SCg+SEFa+SEFg+SEVa+SEVg	101.2	9	11.2	0.805	0.674	0.227
Two-factor (M2)	SCa+SEFa+SEVa, SCg+SEFg+SEVg	21.979	8	2.747	0.970	0.944	0.094
Three-factor (M3)	SCa+SCg, SEFa+SEFg, SEVa+SEVg	70.171	6	11.695	0.864	0.660	0.232
SEM							
M4	M2, +PCC	30.176	18	1.676	0.977	0.965	0.058
M5	M4, +VAL	88.201	32	2.756	0.907	0.872	0.094
M6	M5, +ANX	108.197	48	2.254	0.914	0.884	0.080

Note: Acceptable model fit:1< χ^2/df<3, CFI>0.9, TLI>0.9, RMSEA<0.1.

Discussion

The roles of content domain and affect-related constructs in students' proof-related behavior have been proposed in prior research (Dawkins & Karunakaran, 2016; Weber et al., 2020) but were not systematically explored. Our preliminary analysis of survey data we collected from secondary students in England yielded an empirically-based and theoretically-meaningful model depicting how certain student beliefs about proof construction (affect-related constructs) predict, in isomorphic ways in algebra and geometry, the student participants' PCC (cognitive construct). Specifically, PCC is positively predicted by students' self-perception (a new construct loaded by students' self-concept of ability, self-efficacy beliefs, and self-evaluation beliefs), which is positively predicted by students' value and anxiety. Since the identified model did not differ by content domain, similar implications apply to algebra and geometry for the student participants in our study.

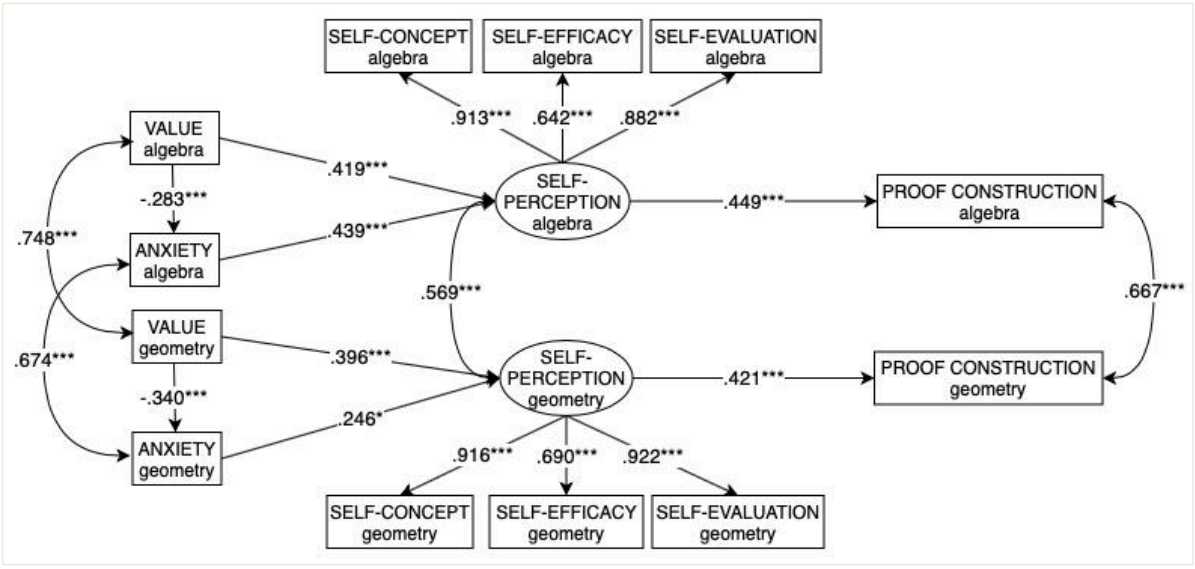

Figure 1: The final (at this point of our ongoing analysis) structural model (M6)

A limitation of our study is that we did not examine the students' prior knowledge and experiences in algebra and geometry. This information is crucial for interpreting our findings, as variations (or the absence thereof) across content domains could be influenced by students' variable experiences with these domains, not only during the year the study was conducted but also throughout their earlier education. Another limitation is that the study included students from a single cultural context, making it uncertain whether the specific model we identified would be applicable in other countries or even in schools located in different regions within England.

The above suggest important directions for future research to more systematically explore the role of the following factors in students' PCC: students' backgrounds in different domains (algebra, geometry, and beyond) and the broader cultural context. In our ongoing research, we are exploring another direction for which we have the required data: whether the relationship between affect-related and cognitive constructs is bidirectional. Irrespectively of this, our identified model supports the argument that instruction in the area of proof construction should consider not only students' cognition (as is typically the case) but also their affective worlds. For example, instruction can help students recognize the value of proof construction by fostering in them an *intellectual need* to learn about proof in general or proof construction in particular (Stylianides & Stylianides, 2022a).

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Reasoning and proving opportunities related to Thales' theorem in selected lower secondary textbooks from four different regions

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In light of the central role proofs should play in mathematics classrooms, this study examines how Thales' theorem, as one of the main theorems in lower secondary geometry, and its proofs are presented in the narratives of textbooks from the Czech Republic, Hungary, Slovakia, and Thuringia. We also analyse the volume and quality of tasks requiring reasoning and proving (R&P) in those textbooks, whose class level differs from 7 to 9 due to varying regional curricula. Our study highlights the diversity of teaching R&P in distinct educational administrations that were once very similar.

Keywords: Proof, secondary school mathematics, textbook research, geometry, Thales' theorem.

Introduction

Because proofs are a central concern within mathematics (Hanna & Barbeau, 2008), mathematics-education researchers and educators agree that reasoning and proof (R&P) should play a significant role in mathematics classrooms (Stylianides, 2009; 2014). However, the implementation of R&P in mathematical classrooms can be challenging (Buchbinder & McCrone, 2022) and the contexts in which proof and proving are taught worldwide differ greatly in terms of curriculum requirements, student age levels, teacher expertise, and other factors (Jones & Herbst, 2012).

In mathematics classrooms, not only do textbooks have an important function as being mediators between the educator and the students, but they also play a key role during classroom activities (Fan et al., 2013) and are crucial in offering students opportunities to engage in R&P (Stylianides, 2009). Therefore, to provide adequate and consistent learning opportunities in mathematics education with regard to R&P, it is of extraordinary importance to investigate how R&P are implemented in existing mathematics textbooks (Stylianides, 2014). In the next section, a short summary of the broad-spectrum national and cross-national textbook research related to the treatment of R&P will be given.

Before the political changes in 1989, the political, educational, social, and economic structures, and efforts for uniformity were paramount for more than four decades in Czechoslovakia (now, the Czech Republic and Slovakia), Hungary and in the German Democratic Republic (now, the new German states: States of Thuringia, of Saxonia, of Saxony-Anhalt, of Brandenburg and of Mecklenburg Western Pomerania). Since 1989, countries have adopted democratic tendencies in all aspects of social life and today they determine their own educational systems independently. Since today in Germany, all states have an individual educational system, tendencies in countries such as Czech Republic, Slovakia or Hungary are comparable with German results on the state level. Thus, in our study, we focus on selected textbooks from those countries as well as from one of the new German states (Thuringia) and aim to detect the opportunities to enhance R&P. Note that differences related to limited mathematical topics have already been discovered between Slovakia and the Czech Republic (Novotná & Slavíčková, 2023) as well as between Thuringia and Hungary (Szűcs, 2023).

Theoretical framework

Reasoning and proving in mathematics textbooks

Research on textbooks with a focus on R&P opportunities for students has become more comprehensive in the last few decades (Fan et al., 2013; Wong & Sutherland, 2018). However, the studies show a wide variety not only in terms of the applied frameworks, but also about the school level and mathematical topic under investigation. Hence, the results depict a high level of diversity.

As an illustration, researchers point often out that the treatment of R&P in textbooks can differ from country to country and from region to region. To give an example, Hoyles et al. (2003) showed that all students in Japan are expected to construct challenging proofs for a wide range of statements related to 2D- and 3D-diagrams. In contrast, in Baden-Württemberg (one of the 16 German federal states) only students in grammar school are expected to face proofs. Further, the focus in Baden-Württemberg's textbooks is mainly on proofs related to the Pythagorean theorem and proofs are treated very much top-down in the textbooks under investigation. Meanwhile, the study by Hoyles et al. (2003) also revealed that in French textbooks students are strongly guided to construct a formal, deductive proof and the focus is on straightforward applications of well-specified knowledge.

According to Wong and Sutherland (2018), a strong variation in the ratio of R&P tasks to all tasks in the textbooks can be detected from study to study. In their review chapters, the authors point out that R&P opportunities in different textbooks vary considerably from 3.7% to 77%. These findings are in line with some recent results: Hong and Choi (2018) compared geometry lessons in secondary school textbooks and found that R&P tasks are more dominant in American textbooks (40%) than in Korean textbooks (20%), for instance. Similarly, Szűcs (2023) examined 23 upper-secondary textbooks from three different German federal states and Hungary related to the law of sines. Her findings suggest that the proportion of R&P opportunities vary between 0% and 87%.

With respect to analytic frameworks, one of the most prevalent schemes is that of Thompson et al. (2012), which not just offers a way of categorising the tasks in textbooks on a conceptualised level but also incorporates the investigation of the proof provided in the narrative part of those textbooks. Their framework encompasses four categories for inspecting the narrative as follows: general justification (GJ), specific justification (SJ), left to the student (LT) and no justification (NJ), as well as seven types of exercises for categorising the tasks: make a conjecture (1), investigate a conjecture (2), develop an argument (3), evaluate an argument (4), counterexample (5), correct or identify a mistake (6) and principles of proof (7). Thompson et al. (2012) developed and implemented their framework to analysing tasks in algebra. Szűcs (2023) adopted this framework to examine tasks in trigonometry and added a new category: (8) investigate a premise. Likewise, besides some other modifications, Otten et al. (2014) extended the framework of Thompson et al. (2012) by coding the statements stated in the narratives using the categories general (g) and particular (p).

Moreover, because geometry traditionally offers numerous opportunities to R&P (Otten et al., 2014), when investigating textbooks, geometry is one of the most preferred mathematical areas. To give an example, Stylianides (2009) examined grades 6-8 secondary school textbooks and found that most of the tasks offering R&P opportunities require either a non-proof argument (60 %) or a proof (19 %).

Consistent with those results, Otten et al. (2014) investigated six geometry textbooks series and detected that about 13 %–25 % of the R&P tasks involve the construction of a proof while about 36 %–54 % of the R&P tasks require to develop a rationale but not necessarily a proof. They also included the study of the textbook narratives and found that general statements are dominant in the narratives, whereas particular statements are prevalent in exercises. Using the same analytical framework, Hong and Choi (2018) compared geometry lessons in five secondary textbooks from the United States and Korea. Similar to the results of Otten et al. (2014), Hong and Choi's (2018) findings suggest that general statements are more common than particular assertions in both American and Korean textbooks. On the other hand, students in the United States have more opportunities for constructing a proof (29 % *of all R&P tasks*) than in Korea (0 % *of the tasks engage in R&P*), while Korean learners are more frequently asked to evaluate an argument (60 %) than American students (45 %). In our paper, we aim to contribute to this research by providing new insights into how textbooks from four regions offer students R&P activities and different types of proofs in geometry.

Thales' theorem (angle inscribed in a semicircle theorem)

According to Alsina and Nelsen (2011), there are two different theorems in geometry called Thales' theorem, one of them states that the angle inscribed in a semicircle is a right angle if its vertex lies on the circumference. In lower secondary geometry classrooms, this theorem is one of the most important theorems students have to face: It not only allows learners to construct tangent lines to circles (French, 2008) and other geometrical objects (Posamentier et al., 2009), it also offers them a far-ranging tool to solve a wide variety of geometrical problems and to prove theorems in elementary geometry. Therefore, in this paper we focus on this theorem, and we will refer to it as Thales' theorem.

Thales' theorem can be proved in different ways: It can be derived as a special case from the inscribed angle theorem (proof A), which states that the angle subtended at the circumference of a circle by an arc is half the angle at the centre subtended by the same arc (French, 2008; Jacobs, 1992). Moreover, a widespread approach (proof B) resorts to the strategy of bisecting the triangle inscribed in a semicircle into two isosceles triangles and makes usage of the interior angle sum theorem in triangles. This approach is found in The Elements of Euclid (book 3, proposition 31) and was adopted by Alsina and Nelsen (2011), French (2008) and Wellstein and Kirsche (2009), for instance. Another approach (proof C) was taken by Gonzalez-Velasco (2001), who, by using point reflection, complemented the triangle in question to a rectangle and concluded that the triangle has a right angle.

It is noteworthy that the converse of Thales' theorem also holds: The hypothenuse of a right-angled triangle is one of the diameters of its circumcircle. As with Thales' theorem, there is a variety of proofs for its converse, too. To illustrate, Wellstein and Kirsche (2009) took an approach which was converse to proof B and proved the converse of Thales' theorem by using isosceles triangles (proof D). They also provided a proof (proof E) by contradiction. Additionally, another approach which is converse to proof C uses the strategy of point reflection, this (proof F) was taken by Alsina and Nelsen (2011).

Research Questions

Because of the prominent role proofs play in mathematics, they should take an essential part in mathematics classrooms, too (e. g. Stylianides, 2009; 2014). Therefore, we aim to investigate the

implementation of R&P in lower secondary mathematics education in the four above-mentioned educational administrations since we are familiar with their curricula, frameworks, teachers, and textbooks. To comprehend the underlying reasons and motivation for learning activities related to proofs, in this paper, similar to that by Szűcs (2023), we will use the term "proof-related-reasoning" in the sense of the work by Thompson et al. (2012) and we take this to mean a wide range of learning opportunities (GJ), (SJ), (LT) and (1)-(8) by that. The following questions led to the study presented in this paper: (RQ1) How does the treatment of proof related to Thales' theorem differ in the textbooks across the four distinct educational administrations? (RQ2) How does the volume and the quality of R&P opportunities for learners vary across those textbooks?

Methods

The sample of our analysis presented in this paper contains three currently accredited and widely used textbooks that entail the topic of Thales' theorem from each of the following four educational administrations: the Czech Republic, Hungary, Slovakia, or Thuringia. Note that, due to curricular differences, the class level varies from 7 to 9 across the 12 textbooks under investigation (the full list of the textbooks in the sample can be downloaded from here: https://www.uni-erfurt.de/fileadmin/fakultaet/erziehungswissenschaftliche/Mathematikdidaktik/Szuecs_Kinga_list_of_analysed_textbooks.pdf. In the following, the textbooks are labelled by CZ1-CZ3, HU1-HU3, SK1-SK3 and by TH1-TH3.

In accordance with Otten et al. (2014), in each lesson that included Thales' theorem, the statement of the theorem as well as of its converse was coded by using the categories (g) and (p). In addition, consistent with Thompson et al. (2012), the proof of Thales' theorem as well as of its converse was categorised by using the codes (GJ), (SJ), (LT) and (NJ). The type of the proof provided in the textbook for both Thales' theorem (proof A-C) and for its converse (proof D-F) was also registered. Further, following Sevinç et al. (2022), and Novotná and Slavíčková (2023), for coding the proving activities addressed to students in the textbooks' narratives, the categories pupils make a conjecture (mc), perform (pe), follow instructions (fi) and use previous knowledge (pk) have been used. In the same way, based on Thompson et al. (2012) and Szűcs (2023), all the tasks in the lesson except review exercises, were investigated according to the categories (1)-(8). To do so, the exercises requiring proof-related reasoning were tagged as R&P-tasks and mapped to exactly one of those categories. Excerpts from the coding manual can be found in Table 1. Further, the number of all tasks was determined and the ratio of R&P-tasks to all tasks was calculated.

Results

Treatment of Thales' theorem and its proof in the textbook narratives (RQ1)

Our analysis of the textbooks revealed several key findings about the teaching of Thales' theorem and its converse. For example, both the theorem and its converse were consistently generally (g) stated across all textbooks. However, while general justification (GJ) of Thales' theorem can be found in all the textbooks under investigation, the justification of its converse varies strongly across the textbooks even within one educational administration. Therefore, no main tendency related to the justification of the converse could have been identified in those regions. Similarly, while the textbook narratives

consistently provide proof B except CZ1, which uses point reflection (proof C) for proving Thales' theorem, when it came to proving the converse, there was a diversity of approaches depending on the region. In textbooks from the Czech Republic as well es from Thuringia only proof E was found, whereas proof D and proof F are present in Hungarian textbooks. Moreover, in none of the textbooks from Slovakia was a proof for the converse of Thales' theorem given.

Table 1: excerpts from the coding manual

Cat.	Definition	standard example
(1)	Learner is asked to generate a conjecture either in general (e. g. about an arbitrary triangle) or of a specific case (e. g. about a specific triangle).	Draw the points A(-2|1) and B(4|1) in the coordinate-system. Where do all the points P lie for which the lines AP and AB are perpendicular? (TH3, p. 101, exercise 2)
(2)	A conjecture or assertion is stated. Learner is asked to determine if it is true or false and provide a rationale. The question does not necessarily use the word conjecture.	Three different points A, B, C lie on the circle k(S, r) such that the center of the circle lies on the segment AB. Mark the false statements: a) Segment AB is the diameter of the circle. b) Angle ABC is right. c) Triangle ABC is isosceles. d) Segment SC is half of segment AB. (SK3, p. 33, exercise 9)
(3)	Learner is asked to provide a proof of a statement, either about a general or a specific geometrical object.	Draw in a circle two different diameters AC and BD. Reason that the points A, B, C and D are vertices of a rectangle. (TH1, p. 126, exercise 16a)
(4)	Learner is asked to determine whether an argument that is stated is valid or not.	Construct the tangent lines to the circle $k(S, r = 2.5\ cm)$ passing through the point R if $\|RS\| = 8\ cm$. Vendulka thought like this: "Given: $k(S, r)$ and R. Tangent $t \perp ST$, ΔSTR is right-angled, and SR is its hypothenuse." Is her thinking correct? (CZ3, p. 54, exercise 4)
(6)	A faulty solution to a problem or an invalid argument in stated. The learner is told there is a mistake and asked to determine the error in the reasoning or to correct it.	Pepa has drawn tangent lines from point Q to circle k, T_1 and T_2 are the points of tangency. The figure did not go too well for him. Draw your own figure and decide whether the line segment T_1T_2 passes through the midpoint of the line segment SQ. (CZ1, p. 25, exercise 10c)
(8)	Learner is asked to identify impacts if premises are changed or, in order to reach a specific impact, to find necessary premises.	The hypothenuse and the height perpendicular to the hypothenuse are given. Construct the triangle. Investigate the conditions of the solvability. (HU3, p. 156, exercise 6)

Additionally, notable differences in the explicitness of the expected student activities were detected. While most textbooks clearly outline those activities, TH3 has only implicit instructions, and all Hungarian textbooks lack proving activities in the narrative entirely. The expected student activities also vary across regions: Textbooks from the Czech Republic focus on (pe) targeting proving statements, while textbooks from Slovakia and Thuringia emphasise (mc). Particularly in Slovakian textbooks, we observed guiding activities (fi) that refresh previous knowledge (pk) and encourage students to use this knowledge by evaluating their conjecture related to Thales' theorem.

Volume and quality of R&P-opportunities in the textbooks' exercises (RQ2)

In our study, the ratio of R&P tasks to all activities in the textbooks varies from 19.61 % (TH1) to 100 % (HU1) (Table 2). The observed strong variation of the data is in line with the results of Wong and Sutherland (2018) and the findings of Szűcs (2023). However, the average ratio of R&P tasks in the regions under investigation (Table 2) offers a picture similar to the results of Hong and Choi (2018): Thuringian students have, on average, approximately 30 % less R&P opportunities than learners in the three other places under investigation (ca. 59 % - 70 %).

Table 2: Ratio and average ratio (in %) of R&P tasks in the textbooks under investigation

CZ1	CZ2	CZ3	HU1	HU2	HU3	SK1	SK2	SK3	TH1	TH2	TH3
61.2	54.55	61.54	100	46.67	29.41	57.89	81.82	70.00	19.61	51.28	28.57
59.10			58.69			69.90			33.15		

Our data suggest that most R&P tasks in all the analysed textbooks require either making (1) or investigating a conjecture (2) or developing an argument (3) (Figure 1). Note that the ratio of tasks involving the development of an argument (3) to all R&P tasks is primarily in Czech Republic (55 %) and in Slovakia (65 %) comparable with the results of Otten et al. (2014) and of Stylianides (2009).

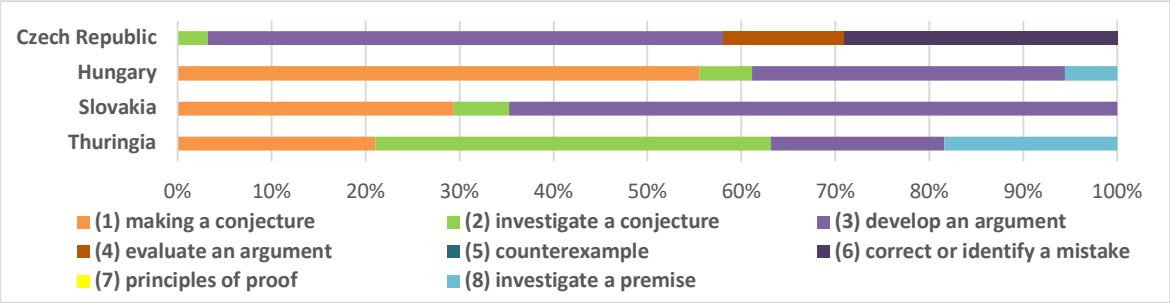

Figure 1: Identified expected students' activities in the textbooks under investigation

Interestingly, we found that tasks instructing students to investigate a premise (8) were present only in Hungarian and Thuringian textbooks, while tasks which involve evaluating an argument (4) or identifying/correcting a mistake (6) appeared exclusively in textbooks from the Czech Republic (Figure 1). Compared with the results of Hong and Choi (2018), the ratio of tasks instructing students to evaluate an argument (4) to all R&P tasks is low (13 %). Further, we did not find any R&P task which requires giving a counterexample (5) or applying the principles of a proof (7).

Conclusions

We found that there is a uniformity in stating Thales' theorem and its converse as well as in proving Thales' theorem across all analysed textbooks. However, students' opportunities to improve their R&P skills vary from region to region. Although students in Hungary have no opportunity for proof-related reasoning in the textbooks' narratives, which is similar to the top-down treatment of proofs in Baden-Württemberg (Hoyles et al., 2003), a high ratio of exercises offers learners R&P activities. In contrast, learners' R&P opportunities in Thuringia are limited in terms of both the volume and the quality of the tasks. Further, as in French textbooks (Hoyles at al., 2003), the emphasis on guiding activities towards the evaluation of conjectures in Slovakian textbooks enables students to create deductive proofs. The facts, that activities in the narratives of Czech textbooks require proving statements and a high ratio of R&P tasks asks students to develop an argument suggest that they are, similarly to Japanese learners (Hoyles et al., 2003) expected to construct challenging proofs. Thus, we conclude, that related to students' engagement with mathematical proofs, varying educational philosophies and underlying teaching strategies currently exist in the four examined educational administrations.

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Collaborative problem solving to foster higher-order thinking skills in mathematics: A systematic literature review[1]

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Keywords: Collaborative problem solving, higher-order thinking skills, systematic literature review.

Introduction

Collaborative problem solving in mathematics lessons may promote higher-order thinking skills (HOTS). However, clear design guidelines for these activities are lacking (Han et al., 2021). Through a systematic literature review, we aimed to identify optimal features for peer group mathematical problem solving such as task selection, ideal group size, and effective group work procedures that can effectively foster HOTS. The research question is: What are the key design features of CPS tasks that promote HOTS and how do communication challenges impact their effectiveness?

Theoretical framework

HOTS in mathematics involve reasoning, mathematical thinking, and metacognition (Makmuri et al., 2021). We use the following definitions in our review: (a) *reasoning*: the process of drawing conclusions in order to achieve goals, thus in forming problem-solving and decision-making behaviour (Leighton & Sternberg, 2004), (b) *mathematical thinking*: a complex blend of thinking processes that involve creating strategies for solving problems, recognizing patterns, making conjectures, and forming and validating arguments and proofs (Stacey, 2006), (c) *metacognition*: the ability to monitor and control one's own thoughts, approach to the problem, choice of strategies to find a solution, or self-questioning about the problem; hence, it can be defined as thinking about thinking (Goos, 2002).

Methods

A systematic review using a language-based machine learning technique (van de Schoot et al., 2021) identified 45 empirical studies on collaborative mathematics problem solving. One reviewer screened the articles, so no consistency measure was calculated, but the iterative approach ensured consistent prioritization. The PRISMA-guided search focused on HOTS, CPS, and mathematics in secondary and tertiary education. The selected articles were analysed using Sandoval's model (2014).

Results and conclusion

In the empirical studies discussed in the reviewed articles, the authors formulated specific conjectures, which, when incorporated into the design, were expected to trigger mediating processes that would lead to the desired outcomes. Our review identified a variety of tools, task structures, and

[1] Link to the poster: https://ap.lc/DGzgm.

participant configurations that are frequently associated with the promotion of HOTS during CPS in mathematics education. Commonly used settings included small group work and educational contexts at the secondary and tertiary levels. The most frequently addressed mathematical subjects were geometry and algebra. Regarding task characteristics, exploratory and inquiry-based learning using technology, along with open-ended problems, emerged as particularly prominent approaches. Additionally, several communication challenges that often impede the development of higher-order thinking are identified. In conclusion, this research underscores the importance of high-quality communication for successful CPS and the development of HOTS. By establishing classroom norms that encourage meaningful discussions (Littleton & Mercer, 2013), teachers can foster "sociomathematical norms" (Yackel & Cobb, 1996), thus creating a learning environment where dialogue drives growth and supports all students.

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Is it possible to assess the impact of proof assistants on undergraduate students' proofs? Preliminary results

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Keywords: Proof, undergraduate students, theorem prover, proof assistant, test.

Introduction

My PhD, started in February 2024, proposes to investigate students' conceptions and difficulties with proof at the secondary-tertiary transition and the potential impact of theorem provers in mathematics classes. This work is part of a larger national project APPAM (Les assistants de preuve pour les apprentissages mathématiques) about "proof assistants for mathematics learning" (14 researchers in didactics, mathematics, and computer sciences from 9 French universities). The need to develop new ways of teaching proofs that consider "theorem provers" has been demonstrated since 2020 (e.g., Hanna & Knipping, 2020). Originally, a theorem prover is a tool used by mathematics researchers designed for the verification of mathematical proofs. In recent years, interactive theorem provers have been developed with different interfaces but with no systematic didactic analysis of the impacts on students' proofs. They are called proof assistants (PAs). In a PA, a set of definitions, lemmas, theorems, etc. is provided, and the aim is to progress through the proof step by step. A PA checks the correctness at each step. The examples of PAs being used with undergraduate students remain rare and isolated but promising (e.g., Kerjean et al., 2022). In particular, there is no large-scale study of the impact of PAs on students' mathematical learning about proof. In this context, this poster presents preliminary results from a quantitative pilot study that addressed the following research question: How can the potential impact of using PAs on undergraduate students' proofs be assessed?

Methods

In February 2024, 135 freshmen from the University of Strasbourg, majoring in mathematics and computer science, completed a pre-test. During the following months, the students attended ten hours of class dedicated to the use of a PA, with this constraint: half of the students followed the course on Verbose L∃∀n (writing code interface developed by P. Massot), while the other half followed the course on D∃∀DUCTION (point-and-click interface developed by F. Le Roux). Finally, 47 of them passed a post-test in June 2024. Unfortunately, among them, a large majority followed the course with Verbose L∃∀n. This imbalance in our data does not allow us to make a comparison of the effects of one PA compared to the other. The pre- and post-tests consisted of three parts, this poster is about the first one. The students did not have access to computers, they were asked to write a "simple" proof (not treated in classes) with accessible concepts similar to existing large-scale studies (Stylianou et al., 2015): the sum of two odd numbers is even (pre-test) and the product of two odd numbers is odd (post-test).

Results and concluding remarks

Our analysis grid of the data starts from the typology proposed by Balacheff (1987) and then develops a detailed description of the objects introduced, the words or symbols that accompany them, the mathematical knowledge that is mobilised and the arguments and conclusions that are made explicit by the students. Toulmin's model, reinvested by Pedemonte (2002), was used for this last part, as well as other criteria and frameworks coming from the literature in education about proof that will be presented in the poster. Qualitative and quantitative analysis with excerpts emphasize students' proving practices and obstacles. In particular, we point out a local phenomenon in the post-tests: the expression of arguments that allow conclusions becomes less frequent. Besides, the formalism in students' productions evolves. Nevertheless, it remains difficult to affirm that PAs have an impact on the students' understanding of the mathematical concepts involved: some students still formalise the sum of two odd integers using the sum of one odd integer with itself. Then, this pilot study gives not only first results on students' proofs, but also an operational grid to analyse students' proofs before and after PAs sessions with pre- and post-tests. It also underscores the strong need of interviews before and after classroom sessions using PAs (designed with the didactical engineering framework) in order to study the evolution of students' proving processes.

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TWG 02: Arithmetic, Numbers and Operations

Introduction to the work of TWG02: Arithmetic, numbers and operations

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Introduction

WG02 is dedicated to research on the presentation and discussion of recent developments in the teaching and learning of arithmetic, numbers and their operations. At CERME14, we retained themes established in previous CERMEs, while also expanding the scope by introducing new topics. Thus, we included, among others, aspects such as conceptual understanding and procedural skills, number and structure sense, problem solving and problem posing, innovative teaching and learning approaches in arithmetic, and the potential use of digital tools.

Our group received 34 submissions — 30 research papers and 4 posters. Of these, 22 papers and 2 posters were accepted for presentation. These contributions were thoroughly discussed both in plenary sessions and in smaller breakout groups. Ultimately, 21 papers and 1 poster will be included in the conference proceedings. The group's work was structured around both whole-group and small-group discussions, and the sessions were organized according to five key themes that emerged from the submitted contributions: (i) *Rational and real numbers* (4 papers), (ii) *problem solving and posing* (4 papers, 1 poster), (iii) *numbers and their operations* (4 papers), (iv) *place value and representations* (3 papers), and (v) *difficulties and understanding in learning arithmetic* (5 papers).

Theme 1: Rational and real numbers

While rational numbers remain a central theme for this group, only a few papers on the topic were submitted to this CERME. *Jérome Proulx* focused on the conceptualization of percentage with the aim of refining its understanding through both theoretical reflections and empirical findings. He reviewed various conceptualizations discussed in the literature which he synthesized into a conceptual frame. This frame was then applied in a case study to investigate students' understanding of percentage highlighting three perspectives on percentage. *Diana Herreros-Torres, Maria T. Sanz, Carlos Gómez-Ferragud, Hélia Oliveira* and *Renata Carvalho* working with Grade 4 students found that the concept of 'fraction as operator' remains a considerable challenge for students. The students engaged with mathematical problems across two contexts —continuous and discrete— and through arithmetic and graphical approaches. The findings suggest the importance of integrating multiple representations to improve students' understanding. *Maria T. Sanz, Carlos Valenzuela García, Sofia*

Peregrina, Emilia López-Iñesta, Francisco Grimaldo, and *Daniel Garcia-Costa* presented an exploratory study focusing on ratios and percentages aiming to evaluate students' performance using software that implements the tree graphs method. The results indicate that tree graphs facilitated students' understanding and improved their performance, particularly in problems involving ratios. The last paper in this group concerns real numbers. *Virginia Garrido-Adame, Olimpia Figueras* and *Minerva Martínez-Ortega* presented a design-based research project focused on developing number sense in the context of real numbers. Using a framework that outlines abilities related to number sense in real numbers, they designed the 'Broken Key' activity and implemented it with twelve students aged 15 to 18. The authors were interested in exploring which number sense abilities students employed while working on the 'Broken Key' tasks. Their findings suggest that the activity effectively activated specific number sense abilities.

Theme 2: Problem solving and problem posing

This is a relatively new theme within the group, which has seen a growing number of submissions over the last two CERMEs. *Mónica Arnal-Palacián* and *Nuria Begué* explored the criteria preservice teachers use to evaluate a 6th-grade student's incorrect solutions to two additive tasks and the levels at which they justify their assessments. Their decision-making criteria fell into three categories: the problem's semantic structure, the use of materials, and the use of diagrams. Overall, the level of their responses was below the required level. *Eszter Kónya* emphasized the need to shift attention from numerical solutions to examining the methods and procedures used during problem solving. She highlighted the significance of identifying and using the arithmetic structures embedded in word problems to solve them. Working with upper primary students she found that certain tasks can effectively be used as a starting point for working with arithmetic structures particularly when the task solution is followed by whole classroom discussion. *Karolína Mottlová* investigated how 5th-grade students approach atypical word problems known as "Fidgets," which are distinctive in that they lack an explicit question. The study revealed that students most frequently employed a Guess-Check-Revise strategy, drawing on mathematical, contextual, and linguistic cues to formulate, test, and adjust their guesses —either systematically or at random. In their poster presentation, *Waqas Bin Sarfraz, Dave Hewitt* and *Ian Jones* shared a study involving two second-grade students using an educational app called 'Stick and Split' to solve multiplication problems. They identified four distinct strategies: known or derived multiplication facts, a two-step multiplication model, skip counting, and trial-and-error grouping. Additionally, one paper in this key-theme focused on problem posing. *Ioannis Papadopoulos* investigated how preservice teachers — with and without problem-solving experience — posed open problems from a closed one. Both groups predominantly interpreted open problems as having multiple solutions and formulated them by keeping the original problem and task environment unchanged or slightly modified. However, those with problem-solving experience demonstrated a clearer understanding of the distinction between open and closed problems.

Theme 3: Numbers and their operations

The four papers in this key-theme explore various aspects of all four operations. *Timo Flückiger* and *Elisabeth Rathgeb-Schnierer* examine the flexibility of Grade 3 students' mental calculation, with a particular focus on how they recognise features of addition and subtraction problems. The results

showed significant differences in both the number and types of problem characteristics the students recognised, largely influenced by the specific structure of the problem. One commonly identified feature was "crossing the ten boundary". *Michael Gaidoschik, Myriam Burtscher* and *Heidi Pechlaner* presented an ongoing teaching experiment, designed to support primary students to understand both partitive and quotative division. The results give evidence that after the first cycle of the experiment, most students were able to comprehend and articulate the dual nature of division. *Éva Fülöp, Hoda Ashjari, Marie Fredriksson* and *Angelika Kullberg* reported on an one-semester intervention focused on teaching two-digit subtraction in Grade 2 and discussed the impact of this intervention on students' learning. Following the intervention, the students demonstrated an increased ability to explain their answers and began to adopt sequential strategies. In the final paper of this key-theme *Sandra Gleißberg*, *Klaus-Peter Eichler* and *Moritz Seibold* investigated German students' mastery of basic multiplication and division facts from the end of Grade 2 to the end of Grade 3 using timed tests. Although overall performance improved by the end of Grade 3, only 42% of the participating students achieved the expected level of mastery, with significant variation across classes.

Theme 4: Place value and representations

Within this key-theme, two papers directly explored aspects of place value while a third one examined a manipulative functioning as a linear representation of number position —also relevant to place value understanding. In their study, *Gerald Wittmann* and *Katja Lenz* validated a test instrument designed to assess German fourth graders' conceptual understanding of place value for numbers larger than one thousand, up to one million. The test focuses on translating pictorial representations into numerals and incorporates non-standard representations and a variety of task types. *Mia Lene Ransiek* and *Thomas Rottmann* investiagted Danish and German second graders' understanding of place value through tasks involving translation between different number representations. Although both languages use inverted number word structures, German students exhibited more inversion-related errors, while Danish students struggled more with irregular tens-related errors. The study shows that additional features of number word formation also seem to play a significant role in influencing these types of error. The third paper, co-authored by *Sarah Hopkins, James Russo, Jane Hubbard, Anne Roche* and *Janette Bobis* examined how to encourage the use of retrieval-based solutions instead of the counting strategy in simple addition through a case study involving a ten-year-old student. The intervention focused on a novel manipulative, the "Keyboard," which visually displays the linear arrangement of numbers. The results showed that after the intervention, the student showed a significant decrease in reliance on minimal counting strategies and a growing use of strategies based on retrieval or derived facts.

Theme 5: Difficulties and understanding in learning Arithmetic

This key-theme is broad and concerns difficulties relevant to (i) the transition from primary to secondary school, (ii) the criteria for evaluating the quality of a student's solution to an arithmetic task as being good or high, (iii) the transition from the real word of quantities to numerical expressions, (iv) the perception and understanding of linearly increasing number patterns, and (v) conceptual and lexical understanding of arithmetic in the context of learning difficulties. More specifically, *Volker Ulm, Annalisa Steinecke* and *Moritz Zehnder* investigated students who, in the

transition from primary to secondary school, have severe difficulties in learning arithmetic. Their findings underscore the challenge faced by the education system in establishing effective structures to identify these students, accurately diagnose their specific difficulties in a differentiated way and provide appropriate support. *Linda Devi Fitriana, Zoltán Kovács* and *Christiyanti Aprinastuti* emphasized hands-on activities by investigating how paper-folding tasks can reveal preservice primary school teachers' number sense skills. Through tasks that offered the chance to explore number patterns, the participants demonstrated their capacity to move from physical to mathematical forms, identify numerical relationships, and intuitively recognize sequence patterns. *Robert Gunnarsson* and *Ioannis Papadopoulos* presented an ongoing study exploring the conventions of what is required for a student's solution to an arithmetic task to be good, or even of high quality. According to 21 secondary school teachers and mathematics education researchers, a high-quality solution must meet three key criteria. It be mathematically correct, it employes an efficient solution strategy and must be communicated transparently — particularly in terms of the syntax of the written expression and its adherence to necessary mathematical conventions. *Darina Jirotková* and *Jana Slezáková* working with Grade 2 and 3 students, analyzed their solutions to a problem involving numerical patterns. They classified the solutions into seven categories according to the accuracy of the answers across various fragments of the pattern in a given task. This classification enabled them to formulate three distinct levels of students' perception and understanding of linearly increasing number patterns. *Charlotte Rechtsteiner* and *Michaela Scheffknecht* focus on children with specific learning difficulties in mathematics. To support the development of conceptual and lexical understanding in arithmetic they employed general Why-explanations as an efficient means. In this paper they followed a Grade 3 student who received supplementary weekly lessons aimed at stimulating general Why-explanations. Their findings indicate that the formulation of general Why-explanations was essential for developing (flexible) calculation competences.

Some concluding remarks

At CERME14, TWG02 participants had the opportunity to engage in productive small- and whole-group discussions on various aspects of arithmetic across all educational levels, with a focus on both teachers and students. This year's well-balanced mix of early-career and experienced researchers significantly enriched the conversations. Discussions emphasized shared interests, such as analyzing students' strategies rather than merely judging the correctness of their answers, deepening our understanding of students' thinking, enhancing learning through appropriate learning environments, and exploring the connections between arithmetic operations. These discussions also laid the groundwork for extending the current work by exploring cultural influences on teaching and learning, examining student competencies and misconceptions more closely, developing alternative instructional approaches, and incorporating interdisciplinary perspectives into arithmetic. Additionally, we recognize the importance of addressing emerging areas in arithmetic, such as patterns involving irrational numbers and the role of arithmetic in STEM education—topics that will be reflected in the upcoming call for papers for CERME15.

Preservice teachers' decision-making: an empirical study using additive problems

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The aim of this paper is to identify the categories on which a preservice teacher (PST) relies when describing decision-making in relation to the resolution of an additive problem, as well as to identify a gradation in the characterisation of this decision according to the chosen category. The study sample consisted of 63 PSTs. They were given a task consisting of two additive problems presented together with a resolution by a child in the 6th grade of Primary Education. Among the results, we found that PSTs take into account in their decision-making the semantic structure of the problem, the use of materials and the use of diagrams, although not in all cases with the same level of precision. In the case of the use of diagrams or drawings, more general decisions are made, while in the case of the semantic structure, the comments are much more precise and elaborated.

Keywords: Decision-making, additive problems, preservice teachers, semantic structure.

Introduction

The literature emphasises the need for teachers to acquire skills and competencies for their classroom practice, which should begin during their initial training. Specifically, Jacobs et al. (2010) describe a theoretical framework, professional noticing of children's mathematical thinking, as a set of interrelated skills, including decision-making based on understanding the child's response.

Stahnke et al. (2016) conducted a systematic review of 60 studies focused on the perception, interpretation, and decision-making of pre-service (PST) and/or in-service teachers (IST). Half of the analysed articles aim to examine at least decision-making, with only five articles focusing solely on decision-making with samples of IST. The results show that decision-making is very challenging for PST. Additionally, the review highlights the limited research focused on decision-making among PST samples, which justifies the need to design studies that seek to identify the criteria on which decision-making is based and the inherent difficulty in doing so. Therefore, we designed an instrument which presents two different additive problems, along with an incorrect solution provided by a primary school student. The assessment instrument requires the PST to describe the reasoning behind their decision-making in order to support the student's learning process in response to this type of problem. The literature highlights the difficulties that students face in carrying out these actions, and we expect a varying level of description depending on their ability to articulate the use of each category. Therefore, the aim of this study is to identify the categories used in describing the type of decision made by the PST and to establish a gradient in the characterisation of the decision according to the selected category. Additionally, the additive problems correspond to different semantic structures, as described in the methodology section. This task variable allows us to explore whether the decisions related to the categories depend on the type of semantic structure.

Theoretical Framework

In the resolution of arithmetic problems that can be solved using addition or subtraction, different classifications have been proposed based on their semantic structure (Carpenter & Moser, 1982; Riley et al., 1983), with the most commonly used being the distinction between transformation, combination, and comparison (Riley et al., 1983). Refining this categorisation further, there is a combination of these categories that results in six semantic structures (Vergnaud, 1982): a) State-State-State (SSS), b) State-Transformation-State (STS), c) State-Comparison-State (SCS), d) Transformation-Transformation-Transformation (TTT), e) Comparison-Transformation-Comparison (CTC), and f) Comparison-Comparison-Comparison (CCC).

The choice of each of these categories impacts the difficulty in solving these problems (Carpenter & Moser, 1982). Despite the existence of these categories, not all of them are presented in the same way or with the same frequency in primary education classrooms (Arnal-Palacián et al., 2023). The difficulty of arithmetic problems, particularly additive ones, can also be explained by the consistency of the language used. For instance, addition is often associated with words like 'increase', 'add', 'more than', etc., while subtraction is linked to terms like 'decrease', 'take away', 'less than', etc. (Orrantia et al., 2005). This inconsistency in language is determined by the position of the unknown in the problem for each of the previously mentioned semantic structures.

In relation to the research, we propose the following research question: Are there differences in decision making provided by PST to student answers to additive problems that allow descriptive levels to be established?

Methods

The intentional sample consisted of 63 PSTs enrolled in the course *Didactics of the Natural Number*, which corresponds to the first Mathematics Education course in their undergraduate degree in Primary Education. The task proposed consisted of two additive problems, each accompanied by a solution provided by a 6th-grade primary school student (11-12 years old) (Figure 1 and Figure 2).

You have 1600 red chips on the table. If you have 900 more chips than your friend Cristina, how many chips does your friend Cristina have?

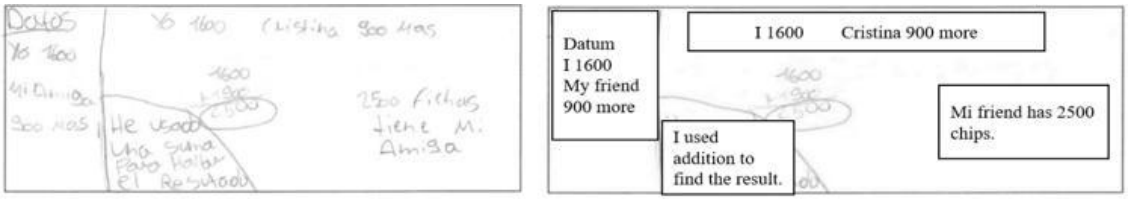
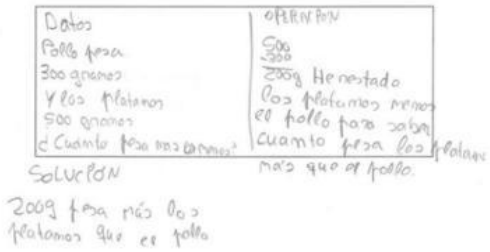


Figure 1: Statement of Problem 1 and Response from the Primary School Student

The chicken weighs 300 grams more than the cake, while the bananas weigh 500 grams less than the cake. How many grams more or less does the chicken weigh compared to the bananas?



Datum
Chicken weighs 300 grams and the bananas 500 grams.
How much more or less does it weigh?

OPERATION

I subtracted the bananas minus the chicken to find out how much more the bananas weigh than the chicken.

SOLUTION
The bananas weigh 200 g. more than the chicken

Figure 2: Statement of Problem 2 and Response from the Primary School Student

The PSTs were assigned the following task: "*Describe the decisions you would make to help the student, if necessary, to progress in their understanding*". The task was presented as a class activity to be completed individually, after receiving instruction on solving and creating additive problems. Therefore, the students were familiar with the categories of analysis described below.

The analysis variables considered were semantic structure, with manipulatives, and the use of diagrams or images. Problem 1 corresponds to an SCS structure, in which an additive comparison relationship is presented between two known quantities. Problem 2 corresponds to a CCC structure, in which each datum of this semantic structure reports the comparison relation that is established between two unknown quantities. In our case, we do not know the amount of mass of the chicken and the pie, but the comparison between those unknown quantities. Therefore, the statement provides information about the positive and negative comparison that is established between the unknown quantities of mass of the chicken with respect to the cake, and the bananas with respect to the cake. Therefore, the question aims to determine the additive comparison that is established between the unknown mass quantities of the chicken and the bananas. Despite the fact that the operations involved in solving these problems do not pose a challenge for students of these ages, as Arnal-Palacián et al. (2023) state, SCS problems have hardly been present in their textbooks, and CCC semantic structure problems have not been presented.

Therefore, PSTs were expected to have this category to explain their decision-making. Moreover, this change in semantic structure has implications for the difficulty of each problem. This explains why the characterisation of this category for Level 3 - levels presented below - differs according to the type of problem. Specifically, a student making a Level 3 decision for Problem 2 identifies the need to work with additive situations corresponding to Problem 1 (SCS). Additionally, the use of manipulatives is considered a potential decision-making factor due to its importance in establishing connections between ideas, concepts, and procedures as a teaching and learning tool (Hodgen et al., 2018). Finally, diagram or pictorial representation is taken into account, as these tools allow for the analysis of a verbal problem (Ramos et al., 2016) by organising and relating its data (Hershkovitz &

Nesher, 2003). The established categorisation is shown below, and student responses are shown in the next section to exemplify the levels defined for each category:

Semantic structure

- Level 0. No mention is made of the semantic structure.
- Level 1. The semantic structure is only mentioned, without any additional information.
- Level 2. It is specified that other problems with the same semantic structure could be used.
- Level 3. It is specified that other problems with the same semantic structure could be used, also specifying a progression in the degree of difficulty.

With manipulatives

- Level 0. No mention is made of the use of materials.
- Level 1. The use of materials is only mentioned, without any additional information.
- Level 2. The use of materials is mentioned, and, additionally, the materials are determined in a specific way: objects or balance.
- Level 3. The use of materials is mentioned, the materials are specifically identified, and it is further specified that their incorporation requires changing the size of the number.

Diagrams/Images

- Level 0. No mention is made of the diagrams or images.
- Level 1. The diagrams and/or images are only mentioned, without any additional information.
- Level 2. Diagrams and/or images are mentioned, and, in addition, one is presented.
- Level 3. Diagrams and/or pictures are mentioned, one is presented, and how to manage their use with the student is explained.

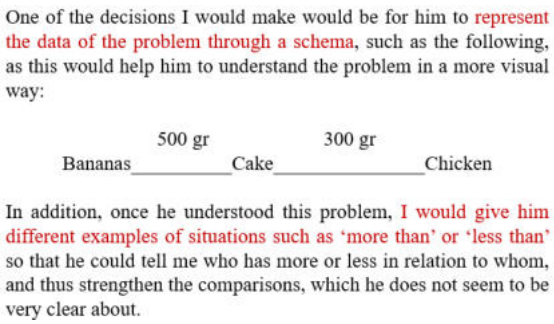
Results

The decision-making made by the PST for Problem 1 resulted in the following results: 26.5% made decisions, but none of the categories were considered, 53.1% dealt with one of the categories, 9.3% with two, 4.9% with three, and 6.2% did not respond to the requested task. Among these decisions, 23.8% attended to the learner with the help of a diagram or drawing, 23.8% took the material into account, for example, PST#32 refers to the use of paper balls to construct the quantity associated with the data in the statement. Finally, 36.5% attended to the semantic structure.

> PST#32: I would re-read the problem with the child and do *a visual example with paper balls* and smaller numbers.

In relation to the results of Problem 2, the analysis of the results shows that 27.4% of the PSTs do not provide an answer that corresponds to a decision-making categorisation. On the other hand, the majority of the sample, 40.6%, refers to one of the indicated categories, while the percentage of PSTs

referring to two or three categories takes the value of 16% (Figure 3) and 2.9%, respectively, and 13.1% do not respond to the requested task.


One of the decisions I would make would be for him to represent the data of the problem through a schema, such as the following, as this would help him to understand the problem in a more visual way:

500 gr 300 gr
Bananas Cake Chicken

In addition, once he understood this problem, I would give him different examples of situations such as 'more than' or 'less than' so that he could tell me who has more or less in relation to whom, and thus strengthen the comparisons, which he does not seem to be very clear about.

Figure 3: PST #48. Two decision-making processes addressed: diagram and semantic structure

As can be seen in PST#59 answer and Figure 3, both PSTs address decision making in which the use of diagrams is taken into account. However, the level of development of each of them is different, being more advanced in the second case. For this reason, the analysis is completed by classifying each type of decision identified according to its level (Table 1). This analysis shows that PSTs are able to provide better decisions for the case of *materials*, in the case of Problem 1, and for the *semantic structure*, for Problem 2. However, despite finding a sufficiently high number of comments for the *diagram* category, this decision-making hardly contains any reflection on what kind of diagram to use or how to manage it with the learners.

Table 1: Number of PSTs at each level for each category and proposed problem

Level	Level 0		Level 1		Level 2		Level 3	
Category	Probl. 1	Probl. 2	Probl. 1	Probl. 2	Probl. 1	Probl. 2	Probl. 1	Probl. 2
Semantic Structure	41	44	0	0	21	15	1	4
Use of materials	48	51	3	8	1	3	11	1
Make a sketch or pictorial representation	48	37	13	21	1	3	1	2

Level 0 is the predominant level in both problems for each of the categories considered. Given the objective of the study, we focus the analysis on the rest of the levels.

The highest number of Level 1 decision making is concentrated in the case of making diagrams or pictorial representations to help solve the problem, while Level 2 could be linked to decisions made with semantic structure and Level 3 is dominated by the use of materials.

Level 1. The results in Table 1 show that for this level, a similar tendency is identified in both problems. At this level, most of the PSTs cite the diagram or pictorial representation, without offering any type of decision for this level in the case of the semantic structure. In the case of Problem 1, most of these comments at this level concentrate on the presentation of a diagram or drawing. The type of decision making for the latter case are of the following type: "I would propose to the student to make a new reading with the help of a teacher that would allow him to understand more precisely and the question and, in addition, if necessary to make a small drawing that would allow him to observe what he should do more easily" (#20), "I think that the easiest way to understand the problem is to use drawings, using them makes it easier for the student to understand" (#25), "I think I would advise him/her to make a drawing or a scheme so that the data of the problem can be appreciated in a more visual way" (#35) or "I would help him/her to make a scheme of the statement" (#59).

In the case of Problem 2, we find similar answers in which the drawing or diagram is mentioned, where the difference lies in who assumes the task of making the drawing or diagram, as can be identified from the answers shown, "And finally, I would help this student above all with a representation through drawings that he makes..." (#19); "I would make a more visual diagram of the weight of the three elements..." (#57); "the student could be encouraged to make a scheme to see more clearly the comparisons that are made,..." (#63).

Level 2. Most of the PSTs provide information regarding the semantic structure at this level for both problems. The decision-making they propose, depending on the problem, is described below:

In the case of Problem 1, 91% of the PSTs that belong to this level for one of the three categories are for the semantic structure. Some of the decisions made by the PSTs were the following: "I would do State-Comparison-State problems with the student to improve his understanding progressively" (#9), "We would do different SCS problems in which the word 'more' also appears but the word 'more' has to be subtracted and the word 'less' has to be added so that the student understands that it is not the same" (#59) or "I would focus on helping the student to improve his/her understanding in relation to the terms 'more than' and 'less than' so that he/she does not associate 'more' always with addition and 'less' always with subtraction and can do the problems without making mistakes" (#55).

In the case of Problem 2, they correspond to answers indicating that the same semantic structure of the problem presented is being worked on, for example, "we could do is to put smaller data and numbers, offering them a material and context close to the child. But, not modifying the semantic structure". (#1), "I would do multiple exercises with semantic category CCC, so that the student understands the differences of comparisons" (#2).

Level 3. The results in Table 1 reveal that the frequency of responses that fall into this category shows different trends depending on the problem.

In Problem 1, most of the PSTs are linked to it, 84.6%, by the decision-making they provide for the use of materials. In the case of semantic structure, only a single PST (PST #53) reached this level for decision-making, where semantic structure was taken into account. The same is true for the

introduction of diagram management or pictorial representation. Relevant is the case of the PST#58, which reaches the highest level for diagrams, as it also reaches this level for the use of materials.

PST#53: Because of the above, *I would go back to the previous step and practice on State-Comparison-State exercises with the unknown in the first term*. When I was sure he/she knew how to do those, I would take those same exercises and *reformulate them to ask the question with the comparison*, instead of the initial state, *changing the position of the unknown and the sign of the operation*. [...]

PST#58: In this case, I think that we should give them a similar problem *with smaller data so that, first of all, they can solve it manipulatively (with cards).* Finally, solve it with the corresponding mathematical operations.

In the case of Problem 2, the degree of complexity of the semantic structure characterising the statement is perceived by 4 students who point out the need to present problem statements involving a progression in semantic structure SCS.

Conclusions

We can conclude that there are certain similarities in the level of reflection that PST develop in decision-making (Jacobs et al., 2010), despite the difference in the semantic structure (Vergnaud, 1982) of both problems, SCS and SSS, and, consequently, the different degree of difficulty (Carpenter & Moser, 1982). The most notable difference in decision-making between the two problems lies in the use of material. While in Problem 1 the decisions are precise, considering reducing the size of the numbers in order to use a tangible material, usually tokens; in the case of Problem 2, the decisions are more generic and involve little more than considering a scale to establish the comparison.

In relation to the link between the categories and development levels, it has been observed that for the semantic structure category, most of the PSTs are at Level 2, while the use of diagrams or drawings is more frequent for Level 1. However, not many PSTs reach Level 3. It is precisely the level that should be reached by all PSTs at the end of their training, both for their degree of reflection and for the variety in the categories addressed in decision-making. Thus, a difficulty in decision-making as a future teacher is identified, as pointed out in the review carried out by Stahnke et al. (2016), which needs to be addressed for adequate professional development from the initial teacher training.

As a future perspective of this research, it is intended to address the creation of teacher profiles based on the decisions considered and to identify factors preventing pre-service teachers from progressing beyond the lowest level despite instruction and design ways to support their competency development. Furthermore, the research can be expanded by analysing combined categories and characterising PSTs based on the levels achieved of the categories, as different levels may be reached depending on the category.

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Unveiling number sense performance in preservice primary school teachers through hands-on activity

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This study explores how hands-on activities, specifically paper folding, reveal the number sense performance of preservice primary school teachers. By engaging in tasks introducing and exploring number patterns, participants demonstrated their ability to transition from the real world of quantities to numerical expressions, recognise relationships among numbers, and intuitively identify sequence patterns. The result discloses varying ability levels to translate between number representations, understand the additive and multiplicative structures of numbers, and grasp the interaction between this translation process and the conceptual understanding of operations.

Keywords: Number sense, paper folding, preservice primary school teachers.

Introduction and theoretical background

Primary school teachers in Indonesia must develop a comprehensive mathematical foundation to effectively convey mathematical concepts to young learners, with number sense being a fundamental component, providing the critical foundation for further learning and cognitive development. While there is still no consensus on the definition of number sense (Pitta-Pantazi et al., 2014), the significance of fostering number sense among students has gained increased attention from mathematics educators globally. Studies by Yang et al. (2009) emphasise the critical role of number sense in children's mathematical growth and advocate for its inclusion in mathematics curricula across primary and secondary education. Therefore, assessing the number sense of preservice teachers is critical (Aktaş & Özdemir, 2017).

The study introduced a hands-on activity for preservice primary school teachers (PSTs) to introduce number sequence, allowing them to explore patterns, transition from tangible to abstract, and intuitively recognize sequences. This approach supports educational practices that link numerical patterns to physical or visual representations, which benefit both students and preservice teachers (Rivera & Becker, 2005; Whitin & Whitin, 2014). PSTs should thoroughly understand number sequence as it underpins essential arithmetic skills, pattern identification, and cognitive development (Griffin, 2009), all critical in early mathematics instruction.

The importance of number sense as a foundational idea in mathematics (NCTM, 2000) necessitates research on how preservice teachers demonstrate this skill. While much research has been conducted on number sense in students, there is relatively limited literature on the number sense of PSTs. This study aims to contribute to the literature by investigating how paper folding may reveal the number

sense of PSTs. Specifically, this study explores PSTs' abilities across selected dimensions of number sense, as defined by Sowder (1992), which are elaborated further in the following section.

There has yet to be a universal consensus on the exact definition of number sense (Pitta-Pantazi et al., 2014). Some studies indicate that the number sense has two dimensions (Berch, 2005; Jordan et al., 2006). The first involves basic skills like counting, number recognition, pattern understanding, and nonverbal calculations. The second pertains to arithmetic, including problem-solving and number combinations. Additionally, number sense is the ability to see numbers as meaningful, involving skills like using different representations, recognizing magnitudes, using benchmarks, composing and decomposing numbers, understanding operations, estimating, doing mental math, and assessing result accuracy (Pitta-Pantazi et al., 2014).

Building on the existing definitions and characteristics of number sense, Sowder (1992) has provided a comprehensive list of behaviours that demonstrate the presence of number sense. This list, while not without some overlap, is a valuable resource in understanding the complex and multifaceted nature of number sense.

> (1) Ability to compose and decompose numbers, to move flexibly among different representations, to recognize when one representation is more useful than another; (2) Ability to recognize the relative magnitude of numbers; (3) Ability to deal with the absolute magnitude of numbers; (4) Ability to use benchmarks; (5) Ability to link numeration, operation, and relation symbols in meaningful ways; (6) Understanding the effects of operations on numbers; (7) The ability to perform mental computation through "invented" strategies that take advantage of numerical and operational properties; (8) Being able to use numbers flexibly to estimate numerical answers to computations, and to recognize when an estimate is appropriate; and (9) A disposition toward making sense of numbers. (Sowder, 1992, pp. 4–6)

The central question guiding this study: how do PSTs perform number sense across Sowder's dimensions 1, 5, 6, and 9? We interpret having good number sense in the context of our study as the ability to precisely translate between different number representations, grasp the structure of addition and multiplication, comprehend how these conversion processes interact, and have a conceptual understanding of operations.

Methods

The study involved 90 PSTs enrolled in their third semester at a university in Yogyakarta, Indonesia. These participants had completed introductory mathematics courses on numbers, spatial reasoning, and symmetry. We conducted the study in the third month of the semester to ensure the participants had a thorough understanding of fundamental mathematics. In this research, participants were tasked with performing a folding activity, in which they received a piece of paper and were instructed to fold it in half multiple times (Mason et al., 2010) (see Figure 1 for clarification of the folding process). The questions were: (1) How many fold line segments are in the first, second, third, and seventh folds? (2) What is the general rule for the nth fold? This paper focuses solely on the first question. The rationale for the seventh fold is the practical impossibility of folding paper seven times and more. The task prompts participants to explore what is happening in the paper folding, express the structure

of how a sequence is generated, and turn to objects to manipulate, make, and justify conjectures (Mason, 2010). Consequently, the task aligns with and supports participants in engaging with the four overlapping number sense dimensions outlined by Sowder (1992). In addition, as noted by Sjaastad (2023), tasks that encourage inquiry activities like exploration, conjecture, and reasoning can enhance number sense.

The lesson was implemented in 90 minutes and divided into three phases: independent work, paired discussions, and a whole-class discussion. This study is part of a broader research project and focuses specifically on individual work to assess the participants' original performance in number sense. Several precautions were implemented during the individual work phase to ensure the validity of the result. Participants were seated individually and reminded to complete their tasks without collaborating. The instructor actively observed the room while a designated observer documented participant behaviour to monitor this process. Data were collected through written responses and researcher notes. The participants' solutions were coded at two levels, namely, (i) the correctness of the final solution and (ii) the correctness of the starting point.

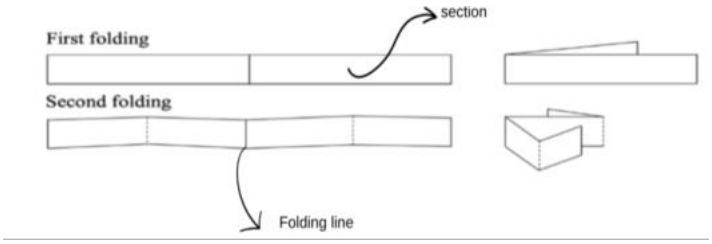


Figure 1: Illustration of the folding

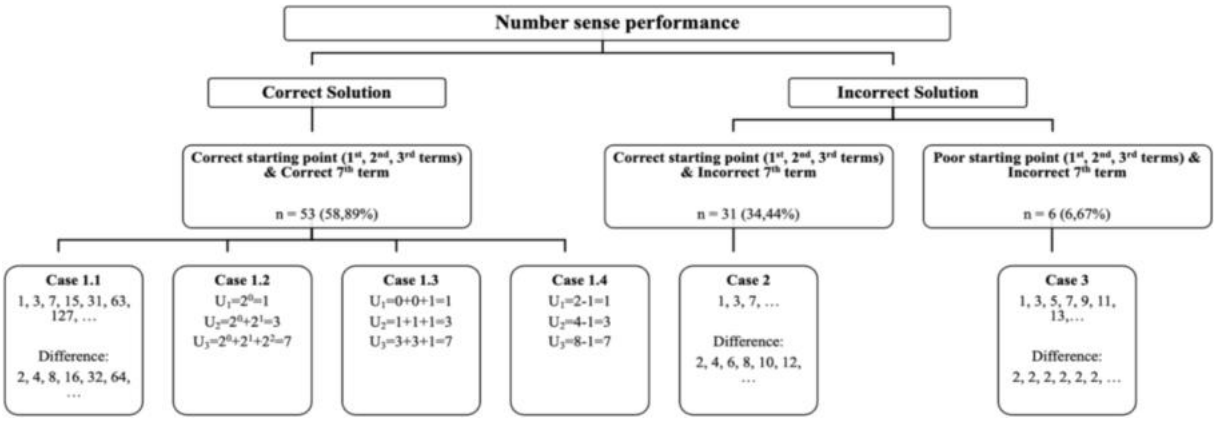


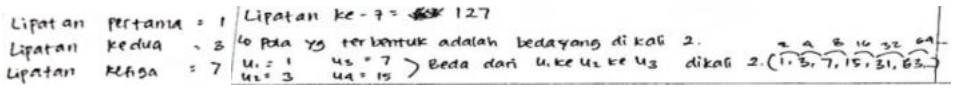
Figure 2: Distribution and details of performance

Results

The participants' performance was analysed by classifying the final results into correct and incorrect solutions, with Figure 2 highlighting the key steps leading to correct or incorrect outcomes. Based on the results, we selected the responses of several participants (HW, VH, AK, MA, NJ, and AB), which represent each case.

Correct solution: Correct starting point and correct 7th term

Case 1.1

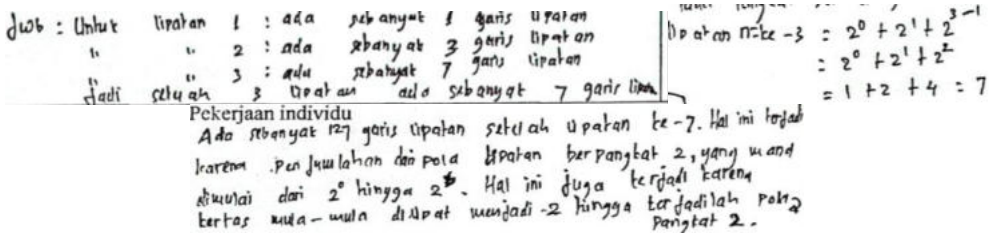


Translation: First fold = 1; second fold = 3; third fold = 7. Seventh fold = 127. The pattern formed is the difference multiplied by 2. $U_1 = 1;\ U_2 = 3;\ U_3 = 7, U_4 = 15$. The difference from U_1 to U_2 to U_3 is multiplied by 2.

Figure 3: Answer by HW

As evidenced by Figure 3, the PST correctly identified the sequence of fold line segments, suggesting she understands the basic idea of observing differences between successive terms, which increase in a structured way, i.e., kept getting multiplied by two. Each subsequent term can be determined by summing the previous term and the continuously doubled difference. Finding the seventh term using this rule remains a reliable method.

Case 1.2

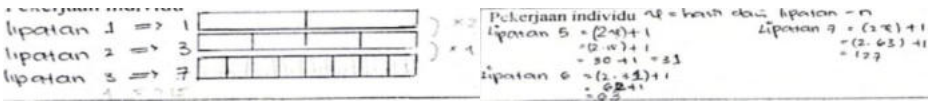


Translation: First fold, there is 1-fold line segment. Second fold, there are 3-fold line segments. Third fold, there are 7-fold line segments. Third fold $= 2^0 + 2^1 + 2^{3-1} = 2^0 + 2^1 + 2^2 = 1 + 2 + 4 = 7$. There are 127-fold line segments on the seventh fold. This is due to the sum of the fold patterns in powers of 2, starting from 2^0 to 2^6. This is also because the paper is initially folded in 2 until a power of 2 pattern occurs.

Figure 4: Answer by VH

Based on Figure 4, the PST accurately recognized that the number of fold line segments follows an exponential pattern with base 2. She employed the formula $2^0 + 2^1 + \cdots + 2^{n-1}$ to articulate how the fold line segments accumulate, performing an advanced and accurate mathematical representation. By identifying the recursive relationship inherent in this exponential sequence, she demonstrated an understanding of how each fold builds upon the previous ones. Her statement, "the paper is initially folded in 2 until a power of 2 pattern occurs," indicated that she made a connection between the physical act of folding and the mathematical pattern.

Case 1.3



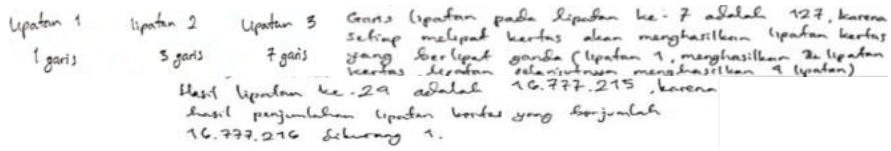
Translation: First fold: 1-line segment, second fold: 3-line segments, third fold: 7-line segments. x: result of the nth – fold. Fifth fold $= (2x) + 1 = (2 \times 15) + 1 = 30 + 1 = 31$. Sixth fold $= (2 \times 31) + 1 = 62 + 1 = 63$. Seventh fold $= (2x) + 1 = (2 \times 63) + 1 = 127$. (Note: The sketch indicates that the number of sections doubles in each fold)

Figure 5: Answer by AK

Referring to Figure 5, the PST attempted to formulate a general rule for each subsequent fold using the expression $2x + 1$ to describe the relationship between the current and the previous terms. The

appearance of this expression suggests that she fully captured the recursive nature of the sequence but struggled with fully articulating it, indicated by x being noted as a "result of the nth fold" rather than "the number of the fold line segments on the previous fold". Even though there is no verbal description regarding the number of sections or fold line segments, her sketch highlights that the number of sections doubles in each fold, which signals a connection to the folding activity.

Case 1.4



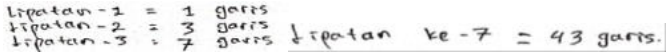
Translation: The number of the fold line segment at the 7th fold is 127, because each fold will produce doubled sections (the first fold produces 2 sections, the next fold produces 4 sections). The result of the 24th fold is 16.777.215 because the number of sections (16.777.216) is minus 1.

Figure 6: Answer by MA

As shown in Figure 6, the PST correctly identified that the number of sections doubles in each fold, reflecting an understanding of geometric progression. He grasped that the total number of fold line segments equals the number of sections minus one and correctly applied this reasoning to the 7th and 24th folds. His statement that "each fold will produce doubled sections" confirms a connection to the folding activity. This manifestation of the PST revisiting the folding activity indicates a deductive closure.

Incorrect solution: Correct starting point and incorrect 7th term

Case 2



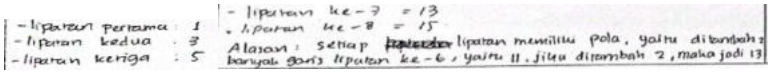
Translation: First fold = 1 line segment, second fold = 3 line segments, third fold = 7 line segments, seventh fold = 43 line segments.

Figure 7: Answer by NJ

As depicted in Figure 7, the identified sequence is $1, 3, 7, \ldots, \ldots, \ldots, 43$, where the difference between two terms might be $2, 4, 6, 8, 10, 12$. While the fourth fold could have been physically performed, the PST could not accurately identify the fourth term, indicating disconnection with the folding activity. The incorrect sequence follows a pattern of adding successively increasing even numbers, reflecting a simple arithmetic progression. This point suggests that the PST overgeneralized his understanding of basic arithmetic sequences, applying a linear pattern where a non-linear pattern was required.

Incorrect solution: Poor starting point and incorrect 7th term

Case 3



Translation: First fold: 1, second fold: 3, third fold: 5, seventh fold: 13, eighth fold: 15. Reason: Each fold has a pattern, which is plus 2. The number of lines on the sixth fold is 11, if it is plus 2 then it becomes 13.

Figure 8: Answer by AB

As presented in Figure 8, the identified sequence $1, 3, 5, 7, 9, 11, 13$ with a consistent difference of 2 between consecutive terms indicates that the PST incorrectly interpreted the pattern as a linear arithmetic progression. This reliance on a linear arithmetic pattern suggests an overgeneralization rooted in familiarity with basic sequences. The sequence that does not match the actual results reflects either a misunderstanding of the instruction or a disconnection with the folding activity. If not due to instructional misinterpretation, this case suggests a gap in his ability to translate concrete experiences into numbers as one of the mathematical representations.

Discussions

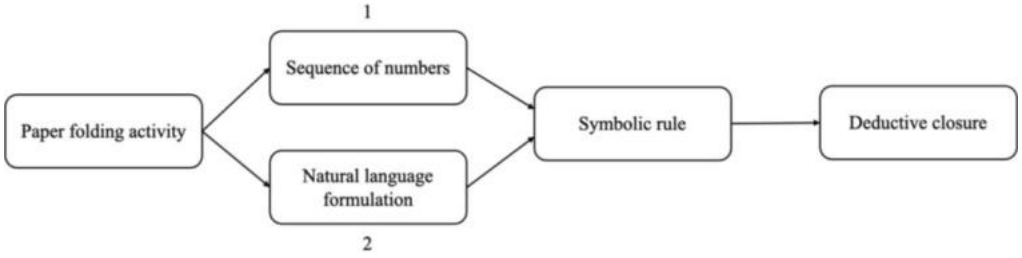


Figure 9: Translation process

Figure 9 depicts two possible approaches for translating the hands-on activity to reach the deductive closure as the optimal step. In this translation, certain visible features of the PSTs' responses are examined and connected to Sowder's (1992) dimensions of number sense. The folding activity could be translated into a mathematical object in two distinct ways:

First, if the fold line segments are simply enumerated, a sequence of numbers $(1, 3, 7, 15, ...)$ emerges without relational understanding from the translation process. This simplest manifestation of number sense aligns with dimension 1 (the ability to move flexibly among different representations), which is evident in how the manual activity produces a number at each step. If this counting is incorrect (Case 3), it could suggest a rudimentary number sense in this dimension or that the PST needed to recognize the need to count the fold line segments after each step. The error might stem from a tenuous grasp of numbers or misunderstanding the instruction. Meanwhile, Case 2 illustrates a specific circumstance where the PST accurately represented the number of line segments only for the first three folds, suggesting an inconsistency between representations.

Second, the translation process can also lead to formulating a rule based on natural numbers' additive and multiplicative structure. This rule could be established after the counting, either by abstracting the activity and identifying the pattern or directly linking the activity to a structured object. This process aligns with the manifestation of dimension 5 (the ability to link numeration, operation, and relation symbols in meaningful ways). Case 1.1 provides a concrete example of this dimension where the PST recognized the difference between consecutive terms. Case 1.3 gives another example, where the PST demonstrated a different approach to recursion, not by analysing the difference but by formulating a direct rule to describe the pattern.

In the latter approach, the relationship could be articulated in natural language: "In each fold, twice as many sections are formed as before, and the number of fold line segments is one less than the number of sections." This formulation necessitates an understanding of the operations' effects on

numbers, reflecting dimension 6. Case 1.2 further exemplifies the presence of this dimension as the PST recognized that consecutive multiplication by two could be expressed as the sum of exponential numbers with base 2, suggesting the formulation of a general rule. This work contrasts with Case 1.1, where the progress stalled in identifying the difference between consecutive terms, and no general term was formulated. Thus, the PST's understanding was confined to the arithmetic level.

Optimally, recognizing the pattern in the number sequence concludes with a deductive reasoning process, connecting back to the folding activity. This manifestation reflects a disposition toward making sense of numbers, which characterizes dimension 9. The emergence of this dimension is evident in Case 1.4, where the PST explained the connection between the folding activity and the rule, providing a clear justification and argumentation for why the sequence is correct. Moreover, the sketch in Case 1.3 serves a visual explanation for the stated rule, simultaneously exemplifying the manifestation of dimension 9.

In summary, some PSTs accurately determine the seventh term, demonstrating their comprehension of the instructions. This performance implies their ability to smoothly transition between representations (dimension 1) from physical to mathematical forms. When PSTs demonstrate procedural knowledge, they often reflect their understanding of how operations affect numbers (dimension 6), as they can execute procedures accurately and make connections between their actions and numerical outcomes. Some PSTs demonstrate a conceptual understanding of exponential growth, recursive patterns, and their connection to the folding paper activity by linking numeration, operations, and relational symbols. It means they understand how the number of folds relates to exponential increases and can articulate these connections meaningfully using mathematical symbols and operations. On the other hand, some of the PSTs needed help to express the rule clearly and tend to overgeneralize arithmetic sequences (dimension 5). Finally, when PSTs connect the folding paper activity to mathematical representation, it shows a disposition toward making sense of numbers (dimension 9). They actively engage with the numerical patterns and relationships, demonstrating an inclination to understand and explore mathematical ideas. Overall, some participants performed well, and some others faced challenges in performing dimensions 1, 5, 6, and 9. In line with this result, another study by Aktaş & Özdemir (2017) highlighted the preservice elementary school mathematics teachers who have lower than expected number sense performance.

Conclusions and pedagogical implications

This study revealed varied performance of number sense among PSTs, particularly in recognising numerical patterns, recursion, and linking concrete actions to mathematical abstractions. To improve these areas, teacher educators should: (1) explicitly encourage PSTs to justify identified patterns, fostering deeper conceptual understanding rather than superficial recognition; (2) use activities like paper folding to strengthen the connections between tangible experiences and abstract mathematical concepts, encouraging PSTs to verbally and visually articulate mathematical relationships.

Acknowledgement

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Students' recognition of characteristics in addition and subtraction problems as a prerequisite for flexible mental calculation

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Flexibility in mental calculation is a central topic of elementary mathematics education. Our study focuses on capturing flexibility in mental calculation through the link between the cognitive elements that sustain the solution process and the solution tools. In this paper, we put emphasis on students' recognition of characteristics in addition and subtraction problems and investigate the repertoire of recognised problem characteristics. Furthermore, we focus on the role of the problem selection. In this context, we analyse in which problems the students recognise more or fewer problem characteristics. The results reveal that there is a broad range regarding students' recognition of problem characteristics. Furthermore, the number of recognised characteristics depends on the structure of a problem and varies between different problems.

Keywords: Flexibility in mental calculation, problem characteristics, number patterns, elementary arithmetic, addition and subtraction problems.

Introduction

Fostering flexibility in mental calculation is a central goal in elementary mathematics education. On the one hand, this is evident in several national curricula (for an overview see e.g., Torbeyns & Verschaffel, 2016). On the other hand, many researchers emphasise the importance of the development of flexibility in mental calculation (e.g., Green & Rathgeb-Schnierer, 2020; Hickendorff, 2022; Torbeyns & Verschaffel, 2016). Different research reports that elementary students often use the standard algorithm when solving multi-digit addition and subtraction problems (e.g., Hickendorff, 2022; Selter, 2001). However, various studies show that it is possible to promote students' flexibility in mental calculation with specific teaching approaches (for an overview of the approaches, look for example, at Heinze et al. (2020) and Hickendorff et al. (2022)). Some of these studies highlight the important role of students' skills to recognise problem characteristics, number patterns, and relationships (abbreviated in the following as 'problem characteristics'; e.g. Rechtsteiner & Rathgeb-Schnierer, 2017).

In general, studies show that students use various strategies while solving problems with different problem characteristics with some strategies being used more frequently than others (e.g., Hickendorff, 2022; Macintyre & Forrester, 2003; Peltenburg et al., 2012). Furthermore, some studies focus on the recognised problem characteristics by students, but not with regard to various problems and their specific characteristics (e.g. Rathgeb-Schnierer and Green, 2013). Peltenburg et al. (2012) report that the proportions of students' problem-solving strategies differ depending on the specific characteristics of a problem. Thus, for assessing flexibility in mental calculation, we need instruments with arithmetic problems that include various problem characteristics. In this context, it is crucial to consciously select problems according to their specific inherent characteristics (e.g., Peltenburg et al., 2012; Torbeyns et al., 2009). Research results suggest that problems which require crossing the ten boundary are more difficult to solve than problems that do not (e.g., Hickendorff, 2022;

Peltenburg et al., 2012). However, Rathgeb-Schnierer & Green (2015) point out that if students recognise further problem characteristics that make the problem easier to solve (such as the second number is close to the next full ten in problem 46 – 19), the ten boundary would not be a challenge anymore. These findings suggest that the problem characteristic of crossing, respectively not crossing the ten boundary, is crucial to distinguish problems and the (successful) use of strategies.

The wide range of research regarding flexibility in mental calculation is contrasted by inconsistent terms, definitions, and operationalisations (for an overview of the definitions and operationalisations look for example at Green & Rathgeb-Schnierer (2020) and Hickendorff et al. (2022)). Despite the different definitions, there is a consensus that flexibility in mental calculation can be understood "as an appropriate way of acting when faced with a problem based on a repertoire of available strategies, which is to say that flexible strategies are adapted dynamically to problem situations" (Green & Rathgeb-Schnierer, 2020, p. 3). Rechtsteiner and Rathgeb-Schnierer (2017) gave an overview of the different approaches to define and operationalise flexibility in mental calculation. We take a closer look at one of the approaches which puts emphasis on the *cognitive elements that sustain the solution process* (e.g., Rathgeb-Schnierer & Green, 2013) and considers the link between those cognitive elements and the exhibited solution tools. In this vein, Rathgeb-Schnierer and Green (2013) emphasise: "Only if the tools of solution are linked in a dynamic way to problem characteristics would we consider as evidence of flexibility in mental calculation" (p. 357). Our definition of flexibility in mental calculation is based on these authors. Specifically, we define flexibility in mental calculation as mental processes that involve the ability to recognise the characteristics of a problem and adapt solution methods accordingly. By problem characteristics, we mean specific number patterns and numerical relationships (for example, see Table 1).

In summary, current research on flexibility in mental calculation shows that students use various strategies with varying frequencies while solving problems with different problem characteristics (Macintyre & Forrester, 2003; Peltenburg et al., 2012). Moreover, there is a conscious that flexibility in mental calculation involves that "flexible strategies are adapted dynamically to problem situations" (Green & Rathgeb-Schnierer, 2020, p. 3). However, no study has yet investigated the role of problems with different characteristics in relation to students' recognition of those characteristics. We address this research gap with our study, which pursues two research questions based on our definition (RQ):

RQ1: How many different problem characteristics do students recognise in various problems with different inherent characteristics?

RQ2: In which problems do students recognise problem characteristics and what differences do occur?

Methods

Sample and procedure

155 third-grade students participated in the videotaped interview. Almost half of the sample consists of female students (n = 77) and half of male students (n = 78). Of all students, the mean age is 9.52 years (SD = 0.37 years), of the 77 girls it is 9.46 years (SD = 0.33 years), and of the 78 boys it is 9.58 years (SD = 0.40 years).

Interview

Data was collected by a semi-structured, guided interview which was detailly introduced at the last CERME (see Flückiger & Rathgeb-Schnierer, 2023). In this paper, we put emphasis on the first part of the interview which includes the sorting and reasoning of 12 addition and subtraction problems (see Figure 1).

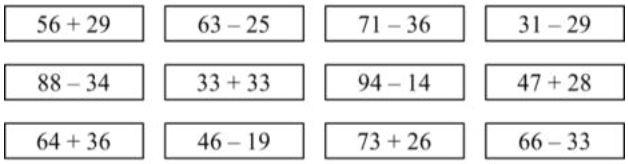

Figure 1: Overview of the problems

In the first part of the interview, we encourage the students to sort the problems into the categories "easy" and "hard", and to give reasons for their sorting. The students' reasoning is a central part of this interview since it provides the possibility of revealing whether students rely on procedures or problem characteristics because they can use both types of reasons (see Table 1 for examples of inherent problem characteristics of three problems). Consequently, that is an appropriate approach to capture students' individual cognitive elements that sustain the solution process (e.g., Rathgeb-Schnierer & Green, 2013). In the further course of the interview, we asked the students to solve the problems.

Table 1: Examples of inherent problem characteristics

Problem	Inherent problem characteristics
63 - 25	Crossing the ten
31 – 29	Both numbers are close together The second number is close to 30 The first number is close to 30 Crossing the ten
66 - 33	The second number is half of the first number The tens and ones of both numbers are the same Inverse problem to 33 + 33 Not crossing the ten

Data analysis

Based on the *Qualitative Content Analysis* by Mayring (2014), we developed a coding system for data analysis. The coding system includes main coding levels with various sub-categories that allow to assign the statements of the students. The main coding levels are: sorting of problems, correctness

of the solution, justifying the sorting, and used solution tools. Table 2 displays a part of the coding system for the main coding level, justifying the sorting with some examples for sub-codes (for more detail, see Flückiger & Rathgeb-Schnierer, 2023). The students' statements were coded at the second sub-code level.

Table 2: Excerpt from coding system

Sub-codes (1st level)	Sub-codes (2nd level)
Numbers at the ones place	The ones are equal One of the ones is twice the other one The sum is less than ten
Relationship of the numbers	Distance between the two numbers

All interviews were independently coded by two trained coders (the first author of this paper and a teacher-student). The coding level for justification of the sorting included 52 sub-codes. The intercoder-reliability for this coding level is $\kappa = 0.81$ (strong intercoder-reliability based on conscious thresholds, e.g., McHugh, 2012). The specific coding process made various analyses possible, such as the calculation of the percentage of problems that were reasoned by problem characteristics as well as the repertoire of mentioned problem characteristics. The repertoire describes the number of different problem characteristics the students have named. When analysing data, it is possible to focus on the entire problem set as well as individual problems separately.

Results

In our last CERME-paper, we provided descriptive results of the first 20 double-coded interviews (Flückiger & Rathgeb-Schnierer, 2023). In this article, we build on this and present the descriptive results of all 155 double-coded interviews. Table 3 reports descriptive results (mean and standard deviation) of three categories: reasoning by problem characteristics as well as the repertoire of those reasons, counting only one reason per problem and all reasons in total. The categories are based on the codes assigned at the second sub-code level.

Table 3: Descriptive results of all of the 155 interviews

Categories	Mean (*M*)	Standard deviation (*SD*)
Reasoning by problem characteristics	6.24	3.44
Repertoire of reasoning by problem characteristics (one per problem)	3.52	1.80
Repertoire of reasoning by problem characteristics (all reasons in total)	4.12	2.21

The values in the lines include correctly and incorrectly solved problems. The first line (reasoning by problem characteristics) exhibits the number of problems whose assignment to a category during the sorting process was justified by problem characteristics. On average, students justified the assignment

to a category with problem characteristics in 6.24 out of 12 problems (52.0%; *SD* = 3.44). The second and third lines refer to students' repertoire regarding reasoning by problem characteristics. Hereby, we identify repertoire by the number of different given reasons. In this context, two different approaches are possible: either to consider one named problem characteristic for each problem or to consider all. We decided to present both aspects, as the 12 problems are different in terms of the number of inherent problem characteristics (see Table 1). Therefore, when all problem characteristics are considered, some problems may have a larger share of the repertoire value. Regarding the repertoire which includes just one reason per problem, the students justified their sorting with an average of 3.52 different problem characteristics (*SD* = 1.80). When focus on the repertoire containing all reasons, the students justified their sorting with an average of 4.12 different problem characteristics (*SD* = 2.21).

Table 4: Descriptive results per problem

Problem	**Number of students who…**	
	reasoned by problem characteristics (percentage)	**referred exclusively to crossing the ten boundary (percentage)**
66 – 33	123 (79.4%)	0 (0.0%)
33 + 33	118 (76.1%)	2 (1.7%)
46 – 19	88 (56.8%)	82 (93.2%)
71 – 36	78 (50.3%)	68 (87.2%)
56 + 29	73 (47.1%)	51 (69.9%)
47 + 28	73 (47.1%)	70 (95.9%)
31 – 29	70 (45.2%)	33 (47.1%)
94 – 14	69 (44.5%)	3 (4.3%)
63 – 25	66 (42.6%)	65 (98.5%)
64 + 36	64 (41.3%)	4 (6.3%)
88 – 34	53 (34.2%)	34 (64.2%)
73 + 26	51 (32.9%)	32 (62.7%)

Table 4 displays descriptive results of all 155 interviews regarding recognised problem characteristics for each problem individually. The first column displays the number of students who recognised problem characteristics as well as the respective percentage of those reasons in comparison to all given reasons (sorted from high to low values). The second column considers only students whose

reasons refer to characteristics, and shows the number of students who only named the characteristic of crossing the ten boundary. The percentages in this column result from the proportion in relation to the number of all recognised problem characteristics in column one.

Regarding all problems, the mean of recognised problem characteristics is 77.17 (49.8%; *SD* = 22.57), and hereof the number of the characteristic crossing/not crossing the ten boundary is 37.58 (24.2%; *SD* = 29.99). The results reveal that whenever students' reason by problem characteristics, there is a large share of the justification by the problem characteristic crossing/not crossing the ten boundary. Regarding and comparing individual problems, it is obvious that the number and percentage of justifications by recognised problem characteristics vary greatly. The problems 66 – 33 (123; 79.4%) and 33 + 33 (118; 76.1%) show the highest proportion, and the problems 88 – 34 (53; 34.2%) and 73 + 26 (51; 32.9%) show the lowest proportion. It is noticeable that the proportion of reasoning by the characteristic crossing/not crossing the ten boundary is very high in some problems. For example, this proportion is over 90% for the problems 63 – 25 (98.5%), 47 + 28 (95.9%), and 46 – 19 (93.2%). At the same time, there are problems such as 66 – 33 (0.0%), 33 + 33 (1.7%), 94 – 14 (4.3%), and 64 + 36 (6.3%), where the percentages are very low. In addition, the percentages vary in problems in which the second number is very close to the next full ten (46 – 19, 56 + 29, 31 – 29). Regarding the problem 46 – 19, most students who reasoned by problem characteristics only mentioned the characteristic crossing the ten boundary. Just five students recognised that the subtrahend 19 is close to 20. In the addition problem with the same structure, 56 + 29 far more students (19) recognised that the second summand 29 is close to 30. Regarding the problem 31 – 29, six students mentioned that 19 is close to 20, and 30 students recognised that both numbers are close to each other.

Discussion and outlook

In this paper, we reported the descriptive statistics regarding students' reasoning for various addition and subtraction problems with specific emphasis on reasons that refer to problem characteristics, and the differences between the different problems. Our approach of sorting, reasoning, and solving problems in a semi-structured interview allows us to reveal the cognitive elements that sustain the solution process, such as recognised problem characteristics or procedures (Green & Rathgeb-Schnierer 2020). Thus, we can see whether students refer to problem characteristics and how many different problem characteristics they recognise (RQ1). In this context, it is interesting that the standard deviation of reasoning by problem characteristics is relatively high (*M* = 6.24; *SD* = 3.44). This means that there is a high scattering around the mean value in the way of justification (via problem characteristics or procedures).

Existing studies show that students use various strategies with varying frequencies while solving problems with different problem characteristics (e.g., Macintyre & Forrester, 2003; Peltenburg et al., 2012). Our method of analysing the data allows us to identify in which problems students recognise problem characteristics and in which they do not. Here, we can see that students also give different reasons with varying frequencies while reasoning about problems with different problem characteristics (RQ2). There is a wide range of the problems regarding the reasoning via problem characteristics. Another aspect of problems is the role of crossing/not crossing ten boundary (e.g., Hickendorff, 2022; Peltenburg et al., 2012). In our results, we can see that reasoning by crossing/not

crossing the ten boundary has a large share in the category reasoning by problem characteristics, and the percentage share also varies greatly between the different problems. However, if students recognise problem characteristics such as the second number is close to the full ten, the ten boundary is no longer a determining factor for the difficulty of the problem (Rathgeb-Schnierer & Green, 2015). In our results, we see that the frequency with which students recognise problem characteristics that do not relate to crossing/not crossing the ten boundary varies greatly between the problems. This result confirms that the problem selection is crucial (e.g., Peltenburg et al., 2012; Torbeyns et al., 2009).

The problem selection is also important for revealing the cognitive elements that sustain the solution process and enables the high scattering in the way of justification discussed above. The chosen method of collecting (interview) and analysing (category system) the data allows further analyses. In the further course of the interview, we asked the students to solve the problems. This conversation, which provides information about the solution process and the result, enables conclusions about utilised solution tools and accuracy. Thus, we will be able to analyse the connection between the cognitive elements that sustain the solution process and the used solution tools. With this approach, we capture flexibility in mental calculation by identifying whether students adapt their strategies dynamically to a problem situation or not (Green & Rathgeb-Schnierer 2020).

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Teaching two-digit subtraction with a structural approach in the second grade

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It has been shown that many students end up in difficulties when solving multi-digit subtraction tasks. This study report on findings from an intervention about teaching subtraction with two-digit numbers in second grade. During one semester four teachers collaborated with two researchers in designing six lessons emphasizing a structural approach, using part-whole relations for solving subtraction tasks. The aim of this paper is to identify what students learned from the intervention. Written pre- and post-tests from three intervention classes and seven comparison classes were analysed with particular focus on the task 204–12=? The results show that the intervention group to a higher extent than the comparison group solved the tasks by decomposing the subtrahend, using a sequential strategy. Two ways of using the sequential strategy in the intervention group was found, decomposing the subtrahend in tens and ones, and decomposing the subtrahend to bridge through ten.

Keywords: Two-digit subtraction, intervention, structural approach, variation theory, elementary.

Introduction

Solving multi-digit subtraction tasks is found to be a difficulty for students in early grades (Heinze et al., 2018; Van Der Auwera et al., 2023). For example, a study found that less than half of the students could solve the task 701–698=? in third grade (Selter, 2001). There is a substantial amount of research on students' ways of solving subtraction tasks (e.g., Selter et al., 2012; Van Der Auwera et al., 2023). However, there are few studies about how to overcome students' difficulties solving subtraction tasks with multi-digit numbers (e.g., Blöte et al., 2001; Heinze et al., 2018). It has been suggested that a structural approach emphasizing part-whole relations for solving addition and subtraction tasks may benefit student learning since it can also be used in higher number ranges (Ellemor-Collins & Wright, 2009). In this study we worked in collaboration with a team of second grade teachers during one semester, designing a series of lessons together. We used a structural approach in the intervention which means for example a focus on mathematical relationships found between elements, for instance part-whole relations of numbers and operations (Reuter, 2023; Venkat et al., 2019). The aim of this paper is to identify what students learned from the intervention. Our research question is: In what ways were students' learning affected by the intervention building on a structural approach?

Learning of multi-digit subtraction

Torbeyns et al. (2009) distinguish between three different types of strategies for calculating subtraction with multi-digit numbers such as 49–25=?. Decomposition is the strategy where tens and ones are divided separately (40–20=20, 9–5=4, 20+4=24), while *Sequential* means that the whole is preserved (49–20=29, 29–5=24). A third type involves strategies where the numbers in the task are changed to facilitate calculation (50–25=25, 25–1=24). The strategy decomposition works well when

solving the task 49–25=? but can create problems when students solve tasks that involve bridging through ten, for example 32–25=? Kullberg et al. (2024a) found that students that used the sequential strategy were more successful than students who decomposed both the minuend and the subtrahend. They argue that students' choice of strategies for solving tasks is influenced by how students perceive and use number relations. To develop powerful strategies for solving multi-digit tasks it has been argued that students need to be able to structure numbers into parts and wholes (Schmittau, 2003). Then they can use part-whole relations to solve a subtraction as an addition 25+5+2=32, or 'see the two and three in the five' in order to bridge through ten e.g., 32–25=, 32–2=30, 30–3=27, 27–20=7. Van Der Auwera et al. (2023) suggests interventions to improve children's understanding of the complement principle (a+b=c thus c–a=b), and to incorporate subtraction as 'determining the difference' in primary mathematics education to provide a more comprehensive understanding of subtraction. In this study the idea was for students to learn to approach subtraction problems considering the structure of part-whole relations using a sequential strategy.

Methods

In this section the method and the intervention and its theoretical foundation is described. Four teachers worked in collaboration with two researchers during one semester designing lessons with the aim to increase second grade students' ability to solve two-digit subtraction tasks using number relations. The teachers and researchers met eight times, whereof six times to plan and analyse the lessons in the intervention. The planning meetings took place at the teachers' school. The teachers had multiple years of teaching experience and had taught their classes from first grade. The intervention group comprised of students from three classes (eight-year-old students) at the same school in a municipality near a larger Swedish city (middle socioeconomic status). The comparison group comprised of seven classes from five schools in similar areas. The participating teachers and the legal guardians of the students in all classes had signed a written consent to participate in the study. In the study video-recorded lessons, student interviews, and written pre- and post-tests were collected as data. All participating students completed the same test as both a pre-test during the first two weeks of the semester and a post-test in the final month. The control group followed standard practices without any researcher-imposed modifications. In this paper we analyse one task (204–12=?) out of fourteen, from the pre- and post-test in more detail to study the learning outcome of the intervention. The task, involving a three- and a two-digit number, was chosen because the intervention focused only on two-digit number tasks. The task allows for multiple ways of using decomposition of numbers while requiring an understanding of place value and the concept of "zero."

The answers to the task were coded by two researchers who first developed a coding scheme, which was then discussed within the research team. Students' solutions were coded according to this scheme, with difficult cases being discussed by the team. The coding included frequencies of correct and incorrect answers, as well as strategies used in the solutions. "No answer" and "No explanation" were also included in the coding scheme. The students were told to calculate the tasks in the pre- and post-test without using the standard written algorithm. The following strategies were identified: single-unit counting, the standard written algorithm for subtraction, decomposition and sequential. If a student had decomposed the minuend and the subtrahend, this was coded *Decomposition* (decomposing both the minuend and subtrahend), and if a student decomposed only the subtrahend,

this was coded *Sequential* (decomposing the subtrahend). The goal of the intervention was to develop a deeper understanding and enable the students to structure numbers using part-whole relations using a Sequential strategy when solving multi-digit subtraction tasks. In this paper we only focus our analysis on two strategies, Decomposition and Sequential.

An intervention based on a structural approach and variation theory

The intervention was based on a structural approach and on the variation theory of learning (Marton, 2015). From a variation theory perspective, to master an object of learning (in this case being able to subtract two-digit numbers) there are critical aspects that students need to discern. Critical aspects can be identified from e.g., a student test, text, interview or during teaching. The critical aspects need be made noticeable for the learners by means of variation, for example by contrast (difference). Insights from previous intervention studies (Björklund et al., 2021; Kullberg et al., 2024b) on addition and subtraction in lower number ranges based on a structural approach and the same theoretical framework was used as a foundation for the intervention in this study, however additional critical aspects were also found during the intervention.

Six lessons were designed in collaboration with teachers in the intervention group and enacted in all participating classes during the same weeks. The first two lessons concerned seeing addition and subtraction tasks as part-whole relations. The triad model was introduced and used to visualize the number relations (whole and parts) when working on tasks creating number sentences. This involved using the same number relation for addition and subtraction while emphasizing the parts and the whole in number sentences. During the task the students were told to circle the whole to see that the placement of the whole shifted in addition and subtraction and in relation to the placement of the equal sign. Lesson 1 focused on single digit numbers (e.g., 7/5/2) whereas in lesson 2 two-digit numbers were used (e.g., 21/18/3). The same critical aspects, e.g., the whole consists of two parts and the whole does not always come after the equal sign, were made possible to experience. The first two lessons and the pre-test showed that the students had difficulties in understanding the meaning of multi-digit numbers, therefore lesson 3 was designed to show the meaning of the positioning system. The teacher discussed and illustrated place values in numbers which were similar and different in place value positions; 222, 202, 226, 27 and 207. However, this lesson was not specifically targeting the object of learning.

The last three lessons particularly targeted the object of learning, solving subtraction tasks with two-digit numbers using number relations. Lesson 4 and 5 focused on the *Sequential* strategy, only partitioning the subtrahend when solving tasks with two-digit numbers. Two ways of subtracting, starting with the ones or the tens, keeping the minuend unaltered (critical aspect), where simultaneously possible to experience. Also, *Decomposition*, decomposing both the minuend and subtrahend which often ended up with an incorrect answer were discussed (e.g., 32–25=?, 30–20=10, 2–5=3, 10+3=13) as a contrast. The different steps taken while solving the subtraction task were also shown simultaneously and on the number line (Figure 1). Special attention was paid to how the ones in the subtrahend can be partitioned to bridge through ten.

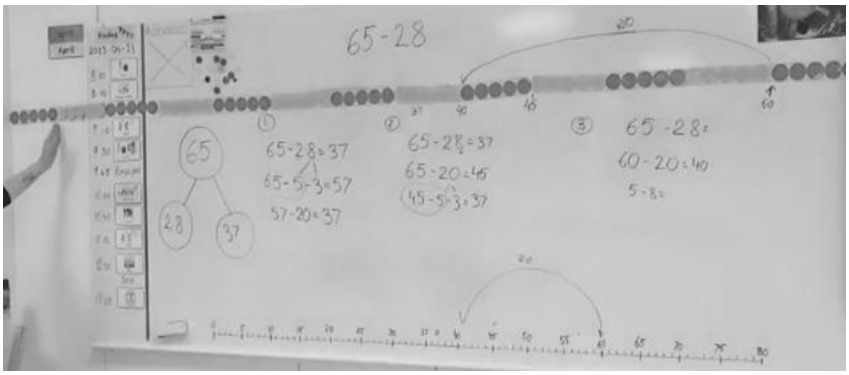

Figure 1: The task 65 –28=? was solved in three ways (1. sequential starting with the ones, 2. sequential starting with the tens, and 3. decomposition of tens and ones)

Thereafter the students worked in pairs with a task where students were to solve several subtraction tasks with two-digits numbers using the two *Sequential* strategies (starting with the ones or starting with the tens) on the same paper. This made it possible for the students to try both strategies for the same task hence to experience a variation in ways of solving. In this way a contrast between starting with the ones or tens were made possible for the students to experience. In lesson 5 and 6 *Subtraction as addition* was introduced as a contrast to the *Sequential* strategy. This was done by contrasting Jonna's and Linus's strategies to solve 35–27=? (Jonna: 35–20=15, 15–5=10, 10–2=8, answer: 8, Linus: 27+3=30, 30+5=35, answer: 8.) The goal was for students to experience the complement principle (a+b=c then c–a=b). Discussion of the different strategies provided an opportunity to experience a contrast between a subtraction task solved as a subtraction or as an addition. Thereafter, the students worked with solving the same subtraction tasks using both strategies. Lesson 6 focused foremost on using the strategy Subtraction as addition for solving subtraction tasks.

Results

First, we present the learning outcomes of the intervention and comparison group on the task 204–12=? on the pre- and post-test and the strategies students display. Following this, we give examples of various ways students applied the sequential strategy. The number of students not providing an answer decreased from the pre- to post-test in both the intervention and comparison group (see Table 1). While the percentage decreased from a higher starting point in the comparison group, from about 28% to 11%, it dropped from about 17% to 6% in the intervention group. It should be noted that there were more students in the comparison group who did not answer the task. In the intervention group the students to a lesser extent answered with no explanation. Although a comparable percentage of students gave no explanation in the pre-tests (intervention group about 17 % correct + about 21% incorrect and comparison group about 19% correct + about 26% incorrect) a more pronounced difference emerged in the post-test. Students in the intervention group providing no explanation while giving correct answers (about 8%) and incorrect answers (about 4%) decreased. In contrast, the comparison group shows a slight increase in correct answers (about 25%) and a decrease in incorrect answers (about 16%). Thus, fewer students in the intervention group responded without showing any form of calculation in the post-test.

Table 1: Student strategies used for solving 204–12=? in percentages (n in parentheses), correct answer in bold, and incorrect answer below correct answer. Note *Other strategies* is not included in the table, which explains the discrepancy in the total number

	Pre-test Intervention group N= 48	Post-test Intervention group N=48	Pre-test Comparison group N=89	Post-test Comparison group N= 89
No answer	16.7 (8)	6.3 (3)	28.1 (25)	11.2 (10)
No explanation	**16.7 (8)** 20.8 (10)	**8.3 (4)** 4.2 (2)	**19.1 (17)** 25.8 (23)	**24.7 (22)** 15.7 (14)
Decomposition- decompose the minuend and subtrahend	**2.1 (1)** 2.1 (1)	**0 (0)** 2.1 (1)	**0 (0)** 1.1 (1)	**2.2 (2)** 3.4 (3)
Sequential - decompose the subtrahend	**27.1 (13)** 2.1 (1)	**64.6 (31)** 8.3 (4)	**18.0 (16)** 1.1 (1)	**24.7 (22)** 5.6 (5)

When students showed their solution (as asked for in the task), we could distinguish that they either decomposed both the minuend and the subtrahend, thus applying decomposition, or kept the minuend preserved and only decomposed the subtrahend. In both groups, only a few students decomposed both the subtrahend and minuend when solving the task in both the pre- and post-test. Far more students solved the subtraction task sequentially. In the pre-tests about 27% of the intervention group did so correctly and about 2% incorrectly, while in the comparison group, 18% solved it correctly and about 1% incorrectly.

A greater portion of the intervention group already displayed a sequential strategy in the pre-test, and more importantly, this difference was amplified in the post-tests. The comparison group increased to about 25% (correct answer) and about 6% (incorrect answer), while the intervention group increased notably to about 65% (correct answer) and about 8% (incorrect answer). In total 73 % (34) of the students in the intervention group solved the task using the sequential strategy in the post-test. Almost half of these students (about 33% (15)) solved the task by decomposing the subtrahend into tens and ones (12 =10+2), while about 42% (19) by decomposing the subtrahend into 8 and 4 (12=8+4). Next, the identified variation within the two sequential strategies from the post-test is illustrated with examples from the intervention group.

Student examples of the sequential strategy - decompose the subtrahend

We identified two different ways of using the sequential strategy in the intervention group. The students who used the sequential strategy all preserved the minuend (204) but decomposed the subtrahend (12) in different ways. The first way was to decompose the subtrahend into tens and ones (Figure 2), either starting with the tens or the ones. This was done either as Student A (Figure 2) subtracting the ones (2) first, or as Student B and C beginning by subtracting the tens (10). Student A solves the task by first calculating 204–2=202 and keeps the minuend unaltered. Next the students calculated 202–10=192. Student B calculates 204–10=194, and thereafter 194–2=192. Student C starts as Student B with the tens and shows the decomposition of 12 with a triad (Figure 2) although

uses the equal sign incorrect in the calculation. During the intervention the students had the possibility to simultaneously experience subtraction starting with the ones or the tens and the critical aspect 'keeping the minuend unaltered'. We infer that this is also shown in their ways of solving.

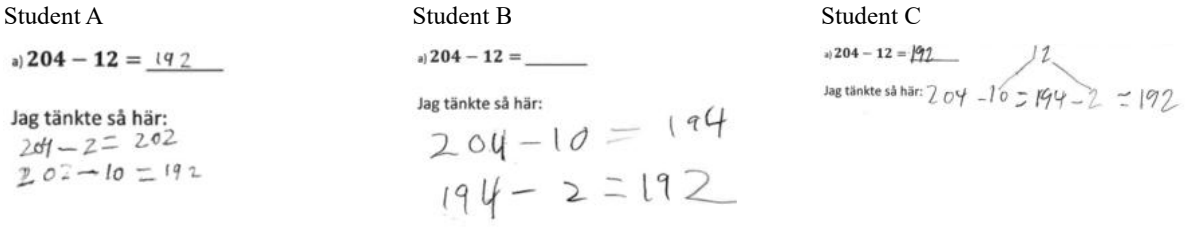

Figure 2: Sequential strategy, decomposing the subtrahend in tens and ones

In the second way of using the sequential strategy, the students decomposed the subtrahend (12) into 4 and 8, 'seeing the four and eight in the twelve', in order to bridge through ten (Figure 3). Hence, these students have likely discerned the critical aspects 'keeping the minuend unaltered', 'ten as a benchmark' and 'parts in the part (12)' brought up during the intervention. Student D calculates the subtraction in two steps, first subtracting 4 to bridge through ten (204–4=200), then subtracts the remaining part (200–8=192). Student E applies the same decomposition strategy, and in addition show the decomposition using a triad (Figure 3) but use the equal sign incorrectly. In contrast, Student F bridges through ten by decomposing 12 in three parts (2, 2, 8). Seemingly Student F decomposes 12 in 2 and 10, thereafter decompose 10 into 2 and 8, subtracting 4 to bridge through ten in the minuend (204–2–2=200).

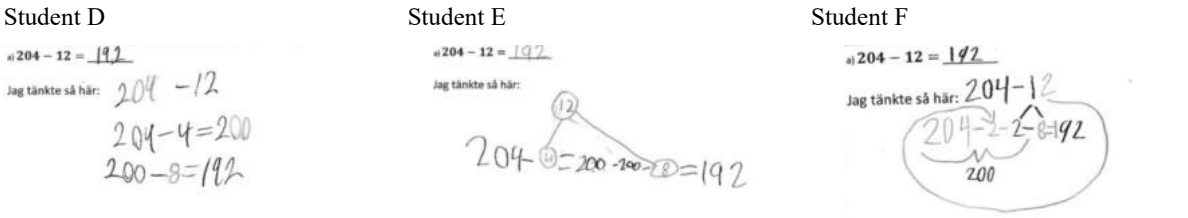

Figure 3: Sequential, decomposing the subtrahend bridging through ten

The post-test revealed that approximately 65% of the students in the intervention group successfully applied the sequential strategy to reach correct answers. However, despite utilizing the same strategy, around 8% (n=4) of the intervention group provided incorrect answers (see Figure 4). These students saw the four and eight in the twelve, indicating that they had discerned the critical aspects 'keeping the minuend unaltered', 'parts in the part (12)'and 'ten as a benchmark'. However, in this case different calculation errors led to incorrect answers.

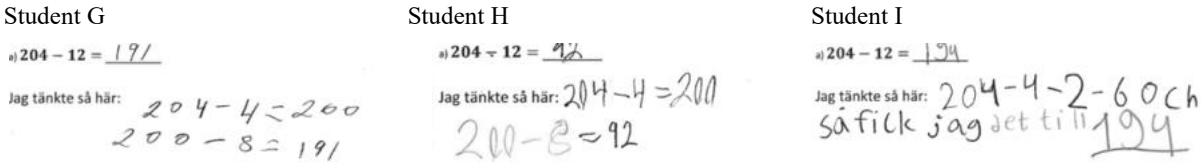

Figure 4: Sequential, seeing the four in the twelve – incorrect answer

Discussion

The results show that the intervention focusing on subtraction with two-digit numbers (65–28=?) had a notable impact on students' adoption of the *Sequential* strategy for solving 204–12=? This was evident in the post-test, where a greater proportion of students in the intervention group employed this strategy compared to the comparison group. Additionally, we found an increase in the frequency in which students provided explanations for their answers. In the post-test, about 88% of students in the intervention group offered explanations, compared to about 60% in the comparison group. This suggests that the intervention not only influenced students' subtraction strategies, but also encouraged greater awareness of their way of solving. By focusing on the structure of part-whole relations when solving two-digit subtraction and having the possibility to experience critical aspects of the object of learning, students in the intervention group most likely developed a deeper understanding for solving the subtraction task with two and three-digit numbers. Our study adds to previous findings (Heinze et al., 2018; Selter, 2001; Torbeyns et al., 2009) by showing how students were influenced differently by the same intervention, hence when solving the same task in another number range they applied sequential strategies in two different ways (Figure 2 and 3). Moreover, Kullberg et al. (2024a) found that how students decomposed numbers when solving 204–193=? reflected the number relations attended to by students in the solving process. Specifically, strategies like 'decomposing the difference' and 'decomposing the subtrahend' led to correct answers, while 'decomposing into digits' and 'decomposing both the minuend and subtrahend' often resulted in incorrect solutions. In our sample few students in both the intervention and comparison group decomposed both the minuend and subtrahend which is a different finding compared to Kullberg et al. (2024a), who found this strategy more common. Our findings suggest that student learning was influenced a great deal by a rather small intervention (only three lessons targeting the subtraction strategies with two-digit numbers). The analysis of 204–12=? contributed to identifying distinct differences between the control and intervention groups in their use of the sequential strategy, as well as revealing variations within the strategy itself. It seems important to continue to analyse several tasks in the pre- and post-tests to determine whether students consistently use the same strategy across different tasks or employ task-specific strategies. Further research is also needed to identify additional critical aspects and explore how these can be effectively implemented in teaching practices.

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Improving primary students' understanding of division

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Teaching division in primary school is a challenging task, not least because the distinction between quotative and partitive division is often poorly understood by students. This paper reports initial findings from an ongoing teaching experiment designed to improve students' understanding of both types of division. The approach integrates direct comparisons of quotative and partitive division to improve students' ability to explain and apply these concepts. The paper reports on how children who participated in the first cycle of the teaching experiment interpreted material representations of division. Qualitative interviews show that most children could interpret three groups of four cubes as a representation of a division in both ways, although some showed conceptual and linguistic difficulties that will be addressed in the next cycle for further refinement of the approach.

Keywords: Partitive division, quotative division, mental models, conceptual understanding

Theoretical Background and Empirical Framework

Concerning the teaching of division at primary and secondary school level, a distinction is made between two types of division and two related concepts, namely quotative division (so-called, e.g., by Fischbein et al., 1985; e.g., Neuman, 1999, uses "quotitive division", e.g., Carvalho & Ponte, 2019, "measurement division") on the one hand, and partitive division (or "sharing"; e.g., Fischbein et al., 1985) on the other. In this paper, we will use "quotative" and "partitive division". In a quotative division $a : b = c$, the divisor b stands for the size of one of the equal groups that the dividend a is divided into, and c for the number of equal groups of this given size. In a partitive division, conversely, b is the number of equal groups into which a is divided, and c gives the size of one group.

Neuman (1999, p. 101) notes that "one problem with teaching division to children is whether they experience this difference and, if not, whether one should draw attention to it". Regarding the latter, there are different and even contradictory recommendations in the mathematics education literature. For example, Downton (2013, p. 242) suggests that "students need to understand the concept of division and its inverse relationship to multiplication rather than whether a task involves partitive or quotitive division". In contrast, Wartha and Schulz (2021) advocate working with children towards a clear *conceptual* distinction, which should also include a clear *linguistic* distinction between the two types of division (Wartha & Schulz, 2021, p. 136).

How do children deal with the dual nature of division as either partitive or quotative? Research on what might be called "individual models" (Prediger, 2008, p. 7) of division and their evolution over the years is scarce. However, studies show considerable deficits in the conceptual understanding of division, both in primary school (e.g., Royar, 2013) and beyond (e.g., Ehlert et al., 2013). Research into whether children are more likely to acquire one or the other of the two basic ideas of division is inconclusive. Fischbein et al. (1985, p. 14) conclude from the difficulties of secondary school students

that the "partitive model" is the "intuitive primitive model for division problems" and that quotative division only later, if at all, becomes an available model for thought. Studies with primary school students do not support this assumption, and Neuman's (1999) study suggests that when children are first confronted with division problems, they tend to favour solutions that follow a quotative interpretation even for partitive problems. In Burtscher's (2022) longitudinal study of 12 Austrian children from the middle of the second to the end of the fourth grade, the children showed a clear preference over time for the quotative interpretation when trying to show and explain a given term using manipulatives. However, the same children were more successful in creating word problems when they had chosen a typical partitive situation as the starting point of their story. Overall, Burtscher (2022) observed several difficulties:

- *Difficulties in creating word problems*: Many children were unable to create any kind of appropriate story for a given division term (only 30% of attempts were successful).
- *Interference between partitive and quotative interpretations*: When inventing a word problem, children often switched between ideas of quotative and partitive division, which prevented them from arriving at an appropriate story.
- *Inconsistencies within explanations*: Inconsistencies occurred when children placed the manipulatives according to a quotative interpretation (divisor as the size of one part) but tried to explain what they were doing with a partitive story (divisor as the number of equal parts).
- *Reluctance to recognize both interpretations*: Children often did not accept the alternative interpretation (quotative vs. partitive) as also fitting after they had initially chosen the other one when asked to explain the meaning of a given division term using manipulatives.

The incidents of interference between partitive and quotative thinking reported above must be seen in the context of how division had been taught in the classroom. The teachers had introduced division in the second school year via quotative problems, and later treated partitive tasks as well, but at no time had they explicitly addressed the fact that there are divisions of two different types, nor had they compared them (Burtscher, 2022).

DiVA – a Teaching Experiment

Given the apparent difficulties that many children have with division, the authors aim to develop and evaluate a design for the initial teaching of division that effectively addresses these challenges. Building on the theoretical background and empirical framework summarized above, we propose that the lessons should not only pay equal attention to both types of division – quotative and partitive – but also explicitly highlight their similarities and differences. We hypothesize that this approach will improve students' ability to model real-world problems in general and, more specifically, prepare them for the conceptual shifts required when extending their understanding of division from natural numbers to rational numbers (cf. Fischbein et al., 1985; Prediger, 2008).

To achieve these goals, we launched the design research project DiVA ("**Di**vision als **V**erteilen und **A**ufteilen verstehen" in German, i.e., "understanding division as partitive and quotative" in English) in the spring of 2023. It involves the development and evaluation of a sequence of teaching activities over several weeks with the explicit aim of helping children to develop a solid understanding of division in both ways. The minimum aim was that by the end of the didactic sequence, the children

should be able to demonstrate and explain the meaning of a division term using manipulatives and to invent word problems that fit a given term, in at least one of the two types of division problems (quotative or partitive). The ultimate aim was for children to demonstrate and explain both types of division on their own initiative when asked to explain a given term using manipulatives, to create word problems of both types that fit a given term, and to explain the similarities and differences between divisions of both types in their own words. However, the aim was *not* for students to verbally categorize problems using *technical terms* commonly used in mathematics education literature, or to create problems on demand based on these terms. Table 1 provides an overview of the lesson design, which consists of five phases and spans approximately 20 lessons.

Table 1: Phases of the Teaching Experiment

Phase	Time	Content
Phase 1: Introduction (quotative division)	4 h	Introducing division as a new type of problem, different from the already familiar addition, subtraction, and multiplication, with problems such as "How many packets of six eggs are needed to pack 24 eggs?"
Phase 2: Consolidation and expansion of the basic concept 1	4 h	Division as creating equal portions of a given size in different contexts (forming groups, spending money, measuring in the strict sense, e.g. filling bottles and jugs of a given volume...); emphasize the common aspect: "We divide a given quantity or number, and we know the size of a part or portion or group."
Phase 3: Solving a division problem by skilful computation	2 h	Finding the multiplication problem that is the inverse as a clever solution method for any division problem (2 hours just to develop this idea; from then on, clever calculation was a matter of constant training)
Phase 4: „Fair sharing" (partitive division)	6 h	Introducing a new type of problem: Dividing when number of parts is given, size of a part not; working out similarities and differences with problems dealt with so far; developing a common way of talking about the two types: "dividing when you know how big a part is" vs. "dividing when you know how many parts you have".
Phase 5: Consolidation of both basic concepts	4 h	Solving word problems of both types; inventing word problems of the two different types that fit the same division term; representing and comparing the two types of division using manipulatives and illustrations.

Four classes participated in the first cycle of the project: three from South Tyrol, Italy, and one from Salzburg, Austria, involving a total of 58 children. With the overarching aims described above, a teaching sequence was designed, refined and implemented in five phases over a period of several weeks (Table 1). The teachers who volunteered to participate in the project decided when to work on each phase and covered other mathematical topics in between. Division was not to be introduced until the children had a solid understanding of multiplication. Also, mastery of at least the core multiplication facts (with factors 2, 5, 10) and the ability to derive other single digit multiplication tasks from these core tasks were considered prerequisites for successful work on division (cf.

Gaidoschik, 2014). For this reason, the teacher of two classes decided to elaborate division only in the third grade. Thus, phase 1 began in May 2023 for two classes at the end of their second year and in October 2023 for two other classes, at the start of their third year.

Teachers were provided with detailed plans for each lesson, including content, objectives, materials, and worksheets for tailored tasks, usually starting with a problem for the children to solve independently, alone or in groups. A key design principle was to include whole-class discussions to help children reflect on the problems and verbalize new insights. The planning included detailed instructions on how to conduct these discussions, including how to formulate prompts and how to deal with the heterogeneity of the class. The objectives and tasks were discussed and reflected upon with the teachers repeatedly, and minor changes to the plan were made based on this feedback over the course of the several weeks that the project was running. Students' worksheets were collected, and lesson preparations were documented. The teachers recorded their experiences and impressions in keywords for each lesson.

As a design research project, DiVA combines "instructional design [...] and educational research" (Gravemeijer & Prediger, 2019, p. 34) by implementing the theory-based initial lesson design in several cycles in different classes and testing its feasibility and effectiveness. The second cycle is currently being prepared. As an essential basis for this, qualitative individual interviews were conducted in all four classes after the completion of Phase 5 with all 58 students (all parents gave the necessary consent). Some selected findings from these interviews are reported below.

Research question and methods

From a wider range of research interests explored in the above-mentioned interviews, we report in this paper, due to limited space, exclusively on the following research question (omitting the parts of the interviews on word problems):

> How do children who have participated in phases 1 to 5 of DiVA perform when they are asked to explain the meaning of a division term with reference to a given representation using manipulatives?

We are particularly interested in whether and in what ways the children show awareness of the double nature of division as quotative and partitive, whether there are indications of preferences for one of the two variants, and whether the existence of two possible interpretations causes misunderstanding or problems of any kind.

To answer our research questions, we interviewed the children individually outside the classroom. All interviews followed semi-standardized guidelines and were videotaped, transcribed and subjected to qualitative content analysis. From the many ways of operationalizing "explaining the meaning of a division term using manipulatives", we chose the following: Children were presented with 12 wooden cubes in three separated groups of four and asked whether they could identify one or more *arithmetic* problems in them, and if so, which one(s). Once they had named a *division* term (e.g., 12:4 for a quotative or 12:3 for a partitive interpretation), they were asked to write it down and explain in more detail why and how this term fits the cubes. If they had not already considered both variants in their answer, they were asked after the explanation whether they could recognize another division in

the same representation. If they did not come up with the alternative term themselves, they were presented with it (written on a flash card) and asked whether it also matched the cubes on the table and why or why not.

The analysis criteria were partly deductive. First, we noted whether a child proposed a division term and whether this term matched the representation, then whether it corresponded to a partitive or a quotative interpretation, and whether the child had needed prompting. Inductive categories were created to capture the variety of behaviours, particularly to analyze how children commented on interpretations they had not initially suggested. We grouped children's responses into four categories: one for children who recognized both interpretations as equivalent and were able to justify this convincingly. Another category was used when a child acknowledged both interpretations as fitting, but there were ambiguities in the explanations. A third included trials where the child stated that the alternative interpretation was also correct but then rearranged the material (see below for a more detailed explanation). The fourth category included responses in which the child presented only one interpretation and considered the other to be inappropriate.

Results

As can be seen in Table 2, about 81% of the children were able to interpret the given picture showing equal groups of cubes as (the result of) a division in both aspects, quotative and partitive division. About 17 % of the children were classified as finally accepting only either the partitive or the quotative interpretation. There was only one child who, when interviewed, was unable to give at least one interpretation of the division represented by the cubes.

Table 2: Interpretations given to three groups of four by the participating students (n = 58)

Both partitive and quotative interpretation		Either partitive or quotative interpretation			Neither partitive nor quotative interpretation
47 (≈ 81 %)		10 (≈ 17 %)			1 (≈ 2 %)
Convincing	Ambiguous	Rearrange	Only partitive	Only quotative	
36 (≈ 62 %)	11 (≈ 19 %)	5 (≈ 9 %)	3 (≈ 5 %)	2 (≈ 3 %)	

Of the 47 children who interpreted the cube display as both 12:4, i.e., as a quotative, and 12:3, i.e., as a partitive division, 11 did so spontaneously, while the other 36 did so immediately after being asked whether a second division fits, having spontaneously named one of the two variants. For all 47 children in this group, the interview further explored whether they could explain clearly how the given representation corresponds to 12:4 and 12:3. 36 of the 47 children then provided a convincing explanation for both interpretations. This group of 36 included all eleven children who had previously suggested both division terms as appropriate spontaneously, and 25 others who had first only mentioned one division but added the second one at the interviewer's request, as to whether another term would also fit. An excerpt from the interview with Gabriela (pseudonym), translated from German into English, is included here as an anchor example for what we rated as a "convincing" two-

fold interpretation. Gabriela was one of the children who spontaneously wrote down both 12:4=3 and 12:3=4, when she saw the cube display. She was then asked to explain what she had written:

Gabriela	[referring to her first interpretation of the cube display as 12:4=3]: Well, you could see that the first package has four cubes [points to a group of four], and then how many packages there are, and there are three packages, and then three times four equals 12. [...]
Interviewer:	And what does the 3 tell us?
Gabriela:	The 3 tells us how many packages there are.
Interviewer:	And for this calculation [points at Gabriela's writing 12:3=4], can you briefly explain how you see the different numbers and what they tell you?
Gabriela:	Well, the 12 is again the total, the 3 means how many equal groups you make, and the 4 means how many cubes are in each group.

In 11 other cases, children who had first given one interpretation and on request also the other, gave justifications that were considered as ambiguous. An anchor example of this is provided by the interview with Damian, who first interpreted the cube representation as 12:4=3 on his own. The following conversation developed:

Damian:	You do this if there are 12 sweets and then three children come, let's say. And they always take the same number of sweets. And then you divide them up into equal numbers, I mean, packages. That's, just how big the group is.
Interviewer:	Okay, and where do you see the 12?
Damian:	When I put them all together.
Interviewer:	And the 4? What does the 4 say?
Damian:	It says how many groups there are, just, how big the group is.
Interviewer:	And what does the 3 say?
Damian:	How many groups there are.
Interviewer:	Do you see another division problem here?
Damian:	Yes, 12:3=4. There are 12 sweets again, then three children come, and they want the same number and then you divide everything fairly into the same number of groups.
Interviewer:	And what does the 3 tell us?
Damian:	How big?
Interviewer:	How big?
Damian:	No, how many.
Interviewer:	And the 4 there?
Damian:	The 4 tells you how big the group is.

Damian starts his explanation with 3 children sharing 12 sweets, which does not fit with his previously written 12:4=3. However, he goes on with "equal numbers" and "how big a group is," which corresponds to a quotative interpretation and therefore is consistent with 12:4=3. He then explains the 4 in 12:4 as "it says how many groups there are", which again points at a partitive interpretation. He immediately rephrases this as "how big a group is." Later, when he tries to explain 12:3=4 as a second appropriate interpretation of the cube display, he first suggests, in a questioning voice, "how big?" as the meaning of the 3 and then switches to "how many" when the interviewer repeats his answer. At this point, the interviewer is suggestive and does not ask enough follow-up questions, but Damian at least shows uncertainty throughout the interview about the assignment of "size of a portion" and "number of equally sized portions" to the numbers of a division term in both variants.

Ten children were assigned to the "either partitive or quotative interpretation" category. Notably, however, five of them initially said that both 12:3 and 12:4 fit the cube-display. However, when asked

to explain how the alternative term also fitted, these five students rearranged the cubes into four sets of three, which they then interpreted corresponding to the either partitive or quotative understanding they had initially demonstrated in explaining the three sets of four.

Discussion

Compared to the children who were given similar tasks by Burtscher (2022) without being specifically taught to differentiate and compare partitive and quotative division problems, the children interviewed for this study showed an overall high level of awareness of the two types of division. Thirty-six out of 58 (about 62%) not only accepted 12:3 and 12:4 as equally fitting interpretations of four groups of 12 cubes but also explained each interpretation in a clear way that demonstrated a sound conceptual understanding. The difficulty that other children had in verbally justifying how the two different terms fit equally with the same representation reflects the significant conceptual and linguistic challenge that DiVA deliberately presents to children. Distinguishing between "four groups" and "groups of four", between "portion size" and "number of portions" is also difficult for adults. Eleven of the 58 children indicated that they were aware of the two possible interpretations but showed uncertainty in explaining them. Ten other children recognized only one division term as fitting. However, at least five of them gave clear indications that they had learned in class that two interpretations were possible. For themselves, however, they seem to have understood this to mean that the same set must be rearranged in two different ways to fit either one or the other term.

This misunderstanding will be addressed directly in a second cycle. Overall, however, we are encouraged by the results. All children, with one exception, were able to make a connection between a division term and a material representation; the vast majority considered both the partitive and the quotative interpretation. This contrasts sharply with the widespread fundamental deficits in the conceptual understanding of division reported on in studies such as those by Royar (2013) or Ehlert et al. (2013).

We will use the second cycle to further refine the design (especially the weighting of individual elements and the provision of clearer guidance for teachers) and, at the research level, to address the methodological weaknesses of this study, especially in the conduct of interviews (standardize the handling of prompts, establish clear criteria for ending interviews to ensure comparability), to strengthen both qualitative and quantitative statements. However, we can already say that, contrary to what some authors suggest for this age group, the children were generally not overwhelmed by the task of recognizing and verbalizing the differences and similarities between the two types of division. The one child who was unable to interpret the cube representation as a division at all – either partitive or quotative – struggled with significant language barriers due to being a migrant child. We are aware that DiVA places high linguistic and cognitive demands on both the children and the teachers who implement the concept in the classroom, but it seems worth the effort. Of course, further research is needed in the form of longitudinal studies to determine whether the DiVA approach also works in the long term when children will be requested to understand and apply division with rational numbers. We assume that the conceptual change (Fischbein et al. 1985; Prediger, 2008) required for this is more likely to succeed if it can be based on sustainable concepts in the field of natural numbers.

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Number sense towards real numbers

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Number sense in basic education has been studied for over three decades. The numbers involved in these investigations are predominantly natural, integer, and rational. However, students aged 15 to 18 use real numbers, and it is most beneficial to develop their number sense regarding these numbers. This paper presents an adaptation of the Broken Keys activity, which aims is to compose and decompose irrational numbers. Data analysis made it possible to identify which abilities related to number sense about real numbers high school students used to solve the proposed tasks.

Keywords: Number sense, Real numbers, Secondary education.

Introduction

Researchers such as Andrews & Sayers (2015), Anghileri (2007), and Yang & Huang (2004) have recognized the importance of developing elementary school students' number sense. At this educational level, the numbers most commonly used are natural, integer, and rational. However, simply calculating the n-th root of a natural number could make students jump to irrational numbers and thus enter the domain of real numbers. Students aged 15 to 18 face these leaps, some merely stand by as spectators because their previous reflections and analyses may have focused solely on rational numbers. Therefore, it is necessary to extend the development of number sense to include real numbers so that students strengthen their flexible thinking and can recognize the properties, similarities, and differences of these mathematical objects and used them properly.

Considering this background, a research project was conducted with the aim of strengthening and developing pre-university students' number sense related to real numbers. In this paper, results obtained to answer the question: What skills related to the number sense of real numbers do students use to solve the tasks proposed in the "Broken Keys" activity? are described. This activity, which is described later, is part of the aforementioned study.

Conceptual framework

Number sense has been the subject of study of several researchers (e. g. Greeno, 1991; Mcintosh et al., 1992; Löwenhielm et al., 2017), whose goal was to define it, characterize it, recognize its elements, or analyse tasks related to this construct. These studies involved natural, integer, and rational numbers. In the Broken Keys activity, in addition to the number systems mentioned above, irrational numbers are included, so it is more convenient to adopt the characterization of number sense related to real numbers described in Garrido-Adame et al. (2022), where it is considered that "number sense is the set of knowledge and abilities regarding real numbers that a person uses to make mathematical judgments and develop numerical strategies when solving problems involving these numbers" (p. 46).

Knowledge of real numbers is related to their properties, as well as the characteristics of the domains they contain: natural, integer, rational, and irrational numbers. These have been considered according to the characterizations made by mathematicians (e.g. Spivak, 1967).

The abilities are related to the use of real numbers and are in Figure 1. These can be used as indicators to observe the use of number sense by students. The first ten abilities are based on behaviours that demonstrate the presence of number sense, as suggested by Sowder (1992). Abilities 11, 12, 13, and 14 were proposed based on observations made during teaching practice. Ability 15 was proposed by Resnick (1989).

Number	Abilities
1	Ability to compose, decompose and recompose numbers
2	Ability to identify which representation of a number is more convenient than another
3	Ability to compare numbers
4	Ability to order numbers
5	Ability to deal with the order of magnitude of a number in concrete situations
6	Ability to use reference points
7	Ability to link symbols of operations and relationships in a meaningful way
8	Ability to recognize the effects of operations
9	Ability to do mental calculations using own strategies
10	Ability to make estimates
11	Ability to associate numbers with the context in which they appear
12	Ability to locate numbers on the number line
13	Ability to recognize given facts
14	Ability to distinguish differences and similarities between number systems
15	Ability to self-regulate

Figure 1: Abilities related to number sense about real numbers

Methods

The methodology used for this qualitative study is known as design research. The design of this research consists of two cycles (see Figure 2) with its three phases (design, experimentation and retrospective analysis). Based on the retrospective analysis of the first cycle new activities were proposed and others were redesigned. In this paper we expose results regarding students' performance in the Broken Keys activity of the second experimental cycle (in grey in Figure 2).

Design Phase. The Broken Keys activity is based on tasks related to number sense proposed to primary school students (Pacio-Quiterio & Juárez, 2024), where it is assumed that some keys on the calculator do not work, so the children must write the numbers of the indicated operations in another way. To answer the questions, the use of calculator was allowed because it is thought that this device can be used as a learning tool to develop number sense (Figueras et al., 2021). The main changes were the numbers proposed, and the type of calculator used. In the first cycle basic calculators were used, and in the second cycle ones that had keys for entering the square and cube roots of a number, as well as fractions and π were used. The operations proposed in the first cycle were with natural and integers numbers. The second cycle's questions can be seen in Figure 3.

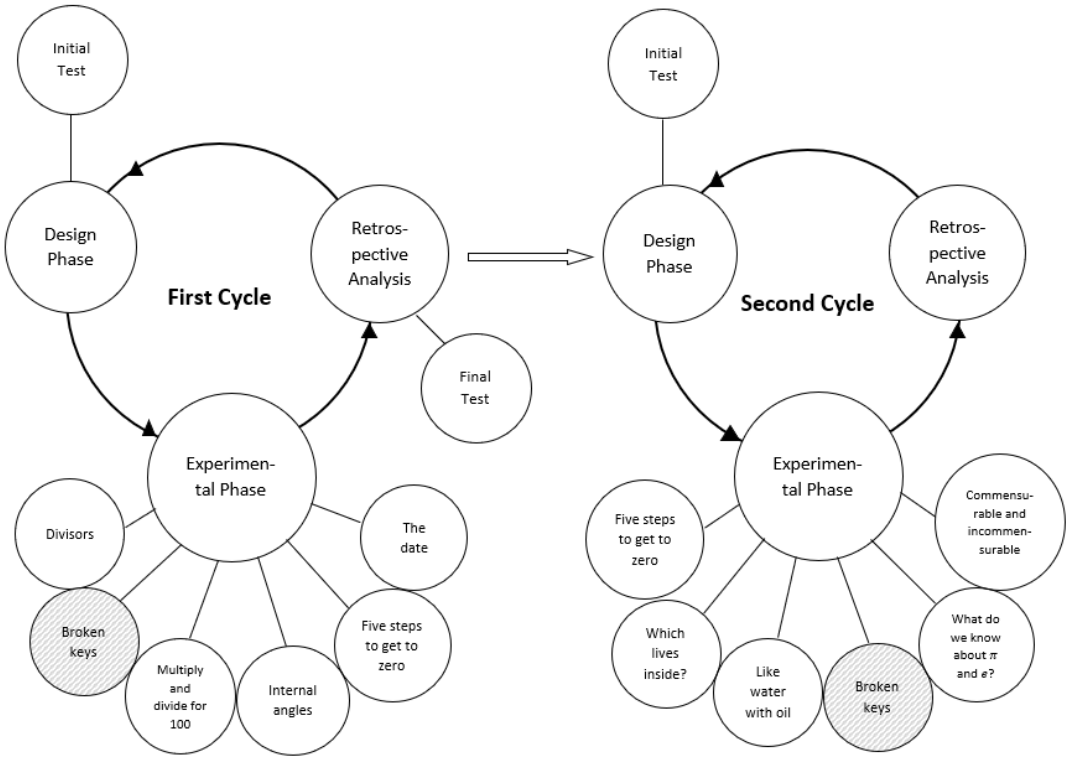

Figure 2: Design experiment cycles

The goal of the activity is to compose and decompose irrational numbers additively and multiplicatively. Students were asked not to use approximations in order to prioritize the use of mathematical symbols for irrational numbers. The activity focuses on irrational numbers, however, it is important to note that an efficient use of the natural numbers (included in the radicands) is required, as well as knowledge about integers in the case of Final Reflections (see the third column of Figure 3). The square root of a negative number is included, with the intention to discuss the existence of different numbers from real numbers.

Initial Reflections	Problem Statement	Final Reflections
How else can you write $\sqrt{20}$? What is the result of $\sqrt{24}+\sqrt{12}$? What is the result of $\sqrt{8}+\sqrt{32}$? What is equal to $\sqrt{50}+\sqrt{30}$? How else can you write $(\sqrt{12})(\sqrt{3})$?	$\sqrt{18}+\sqrt{28}$, broken key [8] $5\sqrt{39}-\sqrt{13}$, broken key [9] $\sqrt{7}+\sqrt{8}$, broken key [$\sqrt{\square}$] $\sqrt{10}-\sqrt{11}$, broken keys [$\sqrt{\square}$] , [$\frac{\square}{\square}$] and [.] $9\pi-\pi$, broken key [π]	How else can you write $\sqrt[3]{54}$? What is equal to $\sqrt{-25}+\sqrt{40}$?

Figure 3: Questions posed in the Broken Keys activity

Experimentation Phase. An initial selection of eight students aged 15 to 16 years was made, according to the results of an initial test on number sense they showed more structured performances regarding the use of numbers. Four additional students of the same age were also considered. Although their

performances did not have the same characteristics as those in the initial choice, they expressed interest in participating in the project, so they were included in the study's sample. This latter group of students was in no way considered a control group; their role was the same as that of the other eight. Therefore, data collection was carried out with a total of twelve students, divided into three teams of four: two from the initial selection, named A and B, and the group of the students who requested to participate, it was named C.

This phase took place during an inter-semester period. The activity was developed in three stages. First, all twelve students were gathered to establish rules such as: do not use approximations or erase their different attempts, record their discussions, and calculators can be used. In the second stage, each team moved to a space where they could discuss and record how to answer the questions in each section, as well as write their answers on a worksheet. In the third stage, they shared their answers with the other teams. Data were collected through answers in the worksheets and the video recordings of the discussions that arose among the team members while answering the worksheets. Additionally, there are videos of semi-structured interviews conducted with each team the day after the activity, aimed at clarifying or complementing any of the written responses.

1. Decompose the radicands (natural numbers)
   1.1. multiplicatively
   1.2. additively
2. Use properties of the square root operation with respect to:
   2.1. Multiplication
   2.2. Addition
   2.3. Addition and multiplication (distributive property)
3. Recognize the square root of perfect squares
4. Write at least one decomposition of an irrational number:
   4.1. Multiplicative
   4.2. Additive
   4.3. Combined
5. Decompositions of irrational numbers involve a natural number multiplied by an irrational number
6. Write an irreducible expression for the indicated irrational number

Figure 4. Code used to analyse the data in the Initial Reflections section

Retrospective Analysis Phase. First, the images of the responses to each question were grouped by section. There were 15 responses for Initial Reflections, 15 for Statement, and 6 for Final Reflections. A description for the strategy used by each team for each question was stated. When the writing on the worksheets was unclear or there was no trace of the procedure the students followed, the information was supplemented by reviewing videos of the peer discussions. Descriptions by the three researchers were triangulated. When the fifth part of these specifications was completed, it was possible to inductively construct a scheme of codes. Codes were interrelated to the abilities in Figure 1. For example, the codes shown in Figure 4 are associated with the ability to compose and decompose numbers. With the schemes of codes, it was possible to classify the students' responses into categories and thus select the images presented in the Findings section.

Findings

This section presents representative examples of the use of the abilities of Figure 1, since putting them into practice encourages their development.

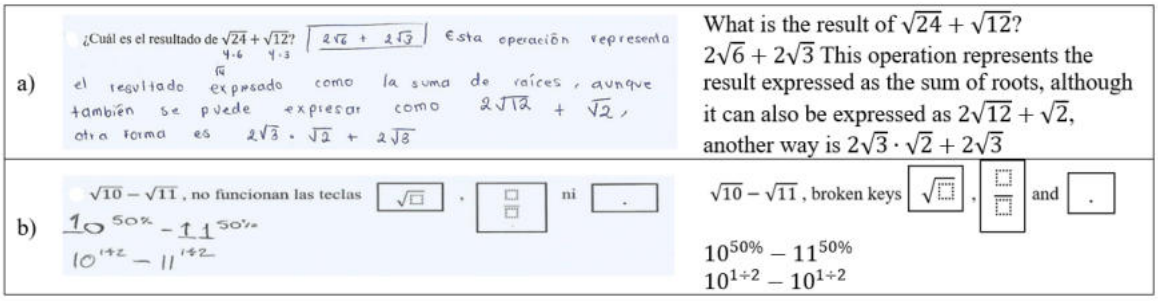

Figure 5: Evidence of the use of ability 1 and 2

Team B used the *Ability to compose, decompose, and recompose number,* with natural and irrational numbers; unfortunately, they were wrong about the decomposition $2\sqrt{12}+\sqrt{2}$. On the other hand, Team A related percentages to fractions (Figure 5b), which is not common, but since the calculator allowed them to perform that operation, they proposed it as their answer. Therefore, it can be stated that they used the *Ability to identify which representation of a number is more convenient than another*.

A behaviour related to the *Ability to recognize the effects of operations* can be observed in Team A's response, as they recognized that to write 5 inside the square root, it was necessary to enter it as 25. Additionally, they noticed that multiplying and dividing by π did not alter the given number. This fact is linked to the knowledge of the multiplicative identity, and even the $1=\frac{\pi}{\pi}$ proposed, also involves knowledge of the reciprocal of a number (see Figure 6). In the team discussion video, it was possible to hear that the students decided to include the number π because "they missed it", and they recalled that irrational numbers were not only square roots.

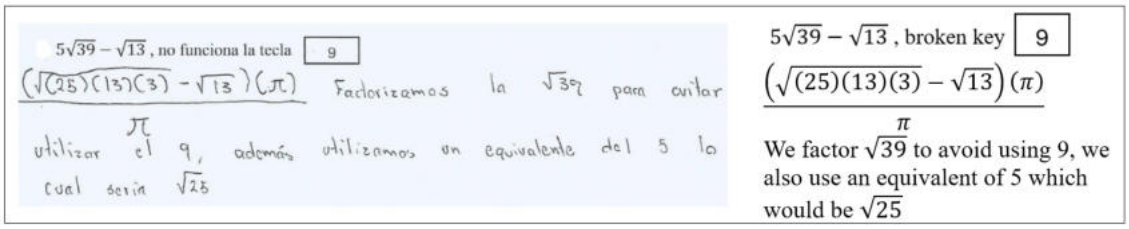

Figure 6: Use of ability 8

The *Ability to link symbols of operations and relationships in a meaningful way* was used by the students in Team B. They first answered that with no other key could they calculate the square root of a number (see Figure 6). However, in the same figure, it can also be seen that in another question, they proposed a way to link the symbols with the square root. Since the origin of this change could not be identified in the team discussion video, the students were asked about it in the interview the following day. They replied that they had made a query to Meta AI (Meta's AI-powered assistant, which can be used via the WhatsApp app in some Latin American countries) during an intermediate time before starting the second discussion. Their query was about the first question. Since in the operation $\sqrt{10}+\sqrt{11}$ there were more restrictions regarding the keys that were not working, the

participants made a combination between the answer provided by Meta AI and their own, verifying with their calculator that linking symbols of operations and relationships involved was correct.

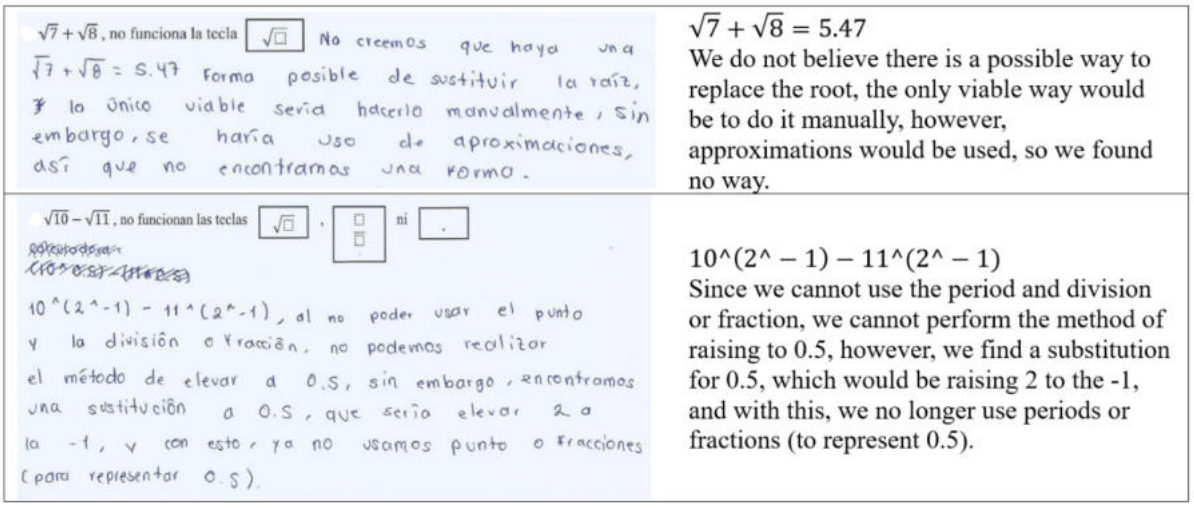

Figure 7: Use of ability 7 to interpret a response from Meta AI

A behaviour associated with the *Ability to use reference points* can be seen in Figure 8. Team C considered π as 3.1416, however, they forgot the instruction not to use approximations. The members of Team A used π as a reference point, understanding it as an irrational number, and they couldn't find a key that would substitute for this number. In the video of their discussion to answer the question, the students argue that π is a different number, that it does not involve any operation like the numbers in the previous questions, and that this is a reason they could not write it in another way. Although Team A knows that π is the ratio of the circumference's length to its diameter, they could not explain why they couldn't use another key on the calculator. In fact, they commented that they wouldn't know if the change would be correct, as they do not know all the decimal digits of π, recognizing that this number has an infinite number of non-repeating decimal digits. The statements that can be made based on these observations will be discussed in the next section of this paper.

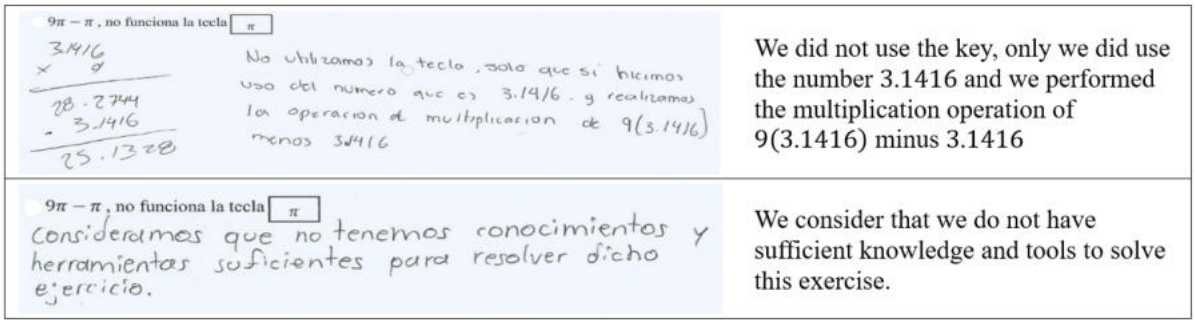

Figure 8: Use of reference points

Discussion and conclusions

In the Broken Keys activity, most of the real numbers used arose from calculating the square root of natural numbers. However, real numbers are very diverse; there are algebraic and transcendental numbers, facts that may go unnoticed if the development of number sense regarding these numbers

is not promoted. A clear example is how the students in Team C converted π into the rational number 3.1416, showing that their number sense remains within this domain.

Therefore, it is necessary to reflect on and analyse with students the properties of real numbers, their different representations, and operations, and not just teach the specific mathematical knowledge of the corresponding subject (for more arguments see Moutinho & Zazkis, 2019). Usually, teachers assume that students can use real numbers fluently, but the reality is that, in the best case, their developed number sense is based on natural, integer, or rational numbers.

Data analysis allowed to answer the posed question: What are the abilities related to number sense of real numbers that students use to solve the tasks proposed in the Broken Keys activity? Referring to the conceptual framework, visualized in Figure 1, those abilities are (1) ability to compose, decompose and recompose numbers, (2) ability to identify which representation of a number is more convenient than another, (6) ability to use reference points, (7) ability to link symbols of operations and relationships in a meaningful way, and finally, (8) ability to recognize the effects of operations. These five abilities revealed during data analysis, and therefore, were evident in the data. However, the use of other abilities, such as (14) ability to distinguish differences and similarities between number systems, should not be ruled out, as students were attentive to the type of numbers they were using in order to decide which operations could be performed and which could not.

There are studies that include activities to promote the development of number sense (Lyublinskaya, 2009; Reys, 1994; Scheiner & Thompson, 2000) which can be adapted or extended to real numbers. The design of the Broken Keys activity in the second experimentation cycle demonstrates that it is possible to adapt or design different activities to promote the development of abilities related to number sense about real numbers. Likewise, activities can be proposed with the intention of reflecting on knowledge associated with these numbers.

Calculators and artificial intelligence are potential tools for promoting the development of number sense, as seen in Figure 7, students made a query to Meta AI and from this they were able to answer three more questions using mainly the Ability to link symbols of operations and relationships in a meaningful way. The use of these resources could be explored in depth because students are already using artificial intelligence tools.

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Mastering the basic facts of multiplication and division till the end of grade 3–selected results of a longitudinal study

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We present the results of a longitudinal study regarding mastery of the basic multiplication and division tasks in Baden-Wuerttemberg, Germany. We used the same speed test at the end of grade 2, at the beginning of grade 3, and at the end of grade 3 in 39 classes in 18 Schools. A total of 535 students who were present at all three measurement points were included in the evaluation. We present and discuss the development of mastery of the basic tasks. It must be emphasised that, despite the many positive developments at the end of grade 3, only 223 out of 535 students met the requirements for mastery of the basic tasks, while 145 students were at the lowest level. Possible reasons and possible directions for further development are discussed.

Keywords: Elementary school mathematics, multiplication, division, basic multiplication facts.

Context of the study and research questions

Multiplication and division are part of the core content of mathematics in the primary school curricula worldwide. The learning objectives for these operations include, firstly, that students have a conceptual understanding of multiplication and division; secondly, that they should be able to determine the value of any product and quotient using various strategies; and finally, that they should know from memory all basic facts of multiplication and derive the basic facts of division[1] at skill level (e.g., Kling & Bay-Williams, 2015; CCSSO & NGA, 2017; KMK, 2022).

Retrieving basic facts from memory is a tool for solving more complex tasks like 27 × 6, shortening fractions, or computing simple percentages mentally. Wong and Evans (2007, p. 91) state, "Many tasks across all domains of mathematics and across many subject areas call upon the recall of basic multiplication facts as a lower-order component of the overall task." Also, memorised facts are essential prerequisites for developing advanced mathematical skills such as problem-solving (Burns et al., 2016). Recalling facts reduces cognitive load (Sweller, 2016) and allows students to concentrate on solving complex tasks. Those who cannot retrieve the basic equations are significantly disadvantaged compared to more practised classmates. If the basic equations are not acquired in primary school, it is unlikely that these deficits will be compensated in secondary school (Steel & Funnell, 2001). Consequently, the Common Core State Standards for Mathematics (CCSSO & NGA, 2017, p. 23) demand: "By the end of Grade 3, know from memory all products of two one-digit numbers". Nevertheless, the meaning of the equations must be understood as Gaidoschik et al. (2017,

[1] Basic fact (also called "basic equation") of multiplication means in this paper an equation of multiplication in which both factors are less than ten or equal to 10. Basic fact (also called basic equation) of division means an equation of division in $\mathbb{N}$ in which the divisor and the quotient are both less than 10 or equal to 10.

p. 2) emphasise: “There seems to be an international consensus nowadays that students should both acquire a sound conceptual understanding of multiplication and eventually solve all basic tasks accurately and effortlessly.”

The crucial starting point for discussing the conceptual understanding of operations is that terms like 3×4 or $12 : 4$ are non-verbal, symbolic representations of concepts, and concepts are mental reflections that describe classes of situations that occur in reality. Conceptual understanding enables students, e.g., to solve word problems. It also allows students to solve every multiplication task, especially those with factors larger than 10, using manipulatives, skip counting, or deriving answers using reasoning strategies based on known facts (Baroody, 2006; Kling & Bay-Williams, 2015).

The work on multiplication and division begins in Germany, usually in the second half of grade 2. Students learn to describe real-life contexts with products and quotients and determine the values of any products and quotients. In doing so, they discover that new tasks can be solved using previously learned ones as an essential method in mathematics. In grade 2, but especially in grade 3, the focus is on memorising the basic multiplication equations. Ideally, students should be able to retrieve the basic multiplication facts from memory in the middle of grade 3. Also, students should be able to solve basic division tasks at a skill level by applying basic multiplication facts.

However, three problems occur again and again up to the secondary school level: *First*, there are deficits in the mastery of basic facts, i.e., in their retrieval from memory. These can be seen, for example, when they cannot reduce fractions like $\frac{48}{56}$ in secondary school, because they cannot identify common factors. *Second*, while students can often reproduce the basic equations from memory, they cannot solve multiplication tasks with a factor greater than 10. *Third*, we repeatedly observe difficulties with the conceptual understanding of multiplication and division: Students know that $6 \times 3 = 18$ but struggle to justify this because they lack the term’s meaning. Often, they argue that 6×3 is a short notation for $3 + 3 + 3 + 3 + 3 + 3$. Similarly, students see division equations often as nothing more than multiplication equations written in reverse: They can only explain the value of quotients such as $24 : 6$ by referring to the inverse equation $4 \times 6 = 24$, but not to real-life contexts.

Considering these facts, we concentrate in this paper on how far students from the end of grade 2 until the end of grade 3 can retrieve the basic multiplication facts from memory and solve the basic division tasks at a skill level. Consequently, we discuss the following research questions:

RQ1: To what extent have students memorised the basic multiplication equations at the end of grade 3, and to what extent can they apply this knowledge at a skill level to solve basic division tasks?

RQ2: How does the level of memorised basic equations and the level of solving division tasks change over one year from the end of grade 2 to the end of grade 3?

Method

To answer these questions, we conducted speed tests with the same basic multiplication and division tasks at three crucial measurement points (MP) in 18 schools with 39 classes in a larger region of Baden-Württemberg. The selected schools included both rural and urban settings with diverse sociocultural backgrounds. The mathematics curricula of the federal states of Germany are

comparable in terms of work on multiplication and division, particularly after the establishment of the educational standards by the KMK in 2004. The same applies to the textbooks used in class, although these are published by different publishers (Gleißberg & Eichler, 2019). It can, therefore, be assumed that our findings, according to the schools and classes, are representative of a larger scale.

A total of 535 students participated in all three measurement points. We assume that excluding students randomly absent due to illness or change of residence did not affect the sample's representativeness. We assessed how far students can retrieve the basic facts from memory at these three MPs. As MP1, we chose the end of the second school year. Following the educational plans for mathematics lessons in the federal states of Germany, we can expect that students understand the concepts "product" and "quotient", can determine the values of products and quotients, and have memorised many basic equations. As MP2, we chose the first weeks of the third school year. We determined the extent to which the students remembered the basic facts learned in grade 2 after the summer vacation. Completing the test in all 18 schools in one week was impossible. Therefore, in some classes, content from grade 2, particularly the basic tasks, may have been repeated in the days before the test. At the end of grade 3, designated as MP3, we assessed how well students mastered the basic facts after one and a half years of dealing with multiplication and division.

In the speed test, the students had to solve 15 multiplication and 15 division tasks in 3 minutes, given as terms on a worksheet (see Figure 1). So, the average time for solving a single task was 6 seconds. We used only 30 tasks in the test to minimise the effect of decreasing concentration. The limited time is crucial because, in just 3 minutes, it is impossible to compute many results. Students can solve only one or two tasks by computing in this short time. The results of the speed tests show to what extent students can retrieve basic multiplication facts from memory and apply at skill level relationships between multiplication and division, as well as properties of multiplication like the commutativity.

We included multiplication tasks with factors 0 or 1, division tasks with dividend 0, and division tasks with divisor 1. These tasks are often solved incorrectly because it is not trivial to build mental images and an understanding of the meaning of terms such as 3×0, 0×6, $0 : 9$, and $5 : 1$ (Van de Walle et al., 2019; Levenson et al., 2004).

$4 \cdot 8 =$ ___	$8 \cdot 5 =$ ___	$5 \cdot 3 =$ ___	$49 : 7 =$ ___	$7 : 7 =$ ___	$5 : 1 =$ ___
$6 \cdot 6 =$ ___	$3 \cdot 6 =$ ___	$6 \cdot 8 =$ ___	$24 : 6 =$ ___	$42 : 6 =$ ___	$16 : 2 =$ ___
$9 \cdot 7 =$ ___	$7 \cdot 8 =$ ___	$0 \cdot 6 =$ ___	$18 : 2 =$ ___	$0 : 9 =$ ___	$56 : 8 =$ ___
$3 \cdot 0 =$ ___	$6 \cdot 9 =$ ___	$10 \cdot 7 =$ ___	$21 : 3 =$ ___	$21 : 7 =$ ___	$28 : 7 =$ ___
$5 \cdot 10 =$ ___	$8 \cdot 1 =$ ___	$5 \cdot 8 =$ ___	$32 : 4 =$ ___	$72 : 9 =$ ___	$30 : 5 =$ ___

Figure 1: The speed test used on MP1, MP2, and MP3

About the matter: Mastering multiplication and division basic facts

Mastering the basic facts of multiplication must be distinguished from fluent computing: Fluently computing means determining a term's value by using strategies and properties of the operation and computing at skill-level (Kling & Bay-Williams, 2015). Mastering the basic tasks of multiplication means the ability to retrieve from memory their values. Mastering the basic tasks of division means

that the result is derived directly and without calculating from the corresponding basic multiplication equation. This means that the relevant number triple is available as a semantic network and can be flexibly retrieved as an equation for multiplication or an equation for division.

Mastering is indispensable by the middle of grade 3 at the latest because this knowledge is needed to solve other tasks. In line with Bülow (1985), we use 4 Acquisition Levels (AL) determined according to the number of correct solutions in the test. These levels apply to the present work:

Acquisition Level I (AL I) means the student solves at least 27 of the 30 tasks correctly (90% or more). The basic multiplication tasks are mainly solved by reproduction, and the basic division tasks are solved by reproduction or applying a basic multiplication equation at the skill level.

Acquisition Level II (AL II) means the student correctly solves at least 24 of the 30 tasks (80% or more). At acquisition level II, most tasks can be solved by reproducing or applying basic multiplication equations at the skill level. Only a few tasks are solved by computing.

Acquisition Level III (AL III) means the student correctly solves at least 18 of the 30 tasks (60% or more). At acquisition level III, there are many incorrect solutions. There is also an increased time requirement because the student calculates and cannot perform these calculations at the skill level.

Acquisition Level IV (AL IV) means the student solves fewer than 18 of the 30 tasks correctly (less than 60%). Many incorrect solutions or unfinished tasks characterise acquisition level IV. The solution procedures cannot be used confidently by the student, or are not available as solution procedures. This results in a high time consumption.

Findings

Figure 2 shows the results of a few of the 39 classes. The significant differences from class to class and from school to school are remarkable. In addition, there are substantial variations in the results in the classes, even at the end of grade 3.

School	Class	n	MP1			MP2			MP3		
			Mean	SD	Mdn	Mean	SD	Mdn	Mean	SD	Mdn
4	1	8	25.62	4.98	27.0	24.25	6.39	26.0	28.75	3.15	30.0
7	1	20	14.70	6.89	14.0	14.40	8.62	11.0	16.25	7.43	15.5
7	2	15	17.13	9.79	16.0	19.80	6.90	21.0	23.47	5.99	22.0
7	3	16	16.88	6.23	15.0	20.56	6.47	20.5	17.12	6.25	15.5
7	4	21	18.67	6.05	20.0	13.71	6.31	12.0	20.90	7.22	23.0
9	1	13	16.54	7.59	13.0	20.31	5.99	20.0	18.38	6.81	16.0
9	2	9	20.33	9.95	25.0	19.33	8.65	16.0	23.00	7.42	26.0
17	2	17	21.82	8.46	24.0	20.18	7.61	20.0	26.00	6.78	29.0
17	3	13	25.31	5.96	26.0	17.85	8.77	17.0	24.15	7.55	27.0
18	1	14	23.57	6.78	27.0	19.71	5.97	18.0	26.64	4.65	28.5
18	2	15	14.67	5.22	15.0	14.53	6.31	14.0	19.53	5.28	18.0

Figure 2: Performance of a few of the 39 classes across the three MPs: Mean, SD, and Mdn

AL I (27 or more correctly solved tasks) represents the learning objective at the end of grade 3. However, based on the mean, only one of the 39 classes achieved this goal. Even if considering the median, which is more stable and robust against outliers, only four classes, e.g., 41, 172, and 181, achieved AL I. At the same time, both the mean values and the medians of classes 71, 73, and 91 are at AL IV. In these classes, most students are far from achieving the learning objective. The differences between schools and different classes within the same school are remarkable, for example, the classes 73 and 74, or the classes 181 and 182.

Figure 3 displays the Average Score (AS) trajectory as the mean of the points achieved by all 535 students. Over the three measurement points (MP), the AS of all students in the test follows an expected path. Before the summer vacation (MP1), the AS was slightly higher (M=18.5, SD=7.57) than at the beginning of the third school year (M=18.3, SD=7.43). At the end of the third year (MP3), the AS was (M=22.4, SD=7.01), which reflects the positive effect of continued learning during the school year. However, it should be noted that the goal, according to the standards, is the red line.

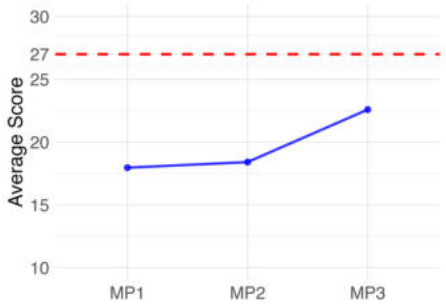

Figure 3: Mean performance of all students across the three MPs

The Sankey diagram (Figure 4) illustrates the flow of students between the four different acquisition levels across the three measurement points.

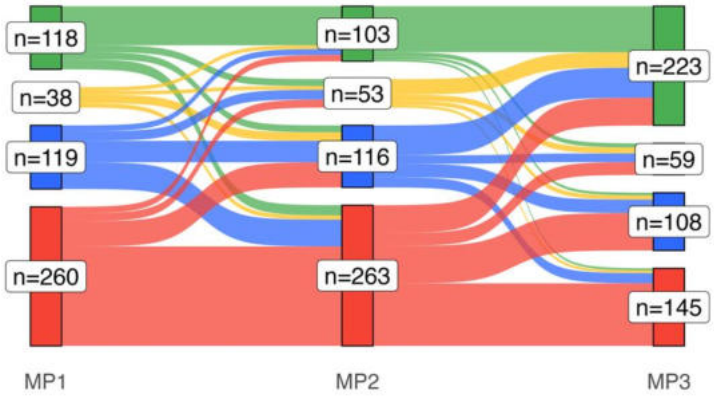

Figure 4: The flow of students across the three measurement points

The coloured bands represent summarising groups of students with the same performance development and their transitions. The width of the bands reflects the number of students in the same category transitioning together from one level to another. The different colours correspond to the

different acquisition levels at the beginning of each of the two intervals: Green represents AL I, yellow is for AL II, blue is for AL III, and red is for AL IV.

The Sankey diagram shows that at the end of grade 3, only 223 students (41,7 %) meet the curriculum requirement. Among these, 69 students (12,9%) are at AL I across all three MP. In addition, there are 145 students (27,1%) who, even after one and a half years of working with multiplication and division, only achieve the lowest acquisition level, AL IV. 103 of the 535 students (19,3%) are at the lowest level across all three measurement points.

Figure 5 shows the performance trajectories of all classes across the three MPs to reveal class-specific dynamics. We found three categories of trajectories of the average scores (AS):

- *Category A*: AS(MP1) < AS(MP2) < AS(MP3) Performance has steadily improved over time
- *Category B*: AS(MP1) > AS(MP2) ∧ AS(MP2) < AS(MP3) Performance dropped after summer vacation (MP2) and improved during grade 3 (MP3)
- *Category C*: AS(MP1) < AS(MP2) ∧ AS(MP2) > AS(MP3) Improvement of the performance after the summer vacation (MP2), followed by a decline during the school year.

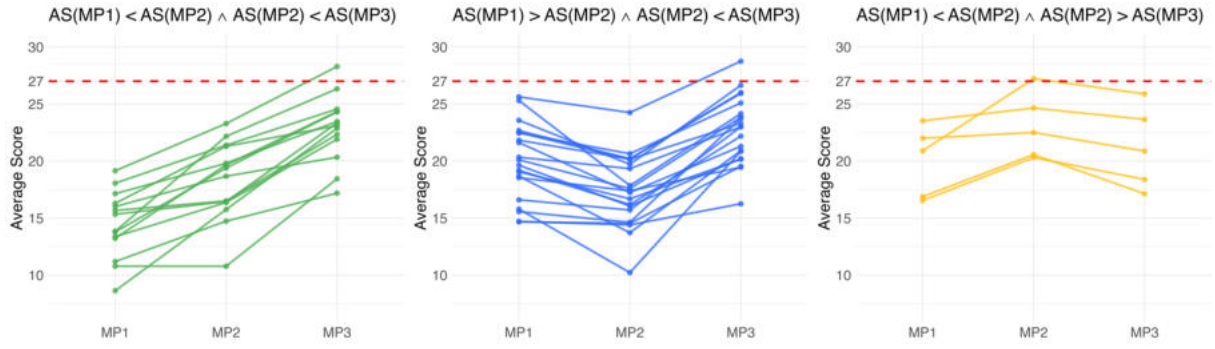

Figure 5: Average score performance by class across MPs

Of the 39 classes assessed, most fell into Category B, with 20 classes showing a recovery in performance following an initial drop in performance after the summer vacations, and 14 classes in Category A followed a continuous path of improvement. In comparison, five classes in Category C declined over the school year.

Discussion

Figures 2 and 4 show that a remarkable number of students moved from lower (AL IV and AL III) to higher levels (AL II and AL I). This indicates an overall positive trend over the year. As good as this trend may be, it was to be expected given one school year of math lessons. Much less was to be expected that, contrary to the positive trend, 312 out of 535 Students (58,3%) could not reach AL I (27 points or more), the learning objective. Figure 5 provides insight into significant differences in performance between classes. At the end of grade 3, only two of 39 classes reached an average score above 27 points (the red line) and met the learning objective.

Because of the similar teaching conditions, our findings are representative and transferable far beyond the classes and schools investigated.

The present study's findings result from the teaching and how students work with tasks. They can be explained by how the teaching is designed and can only be improved by changing this teaching design. Since teaching materials, such as textbooks and workbooks, are the primary medium for teaching, their tasks have a significant impact on teaching and its outcomes. The insufficient quality and quantity of the tasks offered in the grades 2 and 3 teaching materials must be considered as one reason for the reported unsatisfactory findings. Gleißberg & Eichler (2019) showed that tasks to develop a conceptual understanding of multiplication are underrepresented in these teaching materials. Gleißberg & Eichler (2019) also found that the number of tasks in grade 3 is significantly lower than in grade 2, resulting in fewer opportunities in grade 3 to consolidate basic equations. We suspect that the inadequate performance of the students identified in our study may also stem from several reasons that require further analysis.

It is important to memorise basic equations for the long term and be able to reconstruct them in case they are forgotten. Applying the distributive law of multiplication over addition is an important tool for reconstructing forgotten equations. In textbooks, the sequences of multiples of a number are typically presented only up to ten times that number. To develop the ability to solve problems by applying the distributive law, many problems, such as 13×7 or 24×3, where one factor is greater than 10, must be addressed. Such tasks occur rarely or not at all in second-grade textbooks.

Semantic networks help reconstruct basic equations. In almost all textbooks, related tasks such as 5×8 and 6×8 or 3×7 and 6×7 are presented together with the best intentions. It is necessary to examine whether students work on these tasks in the intended way, connecting equations, or whether they solve these tasks in isolation without understanding and utilising relationships. In our opinion, it would be more productive to assign only one task, such as 6×8, and ask students to find other tasks, such as 3×8 or 5×8, that are usable to solve the given task.

Considering these results, the current practice of working through the multiplication table "sequence by sequence" seems questionable. Kaufmann (2019) observed a strong emphasis on addition when students explain multiplication. It is worth investigating whether a stronger orientation towards semantic networks, which students actively construct and use, could help to overcome this one-sided focus on repeated addition. Furthermore, hard, motivated and continuous work on memorising all basic multiplication facts should play a much greater role than it currently does. As Wong and Evans (2007) and Burns et al. (2016) emphasise, memorised facts are a prerequisite for developing advanced mathematical skills and acquiring process-related competencies.

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What is a high-quality solution to an arithmetic task?

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"Show all the work", we sometimes tell students when they solve tasks. But what does this mean? Is a solution better if the students have shown or not shown the component calculations? In this short report, we describe a central part of an ongoing study of conventions about what is required of a student's solution to an arithmetic task to be good, or even of high quality. We made a pilot study where we asked 21 persons, upper secondary school teachers as well as researchers in mathematics education, about what aspects of a solution are important for the quality and what constitutes a high-quality solution. We show that the data can be analysed qualitatively by thematic analysis. The tentative result shows three main themes of arguments for what constitutes a high-quality solution.

Keywords: Mathematics education, arithmetic, structure sense.

Introduction

What triggered the present study is what we have seen in previous research data (Gunnarsson & Papadopoulos, 2019) presented at CERME11. That data was collected amongst Greek and Swedish students at the age of 11-12, who solved different arithmetic expressions. In many cases, we could see the same procedures and calculations in both the Greek and the Swedish data. However, in some specific instances, we could see differences between the two groups regarding the number of correct answers. We could not make any conclusions about the two school systems or curricula, but differences were more likely due to that we collected the data in different classrooms.

The difference we focus on here considers students' evaluation of one single expression, $2 \times 3 \times 5 - 4 \times 2 + 6$. The reason for choosing this example is that it showed a significant difference between the two countries. For this particular expression, the correct answer was given by 46% of the students (51 out of 112) in the Greek classrooms, and the correct answer was given by 22% of the students (27 out of 123) in the Swedish classrooms. That is, we find twice as high a "solution rate" in some classrooms compared to others. With a closer look at how the students solved the task (or rather how the students described their solutions), we see differences in how the solutions are presented. Hence, this is something we wanted to continue exploring. Could it be due to that the teachers considered different types of solutions being better or of high quality?

Theoretical framework

In literature, a considerable number of studies have been devoted to students' ideas about how to solve arithmetic tasks (see, e.g., Blando et al., 1989; Glidden, 2008; Headlam, 2013; Tabak, 2019). The studies show that many students tend to use a left-to-right principle and ignore the rules for the order of operations. These studies indicate that the arithmetic errors made by students are due to

miscalculations, or due to misuse, or ignorance of mathematical rules and conventions, like the rules for the order of operations. In contrast, some studies point out that this kind of errors could be related to a poor "feeling", or perhaps we should call it intuition, about how to calculate (e.g., Banerjee & Subramaniam, 2005; Linchevski & Livneh, 1999; Pittalis, 2023). A concept to capture this feeling is 'structure sense', a term coined by Linchevski and Livneh (1999). Structure sense has mainly been used to discuss the relation between algebraic expressions and arithmetic expressions and has sometimes been used as a predictor of algebraic proficiency based on arithmetic performance (Lüken, 2012) or a precursor to more advanced mathematics (Vincent et al., 2017). Hoch (2003) suggests that a student "who chooses an efficient and elegant method to solve a problem is displaying good structure sense" (p. 2), which could be applied to both arithmetic and algebraic expressions. However, we also know that structure sense and manipulation skills do not always go hand in hand (Hoch & Dreyfus, 2006). Moreover, Banerjee and Subramaniam (2005) have shown that developing a structural understanding of expressions requires a consistent use of the procedures and rules or arithmetic.

Regarding what could be criteria for the quality of students' solutions, a recent review study by Teledahl et al. (2024) has explored what quality can be for students' solutions to mathematical problems. They propose that what is considered to be "good" mathematical writing should cater for semantic issues (whether the answer is coherent and with relevant calculations) and syntax issues (whether the answer is written with correct language and symbols).

Aim of the study

These studies above shed light on the way students solve arithmetic tasks and what can be considered high-quality mathematical writing. But the studies leave aside how the solutions are perceived by mathematics teachers in terms of the aspects considered important for the solution as well as what constitutes a high-quality solution. Therefore, this study aims to explore what is required of a student's solution to be good, or even of high-quality. The research questions are: (RQ1) Is there a qualitative difference between the different types of solutions? In addition, (RQ2) what aspects are important for a high-quality solution?

Background data from the previous study

As input to the present study, we take data from the previous study by Gunnarsson and Papadopoulos (2019) and particularly from the answers to the task $2 \times 3 \times 5 - 4 \times 2 + 6$. Five different types of student solutions could be identified. The different types of solutions are shown in Figure 1. All solutions are mathematically correct (the correct answer is 28, according to the rules for the order of operations). None of the solutions in Figure 1 misuses the equality sign (however, we could see such solutions in the data too). The main difference between the solutions is the way the partial calculations are performed and described. In other words, the main difference is how much of the work is shown by the students. Solution A presents a solution with successive calculations in one line. Solution E is vertical with each step below each other. Solution B accounts for each of the binary operations separately. Solution C accounts for calculation of each term (two and one operation, respectively) and solution D is a short version of solution C but account only for the result of the intermediate

calculations. None of the answers given by the students were just "28", hence all student solutions showed some partial calculations.

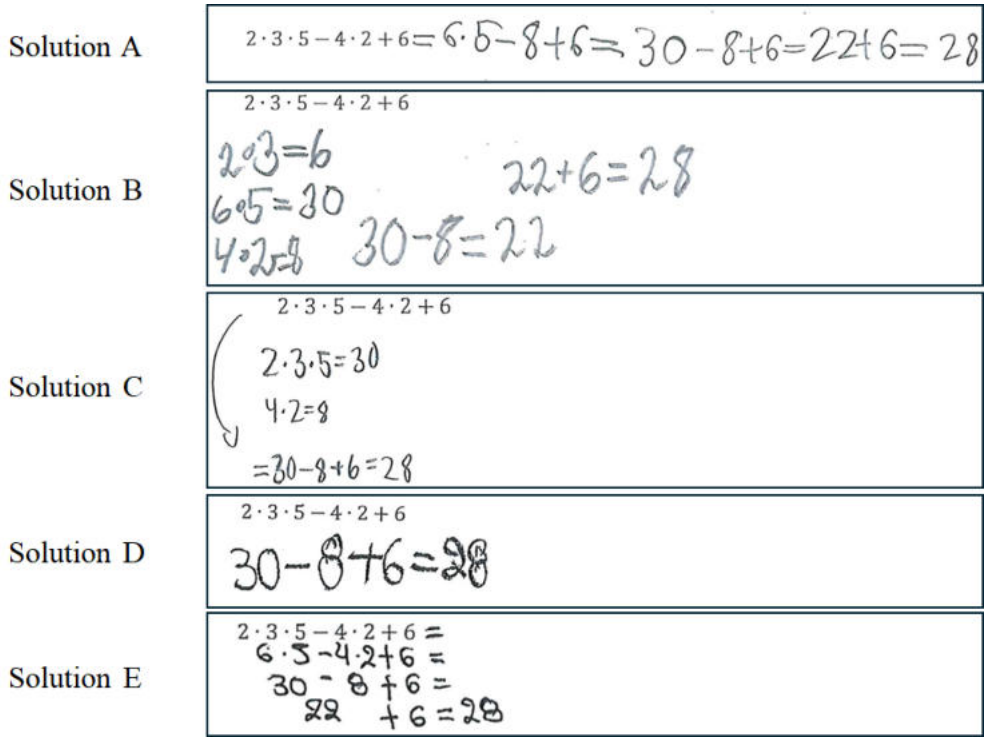

Figure 1: The five different types of students' solutions

Going back to the data and examining how these solutions are distributed among Greek and Swedish students, we found that there is a difference between their solutions. Figure 2 shows the distribution of the different types of solutions (A-E) amongst the correct answers to the task $2 \times 3 \times 5 - 4 \times 2 + 6$ (those who gave 28 as the final result). In both groups, we found answers that we could not easily categorize ("uncat" category in Figure 2), mainly because of a misuse of the equal sign. An example of an uncategorized solution could be $2 \times 3 = 6 \times 5 = 30 - 4 \times 2 = 22 + 6 = 28$, which is a solution that has led to the correct answer and used a procedure similar to type A, but with a misuse of the equal sign.

From Figure 2 we find that a large part of the 27 correct answers from Swedish students belonged to type B (12 students). The 51 correct answers from the Greek students were almost evenly distributed across the solution types A (12 students), D (11 students), and E (19 students). Interestingly, we did not find any solution of type E in the Swedish data set to the task, nor did we find any solution of type B or C in the Greek data set.

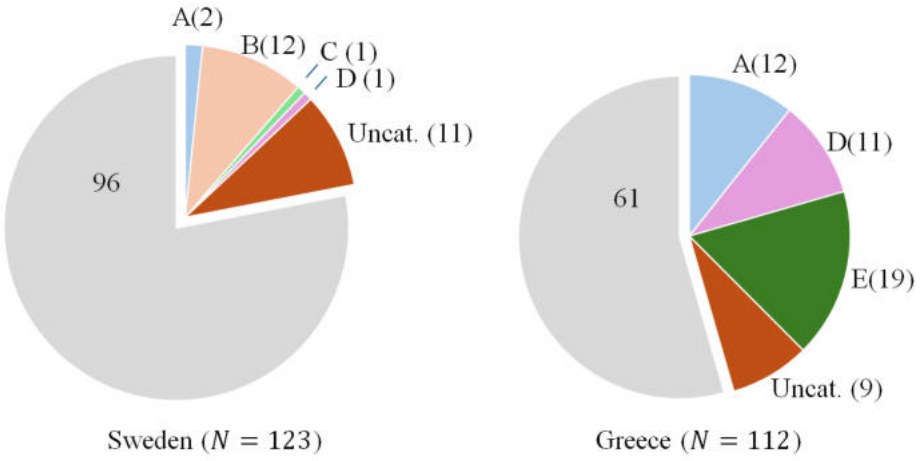

Figure 2: The distribution of the different types of solutions in the data set

This distribution of the correct answers across the five types of solutions became the starting point of the present study. Could it be that the different types of solutions lead students to different arithmetic manipulation skills or different structure sense? Or more fundamentally, are the different types of solutions of different quality, and if so, in what way? Hence, our aim is to explore what a high-quality arithmetic solution is, answering the questions if there is a qualitative difference between the different types of solutions (A-E, Figure 1), and what aspects are considered important for a high-quality solution.

Methods

Design of study. For this pilot study, a written survey was considered to provide adequate data for answering the research questions. We consider it sufficient to validate the qualitative differences between the solutions and start searching for qualitative differences that need to be handled in the bigger study. A written survey was designed, and two groups were identified for data collection.

Survey. The survey was composed of the five different student solutions shown in Figure 1. These are all genuine student solutions to one of the tasks in Gunnarsson and Papadopoulos's (2019) study: $2 \times 3 \times 5 - 4 \times 2 + 6$. The solutions were chosen to represent qualitatively different ways of solving the task. The following questions based on these solutions were posed to the participants: (*i*) Are the solutions equal in quality? (*ii*) If 'yes', why? (*iii*) if 'no', which solution has the highest quality? and which has the lowest quality? (why?) and (*iv*) What aspects do you find important for a high-quality solution? The questions did not specify what aspects of quality were to be considered but left this open for the participants to decide. The questions were designed to be sufficiently opened to allow for a certain "thickness" of the description of themes (see "analysis" below), while still maintaining the survey tool sufficiently compact and handy.

Participants. For the study, participants were chosen by convenience from two groups. *Group 1 (8 participants):* Mathematics education researchers. The data from this group comprise answers from participants at an international mathematics education research conference with a special focus on educational research on arithmetic, numbers, and mathematical operations. *Group 2 (13*

participants): Mathematics teachers. Data from this group comprise answers from upper secondary school mathematics teachers (school years 10-12) in Sweden. Both groups were chosen because of their solid mathematical knowledge and knowledge of the relation between students and the subject matter, from different perspectives and with different academic levels. In total, both groups consisted of 21 participants. The non-randomness of participators is not considered a problem since the aim is not to make a quantitative analysis of the pilot data. The analysis is in progress, therefore at the moment we present the unified data, but in the future, the ambition is to compare opinions between the two groups

Analysis. The collected open answers were analysed by a simplified form of thematic analysis (Braun & Clarke, 2006). The six phases of Braun and Clarke's thematic analysis are followed to get a (thick) description of themes. The "thickness" of the descriptions is limited by the length (the compactness) of the survey. Through the thematic analysis, the themes are inductively found in the data of the open questions in the survey. The themes were extracted from the data, and not predefined. Although the analysis is not yet completed, we allow ourselves to provide a presentation of some tentative results. However, the complete analysis is yet "in progress".

Tentative results

Many participants (14 out of 21) responded that there are qualitative differences between different solutions. Almost all participants also indicate the solutions with the highest and lowest quality and justify their decision. Only two of the participants did not mark any qualitative differences between the solutions and answered "yes" to the question "Are the solutions equal in quality?"

Figure 3 shows the highest and lowest ratings for each participant (1 to 21). Some participants rate more than one solution as the highest or lowest in quality. According to Figure 3, it seems that most of the participants (12 participants) think that the solution E is the one with the highest quality. Then follow solutions A (9 participants) and D (7 participants). The solution with the lowest quality is almost unanimously solution B (chosen by 16 participants). This is strengthened by the fact that in Figure 3(a) none of the participants considered this solution as one with high quality.

Although all phases of the thematic analysis are not yet carried out completely, the qualitative analysis allows some preliminary findings. Given that the participants found qualitatively different solutions, our effort was to identify aspects that contribute to the quality of the solutions. It seems that there are different themes in relation to what aspects are important for a high-quality solution. Until now three themes emerged from the analysis of the data: the correctness of the solution, the strategy efficiency, and whether the solution is communicative or not.

The first theme is the *correctness of the solution*. A high-quality solution is a solution that leads to a mathematically correct answer respecting the mathematical properties (for example, the correct use of the equal sign) and conventions (for example, the rules for the order of operations). Indeed, we were able to collect several answers in this spirit. Some of them were: "the solutions are presented correctly", "the student considers the order of operations", "the student uses the equality signs correctly" and "[the solution] is the best from the point of view of math".

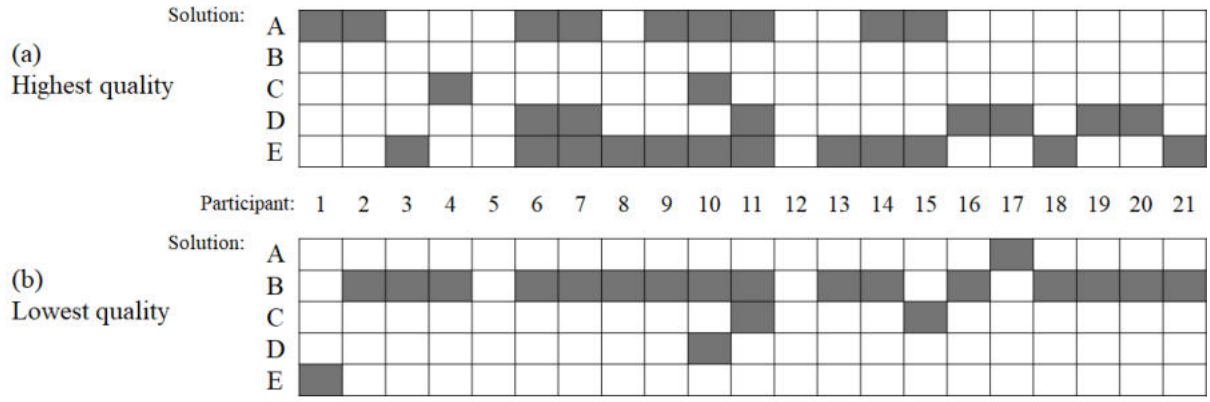

Figure 3: Schematic of what solutions (A-E) were marked as having (a) highest quality and (b) lowest quality, respectively, for each participant (1-21)

The second theme that emerged from the analysis was that the solution should describe an *efficient strategy* or a strategy that is sustainable for the student and can be used in other tasks. Examples of answers within this theme include "the design of the solution is a support for solving the task by not allowing for sign-errors", and "[about low-quality solution strategy] is much harder to apply in future [tasks]".

The third theme is that the solution should be *communicative*. In parts of the data, the communicative role of the solution is manifested as the solution is well structured or easy to follow. In fact, here there are two different objects for this communication. In some answers, the solution should be communicative for the students themselves. But in most cases, a high-quality solution is a solution that is communicative for someone else, for instance, a teacher. Typical answers in this theme contained words like "clearer", "more easy to follow", and solution B that it is "messier".

Discussion

In conclusion, we note that the different types of solutions are considered to have qualitative differences between them (RQ1). This is, as an example, also shown in Figure 3. Moreover, we note that a "criterium" for a high-quality solution (RQ2) can be composed of different aspects, and we found three such aspects in our data; [A high-quality solution should have] (mathematical) correctness, be efficient, and be transparently communicated.

The main aim of this study was to identify what a high-quality arithmetic solution is for mathematics teachers and mathematics education researchers. Is it that the solution leads to correct answers (first theme)? Regarding the qualitative data, mathematical correctness is fundamental for all solutions. Naturally, the data selected for this survey are the ones that have not fallen into the misuse of the order of operations as described by Blando et al. (1989) or Headlam (2013). The participants (teachers/researchers) of this study are quite well synchronized with the data from Sweden and Greece. Solutions A, D, and E are considered as having high-quality, whereas solution B is considered to have low quality (see Figure 3). This is in agreement with the lower solution rate observed in the Swedish data set compared to the Greek data set. Therefore, we could hypothesize that bringing the five different solution types to teachers or students in Sweden along with a discussion on what

constitutes a high-quality solution, could result in the requested goal – to create awareness of what is required of a functioning solution.

Is it important the efficiency of the strategy (second theme)? Of course, we still examine only the correct solutions. For the Swedish data, we have correct solutions achieved by using low-quality solution strategies. In other words, it needs further examination of the fact that the students are finding the correct solutions *despite* using a low-quality solution strategy. This raises the need to examine whether a higher awareness of what constitutes a high-quality solution can improve (increase) the solution rate. This is our aim for the next step which we think would be highly interesting.

Finally, is the communicative character of the solution its most important characteristic (third theme)? The participants argued that the solution could be well-structured. Hence, comparing to the concept of structure sense (as coined by Linchevski & Livneh, 1999) we find that the participants refer to high-quality solutions as "well-structured" or more "efficient". One can say that this structural aspect refers mainly to the syntax of the written expression of the solution and its compatibility with the necessary mathematical conventions. Actually, with the definition of Hoch (2003) that structure sense is related to the ability to choose "an efficient and elegant method to solve a problem", we find a close relation to this communication theme.

This is also in line with the quality criteria by Teledahl et al. (2024), where coherent and relevant calculations and correct language (i.e., communication quality) are considered "good" mathematical writing. However, their study was based on what was considered to be quality criteria for text solutions to mathematical problems. Our study has, when completed, the potential to add research-based information on what high-quality of numerical solutions could be considered to be. Based on the tentative analysis of this pilot study, we make no claims about what high quality is concerning arithmetic solutions or that the list of criteria above is complete. And possibly, hopefully students' awareness of what a high-quality solution could be, can improve the solution frequency of arithmetic tasks.

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Primary school students' ability in fractions as operators

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The learning of fractions improves when they are explored in various ways and their representations and contexts are interconnected. This exploratory study aims to identify the errors and difficulties faced by 213 fourth-year primary school students in developing their ability with fractions as operators. A problem-solving test and a mathematical exercises test are used across two contexts (continuous and discrete) and through two approaches (arithmetic and graphical). The results indicate that the fraction as an operator remains a significant challenge for students, as evidenced by the students' overall low performance, the lack of interconnection between tasks, and their limited ability to approach the tasks from different perspectives.

Keywords: Fraction as an operator, problem solving, primary school students, errors, difficulties.

Introduction

The learning of fractions is a crucial component in primary education, as it lays the foundation for understanding more advanced mathematical concepts and ensures academic success in later stages (Braithwaite & Siegler, 2021). However, teaching and learning fractions, particularly their use as operators, present numerous challenges for both students and teachers (López Arias, 2012).

The fraction as an operator, which involves applying a fraction to modify another quantity, often confuses due to the level of abstraction required and the multiplicity of representations it can take. According to Kieren (2020), mastering fractions requires not only conceptual understanding but also the ability to apply this knowledge in various contexts and through different methods of representation. This study, therefore, aims to identify the errors and difficulties faced by fourth-year primary students when developing their ability with fractions as operators in mathematical problems and exercises, considering different contexts (continuous and discrete) and various problem-solving approaches (arithmetic and graphical). Identifying the specific problematic areas for students in this concept, which is introduced in the fourth year of primary education in Spain according to the guidelines and minimum requirements established in a Royal Decree (Gobierno de España, 2022), may provide insights that could guide its teaching.

Theoretical background

One of the main challenges in teaching fractions lies in the difference between natural and rational numbers, which leads students to perceive a fraction as a whole, without distinguishing between the numerator and denominator (Butto Zarzar, 2013). This difficulty is further intensified when fractions are treated as multiplicative operators, requiring a transformation between their terms and the initial and final quantities (Behr et al., 1983).

In Spain, the teaching of fractions primarily focuses on the part-whole concept, prioritising the partitioning of units, counting, and the mechanisation of rules, which limits their connection to real-world situations (Gómez & Pérez, 2016). As a result, fractions and their arithmetic operations are among the most difficult topics for students to comprehend (Fazio & Siegler, 2011). As noted by Rico (1997), errors are an inherent part of the mathematical learning process and are reflected in the various tasks students perform. In this regard, arithmetic word problems (AWP) provide a key tool for exploring student ability across different contexts and tasks (Rodríguez- Nieto et al., 2023).

Recent studies confirm that the greatest difficulties with fractions arise in multiplication and in operations combining fractions with whole numbers (Sartono & Karso, 2020). Furthermore, when solving problems, the lack of understanding of fraction components hinders students' ability to obtain adequate spatial information, leading to errors in graphical representation. This reveals difficulties in operating correctly and exposes weaknesses in students' skills in analysis, reasoning, and interpretation when reading contextual problems (Herreros-Torres et al., 2022; Padilla-Escorcia et al., 2022). Therefore, errors should be viewed as a starting point for improving learning.

Methods

This exploratory study follows a predominantly qualitative methodology and forms part of a broader project that analyses the teaching and learning process of the fraction as an operator in Primary Education. The methods section includes: sample, instruments and variables, and analysis.

Sample

The study involved 213 fourth-year Primary Education students (aged 9-10) from six public schools and one semi-private school in the Valencian Community, Spain, mostly coming from lower-middle socioeconomic backgrounds.

Instruments and Research Variables

After teachers explained the concept of fraction as an operator, the students completed two paper-and-pencil tests. Firstly, Test 1 was administered, containing these two arithmetic-word problems:

> *Problem 1: A well containing 10 litres of water has three-fifths ($\frac{3}{5}$) of its water emptied to water the plants. How many litres were emptied?*
>
> *Problem 2: I have 10 apples in a basket, and I drop three-fifths ($\frac{3}{5}$) of them. How many apples did I drop?*

Subsequently, Test 2 was provided, featuring two mathematical exercises on the same content but without the context of a word problem:

> *Mathematical Exercise 1: Calculate: $\frac{3}{5}$ of 10 =*
>
> *Mathematical Exercise 2: Colour $\frac{3}{5}$ of the circles. How many circles have you colored? (There were 10 circles drawn).*

In both cases, the fractions operated on a natural number, and students were required to solve the tasks (the different questions included in the tests) using both arithmetic and graphical methods. To

make sure of this, the response section included two separate boxes, one for the arithmetic solution and one for the graphical one.

According to Schoenfeld's (1985) definition, a mathematical problem is any task for which the solver does not possess an immediate, well-rehearsed method, thus requiring exploration and reasoning strategies. In contrast, a mathematical exercise involves the direct application of a known technique or algorithm, without the need for prior investigation. The main difference between the problems lies in the context: Problem 1 is continuous (litres), whereas Problem 2 is discrete (apples). The main difference between the mathematical exercises lies in the method of resolution: Exercise 1 arithmetically, and Exercise 2 graphically.

Six variables were analysed according to the type of task, context, and mode of resolution:1) CPA: Continuous problem, arithmetic resolution; 2) CPG: Continuous Problem, Graphical resolution; 3) DPA: Discrete Problem, Arithmetic resolution; 4) DPG: Discrete Problem, Graphical resolution; 5) AE: Arithmetic Exercise; 6) DEG: Discrete Exercise, Graphical resolution.

Each variable was evaluated considering two subvariables:

1) Type of Success: Whether the answer was correct (1), incorrect (0), or left blank (null).
2) Mode of Resolution: Numerically coded after a classification and categorization guided by the mathematical criteria of a group of experts based on works such as Fazio and Siegler (2011) and León-Robles (2011).

Data Analysis

The data were analysed through: 1) a content analysis, coding and categorising the responses according to success and mode of resolution through consensus among the researchers, achieving an agreement index above 0.8 after three rounds of revision, and 2) a descriptive quantitative analysis, calculating success frequencies and percentages. Additionally, contingency tables were used to explore the relationship between success and modes of resolution, identifying trends and difficulties. Data processing was carried out using Excel, and version 2.3.16 of The JAMOVI Project, 2024.

Results

The results are organised around the subvariables Type of Success and Mode of Resolution, along with a section on Errors and Difficulties in Student Performance.

Type of Success

Table 1 presents the frequencies of student responses according to the success in each variable. The analysis of responses from the 213 students reveals notable variations in their performance, depending on the type of task (problem or exercise), the context (continuous or discrete), and the method of resolution (arithmetic or graphical). Generally, students encountered greater difficulties with graphical resolution compared to arithmetic, regardless of the context or type of task. This is reflected in a higher percentage of incorrect and blank responses in graphical tasks, particularly in mathematical problems versus exercises without verbal context. In Problem 1 (continuous context, litres), graphical resolution (CPG) presented more significant difficulties than arithmetic resolution, with a notable difference in the percentage of correct responses (30.5% in CPA versus 6.1% in CPG).

Table 1: Frequency of values for each variable according to success in responses

	Subvariable: Type of Success		
Variable	Success 0	Success 1	Null
CPA	110	65	38
CPG	188	13	12
DPA	107	61	45
DPG	144	43	26
AE	85	95	33
DEG	140	61	12

In Problem 2 (discrete context, apples), the differences between arithmetic and graphical approaches were less pronounced, although graphical resolution (DPG) still showed a lower success rate (20.2% in DPG vs 28.6% in DPA). Mathematical exercises without verbal context also demonstrated students' better performance in arithmetic resolution (44.6% in AE) than graphical (28.6% in DEG).

In summary, students faced greater challenges in developing the fraction as an operator when solving a mathematical problem rather than an exercise, and arithmetic resolution proved to be more accessible than graphical across all types of tasks, especially in the continuous context problem (Problem 1), where graphical difficulties were more evident.

Mode of Resolution

Table 2 displays the frequency of the most common modes of resolution used by the students, distinguishing between correct responses (success 1) and incorrect responses (success 0) according to whether the method was arithmetic or graphical.In the correct responses (success 1), the method "operator" ($10 \div 5 \times 3$) predominates, applied both in arithmetic resolution (38 to 50 times) and graphical resolution (9 to 61 times). Many students, particularly in arithmetic resolution, provided direct answers without showing their process (10 to 40 times), while a smaller group used the "multiplication" method (10×3 followed by $30 \div 5$). It is also noteworthy that three students employed a non-conventional method, "Subtract and Then Multiply" ($5 - 3$, then 2×3), which indicates that these students do not consider the fraction as a singular mathematical entity but rather treat the numerator and denominator as two independent numbers to operate with. In graphical resolution, 16 students replicated the results obtained arithmetically without applying a proper graphical method, reflecting a tendency towards the arithmetic approach even in graphical tasks.

In summary, Table 2 shows that less than ¼ of the students who participated in the study adequately understand the fraction as an operator. In addition, it should be added that there is a lack of consistency between the solving methods and the types of tasks. Despite dealing with the same mathematical content, more than 20 students change their method of solving arithmetic problems 1 and 2, and more than 10 change their method of graphically solving problem 2 and exercise 2, despite the tasks being similar.

Table 2: Categorisation and frequency of different modes of resolution of tasks according to success

	Arithmetic Resolution				Graphical Resolution			
	Mode of resolution	CPA	DPA	AE	Mode of resolution	CPG	DPG	DEG
Success 1	Operator	46	38	50	Operator	9	28	61
	Direct response	11	15	37				
	Subtract and then Multiply	2	1	0	Copy from arithmetic	2	14	0
	Multiplication	6	5	8				
	Addition	0	2	0	Colour 6 out of 10 without calculating	2	1	0
	Mode of resolution	CPA	DPA	AE	Mode of resolution	CPG	DPG	DEG
Success 0	Direct response	10	6	55	Drawing without correct graphic representation	76	18	0
	Use a data point from the statement	15	15	0	Inadequate consideration of quantities	12	29	0
	Indicate process but do not calculate	11	13	0	Consider only $\frac{3}{5}$	62	38	5
	Subtract fractions	7	7	0	Arithmetic-Based interpretation	12	11	0
	Subtraction with natural quantities	44	39	0	Apply only the numerator	6	21	62
	Incorrect operator	0	0	8	Colour 3 and 5	0	4	19
	Others	18	27	22	Colour 5 out of 10	11	7	26
					Others	9	16	28

Errors and Difficulties in Student Performance

This section analyses the errors and difficulties encountered by students based on the modes of resolution that led to an incorrect result (success 0) as detailed in Table 2. Only the most frequent methods are detailed.

In the arithmetic resolution of mathematical problems, four recurring modes were identified:

1. Direct response without process: Students provided an answer without showing the procedure. For example, answering 2 for $\frac{3}{5}$of 10 without demonstrating the calculation process.
2. Use a data point from the statement: Some students used a given data point from the problem statement as their response. For instance, simply providing $\frac{3}{5}$as the answer.

3. Indicate the process, but do not calculate: Students indicated they needed to calculate $\frac{3}{5}$of 10 but did not perform the calculation.
4. Subtracting with natural quantities: involved subtracting the quantities given in the statement from each other. They used different methods, for example: $5-3$ and/or $10-3$.

In Mathematical Exercise 1, direct responses were predominant (55 cases), alongside the incorrect use of the fraction as an operator. Specific errors were also observed in Problems 1 and 2. The most common errors are illustrated in Figure 1.

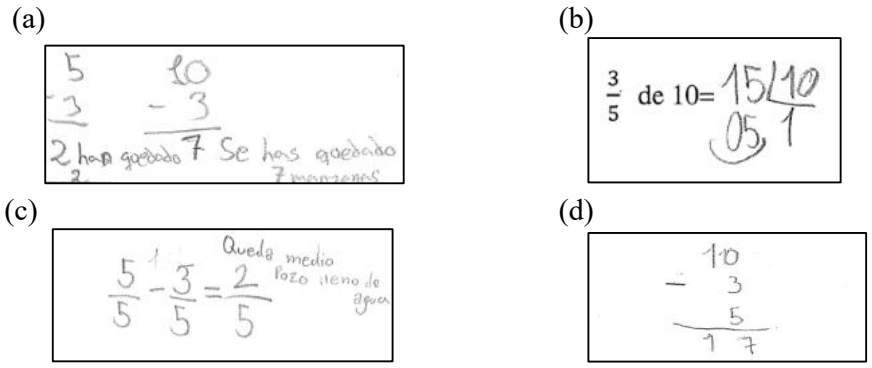

Figure 1: Four example of errors: (a) subtraction with natural quantities, (b) incorrect operator, (c) subtracting fractions, (d) considering natural numbers

In graphical resolution of the mathematical problems, the most common errors were:

1. Drawing without correct graphical representation: Many students drew a depiction of a well instead of a correct graphical representation of the quantities involved in the problem.
2. Inadequate consideration of quantities: Some students applied quantities that were not given in the problem statement.
3. Arithmetic-Based interpretation: Students coloured 6 out of 10 litres or apples by directly using the arithmetic result without an appropriate graphical representation.
4. Consider only $\frac{3}{5}$: Students did not consider the total number of apples or litres to which $\frac{3}{5}$should be applied.

Additionally, across the two problems and Mathematical Exercise 2, recurring incorrect methods included: a) applying only the numerator, b) colouring the values corresponding to the numerator and denominator (3 and 5), and c) colouring 5 out of 10. Finally, notable errors also appeared in Mathematical Exercise 2, where the pupils divided each circle into fifths and shaded three, interpreting the context as continuous rather than discrete (Figure 2).

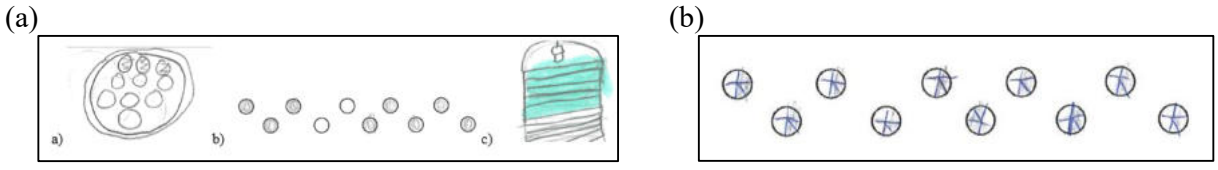

Figure 2: (a) Different graphical resolution modes, (b) dividing each circle into five parts and coloring three

Discussion and conclusion

In this study, two main factors underlie the difficulties in using fractions as operators: 1) the type of task (problem vs. exercise) and 2) the solution method (arithmetical vs. graphical). While the context in which the fraction operates, whether continuous or discrete, also presents challenges, these are less significant and mainly affect graphical solutions. Regarding the type of task, students performed better in exercises than in problems. This aligns with Gómez and Pérez (2016), who note that widely used textbooks often prioritize discrete counting and mechanized rules without real-life contexts, and with Padilla-Escorcia et al., (2022), who observe that primary students struggle to connect everyday language with mathematical language when reading contextual problems. As for the method of solution, the graphical approach proved more complex than the arithmetic one, consistent with the findings of Herreros-Torres et al. (2022), who reported similar findings in Year 6 tasks involving fractions acting on both whole and fractional numbers. This difficulty may stem from Spanish curricular materials (Real Decreto, 2022), which emphasize graphic representations mainly in the early years of primary school (up to 8-9 years old) and not later, before focusing on arithmetic, or to the representational development of students (Bruner, 1964), where many have not yet mastered the iconic stage to obtain precise visual representations.

Table 2 shows that the ability to use fractions as operators is low in the studied group, with fewer than a quarter of the students adequately developing this concept. Moreover, there is an inconsistency between the methods of solution and the types of tasks within the same individual (more than 20 students change their method each time). The alternative approaches to task solutions, along with the recorded errors and difficulties, continue to highlight two significant points: 1) operations that combine fractions with whole numbers still present errors and difficulties (Sartono & Karso, 2020) and 2) the distinction between natural and rational number systems remains a major challenge in learning fractions (Butto Zarzar, 2013).

In conclusion, this study emphasizes the need to focus on reviewing the errors and difficulties in developing fractions as operators across different contexts and representations, to promote a more effective teaching methodology that limits their persistence and enables the deep understanding defined by Kieren (2020): mastering not only conceptual understanding but also the ability to apply this knowledge in various contexts and through different methods of representation.

Acknowledgements

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Advancing the use of retrieval-based strategies with a new manipulative: The case of Nanuka

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In this paper, we describe findings from an intervention study involving a female student (aged 10 years) who appeared to be stuck using counting strategies for simple addition. The intervention involved using a new manipulative called the Keyboard - a tool depicting a linear representation of number position, which makes use of students' subitising skills. Using a microgenetic approach we present findings highlighting the efficacy of the Keyboard intervention in terms of promoting retrieval-based strategies and illustrate how it led to the variable use of derived-fact strategies.

Keywords: Counting, addition, subitising, microgenetic, variability

Introduction

It is imperative that students come to use retrieval-based strategies to solve single-digit addition problems rather than counting strategies (Geary, 2011). Retrieval-based strategies encompass direct retrieval (or recall), where an answer is retrieved from a store of facts held in long-term memory; as well as derived-fact strategies, where answers are derived using a retrieved (known) fact. There are two main types of derived-fact strategies: decomposition strategies and varying strategies (Torbeyns et al., 2009). Decomposition strategies involve partitioning one or more operands to make use of a known fact and include the near-doubles strategy, [e.g., 6 + 7 = 6 + 6 (known) + 1], and the bridging-10 strategy [e.g., 6 + 7 = 6 + 4 (known) + 3]. Varying strategies involve changing an operand to make use of a known fact and making up for this change. These include the overshoot strategy [e.g., 5 + 9 = 5 + 10 (known) - 1] and the balance strategy [e.g., 9 + 7 = 10 + 6 (known)]. Retrieval-based strategies are advantageous as they place fewer demands on working memory than counting strategies and allow for higher-ordered learning to take place (Canobi, 2009; Geary, 2011).

There is growing concern that many students in primary/elementary schools around the globe persist with using counting strategies for simple addition at a time when they are expected to be using retrieval-based strategies. Traditionally in the literature, persistent counting was considered to be uniquely characteristic of learners with mathematical learning disabilities/difficulties (MLD) or learners who exhibited persistent low achievement. The development of retrieval-based strategies was thought to be hindered for students with MLD as, compared to peers without MLD, they made more counting errors due to the use of inefficient counting strategies, and made more retrieval errors due to difficulty inhibiting irrelevant information during the retrieval process (e.g., Geary et al., 2012). Recent research, however, has highlighted persistent counting to be widespread among students in many countries (e.g., Graven & Venkat, 2021), not just among the lowest achieving students. Furthermore, persistent counters often use the most efficient counting strategy (starting at the larger addend and counting on the smaller addend) and make few errors (Hopkins et al., 2022). Hence, there is a burgeoning need for contemporary research to be focused on explaining persistent

counting and on enhancing the teaching of early arithmetic to promote the use of retrieval-based strategies at an appropriate stage of schooling. We are undertaking such a research project using a series of intervention studies designed to help students think about numbers in terms of position.

In this paper, we describe case study findings from one intervention involving a female student in Year 4 (aged 10 years and 3 months) called Nanuka (pseudonym). The intervention involved using a new tool to represent the linear positioning of numbers, called the Keyboard. Microgenetic methods were adopted to address the research question: How does using the Keyboard intervention facilitate the use of retrieval-based strategies for students who appear to be stuck using counting strategies?

Methods

Microgenetic methods require that participants are tested individually and involve an intensive (trial-by-trial) analysis of data collected on multiple occasions to document change as it happens (Siegler & Crowley, 1991). These methods are less common in educational research but have been popular among developmental psychologists to study change as it happens, including the factors that precede change.

The current study builds on the early work of Siegler (Siegler & Jenkins, 1989) who applied microgenetic methods to study how preschool children (4 and 5 years old) developed new strategies for solving addition problems as a result of practice. This seminal work led to the construction of an influential theory of strategy development called the overlapping waves theory (Siegler, 1995). This theory highlights an intermediate stage of strategy variability where children discover and use different strategies to solve the same problem on different occasions, and models how children choose among multiple strategies to eventually exhibit *strategy flexibility* and *strategy adaptivity*. Strategy flexibility involves choosing an efficient procedure depending on the characteristics of the problem (Blöte et al., 2000) and strategy adaptivity involves choosing an effective procedure depending on what works best for an individual (Siegler & Lemaire, 1997). Importantly with Siegler's model, strategy choice is modelled as being an implicit process that is influenced by experience solving specific problems and general problems similar in type, rather than a metacognitive process involving consciousness or deliberateness. The project extends Siegler's early work in two important ways: (i) the participants are older and were expected to use derived-fact strategies and not just counting strategies or retrieval, and (ii) change is studied as a result of teaching interventions (as well as practice).

Participant selection and procedure

Three participants in Year 4 were selected from a suburban primary school in Australia, where curriculum expectations state that by Year 2, students are able to recall and demonstrate proficiency with addition and subtraction facts within 20. These participants were among a cohort of 81 students from the same school who completed an individual assessment (interview) to ascertain the strategies they used for single-digit addition. They were selected for the intervention because they most frequently used counting strategies. Each participant was individually withdrawn from class to work with the research assistant (RA) for the duration of the study that spanned a month. Findings for one

participant (Nanuka) are reported in this paper - the participant who benefited most from the Keyboard intervention.

Nanuka was required to solve a set of 49 single-digit problems on (i) five consecutive school days in the pre-intervention phase (days 1-5), (ii) seven consecutive school days in the post-intervention phase (days 13-19), and (i) one final day, one week after day 19 (day 20). Each session took around 15 minutes to complete and included all single-digit addition problems (49 problems altogether) written in the form 'x + y' where x, y > 2, which were randomly displayed on a computer screen. The intervention phase encompassed learning activities taking around 20 minutes each day for seven consecutive school days (days 6 -12). All phases of the study were video-recorded.

Consistent with a microgenetic approach, a combined method of self-report and observation was used to identify the strategies Nanuka used on a problem-by-problem basis. This approach involved the student answering problems whilst being observed by a researcher (fourth author) and immediately after giving the answer, explaining to the researcher how they had solved the problem. The researcher used computer keys to control the presentation of each problem and record the participant's answer, and the digital program recorded the time taken to provide the answer (reaction time, RT). The strategies reported by the student were coded by the researcher on paper, in situ. The validity of the self-reports was established first by observation, second by RT analysis, and third by video analysis where two additional researchers (authors one and three) viewed the videos (attaining 94.2% agreement with initial coding). The scheme used to code strategies is detailed in Table 1.

Table 1: Scheme used to code strategy use with examples

Strategy Type		Strategy	Example problem and self-report
Counting		Min-counting	3 + 4, started at 4 and counted 5, 6, 7
		Count-from-first	3 + 4, started at 3 and counted 4, 5, 6, 7
Derived-fact	Decomposition	Bridging-ten	7 + 5 = 7 + 3 (known) + 2
		Near doubles	6 + 7 = 6 + 6 (known) + 1
		Fives*	5 + 8 = 5 + 5 (known) + 3
	Varying	Overshoot	8 + 9 = 9 + 9 (known) - 1
		Balance	6 + 9 = 5 + 10 (known)
Retrieval			Just knew it
Other		Not sure	
		Not elsewhere coded	

*The fives strategy was coded separately since the fact 5 + 5 = 10 could be considered either a doubles fact or an add-to-ten fact.

The Keyboard Intervention

The Keyboard comprises a handle and board with 20 "keys" that represent the numbers 1–20. It has no moving parts and no numbers written on it. To use the Keyboard, students use their left hand to hold (or press down on) the handle and their index finger on their right hand to point to the keys. Students move their finger forward (away from the handle) to add numbers and backwards (towards the handle) to subtract numbers. Each fifth key is coloured black (representing 5, 10, 15 and 20) to support students' use of subitising skills when locating and identifying specific keys, and when adding or subtracting numbers. Two or more Keyboards can be joined to represent longer number sequences.

Figure 1: The Keyboard

The intervention involved 7 x 20-minute sessions delivered on consecutive school days. Participants were introduced to the Keyboard in the first four intervention sessions (days 6–9 for Nanuka) using games and activities. The game called "Guess my Number" required the student to guess a number on the Keyboard given a clue, such as, "it is two more than a black key" or "one less than a black key". The activity, Keyboard Race, involved taking turns with the RA to throw a dice and move forward on the Keyboard the number shown on the dice, with the winner reaching the end first. Other activities included using the Keyboard to skip-count forward by 2, 3, and 4 starting from zero, skip-counting backwards, and using different combinations of two or three jumps to land on a given number (e.g., 10 + 5 = 15 or 8 + 2 + 8 = 18).

The last three intervention sessions involved solving the problem set each time using the Keyboard. Minimal explicit instruction was given on how participants could use the Keyboard since we were interested to investigate how students intuitively used the tool. We predicted they would be likely to start at the larger operand given their propensity to min-count. If they did this, the RA showed them how the Keyboard could be used by starting at the first addend (e.g., 4 + 7 = 4 + 6 + 1) for three specified (but different) problems in each session. No self-report data were collected during these three sessions but each session was video-recorded. During the post-intervention phase (days 13–17), Nanuka was provided with a Keyboard and could choose any strategy she wanted to use, including the Keyboard. Trials where Nanuka used the Keyboard were coded simply as having used a Keyboard strategy. (Note, a trial refers to an occasion where a problem was solved by the participant.) Mostly, she touched the Keyboard but occasionally she just looked at the Keyboard. During trials where she looked, she was asked how the Keyboard had helped her. Video of Keyboard trials was later viewed to examine how she had used the Keyboard. On days 18, 19, and 20, the Keyboard was not made available (day 20 was one week after day 19).

Results

In the pre-intervention phase, Nanuka predominantly used a min-counting strategy to solve the problem set (133 trials). On nine trials she used a count-from-first strategy, starting at the first addend and counting on the second addend. Her propensity to min-count meant that she used few retrieval-based strategies during this phase of the study (as shown in Figure 2). She did however consistently and correctly retrieve most doubles (3 + 3, 4 + 4, 5 + 5, 6 + 6, 7 + 7, and 8 + 8, but not 9 + 9) and six problems with smaller sums (3 + 4, 3 + 5, 4 + 5 and their turnarounds 4 + 3, 5 + 3, and 5 + 4). She used a derived-fact strategy on 21 trials. On eight trials, Nanuka made use of an answer she had calculated on a previous trial. For example, to solve 8 + 6 she remembered 8 + 5 was 13 from the previous trial where she had used a min-counting strategy, and added one more. This strategy was coded as a previous-problem strategy (under the Other strategy type), rather than a derived-fact strategy.

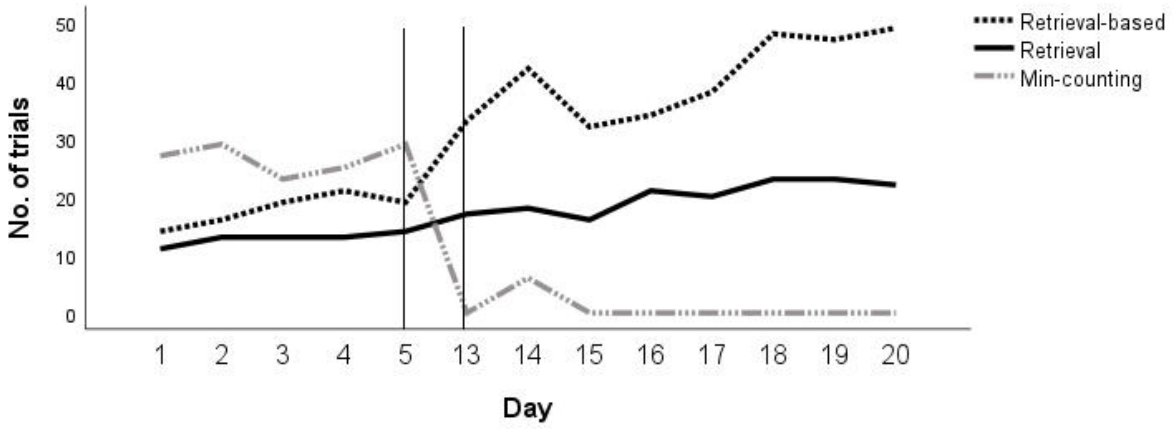

Figure 2: Strategy use for pre-intervention phase (days 1-5) and post-intervention phase (days 13-20)

Nanuka made only four errors during the pre-intervention phase (4/245 trials representing 1.6% inaccuracy): one using a count-from-first strategy, two using a previous-problem strategy, and one where she was "not sure" how she had arrived at the answer. In the post-intervention phase, she recorded 10 errors (10/392 representing 2.6% inaccuracy).

During the post-intervention phase (days 13-20), Nanuka used a min-counting strategy on only six trials (all on day 14). During the post-intervention phase (days 18-20), Nanuka correctly and consistently retrieved answers to six problems in addition to those she retrieved in the pre-intervention phase: 3 + 7, 7 + 3, 6 + 4, 4 + 6, 6 + 5 and 6 + 3. She also increased her use of derived-fact strategies to solve between 14 and 20 problems each day (making only one error that was quickly self-corrected). Most notably in this phase, Nanuka solved problems with an addend of nine using a bridging-ten strategy [e.g., 9 + 4 = 9 + 1 (known) + 5] or an overshoot strategy [e.g., 9 + 7 = 9 + 9 (known) - 2], and problems with an addend of five using a fives strategy [e.g., 8 + 5 = 5 + 5 (known) + 3].

Comparing Nanuka's performance in the pre-intervention phase to that in the post-intervention phase (see Figure 2), her use of retrieval appears to have increased at a similar rate as a result of practice

during both phases. However, her use of retrieval-based strategies jumped from 19 trials in day 5 to 33 trials in day 13, reflecting a notable increase in the use of derived-fact strategies (note, retrieval-based strategies encompass retrieval and derived-fact strategies), and a corresponding decrease in use of min-counting. Furthermore, her use of retrieval-based strategies increased at a higher rate during the post-intervention phase than the pre-intervention phase. It is important to keep in mind here that use of a derived-fact strategy was only recorded for trials where the Keyboard was not used.

When examining video data of trials involving the use of the Keyboard, we found evidence suggesting the Keyboard facilitated the transition from min-counting to retrieval for some problems, from min-counting to derived-fact strategies for some problems, and from min-counting to derived-fact strategies to retrieval on other problems. We illustrate these findings in Table 2 by summarising trials for four problems, including the strategy used (M = min-counting, R = retrieval, D = derived-fact, K = Keyboard, O = Other), the reaction time (RT) recorded, and if an error was made (incorrect trials are shaded grey).

Table 2: Strategy and reaction time (secs) recorded each day for four problems

	Pre-intervention phase					Post-intervention phase							
	1	2	3	4	5	13	14	15	16	17	18	19	20
3 + 7	M	M	M	M	M	K	D	K	R	R	R	R	R
	4.5	2.3	5.3	2.8	2.5	4.6	2.6	7.2	1.7	2.0	1.4	1.8	2.6
8 + 5	M	M	M	M	M	D	M	K	K	K	D	D	D
	5.1	3.3	3.4	2.9	2.8	6.5	7.9	5.9	7.7	12.1	5.1	8.9	3.0
5 + 7	M	M	M	M	M	K	O	K	K	K	D	D	D
	4.0	3.4	6.3	3.0	3.0	6.6	10.6	4.7	4.4	8.4	6.5	9.8	3.0
9 + 7	M	M	M	M	M	K	D	K	K	K	R	D	D
	4.7	3.5	4.4	4.0	3.5	9.4	6.1	6.5	6.6	5.6	2.0	7.8	7.0

Note. M = min-counting, R = retrieval, D = derived-fact, K = Keyboard, O = other. Incorrect trials are shaded grey.

To solve all four problems shown in Table 2, Nanuka consistently and accurately used the min-counting strategy throughout the pre-intervention phase. Differences in RTs for these trials appeared to reflect her efforts to first determine if she knew the answer.

In the post-intervention phase, Nanuka chose to solve 3 + 7 on days 13 and 15 using the Keyboard. On these trials she placed her finger on seven and moved it forward in one movement to point to 10 while at the same time stating that “7 plus 3 is 10”. From day 16 onwards, she confidently retrieved the answer 10. Use of the Keyboard supported her here to make the transition from min-counting to retrieval.

When solving problems 8 + 5 and 5 + 7, the Keyboard supported Nanuka to transition from using min-counting to derived-fact strategies; however, the transition encompassed remarkable variability in strategy choice. To solve 8 + 5 in the post-intervention phase, she used the Keyboard on four occasions (days 13, 15, 16, and 17) by pointing at eight, moving her finger forward five in one movement to 13, and reporting "I imagined two and then three" or "I saw two and then three". On days 18 to 20, she used a derived fact-strategy, but on each of these days, the strategy was different [9 + 5 (known) - 1, 8 + 2 (known) + 3, and 5 + 5 (known) + 3)]. To solve 5 + 7 in the post-intervention phase, Nanuka used the Keyboard four times (days 13, 15, 16, 17) by either pointing or looking at seven, then moving forward five keys, stating that she broke the five up into three and two. She used a derived-fact strategy on the last three days (day 18–20) but it was different to the strategy she had modelled using the Keyboard: on these trials she used a fives strategy [5 + 7 = 5 + 5 (known) + 2].

After the intervention phase, Nanuka solved the problem 9 + 7 using the Keyboard on three occasions by pointing at nine and moving forward seven in one movement (jump), reporting "there's three, two and another two" (day 13), "I counted two, three and another two" (day 15) and "I saw two, and three, and then another two (day 16). On day 17, she pointed to nine, then ten and looked at 15 on the Keyboard reporting "nine plus six is 15 so I plussed one". On day 18 (when the keyboard was not available) she retrieved the correct answer, and on the two final days she used a derived-fact strategy [9 + 1 (known) + 6 on day 19; 9 + 9 (known) - 2 on day 20]. These trials indicate that the Keyboard facilitated the transition from min-counting to derived-fact strategies and retrieval. The time Nanuka took to solve the four problems shown in Table 2 using a derived-fact strategy is noteworthy when compared to the time taken to solve them using a min-counting strategy (discussed below).

Discussion

There is a growing realisation that many students are using counting strategies for simple addition at a time when it is no longer advantageous for them to do so (Graven & Venkat, 2021; Hopkins et al., 2022). Nanuka is an example of such a child. Curriculum expectations suggest that by around Year 2, Nanuka would have replaced counting strategies with efficient retrieval-based strategies for simple addition, but she was still counting on two years on in Year 4. The Keyboard intervention was highly successful in enabling her to replace min-counting with retrieval-based strategies.

From a theoretical perspective, this is an important case study. Nanuka frequently used the most efficient counting strategy (min-counting) to solve simple addition problems and almost never made a mistake when using it. This is intriguing and at first glance appears to challenge Siegler's theory that retrieval will come to dominate performance if students use an efficient and accurate backup strategy. However, the matter is not straightforward when one considers the transitional role derived-fact strategies play in leading to retrieval (Gaidoschik, 2012). Importantly for this student, derived-fact strategies were not particularly efficient in terms of speed. Nanuka could calculate answers more quickly and just as accurately using a min-counting strategy compared to a derived-fact strategy: thus, it could be argued that her propensity to min-count before the intervention reflects the adaptive use of strategies. In relation to the research question, the Keyboard allowed Nanuka to experiment with solution procedures other than counting, which propelled her towards using derived-fact strategies even though these required more time for her to execute. She used these with remarkable variation.

Consistent with Siegler's theory, her choice of strategy appeared to be influenced not only by previous experience solving the same problem, but also by previous experience solving similar problems (i.e., problem types). With practice, Nanuka is likely to become faster at using derived-fact strategies.

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Pupils' understanding of numerical pattern: an analysis of pupils' solutions to similar problems in grades 2 and 3 of primary school

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This paper presents the results of analyses of pupils' solutions to a problem focused on numerical patterns for pupils of the 2nd and 3rd grade of primary school. The theoretical part offers justification of the importance of this topic. We used data from a three-year research project in which one group of pupils (N = 669) was followed for two years. We focused the analysis to detect different categories of the pupils' solutions. We identified 7 categories according to the correctness of the results in the different fragments of the patterns of a given task. Based on this we formulated three levels of pupil´s perception and understanding of linear increasing number patterns. For pupils identified in the project as being at risk of school failure (N = 43), we assessed their success in solving this task.

Keywords: Number series, understanding pattern, relationship, algebraic thinking.

Introduction and theoretical background

The paper deals with the perception of the number series by pupils of the 2nd and 3rd grade and the use of its pattern and possibly the relationship between two series to solve the problem. The task was given to grade 2 pupils and a year later a similar task to the same pupils in grade 3 as part of the project *Teachers' Understanding of the Causes of School Failure and the Effectiveness of Pedagogical Interventions*. The entire didactic test in mathematics, together with 11 other test items (in Czech language, psychology, questionnaire survey of pupils' conditions for learning, their performance at school, etc.) was aimed at identifying pupils at risk of school failure. We are aware that a pupil's possible deficit in understanding the pattern threatens the pupil's development of the ability to generalize, the development of algebraic and functional thinking. "Generalization is an essential component of computational thinking[1] that involves identifying patterns, rules, and relationships that are generally applicable in various mathematical contexts." (Purwasih, et al., 2024, p. 404).

The importance of the concept of pattern for the development of pupils' mathematical thinking is evidenced, for example, by its key place in curriculum documents defining important mathematics-related knowledge and skills learners should develop in the area of algebra (e.g. recognize, describe,

[1] Computational thinking and mathematical thinking differ in their focus, approaches, and goals, although they overlap and complement each other. Computational thinking is based on the principles of computer science and focuses on solving problems that can be formulated and solved using algorithms or computers.

extend, and generate patterns) not only in the Czech Republic[2], but also worldwide – Global Proficiency Framework for Mathematics, Grades 1 to 9 (2020)[3].

The key role of the discovery of patterns in the cognitive process in mathematics is mentioned e.g. by Z. Dienes in his formulation of the six stages of the mechanism of the cognitive process in mathematics (Dienes, 1960). The recognition and creation of patterns creates in the mind of the pupil the basis for the gradual understanding of more complex mathematical concepts and later the transition to abstract thinking. Also, in Grover and Pea (2021) the authors state that pattern recognition along with algorithmic thinking, abstraction, and decomposition are part of computational thinking.

Orton and Orton (1999) explore how to use patterns as an effective tool for teaching algebra as a key and challenging component of mathematics, and they emphasize that working with both numerical and geometric patterns help pupils better understand basic algebraic concepts such as variables, equations, and functions. An extensive investigation of primary school pupils' thinking when working with geometric patterns has been conducted by E. Swoboda (2006) and her team of collaborators (Rozek, 2005). In their research they demonstrate the importance of the concept of patterns for mathematical thinking. They consider them as a key tool for developing pupils' ability to discover patterns and structures. Patterns contribute to the development of geometric reasoning and logical thinking, especially in younger children as they learn to recognize repeating elements or symmetries.

In Blanton and Kaput (2005), the authors emphasize that a good foundation of algebraic thinking in the early stages of education requires the creation of a learning environment that encourages exploration and generalization of patterns. Thus, in addition to learning to solve specific problems, pupils learn to predict other members of a pattern, and they learn to see mathematical relationships and structures that are the essence of deeper mathematical reasoning. Thus, working with patterns allows pupils to move from concrete arithmetic reasoning to abstract algebraic thinking (Cobb, 1998; Lee et al., 2011; Mason, et al., 2010; Mason, 2018; Sophian, 1997). The idea that early cognitive abilities and pattern skills predict future success in mathematics is also introduced in Lee et al. (2012).

Methods

It was noted above that the task we focus on in this paper was part of a large-scale testing of Grade 2 and Grade 3 pupils. In total, 669 pupils took part in the testing. However, not all pupils participated in all items of testing due to reasons such as illness or transfer to another school. The pupils were from 29 classes, so we worked with 29 teachers. The classes were from 22 mainstream primary

[2] A major review of curriculum documents is currently in progress in the Czech Republic in collaboration between recognized experts and the wider teaching community. As far as mathematics at primary schools is concerned, these documents have added one whole area called Algebra with an emphasis on working with patterns.

[3] This document was developed by the UNESCO Institute of Statistics (UIS); the U.S. Agency for International Development (USAID); the World Bank Group; the Foreign, Commonwealth and Development Office (FCDO); the Australian Council for Educational Research (ACER); the Bill and Melinda Gates Foundation; and representatives of many other development partner organizations, including several university professors.

schools, from different areas of the country. The condition for selecting a teacher and thus a class was that the teacher had a good reputation in the way of conducting their lesson at the school and at least three years of experience. The research team of the whole project consisted of 25 experts from the fields of psychology, pedagogy, special education, Czech language, mathematics and statistics. A team of mathematics educators was responsible for creating the didactic test and interpreting the data, but everything was discussed by the entire team. The test in Grade 2 was administered in spring 2021 (N = 632) and the test in Grade 3 was administered in spring 2022 (N = 612).

Task creation

The task for the mathematics didactic test was created based on an analysis of several textbook series of mathematics for 1st and 2nd grades most used in the Czech Republic in terms of expected pre concepts. We found that pupils should have experience with repeating patterns given by namely color, shapes, or symbols, both linear (e.g. a-a-b-a-b-) and cyclical. Pupils continue the patterns or complete them forwards and backwards. Next, they complete the missing elements of a pattern. They also experience a pattern with repeating patterns given by two parameters, e.g. color and shape. Pupils also have experience of completing number series, e.g. completing missing numbers or even marks on the number line and completing two parameter tables in which they add numbers in a relationship. This includes experience of multiplication series as a repeated addition. We have assumed these experiences of the pupils, and our expectations have evolved from this.

Tasks and their apriori analysis

The text of the task describes a story from the pupil's experience, how the missing numbers were created. The pupil's task is to fill in the missing numbers on the fragments. The picture of the three fragments in Figure 1 has a distorted number structure in two linear increasing series.

Text of the problem: *Peter wrote two rows of numbers on a piece of paper. His brother blacked out some numbers and tore the paper. In the picture you can see what was left. Can you find out which numbers are blacked out? Fill them in.*

Figure 1: Assignment for 2nd a 3rd grade

For Grade 2, this is an increasing series of small natural numbers in the 1st row and an increasing series of odd numbers in the 2nd row. For Grade 3, the 1st row is an increasing series of even numbers (multiples of two), and the 2nd row is an increasing series of multiples of three. The pupils' task is to continue the series forwards and backwards from the given number, and in the third fragment in 2nd row it is necessary either to reveal the link with the 1st row, (in Grade 2 task, under the number n in 1st row is the number $2n$-1 in the 2nd row, in the Grade 3 task, under the even number n in the 1st

row is the number $3n/2$ in the 2nd row) or it is possible to find missing numbers between the second and the third fragment. The problem diagnoses the pupil's orientation in two number patterns, the pupil's ability to detect the connection between the number series in the 1st and 2nd row.

Tasks of this type are crucial for the development of generalization skills, which are an important part of mathematical literacy (Dienes, 1960). They can also be considered as propaedeutic for the development of algebraic reasoning, which is the focus of the second stage of primary school (Orton & Orton, 1999).

Data analysis

We considered all the pupils' "mistakes" (any results that differed from the expected one) and we looked for their causes. However, we got the answers only when we watched the whole solution to the problem. We recognized that describing the causes of individual mistakes was not the way to go due to the high variability of the solutions (see Table 2). We decided to divide the solutions into categories according to the results in each fragment of two patterns in the task. We arrived at the following 7 categories:

A pupil

0. does not recognize even one number pattern (incorrectly filled in fields a, f and g),

1.A recognizes only one number pattern (i.e., the entire first fragment in the 1st or 2nd row),

1.B recognizes both number patterns (i.e., the entire first fragment is correct),

2.A recognizes the number pattern and continues it in the second fragment only in the 1st row (a) without, (b) with gap filling,

2.B recognizes the number pattern, continues it and applies the pattern rule in the third fragment in the 1st row (a) without, (b) with gap filling, or in the second fragment in the 2nd row,

2.C recognizes the rule of pattern and continues it in the third fragment in the 1st row and in the second fragment in the 2nd row (a) without, (b) with gap filling,

3. recognizes the rule of pattern in each pattern, completes the missing element and applies the rule of pattern to create the pattern of the 2nd row of the third fragment (a) without, (b) with gap filling.

Findings

According to these categories, we determined three pupil's levels of perception and understanding of the linear increasing number patterns. We consider the pupils whose solutions we classified into categories 0, 1A and 1B on level I. At level II we consider pupils whose solutions fall into categories 2A, 2B, 2C and at level III we consider pupils whose solutions fall into category 3.

Of the group of 43 grade 3 pupils who were identified as at risk in the project based on all 11 test items, 22 pupils were at level I and 21 pupils were at level II, no pupil reached level III.

In the test for Grade 2, the task was among the least successful, in the test for Grade 3 it was the opposite. The average success rate between the grades increased significantly (from 59% in Grade 2 to 78% in Grade 3).

In Figure 2 we show fields coding in the task, which is the same for both Grade 2 and Grade 3.

Figure 2: Coding of fields

Table 1 gives an overview of the success rates in each field.

Table 1: Percentage success rate in each field

	a	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*i*	*j*	*k*	*l*
G 2	92,09	70,25	72,31	66,14	68,83	44,46	54,27	22,94	23,42	5,70	5,85	6,01
G 3	92,65	70,75	73,37	64,54	64,87	91,01	88,89	56,70	56,37	10,78	11,44	10,13

Let us note some interesting phenomena from Table 1. P1. Improvement between fields f and g (the first fragment) in Grade 2 and a slight deterioration in Grade 3; P2. rapid improvement in the field f between Grade 2 and Grade 3; P3. rapid improvement between Grade 2 and Grade 3 in the field h (or in field i); P4. rapid deterioration when moving from the first to the second fragment, i.e. from field g to h in both grades; P5. slight deterioration between Grade 2 and Grade 3 in the fields d and e; P6. significantly poor results in the fields j, k, l in both grades.

To illustrate the variability of the pupils' solutions we offer Table 2 in which we add the information about the most frequent pupils´ mistakes and their frequency to the percentage success rate. The first and the third columns show the expected results in certain fields. E.g. 13 [33] in the second row, second column means that mistake 13 occurred in the field d 3times.

Table 2: Percentage success rate and mistakes in the fields

Grade 2		Grade 3	
d = 15	66,14%; 13 [33], 14 [18], 11 [12]	**d = 30**	64,54%; 32 [46], 26 [27], 28 [20]
e = 16	68,83%; 15 [38], 14 [17], 18 [11]	**e = 32**	64,87%; 33 [47], 28 [19], 30 [15]
f = 7	44,46%; 6 [102], 8 [55], 4 [29]	**f = 12**	91,01%; 1 [9], 7 [4], 11 [4], 14 [4]
g = 11	54,27%; 10 [162], 12 [23], 8 [18]	**g = 18**	88,89%; 17 [12], 14 [6]
h = 19	22,94%; 21 [215], 13 [42], 17 [39], 15 [24]	**h = 30**	56,70%; 28 [38], 24 [37], 21 [29], 34 [22], 32 [21]
j = 29	5,70%; 18 [143], 25 [79], 24 [49], 31 [35]	**j = 45**	10,78%; 39 [130], 40 [67], 42 [44]
k = 31	5,85%; 19 [150], 27 [61], 25 [50], 33 [32]	**k = 48**	11,44%; 42 [138], 43 [50], 45 [35], 36[26], 38 [26]
l = 33	6,01%; 20 [143], 29 [55], 26 [47], 35 [32]	**l = 51**	10,13%; 45 [128], 46 [58], 48 [46]

Discussion and conclusion

First, we will discuss and interpret the 6 phenomena mentioned above.

P1. Better results in field g than in field f in Grade 2 indicate that continuing the number series is easier than filling in the missing number. The slight deterioration in Grade 3 is explained by the fact that numerical mistakes were more frequent: e.g. 15 + 3 = 17.

P2 and P3. The rapid improvement is explained by the fact that in Grade 2 it is a series of odd numbers, which is not worked with as often, and in Grade 3 it is a series of multiples of 3, which is part of the core Grade 3 curriculum.

P4. The deterioration from moving from the first to the second fragment (from g to h field), (in both grades by about 30%) indicates a higher cognitive demand for the application of the rule of pattern.

P5. The slight deterioration in fields d and e (the fields of the third fragment) in Grade 3 is since the difficulty of the rule in pattern in the 1st row and, therefore, the application of this rule has increased.

P6. For all correct solutions in fields j, k, l of the third fragment, the strategy of continuing the row on the same line and filling in the missing numbers in the breakaway part in the gaps between the fragments was applied in Grade 2. In Grade 3, there were already 4% of pupils who did not write any auxiliary numbers in the gap, so we conclude that these pupils were already working with the relationship between 1st and 2nd row.

Now we will provide some illustrations for levels I and II. Figures 3, 4 and 5 illustrate Level I. In Figure 3 we can see an example where all fields are filled incorrectly. The 3rd grader does not recognize even simple number pattern in the first fragment (incorrectly filled in fields a, f and g).

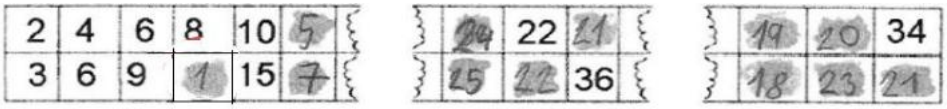

Figure 3: Category 0

In Figure 4 the pupil recognizes only one number pattern (the whole first fragment in 1st or 2nd rows). Although the pupil recognizes the pattern in the first fragment, he/she cannot apply the pattern rule and does not recognize that the series is increasing.

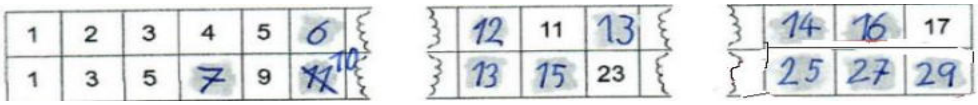

Figure 4: Category 1.A

In Figure 5 the pupil recognizes the pattern in both rows in the first fragment and can continue it but does not apply the rule of pattern in the next fragment. On the positive side, however, he/she is making his/her own pattern within each successive one; in the second fragment the number series grows by 3, in the third by 2. The pupil recognizes both number patterns (i.e. the entire first fragment is correct).

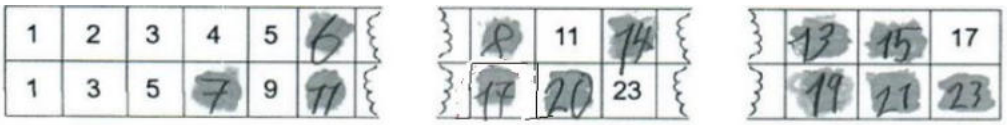

Figure 5: Category 1.B

Figures 6, 7 and 8 illustrate solutions on Level II. In Figure 6 the pupil uses the strategy of filling in the gaps between the fragments. The pupil recognizes the number pattern and continues it in the second fragment only in the 1st row.

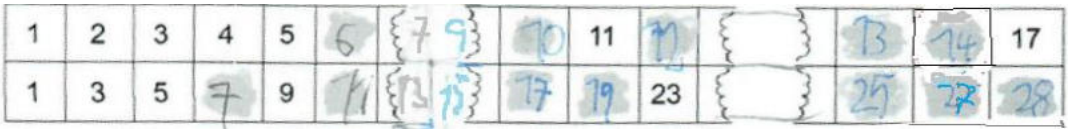

Figure 6: Category 2.A

In Figure 7 the pupil uses the strategy without filling in the gaps between the fragments. In the third fragment there is a number pattern, but it does not follow the second fragment. The pupil recognizes the number pattern, continues it and applies the rule of pattern in the third fragment in the 1st row.

Figure 7: Category 2.B

In Figure 8 the pupil recognizes the rules of both patterns, and can even apply them to the next fragments, but in the third fragment in the 2nd row it does not follow.

Figure 8: Category 2.C

The levels we determined for this task largely correspond to Dienes' (1960) categories of the cognitive process. The pupils in our research showed a great variety of solutions to one task, and in fact, covered the whole range of the cognitive process within one grade. Let us add that if a pupil is at level I in Grade 3, this is a condition that threatens the pupil's further cognitive development (Lee et al., 2012). The 22 pupils out of 43 identified as at risk also confirm by solving the task itself that they belong to this group, as they have only reached level I. With these pupils it is necessary to work more intensively with patterns and to return to simpler patterns (with repetition) given by shape, colour or rhythm presented kinesthetically and acoustically. Only then it is possible to move on to numerical patterns.

To take into account, the wide range of cognitive levels of the pupils when working with patterns, the teacher should be familiar with the content and goals of the pattern tasks and interpret the content in an exaggerated "hundred times different" way based on an understanding of the pupils and their learning processes.

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Word problems as a possible means of raising awareness of arithmetical structures

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In the transition from arithmetic to algebra, it is crucial to move from a focus on numerical solutions to the methods and procedures that led to those solutions. Our research focuses on identifying the arithmetic structures of word problems and using them to the solution. We surveyed 145 upper primary students and analysed their solutions for appropriately constructed word problems. Investigating the solution strategies, we conclude that the tasks presented are suitable as a starting point for working with arithmetic structures. However, a classroom discussion following the task solutions seems to be necessary.

Keywords: Word problem, quantitative reasoning, arithmetic structure.

Introduction

In the transition from arithmetic to algebra, it is crucial to move from a focus on numerical solutions to the methods and procedures that led to those solutions (Warren, 2003). Therefore, many studies (Herscovics & Linchevski, 1994; Kieran, 2018b; Linchevski & Livneh, 1999; Pitta-Pantazi et al., 2022; Warren, 2003) highlight the need for awareness of arithmetic structures and their impact on early algebra learning. Arithmetic operations are interpreted through word problems (WPs) in various contexts, and these tasks also provide practice in performing operations. Researchers generally agree that building a mathematical model for a situation formulated in a WP is more complex than determining the result of operations on numbers. However, some research also argues that a well-formulated and carefully chosen WP can help even lower-ability learners find the correct answer to a sequence of operations (Verschaffel et al., 2020). In this case, the text's connection with everyday life may help them make sense of the operations. Our research focuses on identifying the arithmetic structures of WPs and using them during the solution. The two problems presented in the paper can be solved by simply performing arithmetic operations and reasoning based on the arithmetic structure of the problem. We wonder whether these WPs contribute to identifying the problem situation's arithmetical structure and, thus, indirectly developing early algebraic thinking. Therefore, we formulate two research questions: (1) Can students in grades 3-4 correctly answer WPs whose arithmetic background implies the possibility of identifying the properties of operations and their relations? (2) What problem-solving strategies do they use?

Theoretical framework

Linchevski and Herscovics (1996) suggest that appropriate preparation, i.e., developing the operation sense in upper primary school, would assist students in overcoming many obstacles they face when learning algebra. Mason et al. (2009) emphasise that mastering procedures is essential to exploiting mathematically meaningful opportunities, but recognising mathematical structure makes a significant difference in understanding. They use the term ‘structure’ to refer to the generalised properties of and

relationships between mathematical objects. The arithmetic structure of an expression is, therefore, determined by the properties of the operations it contains and the relationships between them. Maffia and Mariotti (2017) describe the awareness of the structure sense in arithmetic as a set of competencies. Two of the six competencies are particularly relevant to our study. The first is identifying an arithmetical expression as an entity, for instance, comparing two expressions without calculating partial results. The last is recognising which manipulations are possible and which are useful. According to the aim of transformations (comparing or calculating), some manipulations can be more helpful or more effective.

Warren's study (2003) points out that most students leave primary school with limited awareness of arithmetic operations as general processes. From the instances they are experiencing in arithmetic, they have failed to abstract the relationships and principles needed for algebra. Consequently, Warren highlights that preparation should occur in the early years. Pitta-Pantazi et al. (2022) surveyed 6th grader students, where the tasks in the arithmetic test could be solved based on calculation procedures or identifying general properties. It was found that low achievers used strategies based on calculations more often. In contrast, high achievers did not perform calculations immediately, but they analysed and used the embedded structure. The results also showed that students who could manipulate structure within arithmetic expressions could also manipulate the syntactical structure of algebraic expressions where calculations are impossible.

In primary mathematics, WPs play a crucial role in building a situational model of an everyday life context and transforming it into a mathematical model (more precisely, into an arithmetic model). They can be described as "verbal descriptions of problem situations wherein one or more questions are raised that the answer to which can be obtained by the application of mathematical operations to numerical data available in the problem statement" (Verschaffel et al., 2000, p. ix). Traditionally, it was assumed that WPs are solved by identifying and applying the mathematical concept, operation, or rule formulated verbally. Therefore, WPs can only be solved by learners who have already acquired the necessary mathematical knowledge. Consequently, they are more complex than their corresponding symbolic representation. However, an essential result of cognitive analyses of elementary arithmetic WP solving was the discovery and systematic analysis of the various informal ways children successfully solve these problems well before being taught the corresponding mathematical knowledge and skills in school (Verschaffel et al., 2020).

Whereas young children try to process the problem sensibly, older students often apply ready-made strategies. This type of schematic knowledge is only found in proficient WP solvers because schemata are constructed through practice and abstracted from numerous encounters with the same exemplar of a problem category. However, sometimes, a more efficient strategy to solve the problem would be desirable. This strategy supposes that the solver managed to inhibit the activation of typical problem schemata to construct an alternative mental representation of the problem situation. "In other words, individuals seem to be able to weigh the costs and benefits of either blindly applying straightforward schemata or engaging in a cognitively more demanding activity to construct an alternative mental representation" (Thevenot, 2017, p. 16). We should first refer to Vergnaud's work, which distinguished numerical and relational calculus to understand this phenomenon. Numerical calculus involves operations on numbers to find the result, while relational calculus describes transformations

of relationships given in situations (Vergnaud, 1979). The two kinds of calculations are relatively independent and are probably carried out separately. However, they are linked because our decisions about what type of calculation to perform often depend on our understanding of the relation between the quantities involved in the problem (Nunes et al., 2016). Thompson (1993) emphasised the difference between numerical and relational calculation, introducing the term 'quantitative reasoning': "A prominent characteristic of reasoning quantitatively is that numbers and numeric relationships are of secondary importance and do not enter into the primary analysis of a situation. What is important is relationships among quantities" (p. 165). In our view, reason quantitatively helps to construct the situation model of a problem and abstract from the numerical aspects of a particular situation. This way, the focus is transferred from the specific numbers to the generalised features.

We assume that an appropriately chosen arithmetic WP can contribute to recognising the underlying arithmetic structure because its context can draw attention to the relationships between quantities. Based on the research findings above, it also seems possible that these tasks can be solved by students with lower skills or less knowledge of the respective mathematical concepts and procedures, relying on quantitative reasoning.

Methods

The survey involved 76 third- and 69 fourth-grade students (aged 10-11) from a primary school in a large city in Hungary, three classes per grade. The school has above-average academic performance and was in the top 20% of the National Mathematics Competency Assessment in 2022. Students are referred to as S1, S2, ..., S145 in this paper.

The survey took place in May 2023. An exercise sheet with 12 WPs had to be solved individually, each WP based on a particular operation property or a relationship between two operations. No time limit has been set for the written work. A common feature of the problems was that they allowed students to use various solution strategies: arithmetic calculation, quantitative reasoning, or reasoning based on an identified arithmetic structure. The three-digit numbers in the texts were chosen so mental calculations could be easily performed, and using written algorithms was unnecessary. According to the national curriculum, by the end of the second school year, students have mastered the mathematical background of the tasks and have gained practice in solving WPs. In the present study, two comparison problems are discussed.

WP1. Marci and Boti bought a pack of football cards for 350 HUF each. Marci had 850 HUF left after the purchase and Boti 690 HUF. Which of the boys initially had more money? By how much? Justify your answer.

This task is an additive comparison WP. It contains linguistic inconsistency because the keyword 'left' suggests subtraction, but the required arithmetic operation is addition. We must compare the boys' original money (using subtraction) to answer the difference.

– The expected arithmetic answer (AR): Originally, Marci had 850+350=1200 HUF, while Boti had 690+350=1040 HUF. Since 1200 is more than 1040 by 160 HUF (1100–1040=160), Marci had more money by 160 HUF.

– The answer based on quantitative reasoning (QR): Both boys spent the same amount, so whoever had more money initially had more left over. Marci had (850–690=) 160 HUF more money left, so he had that much more originally.
– The answer based on arithmetic structure (AS): If you subtract the same number from a larger number, the difference will be larger by the difference between the two initial numbers. Now, we know that the larger difference belongs to Marci, so Marci initially had more money, by (850-690) = 160 HUF.

WP 2. Luca had 600 HUF, and Sári had a third of that. Their grandmother gave Luca 200 HUF for her birthday while, at the same time, she tripled Sári's money. Which girl now has more money? Justify your answer.

WP2 is semantically complex because it needs to perform two multiplicative and one additive operations. The language is consistent; the keyword 'third' requires division, 'tripled' multiplication, and 'gave' addition.
– The expected arithmetic answer (AR): Originally, Sári had 600÷3=200 HUF. After the grandmother's present, Luca had 600+200=800 HUF, while Sári had 200×3=600 HUF, so Luca had more money.
– The answer based on quantitative reasoning (QR): Grandma supplemented Sári's money to match Luca's original money. Luca, in turn, received extra money from her, so she ended up with more.
– The answer based on arithmetic structure (AS): If you triple a third of a number, you return the same number. If you add a positive number to a number, you get larger than the original one.

To analyse the solutions, we set up a code system. In addition to the correctness of the answer, the chosen solution strategy was also examined. Pre-defined codes were used considering the research aim and the need to code all student answers.
– AR: Correct answer. The arithmetic calculations were done either mentally or using written algorithms. The sub-results were indicated. (Minor calculation errors are ignored.)
– QR: Correct answer. Verbal explanation based on the context of the WP. No calculations appear.
– AS: Correct answer. Verbal explanation based on the structure of the arithmetic model. No calculations appear.
– NE (No Explanation): Correct one-word answer without any explanation.
– MI (Misinterpretation): Incorrect answer. The answer is based on an incorrect arithmetic model without accurately understanding the situation.
– NA (No Answer): Missing answer or wrong one-word answer.

Two coders worked independently and then decided together on the few mismatches. No Answer and Wrong One-word Answer codes were merged after coding because they have similar meanings from the research perspectives.

Result and discussion

The correctness of the problem solution is summarised in Table 1. Proportionally, more correct answers were received for WP1, while more misinterpretations occurred for WP2. Due to the semantic structure of the tasks, their complexity (number of operations to be performed), and the type of operations required (additive, multiplicative), WP1 proved to be easier than WP2 in both grades.

At the same time, it also stands out that both tasks were solved more successfully by the 4th graders. These findings are consistent with our preliminary expectations based on the literature (Daroczy et al., 2015) and our educational experience.

Table 1: Correctness of the answers

		Correct answer (AR, QR, AS, NE)	Misinterpretation (MI)	No Answer (NA)	Total
Grade 3	WP1	84,2% (64 st.)	14,5% (11 st.)	1,3% (1 st.)	76 st.
	WP2	50% (38 st.)	38,2% (29 st.)	11,8% (9 st.)	76 st.
Grade 4	WP1	92,8% (64 st.)	4,3% (3 st.)	2,9% (2 st.)	69 st.
	WP2	75,4% (52 st.)	21,7% (15 st.)	2,9% (2 st.)	69 st.

Although the semantic structures of the WPs are quite complex, the difficulty level of the problems was not an insurmountable obstacle for the survey participants. The high success rate (even in grade 3, half of the students solved the more difficult WP2) confirms the assignment of these tasks; a significant proportion of students interpreted both WPs correctly. Therefore, a deeper analysis of the correct answers seems reasonable. Answers with MI and NA codes are excluded from further investigations.

Table 2 summarises the strategies identified in the solutions where the answer was correct. (We have created a mixed code to characterise solutions with AR and QR features.)

Table 2: Solution strategies

		AS	AR	AR+QR	QR	NE	Total
Grade 3	WP1	0 st.	52 st.	0 st.	8 st.	4 st.	64
	WP2	0 st.	33 st.	5 st.	0 st.	0 st.	38
Grade 4	WP1	2 st.	33 st.	10 st.	11 st.	8 st.	64
	WP2	2 st.	26 st.	13 st.	4 st.	7 st.	52

For both WPs at both grades, there is a strong dominance of expected arithmetic answers, but answers referring to quantitative reasoning also appear. However, QR-answers are expressed verbally at different levels. In WP1, for example, there are 6 of the 8 third-grader students whose quantitative reasoning is indicated only by the subtraction of remainders (850–690). Grade 4 students answered more precisely, e.g., “If you have more money after the shopping, you had more before” (S103). In WP2, the following types of answers (10 students) were also classified as QR: “Luca has more money because she originally had more money”. This reasoning is incorrect, but it may condense the idea that after the grandmother’s gift, both girls’ money increased, and Sári’s just reached Luca’s original amount. In addition, some students provided much more well-articulated answers, for example, AS.

Recognition of the arithmetic structure was indicated clearly by only 2 fourth-grade answers (belonging to 3 students in total) for both tasks. One student, S116, gave this type of answer for both tasks (Figure 1). In the solutions, the role of numbers (350 and 3) was identified, but no operations were performed.

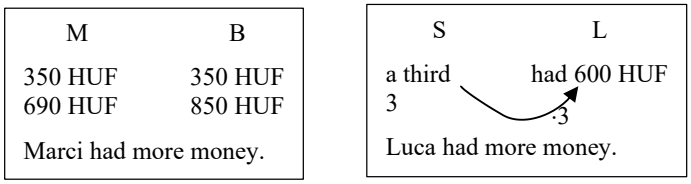

Figure 1: S116 answer based on arithmetic structure (WP1 left, WP2 right)

For both WPs, it was observed that although many students wrote down the proper arithmetic expression, they did not notice the structure this expression exemplifies; they just performed the operations. This may suggest that learners lack practice in deciding whether it is necessary to perform the operations or whether there are other ways of providing the answer. Table 3 shows this phenomenon in more detail regarding the AR-type answers for WP2.

Table 3: Summary of the arithmetic answers for WP2

	600÷3=200, 200×3=600	600÷3x3 600÷3=200, 200×3=600	600÷3=300, 300×3=900	600÷3×3 600÷3=300, 300×3=900
Grade 3	29 st.	3 st.	5 st.	1 st.
Grade 4	23 st.	14 st.	1 st.	1 st.
Total	52 st.	17 st.	6 st.	2 st.

Sixty-nine students calculated 600÷3=200 and 200×3=600 in two consecutive steps without at least noticing the identity of the first and last numbers (Columns 1-2). Although the expression 600÷3×3 or (600÷3)×3 was found in the work of 17 students, none of them determined the final result without calculation (Column 2). The work of S113 illustrates the lack of identifying the relationship between division and multiplication (Figure 2). The missing looking back phase of the problem-solving process (Polya, 1945) is seen exceptionally well by the work of those 8 students who made a mistake in their calculation (600÷3=300) and then accepted 900 as the answer for Sári's money (Column 3-4).

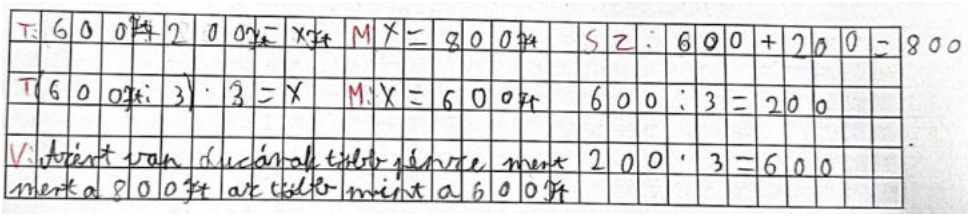

Figure 2: A grade 4 student's (S113) arithmetic answer

Figure 2 also shows traces of schematic thinking: S113 marks the learnt steps – T (Plan), Sz (Calculation), M (Answer), V (verbal answer) – in red. The verbal answer (Luca has more money because 800 HUF is more than 600 HUF.) only confirms the result and contains no new information. We found that all students in S113's class indicated the solution steps in this manner. This suggests that such 'compulsory elements' were practised in the classroom and used as scaffolding. The scaffolding reinforces the need for giving an arithmetic answer and does not encourage students to look for other, sometimes more effective, problem-solving strategies. Therefore, it can be stated that the exclusive use of the "T-Sz-M-V" scaffolding may hinder the recognition of operation properties or relations between operations when solving WPs.

Conclusion

Answering the first research question, we can establish that more than half of the students answered both WPs correctly. Furthermore, a higher proportion of older students (4th graders) answered both tasks correctly. It was also found that the additive task was more successful than the multiplicative one. About the second research question, we found that the strategy most often used, resulting in the correct answer, was to perform arithmetic operations. The students referred to the property between operations (WP1) or the relationship between operations (WP2) only in a few cases. On the one hand, many students do not recognise the structure manifested in the arithmetic model of the task despite having correctly described the model itself (i.e., the arithmetic expression). On the other hand, after 3-4 years of learning experience in school, students typically approach the problem based on a well-established arithmetic solution scaffolding and do not look for an alternative strategy if the previous one is easy to perform. The number of answers that relied on quantitative reasoning was also low.

The participants in previously presented studies were older students (6th graders). Limited awareness of arithmetic operations as general processes was assessed by solving arithmetic sentences, and the need to start this development in earlier grades was expressed. Our study examined younger students, and the arithmetic structure was embedded in the context of WPs. Our results are similar in that we also detected many arithmetic answers. Nevertheless, many other solution strategies appeared, especially among 4th graders. Overall, we conclude that the WPs presented are suitable as a starting point for working with arithmetic structures; however, a class discussion following the task solution seems to be necessary. This is also supported by the findings that the verbalisation of mathematical ideas during quantitative reasoning was often vague; consequently, formulating students' mathematical ideas needs to be improved. In all these activities, the role of the teacher is crucial, but this is a topic for future research.

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Pupils' solving strategies of the word problems Fidgets type

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This paper explores fifth-grade pupils' strategies for solving atypical word problems named as Fidgets, which lack explicit questions and require inserting provided data (numbers, words, etc.) into marked gaps in a problem text. Over six sessions in three weeks, pupil pairs engaged with these problems. Findings show that the most common heuristic strategy is Guess-Check-Revise (GCR), applied through mathematical, contextual, and linguistic senses. Pupils used direct matching or elimination to insert data and checked their solutions via calculative, semantic, and linguistic tests, though not always consistently. Revisions were done either systematically or randomly, with some strategies remaining unobservable. The study highlights the complexity and variability in pupils' reasoning processes.

Keywords: Word problems, problems solving, primary school, solving strategies, heuristic strategies.

Introduction

Word problems (WPs) repeatedly cause difficulties for pupils. Some pupils have insufficient literacy skills (Daroczy et al., 2015). They often do not draw on their real-life experiences to create a situational model of the WP (Palm, 2008) or do not perform a semantic test after solving the WP (Vondrová et al., 2019). The project *Support of the integration of mathematical, reading and language literacy in primary school pupils* aimed to help overcome these difficulties. As an outcome, the project produced an open-access methodology designed for Czech teachers, didacticians, and researchers, which includes four types of WPs along with methodological recommendations for addressing pupils' difficulties. One type is WPs of the type Fidgets. This type of WP is characterised by a larger number of correct solutions. The large number of correct solutions opens space to work with the text of the WP, e.g., to assess which solution is realistic, which one is improbable, and to discuss among pupils the meaning of the numbers used.

Theoretical framework

Among the many definitions and delineations of a WP (e.g., Verschaffel et al., 2014), we use the delineation of Semadeni (1995), i.e., that a WP consists of three components: the mathematical structure, the context, and the format. Mathematical structure refers to the relationships between data (most often numerical) whose precise arrangement in the problem forms the mathematical model of the problem (Goldin & McClintock, 1979). One piece of data can then be omitted from the model, which must be found to solve the problem. Mathematical structure can be divided, for example, according to the types and counts of operations to be performed in the calculation. The context is set of information about the situation in which the problem is embedded (Hembree, 1992). The format of a WP can be seen as a way of wording it. The format is most often a coherent text.

In relation to the solver, we focus on the process of solving WPs and how the three components intertwine the different phases of the solution. We follow the Model of Reusser (1990): 1.

Understanding the text; 2. Understanding the situation and context (creating a situational model); 3. Translating the situation into the language of mathematics; mathematization (creating a mathematical model); 4. Calculation; 5. Creating the answer. Those phases do not go fixed step by step, but the solver continuously returns to specific phases and freely transitions between them.

The solution process is the overarching framework for selecting and applying strategies. When solving WPs, the solver goes through phases and, in each phase, selects and uses different strategies in order to successfully complete the current phase. "When solving problems, the solver first integrates the textual information into an appropriate situation model or a mental representation of the situation being described in the problem, which then forms the basis for a solution strategy" (Daroczy et al., 2015, p. 4). This suggests that the choice of WP-solving strategy starts at the boundary between the second and third phases. The solution can then be approached in three ways: trial without feedback, straight-forward way as a learned solution algorithm, and heuristic strategy. "Heuristics are cognitive strategies that help to solve a task for which one does not have a clear answer or ready-made solution method. Specifically, heuristics are search strategies for problem analysis and transformation that do not guarantee, but significantly increase the probability of, finding a correct solution because they induce an active and systematic approach to the task" (Verschaffel et al., 2020, p. 5). We agree with the research (e.g., Depaepe et al., 2010; Eisenmann et al., 2015) that developing heuristic strategies is a necessary and valuable tool for coping with problems in general. At the same time, we believe that offering WPs that target using heuristic strategies will reduce the likelihood of formal uses of direct paths as surface strategies (Vondrová et al., 2019). Fidgets-type WP supports using these strategies when appropriately implemented by the teacher in the classroom.

Word problem of the type Fidgets

This type of WP comes from the Singapore methodology (Kaur & Har, 2009). WPs of the Fidgets type have three parts: the instruction, the gallery of offered data, and the text of the problem situation with missing data instead of gaps. The pupil's task is to fill in the numbers so that the problem situation makes sense. Each problem situation text must make sense mathematically, contextually, and linguistically. In addition, the specifics of each language govern the words' forms and the numbers that determine the count associated with those words. The following is an example of the WP called "The party" (problem 1).

> *Fill in numbers 32, 2, 80 and 5 in the gap in such a way that the text makes sense.*
>
> There were __ children at a party. Ms Sally bought __ packet of the sweets. There were __ sweets in each packet. During the party, she gave __ sweets to each child and had none left.[1]

Problem 1 has eight possible solutions by filling numbers in the gaps. The difficulty is mainly due to the mathematical structure. The solver may have difficulty finding the relationships between the data in the first (g_1) and fourth (g_4) gaps and between the second (g_2) and third (g_3) gaps. The mathematical model is then, for example, $g_2 \times g_3 \div g_1 = g_4$. Limiting the number of solutions can be achieved either by reasoning based on the context of the problem (e.g., "Only 2 children will not

[1]Adapted from: Shaping Maths (Activity book 3A Part 2, Q4, p. 22) by Pearly Tan (Kaur & Har, 2009)

go to the party.") or, in the case of Slavic languages, the location of numbers is influenced by the ending of the word. In Czech, the word "was" has different endings: for the number 1 [by**l**]; for the number 2–4 [by**ly**]; for the number 5 and more [by**lo**]. This WPs' translation uses "was" to use the numbers five and more before the third gap.

Other WP Fidgets type used were "Jonas is shopping" (problem 2) and "The tram" (problem 3).

> Jonas puts __ cucumbers per __ crowns and cabbage for __ crowns in his shopping basket. He paid __ crowns. (*numbers to fill in: 4, 8, 12, 44*)
>
> When the tram arrived at the stop, there were __ passengers in it. __ got off and __ got on. At the next stop __ people got off and __ people got on. There was then a total of __ passengers on the tram. (*numbers to fill in: 3, 4, 6, 8, 17, 20*)

For those three problems, the solver must use all the numbers offered. There are WPs where fewer numbers are offered, and the solver must determine the remaining numbers himself. Other times, more numbers are offered, and the solver chooses the appropriate ones.

Methods

In this research, we worked with five homogeneous pairs of pupils from two 5th-grade classes (10-12 years old). To create homogeneous pairs, pupils were selected according to their communication ability and their mathematical and reading skills, which were identified based on classroom listening and interviews with the teachers. The 10 pupils were presented with a six series of 2–3 WPs of the Fidgets type they had never encountered before. The solution itself was not preceded by any form of preparation or training. During the six sessions (20 minutes each), pupils solved the problems using the think-aloud method. The 27 video recordings and 140 pupils' written solutions were used to create a research database which was qualitatively analysed using grounded theory techniques (Strauss & Corbin, 1990). The transcripts of each part of the research were divided into thought steps and provided with didactic commentary. Subsequently, using ATLAS ti software, the thought-steps were coded and categorized into phenomena corresponding to the research question: Which strategies do pupils use to solve atypical WPs Fidgets type?

Results

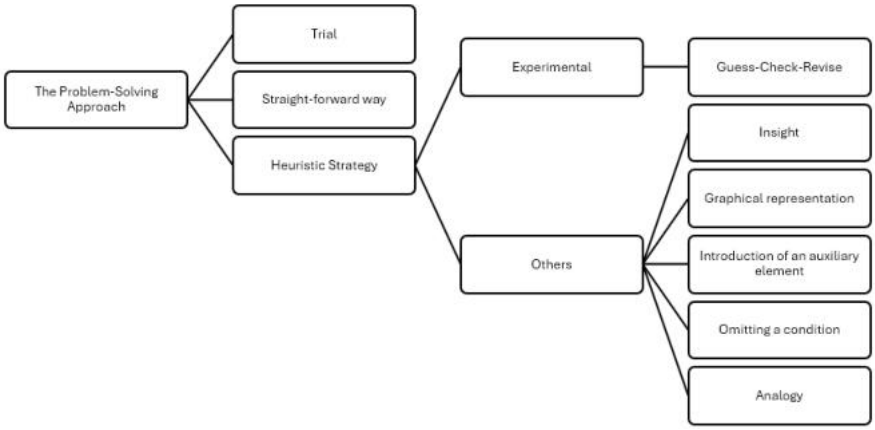

Figure 1. Pupils' problem-solving approach

Figure 1 presents the possible approaches used by the pupils in solving the Fidgets WP. The trial was used only occasionally, as demonstrated in the following example.

5A: I'm going to write Michal's solution because I want to copy it.

Since the pupils had never encountered this type of problem before, they did not have a ready-made solution and could not rely on a straightforward method. Instead, they mostly applied heuristic strategies, which we grouped into two categories: experimental and others. The main approach used by the pupils was experimental heuristics, particularly GCR and its variations, such as systematic experimentation and working backwards. Given that GCR was the most used heuristic, we now explore it in greater detail.

Guess

Guessing can be done systematically or unsystematically, directly or indirectly, i.e., by the exclusion method. The guess is always based on one of three aspects of the WP: linguistic (e.g., declension of words), context (real–life experience of the solver), and mathematics (relationships between data).

4B: ...I wrote 44 very first because of the fact that it's actually the biggest one and it's actually the sum of them all there. (problem 2)

Pupil 4B (she) perceived that focusing on the mathematical structure of the problem is important for maintaining meaningfulness. She deduced from the story and keywords where the largest number needs to be inserted (she placed it directly) and explicitly demonstrated a focus on the mathematical aspect of the problem with the word "sum". Her strategy can also be called the working backwards strategy because she realized the end state.

1A: Because...it's like, I don't know if she'd go to a party with like 2 kids, and I don't know if she'd go to a party with 32 or 80. (problem 1)

Pupil 1A (he) commented on the location of the number based on his experience of the number of children at the party. He eliminated unlikely possibilities and selected an option that made sense within the given context.

2A: Well, I just like looked it up, that the 4 cucumbers in there was probably clear because 12 cucumbers and 8 cucumbers, 44 cucumbers probably wouldn't go in there. (problem 2)

Pupil 2A (he) described his solving process by putting the number 4 directly in front of the word "cucumbers", whose form in Czech corresponds only to the use of numbers 2–4. He performed this action in the first phase of the solving process – comprehension of the text – before reading the whole problem text and creating a situational and mathematical model. He eliminated the remaining numbers, which he proved by using numbers with incorrect word forms[2].

[2]For non–Czech–speaking readers, we can illustrate by the example that we cannot connect the plural to the word "dog". The solver would say: "Here, only the number 1 can be used, because neither 2 dog or 5 dog can be said."

Check

After filling all or most data into the gaps, the pupil proceeded to the check step. Due to the research design, which involved working in pairs, this checking was done either independently or through interaction with the peer or the researcher. The check could be done right or wrong, which could then cause an incorrect answer. The solver made three types of check: a linguistic test (re-reads the text to see if it conforms to the rules of the language), a semantic test (verifies that the information matches the real situation), and a calculation test (whether the data can be put into an equation that fits the mathematical model of the problem situation).

2A: I'm thinking if the calculation works for me. But I'm guessing, again, 32 kids were at the party. She bought 80 packages and there were 5 candies in each package. So that's 400 candies total. And if it's 32, she gave them each 2. That means... 32 times 2 equals 62, so that doesn't work for me. I'll cross that out. (problem 1)

Pupil 2A (he) described the entire calculation process using the think-aloud method after fitting the numerical data into the constructed mathematical model of the problem. He concluded that his attempts were unsuccessful.

3B: Would it make sense if I wrote that the number of children at the party was 80? [laughing]

3A: Like... It seems like a lot, already, ... It wouldn't even fit in an apartment. [laughing]. (problem 1)

The dialogue illustrates the interaction between pupil 3A (she) and pupil 3B (she). Pupil 3B asked a question. Pupil 3A evaluated the completion of the number concerning the context and thus made a semantic test. Although it was unclear from pupil 3B's question whether she is uncertain about the completion of the number specifically concerning the context, pupil 3A, who answered, perceived context as the main aspect determining the use of the number.

R: Do you think you could read the whole text to me again with the numbers just added, thank you.

…

4A: Well, so we're just, we're just stopping there. Now, that it doesn't make sense to read.

After discussing the correct location of the numbers in the mathematical view, the researcher (R) invited pupils to read the whole text with the added data. Pupil 4A (she) then only stated that the solution was incorrect based on the linguistic test, although the completed numbers were correctly located based on another aspect.

Revise

After checking and in case of detecting the incorrect location of numbers in the text, the pupil proceeded to a new guess. This attempt was done either systematically or unsystematically, i.e., randomly. A systematic revise began with an initial guess, followed by a series of deliberate revisions. In the following case, we focus on an example of a systematic revise.

3B writes: first guess [20,17,8,4,_,_]; first revise [20,8,_,_,_]; second revise [20,6,8,_,_,_]; third revise – first completed solution [17,6,8,3,4,20]; second guess [8,6,17,4,_,]; fist revise of second guess – second completed solution [8,6,17,3,4,20]. (problem 3)

Pupil 3B (she) carried out a systematic revision. She filled in the number 20 in the first gap and gradually changed the number by size in the second gap. After three revisions, she placed the second largest number, 17, in the first gap and immediately arrived at a solution that matched the mathematical model of the problem. She then attempted to find a new solution by filling in the number 8 in the first gap—one she had not tried before. By chance, this led to an incorrect solution, but she realized that by changing the numbers in the fourth and fifth gaps, she could reach the correct one.

> 3B: I was thinking, like...32 kids. And, uh... 32, and if she gave each kid the 5 candies, 32 divided by, uh, 5, wait, that's not possible. That's with the remainder. ... It doesn't work. [Laughing] Because 32 divided by 5 equals 30 and the remainder is 2. ... Well. So, theoretically it could be like 80 divided by 5. Oh, I will calculate here. (problem 1)

In another problem, pupil 3B (she) performed an unsystematic revision. She made an error by incorrectly identifying the operation as division instead of multiplication. Despite constructing an incorrect mathematical model of the problem, she made a correction based on mathematical reasoning. Pupil 3B noticed the multiplicative relationship between the numbers and attempted to apply an operation that would result in an integer.

Other heuristic strategies

Other heuristic strategies included insight (created by a sudden understanding of a problem), graphical representation, introduction of an auxiliary element, omitting a condition, and analogy. Each of those strategies was used significantly less than the experimental one. As an example, Figure 2 illustrates the use of the graphical representation strategy in solving a WP[3].

Figure 2: graphical representation of WP "The pheasants" by pupil 3B (she)

According to Reusser (1990), the final phase in problem-solving is providing the answer. In the case of the Fidgets WPs, this phase involves justifying the correctness of inserting specific numbers into the text of the problem situation. The reasoning used to justify the answer is usually grounded in the nature of the test or verification process the pupils carried out.

Discussion

The results show that pupils use various strategies, as also identified in previous research by Eisenmann et al. (2015). The most common heuristic strategy is the experimental strategy, specifically GCR. Students approach the guess from three aspects of the WP: mathematical, contextual, and linguistic. They fill in the data in the text directly, i.e., identifying which data belongs

[3]Children brought pheasants to a rescue station. They brought __ pheasants. They bought grain in __ packages. There were __ kg of grains in each of the packages. That means that if divided fairly among them, each pheasant would get __ kg of grain. (*numbers to fill in: 2, 4, 5, 10*)

to which gap or by the elimination method (elimination of possibilities). Some pupils use a combination of both. They then check according to the three aspects: calculative, semantic, and linguistic tests. However, they do not always perform all three tests simultaneously. They then perform revisions systematically (e.g., starting from the largest number) or unsystematically according to random common attributes. However, it is impossible to say unequivocally for all revisions, whether systematic or not. For example, the pupil revises the change in the first gap according to its quantity, but the other gaps are filled randomly. In some cases, pupils mentally discard certain partial revisions before writing them down, for example, by considering number parity. In addition, some pupils appear to follow internal strategies that are not easily identifiable by researchers.

Conclusion

The paper briefly introduces the Fidgets WPs and the process by which pupils solve them. By analysing pupils' statements and written solutions to the problems, the types of strategies were identified. Notably, the heuristic strategy known as GCR emerged as the most frequently used. The previous findings (e.g., Mottlová & Kilienė, 2023; Mottlová & Slezáková, 2022) show that pupils' approaches to atypical WPs differ significantly from those seen in typical WP solving, suggesting that such problems have strong potential to foster the development of diverse problem-solving strategies. The format of the problems supports cross-curricular relationships between mathematics and language and offers flexibility in difficulty levels. We plan to continue investigating these aspects and to explore how pupils' experiences with Fidgets WPs can support their problem-solving in other mathematical contexts.

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Problem posing by preservice teachers: The case of open problems

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In this study two groups of preservice teachers (PTs) (with and without experience in problem solving) are asked to pose open problems based on a simple given closed problem. The aim was to examine the characteristics of the posed problems and the role of the posers' experience in problem solving. The findings give evidence that the dominant conceptualization of the participating PTs is that open problems are the ones with multiple solutions. To create these problems, they mainly kept both the given problem and the task environment, either unaltered or slightly modified. The experience in problem solving was not a significant factor for the problem posing per se since there were not differences between the groups. However, the participants with experience in problem solving demonstrated a better understanding of the distinction between open-closed problems which was not so clear for the other group.

Keywords: Problem posing, open problems, preservice teachers.

Introduction

Problem posing is recognised as a significant activity in school mathematics which can increase students' access to sensemaking and learning in mathematics (Cai, 2022). This makes evident the need to examine teachers' ability to pose mathematical problems for their students. In the book "Mathematical problem posing: From research to effective practice" (Singer et al., 2015) one can find several studies concerning the kinds of problems teachers pose, i.e., whether the posed problems are solvable, their mathematical complexity, their similarity to the textbook problems, whether they include a variety of mathematical topics, whether they are multi-step problems, etc. Problem posing is also strongly linked to open problems since to a great extend this is what mathematicians do. They not only solve open problems but also formulate their own problems according to their experience and interests (Silver, 1995). Thus, if the aim of the teaching is to provide students with such authentic experiences (working like mathematicians), open problems and problem posing are significant elements. At the same time, problem posing is interwoven with problem solving and can be seen as a special type of problem solving (Cai & Rott, 2024). Indeed, in problem posing the necessary information is given and the 'problem' (goal) of the poser is to form a new problem. In this setting this study aims to combine these three components (problem posing, open problems, problem solving) to examine PTs' ability to pose open mathematical problems. More specifically, the characteristics of the posed problems such as their kind of openness and the way the posers intervene to the task environment when posing open problems will be examined in order to answer the twofold research question: (i) What are the characteristics of the open problems generated by PTs? (ii) How do these characteristics vary according to whether the posers have experience on problem solving?

Theoretical framework and literature review

The theoretical framework of this study builds on the notions of problem posing and open problems. Several definitions of problem posing can be found in the research literature. Papadopoulos et al. (2022) organized them in five categories. This paper is aligned with one of them, "problem posing as both generating new and/or reformulating given problems" which reflects mainly the way problem posing is perceived by Silver (1994). Then it is the notion of open problems. Pehkonen (1997) uses the term "open problems" in a broad sense including problems according to the openness of their starting situation (open-start problems) and/or the goal situation (open-ended problems) defined by the task. Silver (1995) attributes to the term "open problem" three meanings: the problem is susceptible to different interpretations and/or answers, (ii) it invites different methods for its solution, and (iii) it suggests other problems or generalizations. Especially for the second meaning it is worth mentioning that even simple routine tasks afford multiple solution methods, and this multiplicity is not always due to the task itself but also to the poser's mathematical knowledge, abilities, etc. Leikin (2018) elaborated more the meaning of the term by organizing and talking about different types of openness including among others multiple answers tasks, nonroutine problems, insight-based problems, multiple solution tasks, sorting tasks, etc. In this paper we follow the approach of Mamona-Downs and Papadopoulos (2017) which is rather aligned with Silver (1995) (see also Samková, 2019) and according to which a problem might be open in relation to: (i) the interpretations of the task situation, (ii) the variety of methods for its solution, and (iii) its different solutions. Mamona-Downs and Papadopoulos (2017) claim that these three meanings are convenient for encouraging problem posing and describe four different paths the process of posing a new problem takes place: (i) The initial problem and the task environment remain unaltered (the poser's task is to ask new questions that fit to the given data), (ii) the task environment is kept but the problem is changed (It refers to changing some or all the numerical data of the problem, or to interchange the given and the goal of the given problem, thus making the known unknown and vice versa), (iii) the task environment is modified (new properties are added to the given problem objects which are necessary to the solution), and (iv) a new task environment is used (this means that the poser is able to grasp the mathematical idea hidden in the problem and transfers the same idea in another environment).

Silver and Mamona (1990) claim that problem posing can start with transforming closed problems to open. Several studies examine teacher's ability to pose problems without specific emphasis to open problems. Crespo (2003) examining problem-posing strategies of preservice mathematics teachers found that after certain training they were able to shift from posing traditional single steps and computational problems to posing problems having multiple solutions and approaches. Later, Crespo and Sinclair (2008) found that teachers often pose trivial, non-mathematical or poor problems attributing this to the lack of opportunity to engage in a problem situation and explore it before posing. Cai and Hwang (2020) presenting the state-of-the-art in problem posing say that the existing studies "have generally supported the claim that both preservice and in-service teachers are capable of posing interesting and important mathematical problems" (p. 4), although in many cases the problems posed by the participating teachers are routine problems similar to those in mathematical textbooks, or one-step problems.

The link between problem posing and open problems lies, according to Silver (1995), in the nature of problem posing since the process of posing problems can be seen as a problem-solving process where the solution (i.e., the problem to be posed) is ill-defined and consequently several problems can be posed. This makes problem posing, however, an open problem. Given also their link with the way mathematicians work, open problems are seen as having a legitimate place in mathematics classroom. Silver (1995) mentions the "What-if-not" approach as an example of a form of mathematical instruction related to this. In this case open problems and problem posing are combined into the teaching of mathematics by focusing on the creation of new problems through negating one or more elements of the given problem. He also emphasizes the potential of problem posing and open problems for both teaching and assessment purposes. Although important, this link is not specifically examined in the research literature. There is, however, one study (Nicol & Bragg, 2009) where the participating teachers were asked to pose open-ended problems. The researchers identified the teachers' unfamiliarity with this kind of problems which made challenging the problem-posing process itself. This is a difficulty originating from the participants' exposure to (closed) problems having only one solution or requiring a specific known solution process.

Finally, both notions, open problems and problem posing, are closely linked to problem solving. Silver (1995) explains the first based on the idea that open problems are associated with multiple solutions and therefore such problems not only encourage students to solve -when possible- a problem using different solution methods but also to accept the existence of different solutions for the same problem. For the second, he claims that this link stems from the work of Polya who suggests to "think of a related problem", or to think about generating related problems during the "Looking back phase".

In this setting, this paper examines PTs' problem posing on the basis of the three elements of the problems' openness and the four ways of carrying out the process of posing, in relation to whether the participant have or have not experience on problem solving.

Methods

Two groups (A and B) of PTs participated in this study. They were all university students in the same School of Primary Education. They had the same mathematical background coming from their regular mathematics classrooms in secondary education. Both groups (A and B) had a very short introduction on this issue during one teaching session in the context of a compulsory course on mathematics education. During this introduction both groups were presented with the three kinds of openness and the four modes of posing problems. However, Group B had later an extra optional course that lasted a whole semester, dedicated exclusively to problem solving including topics such as heuristics, mental argumentation, experimentation, modeling, control, proof, with a considerable amount of time spent on posing. This gave them the opportunity to deal in depth with problem solving, problem posing and the notion of open problems. Thus, it can be said that Group A 53 PTs did not have any substantial experience in problem solving, unlike the 11 members of Group B. The problem given to all the participants was a very simple closed problem: *I have six 20-cent coins and twelve 50-cent coins. How much money do I have?* The participants were asked to pose three different open problems based on the given problem. The task environment here is that there is a collection of coins and their kind as well as their total number determines the total amount of money. The specific problem,

although simple, can be transformed into a new open problem belonging to any of the three different kinds and these open problems can be generated by intervening with all the four different modes on the given problem. For example, keeping both the problem and the task environment unaltered, an open problem might be: *I have six 20-cent coins and twelve 50-cent coins. How much is 1/3 of all the money?* There is only one solution, but it can be found either by calculating the total amount of money and then dividing it by three, or by calculating 1/3 of each partial amount of money (20-cent coins and 50-cent coins) and then add these two results. This makes the problem open in relation to the solution methods. Another example might be: *I have small cubes with edge length 5cm and 10cm. How many of them do I need to effectively fill a cubical box with an edge length 30cm?* This is an open problem in relation to its interpretation. What does it mean 'effectively fill'? At the same time the task environment is new. Instead of coins we have now edges of cubes but the arithmetical relationship 1:2 between them is kept, namely 5cm and 10cm. The participants had been given as much time as they wanted, but the average time they needed to complete their effort was around 20 minutes. The collected problems posed by the participants constituted the data of this study which were qualitatively analyzed following thematic analysis in the spirit of a priori coding (Mayring, 2014) in two levels: (i) their kind of openness (interpretation, different solutions, different solution methods) and the kind of intervention to the problem and its task environment (keep both unaltered, keep the environment but change the task, modify the environment, new environment).

Results and Discussion

The analysis of the results can be summarized using a double entrance table which combines both the kind of problems' openness and the kind of the posing intervention (Table 1).

Table 1: Summarized results for both groups

Openness / Intervention	Interpretation		Solutions		Solution methods		Closed problems	
	Group A	Group B	Group A	Group B	Group A	Group B	Group A	Group B
Keep environ. keep problem			17 15.32%	2 6.67%				
Keep environ. change problem			32 28.83%	16 53.33%	3 2.7%	1 3.33%	20 18.02%	
Modify environment		2 6.67%	38 34.23%	7 23.33%	1 0.9%	2 6.67%		
New environment								

The numbers in Table 1 refer to the posed problems and not to the participants. As it can be derived from Table 1, 111 problems were posed by the 53 participants of Group A (no experience on problem solving). This means that the average number of posed problems per participant was 111÷53,

approximately 2 problems per participant. The total number of posed problems for the 11 participants of Group B was 30 (approximately 3 problems per participant). Although the size of the samples is small to justify the use of percentages, they are used to facilitate any comparison between the two groups. The first finding as it can be seen in Table 1 is that not all the possible combinations between kinds of openness and interventions were identified in the PTs' posed problems. The collected problems fit to six (out of twelve) such combinations as it will be presented.

Combination–1: Keep unaltered both, the given problem and the task environment, to pose an open problem with multiple solutions: The participants continued working in the context of coins and money, keeping the problem as it was. They only tried to find another question that could be answered using the same data: "I have six 20-cent coins and twelve 50-cent coins. **What other combinations of these kinds of coins are possible to get the same total amount of money?**"

The problem lies in the context of coins and their value (same task environment). It keeps the same data (six 20-cent coins and twelve 50-cent coins) as well as the same total amount of money. But the question now is to use as many as possible different combinations of 20-cent and 50-cent coins to get the specific amount. This is an open problem with multiple solutions. Two of them might be fourteen 50-cent and one 20-cent coins, or ten 50-cent and eleven 20-cent coins.

Combination–2: Keep the task environment but change the given problem to pose an open problem with multiple solutions. In this case the task environment remains the same (collection of coins of certain kind) but some elements (i.e., the numbers) from the initial problem are changed: "**I have some** 20-cent coins **and twice as many** 50-cent coins. How much money do I have?" The ratio between the number of coins has been kept (1:2) and the question of the problem remains the same (total amount of money). But the answer is not uniquely determined since the total number of moneys depends on the number of the 20-cent coins. Each decision for the first component leads to a different solution.

Combination–3: Keep the task environment but change the given problem to pose an open problem that affords multiple methods for its solution. The difference now is that the open problem can be solved in various ways (multiple solution approaches): "I have six 20-cent coins and twelve 50-cent coins. I want to share the total amount equally between 2 kids. How can I do this?". The solution to this problem can be achieved by (i) calculating the total amount of money and then divide by two, or (ii) by calculating separately the amount of money that corresponds to the 20-cent and 50-cent coins, divide each partial amount by two and then add the two results, or (iii) divide each set of coins by two and give each subset to the kids without calculating the amount of money.

Combination–4: Modify the task environment to pose an open problem with multiple solutions. The posers continue working in the environment of the given problem but now they modify it by either adding new components that have an impact on the solution or modifying the given ones. An example was: "I have six 20-cent coins and twelve 50-cent coins. **I need a third kind of coin to get in total 10€. How many new coins do I need**?". In this case, according to the kind of coin we choose the answer will be different. The initial collection makes 7.20€. So, we need 2.80€ more. This might mean fourteen coins if we choose 20-cent coins. The answer will be different if we choose 1-cent coins.

Another example was: “I have **five 2-euro** coins, **five 1-euro** coins and **one 5-euro note. I want to spend all my money to buy oranges (2€ per kilo) and strawberries (4€ per kilo). What are the different combinations of oranges and strawberries amounts**?” The poser is working within the environment of money, but the kind of coins has been modified (coins of 1 and 2 euros), a new kind has been added (notes of 5 euros) and transactions have been also added asking for different combinations to get in total 20€.

Combination–5: Modify the task environment to pose an open problem that affords multiple methods for its solution. Again, the difference here is that the posed problem can be solved in various ways, but all of them lead to the same answer: “I have **ten 2-euro** coins, **five 1-euro** coins and **five 10-euro notes**. What is the 1/5 of my total amount?”

Although the problem seems like the previous one (changing the two kinds of coins and adding a new kind, a note) the difference is that now the answer can be calculated in more than one way. It can be done by either calculating the total amount and then divide it by 5, or by calculating the 1/5 of each sub-collection and then adding these partial results.

Combination-6: Modify the task environment to pose a problem open in relation to its interpretation. In this case the answer is dependent on the way some of the elements of the problem are interpreted. A different interpretation leads to a different solution: “A book necessary for my classroom costs 4€. How much money do I need to buy **enough** copies for **the needs of my classroom**?” The word ‘enough’ must be interpreted by the solver to get an answer. If the interpretation is to get one book per child, then I must multiply the number of students by four. But the problem can be interpreted differently according to what it means “needs of my classroom”. If the intention is to have a book per pair of students or per a group of three or four students, then the solution becomes different.

Finally, and only for Group A, twenty of the posed problems were closed, thus not aligned with the given instructions. An example is: “Student A has six 20-cent coins, and student B has twelve 50-cent coins. Who has more money?”. The solution to the problem is unique and can be achieved in a straightforward manner. The solver can calculate the total amount of each kid and then it is easy to decide who owns more.

It seems that for the openness aspect most of the posed problems (almost 80% for both groups) are problems with multiple solutions. This might reflect the participants’ dominant concept of ‘open problems’ and this is aligned with Chapman (2012) who found that for teachers, openness in problems meant ‘more than one answer’. Moreover, almost 1/5 of the problems posed by the participants of Group A (no experience in problem solving) were closed, which shows also a difficulty in the conceptualization of the notion of open problems. Although there are instances of examples representing other aspects of openness (solution methods and interpretation), their frequency is very small, and they are met mostly in Group B having experience in problem solving. Finally, it seems that the most demanding aspect of openness is to pose open problems in relation to their interpretation. Participants in Group A did not pose any problem in this category, whereas for Group B only two problems were generated through the modification of the task environment. This brings us to the second aspect, the way the posers intervened to the problem and its environment to pose their problems. The most difficult part (zero instances) was to use a new environment. Since this

presupposes for the poser to realize the mathematical idea negotiated within the given problem, a possible explanation might be the not so strong mathematical background of the participating PTs. The participants mainly preferred to (i) keep the problem as it is and pose questions that could fit to the given data, (ii) remain in the same environment but changing the given problem, and (iii) modifying the environment by either adding extra components similar to the given ones or adding new components that should be taken into account for the solution of the new problem. The second research question was focused on the role of experience in problem solving. According to Table 1 the choices the two groups made were more or less similar. What was different was the frequency for each combination. So, Group B exhibited a preference for working in the same environment but changing the problem (changing numbers, reversing given and unknown, etc.) to pose multiple solution problems. Their conceptualization of open problems although limited (no interpretation, very few problems with various solution methods) was in the correct direction (zero closed problems). The picture for Group A is not so much different. The PTs preferred also to intervene in a similar manner with Group B (with different percentages). However, they did not produce open problems in relation to their interpretation and their understanding of open problems is partial (20% of the posed problems were closed).

Conclusions

In this paper the problems posed by PTs were examined in relation to their openness and the intervention approach used. Some of these aspects are rather demanding (open problems in relation to their interpretation or use a new environment to pose a problem). Our two populations had small differences in their problem-posing processes, and it seems that the experience in problem solving contributed more to the PTs' clear understanding of the notion of openness rather on the use of more demanding aspects of openness or more demanding ways of intervention. Perhaps this relates to the posers' mathematical background instead of their experience in problem solving, which is the case that will be examined in the next step of this study. The aim of this step is to work with mathematicians who have a more solid mathematical background compared to preservice primary school teachers. This will help us to examine whether the solid background has an impact on the quality and complexity of the posed problems.

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Toward a conceptualization of %: Some preliminary advances

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This paper aims to contribute to advancing on conceptualizations of the notion of %, as a way of understanding it better as well as making use of it in analyzing data on students' reasoning processes. The first part presents some preliminary developments on a conceptual frame about %. The second part mobilizes that frame to give meaning to preliminary data gathered from individual interviews.

Keywords: Percentage, conceptualization, absolute value, relative value

Introduction

This paper is part of a wider project aimed at investigating students' understanding of %. As a starting point, we aimed at clarifying and conceptualizing what % is, for then engaging with empirical studies on how students understand it. The 1st part of the paper explores and lays out some of the conceptions of % regularly found in articles, textbooks, lexicons, and so forth, as an attempt to offer a robust conceptualization of %. The 2nd part presents preliminary analyses of data gathered from individual interviews, which are aimed to be fed back into the conceptualization developed on % to enrich it.

Part 1: Theoretical reflections on conceptualizing %

The literature solely dedicated to conceptualizing % is scarce, where % has more than often been subsumed or integrated into work on rational numbers (e.g. Behr et al., 1983; Moss, 2000), or tucked in an isolated part in works on general statistics (e.g. Gauvrit, 2014; Huff, 1954; Spiegelhalter, 2019). Although seemingly conceived as an elementary notion, the work conducted on % presents and defines it along numerous and different meanings: as a ratio, as a fraction, as a decimal number, as an operator, as an index, and so on. In front of this literature, one might feel at lost mathematically and in need of gathering this scattered collage of meanings into a coherent frame that succeeds in coordinating and taking advantage of % many facets. This 1st part aims to contribute to this issue.

% as ratio

The ratio conception is probably the most natural one, even considered its purest form in line with % historical origins in taxes and interest (see Parker & Leinhardt, 1995). Interpreted in a literal way, an assertion like 22% is to be read "22-per-cent", pointing directly, as Baruk (1995) mentions, to a comparison between two quantities (here, 22 and 100 or cent). Some dictionaries align with this view:

- Per cent – Literally from the Latin 'in every hundred' (Earl & Nicholson, 2021)
- Percentage – A proportion, ratio or rate expressed with a denominator of 100. (Borowski & Borwein, 2002)

Considered this way, a % is presented as a "specific sort of rate" (Champlain et al., 1996) or a "special ratio involving parts per hundred" (Glatzer, 1984, p. 25), where % is read as "for 100" and "for every 100": 22% is 22-for-each-100. Van den Heuvel-Panhuizen (1994) phrase % as a ratio of so-many-out-of-so-many, hence "so-many-out-of-hundred", leading, as Baruk (1996) explains, to "bring back" or transform a situation in relation to 100. If there are 5 kids in a group of 20 persons, transforming

this in % requires asking "what if it were for 100 persons?", leading to 25 and thus 25% of the group being kids. Here, establishing a % in a situation implies finding equivalent ratios through working with proportions, where e.g. 5 is to 20 as 25 is to 100, represented symbolically by $\frac{5}{20} = \frac{25}{100}$. As well, finding B% of P in a ratio conception is also to be conceived as solving a proportion. Finding 27% of 81 amounts to asking "27 is to 100 as X is to 81" to find X or "if there is 27 for each 100, how much is there for each 81?", represented by $\frac{27}{100} = \frac{?}{81}$. That it be seen as a ratio with 100, a special rate to 100, or any other comparison with 100, the ratio conception of % places an emphasis on the relation that exists between two quantities (one being 100). In other words, in the ratio conception, the key element is the relation between quantities, where the % is an indicator of this relation to 100.

% as fraction

If the ratio conception of % is seen as its more natural one, the fraction conception is assuredly the most encountered one. This conception is generally found in the familiar usage of vocabularies and symbolisms linked to fractions for expressing a % (fraction on, fraction of, numerator, denominator, a/100, etc.). Along this line, a % becomes a fraction out of 100:

- Percent – A percent is a fraction in which the denominator is assumed to be 100. (Downing, 2009).

Transformations from any fraction to one with 100 at the denominator is thus grounded in equivalent fractions. In his lexicon for elementary schools, Grignon (1970) asserts: "The fraction ¼ can be expressed with 100 at the denominator; one says then that it is expressed in percentage. ¼ can be written as 25/100 (twenty-five hundredths) or 25% (twenty-five percent)". This entry on fractions on 100 also enables to consider % as a decimal fraction, linked to hundredths: the % implies a decimal fraction $a/10^n$, where n=2. Thus, 75% becomes 75 hundredths and is written in decimal form as $75/10^2$ or 75/100. Another meaning is to see % as fractions part-of-a-whole, where a whole is partitioned in 100 pieces and the numerator represents the number of pieces taken: 28% means that 28 pieces were taken from a whole cut into 100 pieces, hence 28/100. Here again, converting fractions to % is rooted in equivalent fractions: e.g. with ¾, a whole is cut in 4 pieces and 3 of those are taken; to convert in %, the same whole is now cut into pieces 25 times smaller, giving 100 pieces, and maintaining equivalence requires to take 25 times more of these, that is 75 pieces: ¾ = 75/100 = 75%. In both cases, finding 27% of 81 means that one wants 27/100 of 81, obtained by direct multiplication as in 81 × 27/100. As Deledicq (2004, p. 232) asserts: "Generally, taking *p*% of a number A is multiplying this number by *p*/100". Or, if fraction part-of-a-whole, 27% of 81 means wanting 27 hundredth of a whole of 81: this 81 is split in 100 pieces for which 27 of these are taken (e.g. 81 divided in 100, then multiplied by 27). Whereas the ratio conception focused on the relation between two quantities, with fractions the emphasis is placed on (different) partitioning. Beyond the mechanical conversions between % and fractions, the fraction conception points to % as a partitioning, a part of something.

% as decimal number

The conception of % as a decimal fraction leads to one where % is conceived as a decimal number. If 35% is read as "35 hundredths", it can also be written as 0.35. This conversion in decimal form enables manipulations of the %, obtained by numerically dividing by 100. Converting 35% in decimal form gives 35÷100=0.35, leading to the equality 35%=35/100=0.35. This represents one of the most frequent conversions for operating with %, where e.g. the value of a 35% discount is obtained through

multiplying the initial price by "0.35". As a decimal number, the % can be written as a power of 10, with 10^n and n=⁻2: 25% becomes 25×10^{-2}. When transformed this way, a % can somehow be treated as a number, e.g. by placing it directly on the number line: 35% as 0.35 is between 0.3 and 0.4.

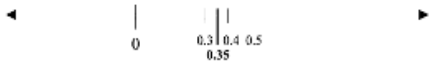

Figure 1: 35% as 0.35 on the number line

Finding 27% of 81 leads to a multiplication by a decimal number, and amounts to 81 × 0.27. As Deledicq (2004, p. 232) asserts: "Taking 12% of a number is like multiplying this number by 0.12". This conception mostly focuses on the numerical representation of %, its number dimension, and the possibilities it offers for operating with it (e.g. comparing, ordering, calculating).

% as operator

Following directly from conversions of % in fraction or decimal form, the operator conception is focused on "reading" the % sign as a request for operating. As Szymanski (1998, p. 101) states: "The words "per cent" mean "per 100", or divided by 100." This operation can also be of "× 1/100" or of "× 0.01". As such, 27% is read as "27 × 1/100", "27 ÷ 100" or "27 × 0,01". This leads the % to be transformed into an operation, where the % is not kept in its original state but instead calls for action; the same when the square root symbol is read as a demand to "take the root": √8 becomes 2.8284… (other conceptions could keep √8 as is, e.g. as a number). Seeing the % symbol in 27% leads to multiply the 27 by 1/100 or 0.01, or to divide it by 100: 27% becomes 27% = 27 × 1/100, 27% = 27 × 0.01 or 27% = 27 ÷ 100. Finding 27% of 81 is multiplicatively obtained by the transformation (27 × 1/100) x 81, (27 × 0,01) × 81 or (27 ÷ 100) × 81. This conception of % as operator is thus mostly focused on calculations, on how to do them and on its result.

% as index

Another "number" conception is one where a % is conceived as an index, always and explicitly linked to another number, that is, its referent number (Berry, 1999). The % is here seen as an indication of what is to by taken from this referent number, insisting less on the conversion of the % than on its connexion to that referent number. As index, a % does not have a predetermined value without that referent number, and it is not possible to operate (compare, order, calculate) without it: e.g. 4% is not a number *per se* e.g. on the number line, not necessarily equivalent to 4/100 or 0.04, where 4% can be worth 12 (4% of 300), ⁻26 (4% of ⁻650), 3217 (4% of 80425), and so forth. As such, this 4% can be smaller or bigger than 12% depending on their respective referent number; 4% cannot be added to 12%, again without knowledge of their respective referent number. Finding 27% of 81 in this conception somehow makes explicit the referent of 81 to which the 27% refers to, and the precise value of 27% of 81 is to be determined through various conversions, be them with ratios, fractions, decimal numbers, etc., as seen above. Seen as index-number, the focus is here on considering % as a relative object, through its connection with a referent number.

Complementarity and flexibility between various conceptions of %

Although presented separately, these conceptions of % are constantly entwined and complement one another. This complementarity is found, as mentioned, in lexicons, textbooks and articles addressing %, where definitions move from one conception to another seamlessly. For example, De Champlain

et al. (1996) define % as a ratio by saying "Rate with a second value of 100". They then offer an operator and a fraction conception with "The symbol for percentage is '%' which means 'divided by hundred' or 'over hundred' and is read 'per cent'". Côté (2002) also aligns sentences interweaving the fraction and operator conceptions: "A percentage is a fraction for which the denominator is 100. The symbol % is read 'per cent' and means '× 1/100'". As well, one can find tons of equalities of the sort "20% = 20/100 = 1/5 = 0.2", joined with mentions of the necessary link of these % with their referent number. Beyond numerical conversions, these all point toward the complementarity existing between different conceptions of %. Here are examples:

- The fraction conception of %, as a decimal fraction, can be linked to decimal numbers through numerical conversions (e.g. 75% = 75/100 = 0.75). This conception can also be related to the index-number one, when considered as fraction part-of-a-whole (e.g. 75/100 of? 75% of?). It can be associated to the operator conception, representing the necessary number to operate on (e.g. 75% becomes 75 × 1/100, or 75% of P becomes P × 75/100).
- The decimal number conception of % can be linked to the operator one, representing as well the needed number to conduct the operation (e.g. 75% becomes 75 × 0,01 or 75% of P becomes 0.75 × P). This conception can point toward an index-number one, when the decimal number is linked to a specific whole, for example when "75% of" becomes "0.75 of". It can be linked to a ratio conception, where 75%=0.75 is meant to be a rate of 0.75 for 1.
- The ratio conception of % is connected to a number conception when related to a rational number a/b with b=100 (e.g. 75% is 75 for 100 symbolized as 75/100). It can be aligned to a referent whole, in which the repartition is linked to a relation to maintain, as in its origin in taxes and interests (e.g. out of 3200, for each 100 there is 75 of these that needs to be handed out), pointing to the index-number and fraction part-of-a-whole conceptions of %.
- The operator conception of % is connected to the fraction and decimal number conception through the operation "× 1/100" or "× 0.01". What is then obtained is a numerical value linked to the %. This conception is linkable to the index-number one, representing the operation for determining the value of the % in relation to its referent number.
- The index-number conception, as mentioned, is linked to the operator one: 75% is 75% of a referent number, and the operation "× 1/100" or "× 0.01" enables calculating its value. Linked to fraction and decimal number conceptions when interpreted along their referent (e.g. 75/100 of, 0.75 of), it can also align with the ratio one (e.g. 75 for each 100 of a referent number).

These associations allow addressing % in a flexible way. In other words, each conception is not sealed and is opened to a transition toward the others. What comes out of these links is their complementarity, a connectedness enabling the passage from a number, operator, fraction, index conception, etc., depending on what is at stake. Along with Janvier's (1983) metaphor, the different conceptions of % can be envisaged as each representing the point of an immersed star (see Figure 2). When one conception is put forward, it is made salient, but it is not isolated: the other conceptions are still there, present and connected to it, waiting and wanting to be made salient themselves. In other words, the other conceptions of % are always connected, but are immersed and represent potentialities that can be exploited and made explicit if need be. To these can be added Parker and Leinhardt's (1995) distinction between a % as a *statistic*, for describing a situation (e.g. 75% of kids wear glasses

vs. 56% of adults), and a % as a *function*, for operating and creating another number (e.g. 75% of 520 kids wear glasses, that is 390). Each of these uses can be mobilized for interpreting or operating on %. This versatility of the nature of the % as a "statistic" or as a "function", lying at the center of the % star, illustrates that the emerged conception can take profit of it in the situation. This star illustrates an array of possibilities for %, through the complementarity between conceptions, and the necessary fluidity between these when working and interpreting %.

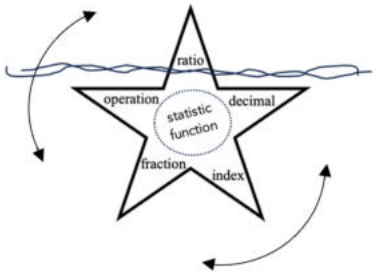

Figure 2: The %-star with its various complementary and interconnected conceptions

Part 2: Preliminary case studies on %

Our research team aims to investigate students' understanding of %. The first phase of our experiments focuses on individual interviews with (voluntary and available) students and teachers, as pilot-testing to prepare the second phase of classroom-wide experiments. This paper reports on aspects of this first pilot-testing phase. Audio-recorded interviews were planned, which consisted of presenting a task and giving students and teachers about 15 seconds to solve it mentally, and then for them to explain their answers (orally, using sheets of paper to leave traces). A series of tasks on arithmetic operations frequently found in textbooks were used: e.g. "50% + 3", "25%/75%; 5% × 12%; 25% + 50%; compare 3%/4 *vs*. 3/4%. These tasks were chosen because they possibly required conversions between the different conceptions of %, and at the same time might raise issues related to an index-number conception since no explicit referent for the % was given in these tasks. Each interview usually lasted between 10 to 20 minutes, depending on the participant's availability. The research data consisted of the audio explanations of the solutions, as well as occasional notes left on paper. The analysis of this preliminary data involved using the %-star conceptions to precisely code the different parts of solutions to the task, as a way for understanding which (and how) the conceptions of % were mobilized in those (and complemented one another). Being preliminary, only excerpts from solutions obtained for the "50% + 3" question are reported (for two participants, namely Marco and Julie, pseudonyms). At this stage, the analysis of Marco and Julie's solutions (1) contribute additional reflections on the nature of %, and (2) enrich understandings about the above complementarity between % conceptions (all interactions are translated from French to English).

Marco's answers

Marco is a 14-year-old boy that just completed Grade-8, and frequently worked on % during his school year. When asked what "50% + 3" gave, his first explanation was as follows:

Marco:	3 and 50. Because 50% is equivalent to ½ and ½ is equivalent to 0.5. And, 0.5 + 3 = 3.5. […] But if you do "of something", it will not render the same thing. Let's say 50% of 10, plus 3, so it is 5+3=8.
Int.:	So is the answer 8 or 3.5?
Marco:	No, this is just an example!

Int.: 3.50 is not an example?
Marco: No, it is the question. This one [8] it is if it would be 50% of something.
Int.: And that's not the same thing.
Marco: No, it's 50% that's all. It is not "of" or "about" something. […] This [50%+3=3.5], is the question, but then *I* decided to act as if it was 50% of something.
Int.: And is there an example that would give 3.50?
Marco: Yes, 50% of 1.

Julie's answers

Julie is an experienced teacher (Grade-7 to 11), who also has been head of her mathematics department (in school districts and at the national level). When asked "50% + 3", she responded:

Julie: If I take it in a purely numerical mode, 50% + 3 could be 350%. […] If one takes 50% in its decimal form, it would give 50 hundredths, and with 3 it gives 3 and 50 hundredths, so 350 hundredths thus 350%.

After being asked if there was other possible answers, she replied:

Julie: If one reasons on the % in relation to a whole, the 3 can also be related to it and it can influence the answer. I would need to have what the 50% refers to. If I have a whole of 6, the 3 is then 50%: 50%+50% would be 100%, or 6. Since 3 is 50% and 50% is 3, hence 6. […] If the whole is 18, 50% of 18 is 9. And, 9+3 gives 12. Twelve is the answer or 12 eighteenths. Or, my 3 in relation to 18 is 1/6 or 3/18, and finding what 1/6 is in %...it's 16.6% so it's 66.6%.

Dialectical relation between absolute and relative dimensions of %

In their answers, both Marco and Julie make use of %, fractions and decimal forms interchangeably, converting from one to the other with ease: one can thus see the %-star used "in action". Both their second answers, that is, Marco's 'examples' and Julie's referral to a 'whole', point to how % can be tied to a referent number and become indexes to be able to grasp its value; hence fluidly transiting toward an index-number conception. Both select examples (8, 1, 6, 18), and use those to illustrate their point. They thus raise two different ways of interpreting the question – although Marco insists that 3.50 is the legitimate answer because of the way the question is posed – from a direct numerical conversion to one variable in relation to a referent number for the %. This is particularly salient in the difference made by Marco between his "3.50", one being *the* answer and the other being simply one example among many. In both Marco and Julie's first answers, the 50% is considered in the absolute as 0.5, a value to operate on or place on the number line. In their second answers, the 50% becomes relative, linked to a referent for obtaining its value. As such, beyond fluidly going from one conception of % to the next, as offered in the %-star, there appears to be a dialectical relation at play that emerges in both Marco and Julie's understanding between work on numerical conversions (from % to fractions and decimal numbers) and the place of the referent of the %. This dialectical relation raises a rich interplay between "absolute" and "relative" dimensions of %, that is, % with or without a referent. Through navigating between conceptions of %, Marco and Julie's work appear to tap directly into that dialectic, transitioning from one to the next with some ease. Their answers show their capacity to go from numerical conversions to referent numbers, to discuss and compare them, showing the difference between each while accepting both as "possible" ways of interpreting the task.

Reflecting on a midway perspective

This absolute-relative dialectic about % that emerges from the initial analyses of Marco and Julie's work can give rise to a subsequent perspective on %; one that in fact appears to have strong ties to

how % is treated in school textbooks. This perspective appears situated midway between an absolute and a relative view of %: it considers the referent of the % by fixating it to "one thing", to one unit, used as a referent, e.g. to a geometrical square cut in 100 smaller squares, to a collection of objects, to a specific number or even to 1. In this midway, the referent is made explicit as one unit, for then undertaking operations (e.g. comparing, ordering, calculating). By stabilizing the referent to one unit, the value of the % is not seen as absolute, like in numerical conversions, but it is not completely relative or variable either. (As such, contrary to Marco who sees one of his "3.50" merely as an example, Julie will assert at one point that 50% is 0.5 as if it was 50% of the number 1, somehow referring to this midway perspective.) This reference to one unit can enable one to compare two %, for example, since both can be linked to the same one unit (25% < 50% of the same referent, which is not necessarily true with different referents; 10% + 5% = 15% of the same referent, which is not necessarily true either, etc.). Lodged within the dialectic, this midway perspective might be seen to emerge in the back-and-forth between relative and absolute, maintaining the link of the % to a referent (by fixating it temporarily to one unit) and taking advantage of the conversions to obtain, temporarily, a numerical value for that %. This offers an additional layer to the (ongoing) conceptualization of %.

Table 1: Three perspectives for working on and considering %

Absolute perspective on %	**Midway perspective**	**Relative perspective on %**
Implicit referent (or not mentioned)	Referent fixated temporarily (for the task at hand)	Variable referent (determined by and through the task)
% as number, its value is obtained by converting it (for operating)	Value of % linked to a fixed unit	% as index-number, its value is relative to a referent (to be known)
50% = 50/100 = 0.5 50% > 25% 50% + 25% = 75%	50% of is 50% of is 50% of is 50% > 25% and 50% + 25% = 75%	The added dotted-square in is both an addition of 50% or $33.\overline{3}$% of the new whole. John spent 50% of his money. Mary spent 25%. Who spent more?

Concluding remarks

The conceptualization of % offered in the first part outlines that conceptual and calculational dimensions are at the heart of interpreting and working with %: interpreting % by making sense of their referent number necessarily implies calculational abilities, and in turn the calculations and numerical conversions with % imply giving meaning to %. Both go hand-in-hand, and this is where the aforementioned dialectical relation illustrated in Julie and Marco's work between %-absolute and %-relative can be seen to situate itself. From this dialectical relation emerges the midway "step" of fixating the referent whole to one unit for the time being while handling the % for the problem at stake. Transiting between these three perspectives (numerical conversions, fixating the referent to one-unit, variable referent) might play a key role in interpreting and working with %. This conceptualization also points to the inherent connectedness and complementarity between various conceptions of %, through the metaphor of the immersed star, allowing flexible navigation from one conception to another (which appears crucial for developing ease with interpreting and working with

%). This (necessary) flexibility might explain the found presence of different but simultaneous conceptions of % in lexicons, dictionaries, textbooks and articles addressing issues of %.

The data presented here is obviously preliminary and this conceptualization is evolving (and Marco and Julie are only two examples, used here to deepen our understanding of %, and in no way are proposed as representative or illustrative of all students or teachers we worked with). More is to be done to better bridge the %-star of conceptions with the dialectic between relative and absolute values of %. It is hoped that our next research phase on classroom experimentations can lead to deepened understandings of % inner workings and in turn bolster this continuously growing conceptualization.

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Effects of spoken number words on place value understanding: A comparative study with German and Danish second graders

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One characteristic of a sound understanding of place value is the ability to switch between different number representations. There are clear differences in the formation of number words in different languages. Both German and Danish use an inverse number word formation, starting with saying ones before tens. This exploratory study compares the abilities and difficulties of second graders from two schools each in Germany (N=96) and Denmark (N=84) in tasks involving translation processes between number representations. Our results show differences between the two groups in terms of both the frequency and type of errors. The German students make fewer errors overall, but in contrast to the Danish students, they show inversion errors particularly frequently. This indicates that it is not only the inverse number word formation that influences the occurrence of errors, but that there are other influencing factors. This also makes the results interesting for studies on other languages.

Keywords: Number concepts, place value, number word inversion.

Introduction

A sound understanding of place value is seen as an important basis for the development of further numeracy skills like written arithmetic procedures. On the other hand, an insufficiently developed understanding is a barrier to successful arithmetic learning and is recognised as one of the main signs of particularly severe and long-lasting difficulties in arithmetic learning (Fromme, 2017). Accordingly, studies show that place value understanding serves as a predictor for later arithmetical performance (Habermann et al., 2020; Moeller et al., 2011).

The development of place value understanding places several demands on children. In order to understand place value, students need to develop knowledge of the fundamental principles of our decimal number system, namely the regrouping principle and the place value principle.

> The *regrouping principle* describes that the same number of units is bundled continuously […]; the *place value principle* states that each digit gives two pieces of information by its position, which indicates the bundling unit it stands for, and its face value, which tells the number of the represented bundling units. (Jensen et al., 2024, p. 3)

In addition, the ability to relate different representations of multi-digit numbers (as a symbolic representation with digits, as a spoken number word and as a quantity; Figure 1) and to switch between these representations is considered essential for understanding place value (Ejersbo & Misfeldt, 2015; Fromme, 2017; van de Walle et al., 2023). Apart from using only digits for symbolic representation (e.g. 32), other notations can be chosen, e.g., with indication of the place values (e.g., 3 tens 2 ones). Quantities can be represented using a variety of materials; one commonly used material is the base ten blocks, which show the place values in a very clear way (Figure 1).

Tasks that require switching between these representations can also be used to assess students' understanding of place value. The mastery of translation processes is an indication of a developed place value understanding. However, the use of different representations places different demands on students. For writing and reading a symbolic representation of a number, it is essential to understand the place value principle, particularly that the position of a digit determines its value in terms of powers of ten (Fromme, 2017; Jensen et al., 2024). For transcoding spoken number words, it is important to have a strong knowledge of the principles and rules underlying the construction of number words.

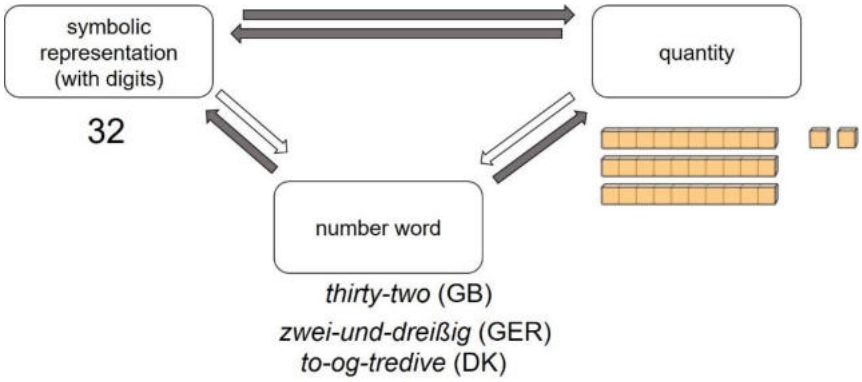

Figure 1: Representations of (multidigit) numbers and translations between the representations

Both the representations by digits and by quantities are independent of the child's language and are (more or less) the same internationally. When it comes to the formation of number words, however, different languages are very different from each other. While Asian languages in particular have a very regular structure, most European languages show some clear irregularities. This can include differences in the number words for ones and tens, but also differences in the order in which place values are given.

In Japanese, as in most Asian languages, the spoken number words have a highly regular structure and directly name the place value (32 is 'three ten two'). In contrast, most European languages show some irregularity in the number words and use specific names for the tens (Fromme, 2017). In English, some of the names for the tens differ from the names for the ones (e.g. 'thirty' instead of 'threety'). Some European languages construct (some) number words for the tens based on remnants of a base 20 system; for example, the French number word for 80 is 'quatre-vingts', meaning 'four (quatre) times twenty (vingt)'. The Danish number words for tens in particular have a high degree of irregularity. For example, the number word for 4 ('fire') differs significantly from the number word for 40 ('fyrre'), while there are clear similarities with the number word for 80 ('firs'). 'Firs' is a hidden remnant of a base 20 system ('four times twenty'), with the additional difficulty that 'twenty' does not appear as part of the number word at all (unlike in French). The same applies to the number word for 70 ('halvfjerds'). In this number, the prefix 'halv' (meaning 'half') encodes 'half times twenty'. Therefore, 'halvfjerds' means 'half four times 20'. "'Half four' is a half taken to four (of three), so it means 'three and a half' (as in German time counting, where 'half four' means 'half past three')" (Sun et al., 2018, p. 108). These construction principles, with many irregularities, are not comprehensible

to students learning Danish number words and therefore have to be memorised (Ejersbo & Misfeldt, 2015).

In addition, some languages show another peculiarity by inverting the position of the tens and ones in the (spoken) number words. This applies to German and Danish, for example. Both languages use a number word inversion by starting with the ones before saying the tens. 32 is 'two and thirty' ('zwei-und-dreißig' in German or 'to-og-tredive' in Danish). The syllable 'und' resp. 'og' means 'and' and is used to connect the number words for ones and tens.

Several studies have already investigated the influence of (verbal) number word formation on the learning of place value understanding and number processing. In general, studies show advantages of regular number word formation, as it is used especially in Asian languages. Most research in this area deals with English and/ or Asian languages (Habermann et al., 2020; Hewitt & Alajmi, 2023). However, some studies also consider languages with a number word inversion, mainly German (Clayton et al., 2020; Moeller et al., 2011; Steiner et al., 2021). As a result of their literature review, Larkin et al. (2024) highlight that "the alignment between the order of writing the digits and the order of naming the numbers, contributes to opportunities or difficulties in learning place value" (p.107). Languages with inverse number word formation have clear disadvantages compared to other languages. In their comparison of English- and German-speaking first graders, Clayton et al. (2020) found significantly more inversion errors (e.g., 35 instead of 53) in the German-speaking students, which also had an impact on mathematical competences in general.

However, it has not yet been investigated in detail whether number word inversion alone is responsible for these differences or whether there are other influencing factors. Previous studies tend to compare languages that differ in the number word inversion. Studies investigating languages that all use number word inversion are still lacking.

In our study, we compare German and Danish second graders' abilities in relating different representations of two-digit numbers. This is particularly interesting because both languages use a number word inversion, but the number words for tens are constructed in a fundamentally different way. In particular, we are interested in students' difficulties and errors when switching between number words, symbolic representations with digits and quantities, with a special interest in inversion errors and errors with the number words for tens. The main research questions are:

RQ1: Which difficulties and types of errors are visible in the translation processes?

RQ2: What are the differences between German (GER) and Danish (DK) students in this study?

Methods

Our study has an exploratory nature to get some initial indications as to whether there are differences between the two languages and whether further research might be worthwhile. We selected one school in an urban setting and one in a rural setting per country to reflect some diversity. However, the groups do not meet the requirement of being representative of the two countries. In total, 96 German and 84 Danish second graders participated in the data collection in autumn 2023. The tasks were given in a paper-pencil test and focused on translation processes between three different representations of numbers: as a symbolic representation with digits (e.g. '32'), as a (spoken) number

word (e.g. 'thirty-two') and as a quantity. For the quantity representation, we chose base ten blocks, presented as an iconic representation (with lines for the tens and dots for the ones), to allow students to make their own representations of the material in the paper-pencil test.

After the introduction of the tasks and the representations used (esp. the iconic representation of the quantity), the students worked on a total of 25 items with the four types of translation (dark arrows in Figure 1): (1) from (spoken) number word to symbolic representation (10 items), (2) from (spoken) number word to quantity (5 items), (3) from quantity to symbolic representation (5 items) and (4) from symbolic representation to quantity (5 items). The number words were given orally by the teacher in German or Danish, respectively, and the students were asked to write down the numbers or to draw them on a pre-structured sheet of paper. The two types of transitions to a (spoken) numerical word were excluded because they could not be implemented in a written test.

Results

Overall, the test (with a max. score of 25) is quite simple, resulting in ceiling effects. The children succeed in the vast majority of translations. The German (GER) students (N=96; M=23.35; SD=2.82) scored higher than the Danish (DK) students (N=84; M=22.32; SD=3.53). To compare the groups, the non-parametric Mann-Whitney U test was used, as there was no normal distribution in either group (Kolmogorov Smirnov $p<.001$). The results show statistically significant differences between the two groups (GER: Mdn=24.00, range=16; DK: Mdn=24.00, range=14), U=3260.0, Z=-2.303, p=.021, r=.17.

However, it is interesting to analyse the errors that occur in more detail. With regard to the required translation processes, both groups make the fewest errors in the translations from (spoken) number words to symbolic representations (errors in 5.1% of the respective translation processes in GER resp. 6.1% in DK) and the most errors in the translations from (spoken) number words to quantity (errors: 10.8% in GER resp. 13.8% in DK; Table 1). Low familiarity with the (iconic) representation of the base ten blocks may potentially contribute to the higher error rate.

Table 1: Percentage of errors in the respective translation processes

	GER	DK
number word → symbolic representation	5,1%	6,1%
number word → quantity	10,8%	13,8%
quantity → symbolic representation	5,6%	13,3%
symbolic representation → quantity	7,3%	12,6%

Even though this was not considered an error, the comparatively frequent use of mirrored digits by Danish students is particularly striking. While this notation was not used at all by the German students, the Danish students used mirrored digits in 8% of the transitions to a symbolic representation (8.5% for transitions from number words and 6.9% for transitions from quantities; Figure 2).

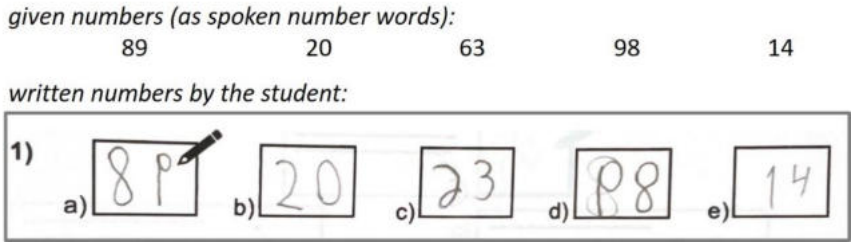

Figure 2: Examples for mirrored digits

In addition to the general errors, we take a closer look at two types of errors, namely *inversion errors* and *errors with incorrect tens*. Several studies show that inversion errors (e.g. 35 instead of 53) are frequent and very typical errors made by German students. The main reason for this is considered to be the inversion of number words (starting by saying the ones before the tens; Clayton et al., 2020).

In our study, *inversion errors* occur more often in German students (number of inversion errors in the 25 tasks: M=0,93; SD=2.31) than in Danish students (M=0,23; SD=0.64), but the difference is not significant (GER: Mdn=.00, range=13; DK: Mdn=.00, range=3), U=3645.0, Z=-1.734, p=.083. Nevertheless, inversion errors are the most frequent type of error made by German students (in 3.7% of tasks, corresponding to 54.4% of errors in total; Table 2). This error is particularly relevant in translations starting from the spoken number word. Danish students, on the other hand, make such errors extremely rarely (in 0.9% of the tasks, corresponding to 8.7% of the errors in total), despite the fact that inversion of number words is also used in Danish language.

Table 2: Percentage of inversion errors in total and in respective translation processes

	GER	DK
inversion errors in total	3,7%	0,9%
number word → symbolic representation	4,9%	1,1%
number word → quantity	6,3%	0,7%
quantity → symbolic representation	0,6%	1,0%
symbolic representation → quantity	1,9%	0,7%

It is also interesting to look at the distribution of this error among individual children. Most students in both countries (80.2% in GER and 88.1% in DK; Table 3) do not make this error in any of the tasks. However, in contrast to the Danish students, there is a relatively high proportion of students in Germany (9.4% in GER resp. 1.2% in DK) who make this error very often (in more than 10% of the tasks).

Table 3: Frequency of number inversion errors among individual students

	GER (n=96)	DK (n=84)
0%	80,2%	88,1%
> 0% - 10%	10,4%	10,7%
> 10%	9,4%	1,2%

Because of the high level of irregularity within the Danish number words for tens, we also focus on *'incorrect tens'* as the result of a task. This category is for answers with the correct number of ones but an incorrect number of tens. Overall, there are only a few errors in this category, and almost exclusively among the Danish students (errors in 1.1% of tasks in DK and 0.1% in GER, respectively). Some of the errors indicate incorrect counting of tens in the iconic representation, as the result differs by 1 from the correct result. However, most errors in this category occur in transitions from spoken Danish number words. This may be due to the specific irregularities of this language.

Figure 3 shows two examples for errors with a close connection to peculiarities of the Danish number words for the tens. In the given spoken number word 'tre-og-**tres**' (63; translated: 'three and three times twenty') in the first example, the Danish number word 'tres' for 60 shows a strong correspondence to the number word 'tre' for three. This similarity could be the reason why the student uses the digit 3 for both the ones and the tens and thus writes down 33 (originally 'tre-og-tredive' in Danish).

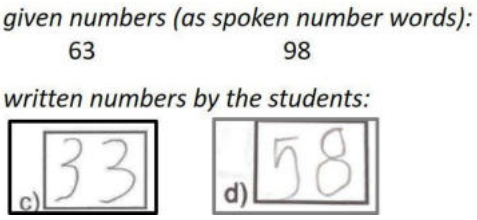

Figure 3: Examples for 'incorrect tens'

The second example shows a similar error. The given spoken number word is 'otte-og-halv**fems**' (98; translated: 'eight and half five times twenty') and is incorrectly written by the student as 58 (originally 'otte-og-halvtreds' in Danish). 'Fem' is the Danish number word for five and is also a component of the number word 'fems'. This probably leads to writing the digit 5 for the tens.

Discussion

The aim of our study was to compare German and Danish second graders' abilities and difficulties in relating different representations of two-digit numbers and to analyse the errors that occurred. The exploratory design of the study limits the generalisability of the results. Nevertheless, the results can be seen as a first indicator of language-related differences between the two countries. Although students in both countries get a high percentage of tasks correct, there are statistical significant differences between the two groups in the success rate. The Danish students make on average more errors compared to the German students. A more detailed analysis of the errors made also shows differences between the two groups. The errors made by the German students show a clear pattern.

By far the most common errors made by the German students are inversion errors (54.4% of all errors). Our data is consistent with the findings of Clayton et al. (2020), who found this error to be particularly relevant for German first graders.

Inversion errors do not play a comparable role for the Danish students in our study (with a proportion of 8.7% of all errors), although number word inversion is also made in Danish. This is particularly interesting because the reason for inversion errors in German is mainly seen in this specific characteristic of number word formation (Clayton et al., 2020; Steiner et al., 2021). However, our study shows that other aspects of number word formation also seem to have a relevant influence on this type of error. This also makes our results interesting for studies on other languages. It seems necessary to take a detailed look at the respective characteristics of number word formation and their differences in different languages (Larkin et al., 2024). One possible explanation for the fact that inversion errors occur less frequently among Danish pupils could be the peculiarities of the number names for tens. Whereas in German, for example, there are clear sound similarities between 'fünf-und-dreißig' (for 35) and 'drei-und-fünfzig' (for 53), this is not the case in Danish ('fem-og-tredive' for 35 and 'tre-og-halvtreds' for 53). The unambiguous distinction between the Danish number names for tens and ones may be a significant factor contributing to the rarity of confusion. However, it is important to note that the difference between the mean numbers of inversion errors in the two groups is not statistically significant. This is probably due to the large number of students in both groups who did not make any inversion errors at all. Nevertheless, it is noticeable that a relatively large proportion of the German students (9.4%) made inversion errors in more than 10% of the tasks, whereas this was the case for only one Danish student.

Unlike the German students, the Danish students do not show any clear patterns of error. Yet it turns out that the formation of number words for tens is a particular challenge for Danish students (Ejersbo & Misfeldt, 2015). Similarities between number words (e.g. 'fire' for 4, 'fyrre' for 40 and 'firs' for 80) seem to encourage confusion between these numbers.

For our further research, we intend to conduct an interview study to provide a more comprehensive and nuanced qualitative analysis of the errors in the two countries. Including students with a different first language could be particularly interesting. It also seems sensible to extend the tasks by including translations into spoken number words and further aspects of place value understanding, particularly the regrouping principle and the place value principle (Jensen et al., 2024). This may assist in the identification of the underlying causes of the discrepancies in the errors of German and Danish students, thereby potentially facilitating the development of intervention strategies. In this context, intervention studies also appear to be valuable. An interesting approach could be to introduce regular number words in addition to the standard number words, as proposed in the design based research project by Ejersbo and Misfeldt (2015), and thus encourage students to reflect on the formation of number words.

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General explanations–conditions and role in the learning process of a child with special difficulties in learning mathematics

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Children with particular difficulties in learning mathematics need to develop conceptual and lexical understanding to achieve in arithmetic. General Why-explanations play a central role in this understanding. This article presents a study in which a third-grader with difficulties in learning mathematics received weekly lessons to support mathematics learning over a period of 15 weeks in addition to regular lessons. The study focuses on the link between the task format used and the language actions encouraged by the teacher to stimulate general Why-explanations. The results show that the teacher's stimuli are of central importance. And they indicate that these explanations can be regarded as a prerequisite for transferring the recognized relationships to the next remedial session as well as to higher number ranges and to tasks with varying characteristics.

Keywords: Going beyond counting, reasoning skills, conceptual understanding, low-achieving students in mathematics.

Introduction

Learning mathematics goes far beyond producing correct results, also for children with particular difficulties in learning mathematics (e.g., Gaidoschik et al., 2021). However, for children to develop a deeper mathematical understanding beyond performing individual tasks, the learning process must increasingly focus on general observations and explanations that are independent of tasks. On the one hand, this involves considering the link between the child's content-related (conceptual) and lexical learning trajectories (macro-scaffolding) in conjunction with stimuli provided by the teacher (micro-scaffolding). On the other hand, it is of interest to consider the role general explanations play in the learning process of a child with particular difficulties in learning mathematics.

Theoretical framework

In first grade, going beyond counting is seen as the central goal of mathematics lessons (e.g., Gaidoschik et al., 2021). To this end, children must overcome both content-related and relationship-oriented challenges. Content-related challenges include developing a comprehensive understanding of numbers and a basic understanding of arithmetic operations (operational understanding and strategic means[1]). Relationship-oriented challenges include developing an understanding of the relationships between numbers, terms, and tasks (Rechtsteiner & Rathgeb-Schnierer, 2017). Ellemor-Collins and Wright (2009) conclude that number structuring represents a bridge between counting

[1] Strategic means are distinct devices used to modify problems to make them easier. They can be flexibly combined in a solution process and include, for instance, composing and decomposing, modifying a problem, deriving the solution from a known fact and using analogies (e.g., Rathgeb-Schnierer & Green, 2013).

and non-counting solution strategies, fostering the process of going beyond counting. Hence, the discovery of mathematical relationships is regarded as one of the most important prerequisites for the ability to flexibly solve unknown tasks (Purpura et al., 2016). Studies on early mathematical education also point to links between the development of numerical skills and children's understanding of patterns (e.g., Wijns et al., 2021). In particular, children with difficulties in learning mathematics partially perceive task characteristics, but then resort to counting strategies to solve tasks (Rechtsteiner & Scheffknecht, 2023). Against this background, it is crucial to focus on teaching children to understand the relationships between numbers and tasks as well as their characteristics in connection with the challenges posed by mathematics-related content in and outside the classroom.

"Zahlenblick" and its development – structure sense

The concept of developing "Zahlenblick" focuses on both content-related and relationship-oriented challenges. "Zahlenblick", as the learning outcome, is the competence to recognize problem characteristics, number patterns, and numerical relationships immediately, and to use them for solving a problem (Schütte, 2004). In some aspects, it can be equated with the terms number sense and structure sense. These concepts are defined in two distinct ways: (1) either as a result of experience-based development (e.g., Linchevski & Livneh, 1999) or (2) as an inherent skill (e.g., Dehaene, 1999). A comparison of number sense, structure sense, and "Zahlenblick" reveals that the meaning of number and structure sense, as an acquired skill, is quite similar to the notion of "Zahlenblick", which can be developed through specific activities. To facilitate understanding, we use the notion of structure sense in this text, as it highlights the focus on structures.

To develop this kind of structure sense, the activities are designed in a way that the pupil's impulse to immediately start calculating is delayed so that he/she recognizes the characteristics, patterns, and numerical relationships of a problem before starting to solve it (Schütte, 2004). Students are also deliberately encouraged to recognize number patterns, problem characteristics, and relations between numbers and problems, as well as to sort and arrange problems by using structural relations. In addition to a specific emphasis on content, all activities include cognitively challenging questions to provoke students' thinking and reflection (Rechtsteiner & Rathgeb-Schnierer, 2017).

Various studies have shown that activities that highlight number and task problems as well as their relationships foster the development of flexible arithmetic competences in all children in a class (e.g., Heinze et al., 2015; Körner, 2024). A study on the development of arithmetic skills in first-grade children with difficulties in learning arithmetic (Rechtsteiner & Rathgeb-Schnierer, 2017) showed that structure sense helped children to both go beyond counting and perceive numbers and task characteristics. Furthermore, children taught with such activities predominantly developed flexible arithmetic competencies.

The activity of "sorting": This study focuses on sorting activities. Sorting activities can be divided into four phases (Rechtsteiner & Scheffknecht, 2023): (1) justifying sorting, (2) developing a solution path, (3) finding further tasks, developing a general description and a Why-explanation, as well as (4) inventing further tasks.

Phase 1: Justifying sorting: In the first phase, children are encouraged to assign term cards with addition tasks, without solving them, to subjective or objective categories. This is followed by a discussion justifying why they sorted the tasks in a specific way.

Phase 2: Developing a solution path: First, the solutions to the tasks that have already been memorized are named by the student, and any strategic means that have already been mastered are discussed. Based on this, the student develops a new solution path and examines whether a task he/she has already memorized can help him/her to solve a task for which the student still needs to do the counting.

Phase 3: Finding further tasks: After a new strategic means has been developed, the child is encouraged to identify any further tasks with the task characteristic of the task just solved in the "I count" category. The aim is that he/she recognizes the same relationships previously perceived in a single task in the other tasks at hand. Then, the general characteristics of this type of task can be formulated to allow grouping on an algebraic level.

Phase 4: Inventing tasks: The child invents and notes down other tasks with the same characteristic. In this way, the more reception-oriented grouping phase (3) moves to the active discovery phase.

In all phases, the mathematical aid (e.g., the Abaco) can be used to clarify and produce Why-explanations (Klein, 2009) regarding the implementation of the relationships under discussion. In this way, links between the levels of representation can be made visible and emerge.

Linking conceptual learning trajectories and lexical learning trajectories

The development of conceptual understanding always takes place based on so-called "hypothetical learning trajectories" (Pöhler & Prediger, 2015, p. 1697). These trajectories describe "hypothetically sequenced learning development" (Götze, 2019, p. 101). Pöhler and Prediger (2015) differentiate between a "conceptual learning trajectory" (p. 1698) and a "lexical learning trajectory" (p. 1698), focusing on the link between these two trajectories. The conceptual learning trajectory can be described throughout the phases of the above “sorting” activity, in which the development of strategic means increasingly involves producing a Why-explanation (Klein, 2009). In terms of the lexical learning trajectory, a distinction can be made between language acts and language tools. Language acts focus on descriptions that are fundamental for developing understanding, ranging from example-based and general descriptions to justifications (Götze, 2019). In contrast, linguistic devices are concrete formulations for technical words and parts of sentences. By linking the perspectives of these two trajectories, the development of theoretical mathematical competency can be described and analysed. Various empirical studies have reconstructed "individual pathways" (Pöhler & Prediger, 2015, p. 1698), showing that children do not follow the theoretically formulated learning trajectories encouraged by teachers all the way to the level of general explanation (Götze, 2019). For this reason, they need support in the form of discursive dialogue. Häsel-Weide (2016) also conclude that targeted stimuli are of considerable importance for learners to see and interpret relationships and thus explain the Why. While theoretical sequential descriptions are classified as macro-scaffolding, support in the form of discussions is referred to as micro-scaffolding.

Methods

The present study considers the last step of the conceptual and lexical learning trajectory, explaining the Why (the connection between task characteristics and strategic means) and generating a related general description in the context of a "sorting" activity. The focus is on the research question: Which situational conditions at the level of the conceptual and lexical learning trajectory, together with support in the form of discursive dialogue, facilitate or hinder a child's description of task-independent characteristics, and what role do they play in the learning process?

The data set consists of ten videotaped remedial sessions with Emi, a third-grader with particular difficulties in mathematics (in the development of number concepts and the competence of calculating going beyond counting) diagnosed with a psychosomatic condition before the study.

The lessons to support mathematics learning (remedial sessions) provided by one of the authors were based on the content- and relationship-oriented challenges and consisted of activities to develop structure sense. At the beginning, the focus was on developing the concept of numbers; this was followed by work on understanding arithmetic operations. The part presented here focused on developing strategic means in conjunction with recognizing and using characteristics and relationships. Only the activities of sorting are considered to allow comparability. The ten videotaped remedial sessions were transcribed, sequenced, and divided into meaningful segments. As a rule, the sequences (S) contain similar content in terms of mathematics didactics (within an activity).

An interpretative research approach was used to interpret the reconstruction of the child's statements (Krummheuer, 2004). For this purpose, we carried out a two-step analysis combining an interactionist and epistemological approach (Steinbring, 2000). The two perspectives complemented each other in the data analysis, making conclusions and interrelationships possible (Korten, 2020). As part of the interaction analysis, alternative interpretations of the individual utterances and turn-by-turn analyses were carried out to analyse (1) the learning processes (Rechtsteiner & Scheffknecht, 2023) and (2) the scenes (Sc) where the child gave general explanations. We compared the different scenes to identify similarities and differences and to reconstruct interpretations to provide explanations (Krummheuer, 2004). These analytical steps were combined with an epistemological analysis to classify the conceptualization (Rechtsteiner & Scheffknecht, 2023). The interpretation was carried out against the background of learning trajectory theory, focusing on the research question. Next, we compared scenes from various remedial sessions in which general explanations were provided or not provided. Drawing on the epistemological triangle, we reconstructed the child's conceptual developments on both levels—regarding the strategic means and the relationship-oriented challenges—in the reference context (Häsel-Weide, 2016).

An analysis of Sc IV, S3, Remedial Session 9 (RS) is presented below as an example. It is described initially as part of the overall remedial process and then as part of RS9. Following an individual diagnosis of Emi's learning needs, the concept of numbers was taught first. From RS5 onwards, she carried out sorting activities. Up to RS9, strategic means were introduced, and she was assigned tasks requiring her to make connections, exchanges, and analogies. Tasks involving a summand of nine did not play a role before RS9. Here, Emi initially sorted tasks in the number range 20 according to subjective categories (Phase 1). She spontaneously tackled related tasks involving doubling and

decompositions of ten, such as 7+6 and 6+6/ 7+7 or 7+4 and 6+4. In S3, the focus turned to tasks with nine. The teacher prompted: "I was wondering if you know what ‘ten plus seven’ is by heart?" (Phase 2). Emi then explained that the task would help her to solve the task “7+9” from the "I have to count" category. Whilst the teacher wanted her to solve the task based on connections with Tasks 10 and 7, Emi solved it by making changes in the opposite direction ("Um, from the nine to ten it's one, then you subtract one from the seven, then you add ten, and then it's sixteen."). Later, after using the visual aid, Emi called this solution a "hopping trick". Next, she was asked to select further tasks that could be solved in the same way from the category "I have to count" (Phase 3). The following excerpt from the transcript starts at the end of this phase:

66	Teacher	Super! And is there another one where that works?
67	Emi	Hm. Yes, nine plus four.
68	Teacher	Hm (affirmative), and then how is it with that one?
69	Emi	Then you take one away from four (.) then you have three, add it to the nine, then you have ten. And then you add the three to the ten and then you have thirteen.
70	Teacher:	What does a task have to be like for that to work?
71	Emi	(...) Um, it always has to be before ten, so nine.

Sc IV began with Emi being asked whether any other tasks could be calculated using the strategic means. After reflecting briefly, she named the task nine plus four, which was confirmed by the teacher, who said "hm" and asked how the solution to the task worked. This can be interpreted as a request for Emi to check, who then explained her solution path. She subtracted one from four and increased the nine to ten accordingly, resulting in thirteen. In Turn 70, she was asked to name the characteristics that a task must fulfil to be solved in this way. Emi stated the characteristic of proximity to ten ("before ten") and specified this by naming nine. From an epistemological point of view, Emi explains the general structure of tasks as their proximity to ten (Figure 1). Her explanation is no longer based on examples, but refers to a task-independent characteristic. However, she lacks the linguistic means to describe this characteristic accurately, as she does not use the term summand, but instead chooses the pronoun "it".

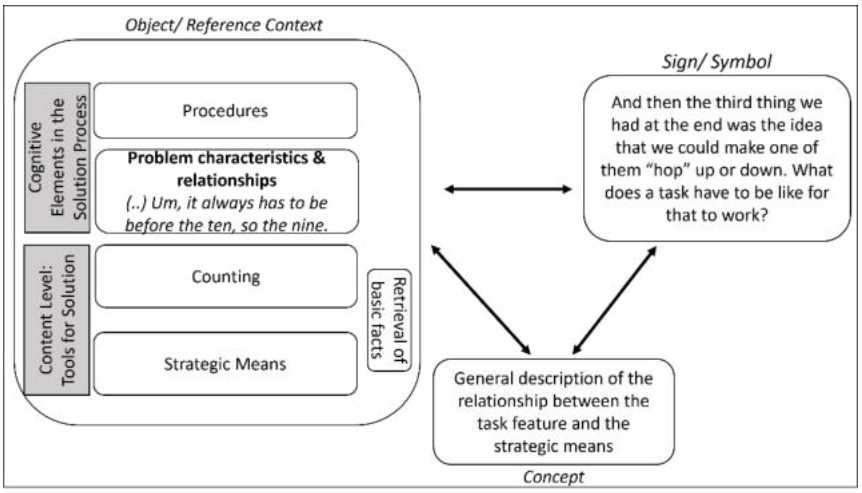

Figure 1: Epistemological triangle, Scene IV, Turns 70-71

Based on analyses of RS6 (Rechtsteiner & Scheffknecht, 2023[2]) and the present scene RS9, we compared the comparisons, interpretations, and reconstructions with a focus on the general Why-explanations relating task-independent characteristics and strategic means within the sequence and with sequences from other RS. All compared sequences were interpreted and reconstructed in the same way as for Sc IV, S3 in RS 9 (see above).

Results

Like in the scene above, the development of a general explanation after completing Phases 1-3 of the sorting activity was also evident in S6, RS6. In both sequences, after the development of the strategic means, Emi was asked to identify further tasks with similar characteristics to the completed tasks. This required different teacher prompts in each case (e.g., "So, among the tasks that you have to count, can you see another one that goes with the 'hopping trick'?" (RS9/S3)). She did not seem to have sufficient conceptual understanding to answer such questions at first. However, after overcoming the initial hurdle, she managed to find other tasks with the same characteristics and to justify their relationship on a lexical level by describing the comparable solution path. Hence, she succeeded in understanding the connection between the task characteristics and the developed strategic means on an algebraic level. However, Emi required teacher prompts to identify and describe the tasks with similar characteristics, reach an algebraic understanding, and formulate a general explanation on her own (e.g., Turns 66 and 70). Furthermore, in analogous scenes, the child did not give general explanations after identifying tasks with the same characteristics without explicitly being asked to do so (e.g., RS9. S1 and S2). Various sequences (RS7, RS11, RS12) indicate that after the child had explicitly formulated a general Why-explanation, she was able to spontaneously recognize the task characteristics and activate the strategic means independently in the next RS. In contrast, in some sessions, she did not recognize the task characteristics, and the strategic means had to be clarified again. This was the case when she had not acquired a sufficiently deep understanding of them to identify further tasks, clarify the general context, and thus provide general explanations (e.g., RS5 to RS6). However, when the child formulated Why-explanations, she was able to transfer the relationship between the generally described task characteristics and the strategic means to the higher number range when prompted by the teacher (RS11). She was also able to perceive slightly varied task characteristics and to adapt the strategic means to them independently (RS11; spontaneous answer to the question as to whether the solution was also possible for the task 65+28: "So from twenty-eight to thirty, there are two, then I subtract two from sixty-five, that's sixty-three, yes, and then you can calculate that.").

Discussion

The interplay between conceptual knowledge and the child's language actions makes it clear how closely they are connected (Pöhler & Prediger, 2015). As described above and in detail by Rechtsteiner and Scheffknecht (2023), scenes in which general explanations are given occur systematically throughout the phases of the sorting activity, which represent the conceptual learning

[2] The analysis in this article focuses on the potential of activities that highlight structures for a student with difficulties in learning mathematics.

path. However, Why-explanations have to be elicited by the teacher (Götze, 2019). These structure-oriented Why-explanations of the relationship between a strategic means and task characteristics appear to be central to the further learning process. Furthermore, the findings indicate that the formulation of general Why-explanations is essential for learning (flexible) calculation competences, and that the prior identification of further tasks with the same characteristics is an important precursor to developing task-independent Why-explanations. At the same time, the child's linguistic resources increase throughout the learning process, albeit not sufficiently to allow the formulation of such explanations. Even if knowledge of the appropriate language does not contribute to achieving a conceptual understanding of content (Götze, 2019), it can allow a more precise description of mathematical relationships.

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Tree graph to improve word problem solving involving rational numbers

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This exploratory (pilot) study aimed to analyse the use of tree graphs to develop problem-solving skills in word problems involving ratios and percentages and evaluate students' performance using software implementing this method. Thirteen students from a public university in Mexico participated. The intervention included three sessions using pencil and paper, applying the analytical readings method by representing relationships between quantities through tree graphs. In a fourth session, students solved similar problems using specialised software designed to implement the method and track their problem-solving processes. Results showed that using tree graphs facilitated students' understanding and improved their performance, particularly in problems involving ratios, highlighting the method's potential for teaching mathematical concepts in diverse educational contexts.

Keywords: Rational numbers, word problem solving, tree graph, graphical representation.

Introduction

Problem-solving is a relevant topic in the educational field, as one of the main goals of all educational systems is to help students improve their ability to solve the problems they and their societies will face. So much so that international guidelines, set by the National Council of Teachers of Mathematics (NCTM, 2000), indicate that problem-solving involves mathematical tasks with the potential to provide intellectual challenges that enhance students' understanding, logical-mathematical development, and have applications in other curriculum areas.

Problem comprehension and resolution take on special significance as they arise naturally when reading the financial section of a newspaper, after election coverage, or while navigating social media (Roy & Roth, 2015).

From the early school years, graphical representation is used to aid in understanding problems (Brenner et al., 1997; Peng & Lin, 2019), and research on this topic is of interest from both educational and research perspectives (NCTM, 2006).

This study underscores the need for tools that assess students' graphical problem-solving processes and reading comprehension through mathematical tasks. To address this, it focuses on two key objectives. First, it aims to evaluate the use of tree graphs as a representation for developing problem-solving skills, particularly in word problems involving ratios and percentages. Second, it seeks to assess students' performance when solving problems using AWPSolver (UV-SW-202475R).

Theoretical background

The theoretical foundation of this work combines word problem-solving, the analytical readings method as a proposed approach for solving problems (López-Iñesta et al., 2021), and rational numbers as the underlying content of the problems. A word problem is a verbal description of a situation in which a question is posed, and its answer can be found by applying mathematical procedures to the numerical data provided (Verschaffel et al., 2020). The process of solving word problems is understood as the student's mental activity from encountering the problem to completing the task. Based on Polya's (1957) ideas, Puig and Cerdán (1988) proposed six phases in the problem-solving process: reading, understanding, translation, calculation, solution, and verification.

One variable associated with the word problem-solving process is the content, which relates to the mathematical meaning of the problem. In this study, the content variable is connected to rational numbers, focusing on their interpretation as percentages and ratios.

Based on the analysis-synthesis and Cartesian methods (Puig & Cerdán, 2014), the analytical readings method is proposed for solving both algebraic and arithmetic word problems. It involves five steps: reading the problem, forming a dictionary of quantities, constructing a tree graph, performing calculations, and verifying the solution. Since word problems often involve multiple steps and relationships, there is no single analytical reading or unique solution process.

In the first step, an initial reading of the complete problem is conducted. Then, the problem is read sentence by sentence, identifying the given data and the verbs associated with them. It is also important to identify the main unknown, i.e., what needs to be calculated. To form the dictionary of quantities, known and unknown variables in the word problem are identified. This information is organized into a table with three columns, corresponding to the second step of the method. For example, for this problem

Problem 1. Normally, a liter of alkyd enamel is sold at the hardware store for 310 pesos, but this month they have a 10% discount on all their products. Additionally, a further 20% discount is applied to all purchases for regular customers. If Ana is a regular customer and needs to buy 10 liters of alkyd enamel this month, how much will Ana pay? Solution: Ana will pay 2,232 pesos.

Table 1 present a dictionary of quantities. Note that, this table depend on the used analytical reading and can be different. The third step is to construct the tree graph by following the analysis-synthesis rules, so that all the numerical data from the problem and the main unknown are obtained. A tree graph has unique decomposition properties that allow for a visual representation of the strategy followed in the problem-solving process (Roy & Roth, 2015). We emphasise that the tree graph serves as a problem-solving tool rather than a pre-established solution. Much like how textbooks guide learners with intermediate questions, the tree graph helps break down the problem into manageable steps. Students are encouraged to formulate their own sub-questions as they build the diagram, ensuring the process remains exploratory rather than purely procedural.

A tree is a data structure consisting of nodes connected by edges, with the unique property of containing no cycles. For the analytical readings method, we specifically use a rooted binary tree,

where there is an upper node, commonly known as the tree's root, and each node is linked to either zero or two successive nodes (see Figure 1).

It should be specified that the upper node represents the main unknown of the problem, and the construction of the tree graph starts from this main unknown. If the problem is arithmetic, only quantities will appear at the base; if it is algebraic, unknowns will be present at the base. The problem will be solved from the base up to the upper node.

Table 1: Dictionary of Quantities for Problem 1

Information	Value	Description
Data	310	Original cost in pesos per liter of alkyd enamel.
Data	10%	Discount applied on all the products.
Unknown	U	First discount in pesos.
Unknown	Z	Cost in pesos per liter of alkyd enamel after the first discount.
Data	20%	Additional second discount applied to the already reduced price.
Unknown	V	Additional second discount applied to the already reduced price in pesos.
Unknown	Y	Cost in pesos per liter of alkyd enamel after the second discount.
Data	10	Liters of alkyd enamel to be purchased.
Unknown	X	Pesos that Ana will pay.

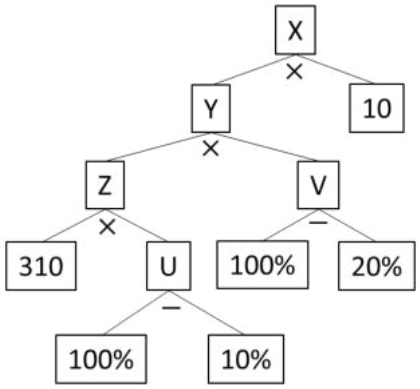


Figure 1: Tree graph of the Problem 1

Methods

This research is framed as an exploratory (pilot) study, intended to identify preliminary trends. It follows a qualitative methodology and forms part of a broader project that analyses the process of learning fractions through educational software, AWPSolver; that incorporates a problem-solving methodology using tree graphs. The software enables tracking the user's solution path, providing detailed performance analysis.

Sample

The study involved 13 students from the mathematics undergraduate program at a public university in Mexico. This group consists of students enrolled in various semesters of the program, all of whom share a common interest in teaching.

Intervention

In the first session, prior to explaining the analytical readings method, students solved isomorph problems to last session to assess their initial performance in this type of task. The session continued with a printed document simulating a student's solution to a word problem, which served as a discussion prompt. Students were divided into teams of 3 or 4 members to discuss the method, followed by a group discussion moderated by the instructor. The instructor explained the phases of the word problem-solving process proposed by Puig and Cerdán (1988) and outlined the steps of the analytical readings method. Students shared observations and questions. To reinforce learning, two additional problems were assigned; one student solved each on the board with feedback from classmates and instructor guidance.

In the second session, students reviewed the analytical readings method by solving a problem in teams and comparing results through group discussion. The instructor introduced different representations of rational numbers, followed by team-based problem-solving using the method. Students worked on four word problems involving fractions, ratios, percentages, and decimals. Each team's dictionary of quantities, tree graph, and final solution were compared and discussed.

In the third session, the relationship between the analytical readings method and the word problem-solving process was discussed in a debate moderated by the professor. Lastly, an anonymous survey was conducted to gather students' opinions on the use of the method.

The fourth session focused on using AWPSolver, where students solved three word problems using the analytical readings method. This report analyses their performance on two problems (percentage and ratio).

Instruments

The two problems analysed in this study both involve the use of rational numbers. The first problem specifically uses percentages (Problem 1 detailed in the theoretical framework). However, Problem 2 explicitly involves ratios, as shown below:

> *Problem 2. Concrete needs to be mixed for a construction project. First, sand and gravel are mixed at a ratio of 2:5. Then, cement is added to this mixture at a ratio of 1:7. If one ton of concrete is required, how many kilograms of sand are needed? Solution: 250 kilograms of sand are needed.*

It should be noted that, according to initial student feedback, Problem 1 is a familiar type of problem, whereas Problem 2 is not commonly encountered, previously, but it is done in the three first session.

The AWPSolver used allows for the implementation of tree graphs, as well as data collection regarding the time the user spends reading, operating, or constructing the graph. As shown in Figure 2a, the problem is presented with its statement shaded, which must be clicked by the student in order

to read it. As seen in Figure 2a, students must insert nodes by clicking "Node" and dragging them into the solution window. The user must fill in all the necessary information: the literal part (e.g., "Children in the park" or "Children going home"), whether the value is unknown (?), or known (value), and if unknown, the operation required to calculate it, as well as the connection between the quantities that allows the unknown value to be determined. Finally, note that the tool does not allow students to submit the solution (Submit) if there is something incomplete and will notify them if this is the case.

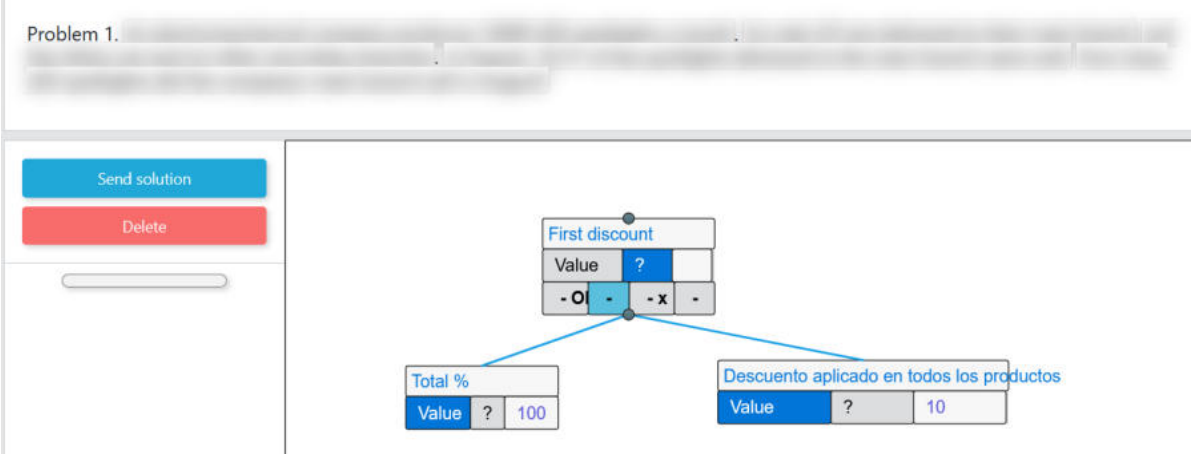


Figure 2: Task Implementation in AWPSolver. Tree graph construction process

In addition to serving as a data-collection tool for researchers, AWPSolver offers educational benefits to students: 1) allow rapid restructuring of branches, letting users easily move or edit nodes; 2) log the user's step-by-step decisions, enabling reflection and deeper insights into the problem-solving process; 3) save time otherwise spent erasing and redrawing, which can enhance motivation and reduce frustration; and 4) integrate modern ICT competencies by encouraging students to work in a digital environment—highly relevant to contemporary educational settings.

Research Variables

Thus, four research variables are extracted to address the study's goals. To answer Goal 1, which is to evaluate the use of tree graphs as a method for developing skills and acquiring knowledge, particularly in problems where rational numbers as ratios are involved, the following variables are used: a) success probability, indicating whether the solution is correct; b) reading time per word (tw_P1, tw_P2) and graph construction time (tg_P1, tg_P2), showing if the challenge lies in reading or building the graph; and c) graph construction moment (mg_P1, mg_P2), indicating if continuous reference to the problem statement is needed.

Times are relativized by dividing specific times by each student's total time. This accounts for differences in time spent reading or constructing graphs. The graph construction moment is calculated as the geometric mean of interventions on the graph: values near 0 indicate construction before detailed reading, around 0.5 indicate simultaneous reading and construction, and near 1 suggest construction after reading.

To address Goal 2, assessing students' performance using AWPSolver, an analysis of their different analytical readings will be conducted. Researchers will identify all possible readings and associated tree graphs to determine where students encounter the greatest difficulties.

Time is critical because constructing a tree graph step by step requires learners to reflect on each decision they make. In our study, we measured both reading and graph construction times to determine whether difficulties arose from interpreting the text (requiring repeated reading) or from structuring the solution (requiring frequent modifications to the tree). By analysing these times, we gain deeper insight into students' cognitive processes and their approach to problem-solving.

Results

The results are organized by goals, dividing this section into Goal 1 and Goal 2.

Goal 1

Prior to using AWPSolver, the researcher evaluated these isomorphic problems, with success rates of 84.6% for percentages and 8.3% for ratios. After intervention, for Problem 1, there is a success rate of 92.3%. The graph construction and reading are done jointly (Figure 3b), although it appears that the graph is somewhat anticipated before the reading (0.44±0.10). This is also observed alongside Figure 3a), which shows that little time is needed per word to read the text (0.39±0.22).

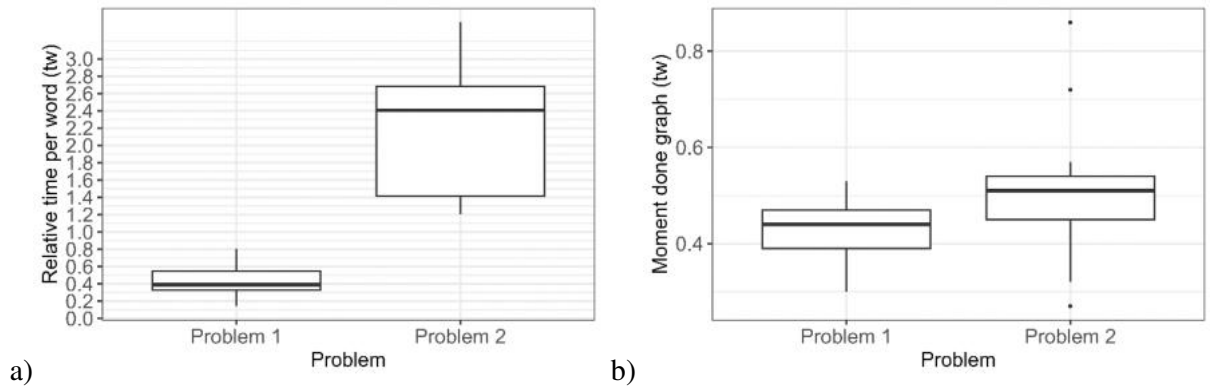
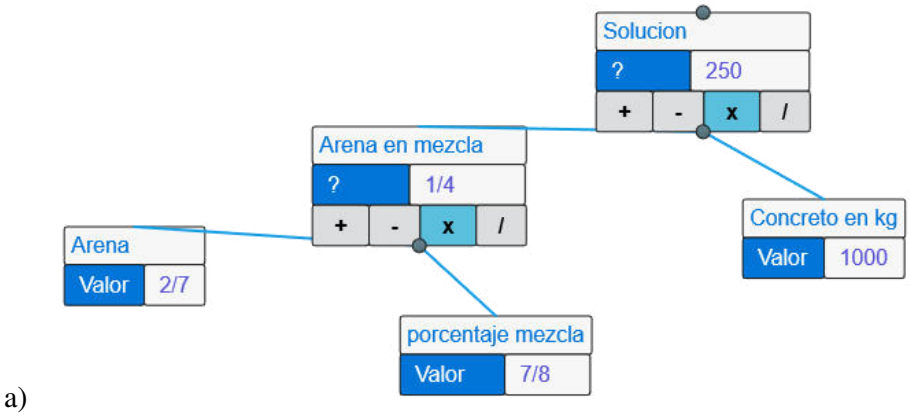


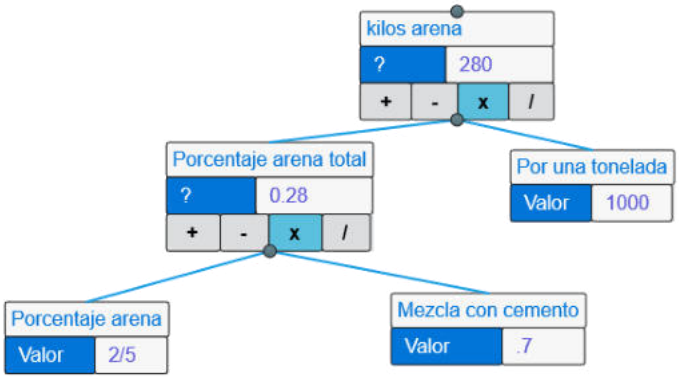
Figure 3: a) time per word; b) moment done graph

Goal 2

Due to space constraints, we focus the study on the graphs created by the students for Problem 2, as it is unfamiliar to them. The use of one graph predominates (Figure 4a), although it should be noted that five possible graphs were identified.



a)



b)

Figure 4: Tree graphs identified in the students' solutions

Among the students who provide incorrect solutions (6/13), three indicate that they do not know how to do it. Another three try to apply the graph shown in Figure 4a), but their difficulties come from not correctly identifying the fractions (for example, Figure 4b).

Discussion and conclusion

This study shows that the analytical readings method, which relies on tree graph representations, allows for the development of skills to implement this new methodology and improve the resolution of a particular kind of problems with the rational as a ratio.

Through an experiment evaluating a tree-graph representation methodology and involving two evaluation phases—the first using pencil and paper and the second using AWPSolver, an educational software—two types of problems were assessed: one involving percentages and the other involving ratios. The former is common in classrooms, while the latter is less familiar to students. High success rates were achieved with AWPSolver, suggesting that the method facilitated the assimilation of the concept of ratio, previously unfamiliar to students. Consistent with the studies by Brenner et al. (1997) and Peng and Lin (2019), the representation enhances students' performance in problem-solving.

The results indicate that for the problem familiar to students (percentages), they perform a quick reading of the statement and construct the graph simultaneously with reading, often anticipating the graph construction process. Nevertheless, for the problem involving unfamiliar concept (ratio), the

reading of the statement was slower, which aligns with research such as Rouet and Passerault (1999), who emphasize that more reading time is needed for tasks involving more complex mental processes, with graph construction lagging behind the reading.

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Pupils with severe difficulties in learning arithmetic: An educational challenge for secondary schools

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The paper addresses the issue of how many pupils in the transition from primary to secondary level have such severe difficulties in learning arithmetic that specific support in secondary school appears to be indispensable. A study is presented in which the scores of fifth-graders in an arithmetic test have been interpreted by experienced teachers. The findings show that the need for support is considerable, but also strongly dependent on the type of secondary school. The results highlight the challenge for the school system to establish structures to identify these pupils, diagnose their difficulties in a differentiated manner, and support the pupils appropriately.

Keywords: Difficulties in learning arithmetic, dyscalculia, numeracy, secondary school.

Introduction

Over the past 25 years, international assessment studies such as TIMSS and PISA have repeatedly revealed that a significant proportion of secondary school pupils in Germany have insufficiently developed mathematical competences. In PISA 2022, for example, 30% of fifteen-year-olds in Germany did not reach competence level 2 (out of 6 levels) in mathematics; these pupils are considered to be particularly at risk of experiencing difficulties in education, employment, and social participation (Diedrich et al., 2023). To find possible approaches to improving this situation, in this paper we focus on secondary school pupils who have not succeeded in developing a sound understanding of natural numbers, the place value system, and arithmetic operations in their first three years of primary school. Without such a basic understanding of arithmetic, these pupils are significantly hindered in achieving the various learning objectives of secondary school mathematics. This paper is, in particular, a statement in favour of becoming more sensitive to pupils at secondary level with severe difficulties in the field of arithmetic. The school system is challenged to diagnose their difficulties in differentiated ways and to set up offers for individual support.

Theoretical Background

There are a variety of terms and concepts to describe particular difficulties in the field of arithmetic, such as developmental dyscalculia, severe numerical learning problems or numeracy disorder (e.g., Santos & Ribeiro, 2024; Kucian & Cohen Kadosh, 2022). In the following, we draw on a conceptualisation of the Society for Didactics of Mathematics from the German-speaking countries (GDM), which describes the special difficulties of pupils in learning mathematics from a didactic point of view (Gaidoschik et al., 2021). We speak of pupils with *severe difficulties in learning arithmetic* when they have serious deficiencies in

- the understanding of natural numbers,
- the understanding of the place value system or
- the understanding of arithmetic operations.

Here, “severe” means that the deficiencies cannot be overcome through undifferentiated further learning in regular lessons. The deficiencies can be in one, two, or all three of the areas mentioned.

From a mathematics didactics perspective, the term “understanding” can be differentiated and concretised in all three areas mentioned above (Steinecke & Ulm, 2025). A key aspect here is that pupils develop the concept of cardinal numbers—in addition to the concept of ordinal numbers. Only with the cardinal number concept can they understand the principles of the place value system and develop sound basic mental models (vom Hofe & Blum, 2016) of arithmetic operations. The above conceptualisation of severe difficulties in learning arithmetic is of high relevance for the school system: On the one hand, pupils’ understanding of natural numbers, the place value system and arithmetic operations can be analysed in differentiated ways. On the other hand, on this basis, pupils can be given individual and appropriate support (Gaidoschik, 2025). We will deepen this in the following sections.

Another way to define and diagnose severe problems in learning arithmetic is based on the International Classification of Diseases ICD-11 and uses the double discrepancy criterion. “The individual’s performance in mathematics or arithmetic is markedly below what would be expected for chronological or developmental age and level of intellectual functioning and results in significant impairment in the individual’s academic or occupational functioning” (World Health Organization, n.d., ICD-11, Code 6A03.2). On this basis, it is common practice in psychological and medical diagnostics, particularly in Germany, to carry out both an arithmetic test and an IQ test. A developmental learning disorder with impairment in mathematics is diagnosed if two criteria are met: Firstly, the individual score of the arithmetic test must be significantly below the average arithmetic test score achieved in the respective age or grade level. Secondly, the score of the arithmetic test must be significantly lower than the score of the intelligence test of the respective person.

However, there is considerable arbitrariness in setting the minimum discrepancy required to speak of this learning disorder (Murphy et al., 2007). A difference of 1 to 1.5 standard deviations is often demanded (e.g., Haberstroh & Schulte-Körne, 2019), some conceptualisations of dyscalculia also require further criteria (van Luit, 2019). Consequently, prevalence data vary greatly. For example, Wyschkon et al. (2009) found a proportion of between 0.1% and 8.1% of children with dyscalculia in a sample of 1970 pupils, depending on the definition chosen. Based on a literature review, Spiegel and Selter (2021) noted that data on the percentage of primary school pupils with severe problems in learning arithmetic ranged from 2% to 20%.

The use of the double discrepancy criterion has been criticised by mathematics didactics for decades. A pedagogical argument: If a pupil performs poorly in both the arithmetic test and the IQ test, the required difference is not achieved. If this pupil therefore does not receive mathematical support, this seems unfair and pedagogically unwise. A more theoretical argument: Intelligence is a multi-faceted construct that also includes mathematical components. Depending on the choice of the tests, the correlation between the intelligence test and the arithmetic test varies (Ehlert et al., 2012). Therefore, the discrepancy between the two test results depends on the quite arbitrary selection of the intelligence test and the arithmetic test. If an intelligence test that does not correlate with any mathematical abilities was used, however, the construct of intelligence would only be partially measured.

Research Questions

As mentioned above, diagnostic procedures based on the double discrepancy criterion do not fully meet the requirements of educational practice in schools. However, the school system needs information on how many and which pupils need specific support at secondary schools because of a lack of basic arithmetic competences from primary level. In the following, we look at pupils in grade 5, as this is usually the first grade of secondary school in the German school system. We study the following research questions:

1. How high is the proportion of pupils in grade 5 who need specific support in arithmetic?
2. How high is this percentage at the different types of secondary school?

Methods

We report on a quantitative study conducted with 2328 pupils in grade 5 in Bavaria, Germany. As background, a glance at the Bavarian school system is useful. Primary school comprises four school years. At the beginning of grade 5, pupils (who are about 10 years old) attend one of the following three secondary school types: basic secondary school, intermediate secondary school or grammar school. The study was conducted at 40 schools of these types (20 basic secondary, 10 intermediate secondary and 10 grammar schools). All fifth-graders at these schools took part in an arithmetic test. This was done at 20 schools at the beginning of the school year 2022/23 and at the other 20 schools at the beginning of the school year 2023/24. The sample can be regarded as largely representative of pupils in grade 5 in Bavaria since the 40 schools are spread across all seven administrative districts and are located in large, medium-sized, and small towns as well as in rural communities. Even though the data come from only one federal state of Germany, the results can provide impulses across Europe and beyond to recognise the difficulties of secondary school pupils in the field of arithmetic more sensitively and to react accordingly.

To assess the pupils' arithmetic competences the "Bayreuth Arithmetic Test" (in German: "Bayreuther Rechentest", BRT) was used. There were several reasons for this choice: This test is based on the above-mentioned conceptualisation of severe difficulties in learning arithmetic according to Gaidoschik et al. (2021). It assesses abilities in the three areas of understanding natural numbers, the place value system and arithmetic operations. Both the test and materials for analysing results are freely available (in German and English).[1] An important aspect of practical use in schools is that the test can be carried out by teachers. It is a standardised paper-pencil test with 24 tasks to be done in 40 minutes. This allows all pupils in a class to take the test during a lesson.

85 teachers from the 40 schools were involved in the study. They took part in a three-year teacher education programme to become experts at their schools in the diagnosis and support of pupils with severe difficulties in learning arithmetic. These teachers corrected the arithmetic tests at their school with standardised instructions. In addition, they then carried out structured diagnostic interviews in one-to-one settings with pupils who had achieved particularly low test-results. The aim of these interviews was to verify the results of the test and to gain deeper insights into the children's individual

[1] https://rechenschwaeche.uni-bayreuth.de/?Materialien_zur_Diagnostik

ways of thinking when dealing with numbers. These diagnostic results served as a basis for specific support lessons for the pupils given by these teachers at the respective school.

The teachers' professional expertise, their experience in diagnostics and their insights into the specific difficulties of pupils at their school were used to tackle the research questions. At a joint meeting, the teachers discussed the question: Up to which test result should pupils receive specific support in arithmetic and how could this be organised?

Results

A maximum of 60 points can be achieved in the arithmetic test. In the following analysis, the percentage of the maximum possible points achieved by pupils in the test is considered; this percentage is referred to as the *test score*.

Distribution of test scores differentiated by school type

Figure 1 shows a classified distribution of the test scores achieved, broken down by school type. The height of the bars indicates the percentage of pupils from each type of school who achieved a test score in the specified interval.

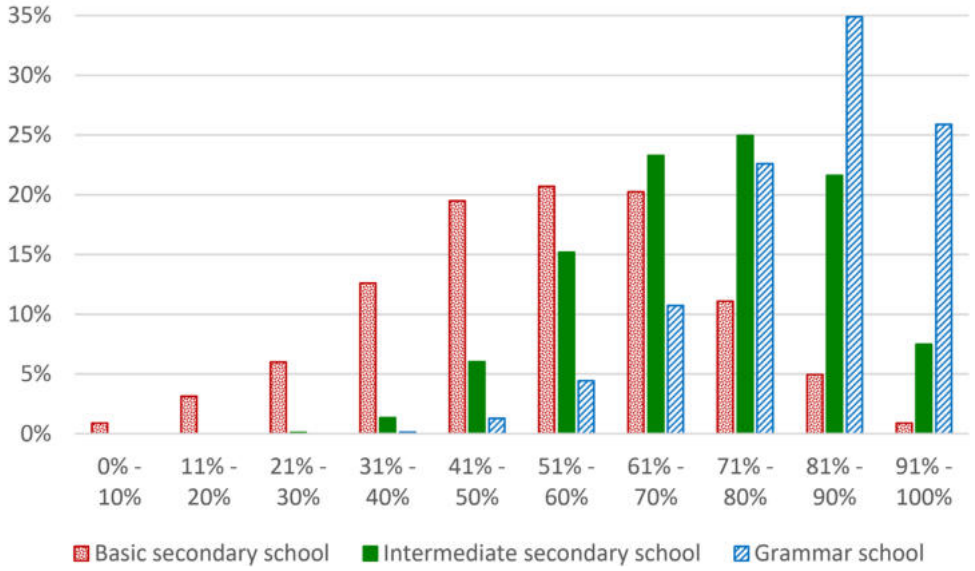

Figure 1: Distribution of test scores differentiated by school type

Comparing the three distributions, on the one hand, it is noticeable that the distributions are—as expected—shifted against each other. The results are better at grammar schools than at intermediate secondary schools and there again better than at basic secondary schools. The medians of the three distributions are 55%, 72% and 83% respectively. On the other hand, there are also considerable overlaps of the three distributions. For example, there are children at all three types of school who only achieve half of the points or less, but also children who achieve more than 90% of the points. Results in the range of 61% to 80% of the points are achieved by 31% of pupils at basic secondary schools, 48% of pupils at intermediate secondary schools and 33% of pupils at grammar schools. These overlaps of the distributions are not surprising, as the decision of which type of school a child attends in grade 5 does not only depend on his or her performance in arithmetic.

Distribution of test scores across school types

To obtain an overview of how test scores are distributed in the overall population of fifth-graders at basic secondary schools, intermediate secondary schools and grammar schools, the three school-specific distributions from Figure 1 are combined to an overall distribution. It is taken into account that the three types of school are attended by different numbers of pupils in grade 5 in Bavaria.[2] Therefore, the three school-specific distributions are weighted accordingly and then added up. As a result, Figure 2 shows the classified distribution of test scores in the population of pupils in grade 5 at basic secondary schools, intermediate secondary schools and grammar schools. The height of the bars indicates the percentage of pupils achieving a test score in the specified interval.

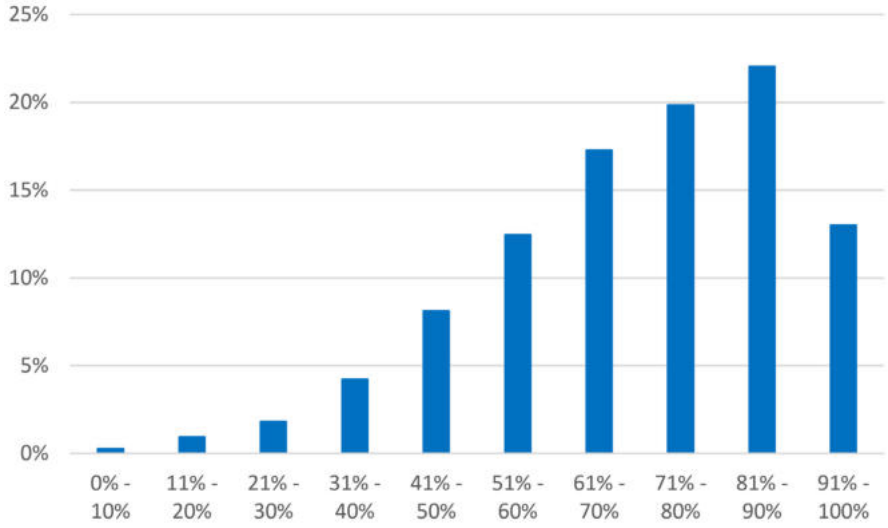

Figure 2: Distribution of test scores across school types

It becomes apparent that the arithmetic test differentiates mainly in the lower performance range. Half of the pupils achieved at least three quarters of all achievable points. The distribution in Figure 2 is particularly useful for relating the test results of individual pupils to the overall population of fifth-graders.

Cut-off values for severe difficulties in learning arithmetic

As described above, in a teacher education event for the 85 teachers at the 40 schools, it was discussed which pupils needed special support to overcome their difficulties in arithmetic. In a joint meeting, the group of teachers was asked orally up to which test score pupils should be given specific support. This question was particularly important for the teachers, as they had the task of setting up appropriate support programmes at their schools. Based on their experience in correcting the arithmetic tests of the 2328 pupils, their insights from the diagnostic interviews and their professional expertise, the teachers' joint discussion resulted in a differentiated recommendation:

[2] In 2023, 29% of pupils in grade 5 attended a basic secondary school, 28% an intermediate secondary school and 38% a grammar school. The further 5% attended schools for special needs or private schools with specific philosophical focus.

- Pupils who achieved less than 55% of the points in the arithmetic test should receive specific support. This can be done through differentiation in regular mathematics lessons or in additional support lessons.
- If only 35% or less of the points were achieved, support in extra lessons in addition to the regular mathematics lessons is urgently required.

Thus, regarding the organisation of support measures, the teachers differentiated between levels of severity of the difficulties.

Proportion of pupils with severe difficulties in learning arithmetic

With these teacher recommendations, the distributions of test scores (see Figures 1 and 2) can be used to directly determine the proportion of pupils with particular needs for support in arithmetic in grade 5 at the three school types and across the school types. This gives the data in Table 1 and thus the answers to the two research questions.

Table 1: Proportion of pupils with particular needs for support in arithmetic among all pupils in grade 5 of the respective school type and across the three school types

	Basic secondary school	Intermediate secondary school	Grammar school	Across school types
Proportion of pupils who need specific support (test score below 55%)	50%	12%	2.4%	20%
Proportion of pupils who need additional lessons (test score of at most 35%)	16%	0.3%	0%	5%

Across all three types of school, teachers judge that 20% of pupils in grade 5 need specific support to overcome their learning deficits in arithmetic; for 5% of pupils, this should take the form of additional lessons. At grammar schools, such additional lessons are not absolutely necessary. Here, 2.4% of pupils require specific support, which can usually be provided through differentiation in class. The situation is different in basic secondary schools, where half of all pupils require specific support in arithmetic. For 16% of pupils in this type of school, it is considered necessary for the support to take place in separate lessons in addition to the regular classes.

Discussion

The results presented show how many pupils at the beginning of secondary school have considerable deficits in their understanding of primary level arithmetic (see Table 1). However, the curricula and mathematics lessons at secondary level presuppose this understanding. They build on this, e.g., in the context of negative numbers, fractions, variables, terms and functions. If these pupils with severe learning deficits in arithmetic do not receive specific support, it is very likely that they will retain these deficits throughout their secondary school years. Furthermore, they will also be substantially hindered in achieving the various objectives of secondary mathematics education. It is therefore highly probable that, at the end of their schooling, they will be among the young people labelled as low achievers by PISA (Diedrich et al., 2023). They are at risk of experiencing barriers in their further

professional development and social participation, as arithmetic competences are required in many areas of professional and private life.

Thus, the results in Table 1 show a substantial challenge for the secondary school system. According to the judgements of secondary school teachers, 20% of pupils in grade 5 require specific support in arithmetic. In some cases, this could be achieved through differentiation in the classroom, but in other cases additional support lessons are necessary (see Table 1). Supporting these pupils requires systemic activities on several levels:

- Firstly, it is a challenge for teacher education. In Germany, prospective secondary school teachers usually acquire professional competences for teaching secondary level content, but not primary level content. They are not made acquainted with diagnosing and developing pupils' basic understanding of natural numbers, the place value system and arithmetic operations at primary school level. To really help secondary school pupils with severe difficulties in learning arithmetic, developments are therefore required in the area of teacher (further) education.
- Secondly, there are challenges at school organisational level. New support offers need to be set up for pupils with severe difficulties in learning arithmetic. For specific support during regular lessons, appropriate differentiation formats are necessary. Support lessons in addition to regular classroom lessons require appropriate organisational planning as well as additional staff capacity and therefore financial resources.
- Thirdly, teachers need concepts and materials for the diagnosis of difficulties and for the support of pupils with these difficulties. Textbooks and workbooks for secondary school mathematics lessons are usually not designed for this purpose.

For tackling these challenges, there are already a variety of successfully tried and tested concepts (e.g., Dowker & Sigley, 2010; Dowrick, 2015; Kucian & Cohen Kadosh, 2022; Prediger et al., 2019). A major task, however, is to put these into practice on a broad basis in teacher education and everyday school life. The data in Table 1 emphasise the need for such developments.

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Assessing students' conceptual place value understanding at the end of primary school

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Place value understanding is a prerequisite for learning mathematics in secondary school. Therefore, students' place value understanding for numbers up to one million at the end of grade 4 is highly relevant. To assess conceptual place value understanding for numbers larger than a thousand, we developed a test instrument consisting of 13 items comprising three task types. All of the items require the transformation of a number's pictorial or symbolic representation to a numeral, whereby non-standard representations are crucial. The statistical reliability and validity of the instrument are demonstrated by confirmatory factor analysis and reliability measures.

Keywords: Place value understanding, conceptual place value understanding, representations of numbers, empirical study, confirmatory factor analysis.

Theoretical Background

Understanding multi-digit numbers in the decimal place value system is a prerequisite not only for everyday life but also for calculating and learning decimals (Fuson et al., 1997; Mix et al., 2023; Verschaffel et al., 2007). For example, crossing a tens boundary in written addition and subtraction is based on an understanding of place values, especially of bundling and unbundling. Likewise, addition and subtraction following both methods of split or jump require an understanding of numbers as a composition of ones, tens, hundreds, thousands, and so on.

Despite its importance, place value understanding remains a vague concept that many authors do not explicitly define. A shared but quite general explication considers place value understanding as the understanding of the meaning of a numeral represented in the decimal place value system, including its principles (especially bundling and unbundling). Following Herzog et al. (2019) and Van de Walle et al. (2023), the distinction between procedural understanding and conceptual understanding is helpful to clarify what place value understanding means.

Briefly summarized, *procedural place value understanding* comprises the reading and writing of numbers ("transcoding", Herzog et al., 2022). More precisely, this means both transfers between number words and the numerals. It consists mainly of semiotic and linguistic insights into the structure of number words and reading skills regarding numerals. Many studies focus on reading and writing multi-digit numbers (cf. Herzog et al., 2022; Nuerk et al., 2014). An essential aspect here is that in German and a few more European languages, the number words are built in the order hundreds-ones-tens, which is different from the order of the digits. As the inversion of tens and ones can cause problems for learners, procedural place value understanding is an important aspect, yet it represents only one aspect of place value understanding.

The core of *conceptual place value understanding* is the understanding of the relation between the decimal units, based on the processes of bundling and unbundling. As stated by Bruner et al. (1967), there are three modes of representation for mathematical knowledge in general and numbers in particular: concrete (e.g., using Dienes blocks), pictorial (e.g., by dots, lines, and squares, as an abstraction of the Dienes blocks, like in Figure 1), and symbolic (e.g., in a place value chart, as a numeral or as a number word). Following on from this, conceptual place value understanding means the ability to perform various transformations between symbolic representations on the one hand and concrete or pictorial representations on the other hand. Such transformations are considered crucial for acquiring a place value understanding (e.g., Resnick, 1989; Treffers, 2001; Van de Walle et al., 2023) and are an approach for assessing place value understanding, too.

According to the principles of the base-ten place value system, a maximum of nine units of each bundle size is permitted to note. Though, in the context of mathematics education, not only representations like "3 hundreds, 4 tens, and 6 ones" following mathematical conventions but also representations like "2 hundreds, 14 tens, and 6 ones" or analogue pictorial representations (cf. Figure 1) can appear. The latter are called non-standard representations (Van de Walle et al., 2023) or non-canonical representations (Herzog et al., 2019), and they are considered highly relevant in the learning process (Ross, 1989; Van de Walle et al., 2023) as well as for diagnostic purposes.

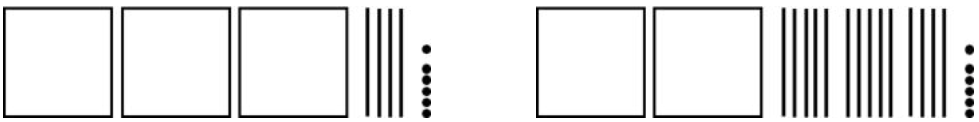

Figure 1: Pictorial representation of 346 (on the right, a non-standard representation)

A relevant empirical approach to place value understanding is the search for developmental models. As their focus is on the early development of place value understanding, most of these empirical studies deal with numbers smaller than a thousand. There are earlier attempts, e.g., Fuson et al. (1997), Resnick (1989), and Ross (1989), who describe some phases from counting to identifying ones and tens to an integration of ordinal and cardinal concepts. Following this tradition, in a more recent study Herzog et al. (2019) set up an empirically validated developmental model for place value understanding that consists of five hierarchical stages from a pre-decadic level to a general decimal-bundling-unit-relation. In keeping with the interest in a developmental model, the sample includes children from grade 2 to grade 5, and the hierarchical model reflects, to some extent, the curriculum.

There are also studies investigating the structure of place value understanding. For example, Jensen et al. (2024) asked third graders who had been systematically taught numbers up to a thousand to transform a representation like 7H 3O 19T into numerals. Analyzing students' errors qualitatively, two aspects of place value understanding could be separated: place value (when the order of the units is not the regular one) and regrouping (bundling in the case of a non-standard representation).

All of the studies show that a substantial part of the students up to lower secondary has severe difficulties in place value understanding, particularly in conceptual place value understanding (Herzog et al., 2019; Humbach, 2008; Jensen et al., 2024; Moser Opitz, 2013; Resnick, 1989; Ross, 1989).

Although there are some studies on conceptual place value understanding, they mostly do not address the understanding of numbers over a thousand. However, only in the higher number range, multiple bundling (basically 10 ones to 1 ten, but furthermore 10 tens to 1 hundred, 10 hundreds to 1 thousand, …) plays a role, which makes it interesting to focus on students' conceptual place value understanding and typical errors in that number range. Besides, most of the published test instruments capture conceptual place value understanding with only a few items.

Therefore, it is necessary to develop a comprehensive test instrument covering the higher number range and different transformations in order to gain deeper insights into the place value understanding of learners who have been systematically taught numbers up to one million.

The present study

The aim of our study is to develop a reliable and valid test instrument to assess students' conceptual understanding of place value for three- and four-digit numbers. In particular, the empirical evaluation of the construct validity and the reliability of the developed test instrument is highly relevant. Accordingly, we examine in this report whether a one-dimensional model representing conceptual place value understanding shows an appropriate fit to the empirical data, and whether a reliable scale can be formed using different transformation tasks.

Methods

Sample

The participants were $N = 630$ students who were spread over 34 classes of grade 4 in German primary school. The age of the participants was on average 10.3 years ($SD = 0.6$). Nearly half of the students were male (46.7%), and just over half were female (53.3%). The test took place at the end of the school year. In Germany, primary school generally lasts 4 years, and all students attend the same type of school. In this respect, it can be assumed that the sample is very heterogeneous. According to the curriculum, the children had been taught numbers up to one million and calculate with them.

Data collection

Data collection was run by specially trained research assistants with a Bachelor's degree during a 45-minute lesson in a whole-class setting. It was approved by the school authorities; students and their parents gave their informed consent.

The test instrument was a paper-pencil test. The students had as much time as they needed to work on it. As mentioned above, our test was designed to capture place value understanding by transforming pictorial and symbolic representations into numerals. We constructed three task types with space for free-text answers (Table 1).

The first task type comprises items where pictorial representations of numbers smaller than a thousand should be transformed into a numeral. This kind of pictorial representation is very common in German textbooks: 15 of the 16 textbooks licensed by the school authorities use it. In the second task type, a number given in a place value chart must be transformed into a numeral. The third task type focuses on the transformation of a base ten-number word into a numeral, each up to one million. In 11 of the 13 items, a non-standard representation was used. When constructing the items, two

aspects were systematically varied: (1) unfilled positions and (2) positions with a noted bundle size larger than nine.

There were two versions, A and B, of the test: Regarding task type 1, the items were the same in both versions but in a different order. Regarding task types 2 and 3, the items were structurally identical but with slightly different numbers.

Table 1: Task types, each with an example item

	Task type	Example item
1	Transformation pictorial representation → numeral (4 items)	Some children represent the number 423 like this: Which number is given here? Write it as a numeral.
2	Transformation place value chart → numeral (4 items)	Write as a numeral: **Th** | **H** | **T** | **O** 2 | 6 | 13 | 4
3	Transformation base-ten number word → numeral (5 items)	Write as a multi-digit number numeral: 3 thousands, 25 hundreds, 1 ten, 7 ones

Data analysis

The data analysis was conducted using the Statistical Package for the Social Sciences (SPSS 28, IBM Corp., 2021) and R (R Core Team, 2021), with particular emphasis on confirmatory factor analysis (CFA), using the lavaan (Rosseel, 2012) and semTools (Jorgensen et al., 2022) packages for structural equation modelling (SEM) and reliability assessment. First, the item difficulty (average response accuracy) was analyzed. Second, the factor structure of the measurement model was investigated. CFA was used to evaluate the model's fit to the observed data. To assess the construct validity of our measurement model, we conducted the CFA using the DWLS (diagonally weighted least squares) estimator, as the items were dichotomously coded (1: correct; 0: wrong or missing). The hypothesized model posited the latent factor 'place value understanding' indicated by three task types: (1) transformation pictorial representation → numeral, (2) transformation place value chart → numeral, (3) transformation base-ten number word → numeral (see Figure 2). Correlations among error terms within the same task type were allowed in the model to account for potential shared variance.

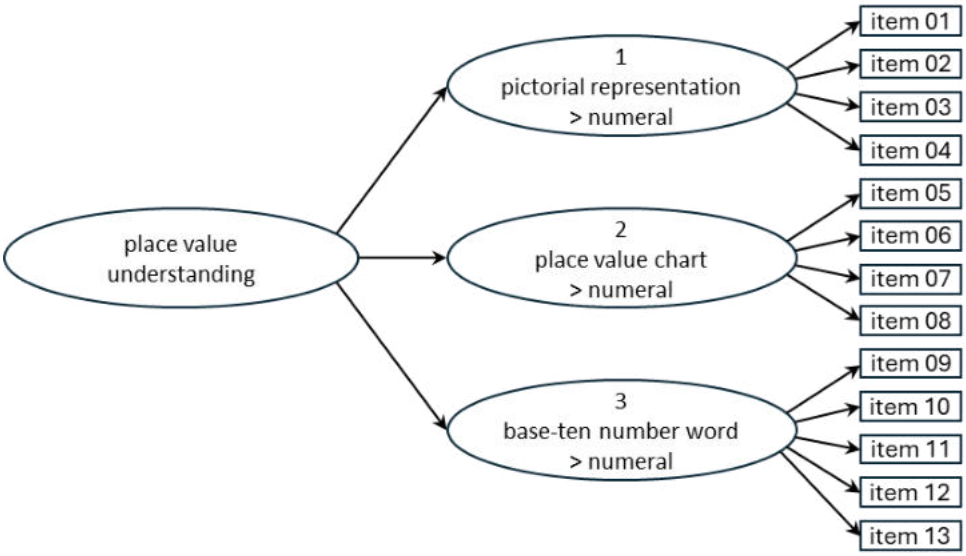

Figure 2: Structure of the model

Results

The item analysis revealed that one item exhibited a difficulty level of 0.87 that fell outside the acceptable range of 0.2 to 0.8. Consequently, this item was excluded from further analysis. Within task type 3, two items were excluded due to their low factor loadings < 0.4 (Kline, 2015). Descriptive statistics, including the mean values for the indicator variables in the structural equation model, are presented in Table 2.

Table 2: Descriptive statistics indicator variables (*N* = 630)

Indicator variable	Theoretical range	*M*	*SD*
1 Transformation pictorial representation → numeral	0–1	0.69	0.34
2 Transformation place value chart → numeral	0–1	0.56	0.41
3 Transformation base-ten number word → numeral	0–1	0.66	0.43

The results of the CFA, which will be detailed below, are based on the empirical covariance matrix. This covariance matrix was derived from the correlation matrix presented in Table 3, which displays the bivariate correlations among the measurement model variables.

Table 3: Correlation matrix of the indicator variables (*N* = 630)

Indicator variable	1	2	3
1 Transform. pictorial representation → numeral	1.00		
2 Transform. place value chart → numeral	0.45***	1.00	
3 Transform. base-ten number word → numeral	0.44***	0.82***	1.00

Note: *** $p < 0.001$

The CFA results show that the overall fit of the measurement model was adequate. The goodness of fit indices are presented in Table 4.

Table 4: Measures of global fit

Fit Index	Value	Recommended Threshold
CFI (scaled)	0.98	≥ 0.95
TLI (scaled)	0.98	≥ 0.95
RMSEA (robust)	0.03	≤ 0.06
SRMR (robust)	0.03	≤ 0.08
Chi-Square ($df = 56$)	51.23 ($p = 0.66$)	$p > 0.05$

The CFI and TLI values are both above the recommended threshold of 0.95, indicating an excellent fit of the model to the data (Hu & Bentler, 1999). The robust RMSEA value of 0.03 is well within the acceptable range of less than or equal to 0.06, further supporting the good fit of the model. In addition, the SRMR value of 0.03 is below the threshold of 0.08, indicating a good fit. The chi-square statistic with $\chi^2 = 51.228$ ($df = 56$) gives a p-value of 0.656. This suggests that the model fits the data reasonably well regarding the standard chi-square test.

All factor loadings for the observed variables were statistically significant ($p < 0.001$), with standardized loadings ranging from 0.41 to 0.95. The factor loadings of the indicator variables on the latent construct factor 'place value understanding' ranged from 0.53 to 0.96, indicating that the latent construct is well represented by its indicators.

The overall reliability of the latent construct was evaluated using McDonald's Omega (McDonald, 1999), which yielded a value of 0.91. An omega value above 0.70 indicates high reliability, suggesting that the latent construct is measured consistently across the included indicators (Raykov, 1997). The average variance extracted (AVE) for the latent construct was 0.57. AVE assesses the proportion of the variance in the indicators captured by the latent construct, as opposed to measurement error. An AVE value above the threshold of 0.50 indicates that the latent construct explains a substantial proportion of the variance in its indicators, thereby confirming the convergent validity of the construct (Fornell & Larcker, 1981).

Discussion

The goal of our study was to develop a reliable and valid test instrument for assessing students' conceptual place value understanding at the end of primary school. Our test instrument is designed to evaluate conceptual place value understanding by operationalizing it through bundling and transforming pictorial or symbolic representations into numerals. Focusing on this, non-standard representations play an important role. While our study addressed bundling and transformation extensively, it did not include unbundling processes—an equally significant aspect of place value understanding (e.g., considering written subtraction). Nevertheless, our test instrument goes beyond previous research, allowing for a more comprehensive assessment of conceptual place value

understanding by systematically varying positions of units more than nine and by focusing on the number range larger than a thousand. In addition, the results demonstrate satisfactory validity and reliability, which is crucial for drawing well-founded interpretations about learners' conceptual place value understanding. The final test instrument comprises 13 items, which can be assigned to three distinct task types, each targeting a specific transformation. As it is brief, it offers a highly time-efficient assessment for research and diagnostic purposes in school.

For future research, there is significant potential to extend the scope of analysis using the developed test instrument. One area of interest is the identification of typical error patterns to gain deeper insights into common misconceptions and difficulties encountered by students. Additionally, investigating the relationship between conceptual place value understanding and other mathematical skills, such as arithmetic fluency, could provide valuable perspectives on how place value knowledge contributes to broader mathematical competence.

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Exploring the strategies utilised by primary pupils to solve equal group multiplication problems using the 'Stick and Split' app

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Keywords: Game-based learning, multiplication, problem-solving.

Introduction

Children often use intuitive strategies to solve multiplication problems before receiving formal education, which evolve into more advanced strategies with age (Downton, 2008). Literature suggests that primary pupils mostly rely on modelling, counting, grouping, repeated addition and multiplication facts to solve multiplication problems (Mulligan & Mitchelmore, 1997). Pupils select these strategies depending on their experiences, problem specificity, and cognitive development at the time (Lemaire & Siegler, 1995). In the last decade, education has embraced technology; as a result, many digital tools (i.e., apps and games) are now available to primary school pupils to help them grasp fundamental arithmetic concepts. Stick and Split is an interactive educational app designed to practise multiplication problems and learn arithmetic facts. A brief introduction to this app can be seen in a YouTube video: https://www.youtube.com/watch?v=lvISZHKWclc.

Stick and Split is developed to help pupils practise equal-group multiplication problems. It has 300 levels, in which pupils either join (stick) different rods of equal sizes into one bigger rod or break down (split) one big rod into several rods of equal sizes. The task at every level is to convert all the available rods into a number (i.e., target number), which is given at the top of the screen by utilising stick and split operations provided at the bottom. This study aims to explore the engagement of primary school pupils with the Stick and Split app to see the following: 1) What strategies do they utilise to solve multiplication problems posed by different levels of the Stick and Split app? 2) How similar are these strategies in relation to the strategies reported by the literature?

Methods

This exploratory qualitative study analyses a 30-minute recorded session of two Year 2 pupils interacting with the Stick and Split app at a primary school in England. In the first five minutes of the session, the pair were introduced to the Stick and Split app on an iPad screen. Following that, this pair interacted with the app for 25 minutes; during this time, they were asked to discuss their strategies while completing the following levels of the app: 78, 79, 106, 160, 161, and 108. These levels were shortlisted from the 300 available levels in the Stick and Split app according to the ready-to-access arithmetic skills criteria of Year 2 pupils outlined in the guidelines of the Department of Education (DfE, 2019, p.52). Thematic analysis was used to analyse the recorded session.

Results

The played levels were categorised into three groups based on their difficulty level, which indicated whether splitting is required and whether the available rods are factors of the target number. Those levels where the available rods were factors of the target number were classed as easy, those that

involved splitting were classed as intermediate, and those where splitting was required and none of the available rods were factors of the target number were classed as difficult (see Table 1).

Table 1: S&S level number, Target number, and the Given rods of the six played levels

Difficulty level	Intermediate		Easy		Difficult	
Level number	78	79	106	160	161	108
Target number	24	24	32	48	48	24
Available rods	8,16	12,18	4, 8	4, 12	20, 44	16, 24

The coding process revealed that pupils utilised four strategies to solve multiplication problems across the six Stick and Split levels: known or derived multiplication facts, a two-step multiplication model (combining one known and one derived fact), skip counting, and trial-and-error grouping. In easier levels, pupils first employed trial-and-error grouping, then known or derived facts, and finally two-step multiplication models when only rods of length 2 or 3 were present. They started with known or derived facts in intermediate levels before switching to trial-and-error grouping. With rods of length 2 or 3 left, they used skip counting in level 78 and the two-step model in level 79. In difficult levels, pupils completed level 161 solely with trial-and-error grouping, while in level 108, they transitioned from trial-and-error to the two-step multiplication model.

Discussion

The literature suggests that pupils' development tends to transition from modelling to counting and eventually to multiplication facts as a problem-solving strategy (Mulligan & Michelmore, 1997; Downton, 2008). Year 2 pupils' experience with multiplication problem solving likely contributed to their use of advanced strategy over basic modelling or counting. The two-step multiplication model emerged as an extension of known multiplication fact strategies. Pupils used these facts when available rods matched target number factors; otherwise, they relied on trial-and-error grouping. Notably, the Stick and Split app supported flexible strategy use, allowing ample practice and opportunities for pupils to deepen conceptual understanding of equal-group multiplication.

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TWG 03: Algebraic Thinking

An introduction to TWG3 Algebraic Thinking

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Overview

In Thematic Working Group 3 on "Algebraic Thinking," 18 papers and 2 posters were presented, with a total of 27 participants from Europe and other continents. The contributions were presented and discussed in working sessions organized around four themes: (1) Functional Thinking, (2) Variables and Expressions, (3) Equivalence and Equations, and (4) Algebraic Thinking from Different Perspectives. Given the interconnected nature of these themes, the assignment of papers was guided by a "best fit" approach. Notably, the contributions collectively reflect progress in the research field of algebraic thinking on both theoretical and practical fronts. Central to this progress are key issues such as the nature and scope of algebraic thinking, the transition from concrete representations to abstract reasoning—and whether this shift constitutes a natural development or a rupture in students' understanding—as well as the pivotal role of teachers in supporting students' algebraic development across grade levels. Below, we discuss the papers presented under each theme. The report concludes with key issues and challenges identified in the discussions during the sessions.

Functional Thinking

This theme was prevalent among the papers presented in TWG3, reflecting a clear interest in function graph tasks as a means of developing algebraic thinking from the elementary grades onward. These papers reveal both students' learning routes and persistent challenges, as well as the crucial role of teacher knowledge and instructional practices. Radford demonstrated that Grade 6 students initially perceived function graphs as static images or maps, neglecting the concept of co-variation. However, through teacher-guided interactions, they gradually developed more sophisticated, culturally situated understandings of the dynamic relationships between co-varying quantities. Liao et al. showed that even Grade 2 students could reason with function graphs when supported by instructional sequences that organized functional relationships into a hierarchy of difficulty—for example, beginning with $y=x$ and progressing to $y=2x$. This finding highlights that an understanding of functional relationships can emerge earlier than traditionally expected. Also addressing early algebraic thinking, Tomaszewski and Nührenbörger illustrated that Grade 4 students were capable of reasoning meaningfully with four types of functional relationships—linear, quadratic, exponential, and inverse—suggesting untapped potential in primary mathematics education, where such topics are often addressed only implicitly. In the high school context, however, Šabaková and Semanišinová found that, despite prior instruction, students struggled to connect tabular data to the graph of a quadratic function. This points to deep-rooted challenges in coordinating different representations of functional relationships. At the instructional level, Belin et al. explored how the depth of teachers'

reasoning about co-variation influenced whether they employed correspondence or co-variation approaches in their teaching, potentially shaping how students encounter and understand quadratic functions. Complementing these findings, Pittalis and Gridos analyzed how Grade 8 teachers introduced the concept of slope. The study revealed that teachers primarily focused on the geometric ratio and the algebraic formula, without allocating time to explore the multiple conceptualizations of slope—thus limiting students' opportunities to develop deeper conceptual understanding. Reinforcing the sociocultural dimension, Reinhardtsen and Carlsen highlighted how the process of "togethering"—the committed engagement in joint activity with a shared ideal or material object between teacher and students—could either facilitate or hinder students' transition from arithmetical to algebraic thinking in the context of pattern tasks. This paper underscores that such transitions are not only cognitive but also deeply entangled with emotions and the joint construction of meaning within the classroom. Finally, Goñi-Cervera et al. contributed to this conversation through a poster focused on young students with Autism Spectrum Disorder (ASD). This work extended the discussion of early algebraic thinking into inclusive education contexts. The study showed that these students experienced similar difficulties to those reported in the literature for typically developing peers in grasping the "unit of repeat". Together, these studies underscore that the development of functional thinking is both cognitively and socially situated. They highlight the need for: (a) well-designed instructional sequences; (b) intentional instructional support; (c) attention to representational connections; (d) sensitivity to the emotional and dialogic processes that shape learning.

Variables and Expressions

The studies under this theme collectively emphasize the persistent challenges students face in algebra—particularly in interpreting variables and expressions—while also highlighting instructional strategies that support conceptual development. Christou et al. investigated how Grade 9 students frequently misinterpreted algebraic expressions with a negative sign, such as assuming that $-2x$ must always represent negative numbers, leading to errors in solving equations and inequalities. Although an instructional intervention using incorrect examples proved slightly more effective than using correct examples, neither approach resulted in long-term conceptual change, underlining the resilience of such misconceptions. Complementing this, Korntreff and Prediger explored how interactive instructional videos supported Grade 8 students in refining their understanding of variables. The interactive features of the videos helped students confront and revise deviant or partial solutions, promoted distinctions between variables as placeholders versus generalizers, and provided language resources to support this conceptual distinction. Similarly addressing students' misconceptions about algebraic letters, Klingbeil and Moons showed that instructional practices explicitly focused on the meaning of variables within algebraic expressions helped Grade 7 and Grade 8 students overcome the widespread "letter-as-object" misconception. Acknowledging the importance of conceptually understanding variables and expressions at the tertiary level, Wladis et al. demonstrated that students' scores on a validated algebra test significantly predicted both mathematics and STEM outcomes at a large, diverse urban university. These scores accounted for disparities in outcomes across race/ethnicity and gender beyond what course grades could explain. Taken together, these studies highlight critical insights: (a) misconceptions about variables—particularly regarding their sign, role, and meaning—are longstanding and persist across grade levels;

(b) distinguishing between different uses of symbols (e.g., variable as placeholder vs. generalizer) is central to conceptual growth in algebra; (c) instructional strategies that explicitly address these distinctions, such as interactive media or conceptually focused tasks, are essential.

Equivalence and Equations

This theme brings into focus the structural thinking dimension of algebraic thinking, particularly in relation to linear and proportional equations. Two studies addressed the challenges students face in solving linear equations, offering complementary perspectives on students' attention to structure. Holmlund and Pejlare's study with 16-year-old vocational students revealed that when tasked with sorting linear equations, they attended to a diverse range of features—both superficial and structural—highlighting the potential of such tasks for assessing students' sensitivity to structure. Weber et al. adopting a semiotic-pragmatic lens and using eye-tracking technology, further illuminated how pre-service teachers interact with the structural components of equations during problem solving. The study found that "adaptive experts" moved rapidly between sub-expressions, operations, brackets, and the equals sign, applying hierarchical structuring which led to more efficient problem-solving. In contrast, "routine experts" relied on more linear, left-to-right strategies, with particular emphasis on areas that support the standard solution approach. These strategies were associated with higher cognitive load and increased error rates. The studies by Tunc-Pekkan et al. and Tuncer et al. shift the focus from linear to proportional equations, further emphasizing structural thinking in a different algebraic context. Tunc-Pekkan et al's study examined how in-service teachers approached problems involving fractions and two varying quantities. The analysis revealed that they had difficulty making meaningful connections between visual models—which are designed to illustrate the underlying structure—with the corresponding algebraic equations. Tuncer et al. examined how gifted 7th-grade students performed on various proportional reasoning tasks and found that they did not experience the common difficulties reported in the literature. Their strong metacognitive skills—such as focusing on relevant information—appeared to support their ability to recognize and articulate structural relationships between quantities. This suggests that their emerging algebraic thinking may have played a key role in strengthening their proportional reasoning performance. Adding a higher education perspective, a poster by Cancec Murillo et al on limit tasks also highlighted the need for reinforcing structural sense in calculus contexts. Together, these studies suggest several key implications for supporting students' understanding of equivalence and equations across contexts: (a) students' ability to attend to the structure of equations—whether linear or proportional—is highly variable and often underdeveloped; instructional tasks that foreground structural features, such as equation sorting, may help build algebraic structure sense; (b) promoting flexible strategies can reduce cognitive load and improve accuracy in solving equations; and (c) the use of representations—particularly in proportional reasoning—requires careful scaffolding.

Algebraic Thinking from different perspectives

The final theme delves into the epistemological, cognitive, and developmental perspectives of algebraic thinking, offering deeper insights into its disciplinary foundations and the challenges it presents in educational practice. Hällback and Viirman, drawing on the Anthropological Theory of the Didactic, developed a Reference Epistemological Model (REM) to map the organization and

content of upper secondary algebra in Sweden. This model systematically described institutionalized algebraic practices and types of tasks valued in the curriculum, serving as a powerful tool for analyzing how algebra is structured and taught at a national level. Extending the theoretical discourse, Akinwunmi emphasized the need to clarify foundational constructs in algebraic thinking by distinguishing between "pattern" and "structure." She argued that while patterns are regularities in visible phenomena, structures refer to underlying mathematical properties and relations. Promoting algebraic thinking requires using representations and prompts that move beyond patterns to help students grasp these deeper structures. Pitta-Pantazi et al. explored algebraic thinking during the transition from primary to secondary education. The study revealed that Grade 6 students performed more successfully on tasks requiring inductive reasoning (e.g., recognizing and extending patterns) than on those demanding deductive reasoning (e.g., justifying general rules based on mathematical properties). This gap between inductive and deductive performance underscores the persistent challenge students face when moving from pattern-based reasoning to formal argumentation and abstraction—thus reinforcing Akinwunmi's call to foreground structural understanding and the role of reasoning in algebra learning. Together, these studies underscore: (a) the value of epistemological tools in analyzing curricular content and (b) the critical need to support students' development of structural awareness and deductive reasoning through purposefully designed instructional strategies.

Concluding Remarks

The studies presented in TWG3 reflect significant progress in algebraic thinking research, both theoretically and practically, while also highlighting critical challenges that call for further investigation. A key theoretical challenge is clarifying the nature and scope of algebraic thinking. There is growing recognition that early algebra cannot be seen simply as "generalized arithmetic." Rather, algebra involves a distinct, analytical form of reasoning, particularly in handling indeterminate quantities—a key difference from arithmetic. This raises questions about how students' generalizations evolve as they move from concrete arithmetic to abstract algebraic expressions, and how their understanding of variables, constants, equations, and graphs develops. For example, the variable-constant distinction signals a deeper conceptual shift in perceiving quantities within dynamic systems. Clearer frameworks are needed to understand these transformations and to inform teaching. Practically, a persistent challenge is translating theory into effective classroom strategies, especially in diverse settings including special education. The role of manipulatives and digital tools remains debated, as their effectiveness in bridging concrete and abstract understanding is complex and context-dependent. Future research should investigate under what conditions these means genuinely foster deeper algebraic thinking and when they might fall short. TWG3 discussions also stressed the importance of starting algebraic thinking development early. Equally important is the role of the teacher in navigating both the cognitive and emotional dimensions of learning. Teachers are not only responsible for facilitating conceptual understanding, but also for creating supportive environments that help students manage the uncertainty and discomfort often associated with abstract reasoning. Their ability to recognize and respond to students' emotional reactions to algebraic tasks—such as frustration, confusion, or disengagement—can significantly influence learning outcomes. By addressing these issues, future research can foster more robust, nuanced, and inclusive understandings of algebra as a foundational part of mathematics education from the earliest grades.

Understanding mathematical structures: A contribution to the discourse about the nature of algebraic thinking

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To constantly sharpen the character of algebraic thinking is a research desideratum, that has intensified in TWG03 in the last years (Hewitt et al., 2022; Chimoni et al., 2023). This paper aims to contribute to the discourse about the nature of algebraic thinking, considering possible perspectives on algebra and algebraic thinking and including a reference to algebra as a mathematical discipline. The paper highlights the important distinction between the terms pattern and structure and emphasises the critical role of representations in generalising. Consequences are derived for the discussion of the relationship between algebraic and arithmetic thinking.

Keywords: Algebraic thinking, patterns and structures, generalisations, representation-based proofs, properties and relations.

Introduction and objectives of the paper

It is an aim that has emerged in the last years, noted in the TWG03 proceedings (Hewitt et al., 2022; Chimoni et al., 2023), to constantly sharpen the nature of algebraic thinking. Following CERME13, Chimoni et al. (2023) emphasise the importance of working out the relation between arithmetic and algebra: "We feel that future work needs to address the differences more explicitly between arithmetic and algebraic thinking, as well as their evolving complementarities" (Chimoni et al., 2023, p. 511).

This paper is intended to continue the discussion at CERME12, where Akinwunmi and Steinweg (2022) presented an empirical study on children's generalisations with focus on patterns and focus on structures. In an exploratory study with fourth graders, we investigated how the theoretical distinction between patterns and structures is reflected in the children's generalisations (against the background the epistemological perspective of Steinbring (2005)) and what consequences this has for the development of algebraic thinking.

The current paper aims to elaborate on the theoretical background more detailed and consider it in the light of the exploration of possible perspectives on algebra and algebraic thinking. In the first section, the meaning of the term structure and its significance for algebraic thinking are derived from the development of algebra as a mathematical discipline. In the second section, based on the respective literature on algebraic thinking, I propose a theoretical distinction between the terms *pattern* and *structure*, which are sometimes used together or even synonymously. I argue, that it is important to differentiate between the two terms for a precise analysis and subsequent promotion of children's algebraic thinking. Section three offers an exemplary insight into previous studies (Akinwunmi & Steinweg, 2022; 2024) to illustrate how the understanding of structures can be promoted and highlights the role of representations in surpassing the focus on patterns. Finally, consequences for a perspective on the relationship between algebraic and arithmetic thinking are drawn from this theoretical framework and the study results.

The nature of algebraic thinking with reference to the discipline of algebra

A theoretically well-founded discussion of the definition of algebraic thinking must be in line with algebra as a mathematical discipline. Radford (2022), for example, expresses his understanding of algebraic thinking by listing elements that are necessarily characterising algebraic thinking. He focuses on semiotic systems and traces the origin of algebraic thinking not to the use of letter variables, but to a specific *deductive way of thinking*, as can already be found in the work of Al-Khwarizmi, the eponym of algebra from the 9th century. When analysing this way of thinking, he can also identify it empirically in primary school children's strategies when solving equations (Radford, 2022). Al-Khwarizmi (translation by Rosen, 1831), who described the solution strategies of different types of equations in continuous text with the help of word variables, is a good indication that symbolism is not necessary for algebraic thinking - even if it simplifies the operational scope enormously from our current point of view. This understanding of algebra is the basis for empirical studies like Steinweg et al. (2018), but also former works of Radford (e.g. Radford, 2010), that show the role of natural language, gestures etc., which take on the role of variables.

A reference to Al-Khwarizmi's algebra is a helpful and important perspective on the nature of algebra. At the same time, algebra as a mathematical discipline has developed continuously since then and has experienced various turning points in history. One such turning point is Galois' theory in the 19th century (Edwards, 1984). Instead of focusing on finding concrete solution strategies for increasingly difficult equations, Galois developed an abstract meta-perspective on the properties of mathematical objects that can be used to determine whether an equation is solvable. On the one hand, it may seem contradictory at first glance to base algebraic thinking at an elementary level on this more advanced and therefore even more abstract *structural algebra*. On the other hand, some of the properties investigated in Galois' theory already appear in the very first encounters of children with mathematics, albeit in an implicit way. For example, when children realize in the first year of school that addition problems can always be solved within natural numbers, but not subtraction problems, they inductively recognize the property of Closure. The power of Galois' theory lies precisely in looking at specific properties and relations across mathematical disciplines at a general level. Commutativity, for example, can be explored in arithmetic, e.g. for multiplication, but also in geometry in symmetry mappings for combining two rotations of a figure, in modular arithmetic for the addition of congruence classes, etc. Therefore, a reference to structural algebra explains why algebraic thinking permeates other mathematical fields as a specific way of thinking.

In this perspective, algebraic thinking involves both deductive and also inductive ways of thinking. Radford argues that “algebraic thinking […] calculates/operates with indeterminate magnitudes *as if they were known*” (Radford, 2022, p. 588). Steinbring (2005) claims that relational thinking means detaching arguments from the concrete numbers and seeing something general in them. Mason and Pimm (1984) call this *seeing the general in the particular*. For primary school, it can therefore be said that children also have to deal with concrete numbers *as if they were general*. Especially when children do not yet know any variables, children can communicate their general ideas through concrete examples and numbers. Steinbring (2005) calls this a necessary balance between *concrete, situated* and *general, relational knowledge*. This creates an approach to algebra in the early grades in which algebraic thinking is considered to be the understanding of structures.

The nature of algebraic thinking with reference to patterns and structures

In literature, algebraic thinking is closely associated with the concepts of patterns and structures. In Germany, the term algebra is not even mentioned in the primary school curriculum, but all processes concerning algebraic thinking appear in the strand “pattern, structures and functional relations” (Steinweg et al., 2018). This section will elaborate on the difference between the terms but also on their important relations. In CERME12, Akinwunmi & Steinweg (2022) showed that this distinction can also be found empirically in children's responses to pattern tasks. For the framework of the study and a more detailed description of the results, see Akinwunmi & Steinweg (2022).

Various studies worldwide describe the core of algebraic thinking as dealing with patterns and generalisations (e.g. Blanton et al., 2019). At the same time, however, patterns are also seen as the heart of all mathematics (Devlin, 1997) and the concept of pattern is intentionally broad. “Mathematics is the classification and study of all possible patterns. […] It is to be understood in a very wide sense, to cover almost any kind of regularity that can be recognized by the mind” (Sawyer, 1955, p. 12).

The sole reference to the recognition and generalisation of patterns can lead to a definition of algebraic thinking that is quite unspecific. In Akinwunmi & Steinweg (2024), we asked fourth-graders to work on tasks with patterns of e.g. the constancy of difference, i.e. subtraction tasks in which the minuend and subtrahend change in the same direction and the difference therefore remains the same. All fourth-graders were able to describe and even generalize the phenomenon. Contrarily, the students could not explain why the results remain the same. They referred to the constancy of difference as a kind of law or rule (“because that’s what we have learned”). Understanding mathematics as a collection of arbitrary patterns must not be the aim of mathematics lessons and lacks algebraic thinking.

This is where the term structure becomes important. Instead of remaining on the surface of the pattern as a perceptible phenomenon, algebraic thinking is characterised as thinking processes to understand the underlying structure of the pattern and to use that structure to justify the emergence of the pattern. “Structural thinking is much more than seeing a pattern, such as ‘when one number increases by three the other goes down by three’” (Mason et al. 2009, p. 23). *Understanding structures* thus forms the core of algebraic thinking and can be defined as the process of recognizing and comprehending mathematical properties and relations.

Kieran (2018) emphasises the desideratum for more precise clarification of the term *structure* because of its importance in algebraic thinking. In their literature review, Venkat et al. (2019, p. 13) note that the term structure is defined very differently and that “we were comforted to find ourselves in good company in terms of our confusions”. It is understandably challenging to define such an important term precisely and, above all, soundly when it is used so differently in the literature. This is where a closer analysis of the development of algebra as a mathematical discipline is helpful, as has been explored in the previous section. With reference to structural algebra and following Mason et al. (2009) and Schifter (2018), Akinwunmi and Steinweg (2024) describe structures as mathematical properties and relations that define mathematics itself as characteristics of abstract objects of thought. Mathematical structures, according to Schifter (2018), refer to characteristics and properties that

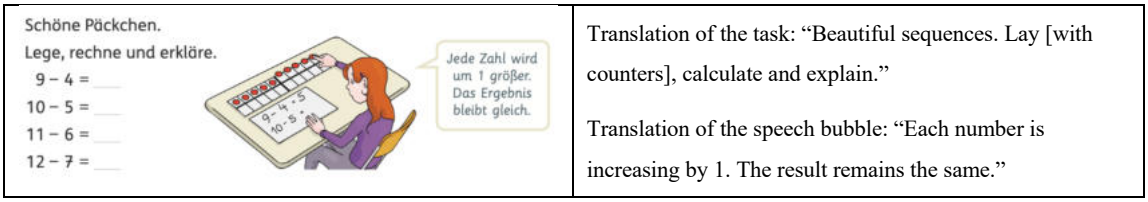
remain constant for all cases. Akinwunmi & Steinweg (2022) list four key ideas of algebraic thinking in arithmetic: understanding structures within *numbers*, *operations*, *equations* and *functions,* each one holding specific mathematical properties and relations to be explored. For example, the central properties of operations in primary school arithmetic are commutativity, associativity, distributivity, neutral elements and inverse operations (which are later extended by inverse elements within rational numbers). This list of central structures (here in the key idea of operations) is manageably short and can therefore be repeatedly addressed throughout the various grades of primary school. Some studies on early algebraic thinking investigate the understanding of such structures, especially in operations (e.g. Pitta-Pantazi et al. (2022) speak of *generalized arithmetic*). The empirical data for various numerical and operational structures have been expanded by Akinwunmi & Steinweg (2024). In summary, brief definitions of patterns and structures can be given:

- Patterns are regularities in visible phenomena.
- Structures are mathematical properties and relations.
- Patterns and structures are inextricably linked: Patterns provide access to structures and structures form the basis of patterns.

Fostering the understanding of structures

In German mathematics education, a fundamental, firmly established didactic principle is the *Operative Principle* (Wittmann, 2021 (original 1987)). According to this, the understanding of mathematical objects includes "to observe which effects properties and relationships are brought about by the operations according to the guiding question "What happens with ..., if ...?"" (Wittmann, 2021, p. 153). At the pattern layer, children can perceive phenomena, for example: the result remains the same if you switch the order of factors in multiplication. To understand, why the pattern occurs and holds for all cases (Schifter, 2018), an *Operative Proof* (Wittmann, 2021) is needed. With appropriate representations, students can e.g. understand that any rectangular array of counters (representing a multiplication) can be rotated by 90 degrees (representing the commutative multiplication) without changing the number of counters. An Operative Proof therefore has an epistemological character and leads to the understanding of structures. Similarly, Blanton et al. (2019) refer to this idea as representation-based proofs (Schifter, 2009), which are useful for *justifying generalizations*.

Akinwunmi and Steinweg (2024) confronted students with visual impulses from German textbooks (for example Figure 1). The students were first asked to re-enact the girl's action and then addressed with the question: Can you use the counters to explain why the result always remains the same?



Translation of the task: "Beautiful sequences. Lay [with counters], calculate and explain."

Translation of the speech bubble: "Each number is increasing by 1. The result remains the same."

Figure 1: Impulse to justify the constancy of difference (Nührenbörger et al., 2017, p. 89, © Klett)

The results of the study show that students use the impulse to move from focusing on number patterns to relating to the operation of subtraction. Nils' explanation is an example:

> Nils: So, if something is added to both, it's one more (*places another tenth counter*), but you also have to subtract one more. So, you can actually take this one straight away (*takes away the tenth counter*).

Some children also generalize, that "if you had more counters, you could always continue like that". With the focus on the general properties of subtraction, i.e. structures, children understand and thus also provide an Operative proof on why the constancy of difference holds for all numbers.

The results of this and further studies (Akinwunmi & Steinweg, 2024) show that suitable representations and impulses from the teacher are sufficient to enable children to reason on the basis of structures. Representations seem to play a crucial role in enabling students to understand structures. Moreover, the studies provide information about possible stimuli (e.g. Figure 1) to encourage children to take a structural view. This does not necessarily imply that every statement made by students can be assigned to the pattern or structure layer. Nevertheless, this distinction has important consequences for the promotion of algebraic thinking. It can promote teachers' awareness of generalisations with a focus on patterns or focus on structures and therefore help to create suitable learning opportunities and make mathematical structures the main focus of classroom communication. The aim of mathematics lessons should always be to go beyond the pattern layer and to understand the underlying structures.

Russell et al. (2011, p. 58) present some impressive successful answers from children to the question "Can you come up with a representation that shows this will always be true, no matter what numbers you start with? Make a picture, draw a model, but don't use any particular numbers". However, one should not underestimate the challenge of selecting a suitable representation. It is precisely the critical point that the children not only represent the numbers from the tasks that are linked with a symbolic sign in between. Instead, they have to find a suitable representation for the embodiment of the operation (e.g. a rectangular array of counters to represent a multiplication). If the children only find patterns in the changing numbers, they cannot think about the property of the operation. Constancy can only be understood as a property of subtraction if subtraction is perceived as an object. The subtraction in this example can be thought of in different ways, such as *taking away* or *determining the difference* (Selter et al., 2012). The impulse presented here uses taking away. A justification based on determining the difference would also be possible but would require different visual impulses (e.g. equidistant intervals on a number line).

Therefore, it is a challenging but essential task of the teacher whilst planning mathematics lessons, to select a suitable representation and to think through possible reasoning and generalising processes, rather than expecting the children to invent it themselves. This makes these challenges an important topic for research in teacher education. The role of the teacher was also recorded as a research desideratum at CERME12 by Hewitt et al. (2022, p. 464), because "the nature of the teacher-student interactions […] can be just as, or even more, significant as the tasks themselves." On the basis of one structure, a variety of patterns can be created for the students to explore. Tasks, suitable representations and impulses can then be developed to lead to the identified structures. It is the

responsibility of mathematics education to identify the essential structures, that are approachable for primary school students.

Consequences for the relation between algebraic and arithmetic thinking

Summarising the discussions at CERME12, Hewitt et al. (2022) raised important questions about the relation of algebraic and arithmetic thinking.

> There was much discussion about the relationship between arithmetic thinking and algebraic thinking. Sometimes these were mentioned as a transition from one to the other. If that were to be the case, then is there a type of continuum between the two or is there an abrupt shift from one to the other? […] Alternatively, are these seem as running in parallel with each other, or are they actually more separate than sometimes imagined? (Hewitt et al., 2022, p. 464).

It is apparent that on the basis of this theory it cannot be a matter of separating algebraic thinking from arithmetic thinking. Arithmetic involves, among other aspects, the conscious and flexible selection of calculation strategies depending on concrete numerical values in order to determine numerical answers to calculations. These strategies only work because of and on the basis of number properties (such as the decomposability of numbers into their place values) or mathematical relations between numbers. Numbers, operations, equations, etc. thus become representative examples that offer patterns for discoveries in order to understand structural properties and relations as universally valid concepts (Akinwunmi & Steinweg, 2024). In this sense, algebraic thinking refers "to think about arithmetic in an algebraic way" (Malara & Navarra, 2003, p. 9). Rather, algebraic thinking can be regarded as a specific way of thinking that contributes to the various areas of mathematics from a meta-perspective and is inextricably interwoven with them - as it is with arithmetic thinking. Algebraic thinking is inherent in other disciplines, just as variables, equations, etc. occur in all fields. Therefore, algebraic thinking should not be considered contrary to, but rather complementary to other fields of mathematics. It is important not to degrade arithmetic, trying to distinguish algebra from arithmetic, as if arithmetic was always just a matter of performing calculations. Algebraic thinking does not replace arithmetic thinking and is not only taught in elementary school as a precursor to algebra in secondary school. Algebraic thinking provides a deeper and richer understanding and is thus at the benefit of arithmetic (Winter, 1982).

This paper is to be understood as *one* possible view in the discourse on the nature of algebraic thinking. It defines algebraic thinking as thinking processes for the understanding of structures based on the term structure as opposed to pattern. Finally, the diversity of possible perspectives on algebraic thinking contributes to a fruitful, lively discourse to enable constant further development.

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Mathematics teachers' correspondence and covariation approaches to quadratic functions in relation to their covariational reasoning

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As part of a year-long multi-case study, in this paper, we investigated two high school mathematics teachers' correspondence and covariation approaches to quadratic functions in relation to their covariational reasoning. We conducted clinical interviews about quadratic functions. Analyzing the data from both teachers' reasoning processes using the code list developed throughout the study based on literature and the data, we found that these teachers showed distinct reasoning in relation to their covariational reasoning. Specifically, the teacher reasoning covariationally up to direct coordination and gross quantification, showed the correspondence approach in explanations of quadratic functions. The teacher reasoning covariationally up to direct and systematic coordination and extensive and intensive quantification, used correspondence and covariation approaches to explain quadratic functions. These findings were discussed in terms of teacher knowledge.

Keywords: Mathematics teachers, correspondence and covariation approaches, quadratic functions.

Introduction

Algebra is a broad branch of mathematics from primary school to college that students need to learn conceptually. In high school mathematics, algebra, as studying relationships between quantities, is the most emphasized conception as students learn functions and families of functions. The literature emphasizes two conceptions of functions: static (correspondence) and dynamic (covariation) (Thompson & Carlson, 2017). The static conception of function relies on the set-theoretic structure and correspondence of the elements in the sets with the criteria that every element in the set of domain should match with one and only one element in the set of range. On the other hand, the dynamic conception depends on covariation, focusing on how variables change together, satisfying the specific criteria between domain and range mentioned previously (Lloyd et al., 2010).

Oehrtman et al. (2008) pointed to the correspondence approach's (action view) indicators in someone's mathematical activity. First, a person might consider that functions are related to a rule. This also leads to considering a single value at a time as input or output by doing calculations (Oehrtman et al., 2008). In relation to this, while finding the sets of domain and range, the person might consider specific rules such as for a rational function, the denominator needs to be different from zero, and for a square root function, the inside of the radicand needs to be greater than or equal to zero. Similarly, the person might tend to relate the inverse of a function to the algebraic and geometric properties. This consideration might stem from the idea that the graph provides a geometric figure of a function rather than representing a simultaneous relationship (Moore, 2016). For instance, a person might have the idea that a quadratic function with real coefficients has a U-shaped curve rather than explaining how the values of the function are changing with respect to the input values of the function. Precisely, the person might reason on the effect of parameters on the function's graph

through the change in the shape of the graph, such as 'the graph opens up/opens down depending on the change in the value of the leading coefficient of the quadratic function.

Oehrtman et al. (2008) also explicated the indicators of someone's actions operating from a covariation approach (process view) while working on functions. Specifically, considering the definition of a function, someone might see a function as a process and a dynamic relationship between input and output values. In relation to this, the person considers that process without the necessity of a formula. This further leads to imagining and explaining the process and relationship between input and output values without calculating single values of the function at a time. Since the person does not need to focus on discrete values, s/he can imagine how inputs and outputs change together and continuously. This is important because the person might be aware that "A function is a transformation of entire spaces" and "Domain and range are produced by operating and reflecting on the set of all possible inputs and outputs" (Oehrtman et al., 2008, p. 10). That is, not one element in the domain, but all the elements produce output values depending on the functional relationship. Additionally, the person might consider the inverse of a function as a relationship between input and output where the process is reversed. Therefore, functions are considered to be dynamic entities. Moreover, a person reasoning with the covariation approach might consider that the graph of a function represents the traces of the points, which are multiplicative objects that refer to the dynamic relationship between variables (Thompson & Carlson, 2017).

Regarding the correspondence approach, Thompson and Carlson (2017) importantly commented that "although the correspondence meaning of function solved problems that arose for mathematicians, introducing it in school mathematics made it nearly impossible for school students to see any intellectual need for it" (p. 422). In this regard, the covariation approach might provide students with the intellectual need to learn functions by considering them through real-life situations. Therefore, there is a need to support students' learning of functions through covariation in secondary school. More importantly, the way teachers reason about functions needs to be investigated. In this study, quadratic functions were taken into consideration for several reasons. First of all, it is the first family of nonlinear functions teachers teach to high school students. It is essential to differentiate it from other function families, such as linear functions, based on their behavior by emphasizing having a linear rate of change (Lloyd et al., 2010). Secondly, none of the research studies included information about teachers' consideration of the covariation approach as a prerequisite to understanding quadratic functions. This is reasonable as Thompson and Carlson (2017) pointed out that "most researchers were operating from the perspective of a set-theoretic meaning of function, so they felt little incentive to attend to… covariational reasoning" (p. 445). In this regard, the findings of the study aim to provide a picture of teachers' understanding of quadratic functions in relation to teachers' covariational reasoning by answering the research question: How do teachers with distinct levels of covariational reasoning reason about quadratic functions through correspondence and covariation approaches?

Method

This study is part of an extensive one-year-long multi-case study investigating how two teachers' covariational reasoning processes support their knowledge of mathematics and teaching practices in quadratic functions. The whole research process included intensive interaction with teachers via a

series of interviews (approximately 17 hours in total) and classroom observations (24 class sessions in total). The participants were two mathematics teachers with distinct teaching experience, educational background, and demonstrating distinct levels of covariational reasoning. Specifically, Deniz, a graduate of a mathematics department, has taught secondary school mathematics for twenty-five years in high-ranking public high schools. Miray, a graduate of a teaching mathematics program, has five years of teaching experience at a high-ranking private high school. Regarding their covariational reasoning, Deniz frequently showed the following pattern: use of secondary variables, uncoordinated way of reasoning, and gross quantification. However, Miray frequently showed the following pattern: use of primary variables, direct and systematic coordination, and extensive quantification (see Belin et al. (2023) for details on teachers' covariational reasoning).

In this paper, with reference to their covariational reasoning, we specifically present the findings of a clinical interview that focused on teachers' knowledge of quadratic functions. The interview questions probed into the teachers' conception of quadratic functions, the properties of the graph of quadratic functions, and the relationship between the rate change and quadratic functions. Teachers were both asked about their definitions of the concepts and given specific cases regarding the graphs of quadratic functions to reason about. Their interpretations were intended to reveal their reasoning processes rather than focusing on their answer for students.

Table 1: Codes for correspondence and covariation approaches

Theme	Category	Codes
Correspondence Approach	Explaining functions through correspondence	Imagining a single value at a time as input or output, function as a relation, focusing on set properties of functions (1-1, on to, etc.), function as a mapping between two sets, function as a rule
	Static interpretation of the relationship between variables	Considering a point (x, y) as static, considering f(x) as a name, focusing on the geometric properties of the graph, finding a pattern through tabular values, using algebraic operations
Covariation Approach	Explaining functions through covariation	Focusing on invariant relationship, considering the inverse of a function as a reversal process, emphasis on the rate of change
	Dynamic interpretation of the relationship between variables	Determining the quantities in problem context as variables
		Variation, covariation, quantification

During the analysis, the researchers categorized teachers' explanations as having both approaches or having one of them. The unit of analysis was a single sentence or a paragraph depicting the teacher's reasoning in detail. Researchers coded the data using an initial list of codes stemming from the literature on functions and quadratic functions and a pilot study with two prospective mathematics teachers regarding quadratic functions interview. While coding the data from Deniz's and Miray's interviews, some codes were merged, some codes were moved to other categories, and some codes were deleted during this process. The final list of codes which are shared in Table 1 were formed in three rounds of the data coding process. Especially codes in the covariation approach had also sub-codes. However, due to the limited space, they are not shared in this paper.

Findings

Table 2 explains teachers' frequency of using functional approaches during the interview. Specifically, for all the content, Deniz's explanations through correspondence and static

interpretation of the relationships between variables were the dominant characteristics of the reasoning. Although scarce regarding quadratic functions, he could dynamically interpret a given functional relationship through direct coordination and gross quantification. In contrast, Miray used correspondence and covariation approaches to explain quadratic functions. Also, while interpreting a functional situation considering the properties of quadratic functions and the relationship between the rate of change and quadratic functions, although she used explanations pointing to a static view, such as explaining geometric properties of a graph and use of algebraic expressions, she mostly dynamically explained the relationships by focusing on the amount of changes in the values of independent and dependent variables and making additive simultaneous comparisons.

Table 2: Teachers approaches for reasoning on quadratic functions

Theme	Category	Explanation of quadratic functions		Properties of the graph of a quadratic function		Relationship between a quadratic function and its rate of change		Total	
		D	M	D	M	D	M	D	M
Correspondence Approach	Explaining functions through correspondence	5	1	0	2	1	0	6	3
	Static interpretation of the relationship between variables	8	7	9	10	0	2	17	19
Covariation Approach	Explaining functions through covariation	0	7	0	0	0	2	0	9
	Dynamic interpretation of the relationship between variables	3	8	0	12	0	21	3	41

The following subsections elaborate on teachers' explanations of quadratic functions, properties of quadratic functions, and interpretations of the relationship between variables, emphasizing the rate of change.

Explanation of quadratic functions

Deniz defined quadratic functions as $f(x) = ax^2 + bx + c$ where a is non-zero and expressed that the graph of a quadratic function is called a parabola. For the content, related to quadratic functions in the mathematics curriculum, he first mentioned quadratic equations and algebraic operations such as factorization since they were necessary to find the zeros of the quadratic function for drawing the graph. Similar to Deniz, Miray gave an algebraic definition of quadratic functions as "the polynomial function… $y = ax^2 + bx + c$ where a is a nonzero real number, b, and c are real numbers because we want the highest degree to be two. x and y represent independent and dependent, respectively." Notably, she thought of correspondence and covariation approaches together in the continuum of her reasoning. Different from Deniz, even though Miray defined quadratic functions emphasizing their algebraic form, she kept in mind how they differ from other functions considering the simultaneous change between independent and dependent variables. She stated:

> Miray: I would say the function is a relationship between two variables, variation and covariation. Since variation is to look at how x value changes, how y value changes so to compare change in x or for every same amount of unit change in x, the corresponding amount of change in the y value will be analyzed... How $ax^2 + bx +$

> c looks in general, what kind of relationship it is, in which situations we use it to represent the relationship between variables, what is the difference between linear functions, what is the rate of change, why we actually think of the function in daily life. When we use it to describe the relationship between two variables, they should be able to distinguish and recognize it. We need to discuss how having two (degrees) affects this rate of change, that is, how degree affects the rate of change so that when I make it three (degrees), s/he will understand what it is.

As the data indicated, Miray expressed that she thought of function as the relationship between variables and emphasized reasoning to analyze the change in y values with respect to the equal changes in x values. This refers to her consideration of the direct and systematic coordination of two variables, which might also be supportive of the additive comparison of changes between variables, leading to extensive quantification of changes in the values of variables. It is also essential that she also included the rate of change as a necessary component for quadratic functions. Miray was aware that seeing that the algebraic expressions differ in terms of the degrees of the function is not enough to interpret the function behavior. For Miray, to understand how the change in the degree of the function affects the function's graph, consideration of the rate of change is essential.

Properties of the graph of a quadratic function

When teachers were asked about the effect of parameters $a, b,$ and c on the graph of a quadratic function $f(x) = ax^2 + bx + c$, teachers' reasoning differed. Specifically, Deniz considered figurative aspects of drawing graphs as he explained geometric properties such as directions and shapes (being narrower or wider). For instance, Deniz thought of the leading coefficient 'a' by saying "Now, for example, a determines the openness and closedness, how open and how closed…If a is bigger than 0, the arms are upward. If a is less than 0, the arms are downward." Similarly, regarding the effect of 'b,' he considered that b determines the place of the vertex together with the values of a. He exemplified that when both a and b are positive values, the value of r, which is $-b/2a$, is placed on the negative side of the x-axis. Therefore, he inferred that the b value changes the place of the parabola to the right and left. Similar to 'b,' for the effect of c, he considered that the graph moves up or down when c is bigger or smaller respectively. His explanations pointed to the fact that he considered the behavior as a rule without explaining its reason. It could be deduced that the only focus was on the appearance of the graph instead of the relationship between x and y. Therefore, it is related to static shape thinking referring to the figurative aspect of a graph.

Miray, on the other hand, first considered the vertex form of the function as $a(x-h)^2 + k$ and by taking the square of the expression she wrote $a(x^2 - 2hx + h^2) + k = ax^2 - 2hax + ah^2 + k$. Then, she matched the coefficients in the standard form with those in the vertex form as $b = -2ah$ and $c = ah^2 + k$. It is important to mention that Miray's writing for the correspondence between coefficients in the standard form and the vertex form is valuable since it shows that she can use both forms simultaneously by relating them to each other. Specifically, she can deduce that $h = -b/2a$ and $k = c - a(-b/2a)^2$. Using both algebraic expressions helped her comment on what happens to the graph of a quadratic function when b is changed since examining the change in parameters a and c is more straightforward than examining the behavior of the function by changing the coefficients b. In contrast, while explaining the effect of changing 'a' on the graph of a quadratic function she focused on the amounts of changes in y values for the same x values. In particular, Miray focused on

the geometric aspects of the graph and the reasons behind them by stating the rate of change characteristics in relation to the change in 'a.'

Miray: Now when I change a, the function... if I think of it as y equals x squared, when a is greater than 1, well, for the same x values, because I'm multiplying the y values by a positive number greater than 1, it's like the y points are actually coming up higher. There is a vertical stretch in the graph. On the one hand, the rate of change also increases. So, the rate of change of x's is increasing. It starts to change faster, x's and y's change together. If a is between 0 and 1, then I decrease the rate of change with the same logic. I decrease the rate of change, and it starts to increase more slowly. While x's change, y's change more slowly.

As the data indicated, Miray had the following knowledge: When the leading coefficient 'a' in $f(x) = ax^2 + bx + c$ changes between 0 and 1 magnitude-wise, the changes in y values with respect to the successive changes (such as unit changes) in x values lead to the conclusion that the changes in y values slowly increase. Therefore, values of the average rate of change get smaller values. Also, when the leading coefficient 'a' in $f(x) = ax^2 + bx + c$ gets values more than 1 magnitude-wise, the changes in y values with respect to the successive changes (such as unit changes) in x values leads to the conclusion that the changes in y values rapidly increase. Therefore, values of the average rate of change get higher values. For reasoning about 'c', Miray considered $y = x^2$ as the parent function, took a point $(1, 1)$ on the parabola, and specified the zero of the function. Then, she took the c value as positive. She concluded that the parabola shifts c units upward, and the new point on the graph is $(1, 1 + c)$, and also $(0, 0)$ becomes $(0, c)$ in the new function. Similarly, she considered c as negative by saying that the graph is shifted downward by c units. Then, she compared the distance between the points in the parent function and the corresponding points in the new function. She concluded that all the y values increased with the same unit as c. For this reason, she considered that the changes were the same. With her explanations, she supported her idea that the rate of change of the function does not change when the parabola shifts upwards and downwards.

Relationship between a quadratic function and its rate of change

Two cases 'The change of rate of change is positive and the function has two zeros' and 'The change of rate of change is negative and the function has no zeros' were given to teachers for their interpretation and drawing for the general structures of the graphs. Deniz stated that there was no special relationship between the rate of change and quadratic functions. However, by using the formula $f(b) - f(a)/b - a$ to find the rate of change, Deniz explained that one could conclude that the value was the slope of the line for a linear function and slope at a point for a second-degree function as follows:

Deniz: Well, I think there is no special relationship, but the formula can be done by using $f(b) - f(a)/b - a$, but the relationship with the coefficient... well, for example, if it is 1st order, for example, when it is a 1st order function, the rate of change is equal to the slope. If the slope is positive, the function is an increasing function. If the slope is negative, the function is a decreasing function. Now when we consider the slope at that point, for example, 2nd order functions, if the slope at that point is positive, it is increasing in that range. If it is negative, it is decreasing in that range.

For both conditions, Miray's overall reasoning indicated that she had the knowledge that 'the change in the rate of change is positive when the leading coefficient of a quadratic function is positive' and

'the change in the rate of change is negative when the leading coefficient of a quadratic function is negative'. To illustrate, for the condition that the change in the rate of change is negative and there are no real roots, Miray directly stated that the leading coefficient is negative by referring to her reasoning on the effect of the parameter 'a' on the graph. Then she drew a graph (see Figure 1) for the given condition. She explained the algebraic properties as follows: a and c are less than zero, and the discriminant is less than zero.

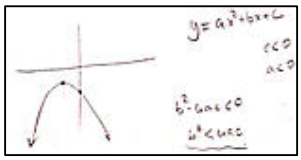

Figure 1: Miray's written work

Furthermore, she interpreted by tracing the graph after reaching the maximum point that as x increases, the y values decrease. She further commented on her explanation for the simultaneous change in x and y values after the maximum point was reached by coordinating the additive changes in x and y as follows:

R: So, uh, can you explain how did you decide?

Miray: ... If I think about the point where it is symmetric, that is if I think about the x coordinate of the vertex, when I increase the x values, that is, the differences between the y values will already open magnitude-wise in the intervals where I increase the x values one by one... In both conditions, let the x's increase one by one, and the y's change for the first example 1 3 5 7. Here, something like -1 -3 -5 -7 is related to the increase and decrease. In other words, nothing about it being quadratic does not change; only that it is looking down, that is, its arms are looking down or up. I highlight that it affects it.

As the data showed, Miray focused on one unit changes in x values to coordinate the changes in y values. She showed an additive simultaneous comparison of changes in x and y values for both conditions. It is important that she compared and contrasted concave up and concave down graphs considering the sign of change in y values as x values change in one unit.

Conclusion and discussion

The study's overall finding revealed that differences in teachers' covariational reasoning predicted their explanations for functions and quadratic functions as they differed in their attention to the correspondence and the covariation approaches in reasoning processes. Specifically, Miray's reasoning covariationally up to direct and systematics coordination and extensive and intensive quantification depicted in her explanations of both correspondence and covariation approaches. Furthermore, she pointed to the covariation approach while explaining her understanding of the quadratic functions. This might provide insight into her primary inclusion of the covariation approach along with the correspondence approach in her explanations (Ayalon et al., 2017). It can be argued that Miray considered the covariation approach as complementary to understanding correspondence approach. Specifically, while explaining the properties of the quadratic function graph, such as openness, symmetric nature, maximum/minimum points, and their relation to the rate of change, she used the changes in the values of dependent variables with respect to the changes in the values of

independent variables. Her explanations also supported her focus on the geometric properties of the graphs. On the other hand, Deniz's reasoning covariationally up to direct coordination and gross quantification is depicted in his explanations as only the corresponding approach. For quadratic functions, while explaining the properties of the graphs, he mainly focused on the geometric properties of the graph, which represented rules for him. This is reasonable as the dominant approach in teaching functions and textbooks is the correspondence approach (Ayalon et al., 2017). From both teachers' data, it might be inferred that covariational reasoning is essential to teacher knowledge (Tallman & Frank, 2020; Thompson & Carlson, 2017). Moreover, Miray's explanations pointed to her flexible reasoning with different forms and representations of quadratic functions, her making sense of quadratic functions linking with rate of change ideas, and the depth of her justifications behind her explanations indicating operational rather than figurative thought (Moore, 2016).

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A 6th grade teacher's and students' togethering in the algebra classroom: A process of overcoming emerging dialectical contradictions

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The aim of this paper is to shed light on the sociocultural process of togethering as a teacher and students in the middle-school classroom attune to each other as they generalize patterns. We address this issue through analyses of data from a Californian 6th grade algebra classroom. Learning algebra is multifaceted because the students are expected to shift their reasoning with respect to both form and function, i.e. from arithmetic to algebra syntax (form) and from calculational to analytical (function). It is crucial to the development of algebraic thinking that students shift from inductive work with numbers and manipulatives to the formulation of deductive arguments. In this paper we discuss how togethering manifests itself as dialectic contradictions in this shift, in terms of how emotions and meaning making permeate the space of joint action in the classroom.

Keywords: Algebraic reasoning, dialectic contradictions, introductory algebra, space of joint action, togethering.

Introduction

Algebra thinking is advocated as a central goal for mathematics teaching and learning as a way for students to learn the algebraic symbolism meaningfully and as a way of bridging different topics in the mathematics curriculum (Kaput, 1998; Kieran, 2018; Mason, 1996; Radford, 2010). Patterning activity, in which students are asked to generalize the growth of figural and numerical sequences, offers students opportunities for developing algebraic thinking (Lannin, 2005; Ellis, 2007). However, research point out challenges in achieving algebraic thinking in the classroom.

Reinhardtsen and Carlsen (2022) showed that the grade 6 students' "previously met calculational perspectives interfere when trying to reason analytically about number patterns and figural patterns" (p. 601). Three characteristics of the students' calculational perspective were discerned: (1) viewing numbers as the main objects of mathematical activity and operations as actions to perform; (2) using problem solving strategies that rely on consecutive series of calculations (not bypassing intermediate calculations); and (3) relying on examples for justification. Analysing the same data, we investigate the opportunities for developing algebraic thinking that arise as participants' perspectives come to the fore when coordinating their efforts solving patterning tasks (involving different mathematical structures such as proportional, linear non-proportional and quadratic).

The reported study stems from an algebra research project called VIDEOMAT (see Säljö & Kilhamn, 2019). Classroom discourse data from patterning activities in Finland, Norway, Sweden and the USA (California) were analyzed. The analyses showed, for instance, that the students, across countries, pursued surprisingly similar approaches to the patterning activities. Furthermore,

the students adopted calculational approaches over analytical ones, irrespective of the fact that the latter had been the object of newly teaching (Reinhardtsen & Givvin, 2019).

We take an activity perspective (Leont'ev, 1978) on the role of interaction in learning. Radford and Roth (2011) argued that interaction is a key component of classroom activity in which participants share and develop their thinking. Students learn by being exposed to cultural forms of thinking as they engage in the activity in which it is actualized through the participants' interactions. The teacher has a central role in bringing algebraic thinking to the fore as the students, left to their own devices, will mainly resort to previously encountered forms of thinking. Radford and Roth explained that for objectification to occur a deep exposure to "dialectic contradictions in the activity is required - one exposing in more detail participants' different perspectives" (p. 237).

Contradictions are neither problems nor conflicts that might be rapidly solved. Rather, they are manifestations of systemic tensions that emerge and are inherent to the situation in question, for instance emerging tensions in teaching-learning situations in mathematics manifested as contradictions between individual and collective perspectives and contradictions between the teacher's guidance and the student's agency. As such, a dialectical contradiction is understood dynamically as a unit of opposites.

Therefore, to investigate the learning potentiality in inviting and making student ideas central in classroom discussions of patterning tasks, we have formulated the following research question: *In what ways do the dialectical contradictions that emerge during Grade 6 students' engagement with a patterning activity facilitate the development of algebraic thinking?*

A sociocultural approach to the study of classroom interactions

We adopt a sociocultural approach to the students' learning of algebraic thinking. Thus, we view learning as a process of appropriating cultural tools, such as algebraic symbols and concepts, collectively nurtured and unfolding through the use of mediating artefacts (Vygotsky, 1986). Radford and Roth (2011), on a sociocultural ground, analyzed the interaction of the teacher and the students in the algebra classroom through two lenses, space of joint action and togethering. According to them, the space of joint action is "a space of relations and embodied reciprocated tunings in the concrete space of interaction" (p. 231). Radford and Roth's lenses are rooted in their theoretical position of the inseparability of individual consciousness and the consciousness of others. An inseparability mediated by language and other sensuous multimodal means of interaction. (Algebraic) thinking is thus viewed as a collective phenomenon.

The space of joint action conceptualizes the inseparability of consciousness, which embeds intercorporeal coordination through artefact- and sign-mediated actions (Radford & Roth, 2011). Students participating in a joint, goal-directed activity (such as developing algebraic thinking) think and act together, contributing to and building on each other's ideas through "tuning of speech, gestures, gaze, and actions" (Radford & Roth, 2011, p. 235). The space of joint action is thus a space in which intersubjectivity and collective thinking among the actors are accomplished.

The complex process of coordination involved in establishing a space of joint action, does not guarantee that students attain the object of the (algebraic) activity. The (mathematics) teacher's

involvement certainly plays a role. Radford and Roth (2011) addressed the involvement of the teacher through the concept of togethering. Togethering addresses ethical issues involved as students and their teacher tune to each other overcoming cognitive and emotional differences. The teacher and the students work inherently together to attain the object of the activity. However, the object appears differently to the teacher compared with the students due to their differing asymmetric positions in the classroom. The teacher approaches the mathematical topic as an expert, while the students solely rely on previous mathematical experience. Learning is therefore seen as the grasping and overcoming of such differing perspectives (Radford & Roth, 2011).

Teaching and learning are from a sociocultural perspective seen as two sides of the same coin. These are not separate, but rather a double-sided, irreducible process. The students and the teacher do shared work in the space of joint action. However, due to their differing positions, it is the students who have to appropriate the culturally bound mathematical meanings. Nevertheless, the teacher has to attend to the students' contributions. In our study, thinking about patterns from an algebraic perspective exemplifies a historical and cultural form of thinking.

For learning to happen, the inevitably inherent dialectic contradictions in the teaching/learning activity have to be exposed, through togethering. As such, togethering becomes "the committed engagement in joint activity with a common ideal/material object" (Radford & Roth, 2011, p. 242), indeed an ethical commitment by all participants to make the object a common object. These dialectical contradictions, opposing tendencies, materialize as different perspectives, calculational versus analytical, counting versus multiplicative structure, words versus signs, etc.

We include affect as intrinsic to cognition and learning, and view emotions as originating in activity. Roth and Radford (2011) argued that "Changes in activity are reflected in changed affective tonalities of the situation as a whole" (p. 21). In its movement the activity shapes the emotions, needs and feelings and the expectations of success culminating in individuals' motivation to act. Roth and Radford emphasized that students' emotions are rooted in the affective tonalities of the situation. Thus, emotions are fundamental as a first step for learning to occur. In analyzing classroom interactions in terms of algebraic learning opportunities using dialectic contradictions, we see these as consisting of affective moments as well as cognitive ones.

Methods

The data presented below come from an international study involving participants from four countries. Data were collected by local research teams and from four introductory algebra classrooms (self-reported by the teacher) in each country and included video-recordings from four consecutive lessons in each classroom. The teachers were asked to teach as they had planned without interference from the researchers. We present data from one Californian classroom in which an experienced teacher, who has a good report with her students, and 30 students worked on eight patterning tasks during the four lessons.

The first four tasks were numerical sequences, the sixth task included a numerical sequence presented in a function table, while the other tasks presented figural sequences. The first task had a proportional structure, tasks 2-4 and 6-8 had a linear, non-proportional structure, while the fifth task had a quadratic structure. In the lessons analyzed the teacher pursued an approach to patterns that

can be seen to involve three main phases of activity: (1) classroom exploration of a pattern; (2) making a function table and developing an explicit generalization based on systematic exploration of the relationship between two sets of numbers; and (3) extending the generalization to all subsequent terms and expressing the explicit generalization using alphanumerical signs.

The four lessons were transcribed, and salient episodes identified. The salient episodes analyzed involve situations in which participants' different perspectives are made evident in the classroom discussion, sometimes leading to tensions and abruption in the flow of interactions, and sometimes passed by as if unnoticed by the participants. We found that the affective tonalities of the situations needed to be considered. We take the same approach to analyze affective tonalities in classroom interactions as we take to analyzing the meaning making of participants. That is, taking an insider perspective to interpret subjective intent and collective meaning making relying on ourselves as fellow human beings and participants in human discourses, and taking an outsider perspective (professional) when determining learning opportunities in algebra. In particular, we study how dialectic contradictions manifest themselves in the shift from inductive work with numbers and manipulatives to the formulation of deductive arguments, a crucial process in the development of algebraic thinking (Ellis, 2007).

Results

There are several dialectic contradictions in the classroom patterning activity that are exposed in the teachers-student interactions: recursive versus explicit generalizations, multiplication in proportional situations versus linear non-proportional situations, functional perspective versus calculational perspective or factorization perspective. In our analysis we found that Ms. Lee's and the students' attunement to each other lead to different exposures of dialectic contradictions: (1) contradiction involving conflicting conclusions, ensuing discussion but not exposing the root causes; (2) contradiction without conflicting conclusions, no ensuing discussion of different perspectives; and (3) contradiction without conflicting conclusions, ensuing discussion that further expose the different perspectives. Due to space constraints, we only present and analyze episodes concerning the first dialectic contradiction in terms of opportunities for algebra learning.

A contradiction involving conflicting conclusions

Leading up to the four algebra lessons recorded, the class had worked with factorization. Presented with the first patterning task (the sequence 4, 8, 12, 16, ...) the students used their knowing of number structure to formulate an explicit generalization as they described the sequence as "multiples of four". The episode analyzed here occurred in the exploration phase of classroom work with the second patterning task (the sequence; 9, 14, 19, 24, …). The affective tonality of the classroom situation appears to change as the activity changes to include a sequence with a non-proportional structure and the students' factoring discourse no longer is sufficient to make meaning of the structure in terms of an explicit generalization.

As the students only presented recursive generalizations in the follow-up discussion, Ms. Lee tried to push them beyond a recursive approach and reminded them that in the previous sequence "they added 4, now they're adding 5." Carlos attuned to the teacher both when comparing the two problems and when attempting to apply a multiplicative approach. He argued: "In this problem, I

saw that you can't multiply anything… Like, uh, like from the beginning it was adding 5, but like, you can't multiply anything to 5 because it's a prime and odd number".

Carlos seems to indicate, using factorization (like the students did in Task 1), that the numbers in the sequence are not multiples of five, neither is it possible to factorize five as it is a prime number and thus multiplication cannot be involved in this number sequence. His argument seemingly is founded on a strong association between multiplication and proportionality. He also points out structural differences between the two sequences, raising the issue of non-proportionality in the latter sequence. Ms. Lee attunes to Carlos in that she confirms that a lot of what he said is correct, but she also voices the possibility of looking at the problem differently:

> I'm not sure if I agree with you. Let's take a look at this information in a different way, and let's see if we, we'll have this conversation in a minute. Okay. I hear what you're saying, you're talking about 5 being prime. You guys agree with that? Yes. 5 is prime, isn't it? And you're saying that because it is prime, and did you say also because it's odd? Then we're not going to be having some sort of multiplying thing going on. Okay, we'll come back to it in a minute and we'll take a look and see if we agree with that. Okay?

Ms. Lee decides to postpone further discussion of Carlos' statements and creates a common point of reference by setting up a function table. To do so, she expresses a possible disagreement with Carlos' conclusion. She reformulates the problem in terms of determining the relationship between two sets of numbers. She takes the lead and shows how two operations are involved and that one of them can be determined from the recursive generalization of the sequence. The strong connection students have formed between proportionality and multiplication is thus only implicitly addressed.

Although the same dialectic contradiction, a factoring perspective versus a functional perspective on a sequence of numbers, also emerged in the solving of the first problem, it was not marked by tension and conflicting ideas as the conclusions formed aligned. In the case of the second task Carlos and Ms. Lee became acutely aware of their very different ways of approaching the problem. Facing a non-proportional structure, Carlos is at a loss for how to involve multiplication as a tool for solving the task, describing 5 as prime and odd only tells him it is not possible. Ms. Lee is committed not only to solving the problem but also to expose the students to a culturally appropriate way of thinking about it. The teacher returns to Carlos' statements in the following lesson, but he is reluctant to participate.

Ms. Lee: Yesterday when we had this pattern, we took the numbers and we put it in our chart, our function table, and we came up with an adding five problem. And we realized that as we did our work, we're going to have to not only, um, use that five like we've been doing by multiplying, but then we had to add on four to get to the first number. So, we had to keep on adding four. You recognize this from yesterday? And so we came up with an expression that looked like that. Yeah? Now I remember Carlos yesterday you had a question. One of the things you had asked about, and I wanted just to come back to it really quickly because we never really answered it. You were saying yesterday you weren't sure that we were going to have any, multiplication happening with five because, you were saying, that five was a prime and five was odd. And so I want you to kind of think back at this, and did we answer your question or do you still have a question about that?

Carlos: Yeah, you answered it.

Ms. Lee: So what do you think? What you now know about this number five and how it works in this problem that maybe you weren't quite sure about here?
Carlos: Um…Um.
Ms. Lee: How is the 5 being used in the math problem here?
Carlos: Um, by multiplying, and I said that you couldn't use multiplication.
Ms. Lee: We are using as multiplication, and can you see why we're doing that?
Carlos: Um-hmm.
Ms. Lee: Can you tell us why?
Carlos: Because, um, I said that, um you can only—the-- for this problem you can only add, but that when you added it didn't work so you had to multiply.
Ms. Lee: Ok, and notice we're adding five over, and over, and over, again. And if I do it over and over and over again it's faster to do it as multiplication. Ok. Okay? It is true that 5 is a prime number? Yes. Is it true that five is an odd number? Yes; but does that mean that we won't use multiplication with five over here? Not necessarily. Okay? So that's-- I kind of just wanted to come back to that, because I know we didn't really address that yesterday.

Although Carlos expresses agreement with the teacher, he is not making an accurate connection between his previous argumentation and the acceptance of multiplication being involved. Carlos seems to have concluded that the factoring discourse, and the issues he raised regarding the nonproportional structure, are irrelevant to patterning activity (indeed, factorization is not mentioned again by anyone in the whole classroom discussions in the following lessons recorded). The affective tonality of the situation is manifested by Carlos' avoidance and compliance rather than curiosity and commitment, in spite of Ms. Lee's eagerness to follow up his previous contribution. Consequently, he does not enter the process of togethering, and meaning making regarding connections between a factoring discourse and the new functional discourse is suspended.

The dialectic contradiction, a factoring discourse versus a functional one, was revealed in the conversation. The participants were made acutely aware of a discrepancy in perspectives because of the conflicting conclusions. However, further commitment was needed to make the differences explicit, a need intuitively understood by Ms. Lee.

Discussion

We set out in this study to come up with answers to the question: In what ways do the dialectical contradictions that emerge during Grade 6 students' engagement with a patterning activity facilitate the development of algebraic thinking? We observed that the dialectic contradiction discussed led to limited opportunities for developing algebraic thinking. Carlos did not enter the process of togethering (Radford & Roth, 2011) and meaning making regarding connections between a factoring discourse (number structure) and the new functional discourse was suspended. Even though Carlos responded to the teacher's questions, he neither addressed nor returned to the nonproportional structure of the number sequence. A reason for this may be, in accordance with Stacey (1989), that students' strong association between multiplication and proportionality (seemingly the root cause for Carlos' and Ms. Lee's differing conclusions) was only implicitly addressed when the activity changed (to involving a non-proportional structure). In an exposure of this dialectic contradiction, there is an opportunity for learning to discern between proportional and non-proportional structures, i.e. thinking algebraically. This could bridge the students' meaning making regarding the two different sequences and a factoring and functional discourse.

The nature of the activity changed from a proportional structure to a non-proportional one, in which the factoring discourse was no longer sufficient. Therefore, the affective tonality of the classroom changed from commitment to avoidance and compliance. Carlos had no more clarity regarding why his initial meaning making led to a wrong conclusion after Ms. Lee's treatment of the problem. Thus, Carlos was not motivated to find out why his previous conclusion did not follow from his true statements regarding the number five. As Radford and Roth (2011) argue, affect is intrinsic to cognition and learning. Thus, the dialectic contradiction without exposing the root causes provided limited opportunities for developing algebraic thinking.

Emotions and meaning making permeate the space of joint action in the classroom, as students and the teacher immerse themselves in a sophisticated evolving coordination of actions (Radford & Roth, 2011). In general, it is in this space of joint action that intersubjectivity and collective thinking are accomplished. However, as we observe in our analysis, the emerging inherent dialectic contradictions cause tensions and violate the students' opportunities to learn, a result that goes beyond the results of Reinhardtsen and Carlsen (2022). The students and the teacher, in their togethering, "engage, respond, and tune to each other" (Radford & Roth, 2011, p. 236). Nevertheless, at least Carlos does not seem to attain the shared object of activity, to discern between proportional and non-proportional structures and thus bridge a factoring discourse held by the students and the functional discourse aimed for by the teacher.

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Employing erroneous examples in refutational text to address misconceptions in equations and inequalities

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This study tested the hypothesis that errors in equations and inequalities may occur due to a tendency to misinterpret the phenomenal sign of the expressions involved (e.g., that $-2x$ can only stand for negative numbers). It also tested if an intervention with incorrect examples presented in a refutation text could help students address their misconceptions better than correct examples. Pre-, Post- and Retention tests were administered to 119 9th grade Greek students. The Experimental Group (N=44) was presented with erroneous examples in refutational text and the Control Group (N=65) with correct ones in non-refutational text. The results showed that students' misinterpretations of the phenomenal sign can influence their errors with equations and inequalities. Although the gains were not maintained over time, both incorrect and correct examples were effective in helping students to address some of their misconceptions.

Keywords: Erroneous examples, phenomenal sign misconception, equations, inequalities.

Introduction

Equations and inequalities are core concepts in algebra, which students often struggle to understand. Many different reasons for these mistakes and difficulties have been suggested in the literature. Inadequate transfer of knowledge acquired in arithmetic to the algebraic context and a deficient understanding of rational and negative numbers as well as the symbolism used in algebra may explain some of students' main misconceptions (Kieran, 1992).

Grasping the meaning and use of literal symbols to stand for numbers is sufficient for students' conceptual understanding of equations and inequalities. However, although variables are introduced in the mathematics curriculum at elementary school (at least in Greece) as letters used to represent any number in a given set, students tend to think of them as natural numbers only (Christou et al., 2022). This tendency may result in the *phenomenal sign misconception* (PSM), which is the tendency to think of the phenomenal sign of a given algebraic expression to be the actual sign of the numbers it may only represent. For example, to think that expressions with positive phenomenal sign, like *5y*, represents exclusively positive arithmetical values, while expressions with negative phenomenal sign, such as *-3x*, represents only negative values. This tendency would be expected to affect students' considerations about the possible solutions, creating specific mistakes, in equations and inequalities that involve algebraic expressions with different phenomenal signs on each side.

Theoretical and empirical background

Natural number bias and phenomenal sign misconception

Understanding the solutions of equations and inequalities are essential skills in mathematics. However, students experience many difficulties when trying to determine their solutions, even when

they know how to solve them using the algorithms (Resnick, 1984). In the process of acquiring conceptual understanding of what an equation and an inequality are and what their solutions actually mean it is essential to understand that there are certain number values which may be substituted for the literal symbols that appear in them that would make the given relation hold. Therefore, students' conceptions about the number values of the literal symbols that are used in algebra to represent the generalized number (i.e., indeterminate number values, one at a time) is regulatory in understanding the solutions of the equations and the inequalities (Christou, 2017).

When introduced to the use of literal symbols to stand for numbers in algebra, students have been usually explicitly told that literal symbols are used to represent any number within a given set, which entails that they may stand for natural as well as rational numbers, positive as well as negative numbers, etc. However, previous studies that focused on the way students understand the specific algebraic symbolism have provided evidence about students' tendency to associate literal symbols only with natural numbers (Christou, 2017; Christou et al., 2022). This tendency has been explained as a byproduct of a more general *natural number bias* (Ni & Zhou, 2005), which is the phenomenon that characterizes students' tendency to use their prior knowledge of numbers, that has been organized around their initial knowledge and experience with natural numbers, in situations where rational number knowledge should be applied. This tendency very often results in specific mistakes and low performance such as in the case of determining the size of rational numbers or performing operations with them (Ni & Zhou, 2005). There is evidence that this bias may also affect students' restricted conception that literal symbols stand only for natural numbers which may also lead them to consider the sign an algebraic expression appears to have as a superficial characteristic of its form (phenomenal sign) to be the sign of the numbers it can only represent, i.e., the *phenomenal sign misconception (PSM)*. Students who would think of literal symbols (e.g., the y in the expression $-4y$) as natural numbers only, they would consequently think that the algebraic expression which contain them would represent only specific kinds of numbers (e.g., negatives in the case of $-4y$) (Christou et al., 2022).

Both the natural number bias and its byproduct, the PSM, may affect students' performance in many different mathematical domains, such as in understanding functions, and may also have certain consequences when trying to reason about the solutions of equations and inequalities (Christou, 2017). For example, students would incorrectly think of *4* as the only possible solution of the quadratic equation $x^2 = 16$, due to a natural number bias that would prohibit them from thinking of x as standing also for the negative number *-4*. Additionally, the solution to $-y = 5$ is *-5*, since $-(-5) = 5$. However, due to a PSM someone could argue that this equation has no solution because a negative number (i.e., $-y$) could never equal a positive (i.e., *5*). In the same line, the PSM could cause students to conclude that the inequality $-y > 5$ is impossible, because, in their head, a *negative* number ($-y$) could not be bigger than a positive one. Thus, students may respond incorrectly because of PSM, especially in cases where expressions with different phenomenal signs appear in the two sides of the relation. A main focus of this study is to investigate such errors in these specific types of equations and inequalities. A second aim is to test the possibility of intervening in a way that might help students to address some of these misconceptions.

Previous studies have shown that such misconceptions are deeply rooted in students' prior knowledge and are therefore difficult to change, especially because students are not aware that they hold these

misconceptions (Christou et al., 2022). Therefore, it is often suggested in the literature that exposing students to their own contradictions that arise from their misconceptions might help them to become aware of their misconceptions and also to address them.

Erroneous Examples in Refutational Text

In recent years, cognitive conflict as a learning strategy has gained renewed interest among researchers exploring instructional methods to address students' misconceptions in mathematics. One approach to creating this conflict is through refutational argumentation, which explicitly identifies and refutes misconceptions while presenting the correct concept (Mason et al., 2019). Refutational texts have been effective in helping students overcome misconceptions and improve performance in both science and mathematics (Lem et al., 2017). A study using refutational texts targeting misconceptions about signs in equations and inequalities showed that cognitive conflict can help students partially overcome these errors (Christou, 2012). The current study builds on this work by adding erroneous examples to refutation texts, focusing on reasoning about solutions to commonly encountered equations and inequalities.

Erroneous examples, which are characterized by intentional errors, are used as learning tools to challenge students and promote deeper understanding (Adams et al., 2014). By studying errors, students can better understand mathematical concepts by analyzing the mistakes of others, an approach that has previously been applied to other mathematical domains such as rational numbers and word problems (Adams et al., 2014; Große & Renkl, 2007). Although there is still some debate about the relative effectiveness of erroneous versus correct examples, studies suggest that erroneous examples can enhance learning, especially when paired with traditional examples (Zhao & Acosta-Tello, 2016). The present study contributes to this discussion by examining erroneous examples in refutational texts to address misconceptions in equation and inequality tasks.

The present study

The purpose of the present study was to investigate whether there are errors that students make when reasoning about the solutions of certain types of equations and inequalities that could be explained by their tendency to misinterpret the phenomenal sign of the expressions that appear in the tasks. Specifically, we hypothesized that a PSM would lead to certain types of errors in reasoning about the solutions of equations and inequalities. In addition, we hypothesized that an instructional intervention with erroneous examples presented in refutational texts would help students address errors in equations and inequalities that reflect a PSM better than correctly solved examples.

Method

Participants

Participants were 119 9th grade students from a public high school in Greece (14-15 years old, 57 girls). The Experimental Group (N=54) and the Control Group (N=65) were from four randomly selected classes. The participants were introduced to equations with literal symbols to represent the unknown from 7th grade, and inequalities from 8th grade. Thus, by 9th grade they had acquired significant experience in both areas.

Materials

There were a pre- and a post-test before and immediately after the intervention, respectively, and a retention test one month later. All three tests had the same structure: nine equations and nine inequalities, with problems that had the same form but used different numbers. Students selected among two alternative given responses whether each equation and inequality had solutions or not, and were also asked to briefly justify their response. Specifically, the question was "Can you find numbers that make the equation/inequality below hold?" There were presented two alternatives: a) Yes, the solutions are…., and b) No there are no solutions because…The questions were posed in this way as a means to direct students to reason about possible solutions of the given equations and the inequalities without necessarily using the typical algorithm for solving them. All tasks were designed to elicit mistakes from students who systematically misinterpreted the phenomenal sign or thought of variables as natural numbers only by presenting algebraic expressions with different phenomenal signs in the two sides of each given relation, or by giving equations which have both positive and negative solutions.

Specifically, the first part of each test included nine equations: a) three with one solution (e.g., $-x = 17$); in these cases it was expected that students with PSM would respond that there is no solution, because a positive could never equal a negative quantity, b) three quadratic equations with two integer solutions (e.g., $y^2 = 25$, with solutions: $y = \pm 5$); in these cases students who showed a natural number bias towards the numbers associated with the literal symbols would erroneously answer by providing only positive numbers as possible solutions, and c) three quadratic equations with no solution (e.g., $-y^2 = 16$; these tasks were used for the alternative response "No there are no solutions because…" to be correct in some cases.

The second part of the test involved nine inequalities with the same structure as the first part presented above. Three inequalities had multiple solutions (e.g., $-y > 7$), and three had no solution (e.g., $3x^2 < -3$). We expected that due to the PSM, students would think that there are no solutions in cases where, for example, a positive-like quantity was smaller than a negative-like one (i.e., in the case of $2 < -y - 2$). The internal consistency of the initial test was acceptable (Cronbach's alpha=0.75).

Teaching intervention

During the intervention, the experimental group read four erroneous examples of a hypothetical student's incorrect response to a solution in inequalities with different phenomenal signs on each side, presented in a refutation text. The incorrect answer would be due to the PSM. The control group read analogous correctly solved examples of inequalities (see Figure 1), which also reminded students that both positive and negative numbers can be associated with literal symbols. It was decided to include only inequalities in the intervention because they are generally more difficult for students (Kieran, 1992). Also, the solution to inequalities can be a range of numbers, which is a useful context for students to better understand literal symbols as a generalised number that can take a variety of

arithmetic values, rather than one or two currently unknown values. This knowledge could be applied to equations, provided that this knowledge is transferred to that domain.

Erroneous example in refutational text	**Correctly solved example in non-refutational text**
Find the possible solutions for: $x + 1 < -x - 1$ A student claimed that the inequality had no solutions because they believed that the positive quantity x + 1 could not be smaller than the negative quantity $-x - 1$. However, this student's answer is incorrect. The variable *x* can have positive values when positive numbers are assigned to it. However, this assumption does not hold true when negative values are assigned to x. For instance, if we consider the value of $x = -5$, then we have: $x + 1 = -5 + 1 = -4$ In this case, $x + 1 = -5 + 1 = -4$ which is indeed smaller than $-x - 1 = -(-5) - 1 = 4$. Thus, $x = -5$ is a possible solution.	**Find the possible solutions for: x + 1 < - x - 1** Keep in mind that the variable *x* can be assigned both positive and negative values. For example: For $x = -2$: $-2 + 1 < -(-2) - 1$ or $-1 < +2 - 1$ or $-1 < 1$, which holds true. Thus, x = -2 is a possible solution. But, if $x = 4$: $4 + 1 < -4 - 1$ or $5 < -5$, which is not true. Thus, $x = 4$ is not a solution.

Figure 1: Examples of the texts used in the intervention *(Text is translated from Greek to English)*

Procedure

The questionnaire was piloted with four 9th grade students from different public schools, which allowed us to finalize the test format and determine the completion time. In the main study, students completed the revised tests in their classrooms in the presence of their math teacher and a researcher. Each test took 20 minutes, which was a reasonable amount of time. The instructional intervention, which took place immediately after the pretest, lasted approximately 15 minutes. During this time, students received individual transcripts (see Figure 1) and reviewed the texts independently. The researcher answered questions and clarified the text as needed.

Data Analysis

Students' responses were scored as incorrect (0), partially correct (1), or correct (2). Answers were rated as partially correct if they provided only one solution for equations with two solutions (e.g., $x^2 + 1 = 5$ has one solution: $x = 2$), or if they gave incomplete justifications for correct answers in equations and inequalities. For example, responses indicating that $x < -x$ has solutions, even without specifying any, were considered more sophisticated than those indicating that there are no solutions or giving no answer, and thus received a higher score. We calculated separate mean scores for the experimental and control groups across equations, inequalities, and each test (see Table 1).

Students' incorrect responses were placed into three main categories: a) PSM mistakes, where students mentioned explicitly the phenomenal sign of the expressions in their justifications (e.g., "*the two terms of the equation have opposite signs and two opposite numbers cannot be equal*);" b) "Trials with integers," where students tested solutions only with natural numbers (e.g., *"the inequality* $2 <$

$-\frac{2}{2x}$ *has no solutions, I tried values such as: 1, 2...*)"; c) "Other," when we could not identify the reason for the mistake.

Results

Categories of incorrect pre-test answers

To address the first hypothesis of the study, we examined the types of incorrect answers the students gave in the pretest as a means to identify the most common causes of students' errors in reasoning about the solution of certain types of equations and inequalities. As shown in Table 1, more than 80% of the incorrect answers in the pretest were PSM errors or trials with integer errors, i.e., errors due to their natural number bias, in both the experimental and control groups, for both the equation and the inequalities. Another interesting finding was that PSM errors decreased between pretest and posttest, while the errors students made by trying only integers increased.

Table 1: Percentages of each category of incorrect responses in the Pre-/Post-/Retention tests

		PSM (%)			Trials with integers (%)			Other (%)		
		Pre-	Post-	Ret-	Pre-	Post-	Ret-	Pre-	Post-	Ret-
Equations	Exp. Group	44.8	34.8	40.8	41.4	47.8	41.4	13.8	17.4	17.8
	Con. Group	50.0	32.5	38.8	31.4	38.6	42.5	18.6	28.9	18.7
Inequalities	Exp. Group	62.3	39.3	45.7	23.2	46.7	24.1	14.5	14.0	30.2
	Con. Group	62.4	32.8	38.4	17.3	57.0	32.3	20.3	10.2	29.3

Effects of the intervention over time

To address the second hypothesis, we calculated mean scores for each test separately for the experimental and control groups (see Table 2). Mixed ANOVA tests were conducted for equations and for inequalities, with mean performance as the dependent variable, group as the between-subjects factor (experimental vs. control group), and test time (pretest, posttest, late posttest) as the within-subjects factors. Analysis revealed a main effect of test time, with students in both groups showing improvement from pretest to posttest on equations, $F(1.84, 196.44) = 15.02$, $p < .001$, $\eta_p^2 = .12$ (Huynh-Feldt corrected for sphericity violation), and inequalities, $F(2, 214) = 34.97$, $p < .001$, $\eta_p^2 = .25$ (see Table 2). The moderate-to-large effect sizes for test time on equations and inequalities (.12 and .25 respectively) indicate a positive trend over time for both groups. There were no significant differences in mean performance between the experimental and control groups for equations, $F(1, 107) = 0.03$, $p = .86$, or inequalities, $F(1, 107) = 0.51$, $p = .48$ across the time points.

We conducted additional paired sample *t*-tests to examine in more depth the effect of the intervention. The *t*-test showed an overall increase in students' mean scores from pre- to post-test in the experimental group on both the equation [($t(43) = 3.109$, $p < .001$] and inequality tasks [$t(43) = 5.560$, $p < .001$]. The same was observed for the control group's mean scores on the equation [$t(64) = 6.894$, $p < .001$] and inequality tasks [$t(64) = 6.351$, $p < .001$]. However, the experimental group did not perform significantly differently in the retention test than in the pre-test on equations [$t(43) = 1.393$,

p = .171] or inequalities [t(43) = 1.499, p = .141], and the control group performed significantly higher in the retention test than in the pre-test only on inequalities [t(64) = 3.281, p = .002] and not on equations [t(64) = 1.658, p = .102].

Table 2: Means (and standard deviations) for each group and task at each phase of the experiment

		Pretest		Posttest		Retention test	
		Mean	S.D.	Mean	S.D.	Mean	S.D.
Equations	Exp. Group	0,74	0,31	0,90	0,33	0,82	0,40
	Con. Group	0,72	0,26	0,92	0,24	0,80	0,36
Inequalities	Exp. Group	0,61	0,28	1,02	0,42	0,71	0,44
	Con. Group	0,58	0,25	0,90	0,32	0,77	0,37

Discussion

The results support the hypothesis that students' reasoning in solving equations and inequalities with different phenomenal signs may be affected by their tendency to misinterpret these signs. This study contributes to the literature by highlighting specific algebraic challenges. In the pretest, students often assumed that expressions such as $-y - 1 = y + 1$ and $-x = 17$ had no solutions, mistakenly believed that expressions with opposite signs could not be equal, and rejected inequalities such as $-y > 7$ based on the misconception that a negative value cannot exceed a positive value. These findings are consistent with previous research showing that students often misinterpret literal symbols as natural numbers and use the phenomenon sign as the actual numerical sign (Christou et al., 2022). In addition, these findings extend previous research by highlighting the consequences of these misconceptions in the specific domain of equations and inequalities.

The results also suggest that presenting erroneous examples in refutational texts helped students address aspects of their phenomenal sign misconceptions in equations and inequalities, consistent with previous studies on erroneous examples (Adams et al., 2014; Große & Renkl, 2007; Zhao & Acosta-Tello, 2016) and refutational reasoning (Mason et al., 2019). However, the experimental group's lower retention test scores suggest that the brief intervention did not significantly alter their underlying misconceptions about arithmetic values in algebraic expressions. This may be because students were not encouraged to self-explain or compare the erroneous examples with correct solutions, as recommended in previous studies (Große & Renkl, 2007). Future research could test the effects of these strategies on addressing these misconceptions. Against our second hypothesis, both erroneous and correct example interventions yielded similar results, consistent with findings that both approaches effectively address misconceptions in equations (Zhao & Acosta-Tello, 2016). This may suggest that reminders through both types of examples can help students overcome the phenomenal sign misconception. The increase in trials with integers from pretest to posttest suggests that after the intervention, students may be more inclined to test with numbers rather than relying solely on the phenomenal sign of the expressions. However, even with this shift, they remain influenced by the natural number bias, which leads them to use only integer trials and not other rational numbers.

Despite their temporary impact, this study provides evidence that using examples of students' errors in reasoning about equations and inequalities- while making students aware of common misconceptions- can partially address these misconceptions and improve task performance. These results contribute to the ongoing debate on the effectiveness of interventions using erroneous examples by providing empirical evidence on their utility, particularly in the domain of equations and inequalities. Students' extensive experience with natural numbers can lead to persistent misconceptions about algebraic expressions, even after exposure to non-natural numbers (Christou et al., 2022). To address these deeper-rooted issues, long-term learning designs are necessary. Incorporating examples that directly target specific misconceptions, such as those used in this study, into textbooks and regular classroom practice could be a promising approach which however, warrant further empirical investigation.

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From Inductive to Deductive Reasoning: Investigating Grade 6 Students' Algebraic Thinking Abilities

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This study examines algebraic thinking among Grade 6 students transitioning from primary to secondary grades. Two tests were administered: the first assessed algebraic thinking through tasks involving questions that prompted students to use inductive and deductive reasoning, while the second assessed the basic mathematical knowledge expected by the end of Grade 6. The results revealed a significant performance gap, with students performing better on inductive reasoning questions than on deductive reasoning questions. This indicates challenges in advancing from inductive arguments to logical deductions. Additionally, it was found that deductive reasoning significantly predicts performance on the basic mathematical knowledge test, whereas inductive reasoning does not. These findings suggest that deductive reasoning is less developed than inductive reasoning by the end of Grade 6, despite its critical role in students' overall mathematical development.

Keywords: Algebraic thinking, inductive reasoning, deductive reasoning, variables.

Introduction

Over the past 30 years, many researchers supported that early exposure to algebra can help mitigate the well-known challenges students face in secondary grades (Kieran, 2018). Consequently, algebraic thinking may progressively develop across all grade levels and become increasingly complex and sophisticated (Ellis & Özgür, 2024; Kaput, 2008). This idea led to the incorporation of early algebra in various mathematics curricula (Radford, 2018), but also sparked discussions about the nature of algebraic thinking and how to make it accessible to younger students (Kieran, 2022).

Despite the various definitions found in the literature, researchers generally agree that algebraic thinking is multi-dimensional (Kieran, 2022). To comprehensively understand algebraic thinking, different dimensions must be considered, including algebra content, mathematical processes that serve the investigation of related concepts and procedures, and forms of reasoning that support these processes (Chimoni et al., 2018). According to Kaput (2008), there are three basic algebra content strands from K to 12: generalized arithmetic, functional thinking, and modeling languages. The mathematical processes related to algebraic thinking involve conjecturing, generalizing, expressing generalizations, justifying, and proving (Kaput, 2008; Blanton et al., 2019; Mata-Pereira & da Ponte, 2017; Stylianides, 2008). Different forms of reasoning unfold within these processes, with inductive and deductive reasoning influencing students' algebra performance across different grade levels (Chimoni et al., 2023). As Blanton et al. (2019) emphasized, algebraic processes are interwoven with reasoning forms that supports justification and proof. Specifically, students often use inductive reasoning to form conjectures and draw generalizations from observed patterns (Radford, 2006; Rivera & Becker, 2007). As they advance, they increasingly rely on deductive reasoning to derive logical inferences based on established premises (Harel & Weber, 2020), which is essential for constructing justifications and proofs (Martinez & Pedemonte, 2014). Similarly, Radford (2014)

suggested that a distinct characteristic of algebraic thinking is "analyticity," referring to the deductive treatment of quantitative relationships that involve variables.

While all dimensions of algebraic thinking are important, and the need for their development as early as primary school is no longer a contentious issue (Freiman & Fellus, 2021), there remains insufficient empirical evidence to assess the effectiveness of early algebra approaches in fostering all of them (Kieran, 2018). Specifically, the transition from inductive reasoning, where students generalize patterns from specific examples to deductive reasoning, where they apply general principles to solve problems, remains a critical challenge in mathematics education. This study addresses this gap by analyzing quantitative data from 75 students who had just completed Grade 6, having been taught a mathematics curriculum and textbooks that emphasized the development of algebraic thinking since Grade 1. Specifically, the study focuses on the dimension of reasoning forms, aiming to understand whether these students can successfully employ inductive and deductive reasoning to solve algebra tasks and whether these abilities are linked to their basic mathematical knowledge. To achieve this, student responses to questions prompting them to induce generalizations are compared with those requiring them to deduce logical inferences. Additionally, students' basic mathematical knowledge is assessed with the aim to examine its relationship with their abilities in inductive and deductive reasoning. Ultimately, the study aims to provide insights into students' achievements as inductive learners and deductive thinkers just before entering secondary school.

In the following sections, we first explore the theoretical underpinnings of inductive and deductive reasoning, highlighting their meaning and roles in the context of algebra education. Then, we present the study's method and the main results from studying students' responses. Finally, these results are discussed, and pedagogical implications are suggested.

Theoretical Framework

Definitions of inductive and deductive reasoning vary across disciplines, but for educational purposes, they can be framed as follows:

(a) Inductive reasoning involves generalizing rules from specific observations to draw broad conclusions from individual cases (Van Vo & Csapó, 2023).
(b) Deductive reasoning involves deriving new information from a set of premises, where these premises establish a rational basis for drawing true conclusions (Harel & Weber, 2020).

In algebra learning, inductive reasoning is often evident when students recognize patterns and regularities within specific instances, formulate a conjecture about a general rule, verify this conjecture with additional examples, and, if the conjecture holds for multiple examples, establish a broader generalization (Rivera & Becker, 2007). For instance, by investigating the sum of three consecutive numbers, such as $1 + 2 + 3 = 6$, $2 + 3 + 4 = 9$, and $4 + 5 + 6 = 15$, students might conjecture that "the sum of three consecutive numbers is always a multiple of 3." To validate this, however, they need deductive reasoning, rather than additional concrete examples. By expressing a general number as n, and then the general structure of its next two consecutive numbers as $n + 1$ and $n + 2$, students can extract a true conclusion: the sum is $n + (n + 1) + (n + 2)$, which, based on properties of operations, equals $3n + 3$ or $3(n + 1)$ where 3 is a factor. In this sense, students move

beyond initial observations to justify the general structure of the sum and prove that it is always a multiple of 3 (Knuth et al., 2019). This approach involves denotation since the symbol n is used in a meaningful way, utilizing the interpretation of a variable as a general number (Usiskin, 1988). Additionally, this process involves analyticity, since students use the properties of operations (i.e., commutativity and associativity) and operate on unknown quantities in the same way they would if these were known (Radford, 2014)). In this sense, the structure of three consecutive numbers is taken as a known premise, from which new information is logically derived— namely, the expression for their sum—ensuring that the conclusion, that this sum is a multiple of 3, is true.

Both the reliance on specific examples to verify general statements and the use of logical reasoning to derive conclusions from axioms, definitions, and previously established results are considered essential proof schemes when examining the truth of mathematical statements (Harel & Sowder, 1998). However, students often struggle with developing deductive proofs (Martinez & Pedemonte, 2014). This difficulty has been attributed to their preference for using arithmetic-based arguments that are immediately evaluable, often excluding algebraic methods (Healy & Hoyles, 2000).

Aim of the Study and Research Question

The current study aims to investigate students' abilities in inductive and deductive reasoning within algebraic contexts, particularly considering that these students graduated from an elementary school that implemented an early algebra program. As a result, they are expected to be better prepared for a smooth transition to secondary-grade algebra. The specific research questions are:

1. What is the students' performance on algebra tasks that require inductive and deductive reasoning steps?
2. How do students' abilities in inductive and deductive reasoning relate to their basic mathematical knowledge?

Method

The study involved 75 Grade 6 students (mean age 11.5 years). This grade level was considered appropriate as at this stage students are expected to begin engaging with both inductive and deductive reasoning. The students received instruction that followed the national mathematics curriculum and textbooks. According to these documents, algebra is a core component of the mathematics curriculum. The topics covered up to Grade 6 include: the equals sign as a relational symbol, properties of numbers and operations, repeating and growing patterns, the concept of variable (unknown quantity, quantity that varies, a general number), algebraic expressions, and one-step and two-step equations.

The data collection took place during the last month of the academic year. Therefore, it was expected that students had the opportunity to develop the mathematical knowledge that was outlined by the national mathematics curriculum and textbooks. Two paper-and-pencil tests were administered. The first test included five algebra tasks, each consisting of two questions: question (a) prompted students to think inductively, while question (b) prompted students to think deductively. By pairing each inductive question with a corresponding deductive question within the same algebraic context, the test aimed to provide insights into students' transitioning from identifying patterns or generalizations

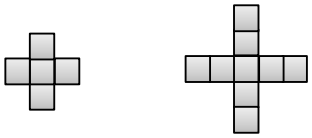
(inductive reasoning) to applying logical rules and principles to reach true conclusions (deductive reasoning). These tasks were adapted from the NRICH website (https://nrich.maths.org), where they are characterized as suitable for students aged 7 to 14 and categorized as medium-level challenges. The algebra test covered all the basic algebra concepts and procedures that students had been taught previously and involved the interpretation of different conceptions of variables: as an unknown quantity, as a quantity that varies, and as a general number (Usiskin, 1988). Figure 1 presents two sample tasks from this test. The first one refers to the sum of three consecutive numbers, described before: question (a) prompted students to observe that the sums of three consecutive numbers are multiples of 3, while question (b) prompted them to validate this conclusion. The second task refers to a figural pattern task: question (a) prompted students to inductively generalize the relationship between the number of the shape and the number of the squares in the shape, while question (b) prompted them to express a general structure that can be used to deduce the number of squares in any shape.

Below the sum of three consecutive numbers is presented.

Consecutive numbers	Sum
2, 3, 4	9
3, 4, 5	12
4, 5, 6	15

(a) Choose three other consecutive numbers and calculate their sum. What do you observe? What is common about these sums?
(b) How can you be sure that your observation is valid for the sum of any three consecutive numbers?

Shape 1 is made of 5 squares.
Shape 2 is made of 9.



Shape 1 Shape 2

(a) Can you predict the number of squares in Shape 5?
(b) Are you sure your prediction is correct? How can you be sure that you can calculate the number of squares for any shape in the pattern?

Figure 1: Examples of tasks included in the algebra test

In the algebra test, question (a) of each task was scored as follows: 0 points for an incorrect answer and 0.5 points for a correct answer, i.e., an answer that demonstrated a correct observation. Question (b) of each task was scored as follows: 0 points for an incorrect answer, 0.25 points for a partially correct answer, and 0.5 points for a fully correct answer. For example, in the task concerning the sum of consecutive numbers, 0.25 points were awarded for a partially correct answer that included a valid argument referring to the structure of three consecutive numbers through verbal explanations, and 0.5 points were awarded for a valid argument that used variables to represent the structure of three consecutive numbers and their sum. The second test assessed students' mastery of basic mathematical concepts and procedures expected by the end of Grade 6, covering numbers and operations, algebra, measurement, and statistics. In this test, 0 points were given for an incorrect answer and 1 point for a correct answer. Figure 2 presents sample tasks: one from the area of numbers and one from the area of algebra.

Which of the following fractions have 24 as a common denominator?
(a) $\frac{1}{20}$ and $\frac{1}{4}$ (b) $\frac{1}{14}$ and $\frac{1}{24}$ (c) $\frac{1}{3}$ and $\frac{1}{8}$ (d) $\frac{2}{3}$ and $\frac{6}{7}$

What is the result of $8.5 - 4.3 - k$, if $k = 3.5$?
(a) 0.7 (b) 1.3 (c) 7.7 (d) 9.3

Figure 2: Example of tasks included in the basic mathematical knowledge test

To analyze the data, we used the SPSS statistical package (SPSS27). Descriptive statistics were used to obtain an overview of students' performance on both tests. We also conducted a paired sample t-test to examine whether students' mean score in the algebra test as an overall is significantly different from their mean score in the basic mathematical knowledge test. To answer the first research question, we employed a paired sample t-test, which allowed the comparison of the mean scores of the students in the two types of questions (inductive and deductive). To answer the second research question, we used regression analysis which allowed the examination of whether students' performance in each type of question predicted their performance the basic mathematical knowledge test.

Results

Table 1 presents students' overall performance on the algebra test and the basic mathematical knowledge test. The results show that students' mean score in the algebra test (M = .35, SD = .18) was lower compared to their mean score in the basic mathematical knowledge (M = .64, SD = .21), while this difference was statistically significant (t = 10.62, p= .00).

Table 1: Students' Overall performance in the algebra test and the basic mathematical knowledge test

	N	*Mean*	*SD*
Overall algebra test	75	.35	.18
Basic mathematical knowledge test	75	.64	.17

Table 2 presents students' mean scores in each set of questions (inductive and deductive). The results show that students' mean score in the inductive questions was higher (M = .55, SD = .26) compared to their mean score in the deductive questions (M = 0.13, SD = 0.10), while this difference was statistically significant (t = 14.77, p= .00). This finding suggests that students had greater difficulties in deductive questions compared to inductive ones.

Table 2: Students' performance in the inductive and deductive questions of the algebra test

	N	*Mean*	*SD*
Inductive questions	75	.55	.29
Deductive questions	75	.13	.10

Table 3 presents the results of the regression analysis, which was conducted to examine whether inductive and/or deductive reasoning abilities in algebra influence students' performance on the basic mathematical knowledge test. This model accounts for 24.7% of the variance in students' basic

mathematical knowledge test performance. The results show that deductive reasoning abilities (β = 0.17, p = 0.02) is a significant predictor of students' performance on the basic mathematical knowledge test. However, inductive reasoning is not a significant predictor (β = 0.37, p = 0.26).

Table 3: Regression analysis results with the forms of reasoning as predictors

Intercept	*B*	*SE*	*β*	*p*
(Constant)	45.39	5.89		<.01
Inductive	.14	.12	.17	.26
Deductive	.84	.35	.37	.02

Discussion

This study provides insights into students' algebraic thinking abilities just before their transition from primary to secondary education, with a focus on inductive and deductive reasoning. A key finding is the significant difference between students' performance on inductive versus deductive questions, highlighting a gap between students' abilities related to these two forms of reasoning. Inductive reasoning, which involves identifying patterns and forming generalizations, seems more accessible to students at this stage. This finding aligns with previous studies indicating that inductive reasoning plays a crucial role in early algebra learning, as it allows students to make conjectures based on observable regularities (Radford, 2008). However, the lower performance on deductive reasoning questions underscores the challenge of progressing from inductive generalizations to logical deductions, revealing that algebraic thinking abilities are only partially developed by the end of Grade 6. One probable interpretation is that deductive reasoning requires a deeper understanding of formal proof structures—abilities that students often struggle to master without targeted instruction (Martinez & Pedemonte, 2014). Another important finding is the lower overall performance of students on the algebra test compared to the basic mathematical knowledge test. This aligns with existing literature suggesting that algebraic thinking is cognitively demanding (Kieran, 2018; Chimoni et al., 2023). A final finding is that deductive reasoning significantly predicts students' basic mathematical knowledge, while inductive reasoning does not. This underscores the crucial role of deductive reasoning for a solid mathematical understadning, including, but not limited to, algebra.

Overall, these findings suggest that there is a significant cognitive shift between inductive and deductive reasoning. Simply incorporating an early algebra approach into the mathematics curriculum and textbooks is not sufficient to ensure the effective development of all dimensions and sub-dimensions of algebraic thinking. Additionally, students' deductive reasoning abilities are crucial predictors of their overall mathematical performance and, therefore, their readiness for more advanced concepts. The weaker predictive power of inductive reasoning indicates that, while essential for early conjecture and pattern recognition, it may not be adequate for successful performance in broader mathematical tasks.

Given the difficulty students experience with deductive reasoning, more emphasis should be placed on developing this ability from the early stages of mathematics instruction (Stylianides & Stylianides, 2008). This could involve integrating more opportunities for students to engage in algebraic processes

that require justification and proof, even in the context of basic mathematical problems (Blanton et al., 2019). By encouraging students to think deductively alongside their inductive exploration, teachers can help them build the logical structures needed for success in advanced algebra and beyond (Martinez et al., 2011). To conclude, this study contributes to the growing body of research advocating for a progressive, well-rounded approach to algebraic thinking in early mathematics education. While inductive reasoning is essential for early algebra, the ability to make deductive inferences is critical for students' overall mathematical development. Future research should continue to explore how these two forms of reasoning interact, especially how students manage to assign meaning to algebraic symbols and use structural properties to deduce conclusions, exhibiting analyticity in their thinking.

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The development of a reference epistemological model of upper secondary school algebra in Sweden

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Reference epistemological models (REM) are employed as analytical tools in research within the Anthropological Theory of the Didactic. This paper provides a methodological description of how the development of a REM of upper secondary school algebra in Sweden has proceeded, and the result of this development. We conclude that the REM gives an accurate and ample description of what upper secondary school algebra in Sweden consists of. The REM is discussed in the light of the opportunities this kind of analytical tool entails, and how this methodological and theoretical approach answers the purpose of the study.

Keywords: Algebra, REM, praxeology, methodology, analytical tool.

Introduction

Epistemological models are employed within many theoretical approaches and fields of research. Already Foucault (1982) discussed the importance of tracing and mapping a specific knowledge area: "We must question those ready-made syntheses, those groupings that we normally accept before any examination, those links whose validity is recognized from the outset" (Foucault, 1982, p. 24).

In mathematics education research, epistemological models are employed for instance in APOS (action-process-object-schema) theory, where they are called *genetic decompositions*, often based on "results from literature, historical knowledge or the experience of teachers" (Bosch et al., 2017, p. 42; see also Dubinsky, 1991). In the Anthropological Theory of the Didactic (ATD), which this study is built on, these models are denoted *reference epistemological models* (REMs) and are "formulated in terms of a dynamical relationship between praxeologies, describing the researchers' interpretation of mathematics and the relation of this interpretation to specific school content domains." (Bosch et al., 2017, p. 43). REMs are models of the given mathematical domain under study, and therefore serve as references for didactic analyses by providing a more unbiased view of the material.

Several REMs in a wide range of mathematical domains have been elaborated by researchers in the ATD community internationally throughout the years (see for instance Barbé et al., 2005; Bosch, 2015). In many cases, the REM is used as an analytical tool to engage with empirical data. In the Swedish context, there are only a small number of empirical studies using REM as an analytical tool (e.g. Asami-Johansson, 2021; Viirman et al., 2022), none of them focusing on the case of school algebra at upper secondary level, the topic of the doctoral project of the first author. We therefore found it necessary to develop such a REM; to fill the knowledge gap, but also to be able to carry out our proposed study. The focus of this particular paper is on the elaboration of a REM of Swedish (upper secondary) school algebra and is therefore of methodological character.

Education is largely characterised by the cultural and social settings and norms of a country. Developing a REM for the Swedish context also helps us distinguish unique phenomena in the Swedish mathematics education, making comparative investigations across countries possible. By

extension, we want to be able to discuss what kind of algebraic knowledge different student groups are offered and how that may affect the teaching and learning of algebra in schools.

Consequently, the main aim of this paper is to describe the elaboration of this REM of school algebra. Correspondingly, the main aim of the REM is to provide a description of what school algebra in Swedish upper secondary school is, from which to analyse text books for the first two courses at upper secondary level. Since the paper is of methodological character, we only provide one example of how the REM is used as an analytical tool. To fulfil the aim of the paper, a detailed and thorough description of the development procedure is presented, followed by a discussion focusing on this procedure and methodological inquires connected to the theoretical framework.

Different methodological approaches to textbook studies

A number of textbook studies in algebra have been conducted around the globe. Since this study is situated in a Nordic context, we focus on a few significant studies from the Nordic countries and contrast them with our methodological approach. There are many possible ways of approaching textbook analysis, depending on the purpose of the study. As the purpose of our forthcoming analysis is to map and characterise the algebra content in Swedish upper secondary textbooks, to be able to make a sociological analysis of what knowledge different student groups are offered, we needed an epistemological ground that captured all nuances of upper secondary school algebra. Hence, we find the ATD (Bosch, 2015), and specifically the REM, particularly useful in this case. Jakobsson-Åhl (2006), for instance, instead used phenomenography to study algebra in mathematics upper secondary text books. Instead of developing an analytical tool to guide the analysis, Jakobsson-Åhl (2006), used grounded theory to detect phenomena in the text books, and form categories inductively. Another example of a similar approach, is Kongelf (2015), using inductive content analysis. Bråting et al. (2019), instead used Blanton's et al. (2015) big ideas of algebra as a base for their analytical tool. Rather than developing a more technical analytical tool, like our REM, they describe how they interpreted these big ideas in the analysis of the textbooks, and adopted these interpretations when examining their units of analysis.

Mathematics in Swedish upper secondary school

To support understanding of the REM and its context, we here outline the structure of mathematics education at the upper secondary level in Sweden. In the upper secondary school curriculum, there are both HEP (higher educational preparatory) and VET (vocational education and training) programs. As a result of a major curricular reform in 2011, Lgy11 (Swedish National Agency for Education, 2011) mathematics as a subject is divided into five courses (six if you count a specialisation course). For the first two courses there are three alignments, according to study programs; a, b, and c, aimed at vocational (VET), social sciences-oriented (HEP), and natural sciences-oriented (HEP) programmes, respectively. In VET programs, Mathematics 1a is compulsory and 2a is optional. In social sciences-oriented programs (HEP), 1b and 2b are compulsory and 3b is optional (compulsory in the economy program). In the technology program (HEP), 1c, 2c, and 3c are compulsory, and 4 and 5 optional, whereas 1c, 2c, 3c and 4 are compulsory in the natural science program (HEP). Mathematics 1a, 1b, 1c, 2a, 2b, and 2c are under consideration in this study.

The national tests in mathematics in upper secondary school are mandatory only for the highest final course within the student's program. This means that the only mandatory national test in mathematics for students in VET programs is for the course Mathematics 1a, whereas the mandatory test for students in the natural science program (HEP) is for the course Mathematics 4. However, a high proportion of upper secondary schools implement the national test for each course and each year, since they are well designed to test the course's core content. They also have thorough assessment instructions for teachers, since they are graded by the mathematics teachers themselves, and results are then reported to the Swedish National Agency for Education.

Theoretical underpinnings – ATD

The ATD is a "programme of research" (Bosch, 2015, p. 52) engaging with mathematical knowledge, such as (school) algebra, from an institutional perspective. A central assumption of the ATD is that the taught knowledge (see Figure 1), is not invented in the school institution (Bosch & Gascón, 2006). Rather, the scholarly mathematical knowledge needs to be made teachable in order to make sense in the school institution and for the learners. To this end, didactical and mathematical choices need to be made, affecting the body of knowledge (school algebra, in this case) in different ways. This process is called *didactic transposition* (Chevallard & Bosch, 2020).

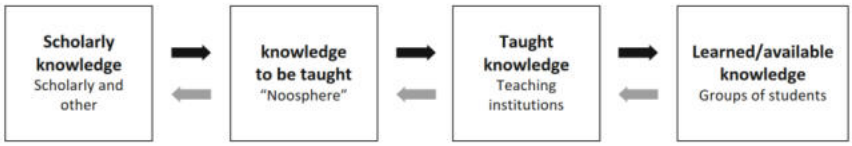

Figure 1: The didactic transposition process (Chevallard & Bosch, 2020, p. 171)

We view the three different course alignments in the curriculum as the result of three different didactic transposition processes, since different objects from the body of knowledge that is (school) algebra are being selected by the "noosphere" – people that have an interest in education (Bosch, 2015) – and put in the curriculum in the different course alignments for the different program types. Using ATD, and specifically didactic transposition, allows us to make differences between the transpositions visible. It also allows us to examine the data material in an unbiased way, since the REM broadly describes upper secondary school algebra in Sweden. The epistemological character of school algebra is epitomised in the REM in a way that makes it easier to detect the specific didactic transpositions in the different programs in Swedish upper secondary school and hence what kind of knowledge different student groups are being offered.

Praxeologies and scale of levels of didactic codetermination

From an ATD perspective, mathematics, like any human activity, can be described in terms of *praxeologies* (Bosch, 2015). The word *praxeology* is built from the Greek words *praxis* and *logos*, praxis being the practical part (k*now-how*) of a mathematical (human) activity and logos being the justifying or knowledge part (*know-why*) (ibid.). The assumption is that one needs both praxis and logos for an activity to hold and make sense. The praxis block is made up of a type of task (T) and one or several techniques (τ) connected to the Ts. The logos block is made up of technologies (θ) supporting and justifying the τs, and a set of overarching theories (Θ) supporting the θs. A praxeology is consequently made up of the quadruple [T, τ, θ, Θ]. This is also called a *point praxeology* (PP) of

an institution. Praxeologies centred on a technology θ are called *local praxeologies* (LP) whereas praxeologies centred on a Theory Θ are called *regional praxeologies* (RP) and can contain several types of tasks, techniques and technologies. Most REMs are formulated in terms of PPs, LPs and RPs, with the aim of giving an overall view of the mathematical domain analysed. By describing the content in terms of praxeologies, we get a REM that describes even how the smallest mathematical elements are linked together. By extension, we get an "aerial photo" of the contemporary discourse of school algebra. The REM in this study is built around PPs and RPs.

As mentioned, there are many factors outside the school institution that influence the teaching and learning, and consequently also the praxeological organisations. To identify what these factors and conditions are, ATD proposes the *scale of levels of didactic codetermination* (Bosch, 2015). In the forthcoming textbook study, we take society, school, pedagogy, discipline, domain, sector, theme and subject levels into consideration when analysing and interpreting the didactical constraints of school algebra, as it is proposed in the REM.

Method – developing the REM

We began the development of the REM by extracting all the algebraic content from the core content for the courses 1a, 1b, 1c, 2a, 2b and 2c in the national curriculum (Swedish National Agency for Education, 2021). The rationale for picking out a specific content area was based on previous studies of what school algebra in Swedish upper secondary school consists of (e.g. Hällback et al., 2022). After extracting the algebraic content from all six courses (1a, 2a, 1b, 2b, 1c and 2c) into a list, a table comparing all of the content was made. Thus, we could detect similarities in content over the three alignments and the two courses. For example, we found content regarding solving equations, but on different levels, in all courses and alignments. This type of content was placed on the same row in said table.

The reason for compiling all three alignments together, rather than developing a different tool for each alignment, was that we needed an analytical tool that captured all aspects of the algebra content, and not just one. Additionally, with one tool for each alignment, detecting nuances of differences between the three alignments in the analysis would not have been possible. This is a serious drawback since the raison d'être of the REM is to be able to make an accurate and close examination of the algebra content in the textbooks.

After compiling all algebraic content, we began formulating type of tasks (T) and corresponding techniques (τ), that corresponded to the core content. After this step, technologies (θ) and theories (Θ) were formulated, so that we had (almost) complete praxeologies. This work was also presented in a table, for the sake of clarity of the analytical tool. At this stage of the development, we noticed that in some cases, detecting techniques (τ) and technologies (θ), and anticipating how the practicing teacher will arrange the teaching, can be difficult. Therefore, we thought of adding national tests from all courses and alignments to the source material. Unfortunately, our analysis of the national tests did not add more details than the core content. We could detect some arithmetical properties, like the distributive law or the commutative properties. However, these were neither mentioned in the task formulations, nor in the assessment instructions for teachers, and therefore we chose to exclude these.

We detected two RPs in the development of the REM; algebra and functions. As a last step in the development procedure, we therefore grouped the PPs in accordance with these RPs.

The REM

The REM was developed in Swedish, since we followed the Swedish syllabi and national tests. As an analytical tool, it will also be used in its Swedish form, as the textbooks that will be analysed also are in Swedish. However, we translated it into English for the sake of academic diffusion. In the illustration of the REM as an analytical tool below, we also translated the analysis into English.

TYPE OF TASKS (T)	POINT PRAXEOLOGY (T)	TECHNIQUE (τ)	TECHNOLOGY (θ)	THEORY (Θ)
T_1: Manipulate algebraic expressions		• Factorisation and multiplication • Managing exponentiation	Arithmetic properties, such as: the difference of two squares, identities of binomial squares, rules for exponentiation, commutative property etc.	RP_1: Algebra
T_2: Express and motivate algebraic relationships	$T_{2.1}$: Formulate general relationships			
	$T_{2.2}$: Motivate the laws of logarithms			
	$T_{2.3}$: Motivate the laws of exponentiations			
	$T_{2.4}$: Motivate the identities of binomial squares and the difference of two squares			
T_3: Solve equations	$T_{3.1}$: Solve linear equations	• Digital methods (to make equation solving more effective) • Graphical and algebraical methods (unspecified)	• Arithmetic properties, such as: the difference of two squares, identities of binomial squares, rules for exponentiation, commutative property etc. • Equivalence, implication and the meaning of the equals sign • The concepts interval and linear inequality	
	$T_{3.2}$: Solve equations with square roots, cube roots etc.			
	$T_{3.3}$: Solve equations with exponentiations			
	$T_{3.4}$: Solve exponential equations			
T_4: Solve inequalities	$T_{4.1}$: Solve linear inequalities			
T_5: Comparisons of different equations	$T_{5.1}$: Comparisons of exponential equations and equations with exponentiations			
T_6: Tasks related to functions	$T_{6.1}$: Create graphs to functions	Digital methods for creating graphs to functions	• The concept of functions (including domain and range) and, representations and properties of functions (in words, function expressions, tables and graphs) • Especially linear, quadratic, exponential and functions with exponentiations o The concept of linear functions (räta linjens ekvation) • Quadratic functions: line of symmetry, extremum and zero of a function	RP_2: Functions
	$T_{6.2}$: Determine linear functions	Graphical and algebraical methods (unspecified)	The concept of linear functions (räta linjens ekvation)	
	$T_{6.3}$: Determine the value of a function	Graphical and algebraical methods (unspecified)	The concept of functions (including domain and range) and, representations and properties of functions (in words, function expressions, tables and graphs)	
	$T_{6.4}$: Comparisons of exponential and linear functions			

Figure 2: The REM, translated into English

In the leftmost column in the REM, we present the types of tasks (T). These Ts are overarching and are our interpretation of the more precise formulations in the syllabi. Next to the overarching type of tasks are the point praxeologies (PP). These are also tasks, but more specific and closer to the actual formulations in the syllabi. Note that T_1 and T_2 are not complete praxeologies. Since T_1 – manipulate algebraic expressions – already is so specific, we could not break it down into PPs. The τ's connected to T_1 are also in line with the formulations in the syllabi, whereas the θ is our interpretation of what mathematically justifies these τs. In T_2, on the other hand, we were not able to detect any τs in the syllabi or the national tests. We decided to leave this space blank, since the θ explains the assumed τs satisfactorily. As visible in the REM, most τs are relatively vague. There were no mentions of specific methods beyond algebraic, graphical and digital, in either the syllabi or the national tests.

In Swedish secondary school, the concept of linear functions is prominent in the teaching and learning of algebra. This concept as a whole is denoted "räta linjens ekvation" ($y=kx+m$ in the slope-intercept form), which would translate to "the equation of the straight line". Consequently, teaching and learning regarding "räta linjens ekvation" not only concerns the straight line, but also gradient, y-

intercept and how analytic geometry is connected to coordinate systems and graphing. This is why we put "räta linjens ekvation" as a θ in our REM.

Illustration of the REM as an analytical tool

In this illustration, we show an example of an exercise from each alignment. Each exercise belongs to the same PP, but we can see how the mathematical content still varies depending on the course alignment. The example shows content from RP_1 (Algebra) and T_1 (Manipulate algebraic expressions). All three exercises are taken from the same book series, Mathematics 5000+ from the publisher Natur & Kultur, belong to similar chapters and have the same level of difficulty.

Table 1: Example of exercises in each alignment, translation done by the first author

Course	Exercise	Interpretation
Mathematics 1a (VET)	2106: A taxi ride costs (75 + 40x), where x is kilometres. a) How much does it cost to go 3.5 km? b) How much more expensive is it to go 2.5 km instead of 1.5 km?	T_1 τ: multiplication θ: arithmetic properties
Mathematics 1b (social science HEP)	1311: Linnea has x kr (Swedish crowns, SEK). Write an expression that describes how much money she has if she a) gets 50 kr, b) gives away 20 kr, c) gets the double amount of what she already has.	T_1 τ: multiplication θ: arithmetic properties
Mathematics 1c (natural science HEP)	1313: A small salmon fry in a fish farming weighs 20 grams and increases its weight by 2.5 g/week. a) Write an expression that describes the fry's weight in grams after x weeks. b) Examine how many weeks it takes for the fry to double its weight.	a) T_1. τ: multiplication. θ: arithmetic properties. b) T_2, and explicitly $T_{2.2}$. τ: Graphical and algebraical methods (unspecified). θ: arithmetic properties and equivalence, meaning of the equals sign.

Note that in 1b and 1c exercises, the student is not only required to manipulate a given expression, but also to construct one. Since this type of task is not explicitly mentioned in the syllabi, it has been introduced by the textbook authors. The 1a textbook, on the other hand, did not contain any such task at this level of difficulty. It appears that the level of mathematical difficulty increases from alignment 1a to 1c. Moreover, the exercise from 1c not only contained T_1, but also $T_{2.2,}$ thus intertwining several praxeologies in the same exercise.

Discussion

In the development process it became obvious that the formulations in the mathematics syllabi are not complete praxeologies, which goes in line with the results of the previous study in this doctoral project, where we found that praxeologies in the VET programs are much more praxis-oriented, compared to praxeologies in the HEP programs which are more evenly distributed between praxis and logos (Hällback et al., 2022).

The core content is generally open for interpretation by teachers and text book authors. However, some paragraphs and some algebraic content are more explicitly written; in the core content covering linear functions, the equation of a straight line is mentioned several times. Content regarding quadratic functions is also relatively explicit; extremum and the zero of a function are for example mentioned explicitly. This suggests that some pieces of content are more important than others, which is not a problem per se, but it raises the question of factors outside the school institution that codetermine the teaching and learning situation. From this study, we cannot say anything about *how* these different levels of codetermination are influencing the school algebra content. We can only suppose that the society, school, pedagogy and discipline levels have influenced the content somehow, making some content appear more prominently. In the forthcoming textbook study, we will be able to discuss these factors more deeply and also to discuss them in the light of the different didactic transposition processes; VET, social sciences HEP and natural sciences HEP.

Jakobsson-Åhl (2006) and Kongelf (2015) demonstrate a more inductive methodological approach to textbook analysis, where the data material gives rise to a number of different categories, whereas Bråting et al. (2019) use a more deductive approach with predetermined categories guiding the analysis. Using a REM as an analytical tool takes the deductive approach even further, since it is a pure composition of mathematical content. On the other hand, although the REM is built around praxeologies, the theoretical framework is less visible in the REM itself, making the analysis less rigidly dependent on the theory. This also shows in the way the data material is approached. Bråting et al. (2019) use their theoretical framework to build categories, thus predetermining which exercises to choose and not to choose. Our analytical tool is instead built on the content in the syllabi, and then formulated in terms of the theoretical framework (praxeologies). Taking this approach, we need to consider every exercise that contains any algebra in any way. If we exclude exercises, we might miss some valuable information about the didactic transposition process (Chevallard & Bosch, 2020). If an exercise displays an incomplete praxeology, that does not correspond to the praxeologies, as expressed through the REM, that result is still valuable in terms of didactic transposition and the levels of didactic codetermination.

To sum up, we do not argue for one right way of conducting a textbook study. The aim and the research questions must, as always, guide the methodological approach. In this case, the ATD and developing a REM proved useful for the kind of inquiries that we had. What the actual textbook analysis will reveal, will be summarised and presented in a forthcoming article.

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Assessing students' emerging awareness of algebraic structures

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To solve linear equations, students need not only to understand algebraic procedures and concepts but also to identify relevant features of the equation and interpret the structure to be able to begin the solving process. This can be challenging if the equation is presented in an unfamiliar format, such as with non-natural numbers or new symbols. In this paper, we explore the features that 16-year-old students in vocational education focus on when sorting equations. The results show that students focus on a diversity of features when grouping equations together. We also discuss the potential of the task to assess students' algebraic structure sense.

Keywords: Algebra, linear equations, algebraic structure sense, sorting task.

Introduction

Students' awareness of structural properties of algebraic expressions, also known as algebraic structure sense, has been identified as an area in need of more research (Rojano, 2022a). This includes both investigating its importance for algebra learning and developing teaching methods that take it into account. This paper explores a sorting task to better understand its potential to reveal what features students consider relevant in linear equations, and its potential as a tool to assess students' awareness of structure.

The term *algebraic structure* is used to denote the internal order in an algebraic expression, referring to the relationship between the parts of the structure, that is either apparent or can be revealed by transformations (Hoch and Dreyfus, 2004). This algebraic structure can be interpreted in different ways, depending on how it is transformed or which parts are focused on; when parsing a linear equation we can either focus on the equality of two expressions $(5x + 3) = (7x + 1)$, or on the grouping of terms $(5x) + (3) = (7x) + (1)$, or on the relation between numbers, variables and operations $(5) \cdot (x) + (3) = (7) \cdot (x) + (1)$ (cf. Rüede, 2013).

When students learn algebra, it is natural that they focus on solving one linear equation at a time. However, as they become more proficient in solving equations, they start to notice general aspects that are central to the solving process. Mason (1996) describes this algebraic awareness as the ability to shift attention – not only to see values and operations but also regard them as expressions and objects. In this process, students need to move their focus from features that might have initially been their primary concern (e.g., the numbers) to the observed generalities, such as the combination of symbols and the relations they represent (English & Sharry, 1996). Hence, as students develop their algebraic thinking, that is the analytical dealing with indeterminate quantities (Radford, 2018), they should also learn to notice features important for the algebraic context. This can be compared to the notion of "saming" described by Sfard (2008) as a central process in learning new mathematical discourses, which involves finding commonalities in what was not previously considered the same.

In line with this, Rüede (2013) found different interpretations of the structure of fractional expressions among 12 novices and 12 experts. Some of the novices looked for visual similarities (syntactical structuring), while others wanted to change the expression (operational structuring). Some saw an initial structure but re-interpreted it (first-order structuring), while all experts considered several structures from the start (second-order structuring).

It is natural that students' awareness of structure is influenced by their previous mathematical knowledge, for example their familiarity with the algebraic sign system and their ability to transform expressions to reveal structure (Rojano, 2022b). Moreover, it has been proposed that students' initial learning of algebra is significantly influenced by surface features they are already familiar with (English and Sharry, 1996). Furthermore, several aspects of equation solving that research has identified as conceptually challenging for students are related to concepts that students already have some understanding of. For example, there is considerable evidence showing that the equal sign is often perceived as something "to do" rather than a symbol for equality (Bush & Karp, 2013). This misunderstanding can make an equation with the equal sign at the beginning (e.g., $480 = 12 \cdot x$) confusing, as the "answer" appears first. Another example, which has been studied extensively (Bush & Karp, 2013), concerns the various ways students interpret letters in algebra (e.g., as labels for objects). Coefficients in linear equations can also serve as a visual feature that guides students toward seeing structure (Rojano & Palmas, 2022), but they also represent an aspect with which students have prior experience, potentially influencing their interpretation of structures (Holmlund, 2024).

This paper is part of a PhD project aiming at developing teaching on structural awareness through a research lesson, in the form of a learning study. The project involves three teachers and one researcher who collaboratively design and evaluate teaching strategies and improve it through three cycles based on variation theory (Marton, 2014). Variation theory explains learning as the discernment of aspects that constitute an object of learning. In this context, *identifying the structures of linear equations* is the object of learning. This is taught by varying other aspects of the equations, such as the coefficients, the letter used for the variable, and the position of the variable, while keeping the operations constant within a specific structure (e.g., $a = b \cdot x + c$, $a = x + c$, or $a = b \cdot x$). One of the tasks used in the pre-test is analysed in this paper. The purpose of the task is to assess how students recognise equations as familiar structures – one of the abilities of structure sense according to the definition of Hoch and Dreyfus (2004) – by asking them to justify their sorting of linear equations. Sorting tasks have previously been used for both teaching and evaluating algebraic understanding. When used for assessment, sorting tasks can observe whether students notice irrelevant or relevant features in equations (English & Sharry, 1996; Rüede, 2009). English and Sharry (1996) used a sorting task to orally assess which features three secondary school students focused on in first- and second-degree equations. They found that, beyond features indicating the degree of the equations, students' attention was drawn to several other aspects: the use of different letters for the unknowns, the presence of one or more unknowns, whether one side contained only a number – particularly linking zero to second-degree equations – denominators with or without variables, and computational aspects, such as the need to group letters. In their study, extra

proficient algebra students identified more features related to the equations' structures compared to less skilled students.

Identifying the features students focus on at an early stage can assist teachers in guiding students towards a more algebraic understanding of equations and helping students to engage with the algebraic context. In this paper, we will discuss the features students attend to in a sorting task and explore the potential of this task as an indicator of students' awareness of structure in the previously mentioned learning study. The research question guiding the study is: What can we learn about students' awareness of structure regarding linear equations from a sorting task?

Method

In this paper we focus on a sorting task in the pre-test of the learning study, answered by students in their first year of upper secondary school in Sweden. 40 students signed a written consent, and the results displayed concern their answers. The students are approximately 16 years old (age was however not controlled in the study). They attend three different schools in two cities and are therefore taught by separate teachers. The students are all in vocational education, focusing on electricity and will likely use linear equations with both decimal and negative numbers as coefficients in their future profession as electricians. They have previously, in lower secondary school, learnt to solve linear equations, but mainly using natural numbers as coefficients. In the present study, the sorting task was adapted to focus specifically on linear equations (see Figure 1). Twelve equations were designed to vary in features that might be relevant for students' attention. Some equations were inspired by equations that had proved useful in probing students' understanding for structure in previous studies, for example $0.12 = 0.4 \cdot x$ and $0.657 = 0.045 \cdot x$ (Holmlund, 2024) and $-24 = 6 \cdot x$ (Vlassis, 2002). Inspiration was also taken from the task by English and Sharry (1996), for example including a zero on one side and varying the letter used for the unknown. Other features were varied among the equations, based on previous research indicating them as central to students in their understanding of linear equations, for example the position of variables on one or both sides of the equal sign (Vlassis, 2002). The task instructed students to "sort the equations you think 'belong together' and resemble each other. Encircle the ones you think belong together and mark the first group as A, the second as B and so on." After sorting, students were asked to justify why they grouped the equations as they did. The response field was divided into four sections, so while students were encouraged to create as many groups as they wanted, the task did not support more than four groups. Also, the task did not encourage students to encircle one equation several times. The free sorting asked for in the task was intended as a tool to learn about the features that students immediately look for in the equations – without there being any wrong answers – and thus not intended to display their understanding of the structures in more depth. However, the open formulation of the task was mentioned as confusing by two students.

A qualitative content analysis was conducted on the output of the sorting task in the pre-test, using students' justifications as the unit of analysis (Graneheim & Lundman, 2004). To identify categories that describe the features students focus on, all answers were carefully reviewed, looking for similarities and differences, and a list of commonalities in students' justifications was compiled. The analysis focused on the meanings conveyed in the students' answers.

Sometimes, a student provided multiple reasons for considering equations as similar, such as noting both the letter used and the operation (e.g., "they have y in the equation and a multiplication sign"). These features were categorised based on shared characteristics. The goal was to identify categories that encompassed all the justifications while remaining mutually exclusive. The first analysis was conducted by the first author. The second author reviewed all student answers and categorizations. A few disagreements were discussed with a third researcher until consensus was reached. The intention was to let the students' perspectives guide the categorization, minimising the influence of the researchers' perspectives.

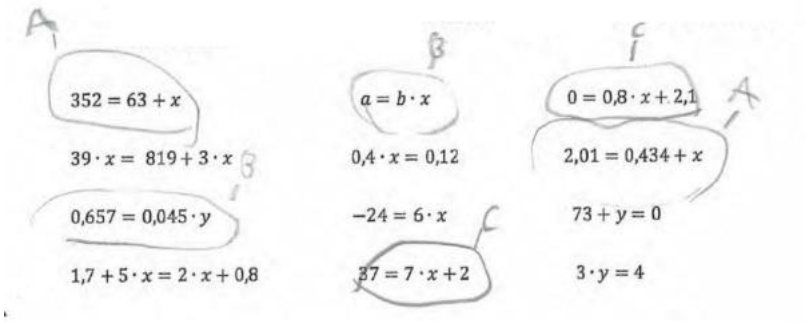

Figure 1: The analysed task with a student's encircled equations

Results

The qualitative content analysis of students' justifications resulted in seven categories, which give an overview of the structural properties of algebraic expressions that the students attend to. The properties, inherent in the equations, are what students notice in their search for characteristics. In parallel with the presentation of each category below, examples from students' answers will illustrate the potential and shortfalls of the sorting task to indicate and describe students' awareness of structure.

Separating different operations

Similarities of operations is a frequent feature that students use to group equations together. Making the type of operations explicit in their justification is distinctive of this category. As an example, one student explains the group of equations as "containing" either addition, multiplication or both addition and multiplication. It is possible to group equations in different ways based on the operations: for example, seeing equations with both addition and multiplication as a third structure or as belonging to the structure of multiplication. Generally, this feature occurs in students' answers with at least the distinction between equations with (only) addition and equations with (only) multiplication. One student explains $a = b \cdot x$ as a formula for multiplicative structure that other equations with multiplication match with.

Separating number of steps

A related feature, that some students mention, is the number of steps required to solve the equation. As an example, one student describes the common feature in group A in Figure 2 as the need for only one calculation on one of the sides to find x or y. In contrast, for group B "two calculations are required on one of the sides if there is no negative number on one of the

sides which can complicate the solution somewhat". Operations are thereby noticed but sorted according to how many they are, rather than by their type (as in the previous category).

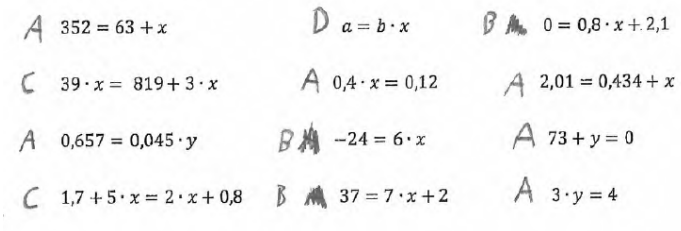

Figure 2: Encircling with justifications based on the number of steps

Variable on one or both sides

A feature considered by some students is whether x is on one or both sides of the equality sign. For example, one student describes one of his groups by stating “There are equations [expressions] on both sides”. Looking at Figure 3, this student explains his choice of group C as “Because you have to calculate on both sides” which also focuses on the variable being on both sides, rather than on the number of calculations (as in the preceding category).

Specific positions

As in the example in Figure 1, several students identify the position of components within the equations as a defining feature. This feature appears in two distinct ways. For some, the position is described, with a focus on either the right or left side of the equation, mentioning whether the side in focus contains only a number, an expression involving a variable, or an expression involving a particular operation. In other cases, students focus not only on one side of the equation but on a specific position within it, such as the final position, sorting equations based on the symbols present at this position (e.g., a number or an unknown). One student places $1.7 + 5 \cdot x = 2 \cdot x + 0.8$ in the same group as $3 \cdot y = 4$ because “they end with numbers”.

Letters

For some students, features concerning the unknown play a central role in sorting the equations. Their justifications involve the specific letter used to denote the unknown, leading them to categorize equations with y separately from those with x. For example, one student groups $0.657 = 0.045 \cdot y$, $73 + y = 0$, and $3 \cdot y = 4$ together, noting “they have y instead of x”. Another aspect of using the letter is to look at the number of unknowns. It is not uncommon for the students to see the equation $a = b \cdot x$ as a category on its own, observing that this equation features multiple letters and “contains only variables”.

Numbers

Some students base their grouping of equations on numerical features. These justifications sometimes involve certain numbers. For instance, some students sort equations that contain only decimal numbers separately from those with whole numbers. One student sorted equations involving negative numbers in one group, explaining that “then you need to calculate and think in a different manner than in the other groups”. Another student identified odd numbers as a common characteristic of one group of equations. Other students focus on more specific

numerical features, such as one noting that "they have very similar digits 3, 5, and 7" or another observing "they are equal to 0". In one instance, the presence of many digits was used as a justification for grouping or excluding equations.

Length of equation

Some students use the length of the equations as a criterion for categorization. Usually, longer equations are those that involve multiple operations, have the variable x on both sides of the equality sign or contain numbers with many digits. For example, one student sorts $0 = 0.8 \cdot x + 2.1$ and $2.01 = 0.434 + x$ together because they are "very long compared to the other [equations]". Another student sorted $352 = 63 + x$ and $3 \cdot y = 4$ in the same group with "short" as the only justification.

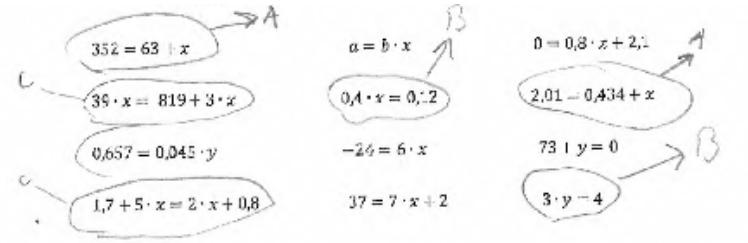

Figure 3: An example where a student focuses on different features

A limitation of the sorting task is occasional weak alignment between students' encircling of equations and the accompanying justifications. For example, some students mention operations as a criterion for the groups but do not encircle all relevant equations, sometimes due to conflict with other features they noticed. This can be seen in Figure 3 where one student justifies groups A and B by mentioning the specific operations without encircling all additive equations for group A, nor all multiplicative equations for group B, as positions of "solutions" and "answers" (which he also mentions) seems to be an important feature to him. On the contrary, the student behind Figure 1 does not mention operations in the justifications but examining the encircling reveals that this is a relevant feature to him. It appears that additional features, such as the positioning of the variable in relation to the equal sign, are significant to this student. Another example is given by a student in Figure 4. He describes group B as "has decimal numbers as an answer and in solution and has one variable" underscoring the importance of numerical features to the student. Group C is characterized as "they look the same, the only difference is that one is integer and the other decimal number". Comparing groups B and C indicates that positioning likely plays a central role in the students' awareness of structure, which is not clear from the justifications.

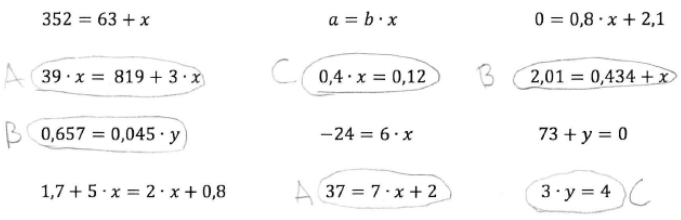

Figure 4: An example where specific numbers guide the sorting

Discussion

To learn more about students' awareness of different features in linear equation, and to be able to use this when assessing how they improve in recognising equations as familiar structures, students in the study were asked to sort a set of equations. When reviewing the seven categories found in students' justifications, we see three features that indicate an emerging focus on algebraic structures. Based on Radford's description of algebraic thinking as analytically handling unknown quantities, it is reasonable to consider the feature of "Variable on one or both sides" as algebraic: as it makes a distinction between equations that requires handling the unknown and those that do not. Focus on the first two features, operations and number of steps, indicates that the students noticed the different parts of the equation. This does not necessarily imply algebraic thinking but could at least show that they are on the way. Focusing on these features can be compared to the operational focus and first-order structuring found in the study by Rüede (2013). However, we interpret the other features mentioned by students (i.e., specific positions, letters, numbers and length of the equation) as not showing algebraic thinking or an emerging awareness of structure. These categories are visually salient, and they are of relevance to students' previous understanding of familiar mathematical concepts. For example, justifications related to specific positions can be partly related to students' view of the equality sign as indicating something to compute, often placing the answer on the right side, without seeing the equality as symmetric. Many of these justifications are similar to those described by English and Sharry (1996). References to the length of the equations generally indicate that the student struggles with these equations, as they rarely solved many equations elsewhere in the pre-test. Students' focus on numbers is supported by earlier research suggesting that numbers are a key feature that students attend to in equation solving (Holmlund, 2024). Thus, regarding saming as a central activity to enter the discourse of algebra (Sfard, 2008), students also need to be guided into what is *not* the same.

To summarize, the analysis of the sorting task has revealed that aligning students' encircling (or lack thereof) with their justifications can be challenging. However, it can display signs of emerging awareness of structure and to effectively use the sorting task to assess students' awareness of structure, it is important to consider both the encircling and the justifications.

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Overcoming the *letter-as-object* misconception in 4 weeks? An analysis of latent class transitions and contributing teaching practices

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While misconceptions regarding algebraic letters have been researched for decades, they still seem to be very prevalent among students learning algebra. Therefore, this study aims to investigate teaching approaches that support students overcome erroneous thinking patterns related to the letter-as-object misconception. Based on latent classes that had been identified for 2220 German Year 7 and 8 students, students' transitions between latent classes were examined and connected to self-reported teaching practices from three interviewed teachers. The results suggest that explicitly focussing on the meaning of variables in the context of algebraic expressions can help students overcome the letter-as-object misconception.

Keywords: Algebra, variables, letter-as-object misconception, online formative assessment.

Introduction

One of the fundamental concepts in algebra are variables. However, students often struggle to grasp this concept, bringing previous knowledge about letters into the classroom that can hinder learning (Arcavi et al., 2017). In addition, also inadequate teaching can foster misconceptions. One of the well-known misconceptions regarding variables is the *letter-as-object* (LO) misconception which is highly prevalent among students (e.g. Akhtar & Steinle, 2017, Lucariello et al., 2014 (termed 'label for an object' there)). Its high prevalence and its potential impact on further algebra learning raises the question how students can be supported to overcome the LO misconception. To investigate the development of students' thinking patterns regarding the LO misconception, we have conducted a Latent Transition Analysis (LTA) based on the SMART test 'Meaning of letters'. For three classrooms, student development was then linked to reported teaching practices.

Theoretical background

Similar to other researchers, Arcavi, Drijvers and Stacey "distinguish five facets of the concept of variable: a placeholder for a number, an unknown number, a varying quantity, a generalized number, and a parameter" (2017, p. 12). Across these facets, variables stand for or refer to one or more numerical values. Yet, algebra learners often struggle with this numerical interpretation, and various typical errors and misconceptions have been identified. One of them, the *letter-as-object* (LO) misconception, has been described already in 1981 by Küchemann as the letter being "regarded as a shorthand for an object or as an object in its own right" (p. 104). Misleading teaching materials and practices, such as 'fruit-salad-algebra' where, e.g., a is used as 'apples' and b as 'bananas', can reinforce this interpretation of algebraic letters as abbreviations for objects (MacGregor & Stacey, 1997). Several varieties of the broad LO misconception can be identified: Küchemann (1981) found that some students seemed to believe they should incorporate a solution into an initial equation, rather than formulate an equation that describes the situation, thus arriving at an equation similar to $35b +$

$10t = 100$ in the *Bikes* item (Figure 1). Here, the letters are used as abbreviations for the involved objects and the coefficients stand for a possible solution ("35 bikes and trikes have 100 tyres altogether"). Hence, we will refer to this LO-subtype as the *solution-as-coefficient* (SAC) misconception in the following. Another special form of LO is called *letter-as-unit* (LU) when the algebraic letter is interpreted as an abbreviation for a unit (Akhtar & Steinle, 2017), e.g., in the *Ducks* item (Figure 1) d in $6d = 12$ would be misinterpreted as standing for dollars (not realising that this would not be a correct equality).

Sometimes, the LO misconception is explicitly taught through inadequate instruction including in textbooks (MacGregor & Stacey, 1997), and sometimes, it arises naturally when students read an algebraic sentence through the lens of natural spoken language and writing conventions. The LO is a significant misconception especially because it leads students to make mistakes when formulating algebraic equations and hence deprives them of the benefits of being able to use algebra to solve problems. As with other misconceptions, conceptual change is necessary (e.g. Strike & Posner, 1985). However, conceptual change is not easily achieved, but "requires particular instructional strategies, such as raising student metacognition and creating cognitive conflict" (Lucariello et al., 2014, p. 32).

The Current Study

In our current study, we have explored typical patterns regarding the *letter-as-object* misconception among 2,220 German Year 7 and 8 students by means of a Latent Transition Analysis (LTA) (Klingbeil & Moons, submitted). Based on the six-multiple-choice item SMART ('Specific Mathematics Assessments that Reveal Thinking') test 'Meaning of letters', we were able to describe five response patterns in terms of latent classes with their particular item response probabilities. For this study, 103 teachers from six federal states of Germany were asked to administer the online SMART test, first, 1–2 weeks into their regular teaching sequence about variables, algebraic expressions and/or equations, and second, after another 4 weeks of teaching. Teachers were encouraged to use the test results from the first test for their teaching. In total, 2220 Year 7 and 8 students (aged 12–14) completed at least one of the tests (877 in Year 7, 1343 in Year 8); 1219 students completed both pre- and post-test. In the following, we briefly describe the used test items and the identified response patterns.

SMART test 'Meaning of letters'

The SMART online test 'Meaning of letters' was designed to assess the *letter-as-object* (LO) by means of six multiple-choice items (see Figure 1 for items of Version A). Evidence for the validity of this test was presented elsewhere (Klingbeil et al., 2024). The first item type (Meaning items) uses only one algebraic letter and asks students to decide on the meaning of this letter in a linear equation in a given context. While the *Ducks* item uses the initial letter of the involved objects and units, the *Bricks* item uses the letter y. The second type of item (Additive items) uses two algebraic letters (corresponding to the initial letters of involved objects), which are additively connected and restricted by the given situation. Students are supposed to choose the correct linear equation for the described context. In the equation of the *solution-as-coefficient* (SAC) option, coefficients correspond to a possible solution to the problem (that has not been required) so that the equation can be read as a solution sentence. The third item type (Proportional items) comprises two variables that are directly proportional to each other. Students are again asked to choose the equation matching the given situation.

While *Biros* contains concrete objects, *Racetrack* deals with more abstract racetrack rounds and the unit minutes.

Meaning items	Additive items	Proportional items
Ducks	*Garden*	*Biros*
Lucy bought 6 ducks for \$12. She wrote the equation $6d = 12$. What does d in Lucy's equation stand for? d stands for... ☐ the cost of one duck (COR) ☐ one duck (LO) ☐ ducks (LO) ☐ dollars (LU)	Payam bought r red rose bushes and l lilac lavender bushes for his garden. One rose bush costs €4. One lavender bush costs €5. Which equation says that the total cost of the plants was €70? ☐ $4r + 5l = 70$ (COR) ☐ $r + l = 70$ (LO) ☐ $10r + 6l = 70$ (SAC)	Biros are sold in 3-packs. Sam bought p packs and has now b biros altogether. Choose the correct equation: ☐ $3p = b$ (COR) ☐ $p = 3b$ (LO) ☐ $p = 3$ (LO) ☐ $b + p = 4$ (LO) ☐ $30b = 10p$ (SAC)
Bricks	*Bikes*	*Racetrack*
Tina stacked 9 identically sized bricks on top of each other making a 99 mm high tower. She wrote the equation $9y = 99$. What does y in Tina's equation stand for? y stands for... ☐ the height of one brick (COR) ☐ one brick (LO) ☐ the bricks in the tower (LO) ☐ millimetres (LU)	At a bike shop there are b bikes (with 2 tyres each) and t trikes (with 3 tyres each). Which equation says that there is a total of 100 tyres in the shop? ☐ $2b + 3t = 100$ (COR) ☐ $b + t = 100$ (LO) ☐ $35b + 10t = 100$ (SAC)	A car takes 12 minutes for one round of the racetrack. In m minutes a driver drives the round r times. Choose the correct equation: ☐ $12r = m$ (COR) ☐ $12m = r$ (LO) ☐ $r = 12$ (LO) ☐ $5r = 60m$ (SAC)

Figure 1: Items of 'Meaning of letters' test (German A version) translated back into English

Identified Latent Classes regarding the *letter-as-object* (LO) misconception

The five identified latent classes are shown in Figure 2 together with their item-response probabilities for each item as well as prevalence in pre- and post-test. Although the order of the latent classes does not play any role in a LTA, they are presented in ascending order based on the average probability of having items correct.

SAC in additive, LO elsewhere is characterised by very high probabilities for *solution-as-coefficient* (SAC) responses in Additive items and high probabilities for LO responses in all other tasks. This indicates a rather consistent interpretation of letters as abbreviations for involved objects in combination with an interpretation of coefficients as solutions in additive equations.

Mostly SAC is characterised by high probabilities for SAC responses. In the Meaning items that do not offer a SAC option, LO is most likely; a correct response in *Bricks* is possible. Since this is the class with the highest probability for SAC in Proportional items, students in this class seem to be quite convinced that coefficients stand for (possible) solutions also in different equation types.

Mostly LO, possibly correct in additive is characterised by LO responses being most likely in Meaning and Proportional items. Even when the initial letter of the involved object is not used (*Bricks*), LO is much more likely than a correct answer. However, in Additive items, correct responses are about as likely as LO responses.

Correct in meaning is characterised by high probabilities for correct responses in Meaning items, high probabilities for SAC in Additive items, and high probabilities for LO in Proportional items. Students in this class seem to be able to identify the correct meaning of an algebraic letter when

directly being asked for it. However, when choosing equations, they still fall into the trap of interpreting letters as abbreviations.

Mostly correct is characterised by correct responses being most likely in almost all items. Only in *Racetrack*, LO is clearly more likely than the correct response. In *Ducks* and *Biros*, LO is also possible but only about half as likely as the correct response. This indicates at least a partial understanding that algebraic letters do not stand for abbreviations.

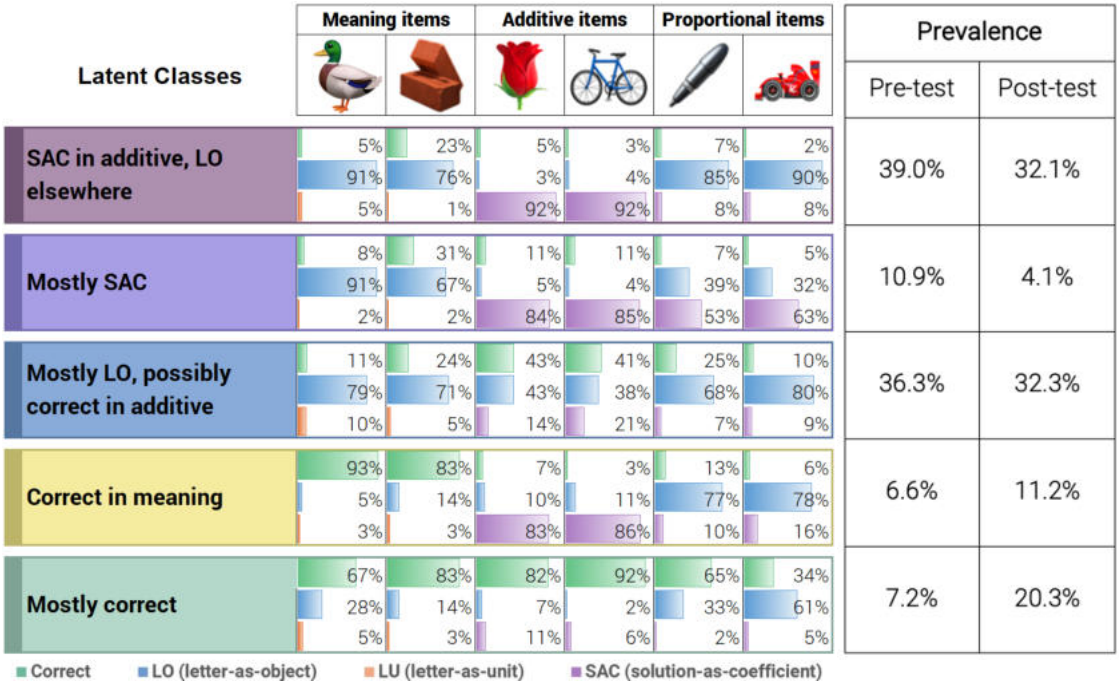

Figure 2: Latent classes with their response probabilities for each item and prevalence

Alarmingly, we found that only 7.2% of students are most likely to belong to the *Mostly correct* class in the pre-test. From there, probabilities for being in *Mostly correct* increased to 20.3% after the 4-weeks teaching period (with 5.8% of students staying in this class and 14.5% transitioning into it). However, 53.6% of all students stayed in one of the other latent classes and 26.1% moved to a latent class other than the best. This observation prompted us to further investigate the question *how (self-reported) teaching during the 4-week period might have influenced different transitions (or stagnations) between latent classes regarding the 'letter-as-object' misconception among students.*

Methods

To answer this question, we were able to draw on information from 9 of our 103 teachers who had volunteered to be interviewed twice. In order to be able to compare the reported teaching and its influence on students' latent classes transitions, we selected a subset of three classrooms from these interviewed teachers who fulfilled the following criteria: same Year level (7), similar distribution of latent classes in pre-test, different distribution of latent classes in post-test. In the following, we are describing the selected classrooms in terms of the students' transitions in more detail.

Figure 3 shows the transitions between latent classes within the three selected Year 7 classrooms in

form of alluvial diagrams. These illustrations of the transitioning between the latent classes at the time of pre- and post-test are based on transition probabilities from the LTA (only including probabilities larger than 3% for better readability).

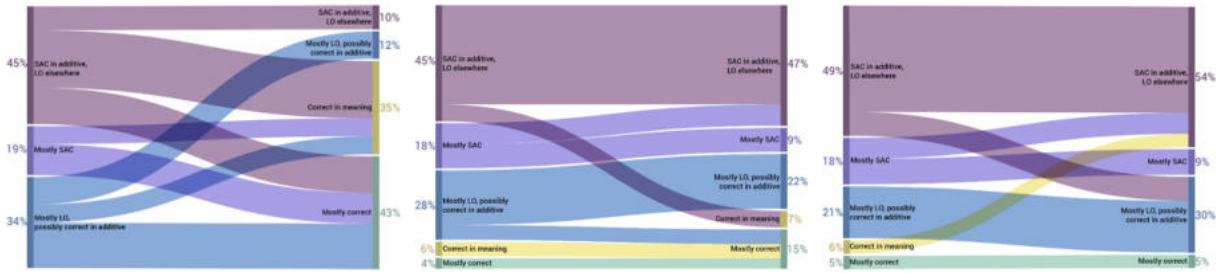

Figure 3: Transitioning between latent classes from pre- to post-test

The three selected classrooms show a similar distribution of latent classes in the pre-test: *SAC in additive, LO elsewhere* is the most prevalent class (45-49%), followed by *Mostly LO, possibly correct in additive* (21-34%) and *Mostly SAC* (18-19%). In addition, Classrooms 2 and 3 each have a single student with *Correct in meaning* and *Mostly correct* as best-fitting class.

In the post-test, however, the distribution of latent classes differs considerably between classrooms. While Classroom 2 and 3 see only a few changes in latent classes from pre- to post-test, Classroom 1 students almost all transition to a different latent class. In the post-test, most of Classroom 1 students correspond to *Mostly correct* (43%), followed by *Correct in meaning* (35%), *Mostly LO, possibly correct in additive* (12%) and *SAC in additive, LO elsewhere* (10%). In Classroom 2, students are less likely to transition, hence *SAC in additive, LO elsewhere* (47%) and *Mostly LO, possibly correct in additive* (22%) are still most prevalent in the post-test. However, *Mostly correct* has increased to 15% with newcomers stemming from *Correct meaning* and *Mostly LO, possibly correct in additive*. In Classroom 3, most students (73%) do not transition with *SAC in additive, LO elsewhere* (54%) and *Mostly LO, possibly correct in additive* (30%) staying the most prevalent latent classes. Unlike in Classroom 2, students do not transition into *Mostly correct* (5%); not even students from *Correct in meaning* who are most likely to transition to *SAC in additive, LO elsewhere* here.

The teachers of these three classrooms took part in two semi-structured interviews each conducted shortly after teachers had deployed the pre- respectively post-test. The questions most relevant for the investigated research question were: Please elaborate on what you have done in the classroom regarding variables and algebraic expressions before the pre-test. What are you planning to do with your students' test results? What has happened during the last 4 weeks in your classroom? To what extent have the test results influenced your teaching? The interviews were inductively analysed for various information on their teaching and case summaries were created (Kuckartz, 2018).

Results

The teacher interviews revealed several differences between reported teaching (see Table 1 for a comparative overview). Teacher 1 administered the pre-test after only one lesson about substituting values and formulating algebraic expressions during which they stressed the importance of defining

what the used variable stands for. Based on the test results, they decided to spend more time on these contents and not to move on to equations yet. In the following weeks, they strongly focussed on the meaning of the variables and required students to be explicit and precise about this (even if tasks from the textbook would not require it on their own). They initiated classroom discussions on differing formulated expressions and variable definitions stemming from group work. For choosing further tasks, they emphasised the importance of switching between different representations and connecting to contexts. As feedback on the test, Teacher 1 discussed the test items of the pre-test in their classroom and addressed substitution as a strategy to check given or formulated equations.

Table 1: Comparison of reported teaching before and after the pre-test

	Teacher 1	**Teacher 2**	**Teacher 3**
Contents before pre-test	formulating algebraic expressions, substituting values into expressions match figure patterns		
	importance of defining what the variable stands for, idea of using variables for generalisations	fruit-salad-algebra used to help, connecting expressions to context situations, no explicit correction of LO statements	simplifying expressions, equality of expressions, variables stand for numbers, same letters stand for same values
Teaching between pre- and post-test	*Contents:* substituting values into expr., same letters for same values, formulating expr. with stating the meaning of the variable, simplifying expressions NO equations	*Contents:* equation solving (scales model & match boxes), equivalent transforming, substituting to check results, facets of variables NO formulating equations	*Contents:* equation solving (scales model), equivalent transforming, formulating equations from a given situation, mostly x used as variable NO focus on meaning of var.
	Tasks: connecting representations: alg. expr., verbal description & table, formulating expr. for patterns and situations	*Tasks:* formulating alg. expr. for variables which stand for letters in name (+ other tasks from SMART teaching suggestions)	*Tasks:* tasks from textbook, e.g. age tasks
	Feedback: no individual feedback, test items discussed, strategy to check by substituting	*Feedback:* no individual feedback, test items/solutions NOT addressed	*Feedback:* no individual feedback, test item solutions shown in classroom

Teacher 2 administered the pre-test after teaching substituting values, and formulating and matching algebraic expressions for two weeks. Reflecting on the SMART test, they acknowledged having used 'fruit-salad-algebra' and handling students' d0efinitions of variables as objects as imprecisions by not explicitly correcting students, but just using a correct formulation themselves. Prompted by the test results, they dedicated two lessons to work on selected tasks from the SMART teaching suggestions focussing mainly on substituting values for letters and facets of variables (unknown, generalised number, varying quantity). Because Teacher 2 was afraid to distort results of the study, they did not discuss the test items with their class before the post-test (afterwards they did and deemed it utterly helpful). For the rest of the 4 weeks between tests, the lessons focussed on equation solving and checking results by substituting, but not formulating equations yet.

Teacher 3's students completed the pre-test after 8 lessons on substituting values, formulating and simplifying expressions, and equality of expressions. While it was addressed that variables stand for numbers, no tasks were included which required deciding what the variables are being used for in a given context. Since the test results were not unexpected for Teacher 3 (because this kind of tasks had not been addressed), they did not adjust their further teaching except for briefly showing the

solutions of the pre-test. Accordingly, the lessons between tests dealt with solving equations, equivalent transforming and formulating equations from a given context without focussing on the meaning of variables. Teacher 3 reported generally using the letter x and no initial letters for variables. They assumed that an understanding of variables would develop in the course of time, e.g. in Year 8 when taking up algebra again, and being able to solve tasks would be sufficient at the moment.

Discussion

As can be seen in the results, Classroom 1 students have improved the most after 4 weeks of teaching. Apparently, Teacher 1's focus on the meaning of variables has indeed helped students to transition to *Mostly correct* (43%) and *Correct in meaning* (35%). Interestingly, equations had not been addressed in lessons yet; thus, a proper understanding of the meaning of variables might be more important than knowledge of equations and equality in order to avoid a *letter-as-object* (LO) interpretation.

Despite dedicating two lessons on tasks from the teaching suggestions, only a few Classroom 2 students progressed to *Mostly correct* (15%) or *Correct in meaning* (7%). This could, on the one hand, be attributed to the focus on substituting values and on facets of variables rather than on explicitly addressing that algebraic letters do not stand for abbreviations of objects. On the other hand, the use of fruit-salad-algebra in previous lessons and the acceptance of students' "inaccuracies" might have made it more difficult to overcome a LO misconception for many students. However, for a few students the reported teaching was sufficient to avoid this interpretation in more test items than before.

Teacher 3 did not change anything in their teaching apart from showing item solutions. Although they are aware of the importance of variables standing for numbers, and avoid initial letters in their teaching, moving on with equation solving and formulating is not enough for their students to improve their understanding of the meaning of variables. The one student with *Correct in meaning* as best-fitting class for the pre-test even deteriorates to *SAC in additive, LO elsewhere*.

Conclusion

Comparing reported teaching with the development of students' latent classes, we can infer that explicitly addressing the meaning of used variables (again and again), discussing the importance of a precise definition (raising metacognition), and comparing different definitions of variables (probably creating cognitive conflict) and their implications for formulating expressions was a successful approach to facilitate conceptual change for many learners. For this purpose, also the discussion of the SMART test items might be helpful since they comprise different subtypes of this misconception (diverse instruction). These findings seem to be in line with instructional recommendations to achieve conceptual change (e.g. Lucariello, 2015). Learning about equations, on the other hand, seems not to be as crucial; on the contrary, moving on to equations too quickly and without stressing the meaning of variables could actually hinder students' improvement since no conceptual change is initiated. Of course, for teachers to be able to employ such helpful teaching, they need to be aware of potential misconceptions in general and amongst their students, but also of appropriate teaching approaches. While our findings are certainly limited by the small number of interviews analysed and the nature of self-reports, we nevertheless were able to gain further information on how teachers could be advised to help students overcome the *letter-as-object* misconception.

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Enhancing students' understanding of variables with interactive instructional videos in a knowledge organization phase

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While research in algebra education has provided rich insights and designs that focus on open-ended exploration activities, instructional designs that support teachers and students in the phase of active knowledge organization after initial explorations have received less attention. This paper explores how interactive instructional videos can enhance students' understanding of variables as generalizers when the videos are used at the beginning of a knowledge organization phase prior to a teacher-led discussion. Snapshots from design experiments with eighth graders in the third cycle of a design research project show that interactive features and follow-up tasks that build on (sometimes only partially correct) student solutions from the exploration phase can enhance students' understanding of the generalizer by making them question prior deviant solutions, helping them distinguish between generalizer and placeholder, and providing helpful language means.

Keywords: Variables as generalizers, instructional videos, interactive features, two-phase instructional designs, design research.

Introduction

Ellis and Özgür (2024), in their review of recent trends in algebra education research, emphasize the need to investigate instructional approaches and materials that help teachers deliver high-quality algebra instruction. Research in algebra education has long supported the use of open-ended inquiry tasks in an initial *exploration phase* in which students re-invent algebraic concepts (Kieran, 2006). While teachers often succeed in initiating such phases, many struggle to productively implement the subsequent *phase of active knowledge organization*, which aims to collect and connect students' ideas from the exploration phase and to organize them into a coherent network of knowledge that conforms to the norms of mathematics (Prediger et al., 2021; Stein et al., 2008): Some teachers lose students' cognitive engagement by simply "telling" the final mathematics after collecting ideas, while others, reluctant to "tell" anything, collect students' ideas but fail to connect them to regular mathematics.

Algebra education research paid little attention to instructional designs for the knowledge organization phase. These designs can draw on effective teaching practices for this phase to maintain students' cognitive engagement (Stein et al., 2008), but should be supplemented with topic-specific materials to support teachers in moderating focused whole-class discussions around typical student solutions. This paper addresses the case of variables as generalizers, a concept essential for generalizing relationships but often difficult to understand for students (Küchemann, 1981; Usiskin, 1988). We report on a design research project investigating the potential of interactive video as an instructional material that incorporates interactive features such as open-ended gaps or self-explanation prompts and can enhance students' knowledge organization after an initial exploration phase in preparation for a teacher-led whole-class discussion (Atkinson & Renkl, 2007; Mayer, 2021).

The research question of this paper is: *To what extent can interactive videos enhance students' understanding of variables as generalizers in a phase of active knowledge organization before entering a whole-class discussion?* After sketching the theoretical background on variables as generalizers, the use of interactive video in the knowledge organization phase, and supporting design features of effective video, and outlining the methods, empirical snapshots of learning processes from the third design research cycle are presented and discussed.

Theoretical background

Understanding variables as generalizers

Generalizing is central to algebra (Kieran, 2006). Students who can use letter variables and algebraic expressions to generalize relationships between quantities and explain their meaning (as in the generalization tasks in Figure 1a) understand letter *variables as generalizers*, i.e., they have constructed the meaning of the letter as representing *all possible* numbers (e.g., driving times). This understanding can be developed in sense-making activities that promote generalization experiences aimed at describing relationships for all relevant cases at once (Usiskin, 1988).

(a) E-scooter task on algebraic expressions

Target task: Till occasionally rents an e-scooter and pays different prices as he drives, sometimes longer, sometimes shorter. Find an algebraic expression with a variable that determines the total price to be paid for every arbitrary chosen driving time.

E-scooter rental
Price per minute 0.15€
Price to unlock 1 €

(b)

Date of e-scooter ride	Driving time (in min)	Costs driving time (in €)	Expression for total price (in €)	Total price (in €)
May, 16	20	20 × 0.15	20 × 0.15 + 1	4.00
May, 19	12	12 × 0.15	12 × 0.15 + 1	2.80
May, 24	27	27 × 0.15	27 × 0.15 + 1	5.05
May, 25	18	18 × 0.15	18 × 0.15 + 1	3.70
May, 29	33	33 × 0.15	33 × 0.15 + 1	5.95
June, 2	42	42 × 0.15	42 × 0.15 + 1	7.30
For every arbitrary driving time:	x	x × 0.15	x × 0.15 + 1	

(c)

Date of e-scooter ride	Driving time (in min)	Calculation	+ unlock costs	€
May, 16	20	20 × 0.15	20 × 0.15 + 1	4.00
May, 19	12	12 × 0.15	12 × 0.15 + 1	2.80
May, 24	27	27 × 0.15	27 × 0.15 + 1	5.05
May, 25	18	18 × 0.15	18 × 0.15 + 1	3.70
May, 29	33	33 × 0.15	33 × 0.15 + 1	5.95
June, 2	40	40 × 0.15	40 × 0.15 + 1	7.00
For every arbitrary driving time:	150	150 × 0.15	150 × 0.15 + 1	23.50

Figure 1: Exploration task for re-inventing variables as generalizers (a), table as instructional support (b), and a typical improvable student solution before working with the video (c)

Developing understanding of generalizers has long been identified as challenging for students (e.g., Bardini et al., 2005, Küchemann, 1981), who often (initially) interpret letter variables through an "arithmetic frame of reference" (Kieran, 2006, p. 25) focused on calculation. Within this frame students tend to interpret *variables as temporally indeterminate placeholders* into which one can put a number to "finally" calculate something (Bardini et al., 2005). Hence, students mistakenly transfer the idea that variables represent something *not yet known* – typical of unknowns in equation-solving activities ($3x+5 = 7$) – to generalization activities. But generalizers do need no further determination, because a general statement is the epistemic purpose (Usiskin, 1988).

Instructional support for understanding the generalizer typically elicits students' calculation-dominated interpretations and then contrasts them with the meaning of the generalizer (Prediger & Krägeloh, 2016). A table can help students appreciate the subtle contrasts (Figure 1b): They initially fill in the e-scooter table for several arithmetic expressions, mostly with a *horizontal* view along rows from left to right, and choose a driving time for June 2 (placeholder). In contrast, the last row calls

for every arbitrary driving time and requires seeing the table *vertically*, using a generalizer to represent all possible numbers that could occur in the driving times column. Providing language that distinguishes between "every arbitrary number" and "a specific chosen number" further supports students in grasping the subtle conceptual differences (Prediger & Krägeloh, 2016).

Students with arithmetic reference frames often do not complete the table as intended. Figure 1c shows a typical improvable student solution that interprets every arbitrary number as a prompt to choose a specific number (e.g., 150min). The exploration phase aims to elicit such typical student ideas, while the knowledge organization phase builds on these ideas, e.g., by contrasting the meaning of the placeholder and the generalizer and leveraging them to understand the generalizer.

Design features of interactive videos that can enhance students' knowledge organization

Knowledge organization phases, typically teacher-led whole-class discussions, pose challenges for teachers and students (Prediger et al., 2021; Stein et al., 2008): Teachers must enact highly demanding teaching practices, such as connecting students' ideas from the exploration phase to the mathematical norms, while students must follow these rich discussions and process ideas not yet developed in their own prior exploration, potentially leading to cognitive overload (Mayer, 2021). Prediger et al. (2021) developed and investigated *knowledge organization tasks*, which are less open-ended tasks launched after students' explorations, that introduce typical student ideas from the exploration phase, explicate connections to regular mathematics, and can support teachers in implementing teacher-led knowledge organization phases. Such knowledge organization tasks, typically completed by students individually or in small groups before being discussed with the whole class, allow students to process typical student ideas at their own pace, can reduce the cognitive load of the subsequent whole-class discussion, and can help students focus on relevant topic-specific aspects.

In this paper, we use interactive videos as knowledge organization tasks and explore how such videos, used after students' individual exploration of the e-scooter task (Figure 1), can prepare a teacher-led whole-class discussion by enhancing students' understanding of the generalizer. In general, instructional videos can enhance understanding, but their effectiveness depends on students' *focused cognitive engagement*, i.e., students' deep processing of the video content with a focus on the relevant topic-specific content elements (Atkinson & Renkl, 2007; Mayer, 2021). Students' focused cognitive engagement with the video explanation can be supported by the following evidence-based *design features* (Atkinson & Renkl, 2007; Mayer, 2021): (1) The video content should build on *typical student solutions* from the exploration phase. (2) The *multimedia design* should take into account students' limited cognitive capacity. (3) Videos should use *interactive video features* and post-video *follow-up tasks* that elicit deep cognitive engagement and focus students' attention on relevant content elements of the video content, such as distinguishing between generalizers and placeholders.

While the general features are well articulated, their topic-specific realization requires design research to meet the key aspects of the topic in view, in our case, variables as generalizers. Therefore, we designed an interactive video on understanding the generalizer and investigate students' algebra-specific learning processes while working on the interactive video in a knowledge organization phase after exploring the e-scooter task (Figure 1).

Methods of the design research study

Research context and instructional design of the video

This paper presents snapshots from Design Experiment (DE) Cycle 3 of a design research project that investigates how videos can enhance students' understanding of variables. In DE Cycle 1, we identified typical improvable student solutions to the e-scooter task (e.g., Figure 1c) and investigated learning processes with a non-interactive video after exploring the e-scooter task. We found that students often overlooked important concept elements, even when they were extensively explained in the video, e.g., the generalizer in the last row of the e-scooter table (Figure 1b). Moreover, students struggled to connect the video content to their solutions from the exploration phase.

Based on these results, we improved the video for DE Cycle 2 by adding interactive features and follow-up tasks to help students process the relevant content elements (e.g., generalizer) and connect them to their prior solutions. For example, after explaining that you can choose a number (placeholder) in the penultimate row (June, 2 in Figure 1b), the video explicates the purpose of the last row as follows: "We don't just want to describe any chosen driving time here, but every arbitrary driving time, in general. So, we no longer specify how long Till drives." In DE Cycle 2, the video speech pauses after this explanation and a written explanation is introduced in silence: "Every arbitrary driving time means: There is no specific number fixed for the driving time." (Figure 2a). In DE Cycle 2, we found that this unusual silence allowed students to see the relevance of the written explanation, and that students who interpret "every arbitrary number" as a request to calculate column sums benefited from being prompted to reconsider the last row of the table after reviewing the written explanation. For DE Cycle 3, we added an interactive feature (Figure 2a) that allows students to enter an answer for the last row and, after pressing the check button, receive corrective color-coded feedback (red: wrong) and explanatory audio feedback by pressing the play button next to their entry.

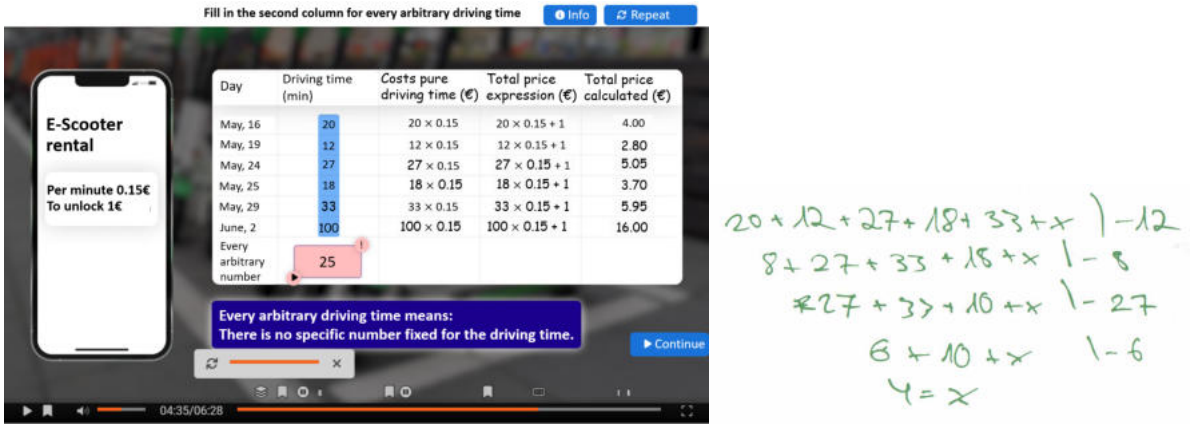

Figure 2: Example of an interactive video feature (a), Gamze's idiosyncratic calculation (b)

Methods of data gathering and data analysis

This paper reports on learning processes from DE Cycle 3 with focus on students' understanding of generalizers in contrast to placeholders. The first author conducted DE with three pairs of eighth-grade students who were selected because they had limited prior experience with letter variables from simple problem-solving activities, but had not yet developed an understanding of the generalizer. We

focus on snapshots of two pairs, Gamze and Arif, and Ilya and Serhat, during the first 90min session: in which they worked on the e-scooter exploration task and then as pairs on the interactive video and its follow-up tasks. Pairs and video screens were videotaped and videos fully transcribed.

To analyze students' learning processes, we identified transcript scenes where they worked on relevant interactive video features and follow-up tasks. Each student utterance was qualitatively analyzed to capture emerging ideas about variables, by deductively coding whether an utterance expressed (a) the generalizer (all possible numbers), (b) the placeholder (a specific chosen number), or (c) other content elements outside the study focus. Coding was done by a trained rater (first author) using a codebook developed in DE cycles 1 and 2 with substantial interrater agreement ($\kappa > .70$).

Empirical insights into students' algebra-specific learning processes with video

Gamze and Arif question their solution from the exploration phase

In the exploration phase, Gamze and Arif produced the improvable solution shown in Figure 1c. They interpret "every arbitrary driving time" as "the driving times that I drive every month", where "every" is interpreted as "throughout the whole month", so they write 150 in the last row of the table to express the sum of all their numerical entries in the driving times column. Their strong arithmetic frame of reference can be seen in their answer to the question of why their expression in the last row is correct for every arbitrary driving time: "Because these are all driving times that Till drives". Thus, Gamze and Arif's response replicates Prediger and Krägeloh's (2016) finding that, for many students, the meaning of the language provided to informally express generalizations ("every arbitrary driving time") is not automatically understood, but must first be constructed in rich generalizing experiences.

After the video introduces the written explanation in silence (blue box in Figure 2a), Gamze and Arif pause the video and write it down just below their own explanation that every arbitrary driving time means "every month". Thus, the design seems to allow them to superficially link the video content to their prior solution (interpreting "every arbitrary number" as a request to calculate column sums), but they do not seem to see any conflict between the two. Hence, they are in the target group that could benefit from the interactive feature shown in Figure 2a. They work on this feature as follows:

202 Gamze: I think it's 25. I added everything up and then divided by 6 [*types 25 into the cell*]
203 Arif: What do you mean? [*confused*] All together?
204 Gamze: Yes. [*Presses check, gets "wrong" as response, clicks on the audio feedback*]
205 Video: As with the other rows, you have entered a fixed number. Think about why this can't be right. Maybe the written explanation from the video will help you.
206-17 [*Students discuss whether they need to enter the sum of all entries, 150, or if the mean 150:6 is correct and they just made a calculation error, while Arif uses his calculator to check 150:6, Gamze listens again to the audio feedback (Turn 205)*]
218 Gamze: [*Reads the written explanation:*] There is no specific number fixed for the driving time. So- Oh my God! Do we have to reckon with this *x*? Because here [*points at 100 in video's e-scooter table*] the 100 is never fixed, it could be another number.

In her prior solution from the exploration phase, Gamze wrote "150:6" below the table (Figure 1c), i.e., the mean driving time, as a second interpretation of the unfamiliar "every arbitrary number". In Turn 202, she refers to this interpretation, while Arif still considers the sum of all entries as an answer (Turn 203). Although Gamze had already copied the written explanation from the video as an explanation for every arbitrary number, it is only after listening to the audio feedback a second time

that she thinks deeply about it (Turn 218) and links it to a vague idea of using "this *x*" – a phrase that, in German, suggests a certain distance. In this scene, we can see that there is a real need for targeted, topic-specific interactive features that elicit students' prior ideas, encourage them to challenge those ideas, and get them to think deeply about the relevant content elements.

Gamze processes the video explanation about the generalizer actively

In Turn 218, Gamze sees the problem that you could choose a number for June 2, so the 100 chosen in the video "is never fixed" and "could be another number", e.g., Gamze's 40. Here, we see how she slowly begins to appropriate the meaning of every arbitrary number, digesting the written explanation from the video. She then "reckons with this *x*" by performing the idiosyncratic calculation shown in Figure 2b. In the first line of her calculation, Gamze writes an expression for the sum of the driving times from the table, taking into account that one number "is never fixed". Yet, Gamze's understanding of "this letter" seems to be dominated by previous equation-solving activities aimed at finding a specific number. So somehow, she still thinks that *x* stands for a specific number. To determine this number, she uses idiosyncratic rules that resemble rules for manipulating equations. Having "determined" $x = 4$, Gamze and Arif proceed as follows:

234 Gamze: I'll try four [*types 4 in the field; presses check, gets "wrong" as response*] Why?
235 Arif: Try- Try 200 [*smiles*], just like that, let's see.
236 Gamze: [*Types 200 in the field; gets "wrong" as response*] Huh? [*clicks audio feedback*]
237 Video: You have entered a specific number again. In this row, we don't want to fix a specific number for the driving time.
238 Gamze: Ahh! *x*! [*types x in the field; presses check, gets "correct" as response*] Did you see, its *x*. I knew it, wait, wait, *x*. I knew it because I also calculated with *x* [*points to her calculation, Figure 2b*] Oh, yes! [*presses play to continue video*]
239 Video: As the driving time is no longer fixed, we cannot enter a specific number here.# Instead we use a variable, for example *x*. [...]## *x* also stands for all possible driving times that Till could ever drive.
240 Gamze: #Yes! ##[*nodes her head*]

In this scene as well as in the previous one, we can see that the interactive feature elicits all sorts of relevant prior ideas that Arif and Gamze have about the last row, ideas about calculating sums, finding unknowns (Figure 2b), and simply choosing a number (Turn 235). The audio feedback (Turn 237) allows Gamze in Turn 238 to finally step back from the idea of calculating column sums and to appreciate that *x* allows her to express that the number "is never fixed" and that this might be of value, even though no "reckoning with this *x*" is involved. What we find particularly remarkable about pairs such as Gamze and Arif, who work intensively on this interactive feature, is how actively they seem to process the video explanation about the generalizer immediately after the interactive feature (Turn 239), e.g., as indicated by Gamze' joy, nodding, and affirmative behavior (Turn 240).

Potentials of targeted, topic-specific follow-up tasks

Follow-up tasks that elicit self-explanations of relevant video content also aim to deepen students' focused cognitive processing (Mayer, 2021). In a follow-up task, which confronts Gamze and Arif with a solution similar to their own from the exploration (Figure 1c), students are asked to explain what is wrong with simply choosing 100min to express every arbitrary driving time:

327 Gamze: [*explains to Arif:*] You always have to use *x* when you think of a number and want to write a calculation [expression]. Then you don't choose a number like 50 or

> 100, because otherwise you get a wrong calculation – so it's not wrong, but it doesn't work because not everyone wants [to use] 50 or 100 or 40.

Gamze's explanation suggests that her understanding of letter variables seems to have changed in the course of working with the interactive video. It seems that she now sees a value in using a letter variable to express generality in the sense that "not everyone wants" to use the same number. This can be a very important first step in appreciating the epistemic purpose of the generalizer: The letter does not need to be specified, because we want an expression that is useful for all possible cases.

Snapshots of another pair, Ilya and Serhat, illustrate how targeted follow-up tasks can encourage students to return to specific video parts to find language means to express generality. For example, when Ilya is asked in a follow-up task to explain why the expression $x \times 0.15+1$ is always correct (not just for the numbers in the e-scooter table), he goes back to the video explanation that x stands for all possible driving times (see Turn 239) and writes: "The variable x [makes the expression] always correct, because x can stand for all [numbers]." Similarly, in a follow-up task where Serhat is asked to explain the structure of the e-scooter table, he returns to an interactive feature (not shown here) that deals with the purpose of letter variables (to distinguish changing quantities from fixed ones), which allows him to express the vertical structure of the table: "In the table, everything is the same except the driving time and the total costs. The total costs depend on the driving time."

Discussion

The empirical snapshots of students' work with interactive videos and follow-up tasks showed that such design features can indeed foster students' understanding of the generalizer. While generic design feature for supporting students' focused cognitive engagement are known (Atkinson & Renkl, 2007; Mayer, 2021), their algebra-specific realization requires design research to specify the relevant content elements that need focus: The *written explanation* introduced in silence took students' limited cognitive capacity into account while focusing a particularly relevant content (the generalizer) missed by students in prior DE cycles. The design of the *interactive feature* shown in Figure 2a was strongly influenced by a careful algebra-specific analysis of students' learning processes in previous cycles, which allowed to build on typical student ideas: In the case of Gamze and Arif, it elicited relevant prior interpretations of letters as placeholders, sums, and unknowns, and encouraged them to focus on the relevant content by articulating ideas about generalizations ("not everyone wants" the same number; "x stands for all numbers"). The design of the *follow-up tasks* was similarly informed by previous DE cycles and encouraged students to engage in cognitively demanding self-explanations focusing the generalizer and making use of algebra-specific language means provided in the video to grasp subtle conceptual differences ("you don't choose a number", "x can stand for all [numbers]").

Thus, interactive videos with targeted topic-specific interactive features and follow-up tasks can be a valuable resource at the beginning of the knowledge organization phase, if they are sensitive to typical student ideas from the exploration to be leveraged and allow students to process challenging algebraic ideas at their own pace and with focused support. Yet, for most students, understanding a concept as complex as the generalizer will require a discussion with the teacher at some point in the learning process, as some fine nuances can only be clarified through discourse. We see interactive video as a useful preparation for such teacher-led whole-class discussions, and initial responses from teachers

suggest that they themselves can benefit from this design by using typical student ideas in the follow-up tasks as a starting point for their whole-class discussion, and by drawing inspiration from the video for targeted language means and ways of connecting multiple representations.

Acknowledgment

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A case study of a seven-year old's reasoning with and understandings of function graphs

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Abstract

This study consisted of Grades 1 and 2 (ages 6-8) classroom teaching experiments (CTE) with a lesson sequence focused on graphical representations of algebraic relationships. We interviewed students before, during, and after the CTE. Here we report on the progression of one Grade 2 (age 7) student's thinking across the CTE. Prior to the CTE, the student had not previously interacted with representations of algebraic relationships. By the end of the CTE, the student was able to generalize about the functional relationships using graphs. This study illustrates how a learning trajectory modeling students' understandings of function graphs can be used to characterize one children's learning and provides evidence that young students are able to reason with function graphs.

Keywords: learning, learning trajectories, mathematics education, algebra.

Introduction

Prior work has illustrated that young students are capable of algebraic reasoning (e.g., Dougherty, 2008), including using variables (e.g., Blanton et al., 2017; Brizuela, Blanton, Sawrey et al., 2015); reasoning about functional relationships (e.g., Brizuela, Blanton, Gardiner et al., 2015), and representing algebraic ideas using tables (Brizuela et al., 2021). A key takeaway from this prior work is that functional relationships are opportunities for young learner to reason algebraically. We build on emerging work (e.g., Strachota et al., 2024) on students' understandings of Cartesian graphs to reason algebraically.

Our study builds on this previous work and assumes that if students have consistent experience with algebraic reasoning earlier in their K-12 schooling, then algebra would be less of a gatekeeper to higher education and subsequent job opportunities.

In this study, we will address the following research questions:

1. How does one student's understanding of function graphs develop over the course of a classroom teaching experiment (CTE)?
2. What aspects of the CTE lessons in which the student participated seem to be associated with shifts in that student's understandings?

Conceptual framework

We rely on the levels of students' understanding of graphs developed in our prior work (viz., Strachota et al., 2024). To develop these levels, Strachota et al. (2024) identified relevant literature on trajectories and progressions in related contexts (e.g., Brizuela et al., 2021; Battista et al., 1998; Gabucio et al., 2010; García-Milá et al., 2014; Martí, 2009; Martí et al., 2010, 2011; Piaget, 1970; Sfard, 1991) and described the ways of thinking about graphs that might occur in the context of functional relationships. Then, using grounded theory (e.g., Corbin & Strauss, 1990) we iteratively revised the levels based on data that was collected through interviews and CTEs with Grades 1 and 2 students. The process resulted in a description of an eight-level trajectory, modelling students' understandings of function graphs (see Table 1); we use these descriptions to characterize students' understandings as well as help explain the shifts we observed in their understandings over time.

Table 1: Descriptions of the levels in the graphing trajectory

Pre-structural	The student does not understand the representation is a graph. They can engage with the graph but not in a way that is unique to this type of representation or that indicates they understand graphs as different from another type of representation.
Interiorization of component(s)	The student understands what the graph is about, what it is referring to, and/or what information is being recorded. They have interiorized (Sfard, 1991) components, such as axes, variables, what quantities are represented by points, etc., but not the graph as a whole. Thus, they do not yet accurately interpret or plot points.
Emergent coordination of quantities	The student interiorizes the coordination of quantities. This might involve gesturing between two coordinated quantities or connecting two coordinated quantities with a line, but not plotting a point or putting their pencil on the point and indicating the precise location. This might include plotting a point at an imprecise, incorrect location.
Condensed coordination of quantities	The student interiorizes the Cartesian coordinate plane (Sfard, 1991). That is, they have or are developing a conceptualization of the graph as two coordinated axes (i.e., uniformed, structured rows and columns according to Battista et al., 1998) and have interiorized the coordination of quantities. Thus, they have condensed a point. They do not however do anything to provide evidence that they understand how two or more points are related.
Emergent recursive inferences	The student can relate two or more points on the graph by describing a recursive pattern. At the emergent recursive inferences level, students describe the change between points in terms of either the *x*-axis or the *y*-axis, but not in terms of both. That is, they describe either the "rise" and the "run" of the slope, but not both.
Condensed recursive inferences	The student can relate two or more points on the graph by describing a recursive pattern. They may interpret consecutive points as always going up by the same amount. They do not however describe the general functional relationship represented in the graph. At the condensed recursive inferences level, students describe the change between points in terms of both the *x*-axis and the *y*-axis. That is, they describe both the "rise" and the "run" of the slope.
Functional inferences	The student can relate two or more points on the graph and generalize about a functional relationship. When students generalize about a functional relationship, they describe a relationship that extends beyond consecutive points. The distinction between this understanding and making recursive inferences, is that students' recursive inferences rely on consecutive points, whereas students' functional inferences are general and connect across points.
Function graph as object	The student understands the function graph as something that extends beyond what is at hand and exists in relation to other function graphs. They can describe how the graph can be transformed or manipulated without having to recreate it or calculate specific values.

Method

We conducted CTEs in Grade 1 and 2 classes at an elementary school in the Northeastern United States. For this paper, we focus on one Grade 2 student's reasoning with algebraic relationships and representations in the CTE. The data collection consisted of videorecording 14 CTE lessons in each of the Grades 1 and 2 classrooms (ages 5-8). The CTEs lessons were about 40 minutes and taught by a former classroom teacher. We detail the development and content of the CTE lessons below. To track students' progress in the instructional sequence, we conducted interviews (Steffe & Thompson, 2000). The interviews took place before, during and after the lesson sequence and were conducted by at least two team members. Lessons and interviews were videorecorded.

Classroom Teaching Experiment

Over the course of the 14 CTE lessons, Grade 2 students reasoned with three different functional relationships within multiple problem contexts: 1) $y = x$; 2) $y = 2x$; and 3) $y = x + 1$. The instructional sequence was designed in the form of cycles of revisiting tabular and graphical representations. For instance, students reasoned with tables and graphs in both the "dog eyes" context ($y = 2x$) and the "cutting string" context ($y = x + 1$). The advantage to this cyclical approach is that it provides opportunities to revise our curriculum plans fluidly to support students in advancing their reasoning (Sandoval, 2014).

The lessons were chosen and adapted from prior research that had included CTEs and instructional sequences (Blanton et al. 2015, 2017; Brizuela, Blanton, Gardiner et al., 2015; Brizuela, Blanton, Sawrey et al., 2015). We intentionally chose lessons that focused on representing, including tabular and graphical, or lessons that could be modified to include these representations to create our instructional sequence. Lessons were piloted with other Grade 1 students prior to the CTE.

Individual Interviews

In addition to the lessons, interviews were conducted before, halfway through and after the CTE to document and understand the students' progress with 1) reasoning with algebraic representations and 2) ability to think recursively and functionally. In the interviews, students were asked to both reason with and create graphical representations to understand the function $y = 2x$ in the context of the relationship between number of birds and bird wings.

Here we report on three interviews from one Grade 2 student, Levi. We selected Levi's interviews for analysis because they supported us in answering our research questions. Levi provided rich verbal responses and a willingness to engage with graphs. For each of Levi's interviews, we noted the highest level within the graphing trajectory that he exhibited.

Data analysis

All of Levi's interviews were transcribed and coded line by line based on Levi's utterances and gestures. using MAXQDA software. Codes were applied irrespective of mathematical accuracy. The codes we used are outlined in Table 1. To ensure reliability, we used a primary/secondary coding approach and addressed differences using third-party resolution (Syed & Nelson, 2015).

Findings

In the following we report the most sophisticated level that Levi exhibited in each of his three interviews and provide examples that illustrate the levels.

First interview

Since students were not expected to have seen the graph before, the graph that Levi received in his first interview was pre-constructed with points representing one through seven birds and the corresponding wings. Figure 1 shows the representation that was given to Levi and his annotations.

Levi interpreted some components of the graph, namely that the x-axis represented the number of birds and that the number of bird wings is represented with the points. For instance, the following exchange occurred when Levi was asked to solve for the number of bird wings if there were three birds.

Levi:	Ok, so we started by three (draws line from [3,0] to [1,2] to [2,4] to [3,6]).
Interviewer:	Can you tell me what you're doing?
Levi:	I'm doing the three and then doing all the way up, with the counting by twos, which number we're starting with.
Interviewer:	Mhm, why did you stop at this point?

Levi: Because it (the point) has a six.
Interviewer: Does it have a three too, or just the six?
Levi: It has a six, you start with the three and then it goes all the way up to six.

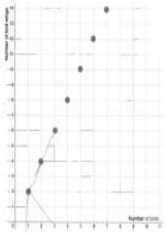

Figure 1: Levi's written work on the graph in his first interview

To solve for the number of bird wings if there were three birds, Levi first marked (3,0) with his pencil, indicating that it is represented three birds on the *x*-axis. Next, Levi drew lines from (3,0) to (1,2) to (2,4) in order to connect (3,0) to (3,6), indicating that (3,6) represented 3 birds and 6 wings. Notably, despite using the *x*-axis as a starting point for his lines, Levi never stated he understood the number of birds is represented by (3,6). When asked if the point "has a three too, or just the six," Levi never stated that three is represented by that point as well. Therefore, we infer that Levi has interiorized *part* of the point, specifically the number of bird wings, evidence of *emergent coordination of quantities* level. He also indicated *emergent recursive inferences* by describing the difference between points by counting up two from the *x*-axis.

At the *emergent coordination of quantities* level, students begin to interiorize how to coordinate quantities in the graph context, such as by drawing lines connecting two covarying quantities. Differing from *condensed coordination of quantities*, students that exhibit this level have not fully interiorized what the points represent.

Second interview

Once again working on the pre-constructed graph, Levi drew lines that connected values on the axes to points on the graph. For instance, when solving for the number of wings if there were two birds, Levi drew a line connecting the *x*-axis (2, 0) to the point (2, 4). This action suggested that he understood two birds were represented within the point (2, 4). When asked where four wings was on the graph, Levi gestured from (0, 4) to (2, 4) and stated that the four is "on this dot," suggesting that he also understood four wings were represented in the point simultaneously. Later in the interview, he drew a line connecting four wings to the point, providing evidence that he was thinking at the *condensed coordination of quantities* level. Levi then accurately coordinated quantities for one, three, and four birds by again drawing guidelines to connect to the points.

Third interview

Similar to his second interview, Levi drew guidelines from the axes to plot points (see Figure 2). The teacher-researcher asked Levi to plot (8, 16) without the use of guidelines. After he accurately plotted the point, the following exchange occurred.

Interviewer: How are the points helping you figure out where the next point is?
Levi: Cause they're going like in a pattern, like 1, 2, 3, 4, 5, 6, and they're going diagonal.
Interviewer: They're going diagonally?
Levi: Yeah, that's how I know.

Levi's statement reflected the *emergent recursive inferences* level, as he is thinking about the points in relation to each other. By stating "1, 2, 3, 4, 5, 6," Levi described a recursive pattern, indicating he has condensed the points and is now acting on the points, reasoning about they are related.

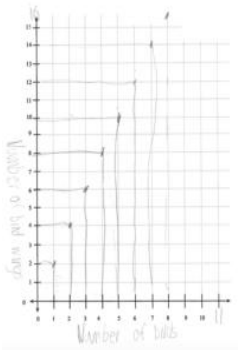

Figure 2: Levi's written work on the graph in his third interview

Summary of findings

To summarize our observations of Levi's progress, we present Table 2, which shows Levi's level of understanding of graphs in each of his three interviews. The table indicates that Levi demonstrated increasingly more sophisticated thinking over the course of the CTE.

Table 2: Levi's levels of understandings of graphs in his first, second, and third interviews

	Interviews		
	1	2	3
Function graph as object			
Functional inferences			
Condensed recursive inferences			
Emergent recursive inferences			
Condensed coordination of quantities			
Emergent coordination of quantities			
Interiorization of components			
Pre-interiorization			

Discussion

The goal of this paper was to describe how one student's understandings of function graphs shifted over the course of a CTE, and to highlight aspects of the lessons of the CTE that seemed associated with those shifts. In the results we presented Levi's understandings of function graphs. Here, we discuss characteristics of the lessons to illustrate how the lessons may have contributed to the shifts in his thinking. For the sake of brevity, we report on two lessons, 4 and 5, that we argue involved content that seemed related to the ways in which Levi reasoned about function graphs.

In lesson 4, students reasoned with a large, human-sized number line taped across the floor that represented the *x*-axis to represent the relationship between number of dogs and dogs' eyes. Students were familiar with a number line but had not yet used the Cartesian coordinate plane, so they practiced using a single number line representing the *x*-axis, with the intention of building on that representation. They reasoned about the relationship, plotting points and stating what quantity

represented. When students plotted points on the human-size number line, they walked them out and stood on the point. The goal was to help students *interiorize components* of the graph, namely the *x*-axis and points.

The lesson was also designed to create a need for the *y*-axis. The idea was that students would notice the challenges of representing two different quantities using a single number line and start experimenting with other ways of representing two quantities simultaneously.

In lesson 5, students continued reasoning about the relationship between the number of dogs and eyes. Students continued with the human-sized representation, but this time they used both the *x*- and *y*-axis. Each student had a paper graph as well to record their representation while the class interacted with the human-sized graph.

Because the axes were human-size, students walked out on the axes and Cartesian coordinate plane to represent quantities. We note this activity because we theorize that it may have contributed to developing a visual link between the axes and the quantities, which manifested as drawing guidelines to plot points on a paper graph.

In lesson 5, Levi plotted points on his paper by drawing guidelines from the axes and walked around on the human graph to represent points. Initially, we did not observe an indication that Levi was relating points or noticing a relationship between points. However, a group of students devised a new strategy of plotting new points based on the placement of previously plotted points. This strategy was shared with the whole class on the human-size Cartesian coordinate plane. We theorize that these moments may have helped students *interiorize the y-axis* and begin to think about the points in relation to each other, helping them reach the *emergent recursive inferences* level.

Later in this lesson we also observed that when Levi plotted the point (4, 8), he used the (3, 6) as a reference to "measure." Levi presented this method to the class, and the teacher-researcher asked the students to work individually and try Levi's method of "measuring." This moment affirmed that Levi's strategy was productive. This strategy also indicated that Levi was beginning to relate points.

We share these summaries of lessons 4 and 5 to highlight the ways in which these experiences contributed to developing Levi's reasoning about function graphs. We presume that students' understandings are intimately connected with the curricular progression in which these understandings were developed (Barrett & Battista, 2014; Clements & Sarama, 2014). That is, the trajectory that models Levi's reasoning is dependent on his engagement in the instructional sequence. Moreover, we highlight that these are just two of the 14 lessons in the instructional sequence while reminding readers that lessons were structured similarly throughout the CTE, using repeating representations and functions, but varying the context of the functional relationships.

Conclusion

Levi's thinking in each of his three interviews grew more sophisticated over the course of the CTE. We are impressed with the levels of sophistication of Levi's thinking throughout the CTE, as he only experienced instruction on graphs for a short period of time. We argue that if algebraic representations were woven more consistently throughout mathematics curriculum, we envision that students would more frequently demonstrate these sophisticated ways of algebraic thinking. Aligning with others, particularly those who have conducted longitudinal studies (e.g., Blanton et al., 2015; Dougherty, 2008), we argue that children would benefit from sustained opportunities to reason algebraically to develop "algebraic habits," such as mathematizing unknown quantities and relationships (Blanton et al., 2015, p. 545).

We also view these results as evidence of the power of representations as a tool for enabling students to see things they may not see otherwise. When incorporated properly into instruction, representations are tools that can help students do things more easily or do things they could not do alone (Hiebert et

al., 1997). In fact, Levi's reasoning confirms that representations "enable some thoughts that would hardly be possible without them" (Hiebert et al., 1997, p. 53). Consider his observations about relationships between points.

We also highlight the power of representations for cognitive scientists. We presume the representations mediated Levi's mathematical thinking and supported us to see Levi's "unseen" ideas (McCormick et al., 1987) about functional relationships. That is, in our study, the graphs were products and interpretations of ideas about how Levi advanced his understandings of functional relationships. With this lens, we can better understand students' developing understandings of the concept being represented through the representation itself.

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Teachers' conceptualizations of slope in teaching Grade 8 students

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This study explores how Grade 8 mathematics teachers introduce the concept of slope and the specific conceptualizations they emphasize. The research involved observing six lessons from two Grade 8 teachers. The findings revealed that the teachers addressed six distinct conceptualizations of slope: geometric ratio, real-life physical property, parameter, linear constant, behavior indicator, and real-life functional situation. Teachers predominantly relied on a visual approach to teach the geometric ratio conceptualization. Despite introducing multiple conceptualizations, there was limited time for students to independently explore and connect these ideas. While both teachers covered the same slope conceptualizations, they differed in their emphasis on explanations, procedures, and connections, such as constructing triangles to visualize the ratio of vertical to horizontal displacement or using the algebraic formula.

Keywords: Slope, functional thinking, covariational reasoning.

Introduction

Slope is a fundamental concept in mathematics education, essential for understanding linear functions, graphical interpretations of functions, and rates of change (Diamond, 2020; Teuscher & Reys, 2012). It plays a critical role in the study of linear functions, typically introduced in Grade 8, and extends throughout secondary school to encompass the description of non-linear functions and pre-calculus concepts (Cho & Nagle, 2017). Grasping the concept of slope is closely tied to a covariational approach to functions, which involves viewing a function as a process of covariation—focusing on how the dependent variable changes in relation to the independent variable (Thompson & Carlson, 2017). Despite its importance, students often struggle with isolated meanings and misconceptions related to slope. These challenges are further compounded by the varying conceptions held by teachers and the procedural emphasis prevalent in many textbooks (Dolores et al., 2020; Lingefjärd & Farahani, 2018).

Although previous research has explored students' and teachers' conceptualizations of slope, there is limited knowledge about how teachers introduce and develop this concept in classrooms. This study aims to investigate how teachers conceptualize and introduce slope in Grade 8, focusing on their instructional strategies, the connections they emphasize, and their approach to balancing procedural and conceptual knowledge.

Theoretical Framework

Conceptualizations of Slope

Slope has been regarded as a measure of various attributes, depending on the quantities involved in the relationship, and involves constructing a ratio or rate to measure the given attribute (Lobato &

Thanheiser, 2002). In secondary mathematics curricula, slope is presented in multiple forms: geometrically as the $rise/run$ ratio, representing the steepness of a line; algebraically as the parameter m in the equation $y = mx + b$ or through the formula $\frac{y_2-y_1}{x_2-x_1}$; trigonometrically as the tangent of the angle of inclination of a line; in calculus as a limit; and in real-world contexts as a measure of the steepness of physical situations, such as mountain roads and ramps, or in functional scenarios where slope represents the rate of change (Stump, 2001). Moore-Russo et al. (2011) expanded on Stump's (2001) model and identified eight different conceptualizations of slope: geometric ratio, algebraic ratio, physical property, functional property, parametric coefficient, trigonometric conception, calculus conception, and real-world situation, along with determining property, behavior indicator, and linear constant. Nagle and Moore-Russo (2013) further refined this framework, distinguishing between procedural and conceptual understanding, as well as visual and analytic interpretations, to propose five general components of slope: constant ratio, trigonometric conception, behavior indicator, calculus conception, and determining property.

The constant ratio component encompasses geometric ratio, algebraic ratio, and linear constant conceptualizations (Nagle & Moore-Russo, 2013). The geometric ratio involves visual methods like $rise/run$ or $vertical/horizontal$ change in a line graph, while the algebraic ratio focuses on the nonvisual expression $\frac{y_2-y_1}{x_2-x_1}$; , independent of specific points on the graph. The linear constant conceptualization, crucial for understanding slope as a constant rate of change between two varying quantities, highlights recognizing slope as a consistent ratio for linear functions. Visually, this is represented by the uniform straightness of a line on a graph, regardless of the segment chosen, and by distinguishing between constant and varying rates of change (Johnson, 2015).

The behavior indicator conception of slope includes a visual approach, identifying whether a linear function increases, decreases, or remains constant, and correlating the slope's sign with the directions of rise and run (Stump, 2011). The non-visual approach links the slope's sign to the relationship between changes in y and x. Additionally, the parametric coefficient, m, in the slope-intercept form $y = mx + b$, represents a non-visual approach to this conception. The determining property conception involves visually assessing whether linear graphs are parallel, intersect, or how they intersect, while algebraically, it assesses whether two linear relationships form a solvable system and the number of solutions. The calculus conception, related to limits and derivatives, is typically studied in upper high school. In real-world contexts, slope is conceptualized as either a measure of physical situations, such as the steepness of roads and ramps, or functional situations where it represents the rate of change between covarying quantities, like in distance-time scenarios (Stump, 2001).

Researchers have highlighted the need to connect different conceptualizations of slope to build a deep conceptual understanding (Hoban, 2021). Each conceptualization of slope is a crucial component of a broader network of understanding (Nagle & Moore-Russo, 2013). Teuscher and Reys (2012) emphasized the importance of helping students distinguish between slope, rate of change, and steepness, and to make the correct connections among them by exploring both visual and non-visual approaches and applying these concepts in real-world contexts.

Teachers' Conceptions of Slope and Teaching Practices

Numerous studies have examined how preservice and in-service secondary school mathematics teachers conceptualize slope worldwide (Byerley & Thompson, 2017). Research by Mudaly and Moore-Russo (2011) found that the most common conceptualizations among practicing teachers were the parametric coefficient and behavior indicator, while real-world situations and physical properties were less emphasized. Similarly, Byerley and Thompson (2017) noted that most secondary teachers focused on additive and formulaic meanings of slope, with few addressing slope as a rate of change. Stump (1999) also noted that few teachers incorporate ratio in teaching slope and tend to prioritize procedural methods over conceptual understanding (Lingefjärd & Farahani, 2018). Consistent with Nagle and Moore-Russo (2013) and others, López et al. (2024) found that teachers often emphasize procedural aspects of slope, making it difficult for students to connect slope with concepts like rate of change and steepness. This focus on procedures, such as the "rise-over-run" mnemonic, contributes to a limited understanding of slope (Birgin, 2012).

Researchers have highlighted a gap between teachers' conceptual understanding of slope and its classroom implementation (Nagle & Moore-Russo, 2013). Although teachers intend to teach slope conceptually, they often default to procedural tasks in practice. Nagle and Moore-Russo (2013) revealed that teachers frequently emphasize algebraic ratios, parametric coefficients, and real-world applications in instructional materials, neglecting other important concepts like determining property and functional property. López et al. (2024) further found that teachers prioritize constant ratio and trigonometric conceptions while overlooking the variational aspects of slope.

Methods

Participants, Setting, Procedure, Data Collection and Analysis

Two Grade 8 teachers from an urban gymnasium in Cyprus participated in this study. Both recently completed pedagogical training at the University of Cyprus, had less than five years of public-school experience, and around 15 years of private tutoring experience. They voluntarily agreed to have their lessons on slope observed. The researcher did not influence lesson planning, allowing the teachers to design their lessons independently. Each teacher taught three classes, involving 116 students. Students were informed about the study's goals and encouraged to freely share their thoughts during lessons. Parental consent and student assent were obtained.

The Cyprus mathematics curriculum for Grade 8 includes attainment targets that emphasize understanding the slope of a line as the ratio of the vertical displacement (Δy) to the horizontal displacement (Δx). It also explores the role of the parameters a and b in the graph of the line $y = ax + b$. The official school textbook defines slope as the ratio of the vertical displacement (Δy) to the horizontal displacement (Δx), referring to this as the rate of change. Additionally, the book provides the formula $\frac{y_2-y_1}{x_2-x_1}$ for calculating the ratio $\Delta y/\Delta x$ and defines the slope for the special cases $y = a$ and $x = b$. Furthermore, the book states that the slope of the line $y = ax + b$ is equal to the coefficient of x. Finally, it illustrates the graph of a line with positive and negative slopes. According

to the official guidelines, three to four teaching hours can be used for the teaching of slope. Slope is part of a larger unit, consisting of around 16 teaching hours on functions.

The two teachers dedicated three lessons, each lasting 45 minutes, to teaching slope, followed by an additional hour for revision, which included linear functions. The researcher observed three consecutive lessons in one class for each teacher to maintain consistency in student interaction. Although both teachers taught multiple classes, they reported that all Grade 8 classes performed at a similar level, allowing them to use the same lesson plans across all groups. Classroom data were collected using an audio recorder placed centrally in the room during each lesson. Two researchers attended each session to take field notes, focusing on whole-class discussions, teacher explanations, and feedback, as group work was minimal. No formal observation protocol was used to preserve the authenticity of the teachers' language and actions.

The study collected audio recordings, observations, and field notes from six mathematics lessons, focusing on the depth and connections in teachers' conceptualizations of slope. Audio recordings were transcribed and enriched with observers' notes, capturing teachers' actions, selected materials, explanations, examples, and class discussions. A qualitative interpretive framework guided the data analysis (Miles & Huberman, 1994), using a coding scheme based on frameworks by Moore-Russo et al. (2011), and Nagle and Moore-Russo (2013). The scheme differentiated between various slope conceptualizations, including geometric and algebraic ratios, physical properties, and rates of change, and was enriched with teachers' explanations and examples.

Results

Analysis revealed that during the three lessons focused on the concept of slope, both teachers covered most of the slope conceptualizations introduced in Lesson 1. Teacher 1 primarily used the geometric ratio approach as the central theme, building upon it when introducing other concepts. In contrast, Teacher 2 initially engaged students with the idea of slope as the physical property of steepness, subsequently connecting this to slope as a geometric ratio through relevant examples.

In Lesson 1, the teachers introduced five different conceptualizations of slope: geometric ratio, real-life physical property, parameter, linear constant, and behavior indicator (see Table 1). Teacher 1 mentioned that slope could be analyzed as a rate of change, a concept to be explored in subsequent lessons. Teacher 2 implicitly referenced the rate of change by explaining that the ratio of vertical to horizontal displacement, as illustrated by a ramp, represents the height gained for every meter of horizontal displacement. Both teachers involved students in understanding slope as a physical property. The activities explored slope as a geometric ratio and as a parameter, while the linear constant was implicitly addressed through tasks involving calculating the slope of a line using different pairs of points. In the lesson's conclusion, the teachers introduced the behavior indicator to explain the slope of special cases of lines. Teacher 1 focused on demonstrating and calculating slope as geometrical ratio, by constructing a triangle on the graph. She introduced this method as the simplest to determine a line's slope. Using the graph, she illustrated that the triangle's vertical side represents the vertical displacement (Δy), while the horizontal one represents the horizontal displacement (Δx). She provided practical advice on how to calculate this ratio, recommending starting from left to right by first constructing the horizontal displacement, and then the vertical

displacement. For a line with negative slope, Teacher 1 noted that a negative sign must be added manually, as the final movement is downward. She emphasized that the triangle should always be constructed on the right side of the line. Teacher 2, on the other hand, emphasized that whether a line inclines or declines determines the sign of the ratio $\frac{\Delta y}{\Delta x}$. Contrary to Teacher 1's approach, Teacher 2 introduced the algebraic ratio as an alternative way to calculate slope. She explained that for any two points A(x_1, y_1) and B(x_2, y_2), the slope is calculated using the formula $\frac{y_2 - y_1}{x_2 - x_1}$.

In Lesson 2, no new slope conceptualizations were introduced. In the introductory activity, Teacher 1 connected the concept of slope as a physical property of steepness with the behavior indicator to explain the slopes of specific types of lines ($y = a$ and $x = b$). She illustrated how the sign of the ratio Δy/Δx correlates with the behavior of the line, indicating whether it is inclining or declining. Graphically and algebraically, she demonstrated that for the line $y = a$, the numerator in the ratio Δy/Δx is zero, resulting in a slope of zero. In contrast, for the line $x = b$, the denominator is zero, which means the slope is undefined. Teacher 2 also focused on the slope of these special cases, using slope as steepness to clarify. She pointed out that the line $y = a$ is flat, neither ascending nor descending, while the line $x = b$ is vertical to the horizontal axis, making it infinitely steep, and thus, the slope cannot be defined. Additionally, she stressed that if the equation of a line is given in the standard form $y = ax + b$, then 'a' represents the slope. She dedicated significant time to tasks that required transforming equations into the standard form to easily identify the slope.

Table 1: Conceptions of Slope as Introduced by Teachers

Slope Conceptualization	Teacher 1	Teacher 2
Visual Geometric ratio: Vertical displacement over horizontal displacement	Formal definition of slope is introduced based on this conceptualization. It includes constructing a triangle to visualize vertical displacement over horizontal displacement, defining vertical and horizontal displacement as Δy and Δx respectively. Students are tasked to calculate the ratio Δy/Δx for lines with positive or negative slope.	Students explore the concept through a textbook activity featuring skateboard ramps. While discussing slope as steepness, they connect it to the geometric ratio, concluding that with equal bases, the steeper ramp is higher, and with equal heights, the steeper ramp has a smaller base. **Linking steepness to the formal definition of vertical and horizontal displacement**, the ratio Δy/Δx represents the slope, indicating the steepness of the line or ramp.
Parametric coefficient	The ratio Δy/Δx is labeled as m. Students are tasked with observing that the ratio Δy/Δx equals the parametric coefficient in the standard form of the line $y = ax + b$.	If the equation of the line is given in the standard form line $y = ax + b$, then a is the slope. Students are tasked to identify whether given equations are given in the standard form to identify the slope.
Physical property – steepness	Students are engaged in the concept through an exploration activity from the school textbook, which presents skateboard ramps. Slope is related to real-life scenarios, such	Students discuss steepness by comparing the four ramps of the school textbook exploration activity. Students are prompted to consider steepness quantitatively as a measure of the meters of height

	as mountain roads, uphill and downhill terrains.	gained in relation to one-meter horizontal displacement.
	Special cases lines $y = a$ and $x = b$: A runner runs on a flat surface, so the slope is zero. A human cannot walk on a walk; thus, it can be supported that slope is undefined.	A line of a graph can be viewed as a large ramp, where the goal is to calculate and express the degree to which the line ascends or descends.
Behavior indicator	Connection of the sign of the ratio $\Delta y/\Delta x$ with the direction of the vertical vector when constructing the triangle that defines the vertical and horizontal dsiplacement.	An alternative method simplifies the process by first finding the absolute value of $\Delta y/\ \Delta x$ and then determining the slope's sign. The slope is positive if the line ascends from left to right; otherwise, it is negative.
Linear constant conceptualization	An implicit connection to linear constant conceptualization by concluding that the calculated slope is the same regardless of the selected points.	Clarification that the ratio of vertical to horizontal displacement is constant for any two points on a line, which is a characteristic feature. The algebraic ratio is provided
Real-word situation: Functional situation	Use of a real-life problem involving a functional relationship. The formula is given, and students are tasked to interpret the intercept and slope.	Students are tasked with identifying the variable and constant measures in the real-life problem make appropriate connections to the slope and intercept.

In Lesson 3, both teachers concentrated on applying the concept of slope to real-world functional situations. Students tackled two problems involving quantities connected by a linear relationship. In the first problem, they were given the formula and needed to identify and utilize the slope. The second problem presented a verbal description of a linear relationship, requiring students to write the corresponding formula. In both scenarios, the teachers emphasized interpreting the slope as the rate of change between the two quantities. They asked questions that encouraged covariational reasoning, such as calculating how the dependent variable changes in response to uniform increments in the independent variable. The increase in the dependent variable for each unit increase in the x-value was also illustrated on a graph. Teacher 1 aimed to link the concept of slope and rate of change specifically to the scenario at hand (e.g., the rate of height increase) by relating it to the vertical-to-horizontal displacement ratio while constructing the graph. Meanwhile, Teacher 2 described the slope as an indicator of how quickly the dependent variable increases within the scenario, while also identifying the intercept value as the constant parameter in the situation.

Discussion

The analysis revealed that teachers addressed six different conceptualizations of slope, either explicitly or implicitly: geometric ratio, real-life physical property, parameter, linear constant, behavior indicator, and real-life functional situation. This finding is consistent with previous research (Stump, 1999). Moreover, it was found that teachers predominantly used a visual approach to the geometric ratio conceptualization. This approach, which involved the ratio of vertical to horizontal displacement, was emphasized in several activities during the first two lessons. Although the teachers

introduced five different conceptualizations in the first lesson, there was insufficient time for students to explore and connect these different conceptualizations or form their own interpretations. The teachers attempted to link different representations and slope conceptualizations through their explanations. This included interpreting the slope's sign as a behavior indicator, calculating the geometric ratio as vertical to horizontal displacement, and then determining the sign based on the line's behavior. Most of the explanations, repetitions, and guidance focused on constructing the appropriate triangle to illustrate the geometric ratio. This finding supports Leong and Chick's (2011) claim that time constraints can lead to direct teaching approaches and reduce opportunities for students to engage in independent exploration.

Although both teachers used the same teaching material, lesson plan, and addressed the same slope conceptualizations, they emphasized different explanations, procedures, and connections. This could lead to students having fragmented or incomplete understandings of the concept and inconsistent comprehension among different groups of students. For instance, one teacher focused on constructing a triangle to demonstrate, calculate, and determine the slope's sign, while the other offered the algebraic ratio as an alternative method. Furthermore, the first teacher spent considerable time calculating the slope of a line using different pairs of values, whereas the second teacher concentrated on transforming the equation of a line into the standard form. Neither teacher's explanations included interpreting the geometric ratio as a rate, such as determining the vertical displacement for different changes in the horizontal displacement (Byerley & Thompson, 2017). This concept could have been reinforced by constructing triangles of different sizes. Additionally, in the third lesson, there was no connection between the contextual interpretation of the rate of change and the geometric ratio studied in the first two lessons. The findings of this study highlight key educational implications for teaching the concept of slope in Grade 8. Slope is fundamental in mathematics, and adequate teaching time is crucial. Rushing through multiple conceptualizations without providing sufficient time for exploration and understanding can lead to superficial learning and poor conceptual approaches.

Furthermore, professional development programs for teachers should emphasize the importance of connecting conceptualizations of slope to real-world applications and rates of change. Teachers should be encouraged to incorporate a variety of teaching strategies, such as connecting the geometric ratio to contextual problems and encouraging students to explore different representations through hands-on activities. Additionally, teacher education programs should focus on equipping educators with the skills to present the geometric, algebraic, and contextual interpretations of slope cohesively. This includes addressing how different approaches—such as graphical or algebraic—can complement each other, reducing the risk of fragmented student understanding.

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Cartesian graphs and co-variational thinking in early algebra

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The aim of this article is to contribute to the existing body of research on the understanding of graphs by young students. To this end, I draw on the theory of objectification and focus on the processes of conceptualizing co-variation between continuous variables when Grade 6 students, aged 11–12, interpret a given graph. The context is a dynamic one, namely, the interpretation of a Cartesian graph representing the movement of a body. The findings suggest that, initially, based on perceptual understandings, the students conceptualize the graph as a map, without co-variation being a central aspect of their analyses. Furthermore, the students' perceptual understandings give rise to some contradictions. However, these are gradually resolved as the teacher engages with the students in exploring the graph and new cultural-historical understandings of co-variation emerge.

Keywords: Motion graphs, theory of objectification, co-variational thinking

Introduction

Although research on early algebra has produced a wealth of results in recent years, research on young students' understanding of Cartesian graphs has not received much attention. Indeed, in their recent literature review, Cañadas et al. (2024) note that studies investigating how elementary students construct, use, or read graphs are almost nonexistent. Yet, graphs are known to provide rich functional contexts for learning about variables and their co-variation (Blanton et al., 2018; Kaput, 2008; Kieran, 2016; Robert, 2024). When studied, Cartesian graphs have often been featured in connection with a new functional mode of representing *discrete* data in x/y tables—see, e.g., Cañadas et al. (2024), Carraher et al. (2008), and Tierney and Monk (2008). In this case, graphs are constructed from a functional relation between x and y variables (e.g., $y = 2x+1$), instances of which lead to points (x, y) in the Cartesian plane. In this paper, I focus on a different type of Cartesian graph: a given Cartesian graph representing the *motion* of a body. Understanding such a graph requires *conceptualizing* the co-variation between the inherent *continuous* variables of time and space. Inspired by Carlson et al. (2002), I consider co-variational thinking as the embodied, semiotic, and material cognitive activity "involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other" (p. 354). The research question I seek to answer is how students come to conceptualize the co-variation of two variables—time and distance—in a given Cartesian graph.

Theoretical framework

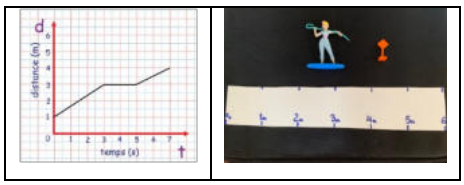
The theoretical framework followed in this paper is based on the Vygotskian theory of objectification (TO) (Radford, 2021). In accordance with the TO, the students' conceptualization is part of a sense-making process: the *process of objectification* (p. 86); that is, the social process in which teachers *and* students engage in sensuous and material activity as they proceed to encounter, examine, and explore cultural-historical mathematical meanings—in the case of this paper,

mathematical meanings related to Cartesian graphs. This teacher-and-students' sense-making process is a creative, imaginative, affective one entrenched in classroom teaching-learning activity. It finds support and expression in language, signs, artefacts, interactions, gestures, and the body more generally.

Methods

The method to investigate the research question is the dialectical materialist one proposed in the TO (Radford, 2015). This means that *the methodological explanatory category* is the *teaching-learning activity* that anchors the various multi-modal and material connections that teacher and students create as they produce ideas to explore and understand the cultural-historical meaning of Cartesian graphs. The teaching-learning activity is designed in a way that allows the students to work in small groups of three or four and encourages the students to work together in the production of what in the TO is called *une oeuvre commune* (a common artistic work): a collectively built idea about how—in the case of this paper—to interpret graphs. We use four or five video cameras and make transcriptions that are then subjected to multi-modal analyses (Radford, 2021). The data presented below come from a recorded lesson conducted in a Grade 6 class (11–12-year-old students) in a French public school in Sudbury, Canada. In this paper I focus on one of the small groups (chosen for its paradigmatic nature)—a kind of case study. Case studies are limited in terms of the generality of their results. However, they allow in-depth explorations of the research problem. The lesson was preceded by a lesson in which the students had access to a video showing a remotely controlled battery-operated train moving at constant speed in a graduated straight line. They were asked to produce a Cartesian graph of the train's motion. In the lesson featured in this paper (which is an adaptation of a lesson conducted in a Grade 8 class; see Radford, 2009), the graph was given. Here is the problem.

The problem and the first graph interpretations



"Tina is 1 metre from a fountain. She walks on a straight-line path. The graph represents her walk. Give a precise description of Tina's walk." Unlike the lesson in Grade 8, the teacher equipped each group with a kit containing a figurine representing Tina, an object representing the fountain, and a flat, graduated cardboard path to enact Tina's walk. I focus here on one of the groups of three students: D., L., and B.—two girls and a boy, respectively (see Fig. 1.1 below) (Fig m.n refers to pic n, from left to right, in Figure m).

The first interpretation:

D: Huh! Tina starts here (*pointing at (0, 1)*). Then the fountain is here (*pointing at (0, 0)) (Fig. 1.1)*).

L: Yes, but how do we find the time?

D: Time is here. (*She slides her finger along the t-axis*) (*Fig. 1.2*) ... So, from 1 (*she points at (0, 1*)) to here (*she points at (3, 3)*), she did 3s (*Fig. 1.3*). Then, from there (*she points at (3, 3), Fig. 1.4*) to there (*she points at (5, 3), Fig 1.5*), she did 5s .... And

then, from here (*she points at (5, 3)*) to there (*she points at (7, 4)*), *Fig. 1.6*) (*she lowers the pen vertically from (7, 4) to (7, 0)*), she did 7s.

Figure 1. Pic 1, from left to right, B., L., and D. The students' first interpretation

In this episode, D. offers a first explanation of the graph based on an interpretation of it as a *map*. She places the figurines of the fountain and Tina at points 0 and 1 of the cardboard, respectively. Then, the attention moves to the graph. The explanation is not clear to L: "You have to explain, I don't understand." D. responds trying to make more explicit her understanding of the graph as a map:

D: Since she came here (*she slides the pen from (0, 1) to (3, 3)*), she went 3m and 3s. She arrived here (*points at (5, 3)*) … 3m and 5s; then, she went straight (*points to graph horizontal segment*).

L: Ok, Ok! Wait! Wait, 1m, it's 7s.

D: No! ... From here (*points at (0, 1)) to there (points at (3, 3)),* it's 3m and 3s. Also, from here (*points at (3,3)*) to here (*points at (5,3)*) it's 5s.

L: Ahhh! So, what's 7s?

D: From here (*points at (5, 3)*) to there (*points at (7, 4)*), it's 7s.

In D.'s interpretation, the t-axis is not conceptualized as a Cartesian variable, the values of which are referred to as a starting point. In her eyes, the numbers on the t-axis are seen as labels or stickers that indicate the duration of the segments that comprise Tina's walk.

L. states that she remains uncertain as to the appropriate explanation for Tina's walk. The teacher comes to see the group. The students explain their current interpretation:

L: (*Referring to the first graph segment*) Tina walks … 3s.

T: So, at 3s, where is she?

D: At 3m. Then (*moving her finger along the horizontal segment*), she walks straight ahead 5s …

T: Does she walk straight ahead? (*She moves her finger across the graph's horizontal segment*) How long?

L: The distance? We can count (*counting horizontal lines on the grid*) 1, 2, 3, 4.

T: (*Pointing to the vertical axis, she calls the students' attention to the fact that the vertical axis indicates the distances. Talking to B. to involve him in the conversation*) What do you see here?

B: The actions Tina did … the meters …

The teacher echoes B.'s remark, emphasizing that the distance is measured along the vertical axis (Fig 2.1) and leaves. The students keep discussing.

The second interpretation: About 5 minutes later, B. suggests a new interpretation based on the vertical axis: "I'm confused, (but) I'll explain. Uh, uh, I don't know. But maybe it's like … like, Tina walked 1 (meter). (*He points at (0, 1*) *on the Cartesian graph; see Fig. 2.2*). She still goes like ... she goes here (*points at (0, 3)*). And then, she goes here (*points at (0, 4)*), and she's here now (*pointing again at (0, 4)*). Maybe it's like this."

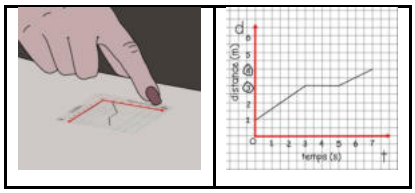


Figure 2. The second interpretation

B.'s interpretation speaks of distance and time in a new manner. Through verbs, B.'s interpretation involves the temporal dimension: Tina *walked*, she *goes here*, but in an unquantified manner. Distance, by contrast is quantified. On the d-axis, B. circles numbers 3 and 4 (see *Fig. 2.2*). It is perhaps the implicit role of time in B.'s interpretation that his teammates find unconvincing; D. and L. go back to the map interpretation of the graph again, as if trying to find something that would so far remain unnoticed.

Third interpretation: L. suggests a third interpretation: "[Tina] started at 1m (*Fig. 3.1*). Then, at three seconds (*she points at 3 on the horizontal axis (Fig. 3.2*) *and slides the pen vertically to point (3, 3); Fig. 3.3*), she continued to 3m (*she points at (0, 3)*) (*Fig. 3.4*); (*repeating*) she continued to 3m (*but now, inspired by what B. said, she moves the pen horizontally to the point (3, 3),*). Then, she continued straight on from 3s to 5s, (*moving the pen along the graph's horizontal segment*), but within 3m (*Fig. 3.5*). Then, she went up ... (*showing the end point of the graph*) to 7s" **(***Fig. 3.6***).**

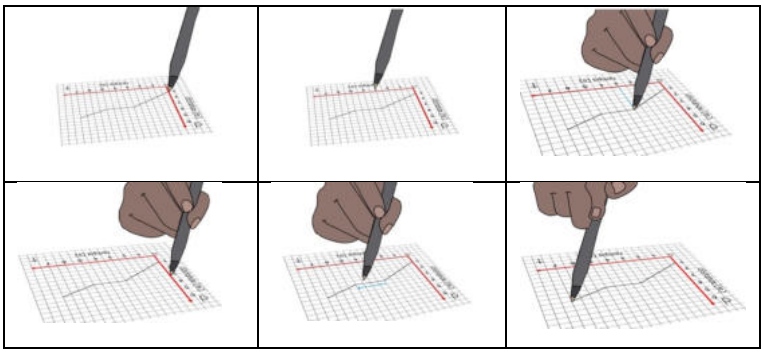


Figure 3. The third interpretation

The interpretation of the horizontal segment persists. But now, *time becomes the main organizer of students' thinking*: at 3s something happened; between 3s and 5s, something else happened, etc. However, the students are not yet satisfied. A few seconds later, L. produces a sharper interpretation: "She started at 1m, ok (*she points at (0, 1); Fig. 4.1*). That's the starting point; and now, at 3s, she was already at 3m (*places the pen between (3, 0) and (3, 3)*; *Fig. 4.2)*. Then, pointing to the horizontal segment, she continued straight on, right? Then at 5s, she started to climb to get to 4m, and at 4m, she is at 7s. Do you understand?"

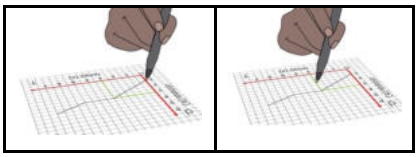


Figure 4. The third interpretation refined

Now, time and distance become related in what we can recognize as a nascent functional interpretation of the graph. Time appears for the first time as a variable; instantiations of the variable (e.g., "at 3 seconds," "at 5 seconds") are put into correspondence with instantiations of the distance seen also as a variable. Yet, the nascent functional interpretation is still haunted by the idea of the graph as a map: the horizontal segment keeps being seen as Tina walking straight on; and in the last segment, Tina is said to climb to 4m. The students start writing their interpretation: "In the beginning, Tina was at 1m and 0s. Then, she walked 3m at 3s. She is at 3m and 3s."

Dealing with the contradictions

When the teacher comes back to see the group, D. offers an explanation that makes the inherent contradictions more visible. D explains: Tina is at "3m and 3s (*she points at (3, 3)*) … Then in the middle of her walk (*she points at (4, 3))* she's still at 3m (*pointing to the horizontal segment*), but she's done 4s (*pointing at (4, 0)*). Then she continued to walk at 3m" (*She moves her pen from left to right on the graph's horizontal segment to indicate Tina's motion*). The contradiction is this: between 3 and 5 seconds, the students imagine that Tina keeps walking while still being at d=3. The teacher encourages the students to keep thinking: "I want you to tell me what happened here (*she points to the horizontal segment*). What do you notice here? What can we add to [the description of] the walk?"

Vygotsky suggested a systemic view of concepts: a concept is formed of dynamic connections between its parts (internal connections) and connections with other concepts (external connections). What distinguishes the concepts that children form in the course of subjective experiences (e.g., "spontaneous" or "quotidian" concepts) from "scientific" concepts is precisely the *nature* of their constitutive connections. Spontaneous concepts often exhibit a *syncretic perceptual understanding* of phenomena. These concepts are based on perceptual connections that form ad hoc *wholes* or *complexes.* Thinking in terms of these perceptual-based connections is what Vygotsky called "complexive thinking" (Vygotsky, 1987, p. 136). What lends "scientific" concepts their particular nature is the cultural-historical character of their internal and external connections. I want to contend that complexive perceptual-based connections are what lead to the perception of Cartesian graphs as maps. Understanding graphs in a Cartesian contemporary mathematical meaning entails

becoming aware of its parts: the meaning of the axis, the identification of variables and their co-variational nature (qualitatively and quantitatively)—among other things. The *movement* from complexive thinking to the Western scientific-mathematical co-variational thinking is not a natural process. In the theoretical framework followed here, the name of the process that makes this movement possible *is teaching-learning activity* (Radford, 2021). In teaching-learning activity, *the teacher works hand in hand with the students*; they collectively generate ideas and novel interpretations of the problem at hand. In this lesson, to help the students deal with the contradictions, the teacher suggested that they use the manipulatives. Initially considered by the students, the manipulatives were ultimately set aside. While manipulatives (or artefacts, more generally) are undoubtedly valuable, they may not always be a stand-alone solution for fostering cultural mathematical thinking. The efficacy of artefacts hinges not only on their intrinsic properties but also on their *integration* and use in the teaching-learning activity. Thus, encouraged by the teacher, the students started placing Tina at 1m from the fountain.

L: She moved forward 2m (*Fig. 5.1*)… at 3s (*The teacher moves Tina to the point marked 3*).

T: Ok, at 3s. Now, what's happening? She's here, isn't she? (*She points at (0, 3) on the graph while pointing to Tina in the manipulative model*). What happens with her walk?

D: She walks straight on, uh … she walks straight ahead in 3m and 5s.

T: She's still at 3m (*see Fig.5.2)*, but she moves forward …?

L: (*As saying something obvious*) At 3m she moves forward 2s!

T: (*Inviting the students to reconsider*) Does she move forward?

L: Yes! She moves forward 2s!

T: There is some time elapsing here (*she points at the t-axis*), but the distance (*pointing to the horizontal segment in the graph*), does it increase or decrease? … Tina is here (*Fig. 5.3*) … Time elapses (*pointing to the t-axis; Fig. 5.4*). But how long?

L: 2s …

T: Ok. 2s. And *here*, (*touching Tina (Fig. 5.5) and pointing at (0, 3) (Fig. 5.5)*); look at Tina's distance when she was at 3m (*making a sliding gesture pointing at the horizontal segment while keeping the figurine motionless (Fig. 5.6)*). Did she move?

B: No!

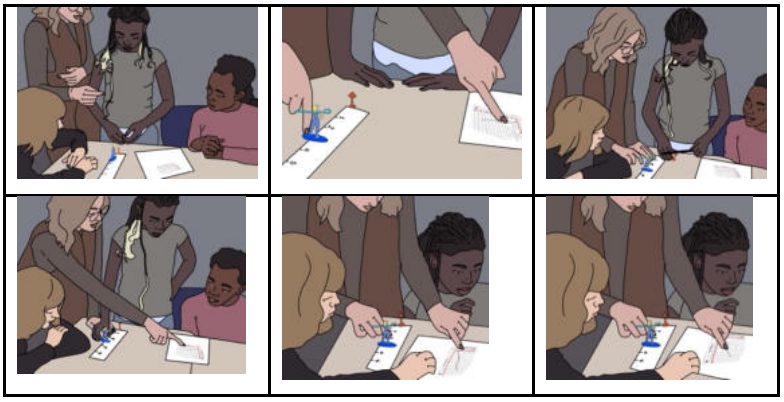


Figure 5. Teacher and students working together

At that moment, the students realized that, in the interpretation of the horizontal segment, changes in one of the variables do not necessarily lead to changes in the other. This new aspect of co-variational thinking had been hindered by the initial percetual sense of the graph as a map. Working hand in hand with the students, the teacher manages to open a new dimension in the conceptualization of Cartesian graphs through a fine-grained link of the enactment of Tina's walk on the cardboard path and the graph. The horizontal graph segment now appears to mean that time elapses while the distance remains the same, thereby overcoming a contradiction that haunted the students' first interpretations of the graph.

The text that the students produced in response to the activity requirements was the following: "In the beginning, Tina did (1m) and (0s). Then she ~~walked~~ advanced (2m at 3s) so she is at (3m and 3s) and 2s pass but she doesn't move so she is at (3m and 5s). Finally, she climbs (1m and 2s) which gives (4m and 7s)." Contradictions still remain. The text is still cast in the rhetorical style of the phenomenological apprehension of the graph as a map: for example, we are told that, in the last segment, Tina climbs. There is still room to move closer to a co-variational description of events. Yet, the ongoing students' encounter with a cultural-historical co-variational understanding of motion graphs paves the way to engage in further reflective mathematical experiences.

Concluding remarks

This paper seeks to contribute to a notorious underrepresented topic in early algebra: the students' understanding of graphs. Cartesian graphs possess an historical epistemological density that may hide its meaning from novice young students, as shown by the example discussed in this article. In our study, which featured a motion graph, drawing on syncretic perceptual understanding of phenomena, the students tended to interpret the given graph as a *map*. D. suggested a first interpretation. In it, thinking about the graph was not organized in terms of Cartesian variables—i.e., variables referred to a certain arbitrary spatial and temporal origin. As the teaching-learning activity unfolded, B. drew on D.'s ideas and on a teacher's hint to formulate a second interpretation where the focus is distance while time remains unqualified. In turn, L. drew on the previous interpretations and formulated a third one, where variables became better identified, although their co-variational nature remained obstructed by the persistent perceptual sense of the graph as a map. At the end of the lesson, drawing on the students' ideas, through a fine-grained link of the enactment of Tina's walk on the cardboard path and the graph, the teacher created room for the students to become conscious of a new meaning of co-variation that discloses the possibility that changes in one of the variables does not necessarily lead to changes in the other. It is this movement of collective ideas that teacher and students form and voice while engaging in teaching-learning activity that we call *l'oeuvre commune*, the collective artistic embodied-intellectual-material work of the participants. The collective idea still requires further refinements, for as noted above, the students' text is still a vivid evocation of the graph as a map walk, rather than a mathematical sign expressing *relations* between variables, the quantities of which "change in relation to each other" (Carlson et al. 2002). "The child's thought [still] must be raised to a higher level for the concept to arise in consciousness" (Vygotsky, 1987, p. 169). This tension between the *perceptual* as the basis of sense that underpins the understanding of the graph as a map, and the *relational* as the basis of sense intended in co-variational thinking is part of the ontogenesis of co-variational thinking. It is

not something to avoid, but rather something to help us, educators, to better work with teachers and students.

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Linking a table and a graph of a function: Slovak case

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This paper investigates whether Slovak high school students can infer properties from the table representation of a function that help them link it to the graph of a quadratic function. 124 students (16-18 years old) solved a task focused on linking a table and a graph of a quadratic function without axes after they had completed learning about functions. The analysis showed that almost 88% of the students link these two representations of a quadratic function incorrectly. Moreover, the students' solutions also showed that they still had some significant misconceptions after learning about functions. Possible reasons for this are discussed in the conclusion.

Keywords: Quadratic function, representations of function, linking representations of function.

Introduction

In the media, in many scientific disciplines, etc., we often come across various graphs and charts that visualise certain dependencies. Data is coded in tables, plotted, and scientists try to find a relationship that describes the data and predicts its behaviour. Various researchers (Duval, 2017; Ronda, 2015; Leinhardt et al., 1990; Pittalis et al., 2020) point to the importance of linking different representations of functions and show that this linking can be difficult for students to grasp.

> ... the teaching of mathematics brutally reminds teachers that the distinction between mathematical objects and their multiple representations constitutes one of the main difficulties of understanding in the learning process. (Duval, 2017, p. 15)

The topic of function is introduced to Slovak students in elementary school (10–15-year-olds). It usually starts with examples of linear functions. This concept is easy for students to follow, but it can create a prototypical image (Hadjidemetriou & Williams, 2002) of a function, which can lead to several misconceptions (e.g., a graph is a line, a function must be a one-to-one function, or it must be only increasing/decreasing, etc.; see Leinhardt et al., 1990; Hadjidemetriou & Williams, 2002; Parhizgar et al., 2021). Later in high school students learn about different types of elementary functions, their representations and properties. The analysis of textbooks used by Slovak teachers showed that the most used transition is from a function equation to a function graph; little attention is paid to the other links (Šabaková, 2024). Together with equations and verbal descriptions, tables and graphs are part of the mathematics curriculum in Slovakia and students are expected to be able to link them in various ways, although the document explicitly mentions only linking graph and equation (ŠPÚ, 2014). This may be the reason that most attention is paid to the transition between the equation and the graph, since the wording of curriculum documents is the starting point for textbook writers and also influences the design of school curricula (Jukić Matić et al., 2022). On the other hand, several researchers emphasise the importance of linking different representations. Duval (2006) explains that the complexity of mathematical thinking is not only due to the concepts themselves, but is influenced by the transformations of the semiotic representations used in mathematics. He identifies

two different types of transformations that are crucial for mathematical understanding, treatment transformations (which involve the manipulation of mathematical objects and representations that students are already familiar with) and conversion transformations (which require students to switch between different semiotic registers, which is more complex and challenging). Although the conversion transformations tend to be more difficult for students, understanding and engaging with both types of transformations is essential for improving mathematical comprehension and learning.

This paper is part of a PhD thesis that focuses on linking different representations of elementary functions. Here we focus on linking two representations from different semiotic registers (a table and a graph) and investigate the depth of this link. We conducted research to find out what is the state of this knowledge in Slovakia. Do our students have the same problems as those described in the works by Ronda (2015) or Leinhardt et al. (1990)? This motivated us to investigate the answers to the following research questions: Which solution method do students prefer when linking a table with a graph of a function? Will the students' solutions reveal misconceptions?

Theoretical background

Learning about functions also means understanding the function as an object with its unique properties that can be identified across its different representations, and successfully linking these representations (Ronda, 2015; Leinhardt et al., 1990; Duval, 2006; Parhizgar et al., 2021; Jukić Matić et al., 2022). Each representation has its strengths and limitations, and "… knowing how to work with different representations of the function does not necessarily lead to the understanding the function." (Ronda, 2015, p. 305). The real understanding of the function lies in knowing what remains the same in what is represented, and what changes when we link different representations. Understanding this is essential for applying mathematical concepts to real-life situations (Duval, 2017; Ronda, 2015). Different representations emphasise different aspects of the function, for example, a graph can show trends over time, or some invariant properties, whereas in a table we can clearly see individual values of a function. Pittalis et al. (2020) highlight four different aspects of a function: 1) **input-output** (e.g., generation of function values from function rule); 2) dynamic process of **covariation** (e.g., noticing constant and/or linear rate of change, or how does the dependent variable change when the independent variable changes by 1 unit); 3) **correspondence** relationship (e.g., the ability to create a function rule); 4) **function as an object** (e.g., using invariant properties of the function). Thompson & Carlson (2017) consider the covariation aspect to be crucial in functional thinking. They describe this aspect in more detail and divide it into two: variation and covariation, each with its own levels of understanding. Correspondence relationship is harder to see in a table independently of covariation. The ability to see different aspects in a given representation can be helpful in linking representations. All aspects are closely related to the task-solving strategies described by Ronda (2015). She describes the perception of links between representations through the process-to-object continuum, using interpretations of function properties. This is closely related to the strategies that students use when solving tasks that focus on linking representations of elementary functions: **strategy (i) –** usually involves generating values (e.g., creating a table) or identifying some of the local properties of the explored function; **strategy (ii)** – usually involves using trends, patterns and global properties and noticing monotonicity or rate of change; **strategy (iii)** – usually involves identifying and using invariant properties of the class of the function (e.g., function transformations). In general, we

associate strategies with aspects in the following way: strategy (i) with the first aspect, strategy (ii) with aspects 2) and 3), and strategy (iii) with the fourth aspect.

Various studies (e.g., Leinhardt et al., 1990; Hadjidemetriou & Williams, 2002; Parhizgar et al., 2021) describe common misconceptions that students can have when learning about functions and emphasise the importance for teachers to be aware of them. Most of them are related to the graphical representation of the function. Some of the reasons for these misconceptions may be the abstract nature of functions or the need to use and link different representations. To answer the research questions, we chose a task inspired by Ronda (2015) that focuses on linking a table and a graph of a function. Many authors (e.g. Leinhardt et al., 1990; Hadjidemetriou & Williams, 2002) deal with a function graph and a function rule, but rarely mention a function table and as far as we know, there is no research in this area in Slovakia. All these reasons motivated us to carry out the research presented in this paper.

Methods

The research consists of three parts:

1) The analysis of two newest Slovak high school textbooks. The analysis focused on links between different function representations. The results of this analysis can be found in Šabaková (2024).
2) Pilot research – pilot test consisted of 8 tasks.
3) Main research – the same test tasks as in the pilot with minor changes to make the tasks clearer.

The qualitative analysis of the data in the pilot and main test was based on the methodology used by Ronda (2015). 43 students (from 2 classes) participated in the pilot research and 124 students participated in the main research (both groups 16-18 years old). We will analyse only the results from the main test in the paper. 124 students were from 5 different classes in 3 different high schools:

- High school in a regional city with more than 10 similar high schools - 2 classes.
- High school in a city with 2 such schools - 1 class.
- High school in a town with only 1 such school - 2 classes.

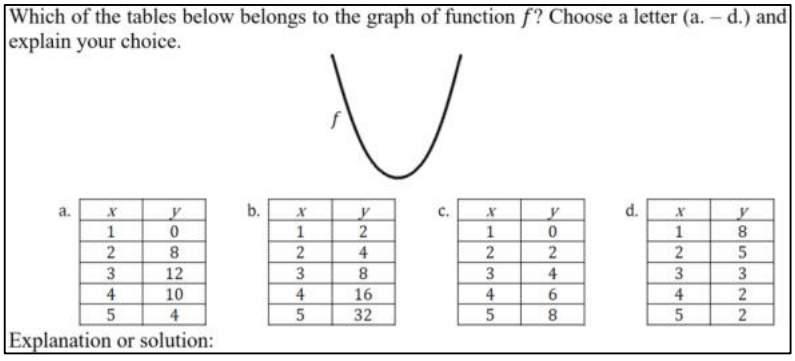


Figure 1: Task 2

All students have already completed the study of functions, including function properties, different classes of functions and their unique properties. Here we will focus on one task out of 8: task 2 (see Figure 1). Students were asked to link a graph of a quadratic function with no axes shown to one of

the four tables. By noticing the monotonicity or rate of change in the tables, we can see that option a) can represent a concave function, option b) an exponential function, option c) a linear function and option d) a convex quadratic function. The only option that fits the parabola is therefore option d).

Three possible strategies of solution were anticipated (as described in the Theoretical Background). In strategy (i) the student draws individual graphs using values from tables and then chooses the answer (see Figure 4) using the input-output aspect. Strategy (ii) concerns describing monotonicity, rate of change, or patterns using 2nd and 3rd aspect, e.g., describing the increase or the decrease of the values (see Figure 5) or noticing the rate of change in the tables (see Figure 2). Using strategy (iii) means understanding a function as an object, seeing invariant properties or, in this case, seeing that the derivation of the option d) is linear. This strategy only appeared in one student's solution in the pilot research (see Figure 3).

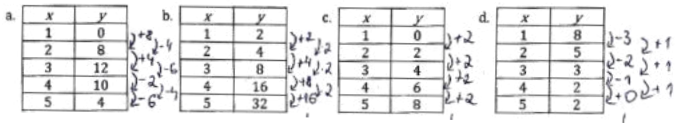
Expected solution	Expected description
	a. first it increases, then decreases – concave function b. exponential growth of values c. constant growth of values – linear function d. linear growth of values – quadratic convex function, first decreases then increases

Figure 2: Expected solution using covariation

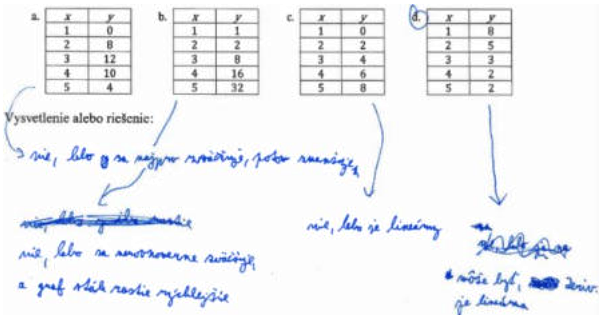
Student's solution	Translation
	a. no, because firstly it increases, then decreases b. not, because it increases unevenly, and a graph constantly increases quicker c. not, because it is linear d. can be because the derivative is linear

Figure 3: Pilot test student's strategy (iii)

Results

106 students solved this task. 18 students did not choose any option or write anything. Most students completed the task within 10 minutes. The solution strategies used by the students to solve this task are shown in Table 1. Strategy (i) was used by 27 students. Within this strategy they evaluated individual points and plotted these points (input-output aspect; see Figure 4). 25 students used strategy (ii) (using a global property – monotonicity, rarely using correspondence; see Figure 5). As for this strategy, students most often mentioned monotonicity. On the other hand, concentrating only on monotonicity prevented many of them from excluding table b) as the correct answer. 15 students used both strategies (i) and (ii) (see Figure 6). Strategy (iii) did not appear in the main research. 8 students seemed to identify linear growth using covariation in option c). However, none of the students explicitly used this strategy for other options (see expected solution in Figure 2). 47 students

(44.34%) chose an option without explaining their solution strategy. Some students chose more than one option, an example of which is shown in Figure 6. 13 students (12.26%) solved the task correctly, of whom three used strategy (i), two used strategy (ii), six used both strategies and two students chose the correct option without any explanation. The rest of the students chose option d) as correct but used an incorrect explanation or did not choose this option as correct. Option b) – a table of the exponential rate of change – was chosen as correct by 57 students (53.77%).

Table 1: Students' answers (mathematics class students [∑=24] + standard class students [∑=82])

Option		Strategy (i)	Strategy (ii)	Both strategies	Without explanation	∑
a.		1	0	0	3	4
b.		15	13	5	24	57
c.		7	4	1	15	27
d.	correct	3	2	6	2	13
	incorrect	0	4	3	0	7
No option		1	2	0	3	6
∑		27	25	15	47	114

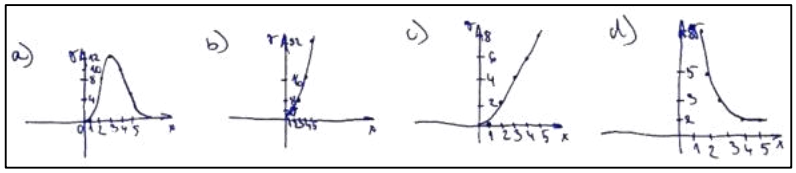


Figure 4: Student's strategy (i)

Student's solution	Translation
a stúpa a klesá, f klesá a stúpa b vyzerá ako 2^x, ktorého graf neprislúcha grafu f c vyzerá ako lin. funkcia d klesá a potom stúpa, čo sedí k f.	a. increases and decreases, f decreases and increases b. looks like 2^x, which graph does not belong to graph f c. looks like a linear function d. decreases and then increases, which fits with f

Figure 5: Student's strategy (ii)

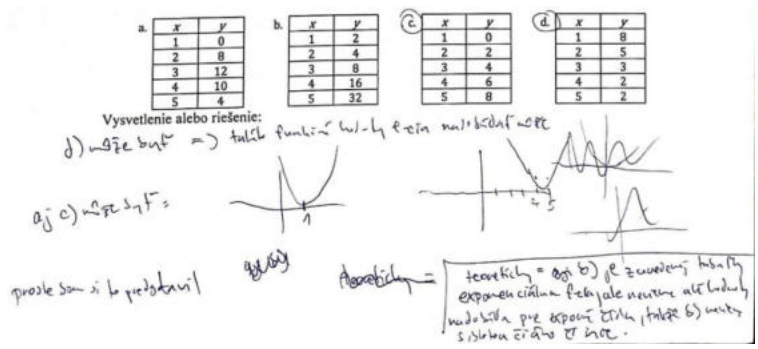
Student's solution	Translation
	d) can be => a function can acquire these values c) can be too I just imagined it
	theoretically – b) is a table of exponential function, but we don't know what values it acquires for negative numbers, so I don't know for sure if yes or no

Figure 6: More than one option, both strategies and inaccurate argumentation

The following misconceptions emerged from students' solutions:

A. The graph must be symmetrical with respect to the y-axis (∑=13). A student added axes to the given parabola assuming that the graph had to be symmetrical (see Figure 7).

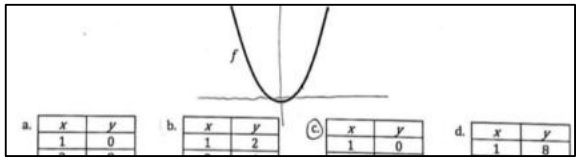


Figure 7: Student's solution – misconception A

B. For two different values of x there cannot be the same values of y, because the function would become constant in this interval (∑=10) (see Figure 8).

Student's solution	Translation
d.) to nemôže byť, pretože x=4 a x=5 nadobúdajú rovnaké y takže na jednom úseku by to bola konštantná funkcia; parabola nie je konštantná	d) can't be it, because x=4 and x=5 have the same y, so in one segment it would be a constant function; parabola is not constant

Figure 8: Student's solution – misconception B

C. Not distinguishing between the rate of change of a quadratic and an exponential function (∑=5). Students were assigned to this misconception if they explicitly wrote that the function is exponential, grows exponentially (see Figure 9) or based on this idea chose option b) as correct. Two students wrote that the function rule is $y = 2^x$.

Student's solution	Translation
Vysvetlenie alebo riešenie: b, aj d, pretože nevieme kde končia body x a vieme že funkcia rastie exponenciálne	b) and d) because we don't know where points of x are, and we know that function increases exponentially

Figure 9: Student's solution – misconception C

Discussion and conclusion

In general, students have difficulty linking a table and a graph of a quadratic function after learning about functions, even after they learned about their representations and properties. The results are in line with claims in Duval (2017) that the distinction between mathematical objects and their multiple representations constitutes one of the main difficulties of understanding in the learning process. Moreover, the results showed that none of the students demonstrated covariation thinking with the table describing the quadratic function (function table d)) and only 3 students showed it with the exponential function (function table b)). They did not explore the variation in the columns and consequently did not look for patterns in the table to link it to the function graph (as described in Pittalis et al., 2020). They limited their exploration to the monotonicity of the function, which was not sufficient for most of them to distinguish between exponential and quadratic growth. The analysis of Slovak curricula and Slovak textbooks (Šabaková, 2024) showed that the covariance aspect of the concept of function is not sufficiently emphasised in the development of concept of function,

therefore Slovak students are not used to work with the covariance aspect. This is in line with Jukić Matić et al. (2022), who stated that the curriculum has an influence on the students' personal definition of the concept. This is in contrast to the claim that covariance aspect is “epistemologically necessary for students to develop useful and robust conceptions of functions.” (Thompson & Carlson , 2017, p. 423). This could be possible reason of students' struggling with the task.

The research revealed three different misconceptions. The first two identified misconceptions point to similar problems of Slovak students as described in the literature. Misconception A is related to a misconception that a graph has to go through the origin and can also be related to the prototypical image of a quadratic function (Hadjidemetriou & Williams, 2002), which is the function $y = ax^2$ or $y = x^2$. Misconception B is related to the common misconception mentioned by Leinhardt et al. (1990) that a function is always a one-to-one function. The last misconception identified, together with the finding that most students chose table b) as the table representation for the parabola, shows that our students have difficulty distinguishing between different rates of change, especially when the function is not linear. This is a very important finding because distinguishing between different rates of change is one of the most important ideas about functions. As mentioned in Cooney et al. (2010, p. 8), “function's rate of change is one of the main characteristics that determine what kinds of real-world phenomena the function can model”.

Preferred solution methods together with the misconceptions identified point to the problems that remain with students after learning about functions and to the need to include tasks that require different strategies for their solutions and working with the table of values. Although we studied only one task involving the linking of table and graph representations of a function, our results showed that one of the important aims of teaching functions, namely that students should be able to characterise different kinds of relationships by noticing the rate at which one quantity changes with respect to another, was not being met in the review of the functions topic.

Limitations of this research lie in the fact that the research was conducted only in high schools in the eastern region of Slovakia, therefore further research needs to be conducted to draw conclusions concerning all high schools in Slovakia and to better understand how students understand these representations, how well they can work with them, and whether they are able to see and recognise differences between elementary classes of functions. The results of this research will be used in the professional development of teachers to help them understand their students' thinking and to better prepare them for the teaching aimed at development of their functional thinking.

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Race among the number line: Functional thinking among primary school students

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In an international context, algebraic thinking is an established subject area in mathematics lessons – even at primary school level. In German primary schools, however, the topic is often addressed unconsciously (Akinwunmi & Steinweg, 2024), which means that a wide range of potentials for stimulating substantive learning processes remain untapped. In the following, a case study will be used to illustrate the thematization of functional thinking in primary school. From an epistemological perspective, it is reconstructed to what extent primary school students demonstrate conceptual understanding with respect to the four function aspects in their engagement with the learning environment 'race among the number line'. Results show that learners as young as fourth grade can find access to functionally characterized situations and can develop and express ideas.

Keywords: Algebraic thinking, functional thinking, functions, primary school, function aspects.

Introduction

"The true mathematical wealth is created by the perspective of function" (Freudenthal, 1983, p. 511). Freudenthal's quote shows how important functional thinking is as part of algebraic thinking in mathematics. However, functional thinking is not only important from a mathematical point of view, but is crucial in society itself.

Although the importance of functional thinking is undisputed, it is far less clear what exactly functional thinking means. Smith (2008) describes functional thinking as a type of

> […] representational thinking that focuses on the relationship between two (or more) varying quantities, specifically the kinds of thinking that lead from specific relationships (individual incidences) to generalizations of that relationship across instances. (Smith, 2008, p. 143)

Functional thinking is therefore not limited to dealing with functions, but describes a broader way of thinking. In primary school in particular, it is of course not about a symbolic examination of functions, but about thinking in *functional relationships*.

> Functional relationships inevitably shift the focus of thinking away from individual solutions to the relationships themselves. The 'solution' of activities to functional relationships usually involves the description of this very relationship in the form of an equation, a term, a verbal rule. (Steinweg, 2013, p. 199; translated by author)

In the following, it is shown how fourth-graders can focus on such *functional relationships* using a suitable learning environment and how they express their intuitive approaches. To do so requires considerations on functions, notions of functions, and the epistemology of mathematical concepts.

Theoretical Background

A function defines a mathematical relationship (relation) between the elements of two sets (X, Y) (Rudin, 1964). An element of the domain (the independent value) ($x \in X$) is uniquely assigned to an element of the codomain (the dependent value) ($y \in Y$), such that $f: x \mapsto y$. A relation is a function if it is *well-defined* and *total on its domain*. Total on its domain: $\forall\, x \in X\ \exists\, y \in Y : (x, y) \in f$; Well-defined: $\forall\, x \in X, \forall\, y,z \in Y : (x, y) \in f \wedge (x, z) \in f \Rightarrow y = z$.

The function types of the linear function, the quadratic function and the exponential function are relevant for this study:

A function f with $f(x) = ax + b$ where $a, b \in \mathbb{R}$ is called a *linear function*.

A function f with $f(x) = ax^2 + bx + c$ where $a, b, c \in \mathbb{R}$, $a \neq 0$ is called a *quadratic function*.

A function f with $f(x) = a^x$ where $a \in \mathbb{R}_{>0} \setminus \{1\}$ is called an *exponential function*.

Understanding functions, however, is not limited to knowing their definitions and formal notations; rather, it involves the active engagement with functions — functional thinking — which constitutes the conceptual core of the concept of function (Warren & Cooper, 2005). Functional thinking is thus not tied to a formal concept of function and can already occur in primary school (ibid.), by "focusing on specific patterns that provide initial access to functions and their structures" (Akinwunmi & Steinweg, 2024, p. 306; translated by author).

Characteristic of this functional thinking are the so-called *function aspects*. In the international context, the four aspects of *input-output*, *covariation*, *correspondence*, and *mathematical object* have become established (Pittalis et al., 2020).

The *input-output* aspect of function is understood as a straightforward computation according to a predefined rule, which does not require the learner to understand the causal relationship between the input and the output. It is often perceived by students merely as a calculative operation, where the insertion of one variable (the input) leads directly to another variable (the output) (Pittalis et al., 2024).

The *covariational* aspect involves analyzing, manipulating, and understanding the relationships between at least two simultaneously changing quantities. This aspect emphasizes the conceptualization of how two or more quantities vary together, aiming to coordinate the changes in the values of these variables (ibid.).

The *correspondence* aspect, frequently employed to formally introduce functions at school, highlights understanding the structural conceptualization of functions. It focuses on grasping the relationship between independent and dependent variables (ibid.).

The *mathematical object* aspect views a function as a mathematical entity that can be depicted through various representations, including arrow chains, tables, graphs, formulas, and verbal descriptions. Each representation offers a unique perspective on the same mathematical object (ibid.).

According to Sfard (1991), there is a gradual shift from a process-oriented perspective, which emphasizes the input-output aspect, to a more structural viewpoint focusing on the object aspect. However, it is important to note that these different perspectives are often intertwined, posing challenges for teaching and learning (Pittalis et al., 2024). In primary school, students

will primarily engage with functions through input-output, covariation, and correspondence approaches.

Keeping these four function-aspects in mind, the research interest for the following case study is: *To what extent do primary school students demonstrate conceptual understanding with respect to the four function-aspects in their engagement with the learning environment 'Race among the number line'*?

Setting & Methodology

The learning environment 'Race among the number line' (Akinwunmi & Lang, 2017) allows learners in primary school to playfully gather initial experiences in handling functions. The basic task is as follows (Fig. 1):

A race is played on the number line with four game pieces. Each game piece moves forward according to specific rules. The game piece that first reaches the number 1000 wins.

- The red game piece is allowed to move 50 steps in each round.
- The blue game piece moves 10 steps in the first round. In each subsequent round, it moves 10 steps more than in the previous round.
- The green game piece moves one step in the first round. In each subsequent round, it moves twice as many steps as in the previous round.
- The yellow game piece moves 100 steps in the first round. In each subsequent round, it moves 10 steps less than in the previous round.

Figure 1: Basic task and specific movement rules

The focus of the task lies on the different movement rules for the game pieces, or the underlying concept of slope. Particularly, the function aspects of *covariation* and *correspondence* are relevant in this context.

The mappings between the number of rounds and the position on the number line represent different types of functions for each game piece (Table 1):

Table 1: Overview of the game pieces and the underlying functions

Red game piece	linear function	$f(x) = 50 \cdot x \; ; \; x \in \mathbb{N}$
Blue game piece	quadratic function	$g(x) = 5 \cdot x^2 + 5 \cdot x \; ; \; x \in \mathbb{N}$
Green game piece	exponential function	$h(x) = 2^x - 1 \; ; \; x \in \mathbb{N}_{>0}$
Yellow game piece	quadratic function	$i(x) = -5 \cdot x^2 + 105 \cdot x \; ; \; x \in \mathbb{N}$

The learning environment was conducted with three pairs of learners from a fourth-grade class (ages 9-10 years) at a German primary school, chosen by their teacher. As part of the survey, which lasted 3 times 60 minutes, the pairs were taken out of their regular lessons. The following will examine the work of the learner pair J. and T. They were described by the teacher as "rather high-performing". The actual survey addressed various follow-up tasks. The analysis only considers individual snapshots.

The sequences were videotaped and transcribed. For analysis, the *epistemological triangle* according to Steinbring (2005) was used. This involves reconstructing the understanding of mathematical *concepts* (e.g., linear functions) through the learners negotiation between specific *signs* (e.g., representations, statements) and *reference contexts* (conceptual reference points of the learners). New mathematical knowledge is ultimately understood as an extension of old knowledge through new, rich relationships, in that possible meanings are transferred from a relatively familiar, or partially familiar *reference context* to a new, as yet meaningless system of *signs* (ibid.). Content knowledge is constructed in interaction. Following this idea, negotiation processes and conceptions of the learners can be reconstructed by understanding expressions, representations, or notations as signs to be interpreted. For instance, conceptions of covariation can be elaborated through the interpretations made by the learners such as when a learner says, *it rises faster and faster*, indicating their perception of the rate of change, or *if this one speeds up, then the other value also rises*, demonstrating their understanding of the relationship between variables in a dynamic system. In this sense, the students processes of constructing new mathematical knowledge in the interaction are reconstructed with the help of the epistemological triangle.

Analysis & Selected Results

After the task was presented to the learners, they were initially asked to write down their first prediction about which game piece would win the race. Both agreed that the red game piece would win (J.: "I believe the red game piece will win because it runs 50 steps in every round."; translated by author) (T.: "I think the red game piece will go further because the yellow game piece runs 10 steps less each time."; translated by author).

Since human thinking and most everyday situations are based on linear processes, exponential growth is a phenomenon that is unfamiliar, especially for primary school students. Consequently, initial guesses are predictably as expected. The difference between repeated addition and repeated multiplication is likely not perceived by J. and T., and certainly not considered decisive. This is evident from the fact that at no point (before the start of the game) was there a discussion about whether the green game piece could win. The small number of "one step in the first round" seems too insignificant compared to the fifty steps of the red game piece. Instead, considerations are made about whether it is better to start with a high number and slow down (yellow game piece), or to run with a consistent stride length (red game piece).

If we consider the red set piece, the following epistemological triangle can be modeled (Fig. 2):

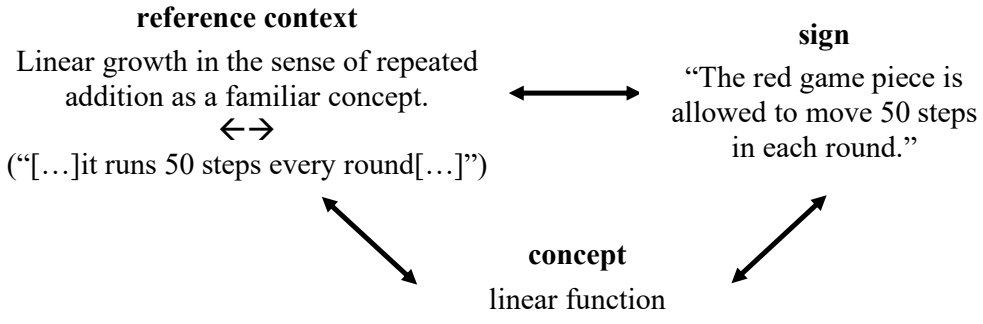

Figure 2: Mediation between specific rule and conceptual reference point for the red game piece

The specified moving pattern for the red game piece ("The red game piece is allowed to move 50 steps in each round.") serves here as the sign that the learners need to interpret. The direct reference to the number of steps per round of the game piece ("[…]it runs 50 steps every round[…]") suggests that J. and T. relate the number of steps to the number of rounds. The emphasis on the phrase *every round* indicates that they recognize the constancy of the slope and identify it as important. Even as their mathematical justifications remain at a more narrative level (see Krummheuer, 2013), they demonstrate conceptual considerations regarding the *correspondence* and, to some extent, the *covariational* aspect in terms of continued addition.

After the learners had noted down their predictions, they were asked to check which game piece would win. J. and T. decided to note the positions of the game pieces step by step for each round and to present them in a table (Fig. 3):

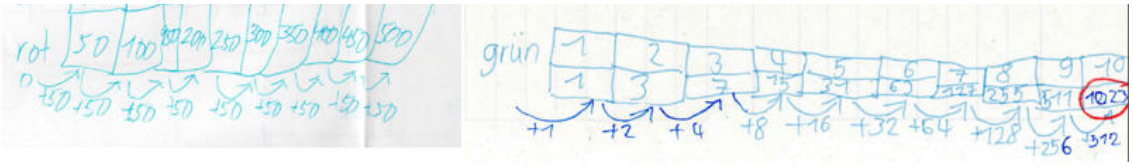

Figure 3: Tabular representation of the game pieces and their position (left: red; right: green)

This leads to the following situation, while J. calculates the position that the green game piece will reach in the 6th round:

J.	[…] and 31 plus 32 are (..) are 63 (*notes result*)
Int.	Mmh.
J.	(...) And doubling thirty-two is (..) well, if I think about it, I really don't know if, if (..) green will make it but (...) well, it could happen that it, it (unintelligible) (..) it will probably happen that, that green takes over a hundred steps.

This passage clearly shows that although J. initially approaches the problem purely arithmetically by calculating the concrete numerical values ("[…]and 31 plus 32 are (..) are 63"), he simultaneously detaches himself from these concrete values ("[…]well, if I think about it[…]"). He is already considering the increase in subsequent rounds without having specifically calculated it ("[…]it will probably happen[…]") and realizes that it grows much faster than initially suspected. This is a clear indication of thinking in *covariation*. After calculating a few more positions of the green figure, they confirm that the green game piece will in fact reach 1000 first. The green game piece not behaving according to their expectations leads to a strong urge to justify. The interviewer asks:

Int.	Why does the green game piece win now, how can that be?
J.	Because (..) eh, because it keeps increasing, if you're already at a hundred#
T.	# it always doubles. And (..) uh and then, you are, you are here (*points to the table*), then you are, then it keeps increasing and if#
J.	# you must not underestimate the smallest one!
Int.	(*laughs*) you must not underestimate the smallest one?
J.	One must not underestimate that if the smallest one, if one starts running very slowly, and then, and then does, then does, then suddenly goes into turbo mode (*laughs*).
Int.	What do you mean by he suddenly goes into turbo mode?
J.	Because, um (...) from one, you think that, you think you're not walking that far, but it doubles. From two to four, from four to eight, from eight to sixteen.
T.	And then suddenly the steps increase very quickly.

In their explanations, they repeatedly refer to "doubling" and “increasing” (*verb*), but the verbalization makes it clear how difficult this is, since the learners naturally lack the algebraic terminology like *slope* or *increase (noun)*.

At the same time, the choice of words (“[…]it keeps increasing[…]”) reveals that the focus is on the change in values (*covariation*), not on the specific values itself. Again they use narrative phrases like “you must not underestimate the smallest one!” or “[…]suddenly goes into turbo mode[…]” to bridge the gap between their available linguistic means and the conceptual ideas they wish to express.

Those phrases, as well as the emphasis on the word "doubling" in their statements, suggest that they see more than just the movement rule of the game piece in this term. They likely see the doubling (or the continued multiplication by two) as the reason for the rapidly increasing growth. This assumption is supported by phrases such as "then the steps increase very quickly", or by listing examples (“From two to four, from four to eight, from eight to sixteen”) that simultaneously have a generalizing character. It is apparent that the steeper slope of repeated multiplication compared to repeated addition is the conceptual core of their argument (function aspect of *covariation*).

If we consider the green set piece, the following epistemological triangle can be modeled (Fig. 4):

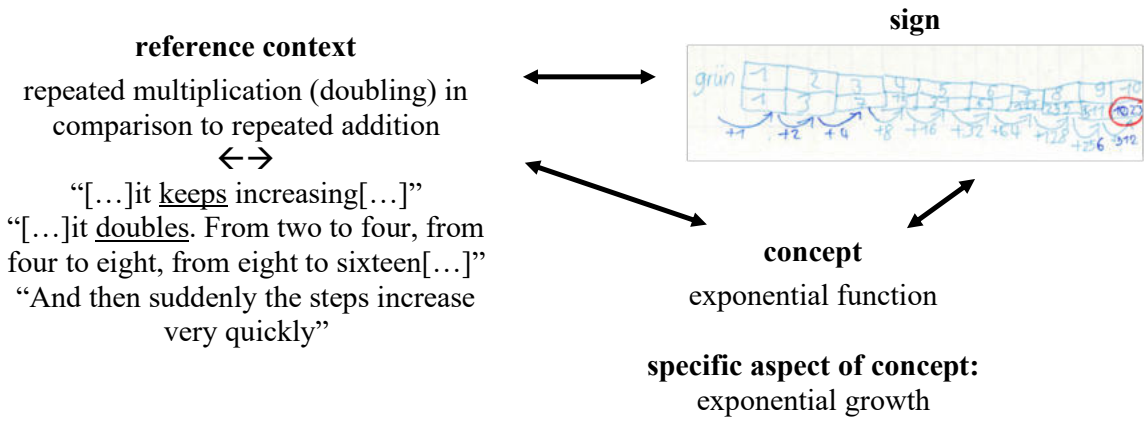

Figure 4: Mediating the irritation by referencing the differences between repeated multiplication and repeated addition

To test these considerations, J. & T. are confronted with a new game piece: this game piece moves 100 steps in the first round. In each subsequent round, it moves 50 steps more than in the previous round ($j(x) = 25 \cdot x^2 + 75 \cdot x$; $x \in \mathbb{N}$). The learners determine some positions of this game piece and quickly recognize that it will reach the 1000 faster than the green game piece. Then the interviewer asks whether this new game piece will also win if the game is played indefinitely, or whether the green figure will catch up again.

J. But it is now questionable (...) uh (...) so in ten rounds, the new figure would win, but if you continue running, then green would overtake again.
Int. Why is that?
[…]
J. Because, because then (*stands up from the chair*), because when you double, steps are, very, very often, are very, when you then, eventually (...) there

> (*points towards the table of the green figure*) you were already at five hundred twelve, and we had calculated that up to here (*points to round 11 of the green figure*), there (...) two thousand forty-seven, that would be, that (*points to round 10 of the 'new figure'*) is indeed more, but here (*points again at the table of the green figure*) the steps are then doubled, but since the steps there are doubled and there (*points towards the table of the 'new figure'*) only, there only fifty is added.

J.'s explanations very aptly underscore the considerations previously worked out: the arithmetic values provide the starting point for further thinking processes that, in the sense of functional thinking, detach from the concrete numbers. In terms of covariation, J. recognizes the difference between repeated addition and repeated multiplication and realizes that in the long run, the latter must always grow faster. His linguistically halting and simultaneously gesturally supported explanations once again show the attempt to express this complex relation with limited linguistic means.

Conclusion and Outlook

By adopting the epistemological approach, insights into students' processing and conceptualization of mathematical functions were gained, confirming the notion that fundamental algebraic concepts, typically introduced later, can be understood by young learners through contextually driven, discursively rich learning environments.

Considering the main research question of this paper "*To what extent do primary school students demonstrate conceptual understanding with respect to the four function aspects in their engagement with the learning environment 'race among the number line'?*", it is evident that fourth graders were able to make conceptually significant experiences with functionally defined situations within the provided learning environment. These experiences particularly allowed the students to express concepts of *covariation* and *input-output*. Although the learners lack the explicit technical vocabulary regarding functions, they are able to express their ideas using the linguistic resources available to them. Similarly, there were moments of generalization observed, indicating a shift from purely arithmetic considerations. The results also emphasize the importance of language (including gestures, objects, drawings, graphs, tables, etc.) for mathematical learning processes (e.g., Moschkovich, 2015), as well as the intertwined use of *mathematical language registers* and *everyday language registers* (e.g., "goes into turbo mode").

For future research, it might be interesting to continue focusing on the approaches and linguistic negotiations of primary school students, as well as to observe student teachers dealing with school-related algebraic tasks. It would be relevant not only to examine which subject-specific concepts the student teachers have developed themselves, but also how they pedagogically assess corresponding tasks for primary school, especially those that provide suitable interfaces for algebraic thinking in primary school.

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Constructing varying relationships of quantities and writing equations with fractions: A preliminary analysis of representations produced by Japanese and Turkish teachers

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We focus on production of representations by two Japanese and two Turkish teachers as they solve problems involving fractions and two types of varying quantities. We identified their multiplicative concepts (MC1-3) and investigated how these constructs could play a role in their problem-solving activities using video recorded semi-structured interviews. Two participants identified as MC2 level drew general pictures to show the problem situation and wrote equations, but they did not relate meaning of equation to production of pictures. Two participants were identified as MC3 level: One recalled the "double number line" (Japan) but struggled in writing equation. The other one correlated her drawing to make the relationship between the two varying quantities with "direct proportion" method (Turkiye) and solve it but did not write an equation. By analysing the process of producing representations, we report their challenges of units coordination in two varying quantities.

Keywords: Equation, multiplicative concepts, units coordination, representation, teachers.

Purpose of the study

The purpose of this study is to investigate the problem-solving strategies of both pre- and in-service teachers in Japan and Türkiye regarding algebraic equation writing that includes fractions. This exploration might be useful for us to understand how adults make connections transitioning to algebraic equation writing including fractions.

Algebraic equation writing involving fractions contains the process of "units coordination" (Norton et al., 2015). The reality of students at different stages of units coordination and the factors that influence their difficulties have been captured (e.g., Hackenberg & Sevinc, 2022) for only one type of quantities such as length. In this study, we aim to understand teachers' problem-solving activities when writing and solving algebraic equations with two varying quantities, dL and m^2. The main reason is that it has been recognized that teachers' content knowledge influences students' learning, and more specifically, that this quality plays a vital role in noticing and responding to students' thinking (e.g., Dindyal et al., 2021). With this in mind, we designed and conducted a qualitative study in Japan and Turkiye. This paper draws on an initial analysis of the video recorded semi-structured interviews with two Japanese pre-service teachers (no teaching experience) and two Turkish in-service teachers (20 years of teaching experience). The research questions are as follows: What kind of drawn representations do the teachers produce in formulating equations involving fractions? And

how are their multiplicative concepts related to teachers' actions and operations in formulating algebraic equations?

Related literature

This study is grounded on the theory of units coordination adopted by Steffe and other researchers (e.g., Hackenberg & Sevinc, 2022; Norton et al., 2015; Steffe, 1994). According to Norton et al. (2015), "units coordination refers to students' abilities to create units and maintain their relationships with other units that they contain or constitute" (p. 111). Researchers have characterized a progression in such abilities by stages that are determined by the number of levels of units students can coordinate in problem situations.

Norton et al. (2015, pp. 112–113) describe three stages of units coordination by using a task: "If the small red bar fit into the medium yellow bar three times, and the medium yellow bar fit into the long blue bar four times, how many times would the small red bar fit into the long blue bar?" Stage 1 students mentally iterate the short bar to know how many times it fits into the long bar. They can assimilate units of 1 and composite units as twelve 1s, but the relationship between these two levels is not multiplicative but additive and figurative. Stage 2 students mentally iterate the medium bar four times, with each iteration corresponding to a 3, by uttering "3, 3, 3, and 3" for example. They can take two-level multiplicative relations (medium and long bars, small and medium bars) as given, but they need to build up the third level of units (small, medium and long bars) in activity. Stage 3 students immediately understand that there are four threes in the long bar by responding "12" and stating that 12 is four 3s. They can assimilate the given relation into a unit of units of units (take three levels of units as given) and can flexibly switch between structures.

Hackenberg and Sevinc characterized students' units coordination in terms of multiplicative concepts (MC1-3), in correspondence with the three stages (Stage 1-3) introduced above. Considering that multiplicative concepts influence students' reasoning in fractions and algebra in middle school mathematics, they examined middle school students' reasoning with fractions and algebraic reasoning together. Concerning the length and fraction units, MC2 students are those who "can imagine a length partitioned into equal parts without making it" (Hackenberg & Sevinc, 2022, p. 179). They can use a unit of units structure. They have an iterable unit and create 3/5 of a bar by iterating three of the one part which was produced by partitioning a bar into five. MC 3 students are those who "can imagine a length partitioned into equal parts, each of which are partitioned into equal parts" (p. 179). They can use a unit of units of units. They can use multiplicative relationship between a unit fraction, the unit, and any fraction made from it, so they produce 5/3 as 5 times 1/3 by disembedding a unit fraction 1/3 and iterating it 5 times, while also maintaining relationship between 1 and 1/3. We are interested how MC levels might play a role in how teachers conceive and formulate equations including fractions and quantities with two different and varying types: dL and m^2.

Methods

We developed an interview protocol with series of 14 problems related to fractions and algebra (see Table 1). We wanted to understand how the knowledge related to multiplicative concepts might help

or hinder teachers to solve algebraic problems including making relationships between varying quantities and writing algebraic equations.

Table 1: Tasks used in the interview protocol

Bar Task	(Given the three bars in picture) Pretend that the small green bar fits into the long orange bar exactly 9 times. Pretend that the small green bar fits into the medium purple bar exactly 4 times. How can you use this information to figure out how many times the medium purple bar would fit into the long orange bar? (Task 7, Norton et al. 2015, p.134)
Seat Task	1. There are blue seats and yellow seats in a train. 1/5 of all the seats are blue seats. 2. There are blue seats and yellow seats in a train. 2/5 of all the seats are blue seats. Can you showcase this situation by drawing a picture? Can you write an equation to express these situations? What if there were 30 blue seats, how many seats would there be in the train?
Plant Task	1. The height of Leyla (Kaede)'s plant measure in cm is 1/5 the height of Serhat (Asahi)'s plant measure in cm. 2. The height of Leyla (Kaede)'s plant measure in cm is 2/5 the height of Serhat (Asahi)'s plant measure in cm.(adapted from Hackenberg & Sevinc, 2022) Can you demonstrate this situation with a picture? Can you write an equation to express this situations and relate to the pictures? What if Leyla's plant was 30cm, what would be Serhat's plant height?
Paint Task	1. With 3/4 dL of paint, I have painted 1 m^2 of a board. With 1 dL of this paint, how many square meters can I paint the board? 2. With 3/4 dL of paint, I have painted 1/5 m^2 of a board. With 1 dL of this paint, how many square meters can I paint the board? 3. With 3/4 dL of paint, I have painted 2/5 m^2 of a board. With 1 dL of this paint, how many square meters can I paint the board? (Fujii & Majima, 2020, p. 55)

The protocol consisted of four parts. Part 1 includes seven bar tasks developed for assessing units coordination structures (Norton et al., 2015: they were translated to Japanese and Turkish by authors): continuous quantities of same type. Part 2 comprises Seat tasks that deal with two unknowns and part-part-whole relationship between blue seats, yellow seats and all seats: discrete quantities of same type. Part 3 comprises Plant tasks that deal with two unknowns and comparison between heights of two plants: continuous quantities of same type. For the Plant task, we modified the tasks from Hackenberg and Sevinc (2022). Part 4 comprises three Paint tasks and deals with two variables and proportional relationship between the amount of paint (dL) and the size of area that can be painted (m^2). The problems in this group are called "third usage of ratio" (Japan Society of Mathematical Education, 2018, p. 314) in Japan. Even though it is taught at Japanese schools, National assessment results show that half of the 6th graders have difficulty in this type of problems (e.g., NIER, 2022). We also hypothesize that the MC stages are contributing factors for students' performances in this type of ratio problems. For each task in Part 2, 3, and 4, we provided the following prompts: Q1 *draw a picture of the situation*, Q2 *write equations for this situation and explain the meaning of your equations in your pictu*re, Q3 *solve the problem*, Q4 *tell mistakes you expect of students making*, and Q5 *tell feedback you will give students who make mistake or have difficulties*. Due to space limitations for this paper, we only present the analysis of one Bar Task and Paint Task 1. We used the "Rubric for Assessing Units coordination Structures" (see Norton, 2015, p. 136) to analyze teachers' actions

and operations in the Bar Problem. Two researchers individually watched the interviews and coded using the Rubric, then researchers came together, discussed their coding, and reached a consensus.

Two Japanese participants were pre-service primary school teachers who would be certified to teach 1-6 grades in their Year-4 (last year) study and have already completed student teaching practice (internship). Two Turkish participants were middle school in-service teachers of at least 20 years of experience (teaching 5th -8th grade). Individual interviews were conducted during summer 2024. Each interview took about 45-120 minutes, and was videotaped; written work was also collected.

Analysis

Teacher A (MC 2 level, Pre-service)

Bar Task: Teacher A first wrote (9÷4=4/9), but she hesitantly said, "I may be wrong." She then drew a line that represented the long orange bar, and underneath, she drew a short line representing the small green bar and counted if the length fits into the orange bar 9 times. Then, she wrote two segments by counting 4 short lines (she first made a counting mistake). By looking at this drawing, she derived the solution "2 1/4." She then changed the answer from 4/9 to 9/4. We conclude that she is in the MC2 level because at the beginning she did not construct multiplicative relationship between three different units such as the purple bar is 4/4, and 9 times 1/4 (one green bar) is the long orange one. It was only after she drew the lines that she found the purple bar fitting into the orange bar 2 1/4 times. She did coordination of three levels of units in activity, so it is categorized as MC2 by referring to Norton et al. (2015).

Paint Task 1: Teacher A drew area diagrams with dL and m^2 (see Figure 1). Area diagrams only showed the problem situation. This tendency of less attention given to the quantitative relationship continued when she was constructing equations. She first wrote "x=," but erased it and wrote "y= ― (fraction bar)." She mumbled, "I don't know which is which (she is trying to decide between 3/4 and 4/3)." She again erased it, wrote "x= ― y," thinking a little while. Finally, she decided on "x=3/4y." When she was asked to explain what happened, she replied, "Since 3/4 dL would paint 1 m^2, I applied it." However, being asked how the equation relates to the diagrams, she replied: *"Relationship with the diagram, …, well, it is difficult*…" She did not explain the equation beyond the numerical correspondences. When asked to make another equation, she quickly constructed y=4/3x. When asked to find the area that can be painted with 1dL paint, she used y=4/3x, calculated y=4/3×1 and got the answer of 4/3 m^2. In this way, for this problem her attention remained numerical correspondences without engaging in units coordination either with amount of paint or with painted area to find the unknown quantity.

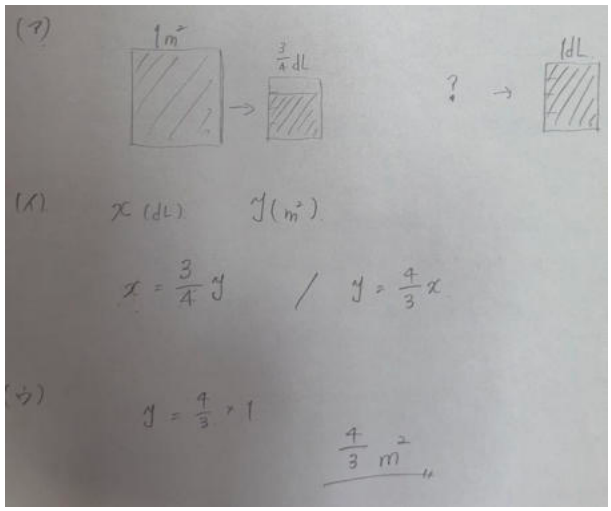

Figure 1: Teacher A's solution to Paint Task 1

Teacher B (MC 3 level, Pre-service)

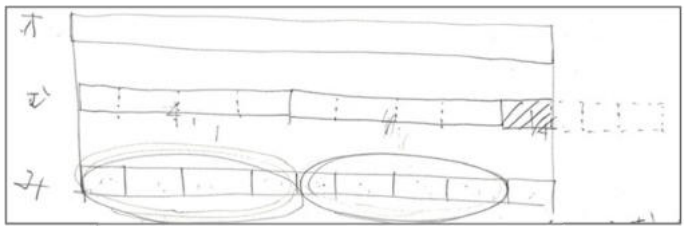

Figure 2: Teacher B's solution to Bar Task 7

Bar Task: Teacher B drew tape diagrams, which she had been drawing in the previous bar tasks and derived the answer: 9÷4=9/4=2 1/4. Compared to Teacher A, she drew three different bars explicitly with units and made relationships verbally, which showed her attention to the three levels of units. She did not conflate the leftover in the purple bar with 1/9. The leftover part was explained, "And this one (pointed the right leftover in the middle purple bar), too (traced the stretched part for emphasis), the one that goes in is green ..., purple (restated) ribbon divided into 4 pieces, and then it comes to the same length as the orange one, so 2 and 1/4 pieces go in" (Figure 2). She appropriately accounted for the leftover part in the middle bar without losing its relationship with the small bar, which we concluded indicated MC3 level by Norton et al. (2015, pp.136).

Paint Task 1: Teacher B quickly drew double number line (dL and m^2) and obtained the answer 4/3 dL (later, she corrected dL to m^2) by using a proportional relationship between dL and m^2 (Figure 3a).

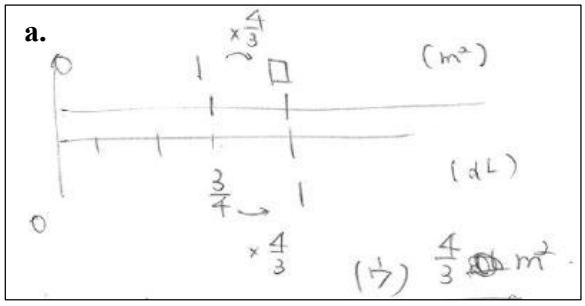

Figure 3: Teacher B's solution to Paint Task 1

However, it was difficult for her to express the relationship between x(dL) and y(m^2). She put x and y on the double number line but could not proceed further. She also drew a rectangle, shaded 3/4 of it, and began to develop parallel expressions between dL and m^2 (Figure 3b). However, in the middle of her writing, she crossed them all and said : *"It doesn't come out."* From her writing, we inferred that to find the whole (4/4 dL) she wanted to use the unit fraction 1/4 dL. However, she did not find the square meters that can be painted by 1/4 dL. Even though she did units coordination within the paint quantity, it was challenging to do it within the area quantity: first of all, she did not come up with 1/4dL equals 1/3m^2. This equality of two different quantities is important to find the solution. One needs to coordinate three levels of units first within the paint quantity and at the same time abstract that a third of 3/4 dL refers to a third of 1m^2. Later, one needs to use the result of this equality as the smallest unit for the m^2 to construct three levels of units coordination: 4 times 1/3 m^2.

Teacher C (MC 2 level, In-service)

Bar Task: Teacher C labeled "1 unit" for the green bar, "4 units" for the purple bar and 9 units for the orange bar, which is 9 times the green bar. She made two individual two-levels of units coordination.

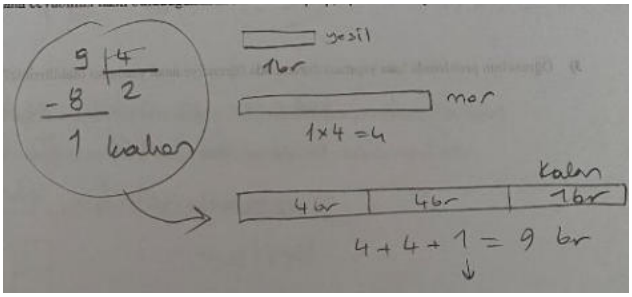

Figure 4: Teacher C's solution to Bar Problem

She also demonstrated these statements by drawing the bars (see Figure 4). To find out how many purple bars would fit into the orange bar, she divided the length of the orange bar by the length of the purple bar so she wrote 9 divided by 4. From the division procedure she had 2 in the quotient and the 1 in the remainder, she concluded that "the orange bar is 1 more than 2 times the purple bar." In her drawing, she indicated that the orange bar would be 9 units with 4 units of the purple bar + 4 units of the purple bar + 1 unit remainder. From Norton et al. (2015, p. 136), we concluded she was operating on level MC2 since she did not use any fraction in her results and especially, she did not articulate what "1 unit" would be in terms of the purple bar (supposed to be 1/4). She did not produce 9/4 as a fraction but she just divided 9 by 4; there was no units coordination other than the initial individual two-levels of unit coordination with whole numbers.

Paint Task 1: For this problem, Teacher C asked for help which was not planned. The interviewer gave help without any other choice. The help she received was as follows: she first had 3/4 dL drawn which was shown as the rectangle having 3 parts and she was told that "this amount painted 1 square meter, each part of the paint then painted 1/3 of the 1 m^2. Since 1 dL consisted of 4 equal parts, so it should be 4×1/3, which was equal to 4/3 m^2" (this was provided). After receiving this help, when she started solving it on the paper on her own, she drew the paint can (3/4) as a rectangle. She shaded 3 out of the 4 parts and assigned a variable "a" to each part (Figure 5). She then added: "*3a paints 1 square meter, so 1 dL…"* Here, since she thought of each part she called "a" as 1 dL, she was about to get confused again, but recalling the previously explained solution, she added: "*Then 'a' paints ⅓ of a square meter.*" She continued, "The equation for this is a + 3a = 4a; 4a becomes 1 square meter + 1/3 square meter," and finished the solution. Similar to what she did in the Bar problem (no coordination of units) she did not have units coordination within the painted area as a multiplicative relationship and she depended on the interviewers. Even though she produced a result of 1m^2 +1/3 m^2, this solution was not hers.

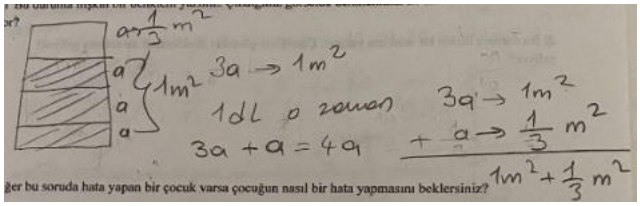

Figure 5: Teacher C's solution to Paint Task 1

Teacher D (MC 3 level, In-service)

Bar Task: Teacher D, after reading the question, stated that "*I am confused* " and re-read the question. On her second reading, she said the answer would be 9/4 times. She got confused again. She read the question as she was pointing out the colored bars: "*purple one is 4 times green; orange is 9 times green so it is 9 over 4 times*". She stated that the orange bar is 9/4 times the purple bar which she said less than 2.5 times and exactly 9/4 times. Here, since she explained how she made 9/4 by pointing to the pictures and she was explicit that the purple was the middle unit for coordinating smallest unit of

green and the biggest unit of orange bar, we categorized her actions as MC3 aligning with Norton et al. (2015, p. 136).

Paint Task 1: Teacher D drew a rectangle for 1 m^2 and had divided it into 4 parts and shaded 3 parts (see Figure 6a). She then said this was not right and she decided to have two figures separately (see Figure 6b): 1 m^2 an unpartitioned wall and 4-part rectangle where 3 parts were shaded for the paint and its unit was dL. At that point she made a direct proportion relationship to help her think about the pictures (see Figure 6c) and the interviewer asked her to show the situation with pictures (instead of continuing with "direct proportion"). She then focused on her picture and she made another set of drawing underneath to show the relationship (see Figure 6d): for extra 1/4 dL she needed second m^2 and interestingly she divided each of the two 1 m^2 board into three. She justified that "if each part of the 3-part (paint, which she explicitly said, "one fourth of paint") makes one of the 3 parts of the wall (1 m^2) then for the extra fourth of the paint we need one (third) of the wall so 1/3 m^2 then it would be four over three m^2" and wrote 4/3 m^2 (see Figure 6d). She said she could not write an equation.

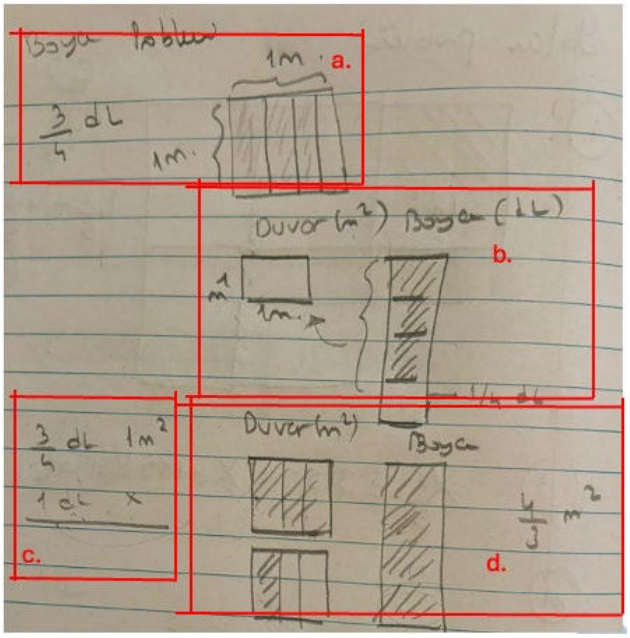

Figure 6: Teacher D's solution to Paint Task 1

Discussion and future research

In this study, we aimed to explore teachers' representations in equation formulation with fractions and the role of their multiplicative concepts. We investigated teachers' understanding of multiplicative relationships between two varying quantities, which have not been reported before to our knowledge. Compared to the other two types of tasks (Seat Tasks and Plant Tasks), the Paint Tasks were found to be the most difficult for all participants. Difficulties were observed in understanding the structure of the problem, and the presence of two quantities in a proportional relationship. This fact seems to indicate that there is a difference in the knowledge required by the teachers between comparing two quantities of one type (only length (plant) or only a discrete quantity (seat)) with a varying relationship and two quantities of two types with a varying relationship.

In the Paint Task, the drawings for presenting volume and area as rectangles, and then dividing them to represent fractions were observed by teachers from both countries. On the other hand, there were some differences between these two groups, one of which was the use of the double number line observed in Teacher B's actions. Although familiar to both Japanese pre-service teachers, the use of double number line was only seen in one of the two participants. The second was the expression of "direct proportion" observed in Teacher D's activities. This is also a familiar expression taught in middle schools in Türkiye, but again, it was recognized by only one teacher. Teachers B and D, who

were able to use other expressions as well as rectangle and square diagrams, were at the MC3 level for Bar Problems when we analyzed their activities using Norton et al.'s (2015) Rubric for Units Coordination Structures. Interestingly, for Paint Task 1, both teachers were able to recall alternative figures or expressions and use them as cues for their thinking when faced with difficulties. However, Teacher B did not develop the equality of the area quantity that corresponds to 1/4dL paint. It was only Teacher D who had two rectangles to represent dL and m^2, and successfully made the varying relationships of the quantities. Making the equivalency relationship of 1/4dL and 1/3 m^2 was a key step in Paint Task 1, and this was very difficult even for MC3 level teachers. It is also noteworthy that neither Teacher B nor Teacher D could write an algebraic equation for Paint Task 1. Writing an algebraic equation that includes two varying quantities, such as paint and board, proved to be a difficult matter regardless of previous teaching experience or educational background. Since only Paint Task 1 was analyzed in this paper, it is necessary to examine each teacher's thought processes for the remaining two Paint Tasks. An interest is on whether the teachers exibit or make progress on units coordination or equation writing at some points because they seem to move between categories on different tasks or different representations. As for the future, we will also explore how teachers with different multiplicative concepts might anticipate potential difficulties their students may encounter when solving a similar set of problems. Lastly, we also hypothesize that if teachers' approaches are symbolic and procedural in writing equations for representing these problem situations or they have different MC levels, their expectations of middle schoolers' mistakes and their responses to them will also be symbolic and procedural.

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Proportional reasoning unleashed: How gifted minds tackle diverse proportional reasoning tasks

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The present study aims to investigate 7th-grade gifted students' performance in alternative types of linear and nonlinear proportional reasoning tasks, namely, missing value, numerical comparison, and qualitative problems. The participants of the present study were three 7th-grade gifted students who attended the Science and Art Center in İstanbul, Türkiye. The findings elucidated that gifted students' performance differed in alternative types of proportional reasoning problems. Specifically, their performance was the highest in missing value problems, while they showed the poorest performance in qualitative problems. In terms of linear and nonlinear problem types, gifted students were more successful at nonlinear problems compared to linear ones.

Keywords: Gifted students, proportional reasoning, problem-solving, illusion of linearity.

Introduction

Proportional reasoning was defined as "detecting, expressing, analyzing, explaining, and providing evidence in support of assertions about proportional relationship" (Lamon, 2020, p. 4). It is a vital concept because it is regarded as the basis of the arithmetic, number, and measurement concepts (Lesh et al., 1988). Furthermore, as proportional reasoning involves expressing the relationships between quantities and covariation, it is closely tied to algebraic thinking (Lesh et al., 1988). Due to being a pivotal concept, a variety of studies were conducted to explore middle school students' proportional reasoning. However, research specifically focusing on the proportional reasoning of special students, such as gifted students, is lacking. Considering that gifted students have above-grade-level ability (Renzulli & Reis, 2021), their proportional reasoning might differ from their peers. Hence, in the present study, we aimed to focus on the proportional reasoning of gifted students.

Theoretical Background

Despite its relevance with diverse topics in mathematics and with other disciplines and its widely accepted importance (Lamon, 2020; Özgün-Koca & Kayhan-Altay, 2009), numerous studies revealed that middle school students have difficulty in developing proportional reasoning (Ayan-Civak et al., 2024). The most frequently cited difficulty in the literature was students' inability to discern the multiplicative and additive situations and select the appropriate strategies for different situations (Ayan-Civak et al., 2024; Tourniaire & Pulos, 1985). Besides, the other difficulty of students was noted as the illusion of linearity (linearity trap), which means students' inability to distinguish proportional and nonproportional situations and overuse of linearity in nonproportional situations (De Bock et al., 2007). A review of the literature indicated that a profound comprehension of (non)-proportionality was not attained by many students (De Bock et al., 2007). The commonality of these difficulties is that both stem from students' inability to recognize and express the relationship between quantities, which is foundational to functional thinking, a crucial aspect of algebraic thinking (Blanton & Kaput, 2011).

Several studies (e.g., Tourniaire & Pulos, 1985) have examined the potential reasons influencing students' difficulty in proportional reasoning. Those reasons can be summarized as problem type,

context, and numerical features (Tourniaire & Pulos, 1985). Moreover, an overemphasis on proportionality and overuse of missing value problems, students' insufficient knowledge of geometry, teachers' inability to connect geometry and proportional reasoning, and the intuitive and straightforward nature of the linear relationship also affect students' performance and illusion of linearity (De Bock et al., 2007). In conclusion, a literature review indicated that problem type stands out as a common reason affecting students' performance and difficulties. For that reason, many studies (e.g., Ayan-Civak et al., 2024; Özgün-Koca & Kayhan-Altay, 2009) were conducted to elucidate students' performances in alternative types of proportional reasoning problems, such as missing value, numerical comparison, qualitative, linear, and nonlinear problems. Nevertheless, there is still room for research on the proportional reasoning performances of special students, such as gifted students.

National Association for Gifted Students (NAGC) defined gifted students as "Students with gifts and talents perform—or have the capability to perform—at higher levels compared to others of the same age, experience, and environment in one or more domain" (NAGC, 2019, p.1). As the definition implies, those students have unique characteristics distinguishing them from their same-age peers. More specifically, those students learn faster (Cross & Coleman, 2014), demonstrate above grade level ability (Renzulli & Reis, 2021), and possess a wide range of knowledge of strategies and adaptability in their use compared to their nongifted peers (Jaušovec,1991). Because of these characteristics that distinguish gifted students from their nongifted peers, gifted students' performances in alternative types of proportional reasoning problems and the possible reasons that affect their performances might show variation. Furthermore, investigating these students' proportional reasoning in alternative problem types holds paramount importance in gaining insight into gifted students' proportional reasoning performance, identifying their difficulties and the underlying reasons behind their performance, and determining the differentiation these students need in proportional reasoning. Hence, the present study aims to investigate the performance and possible reasons for incorrect responses of 7th-grade gifted students in six alternative types of problems: missing value linear problem, missing value nonlinear problem, numerical comparison linear problem, numerical comparison nonlinear problem, qualitative linear problem, and qualitative nonlinear problem. In line with these aims, the research questions that guided the present study were as follows: To what extent are gifted students successful in solving alternative types of proportional reasoning problems? What are the possible reasons behind those students' incorrect responses?

Methods

This case study authorized deeper investigations of a few individuals on the specific topic (Creswell, 2012). In this case study, the participants were three 7th-grade gifted students whose area of giftedness is general mental ability. The participants were selected conveniently and assigned pseudonyms Zenia, Edward, and Andrew. While Zenia attends the Science and Art Center (SAC) for three years, Edward and Andrew attend SAC for five years. SACs are special education institutions for gifted students regulated by the Ministry of National Education. In SACs, gifted students do activities rather than being taught a topic outside school hours. Although the participants in this study learned about ratio and proportion in their schools, they did not receive any instruction about proportional reasoning in SACs. According to the Turkish mathematics curriculum (Ministry of National Education [MoNE], 2018), a 7th-grade student who has learned the subject of ratio and proportion is expected to have skills such as deciding whether two quantities are directly or inversely proportional by examining

real-life situations, expressing the relationship between two quantities, and solving problems related to ratio and proportion.

The researchers of this study generated the data collection instrument. The instrument covers six proportional reasoning problems, namely, missing value (linear and nonlinear), numerical comparison (linear and nonlinear), and qualitative problem (linear and nonlinear). Due to page limitations, one example each for linear and nonlinear problems is given below to shed light on linear and nonlinear problems. The nonlinear numerical comparison problem that was used in this study is presented below:

Simay wants to enlarge two different photos of her with her mother for Mother's Day. The original sizes of the first rectangular photograph are 4 cm and 8 cm. The area of the enlarged version of this photo is 128 square centimeters. The original sizes of the other rectangular photograph that Simay will enlarge are 12 cm and 24 cm. The area of the enlarged version of the second photo is 576 square centimeters.

Which photo is enlarged more?

Following the nonlinear problem example, the problem below represents one of the linear problems, i.e., the linear numerical comparison problem, used in this study.

The true distance between Ankara and Bursa is approximately 388 km, and the distance measured on the map is 2 centimeters. While the true distance between Aksaray and Adıyaman is approximately 582 km, the distance measured on another map is 5 centimeters.

Which map is smaller in scale? (Paper sizes are assumed to be the same.)

The nonlinear and linear problems presented above are both numerical comparison problems in which students were expected to compare two ratios. However, since there is a nonlinear relationship between the side lengths and the area of the rectangle, the first problem was called a nonlinear numerical comparison problem. On the other hand, since there is a linear relationship between the lengths (the true and map distance), the second problem is a linear numerical comparison problem.

While designing the data collection instrument, the researchers pay attention to including problems in real-life contexts and several solution approaches. Once the researchers finalized designing the data collection instrument, they took the opinion of an expert who has previously worked with gifted students, conducted studies on proportional reasoning, and is familiar with the literature on proportional reasoning problem types. The mathematics education expert confirmed the categorization of the problems and the mathematical and pedagogical appropriateness of the problems. Once this process was completed, the researchers began collecting data. Semi-structured task-based interviews were utilized to collect the data. Early in the interview, instead of informing the students that they would be posed with proportional reasoning tasks, it was stated that they were expected to solve real-life problems. Content analysis was employed while analyzing the data, and inductive coding was used to code the data. The researchers individually identified students' correct and incorrect responses and possible reasons for students' incorrect responses. Afterward, they discussed their identifications until they achieved a 100 % agreement. Based on the analysis, the findings will be presented under two main headings: the performance of gifted students, which

included their correct and incorrect responses, and possible reasons for their incorrect responses, which were contextual and conceptual reasons.

Findings

The Performance of Gifted Students in Alternative Proportional Reasoning Tasks

Table 1 presents the correct and incorrect responses of each student to the six alternative types of proportional reasoning tasks.

Table 1: Performances in alternative types of proportional reasoning tasks

Problem Types / Students	Missing Value		Numerical Comparison		Qualitative	
	Linear	Nonlinear	Linear	Nonlinear	Linear	Nonlinear
Zenia	Correct	Correct	Correct	Correct	Correct	Incorrect
Edward	Correct	Correct	Correct	Correct	Incorrect	Correct
Andrew	Correct	Correct	Incorrect	Correct	Incorrect	Correct

As can be seen from Table 1, students provided correct responses to most of the problems. All students answered all missing value problems correctly, regardless of whether they were linear or nonlinear. Moreover, an investigation of students' performances in numerical comparison problems indicated that only one student answered the linear numerical comparison problem incorrectly. On the other hand, when the questions to which students provided incorrect responses were examined, it was evident that they gave incorrect responses most to qualitative problems. Specifically, each student responded incorrectly to one of the two qualitative problems. To be more precise, two students answered the linear qualitative problem incorrectly, and the other answered the nonlinear qualitative problem incorrectly. Due to the page limitation, a few examples of students' correct and incorrect responses will be shared. First, some examples of students' correct responses will be presented.

Figure 1 shows one student's correct solution for the nonlinear missing value problem, where each student provided correct answers.

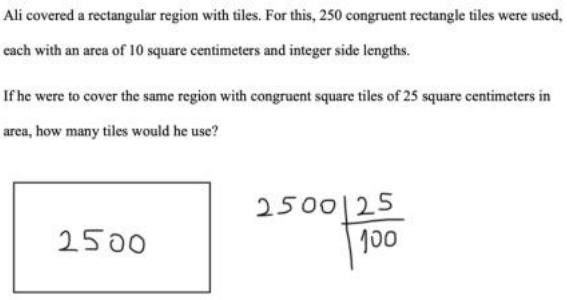

Figure 1: Zenia's correct solution for the nonlinear missing value problem

Zenia: I multiplied 250 by 10 to find the total area. Then I divide 2500 square centimeters by 25 square centimeters; it becomes 100. Therefore, he needs to use 100 of the tiles.

As shown in Figure 1 and supported by the interview excerpt, the student found the total area correctly by multiplying the number of tiles and the area of each tile. Afterward, she divided the total area by the area of the new tiles and accurately found the number of tiles.

Moreover, one of the correct answers to the nonlinear numerical comparison problem, which is presented in the methods, is given in Figure 2.

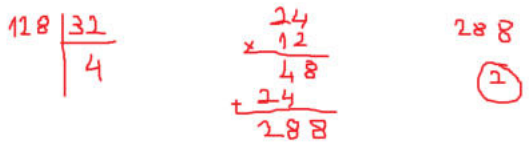

Figure 2: Andrew's correct solution for the nonlinear numerical comparison problem

Andrew: The first photo is enlarged more. First, I multiplied 4 and 8 and got 32. Then, when I divided 128 to 32, the quotient is 4. Afterward, I multiplied 12 and 24 and got 288. When I divided 576 to 288, I got 2. Since the result I obtained in the enlargement ratio of the first photo was larger, the first photo was enlarged more.

In this problem, first, the student found the area of both photographs' original and enlarged versions. Then, he identified the scale factors as 4 and 2. Afterward, he determined that the enlargement ratio of the first photo was larger because its scale factor was larger, and he achieved the correct response.

The Possible Reasons for Gifted Students' Incorrect Responses

In addition to correct responses, analyses of students' answers uncovered incorrect responses as well. The analysis revealed four incorrect answers. When these incorrect answers were examined, it was observed that students' incorrect answers were caused by contextual and conceptual reasons. More specifically, three of them were due to contextual misinterpretation, where the context was scale, and one was due to conceptual confusion. Examples of incorrect answers resulting from each of these two reasons are given below.

The student had a fallacy in interpreting the quotient in the scale context while solving the linear numerical comparison problem, which is presented in the methods.

Andrew: On the map, the actual distance between Ankara and Bursa is 388. When it is divided by 2, I got 194. On the other hand, the actual length is 582 on another map. When I divided 582 by 5, a bigger number, I got something less than 194. Thus, the map showing the distance between Aksaray and Adıyaman is on a smaller scale.

The interview excerpt indicated that the student could not associate the results he arrived at by comparing the divisors in the division process with the scale context. More specifically, although the student correctly calculated the ratio of the actual distance to the length measured on the map, he could not identify which results belonged to a smaller scale. Thus, the student gave an incorrect response stemming from contextual misinterpretation.

In addition to the contextual reason, the conceptual reason, namely conceptual confusion, led one student to provide an incorrect response for a nonlinear qualitative problem (see Figure 3).

When two rectangular prisms, tanks A and B, were filled with equal amounts of water, it was observed that the height of the water in tank A was higher than the height of the water in tank B. The water in the tank has been drained.

Then, these tanks were filled with a certain amount of water again, and it was seen that the height of the water in the two tanks was equal.

In the last case, which tank has the largest volume of water? Explain your reasoning.

Figure 3: Nonlinear qualitative problem

Zenia: I think there is more water in tank A. When an equal amount of water is put into Tank A and Tank B, it looks like there is a lot of water in Tank A. I thought that B was longer and thinner than tank A. When we fill an equal amount of water from here, a larger amount reaches A because it has more depth.

Since the student was confused about the area of base and height, she could not accurately compare the volumes of Tank A and Tank B. In other words, her wrong answer was due to her lack of relational understanding of geometric concepts.

Discussion

The present study aimed to investigate the performance of 7th-grade gifted students in alternative types of proportional reasoning problems and the underlying reasons for those students' incorrect responses. One of the most striking findings of this study was that gifted students had not experienced difficulties reported in prior studies. Specifically, the existing body of research (e.g., Ayan-Civak et al., 2024; De Bock et al., 2007) noted frequent difficulties among middle school students, such as the inability to distinguish additive and multiplicative situations and proportional and nonproportional situations. In contrast, in our study, although gifted students did not know that these problems were proportional reasoning problems, none used additive strategies; instead, they distinguished multiplicative situations correctly.

Moreover, the other compelling result of the present study was that although there was a nonlinear relationship between the attributes, such as side lengths and area in nonlinear problems, none of the students tended to an illusion of linearity because students compared areas with areas and lengths with lengths. This finding implies that gifted students in the present study showed a clear understanding and awareness of the attribute they were coordinating. Gifted students' metacognitive skills might contribute to this awareness. To be more precise, as gifted students know what they know and how to utilize what they know, and they are capable of distinguishing between relevant and irrelevant information (Shore & Kanevsky, 1993), they might be attentive to the specific attributes. Consequently, since the gifted students in this study did not experience the difficulties reported in the literature, it can be concluded that they were capable of identifying and expressing the relationship between quantities. This finding indicated that gifted students in the present study reflect the emergence of algebraic thinking, which might influence their proportional reasoning performance. Therefore, the present study's findings suggest that the unique characteristics of gifted students had an essential role in their proportional reasoning performance.

The other prominent finding was that gifted students' performances differ in alternative types of proportional reasoning problems. Expressly, within the context of the present study, the problem type on which they performed best was the missing value problem. This finding aligned with the findings

of prior studies (e.g., Ben-Chaim et al., 1998) conducted by middle school students. On the other hand, gifted students exhibited the poorest performance on qualitative problems. The reason for this difference in students' performance may be that missing value problems are more common in textbooks than in other problem types (Özgün–Koca & Kayhan–Altay, 2009). This result emphasized the necessity for including qualitative proportional reasoning problems in the mathematics curriculum in Türkiye. Besides, two out of three students had difficulty with linear problems. This might be because students might not make sense of the context of the linear problems, which causes them a *contextual misinterpretation*. This supports that the context of problems affects students' performance (Tourniaire & Pulos, 1985; Özgün-Koca & Altay, 2009) and might explain why they have lower performance in linear problems. Therefore, mathematics teachers should pay particular attention to interpreting the context in teaching and learning proportional reasoning to cultivate the effectiveness of learning for the learners.

Furthermore, the present study's findings revealed that in addition to context, *conceptual reasons* might also have a role in students' performance. Specifically, one student had difficulty understanding and visualizing height despite knowing the volume formula. The reason behind this difficulty may be that the student was exposed to procedural knowledge of the volume formula and/or did not conceptualize the elements of a rectangular prism. This finding underscores the necessity for teachers to prioritize conceptualizing the concept and connecting different branches of mathematics while teaching proportional reasoning.

Based on the valuable findings discussed, the present study highlights the need for differentiation of teaching and learning proportional reasoning tailored to gifted students' unique strengths and difficulties. Moreover, by stressing the disparities between the proportional reasoning performance of gifted and nongifted students, this study might pave the way for future studies to focus extensively on the underlying reasons for these disparities.

Despite the valuable insights and implications the present study offered, it has some limitations. More precisely, since this study aimed to examine gifted students' proportional reasoning performances in-depth, the number of participants was limited. Thus, the findings of the present study illustrated the thought processes of the three participants and shed light on the qualitative facets of problem-solving in this setting, but they were not generalizable. If the findings of this study are desired to be generalized, a quantitative study can be conducted with larger samples to confirm and expand the insights the present study offered.

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Experts' structuring processes of equations that can be solved with varying efficiency: An analysis of gaze paths with eye-tracking

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Solving equations is fundamental to algebra yet remains a persistent challenge in mathematics teaching and learning. Drawing on a semiotic-pragmatic framework, we argue that algebraic equations lack inherent structure; instead, solvers must actively impose their own personal structure, a process known as 'structuring'. Using eye-tracking, we analysed gaze patterns of student teachers solving equations with varying efficiency. Adaptive experts showed global relating, characterised by rapid transitions between subexpressions of both sides of the equation, and hierarchical structuring, enabling faster solutions. In contrast, routine experts exhibited local relating, marked by sequential processing, longer fixations and increased cognitive load, leading to slower and less accurate performance. These findings provide insights into the cognitive strategies of equation solving and offer implications for instructional approaches that promote more effective equation solving.

Keywords: Equations, structuring processes, solving strategies, adaptive expertise, eye-tracking.

Introduction

This paper presents an ongoing study focusing on how experts' reading of algebraic equations is reflected in their gaze patterns. Adopting a semiotic-pragmatic approach, we contend that equations do not inherently possess an (objective) structure that be directly "read off." Instead, equations are strings of symbols devoid of meaning. To make meaning of an equation, we have actively to "read" a structure "into" it—essentially, impose a structure upon it. This process, known as *structuring*, involves identifying subexpressions and relate them to one other (e.g., Rüede, 2013).

Consider the equation $x - 3(x + 1) = 7 - 3(x + 1)$ (based on Hämmerle et al., 2018). It can be solved using two distinct solution strategies: (A) expanding $3(1 + x)$ on both sides and then simplifying the equation, or (B) adding $3(1 + x)$ to both sides. These strategies differ in two significant ways: First, strategy (B) is more *efficient*, requiring fewer computational steps. Second, the process of *cognitive structuring*—the mental framing before beginning to solve the equation—varies between the two strategies: Using the standard strategy (A) requires structuring the equation as $\{x\} - \{3\}\{(1 + x)\} = \{7\} - \{3\}\{(1 + x)\}$, relating the number $\{3\}$ to the subexpression $\{(1 + x)\}$ twice. In contrast, applying the more efficient strategy (B) involves structuring the equation as $\{x\} - \{3(1 + x)\} = \{7\} - \{3(1 + x)\}$, relating the left and right subexpressions $\{3(1 + x)\}$ to each other. Depending on which structuring the solver imposes, the solution will be correct and achieved with varying levels of efficiency.

In this context, the question arises of how structuring processes differ between experts who solve equations with varying efficiency, namely *routine experts* and *adaptive experts* (Baroody, 2003). Routine experts tend to rely on standard solution strategies, whereas adaptive experts apply more

innovative and efficient approaches. This classification is fluid, as individuals may switch categories depending on task-specific factors that influence strategy selection. The present study addresses this question by capturing and analysing the gaze paths of both expert types using eye-tracking methods. Our objective is to reconstruct *ideal-typical gaze patterns* of routine and adaptive experts during equation solving and explore their implications for instruction (e.g., Bikner-Ahsbahs, 2015).

Theoretical Framework

Structuring an equation: A necessary condition to solve the equation

Algebraic expressions and equations are symbol strings requiring interpretation according to established rules. For instance, one rule permits omission of the multiplication sign, while another mandates that multiplication takes precedence over addition. Consider the expression $x - 3(1 + x)$ which, from a mathematical perspective, must be interpreted as $x - \{3 \cdot (1 + x)\}$. However, in algebra education, it is widely acknowledged that algebraic expressions and equations are less self-explanatory than they may initially seem. Much like interpreting mathematical visualisations, reading algebraic equations involves an active interpretative process. Hefendehl-Hebeker and Schwank (2023) describe this as the "dynamic interplay of reading off and reading into" (p. 109). This dynamic implies that a student might interpret the expression $x - 3(1 + x)$ from an entirely different—and incorrect—structural perspective, such as $(x - 3) \cdot (1 + x)$. In this interpretation, the subtraction between x and 3 is prioritised before relating 3 to the subexpression $(1 + x)$, resulting in an alternative structure.

Each way a person relates parts of an expression represents an interpretative achievement of the person. To highlight this process, we use the term 'structuring' to describe how an individual creates an *idiosyncratic structure* for an expression (Rüede, 2013). This cognitive act shapes how the person subsequently manipulates and transforms the expression (Radford, 2010). For instance, a student who perceives the structure $\{x - 3\} \cdot \{1 + x\}$ is more likely to multiply the term. Mastering algebra, therefore, requires students to address two critical questions (Malle, 1993): *What possible idiosyncratic structuring does an algebraic expression allow, and which align with mathematical rules?* And: *Which of these structures is most advantageous for further manipulation?*

Eye-tracking analyses, particularly in the domain of equation solving, offer a promising method for uncovering these cognitive processes (see Strohmaier et al., 2020, for an overview).

Gaze patterns of experts when solving an equation: The eye-tracking approach

The study of eye movements in mathematical activities has gained prominence in recent years through eye-tracking technology (e.g., Lilienthal & Schindler, 2019). Research spans topics like arithmetic, algebra, and geometry, focusing on activities such as proof and problem-solving. Eye-tracking excels at correlating gaze location with overt attention, though the reliability of this correlation with cognitive processing remains debated (Strohmaier et al., 2020).

Few studies examine eye movements during the reading of algebraic expressions and equations. In one expert-novice study, university students rearranged simple equations like $x \cdot a = b$ or $x/a = b$ to solve for x (Susac et al., 2014). Results showed a positive correlation between the number of fixations and the ratio of reaction time to response accuracy. Experts demonstrated fewer fixations—consistent with earlier studies—meaning that experts "knew where to look" (Susac et al., 2014, p. 567). Another

study (Kohlhase & Fürsich, 2016) found that experts located the equals sign in simple linear equations before shifting to operational symbols. Novices, however, often processed equations from the center or, with longer, more complex equations (such as integral equations), from left to right, mirroring text reading. A follow-up study (Kohlhase et al., 2017) showed that experts navigated complex equations by following their expression trees in depth first traversal of the hierarchical structure.

Moreover, eye-tracking has identified ideal-typical procedures in pattern completion tasks (Schwank, 2001) and revealed common erroneous strategies in, e.g., reasoning with tree diagrams and 2×2 tables (Bruckmaier et al., 2019). Eye-tracking highlights task areas that capture attention and maps decision-making and problem-solving processes. Depending on the research focus, data can be distilled into ideal types through typifying content analysis or used to validate and refine existing ideal types.

The current study

Our study explores how eye-tracking can help to reconstruct solving strategies, an approach employed in previous research. We aim to understanding the mental actions taken when routine and adaptive experts encounter an equation that can be solved with varying efficiency. To observe their structuring process, we focus on their gaze paths, and address the following research questions: *Where is the initial gaze directed? Which elements—such as subexpressions, operations, brackets, or the equals sign—are scrutinised more closely* (indicating perceived importance)? *Between which subexpressions does the gaze shift* (suggesting the establishment of relations)? *Which areas does the gaze focus on, and where is it absent?*

Hypotheses

Aligned with our theoretical framework and the overarching research questions, our hypotheses of the current study explore how participants' eye movements—indicative of varying levels of expertise—differ when solving equations of varying levels of efficiency:

- *Adaptive experts*, who solve equations efficiently after an initial scan, will structure the equation following a top-down approach in the corresponding expression tree (see above). They will focus selectively on areas deemed relevant by the expression tree, demonstrating targeted and efficient "initial scanning" within these critical areas.
- *Routine experts*, following a standard strategy, will distribute their attention more broadly across the task areas. Their gaze will follow a more linear, left-to-right pattern, with a particular emphasis on areas that support the standard solution approach.

While eye-tracking studies typically include both qualitative and quantitative measures—such as fixation count or saccade length—our analysis in this report prioritises qualitative aspects. However, in line with the *information reduction hypothesis*, we anticipate that adaptive experts will exhibit fewer fixations compared to routine experts, as they reduce attention to redundant areas and concentrate on task-relevant information. Additionally, we expect adaptive experts to solve the equation more quickly than routine experts, who may take longer to complete the standard procedure.

Method

Sample

Seven student teachers in their early years of teacher education participated in the study. All had completed upper-secondary education and had achieved very good grades in mathematics in their Matura certificate. Their educational background indicates a solid foundation in algebra, making them well-suited for tasks involving equation solving. We could therefore reasonably expect that all participants possessed, at a minimum, the requisite knowledge to solve basic equations correctly and within a reasonable time frame, reflecting a baseline level of mathematical expertise.

Materials

The task set consisted of five equations—both linear and quadratic—selected based on prior studies. Each equation could be solved using either a standard method (e.g., expansion or quadratic formula) or a more efficient strategy. Examples include the linear equation $x - 3(1 + x) = 7 - 3(1 + x)$ (see above), as well as the quadratic equations $(x - 3)(x + 5) = 0$ and $(x - 1)(x - 2) = x - 1$. This task design allowed for variation in strategy use and corresponding gaze behaviour across participants.

Procedure

Eye movements were recorded using an SR Research EyeLink 1000 system (500 Hz sampling rate) with a tripod-mounted camera, offering a spatial resolution of 0.01°. The system tracked a 32° horizontal and 25° vertical range on a screen with a 1920 × 1080 resolution. Participants were seated approximately 55 cm from the screen to ensure consistent tracking.

After receiving instructions and an illustrative example, participants were asked to solve the equations mentally, without external tools or time constraints. As soon as they had found a solution, they were instructed to say "finish" and then state it aloud. At this point, the display and eye-tracking were paused, and participants explained their approach, which was audio-recorded. The full session—including calibration, task performance and interviews—lasted approximately 30 minutes.

Initial Observations

We focused on participants' solution strategies and their gaze patterns (i.e., scan paths). For example, in solving $x - 3(1 + x) = 7 - 3(1 + x)$, adaptive experts showed gaze transitions between x (on the very left side of the equation) and 7 (after the equals sign), indicating efforts to identify structural similarities and apply an innovative strategy. Routine experts, by contrast, exhibited more linear, left-to-right gaze paths, focusing on expanding the expression $-3(1 + x)$ on both sides, consistent with a standard strategy approach. Preliminary findings suggest routine and adaptive experts display divergent gaze patterns, both qualitatively and quantitatively, such as their time progression and focus intensity. These distinctions informed the hypotheses underlying our current study.

Analysis

Traditional heatmaps and gaze plots struggle to capture both fixation locations and order, particularly for linearly structured stimuli like equations, which are read bidirectionally. To address this limitation, we employ *fixation sequence diagrams* (also known as *sequence charts*), mapping fixation order on the x-axis and fixation position on the y-axis.

We began by combining eye-tracking data and interviews to identify participants with consistent strategies for solving $x-3(x+1)=7-3(x+1)$. For instance, *Claudia* consistently employed a shortcut strategy, recognising the identical terms $3(x + 1)$ on both sides of the equation. This adaptive use of her knowledge led us to classify her as an adaptive expert. In contrast, *Indira* adhered to the standard strategy of expanding and simplifying the expressions on both sides. Her consistent use of the standard method across all equations made her gaze patterns a representative for a routine expert.

To explore how solution strategies manifest in gaze macrostructures, we analysed the gaze paths of both participants. By incorporating interview data and existing literature, we reconstructed two ideal-typical gaze patterns: one representing adaptive experts and another for routine experts. However, the extent to which these ideal-typical gaze patterns can be generalised beyond the specific characteristics of Claudia and Indira remains an open question.

Findings

In the following section, we continue to focus on Claudia and Indira, illustrating and contrasting their gaze paths as the solve the previously introduced equation $x - 3(x + 1) = 7 - 3(x + 1)$. This analysis serves to represent broader patterns across equations involving different solution strategies and to address our two hypotheses.

Ideal-typical gaze paths of adaptive experts: The case of Claudia

Claudia solved the equation correctly in just over six seconds, explaining her strategy during the subsequent interview: "*I just saw, on the left and right, there is exactly the same term being subtracted. So there's 7 left over.*" In other words, she structures the equation as $A - C = B - C$, identifying and matching the term C. Claudia not only recognised this peculiarity but also used it purposefully, employing the innovative method by adding C (or $3(x + 1)$, respectively) to both sides, instead of following the standard algebraic strategy.

Claudia's gaze path comprised 39 fixations, each lasting between 50 and 350 milliseconds, amounting to approximately 6 seconds in total (see Figure 1 for her fixation sequence diagram). Her visual attention initially focused on the left side of the equation before shifting to the right. Further analysis revealed approximately a consistent gaze transition between the two sides—roughly one per second—resulting in seven transitions during her solution process. Early in her gaze path, she concentrated on the left-hand side, attending specially to the operators '–', '·' (or '(', respectively), and '+', in this exact order. Notably, her gaze showed no saccades between equivalent symbols on either side of the equals sign, challenging common assumptions about the gaze strategies of adaptive experts.

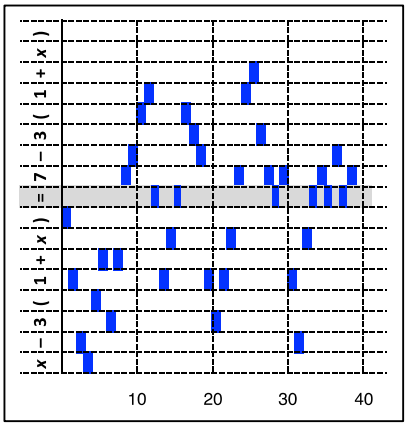

Figure 1: Fixation Sequence Diagram of Claudia (*x*-axis: fixation order)

We interpret these observations as follows: *Adaptive experts* construct relationships that extend beyond the equals sign, relate subexpressions on the left side of the equation to those on the right in

a process we term *global relating*. For these experts, the equals sign does not merely separate the equation into two isolated components; instead, they perceive the equation as a cohesive whole. This integrative perspective allows them to relate partial terms effectively, allowing for remarkably rapid equation-solving. The sequence in which they examine the operators aligns not only with their left-to-right arrangement but also reflects a systematic, top-down traversal of the hierarchical structure within the expression tree of $x - 3(x + 1)$ (Kohlhase et al., 2017). When this expression is written in functional notation—such as $-(x, \cdot(3, +(x, 1)))$— it becomes evident that these experts approach the problem "outside in" (Rüede 2013, p. 402). This way of processing expressions, aligning with our first hypothesis, appears to be a defining characteristic of adaptive experts.

Ideal-typical gaze paths of routine experts: The case of Indira

Indira took 63 seconds to solve the equation, ultimately arriving at the (incorrect) solution $x = {}^{4}/_{5}$. During the interview, she explained her approach: "*I applied the typical math rule. First, I looked at the 3, then at the bracket, and resolved it. After that, I worked it out by hand. Because I didn't have a notepad, it was a bit difficult to remember the numbers. [...] So first the left part and then the same on the right side, and yes, afterwards I looked at both together.*" Her strategy revolved around identifying expressions in the form $A(B + C)$, which led her to follow the standard strategy: distributing, simplifying expressions, and isolating variables sequentially on each side of the equation sequentially before comparing them.

Indira's gaze path (see Figure 2) comprised 150 fixations, ranging from 50 milliseconds to over 2 seconds, totaling approximately 63 seconds. Her visual attention unfolded in four phases: initial focus on the left (~15 s), followed by the right (~14 s), returning to the left (~12 s), and a final revisit to the right (~22 s). Throughout this process, she exhibited minimal gaze alternation between the two sides, with only three transitions across the entire 63-second period. Indira consistently fixated on both operands and operators for extended durations, a pattern observed across all four phases.

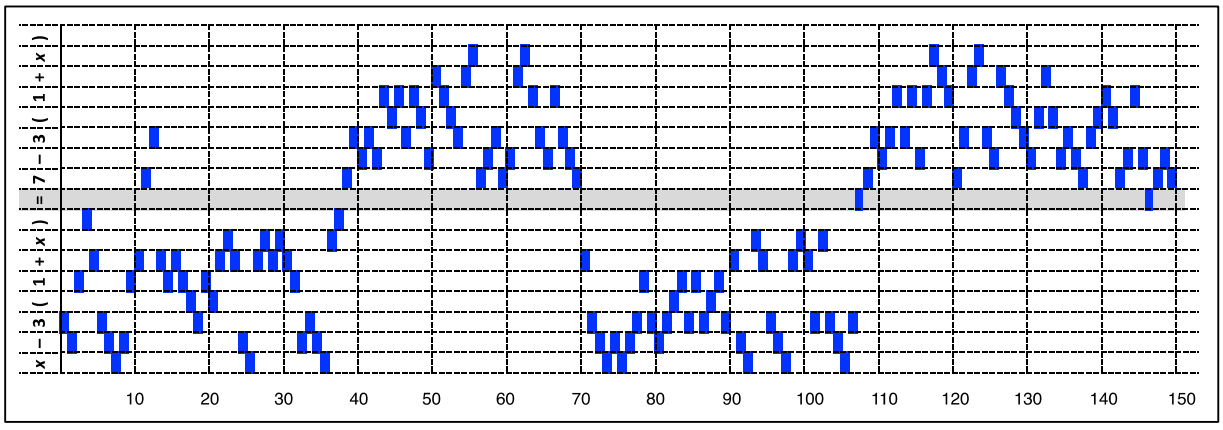

Figure 2: Fixation Sequence Diagram of Indira (*x*-axis: fixation order)

These observations align with our second hypothesis: *routine experts* primarily engage in *local relating*, linking subexpressions mainly within the same side of the equation. They tend to process each side of the equation in isolation, conceptualising the equals sign as a cognitive boundary. This

approach imposes a substantial cognitive load, particularly when mental manipulations are required. Consequently, routine experts must sustain a high level of concentration, investing significant time and cognitive resources to solve the problem. While their strategical approach can handle equations, it also heightens the risk of errors due to prolonged cognitive strain and accumulating inaccuracies.

Discussion and Outlook

The contrasting gaze patterns between Claudia and Indira illustrate distinct cognitive strategies in equation solving. Adaptive experts like Claudia demonstrate *global relating*, rapidly integrating information across the equation through hierarchical processing. In contrast, routine experts such as Indira engage in *local relating*, sequentially focusing on isolated segments with greater cognitive effort, resulting in longer solution times and more fixations. This segmented approach, combined with increased working memory demands, may elevate the risk of errors under cognitive strain.

These findings highlight the importance of structuring in equation solving—an often overlooked aspect of algebra instruction. As Malle (1993) observes, "Many teachers are not even aware that the manipulation of algebraic expressions relies on the recognition of structure [i.e., structuring, C.W.]. As a result, they often focus on 'endless' manipulation practice without addressing this foundational step" (p. 254). Without targeted attention to structuring processes, instruction risks defaulting to procedural practice, with limited development of conceptual understanding.

Furthermore, fixation sequence diagrams offer a promising analytic lens. By visualizing how students structure equations, these diagrams can reveal whether learners adopt globally integrative or locally fragmented strategies. This distinction not only echoes recent work by Schreiter and Vogel (2024) but also aligns with broader expertise literature: adaptive experts draw on flexible pattern recognition, while routine experts tend to follow procedural, step-by-step routines. Our preliminary findings suggest that gaze patterns may serve as predictive indicators of students' solution strategies. As Solstad et al. (2025) note, "eye-tracking can contribute to inferences about mathematical thought processes" (p. 1), though they caution that such interpretations require contextual support and awareness of the limitations of the eye-mind hypothesis. Nonetheless, the integration of eye-tracking into algebra education holds promise for diagnosing and fostering more effective problem-solving strategies.

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Predictive validity of the algebra concept inventory

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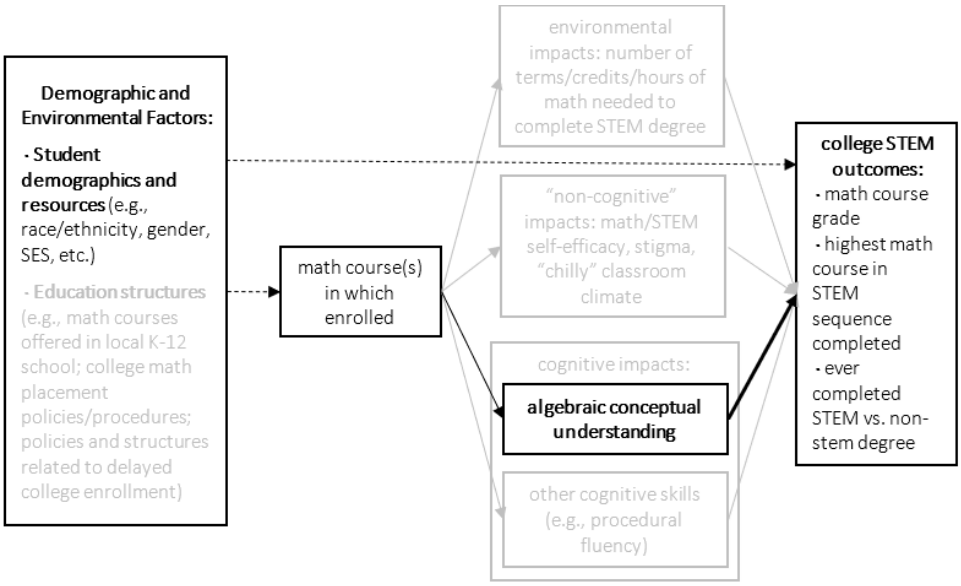
We explore whether outcomes in tertiary mathematics courses and STEM (science, technology, engineering and mathematics) degrees can be predicted by scores on the previously-validated Algebra Concept Inventory (ACI). We also explore whether ACI scores explain gaps in outcomes by race/ethnicity and gender. This research explores a dataset of 6,582 students enrolled at the City University of New York, a diverse, urban, public university in the U.S. Results suggest that ACI scores can significantly predict tertiary math and STEM outcomes, above and beyond math course grades.

Keywords: Algebra, conceptual understanding, tertiary students, STEM majors, equity.

Various research has explored which factors predict student outcomes in tertiary mathematics courses as well as success in pursuing and completed tertiary STEM (science, technology, engineering and mathematics) degrees. Existing literature has explored how various factors predict these outcomes, including: K-12 mathematics preparation (e.g., Cohen & Kelly, 2020) post-secondary school performance (e.g., Chen & Soldner, 2013), or more diffuse assessments of general knowledge, such as GPA (grade point average) or standardized test scores (e.g., Gipson, 2016). Algebra in particular has been recognized as both a barrier to degree completion (e.g., Bailey et al., 2010) and a requirement for earning a STEM degree (e.g., Sithole et al., 2017); however, existing research tends to analyze correlations between grades or levels of algebra completed in secondary school and tertiary outcomes, rather than measuring algebra knowledge directly as a predictor of success in math and STEM in the tertiary context. In particular, no existing larger-scale studies have explored the relationship between higher-level reasoning skills such as algebraic conceptual understanding, and successful math and STEM outcomes in the tertiary context. This research attempts to address this gap.

While conceptual understanding is crucial in mathematics (Richland et al., 2012), it is frequently neglected in algebra instruction (e.g., Hodara, 2011). It is especially important for STEM students, who should be able to flexibly apply their algebraic knowledge in unfamiliar situations (e.g., Quarles & Davis, 2017). Models of tertiary STEM degree attainment have explored various predictors, including environmental impacts from course sequences (e.g., Ryu et al., 2022), placement policies (e.g., Ngo et al., 2018), and general cognitive skills (e.g., Chen & Soldner, 2013). However, there is limited exploration of specific measures of conceptual understanding predicting STEM outcomes, and models that include completion of specific math courses as predictors often obscure the separate impact of different correlated factors related to the course in which a student enrolls (e.g., Quarles & Davis, 2017). Here we adapt and expand existing models by including algebraic conceptual understanding as a core factor (Figure 1). We then use scores on the validated Algebra Concept Inventory (ACI; Wladis et al., 2024b) to explore the extent to which ACI scores can predict students' grades in current mathematics courses; whether they ever completed certain core STEM major mathematics courses; or whether they obtained a STEM vs. non-STEM degree. We also explore the

extent to which ACI score can explain differences in these outcomes by race/ethnicity and gender.



Notes: Grey arrows represent relationships that have been tested in other research. Thick black arrow at far right represents main relationship tested in this research. Thin black arrow represents fixed effects by course (controlled for in all analyses). Dotted black arrows represent control variables included in analyses that use controls.

Figure 1: STEM major access and retention model

Further, there are racial/ethnic inequities in access to high quality K-12 mathematics instruction (e.g., NCES, 2024; Schoenfeld, 2022), which may impact students' opportunities to develop conceptual understanding before tertiary entry. Racial/ethnic minorities are more likely to leave STEM degrees, both because of inequitable access to K-12 mathematics instruction and because of hostile climate in STEM fields (e.g., Riegle-Crumb, et al., 2019). Historically marginalized students often encounter mathematics instruction that provides fewer opportunities to develop higher-level reasoning skills such as conceptual understanding (e.g., Schoenfeld, 2022) and are also more likely to be placed into lower-level mathematics courses in college (e.g., Chen, 2016) where instruction tends to focus more on rote procedures without sensemaking (e.g., Hodara, 2011). This may negatively impact STEM degree progression and completion for traditionally marginalized groups, and thus partially explain inequities in STEM degree outcomes. Yet, this relationship has not been explored in prior research.

Algebraic conceptual understanding

The ACI defines algebraic conceptual understanding as understanding of mathematical concepts, which includes mathematical reasoning and provides the foundation for algebraic thinking. Algebraic thinking is the human activity of recognizing, expressing, and reasoning about generalizations, which serves as the foundation for algebra (Hodgen et al., 2018). Algebraic thinking includes creating expressions and equations to generalize patterns or relationships, manipulating symbols to transform expressions, and using algebra as a tool for analyzing structures or relationships. These activities enable students to engage deeply with mathematical concepts and apply them in real-world and abstract situations. For the ACI, concepts are defined as well-defined mathematical processes, objects, or ideas accepted by the mathematical community (Sfard, 1991). The ACI emphasizes concepts accessible to students in a first-year secondary algebra course (or its equivalent) such as

equivalence, syntactic structure, algebraic properties, functions/variation, and variable. Conceptual understanding is operationalized on the ACI as follows: an item was conceptualized as testing conceptual understanding if it is necessary to use logical reasoning (grounded in mathematical definitions) to answer correctly, and it is not possible to arrive at a correct response solely by restating memorized facts or implementing a rote sequence of actions that could be memorized and correctly applied irrespective of deeper understanding of the mathematical justification. As an illustration of concepts and conceptual understanding, we present two ACI items (see Figure 2[1]).

Item 1. *In the expression* $-7x^3 + 1$ *, which part is being cubed (or raised to the third power)?*

a. $-7x$
b. $7x$
c. x

Item 2. *When solving an equation that contains an* x*, a student gets the answer* $8 = 8$*. This means that:*

a. This could only happen if the student made a mistake somewhere.
b. Every possible value of x *is a solution to the original equation.*
c. There is no solution to the original equation.
d. The solution to the original equation is $x = 8$*.*

Figure 2: ACI example items

For Item 1, students conceptualizing the normative structure of the syntax of the expression would typically choose C. Students who choose A or B typically extract the structure of the syntax from non-mathematically salient features drawn from instructional experiences and may conceptualize the meaning and structure of algebraic syntax differently than those who link it to normative operational precedence (e.g., Wladis et al., 2025). For Item 2, students who understand both that valid transformations preserve the solution of an equation and that all values of *x* would make the equation $8 = 8$ true typically choose B. Students who choose A, C, and D often have alternative conceptions about what it means to solve an equation or for an equation to have a solution; for example, some students who choose D conceptualize solving an equation as a process of performing allowed procedures to get a numerical value as a final result, in a way that is unlinked from the standard definition of a solution of an equation (Wladis et al., 2024c). Both of these items are conceptualized as measuring conceptual understanding because neither can be answered solely by restating memorized facts or performing a rote procedure without understanding its justification.

The ACI is the first large-scale assessment to undergo mixed-methods validation for measuring algebraic conceptual understanding in tertiary students, particularly for all algebra course levels (Wladis, et al, 2024b). Here we use previously-collected ACI scores to explore the relationship between algebraic conceptual understanding and tertiary outcomes such as mathematics course grade, completion of STEM-major mathematics courses, and STEM vs. non-STEM degree completion.

Method

The dataset used for this study included students in mathematics courses at the largest community college at the City University of New York (CUNY) between spring 2019 and summer 2023. This included a wide range of courses, from elementary algebra (focused on objects similar to a first-year secondary algebra course) to more advanced courses such as calculus, linear algebra, and differential equations. We combined institutional records with an existing ACI dataset to explore relationships

[1] For the purposes of test security, numbers and answer option order have been altered.

between students' algebraic conceptual understanding and academic outcomes. This included 6,582 students. The independent variable was a student's score on the ACI that came from various test forms, each of which contain a subset of approximately 25 of the 402 ACI items. The ACI forms used common-item random groups equating design, and analysis has demonstrated excellent reliability and validity for the inventory (Wladis et al., 2024b). ACI scores represent how many standard deviations a student's performance was above or below the mean compared to others in the sample. Dependent variables included: mathematics course grade (0-4.0 scale), successful course completion (grade of C or better), and STEM or non-STEM degree attainment during the study period. "Base" models did not include controls; "full" models used control variables: GPA, prior credits earned, race/ethnicity, gender, age, median household income of zip code, and instructor. Missing data were minimal, affecting less than 2% of the dataset, so we used listwise deletion. Fixed effects were added to relevant models to account for specific course taken, course level/sequence, and instructor.

We used fixed-effects linear/logistic regression models to predict course outcomes and STEM degree attainment. Here we report linear models for easier interpretation. We used the KHB decomposition method (Kohler et al., 2011) for mediation analysis to decompose disparities in ACI scores into "indirect" and "direct" components related to race/ethnicity and gender. Finally, we used entropy balancing (Hainmueller, 2012) to address potential differences between the full population and students with ACI scores; this was used to reweight data and improve covariate balance so that the subset of students with ACI scores matched the larger population on all measured variables.

Results and discussion

Math course and STEM degree completion

ACI scores significantly predicted whether students who attempted, successfully completed, a core STEM major mathematics requirement[2] (Table 1). These differences were not due to correlations between ACI score and current course level because all models included fixed effects for course level (i.e., only compared students enrolled in the same course). Students were 16 percentage points more likely to complete elementary algebra for every 1 SD in ACI score; this was 14, 13, 15, and 11 percentage points for intermediate algebra, precalculus, Calculus I, and Calculus II, respectively.

Table 1: ACI score as predictor of successful completion of STEM major mathematics courses

	Base Models				Full Models			
Ever completed:	**Coeff. (ACI score)**	**SE**	***p***	**Sig.**	**Coeff. (ACI score)**	**SE**	***p***	**Sig.**
Elem. Algebra	0.16	0.04	0.000	***	0.15	0.05	0.001	**
Int. Algebra	0.14	0.02	0.000	***	0.11	0.02	0.000	***
Precalculus	0.13	0.02	0.000	***	0.09	0.02	0.000	***
Calculus I	0.15	0.03	0.000	***	0.09	0.02	0.000	***
Calculus II	0.11	0.02	0.000	***	0.08	0.03	0.008	**
Notes: Linear Probability Multi-Level Weighted Regression: fixed-effects by sequence number of current course, random effects by instructor; *Significance levels:* $*** \alpha = 0.001, ** \alpha = 0.01, * \alpha = 0.05$, $. \alpha = 0.10$								

[2] In the US, as in many parts of Europe, STEM majors are often presumed to begin their studies in Calculus III; however, in reality, many US STEM majors enter college requiring coursework in one or more pre-requisite courses before they can enroll in this course (e.g., Cohen & Kelly, 2020), particularly lower-SES and marginalized students.

ACI scores also significantly predicted whether students who completed a two- or four-year degree during the study period earned a STEM vs. a non-STEM degree. For students in the same level mathematics course when taking the ACI, those with higher ACI scores were significantly more likely to graduate with a STEM degree. ACI score predicted STEM degree completion even more strongly than course grade alone. Students with a 1 SD higher ACI score were 5.1 percentage points more likely to earn a STEM vs. non-STEM degree; accounting for both ACI score and math course grade simultaneously, a 1 SD increase in ACI score corresponded to a 4.0 higher percentage point chance of earning a STEM vs. non-STEM degree for students with the same grades, whereas an increase in one course grade corresponded to only a 1.8 percentage point increase. Thus, ACI scores have predictive power independent of grades, with greater predictive power than math course grades.

ACI score as explaining differential outcomes by race/ethnicity and gender

We also considered whether ACI scores could explain differences in course grades and STEM vs. non-STEM degree completion by race/ethnicity and gender. Critically, we found direct and indirect discrepancies related to course grades highly significant in both base and full models with controls for race/ethnicity (Table 2). In Table 2 we see that ACI score accounts for 13-17% of the difference in course grades by race/ethnicity. For example, URM[3]'s math course grades are on average 0.65 points lower (scale: 1-4), and 0.11 of these points (17%) can be attributed to differences in algebraic conceptual understanding as measured by the ACI (in models without controls). For gender, the pattern is different: while both direct and indirect discrepancies are highly significant in base and full models, they have opposite signs, because women on average earn higher grades in mathematics courses despite having lower ACI scores (a suppressor "effect"). This may be because women on average spend more time on their studies than comparable men (e.g., Wladis et al., 2024a).

Table 2: ACI score as mediator between URM status or gender and mathematics course grade

	Base Models				**Full Models**			
URM status	**Coeff.**	**SE**	***p***	**Sig.**	**Coeff.**	**SE**	***p***	**Sig.**
Total discrepancy	-0.65	0.05	0.000	***	-0.42	0.06	0.001	***
Direct discrepancy	-0.54	0.05	0.000	***	-0.36	0.06	0.000	***
Indirect discrepancy	-0.11	0.01	0.000	***	-0.06	0.01	0.000	***
Female gender	**Coeff.**	**SE**	***p***	**Sig.**	**Coeff.**	**SE**	***p***	**Sig.**
Total discrepancy	0.24	0.04	0.000	***	0.23	0.05	0.000	***
Direct discrepancy	0.29	0.04	0.000	***	0.26	0.05	0.000	***
Indirect discrepancy	-0.04	0.01	0.000	***	-0.03	0.01	0.001	**
Notes: Mediation of Relationship Between URM/Gender and Mathematics Course Grade by ACI Score, fixed-effects by course sequence number, clustering by instructor, using KHB Method *Significance levels:* $*** \alpha = 0.001, ** \alpha = 0.01, * \alpha = 0.05, . \alpha = 0.10$								

ACI scores also explained differences in STEM vs. non-STEM degree completion by race/ethnicity (Table 3). Neither the total nor direct discrepancy is significant in base or full models, but the indirect discrepancy is significant, so that after accounting for the indirect relationship of ACI score as an

[3] In tables, we use URM (underrepresented minority) to indicate students of color who have been traditionally underrepresented in STEM fields; this includes Black, Hispanic, and American Indian/Native Alaskan students.

explanatory variable for differences in STEM vs. non-STEM major attainment by race/ethnicity, URM students are ***more*** likely to complete a STEM vs. non-STEM degree (although not statistically significantly so). Thus, ACI score explains 100% of the total discrepancy in STEM vs. non-STEM degree attainment by race/ethnicity. Exploring mediation models for gender shows the total, direct, and indirect discrepancies are all statistically significant, with ACI score explaining about 10% of differences in STEM vs. non-STEM degree attainment by gender.

Table 3: ACI score as mediator between URM status or gender and degree completion

	Base Models				Full Models			
URM status	**Coeff.**	**SE**	***p***	**Sig.**	**Coeff.**	**SE**	***p***	**Sig.**
Total discrepancy	-0.02	0.02	0.442		-0.00	0.03	0.928	
Direct discrepancy	0.01	0.02	0.812		0.01	0.02	0.806	
Indirect discrepancy	-0.02	0.01	0.000	***	-0.01	0.00	0.010	*
Female gender	**Coeff.**	**SE**	***p***	**Sig.**	**Coeff.**	**SE**	***p***	**Sig.**
Total discrepancy	-0.11	0.03	0.000	***	-0.08	0.03	0.002	**
Direct discrepancy	-0.10	0.03	0.000	***	-0.07	0.03	0.005	**
Indirect discrepancy	-0.01	0.00	0.008	**	-0.01	0.00	0.008	**
Notes: Mediation of Relationship Between URM/Gender and STEM vs. non-STEM Degree by ACI Score, fixed-effects by course sequence number, clustering by instructor, Linear Probability Model using KHB Method; Significance levels: $*** \alpha = 0.001, ** \alpha = 0.01, * \alpha = 0.05, . \alpha = 0.10$								

Discussion and conclusion

ACI scores predicted successful completion of core math courses required for STEM majors. ACI scores not only predicted the likelihood that a student would successfully complete tertiary algebra classes if they enrolled in them, but also whether students would complete courses such as Calculus I and II, where algebra is not the main focus of instruction. This suggests that algebraic conceptual understanding may be critical across tertiary mathematical domains, and not just in algebra courses.

ACI scores also predicted whether students earned STEM vs. non-STEM degrees, independent of math course grades. This is particularly striking, since ACI scores were generated from a 25-item test given on a single day and measures general algebraic conceptual knowledge, whereas course grades are based on a term's worth of assessment specific to the particular course. It may be that improving instruction in algebraic conceptual understanding is a critical component of preparing students for STEM degrees and careers—this is a critical area for future research.

ACI scores also explained differences in math grades and STEM vs. non-STEM degree completion by race/ethnicity. ACI score accounted for 13-17% of racial/ethnic math grade differences and 100% of the STEM vs. non-STEM degree completion gap, suggesting that differences in opportunities to develop algebraic conceptual understanding explain a significant proportion tertiary outcome differences by race/ethnicity. Thus, improving access to substantial opportunities to develop algebraic conceptual understanding in mathematics courses may be particularly critical to addressing racial/ethnic inequities in tertiary outcomes—this is a critical area for future research.

Women on average earned higher grades in math courses despite having lower ACI scores. This may be because women on average spend more time on their studies than comparable men (e.g., Wladis

et al., 2024a), and therefore may earn higher grades because of the added time spent studying. More research is needed to better understand this trend. At the same time, differences in ACI scores explained 10% of the gender gap in STEM vs. non-STEM degree completion. Thus, it may be important to ensure that women receive equal opportunities to develop algebraic conceptual understanding in mathematics courses. This may be related to marginalized experiences that women and girls have in mathematics classes (e.g., Reinholz, 2022), which may negatively impact their opportunities to develop conceptual understanding. Future research that investigates the relationship between equitable learning opportunities in the classroom and subsequent development or improvement or conceptual understanding could be an important area for future research.

Prior studies have identified various factors that impact STEM major enrollment and completion, particularly for marginalized groups, including: K-12 mathematics preparation (e.g., Cohen & Kelly, 2020), postsecondary and tertiary grades (e.g., Chen & Soldner, 2013), standardized test scores (e.g., Gipson, 2016), and experiences in tertiary STEM classes (e.g., Riegle-Crumb, et al., 2019). This study provides evidence of an additional factor that has not been directly measured as a predictor of STEM outcomes in prior studies: algebraic conceptual understanding as measured by the ACI. The ACI measures just one conceptualization of algebraic conceptual understanding; there are likely other conceptualizations, as well as other types of mathematical skills and reasoning that are not measured by the ACI, but which could be measured through other instruments or methods. This study illustrates the potential of exploring the relationship between different types of mathematical skills, reasoning, and knowledge and tertiary STEM outcomes, which could be expanded and built on in future studies.

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Comparison of structural sense in limit tasks between first and final year engineering students

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Keywords: Algebraic sense, structural sense, algebraic errors, algebraic manipulation, higher education.

Introduction

Differential and integral calculus presents difficulties for learners, especially in regard to tasks such as limits, which require factoring skills as well as rationalization and simplification of algebraic fractions. These problems arise due to errors or lack of maturity in algebraic management, which leads to the need to develop or reinforce the structural sense, understood as the ability to perceive the structures of algebraic objects (Rojano, 2022).

Theoretical framework

Authors such as Hoch and Dreyfus (2006) identify three skills for structural sense: recognising basic structures, treating complex terms as single entities, and choosing appropriate manipulations to best utilise algebraic structures. Vega-Castro (2013) extends these skills by adding the ability to distinguish substructures, anticipate the effects of transformations, and extract relevant information from an algebraic expression. Muñoz-Porras (2015) introduces the auxiliary variable technique, a useful tool for recognizing and manipulating structures in algebraic expressions. Descriptors as indicated by these authors have yielded the selection of eight key components: (C1) recognize a familiar structure in its simplest form; (C2) identify substructures within an expression and treat them as a single entity when transforming the expression; (C3) deal with a compound term as a single entity by recognizing a familiar structure in a more complex form; (C4) choose appropriate manipulations to optimally make use of a structure; (C5) distinguish substructures within an entity and recognize relationships between them; (C6) apply substitution, either internally or explicitly; (C7) recognize equality relations in structure of present expressions; y (C8) extract relevant information from the external structure of an expression, relative to the object it represents or to its internal structure. A study by Ruiz-Hidalgo and Fernández-Plaza (2013) on limit solving using algebraic fractions concluded that structural sense is related to analytical complexity and the probability of making errors. In this way, Rojano (2022) highlights the usefulness of applying structural sense in calculus tasks. Therefore, the aim of this study is to compare the structural sense between first and final-year engineering students in limit tasks, a relevant dimension that is absent in previous studies.

Methodology

For this purpose, a content analysis Krippendorff (2019) was carried out with a sample of 16 final-year students and 15 first-year students from a Chilean Higher Technical-Professional Education (TP) institution. The questionnaire used considers 10 items in terms of four parts: A) factorization, B) completion of equivalent expressions, C) equations and D) limits of a function at a point. We analyzed

part D, which includes four items that refer to an algebraic fraction ($\lim_{x\to 2}\frac{x^3-8}{x^2-5x+6}$), rationalization ($\lim_{x\to 3}\frac{x-3}{\sqrt{x+1}-2}$), a trigonometric function ($\lim_{x\to 0}\frac{2sin(3x)}{5x}$) and a contextualized situation (The average revenue for article x in a company is $I=\frac{x}{200x-0,2x^2}$, what will happen if the number of articles is reduced to zero?)

Results

The items most answered by both groups of students were on algebraic fractions and rationalization. In algebraic fractions, both groups recognize numerator and denominator in its simple form (C1), but when simplifying (C5), the senior group had better results. In rationalization, both identify the complex structure (C3), but when simplifying (C5), the first-year group achieved a better result by obtaining an equivalent expression (C7). In the trigonometric limit items and contextualized situation, there was minimal participation, though the senior group responded more.

Conclusions

When comparing both groups, a greater development of structural sense was expected in the senior students, but the results show that, although they answered more questions, this does not necessarily reflect a better understanding of the structure. In mathematics education it appears necessary to enhance the application of structural sense in other contexts than algebra, to be successful in areas such as calculus and modelling, key skills for engineers and researchers.

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Strategies used by students with autism to complete repeating patterns

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Keywords: Repeating patterns, algebraic thinking, autistic students, early algebra.

In kindergarten, children start working with patterns. Patterns are defined as any replicable regularity (Papic et al., 2011). One type of pattern that children engage with is repeating patterns, characterized by a cyclic structure created by the repetition of a smaller segment, known as the unit of repeat (Papic et al., 2011). Generalization occurs as soon as the child recognizes that the structure of the unit of repeat remains the same across the pattern. This kind of generalization appears to provide the basis for structural awareness and, consequently, the ground for the development of early algebraic thinking (Papic et al., 2011). This research study focuses on the strategies that students with autism (ASD) use when engaged with repeating pattern tasks. The identification of ASD students' strengths and challenges in this area is considered essential for better understanding how neurodiverse learners process patterns, and for supporting educators' efforts to develop tailored instructional methods that enhance inclusion and improve learning outcomes. Previous studies on early algebraic thinking of ASD students have shown a tendency towards recursive strategies and difficulties with generalization (Goñi-Cervera et al., 2022), which could possibly be related to certain characteristics of the disorder, such as the concrete thinking style and difficulties with abstraction. However, the studies addressing this student profile are scarce, and those that do exist mainly examined strategies used to solve growing pattern tasks within a functional context (e.g., Goñi-Cervera et al., 2022). The current study focuses on ASD students at the beginning of their mathematical learning, aiming to understand their thinking about fundamental mathematical concepts and procedures, as well as the strategies that they employ when engaged with basic maths tasks, such as repeating patterns.

The theoretical framework is based on previous studies with typically developing students, which provides a foundation for understanding how different strategies affect the identification of near and far terms in sequences (Papic et al., 2011). The participants of the present study were 3 Spanish autistic children, aged 6 years old, who were enrolled in different inclusive schools. Data were collected through a questionnaire that included four tasks involving repeating patterns with two different attributes: colour and shape. Tasks 1 and 3 involved an AB pattern, task 2 involved an AAB pattern, and task 4 involved an ABB pattern. Each task included three similar questions related to identifying terms in the sequence. The first question (Q1) asked the participants to find near terms by completing the gaps at the end of the pattern in tasks 1 and 2, and at the beginning of the pattern in tasks 3 and 4. The subsequent two questions addressed far terms for each task. For example, the questions for task 1 were: (Q2) "If the pattern continues, what figure will be in position 14? Explain how you found your answer", and (Q3) "If the pattern continues, what figure will be in position 21? Explain how you reached your answer". To analyse the responses, we adapted the classification from Papic et al. (2011), taking into account strategies used in tasks similar to those presented in this work. Specifically, we considered the following categories: *Random arrangement* (*RAS*): When students

place items in a random fashion; *Alternation (AS):* When students focus on successive items regardless of the complexity of the unit of repeat; *Basic unit of repeat (URS):* When students identify the unit of repeat and use it to complete the pattern. A category named *No evidence (N)* was used for providing no explanation. The following table shows the strategies applied by the three participants in the different questions and tasks. Answers marked with an '*' indicate incorrect responses.

Table 1: Strategies used by the participants in questions 1, 2, and 3

	Task 1 (AB _)			Task 2 (AAB _)			Task 3 (_ AB)			Task 4 (_ ABB)		
	Q1	Q2	Q3	Q1	Q2	Q3	Q1	Q2	Q3	Q1	Q2	Q3
Student 1	AS*	AS*	AS*	AS*	AS*	AS*	AS*	AS*	AS*	AS	AS*	AS*
Student 2	AS	AS*	AS*	AS	AS*	AS*	AS*	AS*	AS*	AS*	URS*	URS
Student 3	AS	AS*	AS*	AS	URS	AS*	AS	URS	N	AS*	AS*	AS*

As shown in Table 1, the AS occurs more frequently than strategies which involve the unit of repeat. Strategies of alternation only resulted in correct answers for finding near terms in the sequences. In contrast, URS led to correctly identifying far terms in three cases. Errors with this strategy were often due to participants losing count while repeating the basic unit. This finding aligns with studies involving TD students engaged with similar tasks (Papic et al., 2011). Although the small sample size limits generalization, our results highlight a type of visual approach, a common strength among ASD individuals and has previously been evidenced in problem-solving and algebraic thinking, often through strategies based on drawings (Polo-Blanco et al., 2024; Goñi-Cervera et al., 2022).

The findings highlight the predominant use of strategies of alternation by ASD students when engaging with repeating pattern tasks, emphasizing their effectiveness for near-term identification but limitations for far-term accuracy. By recognizing these diverse strengths and challenges of ASD students, instruction can be modified to involve differentiation, such as the support of their visual thinking to move from focusing on successive terms to grasping the structure of the pattern.

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TWG 04: Geometry Teaching and Learning

Geometry Teaching and Learning at CERME14: TWG4 Synthesis Report

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The Thematic Working Group 4 (TWG4) at CERME14 brought together 30 researchers from 12 countries who presented 20 papers and 2 posters, reflecting an acceptance rate of 75%. These contributions spanned various educational levels and showcased a wide range of methodological and theoretical approaches. The work was organised across five themed sessions—General Perspectives and Conceptual Approaches, Curriculum and Task Design, Use of Tools (Physical and Digital), Teachers and Teacher Education, and Advanced Topics and Young Learners, as well as two sessions of group discussions. These sessions enabled focused discussions and also included time for synthesis and collaborative reflection, reinforcing the interdisciplinary and integrative spirit of the group.

Diversity in Research and Mathematical Focus

The TWG4 papers showcased a broad and evolving spectrum of research interests. Examples include argumentation and reasoning, teacher geometry knowledge, analysis and assessment of students' spatial abilities and geometric knowledge, and the use of 3D printing and computer modelling. The mathematical content explored in these contributions also demonstrated considerable diversity. Examples included learning and teaching projective geometry, properties of solids, symmetry, classification of quadrilaterals, and trigonometry, illustrating the wide spectrum of geometrical themes. Applications were distributed across educational levels and often intersected with technological tools and embodied practices, underscoring TWG4's distinctive focus on integrating cognitive, representational, and instructional dimensions of geometry learning. Researchers such as Antunović examined students' transitions between figural and conceptual reasoning using Duval's framework and echoing insights from Fischbein's theory of figural concepts (Fischbein, 1993), while Büttner and Viola explored conceptual breaks in trigonometric contexts, highlighting the interpretive challenges when switching between multiple representations.

This thematic richness reflects the group's commitment to linking foundational geometric knowledge with contemporary educational demands, especially those brought on by technological and epistemological developments.

Tools, Representations, and Their Integration in Learning Environments

The integration of tools—both physical and digital—was a significant focus in TWG4. Rather than treating them as external enhancements, many contributions demonstrated how tools shape the structure, representation, and semiotic richness of geometry learning. Surynková examined the

affordances of 3D printed models in enhancing spatial visualisation and projections. Brunheira analised the contributions to students' spatial reasoning and knowledge about rotation and its relation to the design of the interdisciplinary tasks between visual arts and geometry. Fujita et al. analysed how multiple theoretical perspectives can guide the intentional integration of both physical and digital tools to support reasoning. Okumus and Hollebrands analyzed how teachers use dynamic geometry software (DGS) to focus students' attention on geometric properties in 3D contexts, contrasting the affordances of static drawings and dynamic constructions.

Del Zozzo illustrated how dynamic tools like GeoGebra, when paired with historical texts, can mediate perceptual and semiotic shifts in learners' understanding. Huang and Fujita introduced virtual reality to extend spatial experience in 3D geometry tasks, while Sauerwein introduced a non-digital yet underutilised construction tool that enabled students to explore geometric loci and develop mathematical language in alignment with Van Hiele levels. Together, these contributions reinforce the view that in geometry, the medium itself—whether physical or virtual—modulates what and how learners see, construct, and argue. In line with Duval (2017), these tools engage different modalities of apprehension and require coordinated interpretation across representations. This shift from using tools to designing with them underscores TWG4's evolving vision of geometry pedagogy. The embodied, representational, and semiotic potentials of tools also set the stage for a broader examination of geometry as a multimodal practice—an idea taken up in the following discussions on embodiment and multimodality.

Geometry Through Embodiment and Multimodality

The exploration of embodied cognition took a prominent place in this year's discussions. Palatnik et al. studied a collaborative task involving a life-sized icosahedron, capturing how gestures and bodily orientation were instrumental in constructing geometric arguments. Their use of eye-tracking and multimodal transcription highlighted the emergence of embodied argumentation and its potential to redefine how we understand proof in educational contexts.

Obayashi and Fujita also contributed significantly to this strand by analysing primary school students' reasoning processes during symmetry tasks. Their analysis, grounded in Duval's framework of figural apprehension, illustrated how students progressed from visual intuition to discursive justification, often through teacher mediation.

These and several other contributions—including those by Del Zozzo, Surynková, and the posters by Maschietto and Battilani and Alstad and Berre—suggest exploring how argumentation in geometry is no longer confined to linguistic expression or formal deduction but extends into the perceptual, physical, and gestural domains.

Task Design and Curricular Articulation

The interplay between task structure, medium, and learning outcome was a recurring concern. Multiple studies investigated how tasks designed around digital tools (e.g., dynamic geometry software) or physical artefacts (e.g., printed manipulatives) could support different cognitive pathways. Bímová et al. explored function plotting using both tangible and digital representations, demonstrating how shifting between tools affects students' interpretative strategies. İmamoğlu et al.

examined how DGE-based problem-solving activities can influence students' mathematical performance and thinking strategies, highlighting the importance of adaptive task design. Vízek and Samková developed a framework for analyzing students' conceptual knowledge of quadrilaterals based on dynamic constructions, capturing distinctions among classification, structural relations, and principled reasoning.

Haj-Yahya et al., in their work with prospective teachers, examined how mathematical events could foster an understanding of inclusion relationships among quadrilaterals. They showed that well-designed interventions helped students move beyond prototype-based reasoning to a more definition-oriented understanding. Similarly, Tortorelli and Tortoriello addressed conceptual gaps between learners' figural impressions and formal definitions, highlighting the utility of structured card-based tasks to prompt reflective dialogue without oversimplifying geometric definitions or reducing them to fixed prototypes. In contrast to these classroom-oriented interventions, Miyakawa and Su conducted a comparative curriculum analysis of junior high school geometry in Japan and Taiwan. Using the Anthropological Theory of the Didactic, they revealed how institutional contexts shape both the organisation of mathematical content and the nature of reasoning in varying national settings.

These studies collectively point to a growing sophistication in task design, with careful attention paid to the sequencing of tasks, selection of representational media, and the scaffolding needed to bridge intuitive and formal knowledge. This multi-dimensionality of task design was further captured in a diagram discussed in the group's final session. The diagram highlighted five interconnected aspects: the structure of the task (from activity to objective), the medium used (digital or physical), the source of the task (ranging from historical texts to everyday contexts), necessary modifications (to accommodate learners' needs), and the overall conceptual aim. It reinforced the idea that in geometry, tasks are shaped as much by their design and origins as by the environments and representational tools through which they are enacted.

Insights into Teacher Education

Contributions focusing on teacher education revealed both challenges and opportunities. Pirklová et al. provided a diagnostic framework for categorising pre-service teachers' errors and understanding where instructional gaps occur. Kovalčíková and Semanišinová analyzed teacher responses to geometric problem-solving, underscoring the difficulty of bridging theoretical knowledge with pedagogical practice.

Several papers, such as those by Halverscheid and Harland, focused on how teacher trainers initiate argumentation and proof processes in geometry instruction. Their qualitative study highlighted how subjective theories influence the choice of proving strategies, particularly in the context of classical geometric theorems. Other studies emphasised that strengthening geometry teacher education requires a blend of deep content knowledge, didactical sensibility, and familiarity with both traditional and novel tools. Danişman and Arabacı explored the challenges preservice teachers face in constructing geometric arguments and aligning their teaching practices with curricular goals, offering insight into the need for more balanced teacher education programmes across countries. Notably, embodied approaches and tool-mediated experiences (e.g., Palatnik et al., Surynková) were seen as potential enhancers of preservice teachers' geometrical insight and adaptability.

Theoretical Approaches and Conceptual Frameworks

Across the group, there was a shared recognition of the importance of theoretical clarity and diversity. Haj-Yahya et al. addressed learners' concept images and definitions through tasks involving both prototypical and non-prototypical shapes. Tortorelli et al. contributed to this theme through a card-based task bridging intuitive and formal understandings of quadrilaterals, based on the idea concept image and definition (Tall & Vinner, 1981). Sauerwein built on the Van Hiele model and Winter's Workshop Language to analyse how students' language and geometric understanding co-develop through construction tasks. Vízek and Samková offered a dimensional approach to conceptual knowledge—covering classification, structure, and principles—assessed through dynamic quadrilateral constructions. Del Zozzo applied the Theory of Objectification both globally and locally, emphasising sensuous cognition and semiotic activity. Palatnik et al. extended embodied learning theory to argumentation, highlighting the interpretive role of movement and perception in reasoning (Radford, 2009). These theoretical contributions offered layered insights into how students perceive, construct, and justify geometric ideas, reinforcing a shared orientation toward theory-informed practice that shaped discussions of task design, teacher preparation, and student learning.

Conclusion and Outlook

The 2025 contributions to TWG4 demonstrate a growing coherence in how geometry education research integrates technological tools, theoretical models, and cognitive perspectives. While spatial reasoning, visualisation, and argumentation remain longstanding concerns, their framing has become more nuanced and inclusive of embodied, multimodal, and socially situated approaches.

The group's discussions suggested several future directions: increased exploration of neurocognitive data collection methods (such as eye-tracking), more longitudinal studies on how learners' geometric reasoning develops across media and contexts, and continued theorisation of how physical and digital tool use can converge in meaningful ways within curricula.

As TWG4 evolves, it continues to affirm its role as a forum for theoretically grounded, empirically rich, and pedagogically responsive research on geometry teaching and learning in the 21st century.

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Applying Duval's Teaching Sequence to 7th-Grade Geometry: A Case Study on the Area of a Disc

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This study investigates the application of Duval's teaching sequences within a geometry classroom, with a particular focus on developing 7th-grade students' conceptual understanding of the area of a disc. By engaging students in tasks that bridge perceptual and mathematical comprehension, this research illustrates how Duval's theoretical framework supports cognitive processes in geometry education. Through classroom observations and analysis of student-teacher interactions, the study assesses the impact of Duval's approach on facilitating cognitive apprehension in learning geometric concepts. The findings emphasize the role of structured teaching sequences that guide students in visualizing, deconstructing, and reconstructing geometric figures, alongside verbal articulation, to build a solid foundation for advanced mathematical reasoning.

Keywords: geometry teaching, cognitive apprehension, area of the disc

Introduction and Theoretical Background

Understanding geometry extends beyond simple shape recognition; it requires a cognitive shift from perceptual awareness to formal mathematical interpretation. Duval (2017) observes that students often struggle to transition from initial perceptual impressions to grasping the deeper mathematical properties embedded within visual representations, particularly in tasks involving decomposition and reconfiguration of geometric figures (Duval, 2014).

Teaching geometry, therefore, involves guiding students from basic visual perception to a more sophisticated mathematical interpretation. Effective geometry education supports this cognitive shift through activities that engage students in both physical and mental manipulation of figures. Duval's structured teaching sequences (2017) facilitate this process, focusing on visualization and cognitive processes over measurements and progressing through stages designed to build conceptual understanding of geometry, particularly for complex concepts like the area of a disc[1]. In the first stage, students manipulate physical objects such as 2D and 3D figures by folding, rotating, or overlapping them. This hands-on approach emphasizes exploring geometric properties without relying on calculations or measurements. In the second stage, students analyse 2D or 3D representations to identify figural units such as points and lines and understand their relationships within larger configurations. This stage fosters abstract thinking by enabling students to view complex shapes as assemblies of simpler components. Finally, in the third stage, students articulate geometric relationships using formal language and technical vocabulary, reinforcing their understanding and linking earlier visual and physical observations to mathematical concepts.

[1] The term "disc" means the part of the plane bounded by a circle (Kislev, 2006, p.9).

This structured progression helps students transition from informal exploration to formal mathematical understanding by integrating verbalization with technical language. Central to this approach is the development of various forms of cognitive apprehension, each playing a unique role in understanding geometric concepts (Duval, 1995). Perceptual apprehension (PA) allows students to recognize and identify geometric figures based on visual attributes. Sequential apprehension (SA) involves understanding and describing the step-by-step construction of a geometric figure. Discursive apprehension (DA) encompasses logical deductions and inferences based on given information, theorems, and definitions. Operative apprehension (OA) enables the manipulation or reconfiguration of geometric figures to solve problems or explore proofs.

For a drawing to function as a true geometric figure, it must evoke PA alongside at least one other type, as PA alone is insufficient for mathematical reasoning (Duval, 1995, p. 143). Incorporating activities that engage and integrate these forms of cognitive apprehension enhances students' ability to think geometrically (e.g., Bernabeu et al., 2021). Teachers play a crucial role in designing tasks that promote visualization, decomposition, construction, reconfiguration, and articulation of geometric concepts. These activities provide a strong foundation for understanding formulas, including the formula for the area of a disc.

Significance of the Study

Although many studies have examined Duval's framework of cognitive apprehension (Duval, 1995, 2017) with students across various educational levels (e.g., Michael-Chrysanthou et al., 2024) or focused on pre-service and in-service teachers (e.g., Llinares & Clemente, 2014), there is, to our knowledge, no research on familiarizing teachers with Duval's framework and its integration into the facilitation of geometric concepts in primary education. This gap is particularly significant in the context of developing students' understanding of the area of a disc. As part of a broader collaborative action research project between a teacher and a researcher exploring Duval's framework of cognitive apprehension in classroom settings, this paper focuses on activities aligned with Duval's teaching sequence. These activities, designed by the teacher, aim to guide students toward deriving the formula for the area of a disc. By analyzing these structured classroom activities, the study seeks to provide insights into how teachers can enhance students' geometric understanding using a structured cognitive approach.

Research Question

This study addresses the following research question: How do teachers use Duval's teaching sequence to guide students in connecting figural transformations with the formal mathematical expression of the area of a disc, by coordinating cognitive apprehensions throughout the instructional stages?

Method

Participants

This study is part of a broader doctoral research (Antunović, 2024), designed as collaborative action research focusing on development of geometrical thinking which involved four in-service mathematics teachers and four 7th-grade classes (ages 12–13) from urban schools in County Istria,

Croatia. The experimental group teacher participated in the Development of Geometrical Thinking (DGT) program, which focused on Duval's framework and its application to geometric content, including polygons, circles, and discs. This paper focuses on one lesson designed and conducted by the experimental group teacher with 24 students from the experimental group, integrating Duval's framework to enhance conceptual understanding.

Data Collection and Analysis

As a part of a broader project which employed mixed methods (Ivankova & Wingo, 2018, p. 9) to assess the DGT program's impact on students' geometrical thinking and cognitive apprehension (quantitative data were gathered through pre- and post-tests, while qualitative data derived from teacher questionnaires, interviews, and classroom observations), this paper focuses on qualitative data from classroom observations, analyzed via Duval's framework on teaching sequences and cognitive apprehension. Data sources included researcher notes, transcripts, and observations, with reliability ensured through triangulation using research diaries, student notebooks, photographs, and video recordings.

Thematic coding was applied to identify cognitive processes in line with Duval's four types of cognitive apprehension: PA - recognizing geometric figures or parts of figures; SA - following step-by-step procedures to construct or describe and analyse geometric objects, OA - physically or mentally manipulating geometric objects; DA - articulating geometric relationships and reasoning.

The analysis focused on how these cognitive apprehensions unfolded in the classroom, highlighting the teacher's role in guiding students. Key themes included:

- Transition from Perceptual to Mathematical Understanding: Moving from visual recognition to mathematical interpretation.
- SA and OA: Cognitive processes in deconstructing and reconstructing figures.
- DA and the Role of Language: The shift from non-verbal manipulation to verbal articulation of geometric concepts.

Classroom Activities

Here we present a detailed lesson on the area of a disc, focusing on activities designed by the teacher. To protect the privacy of students' conversations and interactions, fictitious names such as 'Ivona,' 'Lara,' and 'Borna' were used. The term ET refers to the teacher of the experimental group, and ES refers to the students.

Activity 1: From Description to Construction: Creating Inscribed Regular Polygons

Each student brings a disc to class, a 5 cm radius model cut from colored paper. The students work in groups of four.

ET: Choose a regular polygon that you like.

The teacher walks around the class and asks the group representatives about the polygons they chose.

ET: Tell me which regular polygon you chose and what you know about it.

ES: We chose an equilateral triangle. It is a triangle with all sides the same length and all angles the same, each measuring 60°.

ET: And how will you construct your triangle?

ES: I will first construct the central angle.

ET: Great. And then what happens when you construct that angle?

ES: The arms of the angle intersect the circle at two points. I will use a compass to transfer that distance, and each time it will intersect the circle at one point.

ET: And then?
ES: I will connect these points.
ET: So, you will draw the chords, and these chords are the sides of your triangle.

The teacher continues to guide the students through the process of constructing quadrilaterals, pentagons, hexagons, octagons, and dodecagons. Below, we present part of the discussion regarding the octagon and pentagon:

ET: And who chose the octagon? Ana, please tell us something about the octagon.
ES (Ana): It is a regular polygon with eight congruent sides and all interior angles congruent. The measure of the central angle of the characteristic triangle is 360° divided by 8, which is 45°. So, we need to construct a 45° angle and, as we explained before, divide the circle into congruent parts.
ET: Please, Lara, can you continue Ana's explanation?
ES (Lara): Now, we need to connect every two consecutive points of the circle, which represent the sides of our octagon.
ET: And now, we are left with a pentagon. How will you construct your pentagon?
ES: First, we determine the size of the central angle, which is 360° divided by 5. Then we draw that angle using a protractor and transfer the length of the circular arc along the circle.
ET: Which circular arc?
ES: The arc bounded by the points where the arms of the central angle intersect the circle.
ET: Correct, but we transfer the chord, which represents the side of the polygon.
ET: Let's conclude. Borna, can you describe the construction process of a regular polygon if the radius of the circumscribed circle is given?
ES (Borna): The construction starts with the characteristic triangle of the regular polygon. This means we begin by constructing or drawing the central angle and then transferring the length of the corresponding chord along the circle. Now, we construct the points that are the vertices of the polygon. By drawing the chord that connects two consecutive points, we will finally construct our regular polygon.
ET: Great. So, when the circle is divided into congruent parts, i.e., arcs, the sides of a regular polygon can be considered congruent chords of the same circle. Now, you may begin your constructions.

The students construct the inscribed regular polygons on their discs (using the construction method for regular polygons with a given radius and central angle).

Activity 2: Reconfiguring the Disc to Approximate Different Geometric Shapes

ET: Now, let's move on. Please, look at your disc with the inscribed polygon. Divide your regular polygon into characteristic triangles. What do you need to draw with your regular polygons?
ES: We draw the radius, connecting the centre of the circle to the vertex of the polygon, which also lies on the circle.
ET: Excellent. Now, let's look at your discs. What happened to your disc?
ES: We divided it into parts.
ET: Yes, and what do we call those parts?
ES: Circular sectors.
ET: What is a circular sector?
ES: It's a part of a disc bounded by two radii and a corresponding circular arc.
ET: Can you connect a circular sector and a characteristic triangle? What are their similarities and differences?
ES: Well, both are bounded by two radii and have the same central angle. The difference is that the sector is bounded by a circular arc, while the characteristic triangle is bounded by a line segment, which is a side of the polygon.
ET: Exactly. So, if we divide the inscribed regular polygon into characteristic triangles, the disc will be divided into corresponding circular sectors.

The teacher instructs the students to cut the disc into the marked sectors, like a pie, and then reassemble the sectors by swapping them with their peers, ensuring that the pieces

align along the radius and alternate in colour. After rearranging the sectors, students paste them into their notebooks.

Activity 3: Transitioning from Concrete Models to Imagined Configurations

The teacher calls a representative from each group to the front of the class, where they present their work, arranged according to the type of polygon constructed, starting with the triangle. The class then observes and discusses the results. (see Figure 1.a).

ET: Please take a good look at each of the obtained figures and tell me what you notice. What do these figures remind you of? Which geometric shapes do they resemble?
ES: Lara's figure looks like a parallelogram, as does Ivona's.
ET: And what can you tell me about Sara's figure?
ES: It looks like a trapezium.
ET: Great, and the last two?
ES: The same, they look like parallelograms.
ES: But Borna's looks like a rectangle.
ET: Great. And what do you notice when the number of sectors increases?
ES: The figure starts to look like a trapezium or parallelogram.
ES: Or a rectangle.
ET: Excellent. Now, imagine we construct an inscribed regular polygon with 20 sides, or 35 or more, for example 360 sides, and then reconfigure the disc. Can you visualize this in your mind?
ES: Yes.
ET: What geometric figure does this new figure resemble in your mind?
ES: To me, it looks like a rectangle.
ET: Great, but can you explain why?
ES: Well, the sectors look like isosceles triangles.
ET: Do other students understand Karlo? Please look at Borna's figure. These sectors look like the characteristic triangle of a regular polygon. When we inscribe a regular polygon with 20, 36, or more sides in the circle, the base of the characteristic triangle gets closer and closer to the corresponding circular arc. And what happens with the arms of the characteristic triangle?
ES: They are the same, they are the radii of the disc.
ET: Now we can conclude. If the disc is divided into an infinite number of congruent sectors, we can rearrange these sectors into a rectangle.

Activity 4: Verbalizing and Deriving the Area Formula for the Disc

The teacher sketches a rectangle on the board and divides it into many small triangles.

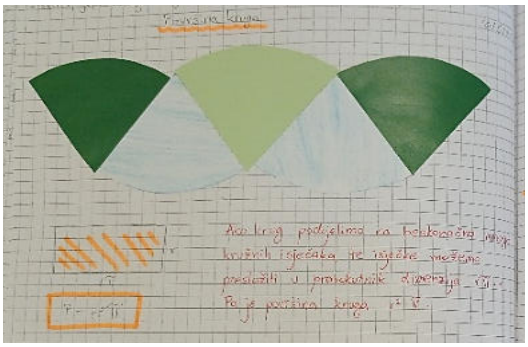
ET: What are the dimensions of this rectangle?
ES: One side is the radius.
ET: Yes, the width is the radius. And what is the length of this rectangle?

The teacher labels one side of the rectangle as r.

ET: Now, let's look at your work. Look at the boundary of each of your figures. Borna's figure is the clearest example.
ES: It's the perimeter of the disc.
ET: Please explain.
ES: If we look at the boundaries, the upper and lower parts are made of circular arcs, which together form the circle.
ET: Great. So, our reconfigured disc is now a new figure bounded by two radii and congruent circular arcs, which together form the circle. What is the length of the upper or lower boundary of Borna's figure?
ES: I think it's half the circumference, half of the perimeter of the disc.
ET: Correct, it's half of the circumference. Now, let's return to our rectangle. Comparing it with Borna's figure, what is the width of our rectangle?
ES: It's the perimeter divided by 2.
ET: And that is...?
ES: It's $r\pi$.

ET: Great.
The teacher is labelling the width of the sketched rectangle rπ.
ET: And finally, now you can tell me what the area of sketch rectangle is.
ES: Area is r multiplying rπ.
ET: Yes. And how do we write r· r which means multiplying a number by itself?
ES: r squared.
ET: Great
The teacher is writing on the board: $P= r^2\pi$.
ET: So, what can you tell me as conclusion?
ES: If the disc is divided into an infinite number of congruent sectors, we can rearrange these sectors into a rectangle with its dimensions r and rπ, so the area of disc is r2π.
ET: Great, that is exactly what this formula is "saying". Please write this in your notebooks (Figure 1.b). Now you can determine the area of your starting disc.

a) Activity 3



b) Students note

Figure 1: Activities of Lesson Area of disc

Findings

This study explores how Duval's teaching sequence helps students connect figural transformations to the formal mathematical expression of the area of a disc. It focuses on how students navigate cognitive apprehensions throughout different instructional stages. The analysis of each activity underscores the teacher's pivotal role in guiding students from perceptual understanding to formal mathematical reasoning.

In this *Activity1*, students construct polygons, progressing from visual recognition to a deeper understanding of geometric properties. They begin by selecting a polygon, describing its features, and then engaging in construction using tools like compasses and protractors. The teacher prompts students to explain their reasoning, helping them move from visual recognition to articulating geometric concepts such as central angles, chords, i.e. from dimensional deconstruction to reconstruction. Through this process, students break down the figure into components and reassemble it, reinforcing their conceptual understanding. The teacher's support encourages students to focus on the ordered steps of the construction, reinforcing their understanding of the step-by-step nature of geometric constructions while enhancing their sequential reasoning and verbal articulation of geometric relationships. One student even generalizes the construction method, deepening their understanding. The dominant cognitive focus in this activity is SA, as students concentrate on the step-by-step construction of the inscribed regular polygon.

In *Activity 2* students begin to view the disc as composed of smaller parts. By reconfiguring sectors to approximate different geometric shapes, such as parallelograms and trapezoids, they move from

visual recognition to interpreting the disc's mathematical properties. This activity emphasizes OA as students divide the disc into congruent sectors and rearrange them, learning how smaller parts combine to form a whole. The physical manipulation of the disc—cutting sectors and reassembling them—reinforces spatial understanding and supports students in grasping geometric transformations. Throughout the activity, the teacher prompts students to articulate their observations and make connections between the geometric shapes they are forming and the disc's properties. This enhances the students' ability to express mathematical concepts verbally and deepens their understanding of the transformations occurring in the disc.

Activity 3 challenges students to move beyond hands-on manipulation and visualize how the disc would appear if divided into more sectors. This shift from concrete manipulation to mental imagery signals a transition towards mathematical abstraction. Students are asked to mentally follow the process of dividing the disc into an increasing number of sectors, engaging in abstract sequential reasoning. As they visualize the transformation (OA), they gain an understanding of its implications. Mental manipulation of images further supports the development of operative reasoning, as students mentally reconstruct the figure and comprehend the changes involved. The teacher encourages students to verbalize these mental images, helping them formalize their thought processes using geometric language (DA). In this activity, the interplay between OA and DA is key, as students move from concrete, physical actions to abstract mental processes, which they then express verbally.

Finally, in *Activity 4* students connect their earlier transformations of the disc to the formal area formula, completing the transition from perceptual understanding to formal mathematical abstraction. This activity involves sequential reasoning as students follow logical progression from visualizing the rearranged sectors to recognizing the rectangle that they approximate. As they apply their earlier physical manipulations to derive the area formula, students engage in symbolic reasoning, transitioning from concrete models to abstract mathematical thought. The teacher prompts students to use formal mathematical language to describe the components of the area formula, reinforcing the connection between geometric properties and formal mathematical expressions. DA dominates this activity, as students verbalize and formalize their understanding of the area of the disc, concluding with the formula $P = r^2\pi$.

In summary, Duval's teaching sequence supports a progressive, multi-stage approach to learning, where PA, SA, OA and DA are woven together to build a deeper understanding of geometry. Students' progress from engaging in hands-on activities (Activities 1 and 2) with concrete models, to developing mental visualization by recognizing figural units within configurations (Activity 3), and finally to achieving a formal understanding of geometric concepts through verbalization and symbolic expression of the area formula of a disc (Activity 4). The teacher plays an essential role in facilitating these transitions, helping students articulate their understanding and move from concrete models to symbolic reasoning. Each activity within the sequence emphasizes different cognitive apprehensions, ensuring that students develop a well-rounded understanding of geometric concepts and their formal representations.

Discussion and Conclusion

This study explores a geometry lesson on the area of a disc, taught by a DGT-trained teacher. It investigates the effectiveness of Duval's framework in structuring lessons to support the development

of geometrical thinking. The findings demonstrate how Duval's teaching sequence helps students transition from perceptual and visual understanding to formal mathematical reasoning. By systematically engaging students in the coordination of cognitive apprehensions, teachers can foster a deeper understanding of geometric concepts, such as the area of a disc and its formula.

A key finding highlights the critical role of teacher guidance in helping students navigate these processes. Through well-planned activities aligned with teaching stages and emphasizing the coordination of cognitive apprehensions, teachers can facilitate transitions from concrete models to abstract reasoning. Strategic questioning further supports this process, underscoring the need for adequate teacher preparation when implementing Duval's framework.

While the research focuses on the area of a disc, Duval's framework applies broadly to other geometric topics, such as axial symmetry (Duval, 2017), and to understanding and deriving various mathematical formulas. This adaptability invites future research to explore its impact across different mathematical contexts and educational levels. Further studies could quantitatively assess this approach across diverse student populations and mathematical topics, providing deeper insights into its effectiveness and applicability.

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Generating 3D virtual models of straight-line helicoids in GeoGebra by animations

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In the paper, we present the results of a study that relates to comparing the level of spatial visualization of students of the 3rd year of the bachelor's cycle of the Mathematics study program with a focus on Education at the FP TUL with their abilities to create 3D virtual models (3DVMs) of straight-line helicoids (S-LHs) in the open-source software GeoGebra using animations. The main goals were to find out, using quantitative and qualitative analyses of students' work and subsequent directed interviews with them, to what extent a lower level of spatial visualization involves their success in solving tasks in a dynamic geometry environment. The next main goal was also to verify what specific possibilities the dynamic software has, according to the students, to increase their spatial visualization.

Keywords: GeoGebra, 3D virtual models, straight-line helicoids, animation, visuospatial abilities.

Introduction

Many studies (e.g., Fitriyani, 2018) highlight the importance of incorporating geometry into the educational content at all levels of schools. One of the goals of teaching geometry is to develop students' geometric thinking. Enhancing the level of geometric thinking contributes to improving students' proficiency in other mathematical skills, which are crucial, for instance, in problem-solving (Clements & Sarama, 2007). Geometric thinking has a significant impact on the correct representation of the magnitude of a number and the understanding of numerical relationships (Geary, 2007) and supports arithmetic transformations (Hubbard et al., 2005). Despite these facts, a considerable number of students and teachers still face challenges in mastering geometry (Seah & Horne, 2020). Another task of geometry is to develop students' spatial visualization. It is clear that acquiring skills in areas such as spatial perception, visualization, and the ability to use positional relationships is not only useful for correctly solving a few problems related to school geometry. It has been shown to be of fundamental importance in many areas of human activity, including scientific thinking, numeracy, arithmetic, and algebraic skills (Castro-Alonso & Uttal, 2019; Georges et al., 2019). Visuospatial abilities also play a crucial role in many aspects of thinking, and it has been shown that their level can be purposefully enhanced (Gilligan et al., 2020).

Spatial visualization and spatial cognition

The description and characterization of spatial visualization have been the subject of many studies, which highlight its important components. E.g., McGee (1985), Carroll (1993), and Gardner (2011) pointed out that the ability to mentally manipulate, rotate, bend, or flip a visualized object is one of the key aspects of human intelligence. An important categorization of students' spatial abilities was published by Maier (1994), who described five fundamental aspects of visuospatial abilities: spatial perception, spatial visualization, mental rotation, spatial relation, and spatial orientation. Each of these components plays an important role in the development and use of an individual's spatial visualization. Mulligan (2015) defined spatial reasoning or spatial intelligence as the ability to recognize and mentally manipulate the spatial properties of objects and the spatial relationships between these objects. One way to enhance students' visuospatial abilities is through the

use of dynamic software for teaching geometry. Baki et al. (2011) highlighted its considerable effectiveness and impact on a deeper understanding of geometry.

Solving spatial tasks, students rely on spatial cognition. Behrmann and Shomstein (2009) understand spatial cognition as a complex, multifaceted set of processes that deal with a wide range of tasks, including, i.e., mental rotation, spatial navigation, and spatial working memory. Gabbard (2015) mentions that spatial cognition does not include one, but many specific abilities. He also adds that these spatial abilities can be applied to the mental representation (imagining) of objects as well as to acting on external stimuli.

Aim of the study and research questions

The aim of this study is to present partial results from comparing the level of spatial visualization of third-year students of the bachelor's cycle of the Mathematics study program with a focus on Education, it means future mathematics teachers for elementary or secondary schools, with their abilities to create 3DVMs of S-LHs in the open-source software GeoGebra using animations. This study seeks to answer the following research questions: "To what extent are students with an average or below-average level of spatial visualization able to independently create 3DVMs of S-LHs in GeoGebra using animations?" If students make errors while creating 3DVMs of S-LHs, we ask: "In the creation of which types of S-LHs do they make the most errors, and what are the primary reasons for these errors?"

Research method

To answer the research questions, we conducted a case study involving 20 third-year students of the bachelor's cycle of the Mathematics study program, focusing on education, over two consecutive academic years. First, each student's level of spatial visualization was assessed using a tool developed by Stolínová (2020). Then, students were tasked with creating 3DVMs of four types of S-LHs in GeoGebra. The 3DVMs of S-LHs created by the students were analysed, and the specific errors each student made during the task were clearly identified. The final step was a structured interview with the students, aimed at uncovering the nature and causes of the errors made during the creation of 3DVMs of S-LHs, as well as gathering their perspectives on the impact of using dynamic geometry software in relation to improving their spatial visualization skills. Harris et al. (2024) described a similar method based on completing a geometry test and a spatial visualization test and exploring the influence of spatial visualization, as measured in psychology, and visualization on geometry tasks.

Results and discussions

Students in the third year of the bachelor's cycle of the Mathematics study program with a focus on Education at the Faculty of Science, Humanities and Education of TUL are studying the course Geometry 2 (GE2) in the winter semester and the course Geometric Software (GSW) in the subsequent summer semester. Various topics of spatial geometry are taught in the GE2 course. The students are asked to fill in the specially prepared test to find out the level of their spatial visualization (Stolínová, 2020) at the beginning of studying the course GE2.

The test was piloted in various classes of the second stage at three elementary schools in the Liberec region and at the secondary school in Louny during the research phase of the mentioned diploma thesis (Stolínová, 2020), as well as during a preparatory course organized annually for first-year bachelor’s students of various faculties at TUL. A total of 127 elementary school pupils, 19 secondary school pupils, and 154 university students participated in the pilot testing. We aim to verify the validity of the test by administering it to students in the Mathematics study program with a focus on Education and subsequently comparing each student’s test results with their academic

performance during the course. The spatial visualization test was administered to these students at the beginning of the winter semesters in the last two academic years. The success of students in solving the individual tasks included in the test reflects the level of spatial visualization of the tested students. The results obtained from the university students are consistent with the results of elementary and secondary school students from the pilot tests, indicating that approximately 60% of the tested students have an average or somewhat below-average level of spatial visualization. Specifically, in the academic year 2022-2023, 5 out of 9 students had an average or below-average level of spatial visualization, and in the academic year 2023-2024, the test results indicated an average or below-average level of spatial visualization in 6 out of 11 students. In analysing the 3DVMs of S-LHs, we focused mainly on the models generated by students with an average or below-average level of spatial visualization.

In the course GSW, students are introduced to the user interface and work in GeoGebra. They create dynamic applets for planar and spatial geometry tasks and also learn about the differential geometry of curves and surfaces. Some kinematically generated curves are drawn using animation. Similarly, students are assigned tasks to model 3DVMs of helicoids in the GeoGebra 3D window using animations. Before this activity, students worked on a task presented at the MEDA4 conference (Bímová et al., 2024). This task involved generating 3DVMs of S-LHs based on their parametric equations, making effective use of GeoGebra dynamic tool – the slider – which was used to set variable values for the corresponding parameters. The created models were then gradually converted into *.stl and *.gcode files, and finally, corresponding physical models of S-LHs were 3D printed. This provided students with a clear understanding of what the different types of S-LHs look like and what the process of 3D printing means. Surynková (2023) similarly mentions that the whole process of 3D printing combines the design, 3D computer modeling with CAD or other computer modeling software, and the process of fabrication itself.

The acquired experiences and visualizations of different types of S-LHs were effectively utilized by students when constructing 3DVMs of S-LHs in GeoGebra using animations. To create such 3DVMs, a correct understanding of the basic principles of screw motion and awareness of the position of the generating curve (in our case, a line segment) relative to the axis of the corresponding screw motion is necessary. This can be quite challenging with a lower level of spatial visualization. We will gradually discuss the observed problems that students faced while creating 3DVMs of the various types of S-LHs. Students created 3DVMs of S-LHs individually, each on their own PC. We saw their work by sharing their screens in the GeoGebra Classroom, where we could immediately analyze any errors that occurred.

Out of 20 students observed, 14 struggled to complete the task on their own, failing to do so successfully. Among these 14 students, 11 achieved an average or below-average level of spatial visualization on the test, while 3 students achieved an above-average level. The students correctly determined that it was first necessary to construct a plane perpendicular to the axis of the created helicoid and that a creating line segment needed to be constructed in this plane, with one endpoint being the point on the axis of the helicoid model Three students attempted to fix this point with the axis, although it was not the correct approach. However, they also knew that the other end point of the creating line segment must lie on a circle that is coincident with the constructed perpendicular plane to the axis of the created helicoid and whose centre is also on the axis of the helicoid model. They constructed the creating line segment as the radius of this circle, placing the second endpoint of the creating line segment as a so-called free point on the circle (see Fig. 1a). When the students began the animation of this point and activated the trace display for the creating line segment, they were intrigued to discover that the animation produced a circle instead of the expected 3DVM of the right-angled closed S-LH (see Fig. 1b). The teacher then guided a thoughtful discussion with these three students to explore the outcome:

Teacher: What was the cause of the circle appearing in your solutions instead of the 3DVM of the right-angled closed S-LH?
Student 1: I found it challenging to fully visualize how to set up the animation.
Student 2: In my imagination, I only created an image of the creating line segment rotating around the axis of the helicoid being constructed, but I completely overlooked that this creating line segment should also move along the direction of the axis.
Teacher: Correct. How will you ensure that the creating line segment moves along the axis of the right-angled closed S-LH?
Student 2: We need to release the endpoint of the creating line segment that lies on the helicoid's axis.
Teacher: That's right, but that alone isn't enough.
Student 3: To make the creating line segment move along the axis of the helicoid, we need to activate the animation for its second endpoint as well, which lies on the axis of the helicoid.
Teacher: Good, then try to correct your solutions according to what you have just mentioned.
…
Student 2: Great, it looks like the 3DVM of the corresponding helicoid is now generated correctly (see Fig. 1c).

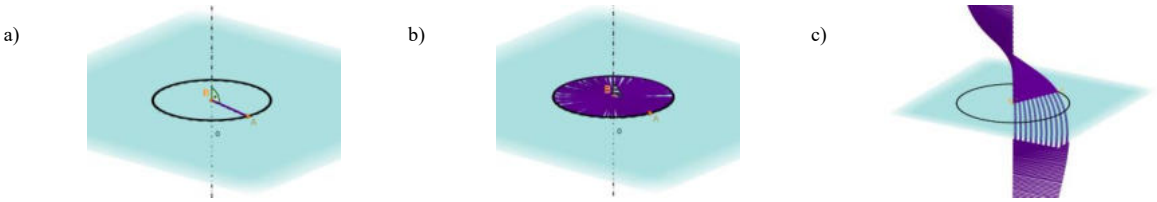

Figure 1: Creation of the circle and 3DVM of the right-angled closed S-LH

The six students mentioned above worked with animations of one or both endpoints of the appropriately positioned creating line segment. They relied on the basic principles of screw motion, which means a uniform motion resulting from the simultaneous combination of translational movement along a given axis and rotational movement around the same axis. They did not consider the curve along which the endpoint of the line segment, not located on the axis of the constructed helicoid model, moves. The remaining eight students revisited the task from the previous lesson and applied the "Curve(<Expression>, <Expression>, <Expression>, <Variable Parameter>, <Start Value>, <End Value>)" command in GeoGebra to generate the helix for the endpoint of the creating line segment that doesn't lie on the axis of the constructed helicoid model. While they made significant progress in comparison with the previously three mentioned students, they did not manage to solve the problem completely successfully. One student displayed only the helix of the considered endpoint but nothing else. Another student displayed the helix of the endpoint of the creating line segment that does not lie on the axis of the constructed helicoid model. This endpoint was inserted on the helix, but no other object was constructed (see Fig. 2a). Four students also placed a free point on the helix and placed an arbitrary point on the axis of the constructed helicoid model, which they fixed. They then constructed a line segment with endpoints at these mentioned points. After that, they started the animation of the endpoint of the creating line segment lying on the helix. Because the second endpoint of the creating line segment was fixed, the animation did not generate the 3DVM of the right-angled closed S-LH as expected. (see Fig. 2c). These four students had initially tried to set the creating line segment so that it was perpendicular to the axis of the created helicoid (see Fig. 2b). The teacher had the following conversation with these four students:

Teacher: Why was the 3DVM of the right-angled closed S-LH not created correctly?
Student 7: I envisioned one endpoint of the creating line segment moving along the helix, but overlooked that the other endpoint must follow its curve. Upon starting the animation, I noticed the line segment's incorrect movement.

Teacher: Constructing the helix for one of the endpoints of the creating line segment of the 3DVM of the right-angled closed S-LH and placing this point on the helix (as a free point) leads to the correct solution. What else must you consider when you know that the final model should be a right-angled closed S-LH?
Student 7: The second endpoint of the creating line segment must lie on the axis of the constructed helicoid.
Teacher: Very good. And what else must be true?
Student 8: For a right-angled S-LH, the creating line or segment must be perpendicular to the axis of the constructed helicoid.
Teacher: Excellent. And how do you ensure that these two conditions are met?
Student 9: We construct a perpendicular line to the axis of the helicoid passing through the chosen point on the helix.
Teacher: Good, but that's not entirely accurate. There are infinitely many perpendicular lines to the axis of the helicoid model that can be constructed by passing through a point on the helix.
Student 9: Ah, so how do we select the correct line?
Teacher: Consider what all such perpendicular lines create.
Student 7: A plane.
Teacher: Correct. And what kind of plane?
Student 10: A plane that passes through the chosen point on the helix and is perpendicular to the axis of the constructed helicoid.
Teacher: Excellent. So how do you obtain the second endpoint of the creating line segment?
Student 8: We get it as the intersection of the constructed plane with the axis of the helicoid.
Teacher: Perfect. What remains to be done?
Student 7: Activate the trace for the creating line segment and set the animations for its endpoints.
Teacher: Yes, that's exactly right. Try solving the task as you've described.

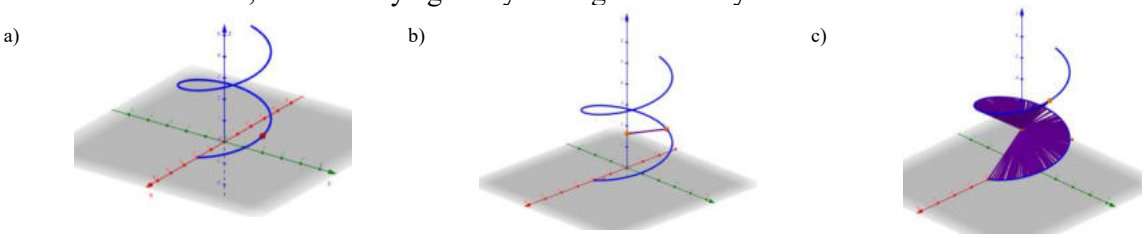

Figure 2: Wrong creation of the 3DVM of the right-angled closed S-LH

Rendering a 3DVM of an oblique closed S-LH using animations initially posed difficulties for all students in both academic years. Students were inspired by the solution to the previous task. They first placed free points representing the endpoints of the creating line segment, both on the generated helix and on the axis of the constructed helicoid model. They then connected these points with a line segment. They activated the trace for the creating line segment and started the animation for both endpoints of this line segment. Even though the speeds of movement for both points were set to the same value and their animations were started simultaneously, the point on the axis of the constructed helicoid model moved faster, especially when approaching the edges of the clipping box in the 3D window of the software, compared to the other endpoint moving along the helix. The result was a model of a surface shown in Fig. 3a). However, this does not represent a 3DVM of an oblique closed S-LH. Consequently, the teacher decided to conduct interviews with all students from both groups. Below, we present a guided interview with students from the academic year 2023-2024.

Teacher: What problem do you see when observing the animation of the creation of the 3DVM of the helicoid as you set it up?
Student 10: I think the problem is that the creating line segment changes its length during its movement.
Teacher: Yes, that's a great observation. Why did the error you made occur?
Student 11: I've didn't realize the importance of visualizing how to position the generating line segment to ensure it maintains a fixed length.

Teacher: How can you ensure the creating line segment's length remains constant during the screw motion?
Student 10: Choose an appropriate plane in which the creating line segment will lie.
Teacher: How would you construct such a plane?
Student 12: One point will be the endpoint of the creating line segment that lies on the generated helix. The other two points will be any two points in space.
Teacher: And how would you obtain the second endpoint of the creating line segment?
Student 13: I believe it would be as the intersection of the constructed plane with the axis of the constructed helicoid model (see Fig. 3b).
Teacher: Honestly, I don't think that is the correct solution. What might be the problem?
Student 13: I can't imagine how the objects will behave once the animation starts.
Student 12: Unfortunately, I can't see it either; I don't understand it.
Student 14: I'm in the same situation; I have no idea what will be rendered during the animation.

The teacher then asked the students to construct their described solution so that they could see the result of their proposed solution in the 3D window and thus gain a clearer understanding of the spatial situation. The resulting model of the surface is shown in Fig. 3c).

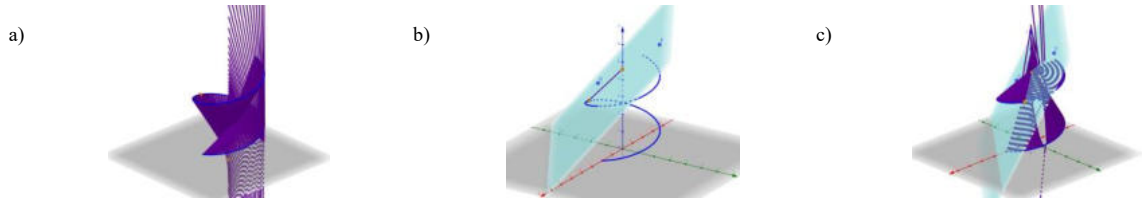

Figure 3: Wrongly created 3DVMs of the oblique closed S-LH

When rendering the 3DVM of the right-angled open S-LH using animation, students made almost no mistakes in either academic year. Based on previous experiences, they generated the helix of one (e.g., point A) of the endpoints of the creating line segment AB using the special command. Then, they constructed a circle along which point A rotates during the screw motion around and in the direction of the axis of the constructed helicoid model. This circle lies in a plane passing through point A and perpendicular to the axis of the constructed helicoid model. The second endpoint B of the creating line segment AB was constructed as the intersection of a spherical surface centred at point A with a chosen radius and the circle passing through point A (see Fig. 4a). They enabled the trace for both endpoints A and B as well as the creating line segment AB, and started the animation of point A. The 3DVM of the right-angled open S-LH began to render (see Fig. 4b). Individual guided interviews between the teacher and students were not necessary in either of the two academic years. Only one student made a mistake in this task by enabling not only the trace of the creating line segment and its endpoints but also the trace of the circle passing through point A (see Fig. 4c). This mistake was immediately noticed after starting the animation, and the student promptly turned off the trace of the circle.

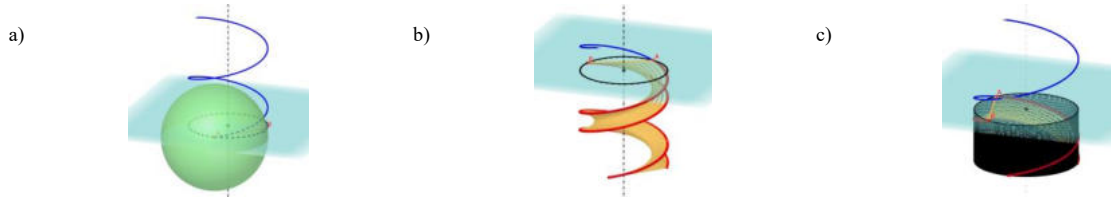

Figure 4: Creation of the 3DVM of the right-angled open S-LH

On the other hand, when generating the 3DVM of the oblique open S-LH, almost all students in both academic years made the same mistake regarding the creation of the second endpoint of the creating line segment. They

tried to create it analogously to the method used for creating the 3DVM of the oblique closed S-LH, specifically by using a spherical surface. The students chose an arbitrary point on the spherical surface centred at point *A* and labelled it *B* (see Fig. 5a). Then, they constructed the creating line segment *AB*. They enabled the trace for both endpoints *A* and *B* as well as creating line segment *AB* and started the animation of point *A* (see Fig. 5b). Initially, they believed that the 3DVM of the oblique open S-LH was rendering correctly. However, the teacher disproved their assumption. The following is the conversation between the teacher and the students:

Teacher:	Look at the created 3DVM from a perpendicular top point of view. What do you see? (Fig. 5c)
Student 4:	A top point of view of the constructed surface.
Teacher:	Yes, that's correct. But do you think this is really a 3DVM of the oblique open S-LH?
Student 2:	I don't see any mistake.
Teacher:	However, there is a mistake.
Student 2:	I can't imagine that the surface should look different!
Teacher:	Try to visualize how the 3DVM of the oblique open S-LH should look from a perpendicular top point of view, or what should be different.
Student 3:	The constructed 3DVM of the helicoid is not symmetrical according to the axis of the screw motion in a perpendicular top point of view onto it.
Teacher:	Yes, that's right. You correctly constructed the creating line segment so that it is non-parallel to the axis of the screw motion. But is there another condition for the path of point *B*? Do you have any idea what it might be?
Student 5:	It is necessary for point *B* to also move along a helicoid whose axis aligns with the axis of the screw motion. In our case, this is not true.
Student 2:	We could achieve this by generating a suitable helix for point *B* as well (Fig. 6a). If point *B* moves along it during the given screw motion, then the 3DVM of the oblique open S-LH should render correctly.
Student 4:	I've tried it this way, and it really works (Fig. 6b).
Teacher:	Look at how skilled you are; you discovered the correct setup of the basic objects and their animations. What geometric object is now displayed in the perpendicular top point of view onto the created 3DVM of the oblique open S-LH?
Student 6:	It is an annulus (Fig.6c).
Teacher:	Yes, very correct. You have successfully created all four types of S-LHs using animations.

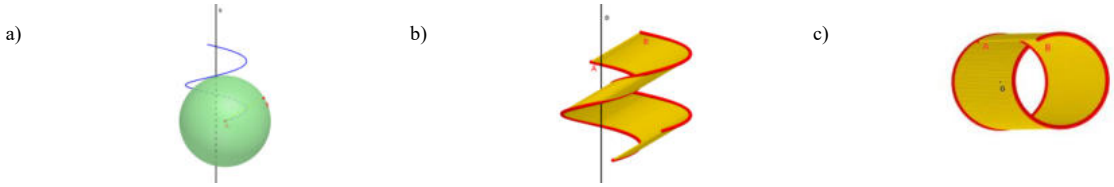

Figure 5: Wrongly created 3DVM of the oblique open S-LH

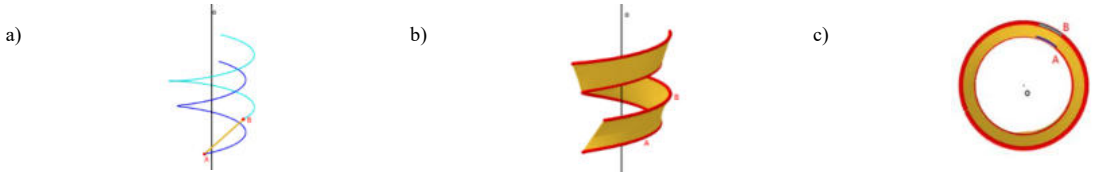

Figure 8: Creation of the 3DVM of the oblique open S-LH

Conclusion

All students with average to below-average visuospatial skills made significant errors when creating 3DVMs of S-LHs. None of these students managed to create an animation that accurately depicted the corresponding S-LH model on their first attempt in GeoGebra. We sought the reasons for these errors through guided interviews with the students. The errors varied, but the origin mentioned by the students was the same for all. Students encountered challenges in fully grasping the concept of the S-LH or in accurately rendering it,

but their efforts to understand and represent it demonstrated engagement and curiosity. All struggled with mentally manipulating the given geometric object, particularly when it was necessary to consider multiple properties and parameters occurring simultaneously. We also asked students whether and how virtual models help improve their visuospatial skills. The responses indicated that visible animation and immediate feedback were helpful. Essentially, all students realized where they had made errors after viewing the animation and were able to describe their errors. The error rate for the first task was the highest among all four tasks assigned. This was due to the fact that it was the first task of this type that the students encountered. In solving the remaining three tasks, students avoided similar errors, although other problematic steps and mistakes emerged, they were discussed in the text. The results of the analysis will enable us to focus on problem areas in teaching and contribute more effectively to improving the visuospatial skills of future mathematics teachers.

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Designing tasks to connect geometry and art – developing spatial reasoning using tilling

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This paper reports a classroom experience in 3rd grade using Portuguese traditional tilling. Our goal is to discuss the contributions to students' spatial reasoning and knowledge about rotation and its relation to the design of the interdisciplinary tasks between visual arts and geometry. Data were collected through observation, audio records of the classroom interactions and students' records. The results show the tasks were meaningful for the students, who learned about rotation and mobilized spatial reasoning. Imagination has an important role for both areas of the curriculum and the tasks should foster that skill, supported by the adequate materials.

Keywords: Geometry, Visual Arts, Interdisciplinary, Spatial reasoning, Task design.

Introduction

In the past two decades there has been a growing interest in spatial reasoning (SR), both as a curricular orientation (NCTM, 2000) and a research field (Davis et al., 2015). Despite some difficulties regarding language in the literature and different perspectives and methods from psychology and mathematics education, there is some consensus about the relevance of SR. For instance, Whiteley et al. (2015) refer that people with strong spatial skills are generally successful at mathematics, that SR supports mathematical understanding and problem solving, it is important in a large range of jobs and daily activities "as the interfaces used to access and manipulate that information are shifting away from the alphanumeric toward the spatial" (p. 9). Regarding specifically mental rotation–a component of SR commonly considered–, as Bruce et al. (2015) state, research shows that people who perform well "also tend to perform well across a range of mathematical tasks, including arithmetic, algebra, calculus, geometry, and word problems" (p. 96).

Concerning the curriculum, SR relevance it is often ignored as a goal and "learning activities for young children that foster spatial reasoning are sorely lacking" (Bruce et al., 2015, p. 85). Another big challenge of education, according to Morin (2002), is to overcome the separation of disciplinary boundaries of knowledge and encourage effective connection, moving towards a unifying approach that understands phenomena in their multidimensionality. Also, Niss (2002) highlights the need to build curricula focused on the meaningful use of mathematical content, seeking to create diverse situations in which mathematics can play a significant role.

Regarding Mathematics and Visual Arts, history has revealed several examples of their fertile interaction, like Leonardo Da Vinci's Treatise on Painting, published in 1651, but this interest has no significant correspondence in the classroom. Despite this, some experiences with children from the age of 4 (Loureiro, 2018) "show a strong potential in visual arts activities for the development of geometric and spatial reasoning and an added value for students to attribute meaning to geometry activities" (p. 572).

In this paper, we will address a classroom experience in 3rd grade using Portuguese traditional tilling to work on mental rotation. Our goal is to discuss: a) What are the contributions to students SR and knowledge about rotation that emerge from the task? b) Which aspects of the design of the interdisciplinary tasks between visual arts and geometry favour students learning?

Conceptual framework

Spatial reasoning and the role of rotation

For Davis et al. (2015), SR cannot be understood simply by reducing it to its components and these should not be considered in isolation. Nevertheless, the authors list a non-exhaustive set of categories, each involving several components, that show critical qualities of the SR. These categories (*interpreting, (de)constructing, altering, moving, situating, sensating*) are presented in a diagram that preferably associates each one with the activity of *understanding* (more "mental") or *transforming* (more "physical") (Davis et al., 2015). In the category *moving*, associated to *transforming*, we find the isometries, in particular *rotating* which, as we mentioned, seems to have an important role:

> What explains the relationship between mental rotation skills and mathematics performance? What are the pedagogical implications of this relationship? We believe that the answer to both of these questions is in part linked to the role of the imagination. In performing mathematics and mental rotations, we must use our imagination to mentally manipulate information . . . In terms of pedagogical implications, we believe that increased efforts are needed to explicitly engage the "mathematical imaginations" of young children. (Bruce et al., 2015, p. 96)

These authors suggest that a curricula that engages children in developing their mental rotation skills promotes a more dynamic and imaginative approach to early geometry instruction and learning. The examples of vignettes they present share some orientations worth noting: they push the boundaries of typical geometry curricula; "they oscillate between and among working with concrete materials, the imagination, and considering abstract ideas in playful contexts for young children" (p. 86); and there is an explicit attention on how multitouch technologies and mathematics tools, which are meant to support the actions of SR, a deeper understanding of spatial ideas, and a more precise discourse.

Interdisciplinary activities of Geometry and Visual Arts

In Portugal, the MARTE1618 project (Loureiro, 2018; Regatão & Loureiro, 2019) has developed several interdisciplinary activities between Mathematics and Visual Arts, with children ages 4 to 6, which highlight the development of distinct aspects, such as knowledge of concepts and procedures, SR and other capabilities that are implicated in two areas. Focusing only on Mathematics, and regarding knowledge, the learning of symmetry "in its wider meaning and with all its richness" (Loureiro, 2023, p. 52) is an example, as well as distinct forms of representation of 3D objects in the plane–two examples with large tradition in art. SR emerges significantly, namely implying the spatial structuring of 3D objects, in which the same object can assume different forms of structuring. Some of these objects may be simple solids such as parallelepiped or solids of revolution, possible to materialize in artistic objects like pop-up books, leading the children "to develop a mathematical view of real objects" (Loureiro, 2023, p. 69). The link between Mathematics and Visual Arts may still be a vehicle for developing transverse skills such as visual communication. As Loureiro argues, "artists

and visual communication experts are the ones who best create and design attractive and meaningful visual contexts . . . they resort to series of organized forms very rich from a mathematical point of view" (2023, p. 63). Regatão and Loureiro (2019) also refer the development of imagination, creativity, aesthetic sensitivity and the ability to solve problems.

Methods

The work we present in this text corresponds to a preliminary stage of the Project IMAVIS. This project aims to promote interdisciplinarity between Mathematics and Visual Arts in Elementary school (Grades 1-6), through collaborative work between higher education teachers and teachers at these levels of education. The main objective is to investigate the design and experimentation of interdisciplinary tasks, falling within the scope of didactic engineering, understood as an empirical science of the didactic phenomena in which empirical validation of outcomes is key (Barquero & Bosch, 2021). These authors focus on the central role of the "situation" and present the four-stage model followed by this researcher: 1) Preliminary analyses, when "the content to be taught and learned is questioned, usually considering different kinds of hypothetical didactic phenomena involved" (p. 251); 2) Design and *a priori* analysis, leading to the characterization of the content and the formulation of problematic questions within a sequence of concrete situations; 3) Implementation, observation and data collection; and 4) A posteriori analysis, validation and development culminates the DE process. In our research, we respect these four working stages, but with small adjustments, particularly in the first one: 1) Characterization of an interdisciplinary situation with the respective epistemological questioning of mathematical knowledge and of knowledge of visual arts.

In stage 3, the implementation of the tasks was carried out in two 3rd grade classes, each one having 21 students. Each class had an experienced teacher with no significant knowledge of interdisciplinary work between Mathematics and Visual arts. Data from classroom experiences was collected through observation, audio records of the classroom interactions and students' records. Concerning SR, we used the framework of Davis et al. (2015), particularly the categories *moving* (which includes *rotating*), situating (which includes *locating*) and *sensating* (which includes *visualising, imagining*).

Results

We organise this section considering stages 1, 2 and 4 of the model previously presented, although in this paper we focus mainly on the geometrical learning.

Characterization of the interdisciplinary situation

Portugal has an important tradition in the art of tiling that goes back to the 15th century and continues to this day, going through different phases, styles and functions. In addition to the functional aspect of covering a surface, this art form has been used to adorn buildings since the mid-19th century. Today, we can still find many examples of this tradition in old buildings, especially in the old parts of major cities such as Lisbon and Oporto (for examples, consult https://azulejosporto.pt/). From an artistic point of view, tiles are particularly rich in their use of colour and the composition of elements that allow them to form captivating patterns. From a geometrical point of view, a typical tile is constructed using a square forming tile panels rich in symmetry. Additionally, when we look at the way of constructing the panel, starting from one tile, other aspects emerge like rotation and

translation. Also, we may classify all tiles in one of three cases (Figure 1): A is invariant under any rotation; B assumes a different orientation[1] if it rotates 90º (or 270º); C has a different orientation if it rotates 90º, 180º and 270º. These three cases lead to different situations when constructing a panel: tile A forms a unique panel, while B and C form many different panels.

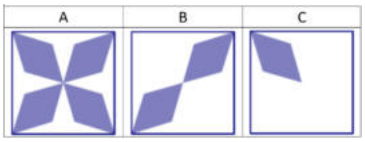


Figure 1: Types of tiles considering variation under rotation

In this analysis, we highlight that the construction of a panel has no relation to reflexion because the back of the tiles is not decorated. Thus, all SR is restricted to rotation and translation.

The recent Portuguese curriculum suggest that, in 3rd grade, students should learn to produce the image of a simple plane figure by rotation, with an amplitude of 90º or 180º, clockwise or anticlockwise. This way, the context of tiling seems to be an interesting and suitable situation to foster geometrical and artistic learning using an interdisciplinary approach.

Design and *a priori* analysis

Figure 2 presents the sequence of interdisciplinary tasks designed. Previously, the teachers presented rotation using a representation of a flag, and students constructed the images corresponding to a rotation of 90º, 180º and 270º, using the base of the flag as its centre.

1. Go to https://azulejosporto.pt/ . There you'll find lots of tiles on the front of buildings in the city of Oporto. Explore the website and see how beautiful the patterns are.
2. In the website, select the tile here presented (AP_20210816_36) and look at the panel. Explain how the tile was rotated to form the panel.

3. Search the website for other examples where rotation was used to create the panels and record their number.
4. Joana has created a tile to make a panel using the same technique. Draw how the set of four tiles will look.

5. Look at the panel formed by the white and blue tile (AP 20160622_08). Rotation was not used in this panel. Using this tile, build a panel in which you use rotation. Record it and explain how you used rotation.

Figure 2: The sequence of interdisciplinary tasks using tiles

[1] To simplify the language, we'll use the term orientation to compare the position of the original figure and its image in relation to the parallelism/perpendicularity of its sides.

The main goal of task 1 was to immerse students into the situation and let them enjoy. While observing the tiles, they could notice the use of abstract forms, the way they were organised in the square and the use of colours, without a necessary attention to the geometrical aspects.

Task 2 focuses on rotation and the formal language (centre, amplitude and direction). The students had to visualise that using 90º turns (clockwise or anticlockwise) around a particular vertex creates a set of four tiles. This new set formed a motif that is repeated in the panel using translation.

Task 3 is interdisciplinary focused, because its purpose is to find how geometry is used to form the panels, how it influences its esthetical dimension, and value this relationship. Students had to develop a mathematical view or, as Fujita and Jones refer, a "geometrical eye" (2002). In fact, the website has many examples of tiles type A, B and C. In the first case, the tile corresponds to the motif, so the panel may be constructed using only translation. Tile in task 2 is type C, as most tiles on the website.

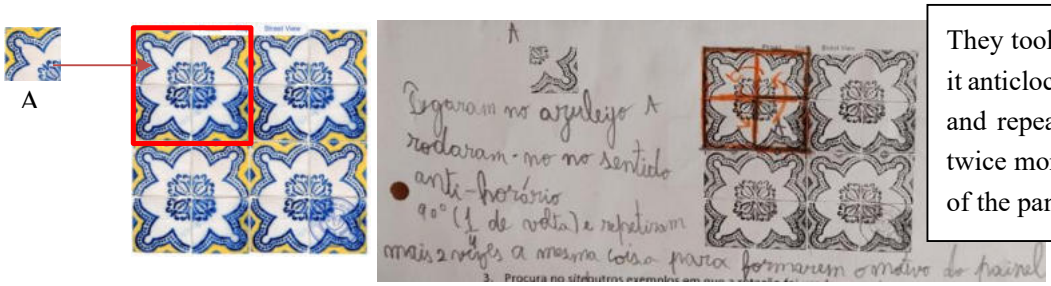
Task 4 was designed to challenge students SR. The sun, star, moon and heart were conceived and placed to allow different levels of difficulty. The star and the sun should be easier because of its rotational symmetry. The moon and the heart will have four different orientations, but we expected that the further away from the centre of rotation, the more difficult it is to anticipate the result of the transformation. Students who showed significant difficulties could use tracing paper and, in the end, the class could confront the drawings with the dynamic construction using GeoGebra (see here).

Finally, task 5 is also interdisciplinary focused, because students should use their geometrical knowledge about rotation to create new panels, using creativity, and communicate their creations. The tile is very suitable for that purpose because. In fact, it is a historical motif, studied in the 18th century by the French monks Truchet and Douat, and allows for the construction of beautiful patterns.

A posteriori analysis, validation and development

In this section, we discuss mainly tasks 4 and 5, based on records and episodes of both classes where the students worked in pairs or small groups. Regarding the other tasks, we will comment briefly.

First, in both classes, the students had a very nice reaction when they observed the tiles and the panels on the website. They made many comments comparing them, inviting their colleagues and teacher to look at a particular one they found especially beautiful, setting the mood for the next tasks. For the next one, it was necessary to give an additional explanation to understand our intention. The teacher asked the class which motif was repeated on the panel and the students identified the four tiles marked (Figure 3a). She then superimposed tile A on the panel and asked how that tile generates the motif, and the students identified the use of 90º turns (Figure 3b).



They took the tile and turned it anticlockwise 90º (1/4 turn) and repeated the same thing twice more to form the motif of the panel.

Figure 3a and 3b: Identification of the motif and the use of rotations

In task 3, the majority of students identified correctly panels that necessarily used rotation, except for one case (see here), a mistake that may be due to its particular configuration.

In task 4, there were mainly three types of responses, illustrated by Figures 4a, 4b and 4c.

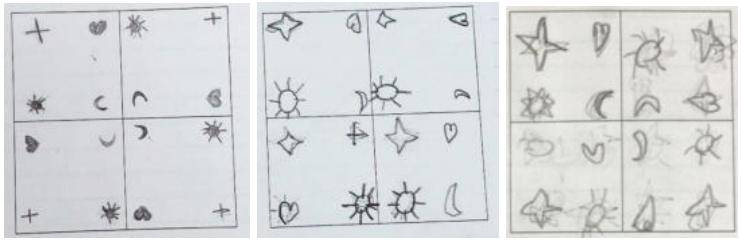


Figure 4a, 4b and 4c: Construction of the motif

Figure 4a shows a right solution. The students were able to *locate* each of the elements in the right place for each of the tiles and also position the heart and the moon with the right orientation, understanding that these two elements would be different when they turned 90º degrees. Considering they solve this question without any tool, this means they *visualise* correctly the movement of the figures when they *rotate* repeatedly 90º. So, these students showed significant skills concerning SR.

Figure 4b was unexpected to us. Considering both the heart and the moon, three of the tiles have these elements with the right orientation if they were to perform 90º turns anticlockwise, although they are not placed correctly in the corner of the tiles. In the fourth tile, it seems the students got lost because there is even no moon. So, this solution may signify that there is some success in *visualising* the rotation of the figures, at least *imagining* its orientation. Concerning *locating*, this solution evidence many mistakes. Again, there are three tiles that are consistent: the elements don't change their position. A possible explanation is that students were considering a path like ⬚ instead of a rotating and got lost in the last tile. Despite the explanation, there is evidence of mobilising SR.

Figure 4c shows the right solution, but clearly after many trials and errors. These students used tracing paper because they were really struggling *visualising* the solution, both the individual *rotation* of each element and its *location* in the tiles. So, this shows how important was the use of the physical tool for these students.

In task 5, the choice of the tile led to different panels, always using rotation. Many of the proposals followed the same construction method as before (Figures 5b and 5c), but there were some surprises.

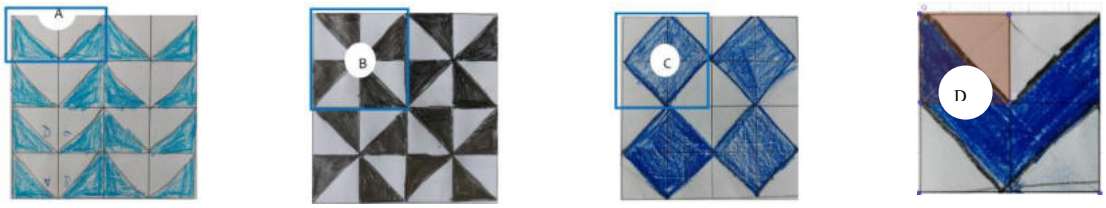


Figure 5a, 5b 5c and 5d: Panels created by the students

The observation of the students' records by the teacher showed the need for synthesis of the elements needed to define a rotation: centre, amplitude and direction. To do this, the teacher selected four of

the motifs constructed by the students (Figures 5a to 5d) and named points A, B and C to facilitate communication and add rigour to it. She also gave each student paper tiles so that they reproduce on the table each motif projected on the board, and moved on to a collective discussion in which the students indicated the centre, the amplitude and the direction of the angle of rotation of the tile.

The fourth motif selected by the teacher (Figure 5d) was surprising, since the centre of rotation of the first tile was not on one of the vertices of the square, but rather the midpoint of one of the sides. One of the students who built this motif came to the board and explained to her classmates the rotations.

Teacher:	Maria, how did you build this one?
Maria:	We took the tile and turned it.
Teacher:	But how?
Maria:	If I put my finger here and turn it anticlockwise, it looks like this. [Without hesitating, points her finger to the midpoint of the side (D) and rotated it].
Teacher:	How much did you rotate?
Maria:	180 degrees.
Teacher:	Notice that Maria didn't rotate from one of the vertices of the square as she did in the other motifs, she fixed a point in the middle of the side of the square. Now, how do you move on to the third tile?
Maria:	I need one more point to rotate it.
Teacher:	Where do you want me to mark the point?
Maria:	Here, at the junction of the four squares.
Teacher:	So, when we explain the rotation of a figure, we always have to refer to what?
Samuel:	How much the figure moves.
Teacher:	I didn't quite understand.
Maria:	If it's 90 degrees, 180...
Teacher:	You're talking about the measure of the angle. What else?
Maria:	We also have to say where the point of rotation is and the direction.

This episode shows how well Maria understands rotation, dominates its aspects and, mostly, how SR supports her knowledge and performance. Naturally, this differs from student to student. Nevertheless, considering the students' records and our observations, it seems that the sequence of tasks helped the students to construct their knowledge about rotation (what it means, how is defined), including the language associated with this transformation. Also the tasks enhance their SR, as they had to *visualise* the image of a figure after rotating 90º turns (task 4) and, on the contrary, identify which rotation would produce a motif they created (task 5). In this case, that implies *locating* the centre of rotation and *imagining* the angle.

Conclusion

Considering the design process, some aspects emerge from this experience. Firstly, as Bruce et al. (2015) point out, it's important to push the boundaries of typical geometry curricula and the interdisciplinarity between maths and visual arts seems to do this, motivating students and favouring the meaningful use of mathematics (Niss, 2002). To do this, the right choice of the interdisciplinary situation is fundamental. It must be powerful from a mathematical and artistic point of view, and students must be able to relate to it. Secondly, we want to notice is the role of imagination, which is present both in the mathematical dimension and in the artistic one. Imagination supports the process of creating and operating with mental images and is at the core of rotation, a transformation frequently used in the construction of panels of tiles, but also necessary to interpret these artistic objects, contributing to developing the "geometrical eye" (Fujita & Jones, 2012) and SR. But imagination is

also an important part of creativity, necessary to the creation of artistic objects. In this experience, students seem to value this aspect and a certain liberty associated with it, which was express when one of the classes suggested the used of the patterns they created to make a gift for mother's day. Finally, we highlight the use of manipulative tools in support of the mental process, also referred to by Bruce at al. (2015), which favour the creation and experimentation, but also the analysis.

Acknowledgment

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Insights into conceptual breaks of individual sine concepts

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This research aimed to investigate students' difficulties in using different representations in trigonometry. For this reason, we classified three types of conceptual breaks based on Tall and Vinner's theory of concept image and concept definition. Analyzing the results of a questionnaire, that aims to investigate individual trigonometric concepts and representations answered by university students, seems to confirm the presence of this classification. Furthermore, we proposed possible implications for teaching that could facilitate overcoming these breaks.

Keywords: Trigonometry, concept image, concept definition, conceptual breaks, sine

Introduction

Trigonometry is a topic in mathematics education in which representations from different content areas interconnect: Right triangles and the unit circle as geometric objects, the Cartesian coordinate system for visualizing trigonometric function curves, and algebraic equations for describing trigonometric relations. A typical task in trigonometry is the calculation of unknown side lengths or angles. Students often memorize the formulas required for these calculations, but their meanings cannot be interpreted (Kendal & Stacey, 1998; Yiğit Koyunkaya, 2016). The graphical representations mentioned are used by students for visualization, but in many cases, they cannot be linked together or do not correspond to mathematically appropriate representations (Brown, 2005). In fact, the lack of linkage goes so far that some researchers speak of *trigonometries* that are perceived as separate from one another, e.g. a right triangle trigonometry and a periodic function graph trigonometry (Chin, 2013). Our study is consistent with this area of research and specifically, aims to investigate students' conceptual breaks within one trigonometric concept. To achieve this, we asked them to complete a questionnaire focused on the definition and representations of sine, and we analyzed the students' answers using the concept image and concept definition theory of Tall and Vinner (1981), identifying three different conceptual breaks.

Theoretical framework

We illustrate the concept image theory which we used to describe individual concepts and we show how conceptual breaks can be seen within this theoretical framework. We also show the results of other research to highlight why a study of conceptual breaks within trigonometry education might be interesting.

Concept Image and Concept Definition Theory

The Concept Image and Concept Definition Theory formulated by Tall and Vinner (1981) provides a theoretical framework to describe individual concepts. In their theory, a *concept image* comprises the "total cognitive structure that is associated with the concept, which includes all the mental pictures

and associated properties and processes. It is built up over the years through experiences of all kinds, changing as the individual meets new stimuli and matures." (Tall & Vinner, 1981, p. 152). Thus, the concept image includes all mental visualizations of graphic or iconic representations, tables, coordinate systems, but also symbolic expressions. These can be both conscious and unconscious. The concept image does not have to form a coherent unit – individual components of the concept image can contradict each other, whereby a mental conflict only arises if these two components are activated simultaneously. The part of the concept image that is activated at a certain time is called the *evoked concept image*.

The *concept definition* is another important element in this theory. It is a verbalized form of a mathematical concept and is distinguished from the concept image. It can correspond to the general mathematical definition or can be an individual description that is based on the concept image and may deviate from an appropriate mathematical formulation. Mental images are also generated when the concept definition is used. These concept definition related images are called *concept definition images* and are part of the concept image.

We use the concept image theory for two reasons: First, in trigonometry, there exist many different graphical representations that have a particular influence on mental images. Thus, it makes sense to use a theoretical framework that focuses on mental images. Secondly, other research about trigonometry is already based on the concept image theory. Thus, our results can be better linked to these results.

Concept Image Theory related research on trigonometry

Trigonometric concepts, their interpretation, and their use by students and prospective teachers have been the subject of several studies. Kendall and Stacey (1998) investigated whether students are more successful in solving trigonometry problems if they use right triangles or the unit circle. Although the group using right triangles performed better in this study, Weber (2005) came to the contrary conclusion in another study: In his case, the students who used the unit circle performed better. Weber (2005) argues that it is not primarily the concrete representation in trigonometry lessons that matters, but how the concept behind the individual representations is understood. The fact that students show low conceptual knowledge of trigonometry has also been the subject of other studies (Chin, 2013; Martín-Fernández et al., 2019). The use of mnemonics to memorize trigonometric ratios (Engl.: SOH-CAH-TOA, whereby sine (S), cosine (C), tangent (T), opposite (O), ankathete (A), hypotenuse (H)) was also associated with procedural understanding in other studies (Brown, 2005; Kendal & Stacey, 1998; Yiğit Koyunkaya, 2016). Based on these studies, there is a call for a stronger focus on conceptual considerations in teaching trigonometry.

Yiğit Koyunkaya (2016) used concept image theory to investigate mathematical educational graduate student's understanding of the relationships between sine and cosine in a right triangle. The right triangles and the unit circle were identified as the students' concept images. The SOH-CAH-TOA mnemonic was identified as the concept definition. Other research work also provides information on the concept image of secondary school students: In addition, to right triangles and the unit circle, students also use coordinates and directed line lengths (in the unit circle) and the periodic function

curve when working on tasks, which means that these can be assigned to their concept image (Brown, 2005; Martín-Fernández et al., 2019).

Description of conceptual breaks within the concept image theory

In this article, an individual concept for trigonometry is understood as the union of concept image and concept definition. A trigonometry-related concept image includes the individually formed graphical representations (right triangle, unit circle, and sinusoid), but also the symbolic functional equations. A concept definition is seen as a verbal explanation of the question 'What is sine?'. The term *representation* is used in a broader sense: Not only graphical representations are meant here but also equations and verbal sentences. Within the concept image theory, we distinguish between different conceptual breaks (Table 1).

Table 1: Different types of Conceptual breaks within the Concept Image Theory

Type	Description	Example
A conceptual break within an evoked concept image	A certain individual representation does not correspond to the appropriate representation.	A student does not center the unit circle in the coordinate origin.
A conceptual break within the concept image	There are linking difficulties between different representations.	It cannot be explained what the representation of the unit circle has to do with the sinusoids.
A conceptual break between concept image and concept definition	There are linking difficulties between individual images and the verbal definitions.	A student defines sin(x) as a ratio of side lengths in a right triangle but uses the periodic function curve to visualize it.

Trigonometry related research about conceptual breaks

In an empirical study (Chin, 2013), student teachers distinguished three separate *trigonometries*: Triangle trigonometry, unit circle trigonometry and analytical trigonometry (series representation). This was made explicit by the fact that most student teachers had difficulties shifting between these *trigonometries* and answering questions in multiple *trigonometries*. These findings indicate a conceptual break within the concept image of trigonometry. In another study (Brown, 2005), conceptual breaks were also observed within evoked concept images. The conceptual difficulties occurred when working with right triangles, the unit circle, and periodic functions.

The following research question arise for us: What conceptual breaks can be identified in university students' concept of sine? We want to answer this question applying a qualitative analysis. We also want to discuss how these conceptual breaks can be addressed in the classroom.

Methods

A questionnaire was designed to investigate students' definitions of a mathematical concept and its representations. The questionnaire consisted of three questions, among others. We only provide an insight into these three questions, as they were the most relevant for answering our research questions during the analysis:

1. Give a definition of sin(x).
2. Provide a representation of sin(x).
3. What does the sin(x) represent in the right triangle?

The questions were designed to explore students' ability to connect different images and the definition of the same mathematical concept (sine). In particular, the first question concerns the definition of the concept. The answers to this question are traded as the students' concept definition or their concept definition images if their answers contain images. The second question aims to find out something about the concept image. There are two reasons why we ask about the representation of the right triangle in the third question: First, we want to further investigate the presence of the misconception based on the identification of the sine as the cathetus of a right triangle (Martín-Fernández, 2019) which would indicate a conceptual break within an evoked concept image based on our definition. Second, this question aims to explore conceptual breaks within the concept image and between the concept image and the concept definition.

The experiment was conducted with 94 freshmen from an Italian university, specifically 68 from the faculty of Architecture, and 26 from the faculty of Engineering. The study took place during the first semester, while the students were attending a Calculus course after the trigonometry review and participated voluntarily in the experiment. Specifically, trigonometry is a topic that they studied during secondary school (present in every Italian high school curriculum) and is revisited by university teachers during Calculus courses. This justifies the selection of these participants. Students had 20 minutes to respond to the questions and we collected their written answers. We conducted a qualitative analysis of the students' responses, focusing on the presence of three conceptual breaks (Table 1). To identify the three different types of conceptual breaks in the students' answers, we analyzed their answers to the different questions in relation to the description in table 1.

Analysis

In a qualitative study, we focused our attention on three cases (students A, B and C) that illustrate the three different breaks (conceptual breaks in the evoked concept image, conceptual break within the concept image and break between concept image and concept definition).

Example of a conceptual break within an evoked concept image (student A)

In Figure 1a, the student A answered question 1 and Figure 1b contains the answer to question 3 (*What does the sin(x) represent in the right triangle?*). A coordinate system, a circle, and an inscribed right triangle can be seen, whereby the angle between the cartesian axes in the first quadrant is the right angle of the right triangle. Next to the drawing, the student wrote, "*The sine in a right triangle represents one of the two catheti forming the angle* α".

The unit circle drawn in Figure 1b contains some specifics that indicate conceptual breaks in the evoked concept image: The position of the inscribed right triangle is at the wrong position and the right angle is labeled by α. The intersection points of the circle with the axes were marked counterclockwise with $0, \pi/2, \pi, 3\pi/2$, which indicate a rotation conception. The x- and y-axis have been swapped. The catheti are labeled with $\sin\alpha$ and $\cos\beta$. These characteristics do not correspond to an appropriate representation of a unit circle which is why a conceptual break within an evoked concept image is identified (Table 1).

a) "$y = \sin(x): \mathbb{R} \to [-1,1]$

$$\sin(x) = \frac{opposite\ cathetus}{hypothenuse}"$$

b)

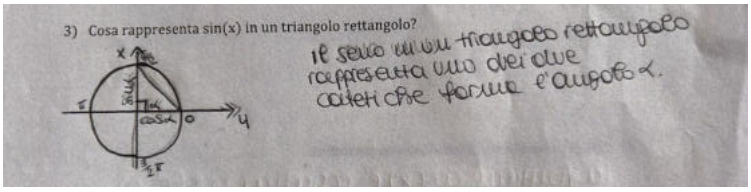


Figure 1: a) definition given by the student to question 1; b) Student's answer to question 3

Additionally, in Figure 1a, it is possible to see that the student A uses another definition, highlighting the difficulty in evoking correlated images. This example can be interpreted as an attempt to relate the right triangle and the unit circumference.

In fact, individual components of the appropriate image are present (circle, coordinate system, right triangle), but they have not been positioned in such a way that they can be used to illustrate trigonometric relations. As a result, no practical problems can be solved with the help of this concept image.

Example of a conceptual break within the concept image (student B)

In Figure 2, we can see the answers of a student B to all three questions. In the first question, the student's answer was "*It's the value that a circle of radius 1 takes on the y-axis.*", additionally the unit circle was drawn. In the second question, the student B drew a sinusoid, and in the third question a right triangle, identifying sin(x) as a cathetus.

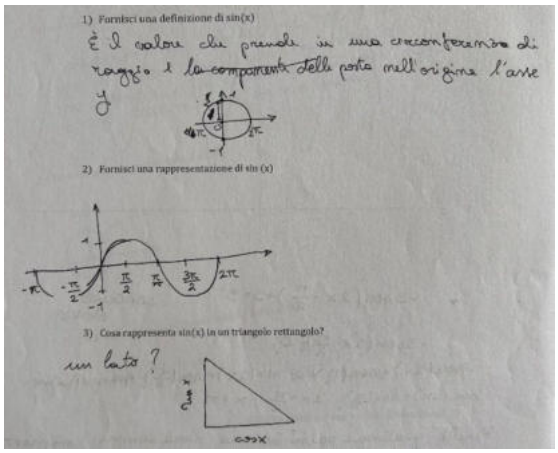


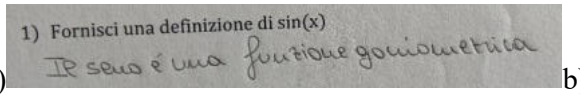
Figure 2: Answers of a student to all questions

Although only task 3 asks for a specific representation, other representations are used to answer the other two questions. In this case, the unit circle is part of the concept definition image, whereas the sinusoid and the right triangle are evoked concept images for the following questions. Especially the use of the unit circle in question 1 and the sinusoid in question 2 could indicate linking difficulties between these two representations which is why we identified a conceptual break within the concept image (Table 1). However, further tasks would have to show whether student B is actually unable to switch between these representations or not.

We interpret the identification of the length of a cathetus with sin(x) as an overgeneralization of the right triangle from the unit circle. This interpretation is also supported by the status of the unit circle as a concept image definition, as well as by the results of other studies (Martín-Fernández, 2019).

Example of a conceptual break between concept image and concept definition (student C)

Sttudent C answered the first question (Figure 3a): "*sin(x) is a trigonometric function*". In Figure 3b, we can see that student C drew a right triangle and wrote: "*Represents the cathetus b, then the ratio of the cathetus a to the hypotenuse c*".

a) 

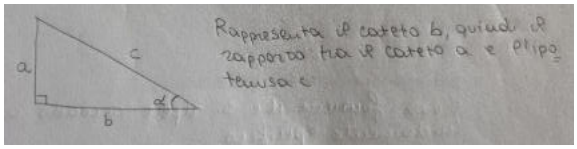
b) 

Figure 3: Student's answer to question 1 and 3

The use of the word "then" is very interesting. We see in it a certain dominant position of the interpretation of sine as a length of a cathetus, while the interpretation as a ratio has a lesser significance. However, the drawing, which represents a part of the concept definition image, only corresponds to the interpretation as a ratio and not to the interpretation as a length of a cathetus or as a trigonometric function. These are indications of linking difficulties between the image and the definition which is why we identified a conceptual break between concept image and concept definition (Table 1). This interpretation supports the hypotheses of other research studies that describe the presence of different *trigonometries* in students' minds (Chin, 2013). However, the concept definition used in task 1 may also have been provoked by the use of "sin(x)" in the task. This notation has a stronger connection to the interpretation as a function and could have provoked a second concept definition.

Discussion

The cases illustrate conceptual breaks by students and thus confirm findings from existing research. We want to discuss potential reasons behind these conceptual breaks as well as present teaching measures that counteract these breaks. It is hoped that the separation into different conceptual breaks will lead to a more differentiated discussion.

Possible reasons for conceptual breaks

The consideration of different types of conceptual breaks (Table 1) enables a differentiated consideration of possible causes.

In the case of conceptual breaks within the evoked concept image, the representations shown do not correspond to the mathematically appropriate representations (Figure 1). This could be due to a lack of prior knowledge: Conceptual difficulties with circles or triangles could be transferred to the concept image of the unit circle. It is also conceivable that only parts from school lessons, and not the whole concept, were understood. For example, the circle and the right triangle are recognized as important components of the unit circle representation, but the incorrect positioning of the triangle (Figure 1) shows the impossibility of representing trigonometric relations and thus an inadequate conceptualization. Thus, a lack of conceptual considerations in teaching in the introductory phase on the topic of trigonometry favors the emergence of such conceptual breaks. Too great a focus on calculus-oriented tasks in lessons could further reinforce existing conceptual breaks.

In the case of conceptual breaks within the concept image, different representations are not adequately interlinked (Figure 2). If the conceptual links between trigonometry on the unit circle, right triangles, and sinusoids are not made transparent in class, the representations cannot be linked together. As a result, separate trigonometries are formed, as already described in the literature (Chin, 2013).

Conceptual breaks between concept definition and concept image could also be demonstrated in this study, but they differed from those already known from the literature: While in Anglo-American studies mnemonics formed the concept definition, in this study the definition of the sine as a function or as the y-coordinate of the unit circle was predominant. On the one hand, the lack of such a mnemonic in Italian could be responsible for this difference, but on the other hand, the illustrations and definitions presented in class could also significantly influence the concept image and the concept definition. The fact that there is no match between concept image and concept definition in the illustrated case (Figure 3) could be due to a lack of support for representation linking in class.

Implications for teaching

The following points should be considered in lessons in order to counteract conceptual breaks at an early stage: 1. Conceptual considerations should be strengthened, especially in the introductory phase of this topic. 2. Linking different concepts and representations of trigonometry should be supported in lessons.

Teaching implementations for the first point are currently being researched within a design research project (Büttner & Erath, 2023b). The developed learning design focuses on the assignment of angles to trigonometric ratios within the context of slopes. This is intended to counteract a conceptual break between trigonometry on right triangles and sinusoids at an early stage.

The link between representations on the right-angled triangle and the unit circle can be supported in class with the help of GeoGebra applets (Figure 4). In the applet shown, the unit circle and sinusoids are shown side by side. By varying the angle using the slider, the corresponding point on the unit circle and the point in the Cartesian coordinate system are moved at the same time. The activated track mode makes the sinusoid visible in the coordinate system. The connections between the two representations can be recognized through the simultaneous dynamics, which support a representation linkage.

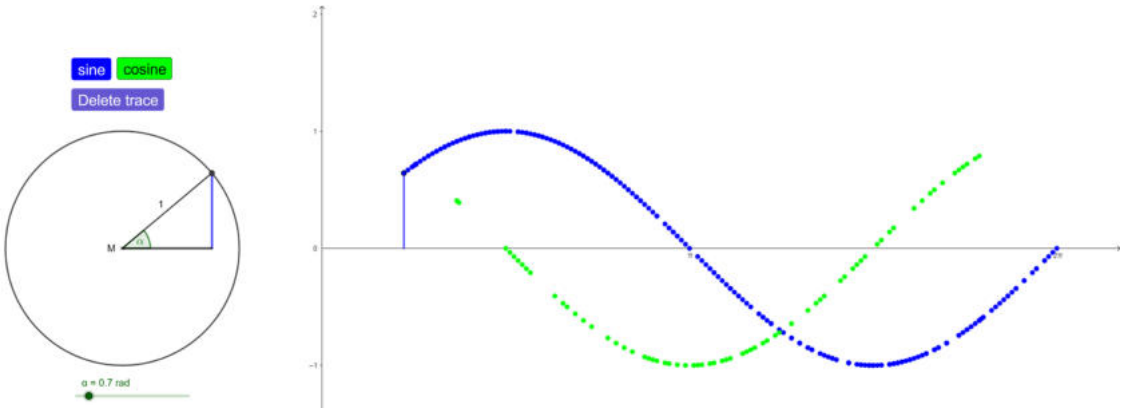


Figure 4: GeoGebra® Applet to support the networking process of different representations, https://www.geogebra.org/classic/awdum8an

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First graders' conceptions and linguistic descriptions of geometric shapes

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This research aims to examine children's conceptions about identification and properties of geometric shapes. This descriptive research was carried out on a research sample consisting of 27 first graders. The data were collected using a printed handout that illustrated models and non-models of planar geometric shapes. The students were expected to identify triangles, squares, rectangles, and circles among the given set of shapes. The findings suggest that the students' preferences were mostly prototypical shapes, and shapes with curved corners or edges that resemble the prototype were also marked. This shows that students refer to the similarity to the original shape, ignore the fact that the edges should be straight line segments, and some students think in three dimensions and mark curved edges as models. The fact that the shape is small, pressed or rotated prevented the correct identification of the shapes.

Keywords: Geometric shapes, conceptions, prototypes.

Introduction

The current state of research demonstrates that the ability to formulate geometric reasoning is closely related to the ability to form mental representations and to manipulate them in the process of reasoning (Pérez et al., 2019). In order to ensure that students are fully able to comprehend abstract geometric concepts, it is necessary to develop both their conceptual knowledge and their language skills, as well as provide hands-on experiences for students as well as clearly define the symbols that correspond to specific terms (Fujita et al., 2020; Leavy et al., 2017). If any of these key elements are neglected in geometry education, then misconceptions will likely develop which may persist throughout the student's mathematical education and make them more difficult to overcome.

Various theories have been proposed for the development of students' concept formation and thinking structures. These theories collectively contribute to our understanding of cognitive development and education, highlighting the interplay between social interaction, language, and structured learning processes. Piaget's Cognitive Development Theory focuses on stages of cognitive development that children go through as they grow. It proposes that children construct knowledge through interactions with their environment, progressing through distinct stages that are age-related. Each stage represents a different way of thinking and understanding the world (Piaget & Inhelder, 1971). Vygotsky disagreed with Piaget's view that the human mind possesses cognitive structures independent of social factors. Vygotsky's Social Development Theory emphasizes the role of social interaction and culture in cognitive development. Language is crucial for the formation of cognitive structures, differing from Piaget's focus on individual cognitive processes. He suggests that learning occurs through social contexts and collaborative dialogue (Vygotsky, 1970). Bruner's Theory of Representations proposes three modes of representation: enactive (action-based), iconic (image-based), and symbolic (language-based). Hence, education of children should progress through these stages to develop symbolic thinking, which is essential for understanding abstract concepts. The theory also highlights

the importance of language in learning and problem-solving, suggesting that different types of representations are necessary for all ages (Bruner, 1966).

Van Hiele's Model of Geometric Thought identifies five levels of geometric understanding, where each level builds on the previous one. It emphasizes that students cannot skip levels; a thorough understanding of one level is necessary to progress to the next. It provides a framework for diagnosing geometric understanding, which can enhance the teaching of geometry (van Hiele, 1986). Knight (2006) summarizes the theory as existence of levels of understanding, properties that are inherent to each level, and occurrence of movement to a level from the previous level. The van Hiele model of geometric thought is a theory that describes how students learn geometry in a hierarchical manner through five distinct levels. These levels are visualization, analysis, informal deduction, deduction, and rigor. In the first level, students recognize shapes and objects based on their appearance. They can identify and name geometric figures but do not understand their properties or relationships. In the second level, students begin to understand the properties of shapes. They can identify and describe the characteristics of geometric figures, such as the number of sides or angles, but they do not yet understand the relationships between these properties. In the third level, students can logically order properties and understand the relationships between them. They can follow informal arguments and understand the significance of definitions and theorems. In the fourth level, students can construct formal proofs and understand the structure of a mathematical system. They can work with axioms, definitions, and theorems in a formal deductive manner. At the highest level, students understand the formal aspects of geometry and can compare different systems of geometry. They can work in abstract and axiomatic systems and understand the role of logic and proof in mathematics. A schematic representation of the van Hiele levels of geometric thought is presented by van de Walle et al. (2013). This representation indicates that the levels are not isolated, but intertwined, and that geometric terms grow in meaning over time.

Understanding geometric shapes is a fundamental aspect of early mathematical education, crucial for developing spatial reasoning and advanced mathematical skills. This research aims to examine children’s conceptions about identification and properties of geometric shapes. In line with this aim, as a part of this study, we are trying to determine the kind of lexical or non-lexical resources the children are using in order to describe the shapes by comparing the children's lingual expression with the geometric terminology used in the shapes.

Methods

This is a descriptive study which seeks for the conceptions of first graders about geometric shapes. The research was carried out on a research sample consisting of 27 first graders. The research was conducted using a printed handout that illustrated models and non-models of planar geometric shapes as a means of illustrating the shapes. These handouts including shape collections were developed by Zilkova et al. (2018). The questions included shapes for identification of triangle, square, rectangle, and circle. Figure in the appendix represents the list shapes among which the requested shapes would be chosen. The students were asked to choose the shapes for triangles, squares, rectangles and circles as shown in the appendix. A researcher guided the students during the implementation. As squares were examined separately and first graders were not expected to know the relationship between a

square and rectangle, correct identification of rectangles included models except squares. All of the choices of the students were coded and frequencies and percentages have been presented separately for each of the shapes examined. Students' choices and discourses for the required shapes were examined altogether for the certain types of shapes. For example, all the data for triangles have been examined synchronously. Pseudonyms (M for males, F for females followed by a participant number) have been assigned for the students.

Findings

The findings have been presented in four sections for each of the shapes.

First graders' conceptions of triangles

Table 1 presents the findings for the conceptions of triangles with descriptive statistics.

Table 1: Descriptive statistics for the conceptions of triangles

Shape	*f*	%	Shape	*f*	%
1e	23	85.2	**2b**	3	11.1
2a	14	51.9	**2d**	2	7.4
3d	10	37.0	**1c**	1	3.7
3e	9	33.3	**2e**	1	3.7
3b	5	18.5	**3a**	1	3.7
3c	5	18.5			

1c, 1e, 2a, 3c, and 3e are correctly drawn figures for the triangle. It is noteworthy that among the triangle models, 1c was marked by only 1 person. This may be due to the fact that the top of the shape is down, and it is a scalene triangle. When asked, a student (F3) stated that it (1c) does not look like the triangles she knows (which are prototypes). 3d and 3b may have been chosen more because they resemble the shape of the prototype triangle. 2b has similar characteristics with these two, but it may have been chosen in smaller numbers due to its outward curvature. Indeed, the dialog with a student (M2) is as follows:

Researcher: What shape do you think this (3e) looks like?
Student: A triangle.
Researcher: Then we will put it on it too. Do you know what a triangle is?
Student: Like this (pointing to 1e).

As seen in the dialog, the fact that the student showed 1e, which represents the most common triangle prototype, may be an indication that they consider prototype shapes when determining shapes.

It was observed that one student marked triangles and all geometric shapes similar to triangles. In the clinical interview with the student, it was determined that the student had partial knowledge about the triangle, but lacked knowledge that the sides of the triangle should be a straight line segment. This situation caused the student to name shapes similar to triangles as triangles. As a matter of fact, when the student was asked about the difference between a triangle and a square, it was seen that he used

the correct mathematical expression that a triangle has 3 vertices, but in the dialog that took place afterwards, it was seen that he named the shapes whose sides are not regular line segments as suppressed triangles. The dialog with the student (M5) is presented below:

Researcher: How do you think a triangle is different from a square?
Student: A square has 4 vertices, a triangle has 3 vertices.
Researcher: Where is the corner?
Student: (Points a corner).
Researcher: Do you think this shape and this shape are the same? (2d-3e) You said the triangle has 3 vertices. Does this figure (2d) have 3 vertices? Can you show me by marking them?
Student: (Points the corners).
Researcher: Here? (concave part)
Student: They pressed it a little bit this way (inward).

First graders' conceptions of squares

Table 2 presents the findings for the conceptions of squares with descriptive statistics.

Table 2: Descriptive statistics for the conceptions of squares

Shape	*f*	%	Shape	*f*	%
1a	21	77.8	**3b**	2	7.4
3c	14	51.9	**1d**	1	3.7
3e	11	40.7	**2b**	1	3.7
3d	7	25.9	**2d**	1	3.7
1b	6	22.2	**3a**	1	3.7
2a	3	11.1			

1a, 3c, and 3e are correctly drawn figures for the square. The fact that the majority of the students' choices were correct models shows that the students correctly identified and recognized the square. In addition, it can be said that most of the students were aware that the position of the square is not important in determining the geometric shape. As a matter of fact, in the clinical interview with one of the students, although the student was initially undecided about whether 3c was a square or not, she decided that this shape was also a square by associating it with the concrete example she encountered in daily life. The dialog with the student (F6) is as follows:

Student: (Not sure whether 3c is a square or not) I couldn't understand that again.
Researcher: Do you think it is a square or not?
Student: I don't know.
Researcher: What do you think about the characteristics of a square?
Student: (After examining it for a few seconds and deciding to choose a shape). This is a square.
Researcher: Why did you change your mind?
Student: I was turning the pillow and it was like this (3c). Normally my pillow is in the shape of a square, looks like this (prototype).

It is also interesting that students did not mark 3e very much even though it was a prototype they knew. The reason why 3e was chosen less frequently among the correct models may be its small size.

In the clinical interview, when a student who did not mark 3e was asked why it was not a square, the student stated that he was not aware of it and then marked it realizing that it was a square. The students' incorrect choices were mostly rhombus and shapes with rounded corners or curved edges similar to a square.

First graders' conceptions of rectangles

Table 3 presents the findings for the conceptions of rectangles with descriptive statistics.

Table 3: Descriptive statistics for the conceptions of rectangles

Shape	*f*	%	Shape	*f*	%
2b	12	44.4	**1c**	2	7.4
3b	9	33.3	**2a**	2	7.4
1d	6	22.2	**2c**	2	7.4
2e	5	18.5	**2d**	2	7.4
3a	4	14.8	**2f**	2	7.4
1a	4	14.8	**1b**	1	3.7
2g	3	11.1	**3c**	1	3.7

2b, 2g, and 3b are correctly drawn figures for the rectangle. The reason why 2g was chosen less frequently may be that it is quite narrow and looks like a line. In general, it is seen that most of the students were unable to identify the correct models of rectangles. The students' incorrect choices were mostly parallelogram and rectangles with rounded or cut corners. One of the students in the clinical interview considered 2c as a rectangle. The student was asked why this shape was a rectangle since it was a different choice from other marked shapes. The dialog with the student (F2) is as follows:

Researcher: What do you think a rectangle is?
Student: A rectangle is a long thing like this, and this one looks short because it is pressed here (at the corners).
Researcher: What about this one? (3c)
Student: It is also rectangular. They shake it like this (Figure 1), that's how it happens. For example, a napkin is rectangular, when you shake it, it becomes like this (3c).

Figure 1: Images of a student shaking a rectangular piece of paper

When the dialog above is examined, it is understood that the student imagines the rectangle in 3D space and thinks about the image of the rectangle in case it moves. Although it is not appropriate, it is noteworthy that the student imagines the shape in 3D space. In the interviews conducted with the student for other rectangles and other geometric shapes (square, triangle, circle), it was noticed that the student had a similar way of thinking.

First graders' conceptions of circles

Table 4 presents the findings for the conceptions of circles with descriptive statistics.

Table 4: Descriptive statistics for the conceptions of circles

Shape	*f*	%	Shape	*f*	%
3a	23	85.2	**3c**	2	7.4
1c	14	51.9	**3d**	2	7.4
2d	11	40.7	**1b**	1	3.7
2a	6	22.2	**1e**	1	3.7
2c	4	14.8	**2b**	1	3.7
2e	4	14.8	**3b**	1	3.7
1a	2	7.4			

1c, 2d, and 3a are correctly drawn figures for the circle. As with the square model, most of the student choices are correct models for the circle. 2d, the correct model with the lowest proportion, may have been chosen less often due to its small size as in the square. Interestingly, 2a and 2c, which are polygons, are the most marked models among the incorrect choices. This may indicate that students refer to the circle as being "round" and are unaware that a circle has no corners. During the interviews, when a student (F2) was asked what a circle was, she defined a circle as "A circle, round." On the other hand, when the student was asked about the two non-circle geometric shapes chosen by the student during the clinical interview, it was determined that although the student had the correct knowledge about the circle, she also marked different shapes because she thought about the movement of the circle by thinking in 3D as in her previous statements.

Conclusion

Students correctly identified triangle and circle shapes with the highest percentage. The rate of identifying the square is also quite high and the lowest percentage of success belongs to rectangles. The students' preferences were mostly prototypical shapes, and shapes with curved corners or edges that resemble the prototype were also marked. This shows that students refer to the similarity to the original shape, ignore the fact that the edges should be straight line segments, and some students think in three dimensions and mark curved edges as models. The fact that the shape is small, pressed or rotated prevented the correct identification of the shapes.

As the literature suggests students commonly fail to correctly define angles and triangles, often due to errors in concept images and reasoning (Kusno & Sutarto, 2022). Young children (5-6 years old) can identify prototypical triangles but struggle with different types, sizes, and orientations, relying heavily on visual prototypes (Dağlı & Halat, 2016). Preschool children also struggle to identify squares in rotated directions and smaller sizes, indicating a lack of flexibility in their geometric understanding (Halat & Dağlı, 2016). Learner-generated examples reveal that prospective teachers often have a limited understanding of what constitutes a proper mathematical definition of a square,

highlighting the need for better pedagogical strategies (Zazkis & Leikin, 2008). Geometric knowledge about squares is developed more through experiential learning rather than developmental stages, suggesting that hands-on activities and real-life applications are effective in teaching this concept (Seah et al., 2016). Many students believe that rectangles must always be regularly shaped and do not recognize irregularly shaped rectangles (Guncaga & Zilkova, 2019; Ningrum et al., 2018). Students' conceptions of circles are often fraught with misconceptions and cognitive challenges.

Research indicates that early and varied exposure to geometric shapes, combined with effective teaching strategies and materials, significantly enhances students' understanding of geometric concepts. Teachers' professional development and the integration of technology further support this learning process. By addressing these factors, educators can better support students in developing a strong foundation in geometry.

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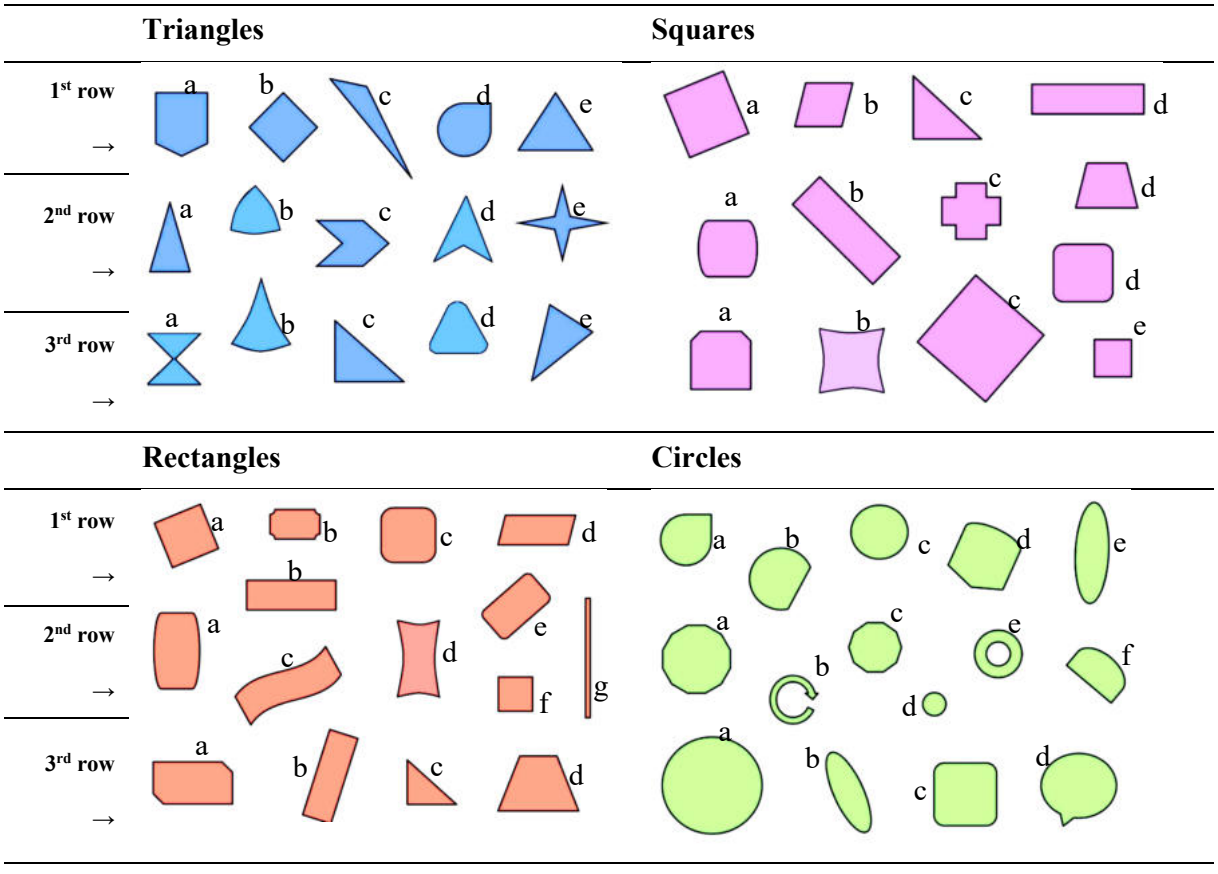
Appendix



Encountering improper points through geogebrization

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Using Radford's Theory of Objectification, this study analyses the learning process of two undergraduate students engaging with a geogebrization activity in the realm of projective geometry, focusing on the operation of projection from improper point as a centre.

Keywords: Geogebrization, projective geometry, improper elements, theory of objectification.

Premise and rationale

This paper stems from my doctoral thesis (Del Zozzo, 2024a), which focused on defining a process called *geogebrization* (GGBZ) *of mathematical texts*[1] (Del Zozzo, 2023; 2024b) and studying its potential through a specific case study involving excerpts from Guido Castelnuovo's (1904) didactic treatise on the foundations of synthetic projective geometry. For the experimental phase, I designed an activity that I proposed to four pairs of participants, each representing an expert perspective on relevant aspects of this process. However, none of the eight participants had previously studied projective geometry using Castelnuovo's synthetic approach or utilized GeoGebra to create dynamic and interactive visualizations. This allowed me to gather insights into how such an approach could enhance the understanding of fundamental concepts in projective geometry, such as improper elements and the operations of projection and section. In this paper, I present an example from the work of two undergraduate mathematics students, analysing an emblematic episode involving the graphical management of projection with improper elements, through the lens of Radford's (2021) Theory of Objectification (TO). This study aligns with international research examining the integration of digital technologies and the history of mathematics, particularly in geometry, where dynamic geometry software and (so far mostly) Euclid's *Elements* serve as focal points. For example, Thomsen and Jankvist (2022) demonstrated how GeoGebra-based activities with 7th-grade students fostered mathematical skills, while Prieto and Arredondo (2021) used TO to analyse prospective teachers' learning through construction tasks based on Euclid's propositions.

This paper aims to achieve two objectives: first, to demonstrate the application of TO in a previously unexplored area, and second, to illustrate how GGBZ activities involving historical texts can influence the learning (in Radford's sense) of advanced mathematical objects that participants have partially encountered before.

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[1] In a few words, it is a process by which one or more learners transform a printed mathematical text into an *appropriate combination of resources* in the GeoGebra Service Platform (GSP, i.e. the website www.geogebra.com). The goal is to create a product made public on the GSP that can be used for communicative, popularization, or educational purposes. By an *appropriate combination of resources*, we mean the set of choices and actions taken by the individuals involved in the GGBZ of the text, who exploit the opportunities provided by the GSP in such a way as to satisfy two requirements: i) unpack and materialize the mathematics embedded in the original mathematical text by leveraging the functionalities of GSP; ii) increase the communicative potential of the original text.

Theoretical framework

The TO is based on the Cultural-Historical Activity Theory and is part of sociocultural theories, which conceive of human beings—what they think, do, feel, imagine, and dream—as deeply interconnected with the culture in which they live (Radford, 2021, in particular p. 17). In TO, mathematical objects are considered to exist as cultural artifacts that are generated and materialized during human activities. Mathematics is consequently conceived as something that is both ideal and material, produced by the activity of teachers and students.

This vision necessitates reconceptualizing, from a cultural-historical perspective, the concepts of *knowledge*—which refers to a general and dynamic entity, a potentiality, the generative capacity to do something or think in a certain way—and *knowing*—which is the actualization and materialization of knowledge through human *activity* into something concrete that can be sensibly perceived. In our case, Castelnuovo (1904) represents a cultural artifact. It is an object in a cultural space-time that actualizes historical knowledge, which is the projective (and analytical) geometry of that period (or at least a part of it).

Such a vision of knowledge and knowing corresponds to (and requires) a particular vision of *learning*. In fact, in TO, the study of learning cannot ignore the specificities of the mathematical knowledge that is learned. Knowledge, as a collective product of previous generations, is something that exists in culture independently of specific individual learning. Therefore, a person can neither build nor possess knowledge. It is something that individuals encounter throughout their lives, and the process of encountering cultural-historical systems of thought is what, in TO, is called the *process of objectification* (Radford, 2021, in particular p. 71). What triggers a process of objectification is human activity—social, intellectual, corporeal, sensorial, and semiotic—during which individuals encounter the cultural object that initially opposes them but which, as the activity proceeds, is increasingly transformed into an object of consciousness.

Within this dynamic framework, noticing occurs through what in TO is called *sensuous cognition*, that is, a broad conception of mind and consciousness which includes not only ideal and mental characteristics but also embodied ones such as perception, feelings, and kinaesthetic activity. Within sensuous cognition, the imaginative, perceptive, and sensorimotor aspects integrate, are produced historically and culturally, and the transformation of consciousness due to the encounter with cultural objects corresponds to (and allows) new ways of perceiving the world. In fact, learning as a process of objectification results in what Radford calls a *domestication of the eye*, a long process that allows individuals—in a cultural-historical activity intertwined with the use of signs and artifacts—to transform the eye (and other senses) into sophisticated theoretical organs capable of becoming aware and giving meaning to the mathematical knowledge encountered through activity.

In the development of the process of objectification, what in TO are called *semiotic means of objectification* are crucial:

> These devices are fundamental to the manner in which things become progressively noticed. These objects, tools, linguistic devices, and signs that individuals intentionally use in social meaning-making processes to achieve a stable form of awareness, to make apparent their intentions, and to

> carry out their actions to attain the goal of their activities, are called *semiotic means of objectification*. They are semiotic in that they are key pieces in the production of meanings embedded in the processes of objectification. (Radford, 2021, p. 100, emphasis in original).

The semiotic means of objectification employed sculpt the forms and modes of activity and are decisive with respect to noticing and the way in which new ways of perceiving the world develop. In general, the role of semiotic means of objectification is so significant that analysing them within the context of the human social activity that gives them meaning allows us to study the objectification process and, therefore, to detect and study learning. The methodological construct that allows us to describe and interpret learning—as a process of objectification mediated by semiotic means—is the *semiotic node* (Radford & Sabena, 2015). A semiotic node is a segment of teaching/learning activity in which a nucleus of the objectification process is identified, evidenced by a rich and complex organization of semiotic means of objectification that allows the emergence of new objects of consciousness. Therefore, the search for semiotic nodes and the analysis of semiotic means of objectification provide important information regarding how learning is taking place (Radford, 2021). The deeper consciousness that subjects acquire in the process of objectification gives rise to a subsequent phenomenon called *semiotic contraction*. Indeed, the process of objectification allows a reorganization of semiotic resources, in which decisions are made about which of them are relevant or irrelevant (Radford, 2021, in particular p. 102). In other words, the process of objectification—evidenced by the semiotic node—occurs through a transformation of the eye with which the mathematical object is perceived, leading to a refinement of the semiotic resources deployed during the semiotic node itself. In TO, semiotic contraction is considered evidence of learning.

Within the framework of the TO, I can now formulate the research question: when engaged in a GGBZ activity involving an excerpt from Castelnuovo's (1904) regarding the operation of projection from a point, how does the domestication of the eye develop if the centre is improper?

Methods

We consider a pair of undergraduate students in Mathematics (LT1 and LT2) engaging with a GGBZ activity structured as a *GeoGebra Book* and delivered via *GeoGebra Classroom*. The activity, described in Del Zozzo (2023, 2024a, and 2024b), was originally in Italian but is available in English at https://www.geogebra.org/m/ksxsjc3v. The entire activity consists in nine problems regarding the basic tenants of projective geometry (improper elements, projections and sections); each problem starts with an excerpt of the original text along with a sequence of related tasks to do. For the sake of clarity in reading what follows, I report in Figure 1 the third problem.

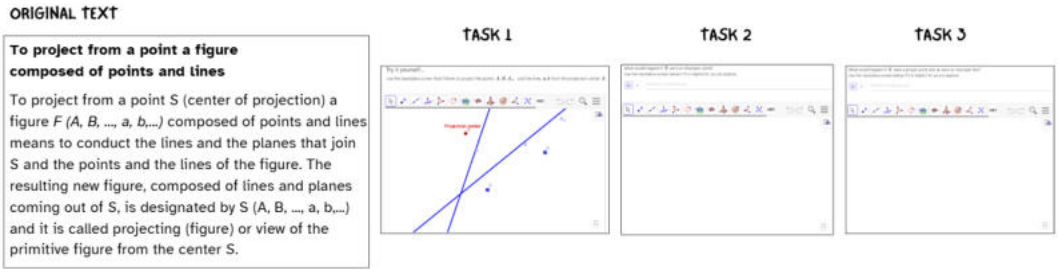

Figure 1: Problem 3 regarding projections from a point

LT1 and LT2 completed the entire activity in four meetings, each lasting between 1 hour and 30 minutes and 2 hours. Each meeting was videotaped from two sources: an external camera and the PC screen used by the participants. The unit of analysis considered is the GGBZ activity segment, and the main research data are the video recordings of the meetings, in a side-by-side edited version, along with all the students working protocols (both digital and material). For the video analysis, I built on Radford and Sabena (2015), and I have outlined an analytical scheme to identify the semiotic nodes: i) Identification of the presence of a conflict that triggers a thoughtful reflection (e.g., silence); ii) detection of an increase in the use of semiotic means of objectification; iii) detection of semiotic contraction indicators. I report in what follow a summary of a semiotic node that illustrates LT1 and LT2's domestication of the eye while managing the operation of projection from an improper centre. For the reader's convenience, I have highlighted specific passages in the transcriptions that are referenced during the analysis.

The encounter with the nature of improper points

In the middle of the first meeting, LT1 and LT2 were engaging with Task 2 of Problem 3 (Figure 1). In this task, building on the situation proposed in Task 1, students are asked to explore the case where the projection centre is an improper point, using an empty GeoGebra screen to perform the construction. To do so, they must manage an improper point as the centre of projection, materializing both the initial construction and the projection of various elements within GeoGebra. In this task, handling the projection of a proper line and another improper point creates a significant conceptual shift from Task 1, representing a conflict between the students' existing knowledge and the one they are encountering. As a first step, LT1 and LT2 construct in the provided GeoGebra screen the initial situation from the one present in Task 1 (two proper points, one improper point, and two proper lines). At this point, they have to insert the centre of projection which, this time, is an improper point:

1 LT1: Ah..now we have to take *S* which we want a point at infinity that is a line basically [...]

An identification between a point at infinity and a line emerges from LT1's words. They insert a new line into the construction, colour it red, and label it with the letter *S* (Figure 2).

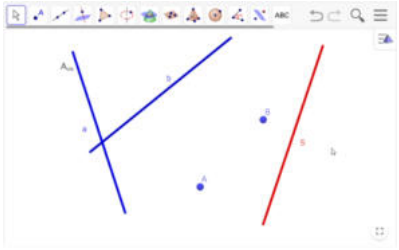

Figure 2: Starting situation

After this, they silently reread the original text (indicator of conflict). After the silent rereading, LT2 opens the conversation and as LT1, she also identifies the improper point with a specific, fixed line. Due to this identification, LT2 share with LT1 her conflict in imagining how to handle the case where the proper line to project is skew with the "line-improper point". Indeed, since it is the projection of a line from a point, one should obtain a plane that contains that point and that line, that results in an impossible operation if the improper point is conceived as a fixed line, which is skew with the proper

line to project. In these first moments, there is an initial increase in gesturing. The discussion continues, and after a while, LT1 again proposes to reread the text but this time does so aloud:

2	LT1:	Let's try to start from here *((places her hand on the printed sheet with the original text that begins Problem 3))* where it tells us... *((starts reading, following the text with her finger))* "to project whatever it is means to draw the lines and planes that join *S* to the points and lines of the figure". So we have *S*, and we take a point—what does it mean to join *S* with a point *((has picked up the mouse and points the cursor at point A on the GeoGebra screen))* ... it means... to draw the parallel line, I'd say.
3	LT2:	Yes
4	LT1:	So you'd have the parallel line like this *((moves the mouse cursor parallel to line S passing through B))* and the parallel line like that... but the points didn't particularly bother us [LT2: Yes]. Now we want the lines, which means *((returns to the text indicating while reading))* […] to draw the plane that joins *S* to the line.

In 2 to 4 new sings appears: verbalizing the words read, paced by the movement of her finger; the subsequent verbal reformulation accompanied by indexical gestures on the screen; and verbalizing the actions that materialize the projection of a proper point from an improper point. The new signs lead to a new perception of the situation, which materializes in the subsequent exchange.

5	LT2:	Yes *((silence))* so we'd want to take a plane that has the same orientation as *S*... but...
6	LT1:	Eh, but *S* is a single direction *((points index finger upward))*; it can't define the orientation of a plane [LT2: Ah, right, true].
7	LT1:	It would mean... drawing the plane that joins... wait *((takes the mouse))* so if with the point we draw the parallel line... with a line *((with the cursor, she points first to point B and then traces the line parallel to S passing through B; then she moves the cursor over line b))*... eh no, but we can't draw the parallel plane *((puts down the mouse))*—the plane parallel to what?
8	LT2:	I mean, the doubt that arises is that *((points to the screen with her hand))* if we could draw a plane that, let's say, contains the plane *((first with a flat hand and then traces S on the screen with her index finger))*—that is parallel to the plane containing *S ((moves her flat hand and then points again to S))*—the intersection *((points to line b))*, let's say *((holds out her flat hand with palm facing upward))*, between the shifted plane *((moves her hand upward continuing to keep the palm facing upward))* toward the other line is a point; that is, if they are not *((flat hand))* parallel, the two lines among themselves *((points on the screen first to line S and then to line b))*, we would have a plane *((flat hand moving as if "slicing" the air))* that intersects *((points to line b))* one of the points of the line.
9	LT1:	Eh, but *((at the PC, moves the mouse cursor along S))* you say the plane that contains *S*, but there are infinite planes that contain *S*, so how do you choose it...? [LT2: Mh] wait *((brings closer the printed sheets related to the problems already solved but doesn't immediately read their content))*.
10	LT2:	I mean, we'd have to choose it based on... no, let's say *((points to the screen))* we choose it by taking a point from the line *((brings her hand closer and points to line b))*, but it seems silly to me *((moves her hand away from the screen and brings it to her chin))*... because obviously, by taking any point on the line, we have a different plane.

We assist here to a new perception of the improper point: between 5 and 9 it is no longer identified with a fixed line but with a *direction*. This shift illustrates the domestication of the eye occurring.

Moreover, in line 7, LT1 seems to act similarly to what was done in line 4, but the phrase she utters presents a substantial difference. Indeed, the case of projecting the points is quickly dismissed in 4, while in line 7, the two situations are related with an "if". LT1's consciousness shifts as she notices the impossibility of drawing a plane with only the condition of parallelism for a line because it's not unique. In line 8, LT2's coordination between the hand gestures and the words "plan" and "plan moved", materializes the possibility of perceiving the situation dynamically. Although the encounter with the cultural object is still partial and in process, the dynamism introduced is refined and clarified in the lines 9 and 10. In fact, responding to LT1's question (line 9), LT2 proposes choosing it "by taking a point from the line", while pointing on the screen to line *b* (10). But she also shows insecurity and after line 10 there is a 13-second silence, which is interrupted by LT1, who again repeats the words describing the type of action required to project a line from a point. From now on, we face a turning point of the activity. LT1 and LT2 then return to their work on previous tasks and problems. In Problem 1 they managed the action of joining an improper point with a proper one (that is conducting a line that passes through the proper point and has the direction indicated by the improper one). In Problem 2 they managed the action of joining an improper line with a proper point (that is conducting a plane that passes through the proper point and has the direction indicated by the improper line). LT1 speaks again:

11	LT1:	Earlier, we were joining a proper point with an improper line...
12	LT2:	While now we are joining... a
13	LT1:	A proper line and a line—and an improper point *((seems to correct herself))*... but if we take a point on a line and we have to join it to the point at infinity as we did here [...] so we make a point [...] *((constructs a point on the proper line))* let's say we fix this point [LT2: Okay], it doesn't matter how we chose it... so we have a proper point; no, here we don't have an improper line—we have an improper point *((had in her hands the printed page related to Problem 2: changes page and takes the one related to Problem 1))* means that we have to... we have an improper point... sorry, uh *((at the PC, scrolls with the mouse wheel to display the instructions of Task 2 and then returns to the GeoGebra screen))* yes, we have an improper point—a point at infinity.

In this passage, the line and the point at infinity are no longer identified: in 13 "a line" is immediately substituted by "a point at infinity" an expression that is retained in the sequel as well, with no longer feeling the need to identify it with a *precise* line. In 13 LT1 is perceiving line *b* in a new way: as *the set of its points*. From now on, the encounter with cultural-historical knowledge related to the operation of projection from an improper point begins to fully take place:

14	LT1:	But here we have the first exercise [LT2: Yes] *((returns again to the original text that begins Problem 1))* "join a proper point with a point at infinity" *((rereads while following the text with her finger))* means drawing the parallel line *((moves her hand, pointing in the air to her left. Then, she takes the mouse back in hand))*.
15	LT2:	Ok
16	LT1:	...So if I choose another... and then it comes out... *((on the GeoGebra screen, she begins to construct the line parallel to S that passes through C, a point on line b to be projected))*.
17	LT2:	It practically results in another line.

18	LT1:	It comes out....*((constructs the black line, see Figure 3a))* yes a pencil of parallel lines because if I now take another point *((constructs a new point she calls D on line b, see Figure 3b))* and....it would come out like this *((constructs the line parallel to S passing through the new point, see Figure 3c))* wait because I perform the magic *((activates the "show trace" command on the parallel line just constructed and begins to drag point D along line b, see Figure 3d))*...show trace…
19	LT2:	Ok
20	LT1:	And I do this *((continues to drag D along b with the trace active on the line, see Figure 3e))* eh yes, [LT2: and so we get] […] an improper pencil of lines […]

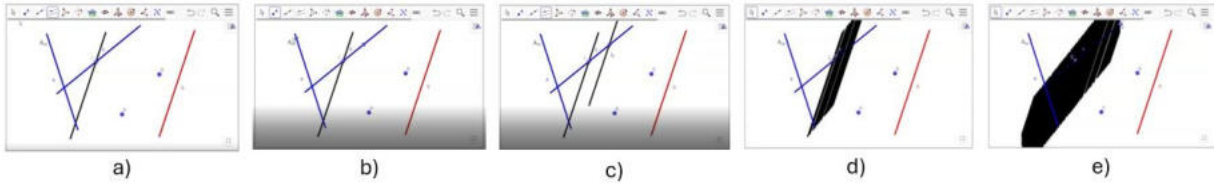

Figure 3: screenshot of lines 18-20, video here https://tinyurl.com/f52224

What happens between lines 14 and 20 is crucial and represents the condensation of everything that has occurred in all the previous activity segments. LT1 is aware that she can choose a point on line *b* to join it with the improper point serving as the projection centre (14); Moreover, perceiving *b* as the set of its points enables LT1 and LT2 to notice in 16 and 17 the possibility of applying the same procedure to another point on *b* generalizing to any other point in 18. Through LT1's interaction with GeoGebra's functionalities described between lines 18 and 20, the process of domesticating the eye becomes completed, and the cultural knowledge materialized by dragging point *D* with the trace active. In conclusion of Task 2, LT1 and LT2 produce a written answer that moves them to a sudden refinement in the semiotic resources used to describe what happens to the operation of projecting a proper line when considering an improper projection centre. We are witnessing a semiotic contraction, evidence of learning, which marks the end of the semiotic node under analysis.

Discussion and conclusion

Through the lens of TO, this study has illustrated how two undergraduate students engaged with the operation of projection from an improper centre in a synthetic approach to projective geometry. The GeoGebra-based activity not only facilitated the students' interaction with historical mathematical texts but also allowed them to actively construct and visualize complex concepts dynamically.

The research question under consideration is: when engaged in a GGBZ activity involving an excerpt from Castelnuovo's (1904) regarding the operation of projection from a point, how does the domestication of the eye develop if the centre is improper? The analysis of the episode shows the students' gradual domestication of the eye, wherein their perception of the improper point shifted from initially identifying it with a precise line, then conceiving it as a direction and finally as the common direction of a pencil of improper line (where we can choose the one we want). This shift occurs within a process of objectification that has been described analysing how the use of semiotic means of objectification develops in the social interaction between LT1 and LT2.

The insights gained from this case study suggest several directions for further research. First, expanding the sample to include a broader range of participants, searching for patterns and invariants. Moreover, it may be argued that GGBZ activity of a historical text on projective geometry could be chosen as an experimental activity to study how learning in this area of geometry develops. Finally, here we are considering only the classical synthetical approach to projective geometry, but it could be interesting to explore a GGBZ activity that includes also a connection with the more modern and analytical-algebraic approach (i.e. homogeneous coordinates).

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Theories for the use of both physical and digital tools for developing geometrical reasoning

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In teaching geometry, the use of both physical and digital tools has been identified as useful ways to develop students' geometrical reasoning. To gain insight into the effective use of these tools for geometrical reasoning, in this paper we reviewed relevant theories in mathematics education. We considered several different theoretical ideas, such as instrumentation theory, the theory of semiotic mediation, geometrical reasoning, figural concepts, etc. By synthesising and combining these theoretical ideas, we provided examples of questions to critically examine task design and learning processes with both physical and digital tools in the context of geometric reasoning.

Keywords: Theories in mathematics education, task design, the use of physical and digital tools, geometrical reasoning

Introduction

In CERME13, we can observe that many papers reported on the effects and influence of the use of different tools, both physical and digital, in the teaching and learning of geometry, e.g. the use of 3D models and GeoGebra (Bímová, et al., 2023), the use of 3D pens (Palatnik, 2023), and so on. On the other hand, as described by Palatnik et al. (2023), the consensus of the CERME geometry WG members is that we still do not know much about how different tools can be combined, especially physical and digital tools, what learning can be achieved or designed, how learners interact with these different tools and what role teachers would play (see also Nemirovsky & Sinclair, 2020).

In this paper, we will address one of the themes identified in the CERME14 TGW04, namely the interplay between different tools and their effectiveness on students' learning. We focus on this topic as it was stated that "the intertwined use of physical and digital tools is a fruitful research area for exploring ways to enhance students' mathematical learning" (Komatsu and Fujita, 2024, p. 274). We take a theoretical approach as "a need for a more coherent understanding of theories from the community" was recognised in the CERME13 geometry working group (Palatnik, 2023, p. 3), and we will critically review some of the relevant theories related to tools in the teaching and learning of geometry.

The aim of this paper is to provide examples of questions to critically examine task design and learning processes with both physical and digital tools in the context of geometrical reasoning. We focus on the promotion of geometrical reasoning as this was suggested as one of the important tasks in geometry teaching (Jayathirtha, 2018). In particular, we explore our working research question "What questions can be considered to critically examine issues related to task design and learning processes with both physical and digital tools in the context of geometrical reasoning?"

What is a theory in mathematics education?

It is probably not possible to define 'what a theory is' in mathematics education. However, debates about theories have played an important role in mathematics education research, and many ideas have been discussed by many researchers (for more details see Komatsu and Fujita, 2024). For example, Mason & Waywood (1996) refer to the following functions of theories: explanatory ("how or why something happened"), descriptive ("providing a language to frame a way of seeing"), informative (informing "practice by sharpening or heightening sensitivity to notice and act in the future"), and predictive ("what will happen in similar situations by stating necessary or appropriate conditions") (pp. 1060–1). They also state that theory provides a language for formulating questions, specifies a methodology, delineates a coherent and consistent view of the objects of study/the nature and role of data collection and analysis, supports a means of validating findings, and formats the reporting of findings (pp.1061–2). Radford (2008) also identified theory as 'a way of producing understandings and courses of action based on' a system, a methodology and a set of paradigmatic research questions or schemata (p. 320).

In this paper, we focus on the theories related to use of both digital and physical tools in the teaching and learning of geometrical reasoning in order to the complexity of the use of tools and geometrical reasoning. Amongst different strategies to network different theories/theoretical ideas (Prediger et al., 2008), our main strategy is 'combining', i.e. combining various theoretical ideas related to the use of physical and digital tools in mathematics education in general, and geometrical reasoning. In what follows, we will first take some theoretical ideas in relation to the use of physical and digital tools, and then those related to geometrical reasoning. We then consider, by combining different theoretical ideas, examples of questions to critically examine task design and learning processes with the both physical and digital tools in the context of geometrical reasoning.

Theories related to the use of physical and digital tools

In this section, we focus on the use of both digital and physical tools for teaching and learning mathematics. In order to design tasks for students and consider learning processes, we need to consider how these tools can be combined or used effectively. Soury–Lavergne (2013; 2021) used the tool Pascaline to teach base–10 place value notation. She designed both tangible and digital versions of the tool and provided examples of how these tools can be used, considering the similarities and differences between these two tools. Bímová, et al. (2023) used both GeoGebra 3D and tangible models to support students' problem solving with 3D shapes. As a slightly different example, we can also consider the use of digital tools and pen and paper. For example, Komatsu and Jones (2020) investigated how students approached challenging proof problems in geometry through the interplay between digital (GeoGebra) and pen and paper. Although the use of these two tools may sound trivial, the interaction between these tools is not straightforward and careful design and intention was required to support students' mathematical thinking and reasoning.

According to Sinclaire et al. (2023), one of the major theories that have been used in the use of technology in mathematics education is the instrumental approach in didactics. This is rooted "in the work of Piaget and Vergnaud (for the concept of scheme), as well as Vygotsky (for the concepts of mediation, and the social aspects of the instrumentation)" (p. 394). The theory distinguishes between

'artefacts' and 'instruments', and the latter is "a mixed entity that pertains to both the subject and the artefact" (p. 394). The developmental process from artefacts to instruments is called 'instrumental genesis'. In this process, the bilateral relationship between the subject (e.g. learner) and the artefact (e.g. dynamic geometry software) is established. One is *instrumentalization* (from the subject to the artefact), which is described by Artigue (2002) as "it is directed towards the artefact, loading it progressively with potentialities, and eventually transforming it for specific uses" and *instrumentation* (from the artefact to the subject) as "leading to the development or appropriation of schemes of instrumented action which progressively take shape as techniques that permit an effective response to given tasks" (p. 250). This theory has been widely used and influential in mathematics education research. In particular, the focus of the study is centred on 'instrumental genesis', and it gives us a point of reflection as to why some technological tools might be less effective than expected (e.g. this is because the instrumental genesis did not happen), or the particular use of tools and behaviours of students so called action/utilisation schemes, e.g. students use the so–called dragging test to check whether the key properties of shapes constructed with a dynamic geometry software will be retained when one of the points on the shapes is dragged (Arzarello et al., 2002). More recently, Pittalis and Drijvers (2023) extended this idea to capture how utilization schemes can be used to examine students' interplay between embodied dragging and geometrical concepts with multi–touch tablet devices.

When considering the use of both digital and physical tools, the duo–artefacts framework proposed by Maschietto and Soury–Lavergne (2013, see also Soury–Lavergne, 2021) is useful. This idea was rooted in the instrumentation theory as well as didactic situations (Brousseau, 1997). Soury–Lavergne has designed and studied the Pascaline artefacts with two different forms, e.g. tangible and digital Pascaline. In considering the use of both tangible and digital Pascaline, Soury–Lavergne did not consider the instrumentation process separately, but a joint genesis, making explicit the continuity and discontinuity of these two artefacts. In particular, the notion of 'antagonism', "properties or functionalities of one artefact that are inefficient or even divergent from the other" (pp. 6–7), will play an important role in this framework, described as follows. In geometry, Voltolini (2018) used both a ruler/compass and an e–book with a dynamic geometry environment and found that the interaction between these tools in a carefully sequenced lesson developed a new concept of a triangle and a new way of using a compass, contributing to a rich understanding of geometric concepts among primary school students.

Another theoretical idea, the theory of semiotic mediation (TSM), integrates both teaching and learning with particular tools (Bartolini Bussi & Mariotti, 2008). This theory is rooted in Vygotsky as background theory, and concerned with how semiotic meanings emerge in learners through the use of certain tools, and how teachers can organise the learning environment to promote the development of personal meanings into mathematical meanings (Mariotti, 2012, pp. 168–9). In this theory, the notion of semiotic potential is used to consider how we can design a learning environment with artefacts, taking into account how the artefacts can be used by both students and teachers for effective semiotic mediation (ibid., p. 170) with the following three components, namely mathematical content, historical references and utilisation schemes (Bartolini Bussi & Mariotti, 2008; Maschietto, 2015). For example, in order to design and implement learning environments with certain tools, it is useful

to critically examine what mathematical content is represented and how it is mediated by the tools, how the tools were invented and developed historically (e.g., construction tools in dynamic geometry software are rooted in Euclid's *Elements*), and how students will use the tools to achieve given learning goals in their learning.

While instrumentation theory considers the process of instrumental genesis and the learner's scheme of use, TSM considers both students and teachers and how mathematical concepts might be mediated through the use of certain artefacts within communities (of course, instrumentation theory considers the importance of interventions, e.g. instrumental orchestrations). The a–priori analysis is also important to consider the semiotic potential of the tools.

Theories related to geometrical reasoning

In this section we shall focus on theories related to geometrical reasoning. Jones (1998) describes theoretical ideas as "As geometry evolves to encompass the understanding of diverse visual phenomena, it is important to be clear about what is meant by the geometrical reasoning necessary to solve mathematical problems involving visual phenomena, and how such reasoning develops" (p. 29) and listed the van Hiele model of thinking in geometry (e.g. van Hiele, 1986), the theory of figural concepts (Fischbein, 1993) and Duval's cognitive model of geometrical reasoning (Duval, 1998). Similarly, Sinclair et al. (2016) listed the following as 'Theories specifically about the teaching and learning of geometry', the van Hiele model, the theory of theory of figural concepts, theories of figural apprehension and dimensional deconstruction (i.e. Duval), and the theory of spaces for geometric work (Gómez–Chacón & Kuzniak, 2015).

Van Hiele's model has been widely discussed and used in the study of geometric reasoning. Briefly, the theory models the development of geometric reasoning as follows: Visual, Descriptive, Abstract/Relational/Informal Deduction, Formal Deduction and Rigour (van Hiele, 1986). For example, while students at the visual level reason about shapes and their properties according to their concrete examples or visual appearance, students at the abstract level begin to see relationships between shapes and make simple logical deductions.

The theory of figural concepts (Fischbein, 1993, p. 140) is also often referred to when considering the nature of geometric figures, i.e. that geometric figures have both visual and conceptual properties. This notion of figural concept is useful when considering students' geometrical reasoning, for example, how students use their visual or conceptual reasoning when attempting to classify quadrilaterals hierarchically (e.g. Fujita & Jones, 2007).

Duval considered there are at least three cognitive processes, visualisation, construction and reasoning (Duval, 1998). Duval's theory of figural apprehension identifies four types of apprehension: perceptual apprehension, sequential apprehension, discursive apprehension and operative apprehension. 'Apprehension' means "the act of understanding something, or the way that something is understood" (Cambridge Dictionary). These 'apprehensions' are useful to consider students' capabilities to solve various geometrical problems. For example, perceptual apprehension is related to the perception of elements of figures. The operative apprehension might be used by students to manipulate, extract or divide certain elements of the given shapes (which might be represented by diagrams), Michael–Chrysantho et al. (2024) describe, for example, "operative

apprehension becomes a starting point in order to investigate other configurations that can be obtained by visual operations" (p. 4). The sequential apprehension might be utilised when students describe how they construct certain geometrical shapes, e.g. constructing an equilateral triangle by using a ruler and compass, or dynamic geometry tools.

Task design and learning processes with the use of physical and digital tools for developing geometrical reasoning

In this section, we shall provide examples of questions to critically examine task design and learning processes with the both physical and digital tools in the context of geometrical reasoning. A theory can be used to consider language to formulate questions, to specify a methodology, to delineate a coherent and consistent view of the objects of study and role of data collection and analysis, to support a means for validating findings, and to format reporting of findings (Mason & Waywood, 1996; Radford, 2008). Due to the limited space of this paper, we will explore what 'language to formulate research questions' can be considered in terms of task design, combining different aspects and ideas from the theoretical ideas reviewed in the previous sections.

As an example, let us consider a problem how to find the centre of spinners with triangles and quadrilaterals using the case of equilateral triangle.

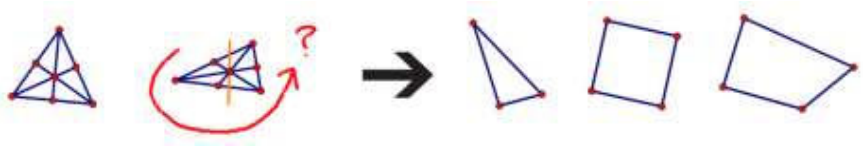

Figure 1: Triangle and Quadrilaterals spinners

Mathematically, a centre of gravity will be an ideal point for spinners and this could be a useful activity as some students may think that a centre of gravity does not exist if a triangle is not equilateral or quadrilateral, relying only on their visual perceptions. Providing a context (spinner) might also encourage students to understand the importance of the centre of gravity from a practical and theoretical perspective. Now suppose we want to design activities and a sequence of these activities to promote, for example, visual to more formal ways of thinking in geometry (van Hiele, 1986). Then, informed by the theoretical ideas reviewed in the previous sections, some questions can be critically explored, which are answers to our working research question as well as our conclusion of this paper.

Task design: when designing tasks from this problem context, we should examine first what semiotic potentials (Bartolini Bussi & Mariotti, 2008) can be considered in order to promote geometrical reasoning by using both physical and digital tools in this particular activities. Historical and cultural contexts might be important as spinners are known as traditional toys in Japan, even though this is not directly related to mathematics. When we consider the different tools to be used, we need to critically examine what 'antagonism' (Maschietto & Soury–Lavergne, 2013, Soury–Lavergne, 2021) between physical and digital tools might be used to promote their conceptual understanding of geometrical figures (Fischbein, 1993) or their geometrical reasoning (van Hiele, 1986). For example, it is possible to physically experiment with physical spinners, but not with digital tools. On the other hand, students can use digital tools to explore various possibilities of the points, and this might

provide some reasoning to explain why a particular point within triangles or quadrilaterals work well as a centre of gravity.

Learning processes: Defining and experimenting with a centre of gravity with/without a dynamic geometry environment could encourage students to use different sequential understandings (Duval, 1998). We can also examine students' use patterns, shaped by the instrumental genesis of the duo–artefact (Artigue, 2002; Soury–Lavergne, 2021), and how these patterns might be observed in their use of conjectures and related to their figural concepts (Fischbein, 1993). For example, some students might assume that an intersection of diagonals could be the centre of quadrilaterals from their visual perceptions (van Hiele, 1986), but dragging one of the vertices might provide opportunities to explore their reasoning further by considering the cases of triangles and the theoretical meaning of the centre of gravity (Figure 2).

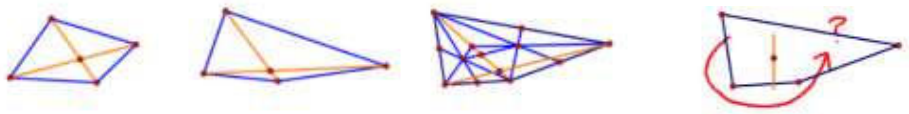

Figure. 2: Investigations with the case of quadrilaterals

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Fostering understanding of inclusion relationships through mathematical events

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This study examined the impact of analysing mathematical events on prospective teachers' understanding of inclusion relationships among quadrilaterals. Initially, most participants struggled to identify these relationships, relying on prototype examples rather than shape definitions. After an intervention involving mathematical event analysis, participants showed significant improvement in recognising inclusion relationships, using geometrical definitions and critical attributes to explain their responses. This approach enhanced their ability to extend concept images to non-prototypical examples.

Keywords: Mathematical events, inclusion relationships, prototypes, critical attributes.

Introduction

Inclusion relationships among geometric shapes are fundamental to advanced mathematical thinking, as outlined in Van Hiele's (1958) model. At the third level of geometric understanding, mastery of these hierarchical connections is crucial. This comprehension serves as a gateway to higher-order reasoning, particularly in constructing geometric proofs. Without a solid grasp of inclusion relationships, students face significant barriers in progressing to more complex mathematical concepts and problem-solving techniques. Research consistently reveals widespread difficulties in comprehending hierarchical relationships among quadrilaterals. Okazaki and Fujita's (2007) cross-cultural study of Japanese students and Scottish teachers found shared challenges in categorising rectangles as parallelograms and squares as rectangles or rhombuses, with subtle differences between these two populations. Zeybek's (2018) study of pre-service math teachers in the United States revealed initial similar struggles, with improvement following targeted instruction, highlight the pervasive nature of these conceptual challenges across diverse groups.

Numerous studies have highlighted the significant impact of non-critical attributes in prototype examples on students' misidentification of inclusion relationships among geometric shapes (e.g., Okazaki & Fujita, 2007). These non-critical attributes, often characterised by strong visual elements, frequently override formal definitions in students' reasoning processes. The powerful influence of visual factors is exemplified in Haj-Yahya and Hershkowitz's (2013) unexpected finding that 10th-grade students performed better on verbal tasks than on visual ones when identifying inclusion relationships between squares. This counterintuitive result underscores the potent effect of visual cues on geometric reasoning. In a recent study, Haj-Yahya and Hershkowitz (2024) uncovered significant challenges in geometric concept formation among 11th-grade students. Their analysis revealed that many students excessively rely on prototypical examples, a tendency that severely impedes their ability to construct proofs related to the attributes of these concepts. This finding illuminates a

significant shortcoming in geometry education: students' limited ability to extrapolate from specific instances to overarching geometric principles.

Various interventions, including dynamic geometry software and concept-mapping techniques, have been employed to enhance students' understanding of inclusion relationships in geometry (Haj-Yahya et al., 2024). While these approaches have shown promise, the specific impact of event analysis remains under-explored. Event analysis offers a unique opportunity to deepen students' understanding. By providing students with contextualised scenarios, event analysis can make learning more engaging and meaningful. Additionally, it can encourage students to actively apply their understanding and justify their reasoning, fostering a deeper level of comprehension. Drawing on prior research, the importance of mathematical defining in teacher professional development is well documented. Studies like Kobiela et al. (2023) and Climent et al. (2024) emphasize that developing teachers' knowledge of mathematical defining requires attention to both pedagogical and mathematical aspects, supporting teachers in understanding not only how to teach definitions but also the mathematical principles underlying definitional practices in geometry.

Mathematical events in educational research

Mathematical events are occurrences in mathematics classrooms that present pedagogical challenges requiring teacher intervention (Markovitz, 2003). According to Conner et al. (2011), a mathematical event can emerge from a student's question, an ambiguous textbook statement, or a teacher's comment. Such events create unique opportunities to explore concepts, make mathematical connections, and reinforce learners' developing understanding. These events have become valuable in teacher education and research (e.g., Tirosh et al., 2019). Engaging with mathematical events benefits teachers' professional development by enhancing their awareness of student thinking patterns and appropriate responses (Markovitz, 2003). Stockero et al. (2020) assert that these events provide opportunities to build upon students' mathematical thinking, facilitating a deeper understanding of critical concepts. Through analysing mathematical events from their classrooms, teachers reported deepening their mathematical understanding, as evidenced by their reflections. This process shows promise as an approach to expanding mathematics teaching expertise (Conner et al., 2011). The significance of mathematical events lies in the discourse they generate, fostering a learning community based on argumentative dialogue where participants articulate reasoning, listen actively, and engage constructively with peers' arguments (Toulmin, 2003). This study examines how prospective teachers analyse mathematical events to develop their understanding and justify the inclusion of relationships between quadrilaterals.

Methods

Research participants and context

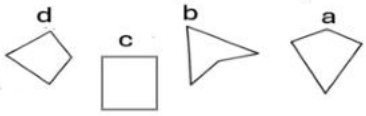
This study involved 20 prospective teachers who are all the participants of a geometry-teaching course. The participants were pursuing their teaching certification for teaching in first and second grades at a teacher training college of the Arab community in Israel. This course serves as a second course in the mathematics education curriculum, alongside a calculus-teaching course designed for first and second graders. The course content was centered on four areas of geometric thinking:

properties of shapes, spatial relationships, transformations and symmetry, and visualization. The teaching and learning process throughout the course was grounded in discussions during mathematical events. The current research specifically focused on three sessions that emphasized inclusion relationships. The three events in the three sessions, highlighting various ways of thinking and addressing common errors in the selected in inclusion relationships. The mathematical events were developed based on literature and the researchers' experience in teaching geometry (see Figure 1) and were presented for all the class. One of the researchers conducted the course, she presented events that encouraged participants to promote understanding of inclusion relationships. Her role was to facilitate learning and motivate participants to independently promote their understanding while providing opportunities for them to explore methods of justification and engage in discussions. She employed a strategic questioning sequence that transformed students' reasoning from visual to analytical thinking, such as "How about examining which features...", "What else do you see about the diagonals?".

Data sources, procedures, and analyses

The data were collected from three sources. The main and first source was observations of class discussions, which were recorded on video and transcribed verbatim. The second and third were pre- and post-questionnaires, which included verbal and visual tasks related to inclusion relationships among quadrilaterals (e.g., Is a square a kite? Yes/No/Don't know. Explain your answer). The pre- and post-questionnaires were conducted within an interval of three months.

Teacher Samir presented the following definition of the kite to his students: "A quadrilateral with two distinct pairs of adjacent equal sides." Then, he showed the following shapes:



Jamil raised his hand and told the teacher that all the shapes are kites except for shape c.
Teacher: Why, Jamil?
Jamil: Because a kite must have two long sides and two short sides.
Teacher: But the definition of the kite is a quadrilateral with two distinct pairs of adjacent equal sides. Does shape c meet the definition?
Jamil: Yes.
Teacher: Can we call it a kite?
Jamil: This is a square, not a kite. This is a square with all sides equal. But in a kite, there is one pair of long sides and one pair of short sides.

Figure 1: The second mathematical event

Initial information was collected regarding participants' understanding through their responses to the pre-questionnaire, followed by participants' engagement in analysing mathematical events. Finally, the post-questionnaire was administered. The pre-questionnaire and post-questionnaire data were analyzed using thematic content analysis (Braun & Clarke, 2006). The researchers categorized the data based on categories provided by Haj-Yahya and Hershkowitz (2013) and additional categories that emerged during the analysis. The observations were analyzed in two stages. The first was based on Toulmin's (2003) argumentation model, beginning with creating an argument log, that based on

observing all discussions in the classroom and highlighting discussions whenever participants drew conclusions. These conclusions were marked, collected, and organised according to the components of the argument model: data, claim, warrant, backing, rebuttal, and qualifier.

Findings

The findings reveal significant shifts in participants' understanding and identification of inclusion relationships among various shapes, particularly squares, rectangles, parallelograms, and kites. This emerged through both verbal and visual tasks administered before and after engagement with mathematical events. The results demonstrate marked improvements in participants' ability to recognize and justify inclusion relationships, moving from narrow, prototypical understandings to broader, more inclusive conceptualisations of geometric shapes. The following sections detail these changes, highlighting the specific improvements in participants' understanding of shape hierarchies and their explanatory abilities.

The findings reveal a significant shift in participants' understanding of inclusion relationships among shapes (see Table 1), moving from misidentification to correct identification. This change is evident in both the participants' responses and reasoning. Participants' judgments were initially based on non-critical attributes exclusive to prototypical examples. However, post-intervention, their reasoning reflects a more sophisticated understanding. Participants began to justify their responses based on formal definitions of the containing group, explicitly explain how one group is included within another, or demonstrate an understanding that the attributes of one shape are fully encompassed within the attributes of another shape.

Table 1: Examples from identification of inclusion relationships - the verbal tasks

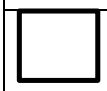
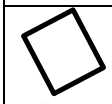
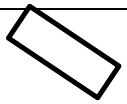
Claim	pre		Post	
	Acceptance of the claim	Examples of justification	Acceptance of the claim	Examples of justification
The square is a rectangle.	4	A rectangle differs from a square because one pair of sides is longer than the other.	17	The attributes of a rectangle are included in the attributes of a square.
The rectangle is a parallelogram.	11	The rectangle has all angles equal to 90 degrees, but the parallelogram does not.	17	The rectangle has all the attributes of a parallelogram.
The square is a kite.	2	Kite sides are not of equal length.	14	A square is a kite with all sides of equal length.

We found significant improvements in participants' ability to identify non-prototypical examples such as squares as rectangles, parallelograms, or kites in the visual task, with substantial increases in correct identifications from pre-questionnaire to post-questionnaire. This trend of enhanced recognition extended to the identification of rectangles as parallelograms, mirroring the positive shifts observed in the verbal identification task and indicating a consistent improvement in participants' understanding of shape relationships across different tasks.

Table 2 shows that participants' identification of the square as a rectangle, parallelogram, or kite improved dramatically from the pre-questionnaire to the post-questionnaire. The same direction was observed regarding the identification of a rectangle as a parallelogram. This tendency was in the same

direction as in the visual identification task. Additionally, as seen in Table 2, the participants identified inclusion relationships that were not directly addressed to them through these mathematical events, particularly the inclusion relationship between a square and a parallelogram. The participants also recognised that while every square is a rectangle, the converse is not true.

Table 2: Identification of inclusion relationships - the visual tasks

The shape	Questionnaire	Identified as a square	Identified as a rectangle	Identified as a parallelogram	Identified as a kite
	Pre-questionnaire	20	7	3	0
	Post-questionnaire	20	20	18	19
	Pre-questionnaire	9	2	0	1
	Post-questionnaire	20	20	15	18
	Pre-questionnaire	2	10	7	0
	Post-questionnaire	0	20	19	0
	Pre-questionnaire	2	20	3	0
	Post-questionnaire	0	20	18	0

Identification of inclusion relationships through analysing mathematical events

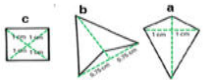
The findings indicate that by analysing the mathematical events, participants developed their understanding according to the identification and justification of the inclusion relationship. Episode 1 describes the development of the discourse regarding inclusion relations between a square and a kite, especially regarding one of the common features: the main diagonal is perpendicular to the secondary diagonal and bisects it. [Due to space constraints, we choose only one episode, and without sequencing the order of speech turns]

Episode 1: One of the common features of kites and squares

1 Rima: I am sure that form a ( ) is a kite, and b ( ) is also a kite, but I am debating about the square ( ) because its appearance is different.

3 Seaham: I want to draw a straight line on the board between all the parallel vertices, which are the diagonals. I will do it like this; it does not matter if it is out of shape. I want to show you this [Seaham drew the diagonals].

6 Maas: The diagonals are perpendicular to each other.
8 Participants: [drawing the diagonals and checking] Yes. All are perpendicular.
10 Participants: The diagonals are perpendicular in all three.
11 Lecturer: What does it mean that the diagonals are perpendicular?
12 Rima: They form a 90-degree angle.
13 Lecturer: Right, they form an angle of 90 degrees. What else do you see about the diagonals?

14	Dalia:	The diagonals bisect.
16	Dalia:	It means that one of the diagonals bisect the other into two equal lengths.
18	Participants:	The longer diagonal bisects the shorter one. 
19	Seaham:	But I have a question. The kite in the middle is not like the others. In all of them, I can clearly see the lengths of the diagonals. But in this kite, the longer diagonal is not bisected into two parts (even if the parts are not equal).
20	Dalia:	It doesn't matter. Actually, this is what they have in common.
22	Rima:	Since the two diagonals are equal in length, it follows that even after they intersect perpendicularly, all their segments are of equal length. I mean, both bisect each other.
24	Seaham:	The other shapes have diagonals of different lengths, and after the diagonals intersect perpendicularly, the segments are also of different lengths.
28	Sema:	Because originally, the square had equal sides.

During the discussion, two arguments were raised Arguments 1 and 2 below, which are presented in Figures 1 and 2. Regarding the inclusion relationship event about a square and a kite, a claim was made by Maas (see Figure 2).

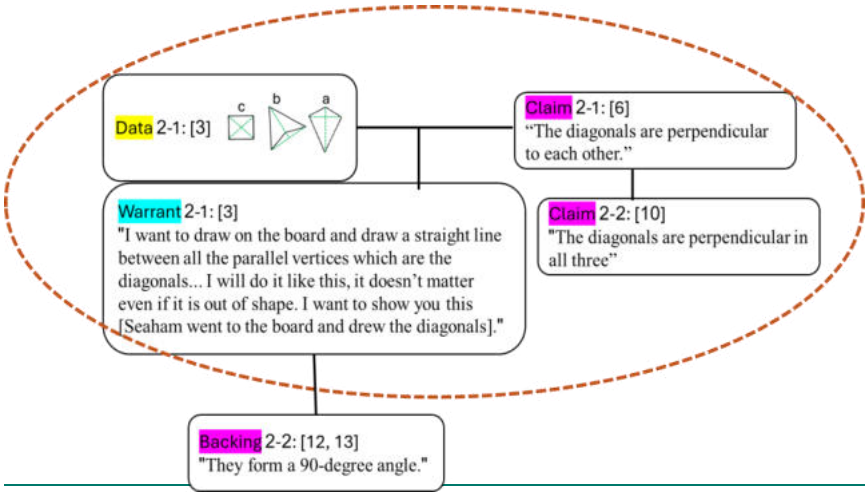


Figure 2: Argument 1- two diagonals are perpendicular to each other

Figure 2 illustrates a process by which the participants identified the critical feature of the perpendicularity of the diagonals in prototypical and non-prototypical kites. By presenting the data, examining it, and concluding that perpendicularity is a property common to the square and the kite, they concluded that it is an important property common to all these shapes.

Figure 3 demonstrates how the participants conclude that one diagonal bisects the other in both the square and the kite. Through the presentation of appropriate data and justifications, they come to the understanding that this is a common and critical feature for all shapes, including the square. Therefore, they conclude that the square has the kite properties. In lines 19 to 23, we see that the prospective teachers discussed what makes a square a special case of a kite, progressing in their understanding of the inclusion relationship between kites and squares. Despite the square being a non-prototypical, the common property that one diagonal bisects another is still present.

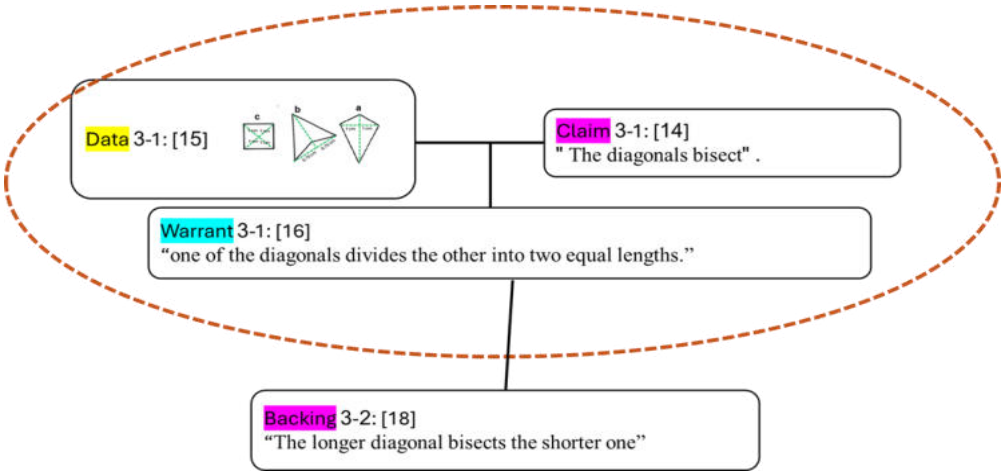


Figure 3: Argument 2- two diagonals, one bisects the other

Discussion

The objective of the current study was to examine how prospective teachers analyse mathematical events to develop their understanding and justify inclusion of relationships between quadrilaterals. The findings indicated that the discussion of mathematical events helped in revealing participants' reliance on non-critical attributes that are exclusive to prototypical examples. Moreover, the need to explain and support their claims led them to use formal definitions or critical attributes, which contributed to them seeing that certain shapes share attributes and belong to broader categories. For example, as presented in Episode 1, the discussion progressed from initial visual uncertainty to a systematic comparison of the properties of a kite and a square, leading participants to identify critical shared attributes and ultimately concluding that a square possesses the properties of a kite. The results that emerged in the current study align with previous studies (e.g., Stockero et al., 2020) that emphasise the effectiveness of engagement with analyses of mathematical events. The application of the Toulmin model effectively highlights how structured reasoning changed, such as presenting warrants and backing in the transition from initial misunderstanding to a more nuanced and accurate grasp of mathematical inclusion relationships between shapes.

Before the intervention, the vast majority of the participants struggled to identify the inclusion relationships between several quadrilaterals in both the visual tasks and verbal tasks. In addition, when they were asked to provide explanations for their responses, the majority of them tended to use the attributes of the prototype examples instead of the definitions of the shapes (see Table 1). In the post-questionnaire, the participants identified the inclusion relationships between the quadrilaterals correctly and provided explanations based on geometrical definitions of the concepts or critical attributes of these concepts. The intervention might have improved their comprehension of inclusion relationships and the development of concept images that are not limited to prototype examples (Haj-Yahya et al., 2024). However, the mathematical events focused on only three inclusion relationships, and not all of the study participants joined the discussion of the mathematical events. The findings revealed improvement in most participants' understanding of inclusion relationships.

Acknowledgments

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Subjective theories of teacher trainers as to how to initiate argumentation and proving processes for geometry in lower secondary school classrooms

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Two experienced mathematics teacher trainers are interviewed to assess the possibilities of proof activities in lower secondary schools that work in everyday school life. The subjective theories of the teacher trainers on how to make the classical proofs of the sum of angles in triangles and the Pythagorean theorem accessible to all are investigated within the research programme of individual curricula. It is striking that their choices of proof procedures are strongly influenced by heterogeneous classroom situations.

Keywords: Secondary school mathematics, Geometry, Classroom Techniques

Introduction

Starting from Styliandes & Styliandes (2007), many CERME activities have been related to argumentation and proof in the classroom (Mariotti et al. 2018) providing teachers and educators with a wide range of tools (Reid &Vargas 2017). In the field of geometry, proof is a classic, but its implementation in the classroom is complex. A broad approach of empirically based discoveries (Jahnke 2006), argumentation and more formal deductive reasoning eventually leads to proof activities (Herbst et al. 2016).

Proving and argumentation

The relationship between proof and argumentation is important for school mathematics (Pedemonte 2002) because proofs are considered difficult and have many facets (Hannah & Jahnke 1996). The important defining characteristics are the formal deductive aspect and the valid reasoning. This means that proof is a process of reasoning from the general to the specific, following strict logical rules, see for example Müller-Hill (2013).

The aim of proof is to deduce the truth of the proposition to be proved from other true propositions. In the process of proof, the freedom from contradiction of the individual propositions, which are linked together during the proof to form a chain of arguments, is central. One of the central aspects of a proof, which plays a particularly important role in the later consideration of proof and proving at school, is the aspect of formal rigour. Perfect rigour is unattainable and undesirable, since too long chains of reasoning would be incomprehensible to the human recipient (Jahnke et al. 2023).

The concept of argumentation, unlike the concept of proof, plays a role in different scientific disciplines and is defined differently in each of them. A widely used definition for 'argumentation' - beyond mathematics education - has been proposed by Van Eemeren et al. (2013) as a 'verbal and social activity of reason that aims to increase (or decrease) the acceptability of a controversial point

of view for the listener or reader, by presenting a constellation of propositions intended to justify (or refute) the point of view before a rational judge.'" (Van Eemeren et al., 2013, p.5)

Duval (1991) sees proof and argumentation as two different activities in the superordinate category of justification. According to him, argumentation follows implicit rules that arise from the structure of language and the utterances of speakers. An argument can occur as soon as arguments are exchanged with someone else, where an argument can be anything that can be used to justify or refute a claim. Proof, on the other hand, is seen as deductive reasoning, in which the operational status of the step is important in the steps of reasoning, rather than the content. In this distinction, argumentation is considered non-deductive (Reid and Knipping 2010). Douek (1999), in response to Duval (1991), presents a further position on the relationship between argumentation and proof. She describes that 'mathematical proof can be seen as a special case of argumentation' (Douek, 1999, p. 129). In this context, her understanding of argumentation, in line with Webster's dictionary, includes both the process of arguing in the form of a logically coherent but not necessarily deductive discourse, and the argumentative text produced by the process of arguing (Douek, 1999).

Beliefs about proof have already been investigated by Furinghetti and Morselli (2011), who conducted an interview study with ten Italian teachers about their beliefs about proof in mathematics education. In a detailed analysis of four teachers of upper secondary school with different specializations, three different main poles could be identified in the teachers' teaching practice regarding proof: the view of mathematics, the perception of the context and the type of attention paid to the students' needs and expectations (Furinghetti & Morselli, 2011).

Methodology of subjective theories: the framework of individual curricula

The idea of the project was to learn from experts in the field of teaching how they would introduce argumentation and evidence into their teaching. The subjective theories research paradigm helps to structure the analysis of the expert interviews. Groeben & Scheele's (2000) subjective theories research programme is a qualitative social psychology research programme concerned with the study of subjective theories and the methodology of collecting subjective theories. The construct of subjective theories arises from the idea of theorizing everyday actions and thus representing the causes and effects of human actions (Groeben & Scheele 2000).

In mathematics education, this has been successfully done in the context of individual teacher curricula, initially developed by Andreas Eichler for the field of stochastics and applied to other areas such as geometry, see (Eichler & Girnat 2011).

Methods

After graduating with a Master of Education degree, every teacher in Germany must attend a teacher training programme for about 18 months, culminating in a final examination to complete their professional qualification. The teacher trainers in these programmes regularly observe lessons in this induction programme and prepare the future teachers for their classroom management. The sample for this study consists of experienced teacher trainers in mathematics in Germany. They come from the same federal state, which ensures that they refer to the same core curriculum.

The study was conducted in three steps:

1. In the first part of the interviews, the interviewees´ beliefs on mathematics as well as on proofs (Styliandes & Styliandes 2007) were investigated.
2. A structured interview on the subjective theories of proving proceeded in four steps:
   2a) proving processes in classrooms,
   2b) proving processes in classrooms on topics on the theory of numbers,
   2c) proving processes in classrooms on angles in triangles,
   2d) proving processes in classroom on the Pythagorean Theorem.
3. After the analysis of subjective theories, a second interview was conducted with each of the respondent to validate the qualitative analysis on the subjective theories based on the first interview.

Study design

A case study is presented here with two participants (teacher trainers A and B) whose attitudes towards mathematical beliefs and proofs (steps 1 and 2a in the steps described in the Methodology section) are comparable. This allows us to focus on their work on geometry in steps 2c) and 2d).

According to the anthropological theory of didactics (Bosch et al. 2005), a mathematical organisation emerges as a response to some kind of question or problematic task and consists of technical, technological and theoretical elements. The idea of steps 2c) and 2d) is to somewhat fix the theoretical block (technologies (θ_c) and theories (Θ_d)) as well as the tasks (T_a) in the practical block somewhat fixed by giving typical schoolbook material for proofs on the sum of the interior angles of triangles and on the Pythagorean theorem. In particular, the theoretical block was well known. The respondents were then asked to describe the techniques (τb) that they would use when carrying out the tasks described in the book material.

Inductive category development (Mayring 2015) was used to work out the teacher trainers' means-end arguments which were subsequently structured in form of a means-end-argumentation (Scheele & Groeben 2000). These are presented for 2c) and 2d) in the form in which they were presented in the validation interviews (step 3).

Analysis of the geometric learning environments on proving

Teacher trainers A and B have similar beliefs about mathematics and proof. They both see geometry as a suitable branch of mathematics because of the structure and clarity of the arguments, of which proofs are an essential part. For the theorem of the sum of interior angles, both teachers illustrate their views by using angle theorems and angle detective tasks to find the necessary arguments.

According to teacher trainer A in 2c), a proof text already requires a higher level of argumentation from the students. In his approach, teacher trainer A provides for graded differentiation measures (see Figure 1), as it may be difficult for some students in a learning group to construct and understand the argument without further help. For strong students it is also possible to give them only the sketch from material I3 and ask them to draw possible conclusions from it.

				Argument chain
			If the students can build arguments for a proof	→ then they can build a chain of arguments
			Building arguments for a proof	
		If the students can retrieve information from sketches on angles in triangles	→ then they can build arguments to be used in a proof	
		Retrieving information on angles in triangles		
	If the students can work with sketches as informative figures	→ then the students can retrieve information from a sketch on angles in triangles		
	Sketches as informative figures			
If the students work at preparing tasks on angle2	→ then the students work with sketches as informative figures			
Preparing tasks for angles				

Figure 1: The means-end-argumentation of teacher trainer A for angles in a triangle

For teacher B, it is important to create a need in the students to prove the theorem of the sum of the interior angles of a triangle by creating a contradiction or a hypothesis in class. Methodically, the teacher has the pupils measure the sum of the interior angles of their own triangle, which they have brought as homework. Teacher B describes how different results are obtained again and again, even though the students are already familiar with the concept of measurement inaccuracy. However, the students always have different 'reasons' why they now get a different sum of internal angles with their triangle.

			Need for proof
		If a hypothesis is generated	→ then a need for proving emerges
		Generating a hypothesis	
	If there are different measuring results	→ then a hypothesis on the sum of inner angles of a triangle occurs	
	Different measuring results		
If the students measure the sum of inner angles	→ then they obtain different results		
Measuring the sum of inner angles in a triangle			

Figure 2: The means-end-argumentation of teacher trainer B for angles in a triangle

These 'reasons' might be that they have drawn a triangle with different proportions or that one triangle is larger than the other. This raises the question of how large the sum of the interior angles really is.

For the Pythagorean Theorem, both teachers prefer decomposition proofs as lines of argument because they make the process clear and understandable. With respect to 2d), according to teacher A, the choice of this proof depends very much on the group of learners and their level of argumentation and ability to deal with symbolic mathematics. Teacher A describes that the independent presentation of a decomposition proof, e.g. with sponge rubber triangles and squares, is very convincing for the students. Furthermore, heuristic strategies such as drawing auxiliary lines, cutting and reassembling are necessary for a decomposition proof. For teacher trainer A, the main aim of the decomposition proof is for the students to understand the principle of the proof and to connect the steps of the argument.

According to teacher trainer B, teachers often choose station learning for the Pythagorean Theorem to do justice to the historical ideas and connections to students' lives on the one hand, and to offer students different approaches to proving the Pythagorean Theorem on the other. Overall, however, the teacher will always emphasize the idea of breaking down the Pythagorean Theorem, as she considers this to be very accessible to the students, in line with the general ideas of simplicity and self-discovery. For example, B claims that students can first discover the statement of the theorem through the tile game.

		If the students perform a dissection proof on their own	→ then the dissection proof plays a persuasive role in the course
		dissection proof for the Pythagorean Theorem	
If the students learn about dissection arguments before	+ when the students can perform a disassembly with foam rubber independently and actively	→ then they can work out independently a dissection proof for the Pythagorean Theorem	
Dissection arguments	**Hands-on work with foam rubber**		

Figure 3: The means-end-argumentation of teacher trainer A for Pythagorean theorem

		Lower obstacle for a symbolic proof	**Understanding the idea of the proof is independent of the symbols**
	If the students work visually and haptically for the proof of the Pythagorean theorem	→ Then the gap to the symbolic level of the proof is not so big.	+ Then even the students understand the idea of the proof who fail at the symbolic level
	Visualising and haptic work	**dissection proof for the Pythagorean Theorem**	
If the students puzzle according to the proof steps on their own	→ then this is a visualising and haptic experience for the Pythagorean Theorem	→ then they can work out independently a dissection Pythagorean Theorem proof	
Material for a dissection proof of the Pythagorean Theorem			

Figure 4: The means-end-argumentation of teacher trainer B for Pythagorean theorem

However, the exploration through the tile game is not sufficient for the learning objectives as it is not a generalizable proof. Furthermore, B describes how the material is also very suitable for teaching because of the internal differentiation it contains.

Discussion

Teacher trainers A and B have similar beliefs about mathematics and proof. But they both share important concerns about the techniques used, which are not at the heart of the discussion about argumentation and proof activities. Although only A explicitly mentions the need for proof, B also pays a lot of attention to it. Similarly, both give a lot of thought to methods of dealing with heterogeneous classrooms in this setting. However, the teachers also set different priorities in the interviews. During the interview, Teacher A repeatedly refers to the meta-level of teaching and the meta-communication with the pupils about the process. At this meta level, Teacher A justifies her planning decisions and involves the pupils in the implementation at the meta level. Teacher B, on the other hand, focused more on the content of the respective sentences in the interview and talked mainly about how to raise the need for evidence, the methodological implementation of the approach and the role of visualization in this. This does not mean, however, that the teachers did not also agree with the other teacher's focus. Some of the validation interviews suggest this. In the original interviews, both teachers addressed the evidence puzzle as a method, but evaluated it differently.

From a methodological point of view, the restriction to the techniques used has not made the discussion very narrow. In previous research on individual curricula, curricular arguments were discussed on a longer scale than a teaching unit. On the contrary, the interviews are more related to classroom situations and the repertoire that teachers have. It will be difficult to conduct many of these interviews, given the particular expertise of teacher trainers, but since evidence is seen as so difficult to bring into the classroom, it might be fruitful to look at variations of evidence and argumentation tasks through the eyes of experts.

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Fostering students' geometric and spatial reasoning in geometry using duo of digital and tangible artefacts

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This study aims to describe and analyse the development of learner's geometric reasoning and spatial skills in geometry teaching and learning with the multi duo of artefact framework in this paper. To achieve this goal, we used duo of digital and tangible tools (Virtual Reality, GeoGebra and traditional tangible artefacts) for design-based research based on our previous study's findings about the instructional design. The data was analysed from our pilot study of spatial geometry experimental learning activities undertaken by grade 7-10 Chinese students. We identified moments of shifts in their geometric reasoning and spatial skills when the students undertook 3D geometry problems with VR, GeoGebra and tangible artefacts. The analysis also showed that students' geometric reasoning levels are effectively improved by using multi duo artefacts.

Keywords: Multi duo of artefacts, Virtual Reality, GeoGebra, spatial skills, geometric reasoning

Introduction

The importance of students' geometric reasoning and spatial skills has been recognized in the teaching of geometry (Fujita et al., 2017), and these can be fostered using a dynamic geometry environment (Sinclair et al., 2016). According to Bímová, et al., (2023), they explored how the students work with the tangible and virtual 3D geometric models. In this context, we will introduce new tools to broaden how students' geometric reasoning and spatial skills can be developed and then help teachers with experienced support and guidance to help students deepen their geometric concepts and properties. The overall goal of this study is using both tangible objects and digital technologies artefact as tools of framework to analysis students' geometric reasoning and spatial skills through their actions and talk related to the task. In this paper, the main research questions we explore are: 'How do students use Virtual Reality, GeoGebra and tangible objects to solve spatial geometry problems?', and 'What geometric reasoning and spatial skills are developed?'

Theoretical Framework

Our theoretical framework focuses on how and what supports a relationship consisting of three major parts: geometric reasoning, spatial skills and the multi duo of digital and tangible artefacts, theorising how the duo of artefacts and geometric reasoning support each other, and how the duo of artefact and spatial skills support each other, and how geometric thinking and spatial skills help students to solve geometric problems. We'll go into more detail about this framework in the coming sub-sections.

The learning of geometric reasoning

Van Hiele's (1986) model proposes that geometric thinking postulates five levels of geometric thinking which are labelled recognition, analysis, logical ordering, formal deduction, and rigor. These different levels of geometric reasoning can be observed. How well shapes are learned and reasoned about depends on the degree of connectedness between the representations used to express the

concept and the individual's ability to visualize and communicate the relationships. For example, Seah and Horne (2019) adapted the development of geometric reasoning moves through levels: (1) visual reasoning, (2) descriptive reasoning, (3) analytic reasoning, (4) relational-inferential property-based reasoning, leading to (Table 1). We identified moments of shifts in their geometric reasoning through qualitative data analysis when they used multi duo of artefacts.

Table 1 Geometric learning progression

Level 1: Visual reasoning 1.1 Pre-recognition. 1.2 Recognition.
Level 2: Descriptive reasoning 2.1 Visual informal reasoning. 2.2 Informal and insufficient formal reasoning
Level 3: Analytic reasoning 3.1 Reasoning about shapes based on properties. However, their definitions are not minimal.
Level 4: Relational-inferential property-based reasoning 4.1 Empirical relations. 4.2 Analysis. 4.3 Logical inference. 4.4 Hierarchical shape classification based on logical inference.

The learning of spatial skills

Lohman (1988) supported the existence of three major spatial skill factors: spatial visualization (Vz), spatial orientation (SO, e.g. remaining unconfused by changing orientations) and spatial relations (SR, e.g. mentally rotating objects). Lohman emphasized that Vz is defined by difficult spatial tasks that require a sequence of transformations of a spatial representation and more complex stimuli. In addition to these general spatial skills, domain-specific knowledge in geometry also plays a kind of coordinating role between spatial visualization and property-based spatial analytic reasoning (Fujita et al., 2020). In this study, we adopted the model of Fujita's which enables us to identify what makes problem-solving successful/unsuccessful in their use of spatial reasoning skills. For example, suppose a student is asked to determine a quadrilateral in a virtual 3D space in VR. To complete this task, the student might use spatial visualization skills to determine its shape and size before s/he started construction, and in the problem-solving process, the learner needs spatial orientation that is not confused by changing perspectives and spatial relation for the length of each side or face to be successful. This illustrates the role of spatial skills in problem-solving.

Duo of artefacts in math education

In this paper, we use both tangible objects and digital technologies to support students' development of geometric reasoning and spatial skills described above. While relationships between tangible and digital artefacts are not straightforward, Soury-Lavergne (2021) extended the instrumentation theory and proposed a duo of artefacts framework. In our study, one of digital artefacts, VR is used to break down the barriers of two-dimensional screen vision and allows students to enter the three-dimensional world of geometry more visually and intimately, while GeoGebra, the other digital tool, allows students to construct and drag the geometric shapes with the changing immediately. So that, a

complementarity between two artefacts may be required to make necessary the system of teaching and learning. However, the example of VR meso-operation and micro-operation in a tangible artefact shows that there must be antagonisms between the two artefacts. Tangible artefacts, our third tool, play an irreplaceable role in this because students and teachers are familied to it and understand it, and results show that students acquire specific ways of reasoning through it. As tangible artefacts, we prepared geometric models of shapes providing opportunities for students to manipulate physically which might be difficult to do with digital tools.

Three artefacts in the learning environment, chosen by the students themselves, our studies in this paper show that two of them are enough to solve the mathematical problems in the study. So, here we will use duo1: Virtual Reality and tangible artefact, duo2: Virtual Reality and GeoGebra, duo3: tangible artefact and GeoGebra, to help address the limitations in cognitive resources that may be imposed by only one artefact.

Models of the spatial skills and geometric reasoning development in multi duo of artefact instructional design

In this paper we use VR and GeoGebra as the digital tools, and a combination of traditional geometry tangible tools to support students' geometrical reasoning and spatial skills. Fig.1 is the framework that we used to summarize the relationships between the use of technological tools, geometric reasoning and spatial skills. In this figure, learners will activate any of the duo artefacts when solving problems, indicated by the support arrows: geometric reasoning and spatial skills are supported through their functional integration into the learners' use of the duo artefacts, and the actions feedback to the duo artefacts serve as functional extensions of the operations to assist in the completion of further tasks. Spatial skills and geometric reasoning in mathematical activities steadily support each other to operationalize the understanding of geometric concepts and properties. Based on this theoretical framework, we have examined each of the geometric reasoning and spatial skills processes of change and artefact manipulation and functioning and found that these different artefacts are gradually integrated into students' use schemes and support their reasoning.

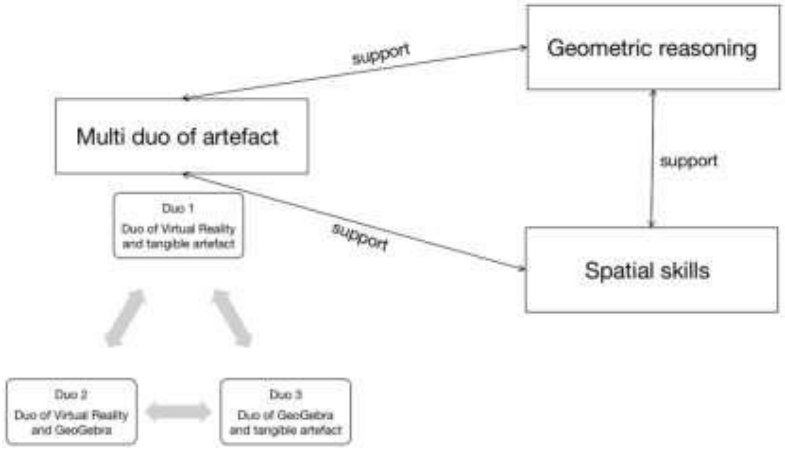


Figure 1 Framework for developing geometric reasoning and spatial skills through multi duo of artefact

Methodology

Study design and data collection

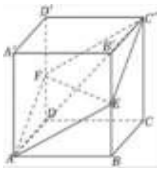
In this paper, we use both tangible objects and digital technologies to support students' development of geometric reasoning and spatial skills. The main research questions we explore are: 'How do students use these tools to solve spatial geometry problems?' and 'What geometric reasoning and spatial skills are developed?' In March - May 2024, we piloted teaching sessions in Shenzhen, China with groups of grade 7-10 learners (both groups of 4 and groups of 2). These learners had prior knowledge of 3D geometric shapes but lacked experience in undertaking geometric problems based on the properties of parallel, perpendicular, and curved faces and edges, as well as combined or planar cut sections.

The table 2 summarises the task used. We specifically designed warm-up problems as our participants had limited experience of using VR and GeoGebra, for example, please create a cube through Virtual Reality. They can get familiarise themselves with artefacts through warm-up problems in 15 minutes. Then they were asked to solve the task in table 2 through the tools just introduced to them.

Table 2: Summary of the main task of episode

The length of a cube is 2, E and F are the midpoints of the edge, the area of a quadrilateral AEC'F is ( ) .

A. $2\sqrt{6}$ B. $2\sqrt{5}$ C.5 D.$4\sqrt{2}$



Data were collected through face-to-face geometry class activities with students' work and their communication and encompassed two types of evaluations: class observation data, pre-test/post-test data (10 questions) and delayed test 4 weeks after the post-test in this paper.

Data – observation data and participants

To examine the development of the student's reasoning with the use of VR, GeoGebra and tangible objects (plastic cube models), we have analysed data collected total of 15 sessions with 18 participants, and in this paper, we selected the following two episodes based on students' thinking-shifting reactions observed through qualitative analysis, as well as the development of geometrical and spatial reasoning during the processes, and their experience with correct/incorrect answers. Table 3 summarises the tool chosen by the two episodes and the participants.

Table 3 Analysed participants' duo of artefacts in this study

Episode	Duo of artefact	Participants
1	Duo3: duo of GeoGebra and Tangible Artefact	Group 1: Sara, Joyce
2	Duo1: duo of Virtual Reality and Tangible Artefact	Group 2: Benjamin, William, Mia, Isabella

Data analysis procedure

After conducting initial examinations of the video data, we selected specific episodes, resulting in a total of 15 sessions with 18 participants. We selected these two episodes because they were closely linked to the use of multi duo of artefacts for understanding, analysing, and solving spatial geometry problems, as well as for interacting with new tools, shifting in their reasoning, and overcoming difficulties in the spatial geometry problem-solving processes. Two of the episodes are presented here as examples of the research.

To begin, we identified the developmental moments of the learners, as evidenced by Seah and Horne (2019). We identified segments of students' interactions geometric reasoning process with four levels: visual reasoning, descriptive reasoning, analytical reasoning, and relational-inferential property-based reasoning. For the spatial skills, we examined the types of spatial ability: spatial visualization (Vz), spatial orientation (SO) and spatial relations (SR), used in their processes of forming their task solutions. To identify such shifting in their reasoning, we conducted a detailed qualitative analysis to determine patterns in problem-solving processes. This analysis serves as evidence to answer both research questions.

Results

Pre and post-test results

The overall observation of 18 students the average pre-test score is 4.18 out of a total score of 10. The scope of the pre- and post-test tasks is the same type as the classroom tasks, and students are scored for each step during the writing process. Some of the students have demonstrated relatively a low level of understanding of 3D geometry. Upon reviewing the pretest data presented in this research, it is evident that students have had varying learning experiences regarding basic knowledge of 3D shapes. This includes understanding cubes, combination or planar cut sections, three views, and how to represent 3D shapes in 2D and 3D. After experiencing the designed sessions, the students' post-test scores reached 8.71, which is a significant improvement. The students then took the delayed test after 4 weeks, and their mean score was 8.00, suggesting that the students' performance remained relatively stable.

Development of geometric reasoning and spatial skills with multi-duo artefacts

For the chosen episode, we did qualitative analysis by using the transcript examined, the task required three main spatial skills and level 2-3 (Seah and Horne 2019) geometric reasoning, finding the area of the quadrilateral. To calculate the area of a quadrilateral, the key lies in identifying the type of quadrilateral. Therefore, after giving the students the question, the teacher asked the students to answer the name of the quadrilateral first. Most students answered as a square, but the quadrilateral is a rhombus.

During the session, for example, in episode one, in Figure 2 a), Sara (S1) and Joyce (S2) activated duo of GeoGebra and cube models in Figure 2b). The analysis suggests that they started from level 1 Visual reasoning in geometric reasoning, but GeoGebra supported spatial visualization and spatial relation skills, and finally reached level 4 Relational-inferential property-based reasoning in geometric reasoning through tangible artefacts.

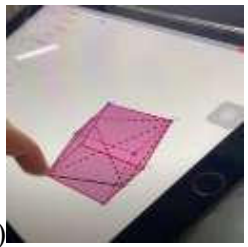
a)  b) c) d)

Figure 2 Artefacts - GeoGebra: a), Tangible Artefact: b), VR: c) and d)

At first, they chose to observe through GeoGebra (digital artefact). They both started by using visual control in decision-making. Sara used visual reasoning stating “kind of like” to indicate that it was a rhombus. Joyce eliminated the cube's appearance and only observed the quadrilateral in the middle, which she thought was a square. In the task, this implies that their geometric reasoning only stayed at the level of pre/recognition of shapes based on physical attributes.

Then, by dragging and converting the size and angle, spatial visualization skill seemed to be supported, combining concept and vision, and even if there was insufficient formal reasoning, they decided that it could be square based on some properties, e.g. four equal sides. It is this that they pointed their fingers at the screen and started to acquire formal language to describe properties, suggesting that this might have allowed them to reach Level 2: descriptive reasoning. At this point they discovered the key point that “The two diagonals are not equal!” with GeoGebra. And then Joyce explained that one of the diagonals was not equal to the other one with GeoGebra, meaning that the shape could be a rhombus. As they were using the fact ‘four equal sides’ and ‘different diagonals’, their reasoning was developing towards analytic reasoning (Level 3). Finally, to explain their thinking process more clearly, they chose the second artefact, i.e. tangible objects. They used diagrams as well e.g. Figure 2 b) and logic to confirm the quadrilateral was a rhombus by explicitly referring to critical attributes of the shape (two diagonals and four equal sides) and expressed their reasoning sufficiency. Because of this use of critical attributes of the rhombus, their reasoning might be seen as relational-inferential property-based reasoning, supported by their spatial orientation skill using tangible objects.

In the second episode, Benjamin (S9), William (S10), Mia (S11) and Isabella (S12) worked with the duo of tangible artefact and Virtual Reality. They started from level 1 in geometric reasoning, started using spatial visualisation and spatial orientation skills with the help of Virtual Reality, and finally reached level 3 in geometric reasoning with tangible objects, using spatial relations skills in the process.

At first, only Isabella said that the quadrilateral was a rhombus, but the other pupils thought it was a square. Because of this disagreement, they used VR, but their reasoning was still visually based, even though they were demonstrating their spatial visualisation skills. Then, by discussing the difference between a rhombus and a square, they proposed that "a square must have four equal angles", but not in this quadrilateral through the perspective-altered VR with the spatial orientation skill. This reasoning based on shape attributes might have supported them to move on to the second level: descriptive reasoning with visual informal reasoning, as they focused on the angels of the

quadrilateral. The other students in the group looked through the pad screen of the VR, taking turns to enter the VR world and observed. Then, "It doesn't have any right angles," William said, looking at the 3D VR shapes. "There are acute and obtuse angles," Benjamin noted. Their observation was further examined as they continued to investigate the diagonals of the quadrilateral using tangible artifacts. They examined the differences in the sizes of the four angles, stating in a quadrilateral where two angles are obtuse and two are acute. They concluded that a quadrilateral with four equal sides, two obtuse angles, and two acute angles is a rhombus. The use of digital and tangible objects creates a new way of thinking about abstract geometric concepts with properties, bringing them not only into the theoretical but also into the reasoning process, indicating that they started demonstrating the third level of analytical reasoning.

Discussion and conclusion

In this paper, we use both tangible objects and digital technologies to investigate the development of students' geometric reasoning and spatial skills. The central questions addressed in this paper are what geometric reasoning and spatial skills are being developed, and how students are using these tools to solve spatial geometry problems. In answering these questions, we identified three activated duos of artefacts. They are duo1: Virtual Reality and tangible artefact, duo2: Virtual Reality and GeoGebra, duo3: Tangible artefact and GeoGebra. When faced with a problem, learners activate different pairs of artefacts and use a new tool to generate a new way of thinking about abstract geometric concepts and properties. The complementarity between the pairs of artefacts may be necessary for the use of each, e.g. a rhombus suggested by GeoGebra, but the tangible objects were needed to confirm that the shape was a rhombus. Thus, by visualising and simulating geometric figures to understand them concretely, the presence of redundancy can contribute to the establishment of continuity. This theoretical framework was tested in data analysis. By improving the level of geometric reasoning (Seah and Horne, 2019), it supports the use of spatial skills (Lohman, 1988) and helps control the perception and recognition of geometric figures. For example, by training their spatial skills in VR, students were able to find the angles of four and form a new spatial understanding of 3D graphics. It allows students to move from informal to formal geometric reasoning through the second object, e.g. tangible objects (Fujita et al., 2020). The study illustrates the potential of the duo of artefacts for learners to improve their geometric reasoning and train their spatial skills. Through our tested theoretical framework, geometric reasoning, spatial skills and three tools' applications interact to help learners improve their understanding of geometry while supporting learners' geometric cognitive construction, improvement and training.

The multi duo of artefact theoretical framework can help mathematics teachers to better understand the role of students' reasoning and actions in grasping important geometric concepts and properties. Because of our findings: judging from the pre- and post-tests, the students' test scores improved significantly. Through our qualitative analysis research, this is because students' geometric and spatial reasoning are effectively improved by 2-3 levels (Seah and Horne, 2019) while active duo of tangible and digital tools. As a recommendation for practice, our findings may encourage curriculum designers to design activities that activate duo of artefact experiences in mathematics spatial problem.

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Working with student solutions to understand geometry knowledge of preservice teachers

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The aim of this paper is to analyse and understand the geometry knowledge of preservice secondary school teachers (PSTs). Fifteen PSTs who had completed a university geometry course focusing on synthetic geometry participated in the research. PSTs were asked to solve a multiple strategies geometry problem and then to examine and report on the students' thinking based on seven samples of student work provided. The results showed that a number of teachers valued the solution using measurement, which was somewhat imprecise, more than the deductive solution, which contained a numerical error. They had difficulty working with the diagram in the solution, which is not an accurate visual representation of the situation; and they did not consider it important to argue and justify the solution, especially if the student had followed the expected solution strategy.

Keywords: Geometry knowledge, preservice teacher, problem solving, students' solutions.

Introduction

Geometry is a very important area of school mathematics that is essential for students' success not only in geometry but also in other areas of mathematics, because many of the ideas of geometry and properties of geometric objects are used throughout mathematics (e.g. transformations of functions, vector space, etc.). Unfortunately, the results of Slovak students in national tests and the results of various competitions for gifted students show that the situation in this area is getting worse. In the research literature we can find several reasons why students struggle with geometry, e.g. Steele (2013, p. 245-246) mentions that an important reason is the limited knowledge of teachers in geometry. Aware of this reason, we have changed the geometry training of preservice teachers (PSTs) at our university over the past three years by adding more geometry courses.

Based on our experience and recent literature, we know that a major challenge in university teacher education is the appropriate design of basic mathematics courses, including geometry, for PSTs. We believe that the link between the content of mathematics courses and future mathematics teaching needs to be clearer and more explicit. This problem has been mentioned by several researchers (e.g. Winsløw & Grønbæk, 2014; Eichler & Isaev, 2023). It is known as the double discontinuity. We focus on the second discontinuity – the transition from academic study (as a PST) to working as a mathematics teacher (Klein, 2016).

To analyse the current mathematics teacher specialised knowledge of geometry, we propose the activity for PSTs with the multiple strategies problem with selected students' solutions to this problem. The aim of the activity was to identify possible discontinuities in the current geometry knowledge of PSTs. The research question formulated within this aim is: How do preservice teachers approach students' solutions that vary in level of argumentation, use different solution methods, and include both correct and incorrect results?

Theoretical Background

There is a consensus among researchers that problem solving should be a key activity in mathematics education. These ideas were among the first to be emphasised almost 80 years ago by George Pólya in his famous book How to Solve It (1945). On the other hand, problem-oriented learning requires the teachers to have a deep understanding of the foundations of mathematics and to be adept at solving mathematical problems (Ronau et al., 2015) in order to deal with both predictable and unpredictable situations, as their students may generate and explore unexpected conjectures about familiar objects.

Several researchers also suggest using problem solving to investigate and better understand teachers' professional competencies (Felmer et al., 2019). The use of open-ended mathematical tasks, as characterised by Klein & Leikin (2020), should enable PSTs to bridge the gap between the mathematics they have learned and the mathematics they will teach (Zaslavsky & Leikin, 2004). In addition, the use of students' solutions provides PSTs with opportunities to experience alternative ways of learning (challenging) mathematics and possibly foster their sensitivity to students and their ability to assess students' mathematical understanding (van Dooren et al., 2002). Research, such as that by Gal (2011), suggests that although PSTs' mathematical knowledge is usually adequate, they often struggle with analysing and understanding students' solutions and identifying their difficulties.

We decided to use the Mathematics teacher's specialised knowledge model (MTSK, Carrillo et al., 2018) to analyse the mobilised specialised knowledge of PSTs. We chose this knowledge model because it focuses on understanding teachers' knowledge rather than assessing it (Carrillo et al., 2018). The MTSK model includes the domain of Mathematical Knowledge and the domain of Pedagogical Content Knowledge. These two domains are composed of subdomains and categories, which allow a detailed distinction of aspects of mathematics teachers' knowledge. This paper focuses on one subdomain of Mathematical Knowledge – Knowledge of Topics (KoT) applied to pre-service teachers' approach to students' solutions that vary in solution strategies and level of reasoning.

Method

Participants and context

The research was conducted with 15 PSTs studying at the Pavol Jozef Šafárik University in Košice, Slovakia, as part of the course Seminar for Mathematics Club. The course is designed for PSTs who will be teachers in lower and upper secondary schools (usually students from 10 to 18 years old). It is recommended for PSTs in the 3rd year of the bachelor's programme. The class consisted of six 2nd year bachelor PSTs (3 male and 3 female) and nine 3rd year bachelor PSTs (1 male and 8 female). The PSTs had completed all the basic mathematics courses from the first year (Mathematical Analysis and Algebra). As for geometry, which is the subject of our research, all PSTs had completed Geometry I (knowledge of classical Euclidean geometry including the axiomatic system) and Geometry II (knowledge of synthetic stereometry – positional and metric properties). Geometry III (dealing with analytic geometry in n-dimensional space) and Geometry IV (dealing with quadratic surfaces and congruent and similar representations) had only been taken by 3rd year bachelor PSTs.

Research instrument

The research instrument consisted of two parts. The first part was a geometry problem from the Slovak Mathematical Olympiad competition for 14-year-old students, presented in Figure 1.

> *In a rhombus ABCD with a side length of 4 cm and a diagonal AC of length 4 cm, the point K is the centre of the side BC. Next, points L and M are constructed, which together with points D and K form the square KLMD. Calculate the area of the square KLMD.*

Figure 1: Geometry problem

The second part consisted of 7 student solutions to the same problem. In order to select different student solutions for the research instrument, we analysed the solutions of 70 students who participated in the competition. 22 students used a correct solution strategy, 19 students used an incorrect solution strategy, 20 students used the strategy "draw and measure" and 9 students had no solution. We assigned each student's solution to a group with the same solution strategy according to the identified auxiliary elements they chose to solve the problem (see Table 1, 2nd column). Then we selected one or two solutions from each group based on the level of reasoning (Jones, 1998) and the correctness or incorrectness of the problem-solving outcomes. This is in line with the methodology used by Palatnik & Dreyfus (2019). The more detailed description of the chosen solutions is in Table 1. Moreover, each of the 7 selected students identified the sub-goal of the problem, find the length of the line segment DK. All correct strategies used the Pythagorean Theorem at least once.

Data collection

Firstly, the PSTs were asked to solve the problem (see Figure 1) themselves during the seminar. They were given 20 minutes to do so. Those who found a solution were asked to look for other solution strategies. We collected written solutions to the problem for further analysis.

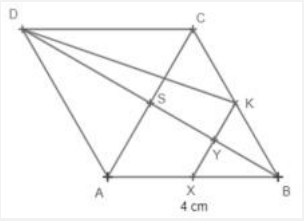
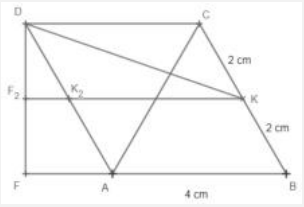
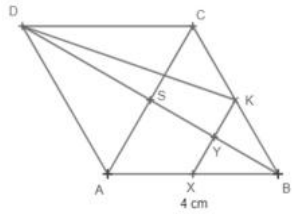
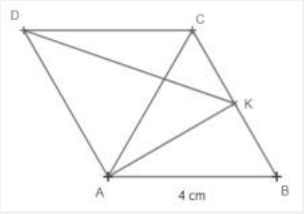
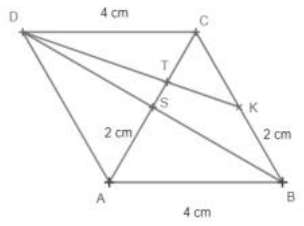
Secondly, the PSTs were given a homework assignment with 7 student solutions to the problem. They were asked with analysing and providing a written report on the students' reasoning based on selected examples of their work. In addition, they were asked to rank the solutions from the best to the worst.

Results

PSTs' solution to the problem

For the qualitative analysis of the PSTs' solutions to the geometry problem we used the same method as for the students' solutions. 4 PSTs used the same solution strategy as Emma (marked as S_E in the 2nd column of Table 1), 2 of them used the same solution strategy as Christian (S_C) and 1 PST used the strategy that is only appropriate for high school students (S_H). For more details, look at the last row of Table 2. The rest of the PSTs did not solve the problem during the seminar. Regarding the PSTs' solution process, we observed that at the beginning of the problem solving, each PST sketched a prototypical image (see Tall & Vinner, 1981) of the rhombus in which the diagonal AC is longer than BD. In comparison with the solutions of the 7 selected students, the PSTs had difficulties in identifying the sub-goal of the problem (only to determine the length of the line segment DK). This was confirmed by the PSTs' questions about the position of the square KLDM ("it is not clear in which half-plane it lies").

Table 1: Description of selected students' solutions

Student	Strategy explanation	Visualisation of identified auxiliary elements	Authors' comment to student's solution
Adam	**Strategy S_{AD}** - Properties of the midsegment of a triangle - Properties of the diagonals of a rhombus - 2x use the Pythagorean Theorem		The solution is without drawing a picture. It is well justified, except for the explanation of the labelling of the two points used. The result is correct.
Boris	**Strategy S_{BG}** - The result is obtained by measuring the length of the segment KD.	No auxiliary elements	This solution begins by observing some invariant properties. The result is obtained by drawing and measuring. The incorrect result is obtained when Boris squares the measured number.
Christian	**Strategy S_C** - Draw the height of the rhombus - Identify similar triangles - Use the Pythagorean Theorem		The solution contains the statement that the height of the equilateral triangle is 7/8 of the triangle's side, which is probably approximation of the true value (√3/2). Some of the statements used were not clearly explained. The incorrect result is due to rounding during calculations.
Daniel	**Strategy S_{AD}** - Same strategy as Adam		Some of the used statements are not clearly explained in the solution. The result is correct numerical expression with irrational numbers, which is not calculated.
Emma	**Strategy S_E** - Calculate angles in triangles AKC and AKD - 2x use the Pythagorean Theorem		The most effective solution. It is almost without any argumentation, explanation of why the properties and relations used are valid. The result is correct.
Franklin	Strategy S_F - Properties of the diagonals of a parallelogram. - Properties of the median of an equilateral triangle - 2x use the Pythagorean Theorem		The picture in the solution is not an accurate visual representation of the situation. For the student, it is a diagram that has the desired properties, although it looks different. Used statements are well explained. A numerical expression containing an irrational number is calculated incorrectly in the last step.
George	**Strategy S_{BG}** Same strategy as Boris	No auxiliary elements	The problem is solved as a construction problem with the construction procedure written down exactly. The result is obtained by precise measurement the segment KD and squaring it.

PSTs' evaluation of students' solutions

As explained above, the PSTs were asked to rank the students' solutions from best to worst and to write at least two sentences to evaluate them. Table 2 gives an overview of how the PSTs ranked the students' solutions.

Table 2: Preservice teachers ranking of the students' solutions

Pre-service teacher (year of study) Ranking of students	1 (3)	2 (3)	3 (2)	4 (2)	5 (3)	6 (3)	7 (2)	8 (3)	9 (2)	10 (3)	11 (2)	12 (2)	13 (3)	14 (3)	15 (3)
1.	E	D	E	D	D	E	E	D	A	E	A	D	E	E	E
2.	D	F	D	F	F	G	A	E	D	D	D	A	D	A	A
3.	A	G	B	A	B	A	D	F	E	F	E	E	G	D	D
4.	F	B	A	E	G	B	C	A	B	A	B	G	F	G	C
5.	C	A	G	C	A	D	F	C	G	G	G	B	C	C	F
6.	G	C	C	B	E	C	B	B	C	B	C	C	A	F	B
7.	B	E	F	G	C	F	G	G	F	C	F	F	B	B	G
PST's solution strategy	0	0	0	S_C	0	S_E	0	0	S_E	S_H	S_E	S_C	0	0	S_E

In the qualitative analysis of the PSTs' comments, we observed whether they distinguished between students' intuitive and deductive reasoning, similar to the methodology used by Jones (1998). Examples from the PSTs' reports on students' thinking, which help us to analyse and understand the thinking of the PSTs in our sample, are presented in Table 3. According to the analysis, we could observe the following:

- Only the solutions of Emma, Daniel and Adam occur in the first place. Emma's solution was effective, written clearly and transparently with a large, colorful picture. The PSTs in her case mostly tolerated the lack of argumentation, perhaps because most of them used the same solution strategy or started solving the problem in a similar way. As for Daniel and Adam, they had the same solution strategy. The solutions given were easy for the PSTs to follow, even though none of the PSTs used it in solving the problem. Both solutions were written in such a way that the steps followed each other logically. We believe that the primary criterion for ranking the solutions was the PST's understanding of the solution itself, or the clarity and structure of the notation.
- For the same reason, in this case misunderstanding, the solutions of Christian and Franklin appeared in the last place next to the measurement solutions. We believe that in the case of Christian's solution, understanding the solution was complicated by the non-standard approximation of the irrational number, which most PSTs identified as incorrect, as well as the lack of clarity in the notation and insufficient justification of the procedure. In Franklin's case, following the solution was complicated by a picture that incorrectly captured the metric relationships between points, but was schematically correct. In addition, Franklin made a numerical error in the last step of the solution, and his result was the furthest from the correct result, which may have influenced the PSTs.
- Two thirds of the students ranked the solutions by measurement (marked in red in Table 2) before at least one of the deductive solutions. They were surprised by this constructive approach to the problem. They often described it as nice and considered it one of the

appropriate strategies for solving such a problem. PSTs tended to tolerate imprecision in measurement. The tolerance for numerical errors in calculations was lower.

Table 3: Preservice teachers report on the solution

Student	PSTs report on the solution
Adam	PST 1 – *I'm missing a picture in the solution. Even if the picture wasn't already there, it would need more comment, e.g. where did we get point S from, etc.*
	PST 8 – *The solution is cluttered and difficult to read.*
Boris	PST 4 – *However, the graphical method above is also a possible way of solving the problem, but it is not completely accurate and using this method would not satisfy the conditions formulated in the text of the problem.*
	PST 6 – *The only error is in the result, because he wrongly squared 5.2, just a numerical error.*
Daniel	PST 6 – *Nice and clear notation. The result is in a confusing expression, it needs to be calculated.*
Emma	PST 2 – *You didn't include an explanation of why the triangles are right-angled and a description of your sketch. How did you determine the angles?*
	PST 6 – *Very nice and clear sketch. Emma has probably come up with the quickest and most elegant solution.*
	PST 10 – *Very nice idea and simple solution to the task. However, there is no comment on the solution of the problem.*
Franklin	PST 8 – *The information used in the solution is nicely outlined in the diagram. What we are starting from is well named and sequentially written down and explained.*
	PST 12 – *There was a fundamental error in this solution, because the rhombus is not drawn correctly. Everything else is based on this error, so the result is also far from reality.*
George	PST 2 – *I like the style/approach of the problem and the mention of the second way of drawing the KLMD square. There is an error in the labelling of the circle k_1 in the construction procedure.*

Discussion and Conclusion

In this study, we were especially interested in how PSTs addressed students' solutions with both correct and incorrect results and with used different solution methods. According to Table 2 and Table 3 we can observe the following:

More than half of the PSTs chose Emma's solution as the best. Her solution has a clear and large colour picture, and important points, segments and angles used are labelled. On the other hand, there is almost no reasoning, no explanation of why the properties and relations used are valid. In the case of other solutions using a less common strategy, teachers sometimes criticised the lack of justification, but in Emma's case most of them did not mind (see PST 6 report in Table 3). This raises the question. Are PSTs aware of the need for justification and reasoning, or do they consider it necessary only to facilitate their understanding of the solution?

As mentioned above, 5 PSTs rated Franklin's solution as the worst. We think there are 3 possible reasons for such a bad evaluation of Franklin's solution: (1) His picture is not an accurate visual representation of the situation but the diagram that has the desired properties, although it looks different. (2) The used method of solution was rare (only 2 students out of 70 used it and no PST). (3) Compared to other solutions, the result is the most different from the correct result. On the other hand, some PSTs appreciated Franklin's solution (see Table 2 and Table 3, PST 8 report).

Our findings on the ranking of Boris and George's solutions also indicated that many PSTs in this study did not see the difference between solutions using measurement and those using deductive reasoning (see PST 2 report in Table 3). Although none of the PSTs attempted to solve the problem using measurement, many of them ranked the solution with a result close to the expected result, albeit obtained by measurement, higher than the correct deductive reasoning. This begs the question. Do the PSTs understand that measurement is an estimation process and that only calculation based on correct deductive reasoning leads to an accurate result?

When working with the selected student solutions, most PSTs showed an ability to understand different student solution strategies. Although no one used a solution strategy like Adam and Daniel in the first part of the activity, the PSTs understood this solution strategy and evaluated the students positively despite the missing picture (Adam's solution) and despite the inappropriate expression with irrational numbers and small deficits in reasoning (Daniel's solution). These results suggest that the proposed activity with students' solutions can help mobilise KoT (Knowledge of Topics) in PSTs, especially in the case of our activity: finding the subgoal of the problem, the different ways of representing the problem, the connections between the different elements of the triangle (midpoint, midline, etc.).

The results of our research showed the challenges we must face in the professional development of PSTs during the last two years of their university studies. They show that PSTs did not apply in practice some of the mathematical knowledge they had to demonstrate during their geometry training, indicating a second discontinuity in the transition from training to practice. We believe that one way to overcome this is to include teacher-oriented tasks, as suggested by several studies (e.g., Eichler & Isaev, 2023). We also believe that the inclusion of open-ended mathematical tasks, such as multiple strategy tasks, multiple outcome tasks and investigative tasks, as suggested by Klein & Leikin (2020) in geometry preparation, could help to overcome some of the identified problems. Of course, more research is needed in this area.

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Geometry to be taught in Taiwanese and Japanese junior high schools

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The aim of this paper is to advance understanding of the geometry curriculum and the arguments that support such a curriculum through an international comparative study of geometry to be taught in Taiwan and Japan. While these countries share a similar educational culture, the geometry curriculum in junior high school is organised in a very different way. Adopting the perspectives proposed by the Anthropological Theory of the Didactic, we analyse curricular documents to shed light on the differences in the mathematical organisation of geometry to be taught and try to identify the arguments (conditions and constraints) that shape such organisation.

Keywords: international comparative study, geometry curriculum, textbook analysis.

Introduction

Geometry is a domain that traditionally occupies an important place in secondary mathematics in any country. Looking at the history of geometry education, the geometry knowledge supposed to be taught at schools, especially secondary schools, is not stable because of some educational reforms or events: for example, in East Asian countries, such as Japan, the reception of Western mathematics, including Euclidean geometry in the 19th century revolutionised geometry teaching, pushing aside traditional mathematics (*wasan*) (Ueno, 2012); New Math movement has led to the downfall of Euclidean geometry; the dynamic geometry software (DGS) changed classroom activities.

Today, geometry curriculum seems to be diversified according to country or region throughout the world. In Japanese junior high school, students learn traditional quasi-axiomatic Euclidean geometry where the proof and congruent triangles play an important role, while this is not the case in some other countries (Miyakawa, 2017). The different arguments could be advanced to justify such a curriculum. For example, some tensions can be identified in such arguments: teaching between mathematical proof (or reasoning) and problem solving (or tools for human activities) (Jones & Fujita, 2013); between intuitive geometry and deductive geometry (Jones, 1998).

In this context, we started working on an international comparative study of the geometry curriculum in Taiwan and Japan. While these two countries have traditionally had a close relationship in terms of culture and distance, junior high school geometry curriculum is organised in a very different way today. It will be interesting to see what geometry is being taught and what kinds of arguments are being developed to teach such geometry. It is expected that this study will provide insights into the discussion of geometry curriculum reform in the future.

The aim of this paper is therefore to present preliminary findings from our comparative study of Taiwan and Japan, focusing on the organisation of the geometry curriculum and the arguments underpinning its structure.

Theoretical perspectives

This study adopts perspectives proposed by *the Anthropological Theory of the Didactic* (ATD; Chevallard, 2019), especially to formulate our research questions. Within this framework, we first consider that the nature of mathematical knowledge differs according to the *institution*. The notion of institution here should be understood in a broader sense, including 'any created reality of which people can be members' (Chevallard & Bosch, 2019, p. xxxi). Second, we consider that what is taught at school originates in other institutions, and the following process of didactic transposition is often considered: scholarly knowledge ↔ knowledge to be taught ↔ taught knowledge ↔ learnt knowledge (Chevallard & Bosch, 2020). In our case, the institutions considered are Taiwanese junior high schools and Japanese ones, in which geometry, a body of mathematical knowledge to be taught, is organised in a different way. Third, we consider that the mechanism of producing such differences is governed by the system of *conditions* and *constraints* prevailing in the institution: conditions supporting the life of a piece of mathematical knowledge and constraints hindering it. These conditions and constraints, called *ecology*, shape the mathematical organisation. Different arguments advanced for geometry teaching imply part of the ecology of school geometry. From these perspectives, the research questions can be formulated as follows:

- What are the mathematical organisations of geometry to be taught in Taiwanese and Japanese junior high schools? How are they organised, and how do they differ from each other?
- Why such mathematical organisations? What conditions and constraints shape them?

Contexts and data

Taiwan and Japan share a similar educational system. In both countries, prior to university, the school system consists of six years of elementary school, three years of junior high school, and three years of senior high school. The ministry of education publishes national curricula which are revised almost every ten years. Mathematics textbooks are published by private publishers and should be approved by the ministry. Their influence in the classroom teaching is significant, because the use of textbooks is required by law (e.g., Article 51 of the Ordinance for Enforcement of *the School Education Act* for Japan) to assure the quality education in any part of the country.

In this study, we analyse curricular documents that determines the geometry to be taught in Taiwanese and Japanese junior high schools: the mathematics textbooks for three years of middle school and the official documents published by the ministry. The textbook series we chose are the one published by Keirinkan publisher for Japan (2021) and the one published by Kang-Hsuan publisher for Taiwan (2023), both of which are the major textbooks series in each country. The official documents we used are *The Guides of the Course of Study for Mathematics* (MEXT, 2017) for Japan and *The Curriculum Guidelines of the 12-Year Basic Education in the Domain of Mathematics* (TME, 2018) for Taiwan.

Method of analysis

While ATD provides some analytical tools such as *the praxeology* and *the scale of levels of didactic codeterminacy* (Chevallard, 2019), we explore, as a preliminary analysis, the curricular documents without explicit tools but broadly aligning with the three types of analysis—horizontal, vertical, and contextual—proposed by Charalambous et al. (2010). The analysis comprises three steps.

The first step involves identifying horizontally the overall organisation of the geometric content as presented in the textbooks. We examined the chapters and sections dedicated to geometry across the three years of junior high school, as well as the proportion of pages devoted to geometry.

The second step investigates vertically how specific concepts are addressed in the textbooks—namely, proof and the Pythagorean theorem. These concepts were chosen because proof plays a pivotal role in geometry learning, while the Pythagorean theorem has been an iconic theorem in geometry since the time of Euclid. This step allows us to elucidate the *habitat* (place or address) and *niche* (role or function) of each concept to be taught in each country, as defined by Chevallard (1994). Together, these two steps address the first research question concerning the organisation of geometry content.

The third step explores the arguments (conditions and constraints) underpinning the organisation of geometry in each country. Specifically, we examine curricular documents to identify key aspects of geometry teaching that are frequently emphasised as supporting arguments without relying on a specific analytical tool. The results of this step, which are discussed in the final section, help to clarify the contextual or cultural factors that contribute to answering the second research question.

Results

Organisations of the geometry in junior high schools

The overall organisation of the geometry to be taught in Taiwan and Japan is presented in Tables 1 and 2. The number of pages in a chapter or section is given in brackets. Each grade in Taiwan has two volumes (two numbers are given under the grade). One may find from these numbers that geometry in Taiwan is mostly concentrated in the second semester of Grades 8 and 9, that is, later in the middle school, while it is distributed almost equally throughout three years in Japan.

Table 1: Geometry content in Taiwanese textbooks

Grade	Chapter	Section
7 (215) (215)	2-6 Geometry in life (30)	1. Perpendicular, line symmetry, and three-view drawings
8 (219) (223)	1-2 Square root and Pythagorean theorem	3. Pythagorean theorem (18)
	2-3 Basic properties of triangles (90)	1. Interior and exterior angles of triangle and polygon 2. Construction by straightedge and compass 3. Congruence of triangles 4. Properties of perpendicular bisector and angle bisector 5. Relationship between sides and angles of triangle
	2-4 Parallel lines and quadrilateral (54)	1. Parallel lines 2. Parallelogram 3. Properties of special quadrilaterals
9 (179) (123)	1-1 Similar figures	2. Proportional line segments (18) 3. Expanding, contracting, and similarity (24) 4. The application of similar triangles (18)
	1-2 Circle (48)	1. Relationships among points, straight lines, and circles 2. Relationships among angles at the centre, angles at the circumference, and the arc
	1-3 Geometry and proof (48)	1. Proof and reasoning 2. The circumcentre, incentre, and centroid of a triangle
	2-3 3-dimensional graphics in life (30)	1. Lines, planes, and shapes in 3D space

Table 2: Geometry contents in Japanese textbooks

Grade	Chapter	Section
7 (263)	5 Plane figures (32)	1. Straight lines and figures 2. Movement and geometric construction 3. Circles and sectors
	6 Spatial figures (36)	1. Solids and spatial figures 2. Volume and surface area of solid figures
8 (205)	4 Investigating figures (30)	1. Parallelism and congruence 2. Proof
	5 Properties of figures and proofs (34)	1. Triangles 2. Quadrilaterals
9 (237)	5 Figures and similarity (40)	1. Figures and similarity 2. Parallel lines and ratios 3. Measurement of similar figures 4. Application of similarity
	6 Properties of circles (20)	1. Inscribed angle theorem and central angles 2. Application of the circle properties
	7 Pythagorean theorem (22)	1. Relationship between sides in right triangle 2. Application of Pythagorean theorem

The geometry content of textbooks from the two countries appears to be very similar: triangles, parallel lines, parallelograms, congruence, similar figures, circles, proofs, the Pythagorean theorem, and so on. In principle, the content of Japanese textbooks is fully covered by Taiwanese textbooks. There are a few topics taught in Taiwan that are not included in Japan (e.g., trigonometric ratios). One notable difference lies in the positions of some mathematical topics, particularly the Pythagorean theorem and proofs. In Taiwan, the Pythagorean theorem is introduced in the first semester of Grade 8 at an earlier stage of geometry learning, whereas in Japan, it is taught as the final chapter of Grade 9. Furthermore, proofs are introduced earlier in Japan than in Taiwan. We explore these topics in detail, focusing on how they are presented and treated in the textbooks of both countries.

Pythagorean theorem

As mentioned above, the Pythagorean theorem is introduced at different stages within the geometry curricula of Japan and Taiwan. This difference implies a difference of the geometry system to be taught in each country.

The position in Japanese curriculum aligns with the axiomatic system of geometry presented in Euclid's *Elements Vol. 1*, where the Pythagorean theorem and its converse are addressed towards the end (Propositions 47 and 48). In the Japanese textbook, the *quasi-axiomatic* system is supposed in geometry (Miyakawa, 2017): beginning from Grade 8, fundamental properties such as parallelism and triangle congruence are introduced; following the introduction of proof, the geometric properties are proved one by one, constructing an edifice of geometry. The Pythagorean theorem and its converse are introduced and proven in the final section of the geometry content (Ch. 7 of Grade 9).

In contrast, the Pythagorean theorem is introduced in Taiwan in the first volume of Grade 8 textbooks, which are arranged in the same chapter as square roots, at the earlier stage of geometry learning. One reason for this organisation can be found in the teachers' guide of textbooks called *Lesson Preparation Book* (Hung, 2022): 'since Pythagorean theorem is often used in future geometry topics, teachers must confirm students' understanding of Pythagorean theorem and should require students

to memorise this theorem' (p. 94). The importance of this theorem for solving mathematical problems lets students to learn earlier. In fact, students are asked to apply this theorem to solve mathematical problems in subsequent chapters, including the areas of geometry and algebra. In addition, the introductory activity in the Taiwanese textbook is to find the relationship between the areas of the squares on each side of a right triangle. The theorem is then derived from an algebraic calculation of the area by placing four identical right triangles. The converse of the theorem is introduced later in the second volume of Grade 8 (Ch. 3, Sec. 5) after learning triangle congruency, and justified by the congruence of two triangles. At this stage, the proof has not yet been introduced in the textbook.

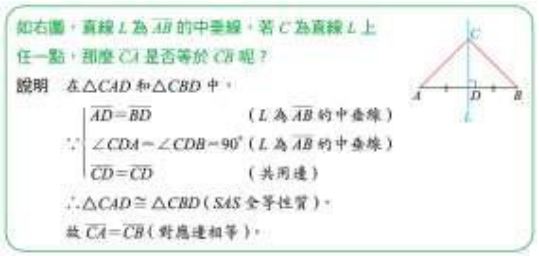
Proof and proving

In Japan, the notion of proof (証明 in Japanese) is officially introduced in Grade 8 as 'a means to show that a proposition is always true' (MEXT, 2017, p. 113). This is the main function of the proof explicitly claimed in an official document. As mentioned above, after introducing some fundamental properties in Grade 8, new statements are proven one by one, even those which have already been introduced in elementary school (e.g., the properties of triangles, parallelograms). Proof plays a role in reconstructing a geometry system axiomatically (while the term *axiom* is not introduced). In the textbook, proof is introduced in Grade 8 as 'Deriving a conclusion from hypotheses, based on something that is already accepted as true, is called proof' (Okamoto et al., 2021, Vol. 2, p. 115). An example of a proof from a textbook is presented in Figure 1. This is a proof for the base angle theorem, which is formulated in the textbook as 'The two base angles of an isosceles triangle are equal' (p. 127).

Proof: By drawing a bisector of ∠A, let D an intersection with BC. In △ABD and △ACD,
because AD is a bisector of ∠A,
∠BAD = ∠CAD ····· (1)
By the hypothesis,
AB = AC ····· (2)
And, because AD is common,
AD = AD ····· (3)
From (1), (2), and (3), because two pairs of sides and the angle in-between are equal,
△ABD ≅ △ACD
Because in the congruent figures the corresponding angles are equal,
∠B = ∠C

Figure 1: Proof in Japanese Gr. 8 textbook (Okamoto et al., 2021, Vol. 2, p. 126) and its translation

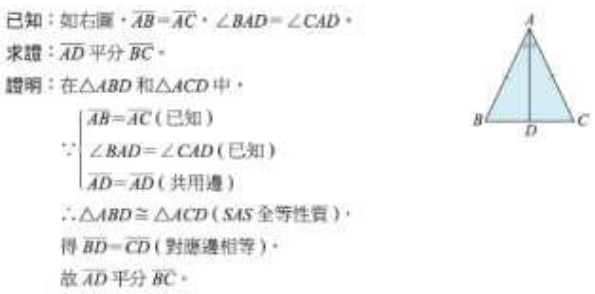
In Taiwan, the notion of proof (證明 in Chinese) is officially introduced in Grade 9 at a point when the junior high school geometry has been almost taught (see Table 1). Unlike the Japanese case, the proof is *not* for constructing a geometry system. However, the lack of proof does not mean that there is no justification for the introduction of theorems. For example, the property of a perpendicular bisector is justified, as shown in Figure 2 in the Grade 8 textbook. At this point, the term 'explanation' is used instead of proof; that is, explanation (not necessarily proof) plays the role of justifying that a proposition is true. The reasons are given in parentheses.



Like the diagram on the right, when line L is a perpendicular bisector of $\overline{AB}$, if C is an arbitrary point on line L, then is $\overline{CA}$ equal to $\overline{CB}$, or not?
Explanation: In △CAD and △CBD,
$\overline{AD} = \overline{BD}$ (L is a perpendicular bisector of $\overline{AB}$)
∵ ∠CDA = ∠CDB = 90° (L is a perpendicular bisector of $\overline{AB}$)
$\overline{CD} = \overline{CD}$ (common side)
∴ △CAD ≅ △CBD (SAS congruency property),
Hence, $\overline{CA} = \overline{CB}$ (corresponding sides are equal).

Figure 2: Explanation in Taiwanese Grade 8 textbook (Hung, 2023, Vol. 4, p. 141) and translation

Proof is introduced in the chapter of "Geometry and Proof" in Grade 9, as "The process of deriving a 'conclusion to prove' from 'known conditions' or 'confirmed properties' is called proof" (Hung, 2023, Vol. 5, p. 135). In this textbook, the term 'proof process' is used for the process from 'known conditions' to 'conclusion to prove' in contrast to the term 'analysis process' for the inverse process of analysing how to prove from 'conclusion to prove' to 'known conditions'. These terms correspond to Pappus' terms 'analysis' and 'synthesis'. An example of the proof process is presented in Figure 3. This proof is very similar to the explanation given in Grade 8 (see Figure 2). The part 'proof' is written almost in the same form as the explanation. The 'known' and properties are given in parentheses. However, a difference is that the proof in Grade 9 consists of three parts, 'known', 'prove', and 'proof'. The students' activity with respect to the proof also differs. In Grade 9, students need to be able to fully express the proof process, while in Grade 8, students are required to be able to write down the reasons for some steps of the geometric reasoning (NAER, 2024). The proof is considered not only a product written in a specific form but also an activity comprising analysis and synthesis.



Known: As in the diagram on the right, $\overline{AB} = \overline{AC}$, ∠BAD = ∠CAD.
Prove: $\overline{AD}$ bisects $\overline{BC}$.
Proof: In △ABD and △ACD,
$\overline{AB} = \overline{AC}$ (known)
∵ ∠BAD = ∠CAD (known)
$\overline{AD} = \overline{AD}$ (common side)
∴ △ABD ≅ △ACD (SAS congruency property)
We get $\overline{BD} = \overline{CD}$ (corresponding sides are equal)
Hence, $\overline{AD}$ bisects $\overline{BC}$.

Figure 3: Proof in Taiwanese Grade 9 textbook (Hung, 2023, Vol. 5, p. 135) and its translation

A comparison of the Japanese and Taiwanese proofs shows that both use a specific geometrical object (e.g., triangle ABC) and step by step derives a new statement on this object from known statements using known properties. In both cases, the deductive structure of propositional logic is shown. However, a specific structure of proof process (known-prove-proof) is required in the Taiwanese textbook. According to this textbook, this is 'a commonly used writing methods of geometric proofs' (p. 134). Both proofs use many symbols. The Taiwanese proof adopts a more formalised language without ordinary sentences. Although the Japanese proof is less formalised, the sentences used in it are far from the ordinary texts that one may read aloud.

Discussion and conclusions

The mathematical organisation of geometry to be taught was different between Taiwan and Japan. The Pythagorean theorem and proof are positioned in different places within the geometry curriculum and play distinct roles. Examining the curricular documents from the two countries implies some arguments or conditions underpinning the organisation of geometry to be taught in each context. We identified five aspects that may be considered as arguments in the geometry curricula of different countries. Those are one of the key results highlighted by our comparative study, the significance of which should be further explored in future research.

The five aspects include *theory/system*, *reasoning/proving*, *problem solving*, *modelling/application*, and *culture/history*, some of which have been mentioned in different ways in previous studies (González & Herbst, 2006; Fujita & Jones, 2014; Herbst et al., 2017). We consider that the organisation of geometry to be taught is shaped according to the importance placed on each aspect in the curriculum, as illustrated in Figure 4, which provides a preliminary overview of to what extent each aspect is addressed. First, *the theoretical aspects* were considered differently between Taiwan and Japan. *The quasi-axiomatic system* of geometry is an object of teaching in the Japanese curriculum. Second, *reasoning and proving* were also considered in a different manner. Although the term 'proof' is not used, most of the new properties or theorems introduced are accompanied by justification in Taiwan, without requiring students to write down formally. Third, *solving mathematical problems* is an aspect highlighted in the Taiwanese curricular document. The Pythagorean theorem plays an important role in later problem-solving activities. This was a reason for introducing it earlier. The fourth and fifth aspects, *the modelling or the application of mathematics in everyday life* and *the culture or history of mathematics*, are not as explicit in the results of the previous section but are often found in the textbooks and curricular documents. Regarding the Pythagorean theorem, the textbooks of both countries similarly refer to the history of mathematics in its introduction and emphasise the application of the theorem for everyday life and for mathematics (e.g., finding the hight, area, distance, etc.). Furthermore, the Taiwanese official document explicitly mentions the fifth aspect as follows: 'The history of mathematics can help us understand the differences regarding how different cultures during different eras have influenced different developments in mathematics' (TME, 2018, p. 1).

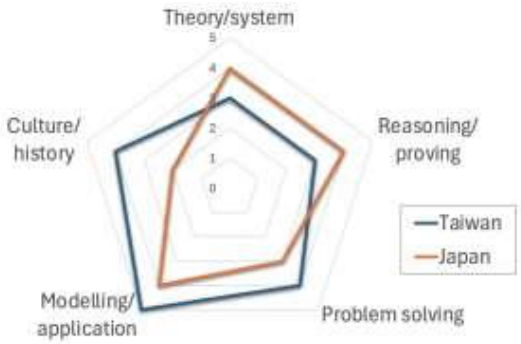


Figure 4: Five aspects dealt with in Taiwanese and Japanese geometry curricula

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Developing geometrical reasoning skills in primary school

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The present study conducted and analysed lessons aimed at improving geometrical reasoning skills among primary schools students. Students could engage with the figure in a generative mode of interaction through the activity of cutting a symmetrical figure in half. Analysis of the group's dialogue revealed that after grasping the figure through Duval's perceptual, operative and sequential apprehensions, they were able to grasp the figure through discursive apprehension and make inferences on the basis of the shape's properties. Additionally, the effect of the teacher's questioning of the rationale in this dialogue enabled students to shift to discursive apprehension. Consequently, students were able to provide a logical explanation using the properties of shapes that the straight line they drew was the axis of symmetry. These results highlighted three critical elements essential for fostering geometric reasoning skills in primary school lessons.

Keywords: geometric reasoning skills, generative mode of interaction, cognitive apprehension

Introduction

The existing literature suggests that students' difficulties in geometry in secondary school mathematics are largely due to the *empirical-deductive* gap between *intuitive and operative activity* oriented teaching in primary schools and *argumentation* oriented teaching in secondary schools (e.g. Knox & Kontorovich, 2022). Particularly, deductive proofs are often employed in geometry, which are difficult to understand (e.g. Fujita & Jones, 2014; Herbst, 2004). To cope with this difficulty, it would be effective not only to improve classes in junior high schools, but also to conduct classes in elementary schools to foster argumentation understanding (Stylianides & Stylianides, 2017).

Thus, in the study of geometry, it makes sense 'to develop the ability to reason in primary school as a basis for teaching argumentation in secondary school'. And such lessons should be conducted. However, the necessary elements for such lessons need to be determined. First, we established a theoretical framework for the development of geometric reasoning based on the previous literature. Then, the research questions and methodology were described. Thereafter, data from students' geometrical reasoning were analysed to examine the contribution of the lessons in developing students' geometric reasoning and obtaining suggestions for improving their geometric reasoning.

The findings of this study can contribute to the CERME discussion paper on students' competence (visual, reasoning, operational, and figural) and issues in geometry classrooms and curricula (e.g. Palatnik et al., 2023).

A theoretical framework for designing lessons that foster geometric reasoning

A useful idea concerning geometrical reasoning is to examine students' skills in dealing with geometrical figures. French psychologist Duval approached geometry from both the cognitive and perceptual perspectives. For example, Duval (1995, pp. 145–147) has provided a detailed framework

for analysing the semiotics of geometric figures. In this framework, he has identified four types of *'cognitive apprehension'* (Duval, 1995; the following descriptions have been taken from Jones, 1998, p. 31).

1. *Perceptual apprehension*: this is what is recognised at first glance; perhaps, for instance, sub-figures which are not necessarily relevant to the construction of the geometrical figure.
2. *Sequential apprehension*: this is used when constructing a figure or when describing its construction. In this case, the figural units depend not on perception but on mathematical and technical constraints (in the latter case this could be ruler and compasses, or perhaps the primitives in computer software).
3. *Discursive apprehension*: perceptual recognition depends on discursive statements because mathematical properties represented in a drawing cannot be determined solely through perceptual apprehension, some must first be given through speech.
4. *Operative apprehension*: this involves operating on the figure, either mentally or physically, which can give insight into the solution of a problem.

He indicated that for a picture or diagram to function as a geometric figure, it must be understood perceptually and by at least one other type of apprehension (sequential, discursive or operative). We observed how these *apprehensions* were manifested in practice. For example, in argumentation lessons, students need to problem solve by using discursive understanding to capture the picture.

When considering the developmental process, it is necessary to consider how students perceive geometric figures. Fischbein (1993) proposed the 'figural concepts' in which, geometric reasoning is driven by the interaction of two aspects: the figure and the concept (Fischbein, 1993; Jones, 1998). Fischbein (1993) describes that the difficulty in understanding geometrical figures is rooted in these two aspects, stating that "the integration of conceptual and figural properties in unitary mental structures, with the predominance of the conceptual constraints over the figural ones, is not a natural process" (p.156). Moreover, it is suggested that to cope with difficulties, it is necessary for teachers to provide learning opportunities to integrate and harmonise the visual and theoretical aspects of geometrical figures.

Next, we considered the learning environment that should encourage the development of geometric reasoning. When designing geometry lessons, the manner in which students engage with diagrams is central to the study of teaching materials. Herbst (2004) considered the interaction between students and diagrams in geometry and proposed four modes of interaction between the actors (A), diagram (D) and geometric object (O). He advocates the *generative mode of interaction* as one of them and describes *generative* as follows (p.134):

> ——— a mode of interaction between actor, object, and diagram whereby an initial, hypothetical identification between object and diagram affords conditions and constraints for the actor to anticipate operations on the diagram and their results, and whereby the actualization of those operations on the diagram constrains the interpretation of the results of those operations in ascribing properties to the object, further differentiating the object from the diagram

Additionally, Herbst (2004) states that 'The work of "deductive guessing" — building reasoned conjectures or using deductive reasoning to find out what could or should be true — can be aided by tasks that engage students in generative interactions with diagrams', stating that 'Therefore, engaging students in "generative interaction" with the diagram could help them to reason' (p. 138). Furthermore, he states that 'engaging students in generative interactions with diagrams requires a teacher to do more than design tasks and manage students' (p. 138). This also emphasises the need for designing lessons through deliberate teacher interventions.

Based on the above theoretical ideas, the following research question was explored in the present study:

What elements are necessary to improve geometrical-reasoning skills as a basis for enhancing argumentation understanding in primary school lessons?

Methods

Using the framework described above, the lessons on symmetry were implemented in a primary school in Japan. Students' use of *apprehension*, interaction between students and diagrams, and the reasoning utilised during actual lessons were analysed. Regarding students' use of *apprehension* for reasoning, their statements were analysed considering Duval's four types of apprehension. Then, students' manner of engaging with the diagrams in class were analysed using Herbst's (2004) mode of interaction.

The study task required students to cut a line-symmetrical garment in half (Figure 1). A useful approach to draw a line to cut the garment presented in Figure 1 in half is to regard Figure 1 as a symmetrical figure and draw an axis of symmetry. The following prerequisites should be identified:

- Figure 1 should be line symmetrical.
- The point at the neck is the middle (point on the axis of symmetry), and the axis of symmetry can be drawn by defining another point.

Figure 1: Shapes presented to students

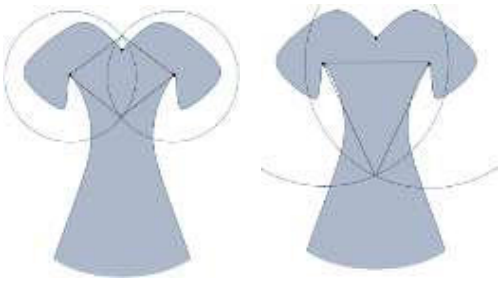


Figure 2: Anticipated drafting

Participants attempted to draw an axis of symmetry using a ruler and compass which are commonly used in Japanese mathematics lessons. They drew curves around various points and shapes by connecting the resulting intersections to the points displayed in Figure 1. Figure 2 presents the drawings expected from the participants. Participants could draw lines that appear as axes of symmetry while drawing various shapes. They discussed why the axis of symmetry could be subtracted and were able to relate the operations performed on the diagram to the properties of the shape.

The above geometry lesson was conducted in a Grade 6 classroom with 30 students in 2022 in a Japanese primary school. By then, students had already learnt about the properties of the axes of symmetry. In the pre-survey, most children were able to draw the axis of symmetry on the figure. However, few students were able to answer in writing about the nature of the axis of symmetry. The study included students from a public school in Japan with an average academic performance. Thus, some children were good at mathematics, whereas others were not. Ethical approval was obtained from an author's affiliated school. All names mentioned in this paper have been pseudonymised. The 45-minute lesson was recorded using three cameras and subsequently analysed.

Results

Students' cognitive apprehensions

In the actual lesson, the activities were conducted individually, followed by discussions in groups of four. First, we examined the *cognitive apprehensions* observed among the students. An example of the group interaction is provided below.

Table 1: Group interactions and participants' views

students	proposal	perceptual	sequential	discursive	operative
Sara	With a compass, I drew around here (flank) and then around here (opposite flank) and then I could draw the line.	○			○
Risa	Is that at 90° ?				
Tomo	Isn't it supposed to be 90° because it overlaps the folded line?	○			○
Teacher	Why is it at 90° ?				
Tomo	90° is 90° because it is vertical. I didn't figure it out, but I feel it would be 90° .	○			○
Risa	Ah! Wait, I get it!(draw a rhombus) Rhombus... look.	○	○		○
Tomo	What, what do you mean?				
Risa	This is a circle. This radius is the same. So, this one has the same length.It's an equilateral triangle over here and an equilateral triangle over there, so they're all the same length and form a rhombus.And because they are diagonals, they are perpendicular.	○	○	○	○

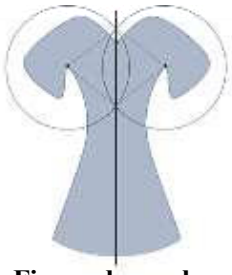


Figure3: Figure drawn by a student

As we can see, students initially attempted to draw different lines using a compass and ruler. Then, as demonstrated in Sara's statement, 'With a compass, I drew around here (flank) and then around here (opposite flank) and then I could draw the line'. She attempted to draw the axis of symmetry by using her *operative apprehension*. Through trial-and-error, students eventually realised that an axis of symmetry could be drawn by connecting the intersections of circles drawn around the corresponding vertices. However, they captured the figure through *perceptual apprehension*, as demonstrated in the statement by Tomo, 'I did not figure it out, but I feel it would be 90°'. Therefore,

they could not determine whether the line was an axis of symmetry and could not reason. Thereafter, the teacher asked, 'Why is it at 90°?'. Students then began to use *sequential apprehension* to recapture the diagram, such as by tracing curves and drawing rhombuses. This can be observed in Risa's statement: 'This is a circle. This radius is the same. Therefore, they have the same length. It is an equilateral triangle here and there; therefore, they are of the same length and form a rhombus. As they are diagonals, they are perpendicular'. Therefore, she was able to use her *discursive apprehension* and the properties of shapes to logically explain the axis of symmetry.

Students' interactions with diagrams

Next, we analysed how the students handled the diagrams. First, the students hypothesised that they should be able to draw an axis of symmetry to achieve the objective of *cutting the clothes in half.* They used a compass and ruler to add various lines to the diagram of the clothes, which added a new meaning of *rhombus* in the figure redrawn by the line. Subsequently, they used the properties of the rhombus to solve the problem through reasoning. Thus, the students engaged with the diagram through a *generative mode of interaction*.

Therefore, students were able to engage in the task using *operative apprehension* as they engaged with the diagram through a *generative mode of interaction*. The teacher's questions triggered *sequential* and *discursive apprehensions* to advance geometric reasoning. These were also for example observed in the following exchanges:

Kazu: The axis of symmetry is at right angles to the line drawn by connecting the corresponding points. I contemplated that the feature of equal distance from the corresponding point to the axis of symmetry was similar in character to the line drawn between the top vertex and the base of an isosceles triangle. Therefore, I drew it using isosceles triangles. (Figure 4)

Teacher: Very interesting. Did you understand that?

Students: I know.

Teacher: What shapes do you see in his head?

Students: Triangles with two equal sides.

Jin: Can we have equilateral triangles?

Figure 4: Kazu's figure

Figure 5: Jin's figure

Teacher: Oh, that is an interesting word. Someone was thinking in terms of equilateral triangles. Please explain the significance of this point.

Jin: Collar point is definite and in the centre. The triangles are of the same length (Fig. 5). If you connect here (collar points) to here (the apex of the equilateral triangle), you can draw an axis of symmetry.

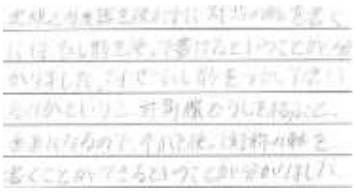
Kazu discovered an isosceles triangle and used it to draw an axis of symmetry. Jin stated that the same could be performed using equilateral triangles. Thereafter, the teacher prompted him to explain, and Jin provided a logical explanation using the properties of the equilateral triangles that he had drawn himself. This interaction was analysed. First, Kazu drew a line using a compass through

operative apprehension. Kazu looked at the line and thought that 'the feature that the axis of symmetry has an equal distance from the corresponding point to the axis of symmetry is similar in character to the line drawn between the topmost vertex and the base of an isosceles triangle'. Then, a new property was added to Fig. 4 as 'a diagram with several isosceles triangles'. This allowed them to explain how an axis of symmetry could be drawn. While listening to the explanation, Jin also thought that it could be done using equilateral triangles, and drew an equilateral triangle (as shown in Fig. 5). Jin used *sequential apprehension* as he deliberately limited the length of the sides using a compass. Here, the teacher asked, 'What is the name of this triangle?'. Then, Jin was able to draw an axis of symmetry using the property that the vertex of an equilateral triangle is *in the middle* and provided a verbal explanation. Thus, the *generative interaction* between the student and diagram could be observed, as the diagram produced by the new manipulation added the new figure properties of isosceles and equilateral triangles. Even in this context, students could use *perceptual, operative, sequential and discursive apprehensions* to explain how the axis of symmetry could be drawn.

Discussion

The results revealed that through children's engagement with the diagrams by means of the *generative mode of interaction*, they were able to use various *cognitive apprehension* to capture geometric objects. This was because the activity of *cutting the clothes in half* allowed them to engage with the diagram through a *generative mode of interaction*. They replaced 'cutting the clothes in half' with the act of *drawing the axis of symmetry* and worked with the diagram using *operative apprehension*. In doing so, students could deepen their understanding of shapes by thinking about questions such as, 'What do the lines drawn with the compass mean?', 'What kind of shapes were drawn?' and 'What are the properties of the shapes they drew?'. During the discussion, they deliberately drew shapes and axes of symmetry using *sequential apprehension*, thinking that they could also use the properties of other shapes to draw the axes of symmetry. This process may have enabled students to provide a logical explanation about drawing the axis of symmetry using the properties of the shape by engaging in *discursive apprehension*.

The worksheet below shows that students in the class began to recognise the axis of symmetry.



I have found that a rhombus is a good way to draw the axis of symmetry without using a measuring tape and protractor. A rhombus can be used because when the diagonals are connected to each other, they become perpendicular.

Figure 6: A student's description

As can be seen from the exchanges in Table 1, in the early stages of their learning, students were able to use their *perceptual apprehension* and judged that the lines were 90° to each other by appearance. In addition, students were able to use *sequential apprehension* to relate the properties of the rhombus to the compass marks through drawing with the compass. This enabled them to write an explanation of how to draw the axis of symmetry (Figure 6). This is the result of concepts controlling images, leading to a state where conceptual and figurative properties are integrated (Fischbein, 1993). A similar students' development can be found in Herbst (2004). This suggests that activities that engage with diagrams through *the generative mode of interaction* are suitable as what Fischbein calls the

'learning opportunities to integrate and harmonise the visual and theoretical aspects of geometrical figures'.

On the other hand, how could they verbalise information that they understood unconsciously? The students perceived the diagram through *perceptual apprehension*, observed through their statements such as, 'It will be 90° when folded' or 'I haven't measured it, but it's probably 90°'. However, the teacher's question, 'Why is it 90°?' prompted them to engage in *discursive apprehension.* Students who had been influenced by the images were able to explain the rhombus by referring to its properties because the teacher's questions created a need to provide a logical explanation. Therefore, an appropriate method of teacher involvement suggested by Herbst (2004) is for the teacher to question the rationale when he or she wants to encourage *discursive apprehension*.

The following three elements were identified as requirements for designing lessons to improve geometric reasoning in primary schools.

(i) Create situations in which students engage with the diagram using the *generative mode of interaction*, activating their *perceptual*, *operative* and *sequential apprehensions*; (ii) provide opportunities for determining logical explanations of the basis of events in the dialogue; (iii) question the rationale and encourage a shift in *apprehension* to *discursive understanding*.

Regarding (iii), the importance of appropriate teacher involvement in improving reasoning skills has been previously noted. (e.g. Stylianides et al., 2019). This study demonstrated the role of linking *operative apprehension* to *discursive apprehension*. This can be considered as a valid perspective for teaching to bridge the *empirical-deductive* gap between primary and secondary schools.

Conclusion

In this study, we analysed a lesson conducted in the area of primary school geometry aimed at developing reasoning skills and discussed the factors that contribute to the development of geometrical reasoning skills. Consequently, the following three elements were derived.

(i) Create situations in which students engage with the diagram using the *generative mode of interaction*, activating their *perceptual*, *operative* and *sequential apprehensions*; (ii) provide opportunities for determining logical explanations of the basis of events in the dialogue; (iii) question the rationale and encourage a shift in *apprehension* to *discursive understanding*.

Lessons that incorporate these elements can provide a foundation for understanding argumentation in secondary school. This allows us to situate lessons in primary schools that bridge the 'experiential-deductive' gap.

The following research possibilities emerged from the conclusions drawn in this study.

- How do students who have experienced the practices described in this study react to proof problems in secondary schools? It is possible to examine the elementary role of argumentation guidance and the need for practices described in this study.
- Applying these three elements to enhance geometric reasoning skills in other areas of geometric learning allows for the verification of the generality of the conditions and demonstrates their practicality in the classroom.

We aim to develop curriculum and learning environment designs that bridge the empirical-deductive gap between primary and secondary education through the advancement of these research.

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High School Mathematics Teachers' Support for Students' Focus on the Properties of Geometric Solids Using Dynamic Geometry Software

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In this study, we examined two high school mathematics teachers' uses of dynamic geometry software (DGS) while solving three-dimensional geometry tasks, and investigated the ways in which they supported students' attention to the properties of geometric solids through the use of a drawing or construction. The data were collected from two geometry classes. In the first geometry class, students used DGS drawings in which they were not able to drag objects. In the second geometry class, students used DGS constructions where they could actively drag and interact with geometric objects. We found that the mathematics teacher who used DGS drawings cited the lack of precision in DGS as a means to resolve contradictions, whereas the teacher who employed DGS constructions relied on task instructions to navigate variables and complete the tasks.

Keywords: Drawing, Construction, Figure, Diagram, Dynamic geometry software.

Introduction

Students often struggle to interpret three-dimensional objects when they are represented in two dimensions. Such representations can be misleading, as crucial information, particularly depth, is lost (Parzysz, 1988). However, with the advent of dynamic geometry software (DGS), students can drag properly constructed objects to explore geometric relationships (Laborde, 2008). Students may also use a DGS drawing without using any function tools (e.g., the Perpendicular Line tool), which messes up when a primitive element (e.g., points) of the geometric object is dragged (Healy et al., 1994). Hollebrands and Smith (2009) describe the drawing and construction in DGS as follows.

> We refer to *drawing* as a process that involves the use of "freehand" tools to create a geometric object…. When a geometric object is created using specific tools, such as perpendicular line command to ensure all angles are right angles, then this process is referred to as *constructing* and the final object is called a *construction.* (pp. 221-222)

In the current study, we focused on high school mathematics teachers' use of DGS drawings and constructions to characterize their teaching activity in which the properties of geometric solids represented within a Euclidean Geometry landscape. Different from prior research on examining students' understanding of Euclidean concepts in DGS where loss of information is minimal, our investigation enabled us to characterize how high school mathematics teachers supported students' attention to the properties of geometric solids represented within plane geometry. The research

question that guides this study is: "How do high school mathematics teachers support students' attention to the properties of geometric solids in DGS, through the use of a drawing or construction?"

Theoretical Background

In a teaching activity, students might be given a DGS construction, a construction with limitations (e.g., soft constructions), or a drawing. Also, students might create their own constructions or draw shapes in DGS. A properly constructed shape preserves its properties upon dragging, while a construction with limitations may not. If a student uses a drawing, then geometrical objects not constructed using DGS tools (such as a midpoint that does not preserve its properties upon dragging) or objects drawn with traditional tools cannot be dragged. Regardless of the tool a student uses, if the student focuses on the properties of the geometric object by viewing its representation, then the student is focusing on the *figure*. However, if the student does not attend to the properties or perceptual characteristics of the visual representation, then the student is working with the *diagram* (Hollebrands & Smith, 2009). Hollebrands and Smith (2009) make a distinction between a figure and diagram as follows.

> When a student views visual representation of a geometric object and attend to its properties, then we state that the student is reasoning about a *figure*. In contrast, we use the term *diagram* when it is not evident whether the student is attending to the properties or the perceptual characteristics of the visual representation of a geometrical object. (p. 222)

Dick and Hollebrands (2011) emphasized the importance of maintaining mathematical fidelity (adhering to accurate mathematical principles) and cognitive fidelity (aligning students' perceptions with the intended mathematical meaning) in dynamic sketches. Prior research investigated students' uses of DGS drawings or constructions and characterized their learning. For instance, Healy (2000) investigated the different types of task constructions employed by students (aged 14-15 years) when working on geometric proofs. One dyad created a robust construction that preserved its critical attributes upon dragging. Another dyad created a soft construction that did not stay true in mathematics, leading to situations where the construction did not preserve properties when its primitive elements were dragged. Healy reports that this second dyad encountered counterexamples due to the limitations of their sketch, which at times resulted in decreased motivation. However, Laborde (2008) appreciates the use of soft constructions, as they allow students to adjust their solutions or reasoning when they gain new insights into the problems.

González and Herbst (2009) describe how DGS enables a functional mode of interaction, where students can explore mathematical relationships dynamically by dragging and measuring elements within a diagram. This dynamic mode contrasts sharply with static, unmodifiable diagrams, allowing students to focus on the properties of figures rather than merely their visual attributes. This flexibility facilitates what they call a "measure-preserving conception" of congruency, where students test congruency by checking for consistency in measures during dragging (González & Herbst 2009). In static diagrams, visual perception often leads students to treat figures based on surface-level visual characteristics, which can obscure deeper geometric relationships. This perceptual approach, typical of static interactions, differs from the measure-based approach fostered by DGS, which allows

students to maintain congruent relationships as they adjust parameters. Herbst et al. (2017) elaborate on these differences, noting that DGS enables students to explore collections of diagrams continuously, rather than isolated, static instances. This approach emphasizes geometry as a dynamic process, where conjectures are grounded in observed invariants across multiple instances. The DGS environment thus provides a foundation for moving from empirical observations to theoretical generalizations, an essential skill for deep geometric understanding (Herbst et al., 2017).

Herbst and colleagues (González & Herbst 2009; Herbst et al., 2017) approach provides a broader classification that includes multiple modes of interaction, accounting for the evolving ways students can engage conceptually with diagrams. Yet, Hollebrands and Smith (2009) emphasize the distinction between students focusing on the properties of a geometric object (figures) and their attention to its visual representation without regard to properties (diagrams) in DGS.

Methods

Mrs. Dabney, and Mr. Taylor (pseudonyms) were two of the participants in a three-year professional development project focused on teaching and learning geometry and algebra with technology. They participated in two weeklong summer institutes and engaged in two years of online professional development, with additional support provided at the classroom level in the southeastern United States. Although the professional development was primarily focused on Euclidean geometry with the use of the Geometer's Sketchpad (GSP), a few teachers used the GSP for teaching three-dimensional geometry. The teachers utilized GSP drawings or constructions in which the geometric solids were not allowed to rotate and examine from different perspectives.

Mrs. Dabney and Mr. Taylor first introduced the tasks to the whole class. Students worked on the GSP tasks individually or in pairs, while the teachers moved around the room, providing feedback as students engaged with the tasks. Finally, they concluded the lesson with a whole-class discussion. Mrs. Dabney implemented tasks accompanied by sketches that were not constructed in the GSP, referred to as drawings. As a result, when students attempted to drag objects within these sketches, the objects did not retain their original properties. Video recordings from a lesson in which students used the GSP to find the surface area and volume of rectangular pyramids were analyzed. Students primarily focused on using the GSP to compute surface areas and volumes in various ways. In contrast, Mr. Taylor had his students study three-dimensional objects using the GSP constructions, which allowed them to drag the objects and change the radius and height of the right circular cylinder. His students engaged in solving optimization problems, such as minimizing the volumes and surface areas of geometric objects. Yet, the cylinder was not allowed to be rotated and examined from different perspectives since the construction was made in a Euclidean Geometry landscape.

During each 90-minute class session, a video camera recorded students' work, including whole-class discussions, individual tasks, and pair activities, as well as teachers' use the GSP. The analysis framework, based on the theoretical distinction between reasoning about a figure and a diagram, guided the examination of students' interactions with DGS drawings and constructions. The researchers focused on the challenges students encountered in preserving mathematical properties, particularly how these challenges varied between static drawings and dynamic constructions. A

comparative analysis of the instructional strategies used by two mathematics teachers highlighted how different approaches to DGS integration influenced students' geometric reasoning.

Results

Focusing students' attention on the properties of geometric solids through a DGS drawing

In Mrs. Dabney's class, students studied geometric relationships using DGS drawings which were not constructed with DGS tools. In other words, students were not able to drag objects, points, etc. and observe the relationships dynamically. Before Mrs. Dabney had her students use the GSP, in a whole-class presentation, she had given directions about finding the areas (the faces and the base of the pyramid), and the volume of the rectangular pyramid as shown in Figure 1, exemplifying on a static pyramid on the board. Mrs. Dabney mislabeled the height and the slant heights. Yet, this did not cause any difficulties since she directed students' attention to calculate the surface area. Students mostly had difficulty understanding the relationships of the pyramid when it was drawn on the board. For instance, in the following dialogue, the student (line 2) could not make the distinction between the slant height and the height of the rectangular pyramid.

1	Teacher:	Mmhmm. 10 is my base. So, 10 would be my base. What would be the height of that triangle?
2	Student:	9.
3	Teacher:	Uh-huh. The 9 goes in the very center. So, the 9 is actually how tall the whole pyramid is. So, look just on the face over here. What is the other number that you see? 8. That's the height of that triangle. Ok, so put an 8. OK, now, if it is a rectangle, what do we know about rectangles?
4	Student:	Opposite sides are the same, aren't they?
5	Teacher:	This side is equal to this side. So, the triangle that makes up this side is the same as the triangle that makes up that side. So, there is how many of them?
6	Student:	2.
7	Teacher:	2, OK, get that area. And, do for like the front triangle, OK. And then, there're two of those. Then, you'll find the area of the rectangle. And, add them all up.

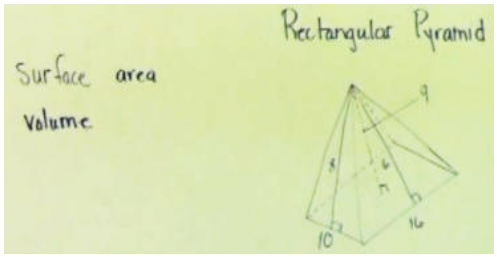


Figure 1: A rectangular pyramid drawn by Mrs. Dabney on the board

The difficulty of using the correct attribute for calculating the surface area was also evident in the GSP drawing Mrs. Dabney implemented. For example, as indicated in the excerpt below in Mrs. Dabney's class, some students used the height of the pyramid instead of slant heights, when they tried to find the surface areas of the triangles on the faces. The exchanges between Mrs. Dabney and a student who worked individually took as follows.

1	Teacher:	OK. For the front one, you did ½ the base. So, *GE*, right?
2	Student:	Mmhmm.

3	Teacher:	Times the height, which is *AC*.
4	Student:	Yeah.
5	Teacher:	Correct?
6	Student:	What? I think, I did the wrong thing.
7	Teacher:	OK. Let's do this. Let's see if I can get the height so we don't get confused.
8	Student:	Well, I made the one of them showing on. I mean, I need…
9	Teacher:	OK. The height of the pyramid, that's the pyramid. I really do not need it right now, that's just for volume. The height of the triangles (pointing at slant heights), I definitely need those.

Mrs. Dabney had her students find the surface area of the pyramid with two methods both of which lacked in mathematical fidelity. The first method that students used was to find the areas of the pink and orange triangles using ½ times base times height and double them, and find the area of the base multiplying two adjacent sides of the rectangle (Figure 2). However, the isosceles triangles representing the lateral surfaces were not created with the DGS function tools. Therefore, they were scalene triangles representing the lateral surfaces. Afterwards, they found the surface area of the pyramid by summing the surface areas and the base. In the second method, students found the areas by choosing the polygon tool of the GSP and placing triangles to the vertices of the pyramid and a quadrilateral for the base. Then, they highlighted the triangles and the rectangle, and found the areas using the area tool of the GSP automatically. These two methods resulted in two different surface areas (77.63 cm^3 vs. 75.18 cm^3 in Figure 2) because in the first method, students used medians of the scalene triangles representing the lateral surfaces, and the base was not a properly constructed rectangle. Students were confused with this mathematical contradiction. Mrs. Dabney provided the following explanation:

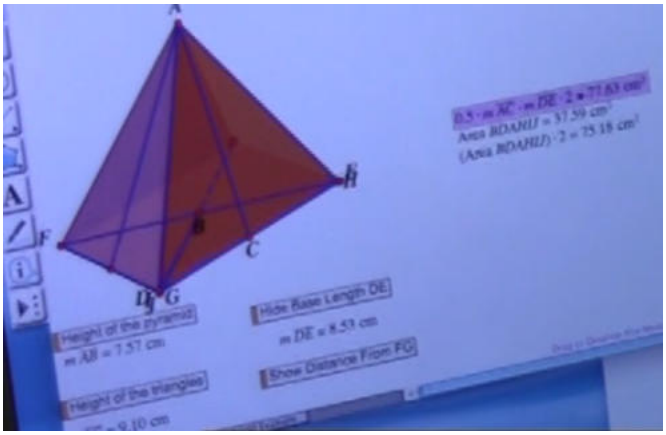


Figure 2: A student's individual work in the GSP in Mrs. Dabney's class

10	Teacher:	They are not exactly the same, but that is ok what we're doing because when we constructed, we're not making it exactly.
11	Student:	So totally different.
12	Teacher:	Oh no, because I want to see that. Because that's how, that's how you found the area only directions. By here it says find the surface area of the pyramid by using the polygons and by using the surface area formula.

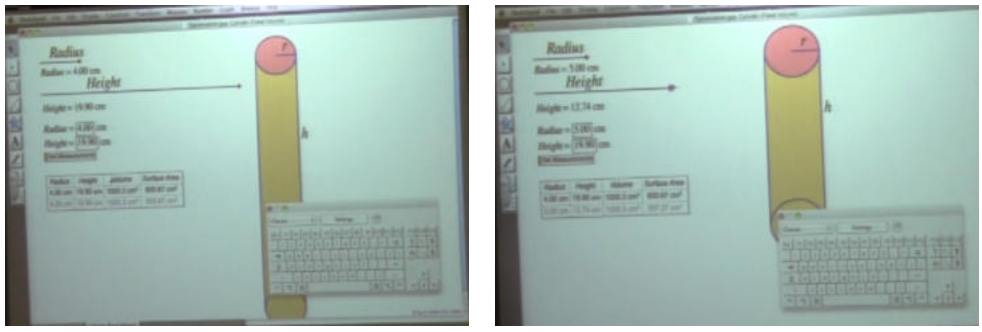
Later, during the whole-class discussion, Mrs. Dabney described this situation to a different student by saying "so, this is one way of finding the area using the polygon tool. This is another way of calculating the area yourself. They should be very close. OK, they're not gonna be identical because

you know when we draw this polygon it is not exact." In both of her explanations, Mrs. Dabney did not refer to the use of the incorrect height leading to the different results. Mrs. Dabney constructed slant heights of the pyramid which were constructed as medians with actions buttons in the GSP. Therefore, students obtained different surface areas. Also, the base was not a properly constructed rectangle, so students calculated different area results with the two different methods.

Focusing students' attention on the properties of geometric solids through a DGS construction

In Mr. Taylor's class, students studied geometric relationships using DGS sketches constructed with DGS function tools. He posed an optimization problem that asked students to find the minimal surface area of a cylinder whose volume was 1000 cm^3. During the task description, he referred to the cylinder with a 1000 cm^3 volume as a liter of shampoo bottle. Students had to maintain a volume of 1000 cm^3 while changing the length of the radius of the base and the height of the cylinder. However, some students dragged either height or radius for observing the change of surface area without keeping the solid at the same volume. In the following example that took during the whole-class discussion, Mr. Taylor asked students to come up with a systematic process by recording some possible radii and volumes.

1	Teacher:	I'm feeling good about 4. Start with 4. So, I wanna make my radius 4.
2	Student 1:	Height?
3	Teacher:	Umm, see radius of 4. Do the math, carry the 1. You have to divide. Yeah, the height is gonna be 19.9. See how I am doing math, I like that.
4	Student 1:	Yeah, you got 19.9 (inaudible).
5	Teacher:	Mmhmm, 19.9. So, I am putting that in as my height (Figure 4a).
6	Student 1:	Why do you say 19.9?



(a) (b)

Figure 4: (a) Mr. Taylor adjust the height at 19.9 cm, (b) He obtains a volume of 1000 cm^3

7	Student 2:	Equals 600 (the surface area).
8	Teacher:	Cause I'm looking for a volume of a 1000. That's why.
…		
13	Teacher:	So, the question is can I do better, or can I do worse? The surface area, I wanna minimize. The volume, I wanna keep constant. Alright? So, I'm gonna show something you can do here. Double click on your table. Again, set it 4, 19.9, and then double click. They'll keep track the volume for you. Alright?
14	Student 2:	So, that will leave you the radius the same?
15	Teacher:	Well, no. We should probably chance the radius to what?
16	Student 2:	5.
17	Teacher:	5. So, what I want you to do is to change the radius 5.

18	Student 2:	It worked!
19	Teacher:	Now, when I go to 5, my volume is too big. What I need to do?
20	Student 1:	Lessen the height.
…		
26	Teacher:	Alright, so let's lessen the height, so we find the volume of a 1000. Ugh, so close. Got it., right? There we go (Figure 4b). The, my surface is going down?
27	Student 1:	(Nods)
28	Teacher:	Good. So, I want you to keep following that procedure, right? And, go ahead and tell me where you think the minimum surface area is going to occur.

In this example, the student could not understand why he should change the height of the cylinder (line 6) because Mr. Taylor computed the height of the cylinder with a radius 4 cm mentally to keep its volume constant. However, when the teacher dragged the height to 19.9 cm, it became clear why the height had to be 19.9 cm. On the other hand, another student (line 14) asked if he had to keep the height the same. However, once Mr. Taylor dragged the radius to 5 cm, the student observed that the volume was larger than 1000cm^3. Later on, a student (line 25) proposed reducing the height of the cylinder to keep the volume constant, which indicated that he was reasoning about properties of the object. Despite his directions, some students had difficulty keeping the volume 1000cm^3 in their individual work with the task. For instance, Mr. Taylor noticed a student's difficulty while he was observing the working students and monitoring the progress. He addressed the student's difficulty as the following:

> That's wrong. Is that gonna hold a liter of shampoo? No!... So, I want you to mess with both (radius and height), right? I want you find the optimal radius for my surface area, alright, and I also want you to umm chance the height so that what do I always need, what's the centiliter becomes for the volume. You have to change the height make sure you always continue with 1000 cm^3 for your volume.

By prompting the student to "mess with" both the radius and height, Mr. Taylor encouraged an experimental approach to refining the shape's dimensions while meeting the volume constraint, aiming to deepen the student's understanding of how changes in dimensions affect both surface area and volume.

Concluding remarks

In this study, we exemplified cases in which students did not focus on the properties of three-dimensional objects using the GSP drawings or constructions. From there, we characterized how the teachers addressed students' difficulty. Mrs. Dabney, who used a DGS drawing, pointed out the inherent limitations of DGS in achieving exactness. Her comments to students, emphasizing that the measurements were "not exact" and that results could differ slightly, reveal her attempt to resolve potential misunderstandings caused by the imprecision of digital tools. This approach suggests that Mrs. Dabney was aware of students' potential confusion and used discussions to clarify the expected level of precision in DGS environments. On the other hand, Mr. Taylor aimed to help students focus on properties by encouraging them to dragg the sliders and observe the dynamically changing volume and surface area outputs.

The differences in instructional strategies between Mrs. Dabney and Mr. Taylor highlight the importance of using dynamic geometry tools to facilitate students' reasoning about geometric objects. Mrs. Dabney's reliance on static drawings and procedural instructions limited students' engagement with the geometric properties, resulting in contradictory calculations and persistent errors. In contrast, Mr. Taylor's dynamic approach allowed students to explore and manipulate geometric objects actively, fostering a deeper understanding of the relationships between different geometric attributes.

The findings of this study further support Herbst et al.'s (2017) emphasis that dynamic geometry environments not only increase student engagement but also promote a shift in how students understand geometric properties. The students in Mr. Taylor's class demonstrated an understanding of the relationships within geometric objects through active manipulation, allowing them to transition from treating diagrams as static forms to viewing them as flexible representations. This dynamic engagement was less apparent in Mrs. Dabney's class, where students were limited to static diagrams. Integrating dynamic diagrams more consistently into classroom practice may thus support students' cognitive development and reasoning skills more effectively than static approaches alone.

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I see what you're thinking: Embodied collaborative argumentation

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A group of 5 pre-service teachers engaged in constructing and exploring a body-scale icosahedron model. Triangulating audio–video and mobile eye-tracking data, we analyzed the roles of motor actions and sensory perceptions in the emergence and propagation of geometrical insight. We present an exemplary episode where Gail multimodally expresses a valid argument for finding the icosahedron's half-height. Another participant's eye-tracking data revealed that his perception of Gail's manual actions helped him interpret and then replicate non-verbal features of her argument. The study contributes to our understanding of embodied cognitive processes in authentic educational contexts, particularly the role of perception–action engagement with concrete media in collaborative mathematics tasks. Teacher awareness of these perceptuomotor micro-processes might enhance the pedagogical efficacy of integrating embodied activities into classroom geometry instruction.

Keywords: Embodied argumentation, collaborative learning, preservice teachers, spatial geometry

Introduction

There has been growing interest in 4E (embodied, enactive, embedded, and extended) approaches to mathematical learning, particularly in geometry (e.g., Abrahamson et al., 2020). These approaches, unlike traditional ones, emphasize physical interaction and sensory experiences in developing mathematical understanding. Geometry—a study of shape and space, where spatial visualization and reasoning are vital components of learning—is particularly amenable to the pedagogical implementation of these approaches. Through manipulating physical models, moving around and through the models, measuring them, and using various gestures, learners engage in embodied argumentation, a collaborative process where action-based experiences are integral for reasoning and proving (e.g., Palatnik & Abrahamson, 2022; Walkington et al., 2019).

Mathematical argumentation is "a line of reasoning that intends to show or explain why a mathematical result is true" (Sriraman & Umland, 2020, p. 65), produced by students and teachers in mathematics classrooms. This paper introduces the concept of *embodied argumentation*—a form of math argumentation, in which learners use physical actions, such as interactions with objects, as well as gestures, to construct, support, and communicate mathematical proofs and explanations. The empirical context of the investigation is a geometric task involving the collaborative construction of a human-sized Platonic solid followed by exploring its global features, such as determining its total number of vertices. Drawing on insights from gestural reasoning, collaborative problem-solving, and Proofs Without Words (PWW, Marco et al., 2022), this study examines how pre-service teachers use their bodies, physical interactions with the model, and gestures to develop and communicate geometric arguments. The goal of this research is to contribute to the field's understanding of how embodied cognition manifests in collaborative mathematical argumentation and to investigate the educational potential of integrating embodied activities into geometry instruction.

Theoretical Background

The late David Tall suggested that mathematical reasoning evolves through three worlds: the embodied, the symbolic, and the formal–axiomatic world. The embodied world, where learners use physical interactions and sensory experiences to grasp mathematical concepts, provides the foundation for math before transitioning to symbolic and formal reasoning (Tall, 2003). He presented a PWW on the sum of the *n* first integers as the manipulation of pebbles (embodied), then proof by arranging a sum of two lines of numbers, one of them in reverse order (symbolic), and finally proof by induction (axiomatic). Tall noted that the first two proofs have "clear human meaning, the first translating naturally into the second. The induction proof, on the other hand, often proves opaque to students, underlining the gap that occurs between the first two worlds and the formal world" (p. 8).

Embodied cognition posits that human thinking is deeply rooted in physical interactions with the environment (e.g., Barsalou, 2008). In mathematics education, this framework has supported claims that learners engage with concepts through gestures, movement, and manipulating physical objects: moving in a new way by perceiving in a new way and then naming in a new way (Palatnik et al., 2023). This embodied view is especially valid in geometry, where physical interaction with geometric figures, such as objects and diagrams, is inseparable from understanding (Herbst et al., 2017).

An embodied approach to geometry instruction aligns with research suggesting that embodied reasoning can support the development of geometric understanding. Nathan et al. (2021) emphasize that intuition and insight in geometry often arise through embodied experiences, such as gestures, that allow learners to transition from informal exploration to more formal proof. Their study provides evidence that dynamic depictive gestures, which involve motion-based transformations of mathematical objects (e.g., dilation or rotation), significantly support the production of valid proofs, whereas static gestures are less effective in fostering mathematical generalizations necessary for proofs. Other studies investigating learners interacting with physical models—such as 3D geometric shapes—show they gain insights through bodily movements (Benally et al., 2022). These actions form the basis of geometric argumentation and reasoning, as learners use their bodies to represent, explore, and communicate geometric properties (Palatnik & Abrahamson, 2024).

Embodiment and dynamism are constitutive to geometric reasoning, whether observable by others or only covertly experienced by individuals through multimodal processes, like mental transformations, or in the form of imaginary auxiliary constructions. We speculate an experiential spectrum between covert and overt multimodal mental activity. We further assume that covert and overt aspects of mathematical reasoning afford complementary contributions: at times, it is helpful to imaginatively manipulate a geometric structure, unbound by available media, whereas, at other times, it could be vital to materially realize these mental figments in the form of publicly inspectable models. A case in point is PWW-based activities, where learners engage in *gap-filling*, which is the process of identifying missing components in visual representation and constructing formal proof (Marco et al., 2022). In the absence of formal symbolic displays, PWW artifacts provoke mathematical thinking and support diverse legitimate forms of argumentation (Nelsen, 1993); they help learners bridge embodied experiences and formal argumentation. Engaging students in interpreting PWW diagrams

fosters opportunities to experience authentic mathematical reasoning, which, as Lakatos (1976) insisted, is the actual lived phenomenological underbelly of the eventual written proofs.

Still, nurturing students' capacity to understand proofs—what they are as epistemic forms—may take holistic entrainment. From a didactical perspective, one might wish to scaffold mathematics students' epistemic practices, cognitive routines, and interpersonal argumentation by way of occasioning for them opportunities first to engage in the *overtly* manifest enactment of geometric reasoning prior to introducing problems more so conducive to *covert* reasoning. One powerful cultural tradition incorporating overt geometric actions is construction activities, for example, the historical pedagogical methodology established by Friedrich Fröbel (Brosterman, 1997). Whereas Fröbel designed his geometry construction tasks for kindergarten students, our philosophical and theoretical commitment to embodiment perspectives, as well as our empirical research, have led us to believe that high-school students, too, should build knowledge through building objects (Palatnik & Abrahamson, 2022). Namely, implicit to the hands-on pedagogical regimens characteristic of our studies into geometry education is an enactivist argument that humans are life-long sensorimotor learners—we never actually graduate the sensorimotor stage Piaget theorized as ending in early childhood (Abrahamson, 2022). Yet these polemic epistemological debates would avail from empirical data. As such, as we now argue, introducing material stuff into geometry pedagogy may shift the focus of *research* on geometry pedagogy, drawing attention to students' authentic praxis and multimodal phenomenology of manipulating concrete objects, possibly revealing the critical cognitive roles of perception and action that embodied mathematics-education researchers allege.

Recent research on embodied learning in proving and argumentation has largely focused on gestures' role in mathematical reasoning. Walkington et al. (2019) show how collaborative gestures support group reasoning, while Nathan et al. (2021) highlight that dynamic gestures may enhance proof production by simulating geometric transformations. While these studies provide valuable insights into the body's role in mathematical reasoning, they focus almost exclusively on gestures as the manifestation of embodied learning. A notable gap remains in understanding other forms of embodied cognition, such as physical interaction with manipulatives, sensorimotor experiences, or spatial reasoning through body movement, and how these contribute to mathematical proving and argumentation. Furthermore, research in authentic classroom settings, where students and teachers collaboratively develop embodied argumentation, is still missing. Further research is needed to investigate these broader manifestations of embodied learning in various educational contexts.

The research questions guiding our study were: What are the roles of motor action and sensory perception in mathematical argumentation? What is the educational potential of fostering opportunities for embodied argumentation in geometry instruction?

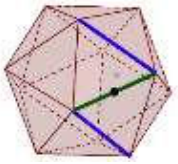
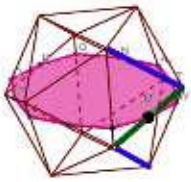
Illustrative case study of embodied argumentation

Our empirical context is a spatial geometry activity in which five pre-service mathematics teachers participated voluntarily: Aya, Gail, Ruth, Tim, and Yoni (pseudonyms). They were tasked to construct and analyze a body-scale Platonic solid (Figure 1). Whereas we had used this activity extensively in researching geometry education (Benally et al., 2022; Palatnik & Abrahamson, 2022, 2025), we had yet to examine experiences related to argumentation. The session was audio-video

recorded. During the activity, three participants wore mobile eye-tracking (MET) devices (Pupil Lab Neon), generating Multimodal Learning Analytics (MMLA) data.

Our data further comprised a stimulated-recall group interview, in which participants discussed their experience of collaboratively constructing the model. We prepared the activity transcript, juxtaposing the three MET sources and adding photos to capture movements, actions, and gestures. We transcribed the SRI interview. We coded data by marking key moments in the construction and exploration processes to capture elements of cooperative action and argumentation. Utterances were transcribed and coded, marking aspects of perception and action. We later juxtaposed key video moments in the construction process with relevant testimonials from the stimulated-recall interview.

We focus our analysis on the episode that begins at the 16th minute of the activity when the participants had already successfully constructed the icosahedron model and answered questions about its number of edges, vertices, and faces. The next question, however, presented a different challenge: "Assume the polyhedron was placed on one of its faces and filled with water. What would the shape of the water's surface be?"

a	b	c	d	  e
Because *this* is here, [slides her right hand down along the edge, starting from the top base (top blue segment on 1e)]	*that* is there. Symmetry. [slides her left hand up along the edge from the bottom base (bottom blue segment on 1e), establishing symmetry]	And then we have… [puts both hands on the ends of the center edge, where her previous movements had ended; slides her hands simultaneously to the middle of the edge]	*this* [The meeting point (M on 1e) is the middle of the edge and precisely at half the model's height when it stands on the face.	

Note: (a)-(d) blue arrows indicate the movement of Gail's hands; a red circle is an eye-tracking marker of Tim's gaze; (e) a scheme of Gail's argumentation and the form of the cross-section—dodecagon

Figure 1: Gail's embodied argumentation

The participants perceived this problem as non-trivial. Aya remarked, "Oh, interesting, because it's not... no, it's not symmetric. Part of (the edges) will be cut in the middle, and part will be cut…" Realizing they needed to establish the height of the cross-section, the participants quickly began approximating the position of the model's half-height above the ground. Yoni intuitively placed his hand on an edge close to the correct height but expressed uncertainty. Other participants offered suggestions on where the half-height might be located and, based on their approximations, attempted to count the vertices of the imagined cross-section. They debated between a pentagon and a hexagon (both suggestions are incorrect) but were unsure of the correctness of the answer.

The group then split into two sub-groups, which, due to the large physical size of the model, became separated: Aya, Yoni and Ruth on one side, Gail and Tim on the other. While partially aware of each other's actions, they worked somewhat independently. Aya first suggested that the cross-section at half-height would include one of the vertices. Aya and Yoni then agreed that two vertices on the same edge were not the same height from the ground. Their argumentation was supported by the observation of the model while touching the vertices, sliding the hands along the edges, simulating and comparing each other actions

Meanwhile, Gail engaged in searching for the half-height in a different way. First, she approximated the half-height of the icosahedron by comparing it to the distance between the palms of her hands put vertically. She took several such measurements with her hands. Then, she stepped back to change her perspective to be able to capture the whole model from a distance. She observed the model from different angles. At 16:08, when the instructor asked, "Where is it, exactly, half of a height?" Tim initially supported the idea that it was located at one of the vertices. However, Gail confidently stepped toward the model and indicated (correctly) horizontally with the palm of her hand: "No. This is half of a height." When the instructor asked, "Why is this half of a height?" Gail produced a concise argument through a series of three coordinated actions (Figure 1).

Tim's eye-tracking data revealed that he followed Gail's gestures closely. His gaze tracked each movement (marked by a red circle in Figure 1). Tim fixed his gaze and pointed at the midpoint of the edge before Gail completed her third movement. Based on this MMLA data, we conclude that he understood the argument before she finished her demonstration. He exclaimed, "Wow, cool!" recognizing the solution.

a	b	c	d
"Yes. Because this [slides a right hand down]	and this are the same height.[slides a right hand up]	And here, [slides both hands simultaneously toward the edge's middle]	this is its middle.
Note:(a)-(d) blue arrows indicate the movement of Tim's hands; a red circle is an eye-tracking marker of Aya's gaze; Gail continues to hold her hands, indicating the half-height point			

Figure 2: Tim's embodied argumentation

Note that the eye-tracking and audio data of Aya and Yoni revealed that they were not yet aware of Gail's embodied proof. Having endorsed Gail's argument, the instructor directed the other subgroup's attention: "Guys, did you hear what Gail said?" This prompted Aya, Yoni, and Ruth to look toward the first group, and Tim quickly took over the explanation (see Figure 2). While Tim's actions enact

the same chain of reasoning as Gail's, his actions are not precisely the same—he captured the essence of argumentation while varying the movements' motorics.

Further, the participants established that the cross-section had 12 vertices: six midpoints of the edges, as Gail demonstrated, and six additional points (see Figure 1e), which they found using a spare wooden edge as a line of intersection of the cross-section plane with the icosahedron's edges.

Discussion

The illustrative case study shows that embodied argumentation is not merely an aid but an integral part of how pre-service teachers construct and communicate geometric proofs. The analysis of the icosahedron task reveals how physical actions and perceptual processes constitute the mathematical argumentation of pre-service teachers.

Firstly, Gail's embodied argumentation exemplifies how a sequence of physical actions can serve as a form of mathematical proof. By sliding her hands along the edges and establishing symmetry, she transformed an abstract geometric property into a concrete perceptual experience for her and the group. The activity in which a group of peers first constructed and then explored the model prepared Gail and other participants to establish a more intimate connection to the mathematical object to externalize their reasoning, supporting both her insight and other arguments of the group (see Nathan et al., 2021 on insight and proof). When she prepared her argumentation, Gail used her hands to approximate distances and moved toward and from the model, thus changing perspective and angles of view (literally zooming in and out of the model). Gail's embodied argumentation emerged gradually through simulation, changes of gaze, and repositioning of the body in relation to the model.

Secondly, Tim's eye-tracking data (following Gail's gestures and anticipating the midpoint identification) reveals how perception is tightly coupled with physical action in embodied argumentation. As Walkington et al. (2019) suggest, gestures can act as collaborative tools, allowing learners to share their reasoning. In this case, Tim's tracking of Gail's physical movements enabled him to quickly understand and replicate her argument for the rest of the group.

Moreover, the internalization, reproduction, and endorsement of Gail's embodied proof by her peer—a future teacher, suggests he finds it valid. Given that the participants are prospective teachers, this acceptance provides a valuable opportunity to influence classroom culture by broadening the scope of argumentation to include movement, manipulation, and gestures involving physical models. In the milieu of the embodied collaborative construction activity, the group accepted the argument as valid.

In contemporary classroom culture, mathematical proof is typically limited to formal, symbolic logic (Herbst et al., 2017). However, embodied argumentation—which includes physical actions, manipulation of artifacts, and gestures—offers a compelling expansion of what constitutes valid reasoning. Rooted in embodied cognition, which posits that human thought is deeply interconnected with physical interactions (Barsalou, 2008), this form of reasoning is particularly relevant in classroom discourse, where participants can perceive each other actions on shared artifacts. Manipulating objects or engaging in physical interactions, such as rotating models, constructing auxiliary elements, or positioning geometric shapes, often reveals mathematical relationships that might be difficult to express verbally (Palatnik et al., 2023). These actions, along with gestures such

as pointing to a diagram or highlighting spatial connections, are not supplementary but integral to mathematical argumentation. Expanding the definition of proof to include embodied argumentation aligns with the interactive nature of classroom learning, where norms are shaped by teachers and students collaboratively constructing knowledge (Stylianides et al., 2022).

Expanding the definition of proof to include embodied argumentation reflects the multimodal ways students interact with mathematical notions, promoting a more inclusive understanding of proof. By fostering a classroom culture that embraces embodied argumentation, educators can better support students' reasoning processes, particularly in spatial and geometric contexts, where physical interaction with objects significantly enhances comprehension. This shift also echoes practices in professional communities, such as in research settings, where manipulating physical models and performing gestures play a role in the collaborative negotiation of meaning and problem-solving (e.g., Roth, 2001). Embodied argumentation may be integrated into sedentary classroom practices in ways similar to PWW-based activities (Marco et al., 2022), where reasoning based on visual information is a stepping stone for the gap-filling process, leading to more formal proving methods.

In conclusion, recognizing embodied argumentation as a legitimate form of proof in classroom discourse expands the cultural norms of mathematics education, making it more reflective and inclusive of how humans do mathematics by naturally integrating its embodied and formal aspects. Future research should explore tasks that best elicit embodied argumentation, methods for training teachers to facilitate it, and strategies for fostering effective group interactions.

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Analysis and categorization of errors in the geometry of prospective primary teachers

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In this paper, we present the results of a quantitative and qualitative analysis of geometry tests taken by prospective primary teachers. The main objective was to develop a classification of the errors made by students and to assess them qualitatively in terms of their nature, severity and potential danger to the understanding of geometry at an appropriate level. The results of this analysis were compared with the results of similar studies in an attempt to reveal which deficiencies in knowledge and skills in geometry future first-grade primary school teachers exhibit and how serious they are. The findings allow us to better focus on important areas and innovate geometry teaching so that it has a high potential to increase the level of geometric thinking of the students.

Keywords: Geometry, errors, prospective primary teachers, geometric thinking.

Introduction

It is obvious that already in the first grade of elementary school it is necessary to pay increased attention to the teaching of geometry. It is during this period that pupils form basic geometric ideas, the first definitions of geometric concepts are introduced, and the relationships between geometric objects are explored. This leads to an important development of geometric thinking (Douglas & Clements, 2011; Robichaux-Davis & Guarino, 2016). Increasing the level of geometric thinking contributes to enhancing students´ proficiency in other mathematical skills that are important, for example, for problem-solving (Clements & Sarama, 2007). Geometric thinking has a significant impact on the correct representation of numerical magnitude and the understanding of numerical relationships (Geary, 2007) and supports arithmetic transformations (Hubbard et al., 2005).

However, many studies have shown that prospective primary teachers do not achieve the level of geometric thinking necessary for their future profession (e.g. Milsaps, 2013). Unal et al (2009) state that one of the reasons for pupils´ low knowledge in geometry is the fact that many primary school teachers lack sufficient geometric knowledge and are unable to provide their pupils with appropriate educational opportunities to facilitate their geometric understanding. Thus, the depth of the teacher's understanding of the taught topic is what most influences the type of mathematics instruction (van der Sandt & Nieuwoudt, 2003). It is obvious that in order to achieve better pupils´ results in geometry, it is necessary to consider the level of knowledge of their future teachers.

Students' mathematical errors

The level of understanding of geometric tasks and the ability to solve geometric problems varies among students. The errors that students make when solving problems also vary. Newman (1977) presented a basic classification of the types of errors that students make during their written solutions. She divided errors into five categories:

- **Reading errors** are caused by the solver's inability to read keywords, variables, or symbols written in the mathematical problem task;
- **Comprehension errors** arise from the inability to understand the meaning of the words in the assignment of a mathematical problem, the solver then uses an inappropriate approach to the solution;
- **Transformation errors** are caused by the solver´s inability to convert the given problem into a mathematically accepted form or to schematically represent the given information;
- **Process skills errors** occur when the solver does not know the steps needed to complete solution algorithm;
- **Encoding errors** occur when the solver incorrectly interprets his results (the solution or answer is correct, but the solver is unable to explain it).

Morales (2014) states that solving mathematical problems requires both conceptual and procedural knowledge. A student´s **conceptual knowledge** refers to knowledge of mathematical facts and properties that are related to each other, whereas **procedural knowledge** consists of a set of rules and algorithms that the student uses to solve mathematical problems. Brown and Skow (2016) also address pupils' errors in mathematics. They divide these errors into three categories.

1. **Factual errors** are caused by a lack of factual knowledge (the student does not understand the meaning of concepts such as line segment, angle, triangle or perimeter);
2. **Procedural errors** are caused by the incorrect use of a rule or solution algorithm (e.g., missing sketch or discussion for a geometric construction);
3. **Conceptual errors** are caused by misconceptions or a misunderstanding of the basic principles and ideas associated with a mathematical concept (e.g., failure to recognize a geometric figure when its position changes).

In his publication, dedicated to the correction of mathematics tests and intended for future teachers, Trávníček (2006) introduces six categories of errors. The first four categories **A-D** are devoted to the types of errors made by students, the fifth category includes grammatical errors, and the sixth category refers to an error made by the teacher during correction.

Aim of the Study and Research Questions

The aim of this study is to find out which errors are made by prospective primary teachers when solving geometry problems in written work and how serious they are, and then to create a classification of these errors. This study seeks to find answers to the following questions: What types of errors do students training to become first-grade primary school teachers make when they solve geometry problems? Based on this, what impact might these errors have on the future practice of prospective primary teachers?

Research method

In order to answer the research questions, a detailed analysis of the writings of students training to become first-grade primary school teachers was conducted, providing sufficient quantitative and qualitative data. The analysis focused on written final exam paper that students take in the 4th year after the completing the two-semester Elementary Geometry course. The analyses were carried out

annually from 2021 to 2024. A total of 161 final tests were assessed over four years. For each test, the authors of the paper carefully reviewed them to determine the number and types of errors in each problem. The types of errors were registered according to the chosen categorization. The identified errors were classified using the categorization according to Brown and Skow (2016) (**BS**) and a modified categorization based on Trávníček (2006) (**T**). The original error categories **A-D** from Trávníček (2006) were supplemented by an additional category **E** called Inattention errors. The modified categorization can be viewed in Table 1.

Table 1: Modified categorization of errors in tests by Trávníček (2006)

Category	Description
A. Clueless errors	The solution of the problem or its essential part is completely missing or is replaced by disjointed, confused notations or random constructions (e.g., confusion of geometric operations, construction of a different figure than requested in the assignment, confusion between formulas for volume and surface area, choosing a formula for a different solid than requested in the assignment).
B. Factual logical errors (semantic)	These errors invalidate the factual and logical part of the solution (a part of the construction is neglected, or an inaccurate analysis of the construction task initiates an incorrect construction, e.g. marking faulty properties of the figure, using incorrect analogies).
C. Factual numerical / graphical errors	Partially incorrect or incomplete knowledge of principles, formulas, and algorithms, or their incorrect use (e.g., misleading sketch, translation by an incorrect vector, incorrect visibility of the solid, faulty conversion of units).
D. Formal errors	These errors may seem minor, but they can sometimes become factual, i.e. serious, errors during the construction (e.g. careless symbolic of the construction, a missing part of the solution in the construction task, inaccurate drawing, or unprofessional expression).
E. Inattention errors	These are errors caused by careless reading of the assignment. The solver overlooks a condition in the assignment and does not apply it during the solution process, but otherwise has the solution completely correct (e.g., incorrect size of the segment, rotation in the plane made around a different point, or by a different angle than given in the assignment).

The combination of these two categorizations was selected because **BS** provides a basic overview of student errors; however, it does not divide them in as much detail as is necessary for the selected analysis. Therefore, the categorization was refined according to **T**. The second reason for using the **T** categorization is its application in detecting errors in student tests in Czech schools. Both of these categorizations then provide a comprehensive overview of errors in student tests. While the error groups in **T** classification are not necessarily disjunctive (Trávníček, 2006), several types of errors may occur in each problem. The decisive factor for assigning an error to a category was its nature and cause. Each error was assigned to one category in the appropriate categorization, **BS** or **T**. The test each year contained the same seven types of problems: **P1**. A constructive positional non-parametric problem solved using locus with given properties; **P2**. the representation of a planar figure in a plane-congruent geometric mapping; **P3**. identifying the intersection or union of two planar figures;

P4. the representation of a solid in free parallel projection; **P5**. the representation of a solid in a spatial-congruent geometric mapping (in free parallel projection); **P6**. the calculation of a solid's measure (volume or surface area); **P7**. determining the set of all points in the plane according to the given properties.

These problems were selected to cover the content of the mentioned course Elementary Geometry, while also theoretically and practically addressing the topics taught in Czech elementary schools. These problems also aim to verify whether prospective primary teachers have the appropriate overview and a sufficient understanding of the theoretical foundations of these topics. More specifically, problem P1 assesses the ability to master and apply the properties of basic geometric shapes when constructing them using loci defined by specific properties. It also evaluates knowledge of all parts of the construction problem, the use of symbolic notation, and correct geometric expression. Problem P2 tests whether the student is able to display planar figures using planar geometric mappings. In problem P3, the test checkes whether the students understand, recognize, and can use operations on point sets, and also whether they have knowledge of the properties of basic geometric shapes and the relevant scholarly terminology. In solving problem P4, students demonstrate their knowledge of terminology, the properties of solids, and ability to project solids onto the plane using free parallel projection. Problem P5 focuses on spatial geometric mappings, understanding the difference between planar and spatial mappings, and also tests knowledge of the use and properties of free parallel projection. Solving the numerical problem P6 requires the solver to recognize the difference between the area of a planar figure and the volume of a solid, to know the properties of solids and planar figures needed for calculating measurements, to be familiar with the appropriate scholarly terminology, units of measurement, and their conversions. The understanding of the metric and spatial relationships of point sets in the plane and the use of geometric operations on point sets are demonstrated when students solve the problem P7. The test as a whole assesses whether the student has sufficient knowledge, understanding, and skills to develop geometric reasoning and spatial visualization skills in their future pupils.

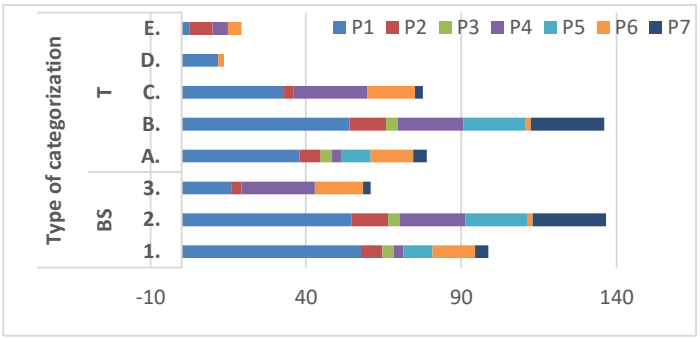
Results and discussion

For each problem described above, the frequency of individual errors within the given categories was determined. The relative frequencies of errors in each problem for the appropriate category are given in Table 2 and Diagram 1.

Table 2: Relative frequencies of errors in problems by categories

problem	Type of categorization							
	BS [%]			T [%]				
	1.	2.	3.	A.	B.	C.	D.	E.
P1	58	55	16	38	54	33	12	2
P2	7	12	3	7	12	3	0	7
P3	4	4	0	4	4	0	0	0
P4	3	21	24	3	21	24	0	5
P5	9	20	0	9	20	0	0	0
P6	14	2	16	14	2	16	2	4
P7	4	24	2	4	24	2	0	0

Diagram 1: Relative frequencies of errors in problems by categories



We will now describe the above-mentioned results for each type of problem, which are sorted in descending order according to the number of errors that students made during the solution process.

P1. The constructive positional non-parametric problem solved using a set of all points of loci

The results of the analysis show that tasks of this type pose the greatest difficulties for students. Factual and procedural errors were made by more than 50% of students. Factual logical errors were the most common, occurring in 54% of cases (e.g. missing part of the geometrical construction, lack of knowledge of important properties of the constructed figures – an example is shown in Figure 1, resulting in a different figure being constructed, or one of the solutions being omitted because, for example, not all necessary intersections were marked), clueless errors appeared in 38% of cases (the problem was not solved at all, or the solver did not know the rules for symbolic notations of the construction, which did not match the construction), and factual graphical errors occurred in 33% of cases (the solution included an inadequate and imprecise sketch, which contained misleading information). These results highlight significant deficiencies in students´ cognitive domain.

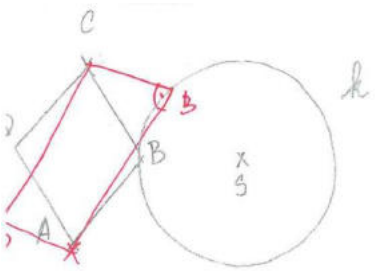
Task: Circle k and two points A, C are given in the plane. Construct a rectangle $ABCD$ with point B on the circle k.		Error: In the sketch, a parallelogram was marked instead of a rectangle. Red note by the teacher.

Figure 1: Example of a logical error in P1.

P4. The representation of the solid in free parallel projection;

This is the second most challenging type of problems, in which more than 20% of students made procedural and conceptual errors. Similarly, there were factual logical errors (students did not follow basic rules for representation in free parallel projection) and graphical errors (incorrectly indicated the visibility of the represented solid). This indicates insufficient understanding of the issue among more than one-fifth of the students.

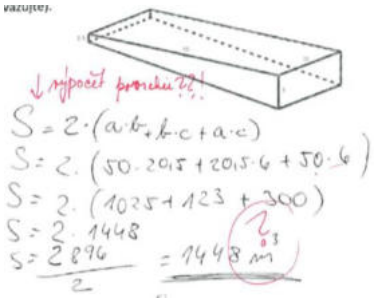
Task: Calculate the volume of the represented solid.		Error: Instead the volume area is calculated. Furthermore, the formula for calculating the area is also chosen incorrectly – for a different solid. At the end of the calculation, an incorrect unit is written. Red note by teacher.

Figure 2: Example of a clueless error in P6.

P6. The calculation of a solid's measure (volume or surface area of a solid)

In this type of problem, both factual and clueless errors occur (e.g., confusion of formulas for calculating volume and surface area, using a formula for a different solid than specified in the assignment

– an example is shown in Figure 2, or not solving the problem at all) (both occurring in 14% of cases), as well as conceptual errors and factual computational errors (incorrect unit conversion) (both 16%). In these cases, students lack the necessary factual knowledge as well as algorithmic and numerical skills. The other categories are minimally represented.

P7. Determining the set of all points in the plane according to the given properties.
This type of problem is close to P6 in terms of the number of errors but differs in the types of errors. Procedural errors and factual logical errors both have the same representation (24%) (students have a poor understanding of the given figures and the relationships between them, leading to failure in identifying the essence of the problem, or students create a correct sketch of the situation but is unable to correctly name the set represented). Other categories are represented in less than 4%. Therefore, students have the greatest difficulties with the process of solving the problem.

P5. Determining the image of a solid in a spatial geometric mapping (in free parallel projection)
The last task type in which any of the categories reaches 20%. These are procedural errors and factual logical errors (e.g., confusion between axial symmetry in the plane and in space – an example is shown in Figure 3, negligence or lack of knowledge, such as representation in homothety with an opposite sign). Again, the biggest complications are in the geometric construction process itself. Factual errors and clueless errors reached 9% (i.e., the problem was either not solved at all or was solved chaotically without knowledge of the properties of the mapping). Other categories are not represented.

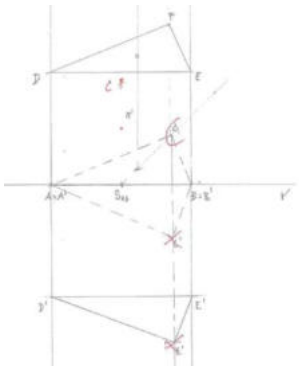
Task: Represent a regular triangular prism $ABCDEF$ in free parallel projection with axial symmetry in space with respect to the axis AB.		Error: The solid is not displayed in spatial axial symmetry (in free parallel projection), but instead with axial symmetry in the plane. Red note by the teacher.

Figure 3: Example of a logical error in P5.

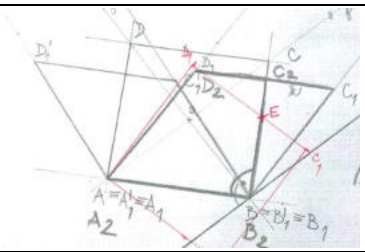
Task: Rotate the square $ABCD$ by an angle of $-30°$ about centre of rotation A.		Error: The result is not a square but a parallelogram $A_1B_1C_1D_1$. Red note by the teacher.

Figure 4: Example of a logical error in P2.

P2. Representation of a planar figure in a planar congruent geometric mapping
The most common errors in the solution were procedural and factual logical errors (both 12%) (due to a lack of knowledge of the principles of mapping and its properties, e.g., the type of figure was not

preserved – an example is shown in Figure 4), followed by factual errors and clueless errors (the problem was not solved) (both 7%). We also observed inattention errors, particularly in reading, where, for example, a different center of rotation was chosen than the one specified in the assignment. Students made such errors in 7% of cases.

P3. Identifying the intersection or union of two plane figures

In this type of problem, students made the fewest errors. None of the categories exceeded 4%. The most common issue was the incorrect naming of the resulting figure from a given operation due to a lack of knowledge of the properties of the figure, or confusion regarding the set operations mentioned.

Conclusion

Procedural and factual errors (**BS**), factual logical errors (semantic), and clueless errors (**T**) were the most commonly observed in the tests. The most significant difficulty students struggled with was their demonstrated lack of knowledge of the basic properties of geometric shapes and their components. These difficulties were evident in the majority of tasks. For example, they led students to construct different figures than required in construction problems solved using sets of points with given properties or in free parallel projection, or to calculate the measurement of a different shape than described in the task. They also incorrectly determined the resulting figure from set operations such as intersection and union. A major issue is the lack of understanding of the properties of geometric transformations, whether planar or spatial. A sign of helplessness in solving problems is, of course, the complete absence of a solution. Student solutions show that, in many cases, prospective primary teachers understand the properties of geometric figures in isolation, lacking a conceptual understanding based on the synthesis of their properties. These findings are consistent with Cunningham and Roberts (2010). The observed difficulties and types of errors made by prospective primary teachers are very similar to those made by students in the Unified Secondary School Entrance Examination (CERMAT, 2023). Primary school teachers build the foundations of geometry and geometric skills in their pupils. If these teachers do not have a solid understanding of these concepts, they cannot effectively pass them on. This is especially true when the teacher does not understand the reasons and significance of the errors described above. In such cases, they will not recognize these errors in their students and will be unable to address them methodologically and didactically. The results of this study will help us to focus on the areas where prospective primary teachers struggle the most during instruction.

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On the mutual relationship of different construction tools and mathematical concepts

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For many reasons, straightedge and compass have been regarded as the classic construction tools of geometry since Euclid's Elements. With increasing experience, their use becomes decoupled from the direct construction activities with straightedge and compass. If this decoupling process is not given space in geometry lessons, there is a risk that pupils will not recognise the underlying structures and will not be able to use individual standard constructions in a meaningful and relevant way to solve geometric questions. The article would therefore like to start before this abstraction process and focus on different construction methods in order to show how the respective construction could shape the view on the geometric object. The interplay of different construction methods in geometry lessons can create opportunities that can be beneficial for both conceptual and language development in geometry lessons.

Keywords: Subject-matter didactics, planar geometry, mathematical notions/concept, development of mathematical language

Introduction

Since Euclid's Elements, geometric constructions have had a firm place both in mathematical culture and in mathematical school curricula worldwide. In the case of Euclid's Elements, geometric constructions always refer to the special tool set consisting of (unscaled) straightedge and (collapsing) compass. 38 such (consecutive) construction problems are presented in the first four books, based on the three postulates:

> "I.) Let it be granted that a straight line may be drawn from any point to any other point. II.) Let it be granted that a finite straight line may be produced to any length in a straight line. III.) Let it be granted that a circle may be described with any centre at any distance from that centre." (Byrne, 1847, p. xxii)

Basic constructions include perpendiculars, bisectors and perpendiculars/parallels to a straight line through a given point. In addition to the aspect of mathematical tradition, numerous other central objectives of maths lessons are associated with geometric constructions with straightedge and compass, such as discovery learning, proving, concept learning and problem solving (Kuzle, 2013). In the game Euclidea, a further aspect (intellectual challenge) is added, as the fastest possible constructions are sought there. For example, constructing the perpendicular to a straight line g through a point A (Book 1, Prop. 11) is possible with only 1 circle and 2 straight lines and thus faster than Euclid, who needed 3 circles and 1 straight line (Berendonk, 2021). A further strength of these construction tools is, on the one hand, their simple describability in terms of the above three postulates, which nevertheless induces an enormously rich mathematical structure, as can be seen in the concept of constructible numbers and their connection to the three classical problems of antiquity (cube doubling, angle division, squaring of the circle).

Different ways of construction

With increasing experience, the use of these tools is decoupled from the direct construction activities with straightedge and compass after the first standard constructions have been recognised as feasible.

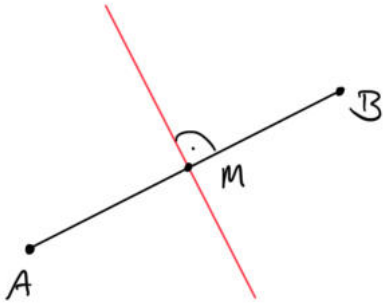

Figure 1: The perpendicular bisector of the line segment *AB*

When you look at a drawing like the one in Figure 1, it is not immediately clear what exactly this red line represents. The view of this red line depends on the context in which it was drawn and the underlying construction:

- 'Centre-perpendicular[1]': A straight line could have been drawn with a geometry set square that runs through the midpoint of the line segment AB and is perpendicular to it.
- 'Iso-distance line': Using a straightedge and compass, two points could have been constructed using circles that are the same distance from A and B respectively. It can then be quickly shown that the straight line g defined by these two points has the following property: $g = \{P | d(P, A) = d(P, B)\}$.
- 'Axis of symmetry': Folding the paper by folding point B onto point A creates this straight fold or mirror axis.

The previous example illustrates that even the concept illustrated in Figure 1 can give rise to different points of view. These perspectives and their interconnection become necessary in a school context, for example, when it is proven that the perpendicular bisectors of a triangle intersect at a point and that this point is the centre of the circumcircle of the triangle. In this article we propose to vary construction tools, thereby varying the focus on the constructed object. With our theoretical considerations we follow the idea of Pandiscio:

> […] a single task completed with a variety of construction tools fosters a greater sense of mathematical contemplation than multiple tasks done with the same tool. The premises are simple: each tool fosters different mathematical ideas, and using multiple tools not only requires understanding of a greater breadth and depth of geometric concepts but also highlights the connections that exist among different ideas. (Pandiscio, 1995, p. 32)

[1] In order to emphasise the perspectives of the construction tool on the constructed objects, we do not use the canonical identifier (perpendicular bisector) at this point, but ad-hoc formulations appropriate to the respective context.

The interaction of different construction methods in geometry lessons can create opportunities that can be beneficial for both concept development and language development in geometry lessons. For the following explanations, we assume that the learners are familiar with straightedge and compass constructions. The change of construction tool should therefore serve to reflect on (and possibly expand) the familiar concepts.

Theoretical background

The van Hiele levels are a classic theory of concept development which is an "important but less frequently used reference" within the CERME geometry group (Palatnik et al., 2023, p. 686). It assumes that a learner starts at the zeroth van Hiele level for each geometric object (independent of the concrete course of instruction). The activity of local ordering of object properties is primarily associated with the second level:

> At the Second Level properties are ordered. They are deduced one from another: one property precedes or follows another property. At this level the intrinsic meaning of deduction is not understood by the students. The square is recognised as being a rectangle because at this level definitions of figure come into play. (van Hiele, 1985, p. 245)

We will see that the constructions and their associated activities presented here create appropriate occasions for questioning the standard constructions with straightedge and compass and the associated terminology with the intermediate level learners. The learning of (mathematical) concepts is strongly interwoven with the development of an appropriate language as van Hiele points out:

> Each level has its own linguistic symbols and its own system of relations connecting these signs. A relation which is "correct" at one level can reveal itself to be incorrect at another. Think, for example, of the relation between a rectangle and a square. Numerous linguistic symbols appear at two successive levels; moreover they establish a liaison between the various levels and assume continuity of thought in this discontinuous domain. But their meaning is different: it becomes manifest by other relations among these symbols. (van Hiele, 1985, p. 246)

Instead of using the generally adopted mathematical terms directly, we will follow Winter's (1978) *workshop language* approach, in which we use appropriate terms for the geometric objects in order to better distinguish between the different nuances of meaning. These different terms should reflect the construction of the named object. Although this intermediate language is strongly context-bound, Winter emphasises that this is an important step in the development of technical terms that should not be taken too quickly (Winter, 1978). The transition from the informal to the formal language register and the space in-between are still being researched (Ingram et al., 2023) and Prediger & Zindel (2017) used the term "school academic register". However, this notion does not reflect the roughcast character of Winter's naming.

The cardboard wedge as alternative method of construction

In the following, we will use a cardboard wedge (as shown in Figure 2) for our geometric constructions. It can be used to mark off a given angle and to draw straight lines. If students made their own cardboard wedge after seeing my cardboard wedge, this would result in different acute angles. In addition to the construction of the straight line in Figure 1, we want to construct the

perpendicular to a straight line g through a point P, the parallel to a straight line g through a point P and the bisector of two legs and illustrate potentials with regard to the development of language and concepts.

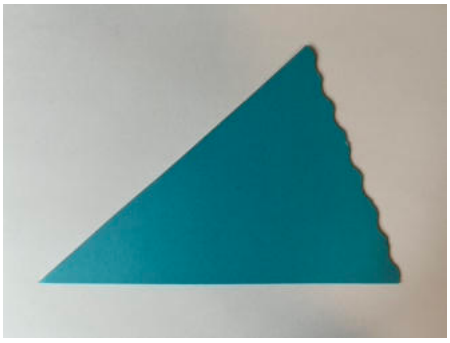

Figure 2: A cardboard wedge without length unit (due to Karl-Heinz Hürten[2])

If we now consider a given line segment AB, we can use the cardboard wedge to take a bearing of a direction. This results in a ray with initial point A consisting of all points P such the angle BAP is the same for all P. If we repeat this angle bearing three more times, we obtain two points P_1 and P_2 such that ABP_1 and AP_2B form isosceles triangle (Figure 3). In the spirit of workshop language, we could name these triangles *isobaseangular* triangles, as they were created by only marking off the given angle. It must now be discussed with learners that the straight line through P_1 and P_2 is independent of the specific angle in the cardboard wedge and that this straight line is an *iso-angle straight line*, that it contains all points P for which A, B and P form an *isobaseangular* triangle.

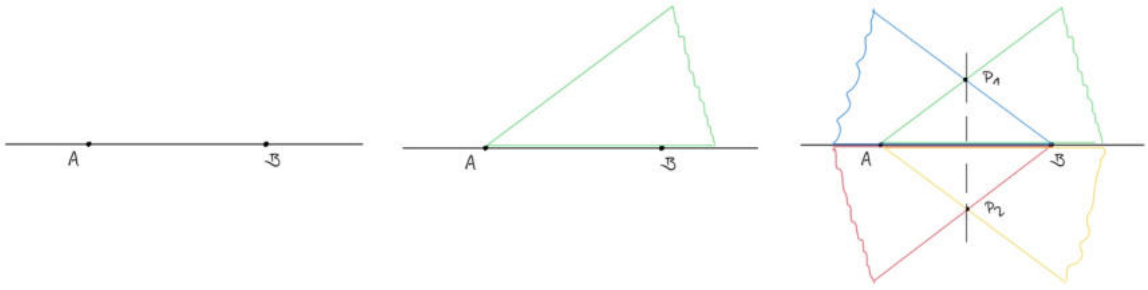

Figure 3: Construction of the *iso-angle line* via a cardboard wedge

The construction of the line in Figure 3 exclusively uses the angular ablation and should therefore also be distinguished from the perpendicular bisector from a linguistic and conceptual point of view. If we now organise the properties ‘iso-angle line’ and ‘iso-distance line’ by considering symmetry, we obtain the base angle theorem in a natural way.

If we want to construct the parallel to a given straight line g through a point A, the corresponding task for the construction with the cardboard wedge requires a different formulation: first we look for the point on the straight line g so that we can aim at point A and the straight line g at the fixed angle.

[2] Karl-Heinz Hürten belonged among other teachers and didacticians to a group which was concerned with subject-matter didactics and was involved in the design and development of teaching materials (Behnke & Stowasser, 1979).

Finally, we aim at the angle from point A again (but rotated by 180°). This gives us the green straight line (in Figure 4), which we do not yet know whether it intersects the given straight line or not. Instead of the true parallel property (having no intersection points), this construction allows us to focus on the alternating angle theorem.

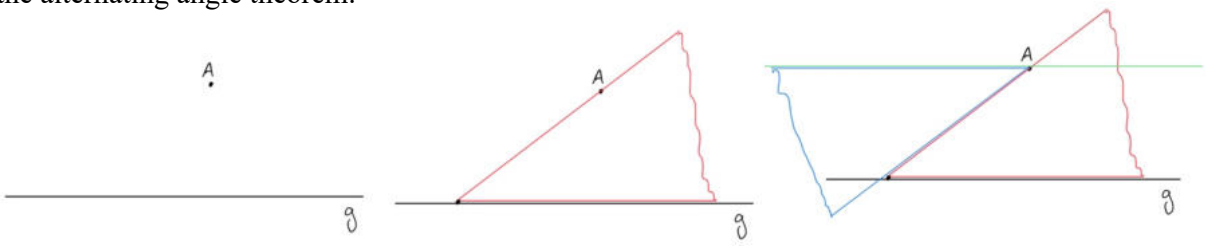

Figure 4: Construction of the parallel through a given point

If we want to construct the perpendicular to a straight line g through a point A, it becomes clear that the cases of whether point A lies on the straight line g or not must be solved differently. If A does not lie on the line, we can first construct an *isobaseangular* triangle ABC by taking a bearing (Figure 5). For the line BC, we can now apply the lower half of the perpendicular centre line construction and finally obtain the perpendicular through point A.

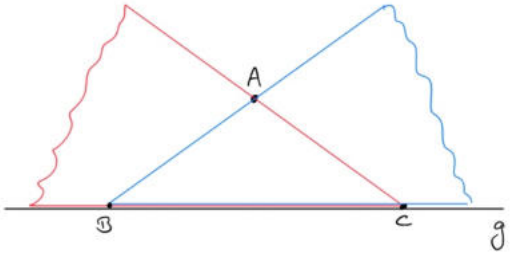

Figure 5: (Auxiliary) Construction of an isobaseangular triangle

Although the cardboard wedge can only be used to mark off angles of a fixed size (instead of distances), constructions were possible that were very similar to the classic straightedge and compass constructions. When trying to construct the perpendicular of a straight line g through a point A on the straight line, it is noticeable that we cannot easily imitate this: we cannot mark off a fixed length. However, it is possible to construct a perpendicular bisector to any two points on the line and then draw a suitable parallel through point A (Figure 6).

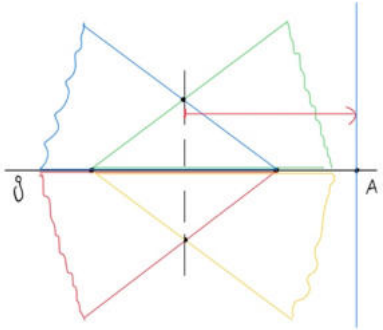

Figure 6: Translation of an arbitrary perpendicular (bisector)

The previous problem has shown that we have not yet been able to mark off any distances. We therefore extend the wedge by a fixed marked length, which is marked off on both sides of the cardboard wedge (Figure 7).

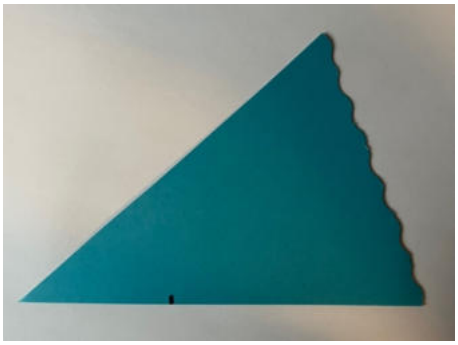

Figure 7: The marked length on the wedge

With this marked wedge, it is now possible to shorten the construction from Figure 6 by placing the wedge with the marked point at point A as shown in Figure 8. With the construction shown, we now obtain the perpendicular bisector to the points that lie at the tips of the blue and green wedge (B and C). The constructed line also passes through A due to its isodistance property.

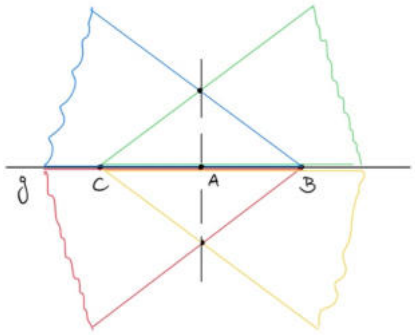

Figure 8: Shorter Construction of the perpendicular

The possibility of marking off a given fixed line segment opens up another previously impossible construction: the angle bisector. To do this, we mark off the fixed distance on both legs of the angle and obtain the points A and B. The perpendicular bisector of AB is then the angle bisector of the angle BSA (Figure 9).

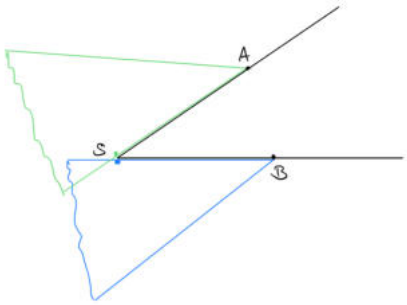

Figure 9: Preparation for the bisector of an angle

Discussion

The different constructions of the perpendicular bisector of a line segment AB and their different terms in the workshop language initially make it necessary to distinguish the constructed lines from each other. Only when learners argue on the second van Hiele level, it is possible to connect these *different* lines by relating their primary properties and agree on a single term (independent of its concrete genesis). The theoretical analysis indicates that the activity of local ordering is strongly connected to the use of appropriate lingual means taking up the genesis of the constructed objects. The presented constructions with a cardboard wedge give rise to tasks and problems which can initially question the canonical approach in German schools via the Euclidean straightedge and compass constructions (in the spirit of Pandiscio). Moreover, it is an easily accessible material and can be easily made from cardboard.

In the afore-mentioned analysis, it was assumed that the learners would be familiar with constructions by straightedge and compass. It should be noted that there is no indication that the presented approach via the cardboard wedge cannot be used as a first encounter with geometric constructions in school. Furthermore, this context offers opportunities to the learners for their own investigations: for example, what would happen if the cardboard wedge had an obtuse angle or a right angle? Hence, the presented construction holds different possible applications for teaching with different goals.

Conclusion

Although the constructions with the wedge are as powerful as the classical constructions with straightedge and compass[3], the focus here is on the contribution to the development of concepts (and language) and less on the discussion of the range (resp. limitations) of these constructions[4]. It has become clear that the use of different construction methods can offer opportunities for the activity of local ordering (second van Hiele level). Thus, by comparing at least two construction methods, a richer conceptualisation can be made possible, which would hardly unfold with an exclusive straightedge and compass construction. The subject-matter analysis in this article can be seen as a first step towards an implementation in a developmental research project (or the like).

Acknowledgment

I would like to thank Alois Görg and Peter Fitting for making me aware of the work of Karl-Heinz Hürten.

[3] Martin (1998, p. 102) calls these wedge constructions “acute-angle ruler constructions” and shows their equivalence to the straightedge and compass constructions.

[4] From such a point of view, the approach via origami would be better as it is more powerful than the Euclidean constructions in the sense that the Delian problem (doubling the cube) can be solved (Geretschläger, 1995).

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3D printed manipulatives as an aid in constructing parallel and central projections of solids

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The main goal of this research was to explore the effect of using concrete 3D printed manipulatives on students' ability to construct projections of solid figures in both parallel and central projections. These projection techniques were introduced to university students in the course "Basics of Projections" at the Faculty of Mathematics and Physics, Charles University, Czech Republic. The use of 3D printed manipulatives in this course was implemented during the winter semesters, first in 2021/2022 with twenty-seven students, and again in 2023/2024 with twenty-four students. This study aimed to assess how manipulating physical 3D printed models could improve students' grasp of different projection methods and deepen their conceptual understanding. The research sought to evaluate and compare students' performance in solving geometric projection tasks, both with and without the use of 3D printed manipulatives. Students were more successful in constructing projections, when using 3D printed models.

Keywords: projections, 3D printed manipulatives, geometry of solid figures.

Introduction

Geometry is often challenging for students at all levels of education, as it requires strong spatial perception and abstract thinking. Instructional aids, particularly 3D models, can help by turning abstract concepts into tangible objects. The use of 3D printing technology has expanded into many fields, including construction, medicine, and cultural heritage (Tsoulfas et al., 2020) and has also opened new possibilities in education by providing innovative methods to support the teaching and learning of STEM subjects (Novak & Wisdom, 2018). Research suggests a positive impact of 3D printed tools on spatial reasoning and mathematical understanding (Herrera et al., 2019), while other studies highlight its role in concrete mathematical concepts like surface area (Corum & Garofalo, 2015). Although 3D printing and the 3D printed objects are becoming more common in education, further research is needed to better understand its impact on teaching.

Building on these findings, our research investigates the specific impact of ready-made 3D printed models on teaching projections in university education. Based on my long-term teaching experience and international studies (Garmendia et al., 2007), it is evident that even university students often struggle with spatial ability which is essential in all areas of geometry, including projections. They find it challenging to visualize spatial situations without physical or virtual models, which can assist them in solving geometric tasks. Although 3D models can be made from various materials, using 3D printing has clear advantages – source files for models can be shared with other students or schools, and models can be reproduced quickly and affordably. Many schools, at least in the Czech Republic, already have 3D printers, allowing them to produce their own models rather than purchasing them.

Our research, therefore, focuses on the following research questions:

Does the use of 3D printed models in constructing projections of basic solid figures help students understand spatial relationships?

Does the use of 3D printed models help determine the visibility of parts of geometric objects in the resulting images of projections?

In addition, we inquire about the specific types of geometric tasks in which 3D-printed models are most beneficial. We explore whether repeated use of 3D-printed models throughout the course enhances students' long-term retention of projection techniques and whether students can maintain their improved skills in projection techniques even after they stop using the models. Finally, we investigate students' perceptions and attitudes toward the use of 3D printed models in their learning process, focusing on how these models influence their understanding.

Theoretical Framework

Concrete and virtual manipulatives in mathematics education

Manipulatives are tools designed to support teaching and learning, engaging students in the learning process while helping them build conceptual understanding and solve mathematical problems. Concrete manipulatives are physical objects, while virtual manipulatives are dynamic, interactive visualizations on computers (Clements, 1999; 2009; Moyer et al., 2002). Pupils and students can interact with dynamic visual representations in ways similar to handling physical objects. 3D printed models, which we use in our teaching and examine for their impact on learning, are a prime example of concrete manipulatives. These models are tangible aids that help students comprehend abstract mathematical concepts.

The effectiveness of manipulatives in mathematics instruction has been widely studied. In a meta-analysis, Sowell (1989) found that the use of manipulatives led to higher achievement at all elementary school levels. However, the study did not clearly identify which specific situations are most suitable for the use of manipulatives. Students who use manipulatives often outperform those who do not (Clements, 1999), a finding that holds across different grade levels, ability levels, and mathematical topics, provided the manipulatives are appropriate for the mathematical topic. Manipulatives are particularly beneficial for helping students to grasp abstract mathematical ideas (Kontaş, 2016). Using models (e.g., 3D printed, wooden, or made of modeling clay) significantly contributes to improving students' understanding of geometry (Bímová et al., 2023).

Action-oriented teaching methods that involve the use of real physical models have consistently shown positive effects on the development of spatial ability. This is linked to the theory of embodied cognition, which emphasizes the role of motor behavior in cognitive processes (Schneegans & Schöner, 2008). Embodied mathematical thinking suggests that mathematical cognition is deeply rooted in physical interactions with the environment and materials (Palatnik & Abrahamson, 2022; Ng & Ye, 2022).

However, not all research supports the use of manipulatives. Some studies have found that manipulatives have no significant effect on students' mathematical achievements (Clements, 1999,

Uribe-Flórez & Wilkins, 2017). It is important to recognize that manipulatives alone do not guarantee success or meaningful learning. If we use it only for rote learning, they may fail to enhance students' understanding of mathematical concepts. Sometimes, teachers use manipulatives without fully considering how these tools support the understanding of the particular mathematical content. In such cases, manipulatives are used merely as a distraction or novelty in the classroom.

Students' Perception of Drawings

Visualizations play a critical role in understanding mathematics, especially in geometry. Mathematical objects, both two-dimensional and three-dimensional, are often represented on a two-dimensional plane. First geometry lessons usually focus on two-dimensional figures, which serve as the foundation for understanding three-dimensional objects. These drawings are crucial tools for comprehending complex mathematical ideas, bridging the gap between our experiences in three-dimensional space and abstract mathematical concepts.

To interpret a drawing that illustrates a mathematical or geometrical idea, it is necessary to establish the connection between the drawing and the concept it represents. Duval (1995) describes this process as involving four types of figure apprehension: perceptual, discursive, sequential, and operative. Each type provides a unique perspective on how a drawing corresponds to the underlying mathematical concept.

Duval (2005) also distinguishes between two forms of geometric visualization: iconic and non-iconic. Iconic visualization involves recognizing shapes that resemble familiar objects, while non-iconic visualization interprets drawings as representations of geometric entities, emphasizing the relationships between components rather than their appearance. Students often alternate between these two forms of visualization when developing geometric understanding. Iconic visualization allows students to identify a three-dimensional object based on its overall shape—especially when the object is presented in a typical, familiar view. Physical models provide tangible representations that support students in connecting their spatial perception with the interpretation of geometric drawings, facilitating the development of iconic visualization. However, non-iconic visualization is essential for a deeper understanding of spatial geometry. In this approach, students move beyond mere recognition of shapes and begin to perceive objects as a composition of interrelated elements and structures. Physical models play a crucial role in supporting students as they physically explore and analyze the relationships between different components of an object, such as edges, vertices, and faces. For example, manipulating a model can help students understand how edges are connected, how different parts of a solid are oriented relative to each other, or how visibility changes depending on the viewing angle. These relationships are often abstract or difficult to visualize from drawings alone. When students work with physical models to project three-dimensional objects, they shift from simply identifying shapes (iconic) to analyzing the relationships and components of the objects (non-iconic).

Parallel and Central Projections

Three-dimensional objects can be represented on a two-dimensional plane using the technique of projection. There are two primary methods of projections: parallel projection and central (or perspective) projection (Carlbom & Paciorek, 1978). In parallel projection, objects maintain their

relative proportions and shapes. A common example of parallel projection is orthographic projection, in which the viewpoint is placed at infinity, and the lines of sight are perpendicular to the projection plane. On the other hand, central or perspective projection reflects how objects naturally appear to the human eye, providing a sense of depth and perspective.

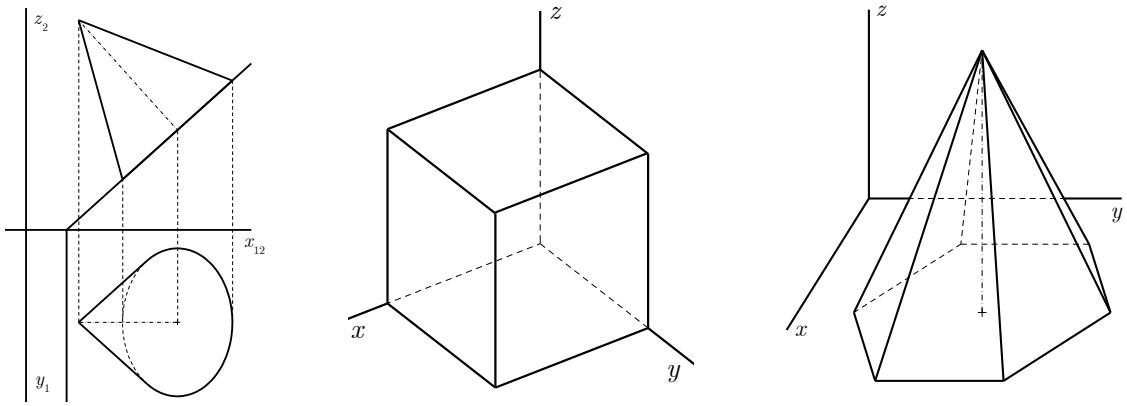

Figure 1: Examples of projections – top and front orthographic view of a cone (on the left), an axonometric orthographic projection of a cube (in the middle), and an oblique projection of a hexagonal pyramid (on the right)

In the university course where this research was conducted, various types of parallel and central projections were introduced. Specifically, we explored top and front orthographic views (see Figure 1, on the left), axonometric orthographic projections (see Figure 1, in the middle), oblique projections (see Figure 1, on the right), and linear perspectives as examples of central projections. In each case, students were tasked to manually construct the projection of a solid figure on paper.

The main problems students face in constructing the projections of solid figures are related to imagining how solid figures appear, determining the visibility of their parts, and understanding spatial relationships between components. Sorby & Baartmans (2000) specifically describe difficulties students experience in spatial visualization tasks using various standardized tests, such as determining cross-sections and cutting planes or performing mental rotations. We believe that the use of 3D physical models can help students overcome these challenges.

Materials and methods

3D Printed Set of Solid Figures

For our experiments, we designed and 3D-printed a new set of solid figures. The set contains right and oblique prisms with regular three- to six-sided bases, cubes, cuboids, right and oblique pyramids with regular three- to six-sided bases, right and oblique cylinders and cones, and spheres. Each set contains all of these types of solid figures and was created for each student to work independently. One such set of solid figures is shown in Figure 2 on the left.

Research context and data collections

The study aims to explore how activities with 3D printed manipulatives contribute to developing students' conceptual understanding of solid figure projections. It also investigates how these models help students understand geometric relationships between the components of objects and determine

the visibility of parts in the resulting projection. The use of 3D printed manipulatives was integrated into the "Basics of Projections" course during the winter semesters of 2021/2022, with twenty-seven students, and 2023/2024, with twenty-four students. Each session lasted 90 minutes and was held weekly throughout both semesters.

Each course was partially structured as a lecture with exercises, demonstrated alongside their solutions. During the lectures, 3D printed models and 3D modeling software were used (by the teacher) to illustrate the projections. This was followed by practical sessions where students constructed the projections of individual solid figures themselves. Students were assigned various tasks, including constructing projections of different solid figures such as polyhedrons, cylinders, cones, and spheres. Additionally, students engaged in geometric exercises involving these projections, such as determining the intersection between a polyhedron and a plane.

The tasks were completed both with and without the use of 3D printed manipulatives or, in some cases, first without and then with—this approach was specified for every exercise. This allowed us to compare students' success rates. For all tasks, students manually drew the projections on paper. The students' work throughout both semesters was continuously analyzed (with selected tasks being evaluated) to assess whether the use of 3D printed models led to increased success in solving geometric problems.

Let us focus on one representative exercise to discuss how students worked:

"Construct the top and front orthographic views (Monge projection) of an oblique prism *ABCDA'B'C'D'* with a square base *ABCD* in the (*x*, *y*)-plane, given the coordinates of *A*, *B* and *S'* (the center of the base *A'B'C'D'*). Determine the visibility of the solid figure." (Semester 2021/2022, submitted by 25 students, the solution can be seen in Figure 2 on the right).

Figure 2: A 3D printed set of solid figures and the right solution of the exercise

The exercise had three subtasks, each evaluated with one point for correct solution: constructing the top view, constructing the front view, and determining visibility. Students initially worked without any visual aids, and the exercise was evaluated by the teacher. Afterwards, the same students were provided with a 3D printed model of the solid and asked to solve the same exercise again. Other exercises were tested in a similar manner.

Students were asked to discuss the benefits of using the 3D printed models as visual aids after each intervention. The feedback from students regarding their experiences with the use of 3D printed models was analyzed (gathered only through oral interviews and discussions).

To pass the course and earn credits, students were required to complete a final test consisting of five exercises on various types of solid figure projections, this time without any visual aids. The final test was administered at the end of the semester. Students were not graded based on their performance during the semester.

Results

The comparative analysis of student success rates for the presented geometric task is summarized in Table 1. This table summarizes the performance of 25 students in three subtasks in the exercise without and with the use of 3D models. The "Score" column indicates whether students achieved 0 or 1 point for each subtask, and the corresponding columns show the number of students who achieved these scores in each category. The last row labeled "Total Success" highlights the overall percentage of correct solutions for each subtask. Additionally, a "Difference" row has been added to show the improvement in success rates between the two scenarios.

Table 1: Comparison of student success rates in geometric exercise with and without the use of 3D printed models

Score	Top view (without/with 3D model)	Front view (without/with 3D model)	Visibility (without/with 3D model)
0	6 / 1	5 / 1	7 / 1
1	19 / 24	20 / 24	18 / 24
Total Success	76.0% / 96.0%	80.0% / 96.0%	72.0% / 96.0%
Difference	20.0%	16.0%	24.0%

From the data in Table 1, it is apparent that when solving the same tasks for the second time with the model, students were more successful, particularly in the subtask related to visibility determination. This was somewhat anticipated, as the use of models can effectively demonstrate both visibility and the front and top views. Only one student remained unsuccessful even with the 3D model; however, with the teacher's guidance, they were able to identify the correct solution. The correct solution to the task was shown to the students only after they had completed the tasks.

Regarding other exercises, in some cases, the success rate did not improve (some students even solved the task incorrectly with the model), while in others, it did. Students' success rates in more complex exercises were lower (in several cases less than 50%). Overall, the results show that when students used 3D printed models, they were more successful in constructing projections of solid figures, particularly when determining the visibility of vertices, edges, or faces of the objects.

Student feedback regarding the use of 3D printed models was overwhelmingly positive. Their responses highlighted the educational advantages and increased engagement that these models provided. For example, one student noted: “The models really helped me understand the visibility of the solid figure. I finally grasped why some edges are hidden in certain projections.” Another student commented on the enjoyment: “The models made even the drawing part more enjoyable—it felt a bit like playing a game.” These responses suggest that 3D models not only deepened students' understanding of geometric concepts but also made the learning more engaging and enjoyable.

The findings of our study indicate that the use of 3D printed models significantly enhanced students' ability to construct projections of solid figures. By working with physical models during the projection process, students progressed from merely recognizing shapes (iconic visualization) to understanding the complex relationships and structures within the objects (non-iconic visualization). At the beginning, students identified only the shape of the solid (iconic visualization) and made errors in projections, particularly in determining visibility. However, through working with 3D models, they were able to distinguish hidden edges and spatial arrangements (non-iconic visualization), which is an example of the development in their visualization skills. Similarly, Mithalal and Balacheff (2019) demonstrated the effectiveness of 3D Dynamic Geometry Environments, which allowed students to explore and manipulate drawings representing three-dimensional geometric objects. This approach played a key role in bridging the gap between iconic and non-iconic visualization. Our findings also align with prior research (Kontaş, 2016), which highlights the value of manipulatives in making abstract mathematical concepts more concrete.

Conclusion and future plans

This study demonstrated the potential of 3D printed models in assisting students with constructing projections of solid figures. Their impact was particularly evident when students determined the visibility of vertices, edges, and faces in projections. Furthermore, the use of physical models significantly enhanced students' ability to solve geometric problems on paper by improving their spatial visualization. These results underscore the value of 3D printed models in helping students transition from iconic to non-iconic visualization, a critical step in geometric comprehension.

In our future work, we plan to expand the collection of 3D printed models and further involve students in the design and creation process. This approach could encourage students to develop a stronger understanding of geometric principles through hands-on 3D printing experience. We also plan to conduct further tests with larger student groups to collect more comprehensive data on the impact of 3D printed models in various geometrical contexts.

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Teaching the concept of minimal definition with the CoD Method

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Solving the difficulties students face in gaining full understanding of definitions is a significant challenge faced by mathematics education: without this, it is difficult, for example, to acquire problem-solving or proof skills. This study, conducted in a 5th-grade class, is a first step towards modelling a teaching-learning experience regarding minimal definitions. The method used is new and innovative (Cost of Definition Method) and consists of a mathematical discussion, orchestrated by the teacher in which a method is taught to identify redundant definitions and, in such cases, to reduce them to minimal definitions. In this activity, the children must carefully analyse each critical attribute and evaluate whether it is necessary or redundant. This process forces them to delve deeply into the meaning of the definition and each of its parts, fostering a more profound understanding.

Keywords: Geometrical definitions, non-economical definitions, quadrilaterals, mathematical discussion, classificatory thinking.

Introduction

Several studies have proven that students and trainee teachers at all school levels encounter various difficulties related to geometric definitions: difficulties in identifying, classifying, and constructing examples and counterexamples of the concept (Haj-Yahya et al., 2022). Numerous studies highlight issues due to a less-than-optimal understanding of the structure and meaning of definitions. In this regard, chronologically, one could refer to the articles by Hershkowitz (1987), Vinner & Dreyfus (1989), Hershkowitz (1989), Hershkowitz et al. (1990), Fujita & Jones (2007), Pickreign (2007), Marchis (2012), Fujita (2012), Haj-Yahya & Hershkowitz (2013). The difficulties associated with defining activities increase significantly in primary school due to the tendency of teachers to provide definitions as a long list of properties of the object. Given these challenges in geometric definition, this study explores the potential of the *Cost of Definition* (*CoD*) *Method* to enhance primary school students' understanding. The research question guiding this study is: 'Can the application of the CoD Method improve the meta-concept of minimal definition in primary school?' The analysis of the results is qualitative, focusing on the examination of transcripts from mathematical discussions, and will be elaborated upon in the following sections.

Theoretical Framework

In this study, following Haj-Yahya et al. (2022), *minimal definitions* are defined as those that include the minimum number of critical attributes necessary and sufficient to define a concept. This criterion is referred to as the *Criterion of Minimisation of Definitions*. It implies that definitions should not include redundant or derivable information from other parts of the definition itself. The principle aims to promote clarity and essentiality, avoiding overwhelming students with unnecessary details. Definitions that do not adhere to the *Criterion of Minimisation of Definitions* are also referred to as non-economical or redundant definitions (Linchevsky et al., 1992; de Villiers, 1998) because minimal definitions generally lead to more economical proofs, resulting in shorter and sometimes less

repetitive demonstrations. These studies show that students and teachers use redundant definitions because they consist of more critical attributes than are strictly necessary. Similar findings are also noted in de Villiers et al. (2009) and Foster (2014), where it was observed that students propose correct but descriptive definitions. Specifically, they tend to list all the properties of the concept. Learning the concept of minimality in definitions and educating students to recognise or produce minimal definitions is considered crucial for developing and enhancing definitional thinking (Tortorelli & Tortoriello, 2024; Haj-Yahya et al., 2022). A clarifying example of a minimal definition can be found in Vinner (1991), where the following minimal definition of a rectangle is presented: 'A quadrilateral is defined as a rectangle if and only if it is a quadrilateral with at least three right angles.' This definition differs from the one typically found in textbooks: 'A quadrilateral is defined as a rectangle if and only if it is a quadrilateral with all right angles.' The latter definition is not minimal because, according to a theorem in Euclidean geometry, the measure of the fourth angle can be deduced by knowing the total measure of the other three. In this experiment, the *Cost of Definition Method* (Tortorelli, 2025), abbreviated as *CoD Method*, will be used. This method is designed to teach students how to recognise whether a definition is redundant. The method also suggests a model of mathematical discussion that could lead to the minimality of definitions. This is achieved through the estimation and comparison of their *costs*. It is now necessary to specify what is meant by the 'cost of a definition.' In this work, a function is proposed that assigns to each definition formulable in a given theory of quadrilaterals a positive integer, which will be called the *Cost of Definition*. The *Cost of the definition D* of a quadrilateral *q* is defined as the number of critical attributes it utilises. The properties considered as possible *critical attributes* are being a quadrilateral or having: 2 parallel sides, 2 perpendicular sides, 2 congruent consecutive sides, 2 congruent opposite sides, 2 congruent opposite angles, 2 congruent adjacent angles, 2 complementary angles, 2 opposite supplementary angles. Each of these properties is conventionally assigned a virtual value of 1 *ca* (initial letters of *critical attributes*). To indicate *Cost of Definition D of a quadrilateral q* we suggest using the notation: $c(D,q)$. For example, denoting by D, the Vinner definition of a rectangle (Vinner, 1991), i.e.: 'A quadrilateral is defined as a rectangle if and only if it has at least three right angles', we have that the cost of the definition is: $c(D,rectangle) = 3\ ca$. The concept of 'cost' in this context serves as a pedagogical tool to help students and teachers recognize the efficiency of definitions. By assigning a numerical value to each attribute, this method encourages a focus on the most essential attributes, avoiding the inclusion of redundant or derivable properties. This not only streamlines the learning process but also fosters a deeper understanding of the role of each attribute in defining geometric figures. The following theorem illustrates the equivalence of two definitions in terms of cost, further supporting the principle that minimal definitions can be derived from more complex ones without losing essential meaning. In this study, the term 'equivalent definitions' is used in the sense of Haj-Yahya et al.: 'Two definitions of the same concept are equivalent if one can deduce one formulation from the other and vice versa.' (Haj-Yahya et. al., 2022, p. 610) *Theorem of the cost of equivalent definitions*: *Given two minimal definitions* D_1 *and* D_2 *of the same quadrilateral q constructed in the Theory of Quadrilaterals, it results that:* $cod(D_1,q) = cod(D_2,q) \Leftrightarrow$ (D_1 and D_2 are equivalent definitions). For example, consider the following two right-angled trapezia definitions, D_1: 'A right-angled trapezia is defined as a quadrilateral having at least one pair of parallel sides and at least two perpendicular sides'; D_2: 'A right-angled trapezoid is defined as a quadrilateral having at least one pair of parallel

sides and at least one straight angle'. It is easily demonstrated that D_1 and D_2 are equivalent definitions and thus the cost of D_2 can be derived from the cost of D_1: $c(D_2,q) = c(D_1,q) = 2$ *ca*. The *CoD Method* is taught through a mathematical discussion (Bartolini Bussi, 1996), which Bartolini Bussi defines as a polyphony of voices (in Bachtin's sense) articulated around a mathematical object (concept, problem, procedure, etc.), which constitutes a motif of the teaching-learning activity.

Method

The experimental context was developed as part of the PhD project entitled *'Development of definitory thinking in geometry through the Geometriko Academy instrument'*. The meetings were managed by the curricular teacher, supported by our research team. The sample consists of a fifth class of an Italian primary school. The class examined consists of 17 students. The class has already studied the theory of quadrilaterals with their teacher. The teacher adheres to a traditional pedagogical approach and asserts that a mnemonic method of definition is the only viable technique at the primary level of education. Furthermore, the teacher believes that children are not yet equipped with the cognitive abilities necessary to grasp the concept of definition. The teacher's traditionalist views of the teacher was what determined the choice of this class for the experiment. The *Teaching Methodological Cycle* comprised four phases: firstly, an entrance test was administered; secondly, a frontal dialogue lesson on the concept of definition was conducted; thirdly, the *CoD Method* was applied using the mathematical discussion method; and finally, the exit test was administered.

Results and analysis of results

The analysis of the results was qualitative and will be carried out by examining the transcripts of the *mathematical discussions* and through the comparative results of an entrance test and an exit test. In this paragraph, some excerpts from the mathematical discussion are given which can help us to answer the question research of this study. The researcher plays the part of the teacher.

010.	Teacher:	Consider the following definition of a trapezium: 'A quadrilateral is defined as a trapezium if and only if it has at least one pair of parallel sides.' This definition costs 1 *ca* and is a minimal definition, or as many say, an economical definition. Let's look at a more complex definition, denoted as D_1: 'A quadrilateral is defined as a parallelogram if and only if it has opposite sides parallel, opposite sides congruent, and opposite angles congruent.' The definition D_1 costs 6 *ca* and is a non-economical definition. What do you think I mean by the *cost of a definition*? What determines the *cost of definitions*? We *pay* in *ca*, which is our official currency used for definitions. So, what is 1 *ca*? In this context, 1 *ca* represents the cost assigned to each *critical attribute* used in a definition. Each attribute, such as parallelism, congruence of sides, or congruence of angles, is assigned a cost of 1 *ca*. Therefore, the overall *cost of a definition* is the total number of independent *critical attributes* it uses. A *minimal definition* is one that achieves the intended concept using the least number of attributes, thereby having the lowest cost in terms of *ca*.
011.	Francesca:	So the number of lines in the definition has nothing to do with it...
012.	Filippo:	No no... We don't have to count characters or words …
016.	Domenico:	I get it! In my opinion money is *critical attributes*... *ca* is the initial letter of the words *critical attributes*! I guessed it teacher?
017.	Marialuce:	True! I also think the coin is the *critical attributes*! In fact in the definition of the trapezium they are because... it's only one... 'at least one pair of parallel sides' and it only costs 1 *ca*. The second definition... that of the parallelogram costs so much because it has so many *critical attributes*... It costs 6 *ca*...

018. Vittoria R. This is also the case for me …
019. Teacher: Well done! Marialuce explained well why the definition of trapezium I chose costs 1 *ca*. Can you explain to me why the definition I have called D1 of a parallelogram costs exactly 6 *ca*?
020. Aldo: We have to count the... the things that the parallelogram has... One... two... three and... after three I don't understand! Why six?
021. Licia: The properties we mention… but I don't quite agree… teacher, I count three!
022. Angela: That's true, I also count three… There are three…

Many children use their fingers to count and raise their hand with three fingers up, almost to reinforce that number. Licia, however, is not convinced and recounts once more out loud:

023. Licia: Teacher, I still count 3! So… 'opposite sides parallel' costs one *ca*, 'opposite sides congruent' costs another *ca*, making 2 *ca* and the last one, 'opposite angles congruent,' costs yet another *ca*. Altogether, it's 3 *ca*, not 6 *ca*!
024. Giorgia: I don't understand… Teacher, why does it cost 6 *ca*? I think it is worth less...
025. Teacher: Let's count them together. If we want to *buy* the definition: 'Opposite sides parallel,' how many *ca* does it cost?
026. Matteo: One *ca*!
025. Teacher: Are you all in agreement? What does 'opposite sides parallel' mean?
027. Lucrezia: Well… 'opposite sides parallel' means that two opposite sides are parallel, and then… and then the remaining ones are also parallel… So…
028. Domenico: So, it costs 2 *ca* because… we first spend one *ca* to obtain what Lucrezia said, 'two opposite sides parallel' and then spend another *ca* to obtain the property that the remaining sides are also parallel. So overall, it costs 2 *ca*.
029. Carlo: That seems like a correct reasoning…
030. Filippo: But if what Domenico says is right, 'congruent opposite sides' also costs 2 *ca* and not 1 *ca* as it seems...
031. Marialuce: Right! So we have to spend 2 *ca* for parallel opposite sides, another 2 *ca* for congruent opposite sides and again 2 *ca* for …
032. Francesca: ... for congruent opposite angles! Two, two more and two more... that's 6 *ca* in all! Here we go at last, the numbers add up....
033. Teacher: In short, we count the properties of the sides and angles. The definition D_1 of a parallelogram is too expensive and we must try to save money. I propose a challenge: take the definition D_1 and, starting from it, can you create a new definition D_2 of parallelogram such that D_2 is cheaper than D_1? I suggest a strategy: we start from the given definition and remove one or more properties at a time and see if the new cut definition is still an unambiguous description of the parallelogram.

The teacher acts based on the teachings of Bartolini Bussi (1996). Their intervention in the discussion [033] occurs at a moment when it is necessary to guide the class in establishing meanings both at a local level (cost of definition) and at a global level (the meaning of a non-economical definition and its reduction to a minimal definition). This is done using a technique employed when a student provides a clear, relevant example or makes a positive contribution to the discussion. In this case, Domenico unlocks the situation [028], which had been stalled until then, and the teacher seizes the opportunity to paraphrase some general concepts related to minimal definitions.

A few minutes pass, and the children try drawing new quadrilaterals on paper. The drawing exercises enable students to test their definitions, leading to a deeper grasp of geometric concepts.

040. Domenico: I have removed a pair of congruent opposite angles and am now testing... Every attempt I always come up with a parallelogram...

041. Lucrezia: I am trying without considering congruent opposite angles. I have tried many times and the drawings are always parallelograms....
042. Carlo: Domenico saved 1 *ca*... Lucrezia saved 2 *ca* together …
043. Filippo: It's the truth! Lucrezia you're a genius, I hadn't thought of taking twin properties together...
044. Marialuce: Filippo, I'm also trying to remove two duplicate properties. First, I removed just one, and then while experimenting, I saw that the other one isn't necessary either. I did many drawings, and whenever I use opposite sides parallel and opposite sides congruent, I always end up with a parallelogram automatically. Opposite angles congruent are unnecessary. So, the new definition is more economical and only includes opposite sides parallel and congruent.
045. Teacher: Perfect! I see that many of you agree that the pair of properties asking for 'opposite angles congruent' are optional. Mathematicians refer to these as 'redundant properties'. Redundant properties in a definition are those that can be derived from the others. By removing the pair of properties asking for opposite angles congruent, definition D_1 is reduced to a more economical definition. We will call this new definition D_2. D_2 can be expressed as: 'A quadrilateral is a parallelogram if and only if it has opposite sides parallel and opposite sides congruent. Do we still have uniqueness? I'll give you 5 minutes to think about it. Take a pencil and graph paper, and draw some quadrilaterals with these properties. This exercise is called 'Find the Counterexample.' The winner is the one who manages to draw a quadrilateral with the properties of D_2 that is not a parallelogram.

This teacher's intervention constitutes an initial phase of institutionalization through a mirroring with integration approach (Bartolini Bussi et al., 1995), where the student's contribution [Marialuce, 44] is repeated with some improvements in the language, expressed in a more academic manner. The definition provided is not necessarily the final one—that is, the one that will be definitively institutionalized and later recorded by the students in their notebooks—because, although it is unique, it is not yet minimal.

Almost 10 minutes elapse during which the children rehearse with pen and paper.

046. Mattia: Teacher, I won the challenge! I found some quadrilaterals that have the properties of the definition D_2 but are not parallelograms. Here they are: a rectangle, a square… there are many more…
047. Angela: Me too... beyond I found the rhombus not to be a parallelogram....
046. Domenico: The cheapest definition doesn't work. I'll try removing something else.
048. Licia: What do we do now?

One by one, all the children raise their hands and present incorrect counterexamples due to serious issues related to a complete lack of classificatory thinking skills [046, 047, 048]. This reveals a clear absence of interplay between the figural aspect and the conceptual aspect, and consequently, the failure to form figural concepts (Fischbein, 1993). This analysis can help interpret the problems with classificatory thinking.

049. Teacher: Do the counterexamples you have drawn verify the D_1 definition?
050. Domenico: Yes, they check it …
051. Lucrezia: If they check D_1 are parallelograms, it is not true?
052. Vittoria R.: True! So we are all wrong... rectangles, rhombuses and squares are parallelograms …
053. Licia: So nobody won... D_2 is valid then!

054. Filippo: How nice if we use D_2 we save a bit!
055. Licia: So nobody won... D_2 is valid then!
056. Teacher: So… has anyone managed to draw a quadrilateral that satisfies the aspiring definition of a parallelogram, D_2, but is not a parallelogram?

This is a suggestive help question in the sense Bartolini Bussi et al. (1995) based on the interpretivist approach; in this case, the tone of voice suggests the answer, which in this case is negative. No one answers.

057. Teacher: Great! None of you were able to draw a quadrilateral with opposite sides parallel and congruent that is not a parallelogram because such a quadrilateral does not exist! We have a new definition. The original price was 6 *ca*, but what is the current cost of this item?
058. Carlo: In my opinion it costs two and two so four …
059. Domenico: That's right! It costs 4 *ca*!
060. Voices: Yes, yes it costs 4 *ca*.
061. Teacher: Perfect! This new definition costs 4 *ca*. Let's continue with the game… Now I'll remove another pair of properties and propose the following more economical definition, which we'll call D_3: 'A quadrilateral is a parallelogram if and only if it has opposite sides parallel.' Do we still have uniqueness? I'll give you 5 minutes to reflect. Again, take a pencil and graph paper, and draw all the quadrilaterals with opposite sides parallel that come to mind. Do you always end up with a parallelogram? I challenge you to find a quadrilateral that is not a parallelogram!

The discussion continues. The children, under the guidance of the teacher, understand (also empirically with the help of drawings) that the definition D_3 uniquely identifies a parallelogram and that its cost is 2 *ca*. The *CoD Method* seems to work. The teacher challenges the children with a further task, asking them to eliminate one of the two remaining properties (a pair of parallel opposite sides) and to check whether the resulting quadrilateral is still a parallelogram. The children immediately realise that this definition costs only 1 *ca* and that such a low cost is unacceptable because the property 'at least two parallel sides' is trivially satisfied by all trapezia. The children realise that the attempt to further reduce the cost of the definition of a parallelogram from 2 *ca* to 1 *ca* has failed. A few students try to use other properties of sides or angles, but fail to invent a definition of a parallelogram that costs only 1 *ca*.

After taking the entrance test, the children did not revisit it again until it was administered again in the form of an exit test. The assessment and the solution of the entrance test were never communicated to the pupils. It was decided to do the two tests the same (minus the order of the items and the order of the answer options) in order to have a homogeneous comparison of results. In addition, in order to prevent the children from remembering answers already given in the entrance test when completing the exit test, it was administered after a sufficiently long time (six and a half months). Figure 1 shows a comparative summary of the results of the entrance test and the exit test and as an example, the text of two of the five items (the first and fourth items). To verify the transfer of skills, it was decided to base the two tests on a different definition of quadrilateral (the rectangle) from those analysed during the mathematical discussion (the trapezia and the parallelogram). Item 1 is the one with the highest number of correct answers in both tests. Lucrezia's answer to item 4 is very interesting because she unconsciously applies the *Theorem of the cost of equivalent definitions*: 'The definition given is minimal because it costs 3 *ca* like the one the teacher gave us who wants at least 3 right angles to give us a rectangle. This also costs 3

ca because it asks us for a parallelogram which we know costs 2 *ca* and then it also asks for a right angle which costs 1 *ca* and so in total makes 3 *ca*'.

Item Number	Entrance Test (March 2023) - N=16		Exit Test (September 2024) - N=16	
	Correct answer rate	Incorrect answer rate	Correct answer rate	Incorrect answer rate
1	37,4%	62,6%	93,8%	6,3%
2	31,3%	68,7%	62,4%	37,6%
3	31,2%	68,8%	50,0%	50,0%
4	18,7%	81,3%	81,2%	18,8%
5	12,5%	87,5%	75,0%	25,0%

1. Andrea: "A quadrilateral is a rectangle if and only if it has all right angles".
☐ It is acceptable as a definition and is also minimal.
☐ It is acceptable as a definition, but it is not minimal. Why?
☐ It is not acceptable as a definition (it is not unique). In this case, draw a counterexample.

4. Francis: "A quadrilateral is a *rectangle* if and only if it is a parallelogram with at least one right angle".
☐ It is acceptable as a definition and is also minimal.
☐ It is acceptable as a definition, but it is not minimal. Why?
☐ It is not acceptable as a definition (it is not unique). In this case, draw a counterexample.

Figure 1: Summary of Entrance Test and Exit Test Results and text of items number 1 and 4

Conclusions

The findings indicate that the *CoD Method* significantly enhances students' understanding of geometric definitions. Through active engagement in discussions, students identified redundant properties and transitioned to more economical definitions. This dialogue effectively connects the results to the study's objectives, illustrating that students can comprehend complex concepts when equipped with appropriate tools. The iterative refinement of definitions—from D_1 to D_2, and potentially to D_3—demonstrates how mathematical understanding evolves through trial and error. These realizations about valid definitions signify significant growth in the student's mathematical reasoning. In the context of the *CoD Method*, teacher questioning is essential for enhancing students' understanding of geometric definitions. It clarifies concepts, encourages reflection, promotes engagement, and facilitates problem-solving. By asking strategic questions, the teacher plays a pivotal role in guiding students through the complexities of mathematical definitions, ultimately fostering greater conceptual understanding and confidence in their reasoning abilities. Consider adding relevant references to support the discussion on teacher questioning and the *CoD Method*. In conclusion, based on the analysis of the discussion dialogues and the comparative results of the two tests, we can reasonably confirm that applying the *CoD Method* in a mathematical discussion context can improve the meta-concept of minimal definition in primary school.

After a short adaptation period, the children seem to handle the *CoD Method* well; it is in line with the reasoning behind minimal definitions. It is assumed that the success of this method is due to the fact that it verifies the real need for each critical attribute within the definition and determines whether it is redundant. The student, in order to achieve the goal of minimal definitions, is compelled by the inherent dynamics of verifying the redundancy of critical attributes to thoroughly investigate the meaning of the definition and each of its properties.

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Dimensions of conceptual knowledge and dynamic geometric constructions

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In this work, we focus on the kind of information on students' conceptual knowledge that can be obtained using dynamic geometry constructions of quadrilaterals. We designed an assessment instrument based on dynamic geometric constructions created in the same way as traditional Euclidean constructions with a pencil, a ruler, and a compass. The constructions are supplemented by printed answer sheets in which various properties of quadrilaterals are listed and, when relevant, asked to be justified. We show how such an approach allowed us to distinguish different dimensions of conceptual knowledge, which we illustrate through authentic examples of students' responses to the answer sheets. We also consider possible stimuli for the contribution to the development of students' conceptual knowledge and, in general, to the teaching and learning of geometry.

Keywords: Dimensions of conceptual knowledge, geometric constructions, dynamic geometry, properties of quadrilaterals

Introduction

In recent decades, conceptual knowledge has received considerable attention in mathematics education, along with procedural skills and flexibility, as they together can form the aims of teaching and learning mathematics (Verschaffel, 2024). Conceptual knowledge is generally regarded as an understanding of mathematical facts and relationships between them that results in a connected network of knowledge (Hiebert & Lefevre, 1986) and refers to a type of understanding whose quality is proportional to the depth of such connections (Star & Stylianides, 2013). In our previous research, we focused on students' reasoning about quadrilaterals (Vízek et al., 2023) and we compared its forms with van Hiele's levels of geometric thinking (van Hiele, 1986). We now follow up on our more recent study, in which we studied what information about the quality of student conceptual knowledge can be obtained using a set of dynamic geometric constructions (Vízek et al., 2024). We developed a diagnostic instrument based on dynamic constructions of quadrilaterals and using it as a data collection tool in qualitative analysis we were able to identify six different subcategories of student comprehension of plane objects. We connected them with the existing general model of conceptual knowledge (Nilsson, 2020), and elaborate on the model in a way that makes it applicable within dynamic geometry environments (DGEs). In this paper, we explain the model in detail and illustrate it by presenting different forms of student thinking that might appear while in DGEs.

We organise the text as follows: at the beginning, the possible distinction of the dimensions of conceptual knowledge is characterised, with the general background for the comprehension of geometric objects and for DGEs. Then the diagnostic instrument is described, and various modes of student understanding are presented through authentic data excerpts. Finally, we discuss the findings and possible implications for educational practice.

Theoretical background

Dimensions of conceptual knowledge

To assess the quality of student conceptual knowledge, Nilsson (2020) differentiates three dimensions of such knowledge in mathematics: *knowledge of classification*, *knowledge of structures*, and *knowledge of principles*. In that general setting, knowledge of classification means identifying a certain object and its properties, which is merely isolated or poorly connected to other facts in mathematics, knowledge of structures refers to an awareness of the interconnectedness of ideas that leads to mapping a conceptual structure of an object or a set of objects, and knowledge of principles represents the ability to connect identified ideas and their structures by establishing principles that are generally applicable and lead to conclusions to mathematical propositions.

To perform an assessment of conceptual knowledge, there are two main approaches to do so. Conceptual knowledge can be assessed implicitly according to students' preferences in solving mathematical problems, comparing different strategies, and evaluating them, or explicitly by letting students define mathematical objects, justify their properties, and explain the nature of solution procedures (Rittle-Johnson & Schneider, 2015). Both ways, but especially the second one, are typically based on assigning non-standard tasks to students, as this approach can lead the students to reveal or invent unusual solution strategies different from routine procedures (Braithwaite & Sprague, 2021).

Geometric objects

In geometry, different levels of understanding are associated with different modes of comprehending geometric objects. They can be grasped as isolated shapes, i.e. classified partially, or ordered in a structure according to their common properties, i.e. classified hierarchically (de Villiers, 1994). The partition classification belongs to primary school and typically describes geometric shapes by their initial definitions using the size of sides and interior angles. In that sense, a square is described as a quadrilateral that has all sides of the same length and all interior angles right, a rhombus has all sides of the same length, and a kite has adjacent sides of the same length.

At the lower secondary school level, individual components of the shapes are studied in detail, especially diagonals and their features. From that perspective, diagonals of a kite are perpendicular and one of them bisects the other (but not vice versa). By adding the requirement that the diagonals bisect each other, we get a rhombus. By adding another requirement that the diagonals also have the same length, we get a square. Therefore, a square can be comprehended as a special case of a rhombus, and a rhombus can be comprehended as a special case of a kite, which makes the structure of these objects hierarchical (de Villiers, 1994). The properties of these objects can be proved from their initial definitions, while such justifications lead to conclusions of general principles and can develop thinking in geometry at the upper secondary school level (Fujita & Jones, 2007).

Dynamic geometry environments

Proper comprehension of planar shapes and exploration of their properties can benefit from the use of DGEs such as Sketchometry or GeoGebra (Bebernik et al., 2022). Within these programmes, geometric objects can be created using points, lines, and circles, so they can be modelled on the same

basis as in traditional Euclidean approach with a pencil, a ruler, and a compass. However, in contrast to the static constructions performed on paper, the constructions in DGEs are not static as they have so-called free points which can be moved and the construction retroactively modified by the movement. This kind of movement is called *dragging* (Leung et al., 2013) and leads to a change in the position of lines and circles determined by the moved point(s). Such dynamic manipulations generate a whole *dragging family* of objects (Forsythe, 2015) with some mutual properties that are invariant to dragging. So, modelling the objects in DGEs can highlight what the shapes within the dragging family have in common, which may be, for example, certain properties of diagonals mentioned in the previous paragraph. In that particular case, the dragging family might comprise a kite, a rhombus, and a square.

It seems that the assessment of students' conceptual knowledge and theoretical understanding in geometry could benefit from tasks that would be designed accordingly in DGEs. In particular, dynamic modifications made to the construction through the dragging of free points can show the users common features of objects in DGEs as phenomena, i.e. on the basis of recognising invariant and changeable properties of the objects (Baccaglini-Frank, 2019). However, the challenge is to relate such phenomenological evidence to the nature of geometric objects that are determined by their construction in DGE and, so, enable to deductive reason about them (İmamoğlu et al., 2023).

The instrument

In the GeoGebra environment, we prepared dynamic constructions based on the compositions of points, lines, and circles that create certain quadrilaterals in exactly the same way as they would have been drawn with conventional hand-by-hand tools on paper. Selected free points from the constructions were given a green colour and each construction was accompanied by a printed worksheet with answer frames (Vízek et al., 2024). We assigned the constructions to the students with a prompt: "Drag the green points to find which different quadrilaterals ABCD occur in the construction. Fill in the printed worksheet for each type of quadrilateral you found." Thus, students did not have to generate their own construction. Their work in the GeoGebra environment required only dragging the prepared model but no previous experience with creating in such an environment.

To introduce the instrument in detail, we focus now on one of the constructions, namely the one containing a kite, a rhombus, and a square (see Figure 1). Its electronic version is available under the link https://www.geogebra.org/m/wgcbybfm. The construction is based on perpendicular lines AT, OP with their intersection S, the line segment BD with the midpoint S, which is part of the line OP, and the point C, which lies on the line AT. The properties of the construction are ensured by concentric circles m, n, and congruent circles k, l. The C and D points are the free points that were coloured green, that is, it is possible to drag them. The point C is movable along the line AT and the point D along the line OP. The default situation (which is also available through the refresh button) produces a quadrilateral ABCD, which is a kite because its diagonals are perpendicular to each other, and one bisects the other one. When point C is placed at point T, its diagonals already bisect each other, so ABCD is a rhombus. Finally, when point D is moved to point G, the diagonals have the same length, so ABCD is a square.

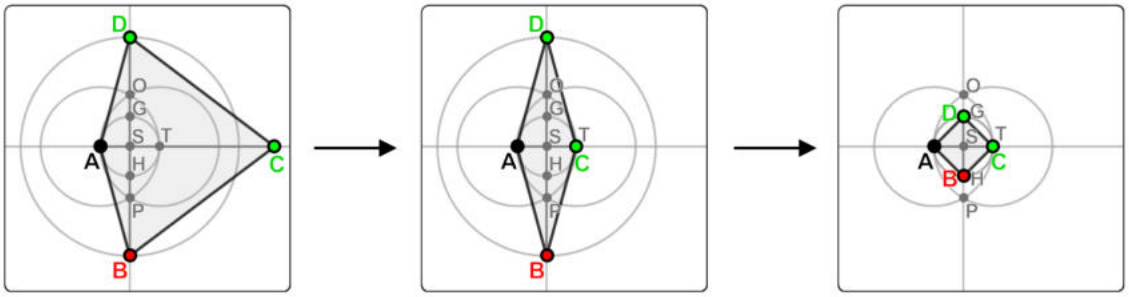

Figure 1: A kite, a rhombus, and a square appearing in the dynamic construction

The fact that a kite is a general quadrilateral in the sense of hierarchy in this particular construction, and a rhombus and a square are its special cases attainable by moving points C, D to specific exact places allowed us to prepare a universal assessment worksheet with answer frames suitable for all shapes appearing in the construction (Figure 2). For the purpose of the assessment, the worksheet is printed on paper, so that respondents can fully focus on the electronic version of the construction and fill in the worksheet at the same time. Each frame of the worksheet first asks respondents to provide the name of a quadrilateral found in the construction and to specify the exact locations of the green points that produce the quadrilateral. This unequivocally defines the shape the respondent is focusing on, and also shows whether the respondent knows the name of the quadrilateral properly. Then follows a list of various properties of diagonals intended for respondents to select and tick the ones that are relevant for the particular shape and fill in their justification besides each tick. In case a respondent noticed a property not listed, it can be specified and justified at the bottom of the frame.

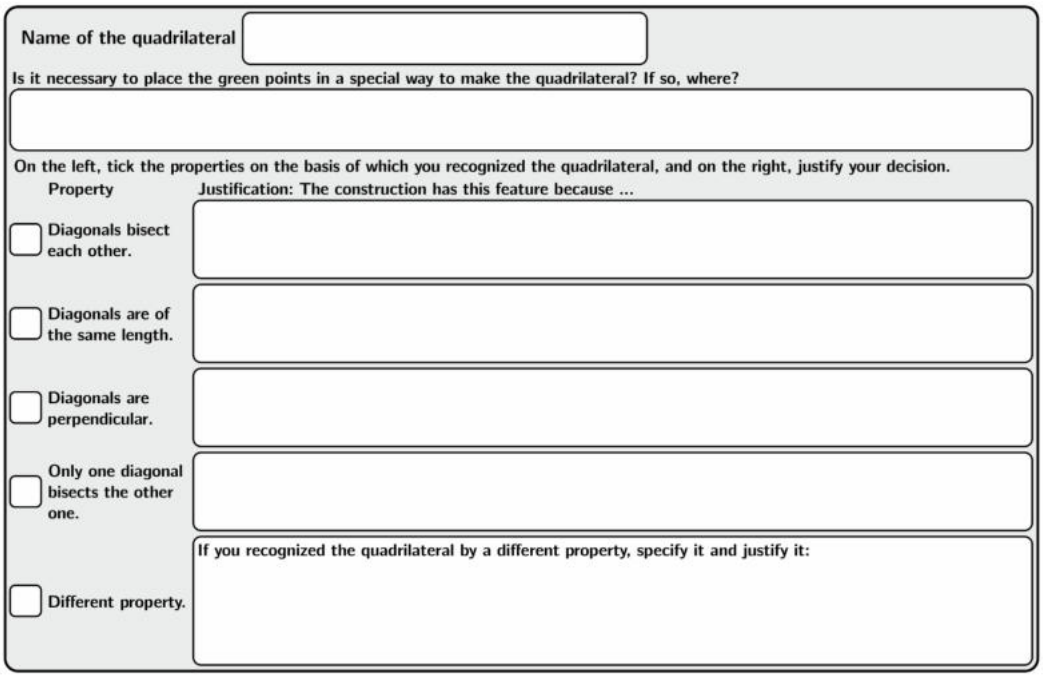

Figure 2: An answer frame for a construction based on properties of diagonals (Vízek et al., 2024)

In this paper, we focus on the kite construction because in the Czech Republic, where the study was conducted, this quadrilateral is not included in curricular documents provided by the Ministry of

Education, Youth, and Sports (MEYS, 2022, 2023). Therefore, working with kites implies a non-standard task and, therefore, the kite construction is suitable to assess the quality of students' conceptual knowledge. Our construction is also non-standard because the quadrilaterals in it are constructed on the basis of the properties of their diagonals and not on the initial definitions referring to the properties of sides and angles.

Student thinking revealed by the instrument

To capture the three dimensions of conceptual knowledge, that is, knowledge of classification, knowledge of structure, and knowledge of principles, we now show samples of authentic student thinking from the filled in answer frames. These samples come from five groups of respondents (labelled B to F); four of them (B to E) were participants of the earlier study (Vízek et al., 2024), the fifth one (F) is new, we included it to obtain a broader perspective. The groups are:

B – 16 future lower secondary school teachers, students of the master's degree program;
C – 27 future secondary school teachers, students of the bachelor's degree program;
D – 25 lower secondary school students of the age between 14 and 15 years;
E – 28 upper secondary school students of the age between 16 and 17 years;
F – 11 future lower secondary school teachers, students of the master's degree program.

To supplement the findings of the earlier study, we present only excerpts not published before. Code names of the respondents consist of a group letter and a randomly assigned order number (e.g. E19).

Knowledge of classification

Knowledge of classification is demonstrated in our context by identifying the quadrilateral contained in the dynamic construction and therefore writing its name at the top of the answer frame. This knowledge is further manifested in the description of its properties, and we distinguish between *a description of a property* and *an alternative explanation of a property*. A description of a property belongs to data excerpts where the respondent filled in a 'different property' at the bottom of the answer frame; however, this particular property was not given by the construction; instead, it just cited an initial definition of the shape based only on the size of sides and angles:

E19/Kite/Different property:	The adjacent sides are of the same length.
D01/Rhombus/Different property:	All sides are of the same length.
B15/Square/Different property:	All sides are of the same length, the angles at the vertices are 90°.

An alternative explanation of a property belongs to data where the respondent provided a justification of a ticked property by mere specifying the given property as such in a different way, without a connection to the construction. For example, some students stated that diagonals of a quadrilateral are perpendicular because they form a right angle or an angle of 90 degrees. Others presented how the ticked property translates to the shape.

F10/Square/Diagonals are of the same length: $|AC|=|BD|$

Knowledge of structures

Student statements that connect geometric shapes with their properties in a logical way represent the knowledge of structure. They can be expressed as an apprehension of *the structure of the shape* and/or

an understanding of *the structure of the construction*. The first of these perceives the shape separately from the construction, as if the lines and circles constructing the shape did not exist. However, individual elements of the shape and the way they are structured within the shape are addressed:

E01/Kite/Diagonals are perpendicular:	↔ DB is the axis of symmetry of the line segment AT => it is perpendicular
F06/Kite/Only one diagonal bisects…:	CS is the altitude of the isosceles triangle ΔDBC
F03/Kite/Only one diagonal bisects…:	Triangle ABD is isosceles, and by constructing the altitude to side a, the altitude = diagonal |AC| will pass through the midpoint of line segment |BD|

On the contrary, knowing the structure of the geometric construction ensures that the shape is perceived as an integral part of the construction, and the justifications reflect this fact by including the constructing lines and circles into the arguments:

C05/Kite/Only one diagonal bisects…:	The circles k, l ensure that BD is the axis of symmetry of AT and S is the midpoint of BD ⋀ AT, but generally |AS| ≠ |SC|.
E14/Square/Diagonals bisect each other:	The diagonals are the diameters of the circle m, bisected at the centre S.
F01/Rhombus/Diagonals bisect each o.:	m, n = concentric circles, common centre S, A, C ∈ n; B, D ∈ m; AC, BD pass through the centre S.

Knowledge of principles

When explaining the properties, *knowledge of principles* means that the student refers to statements which are generally applicable in geometry, that is, not restricted to the particular quadrilateral. These principles can be applied on the shape and its elements only, and/or on the constructing lines and circles. In our data, knowledge of principles was manifested by applying Thales's theorem (all angles above the diameter of a circle are right):

E01/Square/Diagonals are perpendicular:	Δ BCD is an isosceles triangle |DC|=|BC|. The circle m is Thales's circle of DH, where BCD = 90° → 1/2 = 45°.

or by applying the hierarchy of the shapes included in the construction, that is, by justifying a property using a reference to a justification of the same property already provided in a previous frame:

C19/Square/Diagonals are perpendicular: See kite.

Discussion and conclusion

The purpose of this paper was to connect the dimensions of conceptual knowledge to specific forms of student thinking in geometry that emerge when working with dynamic constructions. To complement and particularise our previous study (Vízek et al., 2024), we showed more data samples illustrating how a DGE construction containing a dragging family of quadrilateral shapes (a kite, a rhombus, and a square) can help to highlight and distinguish various quality levels regarding conceptual knowledge in geometry.

We posed the DGE construction as a non-standard task (Braithwaite & Sprague, 2021) by basing it only on the properties of diagonals and including a kite as a default shape, and this approach has paid off. The fact that the initial definitions with the size of sides and angles were not relevant for justifications allowed us to clearly distinguish between mere knowledge of classifications (especially,

knowledge of the name of the shape and its initial definition) and knowledge of structures (i.e. knowledge of individual elements of the shape, their relations, and/or their relation to the entire construction). That is, this approach allowed us to explicitly assess students' conceptual knowledge (Rittle-Johnson & Schneider, 2015). Compared to the dimension model of conceptual knowledge given by Nilsson (2020), we needed to be more specific in some cases, so the elaborated model has also subdimensions. The instrument allowed us to distinguish between students perceiving the shapes partially (especially, when they think about them as separate units) and hierarchically (especially, when they perceive the hierarchy as a general principle and utilise it in justifications), which particularise the meaning of the partial vs hierarchical classification given by de Villiers (1994).

From the perspective of task design, our instrument has a novel design which combines the benefits of DGEs, namely the process of dragging (Leung et al., 2013; Forsythe, 2015), which seems to encourage students to explore the features of the shapes included in the dragging construction (Bebernik et al., 2022); our data confirmed the encouragement aspect. Moreover, our instrument was constructed in a way that is in accordance with classical Euclidean constructions performed on paper. This particular design provides the solvers all relevant information about the constructed shapes, their structure, and the overall structure of the construction, which allows solvers not only to observe the properties of the included objects as phenomena of the dynamic construction (Baccaglini-Frank, 2019), but also to explain their properties in a logical way and, so, stimulate students' theoretical reasoning (İmamoğlu et al., 2023) by connecting the phenomena to their theoretical backgrounds. Future research will show whether the instrument could be also used for investigating the development of geometric reasoning in geometry at secondary school level (Fujita & Jones, 2007).

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The role of perceptual clues in 3D enumeration tasks

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Keywords: Geometry, spatial structuring, visualization, perceptual clues.

Background

This project explores the following research question: How can representations and perceptual clues support children's enumeration of 3D arrays?

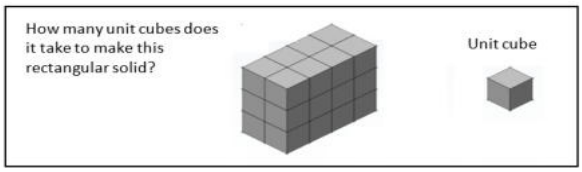

***Figure 1**: Task example from Ben-Haim et al.(1985).*

Research shows that students often struggle with visualizing tasks as exemplified in Figure 1, counting 2D faces instead of 3D units, leading to double or triple counting cubes (Battista & Clements, 1996; Ben-Haim et al., 1985). Battista (2004, 2010) identifies five cognitive processes crucial for enumerating units in 3D arrays: Units-locating, spatial structuring, organizing-by-composites, forming and using mental models, and abstraction. We consider an enumeration strategy as based on and a result of the students' cognitive processes.

Perceptual clues (PC) have shown to support students' enumeration strategies. PCs are a choice of design with the aim of drawing students' attention to a mathematical idea or strategy. For instance coloured cubes (fig. 2) can help students spatially structure the array in layers (Finesilver, 2017; Rupnow et al., 2022); spherical units (fig. 2) can reduce double and triple counting 2D units, thereby allowing students to discover more effective strategies (Alstad et al., 2023).

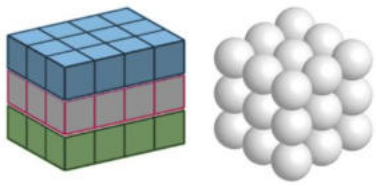

***Figure 2**: Coloured cubes and spherical units as perceptual clues*

Methods

During the fall of 2024 we collected data from 13 3rd grade students. The students were presented with four tasks in which they were asked to enumerate the units in a 3D array. The structure of the array remained consistent across the first three tasks (e.g. 3x4x4 array). Task 1, a 2D representation of the array of cubes; Task 2, a 3D representation; Task 3, a 3D representation with one out of three possible PCs, randomly assigned. Finally, Task 4 reverted to a 2D representation, in line with Task 1. Task 1 allowed us to identify whether students employed common erroneous strategies. Task 2 helped us determine whether children perceived the array differently when represented in 3D

compared to 2D. Task 3 enabled us to examine whether certain PCs supported students in seeing structures in the figure beyond any erroneous strategies. Task 4 assessed whether students could use their experiences and visualization from the 3D representations when encountering a new array represented in 2D.

Analysis and results

Firstly, students' strategies were analysed and categorized using Battista's five cognitive processes and Levels of Sophistication (Battista, 2004). We found minimal, if any, differences in students' strategies when moving from a 2D to a 3D representation. Nine out of 13 students counted squares instead of cubes with both 2D and 3D representations. Secondly, we analysed students' change of strategy after encountering a PC using the constant comparative method of analysis. Six out of 13 students changed strategies after exposure to a PC, indicating that the PC was supportive. The remaining seven students maintained their initial strategy. This variability in response to PCs suggests that PCs sometimes influence students' choice of strategies, though inconsistently. This variability is explained by the core category visuospatial responsivity. Visuospatial responsivity does not explain the underlying causes of these differences but provides a framework describing students' varying capacities to identify relevant figural units and adapt their strategies. We identified four levels of visuospatial responsivity: limited, growth, stable, and flexible.

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Mathematics laboratory on geometric transformations at the museum

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Keywords: Mathematical machines, geometric transformations, laboratory, museum.

Mathematics laboratory and mathematical machines

The idea of the mathematics laboratory has evolved over time in teaching and learning mathematics (Maschietto & Trouche, 2010). In Italian documents for mathematics curriculum (e.g., AA.VV., 2004), the mathematics laboratory is related to the construction of mathematical meanings and a crucial role is played by the use of material and digital artefacts. Mathematical machines are material artefacts related to geometric transformations, conic sections and perspective; they were built for didactic purposes, but they also are significantly related with the history of mathematics. Laboratory sessions based on the use of mathematical machines are designed within the Theory of Semiotic Mediation (TSM, Bartolini Bussi & Mariotti, 2008). They have been studied in educational research, mainly at upper secondary school (Maschietto & Bartolini Bussi, 2011); they are also proposed in out-of-school context laboratory sessions, such as at the university museum. We are interested in designing laboratory sessions for lower secondary school classes at the university museum (Laboratory of mathematical machines, www.mmlab.unimore.it), focusing on the mathematical machine for reflection (Fig. 1a). Our research questions concern: 1) How can the laboratory sessions implemented with upper secondary school students be adapted for lower secondary ones, 2) If the work on the mathematical machine in a laboratory session can foster the emergence of the meaning of reflection. In this poster, we aim to present the experimented laboratory sessions for 13-year-old students.

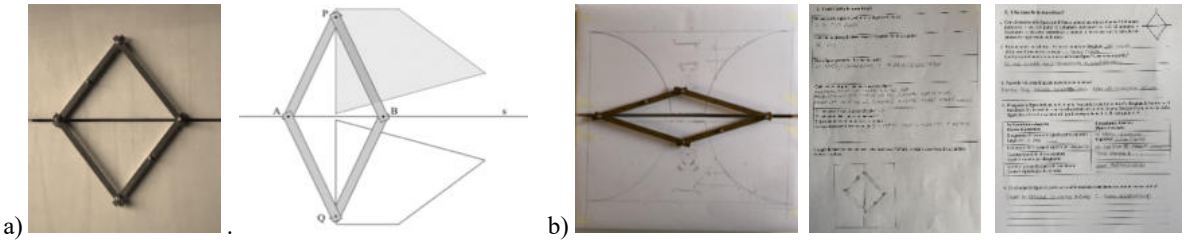

Figure 1: a) Machine for reflection, b) Students' drawing by the machine and worksheets

Methods

In previous research (Maschietto & Bartolini Bussi, 2011), the methodological elements of the TSM for teaching experiments in school have been adapted to a context beyond the classroom, where students go only once and are generally not familiar at all with mathematical machines. The exploration of the machines made through one worksheet and the final collective discussion has been maintained. To answer our first research question, we split the worksheet into two worksheets (the former concerning the description of the machine, and the latter guiding the identification of the reflection), considering linguistic issues, and added some support material to make the work with the machine easier and more pleasant for younger students. We also modified the structure of the

laboratory as follows: introduction of the geometric transformations; small-group work on the mathematical machine (Fig. 1a) guided by the first worksheet (Fig. 1b) with questions referring to the TSM ("How is the machine made?"); collective moment to share answers; group work on the machine with the second worksheet ("What does the machine make?"); collective discussion; final task. Each session lasts a maximum of two hours, including the visit to the exhibition of mathematical machines at the museum.

In spring 2024, we experimented with this new session with six classes of 13-year-old students. For each class, we collected the worksheets filled by the students, and the drawings they produced using the machine (Fig. 1b, left). Our analysis is based on these documents, as well as researchers' reports of collective moments.

Analysis of students' work

Regarding the first worksheet (Fig.1b, centre), students typically identified the rhombus and described its properties. In the last question of the worksheet, students were asked to represent the machine aiming to solicit their semiotic activity. The second worksheet (Fig.1b, right) asks students to draw using the machine and then to interpret/recognise the relationships between the obtained drawings. Although at the beginning of the session most students claimed that they didn't know anything about geometric transformations, words that characterise symmetrical figures ("*equivalent*", "*mirror*", "*same perimeter*", "*overlapping figures*") appeared little by little, as well as descriptions of corresponding figures ("*The fire drawn following the stencil is upwards, while the fire drawn on the sheet is downwards. They are congruent*"; "[the machine] *makes two figures superimposable by reflection*"; "[the drawings] *are the same thing but they are upside down or mirrored*"). In the end, most students correctly identified the transformation, although without providing any justification, with a few exceptions ("*because the diagonals of the rhombus are perpendicular and bisect each other*"). Even if the definition of reflection is not stated with the students, the whole work with the machine allows words and action consistent with the meaning of reflection to emerge in the worksheets and during the final discussion. This suggests a positive answer to the second research question, but it also provides elements to revise worksheets for future work.

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TWG 05: Probability and Statistics Education

Reimagining Statistics and Probability Education in the Age of Data and AI: An introduction to the work of TWG05

Leader: Aisling Leavy (*Ireland*)

Co-leaders: Daniel Frischemeier (*Germany*), Sibel Kazak (*Türkiye*), Orlando Rafael González (*Thailand*), Luca Lamanna (*Italy*), and María Magdalena Gea (*Spain*)

Introduction

This report presents a summary of the activities and key outcomes of Thematic Working Group 05 (TWG05): Statistics and Probability Education at CERME14, held in Bolzano from February 4–8, 2025. The group brought together 30 participants representing 13 countries, who contributed through 18 paper presentations and 6 poster sessions. Guided by CERME's core principles of Communication, Cooperation, and Collaboration, TWG05 cultivated a dynamic and inclusive atmosphere that encouraged meaningful dialogue and constructive exchange throughout the week.

The CERME14 TWG05 team was led by Aisling Leavy (Ireland), with co-leaders Daniel Frischemeier (Germany), Orlando Gonzalez (Thailand), Sibel Kazak (Türkiye), María Magdalena Gea (Spain), and Luca Lamanna (Italy).

The week began with an ice-breaker activity, "TWG05 Colleague BINGO," aimed at encouraging participants to connect and form discussion groups. Sessions were structured to promote active engagement, with each paper allocated 10 minutes for presentation followed by 5 minutes for questions. Broader discussions then took place in small groups, each facilitated by a co-leader and including the presenter. The responsibility for chairing sessions was shared among the co-leaders throughout the week, spanning multiple sessions from Tuesday to Friday, including a dedicated poster session.

Key subthemes identified for discussion across all submissions included:

I. data science
II. visibilizing statistical concepts
III. teacher education
IV. probabilistic reasoning and risk literacy

The identification of these subthemes was the result of a careful examination of the accepted papers and their thematic focus. This mapping process was significantly informed by the themes identified as emerging from CERME13 (see Primi et al., 2024), which included "teacher education," "reasoning about data," "statistical and probabilistic thinking and reasoning," and "data science." Furthermore, it drew insights from two recent and influential research publications: Batanero and Álvarez-Arroyo (2024), and Burrill and Pfannkuch (2024). Batanero and Álvarez-Arroyo (2024) synthesized prominent research in probability education, highlighting areas such as children's probabilistic knowledge, the impact of visualizations on solving conditional probability problems, teacher

education, and probability modelling. Their work also emphasized the growing interest in the concept of risk and its pedagogical link to probability, and called for more research into teachers' mathematical and didactic knowledge of probability. Complementing this, Burrill and Pfannkuch (2024) identified four major emerging trends in statistics education research: data science, visibilizing statistical concepts, social statistics, and new contexts for learning. They underscored the importance of fostering statistical reasoning, argumentation, and the critical evaluation of data-based evidence, including aspects of risk, along with enhancing the accessibility of statistical concepts through interactive visualizations. Thus, the chosen CERME14 subthemes reflect these convergent and critical areas of contemporary research in statistics and probability education. Guiding questions were prepared to facilitate discussions on these subthemes, with groups assigning note-takers to document their discussions on task cards. An online shared space, Task Cards, was utilized to systematically document the content and focus of each session's discussions. This comprehensive documentation was critical for a successful and substantial concluding discussion on the final day of CERME14, where all insights gathered during the week were collectively summarized.

Data science

During the TWG05 discussions on Data Science, several key learnings and open questions emerged from the group conversations. Participants emphasized the importance of introducing data work and data literacy at an early age, acknowledging that children are already exposed to tools that use data. The consensus was that data literacy should be embedded within inquiry-based or problem-based learning, where skills emerge through answering questions. The group discussed the use of different types of data, agreeing that both "dirty" (messy) and "clean" real-world data are important for different learning goals, as real data incorporates a sense of "reality". It was suggested that learners could start by formulating questions and then be presented with existing data, and any disconnects could lead to modifying their questions. Familiar contexts, such as social media or sustainability, were deemed important for relevance, along with using examples like recommender systems to help students understand how their own data can be used for predictions, fostering an awareness of the world. Critical thinking and reflection were highlighted as essential components of Data Science activities. Regarding tools, it was noted that while non-coding applications and technologies can provide support, programming skills (e.g., Python) are important for understanding the "inner workings" of models and fostering algorithmic thinking.

The discussion also outlined the desired knowledge for an "ideal" 16-year-old learner in Data Science. Such a learner should: Be aware of algorithms' impact on their lives (e.g., in insurance); Understand that data-driven decision-making systems can have biases; Recognize that probabilities underpin these systems; Know that models and data are simplifications of the real world and that elements can be omitted; Possess critical thinking skills, asking questions about the data and what is presented to them.

Several open questions and areas needing further investigation were identified:

- Institutional Constraints and Teacher Support: How to overcome institutional limitations and address the vulnerability felt by teachers regarding Data Science education. The need to support teachers in using real-world data, perhaps by providing sample datasets and guidelines, was acknowledged, recognizing the significant responsibility placed on mathematics teachers.
- The "Black Box" Concept: When and how deeply to "scratch" or open the "black box" of algorithms. This includes understanding the underlying mathematics, acknowledging that some models are not fully explainable, and fostering algorithmic thinking through insights into the black box. It was noted that there might be both "pedagogical black boxes" (intentional simplifications for learning) and "real black boxes" (complex systems not entirely transparent).
- Distinguishing Data Science from Statistics Education: Clarifying the differences, particularly the role of context. In statistics, the context and potential models often guide data collection, whereas in Data Science, data and context are often pre-existing, and the task is to formulate questions based on them. The group emphasized the importance of teaching both perspectives.
- Teacher Education: How to effectively equip teacher education students with the necessary knowledge and skills to support Data Science thinking and learning. This includes rethinking the organization of Data Science lessons, potentially promoting more interdisciplinary teaching, and addressing constraints like the need for longer project work blocks.
- Tools and Activities: Developing and making available specialized tools that facilitate understanding, while also considering the benefits of general-purpose tools like Python for understanding inner workings. The challenge of ensuring learners gain insights into models when using simulations was also raised, along with the need for project-based work and activities that encourage reflection.

Visibilizing statistical concepts

Several key learnings and open questions emerged from the group conversations during the TWG05 discussions on Visibilizing Statistics Concepts. Participants emphasized the importance of fostering the ability to represent the same concept or data in different ways (i.e., transnumeration), using different types of representations, in both figural (e.g., scatterplots, tabular displays, decision trees) and non-figural form (e.g., verbal or narrative models, reflective statements, color schemes). Engaging learners in the creation or use of these visualizations, representations, or models of concepts must be done to fit the purpose of making visible previously unknown statistical concepts by the learners (like in the “modeling for” approach, which facilitates a deeper understanding of the underlying mechanisms and abstract ideas that the model represents), instead of adopting a correct/incorrect approach. The creation of visual models or graphical representations by learners, as well as their use of interactive visual data analysis tools (e.g., TinkerPlots, computer simulations), were reported by participants as effective ways to elicit and reveal students’ intuitive reasoning, conceptual processes, and informal notions of statistical concepts such as inference, sampling distribution, sampling variability, central tendency, correlation, enumeration, and uncertainty. Moreover, the participants agreed that learners’ interpretations of visualizations of concepts are socio-cultural and context-dependent, which allow teachers and researchers to delve into and understand more about not only about students’ intuitive reasoning and conceptions, but also about the

experiences and socio-cultural frameworks and schemas that they bring to visualizations (e.g., students' underlying values, norms and semiotic conventions), which are rooted in the learners' socio-cultural background and context.

From the discussion, the new learner's needs and roles in relation to data, and how Visibilizing Statistics Concepts can be instrumental in meeting those needs and roles, were outlined. Today's learners are *data consumers*, individuals who frequently interact with happenstance data or data and visual representations produced by someone else's efforts, usually using digital tools, Software platforms, interactive visual data analysis tools, and meaningful contexts. They are also *data producers*, because they frequently engage in empirical investigations, represent and interpret their own data, and communicate their findings and conclusions, usually making them visible. In addition to these roles, today's learners are also *data prosumers*, because they often have to engage in activities in which they should act as both data producers and consumers.

The participants also highlighted the important role currently played by technology-rich visual learning environments and software platforms that allow students to create, make sense and analyze visual models (e.g., TinkerPlots) in the learning of statistics. Participants also identified a high degree of interconnectedness between activities on Visibilizing Statistics Concepts and the following elements: data science, technology, teacher education, curriculum, affect, context (authenticity), and technology. The interconnectedness between Visibilizing Statistics Concepts and these elements opens new avenues for researchers to explore and gain better understanding of students' reasoning, actions, and conceptual development in learning statistics, while engaged in activities in which statistical concepts are made visible to them.

From the discussions, participants were able to pose several open questions, from which areas needing further investigation are identified:

- How can the interconnectedness of Visibilizing Statistics Concepts with different elements (e.g., technology, context) and domains (e.g., data science, mathematics) be included in the curriculum, in order to make it possible for students to be introduced much earlier to statistical and probabilistic ideas?
- How can the use of technology tools (e.g., TinkerPlots, simulations, case blocky programming with Scratch) be used to visibilize statistical and probabilistic concepts while enhancing both the computational, statistical and probabilistic thinking of students?
- How can AI and machine learning methods (e.g., Support Vector Machine [SVM]) be used to visibilize key mathematical, statistical and probabilistic ideas while contributing to AI literacy and demystification?
- How can simulations and modeling activities be used to make students transition from "modeling of" to "modeling for", while promoting students' reasoning with informal statistical models and probability modeling activities using technology-rich visual learning environments?
- How can researchers, teacher educators, and curriculum developers keep the pace of advances in technology, in order to be able to quickly respond to the current and future developments that will be relevant to students' needs and roles in relation to visibilizing data?

Teacher education

During the TWG05 group conversations on the theme of Teacher Education, one of the main insights was that, in order to engage meaningfully and make interesting use of statistics and probability, teachers should develop deeper contextual knowledge, e.g., in connection with other fields such as politics, medicine, or climatology - an aspect that clashes with actual, mostly outdated, curricula across the countries that do not offer much space for the exploration of real-world applications of both statistics and probability. Many of the presentations and discussions in TWG05 highlighted, on the other hand, concerns that most teachers possess limited statistical content knowledge and pedagogical content knowledge in statistics, largely because they themselves have likely not experienced high-quality teaching of statistics or probability. In connection with this, mathematics textbooks often fail to support interdisciplinary teaching - or, in general, link with other fields - and digital tools are still not widely available or easily integrated into real classroom settings.

A shared understanding emerged during the discussions that data science cannot be meaningfully implemented without the use of (digital) tools. This led to the recognition that, beyond statistical content knowledge and statistical pedagogical content knowledge, teachers also need technological pedagogical content knowledge (TPACK) to effectively incorporate digital tools into their teaching.

Another critical point concerned the need to shift teachers' beliefs about probability and statistics - from viewing them as isolated procedures to be memorized, toward understanding them as ways of thinking about data and context.

All in all, the need for more work towards the above-mentioned directions of research was highlighted, aiming at investigating the teachers' perspectives and knowledge, together with all the institutional setting and constraints, aiming towards a more meaningful teacher training and, in general, teaching of probability and statistics.

Several open questions and areas for further exploration were identified, both for practice and research. In terms of future research directions, there is a clear need for systematic investigation into the development of teachers' specialized content knowledge (SCK), pedagogical content knowledge (PCK), and technological pedagogical content knowledge (TPACK) - in both statistics and probability - building, in particular, on the foundational work of the ICMI study by Batanero et al. (2011). From a constructive and design-based research perspective, efforts should focus on developing textbooks and teaching materials that meaningfully integrate contextual knowledge, data, probability, and professional development. The goal is to create resources that support both classroom practice and in-service teacher learning. Additionally, affective aspects must be addressed, as inspiring teachers to rethink and transform their approach to teaching statistics and probability remains a key challenge -one that calls for further targeted research.

Finally, the following three fields were identified as future issues and points of contact for ways forward:

- Emerging curricular trends, such as data science and STEAM;
- Citizen and democracy education;
- Risk literacy.

Probabilistic reasoning and risk literacy

During the TWG05 group conversations on this theme, the key learnings centered around the topics, such as conceptualization of risk, recurring themes across different papers, and connection of probabilistic reasoning and risk literacy with the other subthemes. As part of stochastic education at school, risk literacy has become an emerging area. Conceptualization of risk from socio-constructivist and realist perspectives was highlighted as useful in fostering risk literacy, as they were found to be interwoven in students' reasoning. There was an agreement that risk literacy would aim at informed-decision making and assessing probabilities would be a crucial part of this process. It was noted that simulations can be used for inquiry-based learning to foster both risk literacy and probabilistic reasoning. When discussing the papers related to "probabilistic reasoning and risk literacy", the group emphasized the links with other topics, including data science (e.g., data generation through sampling or simulation), teacher knowledge to manage risk literacy in the mathematics class (e.g., open discussions, curricula constraints/possibilities), visualization/representation to identify and argue in (with) probabilistic concepts, and relevance of the context.

Several open questions and areas needing further investigation were identified for practice and research:

- Two questions, which are practice-related, were raised with the recognition that subjective interpretation of probability is left out since it is seen as non-calculable and risk literacy is new to mathematics education: How can we address subjective probabilities in school? How can we bring risk literacy into the school curriculum?
- Various suggestions were made in terms of research. Local theories for instruction on risk (e.g., levels of risk literacy and reasoning about risk) can be developed through design-based research. Specifying the conceptualization of probabilistic reasoning and risk literacy in terms of where they overlap and where they are different is needed. Designing meaningful tasks to foster risk literacy was discussed as a way to enable teachers to reinterpret mathematical concepts (e.g., expected value, conditional probabilities, and compound probabilities) already available in the curriculum through a risk lens by connecting them to decision-making under uncertainty. The group also emphasized addressing the emotional influences in students' perception of risk.

Conclusion

The Thematic Working Group 05 (TWG05) at CERME14 served as a dynamic and inclusive platform for exploring critical aspects of Statistics and Probability Education, uniting 30 participants from 13 countries for meaningful dialogue and constructive exchange.

The insights gained from these discussions underscore the imperative to equip learners with essential skills for the modern world. This includes fostering early data literacy and critical thinking within data science contexts, emphasizing the use of both "dirty" and "clean" real-world data, and understanding how data is used for predictions. Discussions also highlighted the importance of enabling learners to interpret and create diverse statistical visualizations**,** recognizing them as data

consumers, producers, and even "prosumers". A consistent focus was placed on teacher education, acknowledging the pressing need for educators to develop deeper contextual knowledge and technological pedagogical content knowledge (TPACK) to effectively incorporate digital tools and navigate these complex topics. Furthermore, the discussions on probabilistic reasoning and risk literacy reinforced their growing significance for informed decision-making in an increasingly uncertain world.

As we stand at the threshold of a new era profoundly shaped by data science, artificial intelligence, and advanced technological tools, this is indeed an exciting time for TWG05 and the broader field of probability and statistics education. While the challenges are substantial—including overcoming institutional constraints, providing adequate teacher support, and adapting curricula to rapid technological advancements—the possibilities for fostering deeper statistical and probabilistic thinking, enhancing AI literacy, and preparing learners to critically engage with data-driven realities are immense. We eagerly anticipate tackling these challenges and harnessing these possibilities to further advance the teaching and learning of probability and statistics.

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Reconsidering Basic Resources for Modern Statistics Teaching and Learning: Opening the Dialogue

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This study discusses the findings of two distinct research efforts that explored how a specific resource, namely dynamic software tools for teaching statistics, was negotiated and integrated into teaching practice. Through a systematic analysis of the findings of these two cases, we draw conclusions about the significance of this resource and the conditions necessary for its effective use. Our findings indicate that dynamic software plays a central role in statistics education. However, challenges such as inadequate infrastructure, limited time, and insufficient opportunities for collaboration within the school setting are persistent barriers to the reform of statistics teaching. These issues highlight the need to reassess contemporary learning goals and reconceptualize the basic resources needed for teaching statistics. Additionally, we discuss how redefining these resources could promote equal opportunities and democratize statistics education in an era dominated by big data.

Keywords: resources, statistics teaching, dynamic software tools, innovative practices

Introduction

In recent years, the teaching and learning of statistics has become a central focus in curriculum reform efforts worldwide (Bargagliotti et al., 2020). These reforms emphasize stochastic reasoning and thinking as key curricular goals, shifting classroom statistical activities towards investigative and model-eliciting approaches. Alongside these curriculum changes, significant efforts have been made to develop effective professional development programs that equip teachers with the necessary knowledge, skills, and competencies to implement the latest guidelines for teaching and learning statistics (e.g. Burril et al., 2023). However, despite systematic research, development of instructional materials, and teacher training initiatives, the practice of teaching statistics often remains entrenched in traditional models.

In Greece, over the past 15 years, there have been two major attempts to reform the school mathematics curriculum. Both reform efforts significantly enriched and strengthened the statistical content within the mathematics curriculum, placing a special emphasis on the development of statistical reasoning and thinking. By adopting a more innovative approach to teaching and learning statistics, statistical inquiry has gained a central role in school statistical activity, while software tools have become essential in supporting this inquiry. During these years, efforts were also made to provide professional development for teachers. Despite these advancements, the integration of modern statistical teaching practices still faces considerable resistance.

Over this period, we have had the opportunity to study the development of practice within different groups of teachers. These studies spanned various time points, different reform efforts, and distinct collaborative contexts. The current study examines these two studies (Bakogianni & Potari, 2019; Bakogianni & Meletiou-Mavrotheris, 2024), both of which share a common focus: the utilization of

dynamic statistics educational software. Specifically, the two studies explored how the software was utilized by participating teachers in designing instructional tasks and in developing teaching practices.

The objective of this discussion paper is to analyze the findings of two distinct research efforts to address the following questions:

1. How were dynamic software tools for teaching and learning statistics negotiated and integrated into teaching practices within two different research contexts?
2. What conditions appear to facilitate or hinder the effective utilization of such tools to advance reforms in statistics education?

Through a systematic examination of the findings, we aim to gain empirical insights into the contextual factors that support or obstruct its use. These conclusions can provide valuable perspectives beyond the specific resource, highlighting the broader role of basic resources that prove critical for advancing the reform movement in statistics teaching and learning.

Theoretical views

Current views on teaching statistics

Statistics stands out from other fields of mathematics because it takes uncertainty into account. The pervasive presence of variability and randomness in modern life enhances the role of statistics in contemporary mathematics education and creates the need to develop new skills for managing the uncertainty inherent in decision-making processes (Burril et al., 2023). The complexities of statistical activity, as well as the nature of statistical problems, are captured in the framework of Wild and Pfannkuch (1999) for statistical thinking in statistical enquiry. This multi-dimensional scheme of Wild and Pfannkuch was central in our conceptualization of DSTR practice (teaching for the Development of Statistical Thinking and Reasoning) in our first study (Bakogianni & Potari, 2019).

Today, the evolution of the field of Statistics and the rapid growth the Data Science domain, introduces new trends in Statistics Education. Contemporary goals for teaching statistics include computational skills, complex forms of data representation, critical risk perception, interactive data visualizations, among others (Burrill & Pfannkuch, 2023). In light of these developments, the American Guidelines for Assessment and Instruction in Statistics Education (GAISE) Report (Franklin et al., 2005), were revised in 2020. The new GAISE II framework (Bargagliotti et al., 2020) emphasizes skills required for navigating the modern world of big data, and key issues in current curricula. Among the topics emphasized in GAISE II were posing questions throughout the statistical problem-solving process, considering different data and variable types, incorporating multivariate thinking, utilizing probabilistic thinking to quantify randomness, leveraging technology, effectively communicating statistical information, and implementing assessments that gauge students' deeper understanding of statistical concepts and reasoning. GAISE II (Bargagliotti et al., 2020) provided the foundation and guiding structure for designing the course in the context of which we conducted the second study (Bakogianni & Meletiou-Mavrotheris, 2024).

A view to resources for teaching statistics

There is considerable discussion in the literature around resources, their use, and their conceptualization as artifacts, tools, instruments, etc. (e.g. Pepin, 2015). Adler (2000) adopts a broad

perspective on resources, considering them both as a noun and a verb. Within this view, she identifies basic resources, which are prerequisites for the practice of teaching (such as buildings, electricity, desks, chairs), as well as resources specifically related to the teaching of a subject. The latter include human resources (e.g. collegiality, everyday knowledge, networks), material resources (e.g. curriculum, technologies, everyday objects), and cultural resources (e.g. time, language). The study of resources *in-practice-in-context* recognizes the complexities of the context and the practice in which a resource is used, developed, or transformed over time (Adler, 2000). Both studies discussed in this paper consider teachers' interactions with dynamic software tools for teaching statistics as an important pursuit in their professional development. Dynamic software tools for data explorations contistute a prominent resource for statistical inquiry at school (Biehler et al., 2013), for their potentiallity to both support statistical reasoning and foster conceptual understanding of stochastic concepts and ideas (e.g. randomness, variability).

The significance of dynamic software tools in the teaching of statistics, along with the challenges they pose to the usual teaching practices, makes this resource central to the development of the studies under discussion and a focal point of the present study.

Overview of the two studies

Developing practice in a teachers' community

The first study (Bakogianni & Potari, 2019) utilizes a community of practice (CoP) context (Wenger, 1998). In this study, eleven secondary mathematics teachers, as part of a CoP, collaborated to explore resources supporting the teaching of statistics, with a focus on promoting the development of statistical thinking and reasoning. Through iterative cycles of exploring tasks, designing, implementing, and reflecting, teachers engaged with various resources, some of which became part of the community's shared repertoire. The role of the researchers in these studies was that of a facilitator, responsible for organizing the meetings, shaping the agenda based on teachers' needs and emerging issues, and providing appropriate materials (e.g., articles, educational resources, critical questions) to support the process.

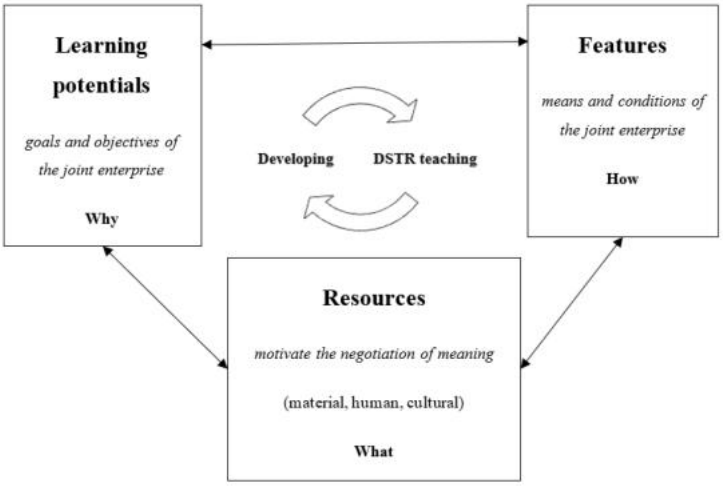

Figure 1: The conceptualization of statistical teaching for Developing Statistical Thinking and Reasoning (DSTR) (Bakogianni & Potari, 2019)

All the theoretical perspectives of this study are condensed in a dynamic triangle that captures three core elements of practice (see Figure 1). Locating this triangle inside a Community of Practice, we see the learning potentials as the purpose of the joint enterprise, the features as characteristics of the teaching that support the negotiated purpose, and the resources as the primary impetus for mobilizing negotiation of meaning.

Focusing on a particular resource – a dynamic digital tool for teaching statistics – the study identified three key phases in the re-sourcing process. The first is the *emergence phase* where the dynamic Software Tool for Statistics Instruction (STSI) is introduced to the CoP and becomes visible among its members. The second is the *exploration phase*, during which the STSI becomes a focal point for discussions and negotiations about its potential uses in the classroom. The third is the *immersion phase*, where the focus shifts to teachers' direct experiences using the STSI in the classroom. Moreover, our findings suggest that this immersion phase often facilitates the consolidation of the resource into the community's shared repertoire.

Progression through the three phases seems to be linked to the interconnectedness of the different elements within the teaching triangle of Figure 1. Particularly, the more explicit and robust these connections become, the more likely it is for the resource to be fully integrated into the practice.

Developing practice in a collaborative learning environment

The second study (Bakogianni & Meletiou-Mavrotheris, 2024) reflects on our experiences in a graduate course aimed at educating teachers in the teaching and learning of Statistics and Probability. Conducted approximately ten years after the first study, this research involved 20 participants—19 secondary mathematics teachers and 1 primary teacher – comprising roughly equal numbers of prospective and in-service educators. The teachers worked in pairs of groups (action team- reaction team) to design statistical projects aligned with contemporary ideas in Statistics Education. We investigated key factors influencing this design process. Unlike the first study, the course in this research placed special emphasis on the use of digital tools, providing students with ample opportunities to familiarize themselves with and experiment with TinkerPlots (Konold & Miller, 2005). Although there was encouragement but no explicit instruction to use the software for their project design, it is noteworthy that all groups chose to incorporate TinkerPlots into their projects. The authors of this article co-taught the course and participated as facilitators during the design phase, providing critical questions and additional resources to support the teachers where needed.

The study highlighted three main themes: the unique role of data in statistics, the importance of technology in fostering statistical reasoning, and the pedagogical challenges associated with teaching stochastic problems. Revisiting the findings, with the dynamic software tool in the center of our focus, we gained new insights into the teachers' experiences within an organized training program.

Synthesis of the key findings

Although the two studies were conducted at different times and within distinct collaboration contexts, they share a common focus on dynamic software tools for teaching statistics, and some implications for the development of secondary teachers' practices. This study draws on data from the reflection phase of the teachers following the design process. For the first study, the data consist of the transcription of a 120-minute discussion that took place after the teacher groups had designed and implemented a statistics activity in their classrooms. For the second study, the data consist of the

transcription of a 180-minute discussion during which the teacher groups reflected on their designs, incorporating feedback received from a peer review team. Using the triangle depicted in Figure 1 as a methodological framework, we explore the connections between the three elements of the triangle in both studies and reconsider our conceptualization of dynamic software tools in statistics education, as well as the integration of such tools into the practice. In the findings we refer with SA(1,2,..) to the participants of the first study and with SB(1,2,…) to the participants of the second study.

Linking Resource to Learning potentials

The establishment of particular links between the dynamic software tool and learning potential was successful and robust in both cases. The findings in both studies showed that all teachers were eventually able to articulate how the tool could effectively support specific learning potentials.

In the first study (Bakogianni & Potari, 2019), forming such connections was challenging and required multiple attempts of negotiating meaning, reading and discussing research papers, and experimenting in the classroom. Despite these challenges, teachers ultimately succeeded in explicitly linking the software tool to specific learning goals in statistics education. The following is a characteristic example of such findings, where SA1, directly linked the software tool to the learning potential of enhancing students' ability to create and understand sampling distributions;

SA1: I mean Fathom or some relevant software. I liked Fathom, I think it is very useful for students. The thing is to give students opportunities to explore the data on their own, to take the samples, to explore, to observe, but this… this requires time. But anyway, I prefer such activities from traditional teaching.

In the second case (Bakogianni & Meletiou-Mavrotheris, 2024), creating these links constituted a central goal of the course. Consequently, these connections were explicitly discussed in the classroom, and when teachers designed tasks as part of their course assignments, they were required to refer to such connections. Although it was not mandatory to use software tools in the course assignments, all teachers opted to integrate TinkerPlots (Konold & Miller, 2005) in their designs. The extract from the primary teacher below (SB1), not only explains why they made this choice, but also demonstrates the clarity with which they (SB1, SB2 and SB3) linked the software tool to the learning potentials.

SB1: For 4th grade students, it's very difficult to talk about probabilities. For example, when they get the same result 3 or 4 times, they start to believe that it's more likely. I had a hard time helping them understand what randomness is. The software helped me a lot in this case. It helped me explain it to them.
SB2: With the software, they become convinced.
SB3: Through the ability to perform multiple repetitions, the idea of randomness and variability becomes clear.

Linking Learning Potentials to Features

This type of connections concerns how specific learning objectives can be supported by particular features, that is, by specific means. For example, how a simulation might help illustrate the concept of variability, and how this feature, which is the engagement of students with simulation processes, is ultimately connected to the resource under study, which in this case is the dynamic software tool.

The last sentence of the previous extract illustrates how teachers connected learning potentials to specific teaching features. Here, SB3 refers to using multiple repetitions of an experiment, which have the potential to support students understand the concepts of randomness and variability.

Although establishing such connections was a primary aim of the course we offered, we found that these links became more apparent and were explicitly discussed by the groups of teachers during the design phase of a statistical task.

In the following extract, SA2 from the first study, discusses how the software's simulation tool allowed him to go beyond the limited data collected in the classroom, enabling him to explore the statistical significance of the result and make a statistical inference.

> SA2: So, the one group tries to recall words with meaning and the other group tries to recall words without meaning, and we note down the number of the words in each case. Then I calculate the mean value for the one group, then the mean value for the other group and I subtract the two values. The point now is to decide whether this difference is usual or unusual. Is it expected or unexpected? If it is usual, there may be many reasons for being usual, maybe it is the way we divide the people into two groups, whether we select them at random or not, etc? …This is where the technology comes.

Linking Resource to Features

This type of connection appeared to be the most demanding. What we sought in this case was to determine how the features of teaching could be linked to specific resources. These connections often depended on linking the features to a set of interconnected resources that are prerequisites for utilizing software tools. The predominant commonality between the two studies lay in the challenges associated with school resources, particularly technological infrastructure, time constraints, and human resource interactions, as key factors impacting the integration of these tools.

In the first study, inadequate technological infrastructure in schools frequently emerged as a barrier preventing teachers from implementing their planned activities in the classroom. In the second study, many teacher groups created alternative versions of their plans to accommodate teachers from schools lacking the necessary resources. However, these alternative scenarios were described as more simplistic: "To make it possible for someone without access to the software to apply it, at least at an initial level of exploration." The difficulty here was in linking these simplified plans with the teaching feature of engaging students in data explorations and randomizations processes.

Time limitations and curriculum constraints also surfaced prominently in discussions with the teachers in both studies. The majority of the designed teaching plans required 2-3 class periods, with all teachers agreeing that it is challenging to conduct a meaningful statistical investigation in just one teaching hour. In this case, the key teaching feature was supporting statistical inquiry to address a meaningful statistical problem.

Finally, one element that seemed to play a particularly significant role in the teachers' design was human resources—collaboration and interaction among them. Collaboration and interaction appeared to be crucial factors in managing the inherent uncertainty underlying statistical concepts and processes, and in addressing the challenges of uncertainty within the statistics classroom. This element was particularly evident in both studies and was noted by nearly all the teachers involved. The following excerpt from SB4 highlights this point:

> SB4: I had to think completely differently. In lesson planning, I always knew what I wanted and could anticipate to some extent what would happen in the classroom when I provided such a scenario. Here, things are very different. This completely took me

> out of my comfort zone. We needed to make up various scenarios, to share million ideas.

Discussion on the implications of the two studies

Biehler et al. (2013) liken statistical reasoning to "travelling" from data to conclusions, with computer tools making this process easier but with a "black box" effect, where some details are missed. Empirical studies have demonstrated that technology enhances students' statistical reasoning, even at young ages, by aiding data exploration and inferential reasoning (Biehler et al., 2013). Data exploration is crucial not only for statistical inference and statistical thinking but also for data science activities, linking statistics to the emerging world of big data, as pointed out in the GAISE II Report (Bargagliotti et al., 2020; Biehler et al., 2022).

The question arises: Is teachers' familiarity with technology and the development of specific learning objectives connected to it, sufficient to transform statistics teaching practices? Using the dynamic triangle of teaching (Figure 1), we aimed to explore how different elements of practice interacted during the process of integrating the tool in two studies conducted at different points in time.

Teachers' professional development and opportunities for negotiating meaning are crucial for developing connections within the dynamic triangle for teaching statistics (Figure 1). However, the implications of the two studies reveal that for the technological tool to be effectively integrated into teaching practice, some prerequisites that enable the connection of the resource with the characteristics and methods of actual teaching are necessary. When the connection between the technological tool (resource) and the teaching characteristics (features) is hindered, integrating the tool becomes more challenging, with negative consequences on the other elements of the triangle, namely, support for learning potentials and adoption of innovative features. Some common factors that seem to influence these links are the school's technological infrastructure, available time, and opportunities for teacher collaboration.

Conclusion

The systematic examination of the two studies helped us to get insights into key issues that appear both important and systematic for the negotiation and integration of dynamic software tools in the teaching of statistics. The explicit and strong connection of the resource with learning potentials prompts us to question whether modern learning goals in statistics can truly be achieved without utilizing such technology. If not, we may need to reconceptualize what constitutes basic resources for contemporary statistics education. For Adler (2000), basic resources constitute the minimum prerequisites necessary for teaching to take place. This prompts the question: What should be considered basic resources for a school where teachers aim to use software like TinkerPlots with their students? Our study highlights that the main and enduring barriers preventing teachers from using dynamic software tools in their classroom seem to be the school's inadequate technological infrastructure, time constraints, and limited opportunities for meaningful human interaction and collaboration.

How can software tools like TinkerPlots be made accessible for all students? How much time is truly required to support open-ended statistical investigations in the classroom? And how can school programs better facilitate ongoing teacher collaboration and student inquiry? These are critical questions that must be addressed to ensure the effective integration of dynamic software in statistics

education. Furthermore, these questions underscore that the reconceptualization of what constitutes a basic resource for school teaching, both generally and specifically in statistics education, also opens important discussions about equal opportunities and the democratization of statistics education. Moreover, as the concept of data evolves in modern society from simple datasets to vast arrays of big data, it becomes imperative for statistics education to clearly define the fundamental resources required to support sustained statistical inquiry at the school level.

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How students justify the usability of self-built machine learning models

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Using machine learning models is a part of the everyday work of many engineers. In this paper, we investigate how undergraduate engineering students use self-built machine learning models in a concrete application example. A qualitative analysis of excerpts from twelve student groups shows that despite similar models, the tendency to use the models in the application example differs. In addition, justifications for the use of the model are provided on two different levels. One level covers the resilience of the model itself. The other level relates to the applicability of the model statement, which is what we call the information that is obtained by applying a model to the new data set.

Keywords: Machine Learning, Model Building, Model Usability, Data Literacy

Theoretical Background and research question

Machine learning (ML) is at the heart of modern engineering and, through the use of mainly statistical methods and automated data processing, an interesting learning content in the context of statistical and data literacy (Engel & Martignon, 2024). ML can be used to optimize complex systems, to predict failures for maintenance, or to accelerate research by data analysis. Besides the building and the evaluation of ML models, which can be outsourced to other experts if necessary, justifying the usability of a model in an application situation is crucial (Bertolini et al., 2021).

The question of justifying the usability of a model in an application situation is closely related to the user reliance on recommendations from ML models. Due to the increasing use of ML-based methods using the label "AI", some relevant publications about reliance on recommendations from ML models also appear with the label "trust in AI" (Bach et al., 2022; Yang & Wibowo, 2022). "Trust in AI" is investigated in the context of various influencing factors on technological, organizational, contextual, social and personal level (Yang & Wibowo, 2022). Looking specifically at ML-based decisions, there is empirical evidence for different types of influencing factors related to the different levels. For example, on the technological level influencing factors can be the properties of a model, measured by different performance metrics (Yu et al., 2019; Yin et al., 2019). On a personal level, the difference between one's own ideas of a model and the model itself seems to be an influencing factor (Bansal et al., 2019). There is also the assumption that personal preferences and expertise regarding the ML-methods or a respective performance indicator have an influence on the reliance in ML-based decisions of users, i.e. laypersons related to model building (Chiang & Yin, 2022; Yin et al., 2019). Another interesting aspect is that trust in the ML-based decisions can be influenced by an appropriate visual representation of the ML-based decision process (Yang et al., 2020).

Our contribution enriches the field of research on user reliance on a recommendation of a ML model with a perspective that deals with a step prior to the recommendation: The usability of a ML model in a specific situation. The question is: *How do engineering students justify the usability of a self-built ML model?*

The situation in which we investigate students' justifications is characterized by two particular aspects: The investigation is based on the processing of a task in a teaching-learning environment in the context of a design research study (Bata et al., 2022). The task, which is described in the methods section, is an example of an engineering application that is intended to represent a situation from professional life, where students apply a ML model to a new data set. Furthermore, the students in our study use models that they have previously developed themselves, which means that they are explicitly not laypersons but beginners related to model building.

Method

Task

In the investigated task, the students have to apply a model to a new data set. The model was developed by themselves beforehand using the k-Nearest Neighbor classifier to predict steel quality (good quality / bad quality) based on two metric features (number of cracks / width of the biggest crack). In Figure 1 there is a visual representation of the models' prediction for the new dataset (containing samples of old and new steel): In the data set, half of the samples consist of old steel (white), and the other half of new steel (black). The background colors separate the areas classified by the model as good (blue) and as bad (red). The representation shows that all samples of the old steel are classified as of poor quality, and about 80% of the samples of new steel are classified as of good quality. Based on the application context and a visual representation such as in Figure 1, the students are asked to make a recommendation for the new steel in contrast to the old.

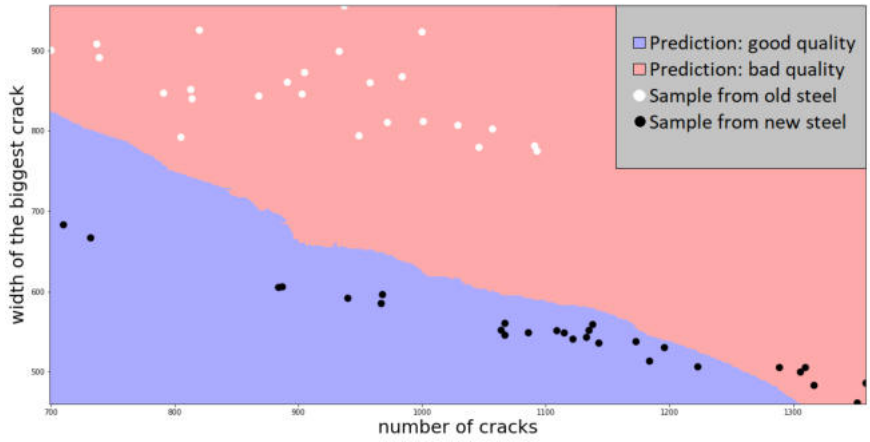

Figure 1: Visualization of a student-built model and the new data set as foundation for the decision

Data from the design experiments

To address the research question, we collected data from three cycles of design experiments in a laboratory setting (Gravemeijer & Cobb, 2006) involving twelve groups with a total of 29

participants. The participants were students from different engineering bachelor programs in the third to seventh semester at a German university, with no previous experience in machine learning. The different bachelor programs had minor discrepancies in the computer science and programming curricula, so a crash course in Python was given to some students before the introductory course started. This was useful in preparing the students for the programming tasks in the introductory course, which were completed using Jupyter Notebooks with Python.

The design experiments were conducted as follows: The twelve student groups were guided individually through the material by the lecturer, which is on top the researcher herself, via an online conference tool in three sessions each (Bata et al., 2022). The work on the material includes both presentation phases by the lecturer and group work on various tasks, one of which is the one described above. The three cycles differed only minimally in their prerequisites, the formulation of the task stayed the same. Each session of each group was documented by video recording.

Qualitative content analysis

The video recordings from the design experiments were transcribed in order to conduct a qualitative content analysis (Kuckartz, 2014). In the course of the qualitative content analysis we developed an inductive category system (Table 1), which was validated by means of consensual coding and by determining the intercoder reliability (Kuckartz, 2014). The intercoder reliabilities, which were calculated on a subset of the material coded by another researcher, of the three main categories range from 0.82 to 1. The value of one is achieved here because the subcategories of the first main category leave little to be interpreted. Taking into account the number of subcategories, the values correspond to a Cohen's kappa between 0.77 and 1 which means a good to very good agreement.

Results

The path from one example to the category system

We will start by showing an excerpt from a transcript of group 2, consisting of the students with pseudonyms Hanna, Wael and Bastian. They are talking about a recommendation looking at the visualization in Figure 1. Wael's statement is an answer to the question "Would you make a recommendation at this time?"

329	Wael:	So I would say, purely intuitively, no, we need much more training data (laughs). But maybe that's also my tic, I don't know.
330	Hanna:	But let's assume that we should now decide on the basis of what we have. Or we don't have no further budget available, then I would say, after 18,000 load changes, the steel, the new steel, is just still good in contrast to the old steel.
331	Bastian:	I think that because the result is so clear it is easier to make a recommendation than if it was a bit scarcer.

In this excerpt the students' decision to use the model for a recommendation is not clear. Two passages are coded regarding their decision, one in [329], coded by "Model is not good enough for recommendation" (C1.3) and the other in [330], coded by "New steel lasts longer / is better" (C1.2). Both codes are part of the first main category "Decision" (C1). This main category distinguishes

whether the students want to make a recommendation at all and if so, in favor or against the new steel. This category serves to frame the further coding with categories C2 and C3 (see Table 1).

Furthermore, we observe that the students provide justifications at two different levels. On the one hand, they give reasons related to the model itself like Wael in [329]. He says that he would like to have much more data to train the model. By mentioning the "Amount of data" (C2.6) he gives an aspect related to the second main category "Model resilience" (C2). Table 1 shows this and the other subcategories of C2. E.g. Students justify with traceability (C2.1), or they state whether their model is good or bad by using some performance metrics (C2.5). On the other hand, the students use justifications that are related to the "Usability of the model statement" (C3). A model statement is the information obtained by applying the existing model to the new data set, e.g. by looking into Figure 1 (all samples of the old steel are classified as of poor quality, and about 80% of the samples of new steel are classified as of good quality). An example for this is, when Bastian points out in [331] that the recommendation using the model is easier because the statement is very clear. This passage is coded as "Clarity of model statements" (C3.2). Table 1 shows the other subcategories of C3. E.g. Students justify with applicability or transferability (C3.3).

Table 1: Category system

Main Category	Subcategory
C1: Decision	C1.1 New steel is not good enough
	C1.2 New steel lasts longer / is better
	C1.3 Model is not good enough for recommendation
C2: Model resilience	C2.1 Clear / Traceable statement
	C2.2 Reality mapping
	C2.3 Parameter selection
	C2.4 Variance and bias
	C2.5 Performance metrics
	C2.6 Amount of data
	C2.7 CRISP-DM
C3: Applicability of the model statements	C3.1 Model quality
	C3.2 Clarity of model statements
	C3.3 Applicability / Transferability
	C3.4 Significance of data / features
	C3.5 Business perspective

Some examples of different subcategories

To illustrate further subcategories, excerpts from the transcripts of two other groups are shown below. First, we look at an excerpt from group 3 with the students Feline and Greta:

182	Lecturer:	[…] Then I have one more question. You wrote that you have a “reliable model”. If somehow, I'll say, something goes wrong in the process at the end. So now you've said we're using the new steel. And it's being used and then something goes wrong. (Greta: Hm (affirmative)). How would you defend that our model was reliable?
183	Feline:	(11) With the uhm criteria. So (..) we were given a data set on the basis of which we were supposed to build a model. And we tried to adapt the model so that the performance indicators were as good as possible and, above all, that the precision was as high as possible. And I would say that you could certainly defend it with that.

In this excerpt, a subcategory of C2 “Model resilience” is coded, namely C2.5 "Performance metrics" in Feline's statement in [183]. Feline explains that the group created the model beforehand with the aim of maximizing the performance metric values, especially the precision. Precision is emphasized here because in the application context, it places work safety above profit, as the students' considerations at another point in the processing show. Another thing, that this excerpt shows, is the lecturer giving a stimulus to motivate the students to give more justifications [182]. Impulses of this kind are discussed later in the paper.

The second excerpt is from group 12 with the students Dieter, Enno and Fabian:

122	Fabian:	[…] Yes, it just depends on the first question, what exactly they want, right? If there are almost no load changes with the steel and it's just permanently installed somewhere. (Enno: Hm (affirmative)) maybe it’s not worth it.
123	Enno:	Or does the company actually want the steel to be replaced as often as possible so that they can sell more parts?

The excerpt shows two subcategories of category 3 “Applicability of the model statements”, namely the two that are particularly broad and cover many different aspects. In Fabian's statement [122] he notes that the given data generated by the simulation of load changes on test components may not be meaningful for the application of the new steel. This formulates the aspect that the data used to build a model or to which a model is to be applied must fulfil certain requirements in order to be meaningful. The statement is therefore coded in category subcategory C3.3 "Applicability / Transferability”. Enno's subsequent statement goes even further than Fabian’s and questions whether the goal of material selection is high quality at all. The associated statement [123] should therefore be categorized as C3.5 "Business perspective".

Both examples highlight the variance in students' justifications, which could already be anticipated by the category system (Table 1): The justifications are at different levels (C2 “Model resilience” & C3 “Applicability of the model statements“) and within the levels concerning diverse topics (e.g.

technical things like performance metrics or contextual things like the business perspective). The next section provides an overview of the results of the coding across all groups.

Summary on the twelve groups

Although all groups start with slightly different models as a result of working with their self-built models, the model statement was mainly similar (most of the old steels as of bad quality and most new steels as of good quality, see Figure 2). The students tend to make a recommendation (C1.3 in only four groups) and when they do, it is more likely to be in favor of the new steel (C1.1 in only four groups), which is indeed the aim of the task. The groups that disagree about using the model or recommending the new steel can be identified by different codes in category 1 (as in the example shown). The disagreeing groups tend to use more and different subcategories in the other two main categories than the agreeing groups. Evaluating all passages coded with the main category 2, we see that not all groups use the resilience of the model (C2) in their justification. Some groups are not even able to do so when asked. A different picture emerges for category 3: The students show a wide range in their ability to justify the usability of the model statement (C3). Each group justifies at least once at the level of main category 3.

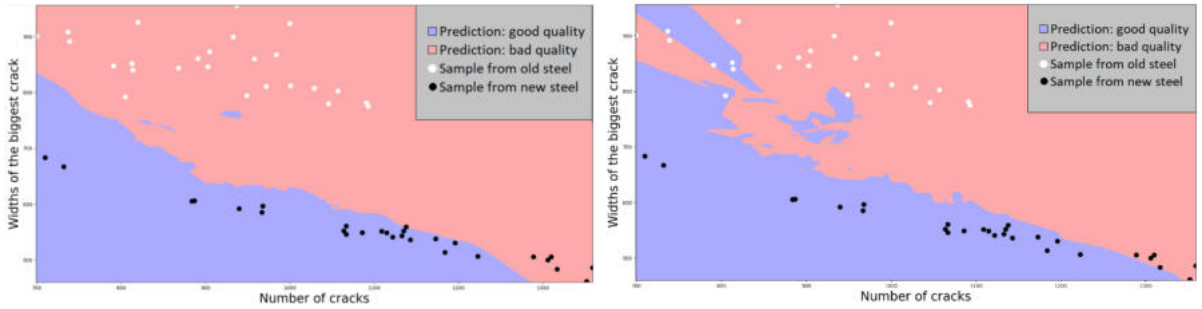

Figure 2: Two different models (group 5 & group 12)

Discussion

The developed category system (Table 1) opens the possibility to identify and analyze the students' justifications for the usability of ML models in an application context. It is interesting to note that although the question is about the application of an existing model to a new data set, the students argue at the C2 level ("Model resilience") in addition to the expected C3 level (**"**Applicability of the model statements"). However, this result is consistent with the existing literature, as the technical characteristics of the models, such as the specification of performance measures, increase the reliance on the model recommendations (Yu et al., 2019; Yin et al., 2019). This is true even when the modelling process has not been carried out by the users themselves in advance, which is the case in the present study. In contrast some groups are not able to argue at C2 level at all even though the lecturer has used a prescribed stimulus (as in the example from group 3) to motivate students to give answers from the category if they have not already done so on their own. Here it would be interesting to understand whether the partial lack of justification is due to the concrete task or to other reasons that lie in the model building processes beforehand (Bata et al., 2024) or in some basic understanding of ML. Despite the partial lack at C2 level, the results show a broad and diverse use of justifications, which is desirable when dealing with recommendations from ML models. This raises the question of

whether it might be useful to involve laypeople in modelling processes or at least to inform them about modelling processes in order to improve their reliance on the model results. This question, or assumption, is in line with Chiang & Yin's (2022) findings that short-term machine learning literacy interventions have an impact on laypeople's interaction with ML models and their outcomes.

Although only twelve groups with 29 students were studied to develop the category system, it is reasonable to assume that the subcategories of categories C2 and C3 are complete and transferable to other applications of supervised learning. This can be assessed from a technical point of view for main category 2, since the usual influences (data, parameters, structured generation) and evaluation parameters (performance metrics, variance and bias, comprehensibility and reality mapping) of supervised learning methods are represented. With regard to the main category 3, usual evaluation parameters of the application of ML models are represented. This is also due to the fact that the categories C3.3 and C3.5 are very broad and cover many different aspects. For example, in subcategory C3.5, the sections vary from “a steel has too long delivery times” to “what is the actual objective of the ordering company”. An investigation of the assumption of completeness in a different setting would be a desirable follow-up study and could expand the picture if necessary.

Although the subcategories of categories C2 and C3 can be assumed to be complete, looking at the differentiation of Yang & Wibowo (2022), there could be other main categories to justify the usability of a ML model. The justifications covered by the category system could, following Yang & Wibowo (2022) be sorted into the "technological" and "contextual" factors. It might be interesting to search intensively for further main categories that analyze the other formulated influences such as personal factors. However, it is quite probable that the learning situation or research format described above is not suitable for such an investigation.

From a design research perspective, there are three further considerations. First, there is the question of how to enable all students to argue not only at C3 level (“Applicability of the model statements”) but also at C2 level (“Model resilience”). Secondly, one could look more closely at whether one can artificially create the disagreement that, at least in the present sample, seems to lead to the use of more and different subcategories. The final consideration, also in relation to the results of Yang et al. (2020), concerns the way in which the model is visualized with the new data set (Figure 1). The question remains as to whether the visualization is suitable for providing students with a basis for justifying the usability of the model and, building on this, whether the type of visualization influences the students' justification.

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A classroom experiment: the impact of teaching correlation and regression on students' knowledge of the equation of a line

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We present the results of a classroom experiment on teaching correlation and regression with 114 students and 6 teachers. The experiment sought to answer the question of how a 6-lesson course in correlation and regression helps students to better understand the equation of a line and to interpret slope. We have seen a general improvement, especially in tasks where the anticipated slope conceptualizations were geometric ratio or functional property. We also found that weaker students were more helped by being exposed to a new perspective on the equation of line. The feedback confirmed the importance for students of a topic that they could easily relate to everyday life.

Keywords: Experimental teaching, correlation and regression, equation of a line, slope of a line.

Introduction and previous research

In the Hungarian National Curriculum (NAT), there is currently no applicable purpose for the equation of a line (NAT). Students are only expected to know the formula $y = ax + b$. In our wider research, we are studying correlation and regression analysis as a possible secondary school application of the equation of a line, which could show students an understandable, everyday application. Previously, we examined the curricula of some European countries to see which approach fits best with the goals of the NAT (Boller, 2024). This research is the basis for the experiment that we present.

We were mainly interested in the impact of learning correlation and regression on students' knowledge of the equation of a line (Research question 1). Our second research question was about the extent to which this topic is accessible to students. The theoretical background for the main question is provided by Zoltán Dienes' perceptual diversity principle, which states that conceptual learning is more effective when learners meet the concept in different contexts and approaches (Dienes, 2015). In our case, we would like to approach the equation of a line from a new perspective of correlation and regression analysis.

Thesis point: "Although students can solve simple related problems, they have only a surface knowledge of the equation of a line, its applicability, the meaning of slope and intercept.

Hypothesis: By teaching correlation and regression analysis, students' understanding of the equation of a line will improve.

Several studies have been carried out on the teaching of correlation and regression analysis. Biehler et al. summarized the major research on reasoning about data. They mention correlation and regression from the perspective of covariational thinking and identify as a major challenge that data-based decision making is not a basic assumption for students (Biehler, Frischemeier, Reading & Shaughnessy, 2018) Gea and colleagues also have described, in the context of covariational thinking,

that the teaching of correlation and regression analysis is under-researched, although it is of great importance (Gea, Batanero, Lopez-Martin & Arteaga, 2016). It was found that some of the biases and misconceptions about the subject are resistant to traditional teaching methods, and the authors therefore proposed a methodological renewal of the subject. Even earlier Walker emphasized the importance of simulations in teaching regression analysis (Walker, 2004). He noted the benefits of simulations in improving understanding of concepts and encouraged interactions between students. As a result, simulations and group work using self-reported data can be found in the course we have developed. Further help was provided by the important research of Engel and Sedlmeier, who reviewed common errors and misconceptions about the concepts of correlation and regression and made four recommendations to avoid them (Engel & Sedlmeier, 2011). Their research shows that reasoning about correlation and regression is not always easy for the layperson, and even researchers sometimes struggle to interpret it correctly. Engel and Sedlmeier suggest helpful approaches to the subject, such as using real data and incorporating modelling experience and technology.

Among the research on the knowledge of the equation of a line, the literature on the interpretation and understanding of slope is significant. Research suggests there are 11 ways that slope can be conceptualized (Moore-Russo, Conner & Rugg, 2011). Linking them is not easy, as it was much more difficult for students to read slope from a graph in Birgin's study than from the algebraic form (Birgin, 2012). And based on the research of DeJarnette et al., the fact that students can quantify the value of slope does not mean that they can identify slope as a property of a straight line (DeJarnette, McMahon & Hord, 2020). In addition to the conceptualisation of slope, context also influences understanding. Planinic et al. showed that it is particularly difficult for students to interpret slope in context, and this is not due to a lack of mathematical knowledge (Planinic, Milin-Sipus, Katic, Susac & Ivanjek, 2012). Studies of university students and pre-service teachers have shown that they do not have an easy time understanding slope in a physical context (Styers, Nagle & Moore-Russo, 2020; Cho & Nagle, 2017).

Algebra and statistics have already been linked in the important research of Casey and Nagle, who showed that students' conceptualizations of slope play a significant role in their reasoning about the line of best fit (Casey & Nagle, 2017). In their study, they carefully investigate how reasoning about slope and the position of the line of best-fit changes in a more accurate way in the case of two students. Their results are important for practicing teachers and for us, as they have shown in detail that with the development of covariational reasoning, the functional and real-world conceptualization of slope has also improved (Casey & Nagle, 2017).

Methods and implementation

114 students participated in the research, with the help of 6 teachers from different areas of Hungary. Most of the students were in grade 11 (16-17-year-olds), except for one group, who were in grade 10.

As the teachers participating in the experiment had not yet studied correlation and regression, we organised a one-day training course on the theoretical background, calculation methods and limits of correlation and regression. The structure of the course was in line with that previously described by Batanero (Batanero, Gea, Díaz, & Canadas, 2014). Their results suggested that teachers also have problems distinguishing between correlation and causation, but the activities they organised helped to develop teachers' knowledge of teaching correlation and regression and enabled them to find out,

for example, if a child draws the wrong conclusion from correlations. This also proved useful for the teachers in our experiment. In our course, we also described in detail the structure of the 6-lesson course and the individual lesson plans.

The experiment had three parts: in the preparation phase, the students wrote the anonymous pre-test, collecting information about their current knowledge of the equation of the line. One week later, the most active phase for the participants followed, when the focus of the lessons was on correlation and regression analysis for 6 lessons. Approximately 2 weeks after the active phase, the students wrote the post-test, also anonymous, in the final phase of the research. This test contained very similar tasks to the pre-test. At the end of the experiment, we received detailed written feedback from the participating teachers on the attitude of students and teachers, the methodology of each lesson and the use of the computing background. The students reported back to us in a student feedback form.

The pre-test was revised based on a previous pilot measure, resulting in the final form (Figure 1). The test consisted of 5 tasks. Tasks 1, 2 and 3, shown in Figure 1.A, measured basic subject knowledge, i.e. whether the student knew the formula and was aware of the basic meaning of the equation of a line (the slope is obtained by looking at how much the function moves along the y-axis while moving in the positive direction by unity on the x-axis). The second part of the pre-test assumed a deep (dynamic) knowledge of the equation of the line (Figure 1.B). This was measured through tasks that required an understanding of the constancy of the slope, and which required a degree of more complex thinking. The pre-test and post-test were both 5-point tests, with each task worth 1 point.

1.A) Task 1 – Find the equation of the line through the points $(-1;\ 3)$ and $(3;\ 11)$.

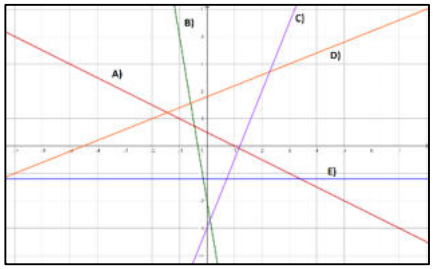
Task 2 – There is a line with the equation $y = -4x + 2$.

a) Does the line pass through the following point? $P = (-1;\ 6)$

b) The point $C = (x_c;\ -22)$ lies on the line. What is x_c?

Task 3 - Write on the line which line has which slope value.

Slope values: $10, 5, \frac{5}{2}, 2, \frac{1}{2}, 5, \frac{1}{4}, 0, -\frac{1}{3}, -\frac{1}{2}, -4, -6$



1.B) Task 4 - Answer the following questions! Give reasons for your answer. Line e has slope 12 and a point $A(27;\ 18)$.. Find the missing coordinates of the points $P(30;\ y_P)$ and $Q(x_Q;\ 14)$ that lie on the line without calculating the equation of the line.

Task 5 - The points $A(-3;\ y_A)$ and $B(3;\ y_B)$ are the points of the line f. What can we say about the slope of f if we know that the second coordinate differs by 1?

Figure 1. – The pre-test. Part 1.A) contains the basic tasks and part 1.B) the more difficult tasks

In terms of methods, we make two comments. First, we introduce the concept of average relative growth, which is used several times in the evaluation. For each student, the change in score on the pre- and post-tests is normalised by the score on the pre-test. This gives a relative improvement, which considers that, for example, a change of 0.2 points does not mean the same for a baseline score of 0 or 0.8. For each task, we calculated the relative growth of the student. The scores for the five

tasks were then averaged to give the average relative growth of a student. The average relative growth of the group was calculated from the scores of the students in the group (Figure 3.). We then looked at the relative growth in each task. And we averaged these values separately for each task, so we got the average relative growth of the students in a task (Figure 4.). The second comment concerns the drop-out rate of the participants in the experiment. The mortality is 11.4%. There were only two groups where the drop-out students scored on average better than the group average on the pre-test, so drop-out probably did not have a significant effect on the result.

The correlation and regression analysis course consisted of 6 lessons. The learning objectives of the topic were given by the conclusion of our previous research. We formulated goals that are in line with the objectives of the NAT. The lesson plans have been developed in a very detailed way so that the teachers can teach the topic in a similar way, exactly as described, and so that the students participating in the experiment receive the same kind of training as much as possible. The topic and the knowledge of the lessons are shown in Figure 2.

Title of the lesson	Knowledge (concepts, principles)
1. Introduction	Visualization of the topic. Relationships between variables (correlation vs. causality).
2. Correlation coefficient	Meaning and definition of correlation coefficient. Determination of different levels of correlation (pairing 'r' values with scatter plots).
3. The regression line	Illustration and purpose of linear regression. Drawing the estimated line by eye.
4. Theoretical background of fitting the line	Theoretical background of linear regression: least squares method (using computer software) Making predictions based on a regression line.
5-6. Application of the regression line in problems	Calculating regression line and correlation coefficient from own data (using computer software to forecast) Interpreting the results obtained by group. Exercise, summarise.

Figure 2 – The topic and the knowledge of the 6 lessons

A lot of basic knowledge was required during the course. The properties of the linear function were strongly relied upon, and the slope and intercept of the line were discussed in many exercises. For the prepared worksheets and lesson activities, we sourced textbooks from the Netherlands, Ireland and Luxembourg, based on our previous research experience, as well as the New Zealand IXL Learning site (www.ixl.com) (Boller, 2024). At the end of the process, students completed a feedback form where, in addition to their basic impressions of the topic, they were asked to rank how useful they found the topic and were also asked what prior knowledge they had to use most in lessons.

Findings

The first striking observation was an outlier group, shown in Figure 3. Most of the students in the experiment performed better on the post-test than on the pre-test, except for one group. The average relative growth of students in the higher-level group was -2%. The reason for this was found to be that members of the group had already scored very highly on the pre-test, with essentially no room

for improvement. In their case, therefore, it did not make sense to look at added value, so their data were not included in the rest of the analysis. The other students in the experiment scored significantly better on the post-test than on the pre-test ($\alpha = 0.05, \mathrm{F-and\,t-test}$).

Average relative growth in the participating groups					
A – grade 11 basic level	21%	D – grade 11 basic level	22%	G – grade 11 higher level	−2%
B – grade 11 basic level	10%	E – grade 11 basic level	9%		
C – grade 10 basic level	10%	F – grade 11 basic level	15%		

Figure 3 – Average relative growth in the participating groups

There was also an observed general improvement by tasks (Figure 4). Here we saw that the standard deviations of the mean of the pre and post tests are the same, with students performing on average $4 - 13\%$ better on each task. Despite our expectations, we observed an average growth of more than 10% in Task 2. This shows that in the post-test, students could calculate the corresponding x coordinate for a given y coordinate more skilfully. The smallest improvement was observed in Task 3. The reason for this is identified as the high input average of the task. The highest relative growth was achieved in task 4. In this task, the missing coordinates of the points of a line with a given slope had to be calculated without writing down the equation of the line, which requires a deep knowledge of the slope of the line. For the fifth task, a low 7% progress was observed, where the difficult wording of the task may have caused a more modest progress than expected.

The greatest improvement was in tasks 2 and 4, one of which we consider to be the basic tasks and the other the more difficult tasks requiring deeper understanding. What the two tasks have in common, however, is that the slope can be interpreted as a geometric ratio or a functional property.

required skills	mean of pre-test (standard deviation)	mean of post-test (standard deviation)	average rel. growth	Anticipated Slope Conceptualizations
Task 1: substitution into equation	0,34 (0,46)	0,37 (0,44)	7%	Algebraic ratio, Parametric coefficient
Task 2: fitting a point to a line	0,57 (0,49)	0,66 (0,47)	12%	Algebraic ratio, **Geometric ratio, Functional property**
Task 3: Pairing slope values with a graph	0,73 (0,30)	0,77 (0,29)	6%	Algebraic ratio, Physical property, Functional property, Behavior indicator

Task 4: calculate the missing coordinates of points on a line	0,39 (0,47)	0,46 (0,48)	13%	Algebraic ratio, **Geometric ratio, Functional property**
Task 5: determine the slope of a line from points with a given property	0,29 (0,40)	0,35 (0,41)	7%	Algebraic ratio, Geometric ratio, Physical property

Figure 4 – Skills expected in the input and output test tasks, average score and average relative growth

Another important finding of the experiment was the relationship between students' performance and their relative growth. Our sample shows a solid but weak correlation ($r \approx -0.32$). The sign of the slope of the fitted line was found to be significantly negative (α=0.05) by statistical methods, F - and t-test. This leads us to conclude that the weaker students made better improvements during the experiment than those who were more skilful in the beginning.

Another mentioned benefit was the introduction of the point cloud diagram in the mathematics lesson and the fact that standard deviation, which had previously had a poor fate, was now given a valuable place with regression analysis. Almost everyone considered that the use of computer software in class could be improved. Several teachers indicated that an extra lesson had to be added because of a lack of knowledge of Excel on the part of the students. Preparation for the lessons generally took a lot of time. As one colleague summarised:

"Overall, it was a very good insight into a very different world from what teachers and students think about mathematics. I love this game (reference to course - author's note) *and when I get the chance, I will definitely bring it to the next groups!"*

The students' feedback was also sought after the experiment. 74.7% of the participants gave a rating of 3 or more on a scale of 5 for how useful they found the topic. Students liked the lesson on correlation and causality the most (57%). Knowledge of the coordinate system (82.3%) and the equation of a line (75.9%) were the most important prior knowledge.

Conclusion and discussion

In our experiment, we investigated the impact of teaching correlation and regression on students' knowledge of the equation of a line. Our experiment supports Nagle and Casey's findings (2017) that the development of real-world examples and covariational thinking has had a positive impact on knowledge of slope (more broadly, the equation of a line). Inspired by their findings from personal task interviews, similar improvements were obtained in a classroom setting with a larger sample of students. Nevertheless, given the many levels of slope and covariational thinking, it is worth further exploring the connections between slope and covariational thinking.

We started from the assumption that students have only a surface knowledge of the equation of a line and the meaning of slope and intercept. From the low scores on the input measures, we saw that in many cases even the simpler problems were difficult to solve. In response to the research question, compared to our expectations, learning correlation and regression not only improved the

understanding of the equation among the participating students, but also led to a higher proportion of students solving the simpler problems correctly. Students performed significantly better on the post-test than on the pre-test. This was largely due to the types of tasks where the coordinates of a point on a straight line had to be calculated. These problems were also similar in that they could be solved with the same slope conceptualizations. We assume that this is related to the conceptualization of slope as a functional property, because in the 6-lessons course, students looked at the regression line as a function and often gave meaning to the slope in each context. We found that lower performing students made more progress during the new topic. Their relative growth was found to be significantly better than those who performed better on the pre-test. In addition to their higher development potential, and the fact that they have an even easier time formulating schemes of knowledge about the equation of a line may also play a role. This could be the subject of another research project.

Based on the experience of teachers and students who participated in the experiment, it would be desirable to place the correlation and regression analysis in an earlier grade. Contrary to expectations, much less knowledge of coordinate geometry was needed in the lessons. Another argument pro earlier publication is the introduction of statistics with standard deviation in grade 10, which may be worth adding. This is reinforced by the role of the higher-level group G in our experiment, where the group average did not improve because of the topic. This decision also includes the role of coordinate geometry, because even if not the concrete knowledge material, the perspective can add a lot to the understanding and the general mathematical skills that result from the maturity should be considered.

The topic of the experiment could be the subject of further research. It may be worth repeating the experiment on a sample 10th grade students with a control group. In the next period, we test our assumption that the teaching of correlation and regression has helped us to understand the slope as a functional property. To do this, we will identify general student misconceptions and model errors and carefully examine student responses to see which slope conceptualization they preferred.

The possible introduction of the topic in the curriculum will also require a rethink of teacher training, since, as the teachers wrote in the reflection, it is a very different world from the one they are used to. However, our experience has shown that the subject has great potential and a lot of interest, is accessible to students and, most importantly: Finally, it is easy to relate to everyday life and helps to make sense of real problems and today's issues, so it is worth engaging with.

Acknowledgment

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Risk literacy – a hybrid conceptualization

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Given the current socio-ecological situation, marked by overlapping crises, the development of risk literacy is a relevant but under-researched topic in mathematics education. An appropriate conceptualization is lacking. We address this gap by elaborating a hybrid conceptualization of risk literacy for stochastics education in school. The concept combines mathematical risk competency with students' perception of risk, thereby combining a realist and a socio-constructivist approach. To illustrate this hybrid conceptualization of risk literacy, we draw upon examples from a design-based research study aimed at fostering risk literacy in stochastic lessons. These examples serve to demonstrate the usefulness of our conceptualization related to our task design. This is done by the method of empirically reconstructing students' interpretations of risk by tracking the social praxis of Game of Giving and Asking for Reasons (GoGAR) in the discourse of a situation of uncertainty.

Keywords: Risk, literacy, competency, instructional research, Inferentialism.

Introduction

The pandemic, climate change and other crises make it clear that citizens need to make well-informed decisions in risk situations that go beyond everyday scenarios. This emphasizes the significance of cultivating students' risk literacy from an early stage in their education (Pratt et al., 2011), but clear concepts of risk and risk literacy have not yet been elaborated in mathematics education (Eichler & Vogel, 2015). If we base risk literacy on Gal's (2002) model of statistical literacy, two components are crucial: statistical/mathematical knowledge and individual dispositions, such as attitudes or beliefs. Pratt et al.' (2011) confirm the relevance of these components in their educational study. They add that reasoning about risk as well as individual and social interpretations of risks are also relevant when risk literacy is to be fostered. However, fostering risk literacy has rarely been considered in mathematics education. Attention has rather been dedicated to risk competence and accessible visualizations of risk (Martignon & Hoffrage, 2019). Furthermore, the topic of risk is not prominently addressed in mathematics curricula (Burrill & Pfannkuch, 2024; Radakovic, 2015). As a consequence, the question of how to foster risk literacy in mathematics education at school remains an under-researched area, particularly for secondary level. To address these gaps, we conducted a design-based research study, named Risk-Design. The study focused on instructional design for developing risk literacy in stochastic lessons at grades 9 and 10. This article deals with a theoretical issue of the study; we develop conceptualizations of risk as a topic for stochastic lessons at school and of risk literacy to be fostered in these lessons. This will answer the first research question (*RQ1): What are the key aspects of risk literacy for stochastics education at school?* We are then going to show that our theorizing is empirically useful by providing two examples from the design study that demonstrate how we use the concepts of risk and risk literacy in our analyses to answer the second research question (*RQ2): How do students navigate risk, and how do they perceive risk thereby?* As this article makes mainly a theoretical contribution, we outline our methodological approach restricted to the analyses of the two examples. Finally, we discuss and reflect our results critically.

Theoretical framework

To elaborate risk literacy, we first conceptualize the term risk for stochastic lessons at secondary school level. This will result in a hybrid conceptualization, in which the two theoretical approaches, realism and socio-constructivism, which are not commensurable, are combined. Based on that, we will map out risk literacy as a hybrid conceptualization of the two approaches as well.

Conceptualization of risk

Although risk is considered in many disciplines, a clear definition of risk is missing in mathematics education (Eichler & Vogel, 2015). Hansson (2010) distinguishes between two perspectives on risk: a realist and a socio-constructivist perspective. The realist perspective assumes an objective world independent of individual and social interpretations. In this view, risk is modelled by probabilistic and statistical means, allowing risk to be treated as a measure of hazard that may help to mitigate risks objectively, for example using statistical data to minimize risk for a health treatment (see Pratt et al., 2011). However, risk decisions cannot solely be based on coordinating risky impacts with likelihood; they also need to encompass the complex conditions of people's lives and their perceptions and interpretations of the risks they meet. As demonstrated by the mathematician Fry (Medeiros, 2023), individuals facing cancer may opt for additional treatments with harmful side effects, even when the probability of positive outcomes is very low. It is the interpretation of risk as a *value-laden harmful event* that makes them decide for the treatment rather than against it based on the very small probability of a positive effect. While certain risks, such as nuclear power, are seen by people as unpredictable, other risks, such as cycling on busy roads, are accepted because risk-control seems possible. Such perceptions of risks cannot be captured by objective measures but rather by reconstructing people's interpretations of risk as it is conducted in socio-constructivist approaches. Hansson (2010), and Hansen and Hammann (2017) argue that both perspectives need to be combined to describe and understand people's handling of risks appropriately. Aven (2023) proposes to define risk as a situation with inherent uncertainty (U) about unwanted consequences (C), briefly (U, C), and distinguishes this from a measure that models risk. With this clarification in mind, we characterize risk as a cluster concept made up of five notions of risk. Related to a socio-constructivist approach, risk can be interpreted by people as a (1) value-laden uncertain and unwanted event, or (2) the cause of it (e.g., pathological gambling leading to financial loss). Related to a realist approach, risk can be understood as measurable, e.g., by (3) the probability of the occurrence of an unwanted event or (4) an expected value, namely, the sum of products of losses and their probabilities, and (5) the spread of a distribution. This cluster concept is particularly helpful for stochastics lessons at secondary school level because the different notions of risk can link the topic of risk to curricular requirements as well as providing space for the students' individual interpretations of risk situations that we have addressed in tasks. If necessary, this concept can be extended to include further risk measures.

Conceptualizing risk literacy as an aim in school

As part of *statistical literacy*, Gal (2012) describes probability literacy as the ability to access and handle "… probability-related information and ideas, in order to engage and effectively manage the demands of real-world roles and tasks involving uncertainty and risk." (p. 4). Here, managing tasks involving uncertainty and risks are crucial. According to Hansen and Hammann (2017), risk

competence is "... the abilities a person needs to address risk-related issues in a real-world setting" (p. 764), including the handling of risk-relevant issues in an individual's life. More concretely, Martignon and Hoffrage (2019, p. 31) explain how this handling can be conducted. For *risk-decision* situations, they distinguish four ways of *handling risks* that influence each other. These actions are not considered to be conducted in a specific order; instead, they clarify the ways in which risk-competent students should be able to handle risk decisions: they should (1) *identify risks*, (2) *analyze and model risks,* (3) *evaluate and compare different options,* and finally*,* (4) *take a decision*. It is not enough to practice these actions in just one context; various contexts should be included. This is considered by Niss and Højgaard's (2019) concept of *competency* from the KOM-project (i.e., *Co*mpetencies (KO) and the learning of *M*athematics (M)), which is the "insightful readiness to act appropriately in response to *a specific sort* of mathematical *challenge* in given situations." (p. 12, emphasis in original), here, situations involving uncertainty and risk. They distinguish three competency dimensions, that are necessary for developing competency: *the degree of coverage, the radius of actions*, and *the technical level*. "The degree of coverage of a competency is the extent to which all the aspects that define and characterize the competency are covered" (p. 21). We call these aspects facets of risk competency, encompassing the four actions introduced by Martignon and Hoffrage (2019), the cluster concept of risk, and all other risk-related mathematical aspects, such as risk-related representations and data handling. The radius of action means the variety of contexts or situations risk-competent people can address successfully. Finally, the technical level describes the "level of the mathematical concepts, results, theories, and methods" (Niss & Højgaard, 2019, p. 21), and risk reasoning that "the individual can bring to bear when exercising the competency" (p. 21). This definition of risk competency predominantly addresses risk from a *realist* perspective. Like Hansen and Hammann (2017) for science education, we also suggest adding a *socio-constructivist* perspective as a complement of risk competency to shape risk-literacy for stochastics education. For this, we use the term *risk perceptions* encompassing students' real-life experience. This hybrid view on risk literacy requires educational designs so that "personal experience and affective responses can be embraced rather than avoided" (Pratt et al., 2011, p. 344) in processes of risk reasoning.

In this elaboration, we answered the first research question by regarding risk literacy as an aim of stochastics education in school, that combines risk competency with risk perceptions, i.e., students' individual and social interpretations and handling of risks. This meets Gal's understanding of literacy and at the same time it allows linking the topic of risk to key concepts of stochastics curricula, i.e., probability, expected value, spread. In addition, the three dimensions of risk competency describe how students' risk competency develops. Fostering risk literacy means fostering risk competency and offering space for the students' risk perceptions by engaging critically and responsibly in risk reasoning and risk decision-making. Thereby, this approach takes up how students perceive, hence, value risk in context, socially and individually. In the next part we show how we have applied this concept in our design study and thereby provide a preliminary answer to the second research question.

Methodology and method

Our hybrid understanding of risk literacy is theoretically rooted in the concepts of statistical literacy, competency and risk-decision handling. We now substantiate our conceptualization empirically by two examples from our design study, beginning with methodological considerations. However, we

cannot describe the entire methodology of the design study, but rather elaborate a methodical account to make the examples accessible, by briefly providing the theoretical background for our focus on reasoning on risk decision-making, and, by a brief methodical description of our data analyses.

Investigating risk reasoning

Risk reasoning in situations of uncertainty, is the key to examine the students' risk notions and their handling of risk. For that we use the inferential approach based on Brandom (2000), which assumes that the meaning of a concept is determined by the ways the concept is used in collective discourses, which establish a *web of inferences of the concept with other concepts*. Brandom sees these collective discourses as an essential social practice (called the 'Game of Giving and Asking for Reasons' (GoGAR)), a process of social interaction towards mastering a concept by the "inferences it is involved in", i.e., by identifying "what follows from the applicability of a concept, and what it follows from." (p. 48). Notions of risk are stipulated in GoGAR related to the context and referring to other concepts. GoGAR is the process of reasoning, in which students may set up a claim, giving reasons, accepting or rejecting a claim while acting together in collaboration (Schindler & Seidouvy, 2019). By tracing inferences, GoGAR allows us to reconstruct processes of concept formation, hence, showing students' risk competency, their notions of risk and risk handling related to the situation.

Method

Our design research has two aims: developing an instructional design to foster risk literacy and theorize the results of the design experiments. The concept of risk literacy as well as GoGAR, both informed task design and data analyses. In the tasks, the students are asked to reason and make decisions in a situation of uncertainty to prompt GoGAR, activate risk competency, evoke risk notions and handle risks. Eight lessons (grades 9/10) were developed by four iterative cycles of revision (Brehm & Bikner-Ahsbahs, 2023). To capture collaborative discourses, video data were collected from two focus groups and the class discussions in each class. Data analyses were conducted in four steps. Step 1: identifying and summarizing episodes. Step 2: transcribing the relevant episodes. Step 3: sequential (turn-by-turn) analyses of the transcripts to *reconstruct* GoGAR, then coding terms of risk competency and the students' notions of risk, to finally build the web of inferences shown in the discourse. Step 4: interpreting/theorizing the results to answer the research questions.

Two empirical examples substantiating our theoretical elaboration

Two examples illustrate how we apply risk literacy and inferentialistic ideas in task design and data analyses (of step 3), how we interpret the results and thereby answer the second research question.

Example 1 (from lesson 1, grade 10): A tossing game (Figure 1)

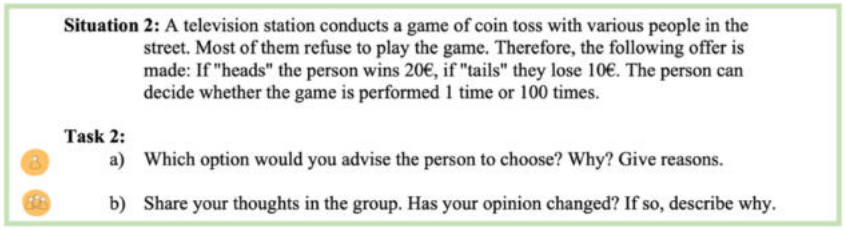

Figure 1: Heads or Tails – Task example 1 (Brehm & Bikner-Ahsbahs, 2023)

The students are familiar with the concept of *probability* experienced in game contexts. At stake in the task is the concept of the *expected value*, that is not yet introduced. The students may choose to play the game of coin tossing one time or 100 times. When heads are up, they win 20 € and if tails are up, they lose 10 €. The expected value is $\frac{1}{2} \cdot 20\,€ + \frac{1}{2} \cdot (-10\,€) = 5\,€$ per coin toss, which leads to an expected win of 500 € for a 100-fold coin toss in the long run.

23	Teacher:	What's riskier, doing it once or doing it 100 times?
24	Umut:	I thought that if I get only "tail", then I need 1000 € [to pay the loss].
25	Sina:	But I wanted to say that the risk is related to single play, then you only have the one-time chance, either you have it, or you don't have it.
26	Teacher:	Fritz and then finally Charlotte.
27	Fritz:	So, I would take one, because the probability is the same for both and with the second you can make more minus, but simply because (?)

As the students are unfamiliar with the Binomial Distribution, they reason informally, where we can observe *value-laden* relations expressed in their *risk notions*. E.g., Umut considers a *worst-case* scenario in the 100-times-play, making the 100 times play *feel riskier* (as he ignores the probability) (#24). Sina associates risk with the *unpredictability* of the one-off, stating, "either you have it or you don't have it" (#25). Implicitly, the risk seems to be more *incalculable* compared to the 100-times-play, in which winnings and losses can be equalized (#25). A similar decision is made by Fritz, but for a different reason, akin to the *worst-case* scenario (#27). Probability seems irrelevant to the decision, as the students can only grasp the probability of heads or tails in the Bernoulli experiment, hence, they state that it is the same in both options (#27). The *focus is the sheer possibility of losing more money* in the 100-times-play. When deciding which play is riskier, the students consider both, probability (#25, 27) and the amount of losses (#27), but coordinating both factors proves to be challenging, as also noted by Pratt et al. (2011). Some students adopt a risk-averse stance, focusing on *potential losses* (#24) and selecting a particular scenario, such as the *worst-case* scenario or the mere *possibility of losing* more money, associated with a particular *sense of high risk* (#27).

This example shows the value of our hybrid conceptualizing of risk literacy. Besides qualitative uses of probability (#25, 27), the students' perceptions of risk are expressed by three *notions of risk*: risk can be *the worst case*, the *possibility of losing much more money*, and unpredictable (*incalculability of risk*). The realist and the socio-constructivist perspectives are *interwoven*. Calculating risks did not appear by itself but needed to be more explicitly addressed in the tasks in the subsequent lessons.

Example 2 (from lesson 3, grade 10): Interpreting headings in the news (Figure 2)

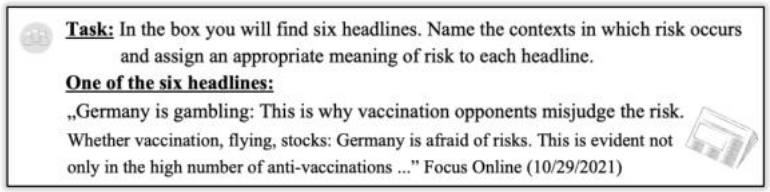

Figure 2: Risk addressed in a headline - Task example 2

The aim of the task (Figure 2) is to apply risk as a cluster concept thereby expand the radius of action and the technical competency level. Drawing on the methodical steps 3 and 4, we show that our conceptualizing of risk appears in the students' discourse, hence, turns out to be useful.

The example focuses on one of six risk contexts (sports, economics, health, workplace, society, climate change) using newspaper headings, thus contributing to expanding the radius of action. We expected two facets of risk competency be expressed: to *identify risks* through *analyzing* the context and *comparing* different risk situations. The following transcript illustrates how students master these facets of risk competency and the definition of the risk as a cluster concept.

1	Christin:	[…] I have cause, but I don't know if that's correct. Because the cause is that Germany is gambling away.
24	Christin:	[…] What do we do now with the headline?
25	Paul:	I have two [cause of an unwanted event].
28	Paul:	[…] "So Germany is gambling, that's why vaccination opponents are misjudging the risk". That is either the risk of contracting Corona or the risk of dying from Corona.
29	Christin:	Yeah. Sure.
30	Paul:	[…] Or the risk of dying from the vaccination, rather than from Corona, because after all, the risk of dying from Corona is much much bigger than dying from the vaccination.
31	Christin:	Ah no, wait, that's what's underneath it too. "Whether vaccinations, flies, stocks, Germany is afraid of risks". That shows, that's […] the unwanted event? Or cause. One of the two.
32	Paul:	Cause.
33	Christin:	Because Germany is afraid of risks. So risks are the cause, aren't they?
34	Paul:	Yes, we can say that.

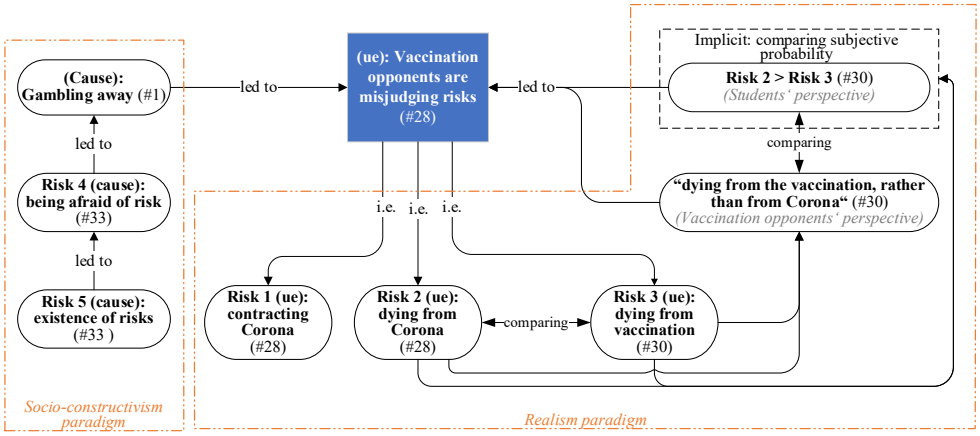

Figure 3: Web of inferences & facets (ue: unwanted event)

One student starts with "cause" as the claim (#1), found to be "Germany is gambling away" (see also Figure 3). The lack of resonance in the group leads to the *question* what to do (#24), *answered* by affirming the *claim* "cause" (#25) and then expressed with two claims (#28). "Germany is gambling away" is *identified* as the *cause* for vaccination opponents' misjudging the risk. This is clarified by *identifying* risks during GoGAR. Three kinds of risks as *unwanted and uncertain events* (ue) are *identified* contextually: "contracting Corona", "dying from Corona" (#28), "dying from the vaccination" (#30), comparing two of them in "dying from vaccination, rather than from Corona" (#30). This misjudging (of the vaccination opponents) demands further, more explicit *comparing* (#30), where risk takes the form of a *subjective probability* (a kind of implicit *modelling*) in "risk of dying from Corona is much much bigger than dying from the vaccination" (#30) as a reason that is given. These comparisons unpack the meaning of "misjudging" and as a consequence of "gambling

away" critically. Finally, being afraid of risks is *identified* as the *cause* underneath the unwanted events (#32), and by *comparing,* "risks" are *identified* as the deeper *cause* (hence risk) of being afraid of risks (#33) (see Figure 3).

In Example 2, the students master risk literacy by establishing a web-of-conceptual-inferences (Figure 3) joining both theoretical perspectives. This web displays two facets of risk competency that are activated: students handle risk by (1) *identifying*, (2) *analyzing and modelling risks* by comparisons. As this task is a reading task (reading headings), we do not find individual notions of risk as in Example 1 (see Pratt et al., 2011). Students rather unpack and interpret *the socio-cultural phenomenon of misjudging risks*.

Discussion and conclusion

We have answered Research Question 1 (RQ1) through theoretically elaborating a hybrid conceptualization of risk literacy by combining two theoretical perspectives as key aspects, a realist and a socio-constructivist perspective, and defining risk as a cluster concept for stochastic lessons in school. In both examples, the two theoretical perspectives come into sight and thus underpin the relevance of our conceptualizations. By analyzing the two examples, we answered how the students navigate risks and how they perceive risks thereby (RQ2). Consistent with Brandom's (2000) view on concept formation, Example 1 (a decision situation) shows that students' perceptions of risk (worst/unpredictable case, loss) arise from navigating risks by comparing them. Example 2 (a reading situation) displays that students navigate risks through activating risk competency, hence, risk literacy, and thereby decode a socio-cultural risk phenomenon within a web-of-inferences. How students navigate risks and how they perceive risks thereby, depends on the affordances of the risk situation confirming that risk must to be regarded in relation to the situation (Eichler & Vogel, 2015).

Gal (2012), Hansson (2010), and Hansen and Hammann (2017) proposed to consider both theoretical perspectives when risks and risk literacy are addressed. Therefore, our approach is not completely new. New is however, the way we have conceptualized risk and risk literacy for stochastics education in school. The added value of these concepts is their practicability for teaching and lesson design, their limitations are that the concepts are unusual and that two *incommensurable* theoretical views come together. Empirically, the two perspectives are interwoven, in research however we need to be aware, that we have hybrid concepts made of complementary, but incommensurable views. Whereas possible measures of risk can be learned and risk competency can be exercised, individual and social interpretations of risks that emerge in social interactions may not be measurable. These interpretations can neither be learned nor exercised. What can be exercised, however, is practicing GoGAR, which can sensitize for risks, help to manage risks and to reflect on risks and risk decisions critically.

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Can you explain the surprising result? About students' arguments in the context of risk literacy before and after working with a simulation

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Risk literacy as the understanding and evaluation of risks for informed decision making is increasingly relevant in today's society. Updating conditional probabilities based on new information, i.e., Bayesian reasoning, is fundamental for risk literacy as risks are understood as negatively connotated events and their corresponding probabilities. In this paper, we provide a novel approach to measuring Bayesian reasoning (and hence one aspect of risk literacy) by asking for explanations regarding surprising results. Our analysis includes three aspects: First, we introduce a coding system which can be used to categorize students' answers into misconceptions or adequate arguments with varying elaborateness. Secondly, we use this coding system to test if working with a computer-simulation affects students' arguments of the surprising results. The results are encouraging to infer that misconceptions can be reduced by using the simulation.

Keywords: risk literacy, Bayesian reasoning, simulations

Risk literacy and Bayesian reasoning

The probabilities of false-positive and false-negative test results along with their potential consequences, were central to the debate over the approval of HIV self-tests in Germany. These probabilities represent risks, with risks defined as negatively connotated events and their respective probability (Eichler & Vogel, 2015; Aven, 2023). Consequently, assessing these probabilities is crucial for risk literacy, which refers to the "ability to understand and evaluate risk, in order to support and make appropriate decisions" (Aven, 2023, p.1). Risk literacy is increasingly relevant in today's citizenship education (Geiger et al., 2023) and involves an internal view, i.e., including the ability for mathematical computations of probabilities, as well as an external view, i.e., putting the probabilities into the context and making sense of them (Eichler & Vogel, 2015). Therefore, risk literacy encompasses not only the ability of calculating probabilities, but also interpreting, communicating and arguing with probabilistic information of risk-related events (Hansen & Hammann, 2017; Operskalski & Barbey, 2016). Thus, measuring risk literacy requires tasks that go beyond the sole calculation of probabilities.

Different mathematical concepts are necessary for risk literacy (Hansen & Hammann, 2017). We focus on Bayesian reasoning in our study, a wide-studied mathematical concept with significant implications for risk literacy (Eichler & Vogel, 2015). Bayesian reasoning involves "combining conditional probability information and base rate information to update a posterior probability" (Reani et al., 2018, p. 63) In the context of HIV self-tests, the so-called sensitivity (e.g., 99.9% probability for an infected person to test positive) and false-positive rate (e.g., 0.2% probability for a non-infected person to test positive) are probabilities which are usually provided in the patient

information leaflet. Furthermore, the base rate refers to the probability that a tested person is infected with HIV and can be deduced from the prevalence of e.g., 0.1% for HIV among people at risk for HIV in Germany. However, users are often more interested in the probability that for instance a positively tested person is actually infected, called positive predictive value (PPV). The PPV, a posterior probability, can be calculated using sensitivity, false-positive rate and base rate information and is therefore an example of the process of Bayesian reasoning.

Estimating the PPV is error-prone. A meta-analysis found that only about 5% of individuals accurately calculate the PPV using sensitivity, false-positive rate and base rate information (McDowell & Jacobs, 2017). A typically erroneous strategy in this context is to assume certainty of the PPV based on the 99.9% sensitivity (Woike et al., 2023). However, despite excellent test-parameters (99.9% sensitivity and 0.2% false-positive rate), the PPV is only about 33%. This result–33% probability for being infected given a positive test result–is often precepted as surprisingly low (Gigerenzer et al., 2007), reflecting a failure to account for base rate information in updating the PPV.

Prior research highlights that translating probabilities into so-called natural frequencies helps improve understanding (McDowell & Jacobs, 2017). Figure 1 illustrates this, by translating the probabilistic information about HIV self-tests (see above) into a sample of 1,000,000 individuals. 1,000 out of these 1,000,000 randomly chosen people (i.e., 0.1%) are infected with HIV. Out of the infected individuals 999 test positive (sensitivity of 99.9%). However, 1,998 of the 999,000 not infected individuals erroneously test positive (false-positive rate of 0.2%). Therefore, a total of 2,997 (=999 + 1,998) individuals test positive, but only 999 of these 2,997 positively tested people (=33%) are actually infected with HIV. This result highlights the importance of the base rate: even a low false-positive rate of 0.2% still results in a large number of false-positively tested people (i.e., 1,998) as there are so many not infected individuals in the tested population. In this case, the false-positives (i.e., 1,998) even outnumber all infected individuals (i.e., 1,000) in the tested population.

Individuals	infected	not infected	total
positive test result	999	1,998	2,997
negative test result	1	997,002	997,003
total	1,000	999,000	1,000,000

Figure 1: 2×2 table with natural frequencies representing the information about HIV-self-tests

Previous research on Bayesian reasoning has primarily focused on asking for calculations of posterior probability, such as the probability of being infected given a positive test result. However, risk literacy entails more than calculation of probabilities only. Asking students to explain the seemingly counterintuitive results can assess how they communicate risks in a situation which requires Bayesian reasoning (Binder et al., 2019), thus using Bayesian reasoning as the mathematical content to measure risk literacy which goes beyond the calculation of probabilities. Figure 2 illustrates how such question can be asked. To date, no empirical studies have explored how students respond to such questions, making it unclear what arguments they provide and whether they recognize the role of the base rate.

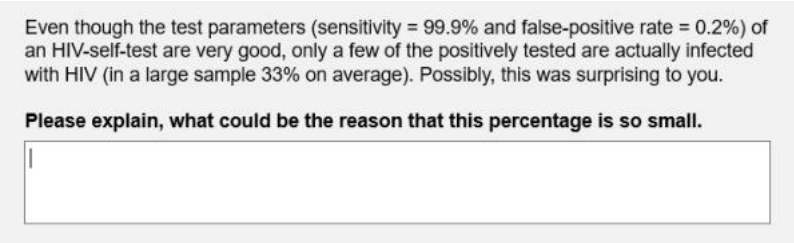

Figure 2: asking students to explain surprising results in the context of Bayesian reasoning

Simulations as a supportive tool

In the context of probability education, simulations are considered a supportive tool for learning (Maurer & Lock, 2016). They enable to re-enact stochastic phenomena (Erickson, 2010) and may also help to reduce misconceptions in the context of probability (Batanero et al., 2005). Particularly in the context of Bayesian reasoning, simulations may support a better understanding of conditional probabilities (Budgett & Pfannkuch, 2018; Engel & Martignon, 2015). Consequently, using a simulation, which imitates repeated testing with an HIV-self-test may be a supportive strategy additionally to the use of natural frequencies spelled out above.

Three characteristics of a simulation are central to our understanding of a simulation as a supportive tool. Firstly, a simulation imitates a (stochastic) phenomenon (Greca et al., 2014), in our case the repeated testing with HIV-self-tests based on the given probabilities. Secondly, a simulation is an interactive tool which "allows students to model and experiment" (Batanero et al., 2005, p.1) in the stochastic situation. Thirdly, a simulation has the purpose to illustrate processes which are otherwise hidden (Batanero et al., 2005; Greca et al., 2014). As such, a simulation (in the context of HIV-self-tests) should illustrate the influence of the base rate on the PPV which is otherwise often overlooked. The question arises, if a simulation with these characteristics can improve students' arguments for explaining the surprising results in the context of Bayesian reasoning.

Research questions

Asking students to explain surprising results in the context of Bayesian reasoning (see Figure 2) can measure understanding of risks and therefore risk literacy (see above). While it has been suggested before to ask this kind of question (Binder et al., 2019), empirical insights into how students answer this question are still missing. This leads to the following research questions:

- R1: What categories of arguments can be identified in answering this question?
- R2: How are different categories in a sample of statistically trained university students distributed prior to using a simulation?
- R3: Can working with a simulation improve the students' arguments?

For answering these research questions, we carried out two rounds of data collections (round 1 for R1 and round 2 for R2 and R3), which are described consecutively in the following.

Data collection and participants of round 1

In the first round, we asked 37 school students from grade 10 and 11 without prior training in Bayesian reasoning to explain the surprising results regarding the PPV in the context of HIV-self-tests. The

answers of the school students in the first round were used to develop a category system for the different arguments.

Data analysis of round 1

For identifying categories of arguments which students use to explain surprising results in the context of Bayesian reasoning we used a qualitative content analysis (Mayring, 2014) and formed categories inductively. The inductive approach was chosen as explaining surprising results in the context of Bayesian reasoning has not yet been empirically studied.

Results of round 1

Table 1 lists the categories of the school students' arguments, three categories were identified, out of which two reflect misconceptions (upper three rows in Table 1) and another category reflected arguments which correctly identify the influence of the base rate.

Table 1: category system for the different kinds of arguments to explain the surprising results

	Reason	**Definition**	**Example**
Misconception	Error of the test	Surprisingly small PPV is explained by errors of the test (e.g., false-positive rate).	"Because sometimes, the test is false-positive."
	Sources for errors of the test	Scenarios which may be the reason for an incorrect test result are stated.	"The test could be used incorrectly." "Possibly, another virus is mistaken for the HI-virus, because the test is not developed professionally."
	Sample size	The sample size is the reason for the small PPV.	"The value depends on how many people are actually tested, thus the sample size."
Identifying the influence of the base rate	Only mentioning the base rate	The base rate is mentioned without further explanation.	"Small probability for infection."

Data collection and participants of round 2

The second round of data analysis included 30 university students (prospective mathematics teachers). The university students participated voluntarily as part of their lecture. During the lecture, first, the university students received a worked-example on how to calculate the PPV followed by an exercise with feedback. This instruction and exercise are all prior training with regards to Bayesian reasoning of the university students. After the exercise, they were asked to explain the surprisingly low PPV in the HIV context (see Figure 2) and thus *before* using a simulation. Next the students were able to use a simulation for the HIV testing scenario for a total of 20 minutes to explore associations, e.g. between changes of the base rate and the PPV. After working with the simulation, they were asked again to explain the small PPV in the HIV context (see Figure 2).

Design of the simulation in round 2

A simulation needs to be carefully designed in order to fully embrace the (possibly supportive) characteristics spelled out in the theoretical background. Consequently, we have made deliberate design choices for the implementation of the simulation (http://bayesianreasoning.de/G1b.html) which we used in round 2 in order to answer R3.

In order to make the (otherwise hidden) influence of the base rate palpable in the simulation (third characteristic), we opted for three design choices. Firstly, we designed the simulation as a *stochastic* simulation. This means that the simulation's outcome depends on probabilistic parameter(s) within the simulation and therefore gathering observations does not always produce the same results (Greefrath & Siller, 2018). This implies that the probabilities inherent in the simulation are translated into (a sample of) frequencies which can be interpreted as natural frequencies which represent expected values. Like that, the support of natural frequencies spelled out above could also be exploited in using the simulation. Secondly, the simulation was set up in such a way, so that the frequencies are not represented all at once but can be sequentially gathered and observed. This may be promising as it has been shown that sequentially experiencing statistical information improves Bayesian reasoning (Schulze & Hertwig, 2022). Thirdly, the simulated frequencies representing the statistical information in the Bayesian situation are displayed in a visualization which is easily understood, in our case a 2×2 table. This choice is based on prior results that frequency-based 2×2 tables (as in Figure 1) are well-known visualizations easing inferences in the context of Bayesian reasoning (Böcherer-Linder & Eichler, 2019). For realizing the second characteristic of simulations (opportunity for interaction), we implemented sliders for the sample size which is drawn as well as for the value of the base rate. However, as interactivity in the context of Bayesian reasoning is not always linked to improved Bayesian reasoning (Mosca et al., 2021), we designed two versions of the simulation, version A in which only the sample size can be varied and version B in which also the base rate can be varied. Like that, version A was used in a first stage to get acquainted with the simulation and version B was used in a second stage, to allow more interaction and covariation of the PPV and the base rate to possibly understand its influence on the PPV.

Data analysis of round 2

For analysing the answers of the university students we used the category system from round 1. Yet, it became apparent that the last category (identifying the influence of the base rate) needed adjustment to reflect different levels of elaborateness in these arguments. While some students simply stated that the base rate was the reason for the surprising results without further explanation (fourth row in Table 1), others more elaborately explained the influence of the base rate on the different subsets (e.g., also including how the false-positives and true-positives were affected by the small base rate). These adjusted categories were used to code the answers of the university students by two authors of this paper and the inter-rater reliability was substantial with $\kappa = 0.76$.

For inferential analysis regarding R3, we carried out a Cochran's Q test to assess whether the distribution of the categories differs from before to after using the simulation. This is chosen based on the nominal outcome variable (category) and the repeated measures (McCrum-Gardener, 2008).

Results of round 2

With regard to R2 and R3, we provide the descriptive results of these categories of the students' answers before and after using the simulation in Figure 3. As evident in Figure 3, the category "Sources for errors of the test" was not used as an argument by the trained university students (and therefore may be unique to less statistically trained students).

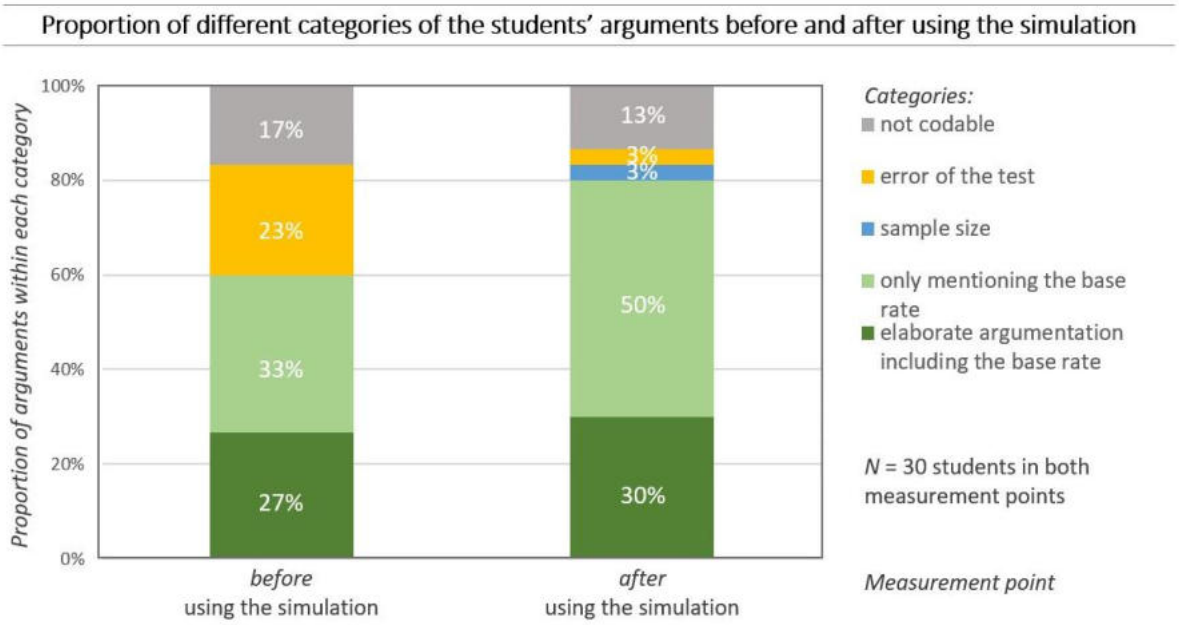

Figure 3: descriptive results of the students' arguments before and after using the simulation

Descriptively, there seems to be an improvement in the students' arguments as after using the simulation more students identified the influence of the base rate. The Cochran Q test is significant (T=18.6, df=1, p<0.01) implying that the distribution of the categories in the students' arguments significantly changed from before to after using the simulation.

Interpretation

The results have manifold implications. Firstly, in developing the category system for the data analysis we have identified misconceptions about the surprisingly small value of the PPV (i.e., attributing errors of a medical test or the sample size as reasons). This resonates with prior research results showing that the base rate is easily overlooked (McDowell & Jacobs, 2017). Secondly, these misconceptions seem to be rare among statistically trained people, but do exist. Thirdly, using the simulation we have designed seems to improve students' arguments, as after using the simulation hardly anyone expresses these misconceptions anymore and the majority (80%) correctly identifies the influence of the base rate. Still, in one case (=3% of the sample) a new misconception in the sample (attributing the small PPV to the sample size) occurs only after using the simulation. This may imply that allowing interactivity in the simulation with irrelevant features may hinder learning.

The presented research has limitations, as it cannot be ruled out that improvement in the students' arguments is based on repeated testing, as there was no control group. Future research could therefore study and compare, how working with and without a simulation changes students' arguments about surprising results in the context of Bayesian reasoning. Further, it would be interesting to study, how less statistically trained people, e.g., students in school, profit from working with the simulation. This is our aim in the larger context of the project siMINT (http://bayesianreasoning.de/br_simint.html) in which we study how working with a simulation can improve risk literacy.

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How do future teachers interpret a pupil solution to a probabilistic task involving proportionality?

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Recent curricular programs worldwide include the teaching of probability from early levels. Consequently, teachers must have the necessary didactic-mathematical knowledge and competencies to solve probabilistic problems and interpret students' responses to these problems. Given the importance of proportional reasoning in solving probability tasks, this study investigates how future primary school teachers interpret mathematical practices in a fictitious student's solution to a probability task involving proportional reasoning. The research uses a qualitative approach to analyse how 133 future teachers interpret a student's solution based on proportional reasoning to calculate the composition of urns instead of calculating probabilities. The results reveal the participants' difficulties in correctly interpreting this strategy, although some improvement is observed compared to previous research.

Keywords: Probability, Proportional Reasoning, Teacher Education, Cognitive Processes.

Introduction

The integration of probabilistic and statistical content from the earliest levels of education requires comprehensive teacher training in probability and statistics. In addition to having the necessary knowledge and skills to solve the tasks they propose to their students, teachers must be able to interpret students' reasoning and effectively manage the challenges and obstacles they encounter with these contents (Franco & Alsina, 2022).

One of the fundamental components of probabilistic reasoning with which students have particular difficulty is proportional reasoning. Proportional reasoning, defined as the ability to establish multiplicative relationships between two quantities and to extend this relationship to another pair of quantities (Lamon, 2007), plays a crucial role in the analysis of the sample space and in the proportional quantification of probabilities (Bryant & Nunes, 2012). Thus, success or failure in the calculation and comparison of probabilities is closely linked to the recognition of the proportional nature of these probabilities (Begolli et al., 2021; Bryant & Nunes, 2012).

Specifically, recent research has analysed the strategies, difficulties, effects of problem variables and achievement levels of schoolchildren in solving ratio comparison problems (Castillo & Fernández, 2022), and to a lesser extent, proportional sharing problems (Yeong et al., 2020). Results indicate that students from primary to the end of secondary school have difficulties in identifying multiplicative relationships, comparing referents and understanding the concept of 'whole' (sum of parts) (Yeong et al., 2020). These difficulties are especially observed in the

probabilistic context, when students solve tasks involving the calculation or comparison of probabilities, often in the context of ballot boxes (Hernández-Solís et al., 2023). Although less attention has been paid to students' compression in sample space construction, it is observed that students with more advanced proportional reasoning skills show better performance in the construction of sample spaces (Hernández-Solís et al., 2024).

On the other hand, these difficulties are not exclusive to students. Thus, numerous studies reveal that both preservice and inservice teachers encounter difficulties with understanding and teaching the components of proportional reasoning (Buforn et al., 2018), as well as in interpreting students' responses in tasks involving proportionality (Fernández et al., 2013). Specifically, trainee (Batanero et al., 2015) and practising primary school teachers (Vásquez and Alsina, 2015) encounter difficulties in correctly identifying the proportional reasoning involved in solving probability problems, as well as in recognising and effectively addressing students' errors in this area (Burgos et al., 2022a, 2022b, 2023). Burgos et al. have specifically assessed the knowledge and skills of preservice primary school teachers in interpreting students' responses to probability comparison (Burgos et al., 2022a, 2022b) and fair play (Burgos et al., 2023) tasks, analysing how they identify proportional reasoning or its absence in the mathematical practices of schoolchildren.

The results show that future teachers have difficulties in identifying possible erroneous strategies behind students' incorrect answers, as well as in detecting proportional reasoning in their answers. In the case of secondary school, Dröse et al. (2022) asked future mathematics teachers to analyse a real solution to a conditional probability problem together with an excerpt from a student dialogue (vignette) to assess their teaching competences. The results revealed that, although preservice teachers could identify procedural errors, they failed to recognise the underlying conceptual difficulties that caused these errors.

In this paper we focus on how future teachers recognise the role of proportional reasoning in constructing sample spaces to create equal probabilities (Supply et al., 2020) when analysing a fictitious student's solution.

The research is framed in the model of categories of Didactic-Mathematical Knowledge and Competences of the mathematics teacher (Godino et al., 2017), which proposes six categories for the didactic dimension of knowledge (epistemic, cognitive, affective, interactional, mediational, and ecologic) to facilitate the categorization and assessment of knowledge and competences and the design of training actions with future teachers to develop such knowledge and competences. In this framework, knowledge is strongly related to the notion of competences. In this paper, we specifically focus on cognitive analysis competence, which allows teachers to interpret students' mathematical practices, i.e., comprehend students' procedures, identify personal interpretations, and recognize any misconceptions that may arise during problem-solving activities but focusing on the presence of proportional reasoning in a probability task.

In this line, we propose to investigate the cognitive analysis competence of future teachers through the interpretation of the mathematical practices in a pupil's solution to a probability task. The research question is: how do future primary school teachers interpret the mathematical practices in a fictitious pupil's solution to a probability task involving proportional reasoning?

Methods

The research follows a qualitative methodology with a descriptive approach. A total of 133 future primary school teachers (ages 20-21) from a Spanish university who had taken a mathematics education course participated voluntarily and in an informed manner. At the time of the intervention, they had already received previous didactic-mathematical training on statistics and chance, including epistemic (main concepts, properties and stochastic procedures), cognitive (analysis of errors and difficulties in the learning of probability) and instructional aspects (design and sequencing of tasks, materials and resources for teaching probability). The task, along with others not analysed in this report, was included as part of a midterm evaluation with an impact on the final grade of the course.

The task proposed to the participants, with a fictitious primary school pupil's solution, is presented below in Figure 1. The task asks future teachers (FTs) to interpret the practices developed by a fictitious primary school pupil and to evaluate both their solution and their justification, providing reasons for their evaluation. This assignment model focuses explicitly on the cognitive facet and has been previously used in research with future teachers (Burgos et al., 2022a, 2022b, 2023). However this particular task has been designed ad hoc to study the competencies of participants to recognize aspects of proportional reasoning involved in a situation of creating equivalent ballot boxes. The proposed fictitious solution is based on reflecting on the proportional composition of the content of box B with respect to box A, assuming that in box A there are only 4 balls, thus calculating the ratio between the white balls and the black balls in box A, a ratio that must also be respected in box B containing 20 balls. Then, the fictitious solution presents a notational error by indicating the multiplication between a fraction and an integer instead of equivalent fractions (1/3x5 as 5/15). The ratio chosen in this solution is different from the calculation of the classic probability as the ratio of favorable cases to possible cases with the Laplace formula, so FT should recognize the particularity of this solution and identify the proportional reasoning behind it.

The task proposed:

We have two boxes, box A and box B, containing both white and black balls. In box A there are three black balls for each white ball. There are 20 balls in box B (black and white). How many balls of each colour are there in box B if it is just as likely to draw a white ball as in box A?

A fictitious primary school pupil's solution:

1/3 **A** i.e. for every 4 balls, 1 white and 3 black

20 balls **B** is just as likely to draw a white ball as box A.

Let us suppose that in box A, there are only 4 balls. If there are 20 balls in box B, then there are 5 times more balls then in box A. Then we multiply 1/3x5 and get 5/15, i.e., for every 5 white balls, there are 15 black balls. If there were 20 balls in box B, then there are only 5 white balls.

Assignment for the future teacher:

Do you think the response given by the pupil is correct? What mistakes has she made? Justify by analysing the pupil's response.

Figure 1: The task proposed to the future primary school teachers

To minimize subjectivity biases, the authors together with an external expert contributor participated in the analysis through a cyclical process of defining, validating, and refining the category system. The three researchers independently analyzed portions of the data (the FTs' written operational and discursive practices), identifying types of responses in the interpretation of the pupils' solutions (Bardin, 2002). They then shared their results, unified the observed response types, and reached a consensus on the emerging categories. This process was cyclically repeated with additional data until all the data had been analysed. Finally, in order to facilitate the visualization of the categories described and the proportion of responses in each one, the information has been synthesized in a frequency table.

Results

The emerging categories identified in the future teachers' responses are described below:

- *Adequate connection between probability and proportionality*. In this category, the future teachers identify and explicitly mention the connection between the use of the ratio and the (equi)probability, demonstrating a correct understanding of the pupil's procedure, so this is a correct interpretation of the pupil's solution.

  FT24: La respuesta es correcta, ya que comprende las relaciones de proporcionalidad a partir de fracciones equivalentes. En ambas [cajas] la probabilidad de una y otra es la misma, aunque varía el número de bolas.
  Translation: *The answer is correct, as she understands the proportionality relationships from equivalent fractions. In both [boxes] the probability of one and the other is the same, although the number of balls varies.*

- *Adequate solution without relation to probability*. In this case, the pupil's solution is considered correct, but it is explicitly indicated a missing relation with probability. We assume that the FTs expected the use of the Laplace rule, which they did not find in the pupil's resolution and they did not understand the pupil's use of proportional reasoning.

  FT11: [La alumna] entiende la resolución del problema a base de fracciones equivalente y supuso la caja B en función de un supuesto de la A. Comprende estos conceptos pero no su relación con la probabilidad.
  Translation: *[The pupil] understands the solution of the problem based on equivalent fractions and assumed box B based on an assumption of box A. She understands these concepts but not their relation to probability.*

- *Correct use of probability*. In this category, the use of probabilistic procedures and concepts is highlighted, due to the use of typical terms of the probabilistic domain, such as favourable cases, possible cases, etc- when in fact the pupil does not make reference to these concepts.

  FT88: [La niña] ofrece tanto una respuesta correcta como una justificación correcta ya que muestra que entiende el concepto de probabilidad diferenciando los casos favorables y posibles.
  Translation: *[The pupil] provides both a correct answer and a correct justification as she shows that she understands the concept of probability by differentiating between favorable and possible cases.*

- *Error with fractions*. In this case, the operation mistake is correctly detected, pointing out that the error lies in the execution of the multiplication of a fraction by a whole number. However, in this interpretation the identification of proportional reasoning is missing.

FT122: Encontramos un problema con el planteamiento de la resolución debido a que la multiplicación 1/3x5 el resultado obtenido no es el que ha contestado la alumna.
Translation: *There is a problem with the approach of the resolution because the result obtained from the multiplication 1/3x5 is not the one the pupil has answered.*

- *Inadequate solution due to absence of probability*. Here, it is found to be an error the lack of use of probabilistic terms or formulas or misunderstanding in the probabilistic reasoning. It seems that the use of probability calculus was expected but did not occur in the pupil's resolution.

FT104: Respuesta y justificación incorrecta. No razona de la probabilidad sino de la cantidad de bolas
Translation: *Incorrect answer and justification. Her reasoning is not based on probability but on the quantity of balls.*

- *Error in the use of the ratio 1/3 for 1/4*. In this case, the use of the ratio 1:3 is considered a misunderstanding in the use of the Laplace formula to calculate the probability of drawing a white ball in box A. The FTs do not identify the fraction as the ratio; therefore, they consider that the error is in the identification of the number of possible cases.

FT64: No es correcto. Aquí el error que comete es identificar que de cada 4 bolas una es blanca, y 3 negras como 1/3 cuando debería ser 1/4.
Translation: *It is not correct. Here the mistake she is making is to identify that out of 4 balls one is white, and 3 black as 1/3 when it should be 1/4.*

- *Incorrect assumption of 4 balls in A*. Here, the particularization made by the pupil in supposing that there are 4 balls in box A is considered erroneous, although it is explicitly stated that this is only an assumption.

FT38: Se ha equivocado ya que parte de una suposición errónea, que es “la caja A tuviera 4 bolas”.
Translation: *She is wrong because she is working on the wrong assumption, which is “box A had 4 balls”.*

Finally, there are two more categories to identify inconclusive responses when FT do not provide any information about their understanding of the strategy used by the pupil and missing answers, when the FT does not give any answer. Here an example of inconclusive response:

FT75: Su respuesta es correcta ya que razona bien el ejercicio.
Translation: *The answer is correct as she reasons the exercise well.*

The following Table 1 shows the frequency distribution (percentages) of FTs' responses according to the categories described. The results reveal that a high proportion (24.81%) of responses are classified as "inconclusive", suggesting considerable difficulties in formulating clear and substantiated interpretations. This may reflect a general struggle with the underlying concepts or the application of proportional reasoning in probabilistic contexts.

The FTs had difficulties to properly interpret the role of fractions in the fictitious pupil's strategy considering, for example, that for the pupil, 1/3 is the probability of drawing white ball in A. It is worth noting the misunderstanding generated by using a different ratio than expected in the classical probability calculation (1/3 instead of 1/4) which has been detected as erroneous by a 15.04% of the participants, as well as the identification of the procedure as erroneous only because the probability is not calculated by a 14.29% of the participants. Despite having purposely avoided

asking FTs for prior resolution of the task, several of the identified categories refer to a prototypical response expected by the FTs that does not correspond to that of the pupil.

Table 1: Frequency distribution (percentages) of FTs' responses. In bold the correct interpretation.

Categories	Frequencies (percentages)
Adequate connection between probability and proportionality	18 (13.53)
Adequate solution without relation to probability	5 (3.76)
Correct use of probability	5 (3.76)
Error with fractions	17 (12.78)
Inadequate solution due to absence of probability	19 (14.29)
Error in the use of the ratio 1/3 for ¼	20 (15.04)
Incorrect assumption of 4 balls in A	12 (9.02)
Inconclusive	33 (24.81)
No reply	4 (3.01)

These results seem to confirm a tendency of future teachers not to go deeper into the analysis of the pupils' answers, preferring to compare the procedures with a single correct expert solution, as already observed in Burgos et al. (2023).

Moreover, this indicates a lack of multiple or relativistic perspectives needed to understand different possible solutions (Buforn et al., 2020), and could be due to a limited or incomplete knowledge of proportional reasoning and its role in the probabilistic context. Actually, only a small percentage of responses (13.53%) managed to establish an adequate connection between probability and proportionality, an essential aspect for a correct understanding of the pupil's solution.

Errors related to fractions, seen in 12.78% of the responses, and incorrect assumptions about the number of balls in A (9.02%) further suggest that future teachers struggled to integrate proportional reasoning effectively, since, although they detect other inconsistencies, they do not focus on what is really important in the pupil's response. The high number of inaccurate and inconclusive responses in interpreting the mathematical practices of the fictitious pupil points to a limited or incomplete understanding of both the probabilistic and proportional concepts involved in the task, indicating the need for a more nuanced and comprehensive approach to teaching these ideas.

Conclusion

A mathematics teacher's ability to interpret the reasoning of his or her students is fundamental to ensuring that both teaching and learning take place with quality. This ability helps teachers identify student understanding and difficulties. In the task analysed, both proportional and probabilistic reasoning play a fundamental role. Compared with the results of Burgos et al. (2022b), a lower percentage of participants recognize the presence of proportional reasoning in the pupil's solution but, in contrast to the results of the mentioned research, when they refer to proportionality, they do it in a correct way, in some cases explicitly mentioning the multiplicative relation between the composition of the two boxes.

The competences of future teachers to recognize and identify pupils' strategies and errors have an impact on the quality of teaching. Thus, it is of great importance to address these didactic cognitive analysis competences in teacher, mainly when unexpected solutions were given and unforeseen

proportional relations -different from the traditional ones involved in formulas- were indicated, thus allowing for a diversity of reasoning (Buforn et al., 2020).

Since many of the interpretations were inconclusive, it would be beneficial in further researches to supplement the data with interviews and follow-up sessions where the FTs can explain their assessments of the students' responses. This could help identifying if deficiencies in mathematical knowledge could be among the causes of these inconclusive or inaccurate interpretations. The study's results may be influenced by the participants' early training stages and limited experience in probability, which affects the generalizability to more advanced groups. Despite our efforts to conduct a rigorous analytical process, the qualitative data analysis retains a degree of subjectivity. These limitations require further research with diverse samples and methods to verify these findings and deepen our understanding of teachers' probabilistic and proportional cognitive competencies.

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Statistical literacy in middle school students' performance of speech acts when reacting to data-based arguments on social media

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In order to adequately deal with possibly misleading statistical information on social media, students need to develop statistical literacy. Especially important for formulating replies to data-based arguments is the communicative component of statistical literacy. However, research provides only little guidance for how to develop this component. This study draws on the framework of communicative rationality for statistical literacy to inform the design of a learning environment on communicating about data-based arguments on social media. The empirical results show that the learning environment can elicit communicative rationality and it shows the variety of the students' communication about statistical information. These insights also motivate a further direction for research, as the students' performance of speech acts when formulating refutations of statistical arguments come into view.

Keywords: Statistical literacy, communicative rationality, disinformation, social media, design research.

Introduction

In today's online society, online media play an increasingly central role for citizens' access to news and the processes of forming opinions on political matters (Tucker et al., 2018). Only about 22% of people in many countries get their news from news media publishers, most people get their news from social media or direct messaging (Newman et al., 2024). Generally, there is a large distrust of online information and 59% report to be troubled with identifying real from fake information online (Newman et al., 2024). This distrust in media poses a challenge for democratic countries, as studies show that heavy social media use is associated with a growing distrust in political institutions (Lorenz-Spreen et al., 2023).

Most of such discourse focuses on the impact of disinformation, meaning false information that is intentionally introduced by an actor to harm the public (Freelon & Wells, 2020). Yet, information shared on social media often also contains data and graphs that can also be used to promote possibly misleading arguments, even if they are rooted in a factual basis. For traditional online news media, Gal & Geiger (2022) show the high critical demands on statistical and mathematical knowledge needed to adequately understand and interpret the news items. The shift from traditional media to social media increases the complexity of the required mathematical and statistical knowledge, as citizens on social media have the opportunity, and possibly the duty, to *react* to the shared information. It therefore becomes an important area of research to identify and to foster the knowledge needed for citizens not only to identify possibly misleading statistical information, but also the knowledge needed to engage in communication on statistical information shared on social media. This study draws on a framework proposed by Büscher (2024) to conceptualize this knowledge needed to communicate about statistical information. Afterwards, insights into the design

of a learning environment are provided to show how task design that uses social media posts can elicit communicative reactions showing students' statistical literacy.

Theoretical framework

Communicative rationality for statistical literacy

Understanding and criticising statistical information on social media requires statistical literacy, for which Gal's (2002) definition specifies two components, which Büscher (2024a) relates to as the (1) *interpretative* component and the (2) *communicative* component: (1) an individual's ability to understand and critically evaluate statistical information (i.e., news articles or social media posts making some quantitative argument) and (2) an individual's ability to communicate their understanding and criticism of such information. The communicative component becomes important for the case of statistical information on social media, as citizens that want to participate in public discourse on social media have to be able to formulate their support or their objections to statistical information.

To conceptualize such a view on communication on statistical information, Büscher (2024a) adapts Jürgen Habermas' works on communicative rationality, which "secures for the participating speakers an intersubjectively shared lifeworld, thereby securing at the same time the horizon within which everyone can refer to one and the same objective world" (Habermas, 1998, p. 315). Communication understood in this way is a process during which interlocutors reach a common understanding of the world. This common understanding might prove especially difficult to reach in the case of statistical information, where contradictions can arise between seemingly objective data-based arguments.

Communicative rationality is exercised by an interlocutor that evaluates an argument based on different *validity claims*, which provide different reasons for why the argument should be accepted or rejected (Habermas, 1998). For the case of data-based arguments, Büscher (2024a) provides a topic-specific adaption of communicative rationality for statistical literacy that specifies four different validity claims. (1) A data-based argument that does not align with conventional rules for formulating statistical arguments, that provides incorrectly drawn graphs, or that misses crucial information on sampling or base percentages might be rejected on the grounds of *comprehensibility*. (2) If the argument is not related to the objective world, if the used data do not adequately describe the underlying phenomenon, or if there are datasets known to the interlocutor that directly contradict the argument, the argument might be rejected on the basis of *truth*. (3) If the interlocutor suspects a hidden manipulative agenda behind the argument or if the reader suspects a lie, the argument might be rejected regarding the validity claim of *truthfulness*. And (4) if the argument violates social norms or ethical standards, it might be rejected because of its missing *rightness*.

Additionally, a Habermasian framework can illustrate that communicative reactions to statistical information posted online are not simple statements that are conducted in a social vacuum. Instead, they can be considered *speech acts* with which the reacting person hopes to achieve specific aims: (1) to inform others that one holds a specific view (a locutionary speech act), (2) to convince others to rationally agree with one's own view (an illocutionary speech act), or (3) to manipulate others into acting or thinking while hiding this intention (a perlocutionary speech act) (Habermas, 1998). An

investigation into the statistical literacy exhibited by students' replies to statistical information on social media thus needs to pay attention to the intended speech acts.

Eliciting and fostering the communicative component of statistical literacy

Empirical studies on students' statistical literacy and on design principles for fostering their statistical literacy are relatively scarce in mathematics and statistics education research. Studies show that statistical literacy seems to be relatively low in middle schools, but that it also can be developed (Callingham & Watson, 2017; Yolcu, 2014). Regarding ways to foster statistical literacy, Büscher (2024b; 2022) illustrates design principles that revolve around using critical contexts, engaging with statistical information that is based on models, and to provide scaffolds for reflecting on statistical information. Yet, these empirical results and design principles focus mostly on the interpretative component of statistical literacy. Regarding the communicative component of statistical literacy, Büscher (2024b) provides some first starting points by illustrating tasks in which students are provided with scaffold criteria for evaluating statistical information. More insights are needed regarding the question how to specifically elicit and foster the communicative component of statistical literacy. This paper engages with following research question: How do students address different validity claims and how do they engage in speech acts when communicating about statistical information found in social media posts?

Method

This study continues earlier work on identifying design principles and investigating students' learning processes on statistical literacy (Büscher, 2024a; 2024b). Earlier studies illustrated students' use of validity claims when criticizing statistical information (Büscher, 2024a), but without focusing on the speech acts committed when reacting to social media posts. This study intensified the work on the communicative component of statistical literacy by designing and investigating a learning environment that uses social media as the arena of statistical arguments.

Design of a learning environment

For this study, a learning environment was designed for use in Grade 7 in Germany. The learning environment consisted of a sequence of four lessons (each 45-90 minutes), during which the students engaged with real-world examples of extremist positions on immigration that are argued on social media using statistics. The lessons were designed by the teacher with the goal of providing the students with the necessary goals for 'debunking' statistical information that one the one hand does have a basis in real data, but on the other hand provides a misleading and manipulative argument.

A full account of the developed materials cannot be given here due to the available space, so that only a short overview is possible. Several design elements were provided in each lesson: (1) A social media post under investigation. (2) A worksheet that instructed the students to analyse the post. Here, students were asked the following questions: "What are your first thoughts on this post? What is the claim of the statistics? How are the data presented? What do you notice?" While working on this work sheet, the students had access to various scaffolds: (3) a list of sentence fragments for scaffolding the students' communication on the social media posts. Sentence fragments included "The statistics shows, that …", "Clues for/against this perspective are…", "If you look closer, you

can notice that…", "my personal opinion is that…", and many different fragments more. (4) A list of 'worry questions' (inspired by Gal, 2002) including "what is fact, what is opinion?", "are there numbers missing?", "is there an intention to mislead?", "what is being described and what is impossible to infer from the statistics?", "are there any events that were not included as a possible reason?". Following work on the work sheet, a whole-class discussion took place. Afterwards, (5) the students were provided with a work sheet for drafting a reply in the form of a social media post.

In the whole sequence of four lessons, three different social media posts were investigated. This study only focuses on the first of the posts, which was the object of the second lesson. Figure 1 shows a reconstruction of the social media post, whose original form that was investigated in the lesson was posted by a far-right group and is reconstructed and anonymized here. This post uses allegedly real data provided by official sources to make a claim that half of asylum seekers in Germany live on social welfare, which is portrayed as very negative and used as an argument for calling for additional measures against immigration. Reasonable rejections of such a claim would pose high demands on statistical literacy, as the post itself seems to be provides an explicit source (i.e., it can be considered *true*) and does not exhibit obvious mathematical errors (i.e., it can be considered *comprehensible*). Students that argue against the *truth* or *rightness* of such a claim could draw on external knowledge of Germany's welfare or immigration systems. For example, they could argue that asylum seekers have a moral right of support (thus challenging *rightness*) or draw on knowledge of the phenomenon by arguing that asylum seekers in Germany actually often are excepted from taking on jobs while their status is being evaluated (thus challenging *truth*).

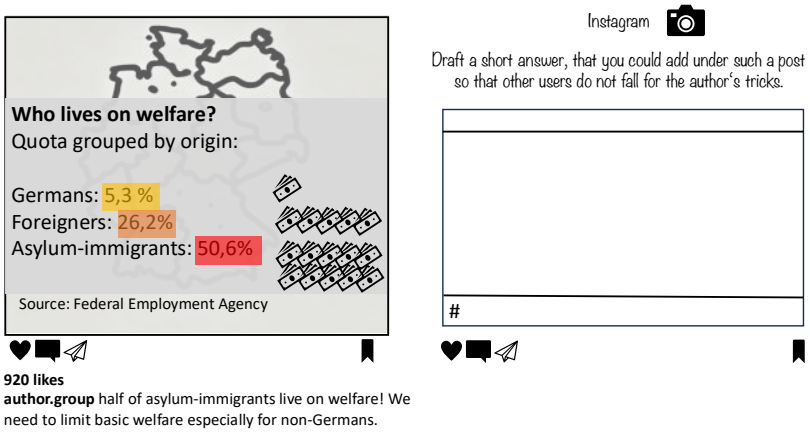

Figure 1: Reconstruction of social media post (left) and work sheet for drafting a reply (right)

Participants

The teaching sequence was carried out with two classes of Grade 7 students in May 2024, with a total of 43 students who participated in the design experiments. The participants were attending a school in a very low-income metropolitan area with an almost 100% rate of family immigration background, which meant that the students were emotionally highly invested in the context and in 'debunking' the social media posts. This means that a lot of care has to be given surrounding the use of this learning environment: care of the teacher to create an environment where students can safely engage with possibly emotionally disturbing content, and care of the researcher to be wary of sharing sensitive

information and of generalizing from highly subjective experiences shown by the students. However, the students' subjectivity is very much the point here: the real-word demand of social media is that students need to be able to adequately express their subjective position regarding statistical information that is possibly perceived as insulting and disturbing.

Data collection and data analysis

Data collection included the students' written replies to the social media posts, as well as videos of small-group discussion and whole-class discussion. For this study, only the written replies of the first class of students were used and only for the first social media post investigated by the students (Fig. 1). Thus, this study illustrates the effects of the design of the learning environment using 16 written replies. These replies do not provide insights into individual students' statistical literacy, as they are in part a product of small-group and whole-class discussions. Nevertheless, they allow insights into the differences in students' communication about data-based arguments that can be elicited by the design of the learning environment.

The written replies were analysed with regard to the students' use of validity claims using the framework for communicative rationality for statistical literacy (Büscher, 2024). Additionally, the written products were compared and contrasted to inductively generate categories for describing the differences in the students' speech acts shown while formulating replies that address different validity claims.

Empirical results

The written replies to the social media post that were created by the students after the whole-group discussion show a high variety of addressed validity claims and speech acts, although there are some commonalities between the students. In the following, translated examples are provided to illustrate the results. Where possible, translations were attempted that were consistent with the numerous orthographic and grammatical difficulties encountered by the students when responding to statistical information.

Some students purely address aspects of percentages and illustration:

S1: The statistics is wrong the percentages do not add up to 100% and the bank notes do not fit the percentages at all.

S2: It is claimed that asylum-immigrants get the most welfare. It is noticeable that the percentages were illustrated wrongly. The darker the colour, the more percentage it is. In summary you can say that it is a bad example because 20% are missing.

S1 and S2 both note that the given percentages do not add up to 100%, which they argue is a mathematical error. S2 also notes that higher percentages are presented in a darker colour, although it remains unclear why exactly S2 considers this wrong. Additionally, S1 argues that the bank notes do not fit the percentages. While the exact reason why S2 thinks this are unclear here, this is often the point of criticism in the whole dataset. These students treat the illustration of bank notes as an algebraic problem: If 5,3% correspond to a single bank note, then ten bank notes should correspond to 53%, not 50,6% as in the post. In this way, both students address violated formal rules of forming and illustrating statistical arguments, making this a criticism of *comprehensibility*.

It is notable that under the perspective of the statistical expert, the students' refutations are not convincing. It is not a mathematical error that percentages do not add up to 100%, because they do not describe the fractions of group members for all people living on welfare, but only the fraction of people living on welfare within each of the three groups – although it is possible this confusion is intended by the authors. A slight vagueness in the meaning illustrations like the bank notes is also often a necessity for creating concise illustrations and can be disregarded. Another notable difference is the level of elaboration of the replies, showing S2's use of the language scaffolds.

Other students focus on different aspects:

S3: [The author group] tries to manipulate us with this post to vote for [far-right party] AfD. The numbers also are not complete because they do not add up to 100 percent. […]

S4: The statistics show that asylum seekers and foreigners apply for a lot of welfare and it doesn't even add up to 100%. [The author group] is just xenophobic and only posts fake posts about foreigners. I also know a lot of refugees who want to work but it is being forbidden. Because of that it does not make sense that [the author group] posts that refugees apply for such and such welfare, if they are not allowed to work they of course have to apply for welfare.

Both students also address *comprehensibility* through the same argument of missing 100% as S1 and S2, but add additional aspects. S3 addresses a suspicion of the author group's intention: they created the post solely to mislead, manipulate and to influence elections. Likewise, S4 posits that the author group is xenophobic. In this way, both students base their evaluations not on validity claims such as truth or comprehensibility, but on issues of the subjective intentions of the authors, and therefore address the validity claim of *truthfulness*. S4, however, also adds knowledge of the social phenomenon of refugees and welfare. Because S4 knows that refugees are often forbidden from working, S4 calls into question whether the statistics adequately represent the phenomenon. By relating to additional knowledge of the phenomenon, this is an evaluation of the *truth* of the data-based argument.

In the replies of both students, speech acts can be observed. S3 appeals to the readers of their reply by making an explicit distinction between *we*, the collective own group of readers, and *them,* the author group who tries to manipulate 'us'. This we-them-framing can be considered a part of a *perlocutionary* speech act, as it provides a subtle way of drawing the readers to one's own side that is not rooted in rational agreement to validity claims. S4 engages in an *illocutionary* speech act through an explicit appeal to personal knowledge of the situation ("I also know a lot of refugees") as a proof of the *truth* of this statement. Other students are very direct in their use of speech acts:

S5: The post is incorrect. If you add up the percentages it doesn't make 100%. Please do not fall for this trick.

In these ways, S3, S4 and S5 seem to attend to the social nature of replies on social media posts by orienting their reply towards an audience.

Still other students provide a more differentiated evaluation of validity claims:

S6: If you add up the numbers it results not in 100% which means that it's wrong. In summary, it is right but also wrong because as mentioned it does not result in 100%.

S7: Actually I think that it would look very different if we would finally let foreigners or 'asylum immigrants' work, because the numbers would look completely different. Finally I want to emphasize that there are numbers missing. About 10-20% are missing. In my opinion, if you want [to influence the elections] then you have to show the full statistics. On the other hand, the numbers are correct. Therefore it would be better to show a real diagram.

Again, both students address the *comprehensibility* of 100%, with S7 adding remarks on the *truth* of statistics that are not taking into account the situation of legal work for asylum seekers. Additionally, both students grapple with the idea that the argument is "right but also wrong" (S6), that "the numbers are correct" (S7) but still objectionable. These students show their ability of understanding parts of the original post while still holding disagreement in other validity claims. These examples, however, also show the difficulties of expressing such a differentiated view, as even S7 – who shows probably the most elaborate and well-formed reply of the whole group – is unable to explicitly and concisely point out what exactly is correct, and what exactly showing a "real diagram" would entail.

Discussion

The students address various validity claims and engage in different speech acts for evaluating data-based arguments on social media. Noticeable is the high frequency of students making a faulty argument regarding supposedly missing percentages. The prevalence of this argument could indicate some kind of automated way of addressing manipulations in data that is carried out without a strong understanding of percentages. In general, the students in this study did not often address validity claims besides understandability. This could indicate a lower grasp on the communicative component of statistical literacy. In contrast, in an earlier study with Grade 5 students, a richness of validity claims was observed (Büscher, 2024a). A reason could be the present study's students' more limited understanding of context and data, as the lesson did not include personal investigations of data. However, requiring students to formulate replies on social media seems to have prompted some students to explicitly engage in illocutionary and perlocutionary speech acts. This was not observed in the earlier study. As such, the results of this study could indicate that designing for the communicative component of statistical literacy should include students' investigation of context as well as explicit formulations of replies to social media posts that make data-based argumentations.

Conclusion

Misinformation and disinformation on social media can erode the trust into democratic political institutions (Freelon & Wells, 2020; Lorenz-Spreen et al., 2023). But besides disinformation, there is also statistical information to be found on social media that literate users need to understand and react to. This requires the communicative component of statistical literacy, which in turn requires communicative rationality (Büscher, 2024). Research provides no guidance on how to develop this component for students and how students engage in speech acts regarding statistical information.

This study provided insights into a learning environment that used real-world social media posts showing positions on immigration as a focus for students' reactions on statistical information. The empirical results show that the designed materials could elicit reactions showing differences in the students' communicative rationality for statistical literacy. Additionally, it could be observed that the students had different success in performing speech acts that could capture an audience on social

media. These results point out an aspect so far undertheorized in research: conceptualizations of statistical literacy such as Gal's (2002) do not seem to include knowledge of forms of discourse on statistical information, of how to form a refutation and how to employ rhetoric. Neither does the framework of communicative rationality for statistical literacy developed by Büscher (2024a). This could be a promising direction for further research.

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Facilitating elementary school students' transitioning from statistical modeling of to statistical modeling for

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The pedagogical potential of the practice of statistical modeling has gained increased interest. However, the practice may be inaccessible for young students who had not yet developed familiarity with the necessary formal probabilistic tools. Engaging young students with informal corollaries has led to promising results, but the reasoning they express is often void of a deep appreciation of the stochastic behaviour of the model that formal probabilistic tools are used to estimate. Encouraging students to engage in modeling for examining the model's underlying stochastic mechanism, rather than solely engage in modeling of the investigated phenomenon, can facilitate the development of its appreciation. I offer a collective case study of three pairs of students' modeling progressions to illustrate how an innovative learning sequence can facilitate young students' transitioning from modeling of to modeling for, and subsequently elicit more mature reasoning with statistical models.

Keywords: Statistical modeling, Informal statistical modeling, Probabilistic reasoning.

Introduction

The pedagogical potential of the practice of statistical modeling has become a main focus of statistics education (Pfannkuch et al., 2018). Despite ongoing discussions, there is much more to elucidate. Specifically, while novices can be supported to develop models to represent what they see in the data, the data representations they create are often not accompanied with a mature probabilistic mechanism (Lehrer, 2017), which is a key characteristic of statistical models. In line with insight from mathematical modeling research, novices tend to engage in modeling *of* a specific phenomenon, rather than modeling *for* developing a deeper appreciation of the more general underlying mechanism of the models they use (Gravemeijer, 2002). This research designed an innovative activity sequence intended to support novices to transition from statistical modeling *of* to statistical modeling *for*, and examined how novices' reasoning with statistical models and modeling can mature throughout it.

Background: Statistical modeling of and for

With the goal of examining how young students can reason with statistical models and modeling, I have adopted the rudimentary view of models as purposeful representations (Hesse, 1962), simplified versions of a more complex phenomenon (Lehrer & Schauble, 2010). The simplification represented by the model may be found to be ill-suited for its intended purpose (Hesse, 1962). Therefore, a model should be continuously re-evaluated and refined. The process of model creation and renovation is what I refer to as modeling. The purpose of the model is key in distinguishing it from a mere representation, and in classifying the type of model. For example, mathematical models can be seen as intended to address a mathematical purpose, while statistical models are intended to address a statistical purpose (Dvir & Ben-Zvi, 2023).

While definitions vary, mathematical models are generally abstract constructs depicting characteristics of a structure or pattern (Lesh & Harel, 2003). The mathematical modeling process

centres on "mathematization", a process in which the original, often real-world, phenomenon is gradually depicted mathematically (Gravemeijer, 2002). The modeling process is iterative, and throughout it the forms of use of the model can transition (Gravemeijer, 1999): the initial use of the model, serving as a model *of* the phenomenon, is intended to identify the underlying mathematical features or structure of the specific real-world phenomenon. However, the model itself and its mechanism can become the subject of interest and investigation. The latter, referred to as modeling *for*, facilitates a deeper understanding of the abstract ideas that the model represents, and its reusability to explain *other* real-world situations with similar underlying mechanisms. This depiction of the mathematical modeling process can similarly apply to statistical modeling.

As statistical models are often formally depicted mathematically (Dvir & Ben-Zvi, 2023; Wild et al., 2018), statistical and mathematical models have similar traits. However, while the purpose of mathematical models is to deterministically describe or explain a structure or pattern, statistical models are intended to represent both a pattern or signal (systematic variation) as well as the noise around the noise (non-systematic variation, Dvir & Ben-Zvi, 2023; Konold & Pollatsek, 2002). The statistical model thus includes a stochastic mechanism (Davison, 2003) that accounts for the sources of the non-systematic variation, formally requiring familiarity with probabilistic tools. These are often not introduced in elementary school. Supporting young students to reason with *informal* alternatives of statistical practices (Makar & Rubin, 2018), involving little or no formal mathematical notations, can serve as an earlier introduction to their formal corollaries.

Various pedagogies have been developed to support novices' reasoning with informal statistical models and modeling. For example, Exploratory Data Analysis (EDA, Tukey, 1977) pedagogies that centre on engaging novices with data investigations, typically focus on seeking patterns in the data. EDA is therefore particularly well-suited to support novices' creation and refinement of deterministic verbal or visual (e.g., graphs or drawings) representations that serve as context-specific informal models *of* the investigated real-world phenomenon. However, the modeling process that EDA activities elicit, seldom leads emergently to students' transitioning to modeling *for* the purpose of examining the underlying stochastic mechanism through which the data were generated (Lehrer, 2017), which could then be applied to additional contexts. This is likely due to the typical engagement with only one real-world sample, whereas appreciating the probabilistic mechanism calls for envisioning all other samples that could have been generated by the same population (Lehrer, 2017). Similar constraints also characterize novices' engagement with Informal Statistical Inference (ISI, Makar & Rubin, 2018) activities, in which the novices' data exploration is guided by the purpose of generalizing beyond the data at hand. The request to formulate generalized claims often leads students to express some uncertainties. These expressions can be considered as initial reasoning with some aspects that would be formally represented stochastically. However, the models that the students generate still tend to be used as models *of* the specific examined phenomenon, and are not used *for* developing a deeper understanding of their underlying stochastic mechanism.

Complimentary, probability modeling activities (Franklin et al., 2007) are well-suited to support novices' creation and refinement of purely stochastic representations. However, these activities typically focus on non-realistic scenarios characterized by basic probabilistic mechanisms (e.g., a repeated coin-toss), challenging to apply to real-world phenomenon. To foster more applicable

insight, some have recently advocated for integrating probability modeling with EDA or ISI (Pfannkuch et al., 2018). Integrated approaches advocate for learning sequences that begin with students' modeling *of* real-world phenomenon, as they engage first with EDA or ISI activities. Probability modeling activities are then introduced related to the real-world phenomenon (e.g., Kazak, et al., 2018). In this study, I focus on a specific Integrated Modeling Approach (IMA, Manor & Ben-Zvi, 2017), in which the probability modeling activities focus on examining the model that the learners had created to represent their real-world ISI by conducting simulations of repeated sampling from it. By doing so, the learners are encouraged to transition from the initial use of the model as a model *of*, to using it as a model *for* examining its underlying stochastic mechanism, even if they are as yet unaware of its existence. To examine if or what shifts in their reasoning occur as a result, in the context of an IMA activity sequence, I examined: *How can novices reason with informal statistical models and modeling as they transition from model-of eliciting real-world activities to model-for eliciting probability modeling activities?*

Method

This study was conducted as part of Connections, a longitudinal design and research project. I employed a collective case study approach (Stake, 1995) to provide multifaceted in-depth accounts of three pairs of sixth grade (11–12-year-olds) students' reasoning, whose classroom participated in the 2016 iteration of the project. The classroom participated in a 19 lessons-long activity sequence inspired by the IMA (Manor & Ben-Zvi, 2017). The sequence included three cycles of ISI-extended EDA activities and, in accordance with the IMA, each was followed by a probability modeling activity that focused on models that the students developed during the preceding real-world data investigations. I focus on the students' participation in the second IMA cycle, in which they examined a sample size 60 that they collected from students in two schools in their district. Their purpose was forming inferences about a larger population of all of the students in their district (n=1300). The students' real-world data investigation began by their formulating research questions and conjectures that served as the students' initial models *of* the phenomenon. The students then created various data representations using TinkerPlots software (Konold & Miller, 2015), again serving as models *of* what the data indicated about the phenomenon. Based on these, they ultimately formulated inferences, verbal representations of their refined model *of* the examined phenomenon.

The need to formulate generalizations led the students to express uncertainty regarding the representativeness of their sample. The follow-up probability modeling activity was introduced as a means to examine these uncertainties. The activity was facilitated by the Sampler TinkerPlots tool, allowing to visually design a population, and simulate generating random samples from it. The students were asked to design the Sampler model according to their refined inference, and by doing so, even if they were unaware of it, attach a stochastic mechanism to their current inferred model *of* the phenomenon. The students were then asked to generate multiple random samples size 60 from the Sampler model and compare between them. The goal was to engage the students in examining sampling variation, the effect of the underlying stochastic mechanism, thus engage in modeling *for*.

The participants, data collection and analysis

The multiple cases study focused on three pairs of students: Noa and Ido, Erez and Orr, Ami and Gil, selected as they communicated their reasoning openly and fluently, and collaborated well. All of the participants' actions were videotaped by two cameras during class discussions, and Camtasia software to concurrently document the students' computer screen and articulations during their pair work. All videos were transcribed and analysed microgenetically (Siegler, 2006), to identify any visual or verbal representation that the students created, used or referred to. If the students assigned a purpose to the representation, it was classified as a model. Using the Reasoning with Informal Statistical Models and modeling (RISM, Dvir & Ben-Zvi, 2018) framework, the models were then classified as either Data models (i.e., representation of what the students saw in the data) or Conjecture models (i.e., representation of what the students believed beyond the data). In the final tier of analysis, the purpose of each model was further examined to identify whether it was used as a model *of* the phenomenon (i.e., focusing on the real-world phenomenon) or as a model *for* (i.e., focusing on the abstract stochastic mechanism of the model). Key scenes and their transcripts were discussed and triangulated (Schoenfeld, 2007) amongst three members of the Connections team.

Findings

Each of the three focus pairs formulated different research questions and created different models whilst investigating them. However, a shared pattern was identified in all three modeling progressions characterizing the students' transitioning from modeling *of* to modeling *for*. To illustrate this pattern, for each pair I focus on one repeated model that they invented and refined throughout the learning sequence. I provide here a more detailed account of one of the pairs' modeling progressions, followed by an overview of the similar trajectory identified in the progressions of the other two.

Erez and Orr chose to examine the relation between arm span lengths and the number of rope-jumps per minute. During their real-world investigation, the pair's original conjecture model was an informal expression of a linear positive relation (holding the rope "near" would make it "more difficult for you to jump"). However, they gradually acknowledged that their data showed otherwise, and consequently formed a new, more data-based, conjecture: "it [arm span] helps", informally representing a weaker positive relation (the students with longer arm span can potentially be better rope jumpers if they practiced, Figure 1, left). To initiate the probability modeling activity, the students designed the Sampler model to represent their refined "it helps" conjecture (Figure 1, right). To do so, they designed two different distributions of rope jumps per minute for shorter and longer arm spans lengths (e.g., the range of the shorter arm span students' rope jumps was much more limited, 0-60, than the range of the longer arm students' rope jumps, 0-200). As expected, all of the models that the students expressed during these stages of the learning sequence served as models *of* the examined phenomenon. However, as the pair began generating simulated samples, they continued to utilize the "it helps" model as a model *of* what they saw in the data (e.g., Erez: "it [the simulated sample] proves that it [arm span] only helps").

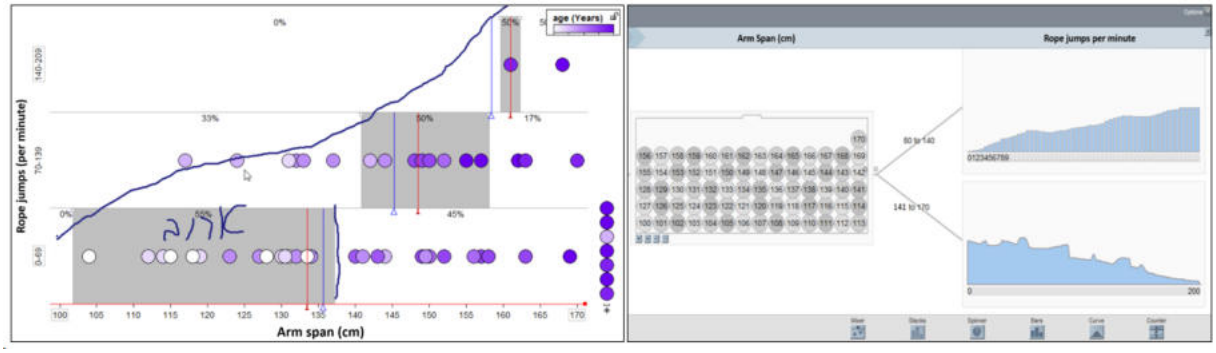

Figure 1: (Left) The real-world data representation of good, medium and bad rope jump results (three rows) according to arm span. The grey rectangles represent the original conjecture. The drawn line creating a triangle with the axis represents the "it helps" refined conjecture. (Right) The Sampler model representing the same "it helps" model, by assigning different distributions of rope jumps per minute to the shorter and longer arm span options.

Examining the initial simulated samples that they generated, the pair also created a new visual representation of a scatterplot, and emergently added drawings of circles, around some of the data but not all. This implies that the purpose of the circles was to represent the pair's "it helps" conjecture, thus again serve as a model *of* the phenomenon, rather than what the simulated data specifically showed. This was evident for example in Erez' explanation of the right upper circles (Figure 2, left):

> Erez: This [top left circle, Figure 2, left] is actually all of those that are, the arm span helps them jump but… they are not so good [rope-jumpers]… These [top right circle, Figure 2, left] are good [have long arm spans] and practice.

As the pair generated additional simulated samples, they continued to utilize the circles' representation. However, they began to articulate a different purpose for its use: comparing between simulated samples (e.g., Erez: "because of the circles that stay we can see it [a new sample] hasn't changed so much"). The shift in purpose is significant. Up to this point the students utilized the "it helps" model solely as a model *of* the real-world phenomenon, but were now starting to utilize it as a model *for* the data that its underlying stochastic mechanism can generate. The new use was further extended as the pair created a refined circles representation during the next lesson (Figure 2, right):

> Erez: We went through several samples and saw that it [the majority of cases] stays sort of in the same circles, it [the majority] doesn't go outside, here [in this simulated sample] it goes out [of the circles] a bit, the most that we have seen… but it is reasonable.

Erez' explanation implies another slightly different purpose of the circles model, representing areas where "it stays sort of in the same circles", meaning what the pair considered stable, or "reasonable" across the multiple samples that they "saw". Although based on what the pair "saw" in their data (multiple simulated samples), the behaviour Erez was now describing by the circles model was more general – "it doesn't go outside", thus serving as a conjecture model describing the limitations of the stochastic mechanism they have been examining and modeling *for*. Erez also utilized the insights gained about the mechanism to evaluate the current simulated sample: although "it goes out a bit",

even “the most that we have seen”, the pair considered it to be “reasonable”, as it is within the scope of the stochastic behaviour that the pair had come to expect.

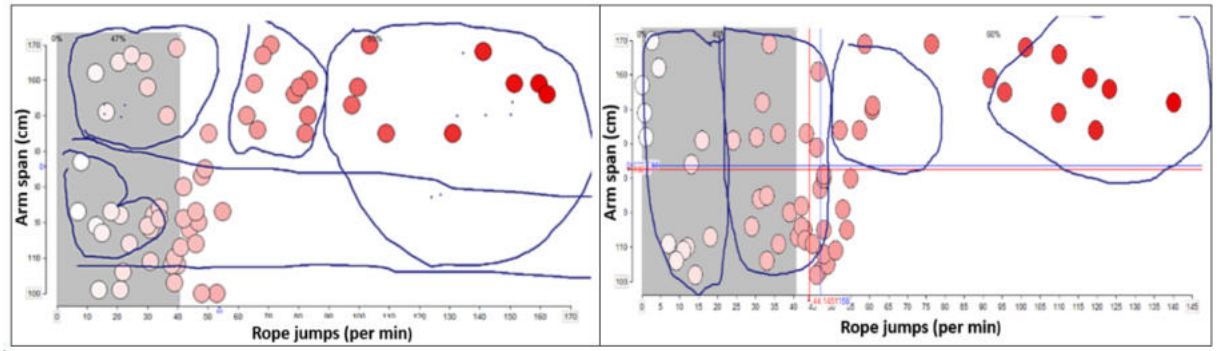

Figure 2: Representations of simulated samples, arm span according to rope jumps (color according to rope jumps) with the hand drawn initial circles model (left) and a refined version (right)

Ami and Gil examined a different type of question (“Who sleeps more, boys or girls?”) and developed a different type of repeated model to examine it. However, their model’s progression followed a similar path to that observed in Erez and Orr’s. Their repeated model focused on comparing between the shapes of the boys’ and girls’ duration of sleep distributions. Throughout the real-world data investigation, the model was only expressed verbally or using hand gestures. It was first visually represented by the shapes that the pair designed for each gender’s distribution in their Sampler model. Again, the model was exclusively used as a model *of* the examined phenomenon during these stages of the learning sequence. After generating their first simulated sample, the pair added drawings circumventing each distribution to represent the shapes they expressed earlier, still using the representation as a model *of* the real-world phenomenon (despite examining a simulated rather than a real-world sample). However, generating additional samples with varied distribution shapes, the pair started adding more circumventing lines (without deleting the earlier lines). The extended representation supported the adaptation of the initial model to a new purpose of comparing between simulated samples (e.g., Ami: “the blue line [I drew to represent the shape of distribution in this sample]… looks exactly the same as the other one”). The pair were therefore beginning to use the model as a model *for* examining the underlying stochastic mechanism. The pair also utilized a further extended version of the representation to summarize their insight about the mechanism, by drawing dots to represent the maximum variation they now expected from the original shapes.

Noa and Ido also examined a different type of question (“Is the stigma that girls run slower than boys true?”) and developed a different type of repeated model to examine it. The pair mostly expressed their repeated model verbally, referring to two medians (the boys’ and the girls’ 600 meter running times medians) and comparing them to determine which of the two genders were better runners. Unlike the other two pairs, they did not utilize the medians representation when designing their Sampler model. This was possibly due to Noa’s persistence that the Sampler model should reflect *no* difference between the boys’ and girls’ running results, despite what the medians in the real data reflected. However, they did continue to use the medians model as a model *of* the real-world phenomenon, as they examined the first simulated sample (Noa: “our [the girls’] median is here

[relatively small], and their median [the boys'] is here [larger]"), as did the other two pairs. Furthermore, when examining their second simulated sample the pair similarly assigned the representation a new purpose of comparing between simulated samples (e.g., Ido: "our [the boys'] median is becoming closer to yours"). The shift in purpose was also accompanied by a shift in the model use – from comparing medians within a sample, to comparing the difference between the medians across samples. These indicate a transition from using the medians as a model *of* what the data indicated about the examined phenomenon, to modeling *for* its underlying stochastic mechanism. The pair also repurposed the model to summarize their insight on the stochastic (here unlimited) behaviour they discovered (e.g., Ido: "It [the difference between the medians] can always change").

Discussion

The commonalities across the three case studies elucidate general aspects of how elementary school students' reasoning with informal statistical models and modeling can transition from modeling *of* to modeling *for* (Gravemeijer, 1999), during an activity sequence that integrates real-world data explorations (Makar & Rubin, 2018; Tukey, 1977) with probability modeling activities (Franklin et al., 2007; Kazak et al., 2018; Manor & Ben-Zvi, 2017) that focus on the students' invented models. Specifically, the initial stages of the probability modeling activity (i.e., designing a Sampler model and generating initial simulated samples) did not elicit the transition. This aligns with prior insight on students' initial naïve views when engaging with a single sample and challenges to envision additional samples that the same population (or stochastic mechanism of the model) can generate (Lehrer, 2017). However, as the students generated additional simulated samples they were provided with a tangible manifestation of the effect of the stochastic mechanism. Although unsolicited, all three pairs utilized the tools, i.e., the models, that they had already developed during the real-world data exploration and extended them to perform the samples comparison task. By doing so, they gradually emergently extended their initial use of the model as a model *of* the phenomenon, to a model *for* examining, evaluating and ultimately summarizing its underlying mechanism (Gravemeijer, 2002). The latter reflected rather mature reasoning with informal statistical models and modeling, despite the students' lack of familiarity with the formal probabilistic tools.

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Novice Students' Engagement with Big Data Modeling

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In today's digital society, supporting students in reasoning effectively with big data has become an important pedagogical purpose. This article examines ways to incorporate big data analysis ideas into teaching, focusing on informal modeling with big data and introducing students to the predictive purpose of a model. We designed a learning sequence and implemented it successfully with a class of 15 tenth graders. An analysis of the students' engagement uncovered two modeling approaches: creating descriptive models that led to overfitting by focusing on detailed data and creating explanatory models that emphasized generalization but neglected crucial data, resulting in underfitting. After reassessment and testing with new data, both groups opted to build a hybrid model, reflecting a more nuanced understanding of balancing model fit and predictive considerations.

Keywords: Informal statistical modeling, big data, predictive modeling.

Introduction

In today's data-driven world, reasoning effectively with and analyzing large datasets has become a crucial skill. The advent of data science has transformed how we approach problems, make decisions, and derive insights from data. With the proliferation of user-friendly tools enabling data exploration and analysis without requiring coding expertise, data science has transformed into a field accessible to non-experts. Consequently, there is a growing need to adapt educational strategies to equip students for this dynamic and evolving landscape. Traditional statistical methods and educational approaches have often focused on classical inference practices, which emphasize understanding underlying data distributions and hypothesis testing. However, in recent years, there has been a significant shift in practice toward predictive modeling techniques, prioritizing predictive accuracy and leveraging large, complex datasets (Breiman, 2001). This study explores a learning sequence that introduces students informally to the fundamental practices of big data analysis. The design is based on prior insights from statistics education literature on informal introductions to classical statistical practices (e.g., Informal Statistical Inference, ISI, Makar & Rubin, 2018), adapted to the different nature and purposes of central data science practices. As proof of concept, we present a learning sequence we designed to introduce a classroom of 15 tenth-grade students (15-16 years old) to the practice of big data modeling and illustrate its utility through the gradual maturation of considerations that the students expressed.

Literature Review

Reasoning with big data

In today's interconnected world, a vast amount of data is generated from diverse sources, such as sensors, smartphones, digital applications, and social media, often presenting unique challenges due to their incomplete, imprecise, misleading, or erroneous nature (Holmes, 2017). Unlike traditional educational statistical data, which is usually precise, comprehensive, and gathered through established sampling methods, this new data type plays an increasingly pervasive role in students' daily lives. Big data are characterized by the five V's: Volume, Velocity, Variety, Veracity, and Value. Volume reflects the sheer

amount of data generated by accelerated technology (Kitchin & McArdle, 2016), while Variety refers to the heterogeneous nature of datasets (Gandomi & Haider, 2015). Velocity reflects the rapid speed at which data are collected and analyzed, often in real time (Gandomi & Haider, 2015). Veracity addresses the reliability of data, focusing on their truthfulness, accuracy, and precision (Lukoianova & Rubin, 2014). Finally, Value pertains to the usefulness of the data, which, despite their typically low-value density, can yield significant insights when analyzed effectively (Gandomi & Haider, 2015). Due to these unique characteristics, reasoning with big data requires distinctive approaches compared to traditional data handling, centering largely on processes like data cleaning, organization, feature examination, and visualization. Modeling with big data similarly has features that distinguish it from classical statistical modeling.

Modeling in data science

Statistical modeling is a powerful tool for developing and testing theories through causal explanation, prediction, and description. It can be categorized into three main types: explanatory, predictive, and descriptive modeling, each serving distinct purposes in scientific research. Explanatory modeling is predominantly used in many scientific fields, particularly in the social sciences, to test causal theories by applying statistical models to observational data (Shmueli, 2010). Predictive modeling, in contrast, focuses on forecasting future or new observations by using statistical models or data mining algorithms. This approach prioritizes high accuracy in predictions over the model's interpretability and is central to fields like machine learning and data science, where models, such as random forests, are used to predict outcomes without necessarily explaining the underlying causal mechanisms. Descriptive modeling aims to summarize and represent the structure of data in a compact form, without necessarily focusing on causality or prediction. While explanatory modeling is used to test theories and predictive modeling aims at precise forecasting, descriptive modeling provides a foundation for understanding the data's characteristics, supporting both explanation and prediction.

There are two primary cultures in statistical modeling, one based on stochastic data models and the other on algorithmic models (Breiman, 2001). Stochastic data models assume a specific probability distribution or data-generating process, typically chosen based on the researcher's understanding or intuition about the underlying mechanism. These models emphasize interpretability and validity and are mainly used in classical statistics for explanatory or descriptive purposes, such as generalizing from small samples to larger populations under controlled conditions (Gould, 2024). While classical models are adequate for understanding causal relationships, they often struggle with complex big data problems, where algorithmic models excel.

Algorithmic models, central to modern data science, do not assume a predefined data structure. Instead, they use computational techniques to discover patterns directly from the data, prioritizing predictive accuracy over interpretability. Methods like random forests exemplify this approach, where multiple algorithms metaphorically "vote" on predictions, and the prediction receiving the most votes is selected (Gould, 2024). The predictive power of the model is typically determined by testing it on a portion of the data set aside prior to the training stage of the model. The main risk stems from overfitting the model to the training data, evident when the model performs lower than expected on the test data (Shmueli, 2010). On the other hand, underfitting might also occur, also evident in poor performances on

holdout sets, indicating that the model is too simple to capture the underlying patterns in the data. Both overfitting and underfitting limit the model's generalizability. The practice of algorithmic modeling therefore differs greatly from the classical practices associated with stochastic data models. However, it is similarly central and representative of the disciplinary (data science) culture and engaging novices with this type of modeling can have great pedagogical potential. A main hurdle might be, however, the computational complexities of the formal practice, which may render it inaccessible for novices who have not yet mastered the necessary formal language and tools.

To address this challenge, we propose devising *informal* corollaries, as had been done before to facilitate novices' engagement with central statistical practices. For example, to raise the accessibility of statistical inference, many have advocated for an informal corollary, Informal Statistical Inference **(**ISI, Makar et al., 2011; Makar & Rubin, 2018), informed by the central characteristics of the formal practice. In more recent years, similar strides have been made to raise the accessibility of the practice of statistical modeling through engaging with Informal Statistical Models (ISMs, Dvir & Ben-Zvi, 2021). Informal models can be defined as purposeful representations. When a representation is used for a statistical purpose—such as describing, explaining, or predicting—it can be considered an informal statistical model (Dvir & Ben-Zvi, 2021). These informal adaptations of central statistical practices serve as a bridge between novice learners' current understandings and formal statistical reasoning. Inspired by these, we designed and implemented a learning sequence based on the *Hypothetical Integrated Data Modeling Learning Trajectory* (Gafny & Ben-Zvi, 2023) grounded in design research methodology (Plomp, 2013), which was initially tested with two students before being implemented with a full class. The learning sequence consisted of informal adaptations of central data science practices and facilitated our examination of the following: *What considerations can novices articulate during informal modeling activity with big data?*

Method

This study was conducted as part Connections, a longitudinal design and research project that examines young learners' statistical reasoning, collaborating with the Taking Citizen Science to School (TCSS) center. We focus on the 2023-4 iteration of the project, in which a 12-lesson-long learning sequence accompanied the participation of a tenth-grade class from a public school in Northern Israel (15-16-year-old students) in the *Sleep-a Third of Life* citizen science project. The students had no prior experience in big data analysis. The students' participation in the project included filling out questionnaires that constitute a "sleep journal" about their sleep habits, *which* comprises some pre-chosen questions formulated by the sleep scientists who accompanied the project and additional questions that the students themselves chose to add. The students were granted access to the data they collected about their class and later to the larger project-generated data set (the "big data"). This dataset was derived from over 1,000 students across Israel who maintained a 14-day sleep diary that recorded 20 key categorical and numerical attributes, including age, gender, and sleep duration. The combined efforts of 38 classrooms resulted in a comprehensive dataset containing approximately 16,000 cases. Due to TinkerPlots software (Konold & Miller, 2015) limitations, the students analyzed 5,165 cases, using 4,165 to develop their models ("training data"), and 1000 cases to later test the initial models they created ("testing data").

The learning sequence. The central goal was to encourage students to emergently articulate the predictive purpose that a model can serve, particularly in the context of big data analysis. Building on the students' prior experience with classical data modeling and their familiarity with descriptive purposes that are more characteristic of classical data modeling, the learning sequence sought to facilitate their gradual discovery that models could be used for prediction. To support these objectives, the first phase began with small samples of self-collected data, following the growing samples heuristic (Ben-Zvi et al., 2012). The activities in this phase were informed by three pedagogical frameworks: Exploratory Data Analysis (EDA, Tukey, 1977), Informal Statistical Inference (ISI, Maker & Rubin, 2018), and the Integrated Modeling Approach (IMA, Manor & Ben-Zvi, 2017). The second phase focused on exploring the big dataset. It began with preparing the data for analysis, followed by creating an initial informal descriptive model of the connection between attributes chosen jointly by the class (net sleeping time and morning alertness). The descriptive purpose was expected to elicit considerations of model fit and potentially overfitting models. Mimicking the formal practice of choosing the best model, the students then began "voting" activities to choose one model out of the seven (each pair produced one model) to introduce to the sleep scientists as the class's finding. The voting process was two-staged: first, we divided the class into two groups of three and four pairs, resulting in two final rivalling models. In the second stage, the entire class selected one of the two. In the first stage, a criterion for what constitutes a "winning" model was not pre-defined, and students were free to express their criteria, use it to advocate for their model to be chosen, and vote accordingly. In the second stage, the predictive criterion, as it was not suggested emergently, was introduced to choose between the two models. The students were asked to test the predictive power of the model they voted for using the 1000 cases set aside earlier, serving at this stage as test data. In a final whole class discussion, the predictive power of the two models was compared, and students were requested to select a "winner" accordingly.

Data collection and analysis. We videoed all the participants' pair work using Camtasia to document the students' computer screens and articulations concurrently. We documented class discussions with two cameras and transcribed and analyzed all videos using the interpretative microgenetic method (Siegler, 2006). This method allowed us to examine the models that the students generated and reasoned throughout their participation; it involves closely examining learning as it unfolds in real time, focusing on incremental changes to understand how and why thinking processes develop. Specifically, we identified any visual or verbal representations that the students created or articulated. If the representation was purposeful, it was classified as a model. The models were then classified by purpose and fitness to data. Key scenes were discussed and triangulated (Schoenfeld, 2007).

Findings

We provide an overview of the types of models that the seven pairs created and transformed throughout the informal modeling activity and the different considerations they expressed alongside their creation or use. We introduce these chronologically, according to the three stages of the activity: a) developing models based on training data; and b) the initial voting stage; and c) the second voting stage informed by testing the models' predictive power.

Developing models based on Training Data. During this activity, each of the seven pairs were asked to develop a descriptive model (i.e., a purposeful representation, meaning they were free to

develop representations in any manner they found appropriate) for the relation between sleep duration at night and the level of alertness the following morning, based on the Training Data. Using the various TinkerPlots visualization options, each pair chose different means and tools, resulting in a diverse range of data representations, each reflecting the pair's final model. Despite these differences, the seven models can be grouped into two groups based on central commonalities. Group 1 consisted of three models, all of which we classified as having a descriptive purpose in line with the explicit guidance that the students received. In addition, we classified the three models as overfitting the data, evident by: inclusion of all cases (even those clearly erroneous); data representations that are difficult to decipher (often due to scaling issues, related to the choice to include all extreme cases, e.g., Fig. 1); and representation mostly of the noise in the data rather than the signal (often obscured in the complex data displays). In contrast, Group 2 consisted of four models, all of which we classified as having an explanatory purpose, despite the explicit guidance they received. We also classified these models as underfitting the data, evident by an omission of a large portion of the cases (restricting the examination only to data considered to be "typical"); readable simplified data representations (e.g., Fig. 1); and a focus solely on the (clearly evident) signal.

The initial voting stage. In this stage, the students worked in two larger groups, divided based on the commonalities identified in the previous activity. We guided the groups to select one model out of the three or four each group had constructed based on a criterion of their choosing. The chosen models reflected the criterion on which they were based. Group 1 chose the overfitting model illustrated by a representation of the distribution of each alertness level across the entire possible range of sleep hours (0-24, Fig. 1). The inclusion of all of the data resulted in a representation in which a clear signal was harder to identify. The group did attempt to represent the pattern by adding a red line to the central clump of each distribution (sleep durations between 4-12 hours). However, in a written explanation they provided for their chosen representation, the students did not express an actual relation between the two attributes; the criterion for the choice of the model implied by the explanation was providing a comprehensive description of what the students observed in the data. Group 2 (Fig. 1) chose the underfitting model illustrated by a representation that focused on the central range of sleep duration, specifically between 3-12 hours. They completely disregarded the values they considered as extreme or atypical (0-3 and 12-24). To represent the pattern that the model focused on, they used averages, boxplots, and a hand-drawn trend line. The representation was chosen as it clearly illustrated that "for most teenagers, the more they sleep, the higher their level of alertness." Their criterion was implied by the focus on the pattern as well as the wording the students chose: the more one sleeps, the more alert they will be, providing a (causal) explanation for the observed pattern.

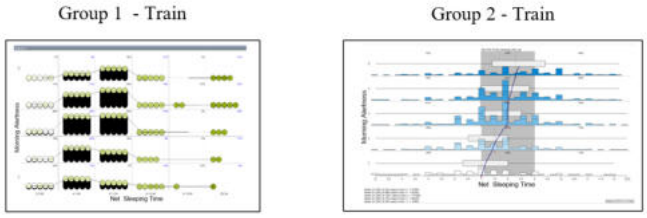

Figure 1. The model representations chosen by Groups 1 and 2, based on the Train Data, with the x-axis representing net sleeping time and the y-axis representing morning alertness.

The second voting stage. This stage was informed by testing the models' predictive powers. Each of the two groups presented their chosen model to the other group. Both groups were then tasked to vote on which of the two "winning" models should be presented to the sleep scientists as the classroom's final finding. A whole class discussion was conducted to consider additional criteria for the selection. The students continuously advocated for either descriptive or explanatory purposes and the overfitting or underfitting models they associated with each, each group advocating for one and dismissing the other. The researcher then suggested predictive power as an optional criterion, and the students accepted it as a reasonable "tiebreaker." The Test Data were then organized according to the two "winning" data displays (Fig. 2), and the students were asked to compare the train and test performance for each. This comparison led to a critical realization that neither model was fully adequate for predicting future data; one was overfitting, and the other was underfitting. This encouraged the students to agree that a hybrid model that could balance the strengths of both approaches—capturing significant patterns (signal) while reducing the impact of overfitting to noise—would better address the predictive purpose.

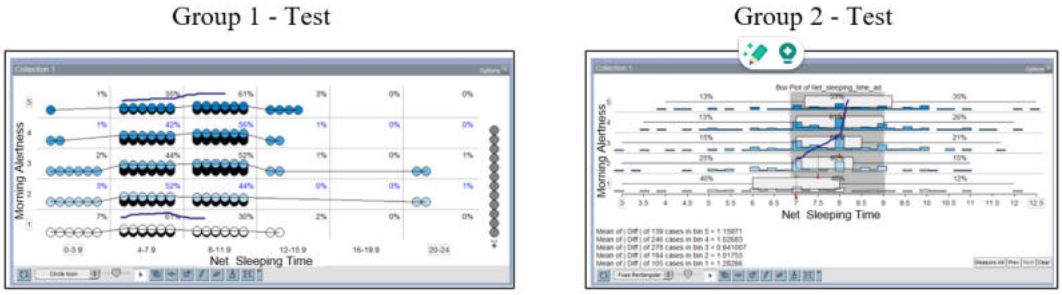

Figure 2: The model representations made by Groups 1 and 2 based on the Test Data, with the *x*-axis representing net sleeping time and the *y*-axis representing morning alertness.

Discussion

A comparison between the two groups reveals distinct differences in their modeling strategies. Although both groups were instructed to build a model to describe the phenomenon, their approaches diverged significantly. Group 1 focused on creating a descriptive model, prioritizing a detailed account of the phenomenon over generalization. They did not ignore the aspect of generalization entirely; for instance, they drew a red line to illustrate the model's prediction versus the actual data. However, their primary goal was to provide a thorough description, which led to an overfitted model that captured too much of the data's specifics (Shmueli, 2010). In contrast, Group 2 adopted an explanatory approach, focusing predominantly on generalization and the pattern while neglecting a significant portion of the data. It is possible that they were drawn to explanatory considerations due to the scientific context of their inquiry and in a different, non-scientific context, they would not have developed an explanatory model. Furthermore, their emphasis was on broad generalization rather than detail. When asked whether their model accurately represented the entire population of adolescents in Israel, they confidently claimed it did, without considering the need for adjustments to account for outliers or atypical cases. This approach resulted in an underfitted model with limited descriptive insight (Shmueli, 2010).

The connection between each group's purposes and their outcomes was evident: the first group's focus on a descriptive purpose led to overfitting, while the second group's focus on an explanatory

purpose resulted in underfitting. Encouraged to assess the predictive capability of their models, both groups discovered the limitations of their approaches. Group 1 recognized that their overly detailed account made their model unsuitable for new data, while Group 2 realized that their overly generalized account did not adequately account for new data. By agreeing to build a hybrid model, the students acknowledged its potential to address the shortcomings of their initial models. This decision demonstrated their evolving understanding of predictive modeling complexities and their readiness to explore more advanced strategies. The hybrid approach allowed them to navigate the trade-offs between overfitting and underfitting, providing them with deeper insights into effective modeling practices in data science.

Through the design of the learning sequence, the students informally experienced several key principles in big data analysis, including: (a) The practice of train and test - dividing data into a training set to teach the model and a test set to evaluate its performance on unseen data (Holmes, 2017). The students built the model with the Training Data dataset and evaluated its performance by comparing it to the results that came from processing the model on the Testing Data dataset. (b) Overfitting and underfitting models: Understanding when a model is too complex (overfitting) and captures noise from the Training Data, or too simple (underfitting) and fails to capture underlying patterns (Shmueli, 2010). The students, by comparing each group's winning model, were able to understand the benefits and limitations of each model in the context of overfitting vs. underfitting. (c) Balancing between model fit and predictability: striking a balance between a model that fits the Training Data well and one that generalizes effectively to new, unseen data (Breiman, 2001). The hybrid model suggestion expressed an appreciation that they can balance fit and generalization concepts. (d) The students' modeling selection – choosing between models based on accuracy and predictability: The process of selecting the best model by comparing different models and finding the one that offers the highest accuracy while maintaining good predictive power (Gould, 2024). The multi-models that the whole class created provided a tangible experience of the practice of building multiple models and "voting" on predictions.

We credit these learning outcomes to the fact that the design of the activity sequence was informed by the formal data science practices but made them accessible through informal adaptations. Specifically, we consider *informal predictive modeling* to be a potential gateway to data science, as stochastic modeling has been acknowledged as a gateway to classical statistics (Dvir & Ben-Zvi, 2021).

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Informal conceptions of appropriate choropleth maps held by in-service East and Southeast Asian teachers

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This paper examines how a group of 14 in-service East and Southeast Asian teachers reason about what makes an appropriate choropleth map, in the context of a task that required from them to choose an effective visual way to communicate the geo-spatial spread of the COVID-19 disease. The findings revealed that, while all the participants reasoned about the task using different visual map components mentioned in the literature (e.g., color scheme, number of classes, class limits, and administrative unit level), most of them focused on the color scheme and the legend's value ranges, and only one considered the administrative unit level. Moreover, contrary to what many specialists recommend on how to create appropriate choropleth representations, a red color scheme was found to be more appropriate for a map content related to a pandemic by the study participants, which may have a cultural connotation to it.

Keywords: Choropleth maps, conceptions, in-service teachers, geo-spatial data.

Introduction

When the COVID-19 pandemic started, a very common way for media to present the panorama of the disease in the world has been through choropleth maps, which have been extensively used by health professionals and general public to identify regional hotspots and to find geo-spatial strategies against the pandemic (Juergens, 2020). Choropleth maps (also known as geographic heat maps, shaded maps or graduated color maps) are graphical representations showing the distribution of statistical data over a geographic region, in relation to a quantitative or categorical variable (Camm et al., 2021). Choropleth maps provide a simplified method to visualize, using color codes, geo-spatial data[1] on how a measurement or attribute (e.g., the absolute count of confirmed COVID-19 cases worldwide) varies across a geographic area or show the level of variability within a region, and are particularly useful when analyzing epidemiological data. That is why the use of these artifacts has become so popular, since most of the public is not trained to read data-driven maps critically, which makes the visualization of the COVID-19 pandemic with easy-to-understand choropleth maps crucial. Moreover, Thai and worldwide health experts believe that the use of geo-spatial data, including representations such as choropleth maps depicting disease density, spreading area, disease prevalence areas, resource distribution, admission rate and COVID-19 distribution patterns, allows health authorities to synthesize large amounts of information that can then be used for the prevention and control of diseases and resource allocation (Laoluekiat et al., 2022).

Due to this growing presence and importance of choropleth maps in our daily life, teachers are expected to promote their students' statistical literacy skill related to this graphical representation.

[1] Geo-spatial data is defined as a composite of descriptive or thematic content (e.g., attributes characterizing a spatial entity) and spatial components (e.g., coordinate system and geometry; Juergens, 2020).

This demands from teachers a specific set of professional competencies, which depend heavily on their professional knowledge and their conceptions of curricular contents (Gal, 2002; González, 2021). Moreover, in the case of Thailand, the use of maps and geographical tools for searching, analyzing, drawing conclusions, and making efficient utilization of geo-spatial data is a key curriculum standard shared across many subjects (Ministry of Education of Thailand, 2008). However, the levels of graph competence and geo-spatial literacy of school teachers have been reported to be worrisome by many authors (see Groth & Meletiou-Mavrotheris, 2018, for a review of such works), and, to the knowledge of the author, there are no reported studies on in-service school Thai teachers in relation to their informal conceptions of choropleth maps.

Consequently, the following research question was posed to clarify aspects of the professional knowledge and conceptions of what makes an appropriate choropleth map held by in-service teachers from different East and Southeast Asian countries working in Thailand: How do in-service East and Southeast Asian school teachers working in Thailand conceptualize what makes an appropriate choropleth map, in the context of looking for an effective visual way to communicate the geo-spatial spread of the COVID-19 disease, and how these conceptions align with previous literature?

The importance of teachers choosing appropriate graphs for the learning of statistics

Reading and interpreting statistical graphs is an essential element in acquiring statistical literacy, which Gal (2002) described as the union of two related competencies: (a) interpreting and critically evaluating statistically based information from a wide range of sources; and (b) formulating and communicating a reasoned opinion on such information (González et al., 2011). Statistical literacy also involves the choice of an appropriate graph in relation to both the situation and the data being represented (González, 2021). Moreover, it is important to consider the possible biases that may be transferred to the graph, voluntarily or involuntarily, especially when choosing the scale or the administrative unit level in a choropleth map (Field, 2020; Juergens, 2020; Lane et al., 2022).

Choropleth maps, as well as other powerful data display artifacts (e.g., dynamic population pyramids or maps of commuter flow) have been reported to facilitate the conduction of statistical investigations, and hence the use of these graphs in the classroom by teachers is highly recommended, so they can engage and enrich student learning of statistics (Promodou & Dunne, 2017). Data visualization is a key pedagogical approach focused on helping students develop data analysis skills and engage with the investigative cycle in the process of exploring data. By using graphical artifacts displaying geo-spatial data, such as choropleth maps, teachers are able to select or create a map that allow students to search for data that may contain relevant information for their analysis task, search for a visual structure that is appropriate for the data, develop further insight, and then make an informed decision and take action (Promodou & Dunne, 2017). However, choosing or creating a task that will trigger the process of formulating a statistical question that can be answered with data, analysing the data with appropriate graphical and numerical summaries, and interpreting the results as they relate to the original question of interest has been found not to be easy for school teachers (Groth & Meletiou-Mavrotheris, 2018; González, 2021; Isoda et al., 2018).

In order to exhibit graphical competence, a data user must be able to read between, beyond, and

behind the data displayed in different sorts of graphs and to interpret meanings from them; to select and create appropriate graphs for specific situations, with or without the support of technology; and to critically evaluate graphs and to distinguish the relative strengths and limitations of particular graphical representations of data (González et al., 2011). Therefore, teachers must master themselves graphical competence before fostering it in their students. Moreover, there is plenty of evidence in the literature of a tendency to underestimate the complexities of learning and teaching graphical representations by both pre- and in-service school teachers, their poor level of graph reading and interpretation skills, their tendency to fail to integrate graphical knowledge with problem context, and their trouble with graph selection and understandings of data types (Groth & Meletiou-Mavrotheris, 2018; González, 2021; Isoda et al., 2018). With the popularity of choropleth maps in media nowadays, it is crucial for teachers to be able to know the geo-spatial data literacy aspects of choropleth representations, and become a sort of cartographer or geo-information expert themselves (Field, 2020; Juergens, 2020), in order to support their students in achieving the educational goals and objectives of the official curriculum documents in relation to the teaching of the statistical contents.

Choropleth maps and COVID-19 visualization

In a choropleth map, each geographic region is color-coded to represent a quantitative value based on a scale with color variation intervals that go from a light tint for smaller values to a dark shade for larger values. This color code allows the reader to recognize the implication of the general patterns of light and dark shades in relation to the variable at hand. This approach makes the viewer's perception of the map's values immediate, since it is quite intuitive (Camm et al., 2021; Kirk, 2016).

Besides the color code scheme, there are other key components of a choropleth map, including the legend, number of classes used in each color code scheme, the class limits or value ranges for each color, and the administrative unit level for data visualization (e.g., school districts, emergency response districts, tax zones, cities, counties, provinces, states, or countries; Field, 2020; Kirk, 2016; Juergens, 2020; Lane et al., 2022)

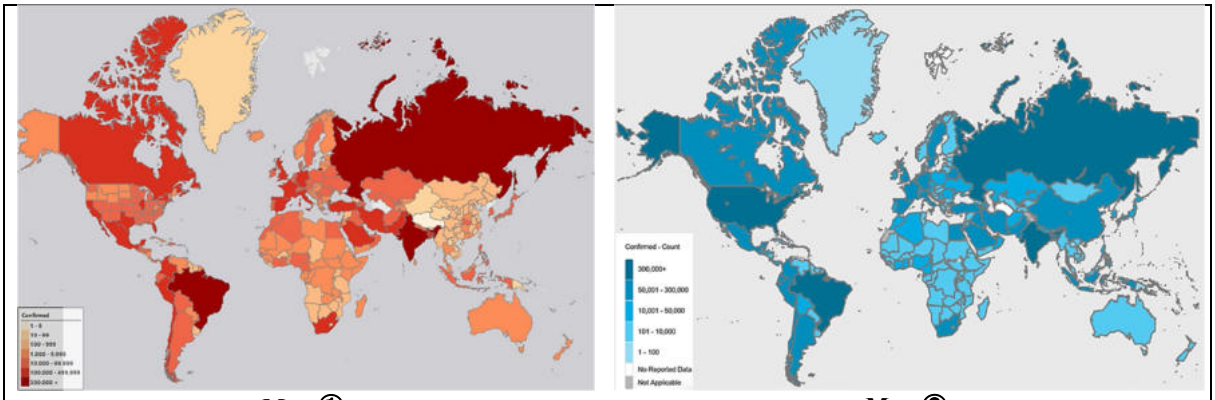
If the quantitative measure being depicted is about, or related to, the consequence of more people living in an area, interpretations of the choropleth map may be distorting and misleading; so, in such cases, it is suggested to consider transforming data to per capita (or other spatial denominator) to standardize the analysis accordingly (Juergens, 2020; Kirk, 2016).

During the COVID-19 pandemic, choropleth maps were a common representation in media to depict the spread of coronavirus cases around the world. Choropleth maps are particularly appropriate for depicting standardized data like rates, percentages or densities, rather than counts or totals. Absolute counts usually increase with area size, and this can be misleading. Many maps for the pandemic have used absolute values to show the number of cases across a country with a choropleth mapping technique. This is known to lead to accuracy issues, as different areas have different numbers of people in it, which leads to inaccurate conclusions when looking at a choropleth map (Field 2020; Franch-Pardo et al. 2020; Lane et al., 2022). Another source of error for choropleth maps is when the map maker uses the "default" classification scheme for creating classes (or different patterns) for the map, instead of figuring out which type of breaks or classes make the most sense for what is being conveyed (Lane et al., 2022; Monmonier, 2018).

Population normalization can solve some of the accuracy problems of creating a choropleth map based on raw numbers (Field, 2020). It is achieved by dividing the number of cases by the population in a specific area to give a percentage of that area that has been infected. In order to make the COVID-19 cases comparable, one has to normalize them against the total population. A widely used rate is cases per 100,000 inhabitants (Juergens, 2020). Otherwise, when visualizing the data with only raw numbers, some areas can appear to have very few cases, but instead have a high percentage of infection if that area has a low population.

Methodology

The task for this study (see Fig. 1) was developed inspired on the discussion made by Juergens (2020) on how choropleth maps are used for effective visual communication of the geo-spatial spread of the COVID-19 disease, and how many map creators seem to forget about the fundamental principles of good and easy-to-read thematic choropleth maps, which requires geo-spatial data literacy. Two choropleth maps from Juergens' (2020) article were chosen, both showing the same information, but with different level of detail: the absolute count of confirmed COVID-19 cases worldwide as of July 6, 2020. Map ① was originally retrieved from the University of Virginia's COVID-19 surveillance dashboard, while Map ② was from the World Health Organization's COVID-19 dashboard. This task was designed to look at the participants' conceptions of what constitutes a good choropleth map, which in turn will reveal aspects of their geo-spatial data literacy and their ability to judge the reliability of spatial data (Field, 2020; Franch-Pardo et al., 2020; Juergens, 2020).



Map ① **Map ②**

Both Map ① and Map ② show the same information: the absolute count of confirmed COVID-19 cases worldwide as of July 6, 2020. Which map (Map ①, Map ②, both, or neither of them) would you consider the most appropriate to use in the classroom to explain the level of worldwide spread of COVID-19 on the given date to your students? Please justify your choice with as much detail as possible.

Figure 1: The task given to the study participants, adapted from Juergens (2020)

The subjects who participated in this study consisted of 14 in-service East and Southeast Asian teachers (four elementary and 10 secondary school teachers, two from Thailand, seven from China, four from Myanmar and one from Vietnam) who were taking a 15-week introductory graduate course in educational research methodology at a private university in Bangkok, Thailand. The main commonality in the statistical preparation of all the participants was their prior learning experiences

with measures of central tendency and variation. All participants were given the study task in mid-June 2024, at the beginning of the first session of the course, for completion in class, prior to any formal instruction or attention being explicitly given to the concept of distribution of a quantitative or categorical variable over a geographic space or within a data set. The task was not given as part of a formal evaluation for the course, but rather as a way of having the subjects show their initial informal thinking on what makes a good choropleth map. All participants completed the task within 30 minutes. The written responses to the task were collected and then transcribed by the researcher.

The study data, comprised of the written responses and transcriptions of the task shown in Fig. 1, were then coded by conducting an inductive content analysis using grounded theory (Strauss & Corbin, 1990), to examine the transcripts of the responses given by the study participants, group similarly-structured answers into categories for analysis, and organize such categories into themes.

Results

About 43% (6 out of 14) of the participants chose Map ① as the most appropriate choropleth map to use in the classroom to explain the level of worldwide spread of COVID-19 on the given date to their students. About 43% (6 out of 14) of the participants chose Map ② as the most appropriate one; about 7% (1 out of 14) chose both of the maps; and about 7% (1 out of 14) chose neither of the maps.

A qualitative analysis of the explanations supporting each participant's choice allowed me to identify different reasons for the choices made by them. Thus, in addressing the study's research question ("How do in-service East and Southeast Asian school teachers working in Thailand conceptualize what makes an appropriate choropleth map, in the context of choosing an effective visual way to communicate the geo-spatial spread of the COVID-19 disease, and how these conceptions align with previous literature?"), results are presented according to responses that focused primarily on color scheme, number of classes, class limits, and administrative unit level.

Color scheme

Out of all subjects, about 43% (6 out of 14) of the participants justified their map choice making reference to the color scheme, preferring the red scheme instead of the blue one. The exemplars that follow show the comments of some of the participants, followed by the initials of the subject.

HS: The color of Map ① indicates very alert for dangerous situation as most countries use red color as a sign of danger or reminding something's terribleness [...] But the color of Map ② is vague.

XY: I would like to choose Map ① because red color will draw your attention at first and the red color may metaphor an urgent situation of one thing. Actually, pandemic is a global issue, which is very urgent to every country. […] When you use blue to indicate the level of worldwide spread of COVID-19 which shows a peaceful vibe to people. They won't be alerted how urgent this pandemic is! When audiences go through Map ② without reading the descriptions, an audience member like me will think like this "Wow! It's more like a world map to indicate where is the water area and where is the land area." Somehow, when you look at the Map ①, I get a sense of something urgent is happening right now, I think I want to know more about it!

Number of classes

Out of all subjects, about 36% (5 out of 14) of the participants justified their map choice making

reference to the legend's number of classes. Both maps have the same granularity of their classes: seven classes each, although only five of those classes are numerical for Map ②. The exemplars that follow illustrate this category of responses.

CP: I would like to choose Map ②, because it […] reports some regions without reporting data and regions that are not applicable. In order to ensure the accuracy of the data, most of the information we can receive at present comes from official platforms and reports, but the data in some places cannot be collected or is not officially displayed. The Map ② marks this point, […] but there are no hints to indicate those areas that have not been reported in Map ①.

SL: Map ② also indicates the regions without reported data and the regions that are not applicable, which seems more rigorous and realistic.

Class limits

Out of all subjects, about 29% (4 out of 14) of the participants justified their map choice referring to the legend's class limits. The exemplars that follow illustrate this category of responses.

AY: The numbers of Map ① is somehow too detailed, and it is less meaningful as there must be mistakes and omissions in the process of information collecting.

SL: The number of people in each color area in Map ② is more reasonable and less complicated for readers.

Administrative unit level

Out of all participants, about 21% (3 out of 14) of them justified their map choice making reference to the map's administrative unit level (i.e., countries, or states or provinces within a country). The exemplar that follows illustrate this category of responses.

EZ: I'm going to choose Map ②, because Map ② is displayed by countries, which is simpler to analyze. Map ① depicts the cases in various regions and countries. There are excessive samples.

RS: Map ② is easier to look at initially because it does not divide US and Chinese states. Because of this, there seems to be some conflicting data on the maps; it appears that in Map ① the US has less cases than in Map ②.

Discussion and conclusion

The study participants were able to reason about important components of a choropleth map, including legend, color scheme, number of classes, class limits, and administrative unit level. From the data analysis, it was also observed that none of the participants mentioned the countries' population or the use of raw counts instead of using population normalization as an issue for making an appropriate choropleth map for displaying COVID-19 information (Field, 2020; Juergens, 2020). Only 1 participant expressed three different characteristics of an appropriate choropleth map, while most of the participants mentioned two. However, only two were able to realize the misrepresentation of spatial information in Map ①, which depicts simultaneously countries, US states, and Chinese provinces, instead of the US and China as countries. This may hint for a need for improvement of the geo-spatial data literacy skills of the participants, which is a key skill nowadays, when choropleth maps are very common in media to visualize and understand not only the geo-spatial spread of a disease, but also election results, population density, healthcare access, and climate patterns (Camm et al., 2021; Juergens, 2020). Moreover, being able to choose the appropriate graph and create appropriate graphs for specific situations, with or without the support of technology; and being able

to critically evaluate graphs and to distinguish the relative strengths and limitations of particular graphical representations, are skills associated to graphical competence and statistical literacy in general (Gal, 2002; González et al., 2011).

The informal descriptions made by the participants of what makes a good choropleth map align well with the characteristics of appropriate choropleth map representations found in the literature (e.g., (Field, 2020; Franch-Pardo et al., 2020; Juergens, 2020; Lane et al., 2022). For instance, map errors and misrepresentation of spatial information (e.g., depicting simultaneously administrative levels of different hierarchy, such as countries, US states and Chinese provinces) is a common way to "lie with maps", influence the reader's understanding in a certain way, and then lead to inaccurate analyses or results by the reader (Juergens, 2020; Monmonier, 2018).

In relation to the color scheme in choropleth maps, red is a common color to use for danger and urgency, that can influence a reader to feel more alarmed about the data (Lane et al., 2022). In fact, red color hues, which are known to be able to also cause anxiety, fear or danger on map readers, are often referred as inappropriate for choropleth map representations, particularly when the map content is related to a pandemic, which is an unpleasant topic by itself (Juergens, 2020). However, in the current study, it was found that all the participants who referred to the color scheme preferred the red over the blue one, claiming that this sense of urgency, alert and alarm about the data triggers the reader's attention to the map information. This conception could have some cultural connotations, since red has been the color of power, status, and repelling evil in Chinese and other Confucian heritage cultures, with copper-red hues having a meaning of difficulty attached to them (Stuart, 2016).

Since the Basic Education Core Curriculum of Thailand mentions as a common standard for many subjects the utilization of maps and geographical instruments for searching, analyzing, drawing conclusions, and efficient utilization of geo-data and information, as well as the ability to explain meanings of data from diagrams, maps and charts in search for data to analyze the physical and social characteristics of Thailand, Asia, Australia, Oceania and the rest of the world, it is necessary the promotion of statistical and geo-spatial data literacy in our teachers, and this ongoing research may shed light on the possible issues held by our teachers in relation to their ability to choose, critically evaluate, and create appropriate graphs for specific real-life situations. Therefore, more information is needed on the teachers' statistical literacy skills related to graphical representations of geo-spatial data, as well as their specialized content knowledge in a context of graphical representation of data, which will provide more evidence on the teachers' readiness to assist students in realizing errors, misrepresentations and (in)appropriate components in graphs, and in properly using their contextual knowledge of graphs (González, 2021).

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Richness of reflective processes on algorithmic decision-making systems by prospective teachers

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As Artificial Intelligence (AI), algorithms, and algorithmic decision-making systems (ADMS) play an increasingly important role in everyday life, it is crucial to incorporate the promotion of data literacy into teacher education. In a seminar for prospective math teachers the functioning of ADMS in general and decision trees in particular was explored. Since in mathematics learning reflection processes play a significant role, the material was designed not only to build basic knowledge on ADMS but to foster reflections on its appropriate use and the associated purposes. In this study the richness of the initiated reflection processes is demonstrated.

Keywords: Data literacy, reflection, learning processes, critical mathematical thinking, teacher education.

Introduction

Due to the steady increase of computing power and the availability of large amounts of data, the use of algorithms and AI-based technologies has significantly increased. As a result, algorithms are increasingly being used in decision-making processes. The use of such systems, called algorithm decision-making systems (ADMS), entails not only opportunities but also significant risks, for example, when personal data is processed and an ADMS makes decisions about individuals based on that data as access to credit, employment, medical treatment and judicial sentences (Castelluccia & Le Métayer, 2019). In the light of these challenges, it is crucial to educate the population to become responsible citizens, so they are not merely passive consumers but can actively shape the digital world through critical questioning. Teachers play a central role in the promotion of mathematical critical thinking and data literacy, as they have a direct influence on how future generations will interact with technology (Casal-Otero et al., 2023). An essential step in the inclusion of data literacy in classroom is therefore the promotion of data literacy among prospective teachers.

The research project presented here focuses on promoting algorithmic literacy among prospective math teachers as one component of data literacy, with a particular emphasis on fostering reflections on the use and purpose of ADMS.

Theoretical background and research question

One component of data literacy is algorithmic literacy (Büscher & Lengnink, in press). According to Lengnink (2020), algorithmic literacy includes, first, the "ability of an individual to interpret algorithmic processing and critically evaluate results generated by algorithms in various contexts" and, second, the "attitude to discuss reactions and opinions regarding such algorithmically obtained results and to critically question their implications" (Lengnink, 2020). Thus, algorithmic literacy is not limited to the understanding of a concept or method but also significantly includes diverse reflections on the significance and implications of concepts, methods, and results.

In mathematics learning, reflection processes are needed that go beyond the acquisition of mere knowledge and focus on the contextualization of this knowledge, as well as the possibilities and limitations of using mathematics (OECD, 2017). In the context of ADMS, this means that not only basic knowledge like a fundamental understanding of ADMS but also reflective knowledge such as the possibilities and limitations of ADMS must be learned.

Reflective knowledge, which refers to the "competence in reflecting upon and evaluating the use of mathematics" (Skovsmose, 1994), can operate on various levels. According to Skovsmose (1998) there are four kinds of reflection. The *mathematics-oriented* reflection focuses on the object (a concept or algorithm) itself. When the appropriate use of the object (relationship between the object and an extra-mathematical reality; validity of the model) is discussed, it is about *model-oriented* reflection. Reflecting on the actual purpose of carrying out the modelling (in the political, social or cultural context) is classified as *context-oriented* reflection. The fourth kind of reflection is the *lifeworld-oriented* reflection, which occurs when the significance of the topic for one's own life is considered.

A structure of various components of a holistic concept of competence can be found in the orientation framework for the learning area of global development (Schreiber & Siege, 2016). This framework lists the competence areas of *recognizing*, *evaluating*, and *acting*. In the area of *recognizing*, knowledge is purposefully acquired. *Evaluating* involves the critical ability to assess the suitability and value of different outcomes, information, or sources. *Acting* entails making (well-founded) decisions and transforming acquired knowledge into responsible actions.

Materials that can be used to learn about the functioning of various ADMS can be found, for example, in the projects "AI unplugged" (www.aiunplugged.org) and "Data Science and Big Data in Schools" (www.prodabi.de). However, the focus of the material is often on conveying basic knowledge, i.e. the understanding of the algorithm itself. Reflections about the use and purpose of ADMS are less actively encouraged. Bilstrup et al. (2022) found in their literature review that reflective discussion with ADMS is underrepresented compared to the demonstration of the performance. Sanusi et al. (2023) identify a similar gap in teaching and research and request that societal and ethical aspects should be included when dealing with ADMS. Correspondingly, the reflective processes of learners in the context of ADMS have been little researched.

In response to this gap in teaching and research, Büscher and Lengnink (in press) developed a seminar concept on ADMS with the goal of initiating the areas of competence *recognizing*, *evaluating*, and *acting* (Schreiber & Siege, 2016) and the four levels of reflection *mathematics-oriented*, *model-oriented*, *context-oriented*, and *lifeworld-oriented* (Skovsmose, 1994) in reflection processes among students. With the help of material from "AI unplugged" (www.aiunplugged.org), knowledge about the functioning of decision trees is built up and reflections are initiated in discussions and written reflection tasks (more details follow in the "Method" section). The material was used in a seminar for prospective math teachers and the written products of the reflection tasks were qualitatively analysed with the aim of specifying the reflection processes in more detail.

In the follow-up study presented here, the material is re-used in a slightly modified version and the written reflection tasks will now be analysed with the focus on the richness of the reflection processes of the students. Specifically, the following research question (RQ) will be investigated:

> (RQ) What are the differences between the students' reflection processes?

Methods

Design of the teaching-learning environment

The teaching-learning environment concerning ADMS and reflections about ADMS developed by Büscher and Lengnink (in press) was slightly modified. In the following the design of the material is sketched.

After a short introduction into decision trees the students were asked to create their own one based on data taken from www.aiunplugged.org (see Figure 1 A). To initiate reflections, the students should justify their choice of features. A possible decision tree can be seen in Figure 1 B (taken from www.aiunplugged.org).

After that, students should validate their decision tree using test data and assess and interpret its accuracy. To foster deeper reflections on the choice of data and features, a monkey with yet unknown features (see Figure 1 C) is shown and the difficulty with the categorisation is discussed.

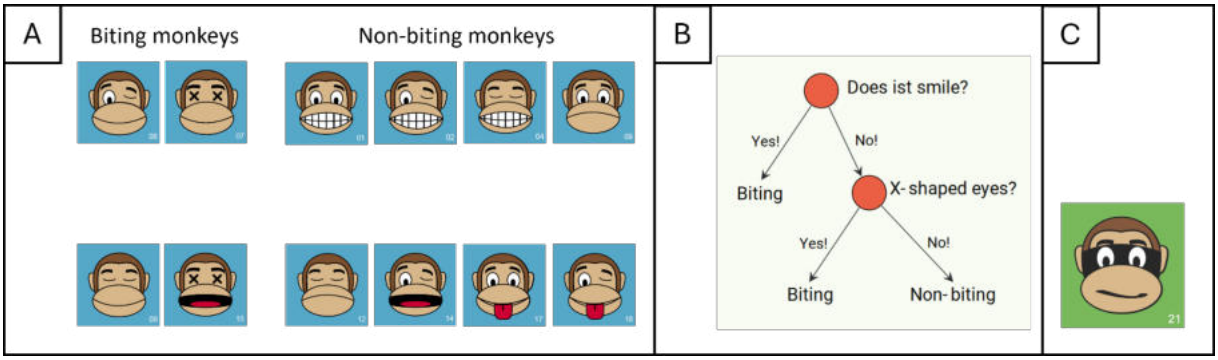

Figure 1: Training data (A), a possible decision tree (B), data with new features (C). Taken from www.aiunplugged.org by Annabel Lindner and Stefan Seegerer. It is licenced under CC BY-NC

Afterward, students were confronted with one real-world scenario where ADMS made decisions about individuals (likelihood of a criminal reoffending in court). The use of ADMS in these situations was to be discussed, and its impacts evaluated. The final reflection task (RT) which was to be discussed and answered in written was:

> (RT) An ADMS uses historical data for decisions that lie in the future. Discuss comprehensively: What are the opportunities and risks of such an approach?

Design of the study

The material was incorporated into a seminar titled "Content-based thinking before calculation and general education in math education" for students pursuing a teaching degree for upper secondary schools at the University of Cologne in the summer term 2024. The students worked in groups of two

to three persons. The reflection processes in the written answers of the students were analysed using a qualitative content analysis (Kuckartz, 2016) applying parts of the category system in Büscher and Lengnink (in press).

Büscher and Lengnink (in press) use the competence areas *recognising*, *evaluating*, and *acting* (Schreiber & Siege, 2016) to differentiate various types of <<reflective activities>>. If the object itself (e.g. the decision tree, the data, ADMS, …) is described, then the activity *recognising* is performed. If results or impacts are discussed (e.g. opportunities and risks of ADMS), an *evaluation* takes place. If a course of action is derived (e.g. discussing how risks of ADMS can be counteracted), the process falls into the category of *acting*.

Additionally, Büscher and Lengnink (in press) elaborate on a distinction between the addressed <<reflective objects>>. In their responses to the reflection tasks, students refer to various objects. If they talk about the data or the selected features, the object *basis* is addressed. When the algorithm, the decision tree, or properties of the model are discussed, reflection on the *process* takes place. The object *result* is coded when the students discuss the model's decisions or its properties. If none of the previously mentioned objects are referenced, but instead the use of ADMS is addressed in general, a *holistic view* is adopted. An overview of the category system is given in Table 1.

Table 1: Category system to capture reflective objects and reflective activities (taken from Büscher & Lengnink, in press)

	Name	Description
Reflective objects	*basis*	There is a reference to data or features.
	process	The algorithm, decision tree, or its characteristics such as accuracy are addressed.
	result	Decisions or their characteristics are addressed.
	holistic view	There is a reference to the use of an ADMS in general.
Reflective activities	*recognising*	Description of the object.
	evaluating	Criteria-based evaluation of the object.
	acting	Presentation of options for action or selection of a specific course of action.

A total of 14 groups participated and completed the reflection tasks. 10 of them answered (RT) and are thus relevant for our analysis. Their answers were deductively coded with the category system described above. The examples presented in this paper were translated from German to English by the authors.

Results of the study

The reflection task (RT) (see in "Method" section) requires the reflective activity *evaluating*, which was undertaken by all students. To uncover the reflective objects of the students, we will first list all addressed objects with their characteristics. Subsequently, the distribution of reflective objects among the students is presented and the richness of the reflection processes is illustrated by the answers of three students.

The students address the three reflective objects *basis*, *result*, and *holistic view*. They mentioned various aspects which were grouped in these categories (listed in Table 2).

Table 2: Addressed aspects in the reflective objects *basis*, *result* and *holistic view*

basis	*result*	*holistic view*
Injustice in the data	Incorrect decisions if the data do not match the situation	Support
Current / future developments not considered	Difficulty in understanding decisions	Objectivity, efficiency
Learning from the past		Cost reduction, resource conservation
Manipulation of data		Predictive accuracy
		Loss of control, job displacement, lack of individuality, injustice

It becomes evident that the reflection task brings forth numerous aspects. However, when looking at the responses of individual groups, a very different richness in their answers can be observed. In Figure 2 one can see that the reflective object *basis* is addressed by all ten groups, whereas the holistic view is mentioned by six and the result only by two groups (called Aaron and Beate).

	Clara	Henry	Marie	Nilas	David	Jakob	Kayla	Laura	Aaron	Beate
basis										
process										
result										
holistic view										

Figure 2: Distribution of the addressed reflective objects among the groups

To illustrate further the richness of the reflection processes we provide a detailed examination of the answers of the groups called Nilas, Laura and Aaron.

1	Opportunity: Past mistakes are not repeated.
2	Risk: Current circumstances are not taken into account (social change, current events such as the climate crisis, politics, war, etc.).

Figure 3: Nilas' answer to RT

The group Nilas highlights positively that "past mistakes are not repeated" (line 1 in Figure 3) but criticizes that "current circumstances are not taken into account" (line 2 in Figure 3). Both aspects are references to the reflective object *basis*.

The group Laura addresses two reflective objects, namely *basis* and *holistic view*.

1	Opportunities: Improved predictive accuracy, cost reduction
2	Risks: Data bias, overfitting to past pattern

Figure 4: Laura's answer to RT

In mentioning an "improved predictive accuracy" and "cost reduction" (line 1 in Figure 4), the group adopts a *holistic* perspective on ADMS. By warning about "data bias" and "overfitting to past pattern", it addresses the data *basis* (line 2 in Figure 4).

The group Aaron provides a very detailed answer to (RT) (see Figure 5).

1	As an assistance (basis) for human decisions;
2	Human vs. machine;
3	Manipulation of test data;
4	Situation does not correspond to the test data --> wrong decision;
5	When it comes to small irrelevant decisions, only minor/no impact;
6	If ADMS should make big decisions, significant impact (especially in case of errors);
7	Can simplify many work steps --> jobs may be cancelled;
8	Risk: People no longer make decisions themselves as they rely on ADMS;
9	"Dehumanization", loss of control;
10	ADMS should not be used in "system-relevant infrastructures" while possibly increasing efficiency;
11	Inequities between those who have access to the technology and those who do not;
12	Especially at the start of testing: good training opportunities for humans (e.g., in chess);
13	Further development in various areas, possibly minimizing anthropogenic errors.

Figure 5: Aaron's answer to RT

A positive *holistic view* in Aaron's answer can be found in the mention of "assistance" and "increasing efficiency" and "can simplify many work steps" resp. (line 1, line 10 and line 7 in Figure 5). Negative holistic aspects are considered in line 2 ("human vs. machine"), line 7 ("jobs may be cancelled", line 9 ("loss of control") and line 22 ("inequities") in Figure 5.

The reflective object *basis* is addressed by mentioning the risk of "manipulation of test data" (line 3 in Figure 5).

The *result* of an ADMS is discussed in line 4,5,6 in Figure 5; there, the risk of "wrong decision" (line 4) is mentioned and the distinction between "small irrelevant decisions" (line 5) and "big decisions" (line 6) is made.

Summary and discussion

To foster algorithmic literacy among prospective teachers, a seminar concept was designed (Büscher & Lengnink, in press) to initiate reflections on the model and the general use of ADMS. In the present study, the variety of reflection processes occurring among the learners is examined. Specifically, the

following research question is posed: “What are the differences between the students’ reflection processes?” A diverse richness can be observed in the students’ answers to the final reflection task: “An ADMS uses historical data for decisions that lie in the future. Discuss comprehensively: What are the opportunities and risks of such an approach?”. The written answers of this reflection task were analysed using the analytical tool from Büscher and Lengnink (in press).

While working with the reflection task, the learners address a different number of reflective objects, namely the data *basis*, the *result* of an ADMS and a *holistic view*. The aspects within these categories are very diverse. But this diversity is addressed by only a few students.

The reflective object *basis* is mentioned by all groups. This was likely elicited by the preceding seminar content and the question (RT) itself: The material on decision trees particularly encourages discussions about the selected features of the monkeys, and the question (RT) itself already hints at the use of historical data in ADMS. In this sense, it is not surprising that all groups address the basis; rather, it demonstrates that one objective of the teaching unit was successfully achieved.

No group discussed the role of the process in ADMS. While working on the decision trees, the model itself and possible changes in the process was indeed part of the discussions among the students, but apparently, this did not lead to comprehensive reflections in the end. In a further developed version of the material, the individual reflection tasks could be better linked.

Six out of ten groups mentioned aspects in the category *holistic view*, and only two groups addressed the reflective object *result*. This shows that the number of addressed reflective objects varies among the groups and illustrates how students can vary in their ability to reflect. This may, for example, stem from the fact that students have varying levels of prior experience with the topic and with reflection, thus leading to different capacities to develop reflective concepts. Another reason for this diversity could be that the teaching-learning arrangement does not equally stimulate the reflection processes of all students. Simply integrating reflection tasks into teaching-learning settings and discussing them in groups may therefore not be sufficient to initiate a certain variety of reflections in all learners. Further research should investigate how reflection processes can be more comprehensively encouraged.

What we missed in the responses was a connection to the students' own lifeworld. They primarily mentioned “generalities” without discussing the impact on their own daily actions. We would like to establish this connection in further developments of the teaching-learning environment.

The analytical tool developed by Büscher and Lengnink (in press) proved to be usable in this second slightly modified iteration of the seminar, indicating that it is quite robust to minor changes in the teaching-learning setting. It would be interesting to explore whether the category system can also be applied in other teaching-learning arrangements related to ADMS, which are further away in terms of the content from the original seminar concept. To address this empirical task, further materials need to be developed and employed, and the reflection processes of students should be further investigated.

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Middle school students' objectification of probability in a computer simulation-based environment

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This study explores the development of students' probabilistic reasoning through their engagement with a computer simulation-based environment. Specifically, this development involves the interpretation of stochastic events, and the social and cultural aspects of the classroom setting are also investigated. Applying Radford's framework of knowledge objectification, we analyse and detail students' developmental processes as their knowledge of probability concepts develops and they conceptualise random events utilising mathematical symbols such as fractions and percentages. Their objectification of probability is analysed and described as processes of objectification: (1) experimenting with software and (2) making sense of symbols or systems.

Keywords: Probability, simulation-based environment, objectification.

Introduction

The development of students' probabilistic reasoning is currently understood as a progression from informal ideas and reasoning to incremental academic and formal interpretations and understanding of probability (Batanero et al., 2016). Effective lesson planning and the teaching of probability in schools necessitate a thorough understanding of students' informal probabilistic reasoning, which has been a significant focus of research for several decades. Typically, middle school students have already acquired an understanding of foundational aspects of probability through prior experience. For instance, they are often capable of identifying a comprehensive set of outcomes for a stochastic event. However, one cognitive challenge for middle school students lies in evaluating and comparing the likelihood of each outcome and utilising mathematical concepts such as fractions and percentages to express these probabilities.

In this study, we investigate 7th-grade students' interpretation of probabilistic reasoning when they encounter formal approaches to expressing their thinking in a school setting. The study is structured around several consecutive lessons focused on the basic concepts of probability, the relationship between theoretical and frequentist probability, and the modelling of compound stochastic events. Specifically, this study examines exploratory lessons to highlight students' initial reasoning and development, which is the primary focus of this article. To effectively analyse and describe students' probabilistic reasoning, we employ the theoretical framework of knowledge objectification proposed by Radford (2002).

Knowledge objectification as a theoretical framework

Radford (2002, p. 14) defines objectification as 'a process aimed at bringing something in front of someone's attention or view'. Building on this, Radford (2003) proposed that students become aware of *knowledge* embedded in a cultural space (e.g., a computer simulation-based learning environment),

a process he refers to as *knowledge objectification* – stating that 'to learn is to objectify something'. We denote mathematical objects as the 'reality of the entity being studied, created, constructed, or investigated by the mathematician, or otherwise of "concepts" when it is desired to emphasise the result of the act of thought [...]' (Iori, 2017, p. 276) during students' interaction within the computer simulation-based learning environment. This refers not only to abstract concepts, but also to culturally and historically created entities, representations, signs, and processes. We view participating students as young mathematicians engaged in the activity of objectifying probability concepts in a computer simulation-based learning environment.

During the objectification process, semiotic tools serve to make abstract mathematical concepts tangible and accessible. These tools – such as words, nonverbal clues, signs and symbols, diagrams, visual aids, and teachers' questions and interpretations – help students concretise abstract mathematical concepts. These artefacts are called *semiotic means of objectification* because they mediate forms of reflexive activity intended to raise awareness of mathematical objects. Semiotic means of objectification have been identified as follows:

> Objects, tools, linguistic devices, and signs that individuals intentionally use in social meaning-making processes to achieve a stable form of awareness, to make apparent their intentions, and to carry out their actions to attain the goal of their activities (Radford, 2003, p. 41).

When interacting with a simulation-based environment, students engage in actions and with artefacts, which facilitates the evolving social practices of doing, thinking, meaning-making, and relating to others and the world (Radford, 2014).

Objectification of probabilistic concepts in information and communication technology environments

Educational research leverages ICT environments to gain valuable insights into students' probabilistic reasoning, typically employing software that models stochastic devices or real-life artefacts in a simplified manner. For example, a study by Pratt (2000) utilises an ICT environment known as ChanceMaker, which is comprised of various gadgets analogous to real-life artefacts, such as spinners, dice, and frisbees. Another notable ICT environment is Probability Explorer, developed and utilised by Stohl and Tarr (2002), which facilitates the exploration of probabilistic reasoning among middle school students.

To investigate and comprehend middle school students' probabilistic reasoning, many studies involve their middle school participants in compound stochastic events. This approach is evident in the work of Pratt (2000), Stohl and Tarr (2002), Abrahamson and Wilensky (2005), and Iversen and Nilsson (2007), where the stochastic events consist of two independent subevents. For example, throwing two dices involves the subevents of each individual die being thrown. Modelling compound stochastic events by connecting two sub-events is typically challenging for students (English & Watson, 2016). To explore students' reasoning about compound stochastic events in depth, Fischbein (1975) developed a set of random devices in which the connection between subevents is particularly transparent. The random devices feature a structure through which one or more marbles traverse a system of crossroads, and each marble has an equal probability of moving either to the right or to the left at each crossroad. In our study, we employed FlexiTree software (Figure 1) to investigate the probabilistic reasoning of middle school students. FlexiTree software is a digital version of the random devices developed by Fischbein, with additional features. For example, FlexiTree can

simulate a large number of experiments, and the probability of going left or right at each crossroad can be set to values other than 0.5.

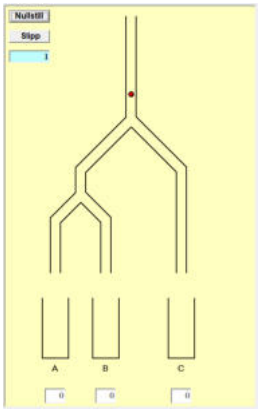


Figure 1: FlexiTree software – System 2

This study focuses on the role of artefacts or semiotic tools in students' objectification of probabilistic concepts. Specifically, the FlexiTree software is highlighted as a semiotic tool, as are the students' use of symbols and various numerical concepts such as fractions and percentages in their meaning-making.

Students and teachers interact to objectify probabilistic reasoning, with the teacher's questions serving as signs or symbols to guide students' thinking, engagement, and responses. Students express their thoughts using signs and symbols, such as percentages and fractions. Together, both teachers and students engage in knowledge objectification, using these signs and symbols to express and understand probabilistic concepts – thereby facilitating the objectification process. Similarly, students' interactions with the FlexiTree software influenced their objectification of probability.

Methods

Teaching probabilistic reasoning and randomness and developing probabilistic models to engage students and foster meaningful experiences are complex tasks. The participants in this study were a group of 7th-grade students who had received only a little prior knowledge of formal probability from school. We approached this phenomenon through well-developed teaching experiments designed to explore and understand the dynamics of how students interact with a semiotic objectification system in a simulation-based learning environment. The teaching experiment comprises a series of lessons that include both whole class discussions and students working in pairs with a computer, as well as teaching episodes and individual interviews during which researchers and teachers closely observe the development of the student's probabilistic reasoning over time (Steffe & Thompson, 2000; Cobb et al., 2003). The lessons were taught by an experienced teacher in a 7th-grade classroom of 17 students. Video recordings were made of the entire class from one corner of the room. In addition, sound and screen activities were recorded for three pairs of students. These data materials, together with the students' work on worksheets, were transcribed and analysed. We conducted a thorough analysis of the data collected using interpretive qualitative analysis methods to analyse the fine-grained development of probabilistic reasoning over a short period (Siegler, 2006), considering verbal, gestural, visual, and symbolic actions in the observed contexts.

The lesson

The first part of the lesson is a whole class discussion that begins with the teacher displaying an image of FlexiTree System 2 layout on the whiteboard (Figure 1). She explains that releasing the marble from the top will result in one of three possible outcomes – A, B, or C – and prompts the students to consider these options and determine the probability of each outcome. This is *the main question* for the students to address during the lesson: *What is the probability that the marble will end up in boxes A, B, or C?* From a formal perspective, we might assume that many students will suggest outcome C, providing arguments based on theoretical probability or the visual appearance of the FlexiTree system. Contrary to this expectation, however, the students offered a variety of suggestions and explanations during the whole class discussion. Next, the teacher asks the students to predict the result of dropping a single marble, followed by a discussion after conducting the experiment. This process is repeated three times. Again, from a formal perspective, one might assume that a majority of the students would propose outcome C based on theoretical probability or intuitive beliefs. In the next part of the lesson, students are required to provide a written response to the main question, along with supporting arguments.

The instructional sequence was designed to facilitate students' active engagement with problem contexts, interaction with pedagogical tools, and the development of their understanding of randomness, probabilistic models, and compound stochastic events. The video data were transcribed, revealing several significant episodes that illustrate students' engagement with problem contexts, their interactions with pedagogical tools, and the evolving processes of probabilistic reasoning about various stochastic events.

Results

Through analyses of the data, we identified *how* or *in what way* semiotic tools impact the students' objectification process. First, the computer simulation-based environment played a significant role as students interpret, reflect upon, and use the environment, which affects their reasoning processes. Second, students' active participation in probabilistic thinking evolved with respect to their use of symbolic representations such as fractions and percentages. In so doing, the students utilised semiotic systems/tools to make abstract probabilistic concepts accessible through their words, nonverbal clues, metaphors, signs, symbols, diagrams, and visual aids.

Objectification as experimenting with software

In the lesson, the teacher initiated the exploratory tasks by asking the following question: 'What is the probability that the marble will end up in boxes A, B, or C?' The students exhibited different forms of probabilistic reasoning about the stochastic event (Julien & Iversen, 2021[1]). The initial response to this question was provided by a student who argued that the probabilities of choosing either the left or right path were equal at both crossroads. Another student concurred, stating that the probabilities are indeed equal at the first crossroads, adding, 'I do not understand why, but I believe

[1] In their 2021 analysis, Julien, A. and Iversen, K. utilize a discursive approach to examine student interaction. In this study we included statements by Charlene and Alphonso and employ Radford's theory of knowledge objectification framework. We obtained permission from the concerned author.

it will proceed to B.' Subsequently, a third student, Charlene, contributed to the discussion by offering a physical interpretation of the scenario.

1.	Charlene:	… it is bent here and then it goes in a straight wall. And if the marble hits the wall, it gets some speed the other way … But … at the same time, the marble has speed … also that way … I don't know.
2.	T:	If I understand this correct … if it comes … you agree that its fifty-fifty, or is it …?
3.	Charlene:	Yes.
4.	T:	So if it comes here [pointing to left side after the first crossroads] then something happens down here?
5.	Charlene:	Yes, you have this wall. If it hits the wall, it will gain speed in the opposite direction.

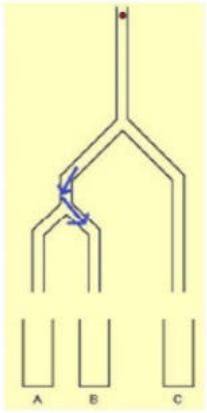


Figure 2: Model of Charlene's physical interpretation of a potential outcome

In the excerpt, Charlene was considering the physical interpretation of the movement of a marble along the slope. She acknowledged the probability of the marble hitting the wall, bouncing in the opposite direction, and thus ending up in box B. We observed a common phenomenon in students' understanding of probability, as they often struggled to separate the physical aspects of the situation from the simulated probabilistic system.

The teacher explained that FlexiTree operates in a computer simulation-based environment without obstructions, unlike in physical space. After this clarification, the students and the teacher agreed to conduct an experiment. Before opening the experiment, the teacher deliberately asked the students to predict the outcome. They were asked to place their foreheads on the desk so that they could not tell what others predicted. The teacher then noted how many students thought the marble would reach boxes A, B, and C. The outcomes predicted by the students' were: [A; 2, B: 9, C: 4]. After that, the teacher ran an experiment, and the marble ended up in box B. This resulted in several students cheering and some shouting, 'I was correct!'

The students showed their eagerness to explore the outcomes of an additional set of experiments. Adhering to previous methods, the teacher noted the potential predicted outcomes. The students predicted the following outcomes: [A:1, B: 5, C: 9]. After conducting an experiment, the marbles ended up in box C. This process was repeated a third time, with the students predicting the outcomes as [A: 4, B: 9, C: 2]. In this round, the result from the experiment was A. To address the main guiding question, the lesson was designed to promote the use of probabilistic thinking, encouraging the

students to analyse the behaviour of probabilistic systems and verbalise the most likely outcome. The students engaged with the interactive FlexiTree software and attempted to analyse the marble's trajectory within the system, thus exploring the interactive visual simulation software. The teacher facilitated by guiding the students to engage in probabilistic thinking processes, pondering and evaluating the variables that influence the probability of each outcome. Many students worked on this task by tracing the path of the marble through the system. However, at the crossroads, which act as a decision point, they found themselves undecided when confronted with an equal chance of the marble going left or right. In these situations, the students felt pressured to pick one of the potential outcomes. After running three experiments, and a subsequent discussion, the students were instructed to submit their written responses along with supporting arguments.

Objectification as symbolising probabilistic reasoning

The overarching objective of the exploratory teaching lesson was to help students objectify their thinking about probability in the context of compound events. The simulation and visualisation in FlexiTree and the teacher's frequent prompts played a crucial role in supporting students in externalising their reasoning about the underlying probability distributions. The teacher slightly modified the main guiding question of the lesson to ask students to calculate the probabilities of the likelihood of a marble ending up in one of three boxes: A, B, or C. The teacher then offered choices to quantify these probabilities as fractions and percentages. Most of the students chose to use percentages, and there were also discussions about dividing the probabilities equally. One student, Alphonso, suggested starting with 100% as the total probability, based on the assumption that the three possible outcomes were equally likely.

6. Alphonso: Since A and B … if you split up … if you split up the chance of one hundred percent, it makes then sixty-six percent for both A and B – chance that (unintelligible) comes to one of them, and thirty-three … thirty-three percent that it comes to C. Because A and B are two.

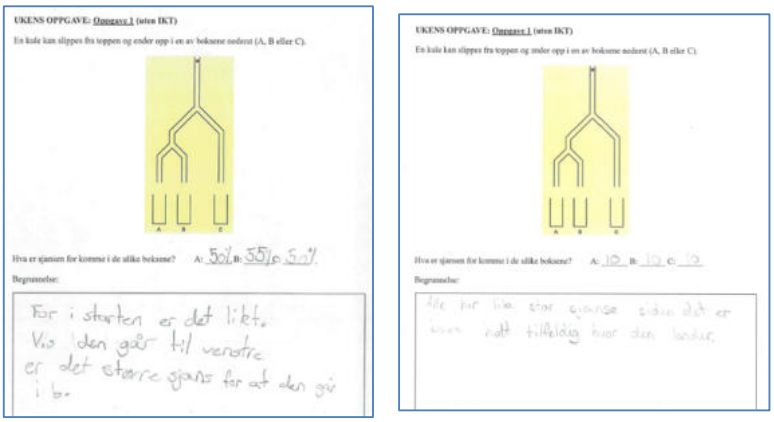
Charlene joined the discussion, disagreeing with Alphonso and arguing that the marbles must pass through two intersections to reach box A or B. Alphonso then continued his argument.

7. Alphonso: Ok … when I said that I didn't mean … There is fifty-fifty chance, and it goes to either right or left … but it is not about whether it goes right or left, what it is about is that there are three alternatives, and two of them are both to the left. So, it is more likely that … that it comes in two out of three than in one out of three. Do you understand what I mean?

Alphonso outlined his argument using percentages (66% and 33%) and used the phrase 'fifty-fifty' to describe equal probability at the crossroads. In addition, he included fractional interpretations of his reasoning ('2 out of 3' and '1 out of 3'). He was the only student who expressed probabilistic thinking in percentages and fractions. Most students prefer using percentages to quantify abstract thinking in numerical values.

Following the experimentation and whole class discussion, the students were instructed to write down their responses to the main question and provide the reasoning behind their ideas. They all used percentages to express probabilities. Six out of the 17 students wrote 'A = 25, B = 25, C = 50', the expected answer. Eight students proposed that each outcome had an equal probability, while two students said that outcome B was the most likely. It is noticeable how the students expressed their

responses. When modelling random phenomena, teachers often stress that the total of all probabilities must add up to 100%. However, some of the answers submitted in this study did not meet this requirement. Figure 3 presents two written responses in which the percentages did not sum up to 100%, but both elucidated their reasoning about the probabilities.



Tom **Anne**

Figure 3: Students' solutions to the main question: What are the probabilities that the marble will end up in boxes A, B, or C?

In the first solution, Tom wrote 'A = 50%, B = 55%, C = 50%', and the explanation was as follows: 'At the start, it is equal. If it goes to the left, there is a higher chance for it to go to B.' Perhaps he understood that the marble might hit the wall and bounce towards B. Anne, on the other hand, wrote, 'A = 10, B = 10, C = 10', arguing that 'They all have an equal chance since it is completely random'. Although the percentages do not sum up to 100%, this solution closely expresses the relative relationship between the probabilities of various stochastic events.

Conclusion

Teaching probability to middle school students is challenging, but implementing an exploratory teaching approach to engage them in probabilistic thinking is even more complex. By using expert-designed software in classrooms, we aim to deepen our understanding of children's experiences and challenges and enhance the development that they yield. In this study, we investigated how students grasp the abstract concept of probability by offering various learning opportunities to students, such as running experiments in simulation software, participating in games, and reflecting on their thinking in writing. The results illuminate the impact of the FlexiTree computer simulation-based environment, which facilitates engagement in probabilistic thinking and reasoning. However, the students often mixed physical interpretations with their explanations of probabilistic events, revealing a common challenge in distinguishing between real-life random generators and computer-based simulations. We assessed the students' development in understanding probability by analysing their dialogues with teachers and their written responses. In conclusion, the computer simulation-based environment enabled students to experiment, reason, and reflect, and it helped them articulate probability concepts mathematically, fostering a deeper awareness of probabilistic thinking.

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Exploring the structure of enumerations in combinatorial problem solving: a case study with Italian secondary school students

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This work proposes a case study aiming at exploring and describing what kind of enumerative processes are developed by a group of Italian secondary school students without previous knowledge in combinatorics. We proposed the students a questionnaire of combinatorial tasks and we focused on the answers in which the numerical answer was computed through an enumeration of the solving configurations, observing the structural elements students introduce in their lists. Our work showed that if, on one side, some of the students develop a structure of enumeration that could be exploited in the development of combinatorial formulas, on the other hand there are some structures that highlight the need of a preliminary work to be done with students before introducing more general solving heuristics.

Keywords: Combinatorics, problem solving, enumeration, secondary school.

Introduction

Combinatorics plays a key role within mathematics and its teaching. On the one hand, it represents a core in discrete mathematics while, on the other hand, in the school curriculum has a role in the development of topics both in mathematics and other disciplines (e.g., computer science or biology – Batanero et al., 1994). Moreover, combinatorics it is a topic that could offer students wide opportunities to develop and train problem solving skills (Kapur, 1970), providing an environment that does not rely on particular mathematical prerequisites and that leaves wide room for the formulation and testing of conjectures (ibid.). As a result, when engaged in the resolution of a combinatorial tasks, students (and people, in general) may develop a multitude of different solving approaches. The literature of research in combinatorics education focused widely and mostly on the study of such approaches – to which we will refer as 'heuristics' following the definition proposed by Schoenfeld (1985) – with an extended description of the possible combinatorial heuristics proposed in Godino et al. (2005) – with university students – and Lamanna et al. (2022) – in the case of secondary school students – or focusing on the difficulties students may encounter while solving a combinatorial task (Hadar & Hadass, 1981) and the variables affecting such difficulties (e.g., Batanero et al., 1997). Another direction of research in combinatorial problem solving is the study of combinatorial thinking and its development, for which a cornerstone research is represented by the model proposed by Lockwood in 2013, where combinatorial thinking is schematized through the interaction of three different dimensions (counting processes, formulas/expressions and set of outcomes), observing that a fully developed combinatorial thinking should take into account all the three dimensions and their interactions. In Lamanna et al. (2022) it was shown that a visible effect of teaching combinatorics is the students' shift from heuristics exploiting many representations of the set of the configurations towards attempts to directly use a single formula or develop arithmetic procedures. This led to the development of a research line focusing on teachers and the ways combinatorics is proposed in Italian schools. As a result, Lamanna & Branchetti (2024) showed that

teachers represent an active variable in students' development of combinatorial thinking and the heuristics they use. In particular it was observed that, in a phenomenon called 'funnel effect', students are encouraged to mostly develop solutions relying only on the use of combinatorial formulas or arithmetical procedures – that are often considered to be more efficient. The effect of such phenomenon is that students' exploration of the dimensions of Lockwood's model is limited – mostly cutting out of the equation the dimension of the set of outcomes. This implies that students are not encouraged to rely on enumerative processes that, according to Lockwood & Gibson (2016), play a key role in the development of a combinatorial formula's understanding. Lamanna & Branchetti (2024) showed that, even recognizing the usefulness of enumerative processes in the resolution of a combinatorial task, teachers mostly consider this heuristic to be 'acceptable' only in early stages of the learning of combinatorics, asking the students to change their approach towards a different choice of heuristics. In this sense, considering the development of a student's combinatorial thinking and the structure of Lockwood's model, it becomes relevant to understand how such change of solving approach is implemented and to what extend it could be anchored on students' developed heuristics, in order not to create a discontinuity in the shift from heuristics relying on the dimension of the set of outcomes to heuristics mostly relying on the dimension of counting processes and formulas. For this reason, it is important to focus on the heuristics students develop before learning combinatorics at school. In the case of Italian students, Lamanna et al. (2022) observed that before learning combinatorics students mostly rely on enumerative processes. In this sense, it is then important to explore how such heuristic is developed and the structures students rely on in order to produce lists of configurations. Therefore, while in the literature of research there are examples of listing techniques developed by students (e.g., English, 1991), fewer attention has been posed on the structure of enumeration that is developed, with almost no focus both on the case of secondary school students' enumerations and the relation between the structure of lists and the passage from an enumeration to a formula. In this paper we will propose a case study conducted on a group of Italian secondary school students, focusing on the enumerations they produce before learning about combinatorics, to explore whether the structures that spontaneously arise could be involved in the process of development of a combinatorial formula.

Theoretical framework

We based our work on the characterisations found in the literature concerning the enumerative processes in combinatorial problem solving. This heuristic consists in the production of the complete list of elements belonging to a certain set of configurations (i.e., the elements to be counted in a combinatorial task). In particular, considering a set C of configurations for a certain task T, an enumeration is a process that produces a complete list of elements of C (Batanero et al., 1994). Enumerative processes could be carried out in different possible ways and described in the works of English (1991), where the heuristics of children and adolescents were studied. We find here a, first, main difference: a-systematic enumeration (or "non-systematic", carried on through a try-and-error procedure) and systematics enumeration (i.e., an enumeration in which it is recognizable a logic in the organisation of the list of configurations). English (1991) also identifies different ways through which children create list of elements, introducing the concept of the "odometer", often identified in literature with different names (e.g., 'pivoting of a variable' or 'fixation of a variable'). The odometer

approach describes a structural use of the variables for systematically creating a list: one of the variables a_i involved in the task is fixed in the configuration to be formed (for instance, in the first position) and the enumeration unfolds by listing all the configurations that have such variable in the position in which is fixed. Once that this process has ended, another variable is fixed in the same position and the respective configurations are listed until all the possible variables become fixed. In 2009, Shin & Steffe, connecting to the description of the odometer approach, conducted a case study in which they described the kind of enumerations produced by two seventh-graders observing how three main enumerative structures were produced: i) multiplicative enumeration, where the list is produced following a multiplicative structure (i.e., the list is organized in a table disposition that would allow the solver to easily count the configurations even not listing them all exhaustively); ii) additive enumeration, when dealing with an exhaustive list that is counted "by-the unit", meaning that each of the elements correspond to a "+1" in the count of configurations and iii) recursive multiplicative, when both an enumeration and a multiplicative structure occur within a same solution.

We can, then, summarize the aim of the work presented in this paper through the following research question: what kind of enumerative processes are developed by a group of Italian secondary school with no previous knowledge in combinatorics?

In particular, we aim at exploring and describing the structures of the enumerative processes that students spontaneously develop in front of a combinatorial task. This is done in order to understand, in a further step of research outside the scope of this work, if and how such structures may facilitate the development and understanding of a combinatorial formula starting from what students actually produced (and therefore, we suppose, interiorized) without inducing discontinuities in their exploration of the model of combinatorial thinking (Lockwood, 2013).

Methodology

To answer our research question we relied on the set of data collected in Lamanna et al. (2022), where a sample of 115 Italian secondary school students was involved, 64 of which did not already learned combinatorics and all belonging to groups of students coming from different schools and backgrounds. In particular, this work will focus on the protocols collected from the 64 students with no previous knowledge on school combinatorics and that consisted in 4 groups of students in grade 10, 11 or 12 (i.e., second, third and fourth year of secondary school with students aged between 15 and 18) with different math teachers. The students were proposed a questionnaire of 13 open-ended tasks (in an Italian translation) adapted from the one used in Batanero et al. (1997) and proposed in two different versions with the same tasks presented in an inverted order, aiming at limiting a possible effect of time on the data collection and to obtain a sufficient amount of data on every item of the questionnaire. Depending on the availability of the school, students had 60 or 90 minutes to complete the questionnaire. We remark, though, that time did not represent an issue, considering that most of the students were able to complete the questionnaire widely before the smaller time limit. In this work we will focus our analysis on one of the tasks proposed in the questionnaire, here presented in an English translation.

> Andrew, Burt, Charles and Dan (A, B, C, D in short) decide to have lunch at the school cafeteria and, after taking the tray, they line up to get the food. Obviously, each of them would like to be the first! We want to write all the possible ways they can get in line. For example, if A is the first, B second, C third, D fourth, we write ABCD. How many ways can the boys be lined up?

We decided to focus on this task because, out of the 13 items in the questionnaire, it represents the task whose solution was attempted the most and that, at the same time, even though with a high percentages of solutions involving enumerative processes, showed a large heterogeneity in students' heuristics when compared to the other items in the questionnaire. The involved task is belonging to the combinatorial model of distribution (Dubois, 1984) of medium dimension (i.e., requiring between 12 and 25 solving configurations), in which students are asked to determine in how many ways 4 people can be distributed in a row and whose solution is $P_4 = 4! = 24$, thus identifying it as a permutation task.

The students' answers were collected in a written form and, out of the 57 answers provided on the selected task, we isolated a total of 45 solutions involving an enumerative process either as a primary or support heuristics (Lamanna et al., 2022). Once that the set of enumerations was selected, we proceeded with a qualitative content analysis (Mayring, 2015) of students' protocols, analysing the nature and structure of the enumeration – that is, systematic enumeration or not (i.e., presence of pattern in the process of enumeration – English, 1991), use of the odometer approach (i.e., fixation of one or more variables – ibid.), structure of the enumeration (i.e., multiplicative, additive and recursive multiplicative – Shin & Steffe, 2009). The occurrences of such elements were recorded and the overall distribution of the codes was analysed.

Results

Out of the 45 protocols collected, we observed that 25 students produced a correct solution by using enumeration either as a primary heuristics or a support heuristics (Lamanna et al., 2023) – meaning that the enumerative process intervene in one of the steps of the resolution, usually to solve a sub-task obtained by fixing one of the variables and analysing the sub-task of reduced dimension. Focusing on such aspect, we can observe in Table 1 the distribution of the use of enumeration either as a primary or support heuristic.

Table 1: Distribution in the use of enumeration

	Primary heuristic	**Supporting heuristic**
Non correct answers ($n = 20$)	0,7 (14)	0,3 (6)
Correct answers ($m = 25$)	0,32 (8)	0,68 (17)
Total ($t = 45$)	0,49 (22)	0,51 (23)

While in the whole sample the use of enumeration is uniformly distributed between the involvement either as a primary heuristic (therefore providing a complete list of all the configurations) or

supporting heuristic (therefore involving enumeration in one of the steps of the resolution), the situation changes introducing a differentiation between correct answers and not. We can, in fact, observe that most of the solutions in which the students attempted a complete enumeration of the configurations produced a non-correct result, with students usually producing an incomplete list of configurations. On the other hand, most of the correct answers involved enumeration as a support heuristic, meaning that students managed to coordinate different heuristics to produce a correct numerical solution (usually involving the arithmetic rule of product – Godino et al., 2005). To delve more in the structures of the enumerations that students produced, we can observe in the following Table 2 a summary of the analysis conducted on the collected protocols.

Table 2: Distribution of structural elements in students' enumerations

			Structure of enumeration		
	Systematic enumeration	**Odometer approach**	***Additive***	***Multiplicative***	***Recursive multiplicative***
Non correct answer ($n = 20$)	0,05 (1)	0,3 (6)	0,7 (14)	0,05 (1)	0,25 (5)
Correct answer ($m = 25$)	0,68 (17)	1 (25)	0,28 (7)	0 (0)	0,72 (18)
Total ($t = 45$)	0,40 (18)	0,69 (31)	0,47 (21)	0,02 (1)	0,51 (23)

We observe that almost all the non-correct answers appeared in correspondence of a non-systematic enumeration. The students approach the solution of the task by a try-and-error procedure that – although relying in some cases on the odometer approach (thus with a hint of systematization of the enumerative process) – does not allow to maintain the control on the configurations that are produced and, therefore, fail in producing a complete list. It is interesting, analysing this sub-case, noting that some odometer approaches occurred in this set of protocols: the students started their enumeration by fixing one (or more) variable and then proceeded in enumerating the relative configurations by try-and-error. For what concerns the structure of the enumeration, we observe that in most of the cases the structure of non-correct answers showed an additive structure, with students aiming at producing lists in order to count configurations by the unit. A smaller fraction of students produced an enumeration with a recursive multiplicative structure but failed in the switch from enumeration to generalisation, thus in the introduction of the multiplicative part of the structure. We finally observe that in this set of answers is included the only one demonstrating a multiplicative structure: the student produced a solution by creating a double entry table with the variables listed on two of the edges and, after listing some random configurations, explained that the total was given by the multiplication of the number of variables listed in the table, thus producing a non-correct calculation.

For what concerns the students that provided a correct answer, we observe that not all the enumerations were systematic, but all the solutions relied on the odometer approach. We suppose that, by fixing one of the variables, the task is reduced to a sub-task whose dimension is low enough to allow students to determine the correct number of configurations – even through non-systematic enumeration. We may argue, considering that there are also non-correct solutions relying on the odometer approach, that the difference here relies on the *a-priori* setting of the solution: on one side, with non-correct answers, the odometer is used to organize the whole listing procedure that, still, needs to produce all the configurations. On the other hand, with correct answers, in most of the cases students introduce the odometer only to reduce the dimension of the task and enumerate a smaller set of configurations. For what concerns the structure of the enumeration, we observe that most of the solutions demonstrated a recursive multiplicative structure that is: the students created a particular partial list (usually by fixing only one of the variables) and then computed the solution of the given task by introducing a multiplicative structure in the list (i.e., once that a variable is fixed, the remaining lists have the same number of configurations and the number of such lists equals the number of possible variables to be fixed).

Overall, we notice that students without previous knowledge in combinatorics tend to widely adopt the odometer approach even though this is not always accompanied by a systematic procedure that, in case the task has a high number of configurations, might induce a loss of control of the process. For what concerns the structure of the produced enumerations, we observe that the students equally rely either on an addictive structure or a recursive multiplication structure with almost no occurrences of multiplication structures. We may argue that this result could be task-dependent, but we find that even the absence of attempts showing a multiplicative structure still provides valuable information regarding the processes of enumeration unfolded by the students. We observe that the use of the odometer approach supported both the process of organising a complete enumeration and setting a resolution computable through product rule, highlighting a multiplicative structure in the process of enumeration. The two approaches, however, when not in correspondence of a try-and-error enumeration, refer to a same organisation of enumeration, with the difference that in one case the enumeration is complete (thus additive) and in the other is partial.

Discussion and conclusion

The analysis conducted on the sample of collected protocols provided an overview of the processes of enumeration that a group of students develop when solving a permutation task. For what concerns our research question, we observed that most of the students spontaneously develop enumeration through the odometer approach, thus fixing one of the variables to simplify the structure of the lists they produce. In spite of this organisation, the use of the odometer approach did not always coincide with a development of a systematic listing procedure. Considering the structure students give to their enumerations, we notice that students mostly rely on additive and recursive multiplicative structures, with different aims. In the former case, the students usually aimed at producing a complete enumeration of the set of configurations (in the sense of Lockwood, 2013) while, in the former case, the overall solving strategy (in the sense of Schoenfeld, 1985) aimed at reducing the dimension of the set of configurations. In this last case, even non-systematic enumerations allowed students to profitably solve the task, in agreement with Lamanna et al. (2022) where it was observed that for

tasks of reduced dimension the non-systematic enumeration did not represent an issue. This aspect, however, would benefit from further analysis, possibly involving students in a think-aloud process of resolution of a combinatorial task to understand which is the reasoning that leads either to an additive or recursive multiplicative structure.

Overall, in spite of the reduced number of students involved and tasks analysed, we observe how our work of research allow us to gain insights on the heuristics that are developed by students with no previous knowledge in combinatorics and, therefore, upon which teachers will build their teaching of the topic. According to Lamanna & Branchetti (2024) teachers accept enumerative processes in the early stages of the teaching of combinatorics but, then, students are asked to drop enumerative processes in favour of more direct approaches. In this sense, also considering the work by Lockwood & Gibson (2016) according to which enumerative processes play a key role in the development of a student's comprehension of a combinatorial formula, it is important – from a teaching perspective – to consider that not all the students develop enumerative processes in the same way. This has an implication on a student's development of combinatorial formulas and, more in general providing a point of interest for educational research on this topic, on the heuristics developed by secondary school students in combinatorial problem solving. In this sense, our work shades light upon the fact that there are some structures that, in spite of being important within combinatorial problem solving, need to be furtherly developed when teaching combinatorics in order to provide a solid base on which students can build formulas and arithmetic rules. In particular, for example, in the shift from an additive structure of enumeration to a recursive multiplicative. In this sense we argue that, exploiting the odometer approach spontaneously introduced by students, it is possible to work on the organisation of students' enumerations and, thus, support the clarification of the numbers that are subsequently involved in combinatorial formulas and arithmetic rules. Such a process would represent a shift in the exploration of dimensions of combinatorial thinking (Lockwood, 2013) passing from the dimension of the set outcomes to the one of counting (and also Formulas/Numerical expressions) without creating a discontinuity in the heuristics that students develop. Similarly, stressing the relevance of the odometer approach, it is possible to provide the students with an approach that would allow them to systematically (and recursively) reason on the configurations to be formed, training a skill that it is not relevant for the sole combinatorics but also for other disciplines (e.g., computer science – Batanero et al., 1994).

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Empowering statistical learning: The impact of growth mindset on students' attitudes towards statistics

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This study explores the effect of a growth mindset intervention on pre-service primary teachers' attitudes toward statistics. Of the 315 participants, the 190 experimental group students received a five-week growth mindset intervention. The SATS-36 and the Implicit Theories of Intelligence Scale were administered before and after the intervention to experimental and control groups. An open-response task and focus group interviews were conducted with students from both groups after the intervention. Preliminary qualitative findings suggest that these growth mindset interventions played an important role in improving students' attitudes towards statistics and that students' overall views of growth mindset interventions are positive.

Keywords: Statistics education, growth mindset, teacher education, attitudes towards statistics, SATS

Introduction

Statistical literacy is a critical skill for citizens in making sense of the world. As statistics is often introduced in primary school mathematics, it is essential in initial teacher education (ITE) programmes. Developing positive attitudes toward statistics in ITE is a critical consideration due to the established link between attitudes and subsequent achievement in statistics courses (Chiesi & Primi, 2010). More importantly, negative attitudes may hinder future teachers' ability to teach statistics effectively, perpetuating misunderstandings or leading to its avoidance in their classrooms (Leavy et al., 2021). While research suggests that pre-service teachers (PSTs) generally hold positive attitudes toward statistics, there is concern relating to the widespread perceptions by PSTs of statistics being difficult (Zientek et al., 2011). Evidence suggests, however, that their attitudes towards mathematics and statistics are malleable, with improvements being observed arising from a range of interventions in initial teacher education contexts (Leavy et al., 2021).

Growth mindset (GM) interventions—the belief that abilities can be developed through effort (Dweck, 1999)—have shown promise in improving attitudes towards mathematics. Applying this framework to statistics education could foster positive attitudes in pre-service teachers. Compared to research on K-12 and college students, research on the impact of GM interventions in pre-service teacher education remains underexplored, leaving a gap in understanding how these interventions might benefit future educators. This study aims to fill this gap by investigating the effectiveness of a GM intervention in improving pre-service teachers' attitudes toward statistics.

Review of Literature

Undergraduate students' attitudes towards statistics, especially those of PSTs, are critical, as they can significantly influence their engagement and achievement (Chiesi & Primi, 2010). Not only are such positive attitudes linked to higher motivation, better understanding, and greater persistence in overcoming challenges, but negative attitudes can lead to avoidance behaviours and poor academic outcomes. The Survey of Attitudes Toward Statistics (SATS-36) (Schau et al., 1992, 2003) is the most widely used instrument to measure attitudes. Confirmatory factor analysis has validated its six-factor structure consisting of *Affect* (feelings concerning statistics), *Cognitive Competence* (attitudes about one's intellectual knowledge and skills when applied to statistics), *Value* (attitudes about the usefulness, relevance, and worth of statistics), *Difficulty* (attitudes about the difficulty of statistics as a subject), *Interest* (level of interest in statistics), and *Effort* (amount of work a student expends to learn statistics). Targeted interventions can improve attitudes. Course design and delivery, active learning strategies, such as collaborative projects and real-world data analysis, have proven beneficial. Interest is growing in GM interventions' impact on attitudes. While an initial study by Zonnefeld (2015) found no significant overall effect, positive shifts in how female students value statistics, compared to their male peers, were observed.

The concept of growth mindset, introduced by Dweck (1999), identifies two belief systems that individuals hold regarding their abilities: fixed or growth mindset. A fixed mindset views intelligence and abilities as static traits, leading to challenge avoidance and feeling threatened by others' success. In contrast, a growth mindset believes abilities and intelligence can be developed through effort and feedback. Mindset is measured using the *Implicit Theories of Intelligence Scale* (ITIS, Dweck, 1999). Compared with growth mindset, a fixed mindset predicts lower achievement among students facing academic challenges (Blackwell et al., 2007). Associations between growth mindset and achievement have been found in meta-analyses (Burnette et al., 2013) and studies of 74 developed nations (OECD, 2019, N=555,458).

GM interventions aim to shift beliefs from a fixed to a growth mindset. Interventions vary in duration and intensity, from single-session workshops to online modules and multi-week programs. Studies show even short interventions can impact learning. For example, over 250,000 students on Khan Academy, who received growth mindset affirmations (e.g. "If you make a mistake, it's an opportunity to get smarter!") solved mathematical problems, with effects lasting months (Yeager et al., 2013). Interventions emphasise developing ability over time through effort (Dweck, 1999), the brain's capacity to develop new neural networks when it learns (Yeager et al., 2016) and reflecting on learning processes rather than solely on outcomes and overcoming obstacles (Yeager & Dweck, 2012). Adolescents in GM interventions demonstrated increased willingness to take on challenges in mathematics (Rege et al., 2020, n=21,000) and improved mathematics grades over a two-year period (Blackwell et al., 2007, n=373). A review of 15 studies found that GM positively affected academic achievement for postsecondary students but no discernible effects on college enrollment (IES, 2022). Other studies of postsecondary students reported increased engagement and achievement (Aronson et al., 2002) and improved attitudes toward statistics among female students (Zonnefeld, 2015).

Debate surrounds GM interventions, with some researchers citing weak evidence and low effect sizes (Sisk et al., 2018). In their review, Yaeger and Dweck (2020) stress the importance of well-designed interventions that emphasise developing abilities through concrete actions, convey neuroplasticity with memorable metaphors, present stories of notable figures using growth mindset and encourage active engagement (Yeager & Dweck, 2012; Yeager et al., 2016).

Methods

Sample and course content: Participants were 476 Irish pre-service primary teachers in the first year of a 4-year Bachelor of Education (BEd) programme. They ranked in the top 15% of Irish college entrants as measured by the national standardised college entrance exams and had studied statistics as a compulsory component of both national primary and secondary curriculum. All were registered in a compulsory mathematics education module, provided with study details and signed consent forms during the first lecture. Each student generated a unique project ID to ensure anonymity. The study took place during the first 5 weeks of a 12-week semester, focusing on statistics, with 2 weekly lectures (Table 1). Statistical content covered big statistical ideas: differences between statistics and mathematics, samples predicting populations, data types (categorical or numerical), distributions describing data, statistics summarising distributions, and comparing group distributions.

Randomisation: At college registration, all BEd students were divided into 7 sections alphabetically by last name and randomly assigned to control or experimental group.

Pretest and posttest: A QR code linked to online versions of the SATS-36 (Schau et al., 1992) and Implicit Theories of Intelligence Scale (ITIS, 8 items) (Dweck, 1999) (Table 1). At posttest, they completed the "Esma Scenario": *Imagine Esma, a 10-year-old child in your class, isn't doing well in the data strand of maths. Esma doesn't want to do homework or ask for help because she doesn't think she is smart enough. She doesn't think anything she'll do will help her better understand data. Type what you would say to Esma explaining how her mindset can prevent her from getting smarter. You can use something you have read or use an example from your own life.*

Intervention: The intervention was guided by a review of mindset literature and guidelines from Yaeger and Dweck (2020). Seven *mindset affirmations* were selected and adapted for statistics (e.g. 'The harder I work [at statistics], the better I get!). *Celebrity testimonials* were collected, and a matching exercise paired each testimonial with an affirmation. Table 1 outlines the intervention schedule for control and experimental groups. The lecturer refrained from commenting on any intervention material or slides.

Statistical Analysis: R, Jamovi and SPSS were used to clean and preprocess the dataset for descriptive and inferential analyses. Qualitative data were analysed with QualCoder. Two independent coders initially coded the focus group interviews and Esma task using 47 codes and 18 codes, respectively, ensuring accuracy and consistency. After reaching 95% consistency, discrepancies were resolved, and the second coder completed the remaining data. This dual-coding method enhanced the reliability of the thematic analysis.

Table 1: Study design for control and experimental groups

Lecture 1	ALL received information and consent forms, SATS-36 and Implicit Theories of Intelligence Scale	
	CONTROL GROUP	EXPERIMENTAL GROUP
Lecture 2	Watched a video on the new Irish Primary Maths curriculum and completed a short reflection on the video content	Read "Growing the Brain" describing adult brain elasticity. Three summary points on the reading were provided. They responded to the prompt: *Think about an example from your own life. What is something you weren't good at at first? Then, you practiced it using a good strategy and became really good at it. For the next 2-3 minutes, write about it and explain how you became good at it.*
Lectures 3-9	Statistics content and pedagogy of teaching statistics Generic reflection at the end of the lecture	Mindset affirmation presented on the first slide GM affirmation appeared at the bottom of each slide containing statistics content and pedagogy of teaching statistics Affirmation slide was re-displayed & celebrity testimonial presented The final slide presented the affirmation and testimonial together and asked students to reflect.
Lecture 10	ALL received SATS-36, Implicit Theories of Intelligence Scale and the Esma scenario assignment	
Lecture 24	Focus group interviews with control group and experimental group	

Findings

Of the 476 participants, 351 (278 females, 69 males) had matched pre-test and post-test data. Seven sections were used: 3 control (n=139) and 4 experimental (n=206). The control group showed positive pre- and post-test scores on Affect, Cognitive Competence, Interest and Effort, and neutral scores on Difficulty, with ITIS scores indicating beliefs that intelligence can improve. In the experimental group, pre-test mean scores for Affect and Difficulty were neutral, while Cognitive Competence, Value, Interest, and Effort were positive. Post-test scores for Difficulty remained neutral and Affect, Cognitive Competence, Value, Interest, Effort were positive. ITIS scores in the experimental group indicated views that intelligence can change at both pre-test and post-test (Table 2).

Multiple linear regression examined homogeneity across gender, section, and pre-achievement groups. Pre-ITIS scores were the outcome variable, with gender, class section, and pre-achievement as independent variables. None of the independent variables significantly predicted pre-ITIS scores. The data were examined by instructor type. Some sections were taught exclusively by one instructor (n=215) whereas two instructors shared teaching on the remaining sections (n=121). Hierarchical multiple regression was conducted to examine the effect of the GM intervention on change-SATS component scores and GM change scores. Pre-SATS scores were entered in Step 1, pre-achievement

and gender entered in Step 2, and group (treatment) entered in Step 3. Our results showed that the GM intervention was statistically significant in predicting the change scores of the Affect ($t(210)=2.943$, $p<.05$, uniquely explaining 3.24% of the variance) and Cognitive Competence ($t(210)=2.58$, $p<.05$, uniquely explaining 2.46% of the variance) components for the sections taught by one-instructor only, after controlling for pre-SATS scores, pre-achievement and gender.

Table 2: Descriptive statistics

		Pre-tests							Post-test						
		Aff	CC	Value	Diff	Int	Eff	ITIS	Aff	CC	Value	Diff	Int	Eff	ITIS
M	Control	4.75	5.07	5.15	4.09	4.93	6.22	4.32	4.89	5.20	5.19	4.23	4.77	5.51	4.36
	Exp	4.37	4.67	5.05	3.73	4.62	6.26	4.32	4.90	5.17	5.20	4.14	4.64	5.71	4.49
SD	Control	1.02	.97	1.05	0.75	1.25	1.45	.78	1.24	1.21	1.09	.80	1.24	1.39	.82
	Exp	1.08	1.10	1.04	0.75	1.29	1.34	.80	1.15	1.10	1.04	.81	1.28	1.27	.78

Analysis of focus group interviews revealed that both experimental (EG) and control (CG) groups reported positive improvements in attitudes towards statistics and the teaching of statistics. Neither group reported negative changes. The EG found affirmations helpful at the beginning of the lecture, but not at the end, when students were tired and eager to leave. They found the reading exercise interesting, stating that it helped them to begin thinking of the brain as a muscle. They reported that the Esma scenario was not only helpful for teaching statistics but for being a teacher in any subject; it made them think about how they would act as a teacher as it put them in the context of the classroom. Some of them also thought they were making these suggestions for themselves (Table 3).

Table 3. (Selected) focus group interviews code frequencies from control and experimental groups

	Code Tree	CG	EG	Example quotations
change in attitudes toward statistics	got more positive	8	8	I think it's better now. Because like, understand this greater meaning behind it, rather than just looking at definitions and learning them off, so things have improved for me (E)
	stayed same	1	2	I was fairly confident anyway. It stayed the same for me (C)
change in attitudes toward teaching statistics	positive improvement	5	5	I say I'd be more confident than I would have been. (C)
	stayed same	0	1	… I don't think my interest in it has changed. ..so not more interested but more confident. (E)
	affirmations helpful	NA	5	I thought they were awesome. Honestlytt (E)

interventions	affirmations not helpful in general	NA	4	I kind of just forgot about it and didn't really read anything after (E)
	beginning affirmations helpful	NA	3	…I agree that they were good at the start of the lecture like to kind of let us know what we were going into (E)
	ending affirmations not helpful	NA	3	…but then at the end of the lecture, like everyone's just tired and we want to go home (E)
	reading exercise was helpful	NA	9	…I thought it was interesting like it did strike me the way it is like muscle that you learn. You can grow. You're not just born with a certain IQ or Yeah (E)
	story about Esma was helpful	NA	5	… scenarios like that are good for us to be learning now and coming up with answers for now. So when we got placement, we will have answers. (E)

NA: not applicable; EG: experimental group; CG: control group

When examining the Esma scenario, both experimental and control groups identified similar sources of success: positive effort beliefs, value of asking for help, changing their fixed mindset, improving their resilience and persistence, and changing their study strategies. Both groups had similar response rates to the view that intelligence can be developed. Experimental groups students were more likely to report that failure should be seen as an opportunity to learn. Both groups emphasised that positive attitude change will help students to succeed. The most frequently mentioned SATS attitude component was Cognitive Competence, with students suggesting that if Esma believed she could do statistics, she would do better in statistics. Control group provided more teacher-centred solutions (providing extra lessons, visuals, step-by-step instructions) whereas experimental group had more student-centred solutions such as exploring a different learning strategy (see Table 4).

Table 4: (Selected) Assignment code frequencies from control and experimental group

	Codes	C	E	Example Quotations
Growth Mindset	Positive effort beliefs	63	105	Explain to Esma that she can get better at maths with continued practice (E)
	Value of asking for help	56	84	I would suggest her to tell someone and let them know she is struggling so they can help her out (E)
	Changing mindset	29	41	… Our mindsets play a huge role .. we need to try to keep them positive (C)
	Resilience & persistence	25	31	… when I was young I used to really struggle at statistics but through shear hard work and determination I got better. U can do this never give up (C)

	Strategy change	14	26	… it seems difficult at first but you just have to adapt your way of thinking and look from a different perspective (E)
	Failure as a motivator	9	23	… it's okay if you make mistakes at the start, you can learn from your mistakes. Always remember it's okay to make mistakes (E)
Attitudes	Cognitive Competence	56	34	I would tell Esma to have more confidence ....believe that she can do it, even when it seems challenging. Believing in yourself is half the battle (E)
	Difficulty	10	3	Do easier data tasks until she can feel confident with them and then work on harder data tasks after with Esma (C)

Conclusions

The GM intervention significantly influenced students' intelligence beliefs and attitudes toward statistics. Qualitative data analysis supported the effectiveness of low-intensity growth mindset interventions (Yeager et al., 2013), in this case, requiring less than 30 mins. Findings aligned with Yeager and Dweck's (2020) recommendations, providing evidence for the value of using metaphors to effectively communicate concepts related to neuroplasticity. The Esma scenario problem revealed that the experimental group made more growth mindset trait-related suggestions, whereas the control group made more teacher-centred suggestions unrelated to growth mindset theory. The study provided insights into experiences with the intervention components, for example, focus group interviews revealed that students preferred to see GM affirmations at the beginning rather than at the end of lectures. In addition, they found the Esma scenario useful because it prompted them to think about how they would behave as teachers in the future. While we cannot discount extraneous influences on study results, the study design ensured a balance across instructors, with each teaching both control and experimental groups, thereby minimizing the potential influence of individual teaching styles on the results. Further studies could investigate the timing and delivery of GM affirmations most especially in relation to student engagement. Building on the findings that the Esma scenario encouraged future-oriented thinking, subsequent studies could assess the long-term effects of scenario-based growth mindset interventions on preservice teachers' classroom practices.

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Bringing math into politics: the Decision Theatre Lab and the influence of model knowledge on the interpretation of stochastic simulation results

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Results of stochastic models in public media are often misunderstood, not least because the uncertainties of stochastic modelling are not sufficiently considered in classical mathematics teaching. This paper first presents a learning environment that confronts students with different forms of mathematical uncertainty: the Decision Theatre Lab. In it, students mathematically explore a professional epidemic model and use its stochastic results for political decision-making. We then present an experimental study that we conducted using this format to investigate how specific model knowledge influences the interpretation of simulation results. The results show that students with specific model knowledge are significantly better at accurately interpreting the stochastic variations in the simulation results than students who had to interpret the results without prior knowledge of the underlying model. The latter, on the other hand, rely more often on contextual knowledge to interpret the model results.

Keywords: Model knowledge, mathematical literacy, stochastic modelling, model uncertainty

Introduction: Mathematics education in the era of vague news

Public media regularly report on the results of scientific modelling, in particular stochastic modelling of climate change. Statements such as "Climate change made US and Mexico heatwave 35 times more likely" (Brosnan, 2024) are prone to misinterpretation, driven by a lack of statistical knowledge and misconceptions about mathematical models. The fact that the underlying models are often inaccessible to readers does not help at all.

Gal and Geiger (2022) have provided a detailed taxonomy of the *Statistical and Mathematical Products* (StaMPs) appearing in public media in the context of the COVID-19 pandemic. They conclude that the interpretation of these StaMPs requires not only an academic understanding of probability but also a literate handling of more subtle and unquantifiable forms of uncertainty that arise, for example, in the case of inadequate data or when making modelling assumptions. They diagnose the need for learning environments that consider these forms of uncertainty.

Against this background, we will first introduce the *Decision Theatre Lab*: a science communication format that enables students to use a scientific epidemic model to make political decisions in a fictitious but reality-based setting. We describe the structure of a Decision Theatre Lab, analyse which forms of uncertainty are inherent in the occurring StaMPs and which aspects of mathematical literacy are important for their interpretation. We then present a case study that used the Decision Theatre Lab framework to investigate how specific knowledge of the model influences students' interpretation of its stochastic results.

Exposing students to uncertainty – a Decision Theatre Lab on infection control

A Decision Theatre Lab – literally and practically – consists of two components: a *Decision Theatre* and a *School Lab*. A Decision Theatre is an IT-supported science communication format in which participants can simulate joint decision-making on various socially relevant topics, guided by the interaction with a scientific model and its simulation results. Decision Theatres have been developed on a variety of topics, from forest planning (Boukherroub et al., 2018) to mobility transition (Wolf et al., 2023) – or infection control (Wiebe et al., 2024), which students attended at Zuse Institute Berlin as seen in Figure 1 and which provides the framework for the study presented here.

In this Decision Theatre, participants take on the role of politicians who must decide on non-pharmaceutical infection control measures at the beginning of a fictitious epidemic. They can draw on the results of GERDA, a GEoReferenced Demographic Agent-based model for the spreading of COVID-19 (Goldenbogen et al., 2022). GERDA simulates the location, the infection probability and the health status of about 12,000 residents of a small town and provides SIRD trajectories for different scenarios. Due to the stochastic modelling, the simulations can differ significantly from each other, even with the same initial conditions. Therefore, the model simulates each scenario almost 100 times and displays the results as a set of curves for the number of susceptible, infected, recovered and deceased. The results should be interpreted as probability distributions that show different degrees of statistical dispersion depending on the scenario. The simulation results depicted in Figure 2, for example, show that dispersion increases the later a potentially imposed lockdown comes into force. This indicates that the prediction accuracy of the model decreases with a later lockdown. Phenomena like these typically emerge in stochastic models (Goldenbogen et al., 2022).

Figure 1: Students in a Decision Theatre

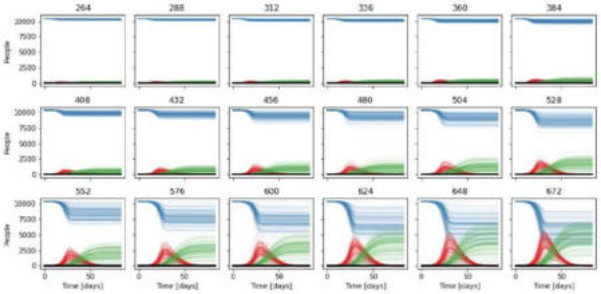

Figure 2: GERDA results for different start times (above graphs in h) of a lockdown

To support their political decision-making, students can simulate three scenarios with different political measures. In the further course of the Decision Theatre, they also reflect on the role of the model for the discussion and evaluate the informative value of the simulation results.

It is important to note that although the participants in a Decision Theatre interact with a mathematical model, this model itself remains a black box for the most part. To put the format in the context of mathematics education, we have therefore developed an optional School Lab that teaches students the mathematical background and the internal structure of GERDA, with a particular focus on the implemented probabilistic concepts: The students derive the probability function for the transmission

of the virus and use it to calculate transmission probabilities in various scenarios. They also use school knowledge of conditional probabilities to determine the probabilities of change in the health status of different people. These calculations, however, are all done once and with pen and paper. It is only after the School Lab in the Decision Theatre that students are confronted with the phenomenon of multimodal distributions resulting from repeated simulations.

The role of model knowledge in the interpretation of stochastic simulation results

The students' use of GERDA exemplifies that the interpretation of Statistical and Mathematical Products (StaMPs) produced by stochastic models is characterised by two types of uncertainty:

- First, uncertainty arises from stochastic modelling itself, particularly from the probability functions and distributions; we refer to this simply as probability. Probability is quantifiable, and its occurrence in the model is intended. It is a basic concept of statistical education, and dealing with it is included in any description of statistical literacy, for example in the influential one by Gal (2002). The correct description and interpretation of GERDAs results, for example, requires basic ideas about the dispersion of probability distributions.
- Secondly, uncertainty is inherent in any form of mathematical modelling. It arises from the fact that models are abstractions, characterised by simplifications and based on certain assumptions. Model uncertainty is a collective term for all reasons that lead to unintended differences between the model and reality and is often difficult to quantify (Thompson & Smith, 2019). An awareness of model uncertainty is therefore an implicit prerequisite for a comprehensive modelling competence, which also includes the evaluation of the validity and scope of given modes, as described by (Niss & Højgaard, 2019), for example. Research conducted during earlier iterations of the Decision Theatre Lab has shown that students tend to explain deviations between model forecasts and actual developments with contextual knowledge rather than with model uncertainty, indicating a low level of awareness of model uncertainty (Lieben, 2023).

Now, one can assume that students depend on appropriate statistical content knowledge and sufficiently developed model competence to accurately interpret such StaMPs – in addition to some general mathematical competencies such as understanding graphs and a positive mathematical identity, as Heyd-Metzuyanim et al. (2021) note. However, the observation of some unsuccessful StaMP interpretations in Decision Theatres without the prior School Lab led us to assume that this framework, as shown in Figure 3, can be extended to include another construct: the specific model knowledge that is acquired in the School Lab.

But since a subject's prerequisites for interpreting (COVID-19-related) StaMPs in the media are usually formulated in terms of statistical literacy and modelling skills – e.g. by Gal and Geiger (2022) or Heyd-Metzuyanim et al. (2021), the influence of specific model knowledge on the interpretation of the model results has so far been underexposed. We, therefore, used the Decision Theatre Lab on infection control to conduct a case study approaching the question: *How does specific knowledge of a model influence the interpretation of its results?*

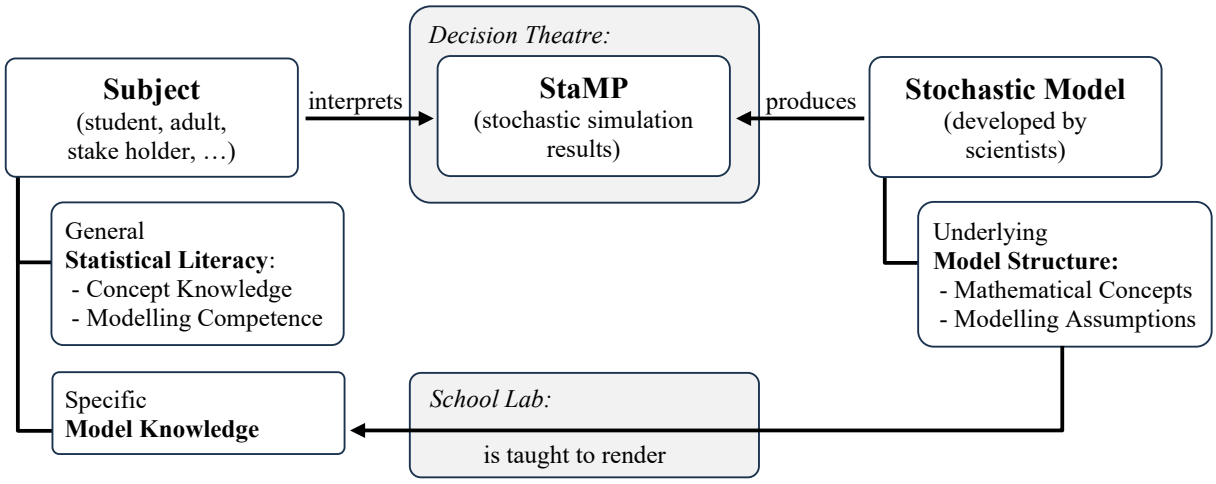

Figure 3: Conceptual framework of a StaMP interpretation in the context of a Decision Theatre Lab

Studying the influence of model knowledge – research design and implementation

To determine the influence of the specific model knowledge acquired in the School Lab on the interpretation of the StaMPs in the Decision Theatre, we conducted an experimental study with the School Lab as an intervention. We defined two groups of students: The experimental group, who first visited the School Lab and then the Decision Theatre, and the control group, who first visited the Decision Theatre and only after that visited the School Lab.

To collect data on how the students processed the stochastic model results in the Decision Theatre, we used the participant observer method according to Weischer and Gehrau (2017) while the students were working on a collaborative group task. For this task, each group received the results shown in Figure 2 from simulation runs with different start times of a lockdown and was asked to interpret the data by completing the sentence "The later the start time of the lockdown, the...". We recorded the group's communication with microphones and analysed the audio transcripts using quantitative and qualitative methods for comparing the experimental group and control group.

Between June 2023 and April 2024, a total of $N = 89$ students from the 10th grade onwards participated in the study. The students were between 14 and 20 years old, and the average grade was 2.31 (on the German grading scale from 1 to 6, where lower grades are better). Of the students, 70 first attended the School Lab and then the Decision Theatre and were thus assigned to the experimental group. The control group, which first attended the Decision Theatre and then the School Lab, comprised 19 students. We randomly assigned working groups. A total of 24 groups were analysed, with 19 assigned to the experimental group and five to the control group.

The data was analysed using a mixed-methods approach, which enables both explorative and comprehensive as well as multi-layered perspectives on the research topic (Kelle, 2019). The material for the analysis consisted of transcripts of the audio recordings and observation protocols. We processed this data according to Mayring's content analysis (Mayring & Fenzl, 2019).

First, we categorised the material inductively, with a particular focus on statements that allowed conclusions to be drawn about the students' statistical concept knowledge and awareness of model uncertainty. To code the data, we used MAXQDA (VERBI Software, 2023). The obtained codes were quantitatively evaluated to carry out a correlation analysis. We calculated the mean values in each category for the experimental and the control group and tested for significance using Student's *t*-test. In the next step, we evaluated the transcripts qualitatively using the categories specified in Table 1.

Table 1: Coding manual for the audio-transcribed student interactions

	Code	Definition	Example
Statistical Concept Knowledge	Description of dispersion	Statements that verbalise the dispersion of the graphs	*"The more people get infected, the greater the dispersion."*
	Correct interpretation as probability distribution	Statements in which the dispersion of the curves is interpreted as a probability distribution	*"The later the lockdown, the more difficult it is to predict precisely."*
	Deterministic Misconception	Statements that provide incorrect non-stochastic interpretations of the results	*"The later the lockdown starts, the more infections there will be."*
Awareness of Model Uncertainty	Contextual Knowledge	Statements based on contextual knowledge that is related to the model results	*"If no one is sick anymore due to the lockdown, then no one can get infected. So, the earlier the lockdown starts, the better."*

Results

The quantitative analysis revealed two (virtually) significant correlations for the inductively selected categories: In the domain of statistical concept knowledge, 50% of the students in the experimental group correctly described or interpreted the model results. This is significantly more than in the control group, in which only 20% answered correctly ($p = .0358$). In the domain of awareness of model uncertainty, the number of context references per minute and person was lower in the intervention group (0.13) than in the control group (0.24) ($p = .0528$). We then proceeded with a qualitative analysis of the answers. We closely examined how the tasks were solved, what led to correct understanding, and what characterised the misconceptions. To illustrate this analysis, we present two exemplary cases below, which show both a typical path to correct interpretation and a widespread misconception in detail.

The following example from the control group illustrates a common statistical misconception, that is then made plausible by contextual knowledge.

Olivia: The later the lockdown starts, the faster the infection rate increases, perhaps?
[…]
Olivia: Because here it has a higher increase…
[…]
John: Yes, but why? Because it's winter?

At the beginning of the conversation, Olivia constructs a deterministic relationship between the start time of the lockdown and the number of infected people. John then tries to explain the model behaviour with real-world contextual knowledge by asking: “But why, because it is winter?”. This statement illustrates a cognitive escape from the actual mathematical task. The real-world reference not only serves the students to validate the graphs, but the students also consider it necessary to interpret the model results.

In contrast, the example from the experimental group shows a typical dynamic within a conversation in which the students move from the same type of misconception to a correct interpretation.

James: The later the time of the lockdown, the greater the number of infected persons.
Emily: The higher the peak of infected people.
[…]
James: The more infected there are.
[…]
Emily: But look. It's also getting thicker. So, the individual lines here no longer agree about what's happening.
[…]
William: Yes, that's just a probability.
[…]
William: So, the later, um, a lock-. The later-.
Emily: The more difficult it is to predict what will happen.

James initially postulates a causal relationship between the lockdown's start time and the number of infected persons. Emily expands on this assumption by referring to the peak of the infection curve. However, both statements are incorrect as they imply a deterministic relationship and disregard the stochastic nature of the underlying model.

A correction process begins as Emily notices that “the lines are becoming thicker”, indicating the dispersion of the curves. This observation marks the start of questioning the deterministic understanding and leads to a reconsideration of the previously assumed cause-and-effect relationship. As the discussion progresses, the participants become more aware of the probabilistic nature of the model. This awareness is reflected in the statement “that's just a probability”, which explicitly acknowledges the connection between the dispersion of the curves and the induced probability distribution. The introduction of the term “probability” then enables the group to move on from the mere description of the dispersion to its correct interpretation. They discern that stochastic models do not provide fixed predictions but rather depict a range of possible scenarios with different probabilities.

Discussion: How could model knowledge affect StaMP interpretation?

The quantitative observation that students in the experimental group interpreted the StaMPs in the Decision Theatre correctly more often could be related to the fact that model knowledge interferes with both the domain of statistical literacy and the domain of mathematical modelling competence.

On the one hand, the use of stochastic concepts and terms for reproducing the model components in the School Lab seems to motivate some students in the experimental group to use the same concepts and terms to interpret the dispersed graphs in the Decision Theatre – which in this case leads them to correct conclusions. Knowledge of the model thus leads to a better understanding of the model results, since the concepts used to describe the model reappear in the interpretation of its results. On the other hand, as students in the School Lab reproduce the structure of the model, they have to follow some of the underlying modelling assumptions. This may contribute to an increased awareness of model uncertainty. The alternation between reality and the mathematical model during the reconstruction process – which is typical for stochastic modelling (Kelter et al., 2023) – could help students to distinguish more precisely between these two worlds. Thus, they are significantly less tempted to use extra-mathematical contextual knowledge from reality to explain mathematical results.

It is not clearly assessable, however, whether the model knowledge has really improved the interpretation of the simulation results in a direct way or whether the *teaching* of this knowledge in the School Lab has rather led to a promotion of statistical literacy and development of modelling competence, which in turn has led to an improved interpretation. Finally, it should be stated that many modern computer models are obviously mathematically too complex to be reconstructed in a School Lab. The promotion of model-independent statistical literacy and modelling skills therefore remains indispensable.

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Problem solving and argumentation in probability context: analysis of some examples of necessary object-level narratives

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The introduction to probabilistic thinking in primary and middle school is one of the goals delineated by the Italian National Curriculum Guidelines and is part of the curricula of other countries. The paper presents activities carried out in Italian classes from grades 3 to 8 in which the introduction to probabilistic thinking is structured through problem solving and argumentation activities. The mathematical claims developed by the students were examined through a qualitative analysis, identifying the necessary object-level narratives and their substantiation. The analysis also considers the role of specific visual mediators in students' justifications when solving probability problems.

Keywords: Probability, primary and middle school, necessary object-level narratives, problem solving, argumentations.

Introduction

In Italy, the National Guidelines, in line with the literature, emphasise the importance in mathematical practice of solving problems, explaining the procedure followed, recognising strategies other than one's own and comparing procedures. This applies to all content areas of mathematics. This paper focus on probability problems. Probabilistic thinking is promoted in the Italian National Guidelines from primary school onwards. Uncertainty permeates many aspects of human life, understanding and managing this uncertainty is hence essential for informed decision-making. It is therefore crucial that students understand probability to make informed decisions when faced with uncertainty (Devlin, 2014; Muñiz-Rodríguez et al., 2020). Research highlights the need to pay more attention to teaching-learning processes related to probability, developing both theoretical and empirical aspects (Batanero, 2020). There are recent studies on the development of argumentative processes and communication in probability contexts in high school (Saccoletto, 2023). Our action-research group, composed of didacticians and teachers, aimed to address this perspective from primary school onwards. Our teaching goals are not only enabling students to acquire knowledge and skills related to specific mathematics contents, but also to promote the development of argumentative processes and communication. To analyse the data collected in our classes, we recognised the need for interpretative lenses to identify and characterize mathematical claims produced by the students. We found that the framework proposed by Valenta et al. (2024) provided tools that met this need. Their study is based on commognition which has close affinities with the approaches adopted by our action-research group. Specifically, we have as common roots the Vygotskian perspective on the social construction of knowledge and the value of communication in mathematics. Our study highlights how the categories identified by Valenta and colleagues can be further characterised by considering the visual mediators proposed by teachers in teaching probability to represent events and evaluate situations of uncertainty. In this paper, we show some tasks developed to introduce probabilistic thinking in the

first cycle of education in Italy. In our work, we will analyse some examples of students' narratives using the interpretative lenses proposed by Valenta et al. (2024) within the field of probability, which we will show as a suitable context for applying these lenses.

Theoretical framework

Research on learning mathematics through communication has been ongoing for several decades. Sfard and colleagues (Sfard et al., 1998) have. addressed this topic since the 1990s. In Sfard's theory of commognition (Sfard, 2008), the study of mathematics learning goes through the exploration and analysis of processes of discourse development. She identifies four aspects of mathematical discourse: word use, visual mediators, routines, and narratives. Of particular interest in this paper are *narratives*, which are utterances (both spoken and written) that describe objects, the relationships between objects, or activities involving or performed by objects. Definitions and theorems are examples of narratives. In the commognitive frame, Jeannotte and Kieran (2017) describe mathematical reasoning (MR) processes as "processes that derive narratives about objects or relations by exploring the relations between objects" (ibidem, p. 9). In this perspective, Valenta and colleagues (Valenta et al., 2024) introduced a framework to analyse opportunities for mathematical reasoning. In their work they investigate the processes of *construction* (which results in new narratives) and *substantiation* (which is related to the endorsement of already constructed narratives) of specific narratives, which they call Necessary Object-Level narratives (NOLs).

> Based on Hewitt, we distinguish between arbitrary object-level narratives (AOLs) and necessary object-level narratives (NOL). AOLs are substantiated by reasonableness (e.g., Kontorovich & Zazkis, 2017), aiming to ease communication or as products of historical development. In contrast, NOLs can be substantiated by proving, following deductive chains of object-level claims. Therefore, NOLs, along with their construction and substantiation, are relevant for investigating opportunities for MR. (Valenta et al., 2024, p. 94)

In their paper Valenta and colleagues analyse discussions among teachers and students in different classroom situations also by talking about written productions. They investigate MR not only in situations in which MR-processes are more easily promoted (such as small-group problem solving), but in all kinds of different situations, such as the introduction of a definition. Table 1 summarises codes and descriptions proposed by Valenta et al. (2024).

Table 1: Classification and description of NOLs construction and substantiation

Construction		*Substantiation*	
Code	**Description**	**Code**	**Description**
unclears	The teacher or the students present an NOL, but the construction process is unclear	no protest	NOL is substantiated due to no protest from either the teacher or the students
NOL presented as a rule/fact	NOL is presented as given and true	short confirmation	NOL is substantiated by a short confirmation of validity (given by the teacher or the students)

NOL presented as a claim	NOL is presented as a claim to be further examined or validated	implicit, by construction	Accepting the validity of the NOL through construction (e.g., by using procedures, properties, or a mixture)
using procedure	NOL is constructed by performing some more or less known actions/ procedures	explicit, using procedures	Presenting the procedure as an answer to the question, "Why is this true?"
using a mixture of procedures and properties	NOL is constructed by a mixture of procedures and properties and relations	explicit, mixture of procedures and properties	Using a mixture of procedure and properties to answer the question, "Why is this true?
using properties and relations	NOL is constructed by using properties and relations (e.g., referring to a definition, comparing, classifying, identifying a pattern, generalizing)	explicit, using properties and relations - valid - not valid	Using properties and relations to answer the question, "Why is this true?"

Constructing, substantiating, and recalling narratives are core routines of mathematical discourse. NOLs can be identified in argumentative practice. Argumentations can serve an explanatory function: the aims are to explain to the teacher or to peers and to build knowledge together (Ferrari & Saccoletto, 2023). As the literature highlights, interactive argumentative processes can support collective learning in mathematics, prompting students to participate in students' exchange, to understand and to anticipate others' perspectives, to refine justifications and motivations, and to connect and evaluate different perspectives (e.g. Asterhan & Schwarz, 2016; Cobb et al., 2001; Krummheuer, 2000). In our study we are interested in identifying NOLs and analysing their construction and substantiation in argumentative processes, i.e. when the students have to communicate both writing and orally problem solving strategies and their motivation to their peers. We stress that we consider written productions as part of the argumentative processes, as they are collectively discussed. In addition, it is important to note that narratives developed during argumentative processes are shaped by the topics under discussion. Our research focuses on argumentation in probability problem solving. Students construct and develop narratives and use visual mediators, which we included in the analysis. The study presented in this paper is guided by the following research questions: How can the study of NOLs be used in the analysis of arguments and argumentative processes in the context of probability? What NOLs can we identify in the argumentations of primary and middle school students in the context of probability problem solving?

The study

The educational activities we present in this paper were carried out in the school year 2022/23 in an Italian Institute, which includes primary school, grades 1-5, and middle school, grades 6-8. We work vertically from primary school to middle school, aligning with the Italian National Guidelines. Approximately three hundred students participated in the study. They were involved in our project because the teachers (five primary school teachers and eight middle school teachers) are part of our research group. Teachers and researchers collaborated in designing the educational activities. Our educational goals were: Acquisition of basic knowledge on probability (classical approach - a priori); Enhanced communication and argumentative skills. The focus was on problematising the experience of rolling dice and the production of arguments (oral and written) within a workshop setting, enabling

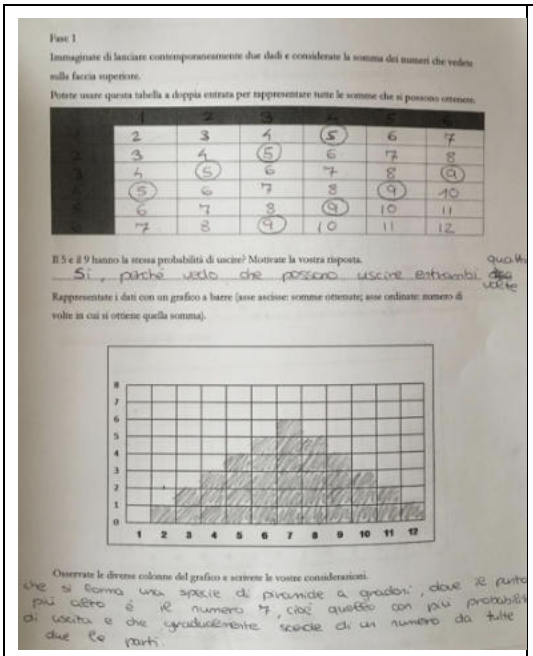
students the opportunity to engage in reasoning and discussions on probability problems. The problems are usually addressed in the teaching of probability and associated with the birth of classical probability. The history of mathematics can be a useful resource for understanding and providing information on the development of mathematical thinking (Radford et al., 2002), helping to predict students' "misconceptions" (Sfard, 2008). The educational activities, starting from the analysis of the outcomes of the throwing of two dice, lead to the solution of the famous problem proposed to Galileo Galilei by Florentine gamblers related to a game of dice known as *zara*. In his work "Sopra le scoperte de i dadi [On the discoveries related to dice]" (ca. 1620/1898), Galilei lists all possible outcomes of rolling three dice and calculates the probability that the sum of the three outcomes equals 9 or 10. A significant point of discussion is the importance of considering the order of outcomes. Although both sums of 9 and 10 are obtained from the same number of combinations, they have different numbers of partial permutations. Specifically, the sum of 10 can be obtained in 27 partial permutations, while the sum of 9 can be obtained in 25. According to classical probability, therefore, 10 has a higher probability of occurring than 9 (27/216 versus 25/216).

In the first phase of our teaching experiment the students worked individually or in small groups to solve problems, write the solution and justify it. Primary and middle-school students faced similar problems. Teachers provided standard graphical representations, double-entry tables and bar charts, to scaffold students' explorations. The teachers then orchestrated a collective discussion about different answers in which the written argumentations were collectively discussed. The data concern argumentations during a social construction of a classroom mathematical knowledge. We collected students' written answers, and we recorded classroom discussions after problem solving activities. We conducted a qualitative analysis of transcripts of class discussions and students' written work. The data gathered were analysed through a content analysis (Patton, 2002). A top-down coding was performed from model to the data, by using the theoretical lenses provided by Valenta et al. (2024). To enhance the trustworthiness and rigour of our findings we used the classification of NOLs (Table 1) to independently analyse the data. Then we compare our results. Through constant comparison, we decided on the classifications and identified new aspects to be underlined in the students' mathematical reasoning.

Tasks and analysis of NOLs

Figure 1 shows the task given to grade 8 students and an example of answers. We cannot identify NOLs during their first genesis, but when the students communicate their strategies and justify them to peers and teachers. In the example shown in Figure 1 we can analyse written answers made by claims, numbers and signs on the table. Using the classification proposed in Table 1, we label the argumentation produced to answer the first question *Substantiation, explicit by procedures, valid* because the justification for the affirmative answer to the question is made explicit, and the argument is valid. We notice that the argumentation is supported by means of the table of possible outcomes and favourable cases are marked with circles on the table. In this case, the procedure is not fully explained by words, but it is clear from the signs made on the table (the numbers to be considered are circled). In our data we also found incorrect arguments such as "no, since the number 9 is greater than

the number 5 there can be more combinations of numbers". It is a *substantiation explicit, using properties, non-valid* because even if it makes explicit a true property that links the two numbers, that is not the correct argument to support the claim that the two numbers do or do not have the same probability of occurrence. In the second part of the task shown in Figure 1 we can see a construction of NOLs that is generated by the exploration of the diagram. Here, NOLs are *presented as facts*.



Translation of tasks and answers shown in the adjacent picture:

Imagine throwing two dice at the same time and consider the sum of the numbers you see on the upper face. You can use this double-entry table to represent all the sums that can be obtained. [empty table] Do 5 and 9 have the same probability of occurrence? Justify your answer.

"Yes, because I see that they can both come out four times."

[empty grid] Represent the sums obtained with a bar chart. Look at the different columns of the diagram and write down your considerations.

"A kind of step pyramid is formed, where the highest point is the number 7, i.e. the one with the highest probability of exit, and which gradually descends by one number on either side".

Figure 1: Example of tasks and grade 8 student's answers

The next activity dealt with the problem inspired by a question posed to Galileo Galilei. This is the text of the task given to the students: "*In the 17th century, the game of three dice was very popular. The game consisted of betting on a number and rolling three unbiased six-sided dice. The winner was whoever correctly predicted the sum of the numbers rolled on the three dice. Some players were trying to figure out which numbers to bet on to have the best chance of winning. After many observations, they realized that 10 came up more often than 9. Not understanding why, they asked Galileo Galilei for help. What do you think Galileo's answer was?"* Also in this case, the teachers provided a graphic representation to be filled in and explored. We report an extract from a grade 4 classroom discussion. A student completed the diagram shown in Figure 2 and explained to the teacher why 9 may come out less than 10.

Teacher: Explain your thinking better...
Student 1: 9, like the number 3 and the number 18, can be made up of the same numbers, i.e. 3+3+3 so it only has one combination.
Teacher: For the score 9, there is only one combination?
Student 1: No, it's just that it can also be formed with two equal numbers and a different one, i.e. 4+4+1 and you can 'spin' them, but not like when there are 3 different addends.

In the transcript above we can see a development of the arguments. Applying the classifications shown in Table 1, we can identify *substantiations explicit, using a mixture of procedures and properties* because the student explicitly expresses the property used to support the fact that there are fewer combinations in certain cases that she also explains how they are generated. When explaining, the student points to the diagram used as visual mediator (see Figure 2).

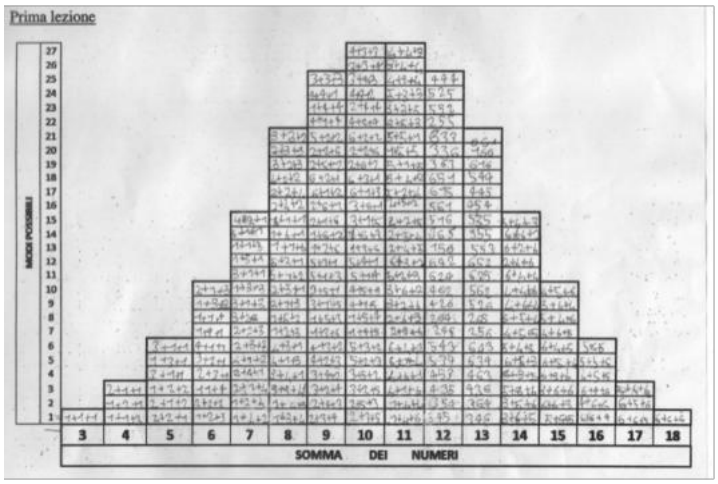


Figure 2: Example of diagram proposed in a grade 4 classroom

The same task was given to middle school students. This is an example of a written answer given by a grade 6 student: "In practice, the number 10 comes out more frequently than the number 9 because even though they have the same number of combinations, it has more variety of possibilities, due to the fact that with the 3,3,3 draw, 9 loses two varieties of possibilities, making 10 slightly more common". Here we can identify NOLs *substantiation explicit, using a mixture of properties and relation, valid* because the arguments generated in support of her statement explicitly mention the properties of the numbers involved and the relationships between their combinations, but the different possibilities are not written down. The specific language on combinations is not precise, but this is due to the knowledge of the students in that school grade. In this case the arguments could be based on the use of the diagram, but we cannot be sure of that.

Discussion and conclusion

This paper shows some tasks for primary and middle school students to introduce them to probabilistic thinking. The goals of these tasks are in tune with Italian National Guidelines. We analyse examples of students' narratives using the interpretative lenses proposed by a framework for necessary object-level narratives (NOLs) proposed by Valenta et al. (2024). We show that NOLs framework provides us with useful tools for identifying and analysing the claims developed by students in order to justify their choices and answers in probability problem solving. In our teaching experiments students are asked to write down the arguments supporting their choices and answers, and then their argumentations are shared and discussed in class. Our data differ from those analysed to exemplify the NOLs framework in the paper by Valenta et al. (2024). Even if we carried out our study in a different country and we analysed problems in different contexts, the NOLs framework

provided adaptable lenses that allowed us to classify not only the narratives developed during oral discussions but also students' written productions to justify their answers developed mainly after probability problem solving activity. By presenting the analysis of our data and the results obtained, we answered the research question: "How can the study of NOLs be used in the analysis of arguments and argumentative processes in the context of probability?". The context is the probability in which the literature tells us that we need to pay more attention to teaching-learning processes (Batanero, 2020). This is only the first attempt to implement the framework for NOLs that we have carried out through the analysis of argumentations in probability context. We show that students in the constructions of NOLs find similarities, differences and identify regularities, and then in the substantiation of NOLs they justify them. The analysis carried out also made it possible to start answering the second question: "What NOLs can we identify in the argumentations of primary and middle school students in the context of probability problem solving?". We have only shown a few examples of argumentations in the area of probability, but the fundamental role of visual mediators already emerges strongly. We show how the categories identified by Valenta and colleagues can be further characterized by considering the visual mediators proposed by teachers in teaching probability to represent events and evaluate situations of uncertainty. We would like to analyse more the genesis and development of NOLs in dialogues during problem solving activities in the context of probability deepening the study of the role of visual mediators. As a matter of fact, Sfard's theory of commognition (Sfard, 2008) identifies visual mediators as crucial aspects of mathematical discourse with word use, routines, and narratives. We hope that our analysis will shed new light on teaching-learning processes of probability by providing refined tools for analysing arguments developed in the classroom, particularly those that rely on visual mediators. However, these aspects require further investigation. This could be the next step in our study which we will develop with our action-research group, to explore how this type of analysis could also be refined as a teaching tool. In particular to develop formative assessment practices and to generate feedback highlighting the characteristics of the NOLs produced and, possibly, the aspects that need to be improved.

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Type of table and reading levels promoted by Spanish textbooks

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In this paper, we analyze tasks related to the different types of statistical tables included in recent Spanish textbooks and, more specifically, the reading levels required of students to solve these tasks. The sample comprises 10 Spanish secondary-school textbooks (12-15 years-old) from two different publishers. We classified the reading levels proposed by Friel and his collaborators into different types of statistical tables. The results show a predominance of first reading levels, and there is no relationship between the type of table and the reading level required to solve the task.

Keywords: Textbooks, reading levels, secondary education.

Introduction

The COVID-19 crisis highlighted the need for citizens to critically evaluate statistical information and thus make informed decisions in uncertain contexts (Gal, 2024). Statistical tables are widely used in the media and scientific fields because of their cognitive flexibility to summarize and communicate information. Therefore, the ability to read and interpret statistical tables is part of statistical literacy. However, their understanding is a sophisticated skill that requires education.

Curricular changes in several countries have promoted meaningful learning to enhance students' literacy competencies. These curricular changes are required to ensure that future generations develop the knowledge, skills, values, and attitudes necessary to build sustainable societies, thus contributing to the goals of the Education for Sustainable Development (UNESCO, 2017). In this sense, recent Spanish curricular guidelines for secondary education (12-15 years-old), stablish ten specific competences to be developed through numerical, spatial, measurement, algebraic, stochastic, and socio-affective sense (MEFP, 2022). The first cycle of secondary education includes 1st to 3rd grade and the second cycle comprises 4th year, with two strands (A and B) with no differences in the basic knowledge established for each option in a stochastic sense.

We focus on the statistical tables, which have been included in the Spanish curricular guidelines since primary education. In secondary education, different statistical tables (with different types of frequencies, data grouped in intervals, etc.) are incorporated, and their study and understanding involve new cognitive demands. This variety of statistical tables, each of which must be read in a specific manner, is difficult (Pallauta et al., 2023a). However, related research is still scarce.

Because textbooks are a didactic resource for teachers and students that constitute an intermediate stage between the official curricular guidelines and the teaching implemented in the classroom, research analyzing the presentation of different topics in textbooks has increased (Schubring & Fan. 2018). We were therefore interested to know whether the new school textbooks published within the framework of the recent educational legislation in Spain will help develop competence in the interpretation of information represented in statistical tables and graphs (MEFP, 2022) because, for the first time, the stochastic sense is mentioned in the Spanish curricular guidelines. More

specifically, we analyze the tasks related to statistical tables included in recent Spanish textbooks and the reading levels required of students to answer these tasks. In this way, we complete our previous research with Chilean and Spanish textbooks in a previous curricular period (Pallauta et al., 2021a, 2021b, 2023b). This topic is relevant because of the scarcity of secondary education research.

Below, we describe the background and method and present the results. We conclude with some implications for teaching statistical tables.

Background

Classification of statistical tables

We selected the classification of Lahanier-Reuter (2003) to base our research because she considered the specific functions and meanings to differentiate the three types of tables: data tables, distribution tables of variables, and two-way tables. In this study, it is important to differentiate these types of tables because each type involves different levels of complexity and forces to work with distinct mathematical objects. In Pallauta et al. (2023b), we subdivided statistical tables in the classification of Lahanier-Reuter (2003), depending on whether cumulative frequencies are represented in the table and whether the data are grouped into intervals or not. Thus, we considered three types of distribution tables: ordinary frequency tables; cumulative frequency distribution tables; and distribution tables with values grouped in intervals. Likewise, we subdivided two-way tables into two types depending on whether or not the values were grouped in intervals. In the following, we describe this classification of statistical tables.

C1. Data tables contain as many rows as elements in the sample, and the heading indicates the variables collected. The first column contains the list of elements, and each table cell contains data associated with different variables. The concept of a variable and its values are implicit in this table.

C2. One-variable distribution tables display the frequency distribution of a statistical variable and can be of three types: 1) C2.1. Ordinary frequency distribution tables incorporating absolute, relative, or percentage frequencies; 2) C2.3. Frequency distribution tables with cumulative frequencies: in addition to ordinary frequencies, cumulative frequencies are included, requiring work with inequalities; 3) C2.3. Distribution tables with grouped data: this method considers the grouping values of variables in intervals and works with intervals, their extremes, and class marks.

C3. Two-way tables represent the joint distribution of two statistical variables with compound, marginal, and conditional frequencies. We consider two types: C3.1 Two-way table, representing absolute, relative, or percentage frequencies; and C3.2. Two-way table with grouped data, including the classification of values into class intervals for all frequency types.

Reading levels of statistical tables

Starting from a statistical table or graph, we can ask questions involving different difficulty levels because the difficulty of interpreting a statistical graph or table depends on the availability of the required information. Friel et al. (2001) established a hierarchy to consider this complexity when reading statistical graphs, which have also been used to analyze statistical tables (e. g. García-García et al., 2019; Pallauta et al., 2021b). We selected this hierarch because it highlights the different

difficulty levels of questions that can be posed from statistical tables.

L1. Reading the data. This level involves only literal reading of explicit information on the table or graph, for example, finding the frequency of a given value.

L2. Reading between the data. In addition to literal reading, it involves comparing data represented in the table or making calculations. An example is determining the distribution mean.

L3. Reading beyond the data. This level involves finding data that are not immediately accessible in the table, for example, interpolating or extrapolating information displayed.

L4. Reading behind the data. This is the most advanced level. In addition to reading the graphs or tables, this involves a critical appraisal of the content and sources from which the information comes.

Previous research

Most previous studies analyzing the reading level required to solve textbook tasks have focused on statistical tables and primary school education books (e.g., García-García et al., 2019), although a few analyzed isolated secondary education grades (Pallauta et al., 2021a). Regardless of educational level, these studies suggest an excess of procedural tasks focused on the most basic reading levels, such as literal reading (L1) or comparing data and performing calculations (e.g., statistical summaries or probabilities) (L2). For example, Pallauta et al. (2021a) analyzed the different types of statistical tables included in 18 Spanish secondary-school textbooks published on the previous educational system from three different publishers. They observed an increasing variety of types of statistical tables as the school grade progressed. Ordinary frequency distribution tables were the most frequently used, while two-way tables with grouped data were the least used. Moreover, Pallauta et al. (2021b) analyzed six Chilean textbooks and six Spanish textbooks for secondary education 1st and 2nd grade published in the previous period. The textbooks in both countries promoted a stronger L2 reading level (over 75% activities), followed by the L1 reading level (about 20%). More sophisticated reading levels (L3 and L4) were presented at lower percentages and were scarcer in Spanish textbooks.

Method

We present qualitative research based on content analysis (Drisko & Maschi, 2016) of mathematical activities in statistical tables proposed in the statistics and probability chapters of a sample of Spanish secondary-school textbooks. Following an inductive and cyclic procedure: 1) We selected all the activities incorporating statistical tables in each textbook; 2) We coded each activity according to the reading level needed to solve it (Friel et al., 2001) and the type of statistical table involved (Pallauta et al. 2023b), which were described in the background section (Type of tables and Reading levels); 3) The coding was reviewed several times by one author and discussed with other colleagues until agreement was reached, to ensure the reliability of the coding; 4) Results were summarized in tables to facilitate the drawing of conclusions.

The sample included ten Spanish textbooks aimed at grades 1st to 4th (12-15 years-old secondary-school students), used in schools and published by two prestigious editorials (Anaya and Santillana) that were widely distributed in Spain. Thus, we analyzed five textbooks for each publisher because in grades 1 to 3, all students study the same content, but in grade 4, students can choose between

academic (4th B) or applied mathematics (4th A). In total, 1217 activities were analyzed using statistical tables: 633 in Anaya and 584 in Santillana.

The reliability of the coding in the research is guaranteed insofar as successive reviews of the textbooks were carried out by the authors of this study, with subsequent discussions of discordant cases, until agreement on the coding was reached.

Results

Type of statistical tables

The following categories were considered:

C1. Data tables. These tables are mainly devoted to list data. Thus, the values of each variable are represented individually for each element in the dataset. Figure 1 presents an example taken from a 4th A grade book, where two quantitative variables (hours of mobile phone used and hours of sleep) are represented.

10 teenagers were asked about the number of hours they use a mobile phone every day and the number of hours they usually sleep each night. Here are the results:

Hours of use the mobile phone	1	2	5	3	6	5	3	1	2	4
Hours of sleep	10	8	4	7	4	5	6	8	6	7

Estimate the number of hours for a day that a student would use his or her mobile phone if he or she usually sleeps 9 hours a day.

Figure 1: Example of data table and reading level L3 (Colera et al., 2023b, p. 219)

C2. One-variable distribution tables. In these tables, each variable value is linked to one or more types of frequency (absolute, relative or percentage frequency). We distinguish three different types:

- *C2.1. Ordinary frequency distribution tables*. These tables usually have different columns to represent the absolute (f_i), percentage (p_i), or relative (h_i) frequencies of the variable values.
- *C2.2. Cumulative frequency tables*. This type of table represents the cumulative relative frequencies (H_i) or the absolute relative frequency (F_i) of a variable (Figure 2, left-hand side).
- *C2.3. Frequency tables of data grouped in intervals.* The values of a continuous or discrete variable are grouped in intervals (Figure 2, right-hand side), which implies dealing with intervals and their extremes to approximate values and calculating summary statistics.

This is the number of books read by students in a class during the semester. Construct the frequency table:0 2 4 2 0 2 0 1 1 3 2 2 1 0 3 2 4 0 1 1 3 4 1 2 3

x_i	f_i	F_i	h_i	H_i	p_i
0	5	5	0,2	0,2	20
1	6	11	0,24	0,44	24
2	7	18	0,28	0,72	28
3	4	22	0,16	0,88	16
4	3	25	0,12	1	12
	25		1		

The table shows the ages of a group of people. It represents the data in a histogram.

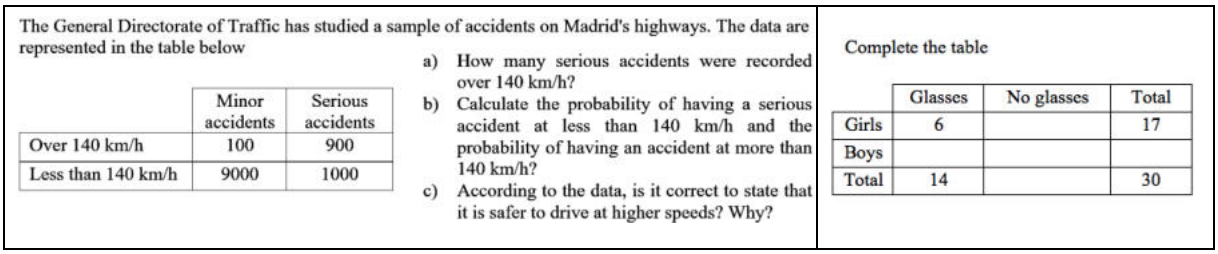
Age	f_i	h_i
[36, 42)	4	0,26
[42, 48)	4	0,26
[48, 54)	5	0,3
[54, 60)	2	0,13

Figure 2: Examples of one-variable distribution tables (Grence, p. 256; p. 258)

C3. Two-way tables. In these tables, it is necessary to identify the values of each variable and the crossing of each category in the table rows and columns. Therefore, three types of distributions

are represented in the same table: joint, marginal, and conditional distributions. We distinguish two different types:

- *C3.1. Two-way tables with ordinary frequencies* represent qualitative or discrete variables. For example, the table in Figure 3 (right-hand side) represents qualitative variables.
- *C3.2. Two-way tables with grouped data* represent continuous variables with data grouped in intervals; for example, in Figure 3 (left-hand side) the intervals are indicated verbally.


The General Directorate of Traffic has studied a sample of accidents on Madrid's highways. The data are represented in the table below

	Minor accidents	Serious accidents
Over 140 km/h	100	900
Less than 140 km/h	9000	1000

a) How many serious accidents were recorded over 140 km/h?
b) Calculate the probability of having a serious accident at less than 140 km/h and the probability of having an accident at more than 140 km/h?
c) According to the data, is it correct to state that it is safer to drive at higher speeds? Why?

Complete the table

	Glasses	No glasses	Total
Girls	6		17
Boys			
Total	14		30

Figure 3: Example of two-way table (Colera et al., 2023a, p. 319; p. 322)

Table 1 presents the percentage of types of statistical tables by school grade and editorial (percentages are computed for each column). Similarly to Pallauta et al. (2021a), we observe the predominance of the distribution table with ordinary frequencies (C2.1), especially for grade 1, although its frequency diminishes after the two first grades because new types of tables are given predominance in the upper grades. For example, distribution tables with grouped data (C2.3) start appearing in grades 3 and 4, although after the percentage diminishes. Two-way tables with ordinary frequencies (C3.1) appear scarcely in Grade 1, but afterwards between 11.6 and 20.9 of the tables are included in this category.

Table 1: Percentage of type of table by school grade and by editorial

Type of table		Grade					Editorial	
		1 N=140	2 N=104	3 N=199	4 A N=344	4 B N=430	Anaya N= 633	Santillana N= 584
C1. Data table		10.0	19.2	11.1	14.2	19.3	22.0	8.4
C2. Distribution table	C2.1. Ordinary	82.9	58.7	35.2	24.7	21.6	30.8	39.4
	C2.2. Cumulative			5.5	20.9	14.4	8.2	15.9
	C2.3. Grouped	2.9		36.7	10.2	24.2	18.8	21.9
C3. Two-way table	C3.1. Ordinary	0.7	20.2	11.6	20.9	20.5	19.9	13.5
	C3.2. Grouped	3.6	1.9				0.3	0.9

Two-way tables with grouped data (C3.2) tasks are very rare and are only included in grades 1 and 2. Similar results were reported by Pallauta et al. (2021a); however, some types of distribution tables were not considered for grades 1 and 2 (C2.2 and C2.3), while two-way tables with grouped data (C3.2) were still more scarce and were not considered for grades 3 and 4.

When analyzing the type of table by publisher, the main differences are that Anaya has almost three

times as many data tables (C1) as Santillana, and Santillana has twice as many distribution tables with cumulative frequency (C2.2) as Anaya. Two-way tables with ordinary frequencies are presented at a slightly higher percentage in Anaya, whereas two-way tables with grouped data are rarely considered by both publishers.

Reading levels

We used the reading levels from the classification by Friel et al. (2001). We describe some examples of activities classified in each of these levels:

L1. Reading the data. At this level, the activity focuses on finding the frequency of a variable value (or the value to which a frequency corresponds). An example is shown in question a of Figure 3 (left-hand side), which asks for the identification of the joint frequency.

L2. Reading between the data. At this level, comparisons or calculations must be performed with the data represented in the table. Question b in Figure 3 (left-hand side) corresponds to this reading level because the probabilities must be calculated. In Figure 3 (right-hand side), this level is also required to complete the table.

L3. Reading beyond the data. This reading level is reached in activities in which the data are not explicitly represented in the table, which includes interpolating or extrapolating data. For example, in Figure 1, students are asked to interpolate the value of one variable according to the data.

L4. Reading behind the data. At this level, the solver critically evaluates the data collection method and its validity or reliability or justifies the obtained conclusions. For example, question c in Figure 3 (right-hand side), the student must assess the veracity of the statement according to the data.

Table 2 presents the reading levels of the activities by grade. Almost all activities required L2 (reading the data) in all school grades like in previous research (García-García et al., 2019; Pallauta et al., 2021b), whereas L1 (reading the data) appeared between 10.6% and 21.4% in different grades.

Table 2: Percentage of reading levels by school grade and by editorial

Reading level	Grade					Editorial	
	1 N=140	2 N=104	3 N=199	4 A N=344	4 B N=430	Anaya N=633	Santillana N=584
L1. Reading the data	11.4	16.3	10.6	15.4	21.4	17.7	14,9
L2. Reading between the data	85.7	82.7	85.4	80.2	74.4	76.0	84.1
L3. Reading beyond the data				4.1	2.6	3.6	0.3
L4. Reading behind the data	2.9	1.0	4.0	0.3	1.6	2.7	0.7

Reading level L3 (reading beyond the data) was very scarce and was present only in grade 4, with no significant difference between the two strands of this grade. The reading level L4 (reading behind the data) was also too rare. Other investigations have also found the scarcity of statistical table tasks that require L3 and L4 skills in both primary education (García-García et al., 2019) and secondary

education (Pallauta et al., 2021b). These results disregard the curricular guidelines and the importance of making inferences and predictions from the data analyzed.

When comparing editorials in terms of reading level, Santillana presents a higher percentage of activities at the L2 reading level (84.1%), which suggests that this publisher places greater emphasis on calculation. Both publishers include a small percentage of L3 and L4 activities; however, they are presented in a slightly higher percentage in Anaya, which promotes statistical reasoning by trying to predict variables that do not appear in the table and reinforces the critical reading of the information.

Conclusions

In this paper we analyze the different types of statistical tables and the reading levels by Friel et al. (2001) required to complete the activities contained in a sample of 10 Spanish secondary-school textbooks published within the framework of the recent educational legislation in Spain (MEFP, 2022). Given the prevalence of statistics tables in secondary schools, our method and variables may serve to analyze other textbooks in that or other countries.

The results, like Pallauta et al. (2021a) demonstrate that the tasks related to different types of statistical tables are focused on distribution tables with ordinary frequencies; however, their frequency decreases as the grade progresses, although no increase was associated with the complexity of the reading levels involved. Moreover, our study revealed a predominance of the first two levels of reading the data, in which only direct reading of the information in the statistical tables (L1) and comparisons or computation with the data (L2) are required, like in García-García et al. (2019) and Pallauta et al. (2021b) and L3 level that promotes the students' ability to interpolate and extrapolate from data which is considered an important part of statistical literacy (Gal, 2024) is almost not-existent in an editorial. These results suggest an excessive emphasis on developing computational abilities rather than focusing on fostering students' statistical literacy or reasoning. We also note the scarcity of L4 level activities, which develop the ability to reason with statistical information, to discuss arguments based on statistical information, and to assess the reliability of such information.

The scarcity of high reading levels activities was also noted in research dealing with statistical graphs (e.g. Friel et al, 2001). Then, it is important to include more L3 and L4 tasks both in graphs and tables, as some research (Sharma, 2013) has shown difficulty with reading at these levels in secondary school students. Furthermore, teachers need to be aware of the variety of the statistical tables, according to the classification described in this paper. The statistical table is not only a powerful tool for statistical literacy but also deepens student's algebraic reasoning (Pallauta et al., 2023b).

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Teaching data-driven machine learning in mathematics education

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Technologies based on data-driven machine learning (ML) methods have taken on a central role in our economy, technology and everyday life. The underlying principles of various ML methods are grounded in data and mathematical concepts often embedded in elementary form within high school mathematics curricula, e.g., distances between points and lines or the dot product. In this paper, we propose using classification problems and the Support Vector Machine (SVM), a supervised ML method, to introduce students to data-driven ML techniques. We provide a didactical analysis of the SVM and present intended learning trajectories for both lower and upper secondary education, enabling students to comprehend key mathematical ideas underlying the SVM, which are representative of many ML methods.

Keywords: Machine learning, AI education, Support Vector Machine, mathematics education.

Introduction

Nowadays, AI technologies significantly impact nearly every aspect of society. Consequently, comprehensive AI education is considered an essential component of contemporary general education. The European Commission emphasises in its Digital Competence Framework for Citizens that for citizens to engage "confidently, critically and safely with new and emerging technologies, including systems driven by artificial intelligence (AI), they need to acquire a basic understanding of such tools and technologies" (European Commission, 2022, p. 77). Since data-driven ML methods, which underpin most AI technologies, are fundamentally based on mathematics and data, mathematics education can play a crucial role in fostering a deeper understanding of AI.

An important class of problems that are solved with ML methods are classification problems. These arise in diverse applications, such as the classification of tumors in CT scans, facial recognition for smartphone unlocking, and fraud detection in financial transactions. Various approaches exist to teach data-driven ML methods to high school students using classification problems as an example. Among others, the k-nearest neighbours method (Hazzan & Mike, 2022; Schönbrodt et al., 2023) and decision trees (Biehler & Fleischer, 2021) have been found particularly accessible. This paper focuses on the didactic analysis of a classification model based on the SVM as learning content to introduce key mathematical concepts of data-driven ML. We will particularly focus on applications and classification problems in the context of autonomous driving and medical diagnosis. Building on prior work describing learning arrangements for secondary education (Schönbrodt et al., 2022), this theoretical paper extends previous works by providing a broader subject-matter didactic analysis.

Methodological approach

First, we will highlight key concepts underlying data-driven ML methods in general. Then, following the 4-level approach for structuring and specifying mathematical learning content by Hußmann & Prediger (2016), we focus on key questions that arise at the formal, semantic, and concrete level. The

formal level addresses mathematical content in their logical structure and formal presentation, the *semantic* level focuses on structural relationships, big ideas, and basic mental models and the *concrete* level deals with the implementation of teaching-learning arrangements through core ideas and problems (Hußmann & Prediger, 2016). The empirical level is not included here, although we have already developed and implemented a variety of materials in different learning groups.

At the *formal level*, we address the mathematical concepts underlying the SVM in their symbolic representation. This leads to the logical skeleton of the subsequently described learning trajectories. At the *semantic level,* we discuss key mathematical ideas and representations that are relevant for a basic understanding of the SVM. We discuss how intended learning trajectories for two age groups (ages 13–14 and 16–18) can be sequenced such that they follow the logical structure of the SVM and ensure that students discover key concepts of data-driven ML. At the *concrete level*, we point out contexts and data that facilitate the discovery of the mathematical aspects of the SVM.

Key mathematical concepts of data-driven machine learning workflows

Here, we briefly outline key mathematical concepts (*highlighted in italic font*) involved in developing data-driven ML methods within the domain of supervised learning (see Figure 1).

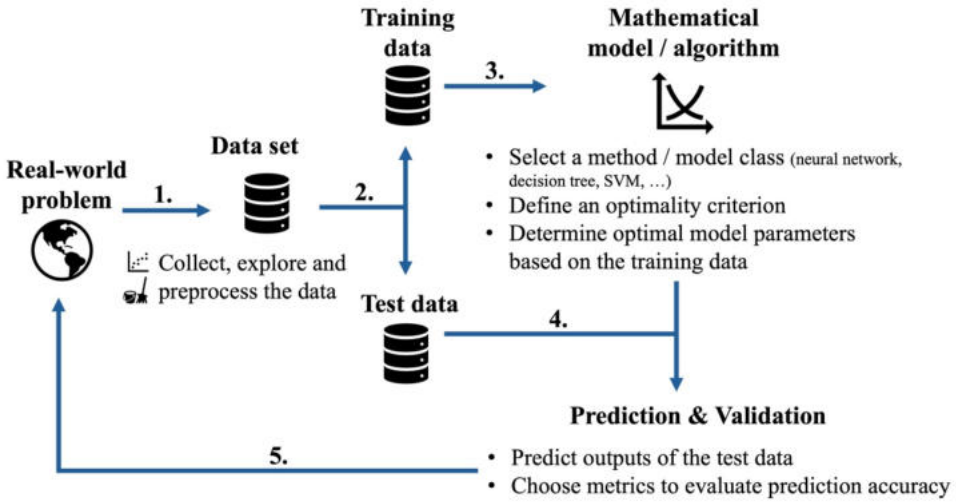

Figure 1: Key concepts when solving problems with data-driven ML (based on Biehler et al., 2024)

Data. The starting point for developing a data-driven ML method to solve a specific real-world problem is a *large amount of data*. This data is typically *explored, analysed* and *pre-processed* (e.g., cleaned by identifying and removing outliers). In supervised learning, the input data $\vec{x}_i \in \mathbb{R}^n$ and the corresponding output data $\vec{y}_i \in \mathbb{R}^m, i = 1, \dots, N$ are known. We restrict the following descriptions to $m = 1$. For classification problems the outputs y_i are categorical (e.g., $y_i = 1$ for class A and $y_i = -1$ for class B). Usually, the available dataset is divided into *training data* and *test data*. The training data is used to develop ('learn') a mathematical model. For most ML methods this means finding a function $f: \mathbb{R}^n \to \mathbb{R}$, with $f(\vec{x}_i) \approx y_i$ that can be used to predict the outputs (here ± 1) of unseen data. The test data is used to subsequently validate the model.

Mathematical model/algorithm and optimisation. In the case of artificial neural networks, SVM, linear regression, and several other ML methods, developing a mathematical model involves choosing a *class of functions* that depend on various parameters. The representative best suited for the given data must be selected from this class. In other words, the function that best describes the unknown relationship between the input data and the output data must be identified. To this end, an *optimality criterion* is determined that defines what is meant by 'best.' An *optimisation problem* is formulated and solved using *numerical optimisation methods*, allowing to determine the optimal parameters of the model. Thus, for most supervised learning methods, the so-called training or learning process boils down to solving a mathematical optimisation problem.

Validation. The mathematical model is used to *predict the outputs of the test data*. The predicted outputs $\hat{y}_i$ and the actual outputs y_i are compared. *Evaluation metrics* are used to quantify the error and thus to evaluate the performance of the model. Using test data to validate a model is a crucial step carried out in mathematical modelling processes beyond ML (Blum & Borromeo Ferri, 2009).

Using the SVM to convey key concepts of data-driven ML

One ML method used to solve classification problems is the SVM (Bishop, 2006). Here, we focus on the linear SVM, which can be applied to (almost) linearly separable data points from two classes (e.g., class A and class B). An example dataset is shown in Figure 2 (left).

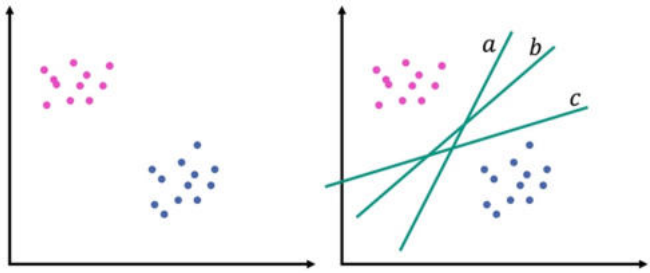

Figure 2: Linearly separable data of two classes (left) and different separating lines (right)

Below, we briefly describe the relevant mathematical foundations of the linear SVM and specify and structure these concepts for both lower and upper secondary education. We discuss to which extend the mathematical aspects of the SVM can be addressed as 'white box' with students of different grades, point out necessary prior mathematical knowledge and highlight links to high school curricula in Austria (Bundesministerium für Unterricht und Kunst [BUK], 2024).

Formal level: key mathematical concepts underlying the linear SVM

The SVM is based on the idea of separating data points of two classes by hyperplanes of the form $\vec{n} \cdot (\vec{x} - \vec{p}) = 0$ (i.e. planes for 3-dimensional and lines for 2-dimensional input data). The maximisation of the data point-free area around the hyperplane, known as the margin, (visualised by dashed lines in Figure 3, left) is used as optimality criterion. This means finding the entries of $\vec{n}$ and $\vec{p}$ such that the distance of the nearest data points of both classes to the hyperplane is maximised, i.e. $\max_{\vec{n},\vec{p}} \min_{i=1,\dots,N} \frac{|\vec{n}\cdot(\vec{x}_i-\vec{p})|}{||\vec{n}||}$. As constraints, the data points of both classes are required to lie on different sides of the separating hyperplane. This is described by linear inequalities. A reformulation leads to

a quadratic optimisation problem under linear constraints that is solved using numerical methods leading to the optimal hyperplane (Bishop, 2006). The class of any data point $\vec{x}_i$ is determined based on the sign of the dot product $\hat{y}_i = \text{sgn}\big(\vec{n} \cdot (\vec{x}_i - \vec{p})\big)$, e.g., class A for $\hat{y}_i > 0$ and class B for $\hat{y}_i < 0$.

Semantic level: intended learning trajectories, key mathematical ideas and representations

We discuss key mathematical ideas of the SVM and representations (verbal, graphical, numerical, and algebraic) that are crucial for developing a basic understanding of SVM-based classification models (Hußmann & Prediger, 2016). We describe the learning trajectory for upper secondary education in detail and discuss how the learning trajectory for lower secondary education differs from it. The intended learning trajectories incorporate the following key mathematical ideas underlying SVM-based classification models, with the corresponding steps of the learning trajectory indicated in parentheses:

Key idea: Using a hyperplane to separate data points of two classes and to classify new data points (steps 1, 2, 6, and 7).

Key idea: Defining an optimality criterion (maximum margin) to find the best separating hyperplane. Based on this criterion, the parameters of the hyperplane are determined optimally (steps 3, 4, and 5).

Key idea: Embedding the SVM into typical data-driven ML workflows (cf. Figure 1), providing a structured approach to classification (steps 1, 8, and 9).

Upper secondary education: When working on a classification problem with 2- or 3-dimensional input data the SVM can be worked out with a direct connection to mathematical concepts from linear algebra and analytical geometry (Schönbrodt et al., 2022). Relevant prior knowledge includes vectors, lines/planes in point normal form, the dot product and distances between points and lines/planes – all embedded in the Austrian upper secondary mathematics curriculum (BUK, 2024). We restrict the following description of the steps that constitute the learning trajectory to lines. However, all mathematisation steps apply to planes (and hyperplanes in general) in the same way.

(1) Exploring input data and separating lines graphically. Based on the graphical representation of the data, the idea of using a line to separate the data points of both classes is derived.

(2) Deriving an algebraic representation of the separating line. Analogously to the formal representation of the SVM the separating line is described algebraically as $\vec{n} \cdot (\vec{x} - \vec{p}) = 0$.

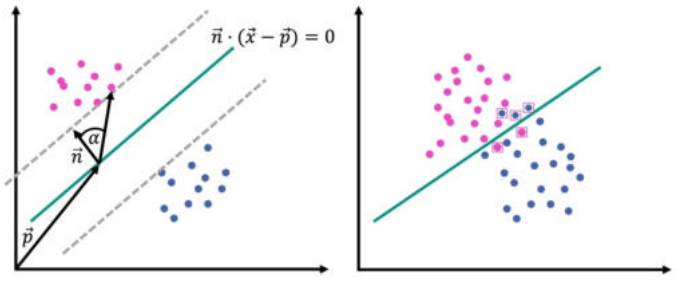

Figure 3: Mathematical description of a separating line (left) and data with overlapping classes (right)

(3) Verbal representation of the optimality criterion. Figure 2 (right) visualises that several lines might be feasible for separating the data. This motivates the definition of an optimality criterion to determine when one line is more suitable than others. If the separating line is very close to the data points of one of the two classes, there is a higher risk of misclassifying new data points belonging to that class. This consideration can be derived based on the graphical representations and leads to a verbal description of the optimality criterion: choosing the separating line with the maximum distance to the closest data points of both classes.

(4) Deriving an algebraic representation of the distance. The distance between any data point $\vec{x}_i$ and the line is described algebraically according to the actual SVM (i.e. $\frac{|\vec{n}\cdot(\vec{x}_i-\vec{p})|}{||\vec{n}||}$) – a mathematical concept embedded in Austrian curricula (BUK, 2024).

(5) Verbalizing the optimisation problem and solving it (black box). The entries of $\vec{n}$ and $\vec{p}$ should then be chosen to maximise the distance between the separating line and the data points closest to the line. Students can conceptually derive the optimisation problem. However, the representation of the optimisation problem does not have to be formalised in detail. Solving the problem can be outsourced to the computer using digital tools (black box approach). Alternatively, the optimal separating line can be approximated through computer-aided variation of the parameters (entries of $\vec{n}$ and $\vec{p}$).

(6) Determining the class of a new data point graphically. Once the optimal line has been found, the next step is to determine how to classify a new data point $\vec{x}_i$. Based on the graphical representation of a new data point and of the optimal line the point is classified by graphically assessing which side of the line it lies on.

(7) Deriving an algebraic term to classify new data. The idea of classifying data points based on

$$\vec{n}\cdot(\vec{x}_i-\vec{p}) = \cos(\alpha)\cdot\|\vec{x}_i-\vec{p}\|\cdot\|\vec{n}\,\| \begin{cases} >0, & \text{for } 0°\le\alpha<90° \\ =0, & \text{for } \alpha=90° \\ <0, & \text{for } 90°<\alpha\le 180° \end{cases} \begin{cases} \text{class A} \\ - \\ \text{class B} \end{cases}$$

is worked out algebraically supported by suitable visualisations (see Figure 3, left) and omitting the symbolic representation of the sign function.

(8) Validating the model statistically. The test data, for which the class labels are known, is used to evaluate the performance of the classification model. The accuracy (the proportion of correctly classified data among all test data) and the confusion matrix are computed as evaluation metrics. Further metrics, such as precision and recall, could be introduced additionally.

(9) Discussing limitations of the model based on graphical representations of data sets. In real-world applications data are often not as neatly linearly separable as previously discussed. During the model development process, it might be necessary to accept data points on the wrong side of the separating function (see Figure 3, right), a fact that should be discussed explicitly to highlight limitations of the model. Furthermore, based on the graphical representation of different data sets, the dependence of the model (the separating line) on the distribution of the training data can be explored.

Lower secondary education. When working with 2-dimensional data and thus using separating lines, the basic building blocks of the SVM for solving classification problems can be introduced in

lower secondary education. Necessary prior knowledge includes handling data (graphically), describing lines as graphs of linear functions, measuring geometrical distances, and, depending on the evaluation metric discussed, understanding absolute or relative frequencies (Schönbrodt, 2024). The learning trajectory follows the same logic as in upper secondary level, albeit with different manifestations in the algebraization steps (2), (4), and (7) as explained in the following.

(2) Developing an algebraic representation of the separating line. Vector calculus and the point normal form of lines are typically not part of lower secondary mathematics curricula. Therefore, we propose describing the separating line using graphs of linear functions f with $f(x) = mx + b$. This is a strong simplification of the original SVM model, which is accompanied by certain restrictions (e.g., no representation of vertical separating lines).

(4) Deriving an algebraic representation of the distance. To find the optimal line, the geometric distance between the data points and the line can be measured manually or by using digital tools – either as black box or as white box by computing the perpendicular base point as the intersection of two lines.

(7) Deriving an algebraic term to classify new data. The location of a data point relative to the separating line and thus the class of any data point (x_1,x_2) can be determined based on the difference $d = x_2 - f(x_1)$, i.e. by comparing the x_2-coordinate of the data point with the function value at x_1. For $d > 0$ the point lies above, for $d < 0$ below and for $d = 0$ on the line (Schönbrodt, 2024).

Concrete level: problems, data and design decisions

At the concrete level, we propose problems, along with data sets, in which students can construct the mathematical concepts in a generic way. We conclude by briefly describing the chosen digital tools of developed learning materials that guide the reinvention of the SVM.

The learning trajectories rely inherently on graphical visualizations of both the data and the separating line/plane. Thus, 2- or 3-dimensional data that can be explored visually is crucial. Publicly available datasets for classification tasks often contain significantly more than two or three features. To address this, larger datasets can be simplified by selecting relevant features or creating synthetic datasets that mimic real-world problems but are simpler to handle.

In *upper secondary education,* we propose using the context of classifying images of traffic lights in the realm of autonomous driving. The input data consists of greyscale images with only two or three pixels. More precisely, from each of the lights (2D: green/red; 3D: green/yellow/red) one pixel is extracted deriving 2- or 3-dimensional input data. The relevant classes are red and green. We chose this context because it allows for systematically increasing the dimensionality of the input data, progressing from 2-pixel images to 3-pixel images to N-pixel images (Schönbrodt et al., 2022) and thus extending the mathematical model from lines to planes to hyperplanes. As digital tools, we chose Jupyter Notebooks that enable various representations of mathematical objects and their interactive exploration. The Notebooks can be accessed in a web browser and are designed such that programming skills are not required.

In *lower secondary education,* we choose the context of medical diagnosis, more precisely classifying patients (classes: sick/healthy) based on two blood values using a synthetic 2-dimensional dataset

(Schönbrodt, 2024). Understanding the data is less complex, than in the case of images represented by the greyscale values of each pixel. For lower secondary education, unplugged materials were developed. Students work with paper-based worksheets. For further exploration, an interactive Gradio app (https://ki-macht-schule.de/ki-und-mathe) is available to investigate the model's performance based on several training and test datasets. The app is operated via sliders.

The SVM in secondary education: practical considerations and educational value

The intended learning trajectories provide a direct link to school mathematics and do not require mathematical knowledge beyond current curricula, theoretically justifying their integration into mathematics education (BUK, 2024). Past implementations with students suggest that these trajectories are also feasible from a practical perspective.

The elementarisation of the SVM for *lower secondary education* significantly differs from the actual mathematical formulation of the SVM due to the use of linear functions. Nonetheless, the key concepts of data-driven ML methods (data, selection of a mathematical model, optimisation, validation) remain conceptually accessible. The formal notation (i.e. symbolic representation) of the optimisation problem is treated as a black box to reduce cognitive load and to focus on conceptual understanding. The elementarisation of the SVM for *upper secondary education* remains close to the original SVM. However, the symbolic representation of the optimisation problem and the numerical optimisation methods are omitted as black box as well.

Value for AI education. Both learning trajectories have educational value in fostering a deeper understanding of key concepts underlying data-driven ML methods, potentially contributing to AI education and demystification. The trajectories reveal the human role in AI system development, which is particularly evident in the model development phase. Using separating lines/planes is an explicit decision made by humans. Similarly, determining the optimality criterion is a deliberate modelling decision. Depending on the context this can lead to fruitful discussions. In the medical diagnosis example, it could be discussed whether it is more problematic to classify sick patients as healthy or healthy patients as sick, and whether the optimal separating line should be shifted closer to the less sensitive group. In this case, a different optimality criterion would be needed.

Value for mathematics education. Incorporating the SVM in mathematics education allows addressing multiple goals simultaneously: (i) It provides an opportunity to connect various mathematical domains, including statistics (data handling and statistical quality measures), analytical geometry (upper secondary level), functions (lower secondary level), and optimisation. (ii) The SVM allows graphically illustrating the interplay and dependence between the ML model ('separating line') and the statistical properties of the given dataset. This demonstrates the crucial role of data in developing ML methods. (iii) It gives students an authentic insight into the role of school mathematics in contemporary, real-world applications while simultaneously highlighting key concepts of data-driven ML (data, mathematical model, optimisation/training, validation/testing).

Outlook

We have seen that for both lower and upper secondary education, intended learning trajectories can be derived that make the key mathematical ideas of SVMs and concepts of data-driven ML methods

tangible. These learning trajectories build upon mathematical concepts embedded in high school curricula for the two age groups considered. Regarding the practical implementation of the learning trajectories in the classroom, we have implemented the described materials in various learning groups – so far without systematically evaluating it at the empirical level. In future research, we are interested in assessing the impact of the teaching-learning arrangements on students' understanding of AI and how they contribute to the promotion of AI literacy.

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A local instruction theory to strengthen students' grasp on graphs

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Keywords: Statistics Education, Dotplot, Histodot and Hatplot, Box-dotplot, Histogram and Boxplot

Theoretical framework

Recent advances in eye-tracking research have provided new insights into how students interpret statistical graphs such as case-value plots and histograms (Bakker, 2004), dotplots (Boels, Lyford et al., 2023; Schreiter & Vogel, 2023) and boxplots (Abt et al., 2024). Building on these studies, this theoretical contribution refines and clarifies a local instructional theory for teaching statistical graphs in grades 4–12. The accompanying poster visually presents this theory with supporting example tasks, contributing to the theme of emerging trends and innovations in teaching statistics.

A key finding from eye-tracking studies is that students and teachers often interpret histograms as case-value plots, despite paying attention to the axis labels (e.g., Boels et al., 2019). Another is that students in Grades 10–12 can easily grasp single dotplots (Boels, 2023). However, both primary and secondary school students struggle more when comparing dotplots (Boels, Lyford et al., 2023; Schreiter & Vogel, 2023). Comparing histograms (Boels, 2023) or boxplots (Abt et al., 2024) turns out to be even more difficult. Both graph types represent aggregated data. The distribution and variability of the same data is represented differently in various graphical representations. Therefore, graphs with different shapes may have the same or similar distributions and variability (Figure 1).

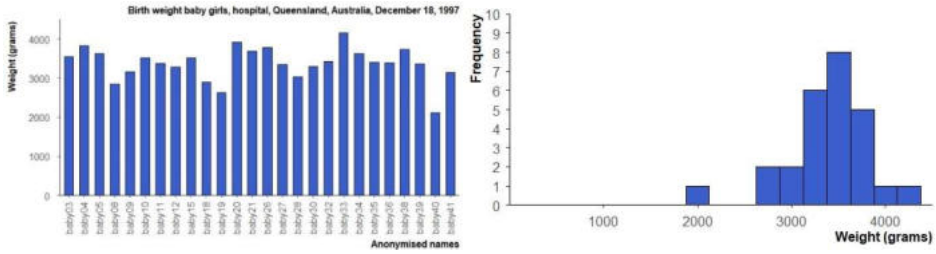

Figure 1 Case-value plot (left) and histogram (right) with the same data and distribution (Boels et al., 2019)

Research on conceptual change provides guidelines for how students can learn key concepts such as distribution and variability. Some important elements are: (1) make students' preconceptions visible [e.g., through eye-tracking], (2) have them predict, (3) create cognitive conflicts, (4) enhance meta-conceptual awareness, (5) increase tolerance of uncertainty and ambiguity (Merenluoto & Lehtinen, 2004). Our poster provides examples of comparison tasks with statistical graphs that illustrate these elements. A task that can prompt cognitive conflict is having students discuss what the data, distribution, and variability are in the graphs shown in Figure 1.

Local instruction theory

Our poster outlines a possible progressive curriculum that exemplifies key elements of a local instruction theory for developing students' understanding of statistical graphs. It shows how dotplots (Frischemeier, 2019) can support a gradual transition from case-value plots (Boels, 2023) via

intermediate graphs such as histodots (Boels, 2023) and hatplots (Fielding-Wells & Hillman, 2018) to histograms and boxplots. The task sequence is based on logical-historical analysis (e.g., Bakker, 2004; Boels, Alberto & Shvarts, 2023) and key findings of eye-tracking research. Researchers can use this overview for further eye-tracking and educational design research studies, and to further investigate how students interpret less-studied graph types, such as frequency polygons, stem-and-leaf plots, and density histograms. For teachers and textbook developers, the overview informs coherent curriculum design for strengthening students' grasp on statistical graphs.

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Transferring responsibilities to students in a statistics inquiry through a study and research path

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Keywords: Investigative questions, Statistics, Study and research paths, Anthropological Theory of the Didactic, Inquiry-based learning.

Introduction and theoretical framework

The choice of a good investigative question that will lead an inquiry or a problem-solving activity in statistics has been proven to be crucial (Arnold et al., 2021; Frischemeier et al.; 2020). This choice usually lies under the teacher's responsibility. We present an activity carried out at the Nelson-Mandela Schule in Berlin with grade 11 students, where the students take an active role finding the investigative questions. The instructional model, study and research paths (SRP), under the theoretical framework of the Anthropological Theory of the Didactic (ATD, Chevallard, 2015) is used for the design and implementation of this instructional proposal. An SRP begins with a generating question Q_0, which guides the investigation. Previous implementations show the students' perspectives when playing an active role in a statistics inquiry (Freixanet et al., 2023).

Research questions and research method

Our research question is: What conditions facilitate the transfer of responsibilities when finding the investigative question in a statistics inquiry and what constraints hinder it? We approach it through an empirical case study based on an inquiry proposal for the teaching of statistics at secondary school level. The research method is an adaptation of the didactic engineering (Barquero & Bosch, 2015). It consists of four phases: the preliminary analysis, which points at a didactic phenomenon to approach (here, the transfer of responsibilities to students), the a priori, in vivo and a posteriori analysis, that is the design, implementation and evaluation of the instructional proposal.

Finding the investigative questions

The main generating question Q_0: "How does our school community contribute to air pollution?" is stated by KlimaLink, a company whose aim is to provide a platform that calculates CO2e emissions from travel components. Grade 11 students are responsible for finding narrower investigative questions and link them to data that can be accessed or collected, and will lead future inquiries. In the a priori analysis, the activity was designed (task responsible in brackets: Y teacher, Z external organisation, x_i students and X_j teams of students) (Figure 1). Google tools were used and shared with the students, and all materials were organised in a common Google Classroom (GC).

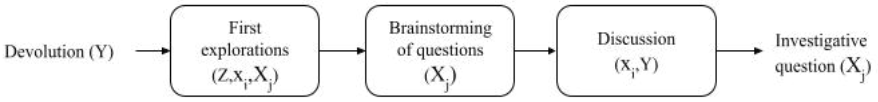

Figure 1. Design of the activity of finding the investigative question

The in vivo analysis showed that two new actions were needed during the implementation, both fell under the teacher's responsibility: (1) the organisation of the questions after the brainstorming, and (2) the creation of an external feedback tool (Google forms) to validate if the questions were good investigative questions, which included videos about "designing a questionnaire" and questions about the availability of data, study purpose, target population, variables of study, etc. (Figure 2):

Figure 2. Final flow chart of implementation of the activity of finding the investigative question

Discussion and conclusions

The a posteriori analysis showed that the prevalence of the traditional didactic contract acted as a main constraint pushing the teacher to assume more responsibilities than planned. The additional didactic tools developed, including Google shared tools and Classroom organisation appear to be useful for the transfer of responsibilities in finding good investigative questions. The evolutions of the didactic contract are weakly described in the literature about the problematisation process in statistics education. Our research aims at contributing to this line of research.

Acknowledgment

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Gaining insight – developing eye-tracking vignettes for teacher training on misinterpretations of statistical graphs

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Keywords: Statistics Education, Teacher Education, Misinterpretation, Eye-Tracking.

Systematic errors in interpreting statistical graphs are often due to underlying conceptual difficulties (Boels et al., 2024; Lem et al., 2013). These errors can be analyzed and made visible through eye-tracking, which provides insights into participants' thinking processes (Boels et al., 2024). For example, a common systematic error is confusing data values with frequencies, such as calculating the mean frequency instead of the mean of the data values. The gaze data in Figure 1 visualize a thinking process leading to this systematic error.

Despite the growing use of eye-tracking in educational research, its application in teacher training, particularly for addressing misinterpretations of statistical graphs, remains limited. The objective of the Erasmus+ project “Eye-teach-stats” is to develop eye-tracking vignettes for teacher training to address systematic errors in interpreting statistical graphs, including histograms, boxplot, dotplots, and case-value plots. Eye-tracking technology can be used to visualize strategies that lead to systematic errors and provide teachers with further representations of these strategies.

This new and innovative type of vignette uses eye-tracking data to combine visual impressions with verbal statements from students. In teacher training, vignettes consist of teaching situations that are linked to a didactically relevant question and require didactic knowledge and skills to answer (Ollesch et al., 2018). These vignettes are designed using the Four-Component Instructional Design (4C/ID) model by van Merriënboer (2020) with the intention to support teachers in diagnosing and dealing with such systematic errors and their underlying conceptual difficulties.

Construction of eye-tracking vignettes according to the 4C/ID model

The four components of the model, *learning task*, *supportive information*, *procedural information* and *part-task practice* offer an approach to design environments for complex learning. In the 4C/ID model, the individual requirements of complex tasks are not explicitly separated; rather they are combined into one task. Starting point for all the vignettes are real-life situations in the classroom where diagnosing systematic errors and their underlying conceptual difficulties is a common task for teachers.

The learning task are intended to encourage teachers to reflect on the student’s solution process presented in the vignette based on their gaze patterns and thus to deepen their diagnostic skills. Within the vignettes, the learning task are divided into different part-task that are worked on in sequential form along the components of the 4C/ID model. Additional supportive and procedural information offers teachers further help with the exercises. They are always available, but must be accessed by

clicking on the round blue and yellow icons in the bottom right-hand corner in Figure 1. The vignette in Figure 1 offers supportive information on dotplots, as graph type, and procedural information on gaze data plots, as the way in which the eye-tracking data is visualized. At the end of the vignettes, they provide the opportunity to practise subtasks in order to gain routine in the everyday task of diagnosing systematic errors.

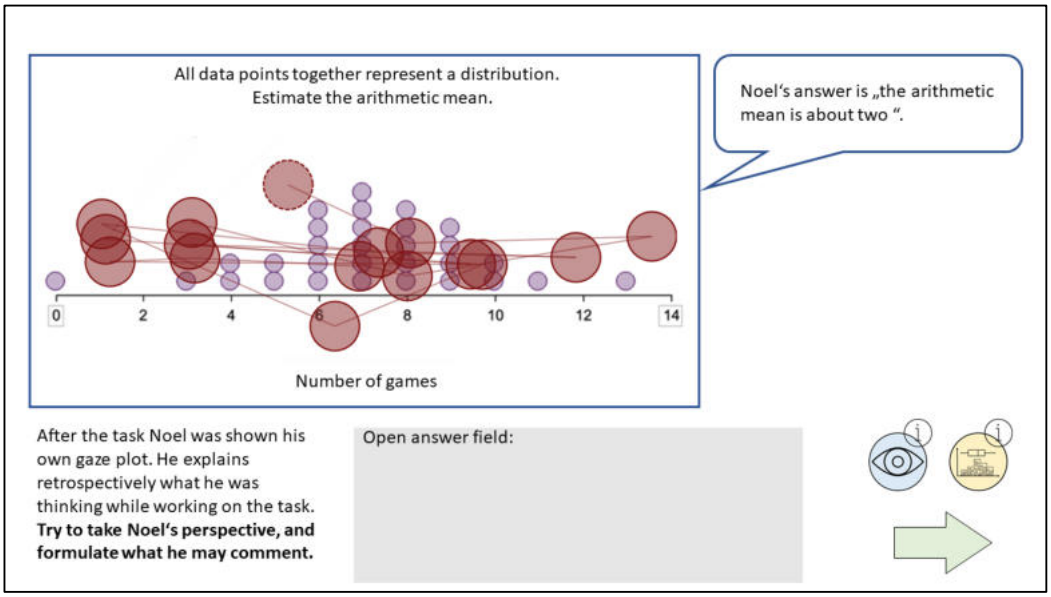

Figure 1: subtask with gaze data in eye-tracking vignette

Following Ollesch et al. (2018), expert surveys will be conducted to review the vignettes and assess their validity. These results will be used for improving and implementing the eye-tracking vignettes in an online learning environment for student teachers from different countries.

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Data science modules for preservice teacher education

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Keywords: Data science education, STEAM education, preservice teacher education.

Introduction

The project Promoting Data Science Education for Teacher Education at the University level (DataSETUP, https://datasetup.euc.ac.cy), part of the Erasmus+ Cooperation partnership, involves institutions from five countries (Cyprus, Germany, Greece, Ireland, and Türkiye) working together to enhance data science education for preservice teachers. An informal needs analysis regarding the preservice teacher programs at the participating partner universities has revealed that almost no STEAM subject includes elements of data science. The project aims at equipping preservice teachers with data science knowledge to effectively educate future generations of informed citizens, a goal that is widely recognized as important (Biehler & Schulte, 2018; Mike et al., 2023). The poster will show an example module developed within the project for use in a STEAM subject.

A framework for preservice teachers' data science education

At the heart of the project is the development of stand-alone data science modules that can be used in various STEAM subjects in teacher education. These modules are built upon a framework currently being developed as part of the project, carefully informed and guided by existing literature from primary to tertiary education (e.g., Adisa et al., 2023; Lee et al., 2022). An initial version of the framework can be found in Figures 1 & 2; this is a work in progress.

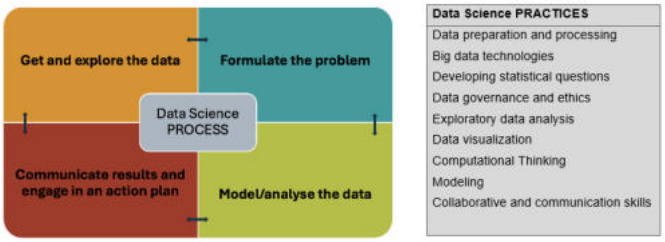

Figure 1: Preliminary DataSETUP conceptual framework for preservice teachers' data science education–Dimension 1: Data science processes and practices

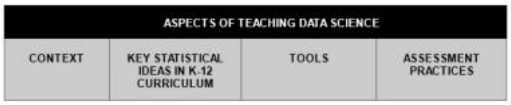

Figure 2: Preliminary DataSETUP framework–Dimension 2: Aspects of teaching data science

The modules center on four interconnected data science processes (Figure 1), each involving practices (right side of Figure 1) that form their core focus. Additionally, the framework incorporates a second dimension focused on teaching data science (Figure 2).

Data Science modules for preservice teachers' data science education

Using a design-based research approach, the objectives are to design independent modules as short courses to enhance preservice teachers' knowledge of data science processes and practices and implement these modules in STEAM teacher education programs. The outcome will be several tested teaching modules. These modules will complement existing teacher education programs and be translated for broader use on the DataSETUP online platform, ensuring a wider impact across European higher education.

The DataSETUP project develops two types of modules. Theoretical modules aim to enhance preservice teachers' essential data science skills and competencies, as outlined in the framework. Application modules will provide participants practical data science skills to tackle real-world problems in various contexts, such as sustainability and civic statistics (Ridgway, 2022).

Sample data science application module on "Gaming habits of young people"

A pilot module was implemented with 19 pre-service primary teacher students in a 90-minute session. Students successfully went through the four-element framework analyzing multivariate data from German young people from the ProDaBi project (www.prodabi.de/en). They explored data, formulated questions, analyzed, and presented their results. Notably, five groups created action recommendations, supporting the development of data-based reasoning.

Acknowledgment

Promoting Data Science Education for Teacher Education at the University level (DataSETUP) project is funded by the European Union's Erasmus+ Cooperation Partnership Programme. All views expressed are those of the authors and not of the European Commission.

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Investigating students' co-construction of statistical argumentation

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Keywords: Argumentation, statistical reasoning, Toulmin's model, Habermas' rationality.

Introduction and rationale

In the poster, I will discuss a preliminary study of a broader project aimed at investigating the argumentation produced by upper secondary school students when working on statistical tasks that foster data-based decision-making process. Specifically, the poster will inform about the use of the combined lens of Toulmin's model and Habermas' construct of rationality (Boero et al., 2010) to analyse the dynamics of co-producing argumentative steps in such a context.

Supporting the development of data-based decision-making is recognized as central for mathematics and statistics education (OECD, 2023). As reasoning is strictly related to argumentation, studying how argumentative processes evolve allows us to explore also how students develop statistical reasoning and thinking, justify their claims, and interact with the data, while also showing how they connect statistical and contextual knowledge and navigate complex reasoning in collaborative settings. In this context, we intend argumentation as presented by Ben-Zvi (2006).

The project focuses on designing and analysing activities that allow for the activation of argumentative patterns to study their characteristics, understanding their specificities and what can be generalized. This also aims to understand better the contexts in which argumentation emerges and to provide insights into what can trigger it. Particular attention is given to how the choice of data representations affects argument, as several different ones may be used to support the argued claim and give different perspectives. Since transnumeration enables access to the understanding of the phenomena beyond the data (Wild & Pfannkuch, 1999), the choice of appropriate representations may have an impact both in the exploratory phase that involves analysing the data set and in producing the argument to communicate and discuss decisions.

Theoretical framework and research questions

We decided to adopt the combined lens of Toulmin's model and Habermas' construct of rationality, because it may efficiently enhance the ability of Toulmin's model to analyse also statistical argumentation, as being able to consider the communicative and goal-oriented components of rational behaviour, which are pivotal in producing statistical argumentations. The Toulmin model has already shown promising contributions in investigating statistical reasoning. For instance, Gómez-Blancarte and Tobías-Lara (2018) reflect on how to make such a model work with informal inferential reasoning (IIR). We believe its application, combined with Habermas' rationality, may give fruitful contributions also in analysing foundational processes for IIR, that do not go beyond the data yet provide the necessary groundwork to make generalizations (e.g., grouping data). With this view, the study contributes to a better understanding and characterization of these forms of reasoning.

In light of the theoretical framework, the research questions are as follows. When upper secondary school students work on statistical tasks that require data manipulation:

RQ1: What roles do the different components of Habermas' rationality play in the construction of argumentation and argumentative steps, and in particular in the selection and construction of warrants underlying them?

RQ2: How can the choice of data representation be analysed and evaluated in warrant production?

RQ3: What is the relationship between the choice and the construction of data representations in an argumentation, and the production of warrants?

Methods

The project focuses on the design and analysis of a teaching intervention that fosters statistical argumentation in classroom settings. The collected data includes video recordings of students while they work in small groups on the assigned task and during classroom discussion, and recordings of the students' laptop screens, as they use software to manipulate data. The analysis is carried out according to the method proposed by Knipping and Reid (2015).

The poster will present the designed teaching intervention and discuss the results we got from the preliminary study conducted last May. The analysis focuses on the work of a triad of 10th grade students from an Italian secondary school and aims at clarifying how to operationalize the theoretical framework within statistics domain (e.g., how to characterize and identify key terms like claims and warrants in statistical argumentation, how to evaluate students' "strategic" statements about the organization of work).

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Case study of a learning environment for collecting and analysing real data in mathematics classrooms with the toolkit senseBox

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Keywords: Data literacy, particulate matter, secondary school, senseBox, sustainability.

Future-orientated mathematical learning with real data

A central topic of digital future education in mathematics classrooms in Germany is the collection, processing, analysis, and, above all, interpretation of a wide range of environmental (real) data. This is subsumed under the term 'data literacy' (hereafter: DL). Due to the omnipresence of data in the current age and the penetration of data science into almost all areas of life, the development of such DL should therefore be a central aim of future-orientated mathematics education (Biehler et al., 2018). Although DL is included in the German curriculum for mathematics, there is often a lack of ideas for implementing it in everyday mathematics classrooms, especially at the lower secondary level. For closing this gap, we introduce the toolkit senseBox (Bartoschek et al., 2018) as a possible and sophisticated tool for dealing with real data in the mathematics classroom. Furthermore, a learning environment in the context of particulate matter from the UmARsense project (project no. 03 02825, Deutsche Telekom Stiftung) is presented, in which the collection, analysis, and interpretation of real data with the senseBox toolkit is central.

Collecting real data with the senseBox within substantial learning environments

The senseBox is an electronic component experiment set that can be assembled and programmed by the students themselves. Its components can be used to collect a wide range of real environmental data (e.g., temperatures, UV intensity, particulate matter, and noise levels). Programming is carried out via an adaptation of the Blockly programming system. Data output is possible by means of measured values on a display and by reading out data with suitable software (e.g., Excel). However, so far there are no concrete substantial learning environments (hereafter: SLEs) for using the senseBox in the mathematics classroom. In order to realise working with real data in mathematics classrooms in a didactically sophisticated way, several SLEs were developed in the project UmarSense. The SLEs were prepared and standardised in the form of didactical situation patterns following Platz (2019). The developed SLEs include environmental data from different areas and fulfil the requirement of collecting, processing, and critically analysing environmental data in order to promote a multifaceted discussion of the environment and aspects of sustainability. Our poster focuses on one SLE exploring particular matter, as this context is highlighted as an important topic in mathematics education literature (Barwell, 2018). The SLE is developed by Josephine Ebert for sixth-grade students at a secondary school. It is structured into 4 steps: (1) Introduction to the topic of particulate matter and air pollution with presentation of the respective characteristic and limit values; (2) Construction and programming of the senseBox tool and the particulate matter sensor; (3) Carrying out the particulate matter measurement and recording the data; (4) Discussion. The SLE was realised with three groups of three to four children. In step 3, the children worked in teams of

two or three to take measurements of chalk dust emissions caused by wiping the blackboard in the classroom. The children were able to take measurements at different, self-selected locations in the classroom. The measured data was collected using the display, noted, and then prepared in the form of a bar chart. During the subsequent discussion with the whole group (step 4), the children drew first conclusions about valid measuring points for particulate matter based on the data they had collected themselves. They hypothesised about how the measuring point influences the values (proximity or distance to the blackboard). When they were presented with values and comments on real measuring stations, the children made a critical assessment in which they referred to the values they had collected and also included the real limit values. Finally, one girl formulated the need to act with regard to particulate matter pollution in cities. She said, „So that we can say that they have exceeded the specified limit and that we now have to do something about it".

Summary of the empirical findings and concluding remarks

In the trial of the presented SLE within the framework of the project, it became clear that working with the senseBox is suitable for interested learners in grades 5 and 6, particularly for fostering (critical) analysis of (environmental) data, promoting data literacy, and enabling digital environmental exploration. Nevertheless, data from the UmARsense project also shows that one challenge of SLEs involving the senseBox is how to mediate between the collected data and consequences for reality. With regard to its use in regular mathematics classrooms, it can be summarised that the senseBox can be used in lower secondary schools primarily in the context of working groups or project work, as the amount of supervision required, particularly with regard to the construction of the measuring instrument, is high. Future research should start here and develop further learning environments and make them suitable for use in everyday classrooms in order to give digital future education more space in mathematics.

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TWG 06: Applications and Modelling

Introducing the CERME14 papers and posters in, and work of, TWG6: Application and Modelling

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This paper presents and gives an overview of the papers and posters discussed in the Thematic Working Group 6 (TWG6) at the CERME14 conference taking place in Bolzano, February 4–8, 2025.

Keywords: Applications, learning, mathematical modelling, models, teaching

Overview and introduction of the CERME TWG6 on application and modelling

Thematic Working Group 6 (TWG6) focuses on applications and modelling. It highlights the importance of exploring how mathematics is used to understand real-world phenomena in educational contexts. TWG6 investigates the interplay between empirical research and theoretical perspectives on teaching and learning applications and mathematical modelling, both within mathematics and across other disciplines, spanning primary, secondary and tertiary education, as well as teacher education.

Since its establishment at the 4th ERME Congress (CERME 4) in 2005, TWG6 has remained an active and evolving group, contributing to each of the subsequent ten ERME congresses up to and including CERME 14. At CERME 14, TWG6 welcomed contributions addressing a variety of theoretical, methodological and empirical issues. These included the application of mathematics and modelling in interdisciplinary contexts; the role of simulations; sustainability; the integration of technology and resources in modelling; and preparing both pre-service and in-service teachers to teach modelling and applications.

The contributions reflect the diversity of research questions, educational levels, theoretical frameworks, and methodological approaches in mathematical applications and modelling research. This highlights the field's dynamic and multifaceted nature. For CERME 14, TWG6 received 45 submissions, 23 of which were accepted as papers and two as posters. These were presented at the congress. Figure 1 provides an overview of the TWG6 papers and posters presented at various CERME congresses. A total of 33 participants from 13 countries attended TWG6 at CERME 14, compared to 20 different countries at CERME 13. The majority of participants (85%) were from Europe, which is an increase on the 67% from CERME 13. However, in addition to the European participants, TWG6 at CERME 14 also attracted participants from North and South America and Asia.

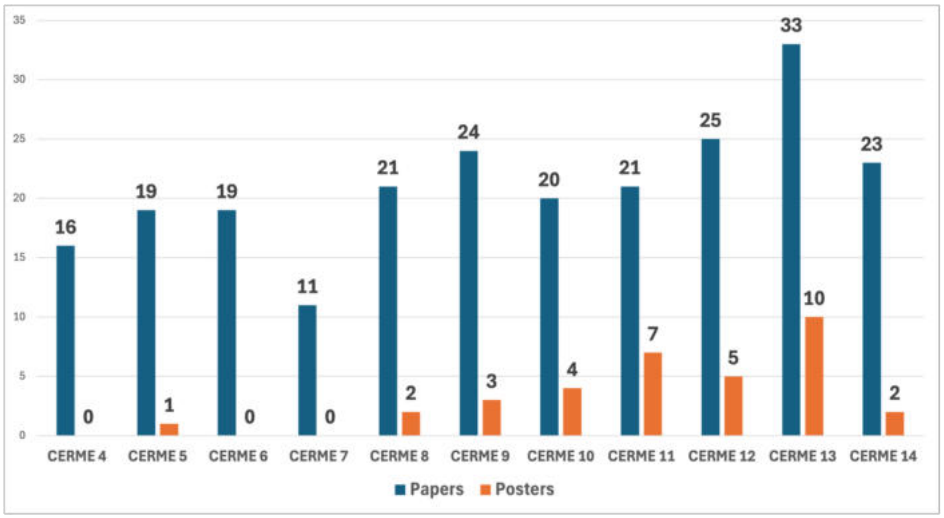

Figure 1: Overview of the presented contributions in TWG6 in CERME 4 – CERME 14

Table 1 outlines the five themes around which discussions and work in TWG6 were structured and organised. The first theme explores ways to support students during Fermi problem solving, including using structuring tools such as Fermi graphs, the importance of estimation and benchmarks, and the potential of AI in modelling processes. The second theme addresses modelling assessment, examining how different tools and theoretical frameworks influence what is assessed and learnt, and how these tools can support research and classroom practice. The third theme focuses on improving mathematical modelling through careful task design, interdisciplinary integration (e.g. with engineering and financial education) and teacher interventions that strike a balance between guidance and student autonomy. The fourth theme considers modelling perspectives within and beyond the classroom, emphasising modelling of societal, educational and research responsibilities and highlighting the role of data science and global challenges. The fifth and final theme investigates the interplay between affect and data in modelling. This includes examining how beliefs, motivation and engagement with diverse data types influence students' modelling experiences and how tools such as ChatGPT may reshape modelling practices and enquiry.

Table 1: Identified themes and number of contributions in the CERME 14 proceedings

Theme	*Papers*	*Posters*
Theme 1: Students' support during Fermi problem solving	4	-
Theme 2: Assessments and models	5	1
Theme 3: Enhancing mathematical modelling – Design, implementation, and interdisciplinary connections	5	
Theme 4: Mathematical modelling perspectives in and beyond the classroom	4	
Theme 5: Mathematical modelling related to affect and data	5	1

Theme 1: Students' support during Fermi problem solving

Fermi problems are a type of reality-based problem that is characterized by a lack of data. This data must be obtained in a meaningful way as part of the processing process. Despite the longstanding focus on Fermi problems within the TWG of Application and Modelling, novel research approaches are continually emerging due to evolving circumstances.

One of the points discussed was the support measures for working on Fermi problems. Fermi graphs are one possible form of support that can be used as a structuring aid during the solution process, as well as to structure the reflection and presentation of the same. In addition to such aids, the role of the teacher is also crucial in facilitating the acquisition of skills in this area. This can include the handling of Fermi problems, as well as the way teachers facilitate class discussions and consolidate results to ensure collective understanding.

Another point that arose in the discussions was the identification of relevant aspects in the processing of Fermi problems. As previously outlined, Fermi problems are distinguished by the necessity of acquiring data to resolve them. One approach to procurement is estimation, which often necessitates the establishment of benchmarks. These benchmarks are not only used for estimation; they are also used to create models and validate results.

A completely different area that has seen significant growth in recent years is the use of AI in the field of Fermi problems. There are various types of interaction with AI during the FP-solving process. In the future, the use of AI in the processing of Fermi problems is seen as having great potential in at least two respects:

- How does the use of AI influence the solution process of Fermi problems?
- (How) Can the use of AI support the evaluation of solution processes of Fermi problems?

A key point of discussion in the TWG's LTs was the contribution to the development of critical citizenship. With regard to Fermi problems, it was highlighted that these problems, although they may appear straightforward, have the potential to contribute significantly to the development of critical citizenship, particularly in the areas of critical thinking and sustainability. In this context, the following questions should be investigated:

- To what extent can the solution processes of students working on Fermi problems be indicative of the development of critical citizenship?
- Which characteristics of Fermi problems are conducive to fostering citizenship awareness?

Theme 2: Assessments and models

Theme 2 is characterized by very diverse explorations of how we can assess students' modelling processes, and what affects the process, but the theme also addresses the question of what is being assessed? Some cases looked at students and others at preservice teachers, none related to exams.

A main issue raised across the discussion of the different papers was the concern that what is assessed may be what students learn. Therefore, we need to discuss further how can portfolio, a priori analyses, criteria, schemes of evaluations scaffold teachers' formative assessment of students modelling processes and therefore their learning? Modelling activities can potentially go in many different

directions and include various mathematical and extra mathematical domains. So a priori map out or consider what knowledge strategies and knowledge expected is valuable. Connected but also different, is the question of how the intertwinement between assessment and theoretical framework (modelling cycle, MEA, Fermi problems, ATD etc.) inform the study, affect the learning potential? Different theoretical frameworks provide specific lenses on modelling and promote or delimits specific features of what can be considered part of modelling processes. Therefore, the frameworks – when guiding the design of assessment activities or explicitly introduced in the teaching – affect what students can learn about modelling and from modelling.

One way to make explicit the consequences of the choice of theoretical framework is the before mentioned a priori analyses or hypotheses about outcomes. Linked to this theme we discussed the importance of the dialectic between designing interventions and analyse hypothetical student strategies when developing assessment tools. This regardless of that being portfolios, tool kits, eye tracking systems etc. Furthermore, for most participants the question of the role of teachers. Are the presented assessment tools suitable for classrooms and everyday teaching? Not all, but some seem to carry the potential of functioning well as tools for students' metacognition regarding modelling processes guided by a modelling cycle, tool kits or explicit encouragement for seeking further resources concerning the extra mathematical domain seem more easily included in classroom practices. Portfolio and eye tracking is deemed more complicated for that.

Thus, more broadly and looking ahead we find it highly relevant to discuss for whom and for what purpose we develop the different tools and methodologies proposed in the papers linked to assessment and models? Which tools are appropriate for teachers and which ones are mainly for research? How can this still inform and improve (someone's) learning?

How to teach teachers to use tools developed for research, which also has potential for practice? Beyond the a priori analysis mentioned, we see research potentials of improving the scaffolding structures for modelling activities. It would be interesting to develop knowledge further on how and why elements like tool kits, empty cycles, eye tracking, potential questioning done during the experiment etc. works? For teachers we see potentials for providing formative assessment, which does not delimit open modelling processes, or which promotes changes of phases in modelling cycles etc. without direct instructions for doing so. Both for research and teaching we see a need for further studies on relevance and linkages between curriculum and assessment, tools, and scaffolding. Also, we see a need to dive further into content related constraints and conditions when designing and implementing assessment tools: What is important to learn and assess? This of course links to the domains of both extra mathematical and intra mathematical modelling, and their intertwinement. When do we expect students to dive into the former or the latter? Or do we wish to see students work in a dialectic between the two, continuously expanding the system and models addressed?

Throughout all initiatives we need to consider time constraints: is there "space" to implement the proposed tools? In research? In teaching and in what form?

During the group sessions we inevitably discussed variations in assessment approaches across different educational systems and how they shape modelling practices. Comparative studies may prove to be productive for further development of this area within mathematical modelling research.

Theme 3: Enhancing mathematical modelling – Design, implementation, and interdisciplinary connections

The discussion on Theme 3 considered different approaches to strengthening mathematical modelling. These relate to the design of modelling courses, their implementation and relationships with other subjects. The discussion focused on the diversity of perspectives in mathematical modelling, the adaptation of contexts in engineering education and the role of modelling in financial education. In addition, the type of implementation and interventions by teachers were discussed in more detail.

Mathematical modelling is approached from *different perspectives*, both theoretical and experimental, depending on the research context. However, a lack of standardised terminology in textbooks and curricula often makes it difficult to understand this complex field. This diversity of perspectives highlights the need to define modelling terminologies more clearly and to clarify their meaning for both teachers and students.

In engineering education, for example, it is crucial to carefully consider and adapt the design principles for modelling problems. This depends very much on the *context* and level of education for which the activities are designed. Especially in higher education, it is important to consider the impact of modelling sequences on the understanding of the intended content. Such considerations are essential to achieve the learning objective effectively.

In addition, mathematical modelling plays a central role in *financial education*. Developing modelling tasks in the context of financial education can improve students' financial literacy. More interdisciplinary research is needed to bridge the gap between mathematical modelling education and financial education.

The *implementation* of modelling tasks by teachers requires a careful characterisation of the conditions that allow a balance between teacher intervention and student autonomy during the modelling process. These conditions must also be transferable to specific contexts, such as lower-achieving groups, and are crucial for the success of the learning process.

Teacher *interventions*, especially scaffolding practices based on the diagnosis of modelling tasks, are essential for effective modelling instruction. Teachers need to integrate diagnostic and instructional skills and design interactive learning environments based on mathematical modelling.

Looking to the future, it is important to adapt perspectives and activities to suit different learning profiles in different contexts. Improving teacher training and professional development in mathematical modelling and strengthening the links between mathematical modelling and current important issues such as financial literacy, sustainable development and social responsibility are crucial.

These broad considerations show the importance of a clear and collaborative approach to mathematical modelling, both in the educational context and in its professional application in different disciplines.

Theme 4: Mathematical modelling perspectives in and beyond the classroom

The theme 4 addresses the discussion of modelling perspectives in and beyond the classroom applications. The researchers highlighted the importance of theoretical approaches that, on the one hand, are drawn from the connotation of its utility in the classroom and, on the other hand, go beyond the classroom applications. Given that mathematical modelling is based on real-world phenomena with uncertainties, challenges, vulnerabilities, harms, and impairments, the modelling perspective has a potential global service to the planet and the society in which we live. In this regard, in CERME14 we articulated three major areas of responsibilities of mathematical modelling: (1) responsibility for society, (2) responsibility for school, and (3) responsibility for research.

First, mathematical modelling provides us a tool to prepare individuals for disruptive times such as pandemic periods or natural disasters due to climate change and global warming. The research investigating how to teach modelling in schools during disruptive times in a data-driven world put forward the opportunities of linking data science and mathematical modelling. In this regard, these societal issues are not merely the real-world contexts to initiate model development, rather they enable problem-solvers to develop solutions for the concerns of the society and promote the awareness and responsibilities of being a world citizen. The data-driven world offers prospects for the integration of data science and modelling activities, which holds the potential for citizenship education.

Second, mathematical modelling puts forward responsibilities for schools. A comprehensive approach for general education along with modelling goes beyond the transmission of specialised knowledge. Considering the development of critical, independent, culturally and environmentally responsible individuals as a part of general education, the researchers called for (re)interpreting mathematical modelling goals, concepts, and tasks. The opportunities given to students in the schools such as teachers' choices of modelling activities or pre-existed activities in the textbooks are needed to be re-evaluated from the modelling perspective. As an attempt of such evaluation, the textbook analysis showed us that modelling processes in French and Norwegian textbooks were often limited with specific cases and types of models, and reversible and recursive work on models were not sufficiently promoted. Yet, our theoretical discussion suggested a need for developing criteria for educational activities, classroom practices, and scaffolding pedagogies, not only for cognitive development but also for identity and personality development in contemporary mathematical modelling.

Third, the TWG in CERME14 discussed the research tools for developing and adapting mathematical modelling for several purposes (e.g., general education, sustainability, etc.). In this regard, the "Wheel of Modelling" introduced in this theme articulated different acts and purposes of mathematical modelling and models, which can be used by both researchers and educators. Similar to the discussion of the TWG in earlier years, the constraints and challenges of researching mathematical modelling (e.g. curricular guidelines, textbooks, lack of measures, etc.) have been on the agenda that we acknowledge as a part of the responsibility for research in mathematical modelling.

The theme on mathematical modelling perspectives in and beyond the classroom also revealed some emerging questions for future developments on this topic:

- How can mathematical modelling be used to address societal and global issues and challenges (e.g., in the context of sustainable development, socio-ecological approaches, or data science)?
- What changes are needed in modelling tasks in a data-driven and technology-driven world for developing critical, independent, culturally and environmentally responsible individuals?
- How do we consider Large Language Models (e.g., ChatGPT) in the emerging modelling-related research issues, trends, and challenges?

Theme 5: Mathematical modelling related to affect and data

Theme 5 addresses two interrelated yet distinct dimensions of contemporary mathematics education: the role of affect in mathematical modelling and the evolving nature of students' engagement with data. On one hand, affective factors such as motivation, interest, and beliefs significantly shape both the teaching and learning of modelling practices, influencing how students approach real-world problems and how teachers design and assess modelling tasks. On the other hand, the growing emphasis on data literacy highlights the need to understand how students perceive, interpret, and work with data in an increasingly digital and information-rich world.

One central issue concerns how teachers' beliefs about mathematics influence the implementation and assessment of mathematical modelling tasks, particularly in relation to assumptions within these tasks. These beliefs shape not only instructional choices but also the ways in which students engage with modelling processes. A noteworthy observation is the positive impact of conducting modelling tasks in outdoor environments, which has been shown to enhance students' motivation and interest. Despite students demonstrating a strong interest in themes such as sustainability, this topic remains insufficiently integrated into mathematics education. The challenge lies in aligning curricular goals with students' affective engagement, thereby fostering a more meaningful connection between mathematical modelling and real-world issues.

In exploring students' knowledge about and with data, several key issues emerge concerning their understanding and use of different data types. Many students appear to hold an implicit belief that only numerical data are valid, often disregarding qualitative information presented through text or other non-numerical formats as irrelevant or less valuable. This tendency affects how they manage data in modelling tasks, potentially leading to the neglect of meaningful contextual information. Moreover, students' perceptions of data are shaped by the nature of the tasks and classroom activities they engage with; when thoughtfully designed, such experiences can help students recognize data sets as powerful tools for interpreting and making sense of the world. Importantly, the incorporation of big data and tools like ChatGPT into mathematics education offers new opportunities to enrich the exploratory modelling process. These digital resources support more dynamic and interactive engagements with data, allowing students to apply statistical methods in ways that are more aligned with contemporary practices than traditional classroom settings typically allow.

In moving forward with Theme 5, several research directions emerge. First, there is a call for deeper exploration of how affect—both students' and teachers'—influences mathematical modelling. This includes examining how modelling settings, task characteristics, and contexts such as sustainability shape engagement, as well as how teachers' concerns may reflect a fear of departing from mathematical traditions. Second, with regard to knowledge about and with data, it is essential to

pursue a more refined understanding of the role data plays in the modelling process. This includes analysing how the presence, absence, quality, and quantity of data influence key phases such as mathematization and validation. Further attention must also be paid to how knowledge about data is constructed and utilized by students, teachers, and textbooks within the modelling process. Finally, the integration of generative AI tools like ChatGPT opens new possibilities for modelling, raising questions about the types of knowledge students seek. Research is needed to explore what kinds of knowledge students seek when interacting with AI, how these tools influence modelling activities—including the validation of models—and in what ways they may encourage or inhibit deeper inquiry.

Discussion/conclusions/outlook

The five themes that emerged from the contributions to TWG6 at CERME 14 reflect both a continuity of interest and a dynamic evolution in the field of applications and mathematical modelling in education. Compared to previous congresses, the discussions at CERME 14 revealed a deepening engagement with established topics—such as Fermi problems, assessment, and interdisciplinary modelling—while also embracing new directions, including the integration of AI tools, the role of affect and data, and the broader societal responsibilities of modelling education.

The thematic strands illustrate a rich diversity of theoretical perspectives, methodological approaches, and educational contexts. They also highlight a growing concern with how modelling can contribute to critical citizenship, sustainability, and data literacy in a rapidly changing world. The discussions underscored the importance of supporting both students and teachers in navigating complex modelling processes, and of designing tasks and tools that are both pedagogically meaningful and practically feasible.

While these developments are encouraging, they also invite us to reflect on how the TWG can continue to evolve. The introduction of new topics such as generative AI, sustainability, and data-driven modelling signals a need to revisit and renew our shared research agenda. Drawing on the collective experience and insights developed over the years; we are now well-positioned to explore how our work can remain relevant and impactful in an increasingly complex educational and societal landscape. This renewal is not a departure from our foundations, but an opportunity to build on them—strengthening our contributions to both research and practice in mathematics education.

Looking back at the trajectory of TWG6 since its inception at CERME 4, and building on the developments seen at CERME 13, the work presented at CERME 14 demonstrates a healthy balance between consolidation and innovation. It is encouraging to see how the group continues to foster a collaborative and reflective research community, committed to advancing the theory and practice of mathematical modelling in education.

Acknowledgment

We extend our gratitude to all attendees of TWG6 at CERME 14 for presenting their work, engaging in the discussions, and constructively collaborating during the intensive working sessions.

The wheel of mathematical modelling: Seeking a new perspective

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In this theoretical paper we present an initial conceptualisation of a new perspective on mathematical modelling. Drawing on seven phases of a creative problem-solving process and making parallels to the purposeful and creative writing process, we first elaborate on the similarities between the modelling process and the writing process. We then reconceptualise the so-called 'Wheel of Writing' in the context of mathematical modelling as a 'Wheel of Modelling' and articulate the acts and purposes of mathematical models and modelling within this perspective.

Keywords: Mathematical modelling, models, theories, wheel of modelling, wheel of writing.

Introduction

In mathematics education research, theories play many important roles such as *defining overarching frameworks* providing views of teaching, learning and mathematics; supporting the *organising of sets of specific observations and interpretations; providing concepts and terminology; and offering methodological approaches* (Niss, 2007). Overall, there are numerous theoretical perspectives within the field of mathematics education reflecting various aims and different social and cultural contexts of the research (Sriraman and English, 2010) – the research field of mathematical modelling is no exception. While a multitude of perspectives enrich the understanding of mathematical models and modelling, they also pose a risk of misunderstandings if researchers are not clear about their theoretical foundations and intentions. This is especially the case, when the theories adapted are taken for granted and are highly specific and specialised with respect to the modelling process or modelling competencies. In this paper, we seek a perspective that is more generic and inclusive when thinking about mathematical modelling, by reflecting on similarities between an inclusive conceptualisation of the mathematical modelling processes along with its purposes and the processes of purposeful writing various types of texts.

Our rationale departs from acknowledging that (i) mathematical modelling is a creative process (Cevikbas et al., 2022) in which modellers are offered opportunities to use, adapt and learn mathematics in different contexts for various purposes; and similarly (ii) that engaging in the process of writing, here generally understood as the process of producing any written text (like a novel, essay, report, biography etc.), is also a creative process (Piirto, 2018). From these premises we draw on selected literature to first express and elaborate on the similarities between the modelling and the writing process, and then to reconceptualise the so-called Wheel of Writing with its *purposes* and *acts of writing* (Berge et al., 2016) in the context of mathematical modelling. In this theoretical inquisition, we asked ourselves the following questions: 1) *What similarities between mathematical modelling and the writing process can be identified when compared from the viewpoint of creative problem-solving components?* and *2) What could a Wheel-of-Writing-inspired conceptualisation of*

mathematical modelling look like and entail? and *What would be the corresponding purposes and acts of modelling be given the similarities in the creative problem-solving process components?*

Before reporting our initial exploration of these questions, we briefly present the main premises of mathematical models and modelling and the wheel of writing.

Mathematical models and modelling

There are numerous approaches and frameworks for mathematical modelling in educational research (cf. Kaiser and Sriraman, 2006). In this paper, we follow Lesh and Doerr (2003) in understanding models as "conceptual systems (consisting of elements, relations, operations, and rules governing interactions) that are expressed using external notation systems, and that are used to construct, describe, or explain the behaviors of other system(s)" (p. 10). In this perspective, mathematical models are "purposeful mathematical descriptions of situations, embedded within particular systems of practice that feature an epistemology of model fit and revision" (Lesh and Lehrer, 2003). That is, we adopt a broad perspective on mathematical modelling in which mathematical modelling is viewed as a general problem-solving process that involves the mathematization of real-world problems, situations, or phenomena, resulting in a model that helps to make sense of the task or situation at hand (Niss and Blum, 2020). All notions and concepts used in research are to varying degrees theory laden (Niss, 2007), and we acknowledge that the above notions of models and modelling are also theory laden. However, we argue that these are generally accepted definitions in our field of research when taken literally without considering deeper underlying theoretical and philosophical considerations. Hence, in this paper, we elaborate aspects of models and modelling with a broader outlook through contemplating about the parallelism of the modelling process and the purposeful writing process.

The Wheel of Writing (WoW)

In context of a curriculum reform on writing competence in Norway, Berge et al. (2016) developed the so-called Wheel of Writing (WoW), which is "a tool for conceptualising the complexity of writing in different cultural and situational contexts that constitute the arenas for writing in a society, including writing in different school subjects" (p. 172). The WoW conceptualises writing as a creative process in which writing is an activity that individuals (or groups) express themselves through different *acts of writing (to reflect, to explore, to describe, to interact, to convince, and to imagine) based on their purpose of writing (identity formation, knowledge development, knowledge organization and storage, exchange of information, persuasion and creation of textual worlds)*; see Figure 1. The blue double arrows in Figure 1 below indicate that the WoW is fundamentally dynamic in the sense that the two concentric circles that make up the model can be freely rotated. This means that different acts of writing can be combined with multiple purposes, and also that a text can be the result of more than one act of writing. It is worth noting that writing as understood in the WoW always takes place in a cultural and situational context.

Similarities between mathematical modelling and writing from the lens of creative problem-solving

As the first step into our theoretical query, we identified the similarities between mathematical modelling and the writing process, using Treffinger's (1995) *Creative Problem Solving* (CPS)

framework. Treffinger (1995) described the CPS process as a mutual interaction between three major components, *Understanding the Problem, Generating Ideas*, and *Planning for Action*, and described six more specific stages of the CSP process along these components. Specifically, Treffinger divided the first component, *Understanding the Problem*, into the stages of *mess-finding, data-finding*, and *problem-finding*; associated the second component, *Generating Ideas*, with the stage of *idea-finding*; and considered the last component, *Planning for Action*, to involve *solution-finding*, and *acceptance-finding*. Neither the CPS components nor the stages are linear in nature in the CPS process, rather they could be engaged in the process as many times as needed. Furthermore, although not listed explicitly as components or stages, Treffinger (1995) discussed that *revisions and refinements,* and *presentation and communication* were implicitly intertwined in the *Generating Ideas* and *Planning for Action* components.

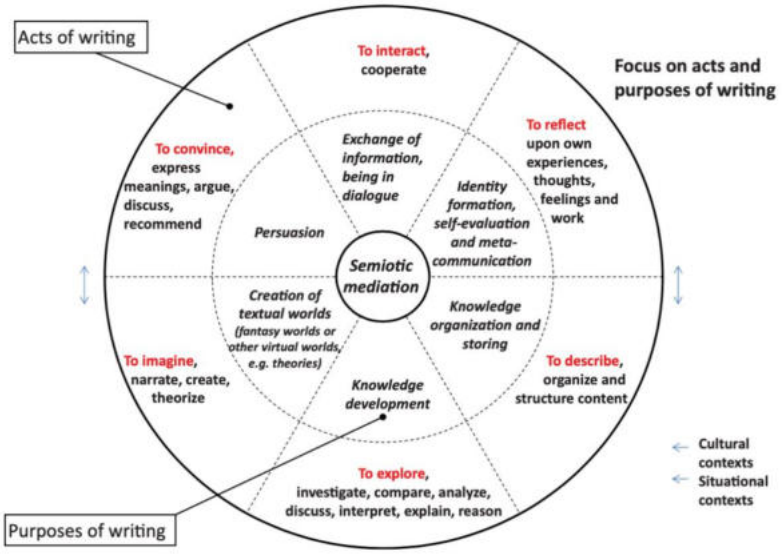

Figure 1: The Wheel of Writing (Berget et al. 2016, p. 180)

Considering the stages in the CPS process, our generally adapted notions of models (cf. Lesh and Doerr, 2003) and mathematical modelling (cf. Niss and Blum, 2020), and the view of the writing process as discussed in Berge et al. (2016), we identified the *categories of similarities* between the modelling process and the writing process, presented in Table 1. We note that since neither the CPS process nor the modelling and writing processes are linear in nature, Table 1 does not present steps of the modelling cycles (c.f. Niss and Blum, 2020) or steps taken in the writing process. Rather, the table structures the similarities between modelling and writing processes from the CPS perspective. To facilitate connections to the existing theoretical discussions on modelling, we briefly use of the notions of *situation-*, *real-* and *mathematical models* (cf. Borromeo Ferri, 2006) to communicate about different phases of the modelling process.

As seen in Table 1 below, we aligned the CPS stages with the modelling and writing processes. To provide a more complete picture of the processes, we additionally considered *revisions and refinements* and *presentation and communication* noted by Treffinger (1995) and named new stages for CPS needed to revise, refine, and present a model and a text. Intrigued by these similarities between modelling and writing that we identified and outlined in Table 1, we asked ourselves the

questions *What could a WoW-inspired perspective for mathematical modelling look like and entail?* and *What would the corresponding purposes and acts of modelling be given the similarities in the creative problem-solving process components?* Hence, the below mentioned identification step served as a methodological organization for us to explore these questions, which report on in what follows.

Table 1: Comparing and outlining similarities between the modelling and the writing processes

Stages in the creative process	Mathematical modelling	Writing	Categories of similarities**
Mess-Finding *Problem-Finding*	Defining and understanding the problem; identifying potential domains, disciplines, potential general ways of working; making assumptions (this akin to creating a situation model	Creating outline or a plan; organizing once's thoughts and ideas into a coherent structure; creating a vision of the text.	**Planning and Structuring**
Data-Finding	Collecting information and (if needed) data to inform the model to be developed. This facilitates the modeller to transitioning toward a real model.	Gathering information, facts, and references to support arguments or narrative in the written text.	**Background research and data collection**
Problem-Finding *Idea-Finding*	Formulating the initial models using prior mathematical knowledge; transforming the real model into a mathematical model.	Creating a first draft; tentatively writing the main ideas (knowing it will need refinement).	**Drafting and initial formulation**
Idea-Finding *Solution-Finding*	Working mathematically with the mathematical model to find mathematical answers.	Working on developing and realising the plan and composing a coherent text or narrative.	**Implementing solution strategies and doing the work**
Acceptance-Finding	Evaluating and validating the mathematical outcome with respect to both the calculations done and their interpretation in relation to the context of the original situation and problem. This can be done by for example collaborating with colleagues, consulting experts, or collecting more data.	Revising the first draft to improve clarity, coherence, and style, often through multiple iterations. This can involve seeking feedback from peers, editors, or reviewers to improve the text.	**Evaluation and Validation**
*(Re)Problem-Finding** *Extension-Finding**	Incorporating the evaluation results and determine the possible revisions and refinements to improve the mathematical model.	In-cooperating comments and feedback from the evaluation and validation to improve the text.	**Revisions and Refinements**
*Communicate-Finding**	Communicating the results and implications of the model and the modelling process in a suitable format (report, presentation, or publication).	Presenting the final written text in a clear and engaging manner (regardless of type of text).	**Presentation and Communication**

**New stages named by the researchers for the CPS process; **Identified categories of similarities*

The Wheel of Modelling - A *WoW-inspired perspective for mathematical modelling*

Drawing on the connections presented above in Table 1, we elaborated on purposes and acts of modelling, in order to construct a Wheel of Modelling (WoM) with the hope to provide a productive perspective to think about models and modelling. Similar to the writing acts and purposes in the WoW, we consider that purposes of modelling aim to describe and capture key aspects of modelling carried out given specific rationales and purposes. Acts of modelling on the other hand aim to describe and capture key aspects of the activities involved in the modelling process. Table 2 below provides an overview and summary of the discussion and elaboration of the parallelism between the purposes

and acts of modelling and writing processes. Figure 2 then pulls all the modelling purposes and acts together, presenting the Wheel of Modelling (WoM). Mathematical models are placed in the centre of the WoM since we argue that it is a mathematical model that constitutes the mediating artifact enabling the realisation of modelling purposes through modelling acts.

Table 2: Comparing and summarising WoW- and WoM purposes and acts

WoW purposes	WoM purposes	WoM acts	WoW acts
Exchanging information and being in dialogue	Using in the context of collaborative interdisciplinary settings	Collaboration and communication across disciplines	Interact, cooperate
Persuasion	Prescriptive modelling	Use the models to convince the clients about their viability; express, argue and discuss the reasonability of the made underlying assumptions and methods used; and additionally, that the recommendations based on the models are ethical and fair	Convince, express meanings, argue, discuss, recommend
Organising, systematising and storing knowledge	Descriptive modelling	Capture, represent and/or describe the situation or phenomenon being modelled	Organise, systematise and store
Knowledge development	Generating understanding, explanations and knowledge	Explore, investigate, compare, analyse, discuss, interpret, explain and reason	Explore, investigate, compare, analyse, discuss, interpret, explain and reason
Identity formation, self-evaluation and meta-communication	Self-conception as a modeller and a more nuanced view of mathematics	Reflect on both the model and the modelling process, self-evaluation, and develop awareness about one's own modelling competencies as well as one's attitudes towards and comprehension of mathematics	Reflect on own work, experiences, thoughts and feelings
Creation of textual worlds	Underpinning/facilitating simulations	Adapt and implement a mathematical modelling so that it can be used as part of a simulation, connect to suitable technology, programming, and use visualization aids	Imagine, narrate, create and theorize

It is important to stress the dynamical relation between the two concentric circles that make up our constructed WoM model in Figure 2. This means that although one specified modelling act looks radially aligned with one specified modelling purpose, this relationship is nor static or one-to-one. The elaboration below is structured based around the six modelling purposes which together constitute our first iteration of the WoM model. In an attempt to be as clear and transparent as possible about the parallelisms identified between modelling and writing, all WoW purposes and acts in what follows are written in *italics*, whereas WoM purposes and acts are underlined.

Collaborative interdisciplinary modelling: The WoW act *to interact* is foremost put forward "with the purpose of establishing and/or strengthening contacts with other people" (Berge et al., 2016, p. 181), but also with the WoW purpose of *exchanging information and being in dialogue* across disciplinary subjects and topics. This latter aspect of the WoW resonates with the modelling purpose of using modelling in the context of collaborative interdisciplinary settings such as STEM projects, cross-curricular collaborations, and STEM education in general. In line with this type of WoM purpose are modelling acts that emphasise collaboration and communication across disciplines. The former WoW act on the other hand aligns with the act of modelling that is reflected in the

collaborative work of groups of students in co-constructing of models in a classroom setting as well as in interdisciplinary teams.

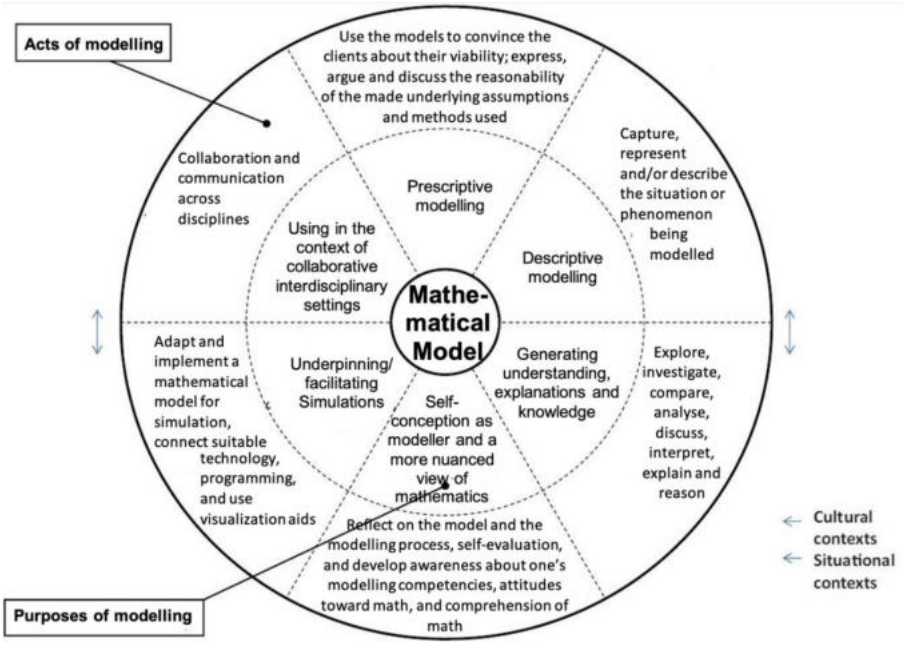

Figure 2: The Wheel of Modelling (WoM)

Prescriptive modelling: The WoW act *to convince* is centred around expressing a position or recommendation about how (some aspect of) the world should be given the WoW purpose "*to persuade* a reader to share our position" (Berge et al., 2016, p. 181, emphasis added). In our conception of the WoM, this is akin to the modelling purpose of prescriptive modelling which aims to design, organise or structure a certain aspect of the real world, often to achieve a specific goal (Niss, 2015). The modelling acts especially coming to the fore in prescriptive modelling are similar to the corresponding acts of WoW, namely, to use the models to convince the clients about their viability; express, argue and discuss the reasonability of the made underlying assumptions and methods used; and additionally, that the recommendations based on the models are ethical and fair.

Descriptive modelling: The WoW purpose *to organise, systematise and store knowledge*, is achieved by engaging in these very acts: *organising, systematising and storing* (Berge et al., 2016). Broadly speaking, these WoW purposes and acts have natural parallels in the WoM. The corresponding modelling purpose is descriptive modelling (Niss, 2015), which focuses on describing events, phenomena or situations given existing conditions. The key modelling acts to achieve descriptive models are to capture, represent and/or describe the situation or phenomenon being modelled.

Generate understanding, explanations and knowledge: In the WoW, the purpose of *knowledge development* is achieved by acts like *exploring, investigating, comparing, analysing, discussing, interpreting, explaining and reasoning*. Berge et al. (2016) describe that the purpose of knowledge development for example involves finding "the most valid interpretation of a phenomena" (p. 182) by acts aiming at critically examine evidence and assumptions. This type of knowledge development in the context of the WoM has the analogue modelling purpose of generating understanding,

explanations and knowledge, and the analogue modelling acts of exploring, investigating, comparing, analysing, discussing, interpreting, explaining and reasoning.

Self-conception as modeller and a more nuanced view of mathematics: The WoW act *to reflect focused on own work, experiences, thoughts and feelings* align with the WoW purpose of *developing self-insight, identity and own understanding* (Berge et al., 2016). This WoW act strives to achieve WoW purposes related to *identity formation, self-evaluation and meta-communication*. In a modelling context, this is similar to the modelling purpose of building one's self-image, confidence and attitudes as a modeller, as well as using models and modelling for students to form a more nuanced view of what mathematics is and its application and role in science, work- and everyday life. The modelling acts through which this is accomplished involve reflecting on both the model and the modelling process, self-evaluation, and developing awareness about one's own modelling competencies as well as one's attitudes towards and comprehension of mathematics.

Underpinning/facilitating simulations: When the WoW purpose is the *creation of textual worlds* the aim is to engage in acts resulting in a described "reality [that] is constituted by the text. That is, a reality that does not exist in other places than the fiction construed and expressed in the text." (Berge et al., 2016, p. 182). The WoW acts connected to those WoW purpose are *to imagine*, *to narrate*, *to create* and *to theorize* (the latter in the sense of speculating). The corresponding WoM purpose is to use models in (doing) simulations, based on the fact that simulations for the most part are run based on mathematical models. This exhibits the fictive dimensions mirrored in the WoW purpose *creation of textual worlds*; simulations are not real, and their existence depend on the models and the actual simulations being executed. In the WoM therefore the corresponding modelling acts heavily rely on aspects of adapting and implementing a mathematical model as a basis or part of a simulation: connecting to suitable technology, programming, and using (technological) visualization aids.

Discussion and outlook

In this theoretical paper, we introduced a perspective on mathematical modelling by exploring the similarities between the modelling and the writing processes, and aligning these with components of the creative problem solving process. By reconceptualising the 'Wheel of Writing' as a 'Wheel of Modelling', we articulated different acts and purposes of mathematical modelling mediated by mathematical models. Using theoretical frameworks from other fields, such as the didactics of language, forces us to clarify our own conceptual notions and their meanings. While different fields may use similar terminology with different meanings, these external frameworks can offer new perspectives and enrich mathematics education. Although this is our first attempt to conceptualise the 'Wheel of Modelling', this perspective and approach opens up several avenues for discussion, future research and development: (i) future work should focus on refining and elaborating the WoM model. This includes introducing additional modelling purposes and connected modelling acts. For instance, 'using modelling to build and develop theory' could be explored whether it fits within the use in/for simulation purpose, the purpose of generating understanding, explanations, and knowledge, or if a new purpose in the WoM is needed; (ii) new ways of thinking about modelling like the one presented in this paper could significantly impact the integration of mathematical modelling into the classroom. By dynamically connecting different acts and purposes of mathematical modelling, researchers and

educators alike can focus on areas that potentially lead to innovative teaching methods and improved student engagement. By addressing and exploring issues raised by these points, we are hopeful that we can continue to develop and refine the WoM model and use it in systematic reviews in the area of mathematical modelling, as well as stimulate further theoretical discussions.

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Modelling indoors and outdoors: A comparison of the student's interest and competence development in two task settings

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Mathematical modelling aims at transferring mathematical knowledge to real-life phenomena. This competence can be promoted in different school settings, e.g. when working with text and pictures related to the task situation in the classroom or when exploring a real object outdoors. While recent research has qualitatively compared the work processes in different task settings, this study aims to address the lack of quantitative results. The study firstly investigates the student's interest in outdoor and indoor task settings. Secondly, it focuses on the development of modelling (sub-)competences. To this end, a quasi-experimental study design was conducted with 10th graders of a German grammar school (15–16 years old; N = 80). The results show that the students have a significantly higher interest in modelling outdoors than in working on similar tasks in the classroom. However, no differences were found between the two settings in terms of the student's competence development.

Keywords: Competence, interests, mathematical modelling, outdoor education.

Introduction

Mathematical modelling aims to investigate real-world phenomena by transferring a given situation into mathematics and vice versa (Blum. 2015). In classroom practice, mathematical modelling activities are usually initiated by a task description, supplemented by a picture of the real-world situation (e.g., Blum, 2015). This approach is referred to below as the *indoor setting*. Also, a modelling process can also take place in an *outdoor setting*, where the real object is investigated on site (Buchholtz, 2021; Hartmann & Schukajlow, 2021; Jablonski, 2023a, 2023b). The present study focuses on a comparison of the indoor and outdoor settings. Within a quasi-experimental study, we investigate (1) the student's interest and (2) their ability to develop modelling (sub-)competences.

Theoretical Background

According to the modelling cycle by Blum and Leiss (2007), mathematical modelling can ideally be depicted as a process of seven modelling steps (also: modelling sub-competences; Hankeln, 2019). A real situation is *understood*, *simplified/structured* as well as *mathematicised* with regard to a given task. The resulting mathematical model is *worked on mathematically* before the mathematical result is *interpreted*, *validated*, and *exposed* in relation to the real situation (Blum & Leiss, 2007). However, as Buchholtz (2021) points out, the nature of modelling activities still depends on the setting.

> In contrast to the photo [indoors], the visual-haptic presence of the [outdoor] object and its actual size would allow for a broader range of approaches […] and an easier interpretation of the mathematical solution […] in relation to the student's data collection (Buchholtz, 2021, p. 144).

Due to the high degree of contextualisation, modelling in the outdoor setting enables students to relate directly to the real object during the modelling steps of *simplification/structuring*, *mathematisation*,

interpretation and *validation* (Buchholtz, 2021). Recent research finding on the nature of modelling processes in the indoor and the outdoor settings are summarised below.

State of the Art

Affective attitudes: Regarding the outdoor task setting, several studies emphasise the high interest of students to perform modelling activities directly at the object's location (e.g., Barlovits & Ludwig, 2023; Cahyono, 2018). On the other hand, by comparing the student's interest in indoor and outdoor settings, Hartmann and Schukajlow (2021) find no statistically significant differences between both settings in an experimental study with $N = 43$ lower secondary students. Also, Jablonski (2023a) reports non-significant differences in the student's interest based on a study with $N = 29$ lower secondary students who worked in both settings. Jablonski (2023a) explains this as follows: Firstly, the study was conducted as part of an enrichment programme for gifted students, which limits its transferability to classroom practice. Secondly, the study shows a ceiling effect, as the students are highly interested in both indoor and outdoor settings. Given the promising results of outdoor mathematics in terms of the student's interest, and the relatively small sample sizes of the studies by Hartmann and Schukajlow (2021) as well as Jablonski (2023a), we see a need for further research on the affective attitudes of learners when modelling in indoor and outdoor settings.

Modelling sub-competences: In a qualitative study, Jablonski (2023b) highlights that working indoors and outdoors is associated with different foci in the modelling competences of *simplification/structuring* and *mathematisation*. In terms of *simplification/structuring*, students are aware of the limitations of using photos to analyse a real situation when working indoors. They accept assumptions and estimates – and the resulting inaccuracies – as an inherent aspect of the indoor setting. In contrast, when working outdoors, students intensively discuss their real-world model, the availability of data and ways to minimise inaccuracies. Consequently, students in the outdoor setting focus primarily on the process of data collection in terms of *mathematisation* (Jablonski, 2023b). In contrast, students in the indoor setting discuss intensively the extent to which the perspectives of the two-dimensional photos might influence their mathematical assumptions about the three-dimensional real-world object (Jablonski, 2023b). These observations are supported by a survey of lower secondary students, who perceive *simplification/structuring* and *mathematisation* as significantly different indoors and outdoors (Jablonski, 2023a). In terms of *mathematising*, significant differences in the use of strategies can also be found for pre-service teachers in the indoor and outdoor settings (Segura et al., 2023).

These findings support the meaning of contextualisation in the outdoor setting described by Buchholtz (2021). So far, however, it remains unclear to what extent the different modelling processes indoors and outdoors lead to a comparable or different competence development.

Research Questions

The present study aims to address the two identified research gaps. Firstly, we investigate how the two task settings affect the student's interest, i.e., whether the work on tasks indoors or outdoors leads to greater interest in the modelling process. Secondly, we investigate to what extent the task settings impact the development of modelling competences. Here, the modelling sub-competences of

simplification/structuring, *mathematisation* and *interpretation* as well as *validation* are focused, as they may differ between the outdoor and indoor settings due to their high degree of contextualisation (cf. Buchholtz, 2021). These considerations lead to the following two research questions.

[RQ 1] To which extent does the student's interest in modelling tasks depend on the setting?

[RQ 2] To which extent does the student's development of modelling competences depend on the setting?

Methods

To answer the research questions, a quasi-experimental study using a pre-post-test design was conducted in Spring 2024. A total of $N = 80$ students from four classes took part in the study, processing up to four modelling tasks in 90 minutes. They attended the 10th grade of a German grammar school (15–16 years old) and collaborated in teams of three students.

Two classes were assigned to the experimental condition, i.e., the *outdoor group* ($N_1 = 43$). The students were equipped with folding rulers and calculators so that they could take measurements and perform calculations directly on the object's side. As a control group, two classes worked on comparable modelling tasks in the classroom (*indoor group*; $N_0 = 37$), equipped with rulers and calculators. The students were supported by pictures of the task situation including a reference object. Both the indoor and outdoor groups worked with identical task formulations on the same objects.

All tasks of the study dealt with the surface area or volume of composite solids. An exemplary task, called "the bench", is formulated as follows. *"The bench around the tree is to be coated with a protective varnish (without the legs). How many square metres of wood need to be coated with varnish?"*. The task is illustrated in Figure 1 for both the indoor and outdoor settings.

Figure 1: The task "The bench" in the outdoor setting (left) and its graphical representation in the indoor setting (all pictures from left to right)

Test instruments

In view of the first research question, the student's interest in mathematics as well as their interest in the task setting was surveyed. In a pre-test, the student's interest in mathematics was assessed using four items on a four-point Likert scale. The items were selected from Rakoczy et al. (2005), such as

"I like mathematics" (from 0 = "I totally disagree" to 3 = "I totally agree"). For the post-test, the items were adapted to assess the student's perception of the setting, e.g. *"I like these kinds of tasks [worked on during the intervention]"*.

With regard to the second research question, two performance tests on modelling competences were completed by the students, namely a pre-test before and a post-test after the intervention. For each modelling sub-competence, four items were selected from Hankeln (2019) that showed high discriminative power for the related construct. From these items, two different performance test versions A and B were created, where both tests contained two items each on the modelling sub-competences *simplification/structuring*, *mathematisation*, *interpretation* and *validation*.

In a rotation design, one class of the indoor group worked on test version A as a pre-test and test version B as a post-test. The other class of the indoor group worked on the test versions in reverse order. This test rotation was also used for the outdoor group. Thus, all participating students completed all selected items exactly once, i.e., half in the pre-test and half in the post-test.

Data processing

The data processing and analysis was carried out in *RStudio*. Due to the rotation design, only half of the test items in both the pre-test and post-test could be processed by the students, i.e. the other items were *missing by design*. In addition, some items that were presented to the students were not answered due to time constraints. To maximise the power of the statistical test for data analysis, all missing values were imported using the *missForest* algorithm (Stekhoven et al., 2012). The *Proportion of False Classifications (PFC)*, estimated by *missForest*, was around 21 %. This value is substantially lower than the expected value of misclassification (60 %). Further, the PFC was at a similar or lower level to the results found by Stekhoven et al. (2012), indicating a successful imputation.

For both the student's interest in mathematics and in the setting, the four items showed a satisfactory internal consistency (Cronbach's α_{math} = .88 and α_{sett} = .89). Therefore, a sum scale (ranging from 0 to 12) was computed for both scales. On the other hand, the four items per modelling sub-competence test did not show satisfactory internal consistency.[1] This issue was addressed by selecting subsets of two to three items per modelling sub-competence. Thus, out of a total of 16 items, seven poorly fitting items were excluded from the data analysis. This resulted in four subsets that achieved at least an acceptable level of reliability of Cronbach's $\alpha \geq .7$ for each, pre- and post-test. Only the mathematisation competence at the pre-test is below this threshold value (α = .61). On this basis, sum scales were computed for all subsets of the four modelling sub-competences (ranging from 0 to 3 for *mathematisation* and 0 to 2 for the other sub-competences).

In addition, all pre-test items analysed were summed to form a scale representing an overall *modelling competence* in relation to the test instrument used (scale from 0 to 15; Cronbach's α = .72). Given the relatively small sample size, which limits the number of predictors for the linear regression, this scale is used as a proxy for all four sub-competences in the data analysis. This is justified because all four

[1] The items were rated according to Hankeln (2019) by 0 and 1 points (dichotomous) or 0, 1 and 2 points (polytomous). In contrast to Hankeln (2019), classic test theory instead of item-response theory was applied for data processing.

sub-competences show a strong correlation with the *modelling competence* scale (*simplification/ structuring*: r = .55; *mathematisation*: r = .81; *interpretation*: r = .54; *validation*: r = .55).

For analysing the data, *robust t-tests* via the *R* package *WRS2* (Mair & Wilcox, 2020) as well as *robust multiple linear regression* via the *R* package *robustbase* (Todorov & Filzmoser, 2009) were used. Robust approaches were chosen because the scales of the modelling (sub-)competences partially violate the test assumptions of normal distribution and homoscedasticity.

Results

Preconditions

In order to check for comparable preconditions between the two groups (indoor and outdoor setting), we compare the mathematical interest, as well as the modelling (sub-)competences via *robust t-tests* (10 % trimmed means; 2,000 bootstraps). No significant differences are found between the two groups, neither for the mathematical interest, $t(64.00) = -0.30$, $p = .765$, nor for their modelling competence in the pre-test, $t(63.68) < 0.01$, $p = .996$. This is also valid for the sub-competences of *mathematisation*, *interpretation* and *validation*. Only for *simplification/structuring*, the assumption of equal preconditions cannot be hold. Here, the *robust t-test* shows a significant difference in favour of the outdoor group, $t(63.91) = -2.23$, $p = .029$, $\xi = .35$ (small to medium effect). Nevertheless, it can be assumed that the two groups are comparable, despite small deviations.

RQ1: Interest in modelling settings

To investigate the student's interest in indoor and outdoor modelling settings, three predictors are considered: the group, the student's interest in mathematics as well as their score in the pre-test of modelling competences. The latter two predictors are centred. The three models are shown in Table 1.

The first model ($R^2 = .09$) shows that the average interest in working on the modelling tasks in the indoor group is 6.11 points (scale from 0 to 12), while the interest of the students in the outdoor group is on average 1.60 points higher than the interest of the indoor group, $p = .009$.

Table 1: Robust multiple linear regression on the student's interest in modelling settings

Predictors	**First model**	**Second model**	**Third model**
Intercept	6.11 ***	6.19 ***	6.22 ***
Group *(indoor setting = 0)*	1.60 **	1.62 **	1.56 **
Interest in mathematics		0.31 **	0.41 *
Modelling competence *(pre-test)*		0.13	0.26 °
Group * Interest in mathematics			-0.23
Group * Modelling competence			-0.22

In the second model ($R^2 = .22$), the student's interest in mathematics and the modelling competences assessed in the pre-test are added. This leads to a significantly higher variance explanation compared to the first model, $F(2) = 15.14$, $p < .001$. While the intercept and the influence of the group are comparable to the first model, the second model confirms that the student's interest in mathematical modelling depends on their general interest in mathematics. It shows that a student with a one-point higher interest in mathematics (scale from 0 to 12) tends to have a 0.31-point higher interest in the intervention, $p = .004$. Remarkably, but possibly due to the relatively small sample size of $N = 80$, the modelling competence is not significant, $p = .124$. The predictor shows that a student with a one-point higher pre-test competence in mathematical modelling (scale from 0 to 15) tends to have a 0.13-point higher interest in the intervention. The standardised regression coefficients indicate that the student's interest in mathematics ($\beta = .32$) is two times as influential as their modelling competence ($\beta = .16$) in predicting the student's interest in the setting.

The third model ($R^2 = .28$) includes the interaction effect of the group and the other two predictors in the analysis. It does not provide a significantly better description of the data than the second model, $F(2) = 4.23$, $p = .120$. Being aware that this might be caused due to a lack of power caused by the relatively small sample size, the third model is briefly analysed: The intercept as well as the influence of the group remain stable, while for the indoor group a stronger influence of the interest in mathematics ($b = .41$), and of the modelling competence ($b = .26$) on the interest in the setting can be found. The model does not show significant interaction effects between the group and the interest in mathematics, $p = .271$, or the modelling competence, $p = .200$. Nevertheless, it can be hypothesised that an interaction effect is likely to be found when a larger sample size is investigated: The student's interest in the setting seems to be about twice as strongly influenced by the interest in mathematics of the indoor group ($b = .41$, $\beta = .43$) compared to the outdoor group ($b = .18$, $\beta = .18$). Similarly, it can be hypothesised that the interest in the setting is more strongly influenced by the modelling competence in the indoor group ($b = .26$, $\beta = .33$) compared to the outdoor group ($b = .04$, $\beta = .05$).

RQ2: Development of modelling competences

To investigate the impact of the setting on the development of modelling (sub-)competences, *robust linear regression models* are used to predict the modelling (sub-)competence at the post-test by the group, the modelling (sub-)competence at the pre-test and the interaction effect of the two predictors.

For the *modelling competence*, the indoor group achieved a mean pre-test score of 7.91 points (SD = 3.18). In the post-test, the score remained almost unchanged (mean = 7.81), while wider range of scores was observed (SD = 4.03). These observations are similar for the outdoor group (pre-test: mean = 7.98, SD = 3.59; post-test: mean = 7.95; SD = 3.73). Consequently, neither a significant group effect, $p = .591$, nor an interaction effect of the group and the pre-test score, $p = .736$, can be found in the regression model. Only the pre-test score for the *modelling competence* is highly significant, $p < .001$. The regression model ($R^2 = 38$ %) predicts that per each added point on the pre-test score a 0.74-point higher result in the post-test score (scale from 0 to 16 each) can be expected.

These results can be maintained for the modelling sub-competences of *simplification/structuring*, *mathematisation*, *interpretation* and *validation*. The regression models each show a significant relationship between the pre- and the post-test, while no group or interaction effect can be found.

Discussion and Outlook

Mathematical modelling aims to investigate real-life situations through the application of mathematical knowledge (Blum, 2015). As mathematical modelling processes can be initiated inside and outside the classroom (Buchholtz, 2021; Hartmann & Schukajlow, 2021; Jablonski, 2023a, 2023b), this study investigated the role of the modelling task setting. It compared the processing of modelling tasks based on photos (indoor setting) with the work on analogous tasks at the location of the real object (outdoor setting). In particular, it examined the impact of the setting on the student's interest (RQ1) as well as on the development of student's modelling (sub-)competences (RQ2).

With regard to RQ1, the outdoor group self-reported a 1.60-points higher interest in the setting compared to the indoor group (scale from 0 to 12), resulting in a significantly higher interest in working in the outdoor setting, $p = .009$. This finding supports the high student's interest in outdoor modelling reported by Barlovits and Ludwig (2023) as well as Cahyono (2018). On the other hand, it contrasts with the findings of Hartmann and Schukajlow (2021) as well as Jablonski (2023a).

This paper went beyond the latter two studies by considering further predictors of the student's interest, namely the interest in mathematics as well as the modelling competence. The results show for both settings that the interest in the modelling setting is more strongly influenced by the student's interest in mathematics than by the student's modelling competence. Moreover, this study can be seen as a first indication that the student's interest in the outdoor setting might be less dependent on their mathematical interest and modelling competence than in the indoor setting. This hypothesis might be supported by the findings of Buchholtz (2021) and Jablonski (2023a), who emphasised that modelling processes outdoors are different from modelling processes indoors. Nevertheless, further studies with larger samples are needed to validate the aforementioned hypothesis. These future studies might also consider additional predictors, such as the student's intra-mathematical competence (as precondition) as well as their performance during the intervention (as process data).

With regard to RQ2, the pre-scores of the modelling (sub-)competences were significant predictors of the corresponding post-scores. However, no learning gain was observed for any of the modelling (sub-)competences. Moreover, no significant group or interaction effects were found. These findings may be due to two limitations of the present study. Firstly, although well-tested items were selected from Hankeln (2019), seven out of 16 items showed a significant misfit with the other items of the related construct and thus had to be excluded from the study. This may limit the variation of the data, which in turn affects the results in relation to RQ2. Secondly, given the design of the study, (i) a relatively small sample size of $N = 80$ students participated (ii) in a short-term intervention of only 90 minutes. These limitations open the field for further research on the development of modelling (sub-)competences in different settings, with a particular focus on the long-term perspective.

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Students managing of modelling problems with superfluous data

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Research has shown that students often encounter difficulties in processing modelling problems with missing data. Modelling problems often also contain superfluous data and differentiating between these data and relevant ones is a key competency for carrying out the modelling process. We thus investigate students' processing of data in a modelling task, wondering to what extent they differentiate between relevant and superfluous data, and how can the anomalous use of relevant and superfluous data be characterized. We perform a qualitative content analysis of the protocols produced by 102 secondary school students, while working in groups. The analysis shows that, for most data, students successfully distinguished relevant and superfluous data. It also provides a first characterization of students' anomalous use of data, which confirms the students' tendency of focusing on numerical data and not considering reality in modelling problems.

Keywords: mathematical modelling, superfluous data, data representation.

Introduction and theoretical background

Problem solving activities in schools often involve the use of word problems. which verbally present problem situations with questions emerging in them. The answer to these questions can be obtained working mathematically with the numerical data available in the problem statement (Verschaffel et al., 2020). Word problems in school settings are mostly well-defined, in the sense that all relevant data are present in the problem statement and no superfluous data is included (Kanefke & Schukajlow, 2022). Authentic problems encountered in the real-world do not often share these features: some real-world problems can be solved with the help of mathematics, but questions are less explicit, and it is often not clear which numerical data and mathematical tools have to be employed for their resolution (Verschaffel et al., 2020). In fact, solving such problems requires performing a complex process of translating between the real-world and mathematics, namely a Mathematical modelling (MM) process (Blum, 2015). Such a process can be represented by the Blum and Leiss (2007)'s modelling cycle, which is composed of seven steps. The first two steps occur within the real-world and enable the transition from the real situation to a situation model (understanding) and then to a real model (simplifying/structuring). The third step (mathematizing) leads to the mathematical model within the domain of mathematics, where it is possible to work mathematically (step 4) to achieve a mathematical result. The next two steps allow for returning to real-world, by interpreting (step 5) and validating (step 6) the results. Finally, in step 7, the results are presented in the real situation. Due to the complexity of the modelling process, MM is highly demanding. Literature (e.g., Galbraith and Stillman, 2006) has shown that each step can be a cognitive barrier for students, even the first steps, namely understanding the task and simplifying/structuring the model situation. Furthermore, many empirical studies (e.g., Verschaffel et al., 2000) have shown that students often treat modelling problems as if they were well-defined word problems, by putting into play superficial solution strategies, for example executing mathematical operations with the numbers presented in the problem, without considering reality in their solution process. For example, in Kanefke and Schukajlow (2022),

this tendency is identified as one of the possible causes for students' difficulties in processing modelling problems with missing data, with students encountering difficulties in identifying that data are missing. Other studies (e.g., Fung Lin, 1990) highlight that the lack of consideration of reality can be the cause of students' difficulties in distinguishing relevant and superfluous information when processing word-problems with irrelevant data.

As highlighted in Maas (2010), the presence of superfluous data is, together with relevant and missing data, one of the features that could distinguish modelling problems from well-defined word-problems. In fact, "to look for available information and to differentiate between relevant and irrelevant information" is, according to Maas (2006), a sub-competency crucial to the initial steps of the mathematical modelling cycle. However, while students' processing of modelling problems with missing data has been the object of research in the past (Kravitz et al., 2018; Kanefke & Schukajlow, 2022), to the best of our knowledge, no study has focused specifically on how students process superfluous data while solving a MM problem. Thus, our aim is to contribute to identifying the reasons for possible students' difficulties in dealing with this particular feature of MM problems. Thus, a MM task, which will be described in detail in the next section, was delivered to students and students' protocols were collected as data. The paper is structured as follows: we first present the research context, the data gathering and the method of analysis. Afterwards, we report the results and, finally, we discuss the findings according to the research questions.

Research context

The modelling task was co-designed by two researchers (one of whom is the author) and 4 teachers participating in a community of practice (Wenger, 1998) on MM. The task was then proposed by the teachers to their students, who worked in groups. Both teachers and students had little experience in MM. The task, translated in English, was as follows:

A new link for the Rimini airport

The city of Rimini is planning to develop a new link to allow passengers to comfortably move from the Fellini airport to the "Metromare" stop (CRT Coastal Rapid Transport). Two alternatives have been identified: (1) Elevated mechanized pedestrian walkway. (2) Shuttle. There is the necessity of estimating the travel time for the two alternatives, to understand which one is the most advantageous in terms of time. Attached are the two projects' sheets. Which one of the two alternatives is most advantageous in terms of travel time? Explain the reasoning you carried out to answer the question through a brief report, to be delivered to the project managers.

- **Sheet 1: Elevated mechanized pedestrian walkway.** The first alternative consists of an elevated mechanized pedestrian walkway (see Figure 1). The key features are the following: it is sheltered and protected (from sun and bad weather); it is elevated; it is mechanized with a moving walkway along 70% of the total path, to reduce the time and inconvenience of pedestrian movement. We report the moving walkway datasheet: energy efficiency class: A+++; connectivity: ok; operating environment: indoor/semi-outdoor; speed: 2m/s; slope: 0°-6°, maximum length: 900m.

Figure 1: blueprint of the project. The path of the pedestrian walkway is highlighted in black

- **Sheet 2: shuttle.** To avoid a pedestrian zone, the shuttle would cover 1.5km. We report the shuttle datasheet: battery capacity: 300kWh; maximum capacity (seated-standing): 40-20 people; average waiting time: 5 minutes; speed:15 km/h.

For the sake of clarity, we now present an a-priori analysis of the task. To this end, we resort to the classification of modelling tasks proposed in Maas (2010), focusing on the categories "Data" and "Type of representation". The category "data" classifies modelling tasks based on the type of data, like, for example, missing data or superfluous data. The category "type or representation" specifies how a task is presented, for example using text or materials.

Concerning the category "Data", the task contained both superfluous and missing data, as well as the relevant data. We distinguish between relevant and superfluous data on the basis of the problem's request, which is identifying the best alternative in terms of travel time. We highlight that, if the problem's request was more open — for example, identifying the best alternative by considering multiple factors such as energy consumption or cost — the distinction between relevant and superfluous data might change accordingly. The relevant data for computing the travel time for the pedestrian walkway are: (1) the total length of the path, which had to be estimated from the blueprint in Figure 1, (2) the fact that the moving walkway covers 70% of the total path, contained in the text, (3) the moving walkway speed contained in the moving walkway datasheet, (4) the walking speed (missing data) and (5) the information about whether passengers walk or not on the moving walkway (missing data). All the data in the moving walkway datasheet, except for speed (4), are superfluous data. The relevant data for computing the travel time of the shuttle are: (6) the distance covered by the shuttle reported in the text, (7) the average waiting time and (8) the shuttle speed in the shuttle datasheet and (9) the getting on/off the shuttle time (missing data). The pedestrian walkway is the more advantageous solution in terms of time. Concerning the category "type of representation", some data were presented as numbers in the text, while other data were presented through materials (blueprint and datasheets). The choice of representing some data using materials was made to increase the authenticity of the task. We highlight that, once the relevant data are identified and proposals for the missing data are made, mathematization and mathematical work are straightforward. This choice was made to make students work mostly on the first steps of the modelling cycle and to ensure that all students held the prior knowledge necessary for solving the problem. Finally, the request to write a short report for the project manager was made to monitor students' resolution processes.

Methods

Participants were 102 students (6 students of grade 10, 82 students of grade 11 and 14 students of grade 12) from 5 classes, belonging to different secondary school tracks (humanistic, scientific and technical), in Italy. Students worked in random groups of 3 to 4 students each. The data of the present study are the 26 groups' protocols, namely the report they produced. The first research question we aim to answer is: to what extent do students differentiate between relevant and superfluous data? Groups' protocols were analysed using a deductive qualitative content analysis (Mayring, 2014). For each relevant and superfluous data, we coded whether students used that data, specifying whether they used the data for computing the travel times or after having computed the travel times. Indeed, we hypothesized that students could correctly distinguish relevant and superfluous data for finding the solution of the problem in terms of travel times and that, afterwards, they could use superfluous data in an attempt of refining the model. Each author interpreted the data independently, and then, we discussed the analyses until consensus was reached. As an example, in the protocol N2 in Table 1, the maximum length of 900m for the moving walkway (a superfluous data) was employed by the group for computing the travel time for the pedestrian walkway. In the protocol M3 in Table 1, the maximum capacity of the shuttle (60 people), an irrelevant data, was considered after having computed the travel times, as a further factor in favour of the pedestrian walkway.

Table 1: examples for the coding procedure for answering RQ1 and RQ2

Prot.		RQ1	RQ2
N2	Through a proportion, we compute the total distance of the path, considering that 70% is moving walkway. $900m : 70\% = x : 100\%; \quad x = \frac{900m \cdot 100\%}{70\%};$ $x = 1285{,}714286$ We compute the time spent on the moving walkway, staying still: $t = \frac{s}{v}$ t=time s=space v=average velocity $t_1 = \frac{900m}{2m/s} = 450s$	Superfluous data for computing travel times	Ignoring
M3	For the same time, we must consider other factors: 1) The time we computed for the first option (moving walkway) considers the case in which one stays still on the moving walkway. However, to reduce the time, it could be possible to walk on the parts covered by the moving walkway, reducing time. 2) The shuttle is subject to traffic and bad weather and the maximum capacity is 60 people.	Superfluous data used after computing travel times	

S5	Moving walkway 70% of 900m (total distance) v=2m/s |---| = 42m → 1cm We did the computations both using the 42m unit and not using it. We present in the following the two different solutions. 1) With the ruler the length of the moving walkway path measures 11.5cm, thus multiplying by 42m we would have 483m, which does not correspond to 70% of the total (900m) $11{,}5 \cdot 42m = 483$ →We do a proportion to show that it isn't the 70%. $483:900 = x:100$; x=53,06% Using the data and not considering the 42m unit, using the proportions: $x:900 = 70:100 \quad x = 630m.$ The time required for the moving walkway to lead a person to his destination $t = s/v \rightarrow x = \frac{630m}{2m/s}$ $x = 315s$ → then, 5,25 minutes However, 270m must be walked. On average, a person walks 270m in 3 minutes, thus it would take 7.25 minutes to get to the metro.	Superfluous data for computing travel times	Misinterpreting Discarding

We then focused on the data that were used by students in an anomalous way, namely superfluous data that were used and relevant data that were not used for the computation of travel times. More precisely we ask: how can the anomalous use of superfluous and relevant data by students be characterized? To answer this second research question, we resorted an inductive qualitative content analysis (Mayring, 2014), identifying in the whole material three categories: misinterpreting, ignoring and discarding. Protocol N2 in Table 1 belongs to the category "ignoring": the way students set up the proportion highlights that the word "maximum", which belong to the data "maximum length: 900m" contained in the moving walkway datasheet, is ignored, thus leading to the interpretation of this data as the length of the moving walkway. Moreover, the blueprint, from which the total length of the path had to be computed, is also ignored, since it is not mentioned in the protocol. Protocol S5 in Table 1 belongs to both categories "misinterpreting" and "discarding". On the one hand, the fact that the group writes "900m (total distance)" highlights that the word "maximum" is misinterpreted as "total", thus leading to the interpretation of this data as the total length of the path. On the other hand, the blueprint is correctly considered by the students, who, however, discard this data due to the supposed inconsistency with the previous one. The students highlight their intentional choice with the sentence: "using the data and not considering the 42m unit".

Data analysis

In the following, we report, for each relevant and superfluous data, two percentages that refer to the groups that used that data for the computation of travel times and after having computed the travel times, respectively. We first present the results for relevant data: total length of the path (60% - 0%); moving walkway on 70% of the path (92% - 0%); moving walkway speed (100% - 0%); shuttle distance (100% - 0%); average shuttle waiting time (92% - 0%); shuttle speed (100% - 0%). We now report the results for superfluous data: energy efficiency class for the moving walkway (0% - 8%); connectivity (0% - 0%); operating environment (0% - 0%); moving walkway maximum length (54% - 0%); moving walkway slope (8% - 0%); shuttle battery capacity (8% - 3%); Shuttle maximum capacity – seated (8% - 23%); shuttle maximum capacity – standing (8% - 19%).

These result show that almost all superfluous data were either not used or seldom used by the groups for the computation of the travel times. An exception is the maximum length of the moving walkway which was used by half of the groups (54%). Some superfluous data (energy efficiency class for the moving walkway (8%), shuttle battery capacity (3%), and shuttle maximum capacities (23% and 19%)) were employed by some groups after having computed the travel times. For example, In the following excerpt from protocol P1, the shuttle limited capacity is considered, after the computation of the travel times, as a further factor in favor of the mechanized pedestrian walkway: “Considering the daily turnout, no queue forms on the walkway, differently from the shuttle for which, every 60 people, the waiting time increases by 5 minutes”. Concerning the use of relevant data, all groups employed the moving walkway speed, the distance covered by the shuttle and the shuttle speed. Almost all groups noticed that the moving walkway covered 70% of the total path and that there was an average waiting time for the shuttle. On the contrary, only 60% of the groups considered the total length of the path, which had to be computed from the blueprint, as a relevant data for computing the travel times.

Concerning the second research question, 5 groups belong to the category “misinterpreting”. In fact, they misinterpreted the word *maximum* in the irrelevant data “maximum length: 900m”, as *total*, thus interpreting 900m as the total length of the path. An example is protocol S1, where the students wrote: “The moving walkway extends for 70% of the path, which measures a total of 900m”. 12 groups can be classified according to the category “ignoring”. In fact, they ignored either the word *maximum* in the irrelevant data “maximum length: 900m” or the blueprint, from which the total length of the path had to be computed. An example of the former case is protocol N4, in which students wrote “s=900m (70% of the total path)”, thus meaning that 900m was considered as the length of the moving walkway and not as the maximum length of the moving walkway. An example of the latter case is protocol N2 in Table 1, in which the blueprint is not mentioned. Finally, one group (S5) belong to the category “discarding”. In protocol S5 in Table 1, the students, along with misinterpreting the maximum length of the moving walkway as the total length of the path, used the blueprint and the scale for computing the total length of the path and pointed out the inconsistency between the two values. For the computation of the travel times students decided to discard the relevant data, underlining their choice with the sentence “Using the data, without the 42m unit [...]”.

Discussion and conclusions

We recall that the present paper addresses the following research questions: (1) To what extent do students differentiate between relevant and superfluous data? (2) How can the anomalous use of relevant and superfluous data by students be characterized? Concerning the first research question, for most data, students successfully distinguished between relevant and superfluous data. In fact, students showed difficulties only in classifying the maximum length of the moving walkway, provided as a numerical data in the datasheet, as superfluous data and in identifying the total length of the path, which had to be measured from the blueprint, as a relevant data.

Regarding the second research question, our analysis provides a first characterization of the anomalous use of relevant and superfluous data by students, that is, when relevant data are not used and/or superfluous data are used. In most cases, the anomalous use of data by students is characterized by the fact that they ignored data or part of the information contained within data. In our study, in particular, the ignored data were either represented as materials or text. In a substantial number of cases, students also misinterpreted certain data or part of the information contained in the data. Again, we emphasize that the misinterpreted information was represented as text. Finally, in one case, students deliberately discarded a relevant piece of data, pointing out its inconsistency. It appears that all three of these student behaviors — where data presented as text and materials are misinterpreted, ignored, or discarded — indicate that these types of data are perceived by students as less significant. In contrast, students seem to prioritize numerical data, as further evidenced by the following statement included in the protocol categorized as "discarding": "using the data, without the 42m unit […]". This statement seems to highlight that the blueprint was not understood as a piece of data and that the only part of the blueprint which was considered as data by students is the scale (i.e., a number). This tendency of prioritizing numerical data, and even not understanding non-numerical data as proper data, aligns with patterns previously identified in the literature, where students employ superficial solution strategies in which they combine numerical data, treating mathematical modelling problems as if they were well-defined word-problems, without considering the real-world context (Kanefke & Schukajlow, 2022; Vershaffel et al., 2000). However, our analysis seems also to be partially in contrast with the just mentioned literature. In fact, some groups, after having computed the travel times, tried to employ superfluous data (e.g., the maximum capacity of the shuttle) as a further input for building a model closer to reality, thus showing to consider the real-world situation.

In Maass (2006), distinguishing relevant and superfluous information is considered as a key competency for successfully carrying out the mathematical modelling process. Our study provides a first indication of students' difficulties in managing relevant and superfluous data, especially when data are non-numeric and are provided through textual information or materials. Despite the limited sample size and the fact that only one modelling task has been analyzed, we believe our study suggests that students must practice processing modelling problems with superfluous data characterized by different representations. As a future work, we aim at testing the same mathematical modelling task presented in this work by providing the relevant data of the total length of the path as a numerical value, to more specifically investigate the impact of the type of data representation on the students' processing of relevant and superfluous data. Furthermore, since Kanefke and Schukajlow (2022) showed the influence of task on how students manage missing data, we aim at extending this study

by considering different modelling tasks which contain non-numerical superfluous data to verify whether our findings can be generalized.

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Portfolio as a form of assessment for mathematical modelling with digital tools – Advantages and challenges from students' perspective

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The assessment of mathematical modelling competencies poses significant challenges, particularly when authentic tasks and digital tools are integrated into the learning process. This study explores the potential of portfolios as an alternative assessment format, focusing on students' perspectives. Using qualitative interviews with eight ninth-grade students, we examined the perceived advantages and challenges of portfolio work in the context of digital-based modelling tasks. The results show that the portfolio offers many advantages such as cooperative working with reduced pressure to perform, combined with the possible integration of real modelling processes and the use of digital tools in mathematics lessons. However, there are also some challenges to consider, firstly in connection with the portfolio as an unfamiliar form of assessment and secondly with the challenges related to modelling tasks and digital tools.

Keywords: portfolio, assessment, mathematical modelling, digital tools, interview study

Introduction

The assessment of modelling competencies is the subject of ongoing intensive debate within the international modelling discourse, as it is very demanding, especially in written performance assessment (Frejd & Vos, 2021). In a study of assessment forms for mathematical modelling, Frejd (2013) concludes that the portfolio has the potential to develop into a valid and reliable assessment method. But what does this potential look like? To find out more about this potential in school practice, the project **AlPMo** (Alternative forms of assessment in mathematical modelling, in German: **Al**ternative **P**rüfungsformen beim mathematischen **Mo**dellieren) was launched in 2024. The aim of the project is to investigate the portfolio as an alternative form of assessment in mathematical modelling. As digital tools can be helpful at different points in mathematics lessons (Geiger, 2011; Greefrath, 2011), it can be promising to combine the portfolio with the processing of digital-based modelling tasks. However, we do not know much about students' perception of using portfolio in the classroom. The aim of the presented study is to determine the advantages and challenges of portfolios in modelling with digital tools from students' perspective.

Theoretical Background

Mathematical Modelling with Digital Tools

Mathematical modelling as well as the use of digital tools are an integral part of national mathematics curricula in many different countries. We understand mathematical modelling as a process of solving a real, non-mathematical task with the help of mathematics (Niss et al., 2007). Modelling tasks are characterized by their authenticity, reference to reality, openness and complexity (Maaß, 2010). Solving modelling tasks is a challenge for students in different phases of mathematical modelling processes (Schukajlow et al., 2023). Digital tools can be beneficial at various stages of solving mathematical tasks or modelling tasks in particular, such as during experimentation, visualization,

calculation, or control (Cevikbas et al., 2023; Geiger, 2011). One example of a digital-based modelling task is the task 'Flat search' (Quarder & Greefrath, 2024, p. 119):

Flat search

Anna and Ben are a couple. Anna works in Aachen and Ben works in Brühl (both in Germany). They want to move into a flat together and are therefore looking for a suitable place to live.

What place would you recommend to Anna and Ben?

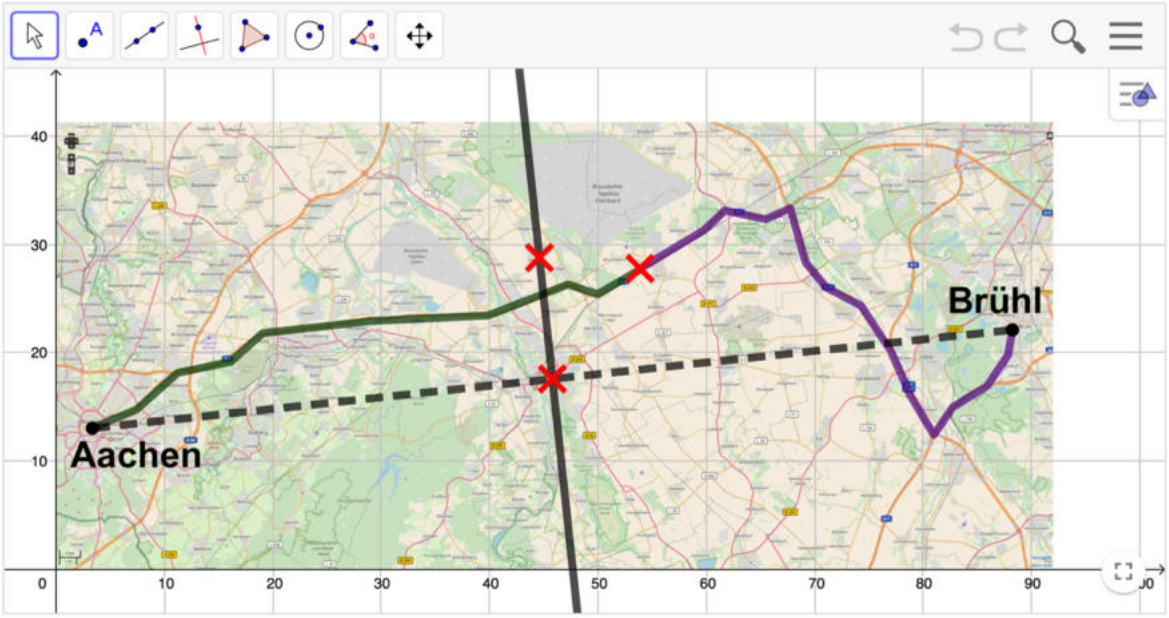

Figure 1: Task 'Flat search' with three possible solutions (Quarder & Greefrath, 2024, p. 119)

(Screenshot: © GeoGebra.org; Map: © OpenStreetMap, www.openstreetmap.org/copyright)

The aim of this task is to determine a common and 'fair' place of residence for a couple who commute between two workplaces. For this purpose, the centre point of the direct connecting line can be formed as one possibility. Alternatively, students can select a point on a previously constructed centre line so that a more rural region can be considered for the housing search. A third option, creating a point between two polygons of equal length that consider the motorway network and not the straight line, is shown in Figure 1. While digital tools offer potential for teaching mathematical modelling, they also present certain challenges such as the necessary translation processes into and out of the digital tool or programme-specific challenges (e.g., technical aspects or the basic ability to operate the digital tool) (Cevikbas et al., 2023).

Assessment of Modelling Competencies by using Portfolio

Mathematical modelling with digital tools is part of curricular standards and should be integrated into assessment of modelling performance. The most used method to assess modelling competencies is the written test (Frejd, 2013). There are mainly conceptual differences in the design of written modelling test items. On the one hand, so-called holistic tasks are used in which the entire modelling cycle must be completed. On the other hand, so-called atomistic tasks are used, in which only one or a few sub-competencies of modelling are activated (Hankeln et al., 2019). However, written tests have their limits when it comes to validity of assessing modelling competences. For example, Frejd

and Ärlebäck (2011) found in a study that collaboration is limited in a class test and that written tests often only consider an atomic view of modelling. In addition, Haines and Crouch (2007) note that written tests do not cover the full range of modelling tasks, such as simplifying a model.

To create alternatives to written tests which may include authentic mathematical modelling tasks, other forms of performance assessment than tests are being discussed in the modelling debate, such as projects, contests and portfolios (Frejd, 2013; Dunne & Galbraith, 2003). Portfolios can be understood as a collection of students' work completed in the classroom or at home and other material that exhibit the student's progress, efforts and performances (Frejd, 2013). As Dunne & Galbraith (2003) point out, portfolios could not only include modelling tasks but also diary notes or written data of achievements. As there is little empirical research on portfolios in connection with digital-based mathematical modelling, it is worth looking beyond modelling to identify the potentials and challenges of portfolios. Portfolios combine possibilities of formative feedback, by collecting it and giving comments during the course, and summative, by assessing it at the end of the course (Francis & Hobbs, 1991). In mathematics education, portfolios foster a more comprehensive evaluation of students' mathematical and "extra-mathematical" competencies (e.g., problem-solving) compared to traditional tests, as discussed by Krall (2023). They present opportunities for improved pedagogical practices, such as reduced test anxiety and enhanced student engagement through metacognition, but also come with challenges related to increased preparation time, questions of reliability, and difficulties in fostering teacher assessment literacy (Krall, 2023; Mak, 2021). In the debate on school digitalisation, digital portfolios are gaining importance as they integrate digital media and technologies into teaching and assessment. One of the challenges posed by digital portfolios is the high level of media literacy on the part of students and teachers (Rachbauer et al., 2023). In this study, we understand digital portfolios as collections of solutions to specific assignments related to the completion of two selected modelling tasks implemented in GeoGebra, conducted within a teaching unit aimed at developing modelling competencies.

Research question

The consideration of the portfolio as an alternative form of assessment is of great relevance, as Frejd (2013) already emphasised, seeing in the portfolio the potential to develop it as a reliable assessment method of modelling processes. To date, there are not yet many studies on the portfolio as a form of assessment in mathematical modelling. The following research question arises from the motivation to look at the student's perspective to work out the conditions for the successful use of a portfolio for assessment in mathematical modelling with digital tools:

What are the advantages and disadvantages of the portfolio as a form of assessment for mathematical modelling with digital tools from a students' perspective?

Methods

To answer the research question, we carried out a teaching unit on mathematical modelling with digital tools at a German grammar school. The sample consists of eight grade 9 students (five females, average age 15.38 years).

Firstly, there was an introductory lesson on the use of GeoGebra because the students had not yet had any experience with this software. During the teaching unit, consisting of five 45-minute lessons, the students created their own individual portfolio on the iPad. The main components of the portfolio were the solution processes of two digital-based modelling tasks, which have been selected from four available tasks. One of these is the task 'Flat search' shown above. In addition to the portfolio, the students worked on an analogue folder with worksheets for a step-by-step introduction to mathematical modelling. These three worksheets were organized in a three-step structure, introducing the steps of (1) simplifying a model, (2) mathematising into a mathematical model and (3) interpreting and validating the solution.

To collect empirical data, we conducted guided interviews at the end of the teaching unit with the aim of capturing the subjective opinions of students. We designed the interviews as one-to-one conversations, as otherwise the students could influence each other. The interviewer followed an interview guideline. The main part of the interview guideline is divided into three sections: general experiences, questions about the portfolio work and questions about the portfolio for assessment in mathematical modelling. The respective workbook and portfolio were used as a stimulus for the students. Transcripts were created based on the audio recordings. These transcripts were coded using Mayring's (2014) qualitative content analysis to inductively uncover advantages and disadvantages concerning portfolios. Categories were developed through four steps: (1) initial transcript review to define coding units, (2) creating a framework with deductive themes from the research question, (3) refining inductive subcategories based on recurring patterns, and (4) validating and adjusting categories. This combined deductive and inductive methods to analyze students' views on portfolio work, assessment, and digital-based modelling tasks. A trained bachelor student and the second author coded the interviews with a high level of intercoder agreement (Cohen's Kappa $\kappa = .84$).

Findings

The results of the qualitative content analysis are shown in Table 1. The categories are structured deductively in perceived advantages and challenges, which in turn are split into the three aspects of portfolio work, assessment and task format. Further, the sub-categories of portfolio work are documentation and working phase.

Portfolio work

Students perceived the documentation in a portfolio as an advantage because it served as a *memory aid*. Ben described the creation of his portfolio as follows:

> Ben: So from my point of view, it was a bit like a kind of college notebook, only digital [...] so that you could read through it again and then not have to look again, ah, how did I do that again? As a kind of memory aid."

Other advantages of the portfolio work are the following: It allows better *traceability for teachers*, especially for modelling tasks; a portfolio can have a *learning effect* due to a better understanding by writing things down and it offers *creative freedom*. However, some students reported disadvantages related to the documentation in the portfolio. On the one hand, students perceived the additional *effort* as a disadvantage of portfolio work. On the other hand, students reported their *lack of experience* with portfolios as a hindering factor. Regarding the working phase for creating the portfolio, students

perceived an advantage in the fact that there was more *autonomy* when working. The *structure* offered via the introductory task as a three-step-sequence for working on a modelling task was evaluated by some students as a strength and by other students as a hindering aspect of the portfolio work. Some students expressed their confusion in the *structure*, as something can get mixed up when creating the portfolio using the three-step sequence. The category of *cooperation* also appears also among advantages and disadvantages. Four students liked cooperation, whereas one student mentioned the difficulties of partner work, as it strongly depends on which partner you get for cooperation.

Assessment

Assessment includes the comparison between the portfolio and a written class test. All students interviewed perceived *reduced performance pressure* as a decisive advantage because of the emotions associated with the assessment. Students reported lower stress associated with this form of assessment and relief due to a smaller amount of preparation. In addition, *time* was another advantage of portfolios in assessment, as Emma described in her interview:

> Emma: Well, I think above all class work you write for 90 minutes, then you hand it in and if you don't have such a good day or somehow, you're not doing so well, then you don't get such a good grade. HERE you don't just work on it for an hour, but a bit longer. That means that if you don't have such a good day, you can make up for it with other lessons.

However, some students experienced *time* also as a disadvantage, as portfolio requires much more time than a test. Further, one student reported that she perceived more time stress in creating the portfolio compared to test. The possibility of *cheating* with the help of the internet was also suggested to be a disadvantage of portfolios.

Task format

Regarding the task format, students positively emphasise aspects that are associated with the characteristics of modelling tasks. This is therefore summarised in a common inductive category of *real modelling processes*. In portfolio work, students positively emphasized ‘more openness in the tasks’ and ‘more reference to reality’ compared to tasks in a class test. Furthermore, students attribute greater relevance and excitement to modelling tasks, as pointed out by Laura:

> Laura: "There wasn’t a single moment where I could say, yes, I didn’t know what to do […]. When I finished the task, I could think of a second solution because it was just fun.”

The *challenges in modelling processes* are summarised as a challenge of portfolios in relation to the task format. The biggest challenge mentioned was understanding the task. Students reported that the modelling tasks led to more frustration as they were perceived as more tedious. Another disadvantage described is that more non-mathematical knowledge was required to solve the tasks, which is also included in the category of challenges in modelling. One student highlights that she prefers written tests over creating a portfolio on mathematical modelling because the test focuses solely on applying class-taught knowledge, while the latter requires integrating basic knowledge from various topics.

Digital working appeared as another advantage as well as another disadvantage of the task format. This category summarises two facets that have emerged inductively in relation to digital tools: On

the one hand, students perceived GeoGebra as making their work easier, especially in terms of working mathematically during the modelling process. On the other hand, students expressed their enjoyment of digital work in general. In contrast, students mentioned technical difficulties with GeoGebra as well as difficulties in using GeoGebra.

Table 1: Perceived advantages and challenges

		Perceived advantages	Perceived challenges
Portfolio work	Documentation	Memory aid Traceability for teachers Learning effect Creative freedom	Effort Lack of experience
	Working Phase	Cooperation Structure Autonomy	Cooperation Structure
Assessment		Time Reduced performance pressure	Time Cheating
Task format		Real modelling processes Digital working Interest	Challenges in modelling processes Digital Working

Discussion and Outlook

This study aimed at the uncovering advantages and disadvantages of the portfolio as a form of assessment for mathematical modelling with digital tools from students' perspective. The analysis indicates that there are many advantages such as the integration of real modelling processes or the use of digital tools in mathematics assessment. Students have a positive attitude towards portfolio work, for example because it enables more independent and cooperative work. The results of our study thus empirically validate the assumptions regarding the benefits of portfolios in mathematics education, particularly with respect to their advantages in mathematical modelling (Krall, 2023; Mak, 2021; Dunne & Galbraith, 2003). The novel contribution of our study is the uncovering of the perceived disadvantages of modelling with digital tools, as the portfolio work shows that modelling is a major challenge for students (cf. Blum, 2011). Furthermore, portfolio work poses a few challenges, such as the potential lack of experience with this form of work. If it were to replace a class test as an alternative form of assessment, the main thing to consider from the student's perspective would be the time and the opportunities for cheating. This is probably related to the fact

that portfolio work allows more collaboration, which fits in with Frejd and Ärlebäck (2011) findings on less collaboration in written tests.

The study has some limitations and strengths, e.g. it has a qualitative methodology that initially reveals findings using a small sample of eight subjects. However, through this study, we have gained deep insights in the success for portfolio work as an alternative form of assessment in modelling. In addition to the student's perspective, the teacher's perspective is particularly interesting and relevant when using a portfolio as a form of assessment, as they are responsible for evaluating student performance. A planned future study will involve conducting a survey of practicing teachers to gather valuable insights. Another forthcoming study will investigate how students' performance in a portfolio is related to their performance in a written modelling test and their overall grade. Students' perspective of the study suggests that mathematical modelling and portfolio work require real-world knowledge and competencies. In future studies, the relationship between performance in creating a portfolio and performance in other subjects should be analyzed. Overall, this study indicates that portfolios are an interesting assessment method in mathematics lessons and that it is worthwhile taking a closer empirical look at them in the context of mathematical modelling.

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Teacher's actions in class discussions after solving a Fermi problem

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This study examines the teacher's role during a whole-class discussion following group work on a Fermi problem, a clear example of a modelling activity that connects mathematics with the real world (Albarracín and Gorgorió, 2014). The problem tasked fifth grade students with estimating how many people could fit in the schoolyard. After working individually and in groups, they presented their solutions and reasoning. Using Drageset's (2014) framework, the teacher's actions were categorized as redirecting, progressing and focusing actions. Results show that the teacher primarily used focusing and progressing actions to guide students through the discussion. While this approach encourages student participation, interactions mainly followed the IRE (initiation-response-evaluation) pattern (Cazden, 2001). The study highlights the importance of teacher's role during class discussions, though further research is needed to explore student contributions in such settings.

Keywords: Mathematical modelling, Fermi problems, classroom discussion, teacher interventions, primary education.

Introduction and research questions

In recent years, there has been a growing interest in introducing mathematical modelling activities at various educational levels to show students the connection between mathematics and their environment (Vorhölter et al., 2014). An example of problems that require a model-based approach and can be explored in primary school classrooms are Fermi problems (Albarracín & Gorgorió, 2014). These are characterized as open-ended questions that offer little or no specific information to guide the problem-solving process (Ärlebäck, 2009; Efthimou & Llewellyn, 2007).

Primary school students can solve Fermi problems by generating mathematical models (Albarracín & Gorgorió, 2019) but it is necessary to explore ways to support them in connecting the main concepts of the different models they can generate. In this study, we analyse the work of 24 fifth-grade students solving the following Fermi problem: How many people can fit in the schoolyard? (Albarracín & Gorgorió, 2014). To encourage student participation, the activity is structured in several phases: beginning with individual work, moving to small group work (with groups of four students), and concluding with a whole-class discussion. In a previous study, we have identified the potential of small group work to generate models to solve the problem and the role of each of the students (Brunet-Brianes & Albarracín, 2024). This study focuses on the final whole-class discussion, in which students present the methods used to solve the problem, discuss the results and justify their answers. This process facilitates the development of their mathematical understanding (Franke, Kazemi, & Battey, 2007). In this class discussion, the teacher plays a crucial role in managing the discussion, including facilitating turn-taking, posing questions, and providing feedback (Smith and Stein, 2016).

The objective of the study is to describe the teacher's role during the class discussion following the resolution of a Fermi problem in groups. The research questions are as follows: (RQ1) What types of

actions does the teacher do during the class discussion? and; (RQ2) How do these actions impact student participation and engagement?

Mathematical modelling and Fermi problems

In the context of mathematical modelling, we adopt Lesh and Harel's (2003) definition of a mathematical model. We understand a model to be a construct comprising a variety of elements, both conceptual and procedural, and that it encompasses different representations of reality, including words, drawings and diagrams. The function of models is to explain a system, which is often complex, that the student seeks to understand. In the primary school context, students do not have the mathematical language tools, such as algebra, to express their models. Nervertheless, they can use natural written language, drawings, diagrams, and corresponding calculations to illustrate their ideas (Albarracín & Gorgorió, 2019). In the context of problem-solving, the model serves to represent the situation presented in the problem and, through the application of mathematical processes, to arrive at a solution.

A category of problems that can be effectively addressed through the introduction of a mathematical model in a primary school classroom setting are those known as Fermi problems. As stated by Ärlebäck (2009), these are "open, non-standard problems that require students to make assumptions about the problem situation and estimate relevant quantities before carrying out often simple calculations" (p. 331). The introduction of Fermi problems in a mathematics classroom is based on the assumption that students can relate to the real-world context presented to them. Efthimou and Llewellyn (2007) defined Fermi problems by the way they are formulated, as they are always posed as open-ended questions that offer little to no information to those attempting to solve them. From the perspective of mathematical modelling, one of the most significant aspects of Fermi problems is the necessity for a comprehensive analysis of the situation presented. The procedure proposed by Fermi was to decompose the original problem into simpler subproblems, and to reach a solution to the original question by making reasonable estimations or educated guesses after first considering the individual subproblems (Carlson, 1997).

In their research, Albarracín and Gorgorió (2019) investigated the models constructed by primary school students to solve Fermi problems. Their findings revealed that there are multiple methods for developing these models, with the most prevalent approaches involving reduction and use of proportion, concentration of measures, reference point, grid distribution, and others.

Class discussion: teacher's role

Class discussions facilitate mathematical reasoning and meaningful learning by enabling students to actively engage in the exchange of ideas and the justification of their perspectives (Drageset, 2014). The development of mathematical understanding requires that students present problem solutions, propose conjectures, discuss a range of mathematical representations, elucidate their solution processes, demonstrate the rationale behind their solutions, and formulate generalisations (Franke, Kazemi, & Battey, 2007, p. 230). Furthermore, it is also crucial to engage in discourse on diverse solution methodologies and to evaluate outcomes (Kooloos et al., 2020).

A substantial body of research has been conducted on the various patterns observed in classroom discussions within the field of mathematics education, with a particular emphasis on the interaction between students and teachers. In many instances, the discourse is dominated by the teacher, following the pattern described as IRE (initiation-response-evaluation) where the teacher asks the questions, the students answer, and the teacher evaluates the response (Cazden, 2001). Although IRE can limit student engagement and focus on procedural tasks (Franke, Kazemi, & Battey, 2007), some researchers highlight its variability (Wells, 1993). More interactive classroom cultures, such as inquiry-based and strategy-reporting, promote student reasoning and justification Wood, Williams, & McNeal, 2006). However, merely increasing student talk is insufficient; teachers must scaffold discussions to make mathematical thinking explicit (Franke, Kazemi, & Battey, 2007). Stein et al. (2008) state that early efforts in mathematical discussions often lacked teacher guidance, leading to superficial exchanges. Brendefur and Frykholm (2000) describe reflective communication, where discussions aim to deepen mathematical understanding by building on students' ideas. Instructive communication further involves teachers guiding and deepening in students' mathematical thinking.

There is a growing consensus that students' contributions should be a significant aspect of classroom communication. However, there is a need to understand how this can be achieved. For students to have this opportunity the class discourse should be dominated by pupils, giving them space to share their ideas and explain their reasoning (Drageset, 2014). Smith and Stein (2016) present five key practices that can assist teachers have control over a class discussion. Smith and Stein (2016) present five key practices that can assist teachers in maintaining control over a class discussion. Their objective is to anticipate potential developments during the activity and thereby afford teachers more time to make decisions. These five key practices are:

- **Anticipate** the possible student responses to challenging mathematical tasks.
- **Monitor** the actual responses of students to tasks (when solved in pairs or small groups).
- **Select** specific students to present their mathematical work during group discussion.
- **Sequence** the student responses to be displayed in a specific order.
- **Connect** the different student responses and link them to key mathematical ideas.

As Smith and Stein (2016) claim, asking the right questions is crucial to promote deeper understanding on cognitively challenging tasks. Effective questions can guide students' attention to previously overlooked aspects of a problem or encourage the discovery of new perspectives. Additionally, such questions facilitate students' ability to articulate their thought processes in a clear and coherent manner, thereby enhancing the comprehensibility of their reasoning to others. This model focuses on students' thinking and how they understand the mathematical content. Teachers have direct and regular access to students' ideas and the details that support them, which is essential information for effective teaching and learning of mathematics (Drageset, 2014; Frank et al., 2007). As Kooloos et al. (2020) emphasise, teachers must strike a balance between student-led contributions and teacher-led guidance in order to ensure meaningful mathematical learning and understanding. While several studies have explored how teachers actively use students' ideas to promote accurate and deeper mathematical understanding, there is a paucity of research on this practice within class discussions where students present their solution strategies for Fermi problems.

Methods

The problem used during the data collection was: how many people can fit in the school yard? In this case, students were required to estimate the number of people that can be arranged across the school yard. The decision to utilise this problem was informed by its previous deployment in the study conducted by Albarracín and Gorgorió (2019). Consequently, the potential strategies that students may employ in response to this problem are already known. The activity presented to the students was set in different stages:

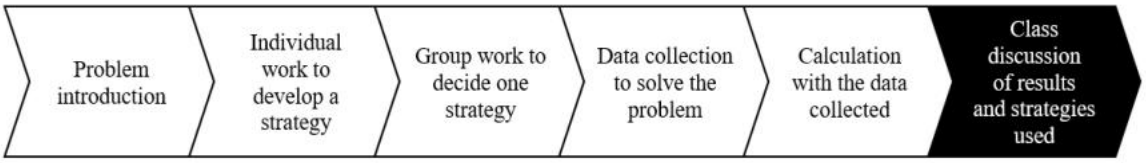

Figure 1: Stages followed during the data collection process

The class discussion was orchestrated following the guidelines of Smith and Stein (2016). The sample was of 24 students in the fifth grade of primary education and their teacher. The activity was developed in two sessions of one and a half hours each. The class discussion was conducted in the second session, with one teacher guiding the conversation. This was recorded in video and audio for subsequent analysis. Two researchers observed and participated in the sessions. Due to space constraints, this paper will focus on the role of the teacher during the class discussion, their interventions, and how they used students' comments.

Analysis

Following the collection of data, the recording of the class discussion was transcribed and structured to facilitate the categorization of the teacher's comments. To ensure the reliability of the data analysis, the audio recordings were reviewed by two researchers to confirm the categorisation of teacher actions. The pre-defined coding categories detailed by Drageset (2014) in the redirecting, progressing and focusing framework were applied. These categories, with minor changes, have been used in other studies, including Kooloos, Oolbekkink-Marchand, Kaenders and Heckman's (2020) work.

- Redirecting actions represent teacher interventions that are designed to guide students towards an alternative approach or strategy. The teacher may acknowledge a student's response but then redirect it, either through corrections, suggestions, or questions that encourage the student to rethink their approach. This category includes putting aside students' comments, explicitly advising students to use an alternative strategy, and correcting questions. An example of this category is when the teacher corrects a student's approach by saying: "Okay, but this part still needs to be filled in, right?"
- Progressing actions are those taken by the teacher to guarantee that the class continues to make progress towards the resolution of the problem. This may entail demonstrating solutions, simplifying tasks, or dividing the problem into discrete, manageable steps to facilitate the ongoing progression of the process. This category includes demonstration of steps or the entire solution, simplifications of the task providing hints, closed progress details by guiding students step-by-step through a process, and open progress initiatives in which the teacher

asks open-ended questions. An example of this category is when the teacher asks students to present their solving process: “Group four, can you tell us how you solved the problem?”

- Focusing actions involve the utilisation of teacher comments or questions that direct attention towards specific elements of the problem-solving or solution process. The objective is to facilitate deeper comprehension by prompting students to elucidate their rationale, substantiate their responses, or emphasise pivotal details. This approach is categorised into two distinct sections:
    - Requests for student input: the teacher asks students to explain details, justify their answers, apply knowledge to similar problems, or seek validation from peers. This category includes enlightening details of the solution process, justifying the reasoning behind the methods used, or applying it to similar problems, and requesting peer-assessment. An example of this category is when the teacher asks students to justify their answer by saying: “How did you arrive at this result?”
    - Pointing out: the teacher highlights or recaps key information. This category includes recap of the solutions or clarification of key points and noticing important details. An example of this category is when the teacher points out one possible strategy for solving problems: “Look, sometimes we have a big problem, one good strategy is to break it into smaller problems. It is easier to find a solution to small problems.”

Results

A key characteristic of the data in this study is that the majority of instances align with the IRE pattern (Cazden, 2001), characterised by the teacher's consistent intervention and response to students' comments. This also indicates that the teacher exercises control over the order of turns, speaking only after the students. Consequently, it was evident that the teacher played a pivotal role in guiding and advancing the class discussion. Considering the categories described previously, the following table presents the results obtained after analysing the data:

Table 1: Teacher actions according to the categories detailed by Drageset (2014) in the redirecting, progressing and focusing framework

Category	Subcategories		Teacher actions
Redirecting	Put aside		2
	Advising new strategy		2
	Correcting questions		3
Progressing	Demonstration		5
	Simplification		2
	Closed progress details		16
	Open progress initiatives		11
Focusing	Request students’ input	Enlighten details	3
		Justification	8
		Apply to similar problems	3
		Request peer-assessment	4
	Pointing out	Recap	19
		Notice	7
Actions of classroom management			9

A new category has been incorporated to the initial list: actions of classroom management. This category refers to the strategies and interventions employed by the teacher to maintain order,

structure, and an effective learning environment during the class discussion. It is important to note that the most prevalent teacher actions are recap (focusing), closed progress details (progressing) and open progress initiatives (progressing). Considering the nature of the activity, the open progress initiatives are often used to give students the opportunity to explain the process they followed to solve the problem. Closed progress details are used by the teacher to follow step-by-step solution plans, with the help of the students he advances to the solution of the problem. Finally, the most frequently used action is recap, in which the teacher often summarizes and rephrases the explanations provided by students. This is a key action as it gives the opportunity to make sense of what students explain and making it available for all the class. Two further significant actions are ask for justification (progressing) and notice (focusing). This first action is used primarily to ask students to explain some parts of the strategy used to solve the problem. Moreover, the action notice is used by the teacher to drive attention to certain details of the different strategies used to solve the problem. Another key action is demonstration (progressing), in which the teacher uses drawings, exemplifies the situation in the classroom, makes references of other areas, etc. This action encourages students to actively participate in the discourse, responding and demonstrating their comprehension of the situation.

In general, the most frequently employed actions in the classroom are focusing actions (52%). These are defined as teacher interventions, either in the form of comments or questions, that direct students' attention to key elements of the problem or solution process Drageset (2014). Given the complexity of the task, the teacher often requested students to provide more comprehensive explanations and justifications. This approach enabled the identification and clarification of their underlying strategies and reasoning processes. Additionally, the teacher regularly summarizes the discussion and emphasizes important aspects to ensure that all students can follow and engage with the discussion. This ongoing process of clarification, recapping, and highlighting key points not only ensured students remained on track but also cultivated a deeper understanding of the mathematical concepts being discussed. Through focusing actions, the teacher guides students in reflecting on their work while ensuring that the entire class remains involved and comprehends the ongoing discussion.

Discussion of the results and conclusions

In regard to the actions of teachers during a class discussion (RQ1), the study highlights that recap (focusing), closed progress details, and open progress initiatives (progressing) are the most frequently employed teacher actions. The frequent utilisation of recap actions indicates a focus on summarizing and clarifying student contributions to ensure collective understanding, consistent with Franke et al.'s (2007) emphasis on making mathematical thinking explicit. Additionally, this can also be connected to one of the key practices suggested by Smith and Stein (2016), connecting students' ideas and linking them between the different approaches. Other pivotal actions observed include justification and notice, whereby the teacher requests students to explain their reasoning and highlights critical aspects of the problem-solving process. Additionally, the utilization of demonstrations by the teacher, such as drawings and references to familiar topics, encourages all students to follow the class discussion and provides them with aspects to reconsider their strategies.

The teacher exerted a considerable influence on student participation and engagement (RQ2). The classroom interactions predominantly followed the IRE (initiation–response–evaluation) pattern

(Cazden, 2001), which is consistent with the findings of Drageset (2014). By balancing IRE sequences with open progress initiatives, the teacher created opportunities for more profound student reasoning and participation, a point also emphasised by Wells (1993) in his work on interactive classroom cultures. Nevertheless, the use of closed questions requiring brief responses resulted in greater student involvement, with a notable number of students offering their responses aloud. However, this type of engagement was less reflective. Therefore, it is essential to find a balance between open progress initiatives and closed progress details to ensure broad student engagement while still encouraging deeper reasoning and self-assessment.

This research offers insights into the dynamics of class discussions and the teacher's role after solving a Fermi problem in groups in the upper cycle of primary education. In the context of class discussions, it is essential that primary school students are provided with clear guidance and structure by their teachers, while also being afforded opportunities to elucidate their reasoning. Teachers may also facilitate comprehension by providing examples and connecting students' ideas to foster collective understanding. These teacher actions are crucial for fostering student engagement, promoting deeper reasoning, and ensuring effective participation in mathematical discussions. The findings of the study are based on a relatively small sample size, which may limit the extent to which they can be generalised. Furthermore, the analysis predominantly concentrates on teacher actions, with comparatively less attention devoted to student comments and interactions. Future research should consider a larger and more diverse sample and include a more comprehensive analysis of student contributions to gain a fuller understanding of the dynamics in classroom discussions. Exploring how students' comments and responses influence and interact with the teacher's actions could provide deeper insights into effective teaching strategies and student engagement in mathematical discourse.

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University students learning to model through large amounts of data, simulations and ChatGPT

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Digital technologies have been studied in the context of modelling for a couple of decades, with recent discussions focusing on the potentials and pitfalls of AI tools such as ChatGPT. In this paper, we present a preliminary analysis of university students' engagement with a modelling activity that encourages them to utilise various digital technologies to develop autonomy in applied statistics, a core field within data science, and a modelling field in itself. The activity is designed to initially draw on students' prior knowledge before requiring the acquisition of new knowledge. We observe varied approaches among students and point to further research needed regarding how to teach students to integrate AI tools in their modelling practices.

Keywords: Study and research paths, digital technologies, simulations, statistical modelling, tertiary education

Introduction

The impact of digital technologies on modelling education has been a recent focus (Siller, Geiger & Greefrath, 2023; Cevikbas, Greefrath & Siller, 2023). Siller et al. (2023) list the roles of digital technologies such as calculators, visualisation tools, dynamic representation tools, simulations, communication, data handling (collection or selection), learning materials developed for specific purposes including full learning environments e.g., virtual reality hardware and software. They identify four distinct ways of drawing on digital resources:

> extending mathematical capability: digital resources specific to mathematics […] digital resources not specific to mathematics […] accessing extra-mathematical knowledge: digital resources for communication and information gathering […] digital resources for simulated, virtual and augmented worlds. (Siller et al., 2023, p. 9).

Despite detailed research and discussions in each category, neither Siller et al. (2023) nor Cevikbas et al. (2023) address studies related to ChatGPT or other AI tools. This paper presents a study examining students' unrestricted use of digital tools to solve an open-ended modelling problem in a first-year applied statistics course for data science. This context is particularly suited to explore students' autonomous tool use, including AI, as it does not face the constraints of limited experience or reluctance to engage in digital inquiries (Cevikbas et al., 2023). The activity integrates statistical knowledge and techniques with extra-mathematical knowledge from a large data set. Students use digital resources beyond those specific to mathematics as well as testing their hypotheses and addressing the modelling tasks through simulations and digital resources specific to mathematics and data science.

Some domains of mathematical modelling are more strongly influenced by digital technologies than others. According to Greefrath and Siller (2017), simulation "serves to investigate an operation, a process or an experiment with the help of mathematical models […] usually in conjunction with a digital tool." (p. 530). When randomness is a factor in the experiment, they refer to these as stochastic simulations. The modelling activity in this paper aligns with this definition. Our aim is for students to employ digital tools and stochastic simulations, leveraging the potential that "statistical modelling can be comprehended holistically as a vehicle and platform for developing statistical competencies, statistical content, and social and societal decision-making" (Kawakami & Ärlebäck, 2023, p. 39). That being a more recent aim in statistics education as well (Pfannkuch et al, 2018) where:

> "The advent of data science has provided new challenges and opportunities for statistics education researchers to re-think what it means to model. […] How can new simulations and visualizations using technological tools increase students' access and agency for telling data stories, and enrich understanding in modeling processes?" (Noll et al., 2022, p. 331).

This aligns with the aims of the bachelor program in data science (IT University of Copenhagen, 2017). To address the holistic nature of modelling in relation to content knowledge, we analyse the activities from the perspective of the Anthropological Theory of the Didactic (ATD) (Chevallard, 2019; Barquero & Jessen, 2020). The notion of Study and Research Paths (SRP) provides a lens to examine the dialectic between study and research processes, showing how digital technologies support modelling from pragmatic and epistemic viewpoints. This approach facilitates students' learning of content knowledge across tertiary and secondary levels (Jessen & Kjeldsen, 2023), encouraging autonomous model construction and refinement through continued questioning and exploration (Jessen, 2017). While we have previously presented the a priori analysis of the task (Carlsen & Jessen, 2024), this paper focuses on two cases from its implementation, preceded by an elaboration on the theoretical framework.

Theoretical background

SRP is an ATD approach to modelling, and is initiated by a generating question, Q_0, which is an open question understandable to students, but it requires the students to engage in a learning process in order to be answered (Jessen, 2017). Learning in ATD is considered the process, where students pursue a question and recall prior or existing knowledge, A_i°, and seek to combine this knowledge in new ways to develop some new answers. However, questions may require the acquisition of new knowledge, meaning students will have to study works, W_j, decompose the knowledge they contain, and put it together with students' prior knowledge, $\sum A_i^\circ$, and reconstruct it when developing a personal answer to the generating question. We denote the final answer, $A^\heartsuit$. The collection of prior knowledge, works to be studied, data, D_n to be produced and analysed, together with questions driving the inquiry process, is called the milieu. Thus, according to Chevallard (2019) and Jessen (2017), we can analyse the learning process by identifying the didactic system (which consists of a group of learners, X_k, guided by one, y, or more teachers, Y, and the generating question at stake, Q_0) and its interaction with the milieu. In short, we describe the elements of the didactic system and milieu using the Herbartian schema: $[S(X_k; Y; Q_o) \rightarrow \mathrm{M}] \rightarrow A^\heartsuit$ The milieu can then be written as the set:

$$M = \{A_1^\circ, A_2^\circ, \dots, W_1, W_2, \dots, D_1, D_2, \dots\}$$

As described by Jessen (2022), both data and digital technologies can play the role of a media to be studied and the milieu in which answers are being developed through the reconstruction process of research (in the ATD meaning). Thus media, being elements of the milieu, can be web searches, videos, textbook materials, answers provided by other students and even newspaper articles. It can also be an existing model students seek to make sense of or a model they have produced in their modelling process, being evaluated, and improved through further study processes. Similarly, data can be both qualitative and quantitative, according to Chevallard (2019), and be something students produce or gain from other sources. As demonstrated by Jessen (2022), this allows students to engage in a dialectic between empirical and theoretical modelling and continued questioning and development of students' answers to the generating question. In particular, the continuously changing roles of digital technologies in a SRP seem to enable students' autonomous modelling processes.

This leads us to the research question of this paper: Using the Herbartian schema, how can we map the modelling process of the stochastic simulations? And how is technology and AI as ChatGPT furthering students' modelling processes?

Context of study

This study was conducted at the IT University of Copenhagen as part of a 15 ECTS Applied Statistics course, involving approximately 80 students in the second week of their second semester. The program aims to provide students with the skills to perform "data-centred, empirical analysis of real-world problems" (IT University of Copenhagen, 2017, p. 2) and develop expertise in statistical analysis, methodologies, and data-driven reflection. The curriculum emphasises cleaning, transforming, querying organizing, visualising data, and interpreting inferential processes. Admission is competitive, requiring a strong background in mathematics and English. The diverse cohort, with about 50% international students, mainly from Europe, reflects varying pre-university experiences. While some students may have prior exposure to stochastic or mathematical modelling, this cannot be assumed for all. At this stage of the course, students have learned basic R for simple data visualization and have used ChatGPT generating code and conceptual assistance but have not yet been introduced to statistical models or model testing.

Data was collected from recordings of plenum discussions, students' logbooks documenting their questions and investigated sources, observation notes, and reports submitted after both rounds of the tasks. This qualitative study uses discourse analysis to identify what questions students raise and how they develop answers, with or without digital technologies (cf. Jessen, 2017, 2022). The implemented activity is presented below.

Presentation of task design

The modelling activity reported here, is an SRP initiated by the generating question, Q_0: Are boys better at math than girls? We previously analysed its relevance for Danish upper secondary students, highlighting how it productively integrates digital technologies and links to a public debate showing the relevance of mathematics and stochastic modelling (Carlsen & Jessen, submitted). In a tertiary education context, this activity bridges applied statistics in secondary and tertiary education. More concretely, it supports teaching phases of statistical analysis and modelling, including exploratory

data analysis, descriptive and graphical examination, and selecting appropriate statistical models based on the data and research question (Ramsey & Schafer, 2012). These phases involve verifying model assumptions, fitting models using software to estimate parameters, performing hypothesis tests, interpreting statistical outputs in context, and assessing both the statistical and practical significance of the findings.

When engaging students with the modelling activity and Q_0, we have altered the design a bit compared to the previous proposal (Carlsen & Jessen, submitted). For the first part of the activity, students are required to design a study to determine if boys are better than girls, but they will not need to carry out the study. In the second part of the activity, students are provided with a data set, D_1, and asked to perform a statistical analysis. D_1 includes all final written exam grades in mathematics A-level from upper secondary school in Denmark, 2019, and the students' sex. Students were strongly encouraged to seek knowledge and inspiration from sources beyond their textbook and use of ChatGPT. When analysing the data set, students will find an average grade difference of 0.2 between boys and girls. The a priori analysis indicates that this will lead to questions about the significance of such a difference (Carlsen & Jessen, submitted). To address this, students must perform statistical analysis, including determining a suitable model that fits the context and data distribution and conducting a hypothesis test. Our a priori analysis anticipated that students would spontaneously utilize ChatGPT, as has been observed with SRPs in upper secondary school (Milthers & Wedderkopp, 2024). ChatGPT will propose various statistical models needing to be studied, emphasising the importance of meeting model assumptions and considering the study's context when selecting the appropriate model.

Results of analysis

Although the first part of the project includes valuable student reflections, our analysis will focus on the second part, where students conduct a statistical analysis. In this part, all groups have performed data visualisation and descriptive statistics. Some groups have explored various partitions of the data such as passed/fail, top-grade/non-top-grade, and more. For selecting a statistical model, many student groups have considered one-way ANOVA, while others have used statistical tests such as the Welch t-test and bootstrapping. In our study, we will analyse two cases: one demonstrating how students utilize their existing knowledge, and the other, where students use ChatGPT and other technology to study new knowledge.

Case 1 – use of existing knowledge

One group of students, X_1, begins by considering the number of data points in each group, male and female. After noting that the groups have different numbers of data points, they proceed by distinguishing between count and proportion respectively, which is existing knowledge $A_1^{\Diamond}$ and $A_2^{\Diamond}$. They first present a histogram, and a segmented bar chart based on count, utilizing previous knowledge, $A_3^{\Diamond}$ and $A_4^{\Diamond}$. The group concludes that, although there are more female students than male students in the data set, the charts show more male students receiving the lowest grade, and more female students receiving grades above. These charts provide insight into the distribution of the grades between the genders. We see that the previous knowledge $A_3^{\Diamond}$ and $A_4^{\Diamond}$ enriches the milieu by providing new insight into the data, D_1. Following, the students observe that the data approximates a

skewed normal distribution, with the second and third-highest grades being the most frequent. Though, the second lowest grade has a significantly higher count than expected in a skewed normal distribution. Based on their prior knowledge, $A_5^{\Diamond}$, regarding visualizations of the normal distribution. This further contributes to the milieu, leading to the conclusion that the data does not follow a normal distribution. Then the students standardise the data, i.e., considering the proportional distribution, transforming the data: $tr(D_1) = D_{2,X_1}$ based on $A_2^{\Diamond}$. Based on D_{2,X_1} they conclude that

> the plot shows that females outperform males in grades 3-5, while males take the lead in the two lowest and highest grades. So, the females' grades are more towards the centre while males are more spread out. (X_1 report).

This observation is confirmed by calculating the standard deviation for the two genders, another piece of prior knowledge, $A_6^{\Diamond}$.

> The standard deviation for males is 1.47 while the standard deviation for females is 1.39. This observation aligns with the pattern observed in our chart depicting the distribution of grades across both sexes. (X_1 report).

The students then describe the average grades for each gender, utilizing yet another element of prior knowledge, $A_7^{\Diamond}$. They conclude that, $A^{\blacktriangledown}$:

> Although the mean grade for males may be higher on average, the higher standard deviation suggests that there is more variability in the performance of the male students, with some excelling and others struggling. In contrast, females may have more consistent performance, with a larger share of grades clustered around the middle range, leading to a lower standard deviation despite a slightly lower mean grade. (X_1 report)

We can depict the study and research process of the student group, X_1, using the Herbartian schema to list how they draw on prior knowledge:

$$[S(X_1; y; Q_o) \rightarrow \mathrm{M}] \rightarrow A^{\blacktriangledown}$$

The milieu engaged with can be described as the set: $\{A_1^{\diamond}, A_2^{\diamond}, \dots A_7^{\diamond}, D_1, D_{2,X_1}\}$

We see that students study their own preliminary model of the data, D_1, the skewed normal distribution, to analyse the data, the uneven number of girls and boys, leading to the need for the transformation of data into visualisations allowing them to complete the analyses based on deviations and nuance preliminary answers, only based on average grades. Thus, the genuine open modelling task in terms of Q_0 and the proposed data, D_1, seemed to be sufficient in nudging the engaged students in the study-research dialectic building arguments for transforming data, and revising models, to better reach conclusions. Still, they did not engage in studying new works, nor explore proposals from ChatGPT.

Case 2 – exploration of technology and simulations

In our second case, we will focus on the use of technology such as ChatGPT, internet pages with distinctive authorship, and simulations, analysing their role in enriching the milieu by recalling existing knowledge. Our analysis focuses on group, X_2, consisting of five other members. According to the self-reported logbook attached to the project, the group used ChatGPT, YouTube videos and

various internet pages. There is a clear distinction between the type of knowledge learned from ChatGPT, a dynamic media, and from webpages and YouTube videos, a (relative) static media.

Initially, students asked ChatGPT broader questions, such as, Q_1"what kind of approaches are there to conduct a statistical analysis on a data set about grades". Based on this prompt, ChatGPT provided a list of ten different approaches: descriptive statistics, data visualisation, inferential statistics, regression analysis, multivariate analysis, longitudinal analysis, cluster analysis, machine learning, survival analysis and text analysis. Each approach and concept were briefly described by its advantages and possible applications. We consider this output from ChatGPT domains of work to be studied, W_1, W_2, W_3, …, W_{10}, which adds to the milieu by presenting an array of approaches and concepts. The group asked $Q_{1,1}$: "expand on inferential statistics", which means unfolding W_3 further. ChatGPT responded along the same lines by listing ten concepts and methods: hypothesis testing, confidence interval, sample distribution, central limit theorem, type I and type II errors, power analysis, effect size and assumptions. We consider this ChatGPT-supported enrichment of the milieu by introducing multiple new works, W_{11}, …, W_{20}, students needed to consider. Thus, we can describe the initial modelling process with the schema:

$$S(X_2; y; Q_o, Q_1, Q_{1,1}) \rightarrow \{A_1^\circ, A_2^\circ, \dots, W_1, W_2, \dots, W_{20}, D_1\}$$

We include also prior knowledge here in the milieu as the background for prompting ChatGPT.

Focusing on hypothesis testing, W_{11}, the students asked ChatGPT $Q_{1,1,1}$:"how do I go about hypothesis testing" (students of X_2). Here ChatGPT outlined the steps involved: formulate hypotheses, select a significant level, choose test statistics, etc., W_{21}, …, W_{30}. In this case, ChatGPT provided an overview of the process but also required further study. For instance, while ChatGPT defined the null hypothesis, it only provided a brief description of test statistics, W_{23}, replying "test statistic depends on the type of data and the hypothesis being tested. Common test statistics include *t*-test, *z*-test, χ^2-test, ANOVA, etc." (ChatGPT output). In the case of the null hypothesis, ChatGPT provided a definition directly applicable to data, D_1, which closed the above described path of studying various works, and students were able to reconstruct the knowledge required to pose the null hypothesis. Thus, we saw how the dialogue with ChatGPT enriched the milieu, and enabled X_2 to produce $A_1^\blacktriangledown$, being the preliminary and partial answer to Q_0, which is crucial for the data modelling. At this point, the students turned to other static media to study specific test statistics in further detail. E.g., they used a YouTube video to answer Q_2 :"What can a bootstrap analysis be used for? $Q_{2,1}$:How to perform it?", and a webpage for learning $Q_{2,2}$: how to "conclude on bootstrap analysis, the importance of confidence intervals" but also $Q_{2,3}$: "what are the shortcomings of a bootstrap analysis". This type of media provided specific, comprehensive knowledge on single subjects, $W_{31} - W_{34}$, enriching the milieu with specific techniques applicable to their context and crucial for data modelling.

In their project report, the students argued for using simulation to determine if the observed difference in means was significant, $A_2^\blacktriangledown$: "multiple random samples based on our already existing population…[will] provide an overall range within which we can expect the mean of our dataset to fall" (X_2 report). They performed a simulation of 10000 data points, generating a new data set, D_{2,x_2}, and two frequency plots of the mean grades for males and females respectively, which became partial

answers produced using R. Thus, we consider this the result of their research process, but also self-produced answer they needed to study further to answer Q_0. Drawing on their prior knowledge, they discussed the confidence intervals of the mean grades for both genders, concluding that the observed difference in grade means is due to randomness, $A^{\blacktriangledown}$. Thus, the knowledge developed by the students is tested against the milieu consisting of their prior knowledge, and new works studied from digital resources including their own productions using computer software.

Concluding remarks

This preliminary work aligns with calls for research on the opportunities for leveraging technology to broaden the learning context and foster autonomous, informed modelling practices (Pfannkuch et al., 2018; Noll et al. 2022) by examining students' free use of digital resources, to analyse complex datasets. The lens provided by the Herbartian schema, allows us to identify 3 types of influence on the milieu from technology: ChatGPT both broad domains to be studied, but also provides definitions. YouTube videos and webpages enrich the milieu with concrete techniques and methods to be used. Simulation enriches the milieu as a technique, and a model, with which we can test our hypothesis. This links to the purpose of digital technologies for modelling education both in terms of extended mathematical capabilities and accessing other resources relevant to applied data science and simulations. In this SRP as a task design tool (Barquero & Jessen, 2020) nudging the continuous questioning done by students (Jessen, 2017), seems productive to engage university students in stochastic simulations. Still, the two cases span a variety of approaches employed by the students in the applied statistics course. More research is needed into our data regarding more detailed analyses of students' study of works, $\sum W_i$, and how they use different digital technologies, but also why some students were reluctant to go beyond their existing knowledge. A hypothesis could be the effects of the didactic contract, which is known to affect students' actions in mathematics courses in tertiary education, especially when it comes to the use of resources beyond lectures and textbooks (Gueudet & Pepin, 2018).

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How do teachers manage the phase of horizontal mathematization during the implementation of a modelling task? A case study

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This paper is issued from a qualitative exploratory study preceding a larger research project aiming to study the implementation of modelling objectives in secondary schools in Geneva. In this paper, we propose to analyse how teachers manage the phase of horizontal mathematization when implementing modelling activities. The aim is to identify and characterize the conditions that foster the transformation of a real-life situation into a problem treatable mathematically. We propose a qualitative analysis of the modalities of the implementation of a well-known modelling task (The Giant's Shoes) in order to enrich the existing empirical results.

Keywords: Mathematical modelling, horizontal mathematising, mental representation, teacher's interventions, lower secondary school.

Introduction

Mathematical modelling is an important part of mathematical activity, however, its implementation and teaching is rather a complex issue (Blum, 2015).

For Blum (2015) "mathematical modelling is a cognitively demanding activity since several competencies involved, also non-mathematical ones, extra-mathematical knowledge is required, mathematical knowledge and, conceptual ideas are necessary, and appropriate beliefs and attitude are required, especially for more complex modelling activities" (p. 78). This may result in students potentially encountering difficulties because

> each step in the modelling process is a potential cognitive barrier for students, a potential "blockage" or "red flag situation" (Goos 2002; Galbraith and Stillman 2006; Stillman 2011). "The weakest link in their modelling chain will set the limits on what they can do " (Treilibs et al.1980). (Blum, 2015, p. 79)

In particular, Blum and Leiss (2007) "describe the transition between real situation and situation model as a phase of understanding the task" and consider it "as an important phase during the modelling process." (Vorhölter et al., 2019, p. 103). In addition, Borromeo Ferri (2006) emphasizes that in this transition "a mental reconstruction of the situation given in the problem takes place, which is rather on an implicit level and mostly unaware for the individual" (p. 92). For Schukajlow et al. (2023) an important characteristic of the situation model "is its holistic structure that allows making inferences during problem solving" (p. 261). Several empirical studies as reported by Schukajlow et al. (2023) "identified students' blockages while understanding the situation described in modelling problems" (p. 261) on the one hand and showed "a positive relation between understanding by constructing a real model and successful solution of modelling problems (Krawitz et al., 2021; Leiss

et al., 2019)" (p.261) on the other hand. On top of that, "neither reading strategy training, nor prompting understanding by using reading comprehension questions, showed positive effects on worded modelling problems in subsequent studies (Hagena et al., 2017; Krawitz et al., 2021)" (Schukajlow et al., 2023, p. 261). These results show that the complexity of constructing a real model and the resulting difficulties cannot be reduced to reading comprehension. It is therefore of utmost importance to continue to identify favourable conditions in which the transitions from a real situation to a mathematical problem take place.

Presentation of the context

In the Swiss education system, the French-speaking cantons, follow a common a curriculum "Plan d'Études Romand" (PER[1]), which was progressively introduced from 2011, with a central place dedicated to problem-solving and modelling. There is, however, a rather scarce documentation of modelling in the PER. Indeed, the only direct comment concerning the objective of modelling is that it is not worked per se but is activated through the other thematic axes. Modelling is also considered as a common objective for mathematics and natural sciences and a clear demarcation line is drawn. In natural sciences the essence of the approach resides in the modelling of the phenomenon, including the problematization of the situation, the formulation of hypotheses, the setting up of an experiment or repeated observations, the analysis of results and the verification of hypotheses by confrontation with reality but the mathematization is left out. On the contrary, in mathematics, the work focuses on the mathematical treatment of the problem which takes place after the "modelling". We thus identified an institutional gap in the mathematics curriculum and are asking what takes place "during the modelling" before the mathematical treatment of the problem. This gave birth to a collaboration with the Department of the public instruction in Geneva and led to the creation of a working group composed of two researchers and four lower secondary mathematics teachers (two of them are also science teachers) in order to develop complementary material and accompanying resources for the implementation of the modelling objective in the Geneva Canton. This project started by an exploratory empirical case study in which we wish to document teachers' spontaneous practices during mathematical modelling as a starting point for the collaboration. In this paper, we present some preliminary results issued from this exploratory study concerning the implementation of two modelling tasks in May and September 2024 in two lower secondary classes (students aged 14 – 15 years old) with two different teachers. The second task, whose detailed analysis is proposed in this paper was conducted with one of the teachers and the same class.

Theoretical background

In this work, we refer to the research of Yvain-Prébiski (2021) and in order to conduct our analysis, we use her adaptation (Figure 1) of the modelling cycle (Blum & Leiss, 2007).

Indeed, Yvain-Prébiski (2021) characterizes *the horizontal mathematization* and, building on the work of Israël (1996), further describes the process recognizing two phases. From an extra-mathematical situation, it is through the selection of *fragments of reality* characterizing it and the

[1] https://portail.ciip.ch/per/learning-objectives/151

identification of their *relevant aspects* in relation to the question that it is possible to elaborate *a problem susceptible to mathematical treatment (PSMT)*. Then, continuing from this *PSMT* and through establishing relationships between the selected aspects, *the mathematical problem* takes shape. According to Yvain-Prébiski (2021), these two phases correspond to stages 1 (construct/understand) 2 (simplify/structure) and 3 (mathematize) of Blum and Leiss's (2007) modelling cycle. This "zoom" on the first three transitions of the modelling cycle provides a more precise characterization of what is done within these steps.

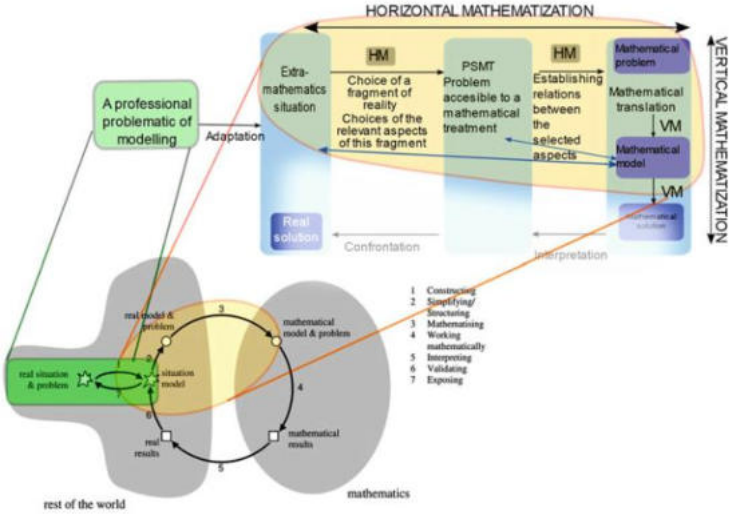


Figure 1: Diagram describing the modelling process (Yvain-Prébiski, 2021, p. 309)

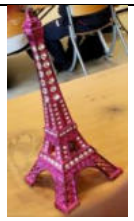
These theoretical elements enable us to specify what is institutionally related to by "after the modelling" in the PER as the *horizontal mathematization* and formulate the following research questions: When proposing a real-life situation, how do teachers manage the horizontal mathematization? More precisely, how can teachers facilitate the construction of a mental representation of this situation, the identification of different fragments of reality, the selection of the relevant aspects leading to the construction of a PSTM, and the establishment of connections resulting in the formulation of a mathematical problem?

Methodology of the research

In this exploratory study we observed the implementation of two tasks: Giant's shoes (Blum & Borromeo Ferri, 2009, p. 45) provided in Figure 2 on the left and the Eiffel tower, where a small souvenir (Figure 2 on the right) was brought to the classroom and the students were invited to question its fidelity towards the real Eiffel tower.

During the preparation, the teachers (Tom and Noah) were aware of the potential difficulties concerning the horizontal mathematization and we left it up to them to manage this phase as well as the possibility to make adaptations to the proposed activities. Tom implemented the Eiffel tower task and the Giant's shoes task in the lowest performing track class whereas Noah explored the Eiffel tower task with his scientifically oriented high-performance students. There were 12 students (14-15 years old) in each class.

Example 1: "Giant's shoes"
In a sports centre on the Philippines, Florentino Anonuevo Jr. polishes a pair of shoes. They are, according to the Guinness Book of Records, the world's biggest, with a width of 2.37 m and a length of 5.29 m.
Approximately how tall would a giant be for these shoes to fit? Explain your solution.



How faithful is this souvenir?

Figure 2: The proposed tasks (Blum & Borromeo Ferri, 2009, p. 45)

Concerning our data, we have an audio-recording of the Giant's shoes task, a video recording of the Eiffel tower lessons as well as the students' work captured through action cameras. This data was completed through pictures of the open board and of the white board and the observer's notes. and all recordings were transcribed. Due to the space available we only propose the analysis of the Giant's shoes task in Tom's class.

Data analysis

Introduction of the situation

Tom decided to allocate one 45-minute period for the implementation of this task. To introduce it, Tom displayed on the open board (OB) only a photo of the shoes and a sketch showing their dimensions (Figure 3) deliberately leaving out the question.

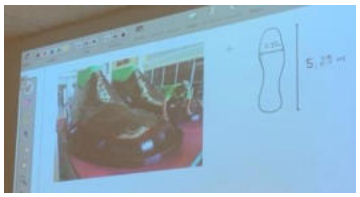


Figure 3: Projection of the real situation

Below is a transcript showing how Tom introduced the task to the students:

1	Tom:	What's important when you're doing math is knowing how to ask yourself good questions, so I'm not going to give you anything... Can anyone describe the picture to me (photo of the shoes displayed at OB)? Eric?
2	Eric:	slightly deformed shoes
3	Tom:	shoes a bit deformed, ok. What do you see in this picture? How do you understand this picture?
4	Students:	big shoes, giant things

This extract clearly shows Tom's vision of mathematics (line 1) as a discipline in which it is important to pose good questions giving thus a "real status" to the horizontal mathematization. Tom asks the students to describe the picture displayed on the OB. At this stage, we are facing an extra-mathematics situation but no question has been asked.

Construction of the mental representation of the situation and constitution of the PSMT

In order to encourage his students to formulate questions about the situation, Tom choses to tell a story providing the students with a sort of a mental experience of the situation (line 12) which

contributed to the construction of the mental representation of the situation and resulted in the formulation of the first questions about the situation (lines 13 and 15). The formulation of questions about the situation created favourable conditions that helped the students to make the first choices of the fragments of reality. Afterwards, Tom guides them to focus on the relevant aspects (line 16). This intervention contributed to a better understanding of the situation on the one hand and enabled the students to start making connections between these aspects (lines 21 – 23) on the other hand. The extract below illustrates our previous points:

12	Tom:	Put yourself in someone else's shoes, imagine you're an archaeologist and one day you find footprints of these shoes in the forest. What questions do you ask yourself?
13	Student 1:	who made them?
14	Tom:	who made them? *Tom writes the question on the board*
15	Student 2:	who's going to wear them?
16	Tom:	who's going to wear them? Why is this an interesting question?
17	Student 3:	no one wears shoes of this size.
18	Tom:	no one wears shoes of this size, so why did we make these shoes? these shoes have to fit someone?
19	Student 1:	it's artistic
20	Tom:	it's artistic all right, but let's put ourselves in the shoes of someone who finds these shoes and wonders who's going to wear them?
21	Student 4:	he must be a size 45.5
22	Tom:	Jay says he's a size 45.5, I'm a size 46.
23	Student 5:	size 100?
24	Tom:	size 100? What's the question about what you're saying?
25	Student 5:	foot size
26	Tom:	what is it? there's a term
27	Student 6:	shoe size
28	Tom:	size. *(writes the question on the board)* obviously 46 no, because I'm a size 46 and my feet should be 5.29 meters long.

Indeed, the mental image of an archaeologist finding these footprints brought up the idea of "someone who would wear them" (line 15) and his foot size (line 21). At this stage, we are witnessing a veritable back-and-forth between the phase of creating the problem, that of building a mental representation of the situation and the beginning of making connections between the relevant aspects of the chosen fragments of reality.

Later on, other questions about the giant's height and weight arise from the students. The choice of fragments of reality such as: shoes → giant → but what kind of a giant? and the focus on the relevant aspects → what size and what weight? provide evidence of the students' understanding of the task and give substance to the problem.

In the intervention below, Tom insists on the importance of posing questions in order to solve this type of problems, provides positive feedback to the students qualifying the questions as relevant and makes a summary on the advancement of the problem:

45	Tom:	So, let's say that what's important in this kind of problem is to be able to ask yourselves questions - I'm showing you shoes like this (*referring to the picture on OB*), there are no questions - but to be able to pose questions like the three you gave me are relevant. What's the size of these shoes? How tall

is he? I think that's interesting, what's his mass, his weight? That's interesting too.

The modality of letting the students pose questions about a real-life situation results in a variety of choices of fragments of reality: some of them are interested in the height of the giant while others mention its weight or shoe size. It also seems that the mental experience (imagine yourselves as an archaeologist) orchestrated by the teacher resulted in the mental image of the "person who would wear these shoes", providing thus an important fragment of reality. Therefore, these two intertwined elements (asking questions about the situation supported by a mental experience) seem to have contributed to the construction of the mental representation of the situation and thus to the understanding of situation and created favourable conditions for a variety of fragments of reality to emerge. Indeed, for some students it was not even relevant to investigate the height of the giant (*I don't care about the height […] I don't bother*) and would probably not engage in the problem in the original version.

The constitution of a PSMT and the mathematical problem

Once the three questions are formulated (line 45) and marked on the OB, Tom asks the students if they could find an exact answer to this problem and the students respond collectively that it is not possible. This intervention shows even though in a very implicit way that Tom wanted to make the students aware of the fact that the answer would depend on certain amount of assumptions. This question also in acted as a trigger for the gradual transformation of the extra-mathematical problem into a *PSMT*, which also continues with Tom's intervention:

51	Tom:	What else? Eric said it earlier, Jay said it, Gill said it, who wears size 40, raise your hand! 40 no one, 41, 42, 43 (*4 students raised their hands*), look around you, so you have the same foot size. Do you have the same height?
52	Students	No
53	Student	Approximately but not exactly the same height

Indeed, Tom asked to students with the same shoe size come up to the whiteboard and made a mark of their height showing a certain variability but also a proximity.

This intervention (line 51) also helps the students focus on the relevant aspects of the chosen fragments (height of a human being, shoe size) and start to make connections between them: making appear that there is a relation between the height and the length of the foot or the shoe size of a human being (line 58). These two elements led students to evoke the concept of "proportion" even if in a very naïve way (line 58):

56	Ben:	So, if the shoe is 2 meters long, for example, you (*referring to Tom*) are 1 meter and …?
57	Tom:	80
58	Ben:	1.80, you have a small foot, he (*the giant*) has a big foot (*inaudible "proportion"*)
59	Tom:	did you hear what Ben said?
60	Students:	yes
61	Tom:	could someone repeat it? It concerns what...?
62	Students:	proportion
63	Tom:	louder
64	Students:	proportion

65 Tom: proportion, what Ben said was that my feet are quite small compared to my height, but these shoes should be in relation to his height. That means he'd have to be really big if the proportions were the same.

This last moment of the collective discussion finalizes the construction of the PSMT and students begin to work in groups of 3-4.

During the group work, various strategies based on the choice of different fragments of reality and different relations between the selected aspects appeared. Some groups focus on the relations between their height (using a meter given by the teacher) and their shoe size while other groups consider the relations between their shoe size and the length of their shoes as shown in Figure 4 (left).

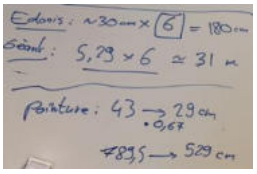


Figure 4: Students' procedures and results

In both cases, students create and solve a *mathematical problem* by using the concept of proportionality (*mathematical model*) as illustrated during the collective moment of sharing different solutions (see Figure 4 on the right).

In this last intervention (line 121), Tom handles the *interpretation* of the mathematical results and validates the coefficient 0,67 as a *real result* and invites the students to check it out on the internet.

121 Tom: his shoe size is 789, he's a 789 and what I found out, and I didn't know it but this is the real thing, is that if you check on the internet how you go from foot size in cm to shoe size, well you have to divide by 0.67. It was exceptional, you've done a great job, I'm delighted, we could go on like this for hours.

The interpretation and validation of the results was quickly handled by Tom at the end of the lesson. Because of the lack of time, Tom did not address the assumptions under which the model was chosen or question the empirical data and their effect on the result, nevertheless, he was aware of it and addressed this issue when implementing this same situation in another class.

Discussion and conclusion

This exploratory study enabled us to observe spontaneous practices of two teachers concerning the implementing two extra-mathematical situations (one of which was analysed in this paper) and identify some practices that created favourable conditions helping students to transform a real-life situation into a mathematical problem. The fact that Tom exposed the real situation without any question ("describe the picture") and supported it by a mental experience ("imagine you are an archaeologist") seemed to have contributed to the construction of a mental representation of the situation. All the more this was reproduced by the same teacher in another class for the situation "Giant's shoes" and in the same class for the situation "Eiffel tower". Another promising path are the interventions in the form of questions that help students sort out the fragments of reality and orient them towards relevant aspects of these fragments, permitting the construction of a PSMT and finally the formulation of at least two mathematical problems. The experimentation of the Eiffel tower

problem further highlighted the importance to dedicate enough time for students to build a mental representation of the situation before selecting the fragments of reality and identifying their relevant aspects that are indispensable for construction of the PSMT. Indeed, in Tom's class students' understanding of the situation affected positively their engagement in the task even though they encountered difficulties because their mathematical knowledge was fragile or not always available. On the contrary, in Noah's scientifically oriented class where the construction of the mental representation was left up to the students, we observed serious difficulties in identifying fragments of reality their relevant aspects leading to the recognition a proportional reduction in the pink souvenir of the Eiffel Tower as key argument of fidelity. This analysis confirms empirical results that a well-aimed balance between students' independence and teacher's guidance contribute to a successful resolution of a modelling task even in a low performing class on the one hand and raises the awareness to not to underestimate the treatment of the horizontal mathematization the high-performing classes.

These results open a new perspective of questioning the transferability of these elements to other tasks and other teaching contexts. To do so, we continue to explore, for other tasks, the effects of an "induced mental experience" on the construction of the mental representation of a situation and the selection of fragments of reality and their relevant aspects on the one hand and to what extent this could be anticipated *a priori*, incorporated in a teaching resource and adapted by other teachers, on the other hand.

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Assessment in a two-dimensional curriculum: The case of solid geometry and modelling supported by an empty modelling cycle

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This paper reports from a study where groups of 9th graders were given opportunities to show their modelling competency. The goal was to create practical feasible ways of assessing both dimensions of a two-dimensional curriculum. An empty modeling cycle was used to support students in the process towards a video they produced for assessment. These first trials show that the empty modelling cycle was used by students to show their insight into modelling.

Keywords: Modelling, assessment, two-dimensional curriculum

Introduction

Many countries now have a two-dimensional curriculum in mathematics, as in Norway, which incorporates both content-oriented objectives and process-oriented objectives, such as mathematical modelling (Højgaard & Sølberg, 2019). In Norway, the legislation was adapted to the two dimensions with the new curriculum in 2020 by stating that the content-oriented objectives should be assessed "in terms of" the process-oriented objectives. The process-oriented descriptions should therefore somehow be assessed together with the mathematical content, without the curriculum explaining how this should be done.

The process-oriented dimension of modelling of the Norwegian curriculum is explicated as:

> A model in mathematics is a description of reality using mathematical language. The pupils shall gain insight into how mathematical models are used to describe everyday life, working life and society in general. Modelling in mathematics means creating such models. It also means to critically evaluate whether the models are valid and what limitations the models have, evaluate the models in view of the original situations, and evaluate whether they can be used in other situations. (Utdanningsdirektoratet, 2020, para. 2).

This description closely aligns with many descriptions or models of modelling processes. In the modelling cycle presented in Blum & Leiß (2007), seven steps are described: *constructing*, *simplifying/structuring*, *mathematising*, *working mathematically*, *interpreting*, *validating* and *exposing*. This model is developed as an aid for cognitive analysis of modelling tasks. These steps do not follow each other linearly or in a tidy cycle, but the students follow different modelling routes. However, the authors clarify that this model of a cycle is not meant to be used as an aid for the student. As a substitute, a simpler cycle is preferred and seems to be more appropriate for students. Beckschulte (2020) and Schukajlow (2015) used such simpler cycles as aid for learning modelling.

We have developed a technique using an *empty* modelling cycle as students' aid for this process of making meaning of modelling (Eriksen & Madsen, 2023). In short, during modelling activities the class do time-outs and students explain what they are doing and choose where to place their current actions in a simplified description of a modelling cycle. As this emerging modelling cycle becomes

a tool for learning modelling, it seems also possible to use it as a tool for students to show their modelling competency. This paper therefore explores: How do the students use the empty modelling cycle to show their insights of modelling competency?

The challenges given to the students were all within the content area of solid geometry. To answer the research question, we then need to describe what we mean by modelling competency, and the nature of the relation to solid geometry.

Assessing modelling x solid geometry competency

Maaß (2006, p. 139) concludes that sub-competencies should be paid attention to in class, but also that her students developed a holistic meta-perspective in the latter part of 15 months of modelling activities. Cevikbas et al. (2022, pp. 207-209) also divides into an analytical bottom-up perspective to modelling and show how this differs from a holistic top-down perspective. Freid (2013) calls for a better theoretical foundation of assessing modelling and we seek to address his challenge by starting from the top-down theoretical landscape of Niss & Højgaard (2019) where modelling competency, and its relation to solid geometry, precisely, but still in a general holistic sense, is combined in a matrix that shows competencies × mathematical areas. We connect our holistic departure to the atomistic descriptions from Blum and Leiß (2007) in Table 1.

The KOMPIS project in Denmark has shown us that it is possible to conduct a large summative oral assessment where students demonstrate their modelling competency by working with a mathematical area (Højgaard, 2023). While the KOMPIS project demonstrates that assessing modelling competency "top-down" is possible, it also describes assessment practices which are both comprehensive and time-consuming. In the project, Højgaard (2023) emphasizes that the main focus is on high validity, i.e. does the assessment measure what it is supposed to measure? This can be achieved by ensuring that what students demonstrate in the assessment situation aligns with the learning objectives in the prior instruction (Biggs & Tang, 2011).

Reliability in this context means making the assessment transparent and independent of who is doing it (Højgaard, 2023). Coherent assessment amongst who is doing it, will necessarily increase reliability, and can therefore be achieved by making explicit the answers to questions like: What is a model of solid geometry and what is modelling x solid geometry competency?

As we know well, an algebraic expression or a formula can be general descriptions of reality, for instance, in solid geometry, $2ab$ or $\sqrt{3}a^2$ could be a model for the surface of a tetrahedron-shaped food container. However, the definition includes more. With a tetrahedron at hand $10\sqrt{27}$ is also a description of the reality, or even the number 32.8 is a model for the same reality. An algorithm in a programming language, or a textual/oral/pictorial description of how to find a surface area would most probably include mathematical language and are therefore also models, e.g. "Four times the area of the triangular sides". However, the students' *actions* leading to these models may not portray the same modelling competency and, as we explain below, the students' *insight* may be different.

Niss & Højgaard (2019) define "A *mathematical competency* is someone's insightful readiness to act appropriately in response to a *specific sort* of mathematical *challenge* in given situations." (italics in original, p.14). They discuss how this definition is oriented towards action and that this action can be

done with or without insight, but the latter do not convey the same competency. The challenge depends on the practice the activity is embedded within, and hence, is both related to the individual and the social and cultural context. Keeping in mind this notion of *challenge*, we define for our project:

Mathematical modelling × solid geometry competency is someone's insightful readiness to act appropriately in response to a challenge of modelling x solid geometry.

In a classroom situation it will be the teacher's role to decide what is appropriate, whereby the students will show their insightful actions. Of course, the duality of action and insight are closely weaved, and the two-sidedness of the conceptualisation in the table below is mainly a construct to create a more reliable assessment. The students' insight is assessed by what they manage to show, but the descriptions in the Table 1 are of a cognitive nature. Hence, in their form in the table, the insights are not assessable, but the students may show their insight by language, writings, drawing or gestures. For example, the upper left cell in Table 1, describing insight into pre-mathematics includes "Knowing that the challenge...." and "Being aware that ..." but none of these are assessable. Nevertheless, they both could lead to some action that can be assessed. In our examples where the student orally explained their actions, often insight was made assessable by language.

Table 1: Modelling competency × solid geometry

Phase	Insight	Action
Pre-mathematics	Being aware of this phase of structuring and simplifying. Knowing that the challenge is to show modelling x solid geometry competency. Knowing how simplifications affect the model and results. Being aware of the phases to come e.g. planning for them. Consider alternative assumptions and choices. Knowing that the physical model provided is just one example.	Make sense of the situation and construct a situation model. Make assumptions and choices to simplify the problem, in order to mathematise into solid geometry, e.g. finding shape rather than weight of a potato. Finding relevant measures.
Mathematics	Being aware that working mathematically is separated from the other. Thinking of calculations and numbers as models, e.g. discussing how numbers change if measures or shapes were simplified differently. Knowing how mathematics is a simplification of the situation and that the model has limitations.	Finding correct measures. Do calculations and do them correctly. Working mathematically with area and volume and using mathematical operations correctly. Using geometrical units and using them correctly. Make a model that is appropriate to measurements and the situation.
Post-mathematics	Being aware of this phase. Knowing that models and results is based on assumptions and choices from earlier phases, e.g. accepting that there will not be one correct answer to this activity. Thinking "what if...". Consciously consider actions from other phases of the cycle.	Using mathematical results to conclude. Interpreting the mathematical results and comparing model and situation. Validating answers by checking calculations and units. Recommend or expose solutions to the real situation.

The table is not meant to be used by the students, but rather by teachers and researchers as an aid to establish what would constitute as good modelling competency regarding solid geometry. The table highlights which insights and actions would be connected to an adequate modelling competency. By translating the steps of the modelling cycle into insights and actions it could help determine what to look for when assessing modelling competency. To align the insights and actions to student work, we vertically divide the table of modelling competency to the simplified empty modelling cycle of just three phases used by the students (Figure 1). And while insights and actions showing modelling x solid geometry competency are described a priori, it still remains for teachers to interpret students' various actions. We have seen a way of operationalizing the holistic definition of modelling competency in relation to a mathematical subject area and the sub-competencies of modelling. In this paper we do not focus on students' actions but their insight.

Method section – the tasks and the empty modelling cycle

Two 9th grade teachers in two different schools participated with three researchers in the study. We conducted three teaching sessions in both schools, six teaching sessions in total, all including one modelling challenge. The students worked in groups of three. In the third session within each school, the groups were given a task in which they had to submit a cellphone video on how they responded to the challenge. The video was assessed by their teacher and the researchers and was given a numeric grade. The students were told that the assessment of their solution of the task would play a part in their year-round grade.

All tasks were invitations to develop mathematical modeling competence, and all tasks were meant for short terms, approximately 60 minutes. All the tasks were selected as knowledgeable situations for 9th graders and, and they were all supported by a physical example. The two training tasks were formulated as "How many dices could you put into a cup?" and "How long strip could you get from a toothpaste container?". The assessment task was given orally as: How many French fries do you get from a potato?

The assessment task is similar to a task found in Reit (2018, pp. 180). The physical example for the assessment task was a potato for each group to use as a starting point, and the students had access to a tape measure and a kitchen scale.

Blum & Borromeo Ferri (2009, pp. 54) states that "[…] teachers have to know ways how to support adequate student strategies for solving modelling tasks". The 7-step model is suitable for teaching and research purposes, but not necessarily as a strategic aid for students in problem solving. We introduced a simplified empty cycle consisting of three phases, i.e. no explanations of the phases were given in the cycle, see Figure 1. The empty cycle was to be used practically in work with modeling tasks. Both in work on the first two tasks, and as part of the presentation of task 3, the students used the empty cycle. In the videos that were to be assessed, the groups showed their modelling cycle and explained what they had done in the various phases. We made a qualitative content analysis of the videos.

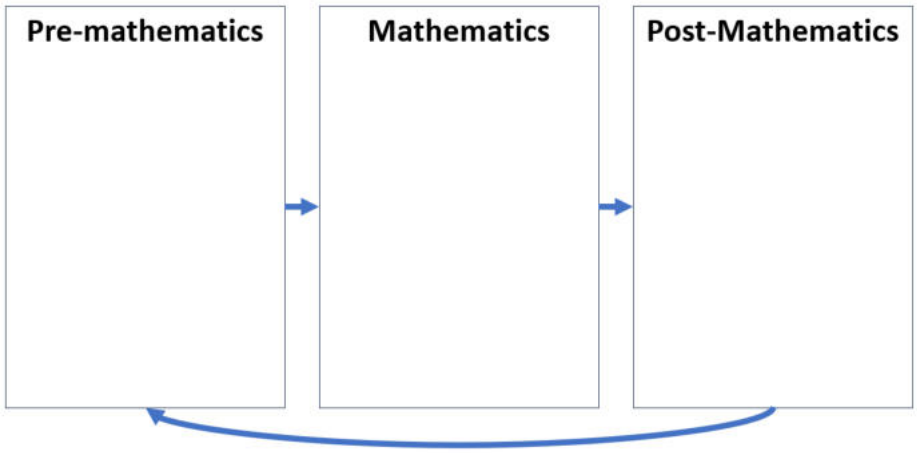

Figure 1: The empty modelling cycle

The empty modelling cycle was introduced as an aid both for the teachers and students. Firstly, we thought that it would help the researchers and teachers to get an insight into the students' modelling competency, and how they worked within the different phases of the modelling cycle. It was therefore useful in the assessment process, as one could see specifically what competence the students showed in the different phases of the modelling cycle, and one could compare this with what we had categorized as satisfying answers beforehand. Secondly, the empty modelling cycle was introduced as an aid for the students to organize and be aware of their modelling activities and prepare for assessment. As Blum & Borromeo Ferri (2009) suggest, a simpler model would be appropriate as an aid for students when working with modelling problems, and the empty modelling cycle is an example of this. By structuring their solution process into the three phases, it would be easier to explicitly illustrate their pre-mathematics work, their mathematical work, and their post-mathematical work.

Results - use of the empty modelling cycle to show insight

The empty modelling cycle was introduced to the students as an aid to both working with the modelling tasks and showing their insights and actions in terms of an assessment situation. We present here three episodes where students plan ahead, consciously navigate in the modelling cycle and show awareness of how their assumptions affect their results.

Firstly, the students used the empty cycle as an aid to determine what they needed to do to solve the problem. In the videos several groups reported on how they made a plan for what to come.

> Student: OK, so before we started the calculations, we started with measuring the potatoes and made a plan to find how many french fries you get from a potato.

When using the empty modelling cycle, it is evident that the students were encouraged to think about a plan to solve the problem. They had to consider what bits of information were needed to fill out the empty cycle, and what bits of information they were missing.

Secondly, it can be hard for both teachers and students to determine which step in the cycle one might find themselves in at a given point during the problem-solving process. Our findings suggest that with

the use of the empty modelling cycle, the students were aware of their whereabouts in the modelling cycle and became self-aware about their movement from one phase to another.

Student 1: And then we went back and started cutting the potato and set new measurements [Points at "pre-mathematics" in the poster].

Student 2: Yes, and then we cut the potato in half, so then it was easier to measure the area in the middle, so we measured new measurements, and then we drew it on the potato, to check, and then we calculated the same way as we had done previously, only with the new measurements.

Student 3: And then we looked at the new answers we had received, and then we looked at the difference between before and after we had cut the potato. And then what [answer] was most realistic.

Here we see that the students actively refer to the cycle, using terms like "we went back" to refer to their position in the cycle. This may suggest that the cycle works as an aid to raise the students' awareness about their own modelling process, and the cycle is also used as an aid to show their insights.

Thirdly, our findings also suggest that by using the empty modelling cycle, the students tend to evaluate their answers in an explicit matter. The box "post-mathematics" encouraged students to explicate their thoughts about their solution process, and what their answers would mean in an authentic context.

Student: It may not be the best model. So, it doesn't work if there are ugly potatoes. They must be round or oval potatoes for it to work.

This group shows insight by distinguishing between the complex aspects of real life and the precise matter of mathematical calculations, and they also proceeded to present a more general formula that could be used for any given potato:

Student: And then the formula. It's the weight of the potato, minus 10%, because it is some redundancy, potatoes are not perfect. And then divided by 4.3, and then you get the number of fries.

The group is both aware of their own model's limitations and criticizes this, and also presents a more general answer, by commenting on how one could solve a similar problem with any given potato, and not just the specific potato the students had in hand.

Ferri (2007) states that the modelling cycle is not linear, and it is not unusual for the problem solver to create their own modelling routes. It can also be necessary for the problem solver to adjust assumptions and go round the loop a second time (Blum & Borromeo Ferri, 2009, pp. 47).

Conclusions

The episodes illustrate how these students used the empty modelling cycle to show their insight into modelling. Firstly, the empty modelling cycle gave the students a predetermined structure that helped

them to plan their work. They showed insight into modelling by putting that insight forward for assessment. Secondly, the empty cycle raised the students' awareness of the last phase of modelling, knowing they were expected to present to the post-mathematics box, and there they evaluated their model against the origin of the situation. Thirdly, we see how the students consciously and insightfully moved back and forth in the modelling cycle, looking at assumptions when validating their model.

The results show that the students use the empty cycle to navigate through the modelling cycle, and that there is a need to go round the loop and adjust the initial assumptions. These results also correspond to the findings from Ferri (2007). The use of the modelling cycle encourages explicitly to this, and the students in this study used it as an aid to navigate through the loop.

As Blum & Borromeo Ferri (2009) suggests, teachers must know how to support adequate student strategies for solving modelling tasks. Introducing and using aids to do modelling tasks are examples of such strategies, and this study shows that the empty cycle can be used as such a tool.

The tasks turned out to be feasible and suitable for the three 60-minute lessons leading to the video for assessment. Other tasks may challenge students more in all phases of the modelling process, especially before and after the calculations are done. The empty cycle seems to be a tool to raise awareness and keep track of modelling activities – both for students and teachers.

When we graded the students together with the teachers, we experienced high reliability through compliance in our gradings and discussions. Our scheme for grading, based on Table 1, seems to be an important factor in this. Explicating insight in a scheme helped us to know what to look for when the students managed to turn their insight into action. But luckily, students continue to surprise us and come up with ideas we did not think of in the first place.

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How do pre-service teachers interact with IA to solve modelling problems? Characterising the use of ChatGPT in solving Fermi problems

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In recent years, the use of AI systems has intensified, particularly in the field of education. It is a tool that could be useful in solving modelling problems, such as Fermi problems. These types of problems are suitable for introducing modelling in primary education classrooms. In the present study, we want to explore how pre-service primary school teachers use ChatGPT when solving two Fermi problems. The qualitative analysis of the interactions with ChatGPT and the written solutions produced by 94 pre-service teachers allows us to categorise the prompts by distinguishing categories that come from the prompting techniques from those that emerge from the phases of the modelling cycle. Through this system of categories, we characterise the use of AI by distinguishing three levels depending on the degree of responsibility given to the AI (from full delegation to instrumental use) in the modelling process: use of ChatGPT as problem problem-solving expert, assistant and support.

Keywords: Modelling, Fermi problems, Artificial Intelligence, pre-service teachers, prompts.

Introduction

Integrating Large Language Models (LLM) in the classroom –on which ChatGPT generative artificial intelligence (AI) is based– is a topic of current debate (Lo, 2023). The novelty and potential of this technology raise challenges and opportunities for education at all levels (Zhang & Aslam, 2021). Its use could facilitate teachers' teaching, for example, through automated assessment of their students' responses, adaptive feedback and content generation (Ausat et al., 2023).

In Mathematics Education, generative AI could be used to support problem-solving, including mathematical modelling activities, thus facilitating its effective implementation in the classroom. One proposal for introducing mathematical modelling in primary education is to use Fermi problems (Ärlebäck, 2009). Fermi problems are a type of modelling problem in which a question is posed in a real context, without data, requiring reasoned estimations and simple calculations to reach the answer.

Consequently, pre-service teachers need to be familiar with Fermi problems, including their solutions and application in the classroom. Our purpose is to explore how pre-service primary school teachers use ChatGPT to solve Fermi problems. To this end, we characterise how they interact with ChatGPT to develop the models generated by solving Fermi problems.

Fermi problems as modelling activities

Ärlebäck (2009, p. 331) describes Fermi problems as "open-ended, non-standard problems that require solvers to make assumptions about the problem situation and estimate relevant quantities before often performing simple calculations". Since these problems ask for an estimation without providing specific data of the real situation, their solution unfolds a process that can be interpreted as

the development of a mathematical model (Robinson, 2008). The process follows the modelling cycle (Blum & Leiß, 2006), as it requires: (i) understanding, identifying and structuring variables of the real situation; (ii) constructing a mathematical model by estimating these variables and mathematising their relationships or structures; (iii) working mathematically within the model to obtain the final estimate; (iv) validating the estimate in reality, being able to revise the model and incorporate new realistic assumptions that, when quantified, can provide greater accuracy to the final estimate. Although Fermi problems' solution process follows the modelling cycle, it can be considered "miniature", as it requires simple calculations, so authors such as Albarracín and Gorgorió (2014) have highlighted its accessibility, recommending its use as an introduction to modelling in Primary education. However, the results of Segura and Ferrando (2021, 2023) show that pre-service teachers have difficulties when solving Fermi problems. Perhaps AI support can help overcome these difficulties (Noster et al., 2024), but the first step is to understand how this tool is used in solving Fermi problems, which to our knowledge has not been studied previously.

Use of generative AI in problem-solving

It has been shown that LLMs can solve some mathematical problems, depending on their complexity and the mathematical domain to which they belong (Schorcht et al., 2024). However, AI can confusingly solve them, due to the lack of transparency in its decision-making processes, and because its answer can be plausible, but incorrect (Noster et al., 2024). In modelling problems, where context plays a major role, research by Spreitzer et al. (2024) concludes their open-ended nature does not appear to have a significant impact on the modelling capabilities of AI, although the greater complexity of the context may affect its reliability.

An important aspect of the use of AI for problem-solving is the interaction through prompts, namely questions asked to the LLM to progressively obtain the solution (Noster et al., 2024). The literature on prompt engineering details the main prompting techniques. In the Zero-shot technique, the AI is presented with the complete problem, seeking to obtain a direct solution (Kojima et al., 2022). The Few-shot technique involves giving the AI some prior instructions to improve its response by conditioning the model with a few examples (Snell et al., 2017). It has been shown that AI's problem-solving efficiency can be improved by "let's think step-by-step" instructions combined with Zero-shot instructions, a method called Chain-of-Thought prompting (Wei et al., 2023). Another type of prompt allows for the personalisation of these responses (Output customisation), by imposing certain rules or specific characteristics (White et al., 2023).

Few previous studies have explored the competence of pre-service teachers using AI tools for mathematical problem-solving (Noster et al., 2024). Indeed, no detailed analysis has been undertaken to characterise the use of these tools by analysing the prompts formulated by pre-service teachers to solve a modelling problem.

Methods

The study was carried out with a sample of 94 pre-service teachers, all of them belonging to the third year of the Primary Education Degree at the Faculty of Teacher Training of the University of Valencia (Spain). All participants had previous experience solving Fermi problems, as they had worked on

solving a sequence of problems in the faculty environment (described in Segura and Ferrando, 2023). None of the participants had previous experience of using ChatGPT to solve mathematical problems.

Data collection was conducted during a 2-hour class session, supervised by two of the researchers. Beforehand, the participants were asked to register in ChatGPT (version 3.5 free access) and to bring their laptops with internet access. At the beginning of the session, we explained that they had to solve individually two Fermi problems, but this time they could ask ChatGPT questions. The problems were contextualised in an environment familiar to the participants but presented hypothetical situations far removed from their immediate experience. Problem P1 asked to estimate how many people could attend a concert held at the Valencia football stadium, and problem P2 asked to estimate the number of people inside the IKEA in Valencia at the time of maximum influx of customers. The statement of these problems was accompanied by an aerial image of the place (Google Maps). The written solution reports produced by each participant and the ChatGPT interaction document generated by using this tool were collected for data analysis.

The results that allow us to address the objective of this study derive from a qualitative analysis of the data collected, specifically from the analysis of the prompts collected in the interaction document with the AI. To categorise the prompts, the written solutions of each participant were also analysed. In this way, we can confirm whether the solution is based solely on delegating the problem-solving to chat GPT or if, on the contrary, the solver has, at some point, followed the phases of the modelling cycle. In the first case, the written productions contain exactly the information provided by ChatGPT, and the prompts are categorised as Zero-shot (Kojima et al., 2022), Few-shot (Snell et al., 2017), Chain-of-Thought prompting (Wei et al., 2023) or Output customisation (White et al., 2023). In other cases, when the solutions contain details (e.g. drawings, diagrams or sentences) that allow us to distinguish the participant's solution from his/her interaction with the AI, we have categorized the prompts following the phases of the modelling cycle. The categories that were finally agreed upon by the researchers combine (albeit in a differentiated way) both perspectives, as described in Table 1. It should be noted that Table 1 does not indicate that there is a correspondence between the prompting techniques and the phases of the modelling cycle, they are two parallel and independent categorisations.

Table 1: Categorisation of prompts

Categories of prompting techniques	**Categories of the modelling cycle**
0.1. Zero-shot: the problem itself is used as input, copying the statement directly into the chat, without having asked any other previous question neither including any specification or additional information. For example: “How many people can fit in the Levante soccer stadium if we want to organise a concert?” in P1.	*1.1. Understanding and structuring the problem:* these prompts seek a better understanding of the problem and an identification of what aspects or data are needed (variables and their relationships), For example: “What data should be taken into account to calculate the maximum capacity of a shop?” in P2.
0.2. Few-shot: unlike the previous one, in this case, specific questions are asked about some aspects of the problem. For example, the dimensions of the space to be	*1.2. Construction of the model:* we include all those prompts related to the estimation (quantification) of the variables that make up the model. They are divided into three types:

occupied ("How many people can fit in 7140 square metres?" in P1), and then directly ask how many people can occupy the total area of the problem. We also include in this category those prompts that seek to refine the solution and make it more accurate by introducing a realistic assumption (without explicit relation to a specific mathematical model) for example: "(...) you should consider that there will be a stage, an area reserved for technical equipment and sound facilities" in P1.	- Questions related to the dimensions of the space to be occupied (*subcategory 1.2.1*), e.g. "What is the area of a football pitch?" in P1 or "How big is the IKEA in Valencia?" in P2. - Questions related to density or the space occupied by a person (*subcategory 1.2.2*), e.g. "How much does one person occupy?" or "How many people fit in one square metre?" - Direct questions about new realistic assumptions to make the participant model more accurate (*subcategory 1.2.3*), e.g. "How many square metres are occupied with shelves and furniture in IKEA?"
0.3. Chain-of-Thought: after a direct (Zero-shot) or quasi-direct (Few-shot) question, the participant gives the ChatGPT several consecutive prompts about the procedures involved in the solution. For example: 'How could you calculate it? [to clarify how the AI can calculate how many people fit in the total area from the data in a smaller area, which requires an explanation of proportional reasoning].	*1.3. Mathematical work:* in this category, we include all the prompts that require mathematical calculations, using ChatGPT as if it were a calculator, e.g. "Multiply 4 times 7140".
0.4. Output customisation: after a direct (Zero-shot) or quasi-direct (Few-shot) question, a specific output format is specified to the ChatGPT. For example, if the AI has explained the solving procedure in a general way, without calculations, the participant formulates the following prompt: "Solve it with mathematical operations", or "I need a quantity".	*1.4. Model validation and review:* these prompts seek to check the validity or conformity of a result or assumption, e.g. "Do you think 420 people can fit on a football pitch if there is a concert?" in P1, or "Is five square metres per customer too much?" in P2.

Results

The first step to characterise the use of ChatGPT in the solution of Fermi problems was to code, for each pre-service teacher, the prompts formulated by applying the categorisation described in Table 1. Table 2 shows the percentages of use of each of the categories of prompts (each category is counted only once in each solution) in the two Fermi problems. For a correct understanding of Table 2, note that the use of prompts is not exclusive: for example, the same participant may have first used a Zero-shot (0.1), not been satisfied with the result and asked questions to detail the process (0.3).

The results in Table 2 show that the most frequently formulated prompts refer to direct questions of the Zero-shot type (appearing in 54% of the productions) and questions about the construction of the model, specifically in relation to the dimensions of the total space available (subcategory 1.2.1, which appears in 69% of the productions). In addition, there is a slight change of trend in the type of prompts

asked in the second problem, as the number of prompts related to the model construction process and mathematical work decreases, while Zero-shot, Few-shot and Output customization prompts increase.

Table 2: Percentage of use of each prompt in the two Fermi problems

	Prompts									
	0.1	**0.2**	**0.3**	**0.4**	**1.1**	**1.2.1**	**1.2.2**	**1.2.3**	**1.3**	**1.4**
P1	50%	27%	11%	4%	4%	70%	51%	22%	20%	2%
P2	59%	30%	7%	10%	7%	68%	18%	12%	13%	4%
Global	54%	28%	9%	7%	6%	69%	35%	17%	16%	3%

From the results analysed in Table 2, together with the analysis of the problems' solutions, we determine three different characterisations of the pre-service teachers' use of ChatGPT to solve Fermi problems.

1) *Using ChatGPT as a problem-solving expert (PS-Expert). S*ome participants delegate the solution of the problem exclusively to the tool. By prompting, they want the AI to eventually come up with a satisfactory solution. In this category of use, pre-service teachers only formulate instructions of the type Zero-shot (0.1), Few-shot (0.2), Chain-of-Thought (0.3) and/or Output customization (0.4). An example of this type of use for solving P1 is the following:

Participant B12: You are an expert in solving mathematical problems, you have to help me by explaining step by step how to solve it and the procedures you would use to reach the solution. [*0.3*]

Participant A26: How many people can fit in the Valencia football field as an audience (only in the grass area) if we want to use that space to hold a concert of a famous artist? [*0.1*]

2) *Using ChatGPT as a problem-solving assistant (PS-Assistant)*. Some participants try to complete the modelling cycle using, for each phase, model prompts (1.1, 1.2, 1.3, 1.4) but also use the prompts from prompting techniques (0.1, 0.2, 0.3 and 0.4) to get ChatGPT to give them an answer. Eventually, they get an assisted solution. The example below shows the interactions of Participant A41 who, in solving problem P1, avoids making decisions about estimating the magnitude value of the model variables, and ends up leaving it up to ChatGPT to decide how many people fit in a bounded enclosure:

Participant A41: What is the size of the grass area of the Valencia football pitch? [*1.2.1*]
How many people can fit in a space of 7140 square metres? [*0.2*]
OK, now if we take into account the size of the stage of the singer who is going to sing in that venue. [0.2]

3) *Use of ChatGPT as a problem-solving support (PS-Support).* Other participants use ChatGPT as a specific tool during the solution process, developing the model autonomously. These pre-service teachers only formulate specific prompts to support their own development of the phases of the modelling cycle (1.1, 1.2, 1.3 and 1.4). An example of this type of interaction can be seen in the following sequence of prompts formulated by a participant:

Participant B09: How many people can fit in one square metre? [1.2.2]
Length and width of the Valencia football pitch. [1.2.1]

Length and width of a standard concert stage on a football pitch. [1.2.3]

Table 3 shows, for each problem, the number of pre-service teachers (and the percentage) who have used ChatGPT according to each of these three characterisations.

Table 3: Proportion of each characterisation of the use of ChatGPT to solve Fermi problems

	ChatGPT as PS-Expert	ChatGPT as PS-Assistant	ChatGPT as PS-Support
P1	25 (27%)	32 (34%)	37 (39%)
P2	24 (26%)	48 (51%)	22 (23%)

Thus, we observe that in both problems, almost a fourth of the pre-service teachers merely 'transfer' the responsibility of finding a solution to ChatGPT itself, as an expert (in the category “ChatGPT as PS-expert”). In those cases, in the productions with the written solution, we observe that many of them simply reproduce ChatGPT's answer directly, without adding any justification. We also find participants who only indicate the process to be followed to reach the solution, but without quantifying it, taking the reasoning provided by the AI.

On the other hand, we identified those participants who make instrumental use of AI (and who are the majority for both problems), i. e. their instructions do not seek ChatGPT to solve the problem directly (in the categories “ChatGPT as PS-assistant” and “ChatGPT as PS-support”). In these cases, the instructions seek the support of the AI in some (or all) phases of the modelling cycle, in particular by quantifying the variables selected in their model. It is the future teacher who reaches the final solution of the problem by choosing the strategy and identifying the relationships between the variables. The responsibility for the solution process thus lies with the solver himself, and the AI becomes a complementary tool. In these cases, a clear distinction is made between those pre-service who use it only as a support, asking very specific questions that allow them to progress in the solution process (as problem-solving support), and those who in some way bypass some phases of the modelling process by delegating to the AI tool (as a problem-solving assistant).

Furthermore, according to the results shown in Table 3, we can observe a certain evolution in the use of ChatGPT in this experience, based on the solution of two Fermi problems. For the problem P1, there is a higher proportion of participants who only use ChatGPT as PS-support. However, for the problem P2, which is slightly more complex and where participants have some experience in using ChatGPT, the majority (51% of participants) use the tool as PS-Assistant, combining instructions related to the solution process. The proportion of participants using ChatGPT as a PS expert remains the same.

Conclusions, limitations and further research

This study presents a categorisation of the prompts given to ChatGPT, during the solution of a modelling problem that requires obtaining the reasoned estimate of the number of elements from a statement without numerical data. The categorisation of prompts is based on the one hand, on prompting techniques described in the process systems literature (Kojima et al., 2022; Snell et al., 2017; Wei et al., 2022; White et al, 2023). But, on the other hand, the experience carried out, in which not only the prompts given to ChatGPT but also the written productions with the solutions to the

problems have been analysed, has allowed us to enrich this initial system of categories with others that derive directly from the modelling cycle. The categorisation of the prompts in Table 1 has enabled the identification of three categories of future teachers' use of ChatGPT in solving a Fermi problem. By combining the exclusive or non-exclusive use of prompts based on prompting techniques and those based on the modelling cycle, three categories of use have been established and are described in Table 2.

From one side, we distinguish future teachers who completely delegate the solution of the Fermi problem to the ChatGPT tool. These solvers never take into account the solution process described in the modelling cycle (Blum & Leiß, 2006), this can be seen from the solution reports in which they simply copy the answers obtained by the LLM-based system. On the opposite side, we have those participants who follow every step of the modelling cycle during the solution process, relying only on ChatGPT with prompts that can be helpful when interpreting the problem, constructing the model, working mathematically and validating the solution. In the middle, we find those participants who also make instrumental use of ChatGPT, avoiding a total delegation of the solution process, but, skipping some phases. Those participants use, in most of the cases, prompting techniques such as Few-shot and Chain-of-Thought, and tend to skip the construction of the model leaving this responsibility to the LLG-based system.

Thus, within these three categories, two attitudes can be distinguished when using ChatGPT to solve modelling problems: delegating responsibility for the solution or making instrumental use of the tool. This allows us to overcome certain reluctance to its use, as well as to provide a culture of AI to future teachers that will enable them to integrate this tool in their classrooms, in addition to developing competencies to evaluate ChatGPT productions. However, a limitation of this study is that it does not rigorously investigate at what phases of the modelling process pre-service teachers feel the need to delegate the solution to the ChatGPT tool. Moreover, the results in Table 3 suggest that the user's experience in solving problems with the help of ChatGPT or the complexity of the problem may be factors that influence the use of prompts and, therefore, the way of interacting with the tool. To confirm whether this is indeed the case, it is necessary to address studies that analyse, on the one hand, the effect of previous experience, and on the other hand, the effect of problem complexity, on the use that solvers make of ChatGPT.

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Mathematical knowledge mobilised by a group of prospective mathematics teachers when developing a modeling project

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The modeling project approach allows apprentices to study a topic of their interest and subsequently develop a project in which they formulate and solve a mathematical problem. The aim of this study is to understand the mathematical knowledge deployed by a group of two future teachers when developing a modeling project. This study analyses a project related to healthy living, specifically, to the number of steps an individual takes per day. The mathematical knowledge deployed in the development of the project was categorised using the Mathematics Teachers' Specialised Knowledge (MTSK) model, focusing on the Mathematical Knowledge (MK) domain. The data collection instruments were the project report with its video presentation, and a narrative. The results show that the modeling projects approach mobilised the subdomains Knowledge of Topics (KoT), Knowledge of the Structure of Mathematics (KSM) and Knowledge of the Practice of Mathematics (KPM).

Keywords: Mathematical modelling, modeling projects, mathematical knowledge, teacher knowledge, prospective teachers.

Introduction

While there are various interpretations of mathematical modeling (e.g. Kaiser & Sriraman, 2006), it typically focuses on the relationships between mathematics and real-world situations, often involving the formulation and solution of problems (Niss et al., 2007). In this paper, modeling is viewed from the perspective of modeling projects that emphasizes student engagement. Learners, working in small groups, select a topic to explore and, based on this topic, pose a problem or question to investigate (Borba & Villarreal, 2005).

The preparation of teachers in mathematical modelling involves fostering the development of their competencies to both teach mathematics through modelling and to teach modelling itself (Blum, 2015; Doerr, 2007). Additionally, it is crucial to equip teachers with the skills to incorporate modelling activities into their lessons, helping them gain confidence in using these activities and the courage to challenge conventional approaches to mathematics teaching and learning. In this context, modeling should address both the knowledge required to engage in mathematical practices and the application of modelling in the teaching process (Doerr, 2007), in line with the fact that the teaching of mathematics in teacher training differs from that of other professions, as it is specialized. This study addresses the mathematical knowledge deployed in the practice of modeling, based on the conceptualization of the Mathematics Teachers' Specialised Knowledge (MTSK) (Carrillo, et al., 2018).

There is a lack of literature reporting on problem-solving practice and specifically modelling, particularly within the framework of MTSK. Studies by Carrillo et al. (2017), Escudero-Ávila et al. (2015) and Galleguillos and Ribeiro (2022) report on mathematical knowledge in relation to problem-solving practices, and Zapata-Jaramillo (2023) proposes a separate category for modelling practices.

The need to delimit these practices related to mathematical knowledge respect to modeling was a key motivation for the present study. Thus, the aim of this study is to understand the mathematical knowledge deployed by a group of two pre-service teachers while working on a modeling project.

Framework

In this paper, modeling projects is seen as a pedagogical approach that emphasises learners, working in small groups, choosing a topic to study of their interest, and from it proposing a problem or question related to that topic (Borba & Villarreal, 2005). Modeling projects approach use problem posing strategy, which consists of formulating and reformulating problems (Cai & Hwang, 2021; Siller et al., 2024) that, in our perspective, allow students to first engage in an extra-mathematical topic, making strong connections between mathematics and reality and, thus, allowing students to visualize the usefulness of mathematics. In the case of teacher training, Ferrando et al. (2025) point out that training in problem-posing can effectively improve teachers' ability to pose and evaluate modeling tasks. An example of this approach with future teachers is found in Galleguillos (2024).

Accordingly, the MTSK model (Carrillo et al., 2018) was adopted to categorise the knowledge dimensions that can be mobilised by working on a modeling project. In this article, focus is given to the Mathematical Knowledge (MK) deployed by a group of two prospective teachers who worked together on a modeling project. The specialized knowledge of MTSK mathematics teachers is used as an analysis framework, in their MK domain, which is divided into three subdomains: Knowledge of Topics (KoT), Knowledge of the Structure of Mathematics (KSM) and Knowledge of Practices in Mathematics (KPM), which are defined below.

- KoT describes what and how the teacher knows about the topics he/she teaches, which implies a deep understanding of mathematical knowledge and its meaning. According to Vasco and Moriel (2022), KoT comprises the following categories: (1) procedures, (2) definitions, properties and foundations, (3) representations, and (4) phenomenology and applications. As an example, the application by teachers of the properties of logarithms shows a knowledge of KoT in definitions, properties and foundations (2) in logarithms topic.
- KSM corresponds to the teacher's knowledge of the connections between mathematical elements. This knowledge considers inter-conceptual connections and the role of mathematical items in the construction of other items. It is related to an increase in complexity or to simplification. As noted by Flores-Medrano (2022), KSM comprises four categories: (1) complexifying connections, (2) simplifying connections, (3) transverse connections, and (4) auxiliary connections. For example, the use of properties of logarithms in conjunction with exponentiation shows a knowledge of mathematics structure in transverse connection (3) between these contents.
- KPM corresponds to any mathematical activity carried out systematically, which is the basis of mathematical creation. This knowledge includes knowing about proof, justification, definition, making definitions and inductions, giving examples, and understanding the role of counterexamples. In addition, knowledge of problem-solving strategies and aspects of mathematical communication such as precision in the use of language is part of the practices in mathematics. According to Delgado-Rebolledo et al. (2022), KPM comprises the following four categories: (1) the practice of proving, (2) the practice of defining, (3) the practice of

problem solving, and (4) the role of mathematical language. For example, using drawings or diagrams to solve a problem shows KPM's knowledge of problem-solving practice (2).

Extensive research has been conducted on the mathematical knowledge of topics (KoT), yet studies focusing on the subdomains of KSM and KPM remain limited (Climent & Montes, 2022; Delgado-Rebolledo et al., 2022). Specifically, studies in which teachers engage with problem-solving, modeling, or teaching problem-solving offer valuable insights into KPM. For instance, Carrillo et al. (2017) examined the development of a mathematics teacher working on a problem, revealing aspects of knowledge related to simplifying the problem to find its solution. Escudero-Ávila et al. (2015) identified the teacher's knowledge in breaking down a problem into specific cases to enable a detailed analysis of the problem. Galleguillos and Ribeiro (2022) explored knowledge related to conjecture justification in an optimization problem. Finally, Zapata-Jaramillo (2023) investigated modeling tasks, highlighting a category focused on modelling practices, such as in explorations and model formulation.

Methods

This study was approached as basic qualitative research (Merriam & Tisdell, 2016), which aims to understand how participants make sense of their experiences. The study was conducted within the context of a mathematical modeling course taught during the seventh semester of initial mathematics teacher training at a Chilean university, where preservice teachers were tasked with developing a modeling project. This article specifically examines one of the projects, developed by Fabiola and Néstor. Course participants were guided through a supporting framework that included the following steps based in Almeida et al. (2016): project title and information about the chosen topic, questions or problems posed, mathematization, explanations, reflections, and conclusions. The project was assessed using a rubric that evaluated elements such as problem formulation and statement, problem solving, interpretation of results, reflections and conclusions, along with a video presentation of the results. The participants chose to collaborate on a project focused on healthy living, specifically examining the number of steps taken daily by an individual. Upon completion of the project, they were asked to write a narrative of their experience, which was written by Fabiola. The narrative was neither assessed nor mandatory. The data related to this project are analysed here using the MTSK model, focusing on the MK subdomain.

Analyses and Results

The project is titled: "A few small steps each day, an investment in the future". It is founded in which: "Physical activity is a preventive measure for the development of non-communicable diseases such as cardiovascular diseases, and it also has a positive impact on mental health, maintaining a healthy weight, and the general well-being of people. Promoting the performance of different types of physical activity, especially in girls, boys, and adolescents, is crucial, since the prevalence of childhood obesity in our country has increased from 9.69% in 2008 to 14.06% in 2022, according to data provided by the Department of Nutrition and Food (2023). Walking, in particular, is an affordable and accessible activity for the majority of the population. But how many steps should we take daily to obtain benefits in our health status? […]" (Project of Fabiola and Nestor).
The problem addressed by the participants is given follow:

A female high school student decides to progressively increase the number of daily steps she takes to improve her cardiovascular health. To do this, she was advised to increase the number of steps taken each day by 5%.

a) If she initially took 2,500 steps a day, how many weeks would it take her to reach a target of 7,000 steps a day?
b) If a person's height is 1.64 metres, how many kilometres would the total number of daily steps they take on the last day of each week amount to?

The Figure 1 shows some slides related to the project, which were presented in a video format.

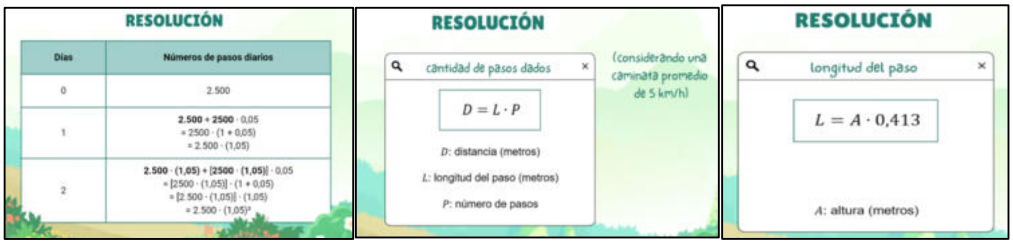

Figure 1. Some slides of the project shown in the video

Solution to Problem (a)

Table 1. Generalisation of number of daily steps

Day	Number of daily steps
0	$2{,}500$
1	$2{,}500 + 2{,}500 \cdot 0.05 = 2{,}500(1 + 0.05) = 2{,}500(1.05)$
2	$2{,}500(1.05) + 2{,}500(1.05) \cdot 0.05 = 2{,}500(1.05) = 2.500(1{,}05)^2$
3	$= 2{,}500(1.05)^2 + 2{,}500(1.05)^2 \cdot 0.05 = 2{,}500(1.05)^2(1 + 0.05)$ $= 2{,}500(1.05)^2(1.05) = 2{,}500(1.05)^3$
⋮	⋮
n	$= 2{,}500(1.05)^n$

In response to problem (a), the group presented a table (Table 1) delineating the number of steps per day, based on a 5% daily increment. To determine the number of days required to reach 7,000 steps the participants formulated the following exponential equation:

$$7{,}000 = 2{,}500(1.05)^n \quad (1)$$

Applying properties of logarithms, the group found that the goal of walking 7,000 steps a day can be achieved in 22 days.

$$7{,}000 = (2{,}500)(1.05)^n \ /\log_{1.05}(\ )$$
$$\log_{1.05}(7{,}000) = \log_{1.05}(2{,}500 \cdot (1.05)^n)$$
$$\log_{1.05}(7{,}000) = \log_{1.05}(2{,}500) + \log_{1.05}(1.05)^n$$
$$n = \log_{1.05}(7{,}000) - \log_{1.05}(2{,}500)$$
$$n = 21.103 \ldots \approx 22$$

The solution presented by the participants show that the participants develop the necessary knowledge to model the problem and answer it, showing knowledge of the three subdomains. Some of these actions are related to mathematisation, reflecting both a knowledge of practices in problem solving and modelling (KPM). First, they applied a 5% increase to the initial number of daily steps, which corresponds to the category of procedures (KoT – *procedures*). Then, they constructed a data table

by applying this increase in the given percentage (KPM – *problem solving*). Thus, they found a general pattern of the situation, which relates to a modelling practice (KPM – *modelling practices*). Continuing, to solve equation (1), they applied different properties of logarithms to find the number of days needed to reach the target (KoT – *definitions, properties and foundations*). Thus, the modelling of an exponential function in a situation of growth from a known percentage was carried out.

In this process, emerge relationships between distinct mathematics elements. A transverse connection results when different mathematical items have features in common (Carrillo et al., 2018). Thus, the participants show their knowledge on a relationship between an exponential equation and logarithms (KSM − *transverse connection*).

Solution to Problem (b)

> b) If a person's height is 1.64 metres, how many kilometres would the total number of daily steps they take on the last day of each week amount to?

As this problem assumes a relationship between a person's height and step length, the solvers established the relationship between distance, step length, and number of steps, as shown below:

$$D = L \cdot P \tag{2}$$

where D is the distance (metres), L is the step length (metres), and P is the number of steps. While this equation can be deduced or found on the Internet, the participants needed to apply it to the height specified in the problem. Fabiola and Nestor produced equation (3) below, where A denotes woman's height and L corresponds to the step length.

$$L = A \cdot 0.413 \tag{3}$$

Therefore, for a woman measuring 1.64 m (height given in the problem statement), step length is approx. 0.68 m. To answer the question, the distance equivalent to the steps taken on the last day of each week, i.e., on the 7 th, 14th, and 21st day, is given in Table 2, while the number of steps is presented in Table 3.

Table 2. Number of daily steps per week

Days	Number of daily steps
7	$2{,}500(1.05)^{7} \approx 3{,}518$
14	$2{,}500(1.05)^{14} \approx 4{,}950$
21	$2{,}500(1.05)^{21} \approx 6{,}965$

Table 3. Distance covered each week

Days	Distance (meters)	Distance (km)
7	$D = 0.68 \cdot 3{,}518 \approx 2{,}392$	$2{,}392 \cdot 0.001 \approx 2.4$
14	$D = 0.68 \cdot 4{,}950 = 3{,}366$	$3{,}366 \cdot 0.001 \approx 3.4$
21	$D = 0.68 \cdot 6{,}965 \approx 4{,}736$	$4{,}736 \cdot 0.001 \approx 4.7$

To solve the problem (b), the participants used the internet to search for information relating the average step length and the distance covered, as well as the height and the step length for a woman, based on the 5 km/h walking speed (Figure 1). Finally, they used the exponential model equation (1) developed for problem (a) to determine the steps taken on the corresponding days. In subsequent steps, they applied equation (2) and (3), which involved the deployment of knowledge of the *practices of modelling* (KPM) on knowing how to apply different models to answer a question. Finally, the participants show how to solve a problem of exponential nature (KPM – *practice of problem solving*). Extracts from the narrative of Fabiola's experience are analysed below.

Fabiola's narrative

By analysing Fabiola's narrative, it is possible to gain reflections of her experience, such as her difficulties and learning.

Fabiola: At first, I found it difficult to select a topic that was of interest to me and that could also be approached from a mathematical point of view. For a moment I thought about using other topics such as sleep time or sugar consumption, but I could not manage to translate this into a concrete question that would allow me to develop a mathematical model. I consider this to be the biggest challenge when starting a modeling project. Once I chose to work with the number of daily steps, I made use of models and knowledge that we had previously worked on in the course, for example, the exponential model. This was a great support. I also enjoyed being able to take the steps problem to a more complex and relevant situation […].

If I could summarise what I learned, I would keep the importance of defining the problem that motivates the modeling project, with all that goes with researching and selecting a suitable topic and being able to pose relevant questions to critically reflect on real life situations or, in this case, habits related to our health and […]. In addition, I feel that I really understood how the elaboration of a project like this one allows us to empower our own learning, since during my basic and middle school stage I always considered this type of work or evaluations as just another responsibility, without taking it seriously.

Fabiola mentions in her experience that it was difficult for her to pose (or formulate) a problem that she could relate to mathematical modelling. She expresses that in the process she developed the ability to propose a problem in which she was able to use the mathematics previously covered in the course and considers that this experience enhanced her learning. Thus, she shows that developed the knowledge of posing a problem based on a real-life, culturally relevant topic of interest to her, in this case a problem of an exponential nature (KPM – *practices of modelling*).

Table 4 summarises the mathematical knowledge deployed by the prospective teachers in their project, both for the project developed and for the experience reported in the narrative.

Table 4. Knowledge prospective teachers deployed in developing a modeling project

Sub-dom	Categories	Knowledge deployed	
KoT	Procedures	KoT-p:	How to increase a quantity in a determined percentage
	Definitions and properties	KoT-d:	Apply logarithm properties to multiplication, addition, and subtraction, as well as logarithms involving powers
KSM	Transverse connections	KSM-tc:	Know logarithmic and exponential expressions are related, since one is the inverse of the other
KPM	Practices of problem solving	KPM-ps1:	How to build a data table by applying a given percentage increase
		KPM-ps2:	How to solve a problem by applying different models
	Practices of modelling	(Indicators) Knowing how to find a model from to identify a pattern. How to apply models for solve a problem Posing a problem based on a real-life in this case a problem of an exponential nature as part of a project	

Conclusions

The modeling project approach, as part of which participants were required to formulate and solve problems, has enabled prospective teachers to formulate mathematical problems, in which their mathematical knowledge (MK) was deployed or developed in the process. The evidence presented here indicates that the prospective teachers developed knowledge in the KoT, KSM and KPM subdomains. Different knowledge emerged in the posing and solving the problem along with important reflections on taking care of the health through daily walking.

The exponential nature of the problem developed by them allowed the prospective teachers deepen their knowledge regarding said topic, as well as to make connections between the chosen topic and the mathematical knowledge involved. In this study, actions of mathematising a situation or problem can be considered as practices of modelling, in the same way that Zapata-Jaramillo (2023) establishes it in a different way of practices of problem solving. Likewise, this study shows that the prospective teachers applied models to solve the problem posed. Thus, the strategy used indicates an interesting way to implement modeling in teacher training. Although modeling project approach is a challenging strategy, it allowed the prospective teachers to pose and solve real-life problems by making concrete connections with the mathematical knowledge with edges and connections between mathematical topics and other models. As this study focused on the development of mathematics knowledge (MK), the dimension of didactic knowledge should be examined in future research in which this approach (or project) can be implemented in classrooms.

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Making the Invisible Visible: Exploring the Situation Model

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It is a central goal of mathematics education to convey practical life skills and to present mathematics as an integral part of culture and society. Mathematical modelling, as the transfer between reality and mathematics, plays a central role in this goal. Such modelling processes can be represented stepwise in a modelling cycle. In this the first three steps focus on the creation of a mathematical model and lead to different stages of development of it: the situation model, the real model, and the mathematical model. Several studies focus on students' development of a real and mathematical model. However, the situation model, in particular, seems elusive, and its concrete formation is not well-researched. In this study, the formation processes of the situation model are made visible by means of eye-tracking. Our results show that the different stages of development are clearly identifiable.

Keywords: Modelling, situation model, eye tracking, outdoor mathematics.

Introduction

Mathematics education aims imparting practical life skills and at presenting mathematics as an integral part of our culture (Greefrath et al., 2013). Furthermore, professional organizations such as the National Council of Teachers of Mathematics (NCTM, 2000) suggest that integrating real-world problems into the curriculum can increase the motivation among students and help with both mathematical contents and strategies (NCTM, 2000). At its core, mathematical modelling consists of translating a real-world problem to mathematics, working with mathematical methods and transferring the mathematical results back to the real world (Greefrath et al., 2013). One way of representing such modelling processes is the modelling cycle by Blum & Leiß (2005). The first three steps (understanding, simplifying/structuring and mathematicising) lead to different stages of development of the model: the situation model, the real model, the mathematical model. The question how the first two stages of development (namely the situation model and the real model) can be differentiated has been discussed for several years with existing arguments for the necessity of such a separation as well as against it (Borromeo Ferri, 2006). Researchers who include the situation model in their modelling cycle focus on the cognitive processes of the solving process (Borromeo Ferri, 2006). Other researchers jump right into the mathematical model and do not distinguish between the first two stages of development. They focus more on the solution processes since they work with more complex models (Borromeo Ferri, 2006). Especially the imagination of the situation model is elusive and has not been visualized yet. In addition, currently only a few information, about how such a mental representation of a situation evolves is available (Leiß et al., 2010). So it is not trivial to explicate in detail which cognitive processes take place in a student's mind while solving a mathematical modelling task (Reusser, 1997), and in particular while processing the first steps in creating a model.

The *Eye-Mind Hypothesis* (Just & Carpenter, 1980) states that there is a direct connection between a person's view and the mental process of the person in this moment. There are studies using eye

tracking in the fields of modelling (i.e. Schmitz & Schukajlow, 2020) but no results on the question how the constructed situation model looks can be derived from them. It is still unclear how the situation model is created. To close the identified research gap, this paper aims at presenting an explorative study that investigates, how mathematical modelling processes can be discovered through eye tracking technology and which differences can be found between the first stages of development of the modelling cycle. Through the explorative character of the study, the modelling processes are limited to real-world objects. This choice is underpinned in the theoretical background.

Theoretical Background

Mathematical modelling

A mathematical model needs to be constructed to mathematicise real-world problems (Blum & Leiß, 2005). Accordingly, modelling is understood as the interplay between reality and mathematics (Blum & Leiß, 2005). Mathematical models should be self-consistent, coherent and expedient (Greefrath et al., 2013). Despite this characterization, they are not explicit because it is just a depiction of the reality (Greefrath et al., 2013). Thus, the modelling cycle by Blum and Leiß (2005) becomes a model of mathematical modelling itself (Greefrath et al., 2013). The modelling cycle describes the modelling process in seven steps, from which this paper focuses on the first three steps: understanding, simplifying/structuring and mathematising. In the first step, the real problem situation needs to be understood. In this process the students have to create a mental model which leads them to a situation model. In the second step, the real model should be created out of the situation model, by separating important from unimportant information. In the third step the real model is converted into a mathematical model, by adding suitable terms, equations, designations, etc., thus mathematising the real model and reducing it to the essential aspects (Greefrath et al., 2013; Blum & Leiß, 2005).

Situation model

The situation model is in the literature synonymously used with mental representation of the situation (Borromeo Ferri, 2006), mental model (Greefrath et al., 2013) or problem situation (Kintsch & Greeno, 1985). It forms the basis for all further modelling activities (Borromeo Ferri, 2006). The situation model is the individual cognitive structure that is built based on the situation understanding (Leiß et al., 2010). It needs to be assumed that the structure of a situation model can also be schematic (van Dijk & Kintsch, 1983) and that it can be described in form of a picture of auditory, visual or tactile information (Leiß et al., 2010). Besides a semantic situation model there is also a communicative one needed (van Dijk & Kintsch, 1983). Blum and Leiß classify the situation model as one of the, if not even the most important phase of the modelling cycle (Borromeo Ferri, 2006). Van Dijk and Kintsch (1983) also postulate that the basis of problem solving is not directly the text, but the resulting mental model. While numerous graphical representations (i.e. Schmitz & Schukajlow, 2020) for real and mathematical models exist, this is not the case for the situation model, which is both more difficult to grasp and has been less researched. Previous studies show that the

difficulty of text tasks cannot only be attributed to logical-mathematical factors and that especially the situation model plays a very important role (Kintsch & Greeno, 1985).

Modelling at real objects

The term outdoor learning refers to educational experiences that take place outside traditional classroom settings, involving activities in which students directly work in natural environments or real-world contexts. Outdoor learning makes it possible to experience mathematics on a daily basis (Cahyono & Ludwig, 2017) and therefore provides possibilities for mathematical modelling at real-world objects (Ludwig & Jablonski, 2021).

Mathematical modelling at real objects has already been investigated in different studies (i.e. Ludwig & Jablonski, 2021). Cahyono and Ludwig (2017) found out that the outdoor activities had a clear positive effect on the motivation of the students. Furthermore, there are different modelling settings, i.e. different types of representing the real situation in modelling tasks (cf. Jablonski, 2023). Next to the traditional way of presenting the text together with a picture of the real object, there is the possibility to use a 3D-printed manipulative of the real object or to solve the modelling task in an outdoor environment right at the real object (Ludwig & Jablonski, 2021).

State of the Art and Research question

As previously stated, it is a challenging effort to characterise the specific cognitive processes that occur within the student's mind while attempting to solve mathematical modelling tasks (Reusser, 1997). To gain a deeper understanding of the cognitive processes involved, Schmitz and Schukajlow (2020) explored the impact of decorative and representative images on modelling tasks. The researchers tracked the eye movements of the students while they were solving the tasks and discovered that the total number of fixations was significantly higher on representative images than on decorative ones (Schmitz & Schukajlow, 2020). The use of eye tracking revealed that, in the analysis of all the modelling tasks, right triangles were identified, indicating the generation of a model (Schmitz & Schukajlow, 2020). From this study, we can assume that eye tracking makes students' (mental) situation models visualisable. Nevertheless, it remains unclear from the study whether this finding reflects the situation model or the real model. Our first research question arises from this:

> [RQ] Is it possible to make the situation model visible (through eye tracking) while solving mathematical modelling tasks outdoors at real objects?

Methodology

The *Eye-Mind Hypothesis* (Just & Carpenter, 1980) posits a direct correlation between an individual's visual attention and the cognitive processes occurring at that moment. To record the eye movements there a multiple different eye tracker to use. We decided on the mobile eye tracker Neon from Pupil Labs because our participants need to move freely around the objects, and we can analyse the data with the different enrichments on Pupil Cloud: Heatmaps and Scanpaths. *Heatmaps* provide information about which objects in the learning environment are viewed. It ranges from not at all (0%) to the most (100%) (Heatmap, n.d.). Heatmaps are placed on a reference image that represents the actual three-dimensional situation, as well as the scanpaths. *Scanpaths* are a visualisation of the interplay between fixations and saccades. *Fixations* arise when the eye pauses on a region and

saccades are the rapid shifts that occur between fixations (Graham et al., 2022). For a certain time lapse, we manually map all the fixations on the reference image and see the numbered fixations (green circles) and the respective saccades (black connecting lines) on the reference image.

The research question is firstly investigated in an explorative study. We asked three lecturers in mathematics education to participate in our explorative study. During the data collection each participant solved three different geometrical modelling tasks on real-world objects. We used this special eye tracker recording during these processes and asked the participants to verbalize their solving processes and to write down their intermediate steps.

Referring to the different representations of modelling tasks, it was our decision to use the outdoor setting due to technical and mathematical reasons: To be able to analyse the eye movements appropriately, we decided to start with big geometrical objects without perspective or scaling. So, we decided for real-world objects that appear in their originality outdoors. For the scanpaths, we manually mapped around 400 fixations on a reference image. The fixations that are mapped onto the reference photo can be located in the period of time from the arrival at the object to the moment when the participants started their sketch.

First results

Observations from the exploratory study show that the solving processes at various stages of development are identifiable. In the modelling task "Stone Circle", participants were asked to calculate how many stones lie in the area framed by the grey ring and the gutter (Figure 1, left). Figure 1 (right) shows the related scanpath of one participant. In this scanpath, the radius and diameter are visible as they were the focal areas of the participant's considerations and the path to the solution. The participant first counted how many stones fit into the radius and the diameter. Since the stone circle is not a perfect semicircle, an average radius was determined from the values for the radius and diameter, and the result was approximated using the area formula for a circle.

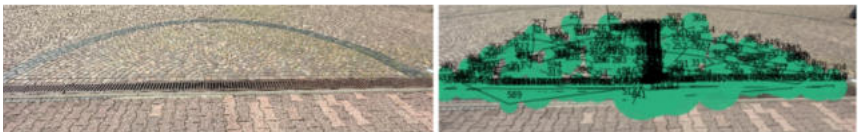

Figure 1: reference image "Stone Circle" (left), scanpath to the task "Stone Circle"

Figure 2 (left) shows the individual real and mathematical model of the task "Stone Circle". As can be observed the scanpath and the real model of the task show similarities regarding the diameter and the radius of the semicircle. The thick black stripes on the scanpath are a consequence of the participant's repeated visual exploration of these areas in both directions, which has resulted in the presence of numerous saccades in these regions. The heatmap in Figure 2 (right) shows the importance of the semicircle's radius in the eyes, respectively, the focal points of the participant. In

the real model we can obviously see the prominent areas from the scanpath building the base for the sketch.

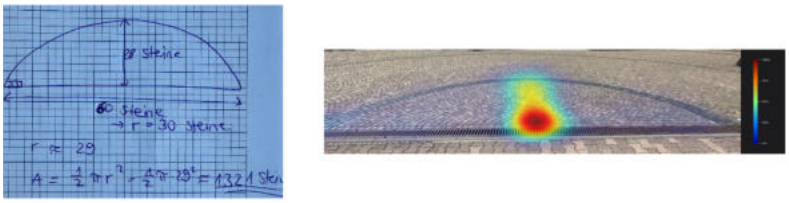

Figure 2: real and mathematical model of the task "Stone Circle" (left), heatmap (right)

In the modelling task "Archway", the objective was to determine the height from the ground to the highest point of the arch (Figure 3, left). The Scanpath (Figure 3, right) of one participant's problem-solving process clearly indicates that the participant identified both the height up to the arch and the diameter of the arch section. With these two measurements, the height was calculated.

Figure 3: reference image "Archway" (left), scanpath to the task "Archway" (right)

In Figure 4, similarities between the real model of the task "Archway" and the scanpath are evident once more. The area in which the participant took all the most significant measurements was observed with greater frequency than other areas of the reference image. This observation is corroborated by the Heatmap in Figure 4 (right). As can be observed in the provided scanpath, the column height and diameter are highlighted on the heatmap, forming a right angle. If the column height and half of the diameter were joined together and stretched, this would result in the archway height.

Figure 4: real and mathematical model of the task "Archway" (left), heatmap (right)

In the task "Subway Window", participants were required to determine the circumference of the large circular window (Figure 5, left). All participants counted the diameter using the wall stones and then calculated the circumference based on this measurement. The associated scanpaths reveal this process, and interestingly, two different approaches were observed: Participant A attempted to

measure the diameter starting from the bottom left corner (Figure 5, middle), while Participant B did the same from the top edge of the circle (Figure 5, right).

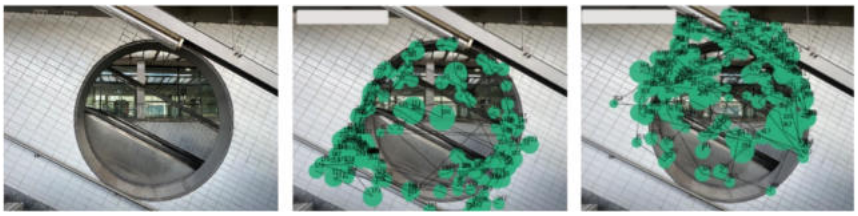

Figure 5: reference image "Subway Window" (left), scanpath A to the task "Subway Window" (middle), scanpath B to the task "Subway Window" (right)

The two different mathematical models of Participant A (top left) and Participant B (bottom left) are shown in Figure 6. Referring to Figure 5, we can observe that the related scanpaths again build the foundations of their real and mathematical models. Also, the heatmaps show the different focuses the participants set. On the top right (Figure 6) we see the focus of Participant A which was on the bottom left of the Subway Window. On the bottom right (Figure 6) we can almost perceive a right angle between the focus areas where Participant B turned the gaze to the top of the Subway Window.

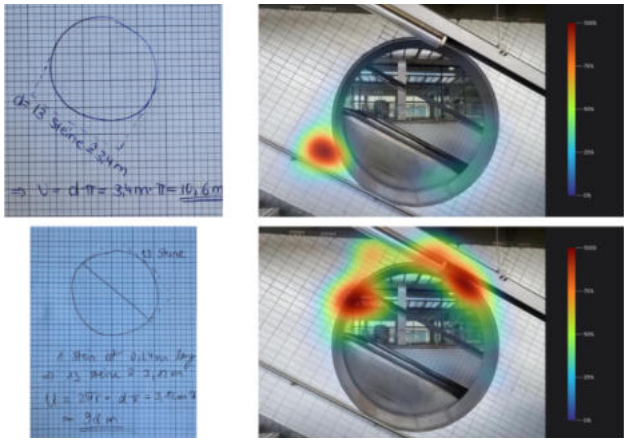

Figure 6: real and mathematical model of the task "Subway Window" of Participant A (top left), real and mathematical model of the task "Subway Window" of Participant B (bottom left), heatmap of Participant A (top right), heatmap of Participant B (bottom right)

In conclusion, the results of the three modelling tasks demonstrate that the scanpaths can be identified in all cases within the sketches, which represent the real models. This formation of the scanpaths within the sketches can be considered the precursors to the real and mathematical models. At the same time, it is possible to identify the mathematical ideas that lead to results of the tasks in the scanpaths. We conclude that the scanpaths represent stages of development of the solution processes.

Discussion and Outlook

Using the mobile eye tracker in combination with verbal and written reports of the solution processes supports our hypothesis that various solution models and cognitive processes can be identified through the analysis of eye movements. One explanation could be as follows: As the sketches can be identified as the real models and the scanpaths function as their precursors, the scanpaths could be

regarded as the situation models of the task. We propose that the scanpaths represent one stage of development of the model, given that the analysis of the eye movements represents a model created from the mental structures that lead to the subsequent sketches (real and mathematical models).

The theoretical background to the situation model emphasises its central importance, as appropriate problem solving of the task is not possible without an adequate situation model. In the present study, all participants achieved correct results. The analysed scanpaths, which are possibly interpreted as visual representations of the respective situation models, indicate an adequate situation model. This study therefore confirms the theoretical assumption that a well-developed situation model is crucial for achieving a correct solution.

In the present study, right-angled triangles, radii and circles were identified in the eye movements of the participants. These results are consistent with the findings of Schmitz and Schukajlow (2020).

As a limitation, it should be noted that the participants as lecturers of mathematics already were familiar with mathematical modelling, which may have influenced their approach to the tasks. Additionally, the study focused exclusively on geometry-based tasks, making the findings less generalizable to other types of mathematical modelling tasks. The use of real objects as a basis for modelling further limits the transferability of the results, as similar analyses have not yet been conducted with purely text-based tasks. Finally, the study examined longer time segments, which, while providing a comprehensive view, may have overlooked the insights gained from analysing shorter, more specific intervals of the problem-solving process.

As an outlook, the next step will focus on conducting a study with students at the lower secondary level. This new study aims to examine whether student's mental (situation) models can also be visualized through their eye movements. To this end, a strictly structured experimental procedure will be implemented, requiring that the task is fully understood prior to the initial observation of the object. This is intended to ensure an integration of task comprehension and object perception during the first viewing. Additionally, it is assumed that the comprehension process will be largely completed by the time participants begin to take notes. A central research question for this study is whether the characteristics of the object can already be mentally structured during observation, indicating a possible convergence of the situation model and the real model.

In addition to further elaborating on the primary research question – whether scanpaths can effectively describe the situation model – future investigations will explore whether eye movements can provide insights into performance and efficiency in solving mathematical modelling tasks.

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Financial literacy and mathematical modelling: an analysis of PISA sample tasks

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This study explores the integration of mathematical modelling and financial literacy within the educational context, analyzing 26 PISA financial literacy items from 2012, 2018, and 2022. Using qualitative content analysis and considering the OECD's framework on financial education, we identify the significant mathematical modelling competencies that are beneficial for understanding financial literacy items. Our findings reveal that competencies such as understanding, mathematizing, working mathematically, and interpreting are often required to solve these PISA items, highlighting a potential for co-developing financial and mathematical literacies. The analysis suggests that while current PISA items occasionally engage mathematical competencies, greater emphasis on modelling could further enrich financial literacy education. This study contributes to the broader discourse on educational synergies between mathematics and financial literacy.

Keywords: Mathematical modelling, financial literacy, PISA, qualitative content analysis.

Introduction

Looking at the term financial literacy from a mathematics education perspective, it is expected that mathematical competences, as contained in the concept of mathematical literacy, defined as

> individual's capacity to formulate, employ and interpret mathematics in a variety of contexts, it includes reasoning mathematically and using mathematical concepts, procedures, facts and tools to describe, explain and predict phenomena. (OECD, 2019, p. 75)

are also required for financial literacy:

> knowledge and understanding of financial concepts and risks, and the skills, motivation and confidence to apply such knowledge and understanding in order to make effective decisions across a range of financial contexts, to improve the financial well-being of individuals and society, and to enable participation in economic life (OECD, 2019, p. 128).

These correlations could also be empirically proven, even if other factors such as reading play a role (Acar, 2023; Ozkale & Erdogan, 2022). It was shown that financial literacy questions have the potential to promote conceptual and process-related competences for mathematics and financial education.

The well-known mathematical modelling cycle is also currently a central feature of the PISA framework for basic mathematical education. It serves to define the mathematical processes that students go through when solving problems (OECD, 2023b, p. 20). In PISA items on mathematics, however, it is often not necessary to go through all phases of the modelling cycle. In practice, significant parts of the mathematical modelling cycle are often performed by others, while the end user only performs some steps of the modelling cycle (OECD, 2023b, p. 25).

The relevance of modelling in relation to contextual problems, especially those from the financial domain, necessitates a consideration of the links between financial literacy and mathematical modelling based on the PISA concept. We therefore consider the overarching question of how mathematical modelling and financial literacy are related. Specifically, we answer the question of which steps in the modelling cycle are most important for the completion of PISA tasks on financial literacy.

Financial literacy competences

The OECD's competence framework (European Union/OECD, 2023) for improving financial literacy for children and young people aged 6 to 18 comprises a total of 244 competences. It focuses on the subject areas of 'money and transactions', 'planning and managing finances', 'risks and benefits' and 'financial landscape' and formulates 55 competences for the 6 to 10 age group, 87 for 11 to 15-year-olds and 102 for 16 to 18-year-olds. In addition to these subject areas, four different context areas are distinguished that are relevant for children and young people, namely 'School and work', 'Home and family', 'Individual' and 'Social'. This considers the fact that some young people are already in paid work and have to make decisions for the future, that financial decisions are often made in the context of family and household, that financial decisions have to be made for personal needs and that many risks have to be taken by individuals. The social aspect, for example, is about sustainable finances or paying taxes. In addition, process-related competencies, i.e. processes, are named, e.g. 'Evaluate financial issues' or 'Analyse information in a financial context'.

There are also competence models for financial education in individual countries; for example, the competence model for general financial education developed by Retzmann and Seeber (2016) in Germany. It comprises the competence areas 'decision and rationality', 'relationship and interaction' and 'order and system'. Each area describes three sub-competencies; they are, for example, 'Analysing situations', 'Evaluating alternative courses of action' and 'Designing possible courses of action'. Such competence models overlap with the European Union/OECD concept (2023), but there are also differences in detail.

Mathematical modelling and financial education

We consider mathematical modelling to be a central mathematical competence of mathematical literacy (OECD, 2023b). Modelling competence can be described as the ability to adequately carry out the necessary steps in the process of moving back and forth between reality and mathematics as well as analysing and evaluating models (Niss et al., 2007).

Modelling cycles describe the various sub-processes of mathematical modelling with varying degrees of detail and focus. The ability to perform such a sub-process adequately is described as a sub-competence of modelling (Kaiser, 2007; Maaß, 2006). They distinguish between the following sub-competences:

- to understand a real-life problem (understand),
- to set up a real-life model (simplify),
- to set up a mathematical model from a real model (mathematise),
- to solve mathematical problems within a mathematical model (working mathematically),

- to interpret mathematical results in a real model or a real situation (interpret),
- to question the solution and, if necessary, carry out a modelling process again (validate).

The connection between mathematical and financial competences has already been established by the OECD (2017). Financial literacy tasks have the potential to promote conceptual and process-related competences for both financial and mathematical literacy (Ozkale & Erdogan, 2022). However, only a few publications address the joint promotion of these competences (Dituri et al., 2019; Ozkale & Aprea, 2024). Modelling tasks in particular have this potential to develop students' financial literacy (Cabrera-Baquedano et al., 2022), because mathematical models can be used to describe and process financial situations in order to make a decision.

Although there are already approaches to developing mathematical tasks that can be integrated into mathematics lessons using financial literacy concepts (Ozkale & Aprea, 2023), these do not explicitly take into account mathematical modelling approaches. For mathematical modelling, however, it was shown qualitatively in a case study that students with experience in financial contexts were more successful in formulating financial comments and linking mathematical knowledge. It is possible that the mathematical modelling tasks encouraged them to discuss and apply different approaches that enabled more advanced financial thinking (Tural-Sonmez & Erbas, 2023).

Research question

In order to work out the connections between the concept of financial literacy from PISA and mathematical modelling as a competence, and to be able to answer the overarching question of how mathematical modelling and financial literacy are related, we focus here on the OECD's concept of financial education (OECD, 2019, p. 128) and the definition of mathematical modelling competence according to Niss et al. (2007) as well as the concretisation in sub-competencies (Kaiser, 2007; Maaß, 2006).

Based on the correlations between mathematical and financial literacy that have already been investigated (Ozkale & Erdogan, 2022), correlations between mathematical modelling and financial literacy in PISA are also to be expected (Tural-Sonmez & Erbas, 2023). What is interesting here is the precise overlap between mathematical modelling and financial literacy. It remains to be seen which sub-competencies of mathematical modelling play a particular role here. We therefore ask the question:

For which sub-competencies of mathematical modelling do the PISA tasks on financial literacy provide an opportunity for promotion and how are these distributed across the various processes of the competence framework for financial literacy?

Method

All 26 published PISA items on financial literacy were analysed. These were 3 items from the 2022 study, 14 items from the 2018 study and 9 items from the 2012 study (OECD, 2023a). The study was conducted as a qualitative content analysis according to Mayring (2015) with the sub-competences of modelling as deductive categories (Kaiser, 2007; Maaß, 2006). For this purpose, all items were rated to determine which sub- competencies they offer an opportunity for. The categories were as follows: understanding, simplifying, mathematising, working mathematically, interpreting and

validating. None, one or more sub-competencies were assigned to each item. The coding was carried out independently by both authors. The interrater reliability is on average almost perfect $\kappa = .88$ (Landis & Koch, 1977). In case of disagreement, a consensus rating was performed. Further categories on subject areas, context areas and processes for financial education as well as the competence levels of the items were taken from the information on the items (OECD, 2023a).

One example is the item 'Music system' from the 2018 PISA study (OECD, 2023a, p. 208). The stimulus of the item describes that Kelly asks her bank to lend her 2000 zeds to buy a music system and that there are two options for the repayment period, with the interest rate being the same in each case. For the repayment period of two years, the information is given in Table 1.

Table 1: information for repayment period of two years

Repayment period	Monthly repayment (zeds)	Total repayment (zeds)	Total interest paid (zeds)
Two years	91.67	2200.08	200.08

The question is then how the conditions for a loan of 2000 Zed over three years differ from the repayment terms over two years. Two statements are to be marked with True or False. These are:

- The monthly repayment will be higher for a loan over three years.
- The total interest paid will be larger for a loan over three years.

The task is therefore about determining the effects of extending the loan repayment period from two to three years on the monthly interest payments and on the total interest payments if the annual interest rate does not change. It is therefore classified in the subject area 'Planning and managing finances', the context area 'Individual' and the process 'Analysing information in a financial context'.

If you want to solve this task with the help of a calculation, the difficulty is that the interest rate is unknown. However, it is possible to compare corresponding models with an assumed interest rate and thus solve the task with the help of modelling. A simplified calculation with annual interest payments and a term of two or three years already clearly shows that the annual repayments are lower in the case of three years, but the total interest is higher with the same annual interest rate. For example, with an interest rate of 10% and annual repayment, the rate for 2000 € would be around 348 € higher, but the interest would be around 108 € lower. These considerations show the potential of using mathematical models for these tasks. This required the modelling steps of understanding, simplifying, mathematising, working mathematically and interpreting. However, no calculation is expected to solve the item, as it is a combined multiple-choice item, meaning that this task could also be answered using experience from previous calculations or 'world knowledge'. This would also require fewer modelling steps. However, this task was coded with the sub-competences mentioned, to recognise the potential of mathematical modelling and because this 'world knowledge' cannot necessarily be assumed for solving the task.

Results

The subject areas, context areas and processes of the 26 items analysed are distributed differently across the categories. The subject area ‘Money and Transactions’ and the context area ‘Individual’ occur most frequently (see Table 2). It should be noted that the ‘Risk and reward’ category and the ‘Education and work’ and ‘Societal’ contexts occur very rarely or rarely. Except for the rare ‘Apply financial knowledge and understanding’, there is an even distribution of processes.

The sub-competence ‘understanding’ of mathematical modelling can be assigned to every item, while ‘validating’ was not explicitly coded in any of the published items. The other sub-competences occur in 10 to 15 of the 26 items, as listed in Table 3. There were 11 items in which none of the four categories from Table 3 occurred and 10 items in which all four of the categories occurred. Working mathematically in a mathematical model occurs least frequently (approx. 40 % of the items) and simplifying most frequently (approx. 60 % of the items).

Table 2: Distribution of financial categories

Content	Number	Context	Number	Process	Number
Money and Transactions	11	Individual	15	Identify financial information	7
Planning and Managing Finances	7	Home and family	7	Analyse information in a financial context	8
Risk and reward	2	Societal	3	Evaluate financial issues	8
Financial landscape	6	Education and work	1	Apply financial knowledge and understanding	3

Table 3: Distribution of the Competency Categories

Simplifying	Mathematising	Working mathematically	Interpreting
15	13	10	11

In order to examine the connection between the various processes of the competence framework for financial literacy and the sub-competences of modelling in more detail, these are presented in detail. If the proportion of the respective sub-competencies is shown individually for the processes of the financial categories, a distribution emerges in which more sub-competencies are required to complete the tasks for the processes ‘Analyse information in a financial context‘ and ‘Apply financial knowledge and understanding’ than for the other two processes (see Figure 1).

Percentages are used here because different numbers of items are assigned to the four processes. In the three processes to which roughly the same number of items are assigned, the proportion of modelling sub-competencies is highest for ‘Analyse information in a financial context’ and lowest for ‘Evaluate financial issues’. The comparison with the ‘Apply financial knowledge and understanding’ process is more difficult here, as this process is based on only 3 items (see Table 2).

In all cases, the sub-competencies of simplifying and mathematising are more strongly or equally represented than working mathematically and interpreting.

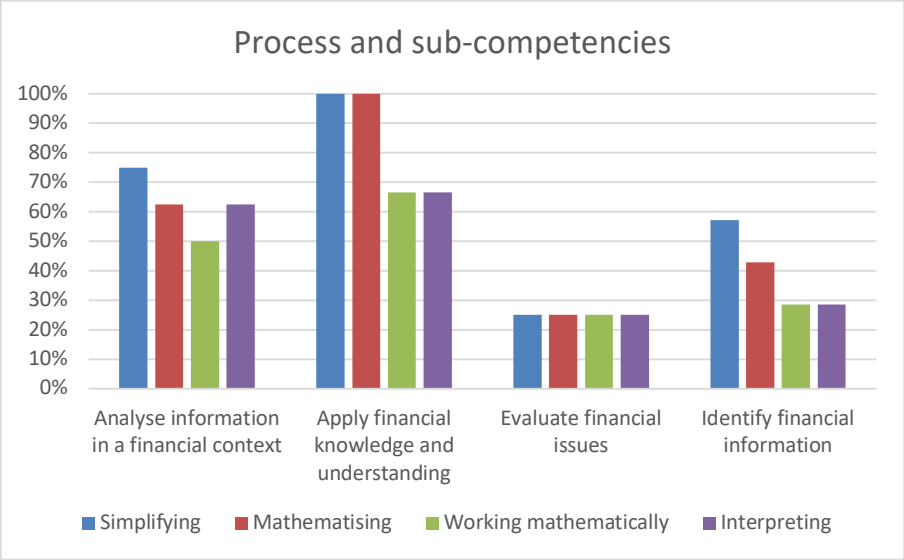

Figure 1: Distribution of the sub-competence categories for the processes

Discussion

The analysis of the published PISA task examples shows that the 26 items for financial literacy from PISA map different subject areas, contexts and processes for financial literacy in varying numbers. On average, the sub-competencies of understanding, mathematising, working mathematically and interpreting mathematical modelling are required or at least useful for solving about half of all items.

The importance for mathematical literacy is therefore likely to be even greater, as only one part, albeit a very important one, is considered here, namely mathematical modelling. It can therefore be assumed on average that mathematical competences are fundamentally useful for the processing of financial education tasks for at least half of the sample items. This is in line with ideas for the simultaneous promotion of mathematical and financial competences (Dituri et al., 2019; Ozkale & Aprea, 2023, 2024). The positive correlation between modelling competences and financial literacy, as seen by Tural-Sommez and Erbas (2023), can also be partially confirmed by these results. However, it should be noted that significantly less than half of the financial literacy items require calculations in the mathematical model at all. It is rather the sub-competencies of simplifying and mathematising that are useful in at least half of the sample items. The reason for the rather low proportion of required mathematical competences could also be due to the design of the PISA tests to measure financial literacy and mathematical literacy separately. However, this could also signal that mathematical competences are only of minor importance for financial education.

It is also interesting to take a differentiated look at the four processes of financial literacy. It can be seen that there is more modelling in the two processes ’Analyse information in a financial context’ and ’Apply financial knowledge and understanding’. This may offer the greatest potential for the joint promotion of financial literacy and mathematical modelling. It should also be noted that the existing

tasks could have a much stronger modelling component with few changes. For example, in the example task 'Music system', the extent of the change in the total interest rate could also be made clear by means of a concrete calculation, thus also enabling validation. Although the PISA documents tend to see financial situations only as a context for mathematical competence, the result reinforces the idea of teaching financial literacy together with modelling competences (Ozkale & Erdogan, 2022). Similar results can also be expected for other models of financial education, such as that of Retzmann and Seeber (2016), because the areas of competence described there overlap with the OECD's concept (European Union/OECD, 2023).

One limitation, however, is that there are relatively few published examples of tasks on financial literacy, which may not fully reveal the connection to mathematical competences and specifically to mathematical modelling. It should also be noted that the items relating to the four processes from the financial categories are not equally distributed, but that three processes occur somewhat equally frequently and one less frequently (s. Table 2). What is even more striking is that only one content and one context occur particularly frequently and there is even a context that only occurs in one item. So there could be distortions here for this reason. In addition, the extent to which learners actually carry out the coded modelling processes during actual processing would also have to be discussed.

In summary, there is potential to promote both areas of competence, i.e. modelling and financial literacy, which is currently not being fully exploited. For some items, it would also be possible to demand more specific mathematical activities in the areas of mathematising, mathematical work, interpreting and validating.

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One step at a time? – An empirical study on the use of graphical representations for solving Fermi problems

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Fermi problems have been established as underdetermined tasks in mathematics education and require students to make their own assumptions. So-called Fermi graphs can be created by the students when solving a Fermi problem and contain information about their activities. In this study, we empirically compare the solution of Fermi problems and the use of Fermi graphs by primary and secondary school student. The results show that both primary and secondary students can solve the Fermi problem completely or with calculation errors only. When creating the Fermi graphs, the eighth graders are more often able to implement the structure of the Fermi graphs, whereas the third graders also create visualisations that cannot be recognised as Fermi graphs. Based hereon, it is possible to identify consequences for the practical use of Fermi graphs in mathematics education.

Keywords: Mathematical models, Fermi problems, primary school, secondary school, graphs

Fermi problems and their graphical representation

Since the introduction of Fermi problems by Enrico Fermi (1901–1954), this type of problem has aroused great interest not only in the scientific community, but also in education, e.g., in mathematical modelling (Ärlebäck & Bergsten, 2010). Fermi problems can be characterised as underdetermined open tasks with "very little or no information given" (Efthimiou & Llewellyn, 2007, p. 254) which therefore – at first glance – appear to be *unsolvable*. Still, through decomposing the problem in several parts and recombining them later on, a solution can be obtained, which is known as the *Fermi method* (Carlson, 1997). Hereby, it is up to the problem solver to consider what data is needed and how it can be collected, which makes data acquisition essential for processing Fermi problems (Meyer & Greefrath, 2023). The missing data can be obtained by research, making estimates and assumptions, but also by using general knowledge or even experimental approaches. Consequently, Fermi problems are characterised by the fact that they do not have one exact result but are approximations that can be represented in a solution interval (Ärlebäck, 2009). This task type is particularly relevant in mathematics education when certain steps of the modelling cycle (cf. Blum & Leiss, 2007), e.g., simplification and mathematisation are to be promoted (Gallart et al., 2017). Further, dealing with inaccuracies can be addressed, i.e., working with unknown quantities and estimating (Ärlebäck, 2009). However, the open nature of the tasks makes it difficult for students to structure their assumptions and approaches as well as for teachers to recognise the students' difficulties.

For this reason, an effort is to make the solution process for Fermi problems visible and analysable. In terms of a tool that supports teachers in analysing their students' processes, Ärlebäck and Bergsten (2010) use the *modelling activity diagram* to connect the processes of students engaging with Fermi tasks to the steps of the modelling cycle by Blum and Leiss (2007). Another example is the *Fermi problem activity template* which aims at supporting the analysis of Fermi problems through the

different methods of data collection that have been acquired (cf. Ärlebäck & Albarracín, 2023). Both examples have mainly been used from the teacher's perspective aiming at an analysis of a Fermi problem in the context of task design and finished solution processes. Besides giving teachers an idea of their students' work, so-called *Fermi graphs* can additionally be used by *students* already during their solution process, aiming at supporting them in terms of structure and interrelation of assumptions (cf. Borys & Hartmann, 2022).

A Fermi graph captures the solution to a Fermi problem using a network structure consisting of *nodes* and *edges* (Borys & Hartmann, 2022). Using the terminology of graph theory (cf. Tutte, 2001), the Fermi graph is connected and acyclic which makes it a tree. The tree is directed with all edges having an orientation to one node that is distinguished as the graph's *root*. To reduce complexity, the orientation is not presented through arcs in the visualisation. One can speak of the *in-tree* type which means that all considerations lead to the Fermi problem's solution. In the Fermi graph, the assumptions made and (intermediate) results achieved are noted on the edges. In the nodes, the data received from the edges is processed and the respective arithmetic operations are noted. The respective *Fermi activities* (cf. Borys & Hartmann, 2022) are located above the nodes, e.g. activate general knowledge (GK), set an assumption (S), extrapolate (EX), estimate (E) or validate (VA). In Figure 1, we present an example of a Fermi graph that summarises a possible solution process for the task: *How many Christmas cookies are eaten in Germany in one year during advent season?*

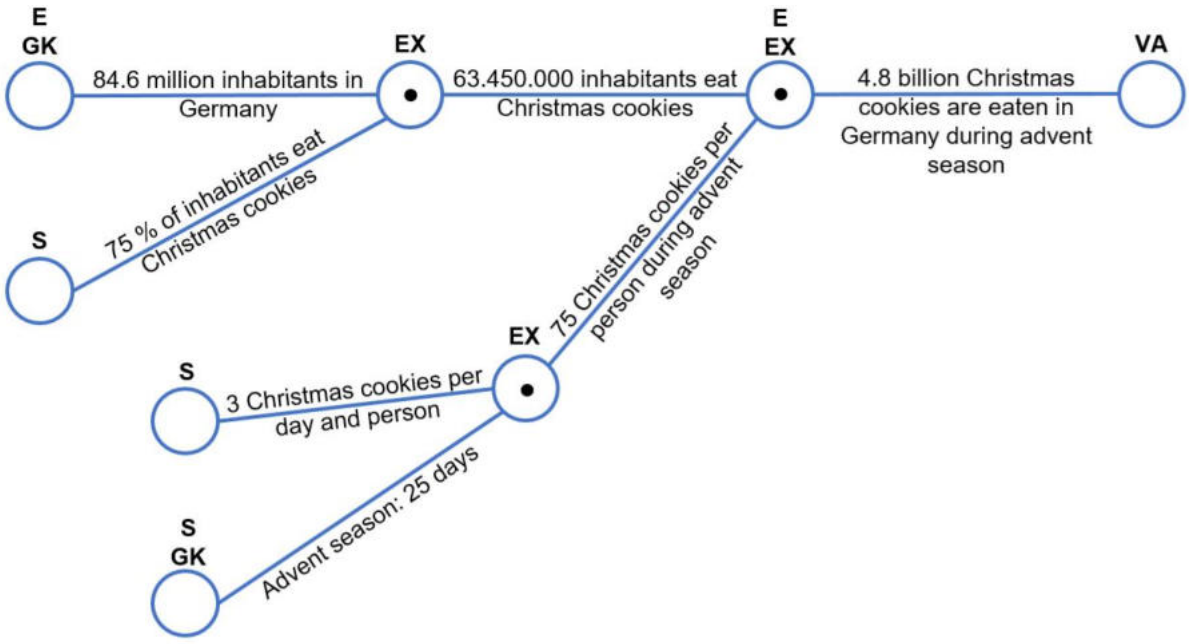

Figure 1: Example of a Fermi graph with assumptions and intermediate results on the edges and arithmetic operations and Fermi activities linked to the nodes

As indicated, this example is just one way of visualising the task's solution process: Based on the students' assumptions and calculations, different graphs appear. This allows us to ask why the graphs differ in appearance, so that students recognise that there is more than just one plausible solution to a Fermi problem (Borys & Hartmann, 2022).

Research question

From previous studies it is known that Fermi tasks can be used in both primary and secondary schools in the context of mathematical modelling (e.g., Ärlebäck, 2009; Ferrando & Albarracín, 2021). Haberzettl et al. (2018) noticed in a study on primary level that grade-three students are interested in the different possible solutions, still, it causes uncertainty at the same time. It turns out that the children make few to no assumptions when working on their first Fermi task with little knowledge of both procedure and documentation of the solution process. In addition, it is unclear to them which values are required for the calculation and how they could be meaningfully integrated into a calculation (Haberzettl et al., 2018). Some of the findings can also be seen in studies on secondary level, too. When students are confronted with underdetermined tasks for the first time, they often find it difficult to make estimates, assumptions and simplifications (Surel & Greefrath, 2023).

In view of the above-named goals of the Fermi graph (cf. Borys & Hartmann, 2022), we assume it to be a tool for primary and secondary school students to visualise their solution process, including the documentation and interrelation of their assumptions made. It could therefore make their own results comprehensible and justifiable for themselves and others. We see the Fermi graph as an opportunity to practically implement the work with underdetermined tasks and to structure them more clearly for the students. Still, studies that empirically analyse its use in general and in a comparison of different age groups are, best to our knowledge, currently missing. In addition, we see that the visualisation tools on Fermi problems presented so far focus on task design and analysis. Findings on how students use Fermi graphs are interesting for the practical use of Fermi problems and Fermi graphs in mathematics lessons. This paper aims to address this field of interest by answering the question: *How do primary and secondary school students use Fermi graphs and which challenges can be observed?*

Method

To answer the research question, an empirical, qualitative study was conducted in the context of an extra-curricular enrichment programme for potentially mathematically gifted students (cf. Jablonski & Ludwig, 2019) at Goethe University Frankfurt during Winter 2023. We chose gifted and interested students for the explorative approach hoping that the children would engage with the potentially unknown type of task and the graph structure. The sample of the study consists of two groups: 19 grade-three students (9-10 years old) and 12 grade-eight students (14-15 years old).

The data collection in both groups happened in the context of a 90-minutes session each. The sessions were conducted identically for both grades but independently of each other. In the first phase of the session, an introduction of Fermi problems in general was given. Based hereon, the students got to know the idea of a Fermi graph via an example and 14 Fermi activities for their solution process (examples are presented in Figure 1; Borys & Hartmann, 2022). In the second part of the lesson, the students were divided into teams of two or three (seven teams in grade three and five teams in grade eight). Each team received material with the above-introduced Fermi problem “Christmas cookies”, an overview of the Fermi activities as well as an example graph. In addition, each team received a laptop for research and a poster for designing the Fermi graph. In the last part of the session, the Fermi graphs were presented and compared in a gallery walk.

With reference to the research question, we particularly focus on a comparison of the processes and challenges in primary and secondary grade. In terms of the data analysis, several steps are undertaken: In a first step, the calculated results are analysed in terms of their completeness. No interval is specified at this point, as the focus should be on the step-by-step justification and calculation of the teams. A result is categorised as *complete* if the result is comprehensible in terms of the assumptions, values and calculations. We also use this category if the procedure is comprehensible, but the handling of (large) numbers is incorrect or if calculation errors occur. A result is categorised as *incomplete* if it is unclear where the result comes from, as, for example, important information (e.g., the number of inhabitants or the duration of advent season) is not taken into account.

In a second step, the designed graphs are analysed. In terms of justifying our analysis of the *structure* of a Fermi graph, we refer to selected definitions from graph theory (cf. Tutte, 2001). Per definition, a Fermi graph should be a directed tree with all edges leading to one node, the graph's root. The formula $|E| = |V| - 1$ (with *E: edges* and *V: nodes*) together with the Fermi graph being acyclic and having a root can prove this condition (*criterion I*). The students' initial assumptions for the task solution are external nodes. We set that a Fermi graph needs to have at least two external nodes in order to involve more than one assumption for the Fermi problem and follow the Fermi method, i.e., decomposition and recombination (cf. Carlson, 1997). Additionally, we refer to the incoming degree of each node which is defined as the amount of neighbour nodes on the left hand side. The incoming degree of each internal node is set at least two (*criterion II*). In addition, each node should be related to at least one (adequately linked) Fermi activity and each edge should summarise an assumption of (intermediate) result (*criterion III*). Based hereon, we distinguish whether a graph can be categorised as a Fermi graph and if so, whether it fulfils the criteria completely or partially. We categorise a graph as a *Fermi graph* if all three criteria are fulfilled. It is a *partial Fermi graph* if Fermi activities are used or named incorrectly, if any activities are used (*criterion III*) and/or if the start and end nodes or connecting edges are forgotten to be drawn (*criterion I*), but the general approach of a Fermi graph can still be seen from the representation. It is *no Fermi graph* if the representation is only a "line", so that the degree of the incoming internal nodes and the number of leaves is not at least two (*criterion II*). From those graphs, in which we found problems, particular challenges of the groups are examined.

Results

We start the report with the *calculated results*. Figure 2 (left) summarises these results. The seven teams from grade three have a span from 928 million – 17.9 billion Christmas cookies and the results of the five teams from grade eight range from 2.1 billion – 28 billion. The mean values are 5.5 billion in grade three and 8.8 billion in grade eight. As an impression, we refer to our example calculation in Figure 1, in which we obtained a result of 4.8 billion Christmas cookies. Included in our result are the population of Germany, a fixed proportion of people who eat cookies, an assumption of how many cookies are eaten per day and the approximate duration of the advent season.

Considering the *completeness* of assumptions as well as the calculation processes of the grade three teams, the results of five of the seven teams are categorised as complete, whereof we saw incorrect calculations in the solution of one team. The five teams used the aspects of the population of Germany, the fixed consumption of Christmas cookies per person per day (ranging from two to ten)

and the duration of the advent season (lasting between 22 to 24 days). Of these groups, three assumed that all people living in Germany eat cookies during advent. The other two teams assumed, based on research or their own judgements, that there are people who do not eat Christmas cookies during advent. The results of the two remaining teams were incomplete. They did not take the advent season into account. For grade eight, the categorization results in four out of five teams having complete results, with incorrect calculations in one team. One team achieved an incomplete result. In general, a greater variation in the assumptions made can be observed, which also led to the greater variation in the group's results: In terms of population, four of the five teams assumed that everyone living in Germany eats Christmas cookies during advent and one team made restrictions here. The fixed consumption per person and day was done by four teams, ranging from one to 15 Christmas cookies. The remaining team instead assumed the consumption of cookies per person throughout the year, meaning that they did not take the advent season into account and answered the task incompletely. The other groups assumed the advent season to be 22 to 25 days long.

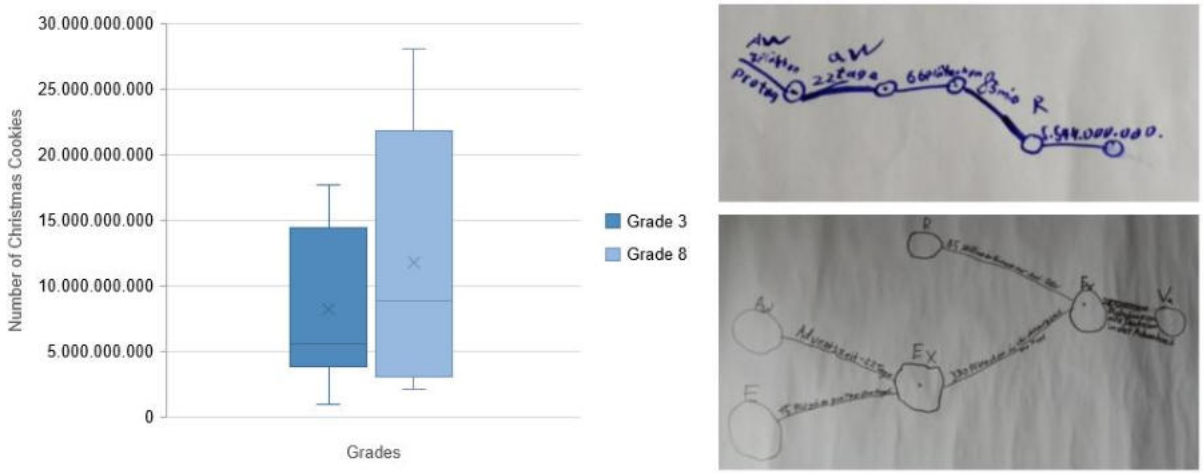

Figure 2: Calculated results (left) and two graphs from grade 3 (right above) and grade 8 (right below)

We focus on the category *structure* of the Fermi graph in the following, starting with grade three. In terms of *criterion I*, three of seven teams had problems in creating the general structure of the Fermi graph with an incorrect difference in the number of edges and nodes. Also three (partly different) teams showed difficulties in fulfilling *criterion II* and the idea of the Fermi method. Instead, the teams created a *line representation* (e.g., Figure 2 right above). Concerning *criterion III*, we can see that, except one team, all teams had problems in the area of linking Fermi activities (adequately) to each node of the graph. Some (internal) nodes were not labelled with activities at all. Similar problems can be seen in the labelling of edges. With the categorization based on these criteria, it can be stated that four teams from grade three created a partial Fermi graph, whereas the graphs of the remaining three teams were not categorised as Fermi graphs. In grade eight, all five teams fulfilled *criterion I* and *criterion II* which means that the general structure of the Fermi graph could be observed. This explains why every graph was categorised as a Fermi graph. Two teams fulfilled all criteria of a Fermi graph (e.g., Figure 2 right below), whereas the other three teams created a partial Fermi graph. The challenges in these teams came from *criterion III* concerning the labelling of nodes and edges with missing or wrongly linked Fermi activities, assumptions and intermediate results.

Linking the results from both categories (see Table 1), it can be reported for grade three that the combination of a partial Fermi graph and a complete result appeared most often. With the partial Fermi graph giving an idea of the overall structure despite missing elements, a potential guidance for a complete result can be assumed. Still, even for the category *No Fermi graph*, results on both levels of completeness were observed. For grade eight, the combination of a complete result and a Fermi graph appeared twice. Also here, the partial Fermi graphs came together with both complete and incomplete results. It is therefore not possible to make a general statement on the Fermi graph's usefulness for a result's completeness on the basis of the data.

Table 1: Combined categorization of the teams' results and graphs (grade 3 / grade 8)

		Fermi graph	Partial Fermi graph	No Fermi graph
Grade 3	Complete result	-	3	2
	Incomplete result	-	1	1
Grade 8	Complete result	2	2	-
	Incomplete result	-	1	-

Discussion

From the results, we can draw conclusions concerning our research question on the use of Fermi graphs in primary and secondary level. The teams from both grades were able to work on the Fermi task and most of the teams achieved a result that was categorised as complete. We can therefore confirm the observation of previous studies (e.g. Ärlebäck, 2009; Ferrando & Albarracín, 2021) that both primary and secondary students are able to meaningfully work on Fermi problems.

Concerning the results, it is remarkable that the teams from grade three calculated more homogenous results in contrast to the teams from grade eight. In addition, we can see that the teams from both grades generally took similar assumptions into account, i.e. in terms of the population of Germany, some people not eating Christmas cookies, the idea to estimate the amount of cookies per day and person as well as the duration of advent season. With the above described exceptions, it can be stated that the teams of both grades worked on a comparable level of completeness and detailedness which is a remarkable result in terms of implementing Fermi problems in primary school. At this point, it has to be limited that the students were participants of an enrichment programme, which might have made them more familiar with open tasks in general. So it might have been easier for them to set assumptions and to deal with the open character (cf. Haberzettl et al., 2018; Surel & Greefrath, 2023). Special knowledge about Fermi problems was not acquired beforehand.

Differences between the grades can be seen in the analysis of the Fermi graphs. It is noticeable that all graphs from grade eight were at least partial Fermi graphs. The reason for this categorisation is that the eighth grade teams only forgot/wrongly named Fermi activities, but the created graphs generally have the character of a Fermi graph. The grade three teams, in contrast, also produced

graphs that were not categorised as Fermi graphs since a line representation arose for some teams. Perhaps the older students' knowledge of probability trees, for example, helped them to realise the graph structure. This knowledge cannot yet be assumed in the third grade. Also, the use of Fermi activities was more consequently done by the grade eight students. This means that these teams placed more mathematical activities on the nodes than the primary school children, who often omitted the node labelling with activities. It can be assumed that the students from the third grade did not understand some Fermi activities, e.g. extrapolation, so that they did not use these mathematical activities or used them incorrectly. Possibly, some teams were already overwhelmed by the graphical representation of the solution process and paid less attention to mathematical activities. Based on this observation, the question arises as to whether the activities could already be adequately understood after a brief introduction. In order to explain the underlying reasons in more detail, it would be necessary to interview the individual groups to find out why they used no or incorrect mathematical activities. Based on the problems described above, it can be assumed that the Fermi graph might be too complex in its structure for primary school children or that the intervention was too short/not incorporating fundamental graph-theoretical knowledge to better understand the idea of the graph.

The results show that a Fermi graph can generally be created and used by eighth-graders after its first introduction. Due to the sample size, the combination with the results obtained does not yet provide any indication of how the graphs can, for example, support the solution quality of a Fermi problem. Similarly, the usefulness perceived by the students is an open question for further research. Both should be examined with a control group (solving the Fermi problem without a Fermi graph) in future research. For grade three, the question arises how the introduction of Fermi graphs can be revised and simplified. For example, it seems reasonable to first practise with the children to transfer their arithmetic solution process into the graph structure before mathematical activities are added (in a reduced form). As an outlook, we will further investigate a long-term intervention on Fermi graphs. In a gallery walk with all produced graphs following the lesson, the children independently mentioned criteria for the complexity and comprehensibility. They recognised, for example, that the line representation cannot fully reflect the solution process. It is possible that these criteria could lead to clearer structures and documentation when creating their next Fermi graph.

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The role of transpositive analyses when discussing modelling in disruptive times

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In this discussion paper we address the potential need for curriculum reforms due to technological development and pressing wicked problems in the disruptive times we live in, and how these may affect what is to be taught and learned regarding mathematical modelling in schools. The paper draws on the Anthropological Theory of the Didactic (ATD) in terms of the notion of didactic transposition. Recent research proposals for including STEM modelling, data modelling, critical and responsible citizenship, and data science point to the need for more detailed analyses of what practices and knowledge can and should be transposed to school mathematics, before imposing radical changes of curriculum, difficult for teachers to implement and navigate.

Keywords: Didactic transposition, curriculum analyses, modelling, interdisciplinarity, technology

Introduction

The most recent ICTMA (International Community of Teachers of Mathematical Modelling and Applications) book series is published under the title "Researching mathematical modelling Education in Disruptive Times – an introduction" (Siller et al., 2024). It states that mathematical modelling is a fundamental activity in general education, and we see from events like the pandemic and political decision making e.g. regarding climate changes (e.g. Strømskag, 2023) that a significant amount of decision making is based on modelling in various forms:

> "Thus, an understanding of mathematical modelling and models needs to be promoted across schooling and in other institutions devoted to education. Recent events and phenomena have also demonstrated the key role of mathematics in forming judgements and making decisions within societal contexts." (Siller et al., 2024, p. 3).

It has been argued that we need further research on data modelling and statistical modelling (Kawakami & Ärlebäck, 2022; Noll et al., 2022), which are key concepts for data (science) modelling. English and Watson (2018) claim that the increasing prevalence of data-driven environments emphasize the importance of teaching students diverse approaches to navigating data modelling presenting a case study differing between mathematical and statistical actions taken by students when addressing a real world situation. Dealing with large amount of data, will often require explicit uses of digital technologies and therefore, they may take a more dominant role in modelling education in the future. But further research is needed in classrooms and for the education of pre-and in-service teachers (Siller, Geiger & Greefrath, 2023). Thus, proposing modelling education capturing modelling in a digitalised and data driven world, needs to consider the skill set of teachers to teach the activities within school mathematics. An issue which only will increase as modelling is considered crucial to citizenship education (Maass et al., 2022), where elements of critical evaluation of one's own and others' modelling are crucial. This relates to 'technocritical mathematics education', which urges mathematics education to let students explore and unpack hidden mathematics in technological

solutions to realise how this impact our daily lives (Jankvist et al., 2023). But does this change how we perceive the notion of modelling?

For decades, different approaches to modelling education have been discussed, leading to different epistemological reference models affecting task design for modelling. The most dominant employed and discussed in TWG06 is modelling competency and modelling cycles, models and modelling, Fermi problems and study and research paths (Ärlebäck et al., 2023). Barquero and Jessen (2020) analyse how some of these theoretical approaches for task design impact what students can learn of and from being engaged with modelling activities. However, the cases analysed do not heavily draw on digital technologies nor large amount of data. For this we may seek inspiration in literature addressing e.g. data science. Heinemann an colleagues (2018) proposes a curriculum for out of school activities held at universities in Germany, explicitly drawing on both mathematics and computer science. But is it sufficient to leave modelling in terms of data science as mere add-ons to ordinary teaching? In Denmark private funding has led to development of initiatives concerning data science for upper secondary mathematics, though still limited to transmission of techniques and methods (Jessen, 2023; Tarp & Frantzen, 2023), not linking data science and mathematical modelling to inquiry based education, though modelling has been considered so in research (Artigue & Blomhøj, 2013). If mathematics education is supposed to prepare students to navigate a data driven reality through modelling activities, curriculum changes may be needed to accommodate other practices than already existing in modelling curricula. Below we elaborate how didactic transposition, from the Anthropological Theory of the Didactic (ATD), can guide and inform such changes, before presenting current challenges linked to transposition efforts, and modelling activities proposed in connection to this debate.

Curriculum reforms and transpositions

If future task design on modelling needs to capture the above stated trends of disruptive times promoting data driven modelling, we must consider how this affect curriculum changes with respect to modelling. The most dominant approach to curriculum analyses and reform efforts is the distinctions presented by Mullis (2019) as part of the TIMSS studies: the intended curriculum (national, societal and educational context), the implemented curriculum (home, school, teacher and classroom contexts), and the attained curriculum (student achievements and attitudes). Despite its wide uses, the curriculum model does not take the dynamic shaping of curriculum into account, as "[…] curricula interact with the conditions and constraints of their functioning to catalyse change in the state of educational systems." (Barquero et al., 2023, p. 193). Few theoretical approaches seem appropriate for analysing coherence (or lack of coherence) in mathematics curriculum reforms in terms of unintended consequences of reforms, implications of teachers being active in developing implemented curricula, rather than mere receivers of curriculum, to identify lost opportunities etc. Moreover, a

> "finding that does stand out and is likely to be generally applicable is that […] many reviews of mathematics curriculum reforms are not well supported by a clear theoretical basis that guides the methodology used. This necessarily limits the robustness of the analyses and, very likely, the impact of the work. […] (Barquero et al., 2023, p. 214)

In the context of mathematical modelling, we need to consider is such analyses are need on how to implement data modelling, stochastic modelling, data science and techno critical mathematics education. More concretely, what does it mean for school mathematics to develop students to responsible and critical citizenship? What actions, should students be able to perform? What methods should they apply? Answers to these questions depends on level of education and we should draw on existing research regarding the shaping of modelling in different school contexts. Barquero et al. (2023) point to ATD and particularly the notion of didactic transposition as one theoretical construct allowing us to provide somehow objective analyses of what and how to teach school mathematics. The process of didactic transposition is depicted in Figure 1.

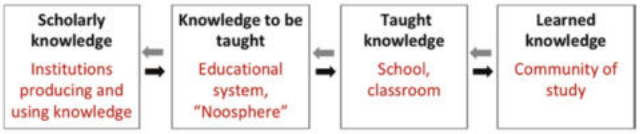

Figure 1: The process of didactic transposition from producing institutions to learned knowledge (Barquero et al., 2023, p. 198)

One of the main contributions of didactic transposition theory when linked to the question of curriculum reforms and implementations is to take scholarly knowledge into account. Bosch (2015) argues that:

> "Before focusing on how children learn and how we can teach them—the viewpoint of the teachers' institution—the attention is first put on what is learnt and taught, its nature (what it is made of), origin (where it comes from) and function (what it is for)." (Bosch, 2015, p. 54-55)

Still, an integral part of the curricular conundrum, as Chevallard (2018) calls it, is the fact that we have to consider the situation from a holistic societal viewpoint, where people in different positions have different relations to the knowledge at stake. In the scholarly context, researchers in public and private positions are developing and using knowledge, such as data science tools and notions, but is that relatable to school mathematics? Most often the external didactic transposition is carried out by the noosphere which is constituted differently from one country to another. The noosphere e.g. covers curriculum developers from ministry of education, politicians, different stake holders, educational researchers, textbook authors etc. (Jessen & Kjeldsen, 2021). Through their production of curriculum documents of various forms, they produce the knowledge to be taught. The internal didactic transposition is mainly managed by the teacher delivering the taught knowledge, based on selections and interpretations of the curriculum documents. The last step in the transposition is learned knowledge within the students. And as all teachers know, this may again be slightly different from what was taught. Thus, to realise the above needs regarding data modelling and practices together with the development of responsible and critical citizenship, transposition of knowledge may need more transparency through well drafted examples making it explicit the knowledge and methods being transposed.

In the following we presents examples of didactic transposition of mathematical modelling, data science and links to tasks proposed to capture and further develop the critical mindset strived for in literature. We do that by pursuing the following research question: What is the potential role of

didactic transposition in research proposing changes of how we teach modelling during disruptive times in a data driven world? And how do the proposed modelling activities link to these findings?

Didactic transposition of mathematical modelling

Didactic transposition analyses of mathematical modelling already exist. Jessen and Kjeldsen (2021) have analysed the external didactic transposition of modelling in upper secondary mathematics, using historic cases of model construction from STEM fields going across disciplinary boundaries reflecting school mathematics and interdisciplinary settings for STEM education. Jessen and Kjeldsen compare the historic representations of modelling with 'knowledge to be taught' in terms of curriculum documents, textbooks and exam exercises, finding a certain impact of e.g. modelling cycles. Still, other crucial aspects of modelling construction such as explorations, uses of analogies and the negotiation of epistemic value across intra- and extra-mathematical domains were missing when compared to the scholarly domain. Frejd and Bergsten (2016) analyse current modelling practices in workplaces, meaning mapping the scholarly knowledge of modelling practices form public and private institutions and how these may relate to school practices. They conclude that models constructed by students are rarely put into use, and the division of labour between expertise is not realised, as students often take same classes and is considered to have similar prerequisites. As such they argue that

> "These major differences between modelling work in educational and non-educational contexts seem to entail that mathematical modelling in school becomes an unrealistic utopia in terms of coherence to non-educational professional practice." (Frejd & Bergsten, 2016, p. 31).

This may also hold true for other domains in school mathematics, but certainly needs to be addressed when promoting future modelling activities or curriculum changes. Moreover, if scholarly knowledge justifies the teaching of school mathematics, rethinking meaningful connections may be relevant (Jessen & Kjeldsen, 2021). Also in statistics education, data science and digital technologies may shape the future of education and research, leading to similar considerations as seen in mathematical modelling:

> "The advent of data science has provided new challenges and opportunities for statistics education researchers to re-think what it means to model. […] How can new simulations and visualizations using technological tools increase students' access and agency for telling data stories, and enrich understanding in modeling processes?" (Noll et al., 2022, p. 331).

A first attempt of transposing data science into a mathematics curriculum for upper secondary mathematics is presented by Boaler and colleagues (2024), who report on a high school applied mathematics course exploring basic elements of data science. Their analyses indicate that the course leads students to take more mathematics courses, and students anticipate carriers in STEM (Science, Technology, Engineering and Mathematics). The course is project based and organised around the data science cycle of inquiry starting with posing questions, gathering and organising data. As such the course aligns with the proposed data science curriculum developed for upper secondary students in Germany proposed by Heinemann and colleagues (2018). However, the replacement of classic mathematics courses with data science has been criticised for hindering students in pursuing higher

education in STEM due to the lack of calculus preparing courses (Conrad & Mazzeo, 2022), though the data science course has proved an important addition to calculus pathways, allowing students to exercise agency while learning mathematics and applications:

> "[…] for all students, data science courses help students develop the ability to read data, create mathematical models, use technological tools, and interpret findings—important understandings for young people taking their place in our newly data-filled world." (Boaler et al, 2024, p. 15).

Despite these descriptions of data science aligns well with the initial justifications for teaching modelling as a fundamental activity in mathematics education generally, none of the new initiatives explicitly link data science with mathematical modelling, which already is part of mathematics curricula across the world. Below we briefly look at example of activities proposed to promote students' critical citizenship in data driven worlds, based on application of digital technologies in various forms.

Proposed cases of modelling with technology and data

Jankvist and colleagues (2023) do not discuss mathematical modelling, but problematise current state of society and mathematics education where hidden mathematics represent challenges in equity in access to democratic decision making. They question if students gain full autonomy over their learning processes due unreflect uses of digital technologies. They argue that:

> "Even though these insights reveal different takes on the consequences of packing and unpacking mathematics for students' learning, none of them talk about the processes of packing and unpacking as goals in their own right. This is the purpose of *technocritical mathematics education*" (Jankvist et al., 2023, p. 10).

They propose public-key cryptography (RSA) as knowledge allowing student to unpack hidden mathematics highly important to society. The case certainly meets the goal of engaging with genuine models drawing on notions like prime numbers, greatest common divisor, but the models also draw on Eurler's theorem, Fermat's little Theorem and the Chinese remainder theorem. This may go beyond school curriculum, and we lack proposals for how this can be taught inquiry based in order to secure autonomoy.

Tarp and Frantzen (2023) has redesigned an activity from on data science from being transmission of knowledge to more inquiry based concerning the notion "k nearest neighbour" (kNN), as a way for "the students to gain insight into the role of modelling in ML [Machine Learning] and broader knowledge concerning the notion of mathematical modelling" (Tarp & Frantzen, 2023, p. 19). Students are given data they need to import in the program Orange, where they conduct their ML activities through block programming. Students are asked to determine gender of a person based on data concerning this person. A priori the students can solve the problem using either ML models, e.g. kNN and a distance measure, or using mean calculations. All are preparing students to revisit linear regression and smallest residuals for finding the best fit of models. The case study show productive uses of the simple programming and different views on models: plots, data tables, the programming, and how this mirror probabilities for certain gender in new data. Still the inquiry approach to modelling challenged some students: "Connecting the pooled data to the tree-model […], they train

the model all over again on the merged data set, not sure what the newly added data looks like." (Tarp & Frantzen, 2023, p. 72). The study show potential of creating coherence and making sense of existing curriculum elements when introducing data science and ML in ordinary mathematics lesson in upper secondary school. However, they argue that this requires knowledge of task design and content knowledge much beyond current requirements.

Jessen (2024) presents a case study of a teacher's attempt to teach students how mathematics can form the basis of technical solutions they know from their everyday uses of e.g. smartphones. The teacher posed the student with the question of which three measures they would need and why, if they were to design a facial recognition system (Jessen, 2024, p. 296). Some students engage in very productive explorations of mathematical concepts calculated based on three measures, when the picture of a face is imported to a coordinate system in GeoGebra. The students used and validated their models taking and importing new pictures to the digital tool and made sure their proposals met the requirements for facial recognition, they found online. Not all groups were equally productive, and none connected the modelling activities to programming, or the mathematics involved in facial recognition systems – the scholar knowledge justifying the activity.

The last example addresses the development of responsible and critical citizenship through modelling of climate changes through the Civimatics project (Strømskag, 2023). The chapter is both an introduction to teachers regarding inquiry based mathematics from the point of view of ATD, but also how to model solutions for climate changes such as carbon storage. The materials meant for teachers (proposed curriculum document), promotes the idea of considering algebra as a modelling tool, while still taking its outset in research on carbon storage. As such it has the potential being genuinely interdisciplinary, though drawing on more classic modelling techniques from algebra – and as such not really exploring the potentials of stochastic modelling through and with digital technologies.

Concluding remarks and looking ahead

In literature we see similar trends regarding how to prepare students for critical and responsible citizenship, but addressed from different theoretical perspectives: statistics education, mathematical modelling with technology, critical mathematics education etc. This will according to Barquero and Jessen (2020) indicate different learning outcomes from and about modelling, though at a first glance the aim of the research trends points in the same directions. Therefore, we see a strong potential insisting on didactic transposition analyses regarding mathematical modelling insisting on knowing the its nature, origin, and function (cf. Bosch, 2015) of these proposed practices before they are implemented in schools. We see initial noospherical works as the proposed curricula (Boaler et al, 2024; Heinemann et al., 2018), proposed teaching material for teachers (Strømskag, 2023), and researchers promoting technocritcal mathematics education content. From classrooms we see mixed success from case studies either imitating the knowledge and practices involved in ML, or cases imitating facial recognition systems reduced. We also see a lack of coherence between what justifies potential curriculum changes linked to data driven modelling in STEM contexts, and what is currently happening in classrooms. Thus, we see a clear need for research aiming at mapping the knowledge justifying the teaching of modelling which prepare students to navigate modelling in disruptive times (Siller et al., 2023). Moreover, this require significant professional development of teachers as well.

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Secondary students' multifaceted perceptions of the role of data from the perspectives of models, modelling and information

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This study explored secondary school students' multifaceted perceptions of the role of data from the perspectives of models, modelling, and information, following two or three lessons focused on a data-informed modelling activity with mathematics and statistics at its core. The results showed that half of the students perceived data not only as a well-organised basis to directly achieve the real goals set in the modelling activity, but also as an unorganised basis to be manipulated or mathematised to generate information and knowledge or wisdom from the data. The students' perceptions were also supported by the real-world contexts of the activity used in the lessons. However, the role of data as an aspect of reality was not found to be naturally recognised and emphasised by the students in the context of the lesson. We argue that students need to experience a range of data-informed modelling activities with a variety of tasks in order to recognise the multifaceted role of data.

Keywords: Modelling, models, data use, information, data literacy.

Introduction

In today's data-informed society, new types of data emerge owing to the expansion of media channels and technology. As such, information and discussions about disruptive events (e.g. the COVID-19 pandemic, climate change, population change, and food disruption) based on complex data-informed models and modelling are becoming increasingly prevalent. To cope with such societal changes, it is necessary to develop data-literate citizens who can recognise what data can and cannot do based on an understanding of the nature and role of data (e.g. Engel, 2024).

Recently, research at the boundary of the mathematical modelling education community and the statistics and data science education community has centred around data-informed modelling with mathematics and statistics at its core to develop mathematical and/or statistical literacy and/or disciplinary learning (Biehler et al., 2024). This was also evidenced by more Congress of the European Society for Research in Mathematics Education (CERME) working groups on "Applications and Modelling" in which the educational use of data, models, and modelling was discussed in relation to statistics and data science education (Barquero, 2023). In this area of research, we seek to connect mathematical modelling education with statistical modelling or statistics and data science education (e.g. Ärlebäck & Kawakami, 2023; Kawakami & Ärlebäck, 2024). In this study, we focused on the role of data as a perspective for elaborating this connection.

Regarding the role of data, Engel (2024) stressed that awareness of the role of data and models in knowledge generation is crucial in the information age, as it helps individuals critically evaluate and interpret the information they encounter. Podworny and Frischemeier (2024) examined primary

school students' perspectives on data through a teaching sequence on statistical modelling, using decision trees as an example. Kawakami and Ärlebäck (2024) identified the role of data in teaching statistical modelling based on a literature review. Despite the importance of data in today's data-driven society and modelling educational community, research focusing on the detailed aspects of secondary school students' perceptions of the role of data in data-informed modelling is sparse. This study examined Grade 9 (aged 14–15 years) students' multifaceted perceptions of the role of data in data-informed modelling from the perspectives of models, modelling, and information.

Theoretical framework

Data

Data are essential in data-informed modelling (Ärlebäck & Kawakami, 2023; Kawakami & Ärlebäck, 2024) and represent 'numbers with context' (Cobb & Moore, 1997, p. 801). Thus, it is imperative to understand and interpret data in real-world contexts relevant to getting information about the data. Furthermore, the data contain signal (regular patterns) and noise (irregular random patterns) structures (Konold & Pollatsek, 2002). Data are subjected to mathematisation to unveil or abstract their underlying structure for various reasons, accompanied by sources of mathematical and statistical models and knowledge (Wild & Pfannkuch, 1999).

The potential role of data from the perspective of models

Data are key in generating, validating, and modifying models in data-informed modelling. Blomhøj and Jensen (2003) emphasised the role of data *as evidence to test model validity* in the modelling process. In addition, data can be considered as a model of reality (Podworny & Frischemeier, 2024). That is, data can *simplify complex reality*, recognising the limitations of data as an aspect of reality from a critical stance (Engel, 2024). Understanding the role of data in modelling reality is also crucial for critically reflecting on the route along which the models have been generated.

The potential role of data from the perspective of modelling purpose

In data-informed modelling, data are generally used to solve real-world problems. To understand the role of data from this perspective, we refer to the following description by Niss (2008), which succinctly summarises the purpose of mathematical modelling as follows:

> (P1) to understand (represent, explain, predict) parts of the world;
>
> (P2) to subject parts of the world to some kind of action (including making decisions, solving problems);
>
> (P3) to design parts or aspects of the extra mathematical world (creating or shaping artefacts, i.e. objects, systems, structures). These purposes are pursued by constructing or utilizing mathematical models to describe or prescribe aspects or parts of some domain. (p. 77)

These purposes align with the aims of data-informed modelling, emphasising the prediction and explanation of phenomena in data science and statistics education (Biehler et al., 2024). P1 requires using data *to understand (representing, explaining, predicting) parts of the world*. P2 requires using data *for decision-making to lead to better actions*. P3 requires using data *to set the standard or norm,* such as best-practice guidelines for lockdowns during the COVID pandemic. From a critical stance,

this role also carries the risk of having the power to shape human perception and behaviour that responsible citizens need to critically reflect on (Skovsmose, 2023).

The potential role of data from the perspective of information

Citizens should elicit information from data and understand the process of generating knowledge from data to judge the credibility of information and its sources in society (Engel, 2024). In information science, the DIKW (Data-Information-Knowledge-Wisdom) hierarchy (Figure 1) has attracted attention as a framework for describing the process of transforming raw data into meaningful information, appropriate knowledge, and useful wisdom for making informed decisions (e.g. Rowley, 2007). The hierarchy has also been emphasised in data-informed modelling (Engel, 2024).

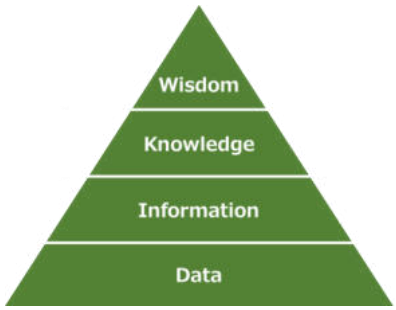

Figure 1: DIKW hierarchy (e.g. Rowley, 2007)

In the DIKW hierarchy, data are the foundation that generates information, knowledge, and wisdom. Generally, data are viewed as unorganised raw materials lacking meaning, context, value, or structure. Information is defined in terms of data and is considered as data with a purpose, context, structure, and meaning. There are various definitions of knowledge, but in this study, knowledge derived from data in the DIKW hierarchy is viewed as the integration of information into existing knowledge, experience, values, and so forth. Wisdom is also multifaceted; however, we view it as a higher level of understanding and the application of knowledge based on thoughtful and ethical judgment (Engel, 2024; Rowley, 2007). According to Engel (2024), moving upwards in the hierarchy is enabled through data-informed modelling. The transition from data to information involves building models to elicit and represent patterns, trends, and relationships in the data, which corresponds to part of the mathematisation in the modelling process. The transition from information to knowledge and wisdom involves interpreting organised information, integrating information into existing knowledge and previous experience, and combining and applying knowledge and models to other situations. In some cases, the explanatory power of a model can be increased by combining multiple data-based models (Ärlebäck & Doerr, 2020). This process corresponds to the interpretation and validation in the modelling process as well as real-world applications of models and modelling. Thus, data could be *the basis for information, knowledge, and wisdom*.

Research question

To examine and elaborate on the theoretical framework using the perspectives of models, modelling, and information, we formulated and addressed the following research question: *What perceptions do Grade 9 students express about the role of data in relation to models, modelling, and information after data-informed modelling?*

Methods

Participants and context

To answer the research question, we used data from lessons (two or three of 50 minutes each) involving data-informed modelling for the population prediction and post-lesson individual report assignments implemented for Grade 9 students (aged 14–15 years). An overview of the lessons and analysis of students' modelling work are given in Kawakami and Mineno (2021). However, the current study substantially differs from this previous work in that it analysed students' perceptions of the role of data in modelling data-rich situations. The participants comprised 151 students from four classes in Grade 9 at a junior high school in Japan. They previously learned descriptive statistics (i.e. representative value, histogram, and relative frequency), probability, and linear functions. However, they did not learn about modelling using complex data.

In the lessons, the students worked in groups and collaboratively tackled *the population prediction task* (Kawakami & Mineno, 2021). This task was to predict the 2010 census population of Japan based on Japanese population data in 1995, 2000, and 2005 by 5-year age groups and other data (e.g. population based on age groups, number of births, birth rate, mortality during 1990–2005, number of births based on mothers' age, and population pyramid graphs). The students constructed, validated, and revised various models while flexibly combining mathematical, statistical, and contextual approaches to knowledge. In the lessons, they shared each other's models going to each group to explain their approach while presenting the model drawn on a small whiteboard; the role of the data was not explicitly discussed in class. In the post-lesson, they were asked to prepare a report including various tasks and questions, one of which was the focus of this study: *What is the role of data in modelling data-rich situations such as population prediction?*

Data collection and analysis

We analysed all participating students' responses to the question about the role of data in their reports. First, seven coding categories for the roles of data (R1–R7 in Table 1) were developed based on the potential role of the data, as explained in the theoretical framework section. As the distinction between knowledge and wisdom in the DIKW hierarchy is not always clear (Rowley, 2007), we combined the two into one category. Then, students' responses were assigned to one or more of the seven categories based on their statements, as indicated in Table 1. The first author performed this coding in Japanese and translated it into English, and the second author reviewed and checked the translated coding performed by the first author. Differences in interpretation between the two authors were discussed until an agreement was reached. The analysis was revised as necessary.

Results

Figure 2 shows the percentages of 151 students' perceptions of the role of the data for each category (R1–R7). Each percentage is calculated based on a sample size of 151. All students described the role of data that fell into one or more of the seven categories. Of these, the most common was description of the roles of the data, classified as R3 (understanding), R6 (information), and R7 (knowledge/ wisdom). Very few students (less than 10%) wrote about the role of data in categories R1 (testing), R2 (simplifying), R4 (decision-making), and R5 (standard or norm).

Table 1: Category for analysing students' perceptions of the role of data

Category	Criterion	Example of description
R1: For testing model validity	Describing the use of data as evidence to test model validity.	"By comparing the predicted values with the actual measured data, you can check whether the method used to make the predictions is accurate." (student C17)
R2: For simplifying complex reality	Describing the use of data to simplify complex reality or the limitation of data as an aspect of reality.	"It is important to check the subject and purpose of the data, and to make sure we understand what we can read from the data." (student B2)
R3: For understanding parts of the world	Describing how data is used to understand (represent, explain, or predict) parts of the world.	"Informing us of the current situation and facts." (student B18); "Clues for prediction." (student D6)
R4: For decision-making	Describing the use of data for decision-making to lead to better actions.	"Leading to the decision-making of national policy." (student D25)
R5: For setting the standard or norm	Describing the use of data to set the standard or norm.	"Enabling methods such as calculating the mean." (student C34)
R6: The basis for information	Describing the use of data to elicit and represent patterns, trends, and relationships in the data.	"The data allow us to understand trends and make evidence-based predictions." (student B5)
R7: The basis for knowledge and/or wisdom	Describing how they use data to interpret organised information, integrate the information into existing knowledge and previous experience, or combine and apply knowledge and models to other situations.	"When we load the data, we can see the changes in society, such as the declining birthrate, the ageing population, the rise in the number of unmarried people, and the trend towards late marriage." (student A15)

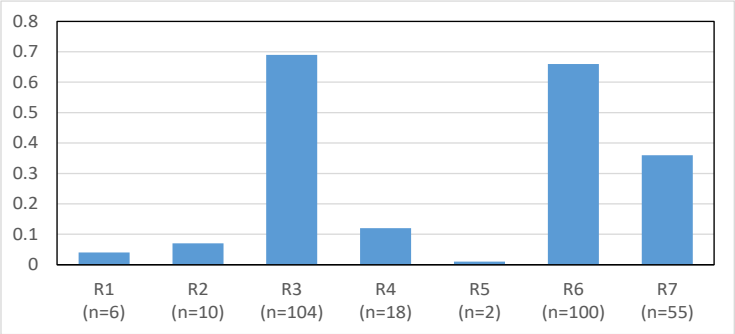

Figure 2: Proportion of students (out of $N = 151$) in each category

First, 69% of the students described the role of data in understanding, describing, and predicting parts of the real world. Most of them related the role to the specific context of the population predictions in the classes (e.g. “Data plays an important role in predicting population trends, as it is possible to predict the future to some extent based on past trends.” [student D2]). In addition, some students described the role in a general way, removing or inducing from the context (e.g. “The data can also be used to make accurate predictions about how it will affect society” [student B27]).

Second, 66% of the students described the role of data as the basis for information. Most mentioned eliciting patterns and trends from data and linking this to using it for prediction assigned to R3 (e.g. “The data gives us clear trends and specific declines. These are the first clues needed to make predictions.” [student A8]). Many students’ descriptions showed they recognised the data as highly objective, accurate, and reliable representations of facts. For example, student B23 explained the robustness of the data as evidence, conveying a metaphor in Figure 3: “Only on the basis of different data can conclusions be drawn and objections of others be defended.”

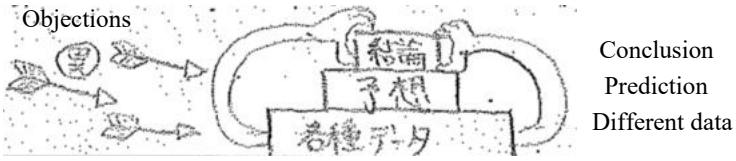

Figure 3: Student’s data metaphor

Third, 36% of the students described the role of data as a basis for knowledge or wisdom. Many mentioned that by using multiple data sets, it is possible to create multiple models (i.e. equations and graphs for prediction), and by using these together, it is possible to make new discoveries and evidentially explain the data from different perspectives (e.g. “If there is more than one piece of data, it is possible to find trends from multiple perspectives, so it is possible to find more realistic trends and increase the evidence.” [student D12]). In addition, some students wrote that by considering multiple data sets, they could view the social context and make predictions based on integrated contextual knowledge (e.g. “If you compare the data with the history of the baby boom and the falling birth rate, etc., the data increases and can be superimposed on current trends in Japan to make detailed predictions.” [student B2]). These student responses reflected the specific context of using different data to predict the population in class.

Fourth, in contrast to the fact that many students recognised the data as factual and highly credible, only 7% mentioned the role of data in simplifying reality. Some focused on the process of generating data from data sources and listed the limitations and points to note about the data (“When verifying data, it is necessary to confirm that it is true. For example, it is necessary to confirm that the person releasing the data is a trustworthy person.” [student B35]).

Finally, 52% of the students (n=79) described the role of the data in R3 and R6/R7. The breakdown was as follows: 45 students were classified as R3+R6, 13 as R3+R7, and 21 as R3+R6+R7.

Discussion and conclusion

This study explored secondary school students’ multifaceted perceptions of the role of data from the perspectives of models (R1 and R2), modelling (R3, R4, and R5), and information (R6 and R7). The

results indicate that half of the students recognised data not only as a well-organised basis to directly achieve the real purposes set in modelling (e.g. R3) but also as an unorganised basis for manipulating or mathematising the data to generate information and knowledge or wisdom for achievement (R6 and R7). In mathematical modelling education, the importance of the interaction between *thinking with models* and *thinking about models* has been noted (Ärlebäck & Doerr, 2020); those results suggest that in data-informed modelling education, the interaction between *thinking with data* and *thinking about data* is just as important as in the case of models. Thus, this study revealed that secondary school students understood the dynamic nature of the data through modelling.

In addition, secondary school students' awareness of these roles was supported by their concrete experience of data-informed modelling, which dealt with the real-world contexts of population prediction. The role of context in understanding these data roles reflects the nature of data as "numbers with context" (Cobb & Moore, 1997, p. 801). This also confirms the results of Podworny and Frischemeier (2024), who found that most primary school students' understanding of the data was reflected in the decision trees used in the lessons. On the contrary, the roles R1, R2, R4, and R5 that were not emphasised in the classroom tasks were not easily identified by the students. In particular, R2 and R5 are essential for data-literate citizens to think critically about the data itself and the process of its generation while participating in society in a well-informed and critical manner (Engel, 2024; Skovsmose, 2023). Therefore, we suggest that students need to experience a range of data-informed modelling activities with a variety of contexts and tasks in order to recognise the multifaceted role of data. Our findings are based on an analysis of students' report descriptions following short-term lessons. It is also necessary to analyse in detail the development of students' perceptions of the role of data through interviews and its impact in the longer term.

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The role of recursivity and reversibility in mathematical modelling: The bakery mould

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This paper focuses on the properties of recursivity and reversibility of mathematical modelling through the implementation of a complete cycle of the didactic engineering process, as outlined by the anthropological theory of the didactic. A preliminary analysis of an algebraic-functional modelling process is provided, and the design and a priori analysis of a study and research path (SRP) about the 'The Bakery Mould' are developed, addressing questions related to the determination of the most suitable mould dimensions for baking a cake. We use the a posteriori analysis of the SRP first implementation to analyse the recursivity and reversibility properties of modelling. Results from students' productions allow us to analyse the emergence and evolution of models and their role in the development of the SRP.

Keywords: Mathematical modelling, anthropological theory of the didactic, didactic engineering, study and research paths, covariation.

Introduction

This research aims to integrate two lines of work from the anthropological theory of the didactic (ATD, Chevallard, 2019). The first line focuses on the study of algebra and covariation in secondary education, based on the reference epistemological models for algebraic and functional modelling (as developed by Bolea et al., 2004; Ruiz-Munzón, 2010). The second line focuses on analysing the conditions that can facilitate the transition of current educational systems towards the paradigm of questioning the world, particularly through the proposal of study and research paths (SRPs, Barquero et al., 2022). We present the analysis of an SRP for secondary school, where these research lines converge. In this paper, we mostly consider the essential characteristics of mathematical modelling when dealing with situations involving the covariation of quantities within the ATD perspective.

Theoretical framework and research method

Mathematical modelling within the framework of the anthropological theory of the didactic

In the ATD, the modelling process is described through the following stages, which may alternate throughout the process (Chevallard, 1989, p. 53): (1) delimitation of the *system* to be studied, understood as the part of the reality to be modelled, specifying its relevant aspects, (2) construction of *model*(s) and work within the model; (3) work with the model to generate knowledge about the system. A fourth stage can be added with the formulation of new problems that can be addressed thanks to the development and work within the model. More generally, modelling is conceived as a tool to structure school mathematics (García et al., 2006) through the use or development of models and systems that succeed one another throughout the modelling process. The models used (if they are already known) or constructed (if they need to be created) are added to the system under study, transforming the model-system set into a new system, which might be modelled again. In this sense, the modelling process constantly evolves generating successively new system-model sets, each step

calling for more complex models. This property can be identified as the *recursivity* of modelling. The *reversibility* property in the modelling process highlights the fact that, at certain steps, systems and models can reverse their functions: for instance, when a previous system is used as a model of a new system or of the used model. As Barquero (2023) has already indicated, there is not much research literature on analysing the recursivity and reversibility properties. In this paper, we aim to investigate the emergence of these properties in the modelling process and their impact on it.

Study and research paths

In the ATD, the proposal of study and research paths (SRPs) is presented with a dual purpose. On the one hand, they serve as a general framework to design and analyse teaching and learning processes, including those based on inquiry and modelling. On the other hand, they are used to propose inquiry-based didactic approaches and to analyse the conditions required for their sustainable implementation. SRPs are structured in such a way that they begin with the proposal of an open question, called the generating question Q_0, which triggers an inquiry process to solve it. This process consists of a series of new questions that the study community (composed of students and teachers) must explore, and whose answers contribute to the joint development of a solution to Q_0. Along this path, the learning of new knowledge and techniques is integrated with information gathering, empirical data analysis, and the creation of new knowledge (Chevallard, 2019). SRPs are presented as a didactic tool intrinsically linked to the teaching of mathematical modelling, which appears as a productive way to develop answers to the questions posed. Research in the context of proportionality and functions (García et al., 2006), interdisciplinarity (Jessen, 2014), or combinatorics (Vásquez et al., 2021) in secondary education highlights this relationship.

Research questions and method

Research on modelling has revealed significant institutional limitations that hinder its integration as a regular activity in schools (e.g. Burkhardt, 2006). These constraints are linked to a predominant conception of modelling in secondary education, where it is often reduced to the mere construction or even application of specific models in predetermined situations. We start from the hypothesis that, to provide answers to the generating questions, SRPs bring forth modelling processes that are sufficiently developed to facilitate and promote the recursivity and reversibility properties of modelling. We can then formulate two research questions (RQ) addressed in this paper:

RQ1: How do recursivity and reversibility appear in the modelling processes generated by SRPs?
RQ2: How modelling process and its recursivity and reversibility properties can be shared and officialised to students along the implementation of an SRP?

Our objective is to address the first question using a specific SRP designed for 9th-grade students (aged 14–15) as a case study. We approach this study through a didactic engineering process (Artigue, 2020) and its four phases: the preliminary analysis of the didactic phenomena at stake; the *a priori* analysis of the educational proposal to be implemented to address these phenomena; the *in vivo* analysis during the implementation; and the *a posteriori* analysis to consider both the phenomenon at stake and the inquiry process implemented. In our case, the preliminary analysis starts from the results from Bolea et al. (2004) and Ruiz-Munzón (2010) interpreting algebra from a modelling

perspective and Barquero (2023) with the interpretation of recursivity and reversibility as central properties of modelling. Here, we focus on the *a priori* epistemological analysis of the proposal (presented in the following section) to highlight the recursivity and reversibility properties of an SRP, which is complemented using some elements of the *in vivo* and *a posteriori* analyses of the first SRP implementation in secondary school during the academic year 2023-24.

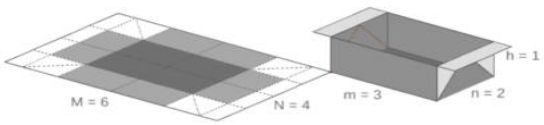
A priori analysis of the study and research path

The SRP starts by considering a type of traditional cake from northern Spain, the 'sobaos pasiegos', which uses a paper mould created through a simple origami technique (Figure 1). The generating question posed is Q_0: 'How can we construct the best mould among those that can be created using the same folding method?' This SRP is an adaptation of the 'cake box' case, from the one proposed by Chappaz and Michon (2003), used in research on mathematical modelling in primary education and teacher education, as presented in Barquero et al. (2024). The initial question in the *sobaos pasiegos* case is more realistic, given that in Spain, these cakes come in various sizes depending on the commercial brands that produce them. The initial system includes the cake mould and the origami technique. Here, we only consider the specific case of making folds in halves and quarters of the sheet in both dimensions, which addresses the question Q_1: How is the mould constructed?

Figure 1: *Sobaos pasiegos* in their paper mould and pattern for mould construction (our production)

From this point, we can consider different initial models: tables with different sheet sizes and the resulting moulds (measurement models, M1), as well as geometric models (M2) (see Figure 2). Working with both types of models leads to formulating and providing partial answers to questions such as: What relationships exist between the dimensions of the mould and the sheet? (Q_2); Given a specific sheet, what moulds can be constructed? (Q_3); Given a specific mould, what sheets create it? (Q_4); What sheet is needed to get a mould with a given volume? (Q_5).

In the search for an answer to question Q_0, we can consider proposals aimed at finding moulds with certain optimal properties: "with the lowest sheet cost", "with the largest volume", "the best-selling one", etc. However, difficulties arise that require measurement and geometric models to be refined or expanded. Individual numerical calculations are needed for each case, which creates challenges in finding and generalising the relationships between the dimensions of the sheet and the mould.



SHEET			MOLD			
LONGER SIDE (cm)	SHORTER SIDE (cm)	AREA (cm²)	LONGER SIDE OF THE BASE (cm)	SHORTER SIDE OF THE BASE (cm)	HIEIGHT (cm)	VOLUME (cm³)
7,7	6	46,20	3,85	3	1,5	17,33
8,3	5,4	44,82	4,15	2,7	1,35	15,13
10,2	7	71,40	5,1	3,5	1,75	31,24
13,1	6,1	79,91	*It is not possible because the flap exceeds the height of the mold.*			
7,2	7,2	51,84	*It is not possible because there is no flap.*			

Figure 2: Example of measurement (M1) and geometric models (M2) (authors' production)

Therefore, it seems reasonable to represent these dimensions and their relationships using algebraic expressions, progressing towards an *algebraic model* (M3) considering N and M as the shorter and longer sides of the sheet respectively, n and m as those of the mould's base, and h the height of the mould. We thus obtain generic answers to the posed questions. For example:

Q_2: *What relationships are there between the dimensions of the mould and the sheet?*

The sheet must satisfy $N < M < 2N$ (it cannot be a square $M = N$ to ensure there is a flap, and $M < 2N$ to ensure the flap does not exceed the height).

From sheet to mould: $h = ¼\ N$ $m = ½\ M$ $n = ½\ N$ (that is to say, $h = n/2$)

From mould to sheet: $M = 2m$ $N = 2n$

Q_3: *Given a sheet, what moulds can be constructed?*

From the previous relationships, we get the length of the flap $p = ¼\ (M - N)$

Q_4: *Given a mould, what sheets can create it?*

Assuming $m > n$, the mould must have $h = ½n$ and, to ensure that the flap exceeds the height, $M < 2N$ (or $m < 2n$). Therefore, if $m < 2n$ and $h = ½n$, we obtain a sheet with $N = 2n$ and $M = 2m$.

The algebraic model can lead to an evolution in the previously posed questions, raising new relevant ones, such as: How do small variations in the sheet dimensions affect the volume and height? (Q_6); What sheets with a given area lead to a mould with the highest volume? (Q_7); Which of the moulds with a given volume have the smallest sheet area? (Q_8). These questions are difficult to address within the algebraic model, as it does not facilitate an easy or detailed analysis of the covariation between variables. However, these questions created the possibility of evolving M3 towards a new model, a functional model (M4), which could be more efficient in relating the dimensions and the sheet area to the dimensions and volume of the mould. For instance, by considering the volume of the mould as a two-variable function $V(M, N) = m{\cdot}n{\cdot}h = ¼N{\cdot}½M{\cdot}½N = (1/16)N^2M$, it is possible to analyse how small variations in the dimensions of sheets influence the volume using different graphs of $V(M)$ or $V(N)$ for given values of N and M respectively. Moreover, given a sheet surface area $A = M{\cdot}N$, the volume can be expressed as a single-variable function $V(N) = A{\cdot}N/16$, enabling the determination of the optimal sheet dimensions to maximize the volume for a fixed surface area.

Classroom implementation and *in vivo* analysis

The SRP implementation took place as a classroom activity in May 2024, involving five groups of 9th-grade students, each with around 25 students, at a school in the metropolitan area of Barcelona where mathematics is taught using a traditional methodology (individual work, board presentations followed by exercises, etc.). Twenty minutes, in the first session, were used to explain the new work dynamics (new student-teacher roles and evaluation criteria for the activity) and to organise the working groups. The SRP was then carried out over seven sessions of 75 minutes each, with five sessions dedicated to the inquiry process and two to preparing a final individual report and taking a test, both of which were evaluative. The standard format for an inquiry session consisted of 10-15 minutes of reviewing the work done so far and discussing the issues to be addressed in the session, 40 minutes of free inquiry teamwork, 15 minutes for presenting the inquiry results and solved questions with a discussion of new questions to be addressed, and, finally, five minutes for writing a session diary. During the sessions, a video on the mould's construction was used along with

recommendations for making moulds and taking measurements. No additional instructions were provided by teachers. The experience was carried out by three teachers non-researchers in their regular mathematics class. Another teacher, the first author of the paper, acted as an observer in one of the classrooms. The *a posteriori* analysis was developed by the observer and two more researchers (all the authors) based on the analysis of students' productions: weekly collected group reports and individual students' diaries collected at the end of the implementation

The generating question was proposed by a pastry company andarrived to the class through an email from the company's CEO. After presenting the generating question, it quickly became agreed that the question to address was "What is the best mould for the company?" The students started working with the initial system, which consists of the cake, its mould, and the origami construction. They attempted to create moulds with different dimensions. During this process, questions about moulds based on the size of the sheet began to emerge. For instance, most students tried to create the mould with a square sheet and discovered that it was not possible as the associated mould did not have flaps. To better understand the relationships between sheets and moulds, they used measurement and geometric models, which coexisted during all the inquiry process (Figure 3).

Figure 3: Examples of the students' work using measurement and geometrical models

At this point, students only considered certain types of moulds (those that met the identified relationships). Then, questions appeared about the sheet properties that lead to the best mould. Several ideas emerged and were developed through the SRP, about the possibility for the company to reduce costs or sell more. Two main lines of inquiry emerged: some groups focused on finding "What mould sells the most?" (searching for dimensions that are more manageable or more appealing to consumers), while others aimed to minimise the mould's cost by using less paper while providing the same volume of cake, that is, "search the sheet dimensions giving a mould of a specific volume". From now on, we will focus on this second line of inquiry.

Throughout this process, students worked with measurement and geometric models, but some students felt the need to generalise the detected relationships using algebraic models. This generalisation seemed to provide a more global understanding of the variables chosen and their relationships, moving from a local level of reasoning about covariation (elements of *one* sheet and *its* mould) to a more general level (variation in the relationships among moulds). In this step towards a more general approach, facilitated by the construction of algebraic models, questions arose about how the height or volume changes as the dimensions of the sheet vary. When students discovered that volume changes even when the sheet's area remains constant, they raised questions including several variables (or variables and parameters): "How does the volume vary for different sheet dimensions that keep the area constant?" and "What is the minimum sheet area required for a specific volume?"

Finally, students conducted a one-by-one case study where they calculated the volume of the moulds constructed from different sheets with the same area but different dimensions. Students did not formally introduce functions or functional models. They tend to continue using the previous ones, comparing the measures or giving different values to the variables of the algebraic expressions. Through this study, students discovered that "the maximum volume is when the sheet's short side and long side are as close as possible" and that "the flaps should be just the right size so the mould can be easily handled." They then propose a reasoned answer to Q_0 of the type: "We concluded the ideal dimensions for us: a sheet of 18 cm x 14 cm. In our opinion, this size will create ideal moulds because the sides are very similar but not identical. This design maximises the volume of the cake while reducing overall costs and ensuring the presence of flaps for mould stability." Finally, students ended up constructing this "best" mould.

A posteriori analysis about recursivity and reversibility

Our investigation aims to highlight the recursivity and reversibility modelling properties and their potential utility in creating questions that help to progress in the modelling process. The recursivity is clearly reflected during the development of the SRP: starting from an initial system S_0 (the cake and the mould construction model that allows answering Q_1), question Q_2 arises regarding the understanding of the relationships between the sheet and the mould to identify the moulds that can be constructed. The measurement and geometric models M_1 and M_2 are used to deduce "one-to-one" relationships between each sheet and its mould. The need to gain a broader understanding of these relationships to obtain general answers through relationships among different moulds and their respective sheets (Q_3 , Q_4 and Q_5) leads to the initial system $S_0 \leftrightarrow M_1 / M_2$ evolving into a new system S_1. The new system calls for a new (algebraic) model M_3 that provides a more general perspective and understanding of the sheet-mould relationship. At this point, the new system S_2 formed by $S_1 \leftrightarrow M_3$ allows addressing questions about volume optimization (Q_6, Q_7 and Q_8), leading to the starting point of a functional model M_4 that helps consider relationships as functions to be optimised. Figure 4 shows the hypothetical development of the inquiry through the consideration of a sequence of models and systems, together with the questions generated.

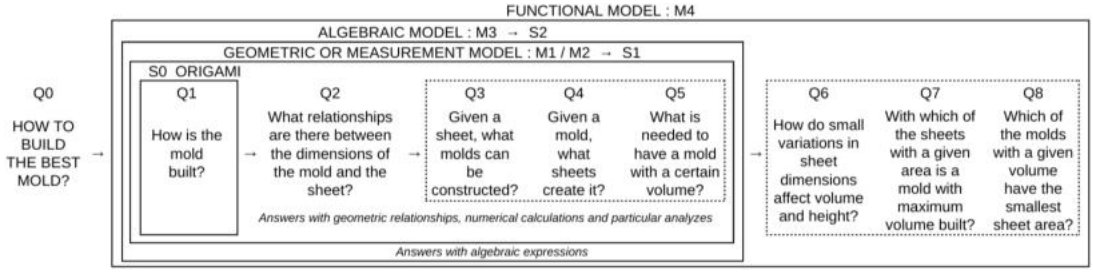


Figure 4: Diagram of the modelling process highlighting its recursivity (authors' production)

The characteristic of reversibility appears consistently in the process of validating the found answers. For example, to provide the final answer, some properties are deduced from the initial system ("the sheet must be close to a square shape to optimise the volume, but not exactly square to ensure the flaps"), but then the length of the flaps and the dimensions of the mould and sheet must be calculated using measurement and algebraic models. Subsequently, the initial system for constructing the mould

is transformed into the model used to build the final mould solution.

$$M_4 \leftrightarrow S_2 \;\rightarrow\; M_1 / M_2 / M_3 \leftrightarrow S_2 \;\rightarrow\; M_3 \leftrightarrow S_3 \;\rightarrow\; M_0 \;(S_0)$$

Functional — *Flaps and proportions* — *Calculations* — *Origami*

On the other hand, we also detect reversibility in the modelling processes as a solving strategy to elaborate answers to the questions that arise. For instance, when addressing the question of how the cake volume changes with small variations in the dimensions of the sheets, which arises when exploring the functional model, students observed that, rather than using a functional model directly, they could reverse the role of system and model to the previously known models (measurement and geometric or algebraic) to propose an empirical answer.

Discussion and conclusion

The analysis presented not only illustrates the characteristics of recursivity and reversibility in action but also highlights them among the fundamental components of the modelling process (questions, systems, models, tools for model validation, etc.). These components need to have a clear and defined status during the development of the SRP to let teachers and students comment on the evolution of the inquiry process. This implies the need to establish a precise terminology to refer to these elements, which are rarely found in the school culture. For example, the *a priori* analysis emphasises the importance of identifying key variables to consider: the sheets' dimensions, the mould, and free or fixed parameters in the sheets or moulds, among others. It also underscores the need to introduce new terms to name and institutionalise the *intermediate models* generated during the modelling process, such as: “origami model,” “geometric and measurement models,” “algebraic model,” or “functional model.” In this regard, this study contributes to anticipating the necessary terminology to be shared and institutionalised with teachers, and then, in their turn, with students in the classroom. Thus, our research tries to intervene in the phenomenon that Wozniak (2012) characterised as the spontaneous “silence” or “mutism” surrounding modelling in the classroom, providing teachers and students with elements to talk about, describe, comment on, explain, and justify modelling processes.

What remains as a future line of research is the in-depth analysis of the relationships between algebraic and functional models, as well as the role of covariation in the models evolution. It is important to note how each model implicitly incorporates the concept of covariation between quantities, a concept that evolves in its treatment depending on what each model enables to do. This might lead to a better understanding of how covariation is interpreted concerning the recursive and reversion process of delimiting and working in and with systems and models.

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Students' use of benchmarks in the process of solving Fermi problems

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In this paper, 9 secondary school students aged eleven to twelve years were observed working individually on two Fermi problems to investigate how students use benchmarks in different activities of solving Fermi problems. Specific focus was dedicated to identifying the sub-activities of processing Fermi problems where benchmarks were used. One of the Fermi problems the students worked on required the estimation of lengths, the other the estimation of weights. The data was analysed using qualitative text analysis. The findings indicate students use benchmarks when working on Fermi problems especially to estimate missing quantities and to validate made estimates.

Keywords: Fermi problems, benchmarks, estimation.

Introduction

In our everyday life, estimation skills are a relevant ability to deal flexibly with quantities (Hoth & Fricke, 2022). For example, during the corona pandemic, it was important for our health to keep a distance of 1.50 m and to estimate this distance (Hoth & Fricke, 2022). To determine missing quantities by estimation, the to-be-estimated object can be compared with another object, about which prior knowledge already exists (use of benchmarks) (Hildreth, 1983). When working on Fermi problems, missing data must be determined by realistic estimates (Ärlebäck, 2009). Thus, benchmarks can be used when working on Fermi problems. In this paper, we present a qualitative study in which the use of benchmarks in students' processing of Fermi problems is examined in more detail. As Fermi problems are suitable for initiating modelling activities in school (e.g. Ärlebäck & Albarracín, 2019), this study focuses on Fermi problems at early secondary school levels.

Theoretical Background

Fermi problems

Fermi problems are named after the physicist Enrico Fermi, who was known for answering seemingly impossible questions with reasonable accuracy (Efthimiou & Llewellyn, 2007). A typical example of such a question is the following: "How many piano tuners are there in the city of Chicago?" (Efthimiou & Llewellyn, 2007, p. 253).

Fermi problems are described as open estimation problems with a real-world connection (Ärlebäck, 2009; Greefrath & Frenken, 2021). They are non-standard problems where it is required to make assumptions and estimate quantities (Ärlebäck, 2009). Furthermore Fermi problems can be characterized as underdetermined, open tasks (Greefrath & Frenken, 2021). Since Fermi problems are characterized by the absence of data and the need to make reasonable estimates, the main focus of Fermi problems is the acquisition of data by estimation (Ärlebäck, 2009; Greefrath & Frenken, 2021). Thus, Fermi problems have a relation to estimation problems (Albarracín & Gorgorió, 2014). At the same time, Fermi problems are described as a special type of mathematical modelling tasks (Greefrath & Frenken, 2021). That's why Fermi problems are mentioned in both the context of estimation and the context of mathematical modelling (Ärlebäck, 2009). Fermi problems are suitable

to introduce mathematical modelling to students (Ärlebäck, 2009; Ärlebäck & Albarracín, 2019; Sriraman & Lesh, 2006). This is also emphasized by the fact that Fermi problems are accessible for all students, they can be solved at different educational levels and can be of different complexity (Ärlebäck, 2009; Ferrando & Albarracín, 2021). Furthermore students do not need pre-mathematical knowledge to work on Fermi problems (Ärlebäck & Bergsten, 2010). At the same time, further research indicates that there are differences in performance on Fermi problems based on prior knowledge (Segura et al., 2024).

Fermi problems can be solved by splitting the problem into smaller sub-problems, so that these problems can be solved by estimating and thus the main problem can be solved (Albarracín & Gorgorió, 2014). For example, the question about the number of piano tuners in Chicago can be broken up into questions about the total number of pianos in Chicago, the average number of piano tunings a tuner can complete in a year, and so forth (Efthimiou & Llewellyn, 2007). According to Greefrath & Frenken (2021) Fermi problems can be described in a more general sense as open tasks that are based on a small amount of information, or even just a picture and a question. This study encompasses such tasks.

Ärlebäck (2009) describes different activities students engage in when working on Fermi problems: *Reading*, *Making model*, *Estimating*, *Calculating*, *Validating*, *Writing*. These activities can be visualised in a graphical representation of the solving process, called *MAD framework* (Ärlebäck, 2009). *Writing* means reading and understanding the task. *Making model* involves simplifying and structuring the task and mathematizing. *Estimating* describes the activity of making quantitative estimates. *Calculating* means doing maths like performing calculations or rewriting equations. *Writing* is the process of summarizing the findings and results and writing up the solution. *Validating* encompasses interpreting, verifying and validating results as well as calculations and the model itself related to the real-life context of the problem (Ärlebäck, 2009; Ärlebäck & Albarracín, 2019). As multiple estimates are needed, there exists no exact answer to Fermi problems (Ärlebäck, 2009; Sowder, 1992). For this particular reason, estimates should be validated (e.g. Segura & Ferrando, 2023). This way, students can be encouraged to critical thinking and discussing variations in estimates (Sriraman & Lesh, 2006). Previous studies have found, among other things, that students validate real results when solving mathematical modelling problems by comparing them to the situation model including real-world experiences or directly to the real model (Czocher, 2018). Other previously research has shown that students validate their results or estimates by using personal extra-mathematical knowledge and experiences from outside schools (Ärlebäck, 2009; Ärlebäck & Bergsten, 2010). Further studies have found, among other things, that pre-service teachers make errors when working on Fermi problems that are related to (missing) estimation skills (Segura & Ferrando, 2021). For example, it has been found that pre-service teachers build incorrect initial models due to inadequate mental representations of benchmarks, or that estimates are made that are incompatible with the real situation (as the mathematical result is not interpreted in the real situation) and this is not validated.

Estimating and benchmarks

Since the estimation of quantities is an essential activity in the process of solving Fermi problems, we will explain the estimation of quantities in more detail. As estimating is defined as the process of determining a measurement without using measuring tools, estimating can be described as a mental process, as contrasted to measuring, which is a physical activity (Bright, 1976). In order to estimate quantities, the object to be estimated can be compared mentally with known objects (Hildreth, 1983). This way, benchmarks can be used as tools. Benchmarks are mental representatives of a certain magnitude that can be used for estimation (Hildreth, 1983; Joram, 2003; Joram et al., 2005). Thus, to determine a measurement, the to-be-estimated object can be compared mentally with an object whose measurement is known by imagining a benchmark (Joram et al., 2005). For example, the height of a person can serve as a benchmark for length and can be used to estimate the size of a room (Joram, 2003).

Research questions

Due to the fact that Fermi problems are characterized by the absence of data and the necessity of estimating quantities, it is conceivable that the use of benchmarks is of great importance in the process of solving Fermi problem. This leads to the following research question of how students use benchmarks for lengths and benchmarks for weights during the process of solving Fermi problems.

In this paper, we focus on the activities engaged in the process of solving Fermi problems to investigate where benchmarks are used in this process. Thus, the following research question arises:

- In which sub-activities of the process of solving Fermi problems do students use benchmarks for lengths and benchmarks for weights?

Method

To answer this research question, we investigated solving processes of nine secondary school students (7 female, 2 male). All in all, we investigated 18 solving processes. The students were between eleven and twelve years old and had good or very good grades in mathematics. Each student worked

Figure 1: Task "thumb"

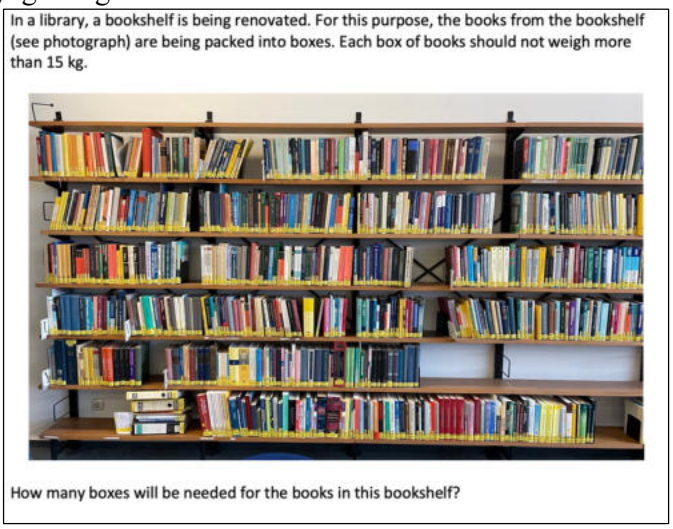

Figure 2: Task "bookshelf"

separately on two Fermi problems outside of regular school lessons. One of the Fermi problems was about estimating lengths (the task "thumb") and the other Fermi problem was about estimating weights (the task "bookshelf"). In both problems it was necessary to work with a given photo. For example, when processing the task "thumb" (Figure 1), the height of the thumb statue, the length of a person's thumb and the height of a person or the person in the photo must be determined. The students were videotaped while working on the Fermi problems. At the beginning the students were instructed to think aloud and practiced this by solving a sudoku before they received the first task. After working on both Fermi problems, the students were interviewed about their solving process. For example, they were asked how they determined their estimates.

The data was analysed with the qualitative text analysis (Kuckartz, 2014). The videos were transcripted and excerpts of the interviews that are relevant for the research question were integrated to the transcripts. The transcripts were subdivided into coding units (each line of the transcript represents one coding unit) and each coding unit was coded with one of the activities students engage in when solving a Fermi problem (Ärlebäck, 2009). In addition, a category was created for the use of benchmarks. This means in addition to the activities, the text passages where benchmarks were used, for example by naming and referring to a benchmark, were also coded. The coding schemes for the activities (in abridged version) is presented in table 1.

Table 1: Coding scheme for the activities when working on Fermi problems (based on Ärlebäck, 2009).

Category	Definition
Reading	Reading and understanding the problem.
Making model	Structuring the task, Simplifying and making assumptions that concern the situation, mathematizing. Also making implicit estimates, for example those that establish a relationship between two quantities.
Estimating	Making estimates of quantitative nature. (Estimates of numerical values.)
Calculating	Doing maths, e.g. performing calculations, also working with the calculator.
Validating	Interpreting, verifying and validating results, calculations, the model itself. Also Re-estimates.
Writing	Summarizing the findings and the results, writing up the solution, entering the result in the answer sentence.
Other	Activities that are not associated with a category but nevertheless relate to the editing process, e.g. statements on the difficulty of the task.

In cases, in which several activities take place at the same time, the activity that is in the main focus has been coded. The data was coded by the first author and a second coder was trained. About 50 % were coded by the second coder and the intercoder reliability for the Fermi activities was $\kappa = .90$, which can be interpreted as almost perfect (Landis & Koch, 1977). No coding units were specified for coding the benchmarks and each coder independently identified in which activities benchmarks

were used. In the case of independently identified coding units, it is more useful to report a percentage agreement rather than Cohen's kappa. The percentage agreement for the benchmarks was 95,65 %.

Findings

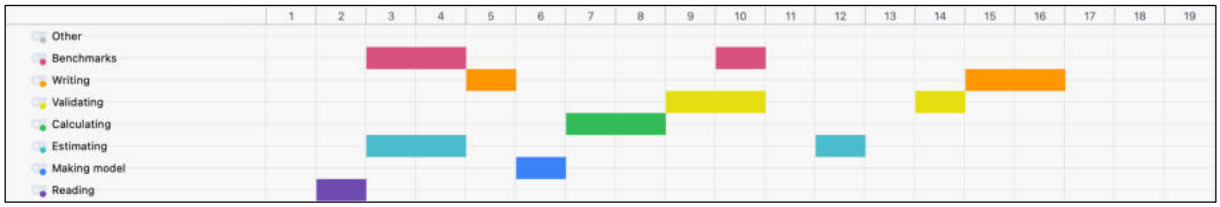

Figure 3: Modelling activity diagram of the process of solving the task "thumb" by Max

To visualize the solving processes of the Fermi problems, we used the modelling activity diagram (MAD framework) based on Ärlebäck (2009). In contrast to Ärlebäck (2009), we decided to represent the activities in the modelling activity diagram using lines from the transcripts. Figure 3 shows an example of a modelling activity diagram. The numbers 1-19 in the header of the diagram represent the line numbers of the transcript. Empty columns are statements made by the interviewer (e.g. at the end asking if the student has finished) or irrelevant statements (e.g. about switching on the calculator). As you can see in figure 3, the student used benchmarks in different activities. On the one hand, he used benchmarks during the activity of *Estimating*, on the other hand he used benchmarks during the activity of *Validating*. In the modelling activity diagrams of other solving processes of other students and the other Fermi problem we can see that many students used benchmarks during the activity of *Estimating*. Overall, benchmarks were used during the activity of *Estimating* in 9 out of the 18 solving processes (6 for the task "thumb", 3 for the shelf "bookshelf"). There is also one solving process where a student used a benchmark during the activity of *Making model* (for the task "thumb"). The student in the aforementioned example is the only individual who used a benchmark during the activity of *Validating*. In other activities than *Making model*, *Estimating* and *Validating*, no use of benchmarks was found in this study. Benchmarks were not used by every student and not in every task, in 8 out of 18 solving processes no benchmarks were used (2 cases in the task "thumb", 6 cases in the task "bookshelf"). In general, benchmarks were used frequently, but more often in the task "thumb" than in the task "bookshelf".

During the activity of Estimating, the students used benchmarks to estimate missing quantities that are needed to solve the problem. An example of using benchmarks for length during the activity of *Estimating* is presented in the following transcript excerpt of the task "thumb":

Sophia: I think the woman is […] a metre 60 […].
Interviewer: How did you find out how tall the woman is?
Sophia: I imagined […] how tall my mum is. Um, and then I compared whether the woman was similar, taller or shorter.

Another example of using benchmarks for weights during the activity of *Estimating* is shown in the following transcript excerpt of the task "bookshelf":

Sophia: 50 grams one book.
Interviewer: How did you find out how heavy a book is?
Sophia: I looked at how much a packet of flour weighs, I think it weighs a kilo, um, and then because I often read books in the evening, I estimated how much a book

weighs, and then I thought um, maybe 50 grams, because all but two or three of the books here aren't that thick.

The activity of *Making model* also includes making implicit estimates that establish a relationship between two quantities. For this, a student also used a benchmark, as the following transcript excerpt of the task "thumb" shows:

Lara: So the thumb has to be bigger than this door here, because the door, that is, it's over 2 metres, because a door like this is 2 metres at most.

A Benchmark were also used for validation to validate made assumptions and estimates. The following transcript excerpt of the task "thumb" is an example of a students' use of benchmarks during the activity of *Validating*:

Max: But if the thumb is 2.50 metres. Then I estimate approximately that the woman is between 1,70 and 1,80 metre (re-estimates the height of the woman). That should be appropriate. There's a door at the back (points at the door in the photo) (…) and doors are normally 2.5 metres tall. Then I would say that the thumb and the door have the same height.

The benchmark that Max used to validate the estimate for the height of the thumb statue is different than the one he used when estimating at the beginning of the problem solving process.

Discussion

The presented study shows that, in general, students made use of benchmarks when working on Fermi problems. Benchmarks were only used in the activities of *Making model*, *Estimating* and *Validating*. Therefore, the discussion focusses on these three activities. They were mainly used during the activity of *Estimating*. Since the estimation of quantities can be described as a mental comparison between the to-be-estimated object and known objects (Hildreth, 1983), it is to be expected that students, if they make use of benchmarks when working on Fermi problems, do so mainly during the activity of *Estimating*. That benchmarks were used for estimation is in line with the results of other studies, in which pre-service teachers used benchmarks for estimation when solving Fermi problems but did not have an adequate mental representation of them (Segura & Ferrando, 2021). As benchmarks can be individual, for example when own body parts serve as benchmarks, (Bright, 1976; Joram, 2003), it can be assumed from the findings of our study that students not only use personal extra-mathematical knowledge and experiences from outside school to make an estimate, as other studies have shown (e.g. Ärlebäck, 2009), but also use benchmarks to make an estimate.

In this study, benchmarks were also used during the activity of *Validating*. Other studies have shown that validating is necessary when working on Fermi problems to prevent the use of unrealistic estimates in the solution (Segura & Ferrando, 2021). As the results of this study indicate, benchmarks might support this process. In our present study, students used benchmarks, which they have not used for estimation in the solving process before, to validate previously made estimates. This may point to an additional aspect of validation processes in terms of students verifying their assumptions and estimates by the use of benchmarks. These findings are in line with prior research (e.g. Ärlebäck & Bergsten, 2010) and extend the description of validation activities in the process of solving mathematical modelling problems in other studies (e.g. Czocher, 2018). To summarize, we can say that students use benchmarks in the activities of *Making model*, *Estimating* and *Validating*. It is

conceivable that benchmarks can support students to make estimates (implicit and estimates of numerical values) and to verify made estimates.

The presented study has some limitations that we would like to address. The use of benchmarks could not be identified in every solving process. However, this does not imply that no benchmarks were used in these processes. It is possible that the students simply did not mention them or were unable to name them, although they were instructed to think aloud and were asked how they estimated quantities. However, it would have been possible to encourage the students to validate their assumptions and estimates more specifically during the interview.

Outlook

Regarding the findings of this study, further research is planned. On the one hand, the benchmarks for lengths and the benchmarks for weights used by the students in this study will be analysed in more detail to classify them. In addition, students' procedures and strategies to estimate quantities when solving Fermi problems will be investigated. To investigate in more detail how students use benchmarks for lengths and benchmarks for weights during the process of solving Fermi problems, we will also examine how realistic the made estimates are. Furthermore, a quantitative study is planned. This study will investigate the knowledge of benchmarks about the quantities length and weight of secondary school students as well as the relation of this benchmark knowledge and the Fermi solving process. Moreover, in this planned quantitative study we will also analyse whether assumptions and estimates are more realistic when benchmarks are used when working on Fermi problems. Additionally, the findings of the study presented in this paper raise the question, whether benchmarks are frequently used in validation. This represents another further future research project.

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Didactic transposition constraints on modelling at upper secondary level: Comparing France and Norway

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This study explores the didactic transposition constraints on teaching mathematical modelling at the upper secondary level in France and Norway, focusing on curriculum and textbook content. Grounded in the anthropological theory of the didactic, the research examines how modelling is conceptualised and implemented in both countries' educational systems. By analysing textbooks, the study identifies variations in how modelling tasks are proposed and how students engage in the modelling process. The French textbook mainly focuses on algebraic tasks, while the Norwegian textbook emphasises computer-based regression analysis. The research also highlights challenges such as the lack of recursive modelling and limited use of parameters, restricting students' opportunities to explore the iterative nature of modelling. The study provides insights into modelling education in both contexts and suggests areas for further development.

Keywords: Didactic transposition, mathematical models, model–system dialectic, praxeology, textbook content.

Introduction

The integration of mathematical modelling in the new upper secondary French curriculum and the reinforcement of modelling as one of six core elements in the new Norwegian curriculum mark a significant shift in mathematics education in these countries. This prompts a new process of didactic transposition (Chevallard, 1991), which refers to how scholarly knowledge is transformed into knowledge to be taught. The resulting transformation involves creating new types of textbook tasks and instructional strategies to incorporate modelling into the teaching repertoire. Generally, in the context of mathematical modelling, didactic transposition faces unique challenges due to the lack of a well-established scholarly consensus and terminology on modelling knowledge (Puchaczewski et al., 2024). Consequently, each country's curriculum develops its own definition and approach to modelling, leading to potential variations in how modelling is taught and understood across different educational systems. The studies briefly summarised below shed light on the situation of mathematical modelling in textbooks at the upper secondary level in some European countries.

Zwaneveld et al. (2017) reveal that the two most frequently used Dutch mathematics textbooks seldom include genuine mathematical modelling. The tasks analysed rarely require students to engage in characteristic modelling activities, such as conceptualising, mathematising, solving, interpreting, verifying, validating, iterating, communicating, and reflecting. Moreover, the majority of modelling tasks in these textbooks lacks appropriate modelling purposes, for instance, explanation, prediction, decision-making, or optimisation. Kröger (2019) examines the alignment of textbook content on modelling with national curriculum goals in Germany and Sweden, finding that textbooks mainly focus on applying and manipulating mathematical formulas within closed systems. This approach fails to meet curricular goals that prioritise applying mathematics to open-ended, real-world

problems. In a similar vein, Berget (2022) uncovers a critical gap in the Norwegian educational system between the theoretical goals set by the national curriculum in force from 2006 to 2020 and the modelling tasks implemented in textbooks and examinations under that curriculum. The tasks analysed tend to emphasise calculation and application of known formulas rather than the creative process of developing and validating models that are new to the students. This situation mirrors the one found in Denmark by Jessen and Kjeldsen (2021).

This study aims to explore and compare the conception and implementation of modelling in the current upper secondary mathematics curricula and textbooks of France and Norway. We examine what aspects of modelling are present or neglected in these educational frameworks and instructional materials. Furthermore, we study what types of modelling tasks and techniques to solve them are proposed in textbooks, along with the explanations and justifications they provide. By analysing curricular guidelines and textbooks, we seek to uncover the prevailing conceptions of modelling and the new responsibilities assigned to teachers and students in both educational systems in the discipline of mathematics. The study of its presence in other school disciplines is beyond the scope of this study. This study provides insights into the didactic transposition constraints on modelling at the upper secondary level in these frameworks, and the comparison between the two countries helps identifying those constraints. The research question we have addressed is:

> What didactic transposition constraints related to modelling at upper secondary level in France and Norway can be inferred from current curricular guidelines and mathematics textbooks?

Theoretical framework: Didactic transposition and mathematical modelling

This study, conducted within the Anthropological Theory of the Didactic (ATD, Chevallard, 2019), draws on the theory of *didactic transposition* (Chevallard & Bosch, 2020), which examines how knowledge is transformed from the scholarly institution where it is created to the educational institution where it is taught and studied (referred to as *external* didactic transposition). Within these systems, knowledge objects—like modelling—are organised around specific types of tasks and techniques (i.e., *praxis* elements), and descriptive discourses and theories (i.e., *logos* elements), collectively termed *praxeologies*. These praxeologies evolve across different institutional systems, where they are generated, reinterpreted, applied, and reshaped. This transformation often fragments and reorganises the original knowledge to adapt to the new institutional conditions and constraints.

The focus of this study is on the *results* of didactic transposition processes in the context of mathematical modelling. We consider two dimensions of modelling. The first is the *epistemological* dimension, which pertains to the dialectic of systems and models and serves as our reference epistemological framework. Grounded in Chevallard's (1989) theorisation, it identifies two primary entities: a system (intramathematical or extramathematical) and a corresponding mathematical model designed to address a question or questions about this system. The modelling process is described in three stages: *defining the system* by identifying its relevant aspects and parameters, *constructing the model* by establishing relationships among these parameters, and *working on the model* to answer the posed questions and generate new insights about the system under study. Two key attributes of mathematical modelling are highlighted (Chevallard, 1989). *Reversibility* involves reversing the relationship between a system and its model, enabling the system to be interpreted as a model of its

own model. *Recursivity* frames modelling as a recursive process, where an existing model can be remodelled at higher levels with more advanced mathematical tools. For example, an initial exponential growth model may develop into a logistic growth model to capture additional dynamics in an ecological system.

The second dimension of modelling is the *ecological*, which relates to how institutional conditions, such as curricular frameworks, teacher preparation and available resources, either enable or constrain the integration of mathematical modelling into educational contexts (Barquero, 2023).

Methods

This study employed praxeological analyses, examining the content of mathematical modelling in mathematics textbooks from France and Norway. Curriculum guidelines from the two countries served as a reference, highlighting the importance of mathematical modelling. The selected textbooks are *Maths 2de* (Gringoz et al., 2023) for Year 10 in France, and *Matematikk 1T* (Borge et al., 2020) for Year 11 in Norway. Note that Year 10 in France (*2de*) corresponds to Year 11 in the Norwegian educational system. These textbooks were chosen as representative examples from a broader sample of textbooks due to their frequent use in their respective countries.

Using the ATD modelling framework, *thematic analyses* were carried out on the textbooks to identify: the types of tasks related to modelling, the systems treated, the construction of models, the work on models, and the presence or absence of recursive modelling. *Maths 2de* was analysed for tasks labelled "Modelling a situation," while *Matematikk 1T* was examined for its dedicated chapter "Modelling."[1] Building on the individual analyses of the textbooks, a comparative analysis of didactic transposition processes in modelling, informed by Puchaczewski et al. (2024), was carried out to examine how each educational system portrays mathematical modelling in textbooks.

Examination of the knowledge to be taught through textbooks

In the French curriculum introduced in 2019, modelling is one of six competencies to be developed, along with researching, representing, reasoning, computing, and communicating (Ministère de l'Éducation nationale et de la Jeunesse, 2019—hereafter MENJ, 2019). Modelling is mentioned in relation to some algebraic notions such as "variables, algebraic expressions, equations or inequations" or "multiple, divisor, even number, odd number, prime number, vectors as a modelling tool for physics, real functions and sequences for continuous and discrete phenomena" (MENJ, 2019, p. 7). Modelling also appears in probabilities, where the curriculum mentions "theoretical models of reference" and construction of models from frequencies, as ways of modelling randomness. To sum up, modelling is not a well-defined category in the French curriculum. It appears as part of the praxeologies to be taught but is partitioned by the type of models to be used—algebra, geometry, analysis, or probability.

Mathematical modelling has been part of the Norwegian upper secondary level curriculum since 1994. In the new curriculum for Years 1–13 (Directorate for Education and Training, 2020—hereafter

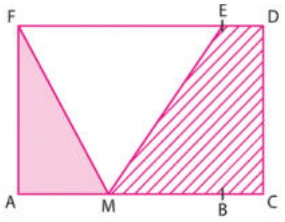
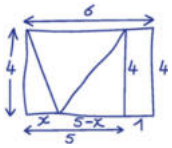
[1] Excerpts from the French and Norwegian documents are translated by the authors.

DET, 2020), it remains central as one of six core elements: "Modelling and applications," alongside "Exploration and problem solving," "Reasoning and argumentation," "Representation and communication," "Abstraction and generalisation," and "Mathematical fields of knowledge." The overarching description of modelling," consistent across Years 1–13, emphasises real-world relevance and critical engagement (DET, 2020, pp. 2–3):

> A model in mathematics is a description of reality using mathematical language. The pupils shall gain insight into how mathematical models are used to describe everyday life, working life and society in general. Modelling in mathematics means creating such models. It also means to critically evaluate whether the models are valid and what limitations the models have, evaluate the models in view of the original situations, and evaluate whether they can be used in other situations.

The French case

When analysing *Maths 2de* (Gringoz et al., 2023), we can see that modelling is explicitly mentioned, through the action of modelling, its process, or its product: "Modelling: Describing a concrete situation using mathematical knowledge, for example, writing an equation or a function for studying the given situation" (*Maths 2de*, 2023, p. 380). In this textbook, modelling praxeologies are specifically developed in tasks labelled as "Modelling a situation," which is introduced by a specimen (Figure 1) with indications about the technique (named "reflexes"), a commented solution describing the solving process in multiple steps echoing the reflexes, and a suggested answer closer to what a student should write.

<table>
<tr><td colspan="2">I learn how to model a situation with a placeholder [indéterminée in French]</td></tr>
<tr><td>Reflex 1–Identify the placeholder and its nature</td><td>Reflex 2–Translate the task using the placeholder</td></tr>
<tr><td colspan="2">Problem statement
ACDF is a rectangle such that AC = 6 and AF = 4.
B is on [AC] and E is on [FD] such that CB = 1 and DE = 1.
M is an arbitrary point on the segment [AB].
Morgan wants to find out at what distance from point A he should place M so that the shaded area is greater than or equal to the area of triangle AMF.
What inequality must he set up and solve to answer this problem?
 
</td></tr>
<tr><td>Commented solution
 

Step 1: I identify that I need to find the possible values of the distance between A and M.
The placeholder is therefore the distance AM, which I will denote as x. These possible values are the "unknowns." (Reflex 1)
Step 2: I look for the formulas "in terms of x" for the two areas mentioned in the problem statement. (Reflex 2)
Draft: Area (AMF) = Base × Height = $\frac{4 \cdot x}{2}$.
Area (MCDE) = Area (BCDE) + Area (MBE)</td><td>Written answer
I set x = AM.
Thus, the Area (AMF) is calculated as:
Area (AMF) = $\frac{AM \times AF}{2} = \frac{4 \cdot x}{2} = 2x.$
And the Area (MCDE) is calculated as:
Area (MCDE) = Area (BCDE) + Area (MBE) =
$4 \cdot 1 + \frac{(5-x) \cdot 4}{2} = 14 - 2x.$
The inequality that Morgan must set up is therefore: $14 - 2x \geq 2x$</td></tr>
</table>

Figure 1: "Model a situation with a placeholder." Example reproduced from Gringoz et al. (2023)

About the task in Figure 1, we note that, first, the system is mathematical, rooted in physical geometry, where distances are used to formulate an inequality. The parameters are the lengths and

areas of the system and those who do not have a fixed value can easily be expressed in relation to those with fixed value and the target length called "x". The model created, an inequality, comes from a subtraction of two areas expressed this way. Solving this inequality, which is not asked in the task, provides the conditions for the placement of the point M in the geometrical system so that the shaded area is greater than or equal to the area of the triangle AMF: $2x \leq 14 - 2x$ has solution $0 \leq x \leq 3.5$.

Studying the inequality would allow to make inferences about the geometrical configuration, such as how changes in x (moving M along AB) affect the geometrical system. For instance, the boundary case $x = 0$ corresponds to the vanishment of the triangle AMF as M coincides with A. There is no invitation to any recursive modelling, even though the potential exists for considering a functional model or to replace the fixed values of the lengths AC and AF with parameters, to study the variation of the studied areas with the variation of the shape of the rectangle.

The Norwegian case

In *Matematikk 1T* (Borge et al., 2020), modelling is presented as a standalone chapter, divided into three sections. The first section, "Mathematical models," establishes connections between models and formulas, as well as between models and functions, linking them to real-world systems. A dedicated subsection within this section addresses parameters, distinguishing them from variables. Parameters are described as fixed, though potentially unknown values, unless data are used to determine them. The second section, "Regression analysis," outlines the making of a function from data with computer assisted regression analysis, introduced as a "method we can use to build models" by "finding a function that best fits a given dataset." Some comments are made by the authors regarding the choice of the model, either based on its "simplicity" (evaluated from the number of parameters), or its connection to known laws governing the system (which can be given by "experts" of the situation to be modelled). The third section, "Modelling in practice," addresses key considerations for selecting and refining models. It discusses aspects such as balancing simplicity with accuracy when choosing a mathematical function and the treatment of "deviating points" that diverge from the general trend.

One of the worked-out demonstrations in the third section examines the relationship between a car's speed and its braking distance, providing a detailed view of the iterative process of refining models. The authors provide data linking these two quantities (see Table 1).

Table 1: Speed vs braking distance (taken from Borge et al., 2020, p. 289)

Speed (km/h)	10	20	30	40	50
Braking distance (m)	0.5	2.0	4.5	8.0	13

The authors then construct models: they first place points representing the data in a coordinate system and approximate them with a linear function, represented graphically as a straight line. They then choose to fit the data with a second-degree function instead, ensuring a better fit to the datapoints. The authors note, however, that "this model gives a [non-zero] braking distance even when the car is stationary, since $f(0) = 0.3$" (Borge et al., 2020, p. 290), prompting them to refine the model again by using a power function. With this third model, they estimate the braking distance of a car at a speed of 80 km/h. Notably, the authors suggest further refinements by adding new parameters, such as the

road dryness or the type of tyres used. Nevertheless, this recursive modelling initiative is not developed further.

Building on the description in the presented example, we recognise the system as being rooted in a mechanical context, providing a dataset that links speed and braking distance for specific values. However, the absence of physical laws in the modelling process, such as those related to friction, represents a significant limitation, as it prevents generating fundamental insights into the system itself. Instead, the regression models are restricted to predicting braking distances within the range of the given data. Moreover, other parameters, such as road surface conditions, are not mentioned, further limiting the dataset's ability to capture more complex conditions inherent in the system.

Some reversibility is present, as the physical system guides model adjustments for boundary cases. Recursivity is also evident when provisional models, influenced by the physical system (e.g., data point positions on graphs), are iteratively refined to create more accurate models.

Discussion and conclusion

When comparing the textbooks of France and Norway, some specificities and commonalities become visible. These observations will be discussed in relation to our reference epistemological model. In both cases, the *modelling process* appears in the textbooks as a special kind of mathematical activity limited to some specific cases or types of tasks. Similarly, *the system to be modelled* is almost always given as a delimited entity, with its elements (quantities or dimensions) either pre-identified, or straightforward to determine. The system is rarely modified or enlarged to evaluate the scope of the model or to enhance its applicability. The *types of models* proposed to be built are also limited. In the French textbook, models consist of algebraic expressions treated as equations, inequalities, or functions. Only one model is typically considered in each case, with little discussion on its relevance to the system and other possible alternatives. By contrast, in the Norwegian textbook, models are predominantly functions derived through regression analysis. In these cases, the textbook encourages the consideration of multiple possible models and selecting the one that either best fits the data or most closely reflects the underlying system. Moreover, some tasks rely on other methods than regression: for instance, solving equations algebraically in the case of determining the equation of a curve passing through given points, or approximating the volume of an object using geometric formulas for composite shapes. The *work on the model* is limited: in the French case, it is largely taken for granted and not discussed, while in the Norwegian case, the model is primarily used to compute function values at specific points. Concerning *reversibility* and *recursivity*, these processes are absent in the French textbook, while the Norwegian textbook implicitly fosters recursivity through tasks that involve refining models, frequently by incorporating a new parameter, as seen in tasks on cooling processes and pulse development following physical exertion.

Our results connect with two established didactic phenomena concerning the ecology of modelling at the secondary school level, along with a curricular constraint identified in this study. The first phenomenon, examined by Strømskag and Chevallard (2022), is the lack of utilising parameters in modelling with algebraic expressions. Parameters, when present, are often assigned numerical values prematurely, which reduces opportunities for symbolic manipulation. This practice obstructs key iterative processes such as model revision (recursivity) and the transformation of systems through

model manipulation (reversibility). In addition to this, the removal of differential equations from the Norwegian upper secondary curriculum in 2020 introduces a significant didactic transposition constraint. Differential equations are crucial for describing and understanding intrinsic laws in dynamic systems, making their exclusion a substantial limitation on modelling practices. This omission likely explains the lack of models based on fundamental principles and the prevalence of modelling processes based on regression analysis. While effective for data fitting, regression does little to foster comprehensive insights into the systems being modelled. It is worth noting the apparent contradiction in introducing modelling as a core element of the curriculum while simultaneously omitting differential equations, a fundamental tool for mathematical modelling.

Finally, the second didactic phenomenon concerns the absence of an explicit and elaborated *logos* of modelling praxeologies (Barquero et al., 2024). In the cases studied, modelling is largely limited to *praxis*: in the French context, two "reflexes" dominate, closely tied to algebraic modelling, while in the Norwegian context, the justification of model choices remains highly specific to individual modelling tasks, lacking generalisation or connection to a broader framework. This limitation poses challenges for teachers and students, as they lack the necessary vocabulary and conceptual tools to articulate, manage, and further develop modelling processes.

We conclude with two observations. First, we note the slowness of the didactic transposition process and the difficulty of modifying the traditional mathematical organisations in secondary school, strongly rooted in the paradigm of visiting works (Chevallard, 2019). In this context, the earlier introduction of modelling in Norway (1994) compared to France (2009) helps explain the more developed presence of modelling in the Norwegian textbook. Second, fully integrating modelling into the knowledge to be taught requires more than introducing new types of tasks. It requires a profound transformation of the mathematical organisations to be taught and a reimagining of the role of mathematics in addressing both extramathematical and intramathematical open questions.

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Modelling in the learning of the definite integral for civil engineering

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In this paper, we show one part of a much broader research focused on designing a modelling sequence to promote the understanding of the definite integral (DI) from the perspective of models and modelling (MMP) through activities that involve accumulation processes. We present part of the design and how it is intended to be implemented with a group of civil engineering students taking the Integral Calculus course. The sequence's design aims to address issues related to the learning of the DI, its interpretation, and its use in specialised courses, particularly in solving problems closely related to the workplace. Furthermore, we review some of the design principles proposed within the MMP and share our reflections on how they might be adapted and used as a guide for designing activities aimed at specific groups of higher education or university students.

Keywords: Definite integral, modelling, civil engineering, teaching and learning.

Why is the definite integral difficult for engineering students to interpret?

We agree with Bressoud et al. (2016) that the teaching and learning process of calculus often focuses on procedural, with an emphasis on the calculation of antiderivatives, while ignoring the meaning of the DI as an accumulation process. Moreover, the teaching is focused on strictly formal aspects, but do students really understand through formal approaches? And is it enough for them to identify the DI in different contexts beyond the calculation of the area under a curve? Empirical studies (Orton, 1983; Kouropatov & Dreyfus, 2014; Serhan, 2015) reveal that although students master integration techniques and algorithmic calculations, they struggle to solve problems in non-mathematical contexts that require the use of the DI. Furthermore, presenting the DI solely as the area may limit students' understanding associated to this concept. To address this, Attorps et al. (2013) and Greefrath et al. (2021) suggest exploring the DI in various contexts that involve accumulation processes.

González-Martin and Hernandes (2018) observed a discrepancy between the mathematics curriculum and the specialisation courses' curriculum. For example, when we examine the integral calculus courses for civil engineers, one of the applications of the DI is the calculation of the centroid of a plane region. But why and for what purpose does a civil engineer need to know how to calculate the centroid of a plane region or a body? By reviewing a Strength of Materials course, we found that calculating the centroid is essential in the design and construction of beams, as it is necessary to obtain the shear force and bending moment diagrams of a beam.

González-Martin (2021) proposes a didactic transposition for of bridging the gap between calculus courses and the specialisation subjects for engineering students. Even though we do not use the concept of didactic transposition, in our proposal we select a specific type of beams of the bridges to contextualise the situation from a MMP (Lesh & Doerr, 2003). Specifically, through the principles of activity design we address the research question: How can we design a modelling sequence that enables civil engineering students to develop an understanding sufficient to recognise, interpret, and apply the concept of the definite integral in various contexts?

The MMP in the teaching and learning process

If the objective is related to the teaching and learning of a particular concept, with a focus from a contextual perspective, where models are understood as systems of elements, relationships, operations, and rules that are useful for describing, explaining, or predicting the behaviour of other systems, and modelling as a set of activities that involve students in the development, understanding, modification, and intentional use of one or more models to interpret a particular context, then, the MMP provides a coherent and systematic framework, allows to reflect on numerous aspects that can support teaching and learning processes through the use of contexts in modelling activities (Ärlebäck et al., 2013; Ärlebäck 2017) and it is characterised by the design of a sequence of models and modelling activities, beginning with Model Eliciting Activities (MEAs), followed by one or more Model Exploration Activities (MXAs) and/or Model Adaptation Activities (MAAs) (Lesh & Doerr, 2003). MEAs are activities characterised by a problematic situation that prompts students to manipulate, predict, construct, describe, and explain a model based on their prior knowledge of a situation. These activities should encourage an iterative process in which students express and represent their initial ideas and thinking about the explicit and implicit characteristics of a problem (Ärlebäck & Doerr, 2017; Türker-Biber et al., 2022). However, since one MEA is not sufficient to achieve the expected learning objectives and students require multiple opportunities to explore and use their mathematical constructions in different contexts, it is suggested to follow the MEAs with the implementation of one or more MXAs, followed by one or more MAAs (Doerr, 2016). MXAs consist of sub-activities that involve different representations using different digital tools, animations, and powerful notation systems, allowing students to manipulate, test, explore, and modify the models that emerged from the MEA. MAAs, on the other hand, are characterised by being designed in such a way that they allow students to extend and apply the models that emerged from the MEAs and were refined in the MXAs to other contexts with similar structures to the initial. It is important to note that MEAs and MAAs differ from the routine mathematical problems commonly found in textbooks and should be designed according to the six principles of the design (Lesh et al., 2003).

Design principles of the MEAs and MAAs

According to Lesh et al. (2000), the main criterion for evaluating and validating the effectiveness of modelling activities is that when students engage with these activities, they reveal significant conceptual models from a mathematical standpoint and powerful ones from a practical perspective. To assess the fulfilment of this criterion, it is suggested that designers of modelling activities ask themselves a series of questions framed in the design principles. However, since the MMP began with a focus on primary level mathematical education, and our activities are designed for higher education, we add questions to the questions suggested by Lesh et al. (2000) for each principle. In the following paragraph, we detail each principle and the reflective questions we suggest.

Model Construction Principle: To check compliance of this principle, the MMP suggests questioning whether the proposed activity for students promotes the need to develop a model to interpret data, objectives, and explore possible solution processes. We consider that it is pertinent to ask: Among the possible solution processes, is there the emergence of models related to, close to, or necessary for studying a particular concept? Is it possible to identify the type of system the students are being

challenged to construct? And can the system focus on a significant mathematical construct? *Reality Principle:* We recognize that one person's reality may not be the same as another's. However, when focusing on a specific university course, we revisit the idea of proposing problems that align with the group's interests and experiences (Lesh et al., 2000). Therefore, we asked the following questions: Could this really happen in a situation close to real life? Is the context of the situation close to the reality of the university student? Is the context or situation relevant to the students' professional career as an engineer? Does the proposed problem or situation allow students to explore different perspectives and solutions that encourage debate and discussion? And does the situation provide opportunities for students to reflect on their own experiences and knowledge and how they relate to the proposed situation? *Self-Evaluation Principle:* It is proposed that designers question whether the activity provides students with the opportunity to assess the validity of the possible model alternatives that emerge, whether these models require refinement in different modelling cycles, and whether the context of the problem allows students to identify whether the proposed problem has been resolved or not. We also suggest reflecting on the question: Does the assessment process enable students to recognise the need to delve deeper into one or more mathematical models to evaluate the effectiveness of the solutions found? (It is likely that this principle can be fully verified once work with the MXAs is underway.) *Documentation Principle:* Designers are advised to consider whether the activity prompts students to explicitly reveal their thinking about the situation, including data, goals, and possible solution paths. It is particularly important that the process provides an "audit trail" that allows examination of the types of systems and patterns the students are considering. We consider that it is pertinent to ask: Is the problem situation clear and sufficient for students to describe types of conceptual systems related to the mathematical object in focus? And does the problem context stimulate and encourage students to externalise their thinking, whether through qualitative, quantitative, and/or symbolic descriptions or graphical representations? We believe that when the focus is on learning, documenting arguments and reflections is important so that during discussions, ideas can be externalised, modified, and refined concerning the concept being understood. It is also likely that this principle can be fully verified only after working with the MXAs. *Reusability/Generalisability Principle*: The idea here is to reflect about the next question, is the model reusable, modifiable, and/or transferable to other situations? We consider it necessary to verify that the proposed situation directs and encourages the student to explicitly deepen the mathematical models that conceptualise and reinterpret a concept in other situations. *Effective and/or Simple Simplicity Principle:* To validate this principle, it is suggested to consider how the model was developed rather than the model itself. That is, whether the ideas from the process leading to the generalisation of the model are applicable in other contexts.

Method and proposal design

Evidence from pilot studies showed that a single MEA is not sufficient to refine students' ways of thinking (citation from articles in press). These findings led us to design MXAs and the MAA. The pilot studies also helped identify modelling cycles and determine the need for a generalisation process, as well as the level of intervention required by the researcher or teacher during the development of the MXAs.

Our proposal is designed for civil engineering students taking the course on integral calculus. Specifically, we seek to address two key issues: the difficulties students face when solving extra-mathematical problems involving DI and the disconnect between integral calculus content and specialised subjects. To tackle this issue, we designed a modelling sequence consisting of two MEAs (MEA1 and MEA2), two MXAs (MXA1 and MXA2), and one MAA. The sequence begins with MEA1, which aims to elicit student ideas on approximation, accumulation, and area through a problem that involves an unconventional way of cutting half a pizza into three equal parts. Next, MXA1 involves the use of a GeoGebra applet to facilitate the graphical and dynamic representation of area approximation via Riemann sums. The aim of MXA1 is for students to explore and refine their initial ideas, while also appreciating and applying conceptual tools related to Riemann sums and limits, which lead them to view the DI not merely as the area under a curve but as accumulation.

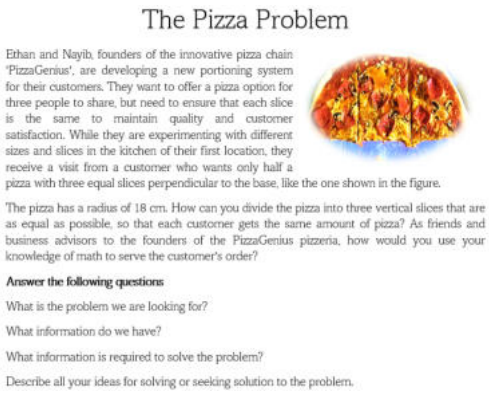
Following MXA1, we propose MEA2, which shifts the problem context to one closely related to the workplace of civil engineering, involving the design and construction of beams. The objective of MEA2 is for students to see the DI as the centre of mass of a load, while the aim of MXA2 is for them to use additional conceptual and digital tools to refine and expand the models that emerged in MEA2, viewing the DI as the accumulation of load that a beam can support and as a moment with respect to an axis. Finally, the aim of MAA is for students to apply this new way of thinking about the DI as a useful tool for calculating shear force and bending moment in beams with trapezoidal loads (accumulation of forces and moments).

In the following, we briefly describe and explain the design of each activity, which was developed and adjusted based on previous pilot studies (Ramírez-Inca & Mejía, in press) that verified the adherence to the aforementioned principles.

MEA 1 and MXA1: Pizza problem

With MEA1 we introduce the pizza problem (Figure 1), which is not related to their future workplace but is connected to their everyday life (*reality principle*). In Mexico City, a pizzeria offers pizzas with innovative shapes and sizes. We chose this context to motivate students to develop ideas and notions related to area approximation (*model construction principle*). When we pose the question, “How can you divide the pizza into three vertical slices that are as equal as possible, so that each customer gets the same amount of pizza?”, we aim for students to reflect on what is accumulating and what is varying, depending on the number of equal pizza slices. Since the restriction is to make perpendicular slices with equal areas, self-assessment *(self-evaluation principle*) can be performed by verifying that the areas are equal.

In Figure 1, it is observed that the activity includes a set of questions aimed at helping students describe and explain their ideas and conceptual systems directed at solving the problem, following the *documentation principle*. We believe that these questions, along with MXA1, guide students to delve into the mathematical models that emerge and reuse them for another similarly structured situation, adhering to the *generalisation principle*. Regarding the *simplicity principle*, although the problem situation requires considerable reflection and data interpretation, we consider that the problem is straightforward and involves integral calculus elements that students are expected to have already studied.



The Pizza Problem

Ethan and Nayib, founders of the innovative pizza chain 'PizzaGenius', are developing a new portioning system for their customers. They want to offer a pizza option for three people to share, but need to ensure that each slice is the same to maintain quality and customer satisfaction. While they are experimenting with different sizes and slices in the kitchen of their first location, they receive a visit from a customer who wants only half a pizza with three equal slices perpendicular to the base, like the one shown in the figure.

The pizza has a radius of 18 cm. How can you divide the pizza into three vertical slices that are as equal as possible, so that each customer gets the same amount of pizza? As friends and business advisors to the founders of the PizzaGenius pizzeria, how would you use your knowledge of math to serve the customer's order?

Answer the following questions

What is the problem we are looking for?

What information do we have?

What information is required to solve the problem?

Describe all your ideas for solving or seeking solution to the problem.

Figure 1: MEA 1. The Pizza problem

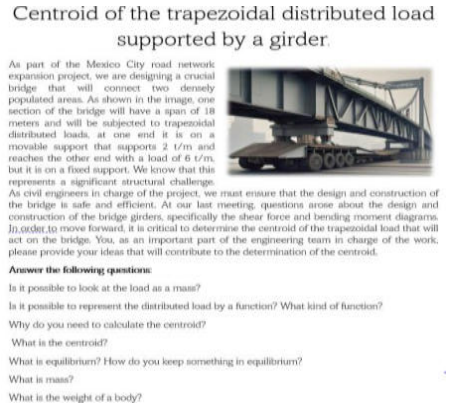
MXA1 involves questioning the models proposed by the students and building upon those ideas that facilitate an "optimal" solution to the problem. Specifically, a GeoGebra applet that allows the translation of problem information into a dynamic geometric representation of area accumulation through Riemann sums. In this dynamic environment, students can manipulate problem parameters while exploring, testing, and refining their conjectures. Additionally, students are asked to organise the new information into a tabular representation, which, through team and plenary discussions, will involve new mathematical knowledge to answer the question: How can they divide half of the pizza into three proposed portions as equally as possible to ensure each customer receives the same amount of pizza?

MEA 2 and MXA2: Centroid of the trapezoid-shaped distributed load supported by a beam

MEA 2 was designed based on a context close to the workplace of a civil engineer, following the *principle of reality*. Gere and Barry (2013) and Hibbeler (2011) explain that the resultant force on a body is equivalent to the area under a curve of the distributed load and acts through the centroid of that area, highlighting the importance of integration in solving engineering problems. As reported by González-Martín et al. (2021), although they recognise the use and significance of integration, in their practical examples they consider it an alternative solution and detail the primary solution as processes involving free-body diagrams and equilibrium equations. Detailed analysis of specific cases, such as the use and significance of integration in the design and construction of beams, shows that its application is straightforward and can achieve the same reasoning, provided the concept of integration is understood.

For the design and construction of bridges, which include the design and construction of beams, shear force and moment diagrams are used. One way to obtain these is through the graphical method, using equilibrium conditions to find the equations and values for shear force and bending moment. Depending on the type of load the beam supports, the first step is to simplify the loads by finding the total area and treating it as a single load applied at the centroid of that area. In this sense, MEA 2 (Figure 2) aims to facilitate the emergence and documentation of concepts related to physical principles and their application for determining the centroid (*principles of model construction and documentation*). Given that the situation involves concepts from other sciences, it is clear that the emerging models will require various cycles of modelling (*principle of self-evaluation*) and challenge

students to deepen and reinterpret a particular mathematical concept in different situations and reuse the idea with which the model is intended to be developed (*principles of generalisation and simple prototype*). This is likely to be fully achieved through an exploratory activity. Therefore, MXA 2 provides students with a set of conceptual and digital tools (Phet simulator) along with a guide of questions aimed at promoting reflection on what is being accumulated (areas, forces, moments).


Centroid of the trapezoidal distributed load supported by a girder.

As part of the Mexico City road network expansion project, we are designing a crucial bridge that will connect two densely populated areas. As shown in the image, one section of the bridge will have a span of 18 meters and will be subjected to trapezoidal distributed loads, at one end it is on a movable support that supports 2 t/m and reaches the other end with a load of 6 t/m, but it is on a fixed support. We know that this represents a significant structural challenge.

As civil engineers in charge of the project, we must ensure that the design and construction of the bridge is safe and efficient. At our last meeting, questions arose about the design and construction of the bridge girders, specifically the shear force and bending moment diagrams. In order to move forward, it is critical to determine the centroid of the trapezoidal load that will act on the bridge. You, as an important part of the engineering team in charge of the work, please provide your ideas that will contribute to the determination of the centroid.

Answer the following questions:

Is it possible to look at the load as a mass?

Is it possible to represent the distributed load by a function? What kind of function?

Why do you need to calculate the centroid?

What is the centroid?

What is equilibrium? How do you keep something in equilibrium?

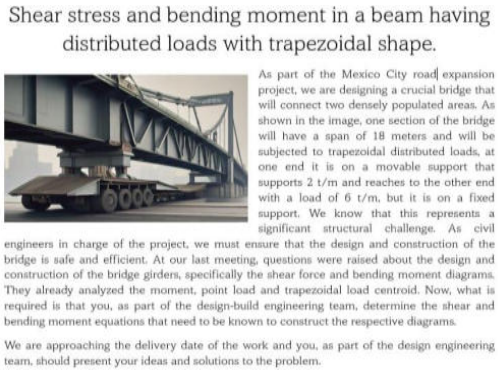
What is mass?

What is the weight of a body?

Figure 2: MEA 2. The centroid of the trapezoidal distributed load

MAA: shear stress and bending moment of a beam carrying a distributed load

Following MXA 2, we propose MAA (Figure 3). Although its context appears similar to that of MEA2, it presents a different issue, adhering to the *reality principle*. This activity requires students to use and refine the models and ideas developed in previous activities. The proposed questions guide students to reflect on and describe their ideas about using integration (*construction and documentation principles*) to create shear force and bending moment diagrams for a beam, ensuring coherence in their graphs (*self-evaluation principle*). In this case, the accumulation involves shear force and bending moment (*generalisation principle*). Furthermore, this reflection can be extended to other structures or engineering problems involving physical concepts and treated as accumulation processes (*principle of simple prototype*).


Shear stress and bending moment in a beam having distributed loads with trapezoidal shape.

As part of the Mexico City road expansion project, we are designing a crucial bridge that will connect two densely populated areas. As shown in the image, one section of the bridge will have a span of 18 meters and will be subjected to trapezoidal distributed loads, at one end it is on a movable support that supports 2 t/m and reaches to the other end with a load of 6 t/m, but it is on a fixed support. We know that this represents a significant structural challenge. As civil engineers in charge of the project, we must ensure that the design and construction of the bridge is safe and efficient. At our last meeting, questions were raised about the design and construction of the bridge girders, specifically the shear force and bending moment diagrams. They already analyzed the moment, point load and trapezoidal load centroid. Now, what is required is that you, as part of the design-build engineering team, determine the shear and bending moment equations that need to be known to construct the respective diagrams.

We are approaching the delivery date of the work and you, as part of the design engineering team, should present your ideas and solutions to the problem.

Figure 3: MAA. Shear stress and bending moment in a beam

Considerations

Based on the analysis of the pilot results and an interview with one of the authors of this theoretical approach, (R. Lesh, personal communication, 12th March 2024), it was considered necessary to

modify and define a warm-up activity that encourages students to reflect on and discuss ideas involving concepts of approximation, variation, and accumulation. Most importantly, it should promote students' ability to engage in modelling activities.

We believe that implementing this sequence in introductory civil engineering courses will foster a better understanding of DI. Through accumulation processes, the aim is to reconstruct and identify the concept of the centroid and recognise distributed loads as a single point load. As Greefrath et al. (2021) point out, this helps to identify that many engineering concepts, like physical concepts, can be understood and reconstructed through accumulation processes.

It should be noted that we are still working on identifying the levels of understanding of the definite integral (ID) that can be achieved through the implementation of the modelling sequence by civil engineering students, and these will be refined in upcoming implementations of the sequence.

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In-service teachers' scaffolding practices based on diagnosing modelling challenges during instruction

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This study focuses on the theoretical relationship between teachers' diagnostic and instructional competencies essential for effective modelling instruction. We explored teachers' scaffolding practices as a key component of their instructional competence, aimed at supporting students, based on their diagnostic competence in identifying modelling challenges. Data from nine observed modelling lessons conducted by teachers during real-time modelling instruction were analyzed. The findings reveal five well-defined themes of scaffolding practices teachers employ to support students in overcoming identified modelling challenges across various modelling phases, while promoting independent student learning. These valuable insights empower the linkage between teachers' diagnostic and instructional competencies and suggest practical recommendations to enhance their practices in modelling instruction.

Keywords: Scaffolding practices, Mathematical modelling instruction, Instructional competence, Diagnostic competence.

Introduction

The significance of mathematical modelling (MM) in mathematics education has seen a consistent rise in recent decades (Kaiser, 2020). The MM process implicates translations between the real-world and mathematics (Blum, 2015; Borromeo Ferri, 2018; Kaiser, 2020) and consists of several phases that form a modelling cycle. This process is cognitively demanding, requiring both mathematical and extra mathematical knowledge not commonly used in regular math lessons, thus presenting cognitive challenges in each step of the process (Wess et al., 2021; Galbraith & Stillman, 2006). Teachers play a crucial role in guiding their students through this complex process (Borromeo Ferri, 2018), which necessitates possessing the essential competencies required for effective modelling instruction (Borromeo Ferri & Blum, 2010).

This study focuses on two key competencies, examining their demonstration in the practical work of modelling instruction in class. The first is the *diagnostic* competence which involves identifying students' challenges in each phase of the modelling cycle, and the second is the *instructional* competence which entails employing appropriate practices to engage students in the MM process, often relying on accurate diagnoses (Borromeo Ferri, 2018; Stender & Kaiser, 2015). Despite their interdependence, scant research exists on teachers' instructional competence that directly stems from diagnostic competence. So far, studies have mainly examined either teachers' diagnostic and instructional competencies as separate entities (e.g., Siller et al., 2023), or the development of pre-service teachers' diagnostic and instructional competencies during teacher training (Wess et al., 2021). Limited exploration exists on in-service teachers' instructional competence, grounded in their diagnostic competence during real-time modelling instruction, particularly across the distinct

challenges of each modelling phase. Scaffolding has become a prominent approach in investigating teachers' instructional practices in educational research over recent decades (Van de Pol et al., 2010), especially in MM research (Schukajlow et al., 2015; Stender & Kaiser, 2015). It comprises providing temporary support to help learners solve tasks independently, aiming to minimize teacher intervention when knowledge or skills are lacking. The current study adopts this approach to investigate teachers' application of scaffolding practices, as part of their instructional competence, resulting from diagnosing modelling challenges during real-time modelling instruction. Building upon previous research that explored the various modelling challenges teachers identify within each phase (Schwartz-Aviad & Kohen, 2023), this study delves into the scaffolding practices teachers use to support students in overcoming these challenges, considering each distinct phase of the MM process.

Theoretical background

The diagnostic and instructional competencies of teachers during modelling instruction

MM is commonly depicted as a cyclical process involving back-and-forth transitions between real-world and mathematical domains (Blum, 2015; Borromeo Ferri, 2018; Kaiser, 2020). Kaiser and Stender's modelling cycle (2013) outlines five major consecutive phases: 1) *Understand simplify*; 2) *Mathematise*; 3) *Mathematical work*; 4) *Interpret*; and 5) *Validate.* To effectively engage students through the MM process and its phases, teachers should possess appropriate competencies, which were identified by Borromeo Ferri and Blum (2010) as: (1) *Theoretical* competence which encompasses knowledge of modelling cycles, aims, and perspectives, (2) *Task-related* competence that includes solving and cognitive analyzing MM tasks, (3) *Diagnostic* competence that addresses the recognition of students' difficulties and mistakes within each modelling phase, and (4) *Instructional* competence which refers to planning and executing modelling lessons, by using appropriate interventions. We address Diagnostic and Instructional (DI) competencies as a unified entity, given their theoretical interconnection (Borromeo Ferri, 2018), encompassing teachers' ability to identify challenges in the MM process within each phase and employ appropriate scaffolding practices to support students in overcoming those challenges. A previous study (Schwartz-Aviad & Kohen ,2023) explored teachers' diagnostic competence, by utilizing Wess and colleagues' framework (2021) for difficulties in the MM process to detect teachers' identification of challenges within each modelling phase. This framework outlines potential challenges in each modelling phase based on studies that have explored students' struggles and challenges in the MM process (e.g., Galbraith & Stillman, 2006). Building on that framework, this study aims to link the challenges teachers detect within each modelling phase with the scaffolding practices they employ to support students in overcoming these challenges.

Scaffolding as theoretical basis for teachers' instructional competence

Scaffolding refers to providing temporary support to help learners solve tasks independently, while minimizing teacher intervention, referring to three main characteristics: 1) *contingency*, which involves the adaptative support to the student's current learning status; 2) *fading*, which includes the gradual decrease of support; and 3) *transfer of responsibility*, which encompasses shifting learning responsibility to the student (Van de Pol et al., 2010). During modelling instruction, teachers should provide adaptive interventions tailored to each student's learning needs to foster independent

engagement in MM. These interventions, while not all explicitly stated, encompass all scaffolding characteristics and are seen as a realization of the scaffolding concept (Stender & Kaiser, 2015). While studies have explored teachers' scaffolding in MM (Schukajlow et al., 2015), including pre-service teachers' activities to promote students' independent learning (Stender & Kaiser, 2015) and MM-oriented practices (Siller et al., 2023), a research gap exists regarding scaffolding practices derived from teachers' diagnostic competence in identifying modelling challenges. This study addresses this gap by examining teachers' scaffolding practices that promote independent student learning during modelling instruction, focusing on challenges specific to each modelling phase.

Research goal and question

Our main goal is to examine the demonstration of in-service teachers' DI competencies, namely their instructional competence, that is based on their diagnostic competence in identifying students' modelling challenges, within the setting of real-time modelling instruction. Specifically, we pertain to answer the following question: What scaffolding practices do teachers use for different modelling phases to assist students in overcoming modelling challenges?

Methods

This study was part of a Professional Development (PD) initiative designed to equip lower-secondary school mathematics teachers with competencies needed for effective modelling instruction. The program spanned 30 academic hours throughout one academic year, emphasizing the essential professional competencies for modelling instruction (Borromeo Ferri & Blum, 2010). It combined theoretical learning through six synchronous online meetings with practical classroom implementations of MM tasks in three asynchronous meetings (Fig. 1).

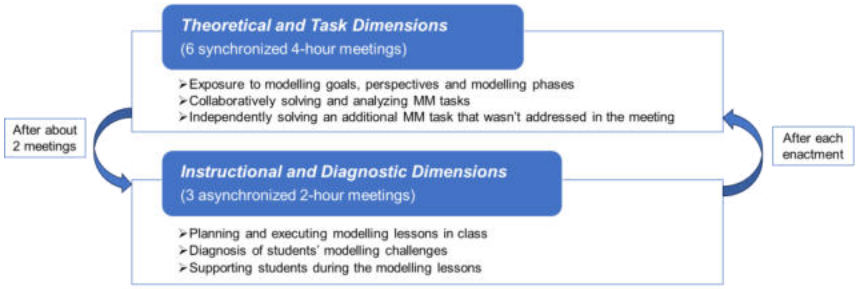

Figure 1: The PD program structure

Throughout the PD program, the teachers were exposed to six MM tasks and were asked to select at least one to implement in their classrooms, according to their preferences. Notably, teachers were not briefed on our conceptualization of the DI competencies. These tasks, rooted in scientific-engineering contexts drawn from authentic tech-related applications (Kohen & Orenstein, 2021), challenge students by requiring extensive scientific-engineering knowledge not typically included in mathematics lessons (Kaiser & Stender, 2013). Thus, students in this study were not required to fully validate their mathematical results, such as improving the model. Further, the execution of MM tasks in formal lessons in this study featured a blend of plenary sessions for real-world context elaboration and model development, group work or teacher-led guidance for mathematical procedures, and plenary sessions for result interpretation.

The study enlisted five Israeli mathematics teachers from lower-secondary schools (8th and 9th grades) who took part in the PD program. Participants comprised four women named Emma, Lea, Lucy, and Rachel and one-man John (pseudo), selected for their readiness to be observed while conducting MM tasks in class. The teachers had no prior knowledge of MM. Research tools included nine observations on teachers' enactments of MM tasks in their respective classes (Table 1).

Table 1: Summary of the MM tasks that were implemented by the teachers

Teachers	MM tasks implemented in class[1]
Emma	YouTube
Lea	YouTube
	Autonomous car
	A Signal from Satellite
John	YouTube
	Google website browser
Lucy	Mobileye (two lessons)
Rachel	Image Resolution

[1] The full tasks can be found at https://mted.technion.ac.il/i-mat-tasks-english/

To thoroughly analyse the challenges that teachers identify within each modelling phase and their corresponding scaffolding practices, we recorded and transcribed each lesson. The first author, an expert in MM, conducted the observations during 90-minute math lessons, focusing closely on teacher-student interactions from a whole class instruction perspective. Building upon our previous study examining teachers' diagnostic competence in identifying modelling challenges (Schwartz-Aviad & Kohen, 2023), this study focuses on their instructional competence that explicitly relies on their diagnostic competence. Using qualitative content analysis (Hsieh & Shannon, 2005), we identified key themes of practices teachers used to address challenges in modelling instruction, focusing on those that promote independent student learning.

Findings

We detected five themes of scaffolding practices, each addressing a specific challenge within a particular modelling phase outlined by Kaiser and Stender (2013) (Fig. 2).

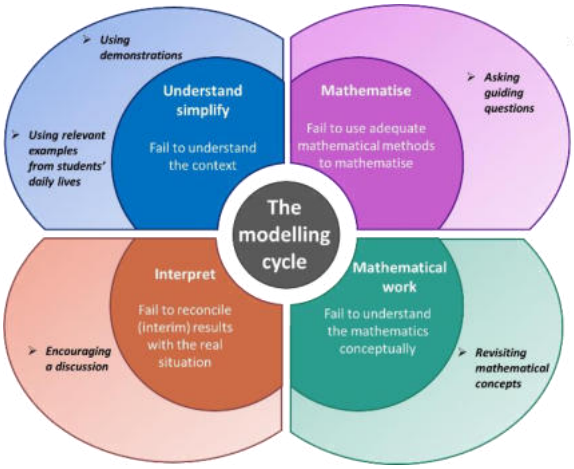

Figure 2: The themes of scaffolding practices, with relation to the modelling challenges within the various modelling phases

In the figure, the inner cycle demonstrates a modelling challenge within each modelling phase that was found to be addressed by a specific scaffolding practice, while the outer cycle depicts the corresponding scaffolding practices identified. In the following, we describe the identified themes according to the different modelling phases.

1. Using demonstrations to base understanding of the real-world context

This theme implicates teachers using various demonstrations to support students establish understanding of real-world concepts to address the challenge of *fail to understand the real-world context*. For instance, the YouTube task addresses video streaming issues, focusing on two indicators while watching a video: the viewing bar (data viewed) and the loading bar (data downloaded). After explaining these terms, Emma noticed students struggling to understand that the viewing bar moves faster than the loading bar, as one student asked, "I do not understand which bar moves faster?". To address this challenge, she used a demonstration with two volunteer students. Emma asked them to stand in front of the class and represent the viewing and loading bars to help students visualize the difference in their movement speeds and the point where they meet. She purposefully chose Eli and Amir, to represent the red viewing bar and grey loading bar, respectively, because they were wearing red and grey shirts. She explained: “Eli represents the red viewing bar, and Amir will be the loading bar. Who is going first? Amir goes first, and then they meet... Why do they meet? Who moves faster? [providing students with time to think and answer before directly answering], Eli." Emma then positioned the students next to each other and instructed them to start walking straight: “Amir, start walking [Amir walks straight], now Eli... stop [they both stop walking]. Who is walking for a longer time? [prompting students to think without specifying the answer directly], Amir [the loading bar takes more time].”

2. Using relevant examples from students’ daily lives to base understanding of the real-world context

This theme comprises teachers providing relevant examples from students' daily lives to support them understand real-world concepts, after identifying the challenge of *fail to understand the real-world context*. For example, John noticed a student struggling to grasp why different movies have varying loading speeds in the YouTube task:

Student Abby: Why does each movie have its own speed?
Teacher John: I'll give you an example. When I was a child, I watched a cartoon called Marko (the Heart) that was simple and quick to load because it had less data. But if you take a Pixar or Marvel movie, which has high quality and more data to download... [encourages the student to draw conclusions independently]
Student Abby: Every second, more data is downloaded because there are more frames, and more frames in each second.
Teacher John: Exactly...

3. Asking guiding questions to select and use an appropriate mathematical model

This theme involves teachers asking guiding questions to support students in selecting an appropriate mathematical model to solve the MM task, after identifying the challenge of *fail to use adequate mathematical methods to mathematise*. For instance, during Lucy's instruction, students struggled to choose using triangle similarities to measure the distance between the two vehicles in the Mobileye

task. Thus, Lucy asked guiding questions to prompt students to think about triangle similarities without directly telling them:

Student Alex: There's some kind of formula between triangles, since the camera captures at a wide angle, so it seems to me that it should work just fine.
Teacher Lucy: What formula between triangles?... what did you mean? [asking guiding questions without answering directly]
Student Alex: I don't remember what it's called. The camera captures from such a wide angle...

Even after the students figured out using triangle similarities, they struggled with its application, and Lucy continued asking guiding questions while maintaining minimal support:

Teacher Lucy: Do we have a connection between the triangle we know, the triangle ECD, and the larger triangle ABC on the vehicle in front of us? What is the connection between these triangles?
Student Jack: It is triangle similarities.
Teacher Lucy: Nice! How?
Student Jack: You have the sides and angles that are equal.
Teacher Lucy: Wait, slowly. What about the sides?
Student Jack: Here you have the angles that are equal...
Teacher Lucy: There are the vertex angles, and what else?
Student Jack: They share common sides.
Teacher Lucy: Common sides? I don't see that. The triangles are connected only at the vertex... EA is not a side in a triangle; it's more than a side... Come, help us think. You said the triangles are similar; now I ask how [prompting students to think].

4. Revisiting mathematical concepts to support their understanding

This theme encompasses teachers revisiting mathematical concepts necessary for the solution to address the challenge of *fail to understand the mathematics conceptually*. For instance, during the Image Resolution task, Rachel intentionally wrote an incorrect Pythagorean Theorem equation on the board and asked students one by one if they thought it was correct, engaging them in identifying the mistake themselves before reinforcing the correct concept. When they did not detect the mistake, she revisited the concept of the Pythagorean Theorem equation to support their understanding: “All 9X should be squared, and all 16X should be squared. I'm not just dwelling on this because it's a very, very common mistake, we forget that each of the factors is squared” (writing on the board the correct equation: $(16X)^2+(9X)^2= (50)^2$).

5. Encouraging a discussion to support understanding of how to reconcile results with the real situation

This theme refers to teachers initiating a class discussion to address a challenge of *fail to reconcile (interim) results with the real situation*. This discussion focuses on the solution process of the MM task, helping students understand how to align their results with real-world contexts. For instance, after achieving a graph representing the loading and viewing bars of YouTube, Emma detected a student who did not understand the meaning of the red line in relation to the real situation. To address the challenge, she facilitated a classroom discussion about the topic to encourage students to think independently through the problem:

Student Mia: Is it possible to continue the red graph so that it is above the grey one?
Teacher Emma: If the red line continued indefinitely, where do you think it would stop? Come show us. I want everyone to listen to the question and please try to answer.

Mia asks if it is possible to continue the red graph so that it is above the grey one [facilitating a classroom discussion to prompt students to think].
Student Zoe: Yes, but you need to wait for the grey line to load
Teacher Emma: Will there be a meeting point here, and will it continue?
Student Zoe: Not now.
Teacher Emma: What do you mean?
Student Zoe: The loading requires time.
Teacher Emma: More delay time is needed so that the grey line will not continue in this way. Because if more delay time is needed, we need more data, right?

Discussion

The findings reveal five major scaffolding practices employed by in-service teachers during real-time instruction, designed to address specific challenges within each modelling phase and promote independent student learning. These practices align with the theoretical scaffolding characteristics of *contingency, fading* and *transfer of responsibility* (Van de Pol et al., 2010). *Contingency* is evident through the tailored approach, with each practice addressing a specific student challenge in a distinct modelling phase. *Fading* and *transfer of responsibility* are apparent as teachers gradually reduce support and shift learning responsibility to students as challenges are resolved.

This study underscores the dynamic interplay between teachers' diagnostic and instructional competencies, where each scaffolding practice directly targets a specific modelling challenge. Theoretically, it suggests that DI competencies form a unified set of skills essential for effective modelling instruction, and their integration as a core component of teachers' professional competencies for modelling instruction (Borromeo Ferri & Blum, 2010). These competencies may be positioned as a critical bridge between diagnostic and instructional competence, given the unique expertise they require, encompassing instructional practices explicitly grounded in diagnosing students' modelling challenges while fostering independent learning in MM.

Further research is warranted to assess the effectiveness of these scaffolding practices on enhancing teachers' DI competencies and promoting student independence and engagement in MM. Future studies may explore the role of scaffolding tools and practices in MM (Schukajlow et al., 2015; Siller, et al., 2023; Stender, & Kaiser, 2015) in advancing teachers' DI competencies. Practical recommendations include designing PD programs that simultaneously address both aspects of teachers' DI competencies, thereby enhancing teachers' practices and their professional competencies for modelling instruction.

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Mathematical modelling as part of a general education in mathematics classrooms - challenge for teachers, responsibility for researchers

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Mathematical modelling is an essential component of the mathematics curriculum in many countries. Intensive research and discussion, also within the TWG Application and Modelling at CERME, have led to well-founded theoretical conceptualisations and frameworks as well as empirical results. However, our world is subject to constant change. It is therefore essential to regularly review the extent to which the formulated goals and perspectives as well as research projects of mathematical modelling (still) meet the requirements of general education or whether they need to be reinterpreted and implemented. Finally, it must be questioned how modelling problems should be designed on this basis. In this article, we put forward hypotheses that we hope will stimulate discussion.

Keywords: Mathematical modelling, general education, aims of modelling, problem situations

Introduction

The challenges of today's world are becoming increasingly complex, and the ability to understand the underlying processes is becoming increasingly dependent on a command of knowledge and skills. A significant proportion of the proposed explanations and derived measures for these complex problems are based on mathematical models. Skovsmose therefore demanded as early as 1994 that students must understand the relevance, but also the limits of mathematics for corresponding situations. However, Greer and Verschaffel (2007, p. 223) highlight that "people uncritically consume the products of mathematical modelling, without having any understanding of the models used, the assumptions on which they are based, or the general concept of modelling itself.". Thus, educated citizens must be able to understand, interpret and evaluate information in order to remain informed and reflective, ultimately contributing to a balanced, just society (Maaß et al., 2019; Vos, 2024).

Irrespective of whether the focus is on understanding stochastic processes when utilising large-language models, forecasting the trajectory of epidemics or the evolution of climate patterns: Each of the models employed is inherently complex and requires differentiated approaches to enable an understanding of the underlying mathematical models. Our following considerations are therefore guided by the questions: (1) What criteria and requirements must be placed on contemporary mathematical modelling in schools? (2) To what extent must the findings, concepts and requirements of recent years be (re)interpreted so that mathematical modelling can contribute to general education?

In this contribution, the term 'general education' is understood in the sense defined by Heymann (1996), who emphasises the importance of education that goes beyond specialist knowledge and promotes comprehensive personal development. He considers general education to be an education that enables individuals to think critically, acquire knowledge independently and act responsibly in a democratic and culturally diverse environment. It aims to turn people into responsible citizens who are able to deal with complex social and ethical issues. This includes the promotion of personal

characteristics and skills such as the strengthening of the student's self and the instruction in critical reasoning, which are related to life preparation, as well as skills that serve the coexistence and orientation in society such as world orientation, the development of a willingness to take responsibility and the practice of understanding and cooperation. Thus, this holistic approach to general education is not merely a transfer of knowledge; rather, it enables students to think and act independently and to navigate a complex world. Furthermore, it is compatible with the concept of higher general education as outlined by Fischer (2001) at the upper secondary level.

The following sections are not structured like a classic research paper. Rather, the first three chapters set out the aims of mathematical modelling, divided into modelling as a method and modelling as a personal capability, with each chapter ending with hypotheses that we believe are worth discussing. We then go into two aspects in greater depth, the aspect of modelling problems and the aspect of models, which we also conclude with hypotheses. The article ends with questions that (at least from the authors point of view) we as a research community should ask ourselves if we want mathematical modelling to contribute to general education today.

Aims of mathematical modelling at school

The aims associated with mathematical modelling have been established in several ways. In general, the formulation of aims can be divided into several levels. On the one hand, mathematical methods and concepts are employed to facilitate a deeper comprehension and mastery of present or future real-world scenarios (Blum, 2015) or at least "to understand central aspects of their world in a better way" (Kaiser, 2020, p. 554) while on the other hand the application references serve "to develop general mathematical competences" (Blum, 2015, p. 81) and "fostering and enhancing the motivation and attitude of learners towards mathematics and mathematics teaching" (Kaiser, 2020, p. 554). While aims of the first type see mathematical modelling as a method, objectives of the second type see mathematical modelling as a personal skill.

Both types of objectives have in common that, although they are formulated for the education of students, the achievement of these aims is targeted at the application of mathematics outside the educational context, i.e. in everyday private, professional and social life. However, the pressing questions of our time are complex and cannot be answered by a single person but require interdisciplinary cooperation (as exemplarily can be seen in expert committees). Subject teaching should therefore give students the opportunity to learn how different disciplines can contribute to finding solutions to global and local challenges of an economic, social and environmental nature (Coles et al., 2024). To this end, it is necessary to examine the extent to which known criteria for mathematical modelling problems need to be reinterpreted so that they also meet the requirement that students learn to question and assess the quality of models.

Mathematical modelling as a method

Mathematical modelling as a method can range from the direct application of known standard models to the independent development of complex models (Siller et al., 2015). At all stages, however, mathematical modelling should focus on developing a simplified description of the extra-

mathematical world within the mathematical world, working within the mathematical model, and then interpreting and validating the results thus obtained in the extra-mathematical world.

An example of the use of a standard model is the well-known application task 'fire brigade' (e.g. Kanefke & Schukajlow, 2022). Students are asked to determine the maximum height from which people can be rescued using a fire engine. Depending on the task, the required information must be read from a text or a table, estimated or researched. The Pythagorean theorem must be used as a standard mathematical model, considering the height at which the ladder is mounted. The use of standard models is essential for dealing with interdisciplinary problems (Goos et al., 2023) but is often not sufficient. It is therefore necessary for students to learn how complex, relevant problems can be solved using mathematical modelling. This can be achieved, for example, by tackling interdisciplinary problems such as the human impact on climate change. (Just et al., 2023). Among other things, students need to mathematically analyse the relationship between different data sets using regression analysis.

For a general education, one of the aims of mathematical modelling at school must be to enable students to understand, at least to some extent, models and the resulting solutions to contemporary problems, to understand and evaluate the measures derived from them and, if necessary, to develop well-founded solutions and measures themselves. This begins with the application of standard models and ends with the independent development of complex models.

***Hypothesis* 1:** *The entire spectrum of modelling problems should be covered at school. While dealing with standard models primarily serves as direct preparation for everyday life, dealing with complex models is also necessary for the development of social responsibility.*

Mathematical modelling as a personal capability

If mathematical modelling is seen as a personal capability, the area of modelling capabilities is affected. In most countries, the acquisition of modelling capabilities is formulated as an objective in school curricula (Kaiser, 2020). However, with regard to the public understanding of mathematical modelling (PUMM), Vos (2024, p. 96) claims: “In schools, the aspects that are highlighted in PUMM may often remain underexposed, whereas aspects often emphasised in modelling education, namely modelling competencies, were apparently absent in PUMM.”

If one applies the definition of competence by Weinert (2001), which many of the relevant definitions of modelling competence do, it consists not only of the capabilities necessary to move from one step of a modelling process to the next (the so-called modelling sub-competencies), but also of a corresponding attitude and metacognitive competencies (Kaiser, 2020; Maaß, 2006).

Current research focuses mainly on the acquisition of (individual) sub-competencies. This often involves isolating a single sub-step of a modelling problem (e.g. making assumptions, e.g. Segura & Ferrando (2023) or in the context of Fermi tasks e.g. Ärlebäck, Sevinc, et al. (2023)). From a research methodology perspective, this can be well justified, as sub-competencies are more reliable to collect and assess than a global modelling competence (Hankeln & Greefrath, 2021). However, Vos (2013) questions the extent to which these sub-competencies measure modelling capabilities in a valid way.

Brand (2014) showed that promoting individual sub-competencies in the sense of atomistic promotion can be more effective than promoting all sub-competencies simultaneously in the sense of holistic promotion. However, it should be noted that, on the one hand, sub-competencies themself are not sufficient to successfully tackle complex modelling problems. It should also be noted that Brand's intervention problems only focused on individual sub-competencies, whereas students were asked to work on complete tasks.

Hypothesis 2: *When promoting modelling capability, a holistic perspective should be adopted to have an affective effect on the learners as well as to develop the learners' cognitive capabilities comprehensively, where the cognitive skills do not consist only of the sub-competencies. This can serve both personality development and the creation of cultural identity by perceiving modelling as a cultural activity.*

Criteria for problem situations

Blomhøj and Kjeldsen (2006) have already presented criteria for good modelling problems for, some of which we believe are particularly relevant if modelling is to contribute to general education: The problem situation must be comprehensible to the students. Together with the fact that students should be able to understand models and the resulting solutions to contemporary problems at least to some extent, to understand and evaluate the measures derived from them and, if necessary, to develop well-founded solutions and measures themselves, this is an indispensable criterion, which at the same time means that the problem must have a connection to the reality of the students. If this is not the case, they will not be able to identify relevant factors and form appropriate models. This means that tasks should be authentic or contain authentic data, where authenticity means that the problem situation is not artificial but is relevant in certain situations outside the school context. For students to perceive mathematical modelling not only as part of the classroom but also as a method outside of school, it is necessary in this context that they can contribute their own experiences and research and use original data. By tackling problems that are either controversially discussed or presented in society, or that initially seem inaccessible to mathematical concepts, students can also learn that the modelling method is suitable for forming and defending well-founded opinions. At the same time, they can experience the dependence of results on the assumptions made and the models developed and thus learn to critically evaluate given models. They do not necessarily have to develop models themselves but can be given different models for a specific problem situation. Finally, Blomhøj and Kjeldsen also stress the need to become familiar with standard models.

This need exists particularly in a social context, as Siller et al. (accepted) emphasise. To this end, they focus on the *capabilities* needed to link the evaluation of media claims through the analysis of large data sets, identifying the skills needed for current and future civic life.

Hypothesis 3: *Students must be given the opportunity to contribute their own experiences and to work with original data that is controversially discussed and presented in a social context. Thus, they can learn to form well-founded opinions with the help of mathematical modelling and to argue for them.*

Quality of models

Dealing with real-life issues in mathematics education makes it possible to take up current problems relevant to the target group and integrate them into mathematics education. Many of these problems are characterised by the fact that they cannot be answered from only one discipline without ignoring important aspects. Questions from the STEM subjects are increasingly popular (Siller et al., 2023), as are questions related to sustainability (Ärlebäck, Carreira, et al., 2023).

A central component of the mathematical modelling of such problems is the application or utilisation of models. Models cannot be considered good or bad per se, but their quality must always be assessed in the light of the objective. Ortlieb et al (2013) identify three criteria for models based on Hertz's explanations. One of these criteria is correctness, which is given if it is consistent with known facts. He points out that correctness always depends on time, as knowledge expands and depends on the subject. Secondly, a model must be internally consistent, which he describes as permissible. The third criterion is the usefulness of the model, which means that the model does not contain parts that are superfluous to the problem being addressed. It is important that a model is as simple as possible and as complicated as necessary to represent the area of reality under consideration, which in turn is related to the purpose for which the model is being built.

An example is the modelling of the development of the corona pandemic and related situations, such as various containment measures, using the concept of exponential growth. This is not sophisticated mathematics at all, but the use of basic procedures that are deliberately chosen to describe, explain or predict situations (Siller et al., 2022; Vasquez et al., 2023).

Hypothesis 4: *For students to experience the consequences of mathematical modelling, it is essential to check the correctness, consistency and usefulness of given models and to question their own modelling as an exercise in critical thinking and the development of a sense of responsibility.*

Discussion points - responsibility of our research community

The extent to which, and the conditions under which, mathematical modelling can contribute to general education have been demonstrated in this article. Consequently, we as researchers must ask ourselves:

- How and to what extent can we, through our research, teaching and communicating science, help to ensure that mathematical modelling contributes to general education - or are we just researching it as an end in itself?
- How can mathematical modelling be used to address current issues and challenges, e.g. in the context of sustainable development and socio-ecological approaches?
- What changes are needed using AI, in particular large language models (e.g. ChatGPT), for example to formulate meaningful tasks?

So far, this paper has only touched the topic of assessment, whether in the context of performance assessment or research studies. However, it is imperative that we fulfil our responsibilities in this regard. The way modelling capabilities are evaluated in large-scale studies, such as PISA, or in national examinations, the tasks that are presented for this purpose, or in teacher training or

publications, exerts a significant influence on the implementation of mathematical modelling in schools and its perception in society. It is therefore imperative to adopt a critical stance.

- How do we measure modelling capabilities? Or to put it another way: What aspect of modelling capabilities do we measure, and do we always do so with this restriction?
- Is it simply the easiest way to meet certain methodological criteria, or does it also make sense in terms of content? What are the alternatives?

In this sense, Vos (2013, p. 488), makes the following request with regard to the measurement of capabilities in schools: "... we do not need to throw out the baby with the bathwater and abolish project-based or portfolio assessment: these formats have a high validity in relation to the creative aspects of modelling; research may assist in improving these formats with respect to test reliability".

- To what extent do the tasks we use in studies meet the criteria
- Is the relevance of the problem situation really given?

The objective of this paper is to contribute to the ongoing discourse on mathematical modelling by addressing current requirements. To this end, the hypotheses and questions presented are intended to stimulate a critical and constructive approach to addressing current and future challenges in mathematical modelling in research and teaching.

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The evanescence of mathematical models and modelling in the teaching of combinatorics

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This paper explores the role of combinatorics under a mathematical modelling approach through the lens of the anthropological theory of the didactic. We analyse three educational approaches to combinatorics, two from classical textbooks and one from a proposal of a study and research path about padlocks and security. This research considers two didactic phenomena in the teaching of combinatorics: the prevalence of applicationism and the evanescence of modelling. It explores the role that intermediate models can play in enriching the modelling process and overcoming significant constraints in learning combinatorics. Through the analysis of different teaching proposals, our research highlights the importance of fostering inquiry-based learning environments where students actively engage with the recursive and reversible nature of mathematical modelling and models.

Keywords: Mathematical modelling, combinatorics, anthropological theory of the didactic, dialectics of systems and models, study and research paths.

Introduction

Combinatorics is a fundamental branch of mathematics that some researchers consider essential for developing key mathematical skills such as problem-solving, formal reasoning, abstract thinking, conjecturing or generalizing. Kapur (1970) and English (2005) emphasize the importance of including combinatorics in mathematics curricula, where its connection to areas such as probability and statistics is highlighted. Lockwood (2013) points out that research in combinatorial education is limited and has not sufficiently explored how students conceptualise counting problems. Often, the teaching of combinatorics is reduced to memorizing formulas and considering standardized counting situations (Batanero et al., 1997), which limits a deep understanding of the underlying concepts.

In this paper, based on the anthropological theory of the didactic (ATD), we examine teaching proposals in combinatorics for secondary education and their interpretation through mathematical modelling. ATD identifies two pedagogical paradigms (Chevallard, 2015). The 'paradigm of visiting works' involves a unidirectional process where teachers present structured knowledge for students to learn and reproduce, often leading to 'monumentalism', where knowledge is valued as an end in itself. The application of this knowledge, if introduced, occurs in artificial contexts, described as 'applicationism' (Barquero et al., 2013). Conversely, the 'paradigm of questioning the world' shifts the focus to the study of questions through collective investigations, where knowledge is introduced as a tool for inquiry. Chevallard (2015) proposes study and research paths (SRPs) to support the transition between the two paradigms. SRPs are instructional formats that start with the consideration of open questions to be addressed collectively by the group of students guided by the teachers, where the search and study of pieces of knowledge is conditioned to their need for the inquiry.

Our starting point is an SRP generated by the question of padlocks' security, aiming to explore combinatorics through real-world scenarios and implemented in Spanish secondary schools since 2019 (Vásquez et al., 2021). Two phenomena are linked to this SRP. First, the dominance of 'applicationism' in current secondary mathematics education, where models (combinatorial formulas) are introduced before their application. Second, the 'evanescence' of modelling activities (Wozniak, 2012) and the difficulty to work with intermediate models that do not have a clear mathematical status even if they play a determinant role in the approach of combinatorics problems.

Theoretical framework and methods

Mathematical modelling in the Anthropological Theory of the Didactic

ATD presents a perspective of mathematical modelling that places it at the centre of mathematical activity. From this vision, doing mathematics involves creating, transforming, interpreting and developing mathematical models. Mathematical modelling activities are described as processes of reconstruction and articulation of mathematical organisations of increasing complexity, generated in response to problematic questions (Barquero, 2023). The modelling process involves the following stages (Figure 1): first, the delimitation of a system S_0, which implies defining the initial piece of reality to be modelled after a question Q_0 is raised. The second stage corresponds to the construction of a model M_1 linked to a mathematical organisation supposed to help produce knowledge about the system. The next stage involves working within the model, where mathematical tools of the model organisation are used to obtain results a produce an answer A_1 to Q_0. Finally, the results obtained in the model are interpreted in relation to the original system, which can generate new questions Q_1, a new delimitation of the system, S_1 (now including S_0, M_1 and A_1), and a recursive modelling process.

A central element of the modelling process lies in the roles played by models and systems. The system refers to the object of study in the modelling process, which can be either intra-mathematical or extra-mathematical. The aspects of the system to be studied depend on the question initially addressed. Meanwhile, the model is built based on a knowledge organisation that represents the system and generate knowledge about it. Being a model or a system is not a property but a function that can be sometimes assumed by the same types of entities.

A crucial aspect in this perspective is the constant questioning of the model's adequacy to the system. This involves evaluating the model's ability to answer the posed questions, identifying its limitations, and, if necessary, redefining the system, adjusting the model or building new models that better fit the system (Barquero, 2009). The ATD emphasizes the importance of intra-mathematical modelling, that is, the modelling of systems that are intrinsically mathematical. This type of modelling has been a fundamental driver in the historical development of mathematics and enables the consideration of modelling processes where a sequence of models is involved. Intermediate models emerge during these processes, serving as bridges between different mathematical organisations.

As explained by Barquero (2023), modelling processes are *recursive* and *reversible*. Recursivity refers to the process where a model becomes the system for further modelling, allowing for successive refinements. This involves creating 'models of models' and redefining progressively the system being studied, integrating previous models and knowledge generated by models (see Figure 1). Reversibility

corresponds to the ability to invert the relationship between a system and its model, where the system can serve as a model of its previously considered model.

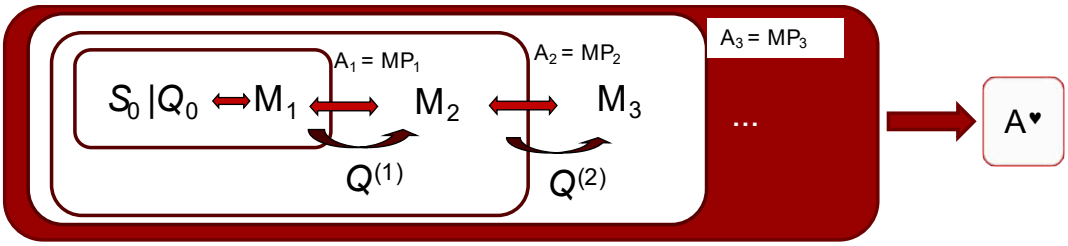


Figure 1: Diagram of the modelling process in the ATD (Barquero, 2023)

Methods

Our research follows the different phases of didactic engineering (DE) as research methodology (Artigue, 2020). The first phase focuses on a *preliminary analysis* of the didactic phenomena we have detected concerning combinatorics: the prevalence of applicationism (Barquero et al., 2013) and the evanescence of modelling praxeologies (Wozniak, 2012) in the teaching and learning of combinatorics. At this stage, we chose two textbooks to analyse their proposals. Following the work of Juberó (2022), we analysed a textbook with a classical introduction to combinatorics (Salvador & Molero, 2022) and a textbook that proposes starting from situations and generates a classification of combinatorics problems and formulas based on sequences of codes as intermediate models (Bosch et al., 1997). The second phase consists of the *a priori* analysis of a teaching proposal that aims to overcome the above-mentioned phenomena. In our case, it is concretised by the design of a study and research path (SRP) about padlocks' security. We started from the design and experiences conducted by Vásquez et al. (2021), where a full description of the SRP about padlocks is provided, followed by two cycles of implementation. From this implementation, the DE continues with the *in vivo* and *a posteriori* analysis of the SRP, which provides empirical data to contrast the initial designs of the SRP and sheds light on the didactic phenomena at stake.

To analyse the inquiry processes followed during the implementation of the SRP, we used the *dialectics of systems and models* to describe, analyse and understand the different tensions and decisions that arise in the modelling processes (Barquero, 2023). We focus on identifying the key elements in a modelling process and their evolution, in particular: the systems, the questions raised, and the models considered, as well as their relationships. With these elements, we describe the potential modelling process deduced from the didactic proposals, and how they integrate the recursive and reversive properties of modelling. We can finally state our research questions with a dual objective:

RQ1: What traits of the dialectics between systems and models characterise different proposals for the teaching of combinatorics at secondary school education?

RQ2: What conditions are established, and what constraints are anticipated, from the different proposals for the teaching combinatorics under a modelling perspective?

Analysis of the modelling process in different proposals

The textbook *Marea Verde* (Salvador & Molero, 2022), analysed by Juberó (2022), exemplifies the dominant mathematical organisation in Spanish secondary education for counting problems, focusing on permutations, variations, and combinations. Proposed strategies include tree diagrams and sequence lists, used to introduce the algebraic formula that correspond to each type of problem. Juberó (2022) highlights several limitations, such as insufficient guidance on selecting methods and a lack of theoretical discourse to interpret the results. The types of tasks proposed are repetitive and closed, with no focus on identifying variables or validating techniques. Combinatorics is taught as an application of pre-defined formulas to fixed problems, omitting key phases of the modelling process. Instead of constructing models, the textbook presents combinatorial formulas directly (M_0 to solve pre-set problems (S_0), reflecting a reduced form of applicationism (Figure 2). This approach treats models as static tools, preventing their evolution or the exploration of alternatives, limiting deeper understanding of their construction and broader applicability.

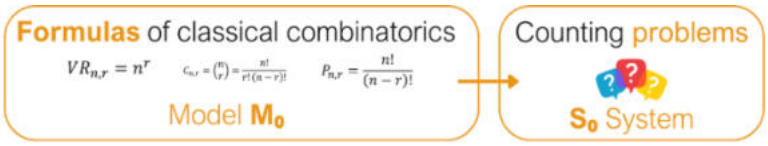


Figure 2: Diagram of the modelling process in the "Marea Verde" textbook

The *Almadraba* textbook (Bosch et al., 1997) offers a different approach, starting with situation-types such as football pools (variations with repetition), posters on a wall (variations without repetition), and prize allocations (combinations). The situations are approached through a modelling process in four phases: translation of the problem statement using a sequence of symbols, qualitative and quantitative analysis of the groups of symbols to see if the order is important or repetitions are allowed, and the generation of a general formula for each case. The book also classifies problems into simple and complex, using a classification algorithm. Juberó (2022) identified the groups of symbols as an intermediate model aiding the identification of permutations, variations, or combinations, with exploratory questions guiding the development of the techniques. However, the approach lacks technique diversity, reverse problems and explicit discussions about the interpretation or justification of the results. Unlike the earlier case, models are developed after studying initial systems (S_0), with exploratory questions facilitating the production of a first model (M_1) using a classification algorithm. Initial situations are categorised, leading to a second model (M_2) structured around symbol sequences (Q_1: *What sequence does the problem admit?*). The sequences obtained form a new system (S_2) addressed by question Q_2: *How can we obtain the total number of possible sequences?* which would lead to a final model, M_3, of numerical formulas (Figure 3).

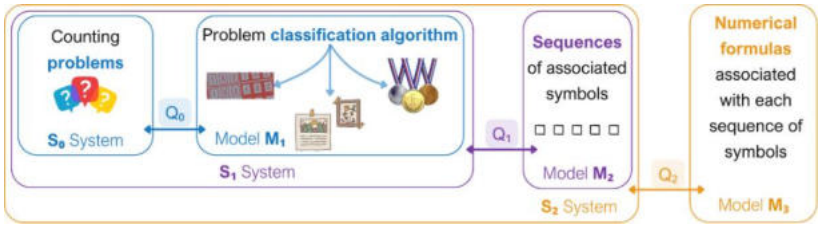


Figure 3: Diagram of the modelling process in the textbook *Almadraba*

In this case, the modelling process is identifiable, with key elements such as systems, models, and questions recurring, at each step as more complex models emerge from studying "types of" counting situations. However, both teaching proposals omit crucial steps in the modelling process, such as model validation or the exploration of opener questions or systems. Their focus remains confined to pre-determined controlled counting situations aligned with combinatorial arrangements.

The third proposal consists of the SRP about padlocks, widely analysed in Vásquez et al. (2021). The initial system presented (S_0) is a set of different padlocks and a generating question (Q_0) about which of the padlocks is the most secure. This initial system is accessible to the students, they can handle the padlocks, experimentally generate lists of codes or passwords, and propose different models to list and count the total number of codes for each padlock. In the first phases of the SRP, initial models M_1 used to be proposed, involving models based on generating lists (with or without structure). For example, students used spreadsheets to make counting more feasible. Quickly a new question Q_1 emerged about *how many codes each padlock does support.* This first model reveals many limitations, especially in validating the result (How can we know if a combination is missing?). The students usually take these initial lists (simulated by the first models) and analysed them. At this step, the set of padlocks and model M_1 constituted a new system S_1 to be again modelled. Then, new models (M_2) can appear, which are more advanced and involve more comprehensive strategies, such as models based on arithmetic calculations to count the total number of codes. This process can be followed by presenting Q_2*: Can we determine the number of codes for new padlocks or those with known information? Is there any possible classification of padlocks, according to the way to count their total number of codes?* The arithmetic calculation models (M_2), along with their corresponding padlocks that comes to be modelled in previous stages, constituted the system S_2 to be studied. Along with the new information, students investigated the validity of the previous models, thereby finding a model M_3 consisting of numerical formulas (pre-algebraic in some cases) associated with each family of padlocks (Figure 4). It is worth noting that M_1 and M_2 act as intermediate models, which are crucial to explain and justify the more advanced models M_3 consisting of numerical formulas.

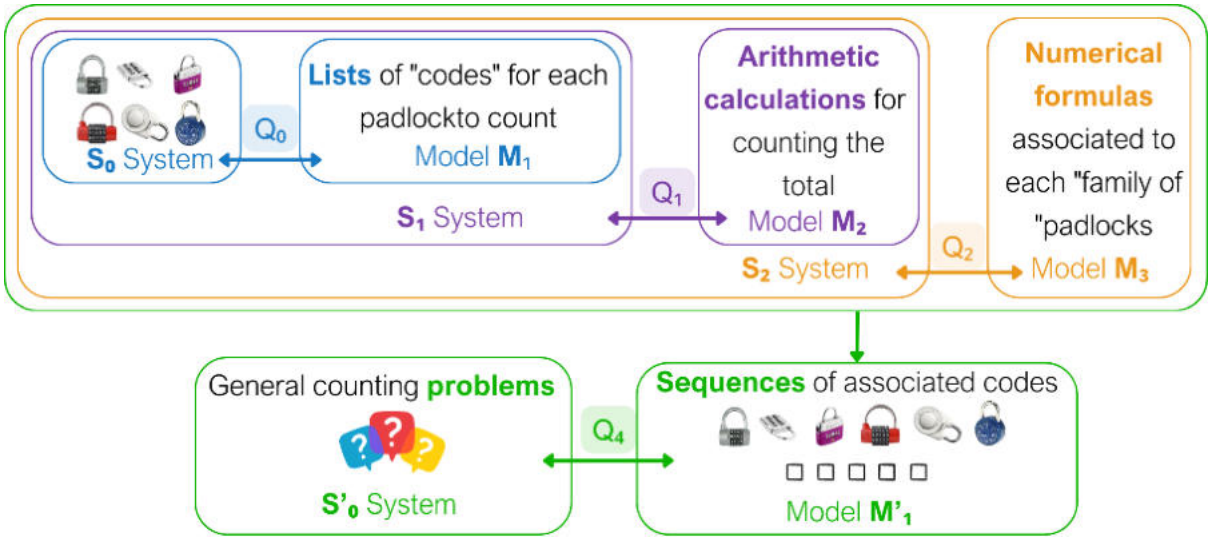
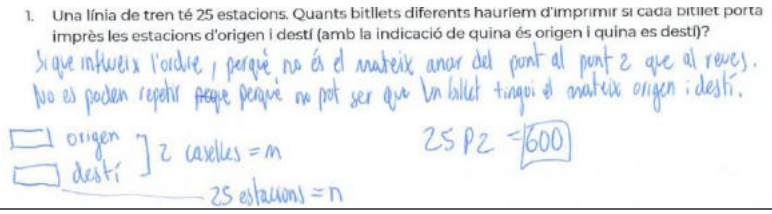


Figure 4: Diagram of the modelling process in the padlock's SRP

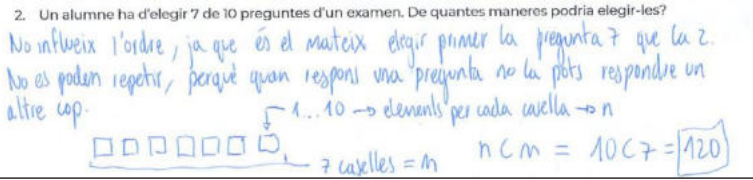
The recursive potential arises when the SRP is expanded to study other (more traditional) counting situations and introduce new questions, such as Q_4*: How can general counting problems be solved?* In this new context, the implementation shows that the padlocks (initial system, S_0) are then used by the students as models to represent sequences of codes, which helps model other counting problems using the previous intermediate models.

Results from the implementation of the SRP about padlocks

Vásquez et al. (2021) analyse students' work to show the various stages of the modelling process shortly described in the previous section. In that study, the upper part of the diagram in Figure 4 is extensively described, but little is said about the latter part about solving general counting problems. The evolution of the SRP implementation within the didactic engineering cycle shows an interesting result to this respect. In the analysis of the first two implementations, when coming to solve general counting situations, students started considering the padlocks to build the initial models. However, this use of the padlocks as intermediate models did not last, and students quickly reverted to applying formulas, without linking them to their previous work and falling, once again, into a more applicationist approach. This may have occurred because too much emphasis was placed on the formulas by the teachers, without relating them to the modelling process. One of the participating teachers described it as "some students see it as two different activities: first, we investigate the padlock problem; but then they forget and go back to using the classical combinatorial algorithm to solve the problems". In the last implementations of the SRP, teachers placed more emphasis on solving problems by always using padlocks as a tool to justify the choice of the resolution method, thus highlighting the role of padlocks as intermediate models. With these last modifications, when the students had to solve general counting problems, they first created a box diagram that represents a padlock adapted to the counting situation and analysed the main variables associated to it: cells, elements in each cell, influenced by the order (or not), repetition of the elements (or not) (Figure 5).



A train line has 25 stations. How many different tickets should we print if each ticket has the origin and destination stations printed on it?
Student: The order does matter, because going from point 1 to 2 is not the same as going from 2 to 1. They cannot be repeated because it cannot be that a ticket has the same origin as destination. (2 cells = m; 25 stations = n)



How many ways can a student choose 7 out of 10 exam questions?
Student: The order does not matter, as it's the same to choose question 7 first as it is to choose question 2. They cannot be repeated, because once you answer a question, you cannot answer it again. (10 elements per cell = n. 7 cells = m)

Figure 5: Example of student's productions

Conclusions

Concerning the analysis of the three proposals on combinatorics, we can conclude that the dialectics between systems and models evolves increasingly throughout the three proposals. In the first case (*Marea Verde*), which corresponds to the most spread teaching strategy in Spanish textbooks, there is an almost direct assignment of the corresponding formulas to the counting situations under study. In terms of the dialectic between systems and models, the formulas come first (acting as systems), and the different counting problems are considered to illustrate the type of situations that correspond to the formulas (thus acting as models). The role is reversed: situations are models of the different combinatorics formulas. In the second case (*Almadraba*), the situation is similar but an intermediate model in terms of groups of symbols and sequences is introduced between the situations and the formulas. This enables more flexibility in the study of counting situations and opens the possibility to address "complex situations" that do not directly align to an established formula (permutation, variation or combination). Finally, the SRP about padlocks initiates a more open inquiry process, which is then used as a model to address a variety of (simple) counting situation. In this case, padlocks serve as intermediate models. However, this role needs to be explicitly reinforced by the teaching strategy.

Regarding the conditions identified from a modelling perspective in the latest implementations of the SRP concerning padlocks, we have emphasized the necessity of providing a clear status to the intermediate models (M1, M2 and the types of padlocks themselves), especially to address the evanescence of the modelling process. In this respect, teachers play a central role, as it is fundamental that they assign and share words about the models, systems and the entire modelling process with their students. That is, it is necessary to develop a didactic (and mathematical) *logos* to describe the modelling process, with its main elements, including recursivity and reversibility. Otherwise, activities related to modelling will end up disappearing to the detriment of the institutionally recognised combinatorics formulas. This is the research line we are developing at present. Explicitly naming and assigning a mathematical status to the system-model dialectic can provide teachers with an epistemological tool to manage the modelling processes that emerge from inquiry processes. It will also help teachers provide new names for the concrete systems and models that are considered during the process. New mathematical developments are needed to overcome the limitations derived from the applicationism and the evanescence of intermediate models.

Acknowledgment

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Pre-service teachers' judgments of students' solutions to open modeling problems: How important is it to make assumptions?

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An important type of mathematical problems are open modelling problems. However, it is a lack of insights into teachers' judgements of students' solutions to open modelling problems. The research question was: Which aspects of the students' solutions do the pre-service teachers emphasise. Twenty pre-service teachers engaged in solving four problems, assessed 14 students' solutions to these problems, and provided justifications for their judgements. A content analysis of the pre-service teachers' justifications identified positive and negative emphases while judging students' solutions. The findings suggest that pre-service teachers emphasised numerical and realistic assumptions positively, while they emphasised unrealistic and missing assumptions negatively. This study contributes to the theory of assessment of open modelling problems by showing what aspects of students' solutions matter when assessing mathematical modelling.

Keywords: Open modelling problems, teacher education, diagnostic competence, judgements, openness.

Introduction

Open problems are important in mathematics education, especially as open modelling problems. Modelling problems are characterised by their connection to real-world contexts and require complex transformation processes between reality and mathematics (Niss & Blum, 2020). A defining characteristic of open modelling problems is that some information needed to solve the problem is lacking. As a result, problem solvers have to make assumptions to account for these missing information. Both students and (pre-service) teachers often struggle with open modelling problems (Stylianides & Stylianides, 2023). Students need to master task-specific requirements while solving open modelling problems and needed to be taught how to solve open modelling problems (Schukajlow et al., 2023). For teaching those problems, (pre-service) teachers should be capable of judging students' performance on open modelling problems (Greefrath et al., 2022). One difficulty in judging students' solutions to open modelling problems is the possibility of multiple solutions and results (Borromeo Ferri, 2022), which makes objective judgements difficult. Diagnostic activities of teachers are a main task of their profession and should take into account the openness of modelling problems. Nevertheless, there is a lack of empirical studies on the judgement of students' solutions to open modelling problems. As part of the DOModA project (Diagnose von Schülerlösungen zu Offenen Modellierungsaufgaben, Diagnostic Competencies of Student Solutions on Open Modelling Problems), we conducted an interview study to fill this research gap. In this study, we analyse which aspects of students' solutions to open modelling problems pre-service teachers positively or negatively emphasized while judging these.

Theoretical background

Open modelling problems

One of the goals of mathematics education is to develop modelling skills (Blum, 2015), for example by working on open modelling problems where crucial information is missing and complex transitions between reality and mathematics are required. In our study, we utilised such problems, where solution-relevant information is missing from the problem statement and it is necessary to make assumptions about it. The improvement of teachers' abilities to deal with open modelling problems must be a focus of attention, as prior research indicated difficulties of students with dealing with openness (Ärlebäck, 2009). Recent research suggested a framework outlining the typical requirements for dealing with open modelling problems (Schukajlow et al., 2023): (1) perceive the missing information, (2) identify the unknown quantities that are needed to solve the problem, (3) make (realistic) assumptions about the unknown quantities and (4) including the (realistic) assumption in the mathematical model. Without recognising the openness, students may approach open problems like closed ones, focusing only on provided numerical data and potentially reaching unrealistic solutions (Galbraith & Stillman, 2001). Identifying the unknown quantities is an important part of solving open modelling problems (Hankeln, 2020). Making numerical assumptions can be challenging, as it often leads to barriers in problem-solving, either due to ignoring realistic considerations or lacking the knowledge to make assumptions (Galbraith & Stillman, 2001). These assumptions must be integrated into the mathematical model, a step some students struggle with even after making the necessary assumptions (Schukajlow et al., 2023). Assumptions are categorised into situational and numerical types (Krawitz et al., 2022), with the latter involving challenging realistic reasoning and extra-mathematical knowledge. Situational assumptions pertain to assumptions about the circumstances, while numerical assumptions involve assumptions about missing values (Chang et al., 2020). Students often face difficulties with estimation skills, resulting in inaccurate estimates (Krawitz et al., 2022). Moreover, assumptions can be evaluated based on their reference to reality, with research indicating that students infrequently make realistic assumptions when solving open modelling problems (Krawitz et al., 2022). The step of making realistic assumptions is essential for achieving a good solution to a problem, as unrealistic assumptions or their absence can lead to unrealistic results throughout the modelling process (Chang et al., 2020).

Diagnostic activities

In our study, we define diagnostic activities as the gathering and interpreting of information related to learning conditions, processes, or outcomes, often through evaluating students' written work (Leuders et al., 2018, p. 4). Diagnostic dispositions encompass the knowledge, skills, motivation, and beliefs of the individuals making judgements (Blömeke et al., 2015; Leuders et al., 2018), and are crucial for effective diagnostic thinking. Teachers often use their own solutions and knowledge to identify students' difficulties (Ostermann, 2018; Philipp & Gobeli-Egloff, 2022). Identifying strengths and weaknesses in students' work is considered the first crucial step in the diagnostic process (Philipp & Gobeli-Egloff, 2022).

The judgement of student solutions in the context of modelling problems is challenging and complex (Blum, 2015; Greefrath & Vorhölter, 2016). Open modelling problems allow multiple correct

solutions, require a combination of several sub-competencies and the modelling process must be evaluated too (Borromeo Ferri, 2022). A potential assessment scheme suggests valuing working mathematically and documentation (Maaß, 2007). One fundamental problem in assessment is how to judge simple, error-free solutions versus complex solutions with minor errors (Borromeo Ferri, 2022). Prior research on in-service teachers revealed different reasons for the judgment of student solutions to an open modelling problem. For example, some teachers see assumptions as necessary for judgement, but they underline that the focus of judgements always dependent on the content and goals of the lesson (Borromeo Ferri, 2022). According to Cai et al. (2022) teachers emphasize strongly positive aspects of students' solutions while giving feedback .

Research question

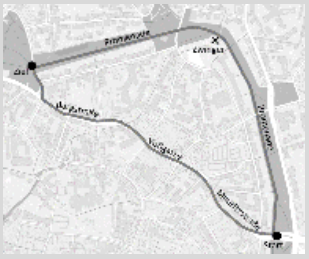
The research question of this paper is: Which aspects of the students' solutions do the pre-service teachers emphasise a) positively and b) negatively while judging students' solutions to open modelling problems?

Methods

Twenty master's students in mathematics education volunteered for this research (13 females, average age 23.35 years). Fourteen were enrolled in the educational program for secondary school teaching and six in the educational program for lower-secondary school teaching.

Promenade

Max is riding his bicycle to school. On his way back home, he is considering whether he should take the route through the bustling city center of Münster, via Mauritzstraße, Voßgasse, and Bergstraße, or if he should bypass the city center by taking the bicycle path along the promenade. To do so, he would have to ride along the promenade for 650 meters until he reaches the historic Zwinger, and then continue for another 600 meters along the promenade.



Which route do you recommend to Max so that he can get home earlier?

Figure 1: Open modelling problem Promenade

We designed an interview-based study to explore how pre-service teachers judge student solutions to open modelling problems, drawing on both theoretical frameworks and empirical evidence. Different modelling problems were addressed in the interviews lasting 60 to 90 minutes, divided into problem-solving and judgement phases. One example is the Promenade problem (see Figure 1), which could be solved by using Pythagorean theorem by calculating the missing side of a right triangle or by using scaling drawing. Participants solved four open modelling problems (problem-solving phase) and then judged various student solutions (judgment phase), justifying their judgements. Key questions were: "What are your considerations about these students' solutions?", "How do these solutions differ from your own approach?", and "How would you judge these solutions in an examination setting?". The interviews were conducted in pairs.

The Promenade problem required students to choose the best path for Max. Students had to perceive missing information, such as Max's cycling speed, and make realistic assumptions, like assuming speeds of 18 km/h along the promenade and 11 km/h through the city centre. We designed four typical student solutions based on the theoretical framework established in a previous study (Schukajlow et al., 2023). These are shown in Figure 2. The solutions vary depending on how effectively the students meet the requirements for dealing with openness. The first solution does not include any assumptions. In the second solution, the student recognised that some information was missing, but did not make a numerical assumption. The third solution contains an unrealistic assumption. The fourth solution contains a realistic assumption.

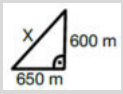
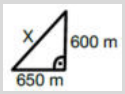
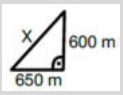
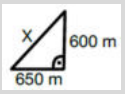
Solution 1: Without any assumption City centre: $x = \sqrt{600^2 + 650^2} = 884{,}59\ [m]$ Promenade: $600 + 650 = 1250\ [m]$  A: The route through the city centre is shorter than the route along the promenade, so Max should choose the city centre route.	**Solution 2:** Perception of the openness (in the answer) City centre: $x = \sqrt{600^2 + 650^2} = 884{,}59\ [m]$ Promenade: $600 + 650 = 1250\ [m]$  A: The route through the city centre is shorter than the route along the promenade, and Max should choose this route. You have to consider how fast Max is riding.
Solution 3: With an unrealistic numerical assumption City centre: $x = \sqrt{600^2 + 650^2} = 884{,}59\ [m]$ Promenade: $600 + 650 = 1250\ [m]$  Assumptions about the speeds at which Max can ride each route: $v_{Promenade} = 5\ km/h,$ $v_{city\ centre} = 10\ km/h$ Let $t = s/v$ be the equation for time, where: $t_{promenade} = \frac{1{,}25km}{5\ km/h} \approx 15\ min$ $t_{city\ centre} = \frac{0{,}885km}{10\ km/h} \approx 5{,}31\ min$ A: The route through the city centre is faster.	**Solution 4:** With a realistic numerical assumption City centre: $x = \sqrt{600^2 + 650^2} = 884{,}59\ [m]$ Promenade: $600 + 650 = 1250\ [m]$  Assumptions about the speeds at which Max can ride each route: $v_{Promenade} = 18\ km/h,$ $v_{city\ centre} = 11\ km/h$ Let $t = s/v$ be the equation for time, where: $t_{promenade} = \frac{1{,}25km}{18\ km/h} \approx 4{,}17\ min$ $t_{city\ centre} = \frac{0{,}885km}{11\ km/h} \approx 4{,}83\ min$ A: The route through the promenade is faster.

Figure 2: Students' solutions of the Promenade problem offered in the study

To answer the question of which aspects of the students' solutions the pre-service teachers positively or negatively emphasised, we conducted a qualitative data analysis (Kuckartz & Rädiker, 2019). The transcripts of the interviews were coded for positive or negative emphasis in the pre-service teachers' judgements. The interviews were double-coded with an interrater-reliability of κ = .92 for positive emphases and κ = .89 for negative emphases. The minimum coding unit was one sentence. Positive emphasis was assigned to statements in which participants explicitly praised or expressed agreement with an idea, approach or solution. This code captured terms such as good, great, interesting or praiseworthy. For example, one pre-service teacher positively evaluated the modelling process of a student, stating:

Lara: Yes, I think it's good that this student has definitely thought about the fact that it's not just a maths problem, but that it's also transferred back into the real world.

Emphases were coded as negative if they referred to the statements that conveyed subjective disapproval or strong negative reactions using terms such as stupid, disastrous, or idiotic. The interview transcripts were first deductively coded to identify pre-service teachers' positive or negative emphases. Then, we identified inductively which aspects of students' solutions were emphasised positively or negatively in the judgements.

Findings

Pre-service teachers positively emphasised the *making of assumptions* in students' solutions, as they saw this as an important aspect of mathematical thinking and praised students for recognising that assumptions had been made. Pre-service teachers appreciated that students had considered the context of the problems by making assumptions. *Making numerical assumptions* was also positively emphasised, regardless of the realism of the assumption, as can be seen in this interview:

Simon: This would also be a perfect solution for me in the sense that an assumption has been made. They may not be exactly right, but at least they have thought about it and made an effort (points to a solution with a numerical assumption).

Furthermore, pre-service teachers positively emphasised *assumptions in text form*, as these did not imply setting exact values and were correctly placed within the context. In an interview, the mention of influencing factors in the text was positively emphasised and, in another interview, making rough estimates without specifying exact values was considered an acceptable and effective approach. In addition, pre-service teachers emphasised positively a *physical approach* of the Promenade problem. For example, the calculation in solutions 3 and 4 of the Promenade problem, which the pre-service teachers expected to be an essential part of any correct solution of Promenade problem ("under any circumstances"). The students' *calculation* was also positively emphasised. It was emphasised that a student solution without an assumption (solution 1) would also be a mathematical achievement because "everything was done correctly" but no assumption was made. The way in which all the student solutions were *written down* and the *use of units* was emphasised positively.

Pre-service teachers particularly emphasised *unrealistic assumptions* that ignored the context negatively, like in interview 4:

Lia: It's also great physically here with T is equal to S by V, but/but the assumption is actually total nonsense.

The pre-service teachers judged the calculation as correct and identified students' problems in interpreting their results, but also noted the *missing transfer to reality* in a negative way due to the lack of assumptions. The *lack of reasons for numerical assumptions* was also emphasised as negative. The pre-service teachers did not perceive the importance of making assumptions in the modelling problems and rated *numerical assumptions* in students' solutions negatively. They explain that "the problem did not require such a complex solution with numerical assumptions" and that solutions without any assumptions were sufficient, which was not correct. The pre-service teachers negatively emphasised solutions, in which students made *assumptions in text form*. On the one hand, because these additional statements were made without a mathematical basis and, on the other hand, because

these statements do not contain any justification. In one interview, this mention was emphasised negatively because it was not mathematically calculated or included. In addition, it is negatively emphasised that the solution perceiving the openness in the answer would be a lazy solution. *Assumptions made in the drawing* were negatively highlighted, because the drawing showed the need to make assumptions, but not the calculation.

Discussion

The pre-service teachers mainly emphasised numerical assumptions positively and showed a general appreciation for students' use of numerical assumptions in their solutions, appreciating the precision and clarity of these assumptions brought to the problem-solving process. This aligns with previous research, which highlights the importance of making numerical assumptions (Chang et al., 2020; Schukajlow et al., 2023). However, judgements of regarding unrealistic assumptions differed significantly, with some pre-service teachers believing that assumptions could be helpful when considered in context, and others believing that they should be kept strictly within realistic parameters. The missing transfer to reality was viewed negatively, emphasising the need for students to make their assumptions explicit. The assumptions in text form, was seen as positive, demonstrating the value of contextualised thinking. This indicates that pre-service teachers value context-specific reasoning (Chang et al., 2020). Assumptions in text form were both positively and negatively emphasised, reflecting different perceptions that depend on the context and plausibility of the assumptions. Following the requirements for open modelling problems, this can be interpreted as an essential step in recognising openness. On the other hand, the pre-service teachers evaluate the answers negatively due to their missing reasoning or missing mathematical considerations. Overall, numerical assumptions, along with realistic and unrealistic assumptions, received much attention from pre-service teachers. This focus indicates that pre-service teachers primarily concentrated on the quantitative aspects of students' solutions. This is in line with the assessment scheme of Maaß (2007) focussing on the calculation and documentation during the modelling processes. This highlights the necessity for the development of assessment schemes that focus on other aspects, such as situational assumptions. In contrast, assumptions represented in calculations were mainly highlighted positively, suggesting a focus of concrete numerical representations as evidence of modelling competencies.

Pre-service teachers tend to emphasise particularly positive aspects of students' solutions (Cai et al., 2022). In line with previous research, our study demonstrated that these emphases differ between pre-service teachers. Furthermore, we observed that the type of students' solutions influenced the emphases. These findings indicate that the accuracy of pre-service teachers' judgments is contingent upon the specific open modelling problems or contexts presented. A novel finding of this study is the identification of specific aspects of students' solutions that pre-service teachers emphasize in their judgments. However, further research is needed to validate these findings. One limitation of this study is that the sample consisted solely of students from a German university. Future studies should assess the judgments of pre-service teachers from diverse educational contexts and analyze the judgments of in-service teachers as well. Another limitation is that four solutions were presented with typical errors which were derived from theory and prior research. In future research, it would be beneficial to adopt a more open approach and inquire directly from students which solutions they consider

effective and how they would evaluate different aspects of these solutions. This aspect warrants further research, preferably utilising qualitative methods or mix-methods, as quantitative approaches might not capture the nuanced differences in pre-service teachers' judgements. By showing what might be important in assessing students' solutions, this study contributes to the theory of assessment of open modelling problems.

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The dishwasher experience: Pre-service teachers' use of extra-mathematical knowledge while investigating societal topics

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Keywords: Mathematical toolkit, water consumption, mathematical modelling, extra-mathematical knowledge, pre-service teachers.

Introduction and previous research

In this poster, we present an ongoing qualitative study focusing on a toolkit for mathematical modelling (MM) of societal issues with pre-service teachers of mathematics (PTs). A toolkit is a collection of different elements, such as readings, prompts, discussion tasks, lectures, etc., related to the societal topic and its teaching (Kacerja et al., 2024). In this class, 30 PTs (4th year) were shown a video as a prompt for some general discussions and a presentation about water scarcity issues in South Africa. Then, they were given a discussion task where they could choose one or more questions about water consumption to answer in their group work for 45 minutes. Here, the focus will be on a videotaped session of one group of four Norwegian PTs working with an MM problem exploring the amount of water used when washing dishes in a dishwasher versus by hand.

Mathematical modelling is a core element in the Norwegian mathematics curriculum (Ministry of Education and Research, 2019), and reality is the source of modelling problems. Several versions of the modelling cycle describe the MM process. Borromeo Ferri (2007) emphasises the activation of extra-mathematical knowledge (EMK) when working on a real-world problem in various MM phases, while Maas et al. (2019) value collecting information as necessary in the process. In addition, Salinas-Hernandez, Kock and Pepin (2023) found that students use general resources, curriculum, social and cognitive resources for solving MM tasks. In our study, we want to explore what extra-mathematical knowledge and what resources the PTs use in MM of societal issues in the different stages of the MM process. In the following, we present some preliminary results of our analysis of data from the group work. The analysis consisted of reading the transcribed data, defining different MM stages of the extended modelling cycle (Maas et al., 2019), and then looking at the kind of resources the students used as in Salinas-Hernandez et al. (2023), and identifying the kind of extra-mathematical knowledge PTs used, according to Borromeo Ferri (2007).

Preliminary results

The PTs were only given the real situation, the issue of water consumption, and they had to pose themselves a problem to explore. Here, the EMK is the PTs' own experience; one of the four PTs had recently bought a new dishwasher, and thus, he had information and details about it. The problem was to compare the amount of water used to wash dishes in the dishwasher with the handwash. In this way, they created a simplified model or a mental representation of the situation. Moving on to the next MM stage, towards a real model, they needed to collect information (Maas et al., 2019) in terms of EMK (Borromeo Ferri, 2007). Internet resources, or general resources in terms of Salinas-

Hernandez et al. (2023), were used to collect data such as how much water a dishwashing cycle uses, or how much water per minute it runs with a full tap. Thus, time and amount of water were naturally pointed out as variables in the real model. In addition, using their experience as extra-mathematical knowledge and as a resource, the PTs agreed that the number of dishes was also a variable to be considered and whether the dishwasher ran half-full or full. These resources were used to help the process of mathematizing. When designing the mathematical models of dishwasher vs handwash to compare the amount of water, they used their experience to decide how long it takes to wash some of the different kitchen utensils by hand and the number of dishes in full vs a half dishwasher.

In the validation phase, the PTs used their experience to validate the correctness of the data and expressed awareness when using the results to look back at their own EMK about water consumption. As one of the PTs said when they found that a full tap pours 6 litres of water: "That's terribly wasteful. Does anyone who washes dishes, if they wash dishes by hand, do it [use full tap all time]?". The PTs found a difference of 50 litres more water for washing by hand for a full dishwasher, which was much more than they expected.

Our findings show that the PTs used extra-mathematical knowledge in many of the MM stages, also more than those pointed out by Borromeo Ferri (2007). Further, the main resources were their own experience, as a social resource, cognitive resources such as their previous mathematical knowledge, and the internet search as a general resource (Salinaz-Hernandez et al., 2024). Further research will focus on analysing what kind of EMK and resources the other groups of PTs used when dealing with MM of societal issues, to be able to support PTs in their MM processes better and advance the research in MM of societal issues.

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Bridging mathematics and sustainability: Exploring the role of sustainability in mathematics education

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Keywords: Mathematics education for sustainability, Education for sustainable development (ESD), Questionnaire, Motivation, Sustainability

Sustainable development is a global priority, as emphasized by initiatives such as the United Nations' 17 Sustainable Development Goals (SDGs). Education for sustainable development (ESD) equips individuals with the knowledge, skills and values needed to foster sustainability. However, not all subjects have integrated ESD to the same extent – a disparity reflected in a year 9 student's remark in this study: "What does this have to do with maths?"

Mathematics plays a pivotal role in addressing sustainability challenges, from climate science to social justice (e.g. Skovsmose, 2023). Sustainability provides a practical context for key mathematical competencies, particularly process-related skills (e.g., modelling) and content-related skills (e.g., functional relationships and data analysis) (Reiss et al., 2016). Mathematical modelling tasks bridge abstract concepts with real-world applications, fostering critical thinking, problem-solving, and engagement (Niss & Blum, 2020). Furthermore, such tasks not only align with ESD goals but can also enhance students' motivation and emotional engagement with mathematics (Maaß & Mischo, 2012). There is a growing body of literature on the integration of ESD in mathematics education (overview in Vásquez Ortiz et al., 2023), but this study addresses a critical gap: understanding students' readiness to engage with sustainability-related tasks in mathematics. It explores students' attitudes, motivation, and perceptions of sustainability within mathematics lessons, with a focus on developing and validating a questionnaire as a tool for further research. The research is guided by the following questions:

1. How suitable are the developed items and scales for assessing students' engagement with sustainability in mathematics education?
2. What do the preliminary results from this questionnaire reveal about students' attitudes, experiences, and perceptions of the connection between sustainability and mathematics?

Method

A mixed-methods approach was adopted, centering on the development and validation of a questionnaire to assess students' engagement with sustainability in mathematics. Items were adapted from the PISA (Mang et al., 2018) and tailored to reflect sustainability contexts within mathematics education. The questionnaire includes Likert-scale items to measure attitudes, motivation, and experiences, as well as open-ended questions to provide insights that can be reconstructed using qualitative methods. This studies sample comprised 192 students (grades 5–12) across academic and vocational secondary schools in NRW, Germany. Data collection took place between December 2023 and May 2024, with all responses anonymised. Descriptive statistics identified trends, patterns, and

anomalies, and Cronbach's alpha evaluated scale reliability. Interpretive analysis of free-text responses is ongoing to explore recurring themes and distinctive perspectives.

Results and Discussion

In addressing Research Question 1, it was found that the scales used in the questionnaire exhibited exemplary to moderate reliability (Robinson et al., 1991). Cronbach's alpha values range from .623 to .859. In addressing Research Question 2, preliminary findings highlight the following:

- Limited integration of sustainability in mathematics: While 71% of students reported encountering sustainability topics in school, only 7% agreed that these topics were addressed in their mathematics lessons.
- Correlation between motivation and sustainability perception: Students who expressed greater motivation and interest in mathematics were more likely to recognize sustainability-related aspects in their lessons.

The questionnaire provides a foundation for assessing students' readiness for these tasks. Future work will expand the sample and refine the questionnaire to aid in developing tailored sustainability-focused tasks for diverse classroom contexts.

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TWG 07: Mathematics for Work, Society and Personal Development: Lifelong Learning

An introduction to TWG7: Mathematics for work, society and personal development - lifelong learning

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DEDICATION

This text is dedicated to Professor Kees Hoogland. At CERME10 in Belfast (Ireland) in 2017, he was among the proposers of TWG7 "Adults learning mathematics" and he was its leader at two subsequent CERMEs. He was a very inspiring mathematics educator with a particular eye for students in the "lower" tracks and for the low numeracy among vulnerable people. Among his scientific contributions is the Common European Numeracy Framework (Hoogland et al., 2021). Despite already being very sick, he still engaged energetically in the TWG7 discussions at CERME14 in February 2025 in Bozen-Bolzano (Italy). A few months later, Kees passed away on 14 June 2025 at the age of 64.

This paper introduces the new Thematic Working Group "Mathematics for work, society and personal development: lifelong learning" (TWG7) within CERME. This research domain focuses on mathematical opportunities, challenges and vulnerabilities that people of all ages face in a wide array of contexts. After a conceptualisation of the domain, we summarise our work at the CERME14 conference. Leading themes were (1) adults learning mathematics, which included mathematics in evening schools, in prisons, and in intergenerational mathematics learning, (2) mathematical literacy and numeracy, and (3) workplace mathematics and the vocational education that prepares for this. Synthesizing the discussions during the conference, we see the richness of theoretical and methodological approaches, commonalities across the three themes and a need for developing new vocabulary. Concluding, we evaluate the work of TWG7 and identify directions for future work.

Keywords: Adults learning mathematics, mathematical literacy, numeracy, vocational mathematics education, vulnerability.

Conceptualizing the research domain of TWG7 within CERME14

The Thematic Working Group with number 7 (TWG7) within the 14th Congress of the European Society for Research in Mathematics Education (CERME) covered the research domain *Mathematics for work, society and personal development: lifelong learning*. This was the first time a TWG had

this title. In this paragraph, we will explain this research domain before summarizing and synthesizing the presentations and discussions at the conference.

Although the title of the research domain of TWG7 was new, it built on work of the TWG7 *Adults learning mathematics* at earlier CERMEs. With the ambition to grow as a research community, it was decided to express the research domain of TWG7 in terms of purposes of mathematics rather than defining the research domain by a target group of learners. Obviously, the new domain still included all research on *adults learning mathematics*, as expressed by *mathematics for personal development* and by *life-long learning* in the new title. The appended terms *mathematics for society* expressed that TWG7 now also explicitly included research on mathematical capabilities needed for participating successfully as citizen in society (also known as *mathematical literacy* and *numeracy*). Finally, the terms *mathematics for work* explicated that TWG7 now also embraced research on *vocational mathematics education* and *workplace mathematics*.

The title change of TWG7 reflects a trend in mathematics education research, in which research foci, perspectives and themes gradually evolve (Inglis & Foster, 2018). Yet, putting different themes together may also lead to confusion and incoherence. Therefore, we (the co-leaders of TWG7) felt a need to discuss the research domain *Mathematics for work, society and personal development: lifelong learning*. We asked ourselves: what characterizes the research domain, what are its boundaries, and what are commonalities across themes? For instance, based on Gal et al. (2020), we agreed that the target group of TWG7 could by and large be connected to *precarity* of life situations (e.g., lacking predictability, job security, material or psychological welfare) and *vulnerability* in using and learning mathematics.

As part of our discussions, we looked at recently published research that may not have fitted the "old TWG7", yet would fit the new. An example was Chapman's (2022) case study of a 9-year-old girl in a voluntary, week-long knitting summer camp, in which she suddenly realises that she needs fractions and proportions for her knitting design. This study would fit the domain of TWG7 being about *mathematics for (hobby) work and personal development.* This study taught us that TWG7 can potentially include youngsters, and that mathematical activities can surface informally and unexpectedly within non-mathematical activities (e.g., knitting). Another example was Sundtjønn (2021), who describes mathematics classrooms with 16-year-old students in vocational education (hair dressing, technical production, etc.). This study would fit TWG7 being about *mathematics for work.* It taught us that vocational mathematics education can be (traditional, pure) mathematics education with few connections to the vocational future of students, albeit offered within vocational education institutions. A third example was from Kalman and Solares (2018), who portray how near-illiterate industrial workers cope with low salaries and their purchases and debts. This study would fit TWG7 being about *mathematics for society*. It taught us that near-innumerate people have complex numeracy strategies, that participants in the research need not be *students* or *learners,* but can be *people* without intention to *learn* mathematics, but still can *use, do, fear,* or *avoid* mathematics. Based on these discussions, we described the research domain of TWG7 as follows in the call-for-papers:

> The domain of *Mathematics for work, society and personal development: lifelong learning* studies the breadth and depth of mathematical challenges that people of all ages face in a wide array of

contexts. These contexts can be at home, in workplaces, in vocational education, in adult education (e.g., evening classes and prison education), in leisure activities, when coping with digital technology and media, and so forth. Throughout their lives, people encounter mathematical challenges for a variety of reasons, such as making informed decisions regarding personal finance and health, quality control in workplaces, or debates on global issues. The challenges can lead to vulnerability, frustrations, or avoidance, but also to finding ways of overcoming barriers. TWG7 concerns mathematical literacy, numeracy, critical mathematics education, using mathematics productively and reflectively in different environments, mathematics in and for work, large-scale surveys about mathematics for work and society (e.g., Programme for the International Assessment of Adult Competencies, PIAAC), and teaching and learning mathematics with adult learners (e.g., curriculum, pedagogy, and didactics).

TWG7 at CERME14 in 2025 in Bolzano (Italy) attracted 23 submissions: 17 papers and 6 posters. After a thorough peer-review process, seven papers were rejected. The resulting rejection rate of 30% was somewhat higher than that of other TWGs. TWG7 was attended by 19 participants coming from Germany (1), Italy (5), Netherlands (3), Norway (3), Sweden (3), Spain (1), and UK (3).

Below, we summarise the research presented and discussed within TWG7 at the CERME14 conference. The leading themes were: (1) *adults learning mathematics*, which includes mathematics in evening schools, in intergenerational family learning, and in prisons, (2) *mathematical literacy* and *numeracy*, and (3) *vocational mathematics education* and *workplace mathematics*.

Leading theme: Adults learning mathematics

Within adult mathematics education, many learners are vulnerable, have experienced discouragement, anxiety, failure, or even dropping out (Gal et al., 2020). Classrooms in adult mathematics education are characterised by age-, language-, culture-, and pre-knowledge-diversity. Reasons to enrol are the lack of learning opportunities during their youth, in particular women who have been historically excluded from educational pursuits due to gender-based caretaking responsibilities. Adults register for mathematics classes, among others, because of the wish to assist children with homework, or to acquire a diploma that opens access to career changes. However, there generally is a substantial turnover in adult classes, since disturbances caused by financial, family- or work-related issues may disrupt studies. However, adults' success in learning mathematics positively impacts not just their own lives, but also their family's well-being.

To this research, *Stacey* added a survey on adults' *anxiety* and *self-efficacy* in relation to their success in obtaining a formal mathematics diploma, the mathematics General Certificate in Secondary Education (GCSE). The results showed that males were just as anxious as females. Older learners were less anxious, had greater self-efficacy, and were more likely to pass than younger adults. Also, those whose first language was not English were more likely to pass than native English speakers.

The study by *Syyeda et al.* departed from the fact that many citizens, in particular in disadvantaged communities, are unaware of free mathematics courses offered in their vicinity. To reach out to immigrant women, they conducted an Action Research, in which they organized workshops with fun mathematical activities for children and their mother (or caretaker). At the end of the workshop,

information was provided to participants about courses at the local adult education college. Interviews confirmed a want for information, an eagerness to enrol, and the positive impact of *intergenerational learning* (or *family learning*) on empowering adults.

The research theme *adults learning mathematics* also includes research on mathematics education in prisons. Within this regulated environment, adult learners are vulnerable, miss their families and their agency, experience depression, and so forth (Ahl, 2020). Just like "regular" adult education, groups show age-, language-, culture-, and pre-knowledge-diversity, but the learning processes are also affected by the variety in lengths of stay (awaiting trial, having short- or long-term sentences), the lack of internet, and the presence of guards. Also, mathematical tasks with real life contexts (e.g., buying a mobile phone) connect to life in freedom, and hence can be frustrating and hurtful. Within TWG7, three studies were presented about the teaching and learning of mathematics in Italian prisons.

Both *Mennuni et al.* and *Regola et al.* presented results from the first phases of an Educational Design Research for mathematics education in Italian prisons, based on various data collections among mathematics teachers working there. *Mennuni et al.* analysed these regarding the intertwinement of learners' motivation, the class-group composition, and possibilities for personalization of the teaching. For instance, the high turnover of learners asked for personalization, and the small classes enabled this. Learners' background variety both hindered and enriched learning processes, and teachers witnessed that pure mathematical problems could make learners temporarily forget their worries or negativity. *Regola et al.* formulated design requirements and proposals, which included, among others, that activities should be exclusively based on pen-and-paper and *imagination*, due to a lack of internet, rulers, compasses, calculators, and so forth. Also, activities needed to cater for the fluctuating group of learners, have a low entry level to be accessible to many, yet also offer challenges to more advanced learners.

Galli et al. conducted an Action Research, in which a mathematics workshop was carried out, evaluated, refined and carried out again. Participants were adult men (18–30-year-old) awaiting trial. They analysed the classroom setting, the implementation of the planning, learners' attitudes and difficulties, and technical-logistical issues. Time management was complex due to learners coming late or walking out for a smoke. The workshops consisted of three parts: an engaging start, a core problem, and a take-away problem (to chew on after the workshop), which was very well-received. Participants engaged actively, testified of feelings of empowerment, and said that the mathematics offered an "escape from the vicious circle of everyday thoughts".

In the TWG7-discussions, we observed that classrooms in adult mathematics education differ in many ways from those in compulsory education. For instance, adults have a long and complex history of mathematics learning and their participation in classes is not obligatory, but driven by motives that can be personal or social. Nevertheless, adult mathematics classes are marked by substantial turnover, which brings challenges to mathematics teachers. We observed that the teacher competences needed in adult mathematics education are largely unaddressed in teacher education. Supplemental modules in teacher education may include the design of mathematics lessons tailored to adult learners' backgrounds, needs and interests, and strategies to flexibly adapt lessons when participation fluctuates. Regarding mathematics education in prisons, teachers need competences to design lessons

that do not require "forbidden" tools, that avoid reminiscing of life in freedom, and that provide "food for thought" for the hours after lessons. We also observed the methodological dominance of intervention studies such as Educational Design Research and Action Research, which testified of a need for research-based teaching approaches tailored to the learning of mathematics by adults.

Leading theme: Mathematical literacy and numeracy

The research theme *mathematical literacy and numeracy* focuses on the mathematical capabilities needed by citizens for participating successfully in society, which some researchers indicate as *mathematical literacy* and others as *numeracy*. Although the two terms differ in their definition, we use these interchangeably according to how presenting researchers used them. According to Gal et al. (2020), many adults with low numeracy are *vulnerable* in our data-driven society, in which citizens must make important decisions regarding health, finances, schooling, and so forth, based on quantitative information. Generally, numeracy levels impact adults' employability, well-being, health, and self-reliance, and also impacts macro-level factors in society.

Since social workers need to use quantitative information to assist their clients, *Mulder et al.* conducted a study on numeracy in social work education. They reviewed curricula in social work programs and administered a survey among students and found that financial and digital literacy were included in social work education, but not numeracy, yet all stakeholders recognized its importance.

Bor-de Vries et al. introduced a methodological study of 17 different research instruments to measure numeracy. The analysis turned out to be more complex than expected and results are still pending.

Hoogland and Díez-Palomar presented preliminary results from the latest international comparative study on adults' numeracy, the Programme for the International Assessment of Adult Competencies (PIAAC), which includes 31 countries and 160 000 adults. The assessment was carried out through tablet-based, door-to-door interviews. The overall results indicate, by and large, increasing skill inequities between and within countries, as well as between genders.

Díez-Palomar and Hoogland studied how *people in the street* understand the term numeracy. They conducted a survey among Spanish adults using an open-item questionnaire. Most participants perceived numeracy to be "about numbers", "performing basic calculations" or "mathematics for beginners". The difference between mathematics and numeracy was explained as the first being advanced and the latter being basic. Apparently, mathematics was associated with "school-like" things, whereas numeracy was not associated to using mathematics and mathematics reasoning for solving problems in everyday life. This research result means that many citizens, and likely also politicians and educational leaders, will be similarly ignorant that the meaning of numeracy is connected to solving real-life problems in order to participate successfully in society.

Bolstad studied theoretically three well-known conceptual frameworks for mathematical literacy by analysing underlying purposes, contexts, outcomes, and practices of mathematical literacy in these. Her results reveal that none of the frameworks captured the distinction between the social practices in which mathematical literacy is needed, and the social practice in which it is learnt. Hence, frameworks for mathematical literacy should consider both formal and informal learning, and also

the practices of teachers and their norms in providing learners with agency, time, and also with real-life challenges that learners care about and want to solve.

In the TWG7-discussions, we observed how the research theme *mathematical literacy and numeracy* overlapped with the theme *adults learning mathematics*, if only because the competences needed for *mathematical literacy* and *numeracy* are expected from adults. Nevertheless, *mathematical literacy and numeracy* is not confined to adults, but needs also fostering with younger learners. Another clear overlap of the themes *mathematical literacy and numeracy* and *adults learning mathematics* is their call for research on the typical teacher competences needed for teaching mathematical literacy, numeracy, adults learning mathematics, and mathematics in prisons.

Leading theme: Vocational mathematics education and workplace mathematics

In many countries, students at a certain age are separated into theoretical and practical streams based on "academic results", students' preferences, and a complex layer of socio-economic, ethnic and other factors (Dupriez et al., 2008). This separation is at the end of grade 4 (e.g., Germany), around age 13 (e.g., Netherlands, England), or around age 16 (e.g., Nordic countries, Italy, Spain). The mathematics curricula in practical streams can be similar or slimmer than in theoretical streams (Italy, Spain, England), they can focus more on future vocations (e.g., Germany), on mathematical literacy (South Africa), or can be a hybrid (Netherlands, Nordic countries). These environments for mathematics education are indicated as *vocational mathematics education*, whilst this term also includes *mathematics for the workplace* training to employees offered by industries (FitzSimons, 2014).

Within this research theme, *Sundtjønn* studied 16-year-old students in education for Technical Production, and their reactions to a paper-based mathematics task on calculating the volume of a moped cylinder. Using Wenger's (1998) theory of communities of practice, she classified the students as *legitimate peripheral participants* in their future vocation. Also, she conceptualized a new verb, *to authenticate,* for the process of assessing whether or not aspects in the task existed in real-life. While working on the task, students used their out-of-school knowledge to assert the given information in the task and the question as authentic, but not the formula-based solution method. Mathematics teachers in vocational education are recommended to recognize and use students' vocational expertise for task solutions, since this raises their motivation to engage with mathematics.

Knutson used tasks on construction work and geometry to interview vocational teachers and mathematics teachers at vocational schools and compare their perception of *educationally significant vocational mathematics knowing*. She derived six categories, which revealed whether and how a teacher foregrounded mathematics or the vocation: procedural memorization, context-bound grasp, conceptual comprehension, vocational-mathematical connection, vocational-mathematical fusion, and mathematics as component in vocational "bildung". For instance, the category "vocational-mathematical fusion" entailed that mathematics was perceived as essential, but insufficient for solving vocational problems, which by their nature cannot be solved without vocational knowledge. The study illuminates complexities in integrating mathematical and vocational knowledge, but also the wide variety in perceptions on mathematics held by teachers in vocational education.

Dalby studied the communities of mathematics teachers at 16 vocational colleges in a 2-year long change project, and like *Sundtjønn*, she used Wenger's (1999) theory of communities of practice. Traditionally, the mathematics teachers operated in isolation within the institutions, which led to divides between mathematics teaching and the mathematical competencies needed in vocational programmes. Through meetings, visiting each other's classrooms and recognizing commonalities, interdepartmental communities emerged that shared responsibilities for developing students' mathematics skills through better connected approaches to teaching mathematics. Simultaneously, the professional identities of mathematics teachers changed, with more emphasis on the educational context and less on the mathematics.

Muhrman and Frejd carried out a survey with learners (n=234) at two schools to statistically compare two teaching conditions of mathematics learning in vocational education and the effects on learner motivation. One condition was textbook-based mathematics learning in traditional settings, and the other was "subject-integrated teaching" with co-teaching by a mathematics teacher and a vocational teacher within a workplace environment. The results showed that students in the latter condition had a significantly higher motivation to engage in learning, more sense of the usefulness of mathematics, and felt better prepared for their professional future.

Another study on co-teaching in vocational education is from *Fiorentino and Montone*, who studied students' experiences in a classroom-based intervention, in which a mathematics teacher and a teacher of Fashion and Design collaborated and learnt from each other's worlds. Using the Theory of Semiotic Mediation (Bartolini Bussi &Mariotti, 2008), the researchers introduced the term *boundary artefact* for vocational objects that can serve the learning of mathematics, such as sewing patterns used in fashion design. They observed very positive reactions from students, who saw usefulness in mathematics and one exclaimed that this helped her see connections between fractions and ratios.

In the TWG7-discussions, we noted the improvements in students' motivation through the creation of "mathematical relatedness" in vocational mathematics lessons. Many students in vocational education have experienced disappointments when learning mathematics, and the TWG7 research showed that this potentially can be curbed when they come to see that mathematics can be useful in their future workplaces and also when they can use their vocational competences in mathematics tasks. We also noted that Wenger's (1999) theory of communities of practice was a two-edged sword in this research, assisting in the analysis of students as already having a vocational identity (in *Sundtjønn's* study), and in the analysis of the isolation of mathematics teachers and subsequent integration into the school community of vocational teachers (in *Dalby's* study).

Cutting across the leading themes and looking forward

In looking back at TWG7-papers, we see that several papers cut across themes. The study by *Mulder et al.* was about *numeracy* within the education of social workers. This study was about learning to use quantitative information in a vocation and therefore, also fitted the themes *mathematics for work* and *vocational mathematics education*. Similarly, the study by *Syyeda et al.* fitted *adults learning mathematics*, being about mathematics learning by female adults with an immigrant background. This study also fitted the theme *mathematical literacy and numeracy,* since it aimed at empowering participants with life skills. This thematic overlap of TWG7-papers revealed cross-overs between

research themes. It showed that the research domain *Mathematics for work, society, and personal development – lifelong learning* was not thematically fragmented, but interconnected.

Another thematic interconnection, in this case between *mathematical literacy* and *workplace mathematics,* came from *Frejd and Vos*. They theorized how citizens and workers interact with advanced mathematics (e.g., calculus), even if they never studied it. Their new concept, the *mathematical co-productivity chain*, describes how mathematical work by professional mathematicians is simplified, transformed and visualized into schedules and diagrams for use by citizens and workers. When using these, the behaviour of citizens and workers is monitored and sent back to the mathematicians for further improvement of the underlying mathematics. Hence, the mathematical co-productivity goes back-and-forth. Their society-based analysis revealed commonalities in workplace mathematics and mathematical literacy.

Frejd and Vos' new concept was just one example illustrating a need within TWG7 for new research vocabulary. Likewise, *Sundtjønn* conceptualized the verb *to authenticate*, *Knutson* defined what teachers perceived as *educationally significant vocational mathematics knowing*, and *Fiorentino and Montone* coined the new term *Boundary Artefact*. This scientific need for new concepts was appended by a need for a better popular understanding of two terms important within TWG7, namely *mathematical literacy* and *numeracy* (see *Díez-Palomar & Hoogland*), which additionally lack satisfactory translations in some languages (e.g., Spanish, German, Scandinavian languages). This urge to develop suitable concepts and vocabulary point at a future scientific agenda for TWG7.

Regarding theoretical frameworks, we noted a rich variety, but also a call by *Bolstad* for practice-based frameworks that differentiate between the "learning environment" and the "using environment". We noted that the first does not necessarily needs to be an educational institution, and eventually, the two can be the same environment. Additionally, the work in TWG7 had a surprising methodological variety, with an international large-scale survey (*Hoogland and Díez-Palomar*), various surveys (*Díez-Palomar and Hoogland, Knutson, Muhrman and Frejd*, *Stacey*), a long-term organisation study (*Dalby*), a curriculum study (*Mulder et al.*), a methodological study (*Bor-de Vries et al.*), several classroom-based intervention studies (*Fiorentino and Montone, Galli et al.*, *Mennuni et al.*, *Regola et al.*, *Sundtjønn, Syyeda et al.*), and two theoretical studies (*Bolstad, Frejd and Vos*). This variety in grain size in research methods, together with the earlier mentioned urge to develop concepts, vocabulary and frameworks, indicate that the research in TWG7 has ambitions and growth potential. Moreover, across all themes there was a call for research and development of specialised teacher education modules that cater for the teaching to *lifelong* learners and users of *mathematics for work, society, and personal development*.

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Mathematical literacy – What is missing?

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In this theoretical paper, I discuss three different conceptualisations of mathematical literacy (ML): the five perspectives of ML proposed by Jablonka, the numeracy model proposed by Goos et al., and the PISA Assessment and analytical framework. After a short summary of each, I compare different aspects included in the three conceptualisations and discuss their commonalities and differences. The three conceptualisations can be related to different phases of education and teaching. However, none of the conceptualisations concern teachers' classroom practice and classroom norms. I argue that a framework for fostering learners' ML could involve aspects related to teacher practice and the social practice of the learners.

Keywords: Mathematical literacy, numeracy, numeracy model, PISA framework.

Introduction

We live in an increasingly data-driven society, in which mathematics plays an important role. Numbers and quantities are everywhere: in medical reports, describing political trends, providing financial advice, and advocating social policy. Also, governmental services present information to the public using mathematical ideas and devices. One recent example is the outbreak of the COVID-19 pandemic in early 2020. Health authorities provided information to the public about the spread of the pandemic in mathematical terms, such as various types of graphs, tables, and concepts, and mathematical modelling was used as a basis for prediction of the virus' transmission speed. Consequently, competences in mathematical communication, mathematical representation, mathematical symbols and formalism, mathematical modelling, and mathematical reasoning were needed to interpret the information provided (Aguilar & Castaneda, 2021). The COVID-19 pandemic is only one example of events contributing to the growing understanding that underdeveloped mathematical competences limit an individual's prospects in terms of health conditions, career opportunities, social well-being, financial security, and political participation. Considerations have been made as to what constitutes the mathematical demands, skills and knowledges needed to work within modern society. The term *mathematical literacy (ML)* denotes the competence to deal with the mathematically related aspects of engaging in society with independence and agency.

It is an educational goal to educate citizens so they can use mathematics to pose and solve problems and, hence, act in society and to further develop their understanding of the world (Hansen & Hana, 2015). Several policy documents and research papers provide definitions, models, and frameworks comprising aspects of ML. The different models and frameworks vary in their level of detail and comprehensiveness and in terms of which group of learners they concern. For instance, the Common European Numeracy Framework (Hoogland et al., 2020) and the PIAAC framework (OECD, 2021) concern adult learners. In this theoretical paper, I discuss three of the most widely cited contributions concerning learners in comprehensive school: the five perspectives on ML proposed by Jablonka (2003), the numeracy model proposed by Goos and colleagues (Goos et al., 2014), and the PISA Assessment and analytical framework (OECD, 2023). I use *conceptualisations* as an umbrella term when referring to Jablonka's (2003) five perspectives of ML, Goos et al.'s (2014) numeracy model,

and the PISA framework (OECD, 2023). I start by briefly summarising the three conceptualisations. Further, I compare them to reveal their commonalities and differences. Finally, I discuss whether there is something missing in the three conceptualisations of ML and what this might be.

Jablonka's (2003) five perspectives on ML

Jablonka (2003) uses the term mathematically literate to refer to a mathematically educated and well-informed individual. She discusses five different, and sometimes conflicting, perspectives on ML which are related to the different ways in which the relationship between mathematics, culture, and curriculum is conceived:

- ML for developing human capital: the ability to analyse, reason, and communicate ideas and results by posing and solving mathematical problems related to individual and social problems
- ML for cultural identity: the use of informal mathematical knowledge involved in cultural practices and artifacts. This also includes practices used in the workplace. Hence, it includes competences that are not developed in school settings
- ML for social change: the use of mathematics in pursuing social and political change.
- ML for environmental awareness: the use of mathematics to analyse, reason, and communicate ideas and results related to environmental problems
- ML for evaluating mathematics: the ability to evaluate the use of mathematical applications and to be critical in terms of its objectivity and its limitations

According to Jablonka (2003), the perspectives vary with the values and rationales of the stakeholders who promote them. She argues that it is impossible to conceptualise ML without promoting a particular social practice.

Goos et al.'s (2014) numeracy model

According to Geiger et al. (2015, p. 611), ML "is the capacity to make effective use of mathematics in contexts related to personal life, the workplace, and in exercising civil responsibilities." Goos et al. (2014) developed the numeracy model as a tool for teachers to plan for, and reflect upon, effective teaching and learning practice with a focus on ML. This conceptualisation of ML incorporates four dimensions that are embedded in a critical orientation to using mathematics (Geiger et al., 2015; Goos et al., 2014):

- contexts: the ability to use mathematics in various contexts in everyday life, work life, and citizenship
- mathematical knowledge: mathematical concepts, skills, problem solving strategies, and estimation capacities
- tools: the use of material, representational and digital tools to mediate and shape thinking
- dispositions: confidence and willingness to use mathematics in engaging with real-life tasks, and the flexible and adaptive use of mathematical knowledge
- critical orientation: using mathematical information to make decisions and judgements, support arguments, and challenge arguments or positions

Goos et al.'s (2014) numeracy model has been used in several research and development projects.

OECD's (2023) PISA 2022 Assessment and analytical framework

In the PISA 2022 Assessment and analytical framework is defined as

> an individual's capacity to reason mathematically and to formulate, employ, and interpret mathematics to solve problems in a variety of real-world contexts. It includes concepts, procedures, facts and tools to describe, explain and predict phenomena. It assists individuals to know the role that mathematics plays in the world and to make the well-founded judgements and decisions needed by constructive, engaged and reflective 21st Century citizens. (OECD, 2023, p. 22).

In this conceptualisation, ML takes place in the context of a challenge or problem that arises in the real world and comprises four categories:

- contexts: personal, occupational, societal, scientific
- 21st century skills: critical thinking, creativity, research and inquiry, self-direction, initiative and persistence, information use, systems thinking, communication, reflection
- mathematical reasoning and problem-solving processes: mathematical reasoning, formulating, employing, interpreting and evaluating
- mathematical content knowledge: quantity, uncertainty and data, change and relationships, space and shape

Comparing the three conceptualisations of ML

In this section, I compare the three conceptualisations of ML. I start by looking at the purposes for which the conceptualisations were developed. Next, I compare the three conceptualisations, focusing on whether, and in what way, the following aspects are included: contexts, mathematical content knowledge and skills, tool use, personal attributes, outcomes (what ML enables one to do).

Jablonka's five perspectives of ML concern societal and political issues beyond formal education. They serve as discussion points to display how it is not possible to conceptualise ML without promoting a particular social practice. Jablonka is not preoccupied with the types of subject knowledge and competences involved in ML or the teaching of these. Rather, her five perspectives concern various ways ML can be used in life outside the classroom. Hence, the purpose of Jablonka's five perspectives of ML is critique, of both society and mathematics, which is something citizens should be able to do. By contrast, Goos et al.'s numeracy model was developed as a response to educational policies in Australia, where ML is considered a general competence to be developed across school subjects. In this context, the development of ML was considered the mathematics teachers' responsibility. The model serves as a tool to support teachers, particularly teachers of other subjects than mathematics, to meet the curricular requirements to teach ML across the curriculum. It aims to capture different aspects involved in ML in a structured way so that teachers can easily use it when designing or evaluating ML tasks. In this way, the purpose of Goos et al.'s numeracy model is education and school, and it is explicitly directed toward teachers' practical work. The third conceptualisation, the PISA framework, aims to assess the extent to which 15-year-old pupils have acquired the knowledge and skills that are essential for full participation in modern societies. Therefore, it is developed to guide the structure and content of the ML assessment. PISA impacts curricula and policy in the participating countries (OECD, 2023), which has also been an explicit aim (OECD, 2004). The purpose of the PISA ML framework is, thus, to assess pupils' outcome of

schooling and to impact school policy. In this way, the three conceptualisations are developed for quite different purposes.

One common aspect of ML in the three conceptualisations, is that the capacity to use mathematics in real-life contexts is central. However, the way contexts are described in the three conceptualisations vary. Goos et al.'s numeracy model includes contexts from personal life, work life and citizenship. These are broad categories describing various areas of life from which mathematical situations may originate. Although the wording is slightly different, Goos et al.'s numeracy model contexts parallel the first three contexts in the PISA framework: personal, occupational, and social. Jablonka's five perspectives also concern contexts from personal, work life and citizenship. Jablonka relates using mathematics in personal, work related, or citizenship contexts to ML for developing human capital and conceptualises ML as "the ability to analyse, reason and communicate ideas and results by posing and solving mathematical problems" (p. 80). In this conceptualisation, the individual is at the centre because ML concerns the individual's ability to use mathematics to change their life prospects. This conceptualisation parallels the conceptualisation of using mathematics in context described by Goos et al. (2014) and the PISA framework (OECD, 2023). The PISA framework also includes a fourth context, scientific, which concerns the application of mathematics in the natural world, science, technology, and pure mathematics. This scientific context has no direct parallel in Goos et al.'s numeracy model, but it is related to Jablonka's perspective of ML for developing environmental awareness. In addition to developing environmental awareness, Jablonka also mentions contexts related to cultural practices, evaluating mathematics, and pursuing social change, which concern using mathematics to understand, challenge and change structures and practices in society. This conceptualisation of using mathematics in contexts does not only concern change for the individual but also a societal change.

Another central issue in ML is mathematical knowledge. Goos et al.'s numeracy model and the PISA framework both include lists of several types of mathematical knowledge needed to solve problems in real-world situations. Goos et al. list mathematical content and processes as one aspect. In the PISA framework, mathematical content knowledge and mathematical reasoning and problem-solving processes are separated. The five perspectives by Jablonka do not include such lists. In fact, Jablonka criticises the idea of universally applicable lists of knowledge and skills. She argues that "it is questionable whether mathematical skills can be separated from the social dimension of action and from the purposes and goals of the activity in which they are embedded" (p. 79) and that the description of transferable methodological or process skills "ignores the interests and values involved in posing and solving particular problems by means of mathematics" (p. 79). When considering societal and political issues and solving problems in everyday life, as described by Jablonka, the type of mathematical knowledge and skills used or needed is not a central issue. However, in school and educational policy settings, the curriculum guides the teaching content and the assessment of learning outcomes. In these settings, lists of knowledge and skills become a way to structure curriculum content and to adhere to curriculum goals. Hence, the educational perspective of Goos et al.'s numeracy model and the PISA framework may explain the inclusion of knowledge and skills lists in the two ML conceptualisations. Appreciating informal strategies and mathematical ideas involved in cultural practices is a form of mathematical knowledge that is usually included in curriculum documents, but not explicitly covered by Goos et al.'s numeracy model nor the PISA framework.

Informal mathematical knowledge relates to ML for cultural identity in Jablonka's perspectives and contributes to a social practice perspective of ML rather than an educational perspective, which is the case with Goos et al.'s numeracy model and the PISA framework.

Tool use can contribute to shape and mediate thinking in all sorts of contexts, within and outside school. Goos et al.'s numeracy model lists different types of tools: physical, representational and digital. The PISA ML framework also includes tool use but embraces the 21st century skill of computational thinking pertaining to abilities to reflect on the evolving role of computers and tools in day-to-day life, pattern recognition and decomposition, designing and using abstraction, and determining whether and which tools could be used. Jablonka does not mention tools as an aspect of ML. As with lists of knowledge and skills, I relate lists of tools to an educational perspective of ML.

A more personal attribute involved in being mathematically literate concerns valuing mathematics and holding the view that mathematics is useful. As many learners experience lack of motivation and meaning, and sometimes also anxiety, related to mathematics as a subject, curriculum documents emphasise the importance of developing positive attitudes toward mathematics (Goos et al., 2014). The view is that since ML involves using mathematics to solve problems in everyday life, a mathematically literate person is someone who considers mathematics important and useful and is willing to engage in mathematical problem solving. In Goos et al.'s numeracy model, this aspect is called dispositions. The 21st century skills in the PISA framework include creativity, self-direction, initiative, and persistence, which can be considered personal attributes involved in ML. The consideration of personal dispositions can be related to the educational perspective for which the two conceptualisations were developed. In Jablonka's conceptualisation, dispositions toward mathematics are not explicitly mentioned as an aspect of ML. Jablonka comments that although practices in everyday life and work involve mathematics, the mathematics involved is not an essential part of these practices. Therefore, positive dispositions towards mathematics are not a prerequisite for using mathematics to solve real-life problems. Rather, interest in the problem itself is important.

Being critical is an essential aspect in all three conceptualisations. In Jablonka's perspectives, ML for pursuing social change involves mathematical argumentation when contributing to debates about social equity issues. In a similar way, ML for creating environmental awareness involves using mathematics to understand and solve environmental problems, and ML for evaluating mathematical applications involves realising that the use of mathematics is not objective and has limitations. The critical aspects in Goos et al.'s numeracy model and the PISA framework relate to several of Jablonka's perspectives of ML. Goos et al.'s numeracy model is grounded in a critical orientation, emphasising that a mathematically literate person must evaluate the results obtained and recognise how mathematics is used for different purposes. In the PISA framework, the critical aspect involves using mathematics to explain and make judgements. In addition, the PISA framework includes critical thinking as one of the 21st century skills. Here, critical thinking involves making justified arguments and defending oneself from lies.

Although they were developed for different purposes and over a span of 20 years, the three conceptualisations have many aspects in common. However, there is one major difference between them related to what ML enables one to do. Goos et al.'s numeracy model and the PISA framework are focused on knowledge and understanding, reasoning mathematically, and using mathematics to solve problems and make decisions. A clear focus in Jablonka's perspectives is the importance of not

only using and understanding mathematics, but also to use mathematical knowledge and understanding to contribute to change, either in one's personal life, work life, or in society. Although one could argue that competence in solving problems and making decisions leads to change because deciding about something is usually followed by some sort of action related to that decision, acting on these competences to impact and change personal, social, work, and environmental situations for the better is not explicitly emphasised in the PISA framework and Goos et al.'s numeracy model. Also, potential conflicts that may arise when solving real-world problems are not discussed.

Table 1: Comparison of ML conceptualisations

	Five perspectives of ML (Jablonka, 2003)	The numeracy model (Goos et al., 2014)	PISA Assessment and analytical framework (OECD, 2022)
Purpose	Outcome (using mathematics in critiquing society and mathematics itself)	Education (supporting teacher practice e.g., in terms of task design)	Education (assessment, aiming to impact policy)
Contexts	Personal, work, citizenship, social, culture, environment, mathematics	Personal, work, citizenship	Personal, work, social, scientific
Mathematical knowledge and skills	Mathematical argumentation and critique	Concepts, skills, problem solving, estimation	Concepts, procedures, facts, reasoning, problem-solving, quantity, uncertainty and data, change and relationships, space and shape
Tools	Not specified	Representational, physical, digital	Not specified
Personal attributes	Not specified	Confidence, flexibility, initiative, risk	Creativity, self-direction, initiative, and persistence
Outcomes	Social and mathematical critique (understand, use, make judgements, argue)	Critical orientation (understand, use, make judgements, argue)	Critical thinking (explain, argue)
Social practice	Critiquing and changing society	Teaching	Testing

A summary of the comparison of the of the three ML conceptualisations is shown in Table 1. The summary shows that the focus of using mathematics in contexts and the grounding in the critical are the two issues in which three conceptualisations mostly align. However, while Goos et al.'s numeracy model and the PISA framework take an educational perspective mostly focusing on mathematical practices involved in ML, Jablonka focuses on social practices. According to Jablonka, engaging in solving real-life problems by means of mathematics is always situated in a social practice. This social practice perspective is not explicit in Goos et al.'s numeracy model nor the PISA framework.

Discussion and conclusions

Educational policy documents have been criticised for only providing descriptions of what ML is and what mathematically literate persons can do without any further support for teaching (Bolstad, 2020). The three conceptualisations of ML compared in this paper can be related to different parts of education. The PISA framework is comprehensive, as it aims to assess ML in a large international test. In this sense, the PISA framework can be compared to an official curriculum, with curricular aims and objectives and consequential assessments, but without guidelines for teaching. Goos et al.'s numeracy model is simpler, aiming to support teachers in conceptualising ML and planning ML teaching and task design. Therefore, Goos et al.'s numeracy model represents a teacher-intended curriculum. Jablonka's perspectives discuss what it means to be mathematically literate in different social practices and describe various ways in which ML can lead to action. These perspectives do not concern teaching specifically, but can be useful for considering the social, political, and cultural values promoted in a task or situation. Therefore, I relate Jablonka's perspectives to intended learner outcomes and how people use ML outside the classroom or learning environment. In this way, the PISA framework and Goos et al.'s numeracy model concern education and learning while Jablonka's perspectives concern the goals of education and the outcomes of learning.

The three ML conceptualisations include mathematical activities, personal attributes, and learner outcomes. Although they may support teachers to conceptualise ML and plan ML learning activities, I claim that something is still missing from discussions about conceptualisations of ML. None of the conceptualisations frame *how* individuals can *develop* ML, and how teachers can *foster* this development in the classroom. The term fostering is often used to denote teaching methods alternative to instructional teaching. Fostering ML requires more than an understanding of the concept and a set of task features. Aspects related to teacher actions, classroom norms, and the learners' social practises should also be considered. This claim is supported by Frejd et al. (2025), who showed that a combination of factors is needed to foster ML. The factors they identified, in addition to task features, were related to teacher actions and classroom norms that fostered pupil agency. Fostering ML requires classrooms governed by democratic norms providing learners with choices, time, challenge, and problems from real life that they care about and want to solve (Frejd et al., 2025). Frejd et al., thus, argue that it is not the tasks in themselves that are important. This means that even though a task is designed according to the aspects in Jablonka's five perspectives, Goos et al.'s numeracy model, or the PISA framework, the task implementation in the classroom may still adopt traditional teacher-centred practices involving traditional classroom norms and expectations focusing on the learning of mathematics. Such traditional practices and norms do not reflect the way mathematics is used by citizens to pose and solve problems which enables them to act and to further their understanding of the world.

Hence, none of the conceptualisations consider the difference between the social practice of the learning environment and the social practice where the ML is needed. A learning environment promoting the social practice of education in terms of traditional teaching and testing will not foster the use of ML in the actual social practice where ML is needed because the social practice of teaching and testing ML is different from, for example, using mathematics in societal critique. Simply setting a task in a real-world context does not educate citizens who can use mathematics to solve problems and engage in society. Fostering ML requires considerations of the learning environment and of the

social practices of the learners. Such considerations are important because different social practices have different norms, and these norms affect how people develop and use ML. A framework supporting the fostering of ML should, therefore, also include teacher practices. I argue for the need to further conceptualise ML in order that the social practice perspective in terms of teacher practices and learner practices is included. Further discussions about what such a framework could look like is encouraged. In this paper, my focus is on formal education. However, fostering ML happens in both formal and informal learning situations. A framework for fostering ML could also consider vocational and adult education, in addition to learning beyond formal education.

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The shaping of mathematics teacher identities in vocational education through a whole organisational approach

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Most post-16 vocational education in England is provided by large Further Education colleges in which specialist teachers of formal mathematics courses are structurally separated from vocational staff. This leads to divisions between formal mathematics and the mathematical competencies embedded into vocational programmes. In this study, by engaging with an organisational change programme to develop a Whole College Approach (WCA) to mathematics, colleges developed new interdepartmental communities of practice with shared responsibility for the development of students' mathematics skills. Participation in these communities empowered staff to address organisational issues collaboratively and develop a more connected approach to mathematics but indications of accompanying changes in professional identities for mathematics teachers, with more emphasis on the educational context, also have implications for those working in vocational education.

Keywords: Mathematics teachers, vocational education, professional identity, communities of practice.

Introduction

In many Further Education (FE) colleges in England, specialist mathematics teachers teach formal courses leading to national examinations whilst vocational teachers are responsible for the mathematics that is embedded into their programmes. Mathematics teachers usually share workspaces with teachers of the same subject and become part of a team in which the primary focus is mathematics teaching. These teams form small communities of practice, held together by shared beliefs that shape the identities of the members (Wenger, 1999). Such communities tend to produce strong subject-focused mathematics teacher identities and can provide valuable opportunities for the professional development of teachers (Dalby & Noyes, 2022) but there are risks of stagnation or inappropriate models of practice when these teacher communities work in isolation from the broader vocational context in which they are situated.

In English post-16 education, there is a longstanding historical division between vocational education and academic subjects (Pring et al., 2009) and this separation is still evidenced within FE colleges in structures, operational processes and pedagogies (Dalby & Noyes, 2016). Although communities of mathematics teachers in these organizations are often tightly focused on subject-specific pedagogy, which is important to classroom practice, there are few opportunities to share across boundaries and develop a more holistic view of students' mathematics learning experiences.

The Whole College Approach (WCA) project sought to design and research an organisational change programme to support the development of a more connected and collaborative approach to mathematics provision, in which vocational and mathematics staff work together to address strategic, operational and pedagogical issues. The aim of this paper is to explore the question: *How does participation in collaborative interdepartmental communities of practice affect the professional*

identities of mathematics teachers? Although this was not the main purpose of the research project, the data available provide interesting evidence for discussion. Further details of the mechanisms and resources used in the WCA project can be found elsewhere[1] and a more complete report of the findings will be published separately.

Context

Most vocational education in England is provided by Further Education colleges. These are typically large and complex organizations that offer a wide range of study programmes, mostly vocational but some academic, to students aged 16 years and over. Students develop mathematics skills related to their vocational pathway in a way that is naturally embedded into their vocational learning but if they did not achieve a specified minimum standard in the General Certificate of Secondary Education (GCSE) mathematics examination taken at age 16 years in school, government policy requires them to attend additional mathematics classes and retake the GCSE mathematics examination. These classes are usually taught by mathematics specialists whilst the mathematics embedded into vocational learning is the responsibility of vocational staff.

Although some FE colleges use a more dispersed staffing structure, many have a centralised team of mathematics teachers who deliver formal courses such as GCSE mathematics to students from across the organisation. Students who are required to retake the GCSE examination would typically attend one or two mathematics sessions a week with these specialist mathematics teachers, often in rooms outside their vocational area, whilst the rest of their study programme takes place in vocational classrooms or workshops. These arrangements tend to reinforce a common perception amongst students that GCSE mathematics is an addition to their study programme with little connection to the mathematics skills they are developing in vocational sessions (Noyes & Dalby, 2020). The structural separation of mathematics teachers from vocational staff is a further barrier that means issues such as students failing to attend mathematics lessons, or exhibiting poor behaviour are difficult to address.

Theoretical perspective

There are two main theoretical areas of relevance to this paper. The first is the theoretical grounding of the WCA programme and the second concerns the interrelated concepts of professional identity and membership of a community of practice.

The WCA programme is based on a framework of theoretical concepts derived from studies of soft systems thinking, complexity and organisational behaviour. Although there are similarities to other whole organisational approaches (e.g. Coben & McCartney, 2016), differences in the aims and context meant that a bespoke theoretical framework was developed, based on five core concepts (contextual, holistic, interconnected, multidimensional, evaluative) from soft systems thinking. Significant contextual differences between colleges evidenced in previous studies (Noyes & Dalby, 2020) suggested the need to consider each college *context* carefully and develop context-sensitive change strategies (Trowler, 2005) for individual colleges. The large and complex nature of FE

[1] https://www.et-foundation.co.uk/professional-development/maths-and-english/cfem/cfem-resources-and-evidence/whole-college-approach/

colleges as organizations meant that a *holistic* approach was appropriate in which the limitations of working in complex situations was recognised (Hawkins and James, 2018). Within such an approach, it is important to understand the *interconnected* nature of the separate parts and value is placed on understanding how different functions interact (Monat and Ganon, 2015). A key element of the WCA programme was the formation of a small WCA team to lead their college intervention, which included a vocational teacher, mathematics teacher, vocational manager, mathematics manager and a senior leader. By involving staff from different levels and sections of the organisation, the aim was to allow complex interactions to be explored from different perspectives and these views then utilized to develop a *multidimensional* analysis (Checkland, 2000). The value of regular feedback when working with complex systems (Hawkins and James, 2018) was also embedded into the WCA programme through formal reviews for *evaluation* and reflection at regular intervals.

The underlying aim of the WCA was conceptualized as journey along a continuum from a position of organisational fragmentation towards coordination, and then collaboration, followed by the active participation of vocational and mathematics staff in efforts to improve students' experiences of learning mathematics. Although colleges started from different points on this continuum, the intended direction of travel was the same. Bringing staff together into purposeful interdepartmental communities and developing a collaborative way of working were considered important steps in the process. Each college commenced with their WCA team working through a set of self-assessment tasks and other materials, guided by an external expert. The dual aims of these tasks were to support the college in the analysis and planning of a college context-sensitive intervention, whilst also facilitating the development of the WCA team into a collaborative community.

The theoretical view of professional identity in this paper is grounded in a socio-cultural position where identity is shaped by social interactions and therefore constantly changing rather than stable (Holland, 2001). Individuals construct identities in their discourse that may be transient (Sfard & Prusak, 2005) but they explain their positioning at a point in time by synthesizing what they think and do into a sense of who they are (Grootenboer & Ballantyne, 2010). Identities are however shaped by social interactions and therefore the communities in which an individual lives and works are influential. The active participation of an individual in a community of practice allows a reciprocal process of membership and identity formation to take place (Wenger, 1999) in which an individual is influenced by the community and its shared beliefs but is also instrumental in shaping its values.

The professional communities that mathematics teachers belong to are important to the construction of professional identity (Wenger 1999). Although the communities that form within organisational structures may not always be active communities of practice as defined by Wenger (1999), a teaching team or department often develops the key characteristics of a domain of operation, a shared interest and commitment to each other. In a mathematics department we might typically see some individuals presenting identities that suggest they are the knowledgeable experts in the community and therefore occupy a central position, whilst others may identify themselves as peripheral members who are still developing skills (Lave & Wenger, 1991). Peripheral members would generally aspire to become more central and therefore the dominant behaviour, skills and professional identities of the 'knowledgeable experts' have a significant influence on the values and practices of others.

Apart from the departmental community, teachers may belong to other communities of practice, taking different positions within them and engaging with varying levels of commitment. The potential of additional communities of practice within a college to shape teachers' identities is the prime concern in this paper but it is also important to consider how such communities might be developed.

Research methods

The findings presented in this paper are based on research carried out during a two-year project (2021-2023) that aimed to improve the understanding, planning and implementation of whole organisational approaches for mathematics in English FE colleges. This involved the design, pilot and research of mechanisms and tools to support FE colleges with the development of a Whole College Approach to mathematics, and with the crafting of their own context-sensitive interventions to improve student engagement with, and achievement in mathematics. The research involved 16 FE colleges, which although self-selected, represented the diversity of Further Education colleges in England in terms of size and location. Each college followed the same basic WCA programme but designed their own college interventions as part of the study.

A case study approach was used, with qualitative data generated from interviews with WCA team members (27), formal written reports by colleges (34) and detailed observation notes from the termly meetings with each WCA college team. Two short surveys for WCA team members and two focus groups were carried out prior to the case studies to identify areas for exploration in the interviews. Monthly online meetings with representatives from the colleges were used to monitor progress and review emerging themes from the analysis. After an initial period, four colleges with contrasting early experiences of the WCA programme and contexts were selected to be the primary case studies.

An iterative grounded approach suitable for qualitative studies was adopted for the analysis, using emergent coding from which categories and key themes were developed. From the initial analysis, four in-depth case studies were developed, followed by further analysis to construct shorter summaries of the other 12 college cases and then a cross-case analysis. Only selected findings from key themes related to this paper are presented in the following section but other findings are discussed elsewhere (Dalby, 2024) and a more comprehensive report of the main study will be published later.

Findings

The reciprocal relationship between identity and membership of a community of practice (Wenger (1999), means there are key points of interest for this paper in two of the emergent themes from the analysis: the formation of inter-departmental communities; and mathematics teachers' identities.

Interviews and college reports showed wide agreement that the development of *an interdepartmental and multi-level WCA team* was an essential element of the programme and vital to the progress of colleges with the development of a WCA. Furthermore, there was strong evidence that developing an appropriate culture within this team was important since this enabled a holistic view of the issues to be constructed. Interview data and college reports showed how WCA team members came to recognise the value of different perspectives and realized that this allowed them to collectively construct a better-informed view of the complex mechanisms and interactions they needed to consider when planning interventions. Several WCA colleges, for example, reported that their early

discussions resulted in the conclusion that the real problem was not the one they initially thought they needed to address, and that recognising this led to more appropriately targeted interventions.

Observation notes of early WCA team discussions provided evidence of some repositioning of individual attitudes and beliefs during this process and shifts in narrative identity. For example, although some teams initially demonstrated divided views, especially between vocational and mathematics staff, their use of language changed after completing the self-assessment tasks, with increased use of "we" in their discussions rather than "us" and "them". This indicated a new shared ownership of the issues with mathematics and a change in identity that reflected recognition of a common interest in students' mathematical development. College reports evidenced agreement that the development of this shared understanding in the WCA team was important for practical reasons since it enabled them to develop more efficient collaborative approaches to operational issues, such as poor attendance at mathematics sessions, rather than trying to address these independently.

The need for more communication between mathematics and vocational staff was identified by most WCA teams as they worked through the self-assessment tasks in the early part of the programme. Different interventions were developed by colleges to facilitate more frequent interdepartmental communication such as: arranging informal meetings over lunch; designing joint mathematics events in vocational contexts; allocating a mathematics teacher to a vocational area to be a 'link' tutor; and visiting each other's lessons. Each of these served to increase communication, but also led to the development of new *interdepartmental communities of teachers* who worked collaboratively on interventions. Interviews and college reports showed how teachers gained understanding of other perspectives by engaging with these interdepartmental communities and discovered benefits from the increased communication, such as new ways of addressing operational and pedagogical issues.

Interviews and observations indicated changes in identity for teachers who participated in these interdepartmental communities. These are divided into three categories to indicate the different aspects of identity that appeared in the data. First, the *narrative identities* presented by teachers themselves provided evidence that the increased communication and better understanding of alternative views that developed within interdepartmental communities led to a softening of previously polarized positions. Mathematics teachers, for example, demonstrated more vocational awareness in their narratives and vocational teachers showed greater interest in students' progress with mathematics. This was empowering since both mathematics and vocational teachers were then less isolated in their attempts to tackle issues such as poor student behaviour or absence from mathematics lessons and could work collaboratively. Interviews with mathematics teachers suggested that they now saw themselves as part of a wider team, working in partnership with vocational teachers instead of acting independently with sole responsibility for students' progress with mathematics.

Observations and interviews also showed how the sharing of knowledge within these communities was instrumental in changes to teacher practice and their *enacted identities*. Mathematics teachers, for example, explained how they gained understanding of authentic vocational contexts that they could use in their lessons, or understood how to synchronize topics and make timely connections to the vocational programme. Vocational teachers referred to benefits such as formal or informal support from mathematics teachers to enhance their teaching of embedded mathematics, and a better

understanding of mathematics examination requirements that enabled them to support students more effectively. The development of enhanced knowledge was often linked to collaborative activities, such as visits to vocational sessions by mathematics teachers where they observed authentic uses of mathematics in context, or to instances when providing support for vocational staff during the general course introduction as simply a member of staff rather than a mathematics specialist.

There was also evidence of changes in *perceived identity* in interviews and reports, where references were made to students' perceptions rather than the teachers describing the changes explicitly themselves. For example, when vocational and mathematics teachers took part in a joint mathematics/sports event with their students that involved playing darts, remarks from students were used by staff to explain the difference in identity they experienced as a co-participant in learning activity outside the mathematics classroom, compared to their normal position as an isolated specialist subject teacher. Similar changes in students' perceptions were also noted by mathematics staff when they entered the vocational environment for other reasons, such as observing lessons or assisting with other planned activities. In some interventions mathematics teachers became actively involved in vocational sessions to the extent that they became just "teachers" to students rather than specifically "mathematics teachers". Although this shift in identity was less pronounced in other cases, there was wide evidence of participating mathematics teachers moving towards a position of being a *mathematics teacher in vocational education*, working collaboratively with vocational teachers, instead of a mathematics specialist independent from the educational context.

Discussion

In summary, the findings show how the development of interdepartmental communities with shared aims is essential to a whole organisational approach since these facilitate knowledge exchange and foster collaboration. Secondly, there is evidence that for mathematics teachers, membership of a mathematics-focused interdepartmental community leads to changes in professional identity. These changes were evidenced in their narratives (narrative identities), changes in practice (enacted identities) and in observations by others (perceived identities) but generally indicated a position of greater sensitivity to the vocational educational context and less isolation.

The benefits to students, mathematics teachers and vocational staff of developing mathematics-focused interdepartmental communities were evidenced in multiple ways. By spanning existing structures, traditional barriers between mathematics and vocational departments were broken down. Changes to practice with positive effects for students and staff emerged from collaborative working in these communities, and differences between vocational and mathematics teacher identities become less stark as each contributed to an increasingly shared ownership of mathematics provision.

A shared interest in improving students' experiences of learning mathematics led the members, over time, to a commitment to each other as they worked collaboratively to tackle the issues that emerged from their discussions. We might consider these as communities of practice since they demonstrated typical characteristics to those described by Wenger (1999) with their domain of operation defined largely by the parameters of the WCA project. It is important to note that although vocational teacher identities were not included within the scope of this paper, they were instrumental in shaping these

interdepartmental communities and so intrinsically linked to the development of mathematics teacher identities. This aspect of the interdepartmental community is worth further consideration.

Despite indications of positive effects on staff and students, the timebound nature of the study means that the sustainability of such communities and identities remains unexplored. Given the instability of identity under the influence of ongoing social interactions (Holland, 2001) and close connection to a community of practice (Wenger, 1999), the issue of sustainability is complex. However, the study provides evidence of the importance of establishing interdepartmental groups across existing organisational structures in England's FE colleges and the likelihood that these would need to be maintained to cultivate and sustain vocationally sensitive identities for mathematics teachers.

This raises questions about the extent to which it is helpful for mathematics teachers in vocational education to develop a vocational identity, and whether there is an optimal balance between this and a subject-focused professional identity. For most of the mathematics teachers in this study, their dominant identity still appeared to be closest to the subject-focused community of their mathematics department, but instances did occur where this became secondary. Positioning mathematics teachers in more than one community means their identities will be influenced by each of these to a greater or lesser extent, so both the nature of the community and level of interaction need to be considered.

Conclusions

As part of a wider study of the effects of the WCA programme, this investigation into teacher identity is only exploratory and the generalizability of the findings is therefore limited. There are, however, indications of multiple benefits to FE colleges from the formation of mathematics-focused interdepartmental communities and these are worth further examination. The shaping of teacher identities to embrace a wider view of mathematics learning than the teaching that takes place in a mathematics classroom opens opportunities for enriching lessons and demonstrating the relevance through connections to students' primary studies. Furthermore, being part of a collaborative interdepartmental community is empowering, allowing mathematics teachers access to different perspectives on the problems they encounter, and mutual support in addressing these. For the organisation, these communities bridge divisions between departments and curricula, strengthening traditional 'silo' structures through new social interactions and harmonizing contrasting sub-cultures.

The sustainability of such communities and associated changes in identity are yet to be explored, which leaves questions about whether the positive effects evidenced in the study can be maintained. However, the findings suggest that a more focused study of how membership of multiple communities of practice might be used to the advantage of both students and staff would be a useful extension to the discussion in this paper.

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Discussing "Numeracy" beyond the scientific community: How adult learners define numeracy and what challenges arise from that for the coming future

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The term "numeracy" is a term whose definition has been changing, particularly over recent decades, since it began to be used in the educational field in the middle of the last century. At an international level, there is consensus that "numeracy" refers to knowledge, dispositions, and cognitive processes (higher-order thinking skills). It also has affective implications, which influence learning. However, no studies investigate how adult learners, who are the end-users of any educational action, define numeracy. In this paper, we address this research gap and investigate its challenges.

Keywords: Numeracy, Adult Basic Education, Adults Learning Mathematics.

Introduction

We inhabit an increasingly interconnected world where "intelligent" devices progressively occupy spaces once reserved for human activity. A clear example of this shift can be seen in the proliferation of applications we install on our smartphones, which now perform mental calculations—ranging from simple to more complex—that we previously carried out in our daily lives. Our cell phones now provide these calculations by pressing a few buttons on a touchscreen. It is not only that we are surrounded by situations where numbers, their properties, relationships, uses, algorithms, and the reasoning associated with them are ever-present—this has been the case many times. What is new is that "intelligent" devices are now performing the calculations and making decisions that we once made ourselves after scrutinising and utilising the numbers or mathematical data presented to us.

In this context, *numeracy* is garnering significant attention. Data from the first cycle of PIAAC (Programme for the International Assessment of Adult Competencies) (2011–2012) revealed that a significant proportion of the population, ranging from 30–60% in low-income countries and 10–40% in high-income countries, scored below Level 1 on PIAAC's five-level scale[1]. But what does it mean to have a low level of numeracy in our current World?

[1] PIAAC uses a five-level scale: below level 1 (below 176 points, which means that respondents carry out simple processes such as counting, sorting, performing basic arithmetic operations, etc.), level 1 (176-less than 226 points, which means that respondents are able to carry out basic mathematical processes in common, concrete contexts where the mathematical content is explicit), level 2 (226-less than 276 points, which means that respondents identify and act on mathematical information and ideas embedded in common contexts with few distractors), level 3 (276-less than 326 points, which means that respondents understand mathematical information less explicit, embedded in contexts that are not always familiar), level 4 (326-less than 376, which mean that respondents understand a broad range of mathematical information

As we anticipate the release of the results from PIAAC Cycle 2, it appears evident that numeracy will assume an increasingly prominent role in the educational policy agendas of nations globally. The challenges we encounter are numerous, many of which diverge from those explored in recent decades concerning numeracy.

Therefore, one of the imminent challenges in adult learning research may be revisiting the concept of "numeracy" itself in this "new *technified* context".

In this paper, we explore the notion of "numeracy" by confronting two points of view: the *scholar's* definition of numeracy and the definition provided by adult learners, which are usually not included in the academic literature.

Theoretical Framework

To improve adult numeracy in contemporary society, we must first establish a shared understanding of what numeracy entails, providing a common framework for action. In PIAAC Cycle 2, numeracy is defined as "accessing, using and reasoning critically with mathematical content, information and ideas represented in multiple ways in order to engage in and manage the mathematical demands of a range of situations in adult life" (Tout et al., 2021, p. 93). This definition represents the most recent evolution of a concept that initially focused on the purely instrumental "ability" to work with numbers—referred to in the 1959 Crowther Report as the "mere ability to manipulate the rule of three" (p. 270)—and has since expanded to incorporate the broader application of numbers and mathematics to solve practical, everyday problems (Baker, 1998; Benn, 1997; Coben et al., 2003; Condelli et al., 2006; Cockcroft, 1982; Díez-Palomar, Ramis-Salas, Močnik, Simonič, & Hoogland, 2023; Gal et al., 2005; Gal, 2000, 2024; Ginsburg, Manly, & Schmitt, 2006; Hoogland, 2023; Hoogland, Díez-Palomar, & O'Meara, 2021; Hoyles et al., 2002; Maguire & O'Donoghue, 2003; Tout, 2020; Tout & Gal, 2015; Yasukawa et al., 2018).

As this concept has evolved, numeracy has become increasingly complex and multifaceted, continually adding new dimensions. The most recent definition of numeracy adopted by the EU (CENF – Common European Numeracy Framework) presents it as a multi-dimensional construct encompassing four key dimensions: context, knowledge and skills, higher-order thinking skills, and dispositions. Numeracy can be observed across various contexts, including everyday situations, work environments, civic engagement, financial management, health and care, recreation, and the acquisition of new learning opportunities.

Moreover, numeracy extends beyond basic arithmetic operations, including knowledge and skills related to dimensions and shapes, patterns, relationships and change, quantity and numbers, data and probability, and digital competencies such as the use of calculators and spreadsheets. Numeracy also involves cognitive processes—manifesting when we analyse situations, process information, reason, mathematise, solve problems, or engage in critical thinking. Lastly, the CENF framework includes dispositions, recognising that factors such as self-confidence, emotions, beliefs, collaboration,

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that may be complex, abstract or embedded in unfamiliar contexts), and level 5 (equal or higher than 376 points, which means that respondents understand complex representations and abstract and formal mathematical and statistical ideas).

flexibility, and maths anxiety significantly influence how individuals engage with and apply numeracy in their daily lives.

Thus, numeracy has emerged as a complex, multi-dimensional concept. All these aspects must be considered when designing training programmes, analysing learning scenarios, or evaluating the specific responses of adult learners to educational activities.

Methods

The study presented here pertains to the case study conducted in Spain within the European *Numeracy in Practice* (NiP) project (2021-1-NL01-KA220-ADU-000 026 292). The primary aim of this project is to promote the definition of numeracy developed in the *Common European Numeracy Framework* (CENF) project (2018-1-NL01-KA204-038913) and to engage in dialogue with potential end-users.

A survey was conducted in the participating countries (Slovenia, Poland, Spain, Italy, Greece, France, Ireland, the Netherlands, and Austria) to explore how they define "numeracy" and what attitudes they have towards it. This paper focuses on the data collected in Spain around the first aspect: participants' definition of numeracy. The research question we discuss in this paper is: How do adult learners who participated in the NiP Project survey define numeracy? The aim is to compare this definition with previous work in the scientific literature, especially with the definition in the PIAAC framework (Tout et al., 2021).

A total of 20 adult learners (8.85% of the sample) responded to the survey, of whom 65% were women and 35% were men. The age distribution of the participants is outlined in the following table (four persons didn't provide their ages):

Table 1: Distribution of the Spanish sample

Age group	Absolute values
From 18 to 30 years old	1
From 31 to 50 years old	6
From 51 to 65 years old	3
From 66 to 75 years old	0
Over 76 years old	6

The survey was carried out through a questionnaire administered online. The link to the questionnaire was sent to various adult schools in Spain (Catalonia) by email and by phone, and the data was collected in a data matrix. Items were translated into Spanish. Since there is no direct translation of the word "numeracy" in Spanish, the term "*alfabetización numérica*" was used. The adult learners who participated in the survey answered the questions through the online questionnaire. The following table contains the items from the data collection instrument used in this paper:

Table 2: Selected survey items

Item 1. Gender // Item 2. Age

Item 3. Experience in the area of mathematics?

- I studied mathematics throughout primary school
- I studied mathematics at lower secondary school
- I studied mathematics at upper secondary school
- I studied mathematics as part of an undergraduate degree
- I studied mathematics as part of a vocational programme or trade qualification
- I studied mathematics in postgraduate studies
- Other (please describe)

Item 4. In what country are you enrolled as an adult learner?

Item 5. As an adult learner, what type of numeracy courses have you engaged with? Please select all that apply.

- Standalone numeracy courses i.e. sole focus of the programme was on numeracy and/or mathematics
- Integrated numeracy courses i.e. numeracy was incorporated into a programme with a different focus such as horticulture, cookery etc.
- I have never engaged in a numeracy course as an adult learner.

Item 6. Explain, in as much detail as possible, what the word “numeracy” means to you. Note: Numeracy can also be referred to as numeracy, mathematical literacy, arithmetic in your country.

Item 7. Do you think numeracy and mathematics are the same thing?

Item 8. Point out most important differences between numeracy and mathematics.

Item 9. Why do you believe numeracy and mathematics to be the same?

EXCEL was used to analyse the survey data. Open-ended questions were categorised and tabulated, with codes generated based on the thematic analysis of responses, following established procedures for the qualitative analysis of open-ended questionnaire data. The discussion in this paper uses the answers to items 1 to 9. First, the responses given by the survey participants to the survey questions (especially items 6, 7, and 8) were collected verbatim. A comprehensive list was compiled of all responses for each item. These responses were then grouped based on the presence of repetitions or similarities. Codes were subsequently developed by clustering participants' responses according to similar themes (see Table 3 for details), ensuring alignment with their original definitions and adhering to the discussion objective outlined in this paper.

Results

Most survey respondents indicated that they had studied mathematics at the primary or lower secondary level (with five adult learners reporting lower secondary education and another five saying

primary education). This suggests that for half of the participants, their formal study of mathematics spanned no more than eight years, with their mathematical knowledge limited to the primary or, at most, lower secondary level.

When asked about the types of courses in which they had studied numeracy, nearly half reported that they had learned numeracy in an integrated manner within courses not focused explicitly on numeracy. Only two respondents indicated they had taken dedicated numeracy courses, highlighting that most participants had yet to engage in specialised numeracy education.

From the profile of survey respondents, it becomes evident that the majority conceptualise numeracy as "basic mathematics." When asked to define numeracy, their responses were as follows (see Table 3):

Table 3: Responses to item 6

Responses
Numeracy is about numbers // To me, numeracy means to have knowledge about numbers // Introduction to the concept of number and its use // Learn about numbers // Any kind of calculations – problems // Different kinds of calculations // Learn different kinds of calculations // Learn to make calculations (arithmetic) // Learn calculations because you need them for the everyday life // Mathematics for beginners // Start studying mathematics // To start to learn mathematics // Yes, it is the beginning of mathematics // Understand deeply what the numbers are for.

The responses to item 6 reveal at least three distinct categories of definitions. Firstly, some respondents associate numeracy with "numbers," interpreting it as "knowledge of numbers" or "everything related to numbers."

Secondly, another group links numeracy to performing calculations, which could be "arithmetic." These individuals associate numeracy with operations such as addition, subtraction, multiplication, and division, with one respondent also including problem-solving in their definition.

Lastly, a further definition of numeracy emerges as "mathematics for beginners," where numeracy is seen as primary, elementary mathematics, serving as the starting point for more advanced mathematical study. Only one respondent referred to numeracy as involving a deeper understanding of the purpose of numbers.

When asked whether numeracy is the same as mathematics, the participants were divided: nearly half (nine of them) believe they are distinct, while the other half (eight of them) view them as synonymous. When asked to explain the differences, the results from item 6 were reinforced: most respondents associate mathematics with more advanced content, while numeracy is understood as "basic mathematics."

Table 4: Responses to item 8. What are the differences between mathematics and numeracy?

Responses
I do not know // In mathematics there are many things like equations, problems. Numeracy is like the alphabet but with numbers, for example, symbols // Numeracy is more basic than mathematics // Numeracy is to start mathematics // Numeracy is to start studying mathematics // The first one is beginners, and the second one is getting more advanced.

Discussion

The scientific community has adopted several definitions of numeracy. One prominent example is found in *The Assessment Frameworks for Cycle 2 of the Programme for the International Assessment of Adult Competencies* (OECD, 2021), where the Numeracy Expert Group (Tout et al., 2021) offers a pragmatic definition, emphasising numeracy's use in responding to real-life adult situations. This definition builds on previous OECD definitions (PIAAC Expert Numeracy Group, 2009). Hoogland and his collaborators (2021) present an alternative definition, highlighting numeracy's complex and multifaceted nature. Additionally, Yasukawa and colleagues define numeracy in sociological terms, framing it as a social practice (Yasukawa et al., 2018).

Across these definitions, one common theme emerges: the complexity of numeracy. This understanding goes beyond the earlier, more limited conception of numeracy as simply "basic mathematics," a definition prevalent in official documents of the mid-1950s (Crowther, 1959).

However, when adults are asked, "What do you understand by numeracy?" their responses tend to reflect a concept more aligned with how numeracy was understood in the mid-20th century rather than the contemporary definitions used by experts in the field.

An analysis of the responses provided (Table 3) reveals that "numeracy" is most frequently associated with concepts such as "numbers," "learning about numbers," "performing calculations," "any kind of calculations," or "mathematics for beginners." These responses suggest that adult learners perceive numeracy primarily as a basic mathematical concept, equating it with the ability to use numbers and carry out fundamental operations. This interpretation is further corroborated by the responses to item 8 (Table 4), where participants differentiate between "numeracy" and "mathematics," characterising the former as basic mathematics and the latter as encompassing "more advanced" topics, such as "equations" and "problems." Additionally, numeracy is metaphorically described as being "like the alphabet but with numbers."

It would be valuable to explore why the participants in this study hold such a view of numeracy. What factors contribute to their association of numeracy with "basic mathematics"? A possible hypothesis for further discussion is that the complexity of numeracy is often "invisible" and instead what we refer to as "mathematics" is "that which is visible." If that is so, then one challenge emerging from this discussion is the need to design training programmes that make the mathematics present in everyday contexts more visible, especially those forms of mathematics that are often not "recognised" as such. It is important to demonstrate that mathematics extends beyond what might be classified as

"advanced mathematics" (such as calculus, topology, differential geometry, vector calculus, numerical analysis, game theory, cryptography, etc.). Everyday mathematics, too, constitutes legitimate mathematical knowledge. These forms of mathematics have their representations (symbols, formulas), connections, rules, properties, and functions. They are used in reasoning and problem-solving processes.

Therefore, a key challenge for the near future is to highlight and make visible these aspects of mathematics, which are increasingly rendered invisible by technologies, artificial intelligence, and other factors. The goal is to raise awareness among adults so that they can use mathematics critically and effectively in various everyday situations.

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Mathematics and vocational disciplines: A co-disciplinary perspective to reduce early school leaving

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The main aim of the research presented in this paper is to address the problem of early school leaving in vocational schools. For this aim we encourage the designing of co-disciplinary training courses, co-designed and co-led in the classroom by mathematics teachers and teachers of vocational subjects to bring out the existing link between the vocational subject and Mathematics. The link between the subjects is based on some vocational artefacts following the Theory of Semiotic Mediation and used as Boundary Artefacts. In this paper we analyse their semiotic potential and how their use has been fundamental in a co-disciplinary intervention which involved Mathematics and Fashion.

Keywords: Co-disciplinarity, early school leaving, boundary artefact, vocational school, theory of semiotic mediation.

Introduction

The main aim of the research presented is to address the problem of school dropout in vocational schools, designing co-disciplinary training courses, co-designed and co-led in the classroom by mathematics teachers and teachers of vocational subjects to bring out the existing link between the vocational subject and Mathematics. To involve students most at risk of dropping out of school, we propose a "meaningful" Mathematics (Wake, 2014) with respect to the vocational subject and vice versa we want to give meaning to the vocational subject through Mathematics. The Research aims to demonstrate how the integration between disciplinary contents and vocational disciplines is essential, which allows us to grasp the importance of the study of disciplines apparently distant from professions, such as for example Mathematics and the Italian Language, aimed at acquiring vocational skills and to improve the quality of vocational development. Often the students in these schools, especially in the first two years, testify to their academic and training difficulties which put the entire structure of institutional vocational training in crisis. Further support for this aim is highlighted by the contradiction that emerges from the national official documents (M.I.M., 2019). It therefore seems necessary to involve teachers of Mathematics and other disciplines in an innovative training course, because it is assumed that on the one hand, mathematics teachers do not know which specific mathematical contents are useful for vocational disciplines, on the other, teachers of vocational subjects need specific training in Mathematics, which is necessary to address some topics of their discipline. The theoretical framework that supported the research starts from the Theory of Semiotic Mediation (TSM) and is enriched by the introduction of Boundary Artefacts (BA), artefacts specially designed with the function of integration between Mathematics and vocational subjects. This article presents the analyses of the semiotic potentials of artefacts and some preliminary results relating to their use in a co-disciplinary intervention which involved Mathematics and Fashion. In this article we present the results of a first pilot study in which, through the voices of the students, the first evidence emerges on the usefulness of the co-disciplinary approach in reference to the phenomenon of school dropout.

Conceptual background

Drawing from the emerging understanding of mathematical activities in work contexts is fundamental to provide new insights into the role of Mathematics and its dual nature both as an area of study and as an increasingly diverse tool with application in and across many aspects of human life (Wake, 2014). In these cases, learning can be mediated by specifically created BA, inspired by the meaning supported by the seminal idea of a boundary object described by Akkerman & Bakker (2011). In addition, our idea of BA has the function to coordinate the different disciplines involved, even if the vocational practices that use them are different. They are suitable enough to maintain their meaning in the disciplines involved, but also flexible enough to be used effectively in each of them (Star, 1995). The BA, for us intended as an artefact in TSM, is any device created and realized by human beings for any objective. The notion of BA, which we refer, enrich the artefact's notion in TSM, in the sense that while using it, it evokes vocational meanings in addition to mathematical concepts. The notion of BA and that of schema of utilization (Rabardel, 1995) refers to the main focus of the TSM's construct of semiotic potential, that is the twofold relationship that the artefact has with the personal meanings emerging from its use, the mathematical meanings that might be evoked by such use and the vocational procedures emerging from its use. These BAs are, in this new scenario, operational tools to implement decisions, to make decisions and to represent a teaching situation intended as a "reference situation" for particular senses of the concepts in play (Vergnaud, 1994). Vergnaud's conceptual field theory, indeed, constitutes a reference theory, in which the proposed problem situations act to give meaning to the concepts in play. To identify such BAs, it seems necessary to bring together members of different disciplines with the aim of realizing a common object/task. Moving in this direction requires a co-learning inquiry approach, where the teachers involved learn together through inquiry, where inquiry is a mediating tool. Sharing Wagner's (1997, p. 16) thinking, in a co-learning agreement,

> Researchers and practitioners both participate to educational processes and school systems. Both are engaged in action and reflection. By working together, each could learn something about the other's world. Of equal importance, however, everyone can learn more about their own world and its connections to institutions and school. (Wagner, 1997, p. 16)

One of the aims of this approach is for researchers and teachers to work together to explore and develop the learning and teaching of mathematics. In our case, using a BA deriving from the vocational world, we aim to deepen and give meaning to mathematical concepts with the help of the other discipline involved and vice versa. BAs, the analysis of their semiotic potential and the activities connected to it, were designed within the TSM (Bartolini Bussi & Mariotti, 2008). TSM has its roots in Vygotskian socio-constructivism, according to which students are guided by their teacher to build mathematical knowledge by solving tasks designed with reference to the use of appropriately chosen artefacts. The use of artefacts and the deployment of their semiotic potential leads students to produce personal knowledge, which through mathematical discussions, orchestrated by the teacher, is gradually transformed into shared knowledge. The notion of semiotic potential expresses the relationship between the personal meanings emerging from the experience of acting with the artefact and the mathematical meanings recognizable by the expert in such actions; its strict dependence on the task, which must be carried out by the students, makes the artefact a key tool for designing

appropriate tasks. The TSM was used both by teachers during the co-planning of the training courses and in the analysis of the semiotic potential of the artefacts, and by researchers to analyse the results of the experimental activity in reference to Mathematics and the vocational discipline. Specifically, in the research proposed here, the tasks are appropriately designed to ensure that during the use of BAs they can bring out their semiotic potential in the same way both in relation to Mathematics and in relation to the vocational discipline involved. Their use and their dual semiotic potential are then made explicit in collective discussions (Bartolini Bussi, 1998) orchestrated by the teachers. In collective discussions it is also fundamental to analyse language, which especially in mathematics sees the use of mathematical registers as extreme forms of evolved registers (Ferrari, 2021). Moreover, from the literature, different approaches emerge in reference to the interactions between different disciplines, for the study presented in this article, it seems necessary to support the co-disciplinary approach. According to Blanchard-Laville (2000) co-disciplinarity is the way in which multiple researchers cooperate generatively on a specific project. In the common work there is the aim of co-thinking, that is designing a space in which different researchers do not have the same idea about something, but something allows them to co-think and bring out ideas to provide solutions. The interaction between the disciplines starts from planning and then materializes in observation, monitoring and reflection on specific training paths. This approach seems appropriate since the dialogue in this study occurs between teachers from different disciplines, in a co-disciplinary way.

The research methods and experimental design

In this article we present the results of a study based on design-based research (Swan, 2020). In particular, the operational phases of the project involved the identification of some problems, the solutions of which are the place where the co-disciplinarity between vocational subject and Mathematics takes place through a BA. The overall aim of the research study is to give meaning to Mathematics in relation to vocational subjects and vice versa, give meaning to vocational subjects through Mathematics. Indeed, our main assumption is that mathematical knowledge is relevant for the fulfilment of professional requirements (Winther, 2016). According with Lave & Wenger-Trayner (2008), we believe that basic skills such as Mathematics are the breeding ground for skills in the vocational field. This concept is based on the theory of situated learning. The specific aim of this research is to discover how the proposal of co-disciplinary educational paths, co-designed and co-led in the classroom by both the mathematics teacher and the vocational teacher, can bring out the existing link between vocational training and Mathematics, with the involvement of students, most at risk of dropping out of school.

We hypothesize that through this connection students give meaning to Mathematics as well as to the theories of vocational disciplines useful to the profession. Furthermore, our research hypothesis is that, by improving learning in the disciplines involved, students are motivated and find school learning useful for their future work. Therefore, the research questions we ask ourselves are:

- Does the co-disciplinary approach, through the use of BA, favor the evolution of signs and therefore learning?
- Can the involvement of students in such designed activities, and the evolution of their learning, motivate them to believe in the usefulness of school in professional development?

In Italy, vocational school is a secondary school (Grade 9 to Grade 13), which permits students both, continue with academic study and vocational career. The following experimentations were conducted in a second class, it means Grade 10, aged 15. Usually in these schools the drop-out rate is very high, quite near to 25%, in the first two years.

The organization of the teaching intervention includes various preparatory design phases before starting the classroom experimentation. Phase 1: Implementation of co-disciplinary work. Vocational and Mathematics teachers discussed the potential of synergistic work between Mathematics and vocational disciplines. Phase 2: Identification of BAs usually used in the vocational field, and analysis of their semiotic potential from both a mathematical and vocational point of view. Several BAs specific to the vocational subject were identified and analysed: the paper model, the quartometer, the flask with balls of different colours. The teachers explored the semiotic potential of the identified BAs. Phase 3: Structuring of educational paths to be tested in the classrooms. The analysis of the "dual" semiotic potential, that linked to the mathematical concepts evoked and that linked to the vocational discipline, allowed the structuring of a co-disciplinary didactic intervention. (Fiorentino et al., 2021). Phase 4: The educational intervention. A class of approximately 20 students was involved in this study and a co-disciplinary teaching path was conducted which involved Mathematics and fashion.

In the teaching paths, an initial problem situation linked to the use of the BA was proposed, the subsequent collective discussion on the first resolutions, subsequent problems to reinforce the evolution of mathematical concepts, and the final discussion. The mathematics teacher and the teacher of the vocational disciplines involved in the experimentation co-designed the activities (the initial and subsequent problems and the collective discussion questions) with the researchers. They co-led the experimental activity and mathematical discussions in class, while the researchers observed, filmed and analysed it. In the classes, approximately five to seven students were at risk of dropping out of school and had negative evaluations in both subjects.

Furthermore, unjustified absences from school were recorded at a rate of 70% in three months. To address our research aim, we analysed the transcripts of the Mathematical Discussions relating to the solution of the proposed problems and the use of BAs. To validate the teaching paths, a qualitative analysis of the transcripts was carried out, according to the criteria of credibility, reliability, transferability and confirmability (Guba, 1981). This study is part of a general project which involved 5 vocational institutes, 15 in-service teachers and around 300 students in the first two years of the schools involved. In the following we will analyse in detail only one of the experimentations, more details about the Chemistry and Mathematics experimentation are on Fiorentino et al. (2023).

"Vocational" artefacts: the case of the paper pattern and the quartometer

The experimentation regarding Fashion and Mathematics was conducted in a second class of the Industry and Crafts for Made in Italy address of a vocational school in the South of Italy. The Fashion teacher had already used a paper pattern to reproduce trousers, the mathematics teacher had addressed the concept of proportion and symmetry. In the first activity, students were asked to carry out the following problem: "Reproduce the pattern for making a bodice on a paper pattern". Students were provided with the artefact and worked in groups of 2 students. At the end of the activity, a collective

discussion was carried out by both teachers, with the aim of bringing out the mathematical concepts of proportion and symmetry linked to the concepts of the vocational discipline. The artefact used allows the modelling and creation of garments considering the algebraic and geometric relationships to create them, providing a flat representation of the garment that will subsequently be created. The identification of a part of the garment to be reproduced evokes the mathematical idea of the axis of symmetry which allows the symmetrical model of a quarter of the garment to be reproduced exactly; from the point of view of vocational discipline, however, it evokes the idea of folding fabric. The identification of specific positions in relation to each other on the paper pattern evokes the mathematical idea of proportion between two measurements; from the point of view of vocational discipline, we recognize the lengths of the circumferences relating to the waistline and the pelvis which are in proportion to each other. Finally, when switching from one size to another, the mathematical idea of proportion between measurements is evoked; from the point of view of vocational discipline, the proportions necessary to move from one size to another are highlighted. The second part of the proposed activity involved the reproduction of the garment created first at 1:4 scale, in the original format, to reproduce it in real size. This step is done using a further artefact: the "quartometer". This artefact is a real ruler built with a 1:4 ratio. The 1:4 ratio is not random, but is necessary, because it is linked to the two axial symmetries of the body. In fact, for example, to create trousers, the axis of symmetry that passes through the center of the crotch and the lateral axis of symmetry that divides the front from the back are considered. This ratio is also transferred to all trouser measurements to create a perfectly proportional model.

Data collection and analysis criteria

The analysis concerns the co-disciplinary discussion carried out in the following phase 4 to group work in which different problem solutions emerged. According to the specific hypothesis of this study, the following analysis attempts to highlight how the co-disciplinary training path has given meaning to the mathematical concepts compared to the concepts of the vocational disciplines and vice versa gives meaning to the vocational concept through the mathematical concepts in play. The analysis of some extracts from the experimentation concerning Mathematics and Fashion is given. In particular, here, the vocational discipline teacher suggests making a bodice with the help of the BA pattern. In doing this the students begin to draw one of the four parts of the bodice and place marks in particular positions. At this point the mathematics teacher intervenes by starting the collective discussion:

Mathematics Teacher: What are you tracking?
Giulia: the flank line
Mathematics Teacher: And why are you drawing it right there?
Giulia: it is central, there is a calculation to define its position.... from F we find G, then divided by two and we get the position… it is not central to everything, but only to this point F. This straight line represents the front and back of the dress…
Fashion Teacher: but are you only doing one side?
Vanessa: yes, I'm only doing the back side, because there is a front part and a back part... but then there is a "roundabout" here.... and then this part overlaps because they are symmetrical. Then the different elements are added, here is the shoulder and here is the neck (point with their finger to the lines on the paper pattern relating to the parts of the dress that correspond to the parts of the body).

Marzia: they are obviously not the same, but there are some symmetrical parts, once they overlap the body from here to here... (and indicates the position on your body of where the dress will be placed)... it is the half circumference bust... divided by the waist circumference and with this calculation we have the recovery... clearly based on the sizes... in 52, it is half of 104... 42 is half of 84 which is valid for size 50. Making (52-42): 4 ... because there are 4 thoughts, because we have four thoughts. It comes out 2.5. This 2.5 represents the gap.

Mathematics Teacher: Why the semi-circumference?

Marzia: it's done half the circumference why we are doing it half of the body... because when we let's go and place (the paper pattern) we place it double… because this is one half of the bodice

Arianna: Why it is reproduced on the double fabric... because the two parts are one front and one rear and then they come together... so the dress is divided into four parts... it is placed on the double fabric, that is, a large, folded fabric... We only make one, because when it is then placed on the fabric two parts come out...

In this extract, it seems clear how much the BA paper model manages to make students dialogue both by providing justifications relating to Mathematics and to the vocational discipline. In the discussion, in fact, in the underlined parts there are explicit references to the mathematical meanings of "symmetrical parts" and circumference, just as the references to the meanings of the discipline relating to what they are representing represents in the body appear to be clear. The discussion continues and the teacher, with the aim of moving on to the second BA, the quartometer, to bring out the concepts of ratio and proportions asks:

Mathematics teacher: how do I go from a small drawing to having a natural drawing?

Giulia: because we make the small drawing which is a quarter of the natural size... we have a particular ruler... for example, 1 cm stands for 4 cm… every centimeter corresponds to 4. Then we use a table with the measurements relating to the sizes… the pattern respects the proportions, everything is magnified 4 times… ¼ is a fundamental fraction for making clothes... starting from the drawing on a small sheet of paper, to move on to the creation of the actual size paper pattern, to finish with the cutting of the fabric and the composition of the dress with these wing the 4 symmetrical parts.

Arianna: I finally understand why we use the symbol ¼ for the ratio! It seems to me like something is getting clearer…

Marzia: I agree with Arianna, by really understanding what I'm doing, I prefer to stay here in class, so that I could better work with paper pattern on my own

In the response reported here, the reference made to the mathematical meaning of the 1:4 reduction scale with which the BA quartometer is made seems evident, indeed Giulia expressly says, "1 cm stands for 4 cm... every centimetre corresponds to 4". Furthermore, the non-randomness of the ¼ ratio also emerges, as it is expressly stated that "¼ is a fundamental fraction for the creation of clothes", referring to the "sewing of the 4…symmetrical parts". Moreover, Arianna highlights how the signs used are useful for her personal understanding of ratio concept. This seems giving an additional value to the mathematics learning. Finally in the last sentences by Marzia it seems to us evident how the involvement of students in such activities, could improve their learning and, above all, believe in the usefulness of school in their personal professional development.

Preliminary results and concluding remarks

Throughout the activity it was possible to observe the presence and interventions of the students most at risk of early school leaving, such as Marzia and Giulia. Furthermore, all of them participated in the

activities, actively intervening in the discussions. These students are often absent from school and, even when they are present, do not participate in class activities. Inspired by some intervention in the discussion, like the Marzia's one, we think that, by involving students in co-disciplinary activities aimed at giving meaning to the theoretical concepts of the disciplines, we can contribute to reducing school dropout. Students seem to demonstrate that they are rediscovering the meaning of symmetry when they encounter the difficulty to explain the position on their body. Furthermore, the reference to mathematical concepts to interpret the vocational concept itself and to the vocational concept to explain the mathematical concepts appears evident in the students' synthesis proposals. These connections seem to emerge thanks to the use of BAs and allow students at risk of early leaving school to achieve the aims set by teachers. In our study, the evolution of signs and learning through co-disciplinarity became evident in how students interacted with the BAs. The semiotic potential is reflected in the process of constructing knowledge, where students reinterpreted the mathematical meaning of proportions and symmetry through the use of BAs. The collective discussion made the dual meaning of the BAs explicit: mathematical concepts emerged alongside vocational reasoning. This process facilitated the transformation of initial student-generated signs into structured mathematical ideas, directly connected to fashion practices.

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Theorizing how citizens and vocational workers interact in co-productivity chains with professional mathematicians

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Not all students need to master advanced mathematics (e.g., calculus) in their later lives. However, citizens and vocational workers may interact with advanced mathematics in, respectively, daily life and workplaces. To analyse such situations, we introduce the concept of "mathematical co-productivity chain", which connects work by (1) professional mathematicians and (2) citizens and vocational workers. We analysed three examples to characterize the concept. Our analysis shows that the mathematical co-productivity chain exists in various situations and be communicated through face-to-face dialogue or boundary objects. The citizens' and vocational workers' co-productivity involves interpreting and handling data, evaluating diagrams, identifying errors, and acting upon these. We discuss how mathematical competences revealed by the mathematical co-productivity chain can inform education for workplace mathematics and mathematical literacy.

Keywords: Mathematical co-productivity chain, mathematical literacy, mathematical models, mathematics education, workplace mathematics.

Introduction and theoretical backgrounds

Advanced mathematics plays an important role in society, such as in engineering and research. Therefore, curricula for general mathematics education often aim at guiding students to advance to the higher levels of mathematics (Mullis et al., 2020). However, many students do not advance to these levels, and when workers in industries are asked what mathematics they use, their answer often points only at primary school mathematics (Andreassen et al., submitted). In this theoretical paper we challenge the notion of separate worlds for advanced and basic mathematics and aim to analyse how people, who did not graduate in advanced mathematics, nevertheless may interact with it.

When mathematics is used in everyday life or at vocational workplaces, there is a network of people involved. For instance, a citizen's evaluation of shopping costs is framed by retailers' prices, the family's income, the landlord's rent, and so forth. Similarly, an administrator's creation of a timetable is framed by schedules and deadlines of others. From this relational perspective, citizens and vocational workers may engage in partial aspects of more advanced mathematical processes (Skovsmose, 2005). Also, they may productively engage and adapt their behaviour in accordance with advanced mathematics in various ways (Niss & Jablonka, 2020).

Skovsmose (2005) distinguishes three roles that people in society can have regarding mathematics. These are *creators*, *operators* and *citizens*. Creators are professional mathematicians employed by civil or industrial institutions (Frejd & Bergsten, 2016). They have a background in advanced mathematics and use data, theory, and existing mathematics to create new mathematics, such as mathematical or statistical models and algorithms. Often, these are hidden in technology but can materialise in spreadsheets, apps or graphs. The models, algorithms and data are used, applied or

adapted to specific contexts by *operators* (vocational workers). For example, a company that produces traffic lights will ask a *creator* to design generic software so a certain brand of traffic light can be used in various situations. An *operator* then installs such a traffic light at a crossroad. There, (s)he follows guidelines adjusting settings for the specific location. The safety provided by the light then impacts *citizens*. But the story does not end there. After installation, the citizens' behaviour is probably monitored and fed back to the creator to adjust the mathematical model and update the traffic system. In such situations, operators and citizens need neither knowledge of the advanced mathematics in the traffic system nor of the statistical analysis of user behaviour data. Hence, we see three groups of people (creators, operators, and citizens) interacting productively with each other in processes where advanced mathematics plays a role. This paper aims to characterise such interactions.

Inspired by Engeström (2001), we perceive the interactions between people in society as multifaceted. When a person engages in a goal-directed activity, this activity is mediated by tools (physical, digital, etc.), by the environment (other people, enterprises, etc.), the division of labour, and rules, norms and ethical issues in that environment. The installation of the traffic light already illustrated a division of labour between creators and operators. Between them, a manual served as a boundary object for their communication (Bakker & Akkerman, 2014). To engage in such an interactivity, each group will have their own benefits. The citizens gain a safe passage through busy traffic, whilst the creator and operator benefit financially through their salary for the work done. To study the interactions between creators, operators and citizens, we define a new concept named *mathematical co-productivity chain*:

> A mathematical co-productivity chain is a sequence of mathematical activities (shackles), in which two or more actors from different groups of people interact (links) in order to facilitate mathematical outcomes that drive social processes (co-productivity).

The rationale for developing this concept is to shed light on advanced mathematical activities in society as a series of interrelated activities divided among people with different mathematical backgrounds. This view goes beyond a perception of mathematical activities as being isolated and carried out from start to finish by individuals who all should have sound mathematical competences. Our study is guided by the following research question: *how can activities within a mathematical co-productivity chain between creators, operators and citizens be characterized?*

The background for our research is the fostering of mathematics, in particular for those who will not need advanced mathematics, which connects to research on mathematical literacy (Niss & Jablonka, 2020) and workplace mathematics (Williams & Wake, 2007). This begs for a description and discussion of operators' use of mathematics in vocational work, and of citizens' use of mathematics in daily life. An analysis of differences across the roles of citizens, operators, and creators in mathematical co-productivity chains can be a means to rethink mathematics education, in particular for those who do not want, need or can learn advanced pure mathematics.

Setting for the analysis

To answer our research question, we selected three authentic cases that we judged as fitting the definition of a mathematical co-productivity chain between creators, operators and citizens. These provided empirical data to characterize mathematical co-productivity chains. The cases were chosen

to represent each of the three perspectives, namely that of a creator, a citizen and an operator. The first case of the mathematical co-productivity chain concerned *the perspective of the creator*. We reanalysed interview data from an earlier study (Frejd & Bergsten, 2016). The study focused on how professional mathematical modellers worked. For the present study, we analysed in detail how one creator communicated with the people who were to use his models for planning their work shifts. The second case concerned the *citizen's perspective* and was based on reports from a family using an app to reduce their electricity bill. The third case concerned the *operator's perspective* and was based on the use of certain charts to monitor safety in laboratories. Using our definition for mathematical co-productivity chains, our analysis was to scrutinise each case and formulate characteristics of (i) the activities involved, (ii) the interactions taking place, (iii) ethical issues, and (iv) boundary objects in the co-productivity chains.

Results

The perspective of a creator

We describe the first case of a mathematical co-productivity from the perspective of a creator, namely an expert in mathematical optimization. He was interviewed to describe how he developed a work schedule in an organisation for health and social care (Frejd & Bergsten, 2016).

> Creator: When you have a problem… and you want to make it better... it's rare that the problem is fixed, packaged and ready to go. Usually there is a fairly long dialogue with the owners of the problem to try to understand the problem, to understand their language, to find out what kind of data they have. That dialogue can be incredibly difficult because there are usually conditions that are …, so obvious that they don't even say them .... Right now, when we're looking at planning in health and social care, these conditions are coming up all the time... We delivered a schedule where someone worked six or seven days in a row. Of course! They hadn't said a word about it, but it was so obvious to them that you can't work more than five days in a row, because then you start to get tired and that can be dangerous... Things like that come up all the time, ... it's an iterative process, you almost always have to do some kind of experiment first to see if you've got it all... You collect their requirements and then you do the test and then the misunderstandings come up. The misunderstandings are a step in the process.

Here, the creator explained that a dialogical interaction was needed for creating his scheduling mathematics. What he called “the owners of the problem” were the health workers and managers who were to use his planning model. The dialogue was complex, because the creator was insufficiently familiar with the operators’ workplace culture and regulations. He missed constraints that were obvious to the operators. This led him to develop a schedule that did not satisfy the operators. Communication back and forth was essential in this mathematical co-productivity chain. The creator described the interaction as an iterative process, whereby the first schedule model was tested by the operators, rejected and revised. In this way, the operators affected the creator’s mathematical process. The prototype mathematical model served as boundary object to mediate this communication.

Below, the creator explains aspects in the dialogue between creator and operators.

> Creator: First of all, you have to leave your own language behind to a very large extent, ... you make mistakes with mathematical terms, you absolutely mustn't do that, but you have to adapt as far as possible to the person who owns the problem... You have to have some kind of mutual respect for each other's competences and realise

that you don't speak the same language, but you have to try to connect anyway. And you can't look down on people, you have to communicate in their language at that level as much as possible, otherwise it won't work.

For co-productivity, the creator had to show respect, avoid complex mathematical terms and adapt his language to the operators. He also emphasizes that co-productivity is based on trust. Yet, the co-productive process can also raise ethical issues, as mentioned in the following excerpt.

Creator: Scheduling in health and social care is quite often done by the staff themselves or by managers. If you come in and do it for them, then people lose power... Because all of a sudden, you are taking away the power from people who have had the power... to set good working hours for themselves... You may need good managers to do that. But there are also ... those who have been given bad working hours in this working environment precisely because they are not so strong and good at speaking up for themselves. Suddenly, when we arrive ... without taking into account who they are and what their names are, it becomes somehow objective, and they are so grateful that they avoid these conflicts.

This excerpt reveals that the co-productivity chain can change the reality of operators; a new working schedule affects not just their own, but also their families' lives. Daily commitments, such as picking up children at school, may be disrupted by a new schedule. Also, power structures at the work floor can change, for instance, when decision making is transferred to others. When an outsider creates schedules without considering existing norms and conventions, some may lose privileges, while others may gain benefits. Then, ethics comes into play in the co-productivity chain.

The perspective of a citizen

For the second case of a mathematical co-productivity chain describes how the second author's family reduced electricity costs by reading mathematical information in an *app*. During three weeks in June 2023, the family logged their hourly electricity consumption (in kWh) and their use of electrical appliances. The electricity company applied hourly pricing, which means their electricity (in kWh) was cheaper at some hours, and more expensive at others. The hourly prices were announced one day in advance through a mobile phone app. A graph from the app screen on 14 June 2023 (see Figure 1) shows the electricity prices for the next day, 15 June 2023, expressed in øre/kWh (an *øre* is Norwegian currency). There is a price peak at 8-9 am, when households and industries compete for electricity, and a slightly lower peak in the late afternoon. By setting higher prices, electricity companies try to encourage a lower consumption when the network is extremely busy. Through the graph, the family could plan and adapt their electricity consumption for the next 24 hours.

Another graph in the app showed electricity consumption for the past days. The bar graph in Figure 2 shows the citizen's electricity consumption on Wednesday 14 June 2023 (in kWh). The citizen's family had increased their consumption and lowered it again before the 8-9 am price peak; according to their logbook, they had programmed their dishwasher and washing machine to run during these hours. Figure 2 also shows a moment in the afternoon with higher electricity consumption that avoided the second price peak. In the logbook, they reported some cooking and vacuuming between 4-5 pm. In this way, the citizens interpreted the price diagrams and adapted their behaviour to reduce the energy bill and simultaneously, contribute to reducing peaks in the electricity network.

Figure 1: Blue line graph of hourly prices (in øre/kWh) for tomorrow 15 June 2023, diagram captured on 14 June 2023

Figure 2: Red bar chart of the family's energy consumption (in kWh) for yesterday 14 June 2023, diagram captured on 15 June 2023

In analysing interactions between citizens and the company, we assumed the company had a unit with mathematical professionals. These mathematicians used the data on citizens' electricity consumption to analyse how their pricing worked out. With their analysis, they could iteratively create new pricing encouragements to further change citizens' behaviour. The underlying mathematics (data, models, algorithms) remained hidden but materialised in a boundary object, namely, the graphs in the smartphone app. The app allowed citizens to collect data (about prices and their own behaviour) and decrease their bill. In this way, citizens and creators interacted in a mathematical co-productivity chain. Unlike the previous case, they did this without meeting directly. And again, there was an ethical aspect, namely, whether companies should be able to spy into citizens' behaviour on an hourly basis.

The perspective of an operator

The third case of a mathematical co-productivity chain relates to operators in laboratories, who use Levey-Jenning charts to monitor and control the quality of chemical and pharmaceutical test results. In clinical laboratories, this type of work can be critical for matters of life and death.

In the 1950s, industries with large laboratories started to hire statisticians to assist in quality control (Karkalousos & Evangelopoulos, 2015). Rather than testing every product, costs could be lower when taking samples, measuring critical characteristics thereof and comparing these to a certain range. The question was how operators (laboratory workers) could contribute to the statistical quality control. Generally, it was believed that operator's mathematical abilities would be low and lead to errors

(Smith III, 2002). To avoid these, pathologists Levey and Jennings designed the Levey-Jennings chart ("Laboratory Quality Control," 2023), see Figure 3. Based on earlier data and quality criteria, experts (creators) determined a mean and standard deviation. These were used to make seven lines in a chart, with the middle line representing the mean, and dotted lines above and below representing distances by one and two standard deviations. The outer lines, which are three standard deviations from the mean, were not to be crossed. In a Levey-Jennings chart, the centre line is usually coloured green (meaning "accurate") and the outer lines are coloured red (meaning "danger"). A laboratory operator was handed an empty chart. At regular time intervals, (s)he was to measure a sample item and plot the result into the chart. They were instructed to alert a supervisor if the measurement approached the outer limits.

Figure 3: Example of a Levey-Jennings chart

Levey-Jenning charts are still in use today, but now digitally with automated alerts to further reduce errors. In Figure 3, days 6 and 7 are coloured red because two consecutive readings were above two standard deviations; this can indicate a handling problem. Also, on days 20 to 27, there is a sequence of several below-average readings, which can indicate a contamination or equipment failure.

The activities of laboratory operators involved taking measurements, plotting these into the chart, interpreting the chart, identifying points or trends that could indicate quality problems, and alerting experts (creators). The interactions between the operator and the creator takes two directions. On the one hand, a creator determines the values on the y-axis and the criteria for the alerts, which are sent to the operators. On the other hand, the operators take measurements, plot these into the chart, and sends these back to the creator for analysis and further statistical calculations. The interaction between creator and operator is facilitated by a boundary object, namely the data visualisation in the chart, which reduces the need to meet face-to-face. However, an operator may be able to interact with the creator if accuracy needs to be further improved, for example if the Levey-Jennings chart needs to be re-coded to display other alerts. As for the ethical aspects, we observed that this co-productivity chain is based on mistrust in operators' mathematical abilities.

Conclusion and discussion

Based on our analysis of (i) the activities involved, (ii) the interactions between operators/citizens and creators (mathematicians), (iii) ethical issues, and (iv) boundary objects, we identified characteristics of the theoretical concept of mathematical co-productivity chains. See Table 1.

Table 1: Characteristics of activities within a mathematical co-productivity chain

Case	Activities involved	Interactions, Gains	Ethics	Boundary object
Creator's perspective	Collecting data, eliciting information, adapting language, creating, testing and evaluating mathematical models	Two-way, face-to-face, long dialogues, iterative process, mutual respect Creator gains salary, operators get a better planning	Transfer of power over planning	Prototype scheduling model
Citizen's perspective	Interpreting statistics, evaluating and adapting own behaviour, feeding back behaviour data	Two-way, do not meet in person. Citizens gain a lower bill, creators reduce power peaks	Loss of citizens' privacy	Data visualizations (in app)
Operator's perspective	Measuring, plotting, interpreting chart, identifying errors, alerting creators	One way, rarely meet in person. Operator and creator gain salary, employer gains higher quality products	Mistrust in operators' skills	Data visualization (digital/paper)

In this paper, we contend that not all students will become creators of mathematical models. Thus, mathematics education cannot only emphasise the perspective of creators. Jablonka and Gellert (2007) argued that citizens and operators engage productively and adapt their behaviour in different ways according to mathematical models. Our results complement their findings by showing that advanced mathematics appears in society in a variety of ways. Advanced mathematics is complex, and with reference to Engström's (2001) notions of division of labour, environment and norms, we conclude from our cases that advanced mathematical activities take place across environments between people. The new theoretical concept of mathematical co-productivity chains provides new input for discussing aims and content of advanced mathematics education. The concept explains that in society, advanced mathematical activities are part of human collaboration and reveals ethical issues of decision making, privacy or mistrust. Also, it puts an issue on the agenda for research on mathematics education regarding those students who will not become professional mathematicians but study more practical mathematics, such as in vocational tracks. Of course, there are drawbacks to the concept, as a mathematical co-productivity depends on perspectives and contexts, such as the cases selected for this paper. We do not claim that all advanced mathematical activities are the same, but we want to point at the variations when considering the different roles that students will have in their adult lives. Based on the results of the three cases, aspects such as interpreting and applying different types of visualisations and diagrams in various technological tools used in workplace situations and in private life for making personal decisions and acting upon these, could be highlighted more in mathematics education. Such visualisations serve as boundary objects (Bakker & Akkerman, 2014) to translate the mathematics across different environments (Williams & Wake, 2007). The first case showed that the creator (a professional mathematician) adapted his language of

communication and showed respect to his partners (operators) when discussing a prototype of the schedule model. It is a goal of mathematical education to provide students with societal situations connected to vocational mathematics and mathematical literacy, and to show that mathematics is human, social and a tool to reflectively act upon and solve problems in society.

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Mathematics workshops in prison: Design and practice

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This paper presents the design and results of an informal mathematics education project for young adults in an Italian prison. The workshops were designed to emphasize problem-solving with the aim to provide inmates with skills transferable from mathematics to daily life. To overcome language barriers and different levels of schooling, a learning-by-doing approach was adopted. Participants' feedback indicate success in fostering mathematical and psychological empowerment: inmates reported increased patience, greater self-esteem, and improved mental engagement. Furthermore, the workshops promoted social inclusion by encouraging collaborative problem-solving among inmates, prison officers, and staff, highlighting the potential of mathematics to empower marginalized individuals and strengthen social bonds in fragile communities.

Keywords: Informal mathematics education, empowerment, problem solving, experiential learning, adult education.

Introduction

Prison education plays an important role in the successful integration of incarcerated individuals and in their social inclusion upon release (Manger et al., 2010; Ferrarello & Mammana, 2022). Mathematics education can be a powerful agency in these contexts due to its capacity to bridge educational gaps and empower marginalized individuals. In this regard, Byrne and Carr (2015) highlight the connection between perceived competencies in mathematics and improved social status. Andrà and Brunetto (2020) discuss how mathematics can trigger psychological empowerment. Here, we refer to it as longitudinal dynamic of development and attainment of a set of insights, and abilities best characterized as participatory competence (Kieffer, 1984). Roughly speaking, psychological empowerment is the process which turns a person from an object (passive and powerless) to a subject who acts on the world (Andrà & Brunetto, 2020). Moreover, as noted by Brunetto et al. (2021), developing problem-solving skills in the "safe space" of mathematics can be useful to gain confidence in daily life problem resolution, fostering mathematical empowerment. Mathematical empowerment is the development of mathematical skills and the ability of applying them both within formal context (e.g. school) and informal one (Andrà & Brunetto, 2020).

However, implementing a mathematics education experience in prison poses immense practical challenges (Byrne & Carr, 2015; Byrne & Harrison, 2022). Inmates typically have different levels of education and speak different languages, and attendance can be sporadic. Learning may also be interrupted because prison officers recall inmates during activities for various reasons, such as meeting with attorneys or communicating with other officers.

Last year, a collaboration between Politecnico di Milano (henceforth: PoliMi) and Casa Circondariale San Vittore di Milano (henceforth: San Vittore) gave our research group the opportunity to design and test an informal mathematics education program in a high-turnover prison context. The aim of our research was to identify suitable designs and practices that would make the program accessible to everyone, despite the aforementioned obstacles, and trigger empowerment. Moreover, we

wondered how mathematics could be used to foster a wider network of social relations within the prison context, reaching not only inmates, but also prison workers.

In this paper, we present design and findings from our workshops, highlighting the potential of mathematics education to enhance inmates' empowerment and promote socialization and collaboration, overcoming social barriers, in the prison context as a starting point for a social inclusion upon release. In this paper, we mean social inclusion as the equitable participation of individuals in social, economic, and political aspects of life (Rawal, 2008).

Methods

Research context

The collaboration between PoliMi and San Vittore started in 2021. The people housed in San Vittore are awaiting trial, after which they are either released or transferred to a permanent prison. The jail section we worked at hosts male adults between 18 and 30 years of age. Due to current policies, inmates remain locked in their cells for most of the day. This condition leads to a strong sense of powerlessness and self-deprecation among inmates (Brown, 2020). Since 2021, San Vittore hosts a PoliMi office (called "Off Campus") inside its premises, where researchers and students involved in prison projects can stay and work during working days. The PoliMi Departments that have so far been involved in this collaboration with San Vittore are Architecture, Design, ICT and Mathematics. Last year, all four of them jointly contributed to a project named "METHEXIS: Mathematics, narratives and generative spaces for extremely fragile contexts", funded by the 2023 Polisocial Award, whose overarching objective was to foster greater participation of inmates and workers in the young adult section in forms of social life and mutual support. As part of the METHEXIS project, an informal mathematics experience was designed to provide a neutral space where people with different backgrounds and with different roles in prison (e.g. inmates and officers) could work together to solve puzzles, setting aside the role barriers of prison life.

Conceptual framework

The research was conducted in accordance with the principles of Design-Based Research (DBR), defined by Wang and Hannafin (2005) as a methodology aimed to improve educational practices through systematic and iterative processes of analysis, design, development and implementation. Findings are based upon collaboration among researchers and practitioners in real-world settings, leading to the formulation of principles or theories. Based on this methodology, we designed a workshop, implemented it (May-June 2024), and collected data that served our analysis. Our subsequent evaluation led to a refinement of the initial workshop design, the implementation of a second version (July-August 2024), and a new analysis of the results.

The initial workshop design was based on previous experiences of mathematical instruction in immigrant communities (Brunetto et al., 2021) as well as existing literature about the teaching of mathematics in prison (Byrne & Carr, 2015; Maffia & Decembrotto, 2022). Furthermore, we considered experiential learning theory and informal mathematics education, where student learning is attained through personal experience and involvement (Kolb, 1984) also out-of-school (Kuş, 2024). In line with Manger (2010) we conjectured that experiential learning activities assume an even more

important role in prison. Indeed, convicted people are often relieved of any work responsibilities. Therefore, having a space where they can learn by doing can be an important incentive. We were also inspired by Maffia and Decembrotto's (2022) recent work, where the utility of assigning tasks with multiple levels of difficulty is highlighted. Finally, both in the way of conducting the meetings and in some of the proposed contents we took some hints from Liljedahl's (2021) thinking classroom.

Design and first implementation

Considering the above-mentioned studies as a conceptual framework, we proposed an informal mathematics workshop based on learning-by-doing principles (Kolb, 1984), with a focus on problem-solving. Our purpose was both offering stimuli to be active thinking about and doing mathematics and opportunities to cooperate regardless of culture, nationality, religion and so on. High turnover, sporadic attendance and differences in languages and mathematical knowledge (average schooling age being around 13) led us to design a short-length workshop consisting of six meetings that were not preparatory to one another and did not require fluency in Italian or previous mathematical instruction. The maximum number of participants was 15 and at least two tutors were always present.

Each meeting was structured into three parts. The first one served as an introduction to buy some time while waiting for all the people to arrive. We proposed intriguing challenges, preferably preparatory or at least related to the next parts. The second part was the core of the workshop: a problem-solving activity, possibly with several steps of increasing difficulty, according with the "flow" theorized by Liljedahl (2021), and following experiential learning principles (Kolb, 1984). In accordance with the experiential learning cycle, the main activity presented a problem in a concrete way (concrete experience) and allowed ample time for participants to reflect on the problem (reflective observation) and to devise solution strategies (abstract conceptualization); once the solution strategy had been identified, participants were presented with further instances of the problem to test their full grasp of the solution strategy (active experimentation). Both in the first and in the second part we invited participants to work in pairs or small groups standing in front of a vertical blackboard, as suggested in the "thinking classroom" by Liljedahl (2021). In Table 1, introductory quizzes and core activities of the first workshop are summarized. The third part was dedicated to a metacognition activity: we will return to this in the section about data gathering. During the final greetings, attendants were given a take-away puzzle that they could try to solve on their own over the following days. We handed over the puzzles also to the prison officers, hoping to create a connection between them and the prisoners.

Table 1: The activities proposed in the 1st edition of the workshop

Day	Introduction	Main activity
Day 1: The desk calendar	Some matches are displayed to form a number as in a digital screen. Moving only 2 matches, what is the highest/lowest number you can get?	Which digits must you write on each face of the cubes to build one of those desk calendars where the date is displayed on the faces of 2 cubes?
Day 2: Symmetry puzzles	Having several cutouts of shapes consisting of 6 squares joined together, identify which of these are developments of the cube and which are not.	Given a square sheet with some dots on it, you can fold it, as many times as you like, so that if you make only one hole, it is at all and only the dots.

Day 3: Points of view	Two players have the same number and type of cubes. One makes a construction, then gives instructions to the other to reproduce it without looking at it.	Solve different skyscrapers grids.
Day 4: Eulerian Graphs	Draw a small house without ever taking the pen off the paper and without going over an already drawn line.	Analyze several graphs to find the criterion for which it does or does not admit a Eulerian path.
Day 5: Eulerian Graphs	Explanation of the historical background of graph theory and the Seven Bridges of Königsberg problem.	Same as in Day 4.
Day 6: Strategy games	Play tic-tac-toe in one of the grids attached to the wall.	Find a winning strategy for the *nim game* starting with 7 bricks, in which players, in turn, can take 1 or 2 bricks. The winner is the last player to move.

Data gathering

In the third part of each meeting, we asked participants a metacognition effort: to fill out a journal answering the questions "What I did and what I learnt" and "What I liked and what I did not like". The purpose was twofold: help them fix ideas in a context where everything is transient and collect feedback. Moreover, at the end of the whole workshop, they filled in a questionnaire where we asked if they liked it, which meeting they liked the most, how they found the level of problems (both in terms of understanding the task and in solving it) and their preferred mode of work (individually, in pairs or in groups). For ease of access, we wrote it in both Italian and Arabic. The Likert scale ranged from 1 to 4, to avoid the medium value, and we opted for a graphical visualization of it, using "stars" as in reviews. After each meeting, the tutors also filled in a journal with observations about the classroom setting, how the activity was conducted, participants' attitudes, difficulties (both logistical and content-related) and interventions. These three data served us for the analysis of the first workshop, as required by the DBR.

The open-ended questions of the journals were analyzed according to the conceptual framework. In particular, the empowerment lens allows us to figure out if and what elements of the workshop trigger the empowerment, namely the participants manifest an experience of control on mathematics, on relationships, or on the future. Examples of such analysis are provided in the next sections.

Second cycle of DBR

We report here the most relevant findings from the analysis of tutors' journals of the workshop's first edition and changes in light of our evaluations. In line with Liljedahl (2021), we confirm that encouraging participants to work by standing in front of a vertical blackboard was very successful both in making them feel involved and in reshuffling working groups, to encourage collaboration between all; hence, we decided to improve this working method in view of the second edition. As

indicated in the literature (Maffia & Decembrotto, 2022), it is useful to give tasks that can be tackled step by step (e.g. the skyscrapers' grids and the Eulerian graphs). We noted that when someone understands the game strategy or the solution, they are enticed to seek challengers; this being a symptom of proactivity, we took care, in the second edition, to stimulate competitiveness to an appropriate and positive extent. We also observed that participants were often not autonomous in grabbing the materials to work on and tutor guidance was necessary. Thus, in the second edition, we looked into ways to encourage participants to be more proactive (e.g. leave the material on a table so that they stand up to get it instead of us distributing it desk by desk). To encourage participants to take better care of the material and to enable them to keep track of the path they would be taking, we planned to provide participants with a cardboard folder to collect all their materials in the second edition. Moreover, since in the previous edition some inmates came to us asking for the solution of the take-away riddle, we decided to put more emphasis on them by asking, at the beginning of each meeting, if anyone has solved the riddle provided the previous time. The metacognition activity of journals turned out to be too difficult for some of them, so in the new edition we decided to ask them just to write on a sticky note a free comment about the meeting at the end of it. Regarding questionnaires, using stars in the Likert scale turned out not to be a good idea both because the language of the reviews was not familiar to everyone and because in Arabic you read from right to left. We thus kept the same questions and same answers' range from 1 to 4 but changed the answers' format into multiple choice.

Considering the findings above, we implemented a second edition of the workshop, whose introductory challenges and core problems are described in Table 2. We made only a few adjustments to the schedule after the first edition, changing the order of the activities and adding a new activity (Day 4) devoted to solving logic puzzles similar to the matches' puzzle provided at the beginning of Day 1 in the first edition, which had proved intriguing for all participants: it seemed appropriate that this activity should not only be introductory but main. We designed this with the aim of supporting a challenge dynamic among participants, that we had noticed as positive, and to stimulate their proactivity, leaving them free to engage in the puzzles that they perceived as most appealing.

Among the changes we introduced in the second edition, having people standing in front of a vertical blackboard worked well, whereas asking them to take good care of the material in the cardboard folder failed almost completely. Referring to the solution strategy of the take-away puzzles at the beginning of each new meeting highlighted continuity throughout the six meetings.

Table 2: The activities proposed in the 2nd edition of the workshop

Day	Introduction	Main activity
Day 1: The desk calendar	Enter the numbers 1 to 8 in a specific grid, satisfying some constraints.	As in the 1st edition, Day 1.
Day 2: Strategy games	As in the 1st edition, Day 6.	As in the 1st edition, Day 6.

Day 3: Eulerian Graphs	As in the 1st edition, Day 4.	As in the 1st edition, Day 4.
Day 4: Mixed puzzles	Card tricks played by tutors.	Six different puzzles on the wall of the room and participants had a study card to mark their solution.
Day 5: Symmetry puzzles	Counting the numbers of triangles in symmetric figures.	As in the 1st edition, Day 2.
Day 6: Points of view	How to fold a sheet into a square and then the square into an equilateral triangle.	As in the 1st edition, Day 3.

Results

According to the student log, the average number of participants and the number of them attending at least 66% of the workshop were respectively 10 and 4 in the first one, 8 and 7 in the second one.

In the final questionnaires the most highly ranked meeting in the first edition was the one about Eulerian graphs, while in the second edition was that on skyscrapers grids. One participant reported the following motivation for enjoying the workshop on graphs the most: “because now I know the street rule”, referring to his having found the criterion for a graph to be Eulerian in his own words. Such a statement is evidence of perceived control on mathematics, at least temporarily. Considering the answers to the second questionnaire only (being it more reliable), the level of problems was judged fair (not too easy nor too difficult) by 87%, and 74% of participants always understood the task (37% at the very beginning, 37% after a while).

The qualitative data collected via journals and sticky notes highlight some notable findings. The role of patience in facing problems (“I confronted my patience and myself by doing these exercises”), participants’ perception of using mind and time well (“I enjoyed pushing myself more and more into more difficult exercises and challenging myself”; “It was stimulating for the mind”), increasing self-esteem (“I found out that I am smarter than I thought”). Such statements refer to controlling themselves: patience, challenging, and feeling smarter. There were also some comments on the opportunity that mathematics gives you in escaping from the vicious circle of everyday thoughts (“There was less noise and more concentration”; “I detached my mind a little, re-entering my logical world”) and some others on purely mathematical topics (“I saw new geometric shapes”; “Mathematical tricks. I learnt the binary system, winning at tic-tac-toe, winning at cubes”). Those are evidence of acting on mathematics, either thinking about or doing. Finally, some wrote about mathematical skills that can be transferred to daily life, such as “I learnt that to everything there is a reason and to everything there is a solution”; “I have learnt not to take anything for granted”; “I learnt to look outside the box” and the utility of mathematical instruction for their future, noting a link between the activities completed in the workshop and problems encountered in real life (“They are useful for our future, even outside [prison]”). Those manifest the control of using mathematics outside the specific context.

As mentioned before, one of the aims of our project METHEXIS was to offer opportunities for social life and mutual support. In this regard, we report the following comments: “I met new people”; “[I spent] a beautiful afternoon with a beautiful family”. Moreover, in both the final questionnaires, all participants indicated they preferred to work in pairs or in groups. Other feedback we had concerns the involvement of other prison stakeholders. As mentioned earlier, the take-away riddles had a twofold purpose: firstly, to give prisoners something to think about to kill time in the cells and, secondly, by also distributing them to prison officers, educators and healthcare staff, to create an invisible network of connections among minds all thinking about the same mathematical riddle. At first, we expected them to maintain a form of detachment and to not put themselves on the same level as the inmates. Over time, however, by making ourselves known as “the mathematicians of PoliMi” and establishing a riddle delivery as a ritual, their attitude changed. Seeing their growing interest, we started hanging a weekly riddle outside our office and left copies in small format to be taken freely by anyone interested. To our surprise, soon after that we observed inmates and police officers solving riddles together. This is evidence of the large potential of mathematics to create a neutral space where to free one’s mind and get involved and it seems to indicate very interesting possibilities.

Discussion and conclusion

In this paper, we presented an experience of informal mathematics education designed for young adults in a high-turnover jail in Italy. Different linguistic and mathematical backgrounds and discontinuous participation have represented crucial challenges for designing a successful workshop and, indeed, have had a strong impact on the way the meetings were conducted. Recalling the aim of our research to find an appropriate design and practice for an informal mathematics educational experience in prison, tutors’ observations and participants’ feedback suggest that, even in extreme conditions of low literacy and high turnover of inmates, there is considerable potential for mathematics activities based on problem solving and learning-by-doing principles to be empowering for them. We observe an important interplay between mathematical and psychological empowerment mediated by experiential learning activities: the comment “I learnt to draw possible and impossible roads, like life” is symptomatic of this. In drawing graphical patterns and identifying rules behind them, the inmate was appropriating a theorem. This could be an indicator of mathematical empowerment. This research indicates that psychological empowerment came from exploiting mathematics, reflecting upon roads which are not only graphical patterns but signify real life.

Researchers’ observation on the dynamics between inmates, prison officers, health workers and educators indicate that mathematics can offer a neutral ground for overall networking within the prison. This was made possible by devising take-away riddles that were accessible to anyone, but at the same time engaging. Further studies on mathematics education through experiential learning can help support its important role in creating a positive and inclusive social fabric in socially marginal and difficult contexts.

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Teachers' perceptions of educationally significant vocational mathematics knowing

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In construction work vocations, geometry evidently plays a significant role. However, the question of how geometry should be understood and used within vocational activities, and consequently how the topic should be treated in vocational education remains an important question of debate. In this study phenomenographic interviews were conducted with the aim to describe vocational and mathematics teachers' various ways of perceiving the core characteristics of "educationally significant vocational mathematics" knowing in geometry. The result shows a set of categories, which describe qualitatively different perceptions of this phenomenon. Taking all of these perspectives into account the study encourages a view of "vocational mathematics knowing" that goes beyond simplified procedures, specific concepts and context bound reasoning, and acknowledges mathematics understanding as an essential component of students' holistic vocational "bildung".

Keywords: Geometric concepts, vocational education, phenomenography.

Introduction

Vocational mathematics teaching and learning is a sub field of mathematics education research in which the context by definition plays a crucial role. Mathematics utilized in workplace activities is often embedded in vocational practice, making it an elusive component of vocational proficiency (Christidis, et al. 2024; Hoyles, 2010). Such tacit knowing, which could be difficult to identify and articulate, naturally poses significant challenges for teaching, and is therefore a critical area of educational research. However, the area has received rather scant attention, partly due to this complexity. Furthermore, there are significant differences in the use of mathematics across various vocations and between vocational education systems in different countries (Bakker, 2014). This study is carried out in upper secondary vocational education in Sweden, where a general mathematics course with a clear intended focus on vocational applications, have been mandatory at the vocational programmes since 2011. Previous research has highlighted both the significance of mathematics as a key component of vocational competence (Christidis et al., 2024; Muhrman, 2016) and the role of vocational relevance in enhancing students' motivation to learn mathematics (Muhrman, 2016; Dalby et al., 2016). Moreover, vocational mathematics knowing has been described as a complex learning objective, involving mathematical as well as vocational aspects, and within both of these areas, theoretical aspects as well as practical. The connection between those aspects, is also referred to as an intricate field of knowledge, which should be in the focus of attention when planning vocational mathematics teaching (FitzSimons & Boistrup, 2017). Furthermore, the fluctuating role of mathematics in vocational education has been highlighted, for example by Dalby and Noyes (2022), who exemplify a variety of requested mathematics qualifications for vocational students, which have been established and later discarded. Vocational mathematics thus seems to be an educational content for which the objectives are not clearly agreed and defined. Therefore, it is urgent to further

investigate, not only the teaching of vocational mathematics, but also perceptions of the very nature of the knowledge to be taught.

Geometry is both one of the main pillars of mathematics and an area of certain interest in construction work vocations. However, addressing real-world spatial problems requires partly different geometrical properties than those typically studied in mathematics, and is therefore a challenge in vocational education (Bessot, 2000). Boistrup et al. (2018) describe geometrical knowledge within a construction work context as an intersection between mathematics and vocational subjects, and pinpoint that mathematical knowledge is used both for application in practical problems, and to explain aspects of the vocational context. Also, Bessot (2000) conclude that geometry could be used both for solving practical tasks, and as a part of the discourse, which enables theoretical justifications of the practices taught. The concept of similarity is a fundamental geometric idea, which is important for further studies in mathematics (e.g., trigonometry and calculus) as well as for vocational practice (e.g., scale drawings and slope of e.g. roof or drainage). Starting from the concept of similarity and its applications in a construction work context, this study aims to provide a multifaceted description of vocational and mathematics teachers' perceptions of *educationally significant vocational mathematics knowing*. This is explored using a phenomenographic approach, which focuses on variation in peoples' perceptions of phenomena. In this way several facets of a phenomenon could be exposed, taking its' complexity, nuances, and contradictions, into account. The research question is: In what ways do teachers perceive *educationally significant vocational mathematics knowing*?

Theoretical framework and method

Phenomenography is an empirical and exploratory research approach, which aims to describe qualitative differences in peoples' perceptions of phenomena (Marton & Booth, 1997). Like phenomenology, phenomenography is based on the assumption that a meaningful phenomenon is constituted in the relation between the experiencer and what is experienced. The phenomenon is thus not a pre-existing entity but is perceived and constituted in this relation. According to phenomenography, a phenomenon can be perceived in a limited number of qualitatively distinct ways. Each individual perception contributes to the understanding of the phenomenon as a whole. That is, the aim of phenomenographic research is not to find a single "truth" about the phenomenon, but to describe the variation of perceptions (also called conceptions or ways of experiencing) that can be observed within a group of people. A phenomenographic result is typically presented as an outcome space of categories, which are often hierarchically related where higher-level categories reflect more inclusive or complex perceptions (Marton & Booth, 1997). A common type of phenomena studied within the phenomenographic tradition, are those that correspond to a specific subject content, e.g. how children perceive number. The study described in this paper, concerns a partly different phenomenon. Here teachers' perceptions of students' appropriate knowing, is considered. The outcome space, thus represent different perceptions of the knowing of a content, rather than different ways of knowing (or understanding) a content. That is, this study does not aim to articulate peoples' actual vocational mathematics knowing, which according to previous studies, is often embedded in vocational practices, but to investigate the ways in which this knowing is perceived at a meta level. The term *knowing* is used here in accordance with, for example, Carlgren et al. (2023) indicating a form of knowledge that goes beyond facts and concepts, encompassing a

disposition for action in real-world contexts. However, *knowing* should rather be seen as a mere labelling of the phenomenon than a pre-determined definition. In a similar way, the term *bildung* serves as a conceptual label for category 6, referring to a process of transformative professional development that encompasses values such as reflection, independence, and active engagement (e.g., Klafki & MacPherson, 2000). For this study, the phenomenographic research process was considered particularly appropriate, since it enables a categorization, which derives directly from the empirical material, and which highlights both important features and significant differences in teachers' perceptions. To suit a phenomenographic analysis, the empirical data should be focused and well constrained, and at the same time exhibit a richness of perspectives (Marton & Booth, 1997).

To this end, the empirical data was generated through in-depth semi-structured interviews. Eight vocational teachers (labelled V) and eight mathematics teachers (labelled M) from seven different schools were interviewed. The selection was based on a combination of purposive and convenience sampling. To ensure a consistent focus, all interviews started from a common set of tasks. Four mathematical textbook-type tasks were designed, aiming to cover different aspects of the similarity concept and at the same time relate to a construction work context. The first task concerned scale drawings. The second task dealt with the slope of a drainage, expressed as a ratio (1:200) and as a percentage (0.5%). In the third task the SSS similarity theorem was applied to enlarge the 3:4:5 triangle and thereby get a right angle. The fourth task involved constructing a porch roof, matching the slope angle of an existing house roof. To achieve this, the SAS similarity theorem, the concept of slope**,** or scaling, could be applied. The intention was that the group of tasks together should involve theoretical understanding of a constrained geometric topic (similarity), as well as potential relations to a broader area of mathematics and to a vocational context. By giving rich opportunities to make connections both within mathematics and within the vocation, the tasks were considered to constitute an appropriate starting point for the interviews. Open ended questions, regarding students intended learning, were asked in relation to the tasks. The interviewees' answers to the questions were assumed to reflect how they made sense of the phenomenon. The interviews lasted about 1h and were audio-recorded and transcribed verbatim.

Drawing on Åkerlind (2005) the analysis process proceeded as follows: To begin with the transcripts were read several times in order to identify a common phenomenon. Then, the sections of the interviews, which were significant for, and maintained a specific focus, in relation to the studied phenomenon, were selected as unit of analysis. The entire interviews and the selected quotes were read in detail to grasp the meaning conveyed in each quote, and to detect distinctive differences in meaning between quotes. Categories were formed based on the reading, and then reformulated, during a process of reading, analysing and comparing interview extracts and category descriptions. To enhance reliability, both authors, together with other researchers were involved at different stages of this process. In several phenomenographic studies it has been observed that the same person can express different or even contradictory perceptions of a phenomenon. The quotes were, thus, not seen as expressions of individual perceptions, but rather as making up a "pool of meaning" (Marton & Booth, 1997). Within the phenomenographic tradition a few different ways of analysing data have been described, all of them aiming to provide a set of categories of description, empirically grounded, clearly distinguished, logically interrelated, and being as few as possible (Åkerlind, 2005). The

current analysis was carried out focusing alternately on the "what" and "how" aspect of the voiced experience, in order to capture key features of the categories. Phenomenography, like gestalt theory, assume that people tend to experience phenomena as meaningful wholes, and that the way they attend to specific parts, reflects how they make sense of the whole. That is, by noticing "what" teachers chose to focus on, and the ways in which these parts are discussed, conclusions could be made about "how" the phenomenon is understood.

Result

As a result of the analysis an outcome space of six categories of description emerged. The categories should not be seen as mutually ranked in relation to a preconceived view of the phenomenon, but rather as complementary perceptions, representing different facets of the phenomenon. *Educationally significant vocational mathematics knowing* is perceived as:

Category 1: Procedural retention of isolated facts and methods

In this category, the phenomenon is perceived as a toolbox of useful, memorized procedures, facts and rules, which could be utilized to perform a specific task. This perception is characterized by reference to isolated skills, number series or techniques which could be used, without necessarily understanding the mathematical rationale behind them. Typical for this perspective is also that mathematical competence, needed in vocational practice, is described as mainly "basic arithmetic skills" (V1, M7). When describing what students need to understand the focus is on knowing how to do, and students learning is discussed in terms of "it's something you have to imprint" (M1) and as "it should become second nature" (V1).

M3: They can learn, maybe, **certain procedures and things like that** (…) if they see 6.5 cm on a drawing and they know the scale, then they can remember to multiply by 50 or multiply by 100, for instance.

V1: You take the height measurements in cm, and then the difference, and then you get, in this table, the roof slope (…) **They ought to be able to learn this:** If you know that you should have one meter in level, like that, and a height there and a height there [pointing]. **That's how you get the roof slope**.

These quotations indicate that, handling the task, is in the foreground. In contrast to the next categories, this category focuses on retention rather than comprehension, and it relates the phenomenon mainly to vocational practice rather than to vocational experience and education.

Category 2: Practice based grasp of specific vocational tasks

Here *educationally significant vocational mathematics knowing* is perceived as context bound understanding, a vocational skill, to be learned by hand and used in a specific vocational situation. The vocational task is in the foreground, perceived as an integrated whole, in which the mathematical part is not discerned from the context. In contrast to category 1, however, students intended knowing is not only referred to as remembering what/how to do but rather as understanding in practice, by grasping it with the hands. This perception is expressed in the interviews through a focus on practice-based knowing, developed in specific situations, using concrete tools and materials.

V3: They learn quite easily to calculate the meter. So, if the pipe is 6 meters, how much should it be from that end to this end then? Then they divide it into meter segments.

(…) They usually work with **6-m long pipes, and 1m parts**. We have made it like that, so that it should become pretty easy for them. (…) When they come out to the workshop, (…) **they can do it with their hands**. Then they get a **completely different understanding**. (…) You need to work with it there and then, like, now we're working with this, and explain it.

That is, when discussing how the concept of slope might be explained, reference is made to specific pipes of certain dimensions, and it is pinpointed that students get a completely different understanding when they might work with their hands and learn in a concrete situation.

Category 3: Conceptual understanding of vocationally relevant mathematics

In this category, generalized understanding of geometric concepts, is in the foreground. This perception is, for example, indicated through an emphasis on the importance to "grasp the essence" (V2) and "explain the connections" in contrast to just "learn things routinely" (M2). Students intended knowing is referred to as "understanding why" rather than just "how to do" (V2), and as "truly understanding concepts theoretically" rather than "understanding in a context" (M6).

M2: I see that many of them may figure out how to do but may still have difficulties to **really understand** what is happening. (…) There are probably many students who struggle to **really, deeply understand the essence of this relationship**. (…) Well, you can learn all kinds of knowledge very routinely by heart, and then you can learn things... well, **learn it deeply**. And that's really what we want.

M6: It doesn't happen automatically that you understand the concept of ratio, that you can apply it to other tasks later (…) it is, somehow contextually learned in some way. (…) **If you truly understand the concept** of ratios (…) you can **solve many different problems** (…) To become a problem solver, you have to understand the concept theoretically (…) to know mathematics, not only understand the concept in a context you need a more theoretical approach.

In contrast to former categories, there is thus a focus on mathematical concepts, generalized across different situations. However, in contrast to both category 2, and the next coming categories, the relation to, and understanding of, the vocational context, is not focused in this category of description. Furthermore, in contrast to category 4, the ability to apply geometric concepts in a vocational context is unproblematized and taken for granted.

Category 4: Conceptual connection between vocational and mathematical contexts

This category focuses on the connection between mathematics used in the classroom, and mathematics used in vocational practice. The application of geometric concepts in a vocational context, is in the foreground and referred to as a non-trivial part of vocational mathematics understanding. Teachers, who voice this perception, typically focus on students' difficulties to understand that mathematics in the classroom and mathematics used in vocational practice is "the same mathematics", and suggest that, seeing this connection, provides an enhanced way of handling vocational tasks, as well as an increased understanding of mathematics.

V4: You have to help them **to see that connection**. And then sometimes, suddenly: 'Oh well! That's right! It's, kind of, the same thing as in the maths textbook, even though we're now in the construction hall, on the floor. (…) When they get this: 'Aha!', as I said, the connection to the maths textbook, like: 'Oh well! Okay! This is how we can use it! Now! Oh! Now I will calculate!'...and that they, **in this way, really understand** what they are doing.

V5: We use percentages, for example concerning humidity in concrete. And then suddenly a student was saying: 'What! **It's the same percentage** as in maths!' So, it is such a difference. It's something that you do in math over there. But **this is for real**. (…) We got rid of this: math over there and the math here (…) it is **the same mathematics** it is actually **not anything new**.

Emphasising these aspects indicates that the transfer of knowledge between mathematics classroom and vocational settings, is seen as an essential, non-trivial part of vocational mathematics knowing.

Category 5: Fusion of vocational and mathematical knowing

In this category, educationally significant vocational mathematics knowing is perceived as an integrative process, where mathematical concepts, vocational understanding, and workplace experiences are combined to address complex vocational challenges. Geometry is referred to as an essential, but insufficient part, which indicates that vocational mathematics knowing is seen as a field of knowledge beyond general mathematics. At the same time geometric aspects are discerned and highlighted. Teachers convey this view by giving examples of complex vocational problems, in which both mathematical and vocational aspect have to be considered, and by emphasising all of these aspects as essential components. That is, various aspects are simultaneously in the foreground, vocational as well as mathematical, theoretical as well as practical. Moreover, the integration of those aspects is also perceived as an essential aspect of vocational mathematics knowing.

V4: When you calculate wallpaper consumption, for example, you can't, simply, use area calculations, because **there are so many other things to take into account** (…) vocational skills, or vocational experiences, things you cannot learn by reading.

V9: I think the context might be the most difficult part. (...) it **adds another dimension**, like. It's really about problem-solving, like in all practical professions (...) [During mathematics lessons] they probably go more in-depth (…) and we focus more on **the matter at hand**. Of course, **the depth is really important** for them to grasp (…) [in the construction site] we have the **theoretical and practical parts at the same time**.

These quotations indicate a focus on vocational knowledge and experience, as well as mathematics understanding, together with the ability to grasp how all of these aspects are related, in order to form a coherent understanding of complex authentic vocational mathematics problems as a whole.

Category 6: A significant part of vocational *bildung*

In the final category, *educationally significant vocational knowing in geometry* is seen as a part of an individual's vocational *bildung* involving, not only factual knowledge and vocational skills, but also reflection, creativity, adaptability, and development. This perception is voiced in the quotes by referring to an increased level of mathematical knowing, as "another way of seeing" (V5). Moreover, a more in-depth mathematical understanding is suggested to entail an ability to handle a variety of situations in creative ways, and to become confident in the vocational role. Compared to previous categories, the focus thus extends beyond immediate demands to a more holistic view of education.

V5: You get another eye (…) It is a matter of **confidence and security** as well. 'I have made the calculations, and it will be all right'. It's like a certificate, **it is about knowledge**. (…) There are various possible ways to go, but the more mathematics you understand… well, **you can see things in a different way** (…) Basically, that is what knowledge really is about.

M2: I would find it valuable to understand the mathematics behind (…) I believe it can help them to **think more broadly**, when everything is not straight (...) If you really understand what you're doing, it's easier to **adapt to various situations**.

M8: …I really think it makes a huge difference, when you get out into the working life. (…) It gives you a **capacity for action** (…) You become more **independent**. (…) As a result of this kind of problem-solving mindset (…) they have a **different way of thinking.** (…) There's an understanding that there are multiple ways into a problem (…) they get **more tools** to handle a task.

The focus on these aspects indicates that the phenomenon is seen as a part of an extended vocational *bildung*, which in turn provides an opportunity to understand, evaluate and solve a variety of vocational mathematics challenges in creative and adaptive ways.

Discussion

This study aimed to explore vocational and mathematics teachers' various perceptions of *educationally significant vocational mathematics knowing*. The result presents six categories of description which illustrate diverse perceptions of this phenomenon. These various perspectives provide insights into the potential role of geometry in vocational activities and may thus inform decisions about mathematical learning objectives. In an interview study, by Christidis et al., (2024) students' experience of mathematics as an aspect of vocational knowing, is explored. The study confirms that while students often recognize mathematics as a part of vocational knowing, they may still lack awareness of its role for future development of the vocation. The authors highlight the significance of curriculum and teaching for preparing students to identify mathematics in vocational activities. How vocational and mathematics teachers identify, understand and articulate vocational mathematics knowing, is thus a critical factor in the enactment and development of vocational education. Previous studies exploring vocational mathematics knowing describe it as a complex competence, frequently embedded in vocational practices (Christidis et al. 2024; Hoyles et al. 2010). Moreover, the importance of considering both mathematical aspects and the vocational context, along with the ability to integrate mathematics into the workplace setting, has been emphasised (FitzSimons & Boistrup, 2017). The result of this study shows that teachers to various extent and in various ways acknowledge both mathematical and vocational aspects of vocational mathematics knowing, and the relation between them. The richness of perspectives in the outcome space shows that vocational mathematics knowing is not merely perceived as basic arithmetic, but that several other aspects are also appreciated, such as relational understanding of geometric concepts, and practice-based perceptions of geometric aspects in vocational tasks. Moreover, the result highlights problems of transfer between the mathematics classroom and the vocational setting and illustrates the complexity of integrating mathematical and vocational knowledge in vocational problem solving. Additionally, it suggests a view of vocational mathematics knowing as an essential component part of students' holistic vocational *bildung*, a means to improve the ability to engage in vocational activities, by developing the eye through which the world is seen. A view, which illuminates aspects of vocational mathematics knowing such as creativity, confidence and adaptability, aspects which in the prolongation might enhance further development of workplaces and vocational competencies.

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Not just threats and weaknesses: Seeking opportunities for the teaching/learning of mathematics in prison schools

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Scientific literature about teaching/learning mathematics within the context of schools in prison is very scarce. Framed by Educational Design Research, the Learning Math in Prison project started with an exploration phase of the context, realised together with Italian mathematics teachers in prison schools. The analysis conducted till now pointed out several categories of characteristics of the process of teaching/learning mathematics in Italian prisons. With the desire to better understand the nature of this process, more than listing and defining these categories is required. In this contribution, we wonder what the relationships between these characteristics are, focusing on some of them in particular. Alongside several weaknesses relating to the specific context, mathematics teachers identify particular opportunities that may prove helpful to support design principles for mathematical tasks within the prison context.

Keywords: Prison education, theory-driven categories, data-driven categories, SWOT analysis, mathematics teachers.

Introduction

The literature on teaching and learning mathematics in prison is very scarce. There is a notable gap in research on mathematics teaching and learning in prison contexts (Gagnon & Barber, 2014), despite mathematics occupying a unique position among the subjects taught, acting as a source of personal satisfaction and a gateway to future educational or employment opportunities (Ahl & Helenius, 2021). In this type of context, many of the usual peculiarities of teaching to adults (Gal et al., 2020) intersect with other issues related to managing the process of learning mathematics for vulnerable adults in a non-native language, dealing with non-homogeneous class groups, and sustaining very different motivations for learning (Gagnon & Barber, 2014; Byrne & Carr, 2015; Gal et al., 2020; Ruhil & Gut, 2019). The literature on teaching/learning mathematics in prison is characterised by discontinuity in class attendance because students come from backgrounds marked by previous failures, low levels of self-efficacy and a negative view of mathematics (e.g., Gagnon & Barber, 2014; Byrne & Carr, 2015) or provides only general data about students' attainment (e.g., Ruhil & Gut, 2019) and does not provide recommendations for educational practice. Consequently, in order to address this gap, two years ago the Italian national research project "Learning Math in Prison" (LeMP) was launched. The LeMP project is an interventionist Educational Design Research project (EDR, McKenney & Reeves, 2018) which involves five universities and teachers from three different prison schools. As usually happens for EDR, our LeMP project is organised into three main phases: analysis and exploration, design and construction, evaluation and reflection (e.g., van den Akker et al., 2006; McKenney & Reeves, 2018). Each phase involves interaction with practitioners and contributes to an intervention's theoretical understanding and progressive development (McKenney & Reeves, 2018). Therefore, the general aim of LeMP is to fill this gap by developing a set of design principles for creating teaching materials for Italian prison schools based on close and

intensive collaboration between researchers and practitioners (teachers involved). Within the LeMP project, the prison context is framed as a context of multi-complexity. This term is used in medical research to refer to patients with multiple (often chronic) diagnoses, whose cure cannot be the simple sum of the therapies for each pathology. In our view, this term only emphasises the high degree of complexity in the prison context, resulting from the complex interaction between personal, relational and structural factors, not necessarily attributable to persons in prison. We are convinced that juxtaposing already available design principles developed in other contexts could not be sufficient.

Starting from these considerations, in this paper, we present the analysis of the initial exploration phase of our research framework, which was based on a systematic literature review and teachers' interviews. Specifically, this article reports on the first phases of a broader work stream that aims to describe practitioners' perspectives on their work context (Maffia et al., submitted). McKenney and Reeves (2018, pp. 78-79) pointed out that "collaboration with practitioners is sought to shape a better understanding of the educational problem to be addressed, the target context, and stakeholder needs". For this reason, we want to fine-grain describe the context, as no such description exists in the literature. In this contribution, we delve, from a theoretical point of view, into some of the categories identified in (Maffia et al., submitted) and ask whether they are disjoint or, if not, what are their mutual relationships. A more precise Research Question is presented after the next section.

Theoretical Framework

Based on the analysis of available literature and the data collected in the first exploration phase of our EDR, we have categorised characteristics of the teaching/learning mathematics process in the prison context (Maffia et al., submitted). Below, we present those categories. Then, to understand the relationship between the presented categories, we use SWOT analysis as a framework for our research.

Theory-driven and data-driven categories

To highlight the aspects that characterise the process of teaching mathematics in prisons, we identified and, at the same time, validated categories linked to the most discussed topics among mathematics teachers; we distinguish between *theory-driven categories* and *data-driven categories*. The *theory-driven categories* are the themes that have been considered a-priori based on the existing literature on teaching in prisons and previous research experiences in prisons (e.g. Maffia & Decembrotto, 2022). In the Methodology section, we will see that the *theory-driven categories* have been used as thought-provoking hints for the teachers' contributions on a Padlet. An example of a *theory-driven category* is the *Motivation to study,* which Maffia et al. (submitted) defined as "students' relationship with study, in general, and with educational institutions, including the purpose of the study".

On the other hand, the *data-driven categories* are the themes necessary to codify the posts in the Padlet, which are characterised by the teachers' observations and are not covered by the *theory-driven categories*. In this paper, we will refer to the following examples of *data-driven categories* defined by Maffia et al. (submitted) as: "*Class group composition*: characteristics of students, […], high turnover, specific needs and any difficulties encountered (of social, cognitive, relational...)";

"*Personalisation*: individualisation and/or personalisation of teaching and/or training paths for students, including their flexibility".

There are more *theory-driven* and *data-driven categories* than those we have listed as examples. Due to space limitations, we have restricted the presentation to those addressed in this contribution.

SWOT analysis

SWOT is the acronym for Strengths, Weaknesses, Opportunities, and Threats (Helms & Nixon, 2010). SWOT analysis provides a clear frame for gathering information from various sources, bringing out an overview of internal factors (i.e., Strengths and Weaknesses) and external factors (i.e., Opportunities and Threats) that could be useful in describing the characteristics of the teaching/learning mathematics process. In this contribution, we will refer to how the reciprocal relationships between the categories identified above (Maffia et al., submitted) could highlight Strengths, Weaknesses, Opportunities, or Threats characterising the teaching/learning of mathematics in prison. In our study, Strengths (S) are seen as the positive characteristics of the school where teachers work and supporting their work. Weaknesses (W) refer to the characteristics that make teachers' work more difficult and hinder their aims. Opportunities (O) refer to the internal or external characteristics offered by the environment in which teachers work that could be exploited to benefit their work. Finally, Threats (T) can be seen as external difficulties independent of the teachers. Still, they undermine the successful functioning of their work in the classroom, thus limiting the achievement of their aims.

Using this theoretical framework, we intend to investigate the characteristics of teaching and learning mathematics in prison. To this end, we analyse teachers' comments on a Padlet (see the next section), addressing the following research question: Which relations between categories (if any) can describe the characteristics (concerning SWOT) of the teaching/learning of mathematics in the prison context?

Methodology

The present study is part of the larger research project LeMP. LeMP is an Italian national project that arose from the collaboration between researchers in Mathematics Education and Special Pedagogy, and three prison schools. Participation is voluntary in the LeMP project, and the research presented in this paper has been proposed to all mathematics and science teachers in the schools involved, all of whom have a master's degree in a scientific discipline but have not received specific training in teaching mathematics and science with adults, foreigners and/or in the prison context. However, our sample is characterised by various work experiences in prison, ranging from a single year to over 20 years. For this study, nine teachers have volunteered to represent all three school contexts involved in the LeMP project. The teachers involved have been asked to fill out a Padlet (described in more detail below) and then to participate in an online meeting to discuss some of the issues that emerged from the Padlet analysis. Due to the space limits, we focus our attention only on the content of the Padlet and its analysis.

To collect the teachers' perspectives, we have created a Padlet by adding some thinking-provoking hints such as *Motivation to study*, *Classrooms*, *Equipment*, *Role and purpose of education in prison*, *School-prison relationship*, *Absence of INVALSI tests*, *Other*, chosen by the researchers as relevant

topics based on the existing literature, their previous research experiences in prison, and the analysis of the institutional documents from the schools. This tool helped us to point out the aspects considered relevant by the teachers to describe the characteristics of teaching/learning mathematics in prison. Beyond this, the choice of using Padlet to bring out aspects relevant to teachers for describing their context is motivated by the desire to collect data remotely from teachers working in dispersed prisons in Italy.

Initially, teachers were asked to access the Padlet individually and post their reflections about internal factors (S and W) and external factors (i.e., O and T). Only then were the posts on the Padlet made visible to all teachers. They were then asked to comment on other colleagues' posts. This phase of work ended with an online meeting to further explore some of the issues that emerged from the Padlet and refine the categories identified in the content analysis of the Padlet (Maffia et al., submitted). Collecting data through a focus group discussion is motivated by the desire to triangulate individual thinking and reflection at the group level (Cohen et al., 2007).

The analysis of the data from the Padlet was divided into several stages in which data-driven categories were identified using the principles of qualitative content analysis (Cho & Lee, 2014). More precisely, the analysis process followed several steps based on an initial parallel analysis of the data collected in the Padlet using the a-priori categories and an initial identification of possible new data-driven categories and then on the comparison and discussion of the parallel analyses of all researchers involved to reach an agreement on the coding of each post and finally, the shared identification of the data-driven categories. The results of this analysis have been presented in detail in another work (Maffia et al., submitted), and we are not introducing that analysis in this paper. In this study, with reference to the chosen theoretical framework, the data analysis aims to describe the relationships between some categories by emphasising the aspects linked to the internal and external factors that the teachers have considered relevant according to their different experiences. In the next section, we describe only the relationships between the following categories: *Motivation to study* (with a focus on the attitude towards mathematics), *Group class composition*, and *Personalisation*.

Results and Discussion

Teachers have described the three categories considered in this study as having positive and negative aspects. The category relating to the *Group class composition* is described by the teachers with critical aspects (5 posts for W and 3 posts for T) and positive aspects (3 posts for S and 4 for O). On the other hand, the category relating to *Personalization* is described mainly by focusing on positive aspects (4 posts for S and 7 posts refer to O) with only 1 post pointing out W. Finally, the category related to "*Motivation to study*" is described by teachers with both positive aspects (3 posts for S and 5 posts for O) and negative aspects (3 posts for W). To answer our research question, in this section, we analyse the data collected with the Padlet by highlighting the relationships between the categories of *Group class Composition*, *Personalization*, and *Motivation to study*, which allow us to describe S, W, O, and T, characterising the process of teaching/learning mathematics in prison also by highlighting the number of posts in which they appear.

Group class composition and Personalisation

A relationship between the categories *Group class composition* and *Personalisation* is highlighted in 2 posts for S, in one post for W, and in 4 posts for O, which could be helpful for us in describing the process of teaching/learning mathematics in the prison context. More specifically, an aspect considered relevant by teachers within the category of *Group class composition* is related to the high turnover of students entering and leaving prison throughout the school year. This aspect presents itself as a challenge that, in most cases, negatively affects teachers' work in the classroom (*High turnover of entries, transfers, and releases: partial or missing information on the educational path. The number of students in each class is constantly changing throughout the year [...]*). However, at the same time, it emerges that this aspect is seen as an opportunity to be stimulated (as teachers) to personalise their teaching practices by students' competencies (The high turnover of inmates, on the one hand, slows down the development of a full curriculum, but on the other hand allows the development of lessons focused on the actual deficiencies/needs of the students in the limited time available). This opportunity stems from the fact that, because of the high turnover of students in the classes, there is also a wide heterogeneity among the students as they have different age levels, different cultural backgrounds, different literacy levels, and different mathematical skills and competencies derived from their different educational and professional experiences. Although the high turnover and the vast heterogeneity of the group of students are considered negatively by teachers, it also emerges from the posts on the Padlet that working in these contexts requires teachers to pay more attention to personalising the educational program that considers the individual student's educational needs. Thus, the need to personalise students' educational programs offers teachers the opportunity to develop teaching strategies aiming at enhancing the mathematical competencies and skills of detained students based on their real and different learning needs (The opportunity to work effectively with multi-class groups of varying levels of proficiency in Italian, ages, and diverse cultures is undoubtedly complex but constitutes a great wealth). However, despite the difficulty linked to the need for teachers to pay specific attention to the needs of individual inmates, it is possible to see that the ability to personalise the education of students by taking into account both their skills and their real learning needs becomes a Strength of the process of teaching mathematics in prison context (The didactical activities are based on the personalisation of the re-educational path of each student [...]and take into account expressed and unexpressed interests and needs).

Group class composition and Motivation to study

It is important to highlight a relationship between the categories of *Group class composition* and *Motivation to study* with the internal and external factors that characterise the process of teaching/learning mathematics in prison. A relationship is highlighted in 2 posts for S, 1 for W, and 3 for O. More specifically, the first negative aspect concerning the category of *Motivation to study* is related to the fact that, in many cases, students have a low motivation towards mathematics. This depends on previous negative school experiences, or in the case of foreign students, it depends on a poor level of schooling (Many students have had negative experiences with school and in particular with mathematics, and also Difficulties with mathematics are more significant in those who have little schooling even in their country of origin). These situations could also provoke feelings of frustration and rejection towards mathematics. From this, it can be seen that different levels of schooling or

different school experiences result from the vast heterogeneity of the class group. Thus, the wide heterogeneity of the class group generally influences motivation towards study and, more specifically, towards the study of mathematics, making it different among students. In particular, in these cases, the students' motivation to study emerges as being associated with other purposes, such as the desire to have better job opportunities after prison (Some students are motivated to obtain a degree to be able to work once they are out of prison); the desire to have opportunities for socialisation (The school in prison is a fundamental element of the rehabilitation pathway for the future reintegration of the imprisoned person into society. It is a place for socialisation, confrontation, acceptance and reflection, as well as learning); the desire to have opportunities for redemption from a previous unsuccessful course of study; the desire to have a new chance in being able to study a subject one has always been fascinated by; the desire to have personal growth with positive effects on the students' self-confidence as they are encouraged by the gratification of studying mathematics and the wish to be distinguished from others; the desire to obtain a school certificate or a certificate recognising their exemplary conduct at school to get a possible reduction in their penalty. From the description of this relationship between the two categories under consideration, it also emerges that this relationship helps describe the S and O of teaching mathematics in the prison context. In particular, in many situations, the psychological and affective aspects of the students emerge as a decisive motivating factor, often having a positive attitude towards the study of mathematics. In these cases, teachers emphasise that this depends on the fact that school, and especially the study of mathematics, offers the students a chance to escape from the reality they are obliged to live by allowing them at that time to forget other negative thoughts (The prisoners are not just students, they must live with guilt, nostalgia, anger, anxiety... a mathematical question can momentarily distract them and keep this burden away for a while, respond to the need to take their head elsewhere).

Motivation to study and Personalisation

Among the Padlet posts, some tried to clarify this relationship by describing internal and external factors characterising the teaching/learning of mathematics in prison. In particular, 3 posts aimed to describe the S and O offered by the context in which the involved teachers work, and that could help improve their work in prison. In these posts, teachers highlighted that motivation towards studying mathematics could be positive when teachers design lessons encouraging students to discover mathematics collaboratively. In these cases, it could be observed that it helps bring out the creative aspect and the students' desire for manual skills and concreteness in an intense way, so that students' motivation might be continually stimulated. To have a greater involvement of the students, the teachers once again highlighted the need to personalise their teaching practices by using more experiential techniques rather than transmissive ones, by designing manual, concrete, and creative activities that – thanks to possible collaborations with experts from outside the institution – could be appropriate for each student, taking into account the real learning needs of individuals (The educational offer could be expanded by allowing and encouraging the entry of external experts who offer their availability also free of charge. [...] or also The didactical activities are carried out with the use of techniques that are mostly experiential as opposed to transmissive and are based on the personalisation of the re-educational path of each student [...] and take into account their expressed and unexpressed interests and needs, to reinforce their motivation to learn). Thus, teachers are given

an additional opportunity to design teaching programs that leverage these kinds of dynamics and are oriented towards personalising the educational pathway of each detained student. Through this classroom activity, the teacher could help students improve their self-confidence and push them to understand that, once the prison experience is over, they could go on in their lives outside with new awareness and tools (The classroom activity aims to recover self-esteem and a sense of self-efficacy, in the direction of a redefinition of oneself, to, once the detention experience is over, resume one's life "outside" the prison with new awareness and new tools). During this type of activity, teachers also pointed out that, to stimulate motivation towards mathematics in the students, they could support them by constantly showing the usefulness of studying mathematics within their educational paths.

Conclusion

The analysis of the individual teachers' posts on the Padlet reveals many weaknesses and threats linked to the difficulties of their work context. However, in this contribution, we have shown that the relationships between the three considered categories also highlight strengths and opportunities as closely connected to some of the characteristics of the teaching/learning mathematics process in the prison context and not always available in other educational contexts. From our analysis, it is possible to note that not only weaknesses and threats emerged as a first preliminary analysis of the categories, but they also seemed to point out mainly (Maffia et al., submitted).

Thus, with the analysis presented in this paper, we enriched the context description (still scarce in literature), also providing positive aspects that may provide a base for designing innovative teaching practices and, consequently, for improving the process of learning mathematics. Such a more detailed exploration and analysis of the context is the necessary first step in any EDR project (McKenney & Reeves, 2018). The results of this first phase of research inform a process of evaluation aimed at refining the problem definition (namely, evaluating the discrepancy between the existing situation and the aim of our project, decomposing the problem into its constitutive parts). As exemplified in this contribution, pointing out opportunities (and not only threats) allows determining the so-called "jurisdiction of change", but that is also, distinguishing what is possible to change from the unchangeable (McKenney & Reeves, 2018). Starting from this awareness, the next phase of our project consists of determining the "design requirements". According to van den Akker et al. (2006), such requirements will become design principles only when based on theoretical arguments (as those developed in this contribution) and empirical ones, which will be based on data to be collected in the future phase of the project.

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Subject integrated team teaching vs. textbook learning: A comparative study on students' experiences in vocational mathematics education

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This study investigates the effects of subject integrated team teaching (SITT) versus textbook-based learning (TBL) on students' experiences of vocational mathematics education. Data from a post-questionnaire of 234 students from two schools revealed significant differences in the effectiveness of vocational preparation between the groups. The SITT group reported positive experiences, high motivation to learn mathematics and identified vocational relevance as a key factor. Conversely, the TBL group had negative experiences and felt that the mathematics they were learning was impractical. The study also explored various motivators and barriers to the implementation of these approaches, highlighting the crucial role of teachers and the impact of grades on student motivation.

Keywords: Subject integrated team teaching, vocational education, vocational mathematics, questionnaire.

Introduction

There is a broad agreement, both at the national and international level, among scholars in the field of mathematics education that it is crucial to implement exploratory teaching and learning methods that link the mathematical concepts students acquire to their prospective professions (Frejd & Muhrman, 2022). Vocational mathematics education serves as a platform where learning occurs, with a clear objective of equipping individuals for their future careers (FitzSimons, 2014). Despite this, it is often observed that students enrolled in vocational education exhibit a lack of enthusiasm towards learning mathematics and fail to understand its relevance (Muhrman, 2016). This disconnect can be destructive to their learning process. However, Dalby and Noyes (2016) propose a solution to this issue. They suggest that incorporating realistic scenarios from the workplace can serve as a motivating factor for students to learn mathematics. This approach can help students understand the practical applications of mathematical concepts, thereby enhancing their interest and engagement in learning.

The idea of incorporating authentic situations into vocational education is not new and there are several ways how to make the teaching realistic (Frejd & Muhrman, 2022). For instance, a student could interview workers on what mathematics they use, or follow a worker a week and identify some mathematics on their own, read about workplace situations in books, work with authentic problems in the classroom and so forth.

In this paper, we introduce and share preliminary findings of our ongoing research project. This project is designed with the objective of exploring how different teaching interventions of mathematics in vocational education can prepare students for their future professional endeavours. Our primary objective is to examine the impact of these interventions on two key areas: the mathematical proficiency of the students and their motivation to learn mathematics. Our project

draws on Ryan and Deci's (2000) theory of motivation, which distinguishes between intrinsic and extrinsic motivation. Intrinsic motivation involves an internal drive to engage in tasks because they are interesting or meaningful, while extrinsic motivation is driven by the desire to avoid penalties or achieve high grades (Ryan & Deci, 2000). Extrinsic motivation is linked to surface learning, whereas intrinsic motivation is associated with deep learning and predicts overall school performance and long-term academic success (Taylor et al., 2014). According to Ryan and Deci's (2000) Self-Determination Theory (SDT), the three aspects of *autonomy*, *competence*, and *relatedness* are considered essential for promoting motivation. Briefly, autonomy refers to the need to feel in control of one's actions and decisions, competence refers to feeling effective and capable in one's activities, and relatedness refers to feeling connected and having meaningful relationships with others.

The project is longitudinal and involves nine upper secondary vocational schools, more than 1000 students and 20 teachers across Sweden. We use a combination of pre- and post-tests, questionnaires, and interviews as our data collection methods. This allows us to gather comprehensive data before and after the implementation of the interventions. Furthermore, we conduct follow-up data collection sessions at 18 months post-intervention. The objective of this paper is to offer preliminary results on students' experiences of a subject integrated team-teaching approach for mathematics learning compared to traditional textbook-based learning. The following questions, connecting to students' feeling of competence and intrinsic and extrinsic motivation, will guide our research:

RQ1: *What are students' experiences regarding the effectiveness of the teaching approaches in preparing them for their future professional roles?*

RQ2: *What experiences do students describe about how the teaching approaches they have encountered motivated them to learn mathematics?*

Subject integrated team-teaching approach in vocational education

Drawing on the work by Cook and Friend (1995) as well as by Frejd and Muhrman (2022), we define a subjected integrated team-teaching (SITT) approach for mathematics in vocational education as a co-teaching approach where two teachers (one mathematics teacher and one vocational and training teacher) jointly organize and deliver a teaching practice in connection to students' training halls.

SITT includes many aspects for qualitative teaching as found in the research literature on vocational mathematics education. Collaboration across disciplines encourages mathematics teachers to collaborate with other professionals and it is essential for developing a teaching practice that consider challenges and requirements of different workplaces (Frejd & Muhrman, 2022), as well as highlights the social and cultural workplace practices (FitzSimons & Björklund Boistrup, 2017). Since the mathematics in the workplace is deeply rooted in traditions and use of artifacts they should also be an integral part of the classroom activities in order for providing authentic learning opportunities (Mouwitz, 2013). A collaboration between the mathematics and the VET-teacher is therefore fundamental in both the design of learning opportunities and in the process of developing students' workplace identity and self-esteem according to Mouwitz (2013). In addition, our definition of SITT includes the learning of mathematics in relation to students' training halls. According to Frejd and Muhrman (2022) learning of mathematics in students' training halls significantly enhanced student

motivation and engagement compared to traditional mathematics classrooms. Vocational tasks in the mathematics classroom alone tend to maintain conventional classroom dynamics, making it difficult for students to apply these tasks to real workplace situations (Sundtjønn, 2021). However, integrating realistic problems linked to vocational contexts in both the mathematics classroom and vocational training hall can redefine classroom roles and alter perceptions of expertise. In training halls, students often view themselves as more knowledgeable compared to their mathematics teachers (Frejd & Muhrman, 2022; Sundtjønn, 2021).

The intervention and data collection

This paper presents initial results from the first year of a four-year study. We compare the responses of students in school A, which uses a Subject Integrated Team Teaching method (SITT, n=106), with the responses of students in school B, which uses a Textbook-Based Learning method (TBL, n=128). The participating students in both schools are enrolled in Mathematics 1a, a compulsory course covering the topics of arithmetic, algebra, linear and exponential functions, probability and statistics. In addition to a focus on academic mathematics it should also include mathematics for the student's future occupation.

School A, applying SITT, has six vocational education programmes and their teaching practice comprise three hours per week. One of these hours is called “vocational mathematics” and is located in the vocational classroom. During this lesson, both the mathematics teacher and the vocational teacher teach together following the dress code for the particular workplace and provide students with mathematics activities related to the workplace. The teaching activities is an outcome of a collaboration between the mathematics teacher and the vocational teacher, with a focus on what students are actually working on in the vocational subject. Examples of activities are connected to, recipe conversion and establishing prices for products (Restaurant management and food programme), driving time and engine displacement (Vehicle and transport programme), the costs of paint and materials for construction (Building and construction programme), etc. The other two lessons are textbook based and take place in the mathematics classrooms. School B, applying TBL, has two vocational programmes (the vehicle and transport programme as well as the natural resource use programme) with different profiles, such as agriculture, forestry, horse breeding, construction machine operator and transportation. In addition, about 20 of the 128 students have chosen a course that includes more mathematics, physics and chemistry, which qualifies them for further education at colleges and universities. School B has allocated four hours per week for Mathematics 1A and the main source of their lesson design is the use of the textbook. The mathematics teachers and the vocational teachers do not collaborate either on content or on teaching activities.

This paper focuses on the students’ post-questionnaire, which were distributed and filled out individually directly after students had completed a post-test in spring 2023. The questionnaire was developed using Microsoft forms for online surveys. It included 23 statements about motivation adapted evaluation forms of PISA (OECD, n.d.) and connected to Self-Determination Theory’s (SDT) aspects of autonomy, competence, and relatedness (Ryan & Deci, 2000), to be ranked on a 10-point Likert scale (1- strongly disagree and 10-strongly agree), and these 23 statements were the same in the pre- and post-questionnaire. In addition, did the post-questionnaire include a question on:

(Q24) what type of teaching approaches the students had met during the year; two statements to be ranked about (Q25) if the mathematics they had learnt had prepared them for their occupation and (Q27) if the organising of the teaching had motivated the students to learn mathematics; as well as two open questions (Q26 and Q28) where the students were asked to explain their reasoning regarding (Q25) and (Q27). The questionnaire question Q25 to Q28 are analysed in this paper.

Analysis and results

The students' responses to the questions in the post-questionnaire, 106 (SITT) and 128 (TBL), were exported into a spreadsheet (Excel). Answers on the Likert items (Q25) and (Q27) were analysed and compared using descriptive statistics and two independent sample t-tests. The students' answers to open questions (Q26) and (Q27) were short in length and most answers were just a few words and even fewer a few sentences. A line-by-line analysis (cf. Chenail, 2012) was used, with a focus on what reasons the students expressed based on their experiences. Briefly, each line in the students' responses was examined to identify units (words, phrases or sentences that convey significant information). These identified units of meaning were then coded with labels describing the essence of the content. These codes were finally grouped into different themes, which are discussed in the next sections.

Students' experiences of the effectiveness of the teaching approaches in preparing them for their future professional roles

In (Q25) the participants in the study ranked the statement*: The math we have learnt has made me prepared for my occupation*. The mean score for the SITT group was 5.92 with a standard deviation of 2.21 and for the TBL group the mean score was 3.98, with a standard deviation of 2.21. The t-statistic was 6.72, with df=224 ($p < .05$) show a significant difference of scores between the two groups. Figure 1 below illustrates the bar charts of frequency of students' rankings in both groups.

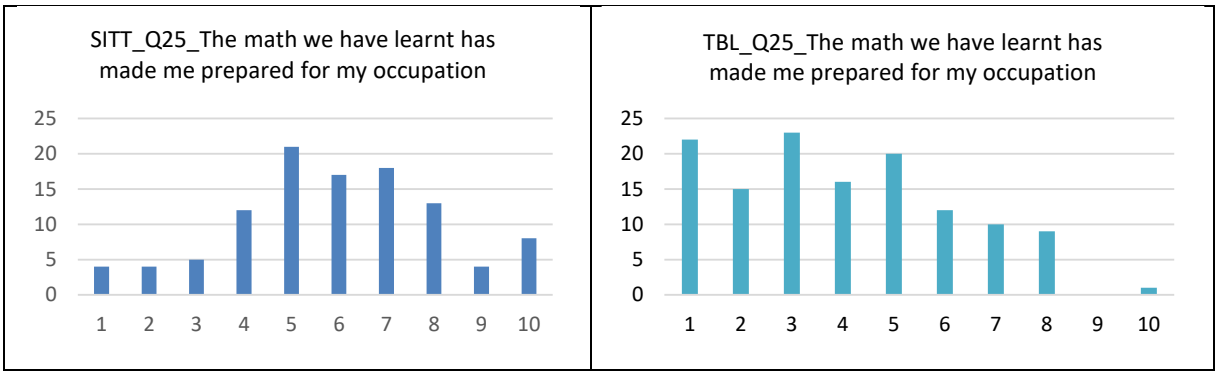

Figure 1: Bar charts of students ranking of Q25, left the SITT- group and right the TBL-group

From figure 1 to the right (the TBL group) it is possible to observe that only one student strongly agreed (ranking 9 or 10) with the statement that the mathematics taught has prepared that student for his or her future occupation, whereas in the TISS-group (in the left bar chart) there are 12 students that strongly agree to the same statement.

Table 1: Examples of students' expressed reasons in Q26 for their ranking of Q25

SITT-Frequent students 'expression with codes in italic	TBL- Frequent students 'expression with codes in italic
We have *vocational math* once a week and we learn about when and how to use math at work. In regular math, we don't really have math that will prepare me for my profession, but we have *vocational math* which is better preparation for the profession itself. However, I don't think we work with it enough to actually be able to remember all the formulas etc. So more vocational math to make it more preparatory. We get to learn *vocational math* that helps with things regular math doesn't and it's more fun.	I am going to drive a truck and do not need to know x*4^5=1 [Code: *unusable math*] I will not need so much mathematics in my occupation. [Code: *not needed*] I think we can work with more things that we can use in our future profession and not a lot of other things that we won't need. [Code: *unusable math*] We use a regular math book with regular math questions, nothing related to the profession. [Code: *unusable math*]

The line-by-line analysis of the students' answers on the open questions (Q26) generated 7 themes. The examples of students' expressions of their experiences in table 1 show the most frequent identified themes, namely *vocational math* (SITT-group) and *unusable math* (TBL-group). About 35% of the students' answers in the SITT-group described the weekly hour when the mathematics teacher and the VET-teacher worked together as a reason for making them prepared for their future occupation. Thereafter, the themes *Yes* (students' expressions that argued for usefulness by only writing yes or yes together with examples of mathematics in the workplace) and the theme *I don't know/no response* were identified 25% respectively. The last 15% of students' answers in the SITT group were distributed between the themes *No* (expressions that argued not being prepared with no argument) [10%] and *Not needed* (expressions connected to that mathematics was not needed for their occupations) [2%], and *other* (single arguments not connected to other themes) [3%]. In the TBL-group almost 40% of the answer describes that students had learn mathematics they express as *unusable* for their profession and a call for learning of workplace related mathematics rather than academic mathematics as well as these answers include expressions that their future occupation does not include any mathematics belonging to the theme *Not needed*. The theme *No* included 5% of students answers, *I don't know/no response* 25%, *Yes* 15%, *General mathematics ability* (expressions that mathematics can be useful in the future life) 10% and *Other* 5%.

Students' described experiences about how the teaching approaches they have encountered motivated them to learn mathematics

How the different teaching approaches have motivated students to learn mathematics was analysed through Q27. The results show that the mean score for the SITT group was 4.93 with a standard deviation of 2.13 and for the TBL group the mean score was 3.87, with a standard deviation of 2.37. The t-statistic was 3.62, with df=230 ($p < .05$) illustrates a significant difference of scores between the SITT-group and the TBL-group. The frequency of students' answer is displayed in Figure 2 in the next page. Notable, the mode for the SITT-group (figure 2 left) is 5, but for the TBL-group (figure 2 right) the mode is 1. The tendency for the TBL-group's answers were compiled on the disagree side of the spectrum (ranking 1-4), whereas the SITT-groups answers are equally distributed on both side of the centre pile.

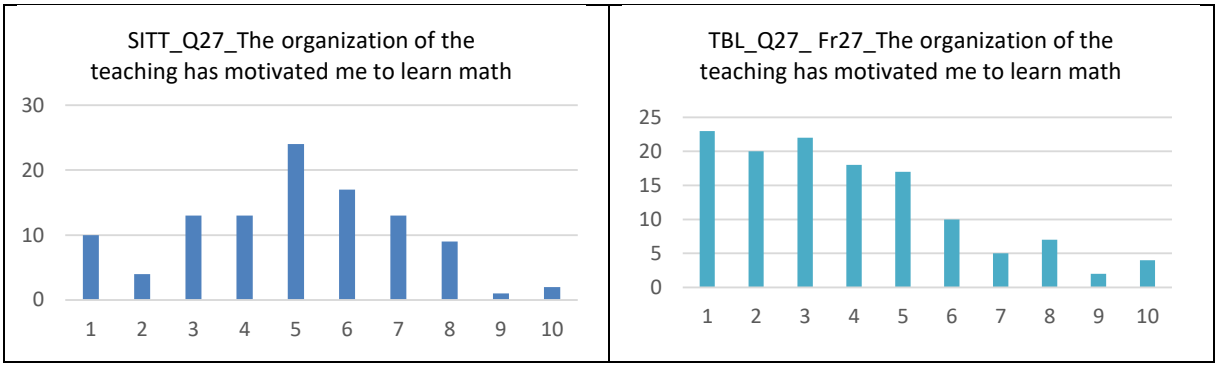

Figure 2: Bar charts of students ranking of Q27, left the SITT- group and right the TBL-group

For the analysis of the open question Q28 the line-by-line analysis resulted in 18 themes. The SITT-group addressed *vocational math* as positive motivator for their learning also in relation to the organisation of the teaching approach, see examples of students' reasons in Table 2 below.

Table 2: Examples of students' expressed reasons in Q28 for their ranking of Q27

SITT-Frequent students 'expression with codes in italic	TBL- Frequent students 'expression with codes in italic
Vocational math [Code: *vocational math*] I had a good teacher who could explain in a good way so you I understood, and I didn't learn anything unnecessary only things I will need. [Code: *teacher*] The arrangement itself does not motivate me. [Code: *other*] It has not been as stressful as in 7-9 grade [Code: *time*]	I would have preferred agricultural related tasks, etc. [Code: *unusable math*] Good teacher [Code: *teacher*] It hasn't been that different from 7-9 grade [Code: *similar to before*] The book is boring grade [Code: *boring*] I just want to get a grade [Code: *other*]

For both groups a good *teacher* was another motivator expressed for learning. Also, were the motivators *small groups*, *small test*, *well defined lesson structure*, and *easy* identified themes in the TBL-group and *time*, *fun and interesting lessons*, v*ariation between workspaces*, and *useful tasks* were found in the SITT-group. However, the analysis also showed that where *other* motivators not really connected to the organisation the seemed to impact on students' motivations, such as grades. In addition, negative motivators were also identified such as *boring* books, *no interest* to mathematics, *lack of variation*, *similar to before*, and *difficult.* Overall, were more negative motivators identified in TBL-group (40%) compared to the SITT-group (20%).

Conclusion and discussion

Our analysis indicates significant differences in the perceived effectiveness of the teaching approaches between the SITT-group and the TBL-group. The SITT-group had a higher mean score of 5.92 on the post-questionnaire and attributed their positive experience to *vocational math*. In contrast, the TBL-group had a mean score of 3.98 and felt they had learned *unusable math*. There was also a significant difference in how the students felt the teaching approach motivated them to learn mathematics. The SITT-group was more positive (4.93) compared to the TBL-group (3.87). However, both groups explained positive motivators related to each approach such as *small groups*,

easy, and *well-defined lesson structure* (TBL-group) and *fun and interesting lessons*, v*ariation between workspaces*, and *useful tasks* (SITT-group). In addition, the *teacher* was an important motivator, but so were *other* aspects not related to the teaching approach, such as grades.

Following recommendations from previous research on teaching approaches where teachers collaborate across disciplines (Frejd & Muhrman, 2022), emphasise social and cultural workplace practices (FitzSimons & Björklund Boistrup, 2017), and provide students with authentic learning opportunities (Mouwitz, 2013; Sundtjønn, 2021), our findings show that students encounter this type of teaching are more engaged and motivated to learn in learning vocational mathematics compared to traditional textbook-based learning. The extent to which the responses to the questionnaire measure effectiveness is of course open to debate, but it does provide an insight into the current situation, although the extent to which these approaches also affect students' academic performance requires further research. The results are the outcome of our design and we only examine two vocational schools in this study, where both participating schools have a fairly long tradition of using the approaches. We cannot generalize our findings, as teaching approaches depend on social and cultural contexts. Further research on the SITT approach is needed to better predict pupils' experiences. Therefore, we will continue and analyse students' experiences in our larger study of 9 schools.

Of the motivators we have identified, it is not only the teaching approach itself that effects students' motivation to learn mathematics, but also how well *teachers* are able to support and communicate with students, and how well students get the *grades* they want. The need for autonomy, competence and relatedness (Ryan & Deci, 2000) such as enough *time* for students to complete tasks on their own, *easy* (and difficult) tasks in terms of feeling competent and *useful tasks* related to the workplace were reported. Our results also show that the students in SITT-group described more intrinsic motivation compared to students in TBL-group who described more motivators connected to external motivation. However, several of the students in the TBL group mentioned *boring* books and *lack of variation* as negative motivators. One possible solution to overcome the lack of variation is to vary not only the content and the teaching method, but also the *workspace*. Previous research (Frejd & Muhrman, 2022; Muhrman, 2016) shows that it is very important for the teacher to consider the didactic questions of *how*, *what* and *when*, as well as *where*, when designing vocational mathematics lessons, as students do not necessarily learn mathematics best in the mathematics classroom. The aspect of relatedness (Ryan & Deci, 2000) is a key need in SDT, and perhaps this need comes to the fore in the teaching of mathematics in the vocational classroom compared to the teaching in the traditional mathematics classroom, but such an assumption needs further research.

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Learning Math in Prison: Preliminary outputs of an Educational Design Research

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"Learning Math in Prison" is an Educational Design Research project that aims to provide practical guidelines for developing structured interventions in mathematics education within Italian prisons and to contribute to the theoretical understanding of this issue. This article presents the results of the initial exploration phase, which outlines partial design requirements and initial design propositions for the specific context. These results are essential for guiding the subsequent phase, in which pilot studies will be conducted to develop design principles and field-test teaching activities.

Keywords: Educational design research, prison education, design requirements, design propositions.

Introduction and theoretical framework

Education is a significant challenge in prison, and mathematics holds a unique position among the subjects taught, serving as a source of personal satisfaction and a gateway to future educational or employment opportunities (Ahl & Helenius, 2021). Despite its importance, there is a notable gap in research on the teaching and learning of mathematics in prison contexts (Gagnon & Barber, 2014). The existing literature highlights that prison context is characterised by discontinuity in class attendance; moreover, students' backgrounds are characterised by previous failures, low self-efficacy levels and a negative view of mathematics (Gagnon & Barber, 2014; Byrne & Carr, 2015). Despite this, active and participatory methodologies can potentially improve mathematical learning in prison (Ferrarello & Mammana, 2022). These single case studies have limited insight into teaching methodologies, and we observed a lack of discussion on the effectiveness and replicability of the approaches used. A significant contribution to this panorama is that of Maffia and Decembrotto (2024), who proposed a set of preliminary design principles for mathematical teaching in Italian prisons: *(I) it is essential to consider a problem-based situation that is accessible to all students, presented through tasks with several levels of difficulty and possibilities for generalisation; (II) it is necessary to refer to fields of experience for students in order to propose problem-based situations that are both contextualized and relevant to their background; (III) it is necessary to set up a didactic contract incorporating various strategies and acknowledging errors as an integral part of the learning process, thus impacting beliefs about mathematics; (IV) it is necessary to build a climate of mutual trust in order to achieve effective interactions between students and among students and teach*ers. Maffia and Decembrotto (2022) also highlighted the extreme need for further research on this issue, making their study the foundation for the "Learning Math in Prison" (LeMP) project. LeMP is an interventionist Educational Design Research (EDR, McKenney & Reeves, 2018) project involving five universities and teachers from three different prison schools to co-construct mathematical teaching units. The project aims to build a theoretical framework for mathematics education in prison embedded with the design and development of educational interventions, which will be disseminated to teachers in Italian prisons. The LeMP project consists of three main phases,

each involving interaction with practitioners (teachers) and contributing to theoretical understanding and gradual development of an intervention (McKenney & Reeves, 2018). The first iteration of the analysis and exploration phase involved identifying and diagnosing the problem. Analysis was done by consulting experienced practitioners and conducting a systematic literature review to obtain theoretical insights related to the issue. A more open-ended exploration was conducted in collaboration with practitioners to examine potential solutions, with the aim of identifying partial design requirements "determined by exploring the opportunities and boundary conditions present; and initial design propositions [which] are generated based on contextual insights" (McKenney & Reeves, 2018, p. 79). The exploration process in design research often remains in the background compared to more formal research activities (such as the literature review). It is often scarcely documented despite its importance for insightful research (McKenney & Reeves, 2018). This contribution aims to document this initial phase of the LeMP research project, presenting partial design requirements and initial design propositions to address the question: What are feasible and appropriate design choices for teaching mathematics in prison?

Data and Methods

The exploratory process in the first phase of EDR aims to identify design requirements and propositions to guide the initial experiments. Subsequent cycles of reflection and evaluation may confirm, refute, or modify these preliminary outputs. The exploration process is an open-ended activity that requires collaboration with practitioners—in this case, nine middle school mathematics teachers working in three prison schools across different Italian regions. The outputs are drawn from diversified sources (see Table 1), offering a multifaceted view of the issue under analysis.

Table 1: Type of sources considered in the analysis and their description.

Source	***Data Description***
Padlet	Written observations on a shared digital platform: practitioners had to share their perspectives on teaching mathematics in prison identifying the related Strengths, Weaknesses, Opportunities, and Threats (SWOT). All teachers were involved.
Focus-group	Notes taken during an online meeting involving 7 practitioners and all the researchers. The meeting aimed to explore the themes that emerged in the Padlet.
Meetings with teachers	Notes taken during face-to-face meetings with practitioners in each school, focused on school practices.
Teaching units	Collection of 4 teaching units made by teachers for teaching mathematics in prison.
Semi-structured Interviews	Videorecording, full transcription and notes of interviews to 4 practitioners (referred to by fictional names) from 2 schools, conducted online by 2 researchers at a time, focused on teachers' identity and professional development needs.

Six researchers qualitatively analysed the comments on Padlet identifying recurring themes. Using the SWOT framework and addressing the questions proposed by McKenney and Reeves (2018, p. 107), we outlined freedoms (aspects of the system that can be changed and offer chances for

improvement) and constraints (aspects of the system that cannot be changed and constitute boundaries for interventions) that form the partial design requirements. Additional insights and practices were gathered from all data collected to refine design requirements and inform the development of initial design propositions. These are based on a sound understanding of the problem and context, and, as McKenney and Reeves (2018) note, they can be heuristic statements that reveal positions on key issues arising from researchers' reflective hypotheses and aim to guide initial design attempts. Design propositions will be evaluated by studying the interventions based on them and revisited after the project's second phase. They are stated here in a simplified conditional schema: "If the goal _, then _. Argument (empirical and/or theoretical)," similar to the one proposed by van den Akker (2010).

Results

Partial design requirements: Constraints and freedoms

Data analysis highlighted several constraints related to teaching mathematics in the prison context. In the Padlet, most teachers stated that the relationship between the prison institution and the school is complex and often conflictual. Teachers complain about a lack of shared goals with prison officers, which makes collaboration difficult. In subsequent meetings, we explored its negative impact on teachers' work. Time dedicated to school, and thus to mathematics, is less than in standard school courses. It can be further reduced by organisational or logistical issues, resulting in students arriving late in the classroom or leaving early and several disruptions during lessons. Classroom spaces are often inadequate, and basic equipment commonly found in ordinary schools is lacking in the prison context. Technological equipment is often absent or inoperative, and access to the internet is not allowed. On security grounds, the prison administration prohibits many everyday objects, including teaching tools (e.g., rulers, squares, calculators). Even fewer objects can be brought by students into their cells to practise individually after the lessons.

The Padlet also offers several freedoms related to mathematics in prison. Since mathematics, as an abstract subject, often requires no specific equipment to be practised—a simple pen and paper, or even just imagination, may suffice—the lack of materials is consequently less critical than in other disciplines. This creates opportunities to devise new ways to teach mathematics with the few materials available in prison. Currently mathematics textbooks used in prison schools are inadequate, but they can be updated and improved to better suit the context; this is an addressable issue.

In both Padlet and meetings, teachers expressed the challenge of constraints related to class-group heterogeneity. In prison, student ages might range between 14 and 25 years old in juvenile detention centres or between 18 and 70 years old in adult prison, while students may have enormously different socio-economic backgrounds (most of them have a poor and/or migratory background). In the same class, teachers work with students with various levels of Italian language proficiency, and those who have received some schooling may have received it in very different cultural contexts. School classes are formed according to logistic issues, meaning they may not share education-related characteristics or needs. Classes are composed of a variable, but generally low, number of students.

This last characteristic also allows for a high degree of personalisation in teaching interventions, compared to what is typically possible. Indeed, teachers often emphasise that they have more freedom

regarding curriculum choices and assessment practices in this context, as they are not subject to much external pressure (as often occurs in ordinary schools). As emerged in Padlet, teachers are *"freer to choose which core topics to cover and which activities to carry out"* and feel less pressure about grading, as they are not subject to external expectations related to student performance.

All the teachers involved in the project agree that one significant constraint is the high turnover of students who continually enter or leave the school. This issue is especially prevalent in "casa circondariale" institutions, dedicated to people awaiting trial or serving short sentences. This characteristic disrupts educational continuity, resulting in fragmented learning paths that are difficult to certify and/or reconstruct. Other factors, such as students' psychological conditions and personal needs (e.g., family visits), further hinder regular attendance.

Also, practitioners voiced other freedoms related to teaching mathematics in prison. Interesting points emerged about motivation to study, which, in this context, is often influenced by external factors; indeed, attending school can bring advantages throughout imprisonment (in terms of good conduct) and a useful qualification for future work purposes; this kind of motivation is fragile and clashes with the fact that students have often had negative school experiences in the past. Motivation to study mathematics seems to be lacking, as it is often perceived as a discipline for "smart ones". Teachers are aware of this, and during the focus group, they stated that it is important to work on students' motivation and change the relationship that many of their students have with mathematics, adopting appropriate teaching methodologies and proposing mathematics contents that students can see as close to their reality and useful for their future. Data from Padlet and interviews highlighted that school has a positive connotation in prison: teachers pointed out that education consists of exercising a right by prisoners, in contrast to the restrictions and deprivations that characterise this context.

Initial design propositions

Based on the design requirements identified in terms of freedoms and constraints and capitalising on practitioners' experience, we go one step further to identify initial design propositions.

a) *If the goal is to actively involve students in learning mathematics, then intrinsic/inner motivation must be aroused by affecting their view of mathematics and sense of self-efficacy.*

Argument: Motivation to study in the prison context is particularly fragile, as highlighted in the design requirements. On the one hand, participation in classes is not compulsory, so it presupposes a good level of motivation; on the other hand, motivation is often extrinsic so the objective is set by others and depends on factors outside the control of both student and teacher. This type of motivation is not strong enough to support continuing studies outside prison (Ahl & Helenius, 2021). Intrinsic motivation, rooted in personal satisfaction, might be more solid and lasting. However, prison classes mainly include students who come from unsuccessful and negative school experiences, especially in mathematics, so intrinsic motivation is scarce and must be encouraged. Several references can be found in the data collected, particularly from the interviews. For instance, Caterina said: "*It has often happened [that students] say 'teacher, I'm not doing anything today!' and, by putting these small problems, questions, even just small logic quizzes […] on the table, the student already starts working without being aware of it."*

Thus, our data confirms that it is important to work on self-motivation and metacognition (Maffia & Decembrotto, 2022). Since these aspects are also related to the purpose of teaching mathematics in prison, a new design proposition emerges:

b) *If the goal is to offer opportunities for redemption, emancipation, and life planning, then educational design must include non-academic objectives.*

Argument: The educative and rehabilitative role of the school in prison emerges strongly from all our data sources; for instance, in the Padlet Arianna wrote: *"The classroom activity aims to recover self-esteem and a sense of self-efficacy, towards a redefinition of oneself, in order to, once the prison experience is over, resume one's life 'outside' with new awareness and new tools."* Some teachers highlighted the central role played by mathematics teaching in the students' educational path precisely because it is often experienced as an obstacle, more so than other subjects. *"Teaching Maths in prison produces a lot; above all, it leads to overcoming these obstacles and, through this, finding oneself. The very fact of getting up and coming to the blackboard to do something (because they have found a stimulus to do so) makes it different from other subjects"* (Salvo). At the same time, there is concern that their work may be mistaken for that of psychologists and social workers, both because teachers are not trained for such roles and because they believe that schooling in prison serves a distinct and essential function. Thus, it is important that these extra-curricular objectives, which are felt to be important to teachers involved in the research, are integrated into the teaching activity. They should not be the focus of separate moments, in apparent suspension of the teacher's work, but should be part of the school practice. The Mathem-Ethics Project (Ferrarello & Mammana, 2022) can be an example of a teaching activity in which mathematical disciplinary objectives were integrated with other objectives, such as ethics.

c) *If the goal is that each student learns mathematics, with their competencies acquired in formal and non-formal contexts, then personalised teaching interventions must be provided.*

Argument: As emphasised in the design requirements, the differences between students in prison are not limited to personal characteristics nor different "levels of schooling". The same class may include students with various life experiences and knowledge acquired in the most diverse contexts. Several aspects of these diversities emerge from the interviews related to mathematics learning; as Salvo says: *"Meanwhile, these people acquired other skills and competencies, they have acquired them necessarily! So, to go back from what […] has become their way of reasoning, and thus, of using mathematical tools (also unconsciously) is very difficult."* Learning mathematics must be integrated with the skills they already possess, which must not be underestimated but enhanced. To do this, it is necessary to personalise the teaching activity as much as possible. This already seems to be part of the practices of the teachers, as Caterina stated in the Padlet: *"Teaching activities are geared towards personalising the re-educational pathway of each student, starting from an individual education pact, which certifies the skills already possessed at entry level and outlines the skills to be acquired for the validation of the pathway."* Prison teaching aims to enhance and consolidate students' previous experiences, adapting the selection of contents and topics according to their interests and needs, both explicit and implicit. Flexible teaching tools, techniques, and methodologies are used to stimulate learning motivation, thus offering solutions best suited to the needs of each student. This is also

possible due to the small number of students in each class, as Anna said: *"Since there are only a few of them, you can really follow them individually; I stand alongside them; I mean, it is easier [to include students with difficulties in maths]."*

Also, a reflection on assessment practices is needed regarding personalised teaching in prison context:

d) If the goal is to adopt personalised teaching approaches suitable for the prison school context, then a formative assessment approach is preferable to a summative assessment.

Argument: We refer to Black and William's (2009) distinction between summative assessment (SA, assessment of learning) and formative assessment (FA, assessment for learning). Through the analysis of the data collected in this research project, it is evident that within the prison context, FA plays a crucial role both as a choice guided by the context of fragility and as a practice shared by most of the teachers interviewed who consider it fundamental to observe the students' entire learning trajectory. Within the focus group, teachers mentioned the possibility of not grading, given that particular context conditions allow informal assessment to occur in a relaxed atmosphere, without pressure about grades as judgment. Naturally, the effects of this type of approach on students also emerge. Although teachers do not explicitly use the term "formative assessment", their descriptions align with this concept, reflecting its practical application in their teaching approach. "*[Here education] does not involve big tests or exercises to be carried out rigidly; it is a continuous exchange that allows us to understand competence levels"* (Caterina). It turns out that FA takes on a personalised and helpful role in the school within a prison, whereas SA risks being perceived as *"another form of judgement that falls upon them,"* as Camilla stated.

Important reflections on time management emerged when the teacher planned a mathematics learning activity; in this case, the specific type of prison where the schooling occurs seems relevant.

e) If the goal is to involve all students who have not consistently attended, then, in a context with higher turnover, it is recommended to plan short activities.

Argument: In some prisons, it is common for students to have fragmented paths due to recurrent releases and incarcerations during the school year or other systemic reasons (e.g. transfers to another prison). In this context, the inclusion of new students in a class, or their abandonment, occurs frequently throughout the school year. The teachers operating in these contexts made explicit the need for the activity to occur in extremely limited time. During the interview, Anna told us: *"There the difficulty is that they run away from you, [...] and you have to make little packets so that at the end of a [teaching activity], they take something home".* The intention, therefore, is to prevent interruptions in the lessons and to ensure that the student can grasp the objective of the teaching activity, integrating it with personal knowledge and thus benefiting from the learning. Teachers suggest that initially, it is better to introduce and partially formalise the mathematical content to involve all students and provide core information. Deepening and reinforcing the topic can be postponed for those who continue to attend classes.

f) If the goal is to involve all students who have not consistently attended, then, in a long-term detention context, scheduling with a flexible timeframe should be provided.

Argument: Class composition is more stable over time in institutions dedicated to prisoners with long sentences. This difference was referenced by Anna: "*At Monza, that was the class; from start to finish, it was always them. [...] so, I remember there wasn't this problem, of this, let's say, dispersion, loss.*" Despite this, the context contributes to a lack of consistency in students' attendance due to personal and systemic needs (e.g. medical visits, family and lawyer interviews, and other activities planned by the prison management). Moreover, school activities often last less than the established timetable due to issues related to the prison organisation (e.g., non-punctuality in moving students between classrooms and cells, interruption of lessons, and inspections). In this case, it is common for a student absent from some lessons to attend the following, necessitating brief review sessions of key concepts to ensure everyone can effectively participate in the teaching activity. It is challenging to predict this type of occurrence in advance. Therefore, a flexible approach to timeframes is advisable, with the expectation that the duration of planned teaching activities may need to be extended or shortened. This is made clear, for instance, in the teaching units collected in the initial phase: "*Note on timeframe: the planned timeframe took into account the various criticalities of the prison […] but, even so, I was not able to respect it and took 4-5 hours longer than planned"* (Camilla).

Finally, the last design proposition emerges with the specificities of the LeMP project:

g) *If the goal is to disseminate the design of a teaching activity among colleagues from other prison schools, then adaptations or alternatives must be provided, as the availability of specific facilities (classrooms and/or equipment) varies across contexts.*

Argument: The scarcity of accessible materials and technologies in prison schools constitutes a significant challenge for teachers. As mentioned in the design requirements, many resources that are standard in every Italian classroom are often either forbidden or inadequate in prison. Consequently, teachers seek guidance on how to adapt teaching interventions when working with limited materials. "*[I would like to develop maths topics] in a more engaging way, one that does not involve the use of technology. Because we at school now are used to doing everything with the Internet, the presentation, the documentary on the Internet... and there nothing, you can't, so you always have to reinvent everything"* (Camilla). In addition, the classroom equipment and tools are highly dependent on the specific institute in which one operates; therefore, in order to make a design replicable and generalisable, one must pay attention to the description of materials and tools used, how to obtain them, where possible, hypothesising different scenarios and envisaging some possible ways in which one could compensate for the lack of a didactic tool without distorting the teaching activity.

Conclusions and further steps

In this paper, we reported the results of the LeMP project exploration phase, aimed at understanding the multifaceted complexity of teaching mathematics in prison contexts, starting from teachers' experiences and perspectives. Data analysis allowed us to identify several design requirements in terms of freedoms and constraints, and thus to reflect on the aspects that teachers and researchers can influence in this context. Starting from the design requirements, seven design propositions were formulated. These expanded upon the design principles proposed by Maffia and Decembrotto (2024), particularly concerning students' intrinsic motivation and the personalisation of teaching interventions. Additionally, new considerations were introduced, including the role of assessment,

the timing of activities, the inclusion of broader objectives, and attention to the availability of materials for replication and generalisation. These results will allow us to design feasible and appropriate teaching activities for mathematics in prison that will be experimented with in the further steps of the project. Furthermore, from a theoretical point of view, design requirements and propositions will be refined and integrated to develop new design principles.

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Complexities when students authenticate aspects of a task in vocational mathematics education

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In this paper I study how students in vocational mathematics education can interact with paper-based mathematics tasks with a vocational context. I used the verb "to authenticate" when students (1) recognized that an aspect in a task really existed out-of-school and (2) gave a convincing argument for this. The data is from a mathematics class for Technical and Industrial Production students that worked on a task related to the volume of engine cylinders. I found that students could authenticate several vocational aspects in the task, such as the real-world context, the question/assignment, the illustration, the data, and the purpose. The authentication process consisted of referring to a vocational origin and showing expertise by describing details not given in the task. The results indicate that when students in vocational mathematics education are given space to use their out-of-school expertise, their vocational identities can raise their motivation to engage with mathematics.

Keywords: Vocational education, authentic aspects, mathematics tasks, communities of practice, students' identity.

Introduction

Mathematics as encountered in schools differs in many ways from mathematics encountered in workplaces. For instance, mathematical workplace problems are not stated clearly on paper but arise practically and messily. Also, solutions in workplaces may not need the working methods, symbols and precision as often required in school mathematics (FitzSimons, 2014; Saló i Nevado & Pehkonen, 2018). These differences pose challenges for mathematics in vocational education for adolescents. In many countries, the mathematical content in vocational education aims to be practical and connected to workplace settings (FitzSimons, 2014; FitzSimons & Boistrup, 2017). These aims can be implemented, for example, through team-teaching by a mathematics and a vocational teacher (Frejd & Muhrman, 2020), through transforming a classroom into a simulated workplace (e.g., a skin care saloon) and ask students to act professionally towards fictive clients (Dalby & Noyes, 2015) or using artifacts from workplaces as starting point for mathematical calculations (Sundtjønn, 2022).

In this paper, I indicate the compulsory mathematics taught within vocational education for adolescents as *vocational mathematics education*. An important constricting factor of this type of mathematics education is that the teaching of mathematics often happens within regular mathematics classroom settings. Frejd and Muhrman (2020) found that students' ways of working on mathematics tasks depended on whether the teaching happened in a mathematics classroom or in a workshop setting. Research, for instance Carotenuto et al. (2021) and Fitzpatrick et al. (2019), shows that students do not use realistic considerations when working on word problems in a school setting. In a regular mathematics classroom with rows of tables and a blackboard in the front, students can interpret tasks with a workplace context as standard word problems they have encountered many times before in their schooling. This suggests that *only* describing a vocational context in a paper-based task does not sufficiently simulate students' future work life.

To better understand how students in vocational mathematics education can interact with paper-based mathematics tasks with a vocational context, I aimed to study the role of *authenticity* in these settings. I follow Vos (2011) in saying that a task created for schooling cannot be fully authentic, since school environments allow for errors to learn from, unlike workplaces, where errors can have serious consequences. However, as Schlüter and Besser (2024) argue, separate aspects within a task can be authentic. Vos (2018) defined that for an aspect within a task to be authentic, it needs to have (1) an out-of-school origin and (2) a certification of originality. The certification can, for example, be a photograph (see Figure 1) or a testimony from an out-of-school expert explaining that the artefact really exists.

The aim of this paper is to study the processes of judging aspects in tasks as authentic by students in vocational mathematics education. I indicate this judgemental process as *authenticating*. So far, the research literature has focused on how authentication can be provided by out-of-school experts, or teachers from the vocational courses. For this paper, I explore how students in vocational mathematics education can showcase their perceptions of authenticity in vocation-related mathematics tasks. The research question is: *how do students in vocational mathematics education authenticate vocation-related mathematics tasks in a classroom setting?*

Theoretical framework

To analyse processes within vocational mathematics education, I use the notion of *communities of practice*. Within a community of practice, members have a common enterprise, they have a shared repertoire of actions and ways of interacting (Wenger, 1998). Newcomers will not immediately fit in and need to be enculturated into the rules and routines of the community of practice. Students in a mathematics class can be perceived as a community of practice. In this particular community, students have many years of experience of how to behave and what is regarded as the "normal" ways of working with mathematics tasks, with learning mathematics and how to deal with teachers. Simultaneously, they will be members of communities of practices in out-of-school domains, such as being a player in a sports team, or working part-time in a shop.

A special feature of vocational education is the aim to prepare students to find their way into workplaces. Through vocational internships, students become what Wenger (1998) describes as *legitimate peripheral participants,* in the community of practice of their future vocation. By receiving increasingly more responsibility on the work floor, they become enculturated into the rules, routines and requirements in workplaces. Through the lens of Wenger's (1998) concept of communities of practice, students in vocational mathematics education are members of a mathematics class with a growing vocational identity. In Frejd and Årlebeck's (2024) literature review, the contribution of mathematics education to this process of vocational enculturation can pertain to learning about 1) tools or artifacts used at the workplace, 2) how workers cooperate, 3) taking a clients' perspective, and 4) developing a worker's identity and professional pride.

As said before, paper-based vocation-related mathematics tasks can never be fully authentic, but they can contain authentic aspects from workplaces. For instance, authentic tools and artefacts used for making calculations or measurements at the workplace can be a start for mathematical activities. Schlüter and Besser (2024) give further explanations and backgrounds of the term *authenticity*. They

created a *Model for Authenticity in Modelling Test Items* (MAMTI), which is a framework for analysing authentic aspects within paper-based mathematical modelling tasks. The MAMTI-framework offers categories for aspects of mathematical modelling tasks that can be judged as authentic or not, such as the real-world situation described in the task, or the tools used in the task. Schlüter and Besser (2024) describe for each aspect, how these can be authenticated. For a *real-world context* to be authentic, the context needs to clearly exist in real life. For an *event* to be authentic, all activities described in the task can potentially occur. For *objects* to be authentic, these must be objects derived from reality, just like authentic *data* are data that are real. The category *question/assignment* is about whether the posed question in the task would be asked in the same way by people in the situation. The category *figures* is about whether illustrations originate from the real world. The category *use of mathematics* is about whether the mathematics is used in the same way as a person in the real-world context would. The category *purpose* is about whether the work done would fulfil a need in the real-world. For further explanations of the categories, see Schlüter and Besser (2024).

The MAMTI-framework can be used to analyse mathematical modelling task before their implementation in a classroom. However, an important tension arises whether and how students judge the authenticity of these aspects in tasks. For an in-depth analysis of this authentication process, I draw on Wenger's (1998) description of communities of practice, in particular on students' identity as a member of a mathematics class and their growing vocational identity.

Methods

In Norway there are many different vocational strands (e.g., building and construction; restaurant and food processing; healthcare, child and youth development; crafts, design and product development, etc.) and all programmes has a compulsory mathematics course in the first year. The data were gathered in vocational mathematics education for Technical and Industrial Production. The class consisted of 16 students, around 16 years of age. Although I do not have the students' mathematics achievement history, many students in vocational education in Norway often have a strained relationship with mathematics.

The data consisted of transcripts from video recordings of students working on the task Engine Cylinders, explained below. These data were transcribed and anonymized (Student A, Student B, Student C, etc., Teacher, Trude). At an earlier stage, these data have been used for Sundtjønn (2021); for the present study they were re-analysed. The task was about calculating volume of engine cylinders, first for moped engines and then for car engines. The start of the task is given in Figure 1. The lesson started with an introduction by the teacher, then the students worked individually, before a discussion in the whole class. During their individual work they talked together, and with the teacher and with the researcher, which yielded episodes to analyse.

The task first offered students two *illustrations*: a table snipped from a brochure on moped engines, and a picture of a moped cylinder block, see left side of Figure 1. On the righthand side, there is a stylized illustration and a series of mathematical questions that the students were asked to solve. These pertain to calculating volumes of cylinders when diameters are given; they aim for students to practice using volume formulae. In terms of the MAMTI-framework (Schlüter & Besser, 2024), a number of aspects in the task can be recognized as authentic. For instance, the type of moped cylinder

and its dimensions are authentic as certified by the given photograph and the photocopied technical specifications. This means the object given in the task and the data are authentic. Likely, the legal regulations regarding a maximum stroke volume for mopeds are also authentic, since they can be looked up. On the other hand, the posed questions seem inauthentic, since there is no evidence that such calculations are carried out by vocational workers repairing or installing moped cylinders.

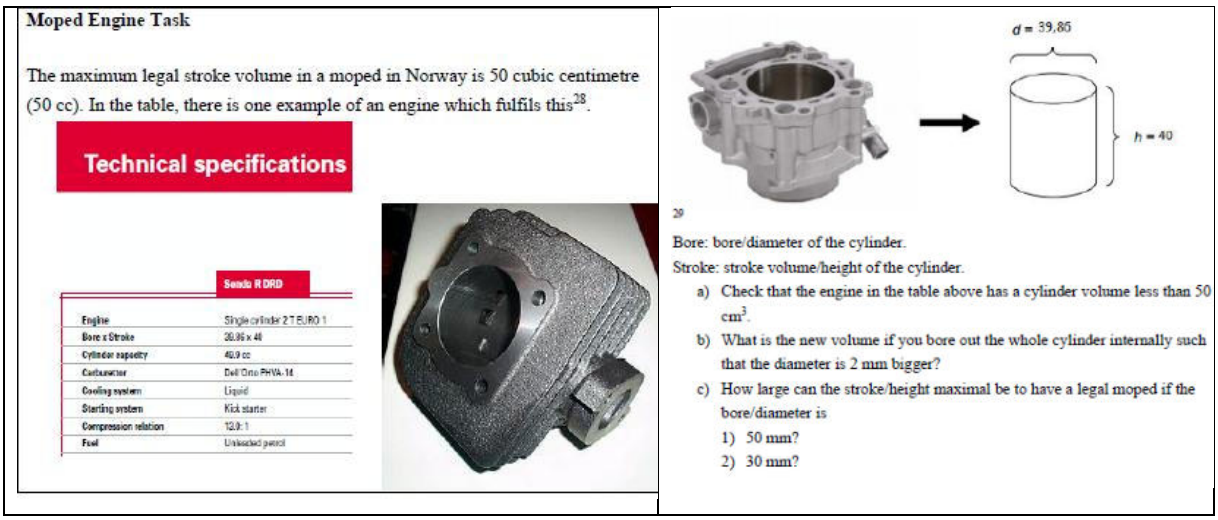

Figure 1: The start of the task *Engine Cylinders*

For the analysis of the students' utterances, I used the MAMTI-framework (Schlüter & Besser, 2024) to categorise the aspects in the task that the students authenticated. However, I did not aim to find that all aspects would be authenticated. I used Wenger's (1998) communities of practice concept to analyse how the identity of students as member of the mathematics class and/or as future vocational worker played a role in the process of authentication.

Findings

In the presentation of findings below, I exemplify how the students authenticated different aspects in the Engine Cylinders task or refuted the authenticity. The aspects from the MAMTI-framework that students discussed were: real-world context, events, objects, question/assignment, figures, and purpose. A potential authentication of the aspect *use of mathematics* did not show in the data. With regards to the *real-world context* in the task, several students owned mopeds. They started talking with each other about their out-of-school experiences with modified moped engines. For instance, during the work on the task in Figure 1, Student A explicitly said: "I'm trying to figure out my own cylinder", and then started calculating the cylinder volume of his own moped. Later, when Student A shared with the class how he had calculated the volume of his own moped cylinder, another student asked "which pistons? How do the pistons go around?". Another student asked which kind of brand the cylinder was. In these utterances, the students collectively judged that the context of the task had an out-of-school origin. Moreover, they showed their knowledge as moped owners, knowing different brands, and understanding that the volume relates to the moped's performance. In this way, their identity as "moped experts" drove the authentication of the task context. Later in the lesson the students worked on calculating volume of car engines cylinders, different sized V4, V6 and V8

engines. Student B said when he explained how he had found his answer that he "thought of another [known] engine and compared them". He had been helping part-time in a car repair shop, and here showed his knowledge and authenticated the *real-world context* in this part of the task.

Student A's knowledge of the context also showcased how another task aspect, namely the *question/assignment* was authenticated by the students as experts who brought out-of-school knowledge into the classroom. When Student A talked about "figuring out my own cylinder", another student asked if he had drilled it to become bigger (a larger volume enhances the motor's performance). Student A replied "it does not help that much", and later confirmed that he had his cylinder block replaced instead. Hence, the students authenticated the questions regarding the volume of the cylinder because this volume question matters to moped owners who want to improve a moped's performance. Simultaneously, the students attested that the calculation of volumes was not an aim in itself but mattered for getting more moped speed through an adjustment of the cylinder (increasing its volume or replacing it). For the students as participants in a community of owners of (illegal) mopeds, the authenticity of a *question* on finding volumes was directly related to the underlying *purpose* of finding more speed.

The task aspect *illustration* was authenticated in the following way. After the class, the teacher told in an interview, that Student A had been shown the printed sheet with the task (see Figure 1) and told "today we are calculating car engines". Student A had immediately said "eh, that is not a car engine, it is a moped engine", with just a short look at the illustration in the task. The student thereby showed to be an expert able to authenticate *the illustration*. When Student A was later asked whether he knew the volume of a regular car engine cylinder, he clearly stated "no". So, he obviously was a moped engine expert but knew his limitations. During the work, the students were critical of the *data* in the task. For instance, when calculating volume of car engines, Student C pointed out that the given volume, 3.5 litres, was not much in a V8 engine. Student D said that in a V4 engine the given number was "yes, I think it is too large. It's 2.8 in my [known engine]". In this way, students showed their expertise regarding engine cylinder volumes of cars without a need to look these up. If real data had been used in the task, the students could have authenticated these.

With regards to the *use of mathematics*, there turned out to be several aspects. In the first place, there was the issue of mathematical *precision*. For instance, Student A did not use his identity as a moped owner but asked the researcher what would be considered as "right". To solve the task in figure 1, he had used the diameter 39.86 from the technical specifications (see Figure 1, on the left), and divided it by 2 to obtain the radius needed for the volume calculations. Then, he rounded off the answer from his calculator to 20.

Student A:	Should it be very precise?
Trude:	What is precise? What should I say?
Student A:	Can I just take 20?
Trude:	You can try it with 20 and be very near, yes.

In this episode, Student A asked for the expertise of the researcher. It means that he could not relate the use of mathematics to his out-of-school moped knowledge. Instead, he considered the researcher as an expert on the use of mathematics, and she did not relate it to out-of-school use either. With no out-of-school origin, the precision of the calculations remained inauthentic. Another aspect in the

category *use of mathematics* was the *tool use* by students for performing calculations. Rather than using a school calculator, the students preferred to use their mobile phones. It had sufficient precision, and they had it readily in their pockets. Using their mobile phones looked like a routine, as if they were not in a mathematics classroom. Being a tool from out-of-the-classroom, and being habitually used by the students, it can be analysed as a non-verbal authentication of the use of this tool for making calculations within vocational contexts.

With regard to this same aspect, *use of mathematics*, another observation was also relevant. The students often tried different numbers. Student A had worked diligently on his calculations to figure out the size of the diameter of a cylinder with height 65.6 mm in a 3.5 l V8 engine. I asked him if he saw approximately how the size of the numbers should be. He then confirmed that he knew moped cylinder sizes. Thus, he again used his expert knowledge on moped engines, also for his calculations. Here is how he explained how he solved the problems.

Student A:	I tried my way to do it.
Teacher:	You tried your way again. You're good at trying; it seems like you mostly get the answer, so that's good. Yes, and how do you do that when you try your way forward then?
Student A:	Take different numbers.
Teacher:	3.14 times some radius times some radius times 65.6 times 8?
Student A:	Yes.
Teacher:	Is there anyone who has calculated differently, or have you [also] done trial and error? (...) Are there any others who have tried and got the same? (Silence 4 seconds). No, but [Student A] has at least figured it out. Okay, there are some who has tried an equation then?
Student E:	I tried, but it didn't go well.

In this exchange between the teacher and the class, Student A explained that he tried numbers. According to his earlier exchange with me, these came from his knowledge on moped cylinder sizes. So, he did not try random numbers, but made *intelligent guesses* to approximate the solution, and thereafter refine the numbers to find the answer. Student A's approach parallels the numerical methods in advanced mathematics for solving ordinary differential equations (e.g., Euler's method, Runge-Kutta method). This is a well-functioning mathematical method often used in vocational communities of practice. Unfortunately, this typical use of mathematics was not explained to the whole class. The teacher named this method "trial-and-error" and judged it as "good" because the correct answers were reached. Yet, by asking for an equation, the teacher showed what solution method which was really valued in this classroom community. We see here two processes that hinder potential authenticating of out-of-school solution methods. First, Student A does not clearly explain his use of pre-knowledge to the teacher and the full class as he explained it to me. Second, the authority of the mathematics teacher closed down opportunities to value a well-functioning method.

Discussion and concluding remarks

My research question was: *how do students in vocational mathematics education* authenticate *vocation-related mathematics tasks in a classroom setting?* Regarding the aspects in the MAMTI-framework (Schlüter & Besser, 2024), I found that students could authenticate several of the aspects in the task, such as the *real-world context,* the *question/assignment,* the *illustration,* the *question,* the *data,* and the *purpose.* The authentication process, as discussed by Vos (2018), consisted of pointing

at a vocational origin and showing expertise in how it was used there. The authentication process was supported by students' obvious, and sometimes extensive experiences and knowledge from their participation in vocational, out-of-school communities of practice. By expressing their knowledge of the vocational context through details not given in the task, they established their expertise. Contradicting Vos (2018), there was no need for an out-of-school expert to convince the students of the authentic aspects. Because of their participation in multiple communities of practice, the students could act as out-of-school experts.

Authentication of the task aspect *event* was not observed because the task did not describe humans undertaking an action, and hence, the tasks did not contain an event. Schlüter & Besser (2024) point out that that there may be differences between if aspects of a task are authentic. In this case, *the questio*n to find a volume from a given moped cylinder diameter is authentic, but the *solution method*, calculating it with a formula and calculator, is not authentic. In the workshop the students would probably look it up from the technical specifications in a catalogue. The authentication of the task aspect *use of mathematics* turned out to embrace several aspects, such as *precision, tool use* and *solution methods.* The analysis shows that all three aspects could have had out-of-school origins, with *precise numbers* being pragmatically rounded off, with *tools* being those that are habitually used out-of-class (mobile phones), and *solution methods* allowing for educated guesses and stepwise approximations. However, there seemed to be too little space in the classroom community for authenticating students' ways of dealing with *precision* and *solution methods.*

The study shows that students' expertise from their participation in out-of-school communities can support the use of paper-based vocation-related mathematics tasks within vocational mathematics education. Students in vocational education can bring highly relevant knowledge that a mathematics teacher may not have (Sundtjønn, 2021). Such expert students can authenticate task aspects to other students and to the teacher, indicate what might be relevant for future learning and hence raise motivation within the classroom community. Important factors that can facilitate such authentication processes are the "space" that students in vocational mathematics education have for judging, for using their experiences, and for working autonomously on what is relevant in future vocations.

The study also shows that task designers and teacher may perceive certain aspects in a task as authentic, whereas the task turns out to also contain inauthentic aspects that work demotivating. For instance, an emphasis on precision and the use of equations in the Engine Cylinders task could demotivate exactly those students who have expertise about the absence of such mathematics in these vocational communities of practice. Further work is needed on tension between the use of authentic real-world context and other aspects, like the questions, that may not be authentic. The results indicate that when students in vocational mathematics education are given space to use their out-of-school expertise, their vocational identities can raise their motivation to engage with mathematics.

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Measuring adults' numeracy: Which instruments are used and what do they measure?

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Keywords: Numeracy, instruments, adults, Common European Numeracy Framework.

Introduction and theoretical background

In our review of research on numeracy and financial difficulties (Bor-de Vries et al., 2025), we identified a variety of instruments used to measure numeracy. Some studies, such as the Berlin Numeracy Test (Cokely et al., 2012), make use of well-tested and validated instruments, while others rely on single questions specifically designed for their own research. These instruments—mostly questionnaires—all aim to measure numeracy, but they approach it in different ways. By examining these instruments more closely, we can better understand and compare findings across studies. This poster is based on the instruments found in the literature review by Bor-de Vries et al. (2025) and addresses two questions: What instruments are used to measure numeracy? What aspects of numeracy do these instruments focus on?

The numeracy aspects of the instruments can be described in terms of the Common European Numeracy Framework (CENF) (Hoogland, 2023) as theoretical framework. The CENF emphasises four aspects that contribute to improving the quality of numerate behaviour: Context, Knowledge and Skills, Higher Order Skills, and Dispositions. Each of these aspects has different sub-aspects, for example, the aspect Context contains: Everyday Life, Work-related, Citizenship, Further Learning, Finances, Health and Care, and Recreation. For more detailed information on the CENF, see: www.cenf.eu/framework.

Method

In our systematic literature review (Bor-de Vries et al., 2025), we identified 24 numeracy instruments and their variations. Certain instruments, such as the PIAAC numeracy assessment or subtests of IQ tests, were unavailable for public access and thus excluded from our analysis. For the remaining instruments, we compiled a list of all items—questions—included and checked for duplicates, resulting in 155 remaining items.

Example item: *Do you think that 10% is bigger than 1 out of 10, smaller than 1 out of 10, or the same as 1 out of 10?*

Each item was coded using a codebook to determine which aspects and sub-aspects of CENF were addressed. To gain deeper insight into the different numeracy instruments, we then aggregated the codes for each specific instrument. To enhance the clarity of the findings, we utilized an iceberg model to visually illustrate the results for each instrument; see Figure 1 for an example. The model illustrates the four aspects of numeracy, with the numbers representing the sub-aspects. The iceberg

model highlights that although certain aspects of numeracy are explicitly captured in numeracy instruments, others are less visible but nonetheless contribute to variations in test scores.

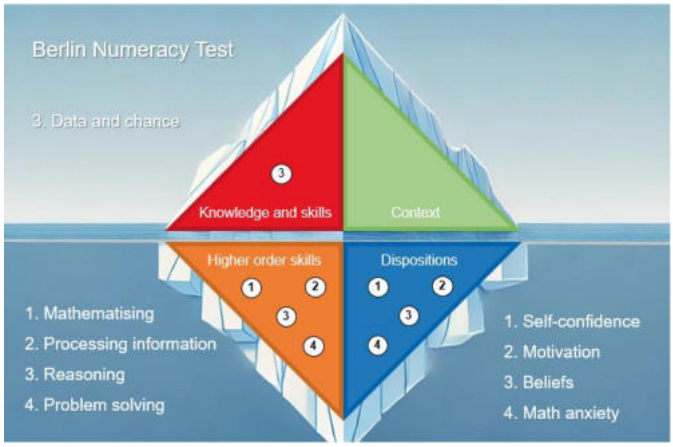

Figure 1: Iceberg model representing numeracy aspect and sub-aspects of Berlin Numeracy Test

Preliminary results and discussion

The instruments vary in length and range from one single item to 23 items, with an average of 6.2 items per instrument. Most questionnaires are short; 19 out of 24 contain five or fewer items. A substantial number of instruments focus on the CENF aspect of *Knowledge and Skills*, particularly the areas of *Quantity and Number*, as well as *Data and Chance*. Many of the items do not include real-world contexts, but when *Context* is provided, it typically relates to *Health and Care* or *Finance*. Some instruments assess *Higher Order Skills*, particularly in the areas of *Mathematising* and *Processing Information*. Instruments not measuring skills do measure *Dispositions*, especially *Self-confidence* and *Math Anxiety*.

Our findings indicate that the instruments used to measure numeracy differ both in their quality and in the particular aspects of numeracy they assess. Since most instruments focus on a limited part of numeracy and include only a small number of items, they are insufficient for assessing the full scope of numeracy. Moreover, comparing results is difficult because the various instruments emphasise different aspects of numeracy. We advise researchers to specify which aspect of numeracy they are evaluating to help other researchers, policymakers, and educators better interpret the results.

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How is Europe doing on numeracy in the 2nd cycle of PIAAC?

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Keywords: *PIAAC second cycle, numeracy, numeracy assessment, adult skills.*

Introduction

Since the significant 'mathematisation' of our society, driven by rapid technological developments and digitalisation, educators and policymakers have increasingly expressed concern. They argue that large segments of the working population must develop and maintain their numeracy skills to participate autonomously and effectively in today's technology-driven, data-saturated world (Hoogland & Díez-Palomar, 2022). A lack of numeracy competence can lead to difficulties in managing everyday tasks and navigating the modern workplace. To gain a better understanding of the current state of adult skills, the first international comparative assessment, the International Adult Literacy Survey (IALS), was conducted in the 1990s, followed by the Adult Literacy and Lifeskills Survey (ALL). These efforts culminated in the global OECD initiative known as the Programme for the International Assessment of Adult Competencies (PIAAC), which assessed skills in literacy, numeracy, and adaptive problem-solving through two cycles conducted in the 2010s and 2020s (Tout & Gal, 2015).

The definition of numeracy has evolved over the last decades (Kaye, 2018; Hoogland, 2023; Hoogland & Díez-Palomar, 2022), culminating in the definition used in the second cycle of PIAAC.

> Numeracy is accessing, using and reasoning critically with mathematical content, information and ideas represented in multiple ways in order to engage in and manage the mathematical demands of a range of situations in adult life. (OECD, 2021, p. 93)

This reflects the broad, social, and holistic perspective now commonly applied to functional mathematical skills. Examples of this approach can be found in the materials developed in the project Common European Numeracy Framework (Hoogland, 2023).

Method

Our research question for this poster is: 'How is Europe doing on numeracy in the 2nd cycle of PIAAC?' The data we use to answer this question is collected through the assessment within PIAAC, known as the Survey of Adult Skills (OECD, 2021). This is a tablet-based household survey of adults in the age range 16 to 65 years. The survey assesses proficiency levels in literacy, numeracy and (adaptive) problem solving. We will use for our study the country scores on numeracy as reported in the OECD overall report and database (OECD, 2024). The 2nd cycle of the Survey of Adults Skills has been conducted in 31 countries and economies so far. A first round of data collection took place in 2022-2023.

Main results

Compared to Cycle I of PIAAC (2012–2018), numeracy scores in the 2nd cycle have increased in 8 of the 31 participating countries, remained stable in 12, and declined in 7. Notably, most of the countries showing a decline are located in Eastern Europe. Alongside these overall trends, the data reveal a growing disparity in numeracy performance within countries—an increase in inequality that mirrors patterns observed in other recent large-scale assessments such as PISA and TIMSS. While the gender gap in numeracy performance—historically favouring men—has narrowed slightly, it remains significant, with men scoring approximately 10% higher on average (OECD, 2024, p. 52).

One particularly relevant indicator for policymakers is the proportion of adults performing at Level 1 or below. In PIAAC terms, individuals at this level are likely to face regular challenges when confronted with everyday numeracy tasks. On average, 25% of adults across participating OECD countries and economies fall into this category, with national figures ranging widely from 10% to 56% (OECD, 2024, p. 63).

Conclusion

Alongside the cognitive scores, a wide range of data was collected from participants through a basic questionnaire. This enables a more in-depth analysis of the skill scores by correlating them with socio-economic factors and the actual behaviours of participants. For example, participants self-report the frequency with which they perform calculations, such as calculating prices, costs, or quantities. These can range from daily to less than once a month or never. The results of such analyses will be disseminated to the scientific community in the coming years and can inform the development of policies around adult education and lifelong learning in various countries.

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Numeracy in social work education

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Keywords: Numeracy, social work education, lifelong learning, mathematical literacy, Common European Numeracy Framework (CENF).

Introduction and theoretical background

Data from the Programme for the International Assessment of Adult Competencies (PIAAC) show that a substantial number of citizens are lacking essential numeracy skills (OECD, 2013). Recent studies (e.g., Gal et al., 2020) have shown that insufficient numeracy skills are a significant risk factor for increased vulnerability in social functioning, manifesting in health and financial problems, and having an even greater impact than insufficient literacy skills. Social workers often encounter clients facing daily challenges linked to inadequate numeracy skills, positioning the social workers to identify such issues early. With sufficient numeracy skills, social workers can better detect and address these problems, ultimately helping clients mitigate or prevent life difficulties. This support enhances social participation, boosts self-confidence, and fosters self-reliance (Gal et al., 2020).

In response to these challenges, this study contributes to the broader discourse on numeracy in professional and societal contexts by focusing on how numeracy is embedded in social work education. We investigated: (1) to what extent Dutch social work education programs focus on developing students' numeracy skills, and (2) how these programs address students' understanding of clients' numeracy skills and needs.

Method

An educational program operates on three levels: the intended curriculum (educational documents and competency profiles), the implemented curriculum (actual lessons), and the enacted curriculum (what students ultimately learn and apply). By combining a document analysis of the intended curriculum in Dutch social work programs with a survey (approximately 160 students) and interviews (10 graduate social workers), we aimed to assess the extent to which numeracy is sufficiently developed within the program. Using a matrix derived from national and international frameworks (e.g., Hoogland et al., 2019) and publications on numeracy (Drijvers et al., 2021), we analysed the intended curriculum for evidence of numeracy integration across four key aspects: context, knowledge and skills, higher-order skills, and dispositions. This approach provided a structured framework to evaluate whether numeracy was explicitly or implicitly integrated into the intended curriculum. The questionnaires explored students' understanding of numeracy, their experiences working with low-numerate clients, and their educational needs related to numeracy. Semi-structured interviews with students offered additional insights. The aim is to verify our findings in the intended curriculum by assessing their alignment with the enacted curriculum, with the results reviewed by a feedback group of students, educators, and numeracy experts to ensure accuracy and consistency.

Preliminary results and outlook

The results of this research provide insights into the extent to which numeracy is embedded in social work programs and how students perceive these skills in relation to their professional practice. Typically, education programs teach aspects of numeracy through generic courses in arithmetic, mathematics or statistics. These courses primarily focus on mathematical content, placing less emphasis on the broader concept of numeracy, as defined in the Common European Numeracy Framework (Hoogland et al., 2019). This indicates that while students are exposed to numeracy, the curriculum may not fully integrate numeracy in ways directly relevant to social work practice. Consequently, recommendations will be made to strengthen the focus on numeracy within the curriculum, specifically tailored to the practical needs of social workers.

In addition, based on preliminary results from the survey, students seem to recognize the importance of numeracy, as reflected in responses to questions such as "I need numeracy in my internship" and "I find it important that my education addresses numeracy in my future profession." However, responses to questions like "Topics in numeracy lessons relate to social work" suggest that many students perceive a lack of connection between numeracy education and practical social work applications. These early findings indicate that current programs may not be fully preparing students to understand and address the numeracy skills and needs of their clients effectively. This highlights the need for curriculum adjustments to better equip students with the skills necessary to recognize and support numeracy-related challenges among their clients. These findings can be incorporated into existing social work education curricula and professional training programs, as recommended by Drijvers et al. (2021). Moreover, this study may contribute to refining or expanding the professional competence profile of social workers, thereby enhancing the effectiveness of interventions addressing sufficient numeracy skills in social work practice.

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Adults studying for a general certificate in secondary education (GCSE) in mathematics in England: self-efficacy, anxiety and examination grades

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Keywords: Adult learning, self-efficacy, anxiety, examinations.

Introduction and Methods

GCSE mathematics is a high-stakes examination necessary for many university courses in England, such as nursing or teaching degrees. In 2020 there were over 30,650 adults enrolled for the examination, approximately 10% of the total. This research was conducted with 21 adults in Further Education (FE) colleges, an under-researched group of learners (Gal, 2024). The research questions were: What are the perceptions of adults enrolling for GCSE mathematics courses, in terms of their self-efficacy and anxiety about the course content, aspects of the classroom environment, and assessments? Are participants' perceptions correlated with their examination results, i.e., are participants with low anxiety and high self-efficacy more likely to pass the examinations? Finally, do participants' perceptions of examinations vary between mathematics and others?

This was a small-scale, mixed methods study and whilst no generalisations can be made from the findings, the research revealed several findings which made an original contribution to understanding the importance of the perceptions of adult learners, which indicates that a larger study is needed. There were 21 participants in total from at least two colleges in England and two academic years.

Findings from other research can lead to beliefs that can be described as stereotypes, which can be detrimental to progress in adults' classrooms. For instance, in other research, older learners were found to be more anxious than younger ones (Watts, 2011), females were more anxious than males according to Szucs et al. (2017), and English speakers can have an advantage over those whose first language is not English in mathematics classrooms (Woolley, 2013).

This research used a questionnaire with two attitude scales on 15 statements: one on self-efficacy (Bandura, 1997; Swan, 2006) and one on anxiety. The statements were mainly taken from the existing Abbreviated Math Anxiety Scale (AMAS) by Hopko et al. (2003), with six additional statements, namely a word problem, a percentage , a fraction , a statement on drawing charts and graphs, one on asking a question in class and another on perceptions of non-mathematics examinations. Each statement required self-efficacy and anxiety rankings, and space for (optional) comments was given.

The data analysis used thematic analysis to cluster both the ranked responses and the comments made on each scale, using main themes, sub-themes, emergent themes, and learner characteristics, to both enhance understanding and protect the identity of participants (Braun & Clarke, 2013).

Findings

For the main themes of self-efficacy and anxiety, it was a mixed picture, but self-efficacy was a better marker for passes in the examinations than anxiety, because most of those with high self-efficacy, as

judged by the median values, passed the examination (8 out of 10). However, some did not (n=2). Some with low self-efficacy and high anxiety also passed (3 out of 9) but both of the learners who withdrew had low self-efficacy and high anxiety (n=2). Tests and examinations were considered the most challenging for all, with lower self-efficacy and higher anxiety rankings than course content or classroom issues.

For the sub-themes, based on statements in the questionnaire, the word problem of 'if it takes three people 5 days to fit a kitchen, how long will it take two people?' had the least self-efficacy and the highest anxiety expressed; algebra statements came second to the word problem. Some participants (4/21) were unable to ask a question in class, even the most able as judged by examination results.

Half of the participants made comments, which allowed for an analysis of emergent themes from the data. The themes showed a belief in innate ability, with comments such as "Maths has always been a failing point for me, I just can't do it", and "I don't have a mind that can learn maths easily". A fatalistic approach to examinations also emerged with "It will be what it will be". There was an awareness of the negative impact of time pressure on thinking and performance, as a participant said, "I read too much into it and panic". The importance of a sense of community was identified by some participants with "We are all in this together", and "I enjoyed being with my peers". A meta-cognitive benefit of participation was also described: "I think that it has been quite a cathartic process for me".

In terms of learners' characteristics, males (n=3) were just as anxious as females (n=18). In addition, older learners (25+) were less anxious, had greater self-efficacy, and were more likely to pass (11 out of 17) than younger ones (one out of four participants passed). Those whose first language was not English were more likely to pass (three out of four) than the native English speakers (10 out of 17).

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Bridging the gap: Facilitating female adult learners in the East Midlands region of England to re-engage with mathematics

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Introduction

Mathematical knowledge and skills are considered essential tools for further and higher education, employment, future earnings, health and well being. However, the numeracy proficiency level of adults in England is quite low, and nearly half of the adult working population have numeracy skills expected of an 11-year-old (National Numeracy, 2017). Over the years, successive governments have initiated many programmes to provide free mathematics education for adult learners, emphasising that good numeracy skills not only unlock job opportunities and lead to higher wages, these are also useful in day to day life such as for budgeting money and helping children with homework. Yet, returning to formal education is a daunting task and potential learners are reluctant to re-engage due to unpleasant prior experiences of maths, stereotype threat related to age, race or gender and other barriers (Syyeda, 2021).

Benefits of re-engaging in mathematics learning

Studying on adult maths courses can be a transformative experience for women, especially among BAME (Black, Asian & Minority Ethnic), migrant and disadvantaged communities. It empowers them to become confident and agentic, and can have a significant impact on their families (Syyeda, 2021). Migrant mothers often use participation in adult learning courses as a crucial tool to integrate their families in the host society (Kelly, 2018). Many women are motivated to enrol on maths courses to equip themselves with the knowledge and skills needed to help their children with homework and enable them to progress in school. Moreover, working on mathematical problems together creates opportunities to spend quality time with children and bond (Syyeda, 2021).

Context and aim of this research

This study is based in a multicultral city in the Midlands region of England and is run in collaboration with a local adult education college. The college has its main campus in the city but also offers maths courses in satellite centres in adjoining areas. There is significant variation in the rate of student enrolment in different areas, with some centres having no intake, which is a cause of concern. The purpose of this research is to understand the reasons for this variation and facilitate potential learners in disadvantaged areas to enrol. The study will be guided by the following research questions:

RQ1: What are the key enabling and disabling factors that impact the participation of women in maths learning, especially in low income, disadvantaged and minority ethnic communities?

RQ2: How does engaging with mothers in primary schools to raise awareness about the provision of maths courses and to facilitate them to enrol influence their participation in these courses?

Methods

This study adopts an Action Research design with data collected through focus group interviews with current and potential female adult learners. Action Research focusses on a problem rooted in a specific context, plans and implements an intervention and evaluates the change; a "process based on a continuous interaction between researcher, action, reflection and evaluation" (Hart, 1996, p. 454). Action research is appropriate because the aim is to address educational inequality and emancipate the voices of disadvantaged women. This study has three phases: in phase 1, we analysed secondary data to map the rate of student enrolment in different parts of the city and conducted three focus groups with current students in areas of high enrolment. In phase 2, we will run workshops in three primary schools and invite mothers/grandmothers and children to participate in fun maths activities. The selected schools are located in disadvantaged areas near adult education centres where enrolment on maths courses is very low. During the workshops we will provide information on courses offered and share success stories of mothers who have previously enrolled and reaped benefits for themselves and their families. The workshops will create opportunities to engage with potential learners and invite them to participate in focus groups to understand their perspective and make the courses more accessible for them. In phase 3, we will look at data on student enrolment for the next iteration of courses to capture any change in student numbers as a result of our intervention in selected areas.

Results

Preliminary findings from phase 1 show that women in these communities have insufficient access to information about the availability of maths courses. Therefore, engaging potential learners in disadvantaged areas through primary schools could be a positive way forward, particularly as this hooks into a key motivation of being able to help with their children's education. The findings of this research could provide useful insights for regional providers of mathematics courses and may also be beneficial for other countries in similar contexts.

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TWG 08: Affect and the Teaching and Learning of Mathematics

Introduction to the work of TWG08: Affect and the teaching and learning of mathematics

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Introduction

The Thematic Working Group 8 (TWG08) "Affect and the teaching and learning of mathematics" has been in CERME since 2003 (CERME 3) and the contributions have led us towards understanding the field better as well as developing new questions that should be answered in the past 22 years. Our call for papers for CERME 14 listed a diverse example of topics and we had contributions with different but related topics. In this section, we would like to introduce the work that has been presented and discussed in TWG8 in CERME 14.

A total of 20 papers and 2 posters had been submitted to TWG8 and 13 papers and 2 posters were accepted, another 2 papers were proposed to become posters, for presentation at the conference after a rigorous revision process. The contributors were from institutions based in a variety of countries, from Europe and from outside Europe. We had several contributors who participated in our group for the first time.

We started with introducing our group and how we planned to work in the sessions, followed by an ice-breaking activity. Considering that there were several new comers, a review of previous work done in TWG8 was presented and contributors were given tasks to discuss the recent directions pointed in the presentation. We planned the next five sessions to have other ice-breaking activities, all of them conducted brilliantly by Hanna Viitala, and the contributions have been briefly presented (10 minutes) followed by a short discussion (5 minutes) for each individual paper, plus a session of 30 minutes dedicated to raise general and overarching themes that emerged from each session of presentations. The 30-minutes discussions have been prepared by Janina Kravitz, Robin Goeller and Chiara Andrà, while Domenico Brunetto prepared and led the final session dedicated to a discussion about the work of the entire WG. The organisation of the sessions allowed us to create a friendly and welcoming atmosphere, together with in-depth discussions about our researches, seeking for links and connections among our research interests.

Reflection on prior research in affect

The focus on affect has gained attention since McLeod (1992)'s review paper on the affect constructs. He addressed the major affect constructs as beliefs, attitudes and emotions and characterized them based on their stability and affective-cognitive properties. Beliefs were considered the most stable over time, the least affective and the most cognitive. Emotions were considered as the least stable, the most affective and the least cognitive, with attitudes in between beliefs and emotions. This categorization was followed by numerous research and elaboration within the affect researchers. CERME5 discussions in this group focused on clarifying definitions of the concepts, enhancing the field by new concepts and specifying the relationships between these concepts (Hannula et al., 2007). Hannula and colleagues (2007) identified motivation, rather than attitudes, as one of the major affect categories. In the following years, based on Hannula (2011)'s analysis of affect research in mathematics education in CERME7, affect constructs beliefs, motivation, and emotions have integrated both state and trait dimensions and social, psychological, and physiological dimensions in the presented contributions (Schukajlow et al., 2019).

Affect can also be viewed by focusing on (a) the object of the affect (such as mathematics or task performance), (b) the subject of the affect (such as teachers, students, or preservice teachers), and (c) valence (such as positive or negative) (Schukajlow et al., 2017). These characteristics can be addressed with acquisitionist or participationist theoretical approaches. Affect constructs are closely related to each other (Goldin et al., 2009; Gómez-Chacón et. al., 2023; Hannula, 2011) in systems and change in one construct results in change in the others.

Contributions in the TWG8

We had the opportunity to learn and discuss interesting studies this year. There were five studies that focused on motivation. Schukajlow and his colleagues, through a quantitative study conducted with lower-secondary school students, examined the motivation behind using the questioning strategy while solving modelling problems (i.e., the strategy to generate questions about the situation described in the task), focusing specifically on the relationships among three components of strategy-based motivation: self-efficacy, values, and costs. Attention has been paid to gender differences in these three components. Interestingly, findings revealed a positive relationship between self-efficacy and values, and lower self-efficacy in the questioning strategy in female students. Viitala's research addresses the descending trend of secondary pupils' mathematics performance (between cohorts) and pupils' mathematics-related affect that develops negatively throughout the school years (within cohorts) by implementing and monitoring a collaborative problem-solving intervention. The results show that while pupils value learning mathematics, they do not think they need to show competence or avoid showing incompetence in mathematics lessons. Krawitz and Wiehe examined the impact of problem-posing prompts on students' motivation and they showed that reformulating easy problems, compared to difficult ones, led to higher expectancies for success, greater utility and attainment value, and lower perceived costs. More broadly, the findings highlight the distinct nature of task-value components and suggest that incorporating prompts for easier problems in problem-posing tasks can positively impact students' motivation. Wang and Zhu investigated the effect of a growth mindset on mathematical performance, considering also mathematics anxiety and mathematics learning

motivation, comparing advantaged and disadvantaged students across six East Asian countries/economies and OECD countries. In East Asia, a growth mindset negatively affects math achievement directly among disadvantaged students but improves it indirectly by reducing anxiety and boosting motivation. In OECD countries, the positive direct effect of a growth mindset is more pronounced, highlighting cultural and socioeconomic differences in its impact. Dell'Anna and Brunetto introduced a theoretical framework that integrates cognitive, motivational, and affective dimensions to analyse how students interact and evolve over time in collaborative settings. The model specifically addresses the role of motivation by examining how students' engagement, goals, and values influence their participation and learning outcomes in group settings.

Identity was the focus for two papers presented this year. Heyd-Metzuyanim explored the unification of the commognitive framework with complexity theory to theorize how students' mathematical identities stabilize to form ritual patterns of participation in mathematical learning. The results suggest that a student's subjectifications cohered with her mathematizing, which was mostly ritual and focused on procedures that she attempted to recall from memory. These subjectifications in turn cohered with her role-identity narratives that revolve around "succeeding" in mathematics for instrumental goals, all forming an attractor state in the complex dynamic system of her role identity. Gildehaus observed that many preservice primary teachers seem to struggle with their compulsory mathematics education at university. Besides cognitive struggle, they report low interest, negative emotions, and an overall rejection towards mathematics. Identity-based approaches shed light on these connections, identifying contrasting relationships and clusters of "either-or" participation for preservice primary teachers in mathematics.

Emotions and values were the focus in another four studies. Turconi and her colleagues consider the students' narrative about the secondary-tertiary transition in mathematics education and aim at evaluating the extent to which artificial intelligence can support the researcher in narrative analysis by identifying relevant themes to categorise keywords and highlight emerging themes from student texts. The model has proven to be a promising tool for automating part of the analytical process for qualitative data, especially in its preliminary stages, without sacrificing the depth of the research. Gómez-Chacón and Bacelo explored the relationship between epistemic emotions and control actions in undergraduate mathematics and computer engineering students in a task requiring the use of computational thinking and linear algebra concepts. The findings reveal that mathematics students give more importance to control elements associated with formalization of mathematical concepts, while computer engineering students give more importance to emotions associated with the instrumental and technological control dimension. Goeller explored the formation of emotions and values situated in university mathematics education and grounded in students' personal epistemology, finding useful hints for theorising the formation of emotions and values grounded in mathematics-specific personal epistemology (PE) with practical implications to overcome alienation and exclusion in university mathematics education. Satyam explores the use of the tool emotion graphing for studying participants' emotions and fostering a larger theorized structure of meta-affect. Emotion graphing, within the range of pictorial methods, can capture shifts in affect through drawing maps or pictures and serves as a vehicle for more effective communication during interviews as well as a form of experience-based sampling, with pedagogical use as exit tickets.

Two studies reported findings related to mathematics teacher affect, with an eye also on their students. Hagenkötter considered the mathematics teacher's conceptions and intention to teach mathematics within an exploratory interview study. The results indicate that the teacher's intention to convey to her students that mathematics is a coherent system and that doing mathematics is a sense-making process does not get through to most of her students, who associate mathematics and doing mathematics primarily with arithmetic and finding results. Hamidy and colleagues examined the impact of mathematics teacher support and mathematical self-concept on the students' perceived value of mathematics. The results show that the indirect effect of instrumental mathematics teacher support on Value via mathematical self-concept is significant, while the indirect effect of emotional mathematics teacher support does not exist. Furthermore, they confirm the framework suggested by the expectancy-value theory and provide a better picture of the relationship between internal and external factors of mathematical resilience.

Lastly but not leastly, four studies investigated feedback and assessment in mathematics. In his systematic review of literature, Daniel focused on teacher feedback practices as a primary mechanism influencing students' appraisal of self-efficacy information, with a particular focus on students' attributions as a potential mediating factor. Spagnolo and Andrà investigated how pre-service and in-service teachers perceive the difficulty of mathematical tasks and examined the factors that shape their evaluations. The findings reveal significant differences between pre-service and in-service teachers, particularly in their handling of open-ended versus structured tasks, shedding light on the influence of teacher experience on perceived difficulty and underscoring the importance of addressing both cognitive and affective factors in teacher professional development. Nicchiotti and colleagues, in a related study, explored the emotional factors influencing primary school students' perceived difficulty when solving mathematical tasks. The results indicate that while emotional factors, both positive and negative, emerged, students often struggled to articulate their emotions. The findings suggest that emotions are meaningful and independent contributors to perceived difficulty. Street and colleagues studied the impact of classroom quality on students' mastery experiences, especially in the context of tasks of different levels of difficulty. The research investigated how quality of student-teacher interactions predicted students' mastery experiences across a sequence of four lessons. The importance of instructional support for students' mastery experiences stood out.

Evolution of the TWG

We had a smaller number of contributions this year but several contributors who visited TWG08 report session indicated that their current studies have an affect dimension, which they plan to present in the next CERME. The number of contributions, on the other hand, enabled us to discuss the ideas presented in the studies in depth. Most of the studies were about students but there were also those about teachers and on the relationship between teachers' and students' affect. Gender continued to be a part of our discussions in some studies. Several studies utilized sophisticated quantitative methods but we also had mixed methods studies with several instruments, as well as emerging and valuable qualitative studies.

Our discussions focused on different aspects of the studies in terms of both content and method. We are still dealing with the definitions of newer terminology, such as engagement, which is enriching

the field with fruitful discussions. We are still bounded by our data collection tools; however, this brings a positive trigger and leads us to think about other possibilities as well as contextual issues that result in such limitations. Identifying limitations helps us to develop improved tools to address affect in our studies. We still call for more research on the relationship between affect and cognition, theorising, deepening new concepts, longitudinal and comparative perspectives, and research focusing on different groups of students such as students with special needs.

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The COIF model: Modelling the evolution of student states during collaborative sessions

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This article introduces the COIF Model, a theoretical framework developed to contribute to the understanding of student working group. The model builds on cognitive and motivational aspects, perceived competence, and emotional disposition to analyse how students' states evolve over time. The model specifically addresses the role of motivation by examining how students' engagement, goals, and values influence their participation and learning outcomes in group settings. After presenting the theoretical background, the paper outlines the model's central components, introduces the representative schema used, and demonstrates how the model can characterize the evolution of student states, including their motivational aspects.

Keywords: Group dynamics, student participation, student motivation.

Introduction

In recent decades, research in mathematics education has recognized the crucial role of social interactions in student learning (e.g., Sfard, 2001) as well as the interplay of cognitive, motivational, and emotional dimensions as key factors influencing behavior (Hannula, 2012). Affective factors shape the educational experience by influencing students to form meaningful connections with both the mathematical content and the social context of the classroom. Andrà and Liljedahl (2014) highlights how affective moves influence group dynamics in classroom interactions.

Liljedahl (2020) proposes pedagogical practices to transform the classroom into a space of active engagement, based on participation and collaboration, where teachers have the discretion to intervene in small group activities, with their interventions being either cognitive or affective in nature. This perspective aligns with the work of Andrà et al. (2015, 2020), who identified four fundamental modes of participation during group activities based on the levels of "I Can" and "You Can". Additionally, motivation plays a pivotal role in this context, influencing student engagement, goals, and values, which in turn affect their participation and learning outcomes (Eccles & Wigfield, 2002).

Despite the importance of group work in mathematics education, research often examines it in fragmented ways, focusing on individual factors, yet students are influenced by cognitive, motivational, emotional, and social elements simultaneously. As Hannula (2012) observes, many studies in mathematics education foreground one dimension—cognition, motivation, or emotion—while relegating the others to subordinate roles, thus neglecting their interconnected nature. This fragmented approach presents challenges for understanding the complexity of group dynamics.

The COIF Model aims to address this gap by integrating these dimensions within a unified analytical framework. We use the term "model" in the scientific sense, as defined by Schoenfeld (1999, p. 248), who states that "a representation of a particular phenomenon, in which objects and relationships characterized in the model correspond to selected objects and relations in the phenomenon being represented". Specifically, a model is defined as a triple (D, M, f) consisting of a domain D of the

real world, a subset M of the mathematical world and a mapping from D to M (Blum, 2015). Following Blum's (2015) framework of mathematical modelling, which outlines seven key steps, this study focuses primarily on step 2 (structuring), step 3 (mathematizing) and step 4 (working mathematically), thus we started from the situation, proposing a real model, which is transformed into mathematical-model, and provide some mathematical results. The COIF Model is tailored to secondary school contexts, with future research aimed at validating (step 6) its practical applicability within this setting.

This paper begins with a detailed *Theoretical framework*, reviewing key literature that provides the foundation for the COIF Model. This is followed by the *Key hypotheses for mathematical model construction* section, followed by which outlines the theoretical assumptions of the model. The next section, *Results - COIF model*, explains how the model was structured and provides two examples of a simulation to demonstrate how student state transitions are analysed. Finally, the *Discussion and conclusion* reflect on the interpretation, such as the educational implications of the model, its limitations, and potential directions for future research.

Theoretical framework

The focus on affect in educational research has grown significantly since McLeod's (1992) review, which introduced a taxonomy that organizes affective constructs along a spectrum from beliefs (most stable and cognitively oriented) to attitudes (intermediate stability and involving both affective and cognitive components) to emotions (least stable and primarily affective). Building on this, Hannula (2012) emphasizes the interconnected nature of cognition, motivation, and emotions. Cognition involves processing information, while motivation directs behaviour through goals and choices, and emotions provide feedback for both. Hannula (2012) also suggests distinguishing between the stable, trait-like nature of affective variables, and their more variable, state-like nature. Di Martino and Zan (2011) highlight the complex relationship between beliefs, emotions, and attitudes in mathematics learning, showing how perceived competence (e.g., 'I can/can't') and emotional disposition (e.g., 'I like/dislike mathematics') influences students' engagement and group interactions. This framework provides valuable insights into how attitudes influence behaviour, particularly in group settings, where belief states significantly influence the development of the activity (Roth & Radford, 2011).

Likewise, student emotions fluctuate during problem-solving, shaping the mathematical activity itself (Roth & Radford, 2011). Research shows that students' motivations, emotions, and perceived competence can change not only throughout the academic year (Andrà & Scalvini, 2023) but also within the same collaborative session (Liljedahl, 2020). These findings underscore the variability of motivation, emotions, and perceived competence, highlighting their critical role in shaping student engagement and collaborative interactions. Di Martino and Zan (2011) highlight how perceived competence influences participation, with "I can" fostering engagement and "I can't" leading to passivity (Andrà & Liljedahl, 2014). Recognizing peers' abilities ("You can") also shapes group dynamics (Andrà et al., 2015). In their study, Andrà et al. (2020) identified four fundamental modes of participation during group activities, each characterized by different levels of "I Can" and "You Can" (Andrà et al., 2015; 2020). These modes include *cooperative* participation (high levels of both "I Can" and "You Can"), *obstinate* participation (high "I Can" but low "You Can"), *isolated*

participation (low levels of both "I Can" and "You Can"), and *follower* participation (high "You Can" but low "I Can"). Building on Hannula's (2012) concept of states as variable and momentary expressions of affective variables, and acknowledging the documented fluctuations in perceived competence, we define student states in our model as characterized by three key dimensions: "I Can", "You Can" and motivation. This framework allows us to examine the dynamic interplay of these factors during collaborative activities. By focusing on the variability of "I Can" and "You Can", we aim to study the evolution of participation modes (Andrà et al., 2020) in group settings. Additionally, the model integrates a motivational component to reflect the interdependence between perceived competence and engagement, modeling transitions between states based on the success or failure of collaborative efforts.

Given the importance of group work, as well as the impact of emotional states and motivation on educational outcomes, this study seeks to address the following research question: How can we model the evolution of student states and their reciprocal relationship with motivation during collaborative group work?

Key hypotheses for model construction

The model is based on key hypotheses regarding student transitions between different states during collaboration. Specifically, when a student solves a task independently, their "I Can" confidence increases, reflecting a sense of self-efficacy, while reliance on peers ("You Can") decreases. This dynamic can lead to obstinate behavior, as the student perceives less need for group input. Conversely, when a student struggles to solve a task independently but observes peers succeeding, their "I Can" confidence may decrease, while their reliance on peers ("You Can") increases. This adjustment fosters follower behaviors, where the student values and depends more on the contributions of others. Success in group activities is hypothesized to increase both "I Can" and "You Can", encouraging cooperative behaviors and enhancing group cohesion. On the other hand, failure may result in frustration, disengagement, and a decline in both "I Can" and "You Can" perceptions, which can manifest as withdrawal or isolated behaviors. These changes highlight the interplay between confidence, motivation, and group participation during collaborative tasks. In addition to these dynamic elements, the model incorporates the level of preparation as a stable cognitive trait that shapes the baseline conditions for group interactions. This trait influences the group's capacity to succeed and the dynamics of collaboration. Building on these key hypotheses, we formulated further theoretical hypotheses regarding the outcomes of collaborations among students, taking into account their initial states, including their initial modes of participation and levels of preparation.

For clarity, Table 1 illustrates the hypotheses regarding the outcomes of collaborations involving a cooperative student paired with others (cooperative (C), obstinate (O), isolated (I), or follower (F)) at different levels of preparation (high or low). When both participants are well-prepared, group success (1,1) is achieved, leading to an increase in both "I Can" and "You Can", which foster self-efficacy and group cohesion. Conversely, when neither is well-prepared, group failure (-1,-1) occurs, decreasing both metrics and potentially reducing motivation and engagement. The dynamics vary when one participant is well-prepared while the other is not. In collaboration between two

cooperatives (or one cooperative and one follower), success (1,1) is likely even if only one participant is prepared, as mutual trust and perceived competence foster effective collaboration. In pairings between cooperative and obstinate students, success (1,1) is achieved if the obstinate student has high preparation, as their competence, combined with the cooperative student's encouragement, facilitates collaboration and can lead to group success. In contrast, if the obstinate student is unprepared, group failure (-1,-1) is likely, as low recognition of peer ability and obstinance hinder progress. Collaboration between cooperative and isolated students presents more complex outcomes. When only one has studied, the prepared student achieves individual success (1,-1 or -1,1), while the unprepared student does not, reflecting individual success but group imbalance. In these cases, the prepared student increases their "I Can" and decreases their "You Can" for their partner, while the unprepared student experiences the opposite.

Table 1: Collaboration outcomes with cooperative student

C|C	High	Low
High	$(1,1)$	$(1,1)$
Low	$(1,1)$	$(-1,-1)$

C|O	High	Low
High	$(1,1)$	$(-1,-1)$
Low	$(1,1)$	$(-1,-1)$

C|I	High	Low
High	$(1,1)$	$(1,-1)$
Low	$(-1,1)$	$(-1,-1)$

C|F	High	Low
High	$(1,1)$	$(1,1)$
Low	$(1,1)$	$(-1,-1)$

These assumptions ensure the model captures the dynamics of different participatory states and their impact on group work success or failure. Furthermore, successful outcomes (1,1) and (1,−1) increase motivation by boosting confidence, while unsuccessful outcomes (−1,1) and (−1,−1) decrease motivation, as frustration reduces effort and engagement.

Results - COIF model

Initially, the COIF model was structured by focusing exclusively on the evolutionary states of students, represented by the four primary states: Cooperative (C), Obstinate (O), Isolated (I), and Follower (F). This initial phase of the model aimed to describe transitions between states based on group interactions and task outcomes, utilizing a Markov chain (Norris, 1997) to map the transition probabilities. As illustrated in Figure 1, the links between nodes represent the probabilities of transitioning from one state to another (e.g., from Cooperative to Obstinate), while self-loops represent the probability of remaining in the same state (e.g., a Cooperative student staying Cooperative). The transition probabilities for each state j = C, O, I, F, are defined as follows:

- p_j: Probability that a student in state j solves the task independently, leading to an increase in their "I Can" and a decrease in "You Can", making them obstinate.
- q_j: Probability that a student in state j does not solve the task, while another student does so independently, leading to a decrease in their "I Can" and "You Can", making them a follower.
- r_j: Probability of a student in state j achieving a solution collaboratively within the group, leading to an increase in both "I Can" and "You Can", making them cooperative.
- s_j: Probability that neither a student in state j nor their group collectively solves the task, leading to a decrease in both "I Can" and "You Can", making them isolated.

We stress that these probabilities are based on theoretical assumptions about the possible outcomes of collaborations and are not intended to represent objective quantitative measurements but are instead meant as trends in the evolution of student states.

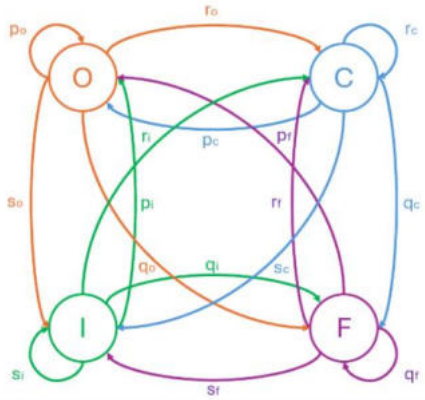

Figure 1: Representation of student state transitions

The evolution of the number of students in each state is modelled using a system of differential equations. The rate of change for a state j is calculated as the algebraic sum of the inflow (students transitioning into state j, weighted by the corresponding transition probabilities) and the outflow (students leaving state j, weighted by the probabilities of transitioning out of that state). For example, if C represents the number of cooperative students, the rate of change in cooperative students is given by: $dC/dt = r_O \times O + r_I \times I + r_F \times F - (p_C + s_C + q_C) \times C$, where I, F, and O represent the number of isolated, follower, and obstinate students, respectively.

To incorporate motivation into the model, we introduce the new variable M, representing the level of student motivation, ranging from 0 to 1. In this framework, motivated students who achieve the correct solution remain motivated, while students who fail to find the solution may decrease their motivation and potentially experiencing frustration. Conversely, initially unmotivated students who succeed can increase their level of motivation. The rate of change in motivation (dM/dt) is calculated by considering the transitions between high level of motivation ($M \geq 0.5$) and low level of motivation ($M < 0.5$), weighted by their respective probabilities. This formulation dynamically captures the interplay between student motivation and task outcomes in collaborative settings.

This system of differential equations provides insights into the evolution of the number of students transitioning from one state to another, representing the group's overall dynamics. However, our primary focus is on understanding how the individual student's state evolves in terms of “I Can”, "You Can” and motivation variables. To that end, we define three global quantities: the average levels of “I Can”, “You Can” and motivation across the group. The average of I can at the time t (EI(t)) is the mean number of cooperative (C(t)) and obstinate (O(t)) students at that time t: $EI(t)=[C(t)+O(t)]/N$, where N is the total number of students. Similarly, the average of You can (EY) and motivation (EM) are: $EY(t)=[C(t)+F(t)]/N$, and $EM(t)=[M(t)]/N$, where M(t) is the number of students in Layer 1 at time t (students with high motivation). We then compute the global variation (Δ) for each of these quantities over a time interval Δt: $\Delta I=EI(t+\Delta t)-EI(t)$, $\Delta Y=EY(t+\Delta t)-EY(t)$, $\Delta M=EM(t+\Delta t)-EM(t)$. This allows us to model level of the states of each student at time $t+\Delta t$ by

summing their state variables at time t with the corresponding global variation. For example, the level of I can of student *i* at time t+Δt is: I can$_i$(t+Δt) = I can$_i$(t) + ΔI.

To clarify the behaviour of the model, we provide two examples. Figure 2 (left) shows the dynamics of the first group of four students the cooperative (C) and obstinate (O) students start with high motivation, while the isolated (I) and follower (F) students have lower motivation. As we can see in the figure, both "I Can" and "You Can" levels steadily increase, indicating growing confidence and enhanced student cooperation. However, the zig-zag pattern indicates fluctuations in motivation. According to the model's hypothesis, which links motivation closely to task success, this dynamic suggests that the group faces obstacles, delaying progress and making it challenging to reach the correct solution quickly. These challenges manifest as alternating periods of high motivation during progress and low motivation when frustration arises from setbacks. Despite these fluctuations, the overall increase in "I Can" and "You Can" suggests the enhance of group's cohesion and cooperative efforts. In the Figure 2 (right), the isolated and follower students now exhibit higher motivation than the cooperative and obstinate students. As observed in the figure, the trajectories of all variables, "I Can," "You Can," and motivation, increase monotonically. In particular, the greater overall increase in motivation and the more significant rise in "I Can" levels suggest that fostering individual confidence through achievable tasks can enhance engagement. However, the smaller increase in "You Can," compared to the first scenario, could be attributed to the presence of a motivated isolated student, which may slow down cooperation. This aligns with findings by Andrà et al. (2020), which emphasize that isolated participation can act as a barrier to group cohesion, even in scenarios of high individual motivation.

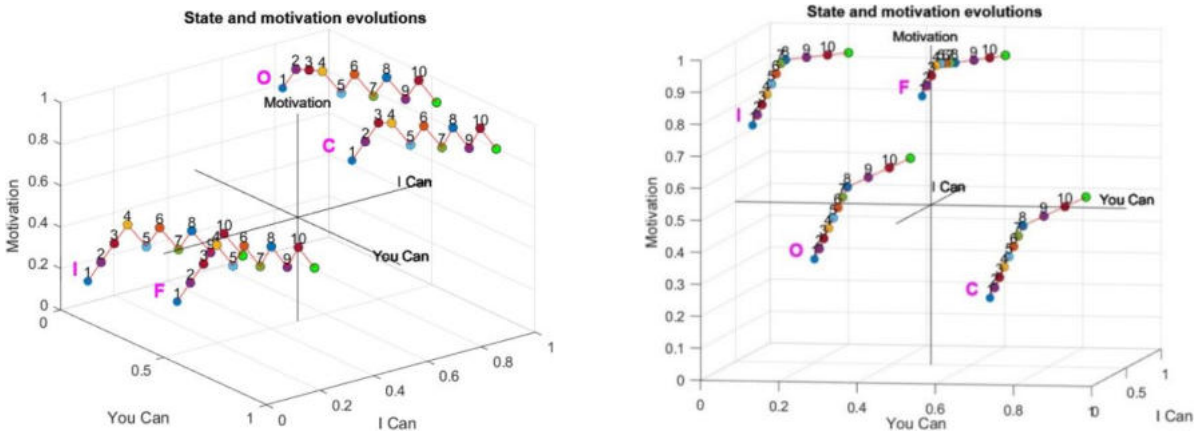

Figure 2: State and motivation evolution of two work groups

Discussion and conclusion

The two reported examples and their relative analysis drive attention towards the outcomes of the group work, wondering what a positive outcome means. We claim that positive outcomes occur when students become cooperative (i.e., increase both "I Can" and "You Can") with high motivation. As consequence, the success of the task is left on background as variable which is influenced by high motivation. Namely, the priority of the positive outcome is not the success of the task itself, but the

high motivation and cooperation which can bring students towards the correct solution. The results confirm that this model aligns with the literature and fulfills our objective of mathematical modeling.

In the first example, students with high "I Can" levels also exhibited high motivation, demonstrating that, even after setbacks that could decrease their motivation, they were able to improve their collaboration and self-confidence. This supports the idea that collaborative group work, even during fluctuating periods of motivation, fosters both individual and collective confidence (Liljedahl, 2020). In the second example, favourable dynamics emerged when students with lower confidence levels exhibit higher motivation. Increased motivation helps balance their lower 'I Can' levels, enabling them to contribute more positively and boosting their confidence. This aligns with the expectancy-value theory, which posits that students' expectations for success and the value they assign to tasks significantly influence their motivation (Eccles & Wigfield, 2002). These two scenarios reinforce the concept that the variables under analysis are strongly interconnected, with each influencing the others (Hannula, 2012).

In this theoretical work, we aim at providing a model for the evolution of the students' states during collaborative work. Contributing to the need for a more integrated analysis of classroom interactions and the research question, the development of the COIF model aims at bridging the gap identified in the literature by offering a framework to analyse the evolution of student states within a collaborative classroom setting. The presented examples indicate that the model is on the right track as a descriptive tool for scenarios already explored in the literature. Our ambition is for this model to evolve into an analytical and informative instrument for researchers, guiding the investigation of yet uncharted scenarios, and a valuable tool for advancing insights into classroom dynamics.

To that end, we will refine the parameters within the COIF model to better mirror real-world classroom scenarios. The other that requires further reflection is the role of motivation, which should be integrated in more holistically way, considering it not only as a consequence of interactional outcomes but also as a driving force that shapes student engagement and participation from the outset. Finally, empirical validation could prove challenging, as motivation could be more difficult to capture compared to the "I Can" and "You Can" dimensions.

While empirical validation, comparing the model's predicted state transitions with observed classroom data, is a longer-term goal, the current model provides a solid foundation for representing the evolution of student states in collaborative contexts. Despite its limitations, the COIF model serves as a promising starting point for further research and development.

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You are what you feel? – How primary teacher students' identification as mathematicians and teachers relates to other affective variables

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Many preservice primary teachers seem to struggle with their compulsory mathematics education at university. Besides cognitive struggle, they report low interest, negative emotions, and an overall rejection towards mathematics. Identity-based approaches shed light on these connections in recent years from a qualitative perspective. In this paper, I aim to investigate the relation between one's identification with mathematics (and teaching) and other affective variables from a quantitative perspective. A correlation, as well as cluster analysis for n = 99 preservice primary teachers, was conducted, identifying rather contrasting relationships and clusters of "either-or" participation for preservice primary teachers in mathematics. Theoretical implications about relations of affective variables, as well as practical implications for teacher education, are being discussed.

Keywords: Preservice primary teachers, affect, identification, relations in affective variables.

Affect and identity in preservice primary mathematics teacher education

While mathematics is compulsory for most preservice primary teachers worldwide, many seem to struggle with studying and learning mathematics during their teacher education (Chen, 2017; Ingram et al., 2018), which may also hinder their later teaching and influence their future students' mathematical learning (Beilock et al., 2010). Recent studies also point towards affective struggles next to cognitive challenges (Coppola et al., 2012; Ingram et al., 2018). For example, Fray and Gore (2018) found in their meta-study on study choice motivation that preservice primary teachers are more motivated for teaching per se and less motivated for the subjects to teach, such as mathematics. Regarding subject-specific content, many preservice primary teachers question the relevance of their obligatory mathematics courses at university and show low motivation for them (Gildehaus, 2024). However, many preservice primary teachers also report high negative emotions, such as anxiety or frustration when learning mathematics at university, as well as rather low self-efficacy about their mathematical abilities (Jenßen, 2023). Thus, their struggle seems not only indicated by less motivation for the subject but also by deeper underlying social structures that cause negative feelings and low self-esteem in mathematics.

In recent years, identity-based approaches to preservice primary teachers' relation with mathematics investigated these underlying structures. They showed a broader yet worrying picture of preservice primary teachers' mathematical identities: Many students reported very negative experiences with mathematics during their school experiences (Hodgen & Askew, 2007). Stories of failure and exclusion were being narrated, told about not knowing or understanding mathematics (directly) and therefore being pointed out (Lutovac, 2020). Such failure experiences play a significant role in becoming a teacher during university education (Lutovac, 2020). They may thus be related to or even be part of possibly underlying reasons for low self-esteem, motivation, and negative emotions of preservice primary teachers during their university education – specifically when also during university education experiences of not understanding are quite common (Gildehaus et al., 2024; Ingram et al., 2018).

However, while such a relationship between identity and affective variables seems likely from a theoretical perspective, its empirical evidence is rare. Moreover, there is evidence that also preservice primary teachers who do not develop a positive relationship towards mathematics but their overall pedagogical competence and teaching can show a highly functional teacher's identity (Alderton, 2020). On the opposite, Lutovac and Kaasila (2014), for example, investigated how specifically preservice primary students with negative experiences from their school education were able to overcome these negative experiences and build up a more positive mathematics learner identity. Coppola et al. (2012) also stated that besides the dominance of negative emotions in their study about how preservice primary teachers feel about mathematics, positive emotions, such as enjoyment and satisfaction, had been mentioned simultaneously. Thus, the overall situation of different affective variables may be more complex than theoretically anticipated.

In the following, I aim to bridge the social context of identity with preservice primary teachers' overall affect from a quantitative perspective. I will first investigate possible relations between one's "identity" (operationalized as identification with a certain community) and different affective variables (operationalized as self-efficacy, emotion, and motivation). Secondly, different clusters of students' affective profiles at the end of their mathematics university education are being identified to investigate possibly different states of relations between these variables.

Theoretical background

Following this aim, I first clarify how to operationalize identity towards a quantitative framework and further investigate relevant affective variables and their relatedness to each other as well as to the social context. Hereby, different objects of preservice primary teachers' affect are also considered.

Operationalizing identity as identification

The cited identity works above stem from a qualitative and mostly discursive paradigm. While this reconstructs the rich complexity of underlying structures, it cannot be operationalized in a quantitative study design without immense reduction. Yet, given that identity has been intensively researched during the past years in mathematics education, we do have a great overview of *how* students see themselves as belonging and identifying with a mathematics community (c.f. Wenger, 1998 on how building one's identity is becoming a member of a certain community). Instead of further investigating this *how*, I will in the following focus on the *if,* in terms of *if* students do identify themselves with a mathematics community and *if* they feel a sense of belonging towards this community. In this sense, I am stepping away from the relevant process given rise to the *if* but may still observe a situated positioning of identification directly related to the social context.

Moreover, preservice teachers are also known to be torn between different communities, for example, that of theory and practice (Solomon et al., 2017) or that of mathematics and mathematics teaching (Gildehaus et al., 2024). In addition to identifying with the mathematics community, I am also operationalizing one's identification with the teaching community.

Self-efficacy and shame as affective variables related to the social context

Following this aim of staying related to the social context, Hannula (2012) argued that affective variables can also be seen from a social (additional to a psychological) perspective. While a psychological perspective perceives affective variables as individual mental processes, a social

perspective may see them as aspects of social interaction and structure (Hannula, 2012). I will, at least to some extent, bridge those two perspectives by investigating affective variables from a psychological perspective but still interpret them in relation to the findings of students' identification with mathematics. For example, self-efficacy, an affective variable closely related to cognition and performance, may also be related to one's identification with the community in which one perceives oneself as efficient.

Furthermore, emotions play a crucial role in mathematics, and Hannula (2012) states them as the most social and situated affective variable. Hereby shame can also bridge our psychological and social perspective since it is known to be a social emotion. In contrast to anxiety or frustration, it only concerns a given social context – one feels ashamed of something in front of someone. Yet, shame is often related to general beliefs about one's capability, for example "I am just too stupid to ever learn mathematics" (Jenßen, 2023), making it individually accessible and measurable. Therefore, I would anticipate a (possibly negative) relation towards one's identification with the mathematics community.

Different objects in preservice primary teachers' motivation

Taking our relatedness to the social context further, Schukajlow et al. (2023) stress the importance of explicating possible different objects towards which affective constructs can be directed. This, again, seems of specific importance in the context of teacher education. For example, as I discussed earlier, preservice primary teachers' motivation in teaching is often higher than in the subjects. Thus, the objects of teaching and the subject can be differentiated in their motivation (Fray & Gore, 2018).

Taking a closer look at preservice *mathematics* teachers, they often seem to differentiate between the objects of mathematics as they know from school and mathematics at university (cf. Ufer et al., 2017), where the first is more related to the general object of teaching and the second to the subject. Furthermore, preservice teachers report different objects they value in relation to how they identify (e.g., as stated earlier, mathematicians and future teachers) – at least in some qualitative studies (cf. Gildehaus et al., 2024). While "understanding mathematics deeply" seems to be relevant for students who see themselves as part of a mathematics community, "nicely explaining mathematics to children" seems of great importance for students identifying as future mathematics teachers (Gildehaus, 2024). Operationalizing such different objects in preservice teachers' motivation may again enable us to investigate such possible social relations from a more quantitative perspective.

Research Questions

Given this overall setting, I pose the following research questions. While RQ1 focuses on the concrete relations between different variables, RQ2 aims to further investigate different patterns of relatedness in these variables for different students, following the earlier discussed aim of getting insights into students' different clusters of affective profiles at the end of their mathematics university education.

RQ1: How do identification with mathematics and teaching relate to the affective variables self-efficacy, shame, motivation for school and university mathematics, and motivation for understanding and teaching mathematics?

RQ2: What different clusters of students can be found throughout these variables, and what characterizes them?

Methods

Sample

The sample consisted of a total of $n = 99$ students from a medium-sized German university. The surveyed course, "Elements of Mathematics", focused on basic ideas of formal mathematics, such as logic and (generic) proofs. It is the last mathematical course the students attend during their studies before they finish their university education. The course was obligatory for preservice primary teachers in this context; however, they could voluntarily intensify courses in mathematics (8.2 % reported to have chosen this option). Furthermore, the course could be visited by preservice teachers for special education as well, who would have also chosen the course voluntarily (14.1 % reported to have chosen this option). Of all students, 88.7 % reported identifying as female, 11.2 % as male, and 0 % as non-binary. Most surveyed students (66.3 %) were in the last year of their master's program, 21.4 % had just started their master's, and 14.3 % were in the last year of their bachelor's degree.

Data collection

The data was collected with an online questionnaire during lecture time in the third week of the semester. Students were guaranteed anonymity and participated voluntarily. As discussed earlier, students' "identity" was operationalized as identification with a mathematics community (Math I.) and identification with a teaching community (Teaching I.). The former scale had recently shown great reliability (Kempen & Liebendörfer, 2021), while the latter was adapted for this study for the first time.

Adapted scales from PISA were used to measure self-efficacy, which had often been used in university mathematics contexts before (c.f. Gildehaus, 2024; Kempen & Liebendörfer, 2021). The well-established SHAME-Q was used to measure shame (Jenßen, 2023).

To measure motivation for different objects, I used the scales recently evaluated by Gildehaus (2024). Motivation is understood in terms of expectancy and value (Eccles & Wigfield, 2020) and different objects for the values are addressed: school and university mathematics (in line with Ufer et al., 2017) as well as understanding and explaining mathematics (in line with Gildehaus et al., 2024). This resulted in four scales in total for intrinsic value for school mathematics (IValue school) and university mathematics (IValue Uni), as well as attainment value for understanding mathematics (AValue Understand) or explaining mathematics (AValue explain).

The original 5-point Likert scale from 1 (= strongly disagree) to 5 (= strongly agree) of the SHAME-Q was kept, while the other variables were measured on a 6-point Likert scale accordingly. Example items are displayed in Table 1. All the scales showed great internal consistency (Cronbach's $\alpha > .83$ for all scales) as well as discriminatory power between the different items (item-rest-correlation $> .55$ in all cases). However, a potential ceiling effect for the scale "attainment value explaining" was identified (*mean*: 5.6), which I keep in mind for the following analysis.

Data analysis

To answer RQ1, a correlation analysis (PCC) of all variables was performed. For RQ2, a *k*-means cluster analysis was performed. The analysis was conducted for $k = 1, \ldots, 10$ clusters with different starting points, to avoid dependences of those, as well as with *z*-standardized variables to take care of the different scales. To identify the optimal number of clusters, the gap statistic approach was used.

It compares the total intra-cluster variation for different values of *k* with their expected values under null reference distribution of the data. The estimate of the optimal *k* will be the value that maximizes the gap statistic (Tibshirani et al., 2001). All analyses were performed using Jamovi 2.3.2.1.

Table 1: Overview of the Scales (number of items in parentheses), example items and their Cronbach's α

Scale (number of items)	Example Item	α
Math I. (5)	I would describe myself as a mathematician.	.87
Teaching I. (5)	I would describe myself as a primary/special needs teacher.	.83
Self-Efficacy (4)	I am convinced that I can perform well in homework and exams.	.91
Shame (6)	I am ashamed that I am not as good in math as I would like to be.	.93
IValue School (4)	I enjoyed mathematics in school.	.95
IValue Uni (4)	I like the mathematics at university.	.89
AValue Understand(4)	I want to deeply understand mathematics.	.89
AValue Explain (4)	I want to be someone who can explain mathematics really good	.87

Results

Research Question 1

Table 2 provides all correlations between the variables. Math Identification is highly correlated with Self-Efficacy, Intrinsic Value for School and University, and Attainment Value for Understanding. It is also negatively correlated with Shame, as well as with Teaching Identification. The latter, in contrast, shows negative correlations with Self-Efficacy, Intrinsic Value for University, and Attainment Value for Understanding, but no correlation with Attainment Value Explaining. Moreover, Self-Efficacy is highly correlated with all Values except Attainment Value Explaining and is negatively correlated with Shame. Shame, in line with that, is negatively correlated with these Values. Both Intrinsic and Attainment Values are positively correlated.

Table 2: Correlations of the different variables $^{*}p < .05$, $^{**}p < .01$, $^{***}p < .001$

Variable	1	2	3	4	5	6	7
Math I. (1)							
Teaching I. (2)	$-.37^{***}$						
Self-Efficacy (3)	$.54^{***}$	$-.30^{**}$					
Shame (4)	$-.28^{**}$	.12	$-.51^{***}$				
IValue School (5)	$.40^{***}$	-.20	$.47^{***}$	$-.45^{***}$			
IValue Uni (6)	$.46^{***}$	$-.26^{*}$	$.55^{***}$	$-.33^{***}$	$.49^{***}$		
AValue Understand (7)	$.38^{***}$	$-.26^{*}$	$.50^{***}$	$-.18^{*}$	$.24^{**}$	$.65^{***}$	
AValue Explain (8)	.17	.10	.05	-.14	.17	$.18^{*}$	$.30^{***}$

Research Question 2

The gap statistic revealed two different clusters as the optimal number *k*. This provided two rather dichotomous clusters that highly contrast each other in all variables. Cluster 1 (44.9 % of all students) shows, in comparison to Cluster 2 (55.1 % of all students), low Math Identification (*z-standardized mean*: -.66) but high Teaching Identification (.30). In line with that, students in Cluster 1 report low Self-Efficacy (-.76), high Shame (.64), low Intrinsic Value for School (-.69) and University (-.64) as well as low Attainment Value for Understanding (-.50). In contrast Cluster 2 shows higher Math Identification (.51), but lower Teaching Identification (-.23). Students report higher Self-Efficacy (.59), less Shame (-.49), higher Intrinsic Value for School (.53) und University (.49), and slightly higher Attainment Value for Understanding (.38). They also report slightly higher Attainment Value for Understanding (.31) compared to Cluster 1 (-.41). However, given the ceiling effect in this scale, I would not necessarily interpret this as an actual difference.

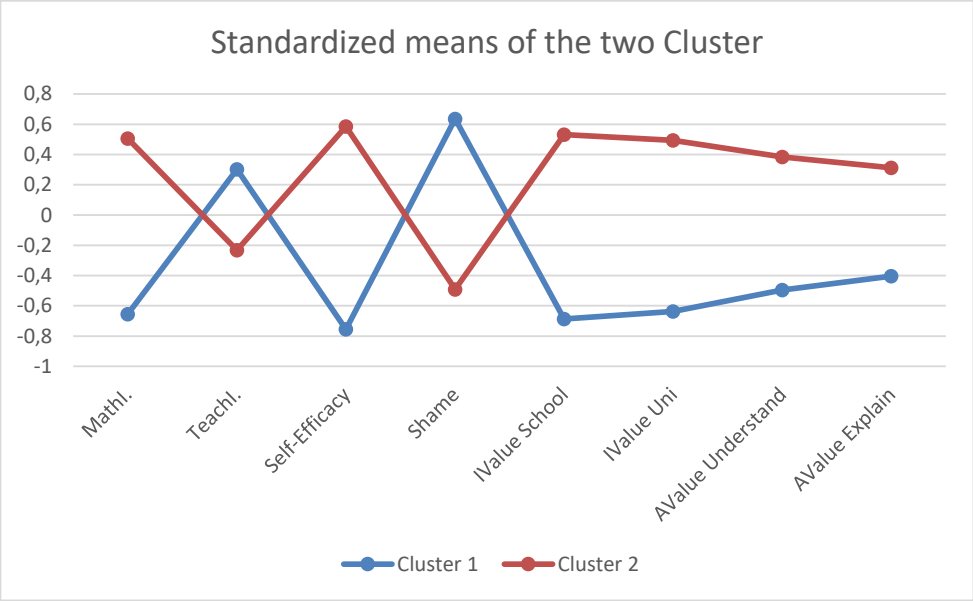

Figure 1: Display of the standardized means of the two Cluster

Preservice primary and special needs teachers were almost equally distributed through the clusters, as well as female and male students. However, all preservice primary teachers who opted to intensify mathematics had been distributed to Cluster 2.

Discussion

My aim in this paper was to bridge the social context of identity (operationalized as identification) with preservice primary teachers' overall affect from a quantitative perspective. In line with that, RQ1 investigated correlations between one's identification as a mathematician and as a teacher, where rather contrasting pictures could be found: While the latter was negatively correlated with Self-Efficacy and Motivation but positively with Shame, Identification as a mathematician was positively correlated with Self-Efficacy and Motivation, but negatively with Shame. Furthermore, Math and Teaching Identification were negatively correlated, indicating a dichotomous rather than an inclusive picture. In RQ2, this dichotomy was further established, given the cluster analysis revealed two contrasting clusters of students reporting either high Math and low Teaching Identification or vice

versa, accompanied by high Self-Efficacy and Motivation but low Shame or vice versa. At first, this confirms the theoretical understanding and sporadic evidence that low identification with mathematics seems to be related to overall low positive affect (cf. Gildehaus, 2024; Ingram et al., 2018). In contrast, higher identification indicates higher means in positive affective variables.

Surprising, if not yet striking, in these findings, is the clear dichotomy of the clusters, as well as the negative correlation between math and teaching identification. This suggests that preservice primary teachers in mathematics at the end of their university education have developed an "either-or" identification. They either see themselves as somewhat part of the mathematics community or they do not – which goes hand in hand with an overall problematic affective relationship to mathematics that may make it challenging to teach mathematics. It remains unclear if these "either-or" identifications are directly related to the earlier discussed often negative mathematics experiences in school (cf. Hodgen & Askew, 2007). However, the significant difference in the motivation for school mathematics between the clusters at least indicates that. Furthermore, these findings underline the importance of identity work in preservice teacher education and the relatedness of affect with the social context. Practical implications for teaching preservice primary teachers may point towards pedagogical practices that could help preservice teachers overcome their negative experiences (Lutovac & Kaasila, 2014) and build a ground for identification with the mathematics community.

Even though it is a strength of this analysis to provide empirical evidence for often theoretically assumed relations, our data sample was still small and from a specific national context. Its representative power is thus very limited, and further investigations towards the relation of affective variables and one's identification seem desirable – not only from an empirical perspective.

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Emotions grounded in mathematics-specific personal epistemology

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This paper explores the formation of emotions and values situated in university mathematics education and grounded in students' personal epistemology. To do so, a case study based on four interviews with Alex, a high-achieving mathematics preservice teacher reporting predominately negative emotions towards her mathematics study, was analysed, revealing how her negative emotions emerge from discrepancies between her beliefs and her experiences at university regarding mathematics, the learning and teaching of mathematics, as well as the learning environment and the herein incorporated roles of students and lecturers. The findings give hints for theorising the formation of emotions and values grounded in mathematics-specific personal epistemology (PE) with practical implications to overcome alienation and exclusion in university mathematics education.

Keywords: Emotions, values, beliefs, personal epistemology, secondary-tertiary transition.

Introduction

The secondary-tertiary transition is accompanied by several difficulties (e.g., Gueudet, 2008), which are often linked to strong negative emotions (Di Martino & Gregorio, 2019; Göller & Gildehaus, 2021). Consequently, affect, and especially students' emotions, has emerged as a crucial topic that, however, is still rarely researched (Di Martino & Gregorio, 2019; Göller & Rück, 2022; Gueudet, 2008). The differences between school and university mathematics (cf. Gueudet, 2008) entail epistemic challenges regarding, for example, the nature of mathematics and mathematics learning, which has been described as "beliefs overhang" (Daskalogianni & Simpson, 2001), leading some students to disengage from mathematics portrayed as alienation, missing sense of belonging, or positioning as outsiders (Gildehaus et al., 2024; Lahdenperä & Nieminen, 2020; Solomon & Croft, 2016). Additionally, affective variables, such as mathematics-specific beliefs, values, and self-concepts, are associated with exam achievements, study satisfaction, and study dropout (Benden & Lauermann, 2023; Geisler, 2023; Göller & Gildehaus, 2024) and are closely related to emotions students experience when studying mathematics at university (Göller & Gildehaus, 2021).

This interplay of control and value appraisals with emotions is theorized and empirically proven by control-value theory in great detail (Pekrun, 2024). However, although social environments, sociocultural contexts, and personal beliefs are considered in control-value theory as influencing factors for students' control and value appraisals, the processes of assigning value to a specific situation are described and empirically analysed in far less detail. Therefore, the present paper aims to investigate the formation of students' mathematics-related values in light of their personal beliefs situated in the social environment and the sociocultural context of university mathematics education.

Theory

The theoretical perspective on mathematics emotions presented in this paper is vastly influenced by considerations of Op't Eynde et al. (2004; 2006a, 2006b). Learning is considered a form of engagement that enables students to actualise their identity by participating in activities in a specific

context (Op't Eynde et al., 2006a). This means that students are continuously challenged to integrate their past experiences with the way they experience themselves in the given context (here, the university mathematics learning environment) and its practices to discover meaning and to renegotiate or construct new meanings through their way of participating (Op't Eynde, 2004). These past experiences are thought of as being condensed in students' beliefs, which are defined as their implicitly or explicitly held subjective conceptions that are considered true (Op't Eynde et al., 2006b). Mathematics-related beliefs can refer to different objects, for example, to *mathematics*, to *mathematics teaching*, to the *self as a mathematician*, or to the *environmental context*, such as socio-mathematical norms or the role of the self and others in learning mathematics (Op't Eynde et al., 2006b). Through the personal relationship towards such beliefs, beliefs can be valued in the sense that they, for example, do not only refer to what mathematics, mathematics teaching, or the role of a mathematics teacher is but also to which aspects of mathematics or mathematics teaching are preferred by students or what they think the role of the teacher should be. Indeed, some students seem to value *instrumental/static* aspects of mathematics, such as knowing the rules and applying them correctly by performing the (schematic) calculations, while others rather value *relational/dynamic* aspects, such as figuring out why the rules work or puzzling over mathematics problems (Di Martino & Zan, 2010; Gildehaus et al., 2024; Göller, 2023). This system of mathematics-related valued beliefs is subsumed as *personal epistemology* (PE, cf. Hofer & Bendixen, 2012) in the following.

Students' *emotions* are theorised to arise from their evaluative interpretations – their appraisals – of the specific situation, which are grounded in their personal epistemology (Op't Eynde, 2004). This importance of appraisals for students' emotions aligns with psychological emotion theories (Pekrun, 2024; Scherer, 2019). Control-value theory, for example, elaborates in detail on the effects of control and value appraisals on different emotions (Pekrun, 2024). However, their personal epistemology is not always transparent to individuals. Often, they do not know a priori what they will find worthwhile acting upon; it emerges in the situation (Op't Eynde, 2004). Herein, it is the emotions students experience that reveal to them who they are and what they value in this specific situation (Op't Eynde et al., 2006a). Emotions are perceived as an act of participating in certain practices and grounded in the way students experience themselves in that context (Op't Eynde et al., 2006a), which in turn is theorised to build on past experiences and is manifested in values and beliefs that mirror students' personal epistemology. This means that studying, for example, joy or frustration implies analysing joyful or frustrating situations in the respective context (Op't Eynde, 2004) and to study students' personal epistemology, it appears fruitful to analyse joyful, frustrating, positively or negatively appraised situations for underlying values and beliefs that become apparent in this situation.

Research aims and research question

This paper aims to situate students' emotions in the specific context of university mathematics education by analysing the case of Alex for the interconnectedness of students' emotions, values, and beliefs and, building on that, exploring the formation of emotions and values situated in university mathematics education grounded in students' personal epistemology.

Research question: How are the emotions Alex experiences in her university mathematics study grounded in her mathematics-specific personal epistemology?

Methods

The case of Alex is theoretically chosen due to her rare combination of high mathematics achievement and negative emotions found in previous analyses of the data (e.g., Göller & Rück, 2022). According to control-value theory (Pekrun, 2024), this holds the potential to analyse the formation of emotions and their corresponding values and beliefs while mitigating the (often dominant) effects of achievement issues and control appraisals. Alex, a mathematics preservice teacher, was interviewed four times during her first year of study on her self-regulated learning processes and attitudes towards university mathematics. The first time, she was interviewed two weeks before the start of the lectures of her first semester while attending a preparatory mathematics course; the second time in the middle of her first semester, the third time at the end of her first semester, and the fourth time in the middle of the second semester of her university study. In her first semester, Alex attended two theorem-proof-focused pure mathematics lectures on linear algebra and mathematical foundations (logic, proof types, sets, finitary relation, functions, …) together with mathematics major students; in her second semester, a follow-up course on linear algebra together with mathematics major students and an elementary geometry lecture for preservice mathematics teachers. To pass these courses, students had to achieve 50 % of the points given for weekly exercises and pass a written exam determining the grade for the respective course at the end of the semester. Alex achieved the most points of all students in the weekly exercises of her first semester in the lecture on mathematics foundations, the third most in linear algebra (out of more than 200 students), and passed all her mathematics examinations with good grades. Nevertheless, Alex reported predominately negative emotions towards her mathematics study in the interviews.

All interviews were audio recorded, transcribed verbatim, and analysed using MAXQDA24. Following the considerations in the theory (cf. Op't Eynde, 2004), as a first step, all interview data of Alex were screened for words indicating positive (e.g., good, like, great) or negative (e.g., bad, annoying, stupid) appraisals as well as positive (fun, happy, (en)joy) or negative (e.g., frustrating, depressing, no fun) emotions. In the second step, the passages identified in this way were analysed regarding their underlying values and beliefs presented in the theory. These were then analysed in terms of the interconnectedness of emotions, values, and beliefs in the third step.

Findings

In the following, we first look at Alex's personal epistemology at the beginning of her study to set the ground for analysing the interconnectedness of emotions, values, and beliefs, which will be presented in the second part of the findings.

Two weeks before the lectures started: Alex's personal epistemology at the start of the study

Regarding the choice of mathematics as a study subject, for Alex, it had been clear "since the fifth grade that I had to continue (maths) somehow because I can't do without it," indicating a high personal value of mathematics (which here appears to be relatively general and independent of other persons). Regarding the choice of her second subject (physics), the prominent role of the respective teachers becomes apparent, especially regarding her emotions towards the respective subjects.

> I had actually planned to study biology, but I had a horrible biology teacher in secondary school. She literally took the joy out of it. And I had a great physics teacher, and since then, it's been fun.

As important skills for herself as a mathematics teacher, she mentioned "that I can explain well and have my class under control […] and that I master the content." Alex's personal epistemology regarding *mathematics teaching* here identifies good teaching through a prominent role of the teacher, who is competent, able to explain well, and controls the class.

More details about Alex's *valued beliefs about mathematics* can be found in the following excerpt:

Interviewer:	Can you describe what you find interesting about maths, what you enjoy about it?
Alex:	Hm, the numbers and calculating, that I somehow find the solution or get there in a roundabout way. [...] Hm, the graphs, I actually quite liked that. We used to draw things, that was kind of fun. And in analysis, with the functions, with the curve sketching [Kurvendiskussion], it was always the same. You had your scheme and could work according to it.
Interviewer:	And you found it important to have a scheme?
Alex:	Yes.
Interviewer:	Why?
Alex:	Otherwise, I wouldn't have known what to do with the function.

This excerpt indicates that Alex values *instrumental/static* aspects of mathematics, especially specified here by valuing calculating and working according to a predefined scheme. This suits with valuing a prominent role of the teacher, who explains what to do (here "with the function").

However, in this first interview, it seems that Alex could also develop positive emotions towards *relational/dynamic* aspects of mathematics, as is indicated here by her appraisal of proofs in the preparatory course: "They [the proofs] were actually fun here in the preparatory course." These positive appraisals can be traced back to her experiencing herself as being able to perform (possibly referring to *instrumental/static* aspects of mathematics), that "it worked out" when "something was given, and then you came to the same result as you wanted to prove." As we will see later, these positive experiences rely on a way of teaching in the preparatory course that fits her valued beliefs.

During the first year of study: A clash of valued beliefs and experienced learning situations

Noteworthy, Alex's personal epistemology hardly changed during her first year of study. On this basis, Alex experienced most learning situations with negative emotions, which can be traced back to discrepancies between her experiences and her valued beliefs, as will be illustrated in the following. Regarding the *environmental context* of learning mathematics at university, such discrepancies could be found in her personal relationship with the professor who gave the lectures, the style of the professor, his teaching style with the herein incorporated roles of students and lecturers, the style of the tutorials, as well as her perception of goals, valued beliefs, and strategies of her peers. As an example, we look at the very beginning of the interview in the sixth week of Alex's first semester:

Interviewer:	How are your studies going so far?
Alex:	It's quite exhausting and too much. [...] Actually, if mathematics weren't so bad, then it would be better because the new lectures are kind of stupid.
Interviewer:	Can you elaborate a little on that?
Alex:	I don't know; maybe if another professor were doing it, it would be better. [...] you sit there with the exercises: Yes, what does he actually want from me now? And we have to find our way through somehow. And in the lecture, he tells you what he wants, but you're not really allowed to ask questions, apparently. Or at least, if you ask questions, you make a complete fool of yourself. And I don't really need that.
Interviewer:	Why do you think asking questions makes you a fool? Do you have an example?

Alex: He always makes a mockery of it. Or: "Yes, that's trivial", or some weird stuff. And he also writes far too little on the blackboard. He says something but doesn't write it down. And somehow, you only have half notes. And in the end, you can't do much with them.

Firstly, the mathematics study is experienced as too "exhausting and too much", indicating high cost (negative value); it is appraised as "bad" and the lectures as "stupid". These appraisals are first directly referred to (the personal relationship with) the professor, which aligns with the prominent role of the teacher in Alex's personal epistemology (PE). In this personal relationship, Alex feels excluded and not valued, as becomes apparent later in the interview: "He said that he isn't interested in teacher training. He mainly refers to math majors and everything else. You notice that: 'Oh, you're just a preservice teacher.' That's somehow a bit annoying."

The teaching style Alex experienced contradicts her PE regarding the ability to explain well. Here, she experienced that the professor "writes far too little on the blackboard" so that "you can't do much with" the notes, indicating bad explaining (i.e., bad teaching in her PE). So, she experiences that in this university mathematics setting, students "have to find [their own] way through" and that there are no clear instructions on what "he actually wants from me", which contradicts her PE of teaching and also her *instrumental/static* valued beliefs regarding mathematics (see below).

All this, as well as the importance of asking questions in her PE, can be found in the following excerpt, especially in comparison to the preparatory course:

Alex: Maths is no longer fun here. It was fun in the preparatory course, but now, when you get a professor like him, and then in both maths lectures, you're really fed up. I hope to get a better one in the third semester because otherwise, I don't know if it still makes sense.

Interviewer: Can you tell me why that is? [...] That it's no fun, or what was better or different in the preparatory course?

Alex: We could ask questions to [the teacher of the preparatory course], and she explained things to us in such a way that we could do it. And she also wrote it down in such detail when you looked at it again at home that it was all okay. But here, you kind of miss the connections, and then, at some point, you get frustrated because it's just stuff you no longer understand.

Asking questions is emphasized here as an important part of Alex's PE, while the learning environment is experienced as obstructing these. In the lecture, the feared "mockery" of the professor and the risk to "make a complete fool of yourself", but also in tutorials and peer learning situations, she experiences others as "annoyed because I hold up everything, I hold up the business, and they're not so enthusiastic about that". Overall, Alex experiences the social environment at the university as not conducive to learning:

Alex: In the tutorials, I somehow always have one-to-one conversations with the tutor because the others don't give a shit or leave after half the time or right at the beginning. And when I ask a question, there's moaning at the back the whole time.

Interviewer: Really?

Alex: It's really annoying when you ask a question, and they start moaning. If they're not interested, then they can leave. I thought the tutorials were there to ask questions.

Interviewer: Yes, but why do you think the others don't ask questions in Linear Algebra? Why do you believe that is?

Alex: Because they're just trying to get above 50 per cent, and after that, they don't care. They just want to get through somehow. And they don't want to understand everything so precisely. They just want to pass somehow.
Interviewer: And how is that with you?
Alex: I want to understand it because if a teaching degree doesn't turn out to be my choice after all, then perhaps I would still have the option of a Bachelor's degree or, in general, I would need it for the final examination.

Because of these different goals, peer learning appears "useless" to her because there she experienced that "one person solves it and the rest copies it", while in her PE "it's better to discuss it".

Regarding her valued beliefs about mathematics, Alex enjoyed doing calculations – "Gauss [algorithm], it's fun if you can do a bit of calculating for once" – and disliked proofs, which is in line with her PE in favour of *instrumental/static* aspects of mathematics but also refers to the way she experiences herself in the given learning environment (cf. her experiences in the preparatory course).

Alex: Right now, with the stupid proofs. We now have an exercise sheet with proofs. And nobody knows how to do the proofs. I asked someone in the third semester, and he couldn't help me either. And that's quite depressing. I know a lot of people who are quitting because they can't cope with it. […]
Interviewer: Were there any tasks that you enjoyed that you liked?
Alex: Yes, when I had a matrix and had to do some calculations, a bit of calculating and not just proofs.

The frustration about this excluding learning environment and being confronted with "stuff you no longer understand" seems to crystallise in her opposition to proofs that "nobody knows how to do", which are identified as an essential feature of this university mathematics learning environment. All this leads to a situation in which Alex experiences that studying mathematics is "no fun anymore" and especially a devaluation of proofs, which are appraised as being "pointless" and something "you never need at school". We conclude with an excerpt exemplifying this argumentation:

Interviewer: Has your attitude towards mathematics changed during your studies so far?
Alex: It's shittier.
Interviewer: Yes?
Alex: I liked it at first, but now/ Every time I go to [linear algebra], I think to myself: Oh my God, I hope it's over quickly. [...] It's no fun anymore. […] There were mainly proofs, for me that's something, I can't stand that, I don't need that somehow, it's pointless for me. Because moving letters back and forth, that's not really my thing.
Interviewer: Well, is proving only moving letters back and forth? Is that so?
Alex: Yes, I see it that way. […] I think it's pointless; you can't do anything with it.

Discussion

The presented case study of Alex, a high-achieving mathematics preservice teacher who reported many negative emotions toward her mathematics study, focussed on the formation of mathematics-specific values grounded in personal epistemology (Hofer & Bendixen, 2012; Op't Eynde, 2004) to better understand students' emotions in the secondary-tertiary transition. Thereby, our theoretical considerations regarding the analysis of emotions alongside students' valued beliefs about mathematics, mathematics learning, the self as a mathematician, and the mathematics learning environment (Op't Eynde, 2004; Op't Eynde et al., 2006a) have proven fruitful for further theorising the formation of values and emotions. Indeed, the findings reveal that Alex experienced the university mathematics learning environment as opposing her personal epistemology in all these aspects.

Regarding her valued beliefs about mathematics, Alex's negative emotions can be traced back to a mismatch of her positive appraisal of instrumental/static aspects of mathematics that did not change during her first year of study and conflicted with the experienced focus on proofs and the way they were taught and learned in this specific university mathematics learning environment. The findings indicate that this "beliefs overhang" (Daskalogianni & Simpson, 2001) that was also found in other studies is not only a cognitive issue that can be solved by acquiring knowledge but additionally requires an adaption of values attached to such beliefs for positive appraisals towards university mathematics (cf. Göller, 2023). The results on Alex's experience of proofs in the preparatory course indicate that with a teaching environment aligned to Alex's other valued beliefs, this adaption could have been possible but did not materialise, for example, due to her experiences of exclusion in the mathematics university learning environment. Indeed, the way she experienced teaching and the herein-incorporated roles of students and lecturers at the university that obstructed a realisation of her valued beliefs about teaching and learning mathematics (e.g., asking questions, discussing exercises) highly contributed to a missing sense of belonging and a positioning as an outsider (Gildehaus et al., 2024; Lahdenperä & Nieminen, 2020) and even seemed to play a bigger part in her emotional alienation (Solomon & Croft, 2016) from university mathematics than the mathematics difficulties and challenges of her valued beliefs about mathematics. This gives hints on how changes towards more inclusive teaching practices have the potential to ease transition difficulties without changing mathematics contents (Lahdenperä & Nieminen, 2020).

In line with our theoretical considerations, Alex's emotions and appraisals of the mathematics learning environment are highly individual and based on her personal experiences and epistemology. Such results should not be generalised without further similar studies. Nevertheless, the findings call for a fundamental reflection and reconsideration of university mathematics teaching practices, which create such toxic-experienced learning environments even for the highest-achieving students.

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Connections between epistemic emotions and control actions in linear algebra

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Emotions related to knowledge acquisition (epistemic emotions) play a key role in epistemic processes in linear algebra. In this study we explored the relationship between epistemic emotions and control actions in 99 undergraduate mathematics and computer engineering students in a task requiring the use of computational thinking and linear algebra concepts. Control (theoretical, semiotic and technological) and appraisals as antecedents of epistemic emotions are analyzed. It is shown that mathematics students give more importance to control elements associated with formalization of mathematic concept while computer engineering students give more importance to emotions associated with the instrumental and technological control dimension.

Keywords: Epistemic emotions, control-value theory, linear algebra, mathematical working spaces.

Introduction

The research we present arises from the need to connect mathematical thinking and computational thinking in university teaching with students of the Bachelor's Degree in Mathematics (DM) and the Bachelor's Degree in Computer Engineering (DCE). Research has shown that engineering students have less interest in mathematics, even though it is essential knowledge for their studies. Mathematics contents (e.g. linear algebra) taught in the first courses of computer engineering present a great complexity for the student due to their high level of abstract reasoning (Stewart et al., 2018; Gómez-Chacón et al., 2015), so it is necessary to propitiate actions and resources that materialize reflection on contents and highlight actions of "control" of mathematical work.

To study students' emotions when performing a mathematical tasks related to computational thinking in algebra, the control-value theory of emotions of Pekrun (2006) was used in coordination with the characterization of the control action in didactics carried out from the Theory of Mathematical Working Spaces (Kuzniak & Nechache, 2023; Gómez-Chacón & Marbán, 2024). This coordination of theoretical frameworks allows categorizing the type of epistemic emotions experienced by students when they engage in processes of solving computational thinking tasks related to concepts of the subject of linear algebra. This study addresses the following research question: What evidence is there of relationships between epistemic emotions and mathematical control actions experienced by DM students and by DCE students in solving computational thinking task?

Theoretical framework

Epistemic emotions

Pekrun (2006) considers academic emotions as the set of emotions that are triggered in academic contexts, classifying them into: social emotions, thematic emotions, epistemic emotions and achievement emotions. Epistemic emotions are defined as emotions that arise when their object is knowledge and the process of knowing (Gómez-Chacón, 2017; Pekrun et al., 2017). They are

triggered by knowledge, the characteristics of the tasks and the cognitive activity of the subject when solving them. These investigations agree in pointing out control, value, novelty, complexity and achievement (or epistemic goals) as antecedents of epistemic emotions. Epistemic emotions are related to knowledge characteristics of cognitive activities. Empirical results in mathematics education support the idea that epistemic emotions can influence learning and performance (D'Mello & Graesser, 2012; Gómez-Chacón, 2017; Gómez-Chacón & Marbán; 2024; Schukajlow et al., 2017; Schubert, Pekrun, & Ufer, 2023).

Control-value theory provides an integrative framework for analyzing the antecedents and effects of epistemic emotions (Pekrun, 2006). This theory focuses on two variables that influence internal evaluations of an epistemic process and cognitive processes: (1) the control (control appraisals) they have over the activity and (2) the value (value appraisals) they assign to the activity. Control appraisals indicate the extent to which learners see themselves in a position to construct and solve the problem, relating it to the knowledge needed and related skills. Value refers to the importance the learner attaches to the activity.

Control actions in didactics

The use of control in mathematics teaching has its roots in Polya's (1945) reflections on mathematical problem solving and, later, in Schoenfeld's problem-solving approach to mathematics education (Schoenfeld, 1985). More recently, and in connection with didactic theories, the nature of controls is being deepened. Thus, in the development of the methodological aspects of the *Theory of Mathematical Working Spaces* (ThMWS), control activities have been highlighted as an aspect to be deepened, being developed from different approaches. Kuzniak and Nechache (2023) from the ThMWS characterize the nature and role of controls in mathematical work, proposing an analysis methodology based on different criteria of the ThMWS: conformity of the resolution processes to a mathematical paradigm, the completeness of the work flow in the MWS and, finally, the correctness of the results obtained. In the ThMWS framework was introduced the concepts of semiotic, technological and theoretical controls: *Semiotic control* refers to the selection and use of a semiotic tool based on the use of signs and representations. It ensures that these signs have been properly chosen and that they have been interpreted. Associated with semiotic genesis, it guides and validates the visualization process.

Technological control refers to the decision taken to select one or more technological tools (physical or digital artifacts, algorithms). This type of control is used to ensure the correct use of these tools as a technological instrument. Combined with instrumental genesis, it guides and validates the construction process.

Theoretical control refers to decision-making on the choice of theoretical tools (theorems and properties of the theoretical frame of reference) and the forms considered valid in the paradigm adopted in the reference ThMWS. Associated with the discursive genesis, it orients and validates the demonstrative discourse of the proof.

These three types of control work together to validate the interactions between the different genesis of the mathematical working space.

Objectives

The following objective is proposed: To identify what connections are established between epistemic emotions and control actions experienced by DM students and by DCE students when solving computational thinking tasks in linear algebra.

Method

The study is part of a teaching experiment (Cobb, et. al., 2003) carried out with two groups of students, one from the DM from the UCM university (55 students) and the other from the DCE from the URJC university (44 students). Both groups of students were asked to form teams containing students from both degrees (interdisciplinary groups) to design and develop a digital tool for teaching matrices to 5th and 6th grade students.

Task statement:

> The objective of this practice is to make a first prototype (web and mobile) of a digital tool, serious game or video game that works mathematical skills. To do so, we will consider a mathematical object highly used in many aspects of our life: the matrix. [...] Therefore, this tool is going to be asked to explain in an exhaustive and formal way the concepts of matrix, matrix addition, matrix subtraction and matrix transposition. Specifically, it is requested to include activities or resources that allow the user to assimilate all of the above in a simple way, but without losing the mathematical background by working from real world examples.

The most advanced concept considered is the transpose of a matrix. This requires the associated theoretical foundations: operation, properties and applications. In addition, the mathematical and computational algorithms used must be considered.

The development process followed was as follows:

- Groups of 4 or 5 people are formed in each degree and interdisciplinary teams are formed with a group from the Degree in Mathematics and a group from the Degree in Computer Engineering.
- The different interdisciplinary teams were given the task statement and additional information to help them in the process of creating the digital tool.
- The teams begin to create a prototype of the digital tool. There is constant communication between the DM students and the DCEs that make up the interdisciplinary teams.
- When there is a first version of the tool, the DM students carry out an expert evaluation.
- With the data obtained, a second version of the tool is made. The evaluation of this prototype is carried out by the DCE members of the interdisciplinary teams.
- A third version is made based on all the comments received and a heuristic evaluation of the prototype is carried out by the DM students of the team itself.
- The final version of the tool is made, and a list of improvements is given.

The sources of data collection are varied: the different prototypes of the digital tools developed by the interdisciplinary teams, partial and final reports on the computational development of the tool, questionnaires on user interaction with the digital tool, audio and video recordings of user interactions

with the tool, communications via WhatsApp and/or email between the members of the interdisciplinary teams and individual self-assessment questionnaires on epistemic emotions and control actions. Figure 1 shows some examples of items from the latter questionnaire.

3. Indicate how often you have experienced some of these emotions: Surprise (E1), Curiosity (E2), Enjoyment (E3), Interest (E4), Anxiety (E5), Confusion (E6), Confidence (E7, Frustration (E8), Boredom (E9), and it is given as response options: Not at all, Rarely, Sometimes, Often, and Always.

4. Choose the one you have experienced most frequently:
Emotion: ----------

4.1 Now focus on the emotion you have chosen What were you doing in the problem when you experienced *that emotion*?

4.2. Indicate how often you have experienced the following processes when experiencing the chosen emotion:

I controlled the situation (C1), I understood the situation well (C2), I made enough effort (C3), I thought I could influence to solve it (C4), I was able to predict what was going to happen in that situation (C5), I was attentive and focused on the problem (C6), I was able to break it down into simpler problems (C7), and it is given as a response option: Not at all, Rarely, Sometimes, Often and Always.

Figure 1: Examples of self-assessment questionnaire items

Since two types of data were used (discursive responses to the questionnaire and scale-type responses), two analytical approaches were employed. For the discursive data, a qualitative approach based on cross-checking the responses by three researchers was used. The inductive analysis process generated categories organized in two dimensions: epistemic emotions and control actions (semiotic, technological and theoretical) in order to identify types of control that are encompassed in the set of interactions between students of the Degree in Mathematics and the Degree in Computer Engineering that can be described thanks to the circulation of work in computer applications. And for the quantitative analysis, the implicative statistical analysis is applied for modal variables (Lagrange, 1998), with 5 modalities (respectively to the values 1, 0.75, 0.5, 0.25 and 0.). The variables emotions (E1, E2,..., E9) and control actions (C1, C2,..., C7)). The cohesive tree, which allows us to obtain a set of rules and meta-rules that express causal relationships between the epistemic emotions and control actions variables used by the students in the task. Cohesion arises as a measure of the propensity quality of a rule of type R: a → b as an asymmetric relationship between two variables, where their values lie within the range [0, 1].

Results

Relationships between epistemic emotions and control actions

We analyzed the responses of the questionnaires related to epistemic emotions and control dimensions, exploring the similarities and cohesion between variables in order to identify clusters and establish R-rules that allow us to affirm associations between variables and between dimensions.

In the DM, the variables gathered up to level 14 of classification have a low similarity index and the significant node is the union of emotion and control variables: (((E7 C7) (C1 C2)) ((C3 C4) (C5 C6))). Thus, the deduction we can draw from these relationships is that “people who have confidence (E7), know how to decompose into simpler problems (C7) and understand the problem-situation (C2) tend

to think they can influence the problem (C4) and make an effort (C3) because they can predict the resolution (C5) and are attentive and focused (C6)". The variables with the highest similarity index are at level 1 with an index of 69% (E5 E8), allowing us to affirm that those with anxiety (E5) tend to be frustrated (E8).

In the DCE, the variables gathered up to level 14 of classification have a low index of similarity and the significant node is the union of variables of emotions and control: (((E7 C7) (C3 C4)) ((C1 C2) (C5 C6))). Thus, the deduction we can draw from these relationships is that "people who have Confidence (E7), know how to decompose into simpler problems (C7), make an effort (C3) and think they can influence the problem (C4) tend to think they are able to control the situation (C1), understand the problem-situation (C2), predict a resolution (C5) and be attentive and focused (C6)". The variables with the highest similarity index are at level 1 with an index greater than 90% (E5 E8), allowing us to affirm that those with anxiety (E5) tend to be frustrated (E8).

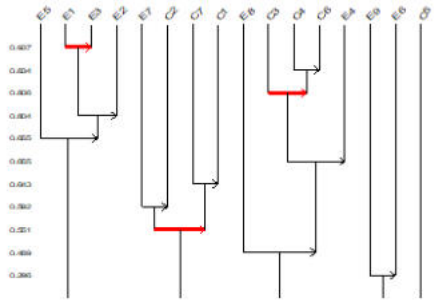
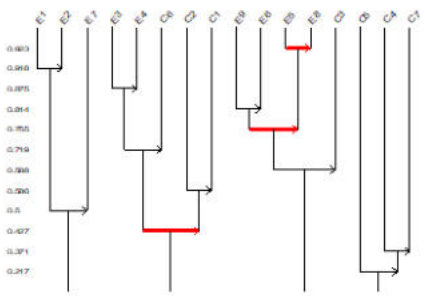
Bachelor's degree in mathematics (DM)	**Bachelor's degree in computer engineering (DCE)**
	
Level 1 of classification: (E1 E3) cohesion: 0.937 Level 3 of classification: (C3 (C4 C6)) cohesion: 0.806 Level 9 of classification: ((E7 C2) (C7 C1)) cohesion: 0.551	Level 1 of classification: (E5 E8) cohesion: 0.923 Level 5 of classification: ((E9 E6) (E5 E8)) cohesion: 0.755 Level 9 of classification: ((E1 E2) E7) cohesion: 0.5 Level 10 of classification: (((E3 E4) C6) (C2 C1)) cohesion: 0.427

Figure 2: Cohesive tree task by students of the Bachelor's DM and Bachelor's DCE

In the DM this non-symmetrical classification of the classes brings us closer to the reality of the data. We indicate that only those variables with an implicative intensity greater than 0.5 have been joined and then organized according to cohesion (Fig. 2). The cluster with cohesion greater than 0.5 is ((E7 C2) (C7 C1)), whose meaning is located in the association of epistemic and control emotion variables and allows us to state: "people with confidence (E7) and who understand the problem-situation well (C7) tend to decompose the problem into simpler problems (C2) and control the situation (C1)". Also, we highlight the cluster (C3 (C4 C6)) established at a level higher than 0.806 of cohesion and in red in the tree, since it is a significant node. Its meaning is located in the association of variables of the control dimension and allows us to state: "In general, people who have made an effort (C3) usually think that they could influence to solve it (C4) and have been attentive and focused on the problem (C6)". The cluster with a cohesion index of 0.937 (E1 E3) refers to epistemic emotions and allows us to state that: "those who have experienced surprise (E1) usually experience joy (E3) in the resolution". In the DCE, the significant nodes are at levels 1, 5, 9 and 10 (Fig. 2). We highlight the cluster ((E9 E6) (E5 E8)) established at a level of almost 0.75 of cohesion and significance. This cluster integrates

variables of epistemic emotions and the meaning of the class is: “those who experience boredom (E9) and confusion (E6) often experience anxiety (E5) and frustration (E8)”. Also, in relation to the epistemic emotions experienced we find the cluster (E5 E8): “those who experience anxiety (E5) often experience frustration (E8)”. Referring to the relationship between epistemic and control variables, the cluster (((E3 E4) C6) (C2 C1)) stands out with an index of 0.42. This allows us to state that “those who experience joy (E3) and interest (E4) and pay attention and concentration (C6) tend to understand the problem (C2) and think they can control the situation (C1)”.

Control actions

An overall qualitative analysis of the responses of the interdisciplinary teams and the transcription of communications via WhatsApp and/or email between the team members allows us to qualify some aspects of control by the DM students when they review what has been elaborated by the DCE students within the interdisciplinary team itself. Related to the concept of matrix itself, they point out the need to explicitly include the definition of matrix in the prototype. Several of them indicate that it is not enough to put the image of the matrix, but that it must be written formally. This constitutes an element of semiotic control.

A theoretical control element would be the indication that the basic elements that form a matrix must be defined: row, column and order. It is emphasized that these properties are necessary not only to understand what a matrix is, but also for the proper development of a transposition algorithm. Another element of theoretical control is the inclusion of real applications and examples, which help to better understand the concept and the elements associated with it.

If we analyze the control elements related to the transposition itself, we can find that by not being clear about the concepts of matrix, row, column and order, this leads to a poor understanding of how the transposition works. Therefore, this again reflects an element of theoretical control. In addition, the explanation of the process itself is not understandable and is requested to be clearer, more formal and supported by practical examples. This is reflected in the following dialogue between students from one of the interdisciplinary teams regarding the explanation of the transpose matrix:

DM student 1:	Okay, you just told us that you don't understand the explanation of the transpose.
DCE student 1:	Could you give us an example of how to explain it easier?
DM student 1:	We have explained it first with a 2x2 matrix and in steps and then he did it well.
DM student 2:	I think that in general it is fine, but the most important thing is to reinforce the explanations at the beginning. Because otherwise they get very lost, like what columns and rows are and how they work.

One element of technological control that is indicated is to drag the elements instead of simply pointing to them. This clearly aids the user in their resolution work because it is more intuitive to make the change they need to make.

Discussion and conclusions

In this study we investigated the connections between epistemic emotions experienced by computer engineering and mathematics students when solving tasks that require control in mathematical work in computational thinking tasks in algebra. To study the connections between epistemic emotions and

reasoning, theoretical notions from control value theory (Pekrun, et al., 2017) and mathematical work theory (Kuzniak & Nechade, 2023) were combined.

The data show that the study of the similarities between the variables in the DM and DCE students does not show any difference between them. In both groups, the control variables are linked to confidence in knowing how to solve adequately. Anxiety is linked to frustration in the failure to solve.

In the case of the causal relationship between variables -expressed by the cohesion index-, we can observe differences between the students of both grades. In the case of the GMs, there is a strong association of control variables, relating effort, ability to influence the problem and concentration. This can be explained by the importance given by mathematicians to formalization when explaining mathematical concepts. In addition, it reflects the importance underlying the training they receive that concepts must be constructed from the most basic and without ever losing sight of the objective to be achieved. In addition, it appears related to control variables the emotion of confidence, which, as we have indicated above, has to do with a good result in the work performed (Gómez-Chacón, 2017).

In the case of DCEs, significant associations appear between negative emotion variables. Boredom, confusion, anxiety and frustration are related. And with high cohesion, anxiety and frustration in relation to mathematical content (Stewart et al., 2018; Gómez-Chacón et al., 2015). It should be noted that the relationship between joy and interest (positive emotions in this case) and the elements of concentration control, understanding and control of the problem (which also appeared in a certain way in the DMs) appears in a slight way. Implicative statistical analysis reveals that the DCEs give greater weight to emotions, and this leads them to the mathematical control elements in the task. Qualitative analyses of conversations between the student groups about intermediate control steps during solving allow us to qualify the interaction between emotion and type of control. Frequently the indications by DMs are related to the theoretical contents associated with the transposition of matrices while DCEs refer to the instrumental and technological control dimension. Therefore, we can conclude that DMs give more importance to control elements linked to formal content when facing the task while DCEs give priority to the emotions they feel and play a more important role when developing that same task. Finally, we note that for an extension of the research and that the results may allow generalizations about the elements compared, it is important to consider the limitations derived from the roles in the interdisciplinary groups and their involvement in the task; and the validity of the conclusions by triangulating the responses in different contexts.

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"Calculating with numbers and then finding a specific result" – Exploring students' conceptions about mathematics and doing mathematics and possible relations to mathematics teaching

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Based on the student conceptions of mathematics framework, the present paper aims to explore students' conceptions about mathematics and, especially, doing mathematics as well as possible relations to mathematics teaching by also considering the mathematics teacher's conceptions and intention to teach mathematics. For this purpose, I analysed additional data from 12 9th graders and their mathematics teacher from an exploratory interview study. The results indicate that the teacher's intention to convey to her students that mathematics is a coherent system and that doing mathematics is a sense-making process does not get through to most of her students, who associate mathematics and doing mathematics primarily with arithmetic and finding results. Possibilities for counteracting this are discussed.

Keywords: Students' conceptions, mathematics teachers' conceptions, doing mathematics, mathematics teaching.

Introduction

Students' conceptions about mathematics, which represent a "set of conscious beliefs" (Furinghetti, 1996, p. 20), are likely to be interrelated with their learning, motivation and self-concept (e.g., Grouws et al., 1996; see also, e.g., review by Østergaard, 2024). For example, the results of a study by Grouws et al. (1996) reveal differences in students' conceptions of doing and learning mathematics between mathematically talented and average high school students: While mathematically talented students consider doing mathematics as a sense-making process in which one must rely on personal thought and reflection to establish the validity of knowledge, for average high school students doing mathematics often means implementing known procedures and formulas and accepting mathematically truth as established by others than depending on logical thought to deduce mathematical knowledge. As, for instance, the results of the Third International Mathematics and Science Study (TIMSS/III) for Germany (Köller et al., 2000) show, such limited conceptions of students about mathematics as a mere application of solution algorithms may negatively influence their interest, learning and, thus, achievement.[1] The conceptions of students about mathematics are probably largely based on their experiences in mathematics class (e.g., Schoenfeld, 1994), which, in turn, consist of perceptions that originate from mathematics teachers, schoolmates and learning materials. In particular, mathematics teachers have a major influence, among other things, through their curricular decisions (e.g., Furinghetti & Pehkonen, 2000). With regard to mathematics teaching,

[1] Please note that Köller et al. (2000) actually investigated students' epistemological beliefs about mathematics. However, following Thompson (1992, p. 130), "the distinction [between beliefs and conceptions] may not be a terribly important one". Therefore, the present paper also draws on research on beliefs about mathematics.

Schoenfeld (1994, p. 53), for example, emphasizes the importance of engaging students "in the practices of doing mathematics and, as a result, […] [having] them develop a sense of discipline (i.e., a mathematical perspective) consistent with that held by mathematicians." This seems to be of particular importance when considering mathematic classes in terms of figured worlds (e.g., Boaler & Greeno, 2000) in which students are not only viewed as learners of mathematics, but as individuals negotiating a sense of self. Against this background, the present paper aims to explore students' conceptions about mathematics and, especially, doing mathematics as well as possible relations to mathematics teaching by also taking the mathematics teacher's conceptions and intention to teach mathematics into account.

Theoretical background

According to Schoenfeld (1994, p. 58), "mathematics is a 'hands-on', data-based enterprise for those who engage in it […] [and] doing mathematics is doing science", meaning systematizing knowledge based on observations, study and experimentation carried on in order to determine the nature or principles of what is being studied. What makes it mathematics for him, compared to the natural sciences, is the unique character of the objects being studied and the tools of the trade. However, as the results of a review by Østergaard (2024) on students' beliefs about mathematics as a discipline – in which the terms beliefs and conceptions were considered as similar – show, students often do not view mathematics as described by Schoenfeld (1994), but as a static, rigid and rule-based discipline. For example, students often believe that mathematical problems only have one correct answer and can only be solved using the correct rule, formula or procedure, usually shown by the teacher. Moreover, students regard mathematics as a fragmented set of rules and formulas that they have to memorize and apply appropriately. This represents the dualistic perspective on mathematics as opposed to the relativistic perspective, from which mathematics is seen – in line with Schoenfeld's (1994) description – as a dynamic, relativistic and applicable "science of patterns". The results of the review further reveal that these conceptions about mathematics are not conductive for students' learning, motivation and self-concept. For example, as already mentioned, in their study on students' conceptions about mathematics, Grouws et al. (1996) found differences in students' conceptions of doing and learning mathematics between mathematically talented and average high school students. Referring to Schoenfeld's work, among others, they distinguish in their *student conceptions of mathematics framework* seven dimensions of students' conceptions about mathematics, namely composition of mathematical knowledge, structure of mathematical knowledge, status of mathematical knowledge, doing mathematics, validating ideas in mathematics, learning mathematics and usefulness of mathematics. For each dimension, they distinguish between two poles, which correspond to the dualistic and relativistic perspective on mathematics described above. Figure 1 provides an overview of the dimensions and their poles. For more detailed descriptions of the single dimensions and their poles, see Grouws et al. (1996). With regard to students' conceptions about doing mathematics, the results reveal that most mathematically talented students view doing mathematics as a process of sense-making that involves figuring things out more than just applying known procedures and facts. Moreover, most of them do not believe that the ability to use a formula accurately is sufficient to understand the mathematical concepts underlying the formula. In contrast, for the majority of the average students, the accurate use of formulas represents understanding and

good problem solving depends on the availability of formulas. Regarding students' conceptions about learning mathematics, the results further show that the average students associate learning mathematics more strongly with memorization than the mathematically talented students.

Dualistic	⟷	Relativistic
1. Composition of mathematical knowledge		
Facts, formulas and algorithm	⟷	Concepts, principles and generalisations
2. Structure of mathematical knowledge		
Collection of isolated pieces	⟷	Coherent system
3. Status of mathematical knowledge		
Static entity	⟷	Dynamic field
4. Doing mathematics		
Results	⟷	Sense-making
5. Validating ideas in mathematics		
Outside authority	⟷	Logial thought
6. Learning mathematics		
Memorizing	⟷	Constructing and understanding
7. Usefulness of mathematics		
School subject with little value	⟷	Useful endeavor

Figure 1: Overview of the dimensions of the student conceptions of mathematics framework by Grouws et al. (1996) and their poles ranging from the dualistic to the relativistic perspective on mathematics (see also Østergaard, 2024, p. 54)

As emphasized, for instance, by Schoenfeld (1994), students' conceptions about mathematics arise from their experiences with mathematics that predominantly take place in the mathematics classrooms. Therefore, he claims:

> […] if we believe that doing mathematics is an act of sense-making; if we believe that mathematics is often a hands-on, empirical activity; […] if we believe that learning mathematics is empowering and that there is a mathematical way of thinking that has value and power, then our classroom practices must reflect these beliefs. (Schoenfeld, 1994, p. 60f.)

However, as, for example, the results of an interview study by Boaler and Greeno (2000) reveal, students often depict a different picture of their mathematics lessons. Most of the students surveyed describe their mathematics lessons mainly through their teacher reviewing homework, explaining methods at the board and assigning questions to be completed and, thus, portray an image of mathematics teaching that afforded received knowing. Given that "students do not just learn mathematics in school classrooms, they learn to *be*" (Boaler & Greeno, p. 188), the practices of teaching do not only define the knowledge that is produced, mediated by the beliefs that students develop about mathematics and learning in response to different teaching methods, but also their sense of self. As a result, many able students may leave mathematics because they do not want to author their identities as passive receivers of knowledge. At the same time, students may choose

mathematics as their main field of study based on their experiences in class and their limited conceptions about mathematics, which may lead to significant problems as studying mathematics in upper grades, such as secondary schools or at university, requires different components of mathematics (see also, e.g., Pehkonen & Törner, 1999). How mathematics classes are organized and conducted depends, in turn, on how mathematics teachers view mathematics (e.g., Grouws et al., 1996) and on their curricular decisions (e.g., Furinghetti & Pehkonen, 2000). For example, the results of a case study by Cooney (1985) indicate that the teacher's conception of mathematics as problem solving influences his instructional decision and the curriculum he implemented. However, mathematics teachers likely choose to become school teachers of mathematics because they preferred the practice of mathematics teaching they had experienced in school, namely mainly receiving knowledge and, thus, continue teaching received forms of knowing (e.g., Boaler & Greeno, 2000).

Against this background, the present paper aims to obtain a (more) comprehensive view of students' conceptions about mathematics and possible relations to mathematics teaching. For this purpose, students' and their mathematics teacher's conceptions and intention to teach mathematics are considered together. In light of, on the one hand, the differences between mathematically talented and average students' conceptions about doing and learning mathematics found in the study by Grouws et al. (1996) and, on the other hand, the importance of engaging students in the practices of doing mathematics in order to develop a mathematical perspective (Schoenfeld, 1994), I focus especially on the conceptions about doing mathematics.

Methods

To address this research objective, I analysed additional data from an exploratory interview study (see Hagenkötter et al., 2022) that has not yet been analysed. Here, I focus on a subsample of 12 9th graders (age: $M = 14.08$, $SD = 1.31$; 83% male; interest in mathematics between 1 = no interest and 6 = high interest: $M = 4.08$, $SD = 1.31$; school grade in mathematics between 1 = very good and 6 = unsatisfactory: $M = 2.08$, $SD = 0.90$) and their mathematics teacher (female; 5 years teaching experience). I interviewed the students as well as the mathematics teacher in June and July 2021 using semi-structured interviews. The whole interviews with the students lasted between 9 and 16 minutes and with the teacher around 30 minutes. The interview protocol contained the following three parts, with additional questions for the mathematics teacher about her teaching of mathematics: (1) View on mathematics and doing mathematics, (2) Typical mathematical activities of mathematical scientists, (3) Typical mathematical activities during mathematical hands-on experimentation. To address the research objective of the present paper, I analysed students' and the teacher's responses to the questions of the first part of the interview (which have not yet been analysed).

In the first part of the interview, the students and the mathematics teacher were asked the following questions: "What is mathematics for you?" and "What does it mean to you to do mathematics?" I also asked the mathematics teacher what it means to her to teach mathematics. In addition, both students and the mathematics teacher were asked to present at least one example of a task in their schoolbook that they felt required typical mathematical activities and to explain their choice. I analysed the data using qualitative content analysis following Kuckartz (2018). In doing so, I used a deductive-inductive category system, in which the deductive categories arise from the student conceptions of

mathematics framework by Grouws et al. (1996; see also Figure 1). More specifically, I first marked the relevant parts (i.e., words, lines and segments) of the students' and teacher's answers for achieving the research objective of the present paper. I then categorized these parts into the dimensions of the student conceptions of mathematics framework and assigned them, if possible, to one of the poles. For this, I did not consider the single statements separately, but in the overall context of the interview. Finally, I related the students' answers to the teacher's answers and formed inductive categories.

Results

Mathematics teacher's conceptions about mathematics and doing mathematics and intention to teach mathematics

With regard to the mathematics teacher's conceptions about mathematics and doing mathematics, the results show that the mathematics teacher interviewed regards mathematics as the "language of science". For her, mathematics is a coherent system in which "everything builds on each other somehow […] and it's not just knowledge that exists side by side, so it's not knowledge that can be simply memorised". Accordingly, with regard to the teaching of mathematics, it is important for her to make it clear to her students that "everything builds on each other […] [and] how much it's all connected". When asked what she considers to be typical mathematical activities for her grade 9 students, she listed the mathematical subjects they had dealt with during the school year: "We have done the Pythagorean theorem and the intercept theorem. We have discussed linear functions and systems of linear equations. Now, we are currently doing power. […] And the circle […]. So, a lot of geometry and a bit of functions." For her, especially application tasks with real-world contexts, which they deal with "relatively often in class, but not nearly as much as we should", are tasks in which students carry out typical mathematical activities. In addition, she named proving as typical for doing mathematics, but also mentioned that this "doesn't happen very often in lower secondary education". Nevertheless, in her mathematics lessons, the students, for example, carried out a proof of the Pythagorean theorem and a hands-on exploration of Pi.

In summary, the mathematics teacher interviewed shows conceptions that can be assigned to the relativistic perspective of mathematics. In her mathematics lessons, she particularly aims to convey to her students that mathematics is a coherent system and uses tasks demonstrating that doing mathematics is a sense-making process (i.e., proof of the Pythagorean theorem, hands-on exploration of Pi) and that mathematics is a useful endeavour (i.e., application tasks with real-world contexts).

Students' conceptions about mathematics and doing mathematics

With respect to students' conceptions about mathematics and doing mathematics, I observed that all of the students interviewed ($n = 12$) associate mathematics and doing mathematics primarily with numbers and arithmetic. For example, S4 answered to the question of what it means to do mathematics: "Calculate as much as you can in your head and if you can't do it anymore, then use the calculator if there are too many numbers." In this context, many students explicitly refer to finding results ($n = 8$), which corresponds to the dualistic perspective of doing mathematics. S5, for instance, states: "Doing mathematics is actually just calculating the task you're given. […] When I do that, I simply calculate the task for a very long time until I have a good way of calculating and then, finally,

also the correct solution." Similarly, S7, as quoted in the title of the present paper, answers: "Calculating with numbers and then finding a specific result". Students' choices of exemplary tasks that they felt require typical mathematical activities also demonstrate that they associate doing mathematics mainly with arithmetic, as most of them chose exercises which merely require arithmetic, such as solving systems of equations or calculating areas. Regarding students' conceptions about the usefulness of mathematics, the results reveal a mixed picture. While three students consider mathematics only as a "school subject" (e.g., S3), five students emphasise the relevance of mathematics and working mathematically outside of school. In doing so, the students refer to concrete everyday life situations, such as "calculating prices in the supermarket" (S5), or more generally to the fact that they learn "very important basics for everyday life" (S8). Furthermore, two of them also point out that mathematics is a "science that involves working with numbers in some way" (S8). Moreover, with regard to the learning of mathematics, two students explicitly refer to memorizing. For example, when presenting an example of a task in the schoolbook that he felt requires typical mathematical activities, S13 noted: "Above all, you have to know the formulas […] you have to learn them by heart". In contrast, individual students refer to "logical thinking" (S5) when solving tasks or to "brain teasers" (S11).

In summary, most of the students interviewed tend to hold a rather dualistic perspective of mathematics associating mathematics and doing mathematics primarily with calculating and finding results. Nevertheless, some students also demonstrate a partially relativistic perspective of mathematics by emphasizing the relevance of mathematics and working mathematically outside of school and considering mathematics as a science. However, overall, many of the students interviewed had difficulties answering the questions about what mathematics is and, in particular, what doing mathematics means to them. When asked about typical mathematical activities, they often referred to mathematical subjects, their general approach to solve tasks or listed professions that require mathematics. Furthermore, when explaining their choice of the exemplary tasks that they felt require typical mathematical activities, they often did not refer to typical mathematical activities involved in completing the tasks, but chose them for other (personal) reasons, such as "because that was the last topic" (S6), "because I was actually very good at it" (S4) or because "that is, for example, the topic I'm not very good at" (S13).

Discussion and outlook

The present paper aimed at obtaining a (more) comprehensive view of students' conceptions about mathematics and, especially, doing mathematics as well as possible relations to mathematics teaching by simultaneously also taking the mathematics teacher's conception and intention to teach mathematics into account. For this purpose, I analysed additional data from an exploratory interview study with 12 students and their mathematics teacher. The results indicate differences in students' conceptions about mathematics and doing mathematics – even though they all attended the same mathematics class. Although the mathematics teacher attempts to make it clear to her students that mathematics is a coherent system and states that she often implements application tasks with real-world contexts or tasks demonstrating that doing mathematics is a sense-making process, all of the students from her mathematics class interviewed associate mathematics and doing mathematics mainly with numbers and arithmetic. This result reflects the findings of previous research on students'

conceptions about mathematics (see, e.g., review by Østergaard, 2024). Some students also demonstrate a partially relativistic perspective of mathematics by emphasizing the relevance of mathematics and working mathematically outside of school and considering mathematics as a science. Nevertheless, in summary, the results indicate that the teacher's intention to convey to her students that mathematics is a coherent system and that doing mathematics is a sense-making process does not get through to most of her students. This could be related to the fact that it was difficult for her to answer the question of what doing mathematics means and to name typical mathematical activities, meaning that she likely did not address this (explicitly) in class. Similar to some of her students, she instead listed the mathematical subjects they had dealt with during the school year. This could possibly be related to her own, likely limited experiences with mathematics as a student (see, e.g., Boaler & Greeno, 2000). However, in the present paper, only the conceptions and intention to teach mathematics of one individual teacher and her students' conceptions were analysed. The small sample size and, thus, the limited generalizability of the findings constitute a limitation of the present paper, which should be addressed in future research. Moreover, especially students with a high interest and good grades in mathematics participated in the study. Thus, the findings appear particularly problematic, as these students may choose mathematics as their main field of study based on their limited conceptions, but studying mathematics in upper grades requires more components of mathematics than numbers and arithmetic (see, e.g., Boaler & Greeno, 2000; Pehkonen & Törner, 1999). Another limitation of the present paper relates to the fact that only the teachers' personal statements from the interview could be taken into account. In future studies, these insights may be further extended through, for instance, classroom observations in order to capture how teaching practices align with or deviate from the teacher's intention.

Despite this, the results of the present paper highlight the need to foster both students' and mathematics teachers' conceptions about mathematics and, in particular, doing mathematics. In line with Schoenfeld's (1994) claim quoted earlier, this requires mathematics teachers to expand their own experiences gained as a student and become aware that mathematics is an act of sense-making and often a hands-on, empirical activity. Only then they will be able to increasingly convey this in their mathematics lessons and, thus, enable authentic experiences for their students. For this purpose, for example, teacher training is needed that specifically focuses on what mathematics is and what it means to do mathematics. Furthermore, with regard to mathematics teaching, it seems necessary to teach mathematics as a process rather than as a finished product. Another possibility to offer both students and teachers (more) opportunities to gain authentic experiences with mathematics and, thus, foster more adequate conceptions relates to cooperating with mathematicians who describe their perspective on mathematics and their work to teachers and students. In addition, it would be promising to have students carrying out activities similar to those of mathematicians and reflecting on their approach in mathematics lessons.

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Mathematics teachers support on valuing mathematics: The role of mathematical self-concept

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This study examines the relationship between mathematics teacher support (MTS), mathematical self-concept, and the value facet of mathematical resilience (Value). Using self-report data from 863 Indonesian students, we applied Maximum Likelihood Structural Equation Modelling with robust standard errors to analyze how emotional and instrumental mathematics teacher support and self-concept contribute to the development of Value. The results show that the indirect effect of instrumental MTS on Value via mathematical self-concept is significant, while the indirect effect of emotional MTS does not exist. Nevertheless, emotional MTS directly affects Value. The results particularly confirm the framework suggested by the expectancy-value theory and provide a better picture of the relationship between internal and external factors of mathematical resilience, especially on valuing mathematics.

Keywords: Mathematics teacher support, value, mathematical self-concept, expectancy-value theory.

Introduction

The global prevalence of mathematics anxiety indicates the necessity for a psychologically safe mathematics learning environment that promotes well-being (Johnston-Wilder & Lee, 2024). In the classroom context, mathematics teachers play a vital role in realizing this. For instance, teaching practices that cultivate mathematical resilience "ensure that the relationships between teachers and students are nurturing, respectful, and warm" (Johnston-Wilder & Lee, 2024, p. 3). Such teacher support enables students to persevere, progress in mathematics, and address negative attitudes when facing challenges.

In a previous study (Hamidy et al., accepted), we found a positive direct relationship between mathematics teacher support (MTS) and the value facet of mathematical resilience (*Value*). This implies that mathematics teacher support enables students to unveil the benefits of mathematics for their current and future goals and then motivates them to keep learning mathematics despite the challenges. Since teacher support can be categorized into emotional and instrumental support (Federici & Skaalvik, 2013), exploring these two types of support may highlight each of their unique contributions.

Furthermore, the finding also aligns with the expectancy-value model of achievement (Eccles & Wigfield, 2020). It proposes a relation between socializers' beliefs and behaviours (here: MTS) and one of the subjective task values: utility value, which is similarly conceptualized as *Value*. Yet, this effect is indirectly regulated by the perception of the socializers' beliefs and behaviours as well as one's self-concept and abilities. Based on this framework, we enrich the significance of teachers in promoting *Value* by examining the relationship between each type of MTS and *Value* via

mathematical self-concept. This, in turn, could enhance the understanding of how teacher support ensures their students keep progress in mathematics.

Theoretical framework and research questions

Mathematical resilience is the ability to adjust to emotional struggles when facing any challenges in mathematics (Johnston-Wilder & Lee, 2024). This positive attitude helps navigate mathematics difficulties and keep engaged and persistent in mathematics (Neumann et al., 2021). Mathematical resilience is developed from four aspects: *Value*, *Struggle*, *Growth*, and *Support* (Kooken et al., 2016; Lee & Johnston-Wilder, 2017). In this contribution, we focus on *Value*, which refers to the capacity to persist in learning mathematics by recognizing its importance to current and future goals, for instance, by being useful (Kooken et al., 2016). Being useful is also central to utility value, one of the subjective task values in expectancy-value theory (EVT), that Eccles and Wigfield (2020) conceptualize as the subjective fit of an activity or task with individuals' current and future goals.

In our previous study (Hamidy et al., accepted), we could show that students' *Value* is positively influenced by their perceived support from their mathematics teacher. However, support provided by mathematics teachers may have several dimensions. For instance, Federici & Skaalvik (2013) suggest that teacher support can be differentiated into emotional and instrumental support. Emotional support is characterized by nurturing relationships such as caring empathy, encouragement, esteem, and friendliness, while instrumental support is closely related to action-facilitating support that enhances students' understanding, such as clarifying, correcting, elaborating, and demonstrating (Federici & Skaalvik, 2013; Semmer et al., 2008). Subsequently, operating emotional and instrumental support could provide a more detailed understanding of the relationship between MTS and *Value*. For English as a subject, the findings of Han et al. (2022) suggest that students who believe that they receive stronger emotional support tend to believe that their teachers value the subject more. They, in turn, benefit from these beliefs in the form of increased utility value and achievement in the subject. In addition, Federici and Skaalvik (2014) found that students' perception of the utility value of mathematics was positively related to the perceived instrumental support from their mathematics teachers. In the current study, we simultaneously examine the role of both types of MTS. Thus, our first research question is RQ1: How do students' perceived emotional and instrumental support from mathematics teachers relate to the students' *Value*?

In addition, according to the EVT framework, subjective task values are connected to one's goals and self-beliefs (Eccles & Wigfield, 2020). As one of the self-beliefs, self-concept in an academic context refers to the individuals' self-awareness and beliefs regarding their performance (Bong & Skaalvik, 2003). Despite the relationship between competence beliefs and subjective values is not clearly specified (Pintrich & Schunk, 2002; Wigfield & Eccles, 2002), studies found that self-concept is one of the individual factors influencing the development of mathematical resilience (Xenofontos & Mouroutsou, 2023). Therefore, we argue that an individual's mathematical self-concept contributes to *Value* development. We investigate this relationship in RQ2: How does mathematical self-concept relate to *Value*?

Furthermore, the EVT framework suggests that one's beliefs in competence are related to one's perceptions of socializers' behaviour (Eccles & Wigfield, 2020), such as social support. Lee and

Simpkins (2021) found that MTS is critical to developing students' mathematical self-concept, while Oda et al. (2021) showed that instrumental and emotional MTS positively correlated with mathematical self-concept, albeit weakly. Specifically, Wang (2023) found that instrumental support such as cognitive activation has a positive impact on students' mathematical self-concept. Then, we re-examine the relationship in RQ3: How do emotional and instrumental MTS relate to mathematical self-concept?

The past paragraphs imply that mathematical self-concept may also regulate the relationship between MTS and *Value*. That is, the quality of support provided by mathematics teachers affects students' beliefs in their mathematics competence, shaping their perceived value of mathematics. A study of Lyakhova & Neate (2019) supports this idea, indicating that there is an association between students' value of Further Mathematics, their mathematics teacher support and confidence in their own mathematical abilities. In addition, the characteristics of the two types of MTS may cause different dynamics between the constructs. Thus, RQ4 is: What is the role of mathematical self-concept on the relationship between emotional and instrumental MTS and *Value*?

Methods

We conducted a correlational study to answer RQ1-4 using data obtained from self-report scales. To measure perceived emotional and instrumental MTS, we used Federici and Skaalvik's (2013) MTS scale consisting of two subscales: 6-item emotional support (e.g., "My math teacher is friendly") and 6-item instrumental support (e.g., "My math teacher is good at explaining challenging math problems"). To measure students' mathematical self-concept, we used the 5-item mathematical self-concept scale from PISA 2012, for instance, "I learn mathematics quickly" (OECD, 2014). Finally, we used the 8-item *Value* subscale of the mathematical resilience scale of Kooken et al. (2016), for instance, "Math is essential for my future". We made some wording adjustments to the scales to better fit the context of secondary school students and translated them into Indonesian. The translation was reviewed by an experienced researcher fluent in both English and Indonesian, together with several Indonesian mathematics teachers. All items were measured using 5-point Likert scales, with high values indicating high levels of agreement except for one item on the mathematical self-concept scale, which was negatively worded and recoded prior to analysis. We used the final scales to collect data from 877 Indonesian 9th and 10th graders. Their participation was anonymous and voluntary. To ensure data accuracy, we applied a data screening procedure by Ward and Meade (2023) to identify students who responded carelessly to the survey. Fourteen students were excluded from the data set. Finally, 863 (60.8 % female, 2.9% did not answer) respondents were included in this study (average age: 15;2).

For data analysis, we conducted confirmatory factor analyses (CFA) and omega reliability estimation to check the psychometric quality of the scales and the dimensionality of the MTS scales. We calculated the descriptive statistics and correlations as preliminary results. Finally, we developed a model for each research question (due to page limitations, only the model for RQ4 is presented in the results section). We analysed them by structural equation modelling. For estimation, we applied maximum likelihood with robust standard errors to control for missing values. We also used a bootstrap procedure with 1,000 subsamples for the mediation effect. The criteria for assessing model

fit included CFI > 0.90, TLI > 0.90, RMSEA < 0.08, SRMR < 0.08, and factor loadings > 0.4. The lavaan package (Rosseel, 2012) in R (R Core Team, 2023) was used for analysis.

Results

The results of the CFA indicate that one item from the mathematical self-concept scale should be excluded because the factor loading was less than 0.4. In addition, both one-dimensional and two-dimensional (emotional and instrumental support) models of MTS meet the criteria for a good fit, yet the two-dimensional model is considered better. We excluded the item and adjusted the MTS construct into a two-dimensional model so that the overall measurement model is improved and meets the criteria for a good fit.

Table 1 shows the omega reliability coefficients, means, standard deviations, and correlation matrix. The scales are acceptably reliable (ω > 0.7). Overall, participants perceive high levels of emotional and instrumental teacher support, as well as a strong sense of *Value*. Yet, their mathematical self-concept is comparatively lower. Furthermore, the correlation matrix shows positive and significant associations between the constructs, especially between the two facets of MTS, which is also found in the previous study (Federici & Skaalvik, 2013).

Table 1: Descriptive statistics and correlations

Variable	ω reliability	M	SD	1	2	3
1. Emotional MTS	0.848	4.084	0.575			
2. Instrumental MTS	0.905	4.208	0.581	0.916***		
3. Mathematical self-concept	0.851	3.135	0.735	0.311***	0.377***	
4. Value	0.926	3.895	0.669	0.495***	0.496***	0.485***

Note. *** $p < 0.001$; Likert scales ranging from 1 to 5 (5 indicating high agreement); MTS = mathematics teacher support

To address RQ1-3, we examined three structural models (Model 1-3) separately. All of the models demonstrate an acceptable fit (see Table 2). The results of Model 1 indicate that emotional and instrumental MTS are positively related to *Value* (Emotional MTS: $\beta = 0.213$, $p < 0.05$; Instrumental MTS: $\beta = 0.282$, $p < 0.001$). The result of Model 2 shows that mathematical self-concept is also a positive, significant predictor of *Value* ($\beta = 0.440$, $p < 0.001$). Furthermore, there is a positive relationship between instrumental MTS and mathematical self-concept in Model 3 ($\beta = 0.403$, $p < 0.001$), while emotional MTS is not significant ($\beta = -0.075$, $p > 0.05$).

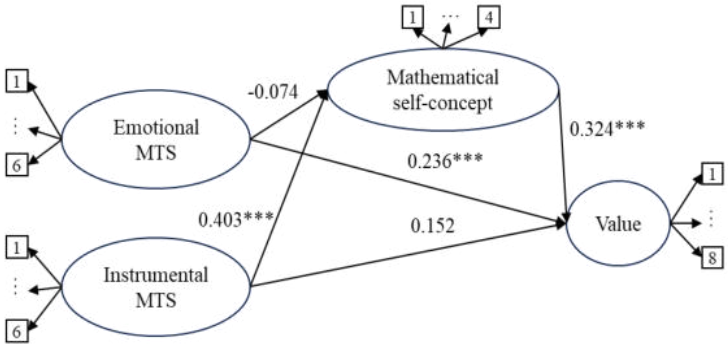
Table 2: Fit Statistics for Models 1-3

Model	*Df*	χ^2 (N = 863)	CFI	TLI	RMSEA	SRMR
1	167	613.523	0.946	0.939	0.063	0.039

2	53	249.691	0.957	0.946	0.077	0.036
3	101	342.026	0.960	0.953	0.058	0.036
4	246	759.629	0.950	0.943	0.054	0.038

Note. CFI = comparative fit index, TLI = Tucker-Lewis index, RMSEA = root mean square error of approximation, SRMR = standardized root mean square residual

To address RQ4, we examined Model 4 in Figure 1. The SEM results show an acceptable model fit (see Table 2) and confirm the results of Models 2 and 3. Yet, the direct effect of instrumental MTS in *Value*, as identified in Model 1, was reduced when mathematical self-concept was included (β = 0.152, p > 0.05). Nevertheless, it is found that the indirect effect of instrumental MTS on *Value* was found to be significant via mathematical self-concept (β = 0.131, p < 0.001). In contrast, the indirect effect of emotional MTS is absent (β = -0.024, p > 0.05).



Note. ***p < 0.001; latent constructs are shown in ellipses.

Figure 1: Model 4 and its measured relationships

Discussion and Conclusion

This study aims to examine the relationship between emotional and instrumental MTS, mathematical self-concept, and *Value*. The results of Model 1 show that both students' perceptions of emotional and instrumental MTS are significant predictors of students' understanding of the benefits of mathematics, thus enriching our previous study (Hamidy et al., in press). The significance of emotional MTS confirms that the findings of Han et al. (2022) are transferable to mathematics. The importance of instrumental MTS is also expected. Federici and Skaalvik (2014) explain that it may increase the utility value of mathematics by enhancing the understanding of mathematics problems and improving mathematics skills. Furthermore, the significance of both types of MTS on Value may also suggest that MTS does not need to be distinctly categorized into emotional and instrumental support, as they are also highly correlated. Federici and Skaalvik's (2013) argue that the strong correlation between both types of MTS means that the supportive mathematics teachers typically provide emotional and instrumental support. Additionally, students may perceive their instrumentally supportive mathematics teachers as being emotionally supportive as well. Nevertheless, the unique role of both types of MTS is found in the Model 3 and 4, which we discuss later. To sum up, the findings highlight that the social environment plays a crucial role in fostering mathematical resilience, particularly regarding *Value* (Eccles & Wigfield, 2020; Johnston-Wilder & Lee, 2024; Xenofontos & Mouroutsou, 2023).

In addition to the social environment, self-concept plays a significant role in predicting *Value*, as shown in Model 2. The more students believe in their mathematical abilities, the more they value mathematics for their current and future goals and perceive the importance of excelling in it. The results contribute to the clarification of the relationship between competence beliefs and subjective values (see Pintrich and Schunk, 2002; Wigfield and Eccles, 2002), and confirm the importance of psychological factors in the development of mathematical resilience (Xenofontos & Mouroutsou, 2023). To conclude, helping students to develop confidence in their mathematical abilities will help them to see the value of mathematics and then feel motivated to progress through challenges.

Instrumental MTS is one way to facilitate students' development of their mathematical self-concept, as shown in Model 3. Providing support such as clarification, correction, elaboration, and demonstration (Federici & Skaalvik, 2013; Semmer et al., 2008) not only helps students to understand and solve mathematical problems, but also develops their confidence in their mathematical competence. On the other hand, emotional MTS does not have a significant effect on students' mathematical self-concept. This suggests that the different characteristics of each type of MTS are related to the development of students' belief in their mathematical competence. Friendly, caring, and warm mathematics teachers may create a safe environment in the classroom (Federici & Skaalvik, 2013), but may not have a significant effect on students' belief in their mathematics competence as these characteristics are not strongly related to understanding. The findings clarify the weak correlations between instrumental and emotional MTS and students' mathematical self-concept found by Oda et al. (2021), support the study of Wang (2023) and contribute to a deeper understanding of the findings of Lee and Simpkins (2021).

The results of Model 4 show that the significance of instrumental MTS found in Model 1 is missing when mathematical self-concept is taken into account, while emotional MTS retains its significance. Given that both mathematical self-concept and instrumental MTS are related to students' competencies in mathematics, it is plausible that these constructs overlap on *Value* so that their effect is reduced: mathematical self-concept remains significant, whereas the significance of instrumental MTS disappears. Although the direct effect is not significant, the results show a significant indirect effect of instrumental MTS on *Value* via mathematical self-concept. Hair et al. (2021) refer to this situation as indirect-only mediation. The results underline the significant role of mathematical self-concept as a regulator of the indirect effect of instrumental MTS in promoting students' value of mathematics. This clarifies the relationship suggested by Lyakhova & Neate (2019) that providing students with tangible support when they encounter difficulties in mathematics helps them to grow in their belief in their mathematical competencies and then to value it as an important asset to be developed. Furthermore, the results suggest teachers should ensure that their support emphasizes students' competence rather than providing quick-fix and path-smoothing so that students really perceive their competence and are then motivated to progress in mathematics.

In addition, the lack of significance in the indirect effect of emotional MTS may be due to a lack of relevance between emotional MTS and belief in competence. This indicates that each type of MTS has a unique mechanism to develop students' Value despite the high correlation between them. Furthermore, regarding the indirect effect of instrumental MTS, the results confirm the EVT framework that self-belief regulates the relationship between social environment and subjective task

values (Eccles & Wigfield, 2020). Moreover, the results provide a more detailed picture of the interplay between external and internal factors in the development of mathematical resilience as suggested by Xenofontos and Mouroutsou (2023).

Yet, this study has several limitations. As it was conducted in a cross-sectional setting, the result could not provide sufficient support for causality. Therefore, longitudinal studies of the same model should be considered for further research to address the limitations of this study. In addition, the gender imbalance of this study may also affect the results, so that future research should be based on a gender-balanced sample. Finally, conducting the same or more complex research framework on the other facets of mathematical resilience and social support should be considered to provide a deeper understanding.

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Bridging commognition and complexity theory for the study of identity and ritual participation in mathematical learning

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This paper integrates commognitive and complexity theory, specifically, the Dynamic Systems Model of Role Identity (DSMRI), to analyze an eighth-grade student's mathematical identity. I start by outlining the main concepts taken from the two frameworks. I then exemplify the utility of the unified framework by analyzing one middle-school student's discourse during a problem-solving task, taken from a mathematical interview followed by an identity interview. The analysis suggests that the student's subjectifications cohered with her mathematizing, which was mostly ritual and focused on procedures that she attempted to recall from memory. These subjectifications in turn cohered with her role-identity narratives that revolve around "succeeding" in mathematics for instrumental goals, all forming an attractor state in the complex dynamic system of her role identity.

Keywords: Identity, mathematical discourse, commognition, complexity, rituals.

Introduction and theoretical background

The intersection of identity and learning has become a focal point in mathematics education research, especially as it relates to how students develop a sense of self within the discipline, in relation to various norms, teaching practices and cultural narratives (Graven & Heyd-Metzuyanim, 2019; Gresalfi & Hand, 2019).

The commognitive framework provides a robust toolset for analyzing the construction of mathematical identities through discourse (Heyd-Metzuyanim, 2015; Sfard, 2008). However, it offers limited tools for explaining how certain identity narratives stabilize into persistent patterns over time. To address this, I turn to complexity theory (Hilpert & Marchand, 2018), particularly the DSMRI framework (Kaplan & Garner, 2017), as a complementary lens for understanding the dynamic evolution and solidification of mathematical identities. The goal of this paper is to initiate the theoretical integration of commognition and DSMRI by illustrating how their combined use can enhance our understanding of the co-shaping of identity and mathematical learning. This integration is demonstrated through the case study of an eighth-grade student, highlighting the potential of this approach for future research and practice.

According to Sfard & Prusak (2005), identity is a collection of reified, significant narratives about a person, which can be told by the person about herself (1st person narratives), or by others about her (2nd person or 3rd person). These narratives are further divided into two complementary subsets: **current identity narratives** and **designated identity narratives**. Current identity includes the stories individuals tell about themselves in the present—how they see their mathematical abilities and roles at this moment. Designated identity includes the narratives they construct about who they expect or desire to become in the future.

More generally, the commognitive theory (Sfard, 2008) is based on the premise that thinking is based on individualizing socio-culturally established discourses with mathematical thinking being viewed

as a specific type of discourse. In this approach, learning is a process of becoming a participant in the mathematical discourse, and one's mathematical identity is constructed and shaped through participation (or non-participation) in this discourse.

In order to examine the process of becoming a participant in the mathematical discourse, commognition suggests to examine not just students' mathematizing (their communication about mathematical objects) but also their subjectifying (communication about themselves). **Subjectifying** statements are the transient, more local and non-reified statements associated with one's identity narratives. In 1st Person, subjectifying involves the attribution of characteristics or capabilities to oneself, which can range from descriptions of specific actions (e.g., "I forgot this") to broader assessments of personal traits (e.g., "I'm bad at math").

Previous commognitive research has shown that students' identities are linked with specific patterns of participation (e.g. Heyd-Metzuyanim, 2015). For examples, identities of failure or having "learning difficulties" in mathematics were linked with **ritual participation**, which is focused on the execution of procedures, rather than on the product of these procedures (namely, the narratives produced by them) (Lavie et al., 2019).

The main constituents of participation according to commognition are routines, defined as a pair consisting of a task (the goal of the routine) and a procedure. This procedure often takes form of a series of sub-routines (or sub-tasks) that the performer enacts to achieve the overall task. Routines can be characterized along a continuum from rituals to explorations (Lavie et al., 2019). Novices often perform ritual routines, while experts perform explorative routines. Ritual routines have several defining characteristics:

Procedural focus: Learners prioritize the steps of the procedure over its overarching goal.

Lack of Agentivity: Learners rely on external authority (e.g. teachers or textbooks) for validation rather than independently producing and endorsing mathematical narratives.

Lack of bondedness, substantiation and flexibility: Ritual routines often lack connections between steps, substantiation of conclusions, or flexibility in applying the routine to different contexts or using different procedures for the same task.

For example, a student solving a fraction problem might follow a memorized procedure like "invert and multiply" without producing or endorsing narratives that explain why this procedure is mathematically valid. The student focuses solely on completing the steps correctly and rely on the teacher for confirmation. Such ritualized participation, if repeated, can foster narratives like "I'm bad with fractions", particularly when the memorized steps fail to produce correct results.

So far, commognition has documented the ways by which certain identity narratives of disability and lack of competence go hand in hand with ritual participation (Heyd-Metzuyanim, 2013, 2015). However, commognition lacks theoretical tools for modeling how specific patterns of mathematizing stabilize and get reified into maladaptive identity narratives. A potential theoretical enhancement, which may aid in filling this gap, can be found in complexity theory. **Complexity theory**, often applied to fields ranging from biology to social sciences, emphasizes the non-linearity, dynamic and emergent nature of learning systems (Hilpert & Marchand, 2018). This perspective offers a way to

conceptualize how identity narratives and mathematical participation might stabilize into 'attractor states'.

Attractor states, a concept from complexity theory, refers to stable patterns of action within a dynamic system. These states emerge through the interactions and feedback loops within the system, which gradually reinforce certain configurations and make them resistant to change (Hilpert & Marchand, 2018). In the context of identity stabilization, attractor states represent recurring patterns of discursive participation that solidify over time, shaping how an individual engages with particular tasks. To further explore the stabilization of identity narratives within a dynamic system, Kaplan and Garner's (2017) **Dynamic Systems Model of Role Identity (DSMRI)** offers a framework that connects identity elements – such as self-perceptions and goals – with the broader dynamics of participation and discourse.

DSMRI conceptualizes role identity as a complex dynamic system comprising self-perceptions, goals, beliefs, and action possibilities. These elements can be mapped onto Sfard and Pruask's (2005) "collection of narratives" that constitute identity: self-perceptions align with **1st-person identity narratives** about current mathematical abilities; goals correspond to **designated identity narratives** about future aspirations. Ontological and epistemological beliefs map to **endorsed narratives about mathematics and how it is learned;** and action possibilities correspond to **precedents of routines of interaction** (Lavie et al., 2019) - recurring ways of engaging in mathematical discourse that learners consider relevant in a given situation. Together, these narratives interact dynamically, stabilizing into attractor states that form learners' patterns of participation in mathematics.

In the following case study of one student named Liat (pseudonym), I use the unified commognitive-complexity toolset to ask: in what ways did Liat's mathematizing, subjectifying and role-identity narratives cohere into an attractor state of ritual participation?

Methods

Liat, a beginning eighth-grade student, participated in a larger study on the development of algebraic discourse in middle school students. She was selected for the present analysis due to her relatively explicit subjectifying talk during her engagement with mathematical tasks. Data were collected through a "think-aloud" interview in which Liat solved a series of algebra problems, followed by an "identity interview" which asked Liat about her interests, goals and evaluations of herself as a mathematics student. The interview was recorded by two cameras and fully transcribed.

The analysis proceeded in three stages. First, Liat's discourse was organized into tasks and sub-tasks, identifying the routines she enacted at each stage of the problem-solving process. This provided the foundation for the second stage, where Liat's mathematizing routines were analyzed for their ritual and explorative dimensions. In parallel, subjectifying utterances – where Liat referred to herself or her actions (e.g. "I think", "I forgot", "I'll do") – were highlighted in relation to the routines and sub-routines she enacted. Third, these patterns were synthesized with the commognitive counterparts of DSMRI elements, such as 1st-person and designated identity narratives. These were compared to assess the coherence between Liat's mathematizing, subjectifying during the problem-solving task, and her more reified role-identity narratives. This comparison enabled an exploration of whether

ritual mathematizing was contributing to the formation of an attractor state with Liat's identity narratives.

Analysis and findings

I use the episode where Liat attempted to solve one problem from the mathematical interview, to exemplify the ways by which Liat's identifying, subjectifying and mathematizing cohered as an attractor state. The analysis is structured into three stages: detailed examination of Liat's mathematizing and subjectifying, The task posed to Liat, named "the student council problem", is presented in Figure 1.

> In the student council elections, Yonatan received 1/3 of the votes and Nadav received 1/4 of the votes. Naama received 40 votes which are all the remaining votes. How many students voted in the student council elections? Present a solution.

Figure 1: The student council problem (translated from Hebrew)

The total time in which Liat engaged with the problem was 10 minutes. This time was divided into two parts: 5 minutes during the first trial, then 5 minutes at the end of the interview, where she revisited the problem. In both of these, Liat was unable to solve the problem. During this time, Liat posed to herself no less than 9 sub-tasks which are listed in Table 1.

Table 1: The series of sub-tasks Liat posed to herself

1	$\frac{1}{4}+\frac{1}{3}+40=$	**4**	$27x12=$	**7**	$\overline{73)}\,3$
2	$40x12=$	**5**	$12\,x\,37=$	**8**	$\overline{73)}\,4$
3	$\overline{487)}12=$	**6**	$\frac{73}{120}=$	**9**	$\overline{73)}\,5$

The next excerpt shows the beginning of Liat's engagement with the problem

157 Liat (Reads the question out loud, twice). Oh, OK. A quarter plus a third plus forty equals ...I need to make a common denominator now, right?
158 Liat I forgot how to make a common denominator. I know how to make a common denominator ..
159 Liat (Starts writing, see Figure 2). It's one, so it's like twelve ,the closest to both and now if I do it then it will be..
160 Liat If it's twelve and it's twelve.. oh then I multiply that by twelve too?
161 Liat I forgot it It's double this ..Yes ,I also multiply it by twelve

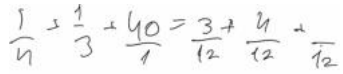

Figure 2 - Liat's writing during her initial engagement with the student council problem

This excerpt illustrates key characteristics of Liat's participation that were eminent in her solution of the task.

Liat's mathematizing: In general, Liat's mathematizing is highly ritual in this excerpt. First, there is a clear focus on procedures ("make a common denominator" [157], "If I do it" [159], "I multiply that" [160]). Second, there are indicators of reliance on external authority. These can be found in the "I need to" [157] which implies accountability to externally given rules, as well as the turns to the interviewer for some assistance ("right?" [157], "I multiply that by twelve too?" [160). Another ritual characteristic of Liat's discourse is lack of bondedness between the situation described in the written task and the chosen procedure. In particular, this lack of bondedness is related to the syntactic treatment of fractions. Whereas the written task refers to "1/3 of the votes", Liat treats the 1/3 as a standalone number, ignoring it's function in the task-wording as an operator on a quantity ('of the votes'). Moreover, her treatment of numerators and denominators is unrelated to their roles as part of a rational number. Rather, she only relates to the numbers as parts of a procedure of "making a common denominator", the rationale and substantiability of which seem to remain hidden to her.

Liat's subjectifying: Table 2 presents a collection of the subjectifications produced by Liat during her attempts to solve the "student council" problem. This Table is not exhaustive of all of Liat's subjectifications, but it represents the main stories she told about herself.

Table 2: Liat's subjectifying during her engagement with the students' council problem

Line no.	What is said (implied/context)
157-158	I need to … I forgot how to… . I know how to**..**
169	Oy, I don't think (that works) … I don't like it
171	Naama (her previous math teacher) would have killed me now over this
176	(while performing the long multiplication) at least this I know how to do
183	(After multiplying and producing an answer to a sub-task) Good, Liat. Now go on
189	(When multiplying 12x47 and hoping to find the result is 487). Yes! Boom! (excited) … No (disappointed) Yes! (excited) Boom! (higher tone)… Oh, no, no… No.. (lowered voice, sighing)
323-325	I know what I did wrong here! … I just did all this calculation for nothing. OK, I think I know what to do. If I succeed then I succeed (meaning: let's give it another chance).
338	(Facing a sub-task she posed to herself of 37/120, after requesting and being denied a calculator): How I don't like mathematics. Who created this?!

As can be seen in Table 2, Liat's subjectifications generally cohered with her ritual mathematizing. She talked about herself as "knowing how to do" certain procedures, or as "forgetting" them [157-158]. These statements subjectified her as either remembering or not remembering a certain procedure, but not as able to figure out or make sense of the problem in new or creative ways. Her emotional responses, including the "Boom!" "Yes!" "no..", with the highly excited, followed by

disappointed and slightly depressed tone, all revolved around succeeding or failing to enact procedures. Her evaluation of this failure or success was not related to the situation described in the written task. Thus, for example, the excited "Yes! Boom!" in line [189] appeared when, for a moment, Liat expected her long multiplication procedure to produce 487, a number she expected to receive when she decided to divide 487 (which was the result of 480+3+4, the numerators in her calculation) by 12. This expectation was part of a procedural sub-task Liat posed to herself, but was not bonded to the original task. That is, Liat emotional responses related to her expectation that a solution to the 487:12 task, if whole, would somehow prove that her procedure was correct. Yet this expectation had little to do with the original numbers described in the task.

Liat's role identity narratives

Following is a summary Table of Liat's role identity narratives, retrieved from the identity interview.

Table 3: Role identity narratives authored by Liat during the identity interview

1st Person identity narratives (general participation evaluations, attributes and membership)
I like succeeding in mathematics but **I don't like mathematics** [377] But when I don't succeed then it's like, **I get tired of it and I give up** [390] **I'm a good (sababa) student**. I was on the honors roll, despite the Advanced Placement class (being harder). And I think **I am capable of that** (of practicing hard) but one needs a lot of will power for that [396] I think **in math I have a bit of an ADD problem** and sometimes that lowers my grade [398] **I once had a fail** (on an exam) but say **the classwork and homework that I always do,** that raises my grade
Narratives about purpose and goals of learning mathematics, designated identity narratives
If I'd known better mathematics, and I'd better understand, then it's fun to succeed [376] (*implied: I don't know and understand math as much as would like)* (What is the subject in school **most important for you to succeed in?) I think that math** [413] **Because with math I can get into many things**. [415] (in relation to the subjects she will want to take in high school) I think that, for instance **Math 5 units I will take** (5 units is the highest math track)
Narratives about math learning and what it takes to succeed in math
As much practice as possible, **that will always work**, but one doesn't always like to practice a lot (Interv: right). Right? You don't like doing homework and that, so it's like **whoever can persist in it (would succeed**). [392-394] And I think I am capable of that (**of practicing hard**) but **one needs a lot of will power for that** [396]

In general, there is much coherence in Liat's system of role identity narratives. Most of them cohere around "succeeding" as the ultimate goal in mathematics. There is some gap between Liat's current and designated identity narratives in that her current identity narratives describe her as only partially successful, suffering from a "bit of an ADD problem" and getting tired when she is unsuccessful. In contrast, her designated identity narratives are about enrolling in the highest mathematics track, and about mathematics being one of the two most important subjects in school. In that sense, the gap between Liat's current and designated identity may be conducive for learning as it can propel Liat to

work hard on her mathematics studies. Yet when closely examining Liat's narratives about what succeeding in mathematics means, one can see that closing this gap means for Liat to simply exert more effort and "practice more". This emphasis on practicing and making effort is coherent with Liat's repeated attempts to solve the student council problem, even in the face of much obstacles. There is no mention in Liat's role identity narratives about "learning", "enjoying" or being interested in mathematics for its own sake. The only slight disruption of this coherent system is Liat's mention of wishing to "better understand" mathematics [376], yet it is unclear if by "better understanding" she means sense making, or simply "knowing how to do" the procedures**.** The general coherence between Liat's mathematizing, subjectifying and role-identity narratives indicates that her ritual participation may indeed function as an attractor state.

Discussion and conclusions

The integration of commognitive theory and complexity theory enhances our understanding of how students' mathematical identities and mathematical learning shape each other, particularly in cases of stabilizing of ritual participation. The findings in this study, exemplified by the case of Liat, illustrate how identity narratives—shaped by both individual experiences and socio-cultural narratives—become entrenched in unproductive patterns of mathematical engagement.

From the perspective of complexity theory, Liat's identity as a ritual participant in mathematics can be seen as an attractor state—one that achieves coherence through alignment of the role identity narratives, subjectifications during mathematizing, and enacted mathematical routines. Moreover, there is good chance that this system is resistant to change. Without role identity narratives that value sense making and producing mathematical narratives for their own sake, there is little chance that Liat's mathematizing would shift towards more productive, explorative forms of participation. This is unfortunate, since based on the non-canonical results that Liat's rituals are producing already at this stage, there is little chance that she will qualify for the highest mathematics track she aspires to become a member of in the long run. In that sense there is a *mathematizing-identifying barrier* that prevents Liat's role identity system from cohereing towards her designated identity of becoming a "5 units student". She is unaware (that is, she is not subjectifying herself) as someone who is malfunctioning in those particular aspects of her mathematizing that need to change. Rather, she designates herself to "practice more" or to show "more persistence", actions which would probably not propel her towards better success in mathematics in the long run.

It is important to note that this study is based on a single case and examines identity stabilization at a single point in time. While the findings offer valuable insights into the potential of a combined commognitive-complexity theory for understanding identity stabilization, they remain preliminary and require further investigation. Future studies should build on this framework by examining identity dynamics longitudinally and across diverse contexts to further validate and refine the proposed theoretical integration.

The integration of commognition and complexity theories may have broader implications for mathematics education. It suggests that interventions aimed at disrupting unproductive learning patterns must operate on multiple levels. Simply attempting to enhance conceptual understanding, or providing more explorative opportunities to learn, may not be sufficient since they do not attend to

identity narrative that preserve ritual participation. At the same time, interventions that only target identity narratives may be similarly ineffective, since they do not attend to the students' patterns of mathematizing. Instead, interventions need to target the full system of role identity narratives and mathematical participation, as well as the broader systems in which these role-identity narratives develop over time. Multi-level changes in tasks (or more broadly, curriculum), changes in socially produced narratives about what it means to succeed in math, and better assessment practices, which would allow students to better align their current and designated identities around doing mathematics, may be necessary to allow students to break away from attractor states of ritual mathematizing.

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Problem posing and motivation: Does reformulating problems into easier versus more difficult versions affect students' expectancies for success, task value, and costs?

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Problem posing was found to be a promising approach for enhancing motivation in mathematics classes. This study examines the impact of problem-posing prompts on students' motivation. A sample of ninth- and tenth-graders (N = 78) was asked to reformulate given real-world problems into an easier and a more difficult version. Students then evaluated their expectancies for success, task values, and perceived cost associated with reformulating problems into easier versus more difficult ones. The results showed that reformulating easy problems, compared to difficult ones, led to higher expectancies for success, greater utility and attainment value, and lower perceived costs. However, no significant effect was found on intrinsic value. These findings highlight the distinct nature of task-value components and suggest that incorporating prompts for easier problems in problem-posing tasks can positively impact students' motivation.

Keywords: Expectancies for success, task values, cost, problem posing, task-based motivation.

Introduction

A key question in problem-posing research is how different prompts influence the process and outcomes of problem posing (Cai et al., 2023). Recent findings suggest that encouraging students to pose problems of varying difficulty levels (e.g., easy, medium, and difficult) leads to greater complexity in their self-generated problems and thus improved problem-posing performance compared to asking them to pose three problems without any conditions (Cai et al., 2023). While prompting students to pose a single difficult problem may be sufficient to improve achievement-related outcomes, asking them to pose easy problems could play an important role in enhancing their motivation. While some studies have investigated the impact of prompts in problem posing on achievement, there is scarcely any research focusing on their effect on motivation. Initial findings on this topic come from a study focusing on problem generation (Krawitz & Wiehe, 2024). The present contribution extends this research by examining problem reformulation. The aim is to explore whether students are able to change given real-world problems into easier and more difficult versions and investigate how prompts to reformulate problems of varying difficulty impact students' motivation, including their expectations for success, task values, and perceived costs (Eccles & Wigfield, 2020).

Theoretical background and research questions

Expectancies for success, task value, and costs

There is widespread agreement on the necessity of supporting student motivation. Expectancy-value theories highlight the importance of students' expectancies for success and their perceived value of

academic tasks in determining their motivation, performance, and educational choices (Eccles & Wigfield, 2020). Expectancies for success are defined as students' beliefs about their future performance on upcoming tasks, while task values reflect their personal importance they place on tasks. Both of these constructs are situated and sensitive to the objects targeted, which means they vary depending on the task and context (Schukajlow et al., 2023). Eccles and Wigfield (2020) differentiate between intrinsic value (the enjoyment and interest derived from the task), attainment value (the task's significance to personal identity), utility value (its relevance to current or future goals), and costs (the negative aspects associated with the task).

Several factors influence the development of expectancies for success and task values. One critical factor is task complexity. The way students perceive the difficulty of a task can significantly impact their expectancies for success and the values they assign to it (Eccles & Wigfield, 2020). Higher perceived difficulty often leads to lower success expectations and greater perceived costs. Since task values and expectations of success are positively related, higher difficulty can be expected to be associated with lower task values (Eccles & Wigfield, 1995). Consistent with these considerations pre-service teachers (Böswald & Schukajlow, 2022) and high school students (Krawitz & Schukajlow, 2018) were found to report lower expectancies for success and lower task values for complex modelling problems compared to simpler intra-mathematical problems. However, too easy problems can lead to boredom (Schwartze et al., 2024) and thus to lower intrinsic value. In addition, studies have shown that the complexity of reality-based mathematical problems is a significant contributor to intrinsic value for both university students (Fielding et al., 2022) and primary school students (Russo & Hopkins, 2017). Therefore, tasks perceived as complex and challenging may actually increase intrinsic value.

Mathematical problem posing

A commonly accepted definition of problem posing encompasses both the reformulation of existing problems and the generation of new ones (Silver, 2013). In reformulating a problem, the problem solver modifies a problem into a revised version. In generating a problem, the student constructs a new problem e.g. by finding mathematical questions for a given real-world situation. The broad application of the term problem posing, which includes conceptually distinct activities like reformulation and generation, complicates the interpretation and comparison of research findings. Consequently, there is a need to systematize the various types of problem-posing tasks and approaches (Silver, 2013). Here we focus on reformulating problems.

A problem-posing task comprises two components: the problem situation and a problem-posing prompt (Cai et al., 2023). The problem situation is basis for problem posing and can be for example, a mathematical figure, an equation, a real-world situation, or a prior problem. Problem-posing prompts tell the students what to do with the problem situation. For example, they can ask for finding a mathematical question for a situation (problem generation) or to reformulate a given problem (problem reformulation). In addition, prompts can also specify certain conditions that should be met for the posed problems. A typical prompt is, to vary the level of complexity in the posed problems like the prompt given in the Swimming Pool Task in Figure 1.

Swimming Pool

After the winter break, the outdoor swimming pool in Münster is being prepared for the upcoming swimming season. Soon, visitors will be able to train again on the three lanes of the 50-meter long sports pool. After a thorough cleaning, the pool will be filled with water. There are water pipes available for this purpose, which can be operated independently. It takes 6 hours to fill the pool using the first water pipe and 9 hours using the second water pipe.

How long will it take to fill the pool if both pipes are used simultaneously?

Change the task to create one easy and one difficult mathematical problem for your classmates

Figure 1: Swimming Pool Reformulation Task adapted from Schupp (2002)

Here, we focus on problem situations that are based on real-world situations. Reformulating such problems is related to mathematical modelling, as it requires transformation processes between the real world and mathematics (Hartmann et al., 2023). This approach allows students to explore the real-world situations by modifying problem conditions, deepening their understanding of these situations and the importance of making assumptions. Additionally, prompting students to reformulate problems into versions with varying levels of difficulty in addition challenges them to analyze and adapt the problem's structure.

The use of different problem-posing prompts was found to affect learners' achievement-related outcomes, such as the complexity of their posed problems (Cai et al., 2023). However, research also indicates that problem posing can be challenging for students (Hartmann et al., 2021), and it remains unclear to what extent they can carry out problem-posing prompts.

Further, problem posing is considered as beneficial for students' motivation (Cai & Leikin, 2020), the impact of problem-posing prompts on students motivation is rarely studied. Krawitz and Wiehe (2024) showed for problem generation, that prompting students to pose easy compared to difficult problems has a positive effect on students attainment value and utility value, no effects on intrinsic value and negative effect on the perceived costs. Since, problem generation and reformulation are different constructs that also vary in their relation to other variables, it is an open question whether these results also hold for problem reformulation.

Research questions

These considerations led to the following research questions:

RQ1: Does prompting students to reformulate easy problems, in comparison with difficult problems, affect the complexity of their reformulated problems?

RQ2: Does prompting students to reformulate easy problems, in comparison with difficult problems, affect (1) their expectancies for success, (2) their task values, and (3) their perceived costs?

Regarding RQ1, we expect on the basis of prior research (Cai et al., 2023) that students reformulate more complex problems when they are prompted to pose difficult problems. Regarding RQ2, based on expectancy-value theory (Eccles & Wigfield, 2020) and prior research (Eccles & Wigfield, 1995; Krawitz & Wiehe, 2024), we expect a positive relationship between perceived task difficulty and students' expectancies for success, attainment value and utility value, as well as a negative relationship with costs. Since perceived difficulty may also increase intrinsic value, we do not have a specific hypothesis for this relationship.

Methods

Sample, instruction, and procedure

The study involved 78 ninth- and tenth-grade students from two middle-track and one high-track school in Germany (50% female, 48.7% male), with an average age of 15.6 years. Students received a brief instruction on problem reformulation before the test. The aim of the instruction was to explain, how they can reformulate problems by adding or removing information to the problem statement. Therefore, the Toothpaste problem ("How many centimeters of toothpaste are used in one month?") and a reformulated version of the problem ("How many centimeters of toothpaste are used in one month, if a person brushes his teeth two times a day and uses 1.5 cm toothpaste each time?") was shown and briefly discussed in class. After the instruction, each student was provided with a test booklet containing four real-world problems including the Swimming Pool Task depicted in Figure 1. For each real-world problem, students were asked to reformulate the problem to an easy and a difficult new mathematical problem ("Change the task to create one easy and one difficult mathematical problem for your classmates."). After students completed reformulating an easy and difficult problem for each given real-world problem, they filled out a questionnaire assessing their expectations for success, task values, and perceived costs related to reformulating the easy and the difficult problem.

Measures

To assess students' expectancies for success, task values, and perceived costs when reformulating easy versus difficult problems, students rated their agreement with statements on a 5-point Likert scale (1 = not at all true, 5 = completely true) corresponding to these constructs. Task-based measures were used, which means that for each given real-world problem such as the Swimming Pool Task, students were asked to report their expectancy of success for reformulating the Swimming Pool Task into an easy version and additionally into a difficult version. Responses related to reformulating easy and difficult problems were aggregated across the four given real-world problems into ten separate scales: expectancy of success, intrinsic value, attainment value, utility value, and costs related to reformulating easy problems and related to reformulating difficult problems. Each scale consisted of four items, one for each of the given real-world problems. The following statements were used: For expectancies for success, "I am sure that in the future I will be able to change the problem on the subject of [problem situation] into a(n) [easy/difficult] problem"; for intrinsic value, "I find it interesting to change the problem on the subject of [problem situation] into a(n) [easy/difficult] problem"; for utility value, "I find it useful to be able to change the problem on the subject of [problem situation] into a(n) [easy/difficult] problem"; for attainment value, "I find it important to be able to

change the problem on the subject of [problem situation] into a(n) [easy/difficult] problem"; and for costs, "Changing the problem on the subject of [name of task] into a(n) [easy/difficult] problem exhausted me." All scales showed satisfying levels of internal consistency (Cronbach's $\alpha \geq .72$).

To assess the complexity of the reformulated problems, we coded whether the problem was modified to have lower (scored 0), the same (scored 1), or higher complexity (scored 2) compared to the original problem. For complexity, we focused on the complexity of the mathematical model, necessary to solve the problem (see sample problems in the results section). Additionally, other types of responses, including non-mathematical problems, no changes, or minor changes that did not affect the complexity of the mathematical model (such as adding unnecessary information), as well as missing responses were coded as "other" (all scored 0). The interrater reliability for the coding was satisfactory ($\kappa \geq .67$, $M = .83$).

Results

For answering RQ1, we compared the complexity of reformulated problems posed in response to the prompts to pose an easy problem compared to the problems posed to the prompt to pose a difficult problem. McNemar-Bowker tests revealed a significant difference in the frequencies in terms of complexity across all given problem situations (all ps < .04) (see Table 1).

Table 1: Frequencies of problems with lower, same, or increased complexity reformulated in response to the prompts to pose easy vs. difficult problems

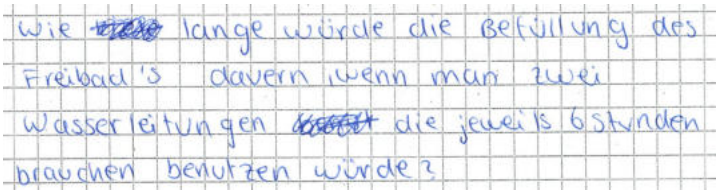
Problem situation	Renovation		Swimming Pool		Promenade		Greenhouse Gases	
	Easy	Dif.	Easy	Dif.	Easy	Dif.	Easy	Dif.
New problems with lower complexity	15	5	29	16	21	12	14	8
New problems with same complexity	30	31	6	23	10	17	15	15
New problems with increased complexity	2	9	1	2	0	3	7	18
Other	31	33	42	37	47	46	42	37

Note. "Easy" = Reformulate into an easy problem and "Dif." = Reformulate into a difficult problem.

Results from a paired sample t-test showed that students posed more complex problems in response to the prompt to pose difficult problems compared to the prompt to pose easy problems (M_{easy} = 0.26, SD_{easy} = 0.29, M_{diff} = 0.48, SD_{diff}= 0.39; $t(77) = -5.52$, $p < .001$, $d_{Cohen} = 0.34$). Figure 2 presents three student solutions illustrating reformulated versions of the Swimming Pool Task with varying levels of complexity. In the first solution, the complexity is reduced compared to the original problem; in the second, it remains the same; and in the third, the complexity is increased.

A high number of students kept the original problem and made only slight changes, such as adding superfluous information, or kept the problem completely as it was. This occurred particularly for the prompt to reformulate into easy problems, with 22.13% of problems responding to the prompt to reformulate into easy problems, compared to 10.25% for the prompt to reformulate into difficult problems.

Table 2: Examples of students' reformulated problems to the Swimming Pool Task with lower, maintained, or increased complexity than the initial problem

Complexity	Example
Lower complexity:	 *[How long would it take to fill the swimming pool if you used two water pipes, each taking 6 hours?]*
Same complexity:	*[How long does it take to fill 1/3 of the pool?]*
Increased complexity:	*[How long does it take to fill the pool if the first water pipe stops working after 3 hours?]*

Regarding RQ2, the results showed that learners reported significantly higher expectancies for success when reformulating the problems into easier versions compared to more difficult ones (M_{easy} = 3.01, SD_{easy} = 1.05; M_{dif} = 2.77, SD_{dif} = 1.10; $t(76)$ = -2.88, p = .003; d_{Cohen} = 0.71).

For attainment value and utility value, students reported that they find it more important (M_{easy} = 2.35, SD_{easy} = 1.15; M_{dif} = 2.06, SD_{dif} = 0.97; $t(76)$ = 2.62, p = .011; d_{Cohen} = 0.97) and more useful (M_{easy} = 2.33, SD_{easy} = 1.07; M_{dif} = 2.04, SD_{dif} = 0.98; $t(76)$ = 2.74, p = .008; d_{Cohen} = 0.66) to be able to reformulate problems into easier version compared to more difficult ones. However, for intrinsic value, students did not perceive reformulating problems into easier version as more or less interesting than reformulating them into more difficult ones (M_{easy} = 2.33, SD_{easy} = 1.14; M_{dif} = 2.25, SD_{dif} = 1.09; $t(76)$ = 0.89, p = .378; d_{Cohen} = 0.66).

Consistent with our expectations, learners reported higher perceived costs when reformulating problems into more difficult versions compared to easier ones (M_{easy} = 1.96, SD_{easy} = 1.06; M_{dif} = 2.34, SD_{dif} = 1.20; $t(76)$ = -3.84, p < .001; d_{Cohen} = 0.99).

Discussion future directions and limitations

The goal of our study was to examine how prompting students to reformulate problems with varying difficulty levels impacts the complexity of their reformulated problems and their motivational outcomes, specifically their expectations for success, task values, and perceived costs.

Our findings demonstrate that prompting students to reformulate problems into more difficult versions, compared to easier ones, increases the complexity of the reformulated problems. However, many students struggled to follow the prompt, indicating that reformulating real-world problems and specifically addressing problem complexity is challenging for students. Future research could explore scaffolding techniques or instructional strategies to better support students in problem reformulation tasks.

Previous research has emphasized the importance of problem-posing prompts on achievement-related outcomes (Cai et al., 2023) and on motivation when problem generation is focused (Krawitz & Wiehe, 2024). Our findings extend these results by showing that students' motivation varies when they reformulate problems into easier versus more difficult versions. Consistent with prior findings, we found that reformulating easy problems, compared to difficult ones, leads to higher expectations of success, utility value, and attainment value as well as to lower costs supporting theoretical insights from expectancy-value theory (Eccles & Wigfield, 1995).

In line with research on problem generation (Krawitz & Wiehe, 2024), we found no significant relationship between complexity and intrinsic value. Intrinsic value may differ from other value components since complexity can be a source of interest (Fielding et al., 2022; Russo & Hopkins, 2017). Moreover, the differing impacts on intrinsic value compared to other task-value components highlight the need for a nuanced understanding of the task value construct and the importance of differentiated assessments. The research primarily focused on the reformulation of real-world problems. Including a wider variety of problem types (e.g., intra-mathematical problems) could help determine whether this finding holds across different contexts and problem formats. In addition, qualitative research on this issue could provide further insight into the differences between value components in problem posing.

On a practical level, our findings suggest that prompting students to reformulate easy problems can be beneficial, even if it does not improve their problem-posing performance, because it can enhance their motivation.

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Decoding emotions in relation to perceived difficulty: How primary students engage with mathematical tasks

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This study explores emotional factors influencing primary school students' perceived difficulty when solving mathematical tasks. The research involved 49 students aged 7 to 9, with 20 of them being interviewed. The aim was to identify emotions related to perceived task difficulty through qualitative analysis of semi-structured interviews. Students engaged in collaborative, geometry-focused activities using Lénárt spheres. The results indicate that while emotional factors, both positive and negative, emerged, students often struggled to articulate these emotions. The findings suggest that emotions are meaningful and independent contributors to perceived difficulty. Further research is required to deepen understanding of the role emotions play in students' learning experiences.

Keywords: Mathematics education, qualitative study, perceived difficulty, emotions, primary school.

Introduction and theoretical framework

Research in mathematics education has long dealt with difficulties in mathematics. Focusing on students' difficulties related to solving mathematical tasks, factors such as the mathematical content and affective factors have an influence on them (see, for example, Zan, et al., 2006). On the contrary, the issue of perceived difficulty has not been analysed with the same attention. The *difficulty* and the *perceived difficulty* of a task are deeply interconnected; however, they are different. In fact, the difficulty of a task is usually objective and evaluated in retrospect based on the results obtained from the sample to which the task was proposed (Mehrens & Lehmann, 1991), while perceived difficulty is related to individual perception, and it is more personal. In mathematics education, there is currently no agreed-upon definition of the perceived difficulty of a task, although this aspect has been analysed in different fields, including cognitive psychology (see, for example, Efklides & Touroutoglou, 2010).

Concerning students, there are very few studies analysing their perception of difficulty and the factors influencing it and, in those studies, difficulty and perceived difficulty are not explicitly distinguished and the focus is on the cognitive side of the process, meaning that the factors described are merely linked to the objective characteristics of the task (Lee & Heyworth, 2000). Recently, some qualitative research highlighted the factors that contribute to the perceived difficulty according to students with respect to mathematical tasks (Saccoletto & Spagnolo, 2022; Spagnolo & Saccoletto, 2023). Those factors were organised into five non mutually exclusive macro-categories (Resolution Strategies, Capabilities and Experience, Emotions, Task Formulation, Self-considerations), further detailed in more specific subcategories. The macro-categories come from textual analysis of secondary students' open-ended answers, when asked to solve mathematical tasks, evaluate their difficulty and explain their reasons. The analysis was performed according to Constructivist Grounded Theory (Mills et al., 2006), therefore the macro-categories of factors influencing perceived difficulty have been

determined inductively from data. The first macro-category is "Resolution strategies" and it refers to the strategies necessary to solve the task or the fact that elements were needed to reach a solution. The second macro-category is "Capabilities and experience", and it includes students' self-perceived competence, skills as well as eventual prior experiences in solving similar tasks, and therefore their familiarity with them. Within this macro-category, we can find also students' reflections on their doubts and challenges faced during problem-solving and the time spent doing it. The third macro-category, named "Emotions", regards the emotional aspects. The fourth macro-category, "Task formulation", refers to the formulation of the task itself, both in its textual and non-verbal elements. Finally, the fifth macro-category is "Self-considerations", and it includes references to students' personal reflections related to their own success and effort in mathematics. Qualitative research involving secondary school students, carried out using the macro-categories, showed that the "Emotions" one results systematically underrepresented or not represented at all (Spagnolo & Saccoletto, 2023; Nicchiotti & Spagnolo, 2024). In fact, secondary school students seem not to give importance to subjective aspects when dealing with mathematics and tend to mention the emotional side only when directly asked about it (Nicchiotti & Spagnolo, 2024).

Research in mathematics education has shown an increasing interest in affective aspects concerning mathematics from the beginning of the 2000s while, until then, it had focused mostly on cognitive factors, with rare exceptions such as mathematics anxiety (Schukajlow et al., 2023). McLeod (1992) conceptualizes emotions in mathematics learning considering them as part of an affective range together with belief and attitudes, with emotions being the most intense but the least stable among the three. Furthermore, Ekman's theory of basic emotions (1992) has been used in research about problem solving in mathematics education (Hannula, 2015), but it emerged that basic emotions would be rare in learning context. Therefore, there is the need to investigate emotions emerging strictly in the educational context (Lehman et al., 2008). We referred to achievement emotions (Pekrun, 2006, 2011), which are defined as emotions linked to activities or outcomes, evaluated according to achievement-related quality standards. In particular, we adopted the novel three-dimensional taxonomy of achievement emotions (Pekrun et al., 2023), which includes the following 12 emotions: enjoyment, hope, pride, relaxation, assurance, relief, anger, anxiety, shame, boredom, hopelessness, disappointment. According to Pekrun and colleagues (2023), individual perception, including perceived difficulty of a task, can influence emotions. However, emotion can mutually impact the environment and the perceptions associated with it, even though this side of the reciprocal causation still needs to be investigated (Pekrun et al., 2023).

Research aim

Under these premises, it appears that students' perceived difficulty needs to be investigated further, focusing on the possible emotional factors influencing it. Antognazza and colleagues (2015) show that primary school students, when asked straightforward questions about the emotions felt before and during problem solving, mention different emotions, both positive and negative; these emotions are related to their assessment of the difficulty of the problem (Antognazza et al., 2015). We are interested in exploring whether emotions would still emerge with different and more open prompts, not explicitly mentioning them, and check if students' emotions would allow us to characterize more precisely the concept of perceived difficulty.

In this first explorative study, we focused on primary school students, as research still needs to explore the emotions experienced by children in that age group (Lichtenfeld et al., 2012). Our research question can be expressed as follows: which types of emotion related to perceived difficulty emerge, if any, from primary school students when dealing with mathematical tasks?

Methods

We carried out a qualitative exploratory study, involving 49 primary school students aged 7 to 9, who were attending an optional after-school course. The students were 21 girls and 28 boys, and they were from different classes and had been grouped into two new classes for the course (hence, each of them knew only some of the other children). They participated in five sessions, lasting two hours each, held over the course of a week. The activities were guided by two researchers in mathematics education; the mathematics class teachers were present only to supervise the students and they interacted sporadically to voluntarily participate in some of the activities with the students. The course consisted of activities with active involvement of the learner, in the sense proposed by Emma Castelnuovo (see, e.g., Castelnuovo, 1963).

The course was titled "The geometry of soap bubbles" and was focused on activities regarding comparative geometry, namely the comparison between Euclidean and spherical geometry in our case. During the sessions, the students worked in small groups collaboratively using Lénárt spheres to reflect on geometrical concepts and their characterisation on the plane and on the sphere; they investigated and built geometrical objects and definitions. The tasks proposed were selected among the ones experimented with children from different school grades and pre-service teachers using the Lénárt spheres, already present in the literature (Gambini, 2021; Gambini & Lénárt, 2021). Some of the tasks required children to discover the counterpart of segments, lines and triangles (which they knew on the plane) on the sphere. The tasks were introduced in a discursive and playful manner: we started recollecting the knowledge the students already had about plane geometry; then students were asked what they thought would happen if they tried to construct the same geometric entities on the sphere. They found out that lines on the sphere are maximum circumferences, and therefore on the sphere it is indeed possible to build three perpendicular lines, and that on the sphere the sum of the internal angles of a triangle could be greater than 180°. Much time was dedicated to their discussion and explanation of the construction and reasoning of each group.

At the end of the last session, we randomly selected 20 students and interviewed them in 7 small groups, keeping together the ones that had worked together during the sessions. Each group of students was interviewed separately; we let them express their opinions freely and sometimes asked clarifying questions. We asked them how they perceived the tasks, if they were easy or difficult and why, then we asked what they would change, being the teacher, to make the tasks easier. We decided to evaluate perceived difficulty after solving the task as we were interested also in the elements related to strategies implemented to solve it. We recorded the audio of the interviews and later analysed the transcript qualitatively, focusing on the answers to the first questions. The analysis was carried out using the software MAXQDA Analytics Pro 24 (v. 0.4). Specifically, we imported the interview transcripts into the software, where they appeared in the "Documents" section. Subsequently, we created 17 distinct codes within the "Code" section, each corresponding to a specific subcategory

within the five macro-categories of factors influencing perceived difficulty (Spagnolo & Saccoletto, 2023). Each macro-category's code was assigned a unique and casual colour and each code was labeled according to its respective subcategory. Following this preparatory phase, we proceeded with the analysis of each interview by reading and identifying words or sentences that were relevant for our research, and assigning each excerpt an appropriate code. Upon completion of this step, the "Code" section displayed numerical values next to each code, indicating the frequency of the code application throughout the dataset. Finally, we analysed in depth any reference to the macro-category "Emotions", using achievement emotions (Pekrun, 2023) as a reference, highlighting each specific emotion expressed by students. The coding phase was carried out independently by two of the researchers, who then compared their results with each other and with the third researcher to reach a consensual coding.

Results and discussion

We started analysing children' perceived difficulty of the tasks proposed. The answers varied from "some [tasks were] easy and some difficult", "medium difficulty" and "differently easy". None of the children stated that they were all easy or difficult and most of them then provided punctual examples of tasks solved during the sessions that they found easy or difficult. We noticed that there was no accordance between the children in defining the same activity easy or difficult, even with children from the same small group. Each student had their own perceived difficulty regarding each one of the tasks, and their perceived difficulty was influenced, but not determined, in absolute terms by the group they worked in.

Regarding the factors influencing perceived difficulty, we could find reference to each of the five macro-categories. In Table 1 we reported a summary with the number of references classified in each of them and the percentage. Being the macro-categories non mutually exclusive, the same part of the transcript might be categorised in more than one of them and therefore coded with different codes. It occurred with high frequency to find short statements in which elements pertaining to different macro-categories were mentioned. A representative example is the phrase "[Drawing three parallel lines on the sphere was difficult] because I had drawn a circle, another one on it and one more had to be done crossway, and I had done it kind of sideways", which belongs to the macro-category "Capabilities and experience" because of the reference to the obstacle encountered but also to "Resolution strategy" because the child explains the procedure needed to solve the problem.

Table 1: Distribution of the references among the macro-categories

Macro-categories	Number of references	Percentage
Resolution strategies	11	22,45%
Capabilities and experience	17	34,69%
Emotions	6	12,24%
Task formulation	5	10,20%

Self-considerations	10	20,41%

As shown in Table 1, we identified 6 references to the macro-category "Emotions". The emotional factors were mentioned by 4 out of 7 groups interviewed: only once, the fact that a student mentioned emotions during the interview sparked a discussion in the group about the theme, resulting in more children considering them. Moreover, the emotional factors do not seem to be systematically related to other specific macro-categories, meaning that the children who mention emotions consider also other factors when explaining their perceived difficulty, but those factors vary depending on the child.

Moving on to describe the emotions emerged, we identified both positive and negative emotions, which coexist even within the same group of children. In Table 2 we show the parts of the transcript coded with the code "Emotion", associated with the specific achievement emotion that we assigned during the analysis.

Table 2: Interview excerpts referring to emotions, associated with the achievement emotion involved

Excerpt	Achievement emotion
[The task regarding the spherical triangle was difficult] because Child A did everything, because I had not understood [how to do it].	Shame
[The activities] were fun.	Enjoyment
Because I like geometry, well, the subjects I like are also easier in my opinion.	Enjoyment
[Working] alone is easier but together is more fun.	Enjoyment
[Working] alone [is easier] because alone you are more satisfied	Pride
[It was hard because] the first time I said oh my dear! spherical geometry what is that?	*Hopelessness*

Not all the achievement emotions have been identified in the interviews; however, the model seems suitable for our analysis as the emotions expressed by children were classifiable as such. The only exception is the last excerpt. It was part of a student's longer and broad discourse regarding also their previous experiences with mathematics and other school subjects, in which they said that when you see something for the first time you might think it is very hard but then you look back at it, you study and understand it perfectly. In this case, the emotions are conveyed through interjections and exclamations rather than a specific word or expression. We could identify this emotion as *hopelessness*, but it does not seem to fit precisely, as the phrase in its context seems to communicate something more similar to surprise, which is not part of the achievement emotions.

Another aspect to consider is the influence of the type of task on the emotions experienced by children. Firstly, not all children consider the type of task when motivating their perceived difficulty.

They discuss the task, but do not really consider explicitly their general characteristics, such as the fact that it is very distant from their everyday schoolwork, even though they are absolutely clear about that when asked if they have already done something similar. Secondly, the students that elaborate on the type of task and link it to emotional aspects, do it in different ways. For instance, two children say that working alone is easier: one of them links the autonomous work with a positive emotion (pride), while the other links the group work with a different positive emotion (enjoyment), even if they find it more difficult. Hence, the type of activities (in the sense of Emma Castelnuovo) seems to encourage the development of positive emotions, but it is not the only condition to do so.

Analysing the coded excerpts as a whole, we could finally make a general observation: children seem not to be able to describe their emotions, nor to give them a name. To communicate them, they rely on stories, mimical expressions and tones, more than words. This might be for two main reasons: the first possibility is that their young age makes it difficult to them to understand their emotional states; the second one is that their emotions are clear to them, but they lack the verbal skills and terminology to communicate them. Obviously, since the sample is very small, we cannot generalise our findings; however, they might be hypothesis to be taken into account and investigated with further research.

Concluding remarks

In this paper, we presented results of a first qualitative study aimed to explore if and how emotions are mentioned in relation to perceived difficulty by primary school children. To do so we focused on tasks which are far from the usual schoolwork, and we investigated children's opinions through semi-structured interviews. We could conclude that emotions are expressed among the factors influencing primary school children's perceived difficulty. In addition to that, both positive and negative emotions are considered (in accordance with Antognazza et al., 2015), and they do not seem to be subordinated to other factors. This means that emotions were present in children's responses independently of the presence of other factors and not only associated with some of them, suggesting that they are one of the independent macro-categories of factors influencing perceived difficulty also for primary school students. To generalize our findings, we need to be cautious due to the small sample size, and further research is required to validate and strengthen the results.

The need for the study originated from the underrepresentation of the macro-category "Emotions" in other qualitative studies (Spagnolo & Saccoletto, 2023; Nicchiotti & Spagnolo, 2024), while the findings of the present research are contradicting that. This might depend on several elements: the activity proposed, the age of the participants and the method of data collection. In fact, in the other studies, the tasks were not collaborative, with active involvement of the learner and distant from the usual ones done in class. Secondly, we involved primary school students, while in the other studies only high school students participated. Finally, in the other studies perceived difficulty had always been analysed through questionnaires of different types, and not always associated to interviews. We do not want to reduce the complexity of such an image isolating one key element to unlock the expression of emotions, nor we think it would be useful or possible. Anyway, the elements discussed have to be held present in successive investigations.

Finally, we see that the research on emotions as factors influencing perceived difficulty might be developed in many directions. Looking at the whole picture, we have primary school children which

express emotions but often struggling and they have difficulties in describing and naming their emotions. On the other hand, previous research shows that secondary school students often neglect emotions, not even mentioning them at all (Nicchiotti & Spagnolo, 2024). Therefore, there is the need to close the gap studying what happens in between the two moments. It is important to understand whether students simply lose or not develop the ability to describe their emotions as they grow older, or if they do not express emotions because they have still difficulties doing so as in primary school, but growing up they come to disregard emotions, because anything they have difficulties expressing basically does not exist.

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Strategy-based motivation regarding questioning: Self-efficacy, values, costs, and gender

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Motivation and strategies play a crucial role in students' learning. In this study, we examined the motivation behind using the questioning strategy while solving modelling problems (i.e., the strategy to generate questions about the situation described in the task). Using the expectancy-value theory, we analyzed the relationships among three components of strategy-based motivation: self-efficacy, values, and costs, as well as gender differences in these three components. To address our research questions and test hypotheses, we collected data from a study involving 121 lower-secondary school students, assessing their motivation. The findings revealed (1) a positive relationship between self-efficacy and values, but no relationship between costs and other components, and (2) lower self-efficacy in the questioning strategy in female students. These results suggest that gender differences should be considered in research and practice when examining strategy-based motivation.

Keywords: Motivation, gender differences, learning strategies, self-efficacy, value.

Introduction

Motivation is a key concept in theories of affect (Hannula et al., 2019). It serves as the reason behind human behavior (Middleton & Spanias, 1999) and includes a variety of affective constructs. One of the most influential motivational theories in educational psychology, which received recently increased attention in mathematics education (Rach, 2023; Schukajlow et al., 2023), is Eccles's expectancy-value theory of achievement motivation (Eccles & Wigfield, 2020). This theory posits that expectations of success (including self-efficacy expectations), task values and (perceived) task costs are key determinants of students' educational choices and academic outcomes, such as the decision to pursue an advanced mathematical career, their school grades, performances, and competencies.

Motivation—similar to other affective constructs—can be directed toward various aspects such as learning in general as well as mathematics, specific mathematical topics or competencies, strategies, or tasks like problem solving and modeling tasks (Schukajlow et al., 2023). This study focuses on the concept of strategy-based motivation, which involves the characteristics and use of specific strategies (Karabenick et al., 2021; Schukajlow et al., 2022).

Prior research on solving modelling problems (i.e., problems that required demanding transition between real-world situation and mathematics) demonstrated that students already have difficulties with understanding the situation. Using strategies, such as generating a question to a situation described in the task, can improve learning. Prior studies indicate that learner-generated questioning can enhance reading comprehension (Song, 2016; Weinstein et al., 2010). As reading comprehension is a crucial ability in solving modeling problems, generating questions for a modelling task can improve modelling performance. Thus, this study focuses on strategy-based motivation regarding

learner-generating questioning while solving modelling problems. Additionally, we explored gender differences in questioning strategy. Specifically, this study aimed to analyze the relationships between self-efficacy, task values, and costs related to the questioning strategy and to examine gender differences in motivation for using this strategy.

Theoretical background

Self-efficacy, values, and costs

According to expectancy-value theory, students' subjective beliefs play a crucial role in their career choices, performance, and strategy use in the context of education and learning (Eccles & Wigfield, 2020). Expectancies refer to one's beliefs about their ability to perform a task, such as generating a question about a given situation. Expectancies are closely related to self-concept of ability and self-efficacy expectations, which were used in the past to assess expectancies. The construct of self-efficacy expectations was introduced by Bandura (2003) in social-cognitive theory. For the questioning strategy, students with high self-efficacy expectancies would respond positively to the question "Can I generate a question?". Other key components of expectancy-value theory are task values. Students who place high values on a strategy are more likely to use it frequently. Those who value the questioning strategy would respond positively to the question "Do I value generating a question?". Task values can be categorized into three types: attainment value, intrinsic value, and utility value. Attainment value suggests that an activity, like generating a question to solve a problem, is integral to one's personal identity. Intrinsic value reflects the enjoyment derived from the activity, such as the pleasure of generating questions. Utility value indicates the perception that the activity will help achieve goals, for instance, using a questioning strategy to solve a problem. Task costs are another critical aspect of motivation, encompassing the time cost and effort cost which students are willing to invest in an activity. Students who have low task costs associated with the questioning strategy might affirm, "Am I free of blockages that prevent me from investing time and effort into generating questions?". Originally, costs were considered to be a part of values (Eccles & Wigfield, 2020). In recent discussions, researchers have shown that cost is a distinct factor, have differentiated between different types of costs, and have analyzed the relationships of cost to values and expectancies as well as to other motivational and achievement constructs.

Expectancies, values, and costs have been found to predict learning outcomes, including educational choices and academic achievement in mathematics (Eccles & Wigfield, 2020; Rach, 2023). In one of the studies, researchers observed that self-efficacy expectations and the task costs of using a drawing strategy influenced the quality of students' drawings and their performance in solving geometry modelling problems (Schukajlow et al., 2022). This result indicates that strategy-based motivation can be an important predictor of strategic behavior and academic achievement.

Relationships between expectancies, values, and costs

The relationships between self-efficacy, values, and costs have implications for whether these components can be distinctly measured. However, these relationships have not yet been the central focus of research and require further exploration of different motivational objects. Whereas several studies included information about the relationships of self-efficacy, values, and costs of

mathematics, there is barely any evidence about the relationships of motivational components regarding other objects such as strategies. For example, with the questioning strategy, a key question is whether students who believe they can successfully generate a question also value this strategy more and perceive lower task costs associated with it compared to those who are uncertain about their ability to use this strategy effectively in problem-solving.

Based on the expectancy-value theory, one can posit that students who assign high values to an activity will more deeply and more often engage in it, develop greater proficiency in this activity, and subsequently report higher self-efficacy regarding that activity. Additionally, self-efficacy may be linked to task costs: students who see an activity as less costly may engage in it more frequently, achieve greater success, and, as a result, improve their self-efficacy. Expectancy-value theory (Eccles & Wigfield, 2020) suggests that several factors influence students' task values and costs. For instance, students' past experiences, which are related to self-efficacy expectations, can affect their emotional responses (such as anxiety toward the activity), which in turn influence task values and costs. The theory posits a close relationship between values and costs, with Eccles and Wigfield (2020) assuming that the overall value of a task depends on its costs. If students perceive an activity as too time-consuming or challenging, they may perceive it as less valuable. For example, if students believe that generating a question is not worth the time or effort, they may value the questioning strategy less than students who perceive it as less costly, regardless of the benefits.

Most studies have found significant relationships between the three components of motivation. For example, the relationship between expectations and the intrinsic value of mathematics for U.S. school students ranged from .36 to .68 (Gaspard et al., 2020). In a study of adolescent students in Korea, cost was negatively related to both expectancies (-.24) and task values (-.23) in mathematics (Jiang et al., 2018). To the best of our knowledge, only one study analyzed strategy-based motivation for drawing strategy, and no study focused on motivation regarding questioning. In the case of learner-generated drawings, self-efficacy was found to be positively related to values (.39 and .48) and negatively to costs (-.39 and -.15) (Schukajlow & Rellensmann, 2022).

Gender and motivation

Gender differences have been a central focus in the development of expectancy-value theory, which suggests that disparities in STEM choices between genders can be attributed to variations in expectancies and values, rooted in societal gender stereotypes (Eccles & Wigfield, 2020). Even in some recent studies, gender was recognized as a continuum, many studies still use gender as a binary concept (Becker & Hall, 2024). The reason for this conservative view is lower girls' performances in STEM subjects—especially among high-achieving students—as well as girls' lower self-efficacy in mathematics and higher anxiety across different countries and cultures (Encinas-Martin & Cherian, 2023). Thus, gender differences remain a great concern and that must be addressed in research, educational policy, and teaching practice. Students' motivation is suggested being one of the prevalent reasons for the gender gap in mathematics (Becker & Hall, 2024; Encinas-Martin & Cherian, 2023).

Expectancy-value theory posits that students are more likely to value activities that align with their social and personal identities (Eccles & Wigfield, 2020). Participating in these activities allows them

to express their identities. In many societies, mathematics is often perceived as a male domain, leading many female students to feel that mathematics is not a part of their identity, and thus they tend to value it less than their male counterparts. Gender role stereotypes were found to be related to lower expectations of success in girls and higher expectations of success in boys (Wolff, 2021). This contributes to the observed gender gap in expectancies and values. For instance, in the 2022 PISA study, female students in Germany reported lower self-efficacy in mathematics compared to male students, with Cohen's *d* values of 0.34 vs. 0.45 for easy vs. demanding tasks, respectively (Diedrich et al., 2023, p. 100). Similar results were observed also for value-related constructs such as interest (Schiepe-Tiska & Schmidtner, 2012).

However, it remains unclear how gender differences in self-efficacy, values, and costs are specifically related to strategy use, such as generating questions for problem-solving. To the best of our knowledge, only one study has analyzed strategy-based motivation by assessing students' use of the drawing strategy (Schukajlow & Rellensmann, 2022). In that study, female students reported lower self-efficacy in making drawings but placed a higher value on this strategy compared to male students. The task costs did not differ between female and male students.

Given that questioning is integral to the problem-solving process in mathematics, it is reasonable to hypothesize that female students may have lower expectations of success in this area. This hypothesis is supported by evidence of gender differences in anxiety and self-efficacy related to the drawing strategy (Schukajlow et al., 2019; Schukajlow et al., 2022). However, we do not have specific predictions regarding differences in values and costs. It is possible that female and male students may value the questioning strategy similarly and perceive a comparable level of costs when using it.

Research questions and hypotheses

In this study, we analyzed the following research questions and posed the following expectations:

Q1. Are self-efficacy, task values, and costs related to each other with respect to the questioning strategy?

H1. We hypothesized a positive relationship between self-efficacy and task values and a negative relationship between both self-efficacy and task values with costs.

Q2. Do female and male students differ in self-efficacy, task values, and costs with respect to the questioning strategy?

H2. We expected that female students will report lower self-efficacy expectations for the questioning strategy than male students. No clear hypotheses on gender differences regarding values and costs with respect to the questioning strategy could be derived from motivational theories.

Methodology

Sample and procedure

To answer the research questions and examine the hypotheses, we developed questionnaires for strategy-based motivation regarding questioning strategy and distributed them to low-secondary students. One hundred and twenty-two students from three middle-track schools (German Realschule) participated in this study. 82 students were males, 34 were females, three reported being of non-binary

gender and two did not report the gender. Thus, the sample utilized in this study was one hundred and sixteen students ($N = 116$). The mean age of participants was 14.3 years of age.

Students received modelling problems and were asked to generate two questions about the situation. After that, they estimated their understanding of the situation and filled out questionnaires on self-efficacy expectations, task values, and costs with respect to the questioning strategy.

Measures

We employed 5-point Likert scales to measure self-efficacy, task values, and task costs regarding the questioning strategy. The responses ranged from 1 (not at all true) to 5 (completely true). The items were adapted from previous studies on motivation, learning strategies in mathematics, and the drawing strategy in modelling (Schukajlow et al., 2022), tailored to focus specifically on generating questions for solving challenging word problems. Self-efficacy in using the questioning strategy was assessed through three items (e.g., "I am confident that I can always ask helpful questions about difficult word problems"; internal consistency Cronbach's $\alpha = .467$). Task value was measured across three dimensions: interest value (2 items; e.g., "I enjoy asking questions about difficult word problems"), attainment value (3 items; e.g., "Asking questions about difficult word problems is typical for me"), and utility value (3 items; e.g., "I believe asking questions is important because it helps me solve difficult word problems"). The overall task value scale demonstrated sufficient internal consistency (Cronbach's $\alpha = .75$). Personal cost was assessed using three items which refer to the time and effort needed for generating questions (e.g., "Asking questions about a difficult word problem requires significant effort from me"; Cronbach's $\alpha = .516$). Although the scales for assessing strategy-based motivation regarding questioning were adapted from prior studies, the internal consistency for value and cost was unexpectedly low. Thus, the results should be interpreted with caution. The low internal consistency is a significant limitation of the study.

Results

Relationships between self-efficacy, values, and costs with respect to the questioning strategy

Firstly, we explored the relationships among three components of motivation based on the expectancy-value theory: self-efficacy, values, and task costs. To investigate this, we analyzed Pearson's correlations. Consistent with hypothesis H1, the analysis revealed a positive relationship between self-efficacy expectations and values with respect to the questioning strategy (.581; $p < .001$). However, unexpectedly, we did not observe a negative correlation between self-efficacy and cost (-.065; $p = .250$). Furthermore, task values and task costs were not negatively related (.118; $p = .209$) as assumed in expectancy-value theory. These findings partially supported initial expectations.

Differences in self-efficacy, task values, and costs between female and male students

We addressed the second research question on gender differences in motivation by calculating multivariate ANOVA with independent variable gender and dependent variables self-efficacy expectation, values, and task costs (see descriptive statistics in Table 1). The statistical analysis confirmed differences between female and male students regarding motivation, $F(3, 111) = 3.161$, $p = .013$, $\eta^2 = .079$. As expected, we observed lower self-efficacy expectations in female students $F(1,$

113) = 3.087, p = .011, η^2 = .045. Values and costs did not differ between female and male students (values: $F(1, 113) = 0.042$, p = .784, η^2 = .001; costs: $F(1, 113) = .185$, p = .668, η^2 = .002).

Table 1: Gender, self-efficacy, task values, and costs with respect to questioning strategy

	Self-efficacy *Mean (SD)*	Values *Mean (SD)*	Costs *Mean (SD)*
female	2.07(.77)	2.47(.76)	3.26(1.01)
male	2.43(.75)	2.43(.76)	3.18(.89)

Discussion

This study explored the relationships between motivational components and examined gender differences. Grounded in the expectancy-value theory (Eccles & Wigfield, 2020) and initial findings about the possible positive effects of generating questions on reading comprehension (Song, 2016; Weinstein et al., 2010) and modelling, we focused on questioning strategy as an object of motivation (Schukajlow et al., 2023). An important limitation of this study is addressing a strategy that students might not be familiar with, which suggests the need for qualitative methodology in future research.

Our findings revealed a positive correlation between self-efficacy and task values. Students with higher self-efficacy for using the questioning strategy tend to prior findings about questioning strategy, highlighting a potential feedback loop where increased value leads to greater engagement, enhancing performance, and boosting self-efficacy in questioning strategy. Interestingly, our data did not confirm the hypothesized relationships between self-efficacy and cost and between values and costs. One explanation might be that students had difficulties estimating how costly is this strategy and because of that, their reports about costs were biased. Even though we asked students to generate questions about a given situation before assessing motivation in this study, this could be not sufficient, as the questioning strategy is not a typical strategy that students apply in mathematical classes. This finding, along with the generally weak correlations involving costs, suggests that cost should be considered a distinct motivational component in future research. Prior findings indicated that costs' relationships with self-efficacy and value are not stable (Schukajlow & Rellensmann, 2022) and might be influenced by students' problem-solving experiences with the task that students attempt to complete such as generating questions in this study. Our study contributes to the broader understanding of motivational components by extending the predictions of expectancy-value theory to the context of self-generating questioning in problem-solving. The results of this study add to prior findings on the relationship between motivational components in mathematics (Gaspard et al., 2020) and strategy-based motivation (Schukajlow et al., 2022; Schukajlow & Rellensmann, 2022). The investigation of strategy-based motivation in this specific context is a novel aspect of our research and offers fresh insights into the dynamics of self-efficacy, values, and cost. In future research, factors such as prior questioning experiences and prior knowledge should receive more attention.

The analysis of gender differences revealed significant disparities in self-efficacy between male and female students. Female students reported lower self-efficacy compared to their male counterparts,

consistent with previous research on gender gaps in mathematics-related self-efficacy (Becker & Hall, 2024; Diedrich et al., 2023; Encinas-Martin & Cherian, 2023) and in drawing strategy-related self-efficacy (Schukajlow & Rellensmann, 2022). The explorative analysis of gender differences in values and costs revealed that students value the questioning strategy independently of gender and that female and male students estimate the strategy as similarly costly. This finding suggests that traditional gender stereotypes in mathematics do not uniformly apply across all motivational components and types of problem-solving activities, particularly those involving value and cost with respect to creative strategies such as questioning in this study (see also results for value of the drawing strategy in Schukajlow & Rellensmann, 2022). These findings have important practical implications for mathematics education. Female students' self-efficacy needs stronger support in the classroom not only in mathematics in general but also with respect to strategies.

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Perceived difficulty of a mathematical task: do in-service and pre-service teachers have a common view?

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This qualitative study investigates how pre-service and in-service teachers perceive the difficulty of mathematical tasks and examines the factors that shape their evaluations. Using two tasks from the national large-scale assessments INVALSI, we aimed to identify whether perceived difficulties align with those of students and to explore any discrepancies between the two teacher groups. Our findings reveal significant differences between pre-service and in-service teachers, particularly in their handling of open-ended versus structured tasks. In-service teachers generally demonstrated alignment with students' evaluations for simpler tasks, while pre-service teachers exhibited emotions with respect to complex tasks. In our interpretation, these results highlight both the influence of teacher experience on perceived difficulty and the importance of addressing both cognitive and affective factors in teacher professional development.

Keywords: Mathematics education, perceived difficulty, pre-service teachers, in-service teachers, qualitative analysis.

Introduction

Difficulties in mathematics have been investigated in several researches, being one of the most open and studied topics in mathematics education. Difficulties can be classified into different categories depending on the focus: difficulties related to mathematics as a discipline or difficulties related to students in terms of deficits or their specific relationship with mathematics (Zan et al., 2006).

In our research, we focused on the difficulties of mathematical tasks, specifically on the *perceived difficulty* (PD) of in-service and pre-service teachers after solving a mathematical task. Literature distinguishes between difficulty *before* versus *after* a task has been attempted or solved (Eccles & Wigfield, 2020; Efklides & Touroutoglou, 2010); this is an important distinction in terms of perception of difficulty, yet in what follows we articulate a definition of PD that works in either cases. The difficulty of a mathematical task seems to depend on a set of factors, including the mathematical content (Thevenot & Oakhill, 2005; Radmehr & Drake, 2017), the formulation of the problem (Bolondi et al., 2018), text comprehension (Spagnolo et al., 2021), students' abilities or capabilities (Vicente et al., 2007), and affective factors (Zan et al., 2006). During the task-solving process, in fact, a student may encounter multiple difficulties that can depend on both the student's individual characteristics (such as abilities, knowledge, beliefs, and attitudes) and the specific characteristics of the task (such as the text or the mathematical content involved). These features of the mathematical task can influence both the student's and the teacher's perception of the task and, consequently, its PD. Thus, while closely connected, difficulty and PD are two different aspects (Spagnolo & Saccoletto, 2023). Furthermore, a student's PD does not necessarily coincide with the teacher's PD, as shown by Arzarello and Ferretti (2021). The mismatch between teachers' PD of a mathematical tasks and the actual success rates of those tasks opens up reflections related to research on teachers' beliefs about what students think (Philipp, 2007), as effective teaching and learning are believed to

develop from the alignment between teachers' and students' beliefs (Kumaravadivelu, 1991). Our goal is to highlight how pre-service and in-service teachers perceive a task—whether it is easy or difficult—and what factors lead them to this evaluation, while also identifying potential mismatches. Specifically, we aim to investigate which view, between pre-service and in-service teachers' ones, is closer to the students' one.

Theoretical and context background

Currently, in Mathematics Education, there is no widely accepted definition of PD, despite its recognized importance as a factor influencing students' behavior when approaching and solving tasks (Eccles & Wigfield, 2020; Saccoletto & Spagnolo, 2022; Doz et al., 2023). In contrast, in metacognition research, the issue of subjective difficulty has been explored over the past 30 years under various terms (e.g., Eccles & Wigfield, 2020; Doz et al., 2023). At times, PD has been considered a form of self-efficacy, though this connection does not seem entirely appropriate (Eccles & Wigfield, 2020). However, related or overlapping concepts have been defined, such as the "feeling of difficulty" (FOD), which was elaborated by Efklides at various points between the late 20th century and 2011. FOD is described as a "metacognitive experience that monitors cognitive processing as it takes place" (Efklides & Touroutoglou, 2010, p.172) and is explicitly distinguished from PD due to its "experiential nature".

Although conceptually distinct, "feeling of difficulty" and PD are sometimes treated interchangeably (Nuutila et al., 2021). Acknowledging their differences, in this paper, we have drawn on some characteristics of both concepts and adapted them to the context of mathematics education research. From this perspective, the synthesis proposed by Doz et al. (2023) was especially helpful and aligned with our viewpoint. It suggests that the nature of the "feeling of task difficulty" is metacognitive, stemming from the monitoring of ongoing task processing, and awareness of this process influences self-regulation, effort, emotions, and strategy use.

The factors influencing PD of math tasks have mainly been studied in relation to students' perspectives (Saccoletto & Spagnolo, 2022). For our analysis, we began with the factors characterizing students' PD, as qualitatively identified by Saccoletto & Spagnolo (2022), and compared these to those reported by teachers in our data. According to this study, PD in mathematics can be classified into five categories: *Resolution Strategy*, *Capabilities and Experience*, *Emotions*, *Task Formulation*, and *Self Consideration*. Note that these categories are not mutually exclusive.

The category *Resolution Strategy* includes responses that explicitly mention the approach or method needed to solve the task, according to students, or highlight key elements like calculation or reasoning required to reach a solution. The category *Capabilities and Experience* encompasses answers that reflect students' views of their capabilities and abilities, as well as their prior experiences with similar tasks, which contribute to their familiarity with the problem. Answers that suggest a problem is easy because it resembles something previously encountered also fall into this category. Additionally, it includes responses related to students' self-perception, doubts about their solutions, and difficulties they faced, including the time spent on the problem. The category *Emotions* pertains to responses involving emotions. The category *Task Formulation* comprises responses that refer to the task's formulation, particularly its textual aspects. Finally, the category *Self Consideration* relates to students' personal reflections on their success in mathematics.

Research aim

Under these premises, our goal is to highlight how pre-service and in-service teachers perceive the difficulty of a mathematical task and to examine the factors that contribute to their evaluations. Furthermore, we aim to identify any discrepancies or differences between the perceptions of these two groups of teachers. A key focus of our investigation is to determine which of the two perspectives is more aligned with students' views of task difficulty, providing insight into how teacher experience influences their understanding of student challenges.

This study seeks to elucidate the multifaceted interplay between teacher experience and the PD of mathematical tasks. By employing a qualitative approach, the research not only investigates the cognitive dimensions of task evaluation but also delves into the affective responses that underpin teachers' judgments.

The research questions guiding this study are:

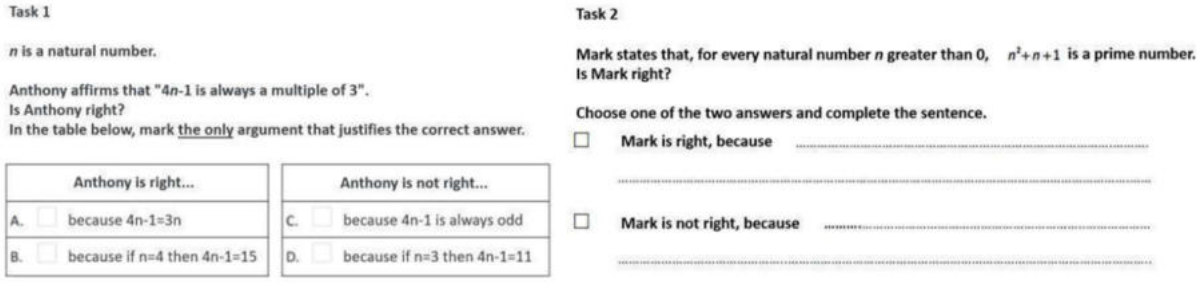
RQ1: How do pre-service and in-service teachers perceive the difficulty of a mathematical task, and what factors contribute to their evaluations?

RQ2: Are there notable discrepancies between the two groups of teachers?

RQ3: To what extent do their perceptions align with those of students?

Methods

We carried out a qualitative deductive study, involving 52 pre-service teachers and 49 in-service teachers of secondary school. All the teachers were from different Italian regions. Teachers filled in an online questionnaire through Google Forms, containing two tasks, some specific questions related to each task and three general questions. Each teacher worked on the questionnaire on their own, without knowing the other participants' answers. The questionnaire was the same used for students in Saccoletto & Spagnolo (2023), as we considered the categories determined by them for students, as an initial reference for teachers' results. We asked the teachers to solve the tasks (Figure 1) and, for each one, answer specific questions related to PD. Additionally, there were specific questions aimed to investigate attitudes and beliefs towards mathematics.


Task 1

n is a natural number.

Anthony affirms that "4*n*-1 is always a multiple of 3".
Is Anthony right?
In the table below, mark the only argument that justifies the correct answer.

Anthony is right...		Anthony is not right...	
A. ☐	because 4n-1=3n	C. ☐	because 4n-1 is always odd
B. ☐	because if n=4 then 4n-1=15	D. ☐	because if n=3 then 4n-1=11

Task 2

Mark states that, for every natural number *n* greater than 0, n^2+n+1 is a prime number.
Is Mark right?

Choose one of the two answers and complete the sentence.

☐ Mark is right, because .....

.....

☐ Mark is not right, because .....

.....

Figure 1: Task 1 administered to Grade 08 Italian students by INVALSI in 2017 and task 2 administered to Grade 10 Italian students by INVALSI in 2014, www.gestinv.it

We choose two mathematics tasks from the INVALSI tests, as they are statistically validated and were also used in Spagnolo & Saccoletto (2023). In the Italian educational context, INVALSI tests—administered since 2008 in grades 2, 5, 8, 10, and 13 by the National Institute for the Evaluation of the Educational System—allow us to track some of the difficulties students experience over time.

These tests, designed to measure student competence in relation to the Italian curricular guidelines, offer a quantitatively validated tool that complements our qualitative analysis. Our focus was on argumentative questions within the Numbers area. The two tasks (Figure 1) share mathematical similarities, while also exhibiting some differences. Task 1 was a multiple-choice question requiring the identification of a correct argument, whereas Task 2 was an open-ended question that asked for the creation of an argument. Both tasks, however, were based on literal calculation and could be solved using the same strategy—proving the falsity of a statement by providing a counterexample. We remark that the two tasks have been given to students in grades 8 and 10 and they turn out to be difficult in the INVALSI tests, namely a few number of students gave the correct answer. In the study of Spagnolo and Saccoletto (2023) these tasks were administered to students at different grades: grade 8 students (at the end of the school year) and grade 9 students (at the beginning of the school year). The tasks were selected together with the class teachers.

The questions in the questionnaire for tasks 1 and 2 were identical and aimed at probing teachers' ideas, linking them—strictly in a qualitative way—to teachers' attitudes, beliefs, or specific features of the INVALSI tasks. The study adopted a deductive approach to data analysis, starting from the categories identified in Spagnolo and Saccoletto (2023) and working towards a broader understanding of overarching patterns and trends.

All analyses were carried out with Excel for the calculation of average values and for the graphical representation of the data, while the qualitative analysis of the teachers' evaluation of motivation was developed with the software MAXQDA 24 (v. 4.0), using as codes the macro-categories of factors influencing PD (Spagnolo & Saccoletto, 2023).

Results and discussion

Qualitative analysis of teachers' answers highlight important distinctions between how pre-service and in-service teachers perceive the difficulty of mathematical tasks, particularly when comparing Task 1 and Task 2.

Task 1, a multiple-choice question, was generally perceived as less challenging by both pre-service and in-service teachers, though the reasons behind their evaluations differed. In-service teachers, being more familiar with structured tasks, found Task 1 relatively straightforward. Pre-service teachers, while acknowledging the simplicity of the task, raised concerns regarding the clarity of the task formulation. Both groups perceived task 2, an open-ended question, as more difficult. On one hand, in-service teachers found the need for constructing a mathematical argument challenging, but their experience helped them cope with the complexity. Pre-service teachers, on the other hand, attributed much greater difficulty due to the open-ended nature of the task, reporting a reflection focused on the anxiety of answering an open-ended question. Below (Table 1), we present the distribution of responses in percentage terms for each category.

Student's PD does not seem to be related to the ability to answer the question correctly (Saccoletto & Spagnolo, 2022) and seem to refer much more often to the text, personal preparation, specific content, or aspects related to the production of explanation or argumentation to support their results. Also completely missing are aspects related to emotions, but much more present are self-considerations on how to improve one's performance.

We report the distribution of the categorized responses of students, pre-service and in-service teachers in the table below. In particular, for students we refer to the analysis in Spagnolo & Saccoletto (2023), so we report number and percentages for teachers, and only percentages for students.

Table 1: Distribution of the responses to Tasks 1 and 2 of students, pre-service and in-service teachers according to macro-categories identified in Saccoletto & Spagnolo (2023). The column with students' percentages is derived from Saccoletto & Spagnolo (2023).

Macro-categories	Students Task 1	Pre-service teachers Task 1	In-service teachers Task 1	Students Task 2	Pre-service teachers Task 2	In-service teachers Task 2
Resolution strategies	51,8%	40,38% (21 teachers)	59,18% (29 teachers)	33,6%	36,54% (19 teachers)	57,14% (28 teachers)
Capabilities and experience	39%	21,15% (11 teachers)	26,53% (13 teachers)	41,2%	30,77% (16 teachers)	46,94% (23 teachers)
Emotions	0%	36,54% (19 teachers)	6,12% (3 teachers)	2,9%	42,31% (22 teachers)	4,08% (2 teachers)
Task formulation	15,3%	28,85% (15 teachers)	18,37% (9 teachers)	5,3%	19,23% (10 teachers)	14,29% (7 teachers)
Self-considerations	47,8%	11,54% (6 teachers)	2,04% (1 teacher)	44,8%	5,77% (3 teachers)	4,08% (2 teachers)

Both for Tasks 1 and 2, pre-service teachers focused mostly (respectively 40,38% and 36,54% of teachers in each task) on *Resolution strategies*, identifying the correct strategy to solve the task, using logical elimination methods common in multiple-choice formats for Task 1, whereas for Task 2 emphasized the challenge of developing a strategy to construct an argument. For example, one teacher states that "if you often solve multiple choice questions, it becomes easier for you. That way you know it's important to start with the options and go by exclusion using logic". This answer was also categorised under *Capabilities and experience*. In-service teachers give also importance to experience and prior familiarity with similar tasks. 6% and 4% in-service teachers referred to the *Emotions* commenting Tasks 1 and 2 respectively, and 2% and 4% on *Self-Considerations* categories, commenting that "if I would have given more open-ended questions to my students during classes, they would have been less in trouble". In short, a few in-service teachers elicited emotions and self-considerations. Some teachers hypothesised problems in comprehension of the task (for both Task 1

and Task 2) connected with *Task formulation*. For example, one teacher stated, "For both tasks, I noticed that the wording could potentially cause some confusion, especially for students who struggle with calculations with letters. However, the instructions are clear enough for those with some experience in solving similar problems".

Pre-service teachers also focused on problem-solving strategies, in particular the answers of 21 pre-service teachers (for Task 1) and 19 pre-service teachers (for Task 2) were categorised below *Resolution strategies*. In both tasks, also *Capabilities and experience* and *Task Formulation* were dominant factors influencing PD of pre-service teachers. However, pre-service teachers frequently elicit emotional factors. Most of the answers under macro-category *Emotions* are related to negative emotions; for example, "Arguing something that has never been seen before in class [referring to '4n-1 is always a multiple of 3'] could generate a lot of anxiety in students", or "If you don't think of using a counterexample, proving a sentence in general can start panic". The emotional impact was most pronounced in Task 2 for pre-service teachers, highlighting the importance of preparing their future students to handle the cognitive and emotional demands of open-ended tasks.

In terms of alignment with student perspectives (for results of students see Spagnolo & Saccoletto, 2023), Task 1 was more easily aligned with students' evaluations, particularly by in-service teachers. This suggests that experienced teachers have a clearer understanding of how students approach and evaluate simpler, structured tasks. Task 2, with its open-ended format, revealed a greater mismatch, especially among pre-service teachers, who overestimated the difficulty, likely due to their limited experience in such tasks with students.

In direct response to our research questions, the results indicate that for RQ1, both pre-service and in-service teachers exhibit distinct perceptions of mathematical task difficulty, with pre-service teachers generally overestimating difficulty due to a greater emotional response and limited experience, whereas in-service teachers demonstrate a more balanced approach rooted in cognitive strategies and familiarity with task structures; in addressing RQ2, clear discrepancies are evident between the two groups, with pre-service teachers' evaluations being more influenced by affective factors and in-service teachers' judgments aligning more closely with analytical reasoning; and concerning RQ3, the alignment between teachers' and students' perceptions is notably stronger among in-service teachers, suggesting that practical experience enhances the ability to accurately estimate student difficulties.

Concluding remarks

The aim of the research is to compare how in-service and pre-service teachers perceive the difficulty of a task, and to see how much they differ from students' PD. PD has a strong affective connotation, as it entails beliefs, self-efficacy and emotions. The categories of resolution strategy, capabilities and experiences (which in our view are linked to beliefs), emotions, task formulation and self-consideration emerge in a previous study (Saccoletto & Spagnolo, 2023) and it has been observed that they might overlap, namely that a same teacher might consider more than one category when formulating their account for why they perceive a task as difficult or easy. Even if overlapping also in our study, these categories emerge with different intensity among groups of in-service and pre-service teachers. We have noticed, in fact, that resolution strategies, capabilities and experiences, and task formulation categories emerge in in-service teachers' accounts. This is also in line with literature

on *difficulty* of mathematical tasks, as Vicente *et al.* (2007) stress the central role of students' abilities and capabilities, Bolondi *et al.* (2018) and Spagnolo *et al.* (2021) maintain that difficulty depend on the formulation of the problem and text comprehension, respectively, and Thevenot and Oakhill (2005) and Radmehr and Drake (2017) link difficulty to the mathematical content, which can be related to the category of resolution strategies.

We have also highlighted that Zan *et al.* (2006) argue that affective factors play a central role in the difficulty of a task, and pre-service teachers in our study focus on emotions quite significantly.

The students' perceptions are more in line with in-service teachers' ones, as if, on one side, as Kumaravadivelu (1991) posits it, there is a (positive, in our view) alignment between students' and teachers' beliefs, but on the other side it seems that emotions are elicited by those who are somehow further from everyday life in mathematical classrooms, namely pre-service teachers.

The relatively small sample size and qualitative nature of the study limit the generalizability of the findings. However, the consistent trends identified across both tasks provide a useful foundation for further exploration into how task type and teacher experience influence PD of mathematical tasks. Moreover, despite the differentiation of factors influencing PD, our definition appears to be adequate in capturing the perception of difficulty across different cases. The categories we identified, while with different distributions, help us categorize the responses of both groups of teachers (pre-service and in-service). They highlight the multifaceted nature of the construct, reinforcing the idea that PD encompasses both cognitive and affective dimensions, as well as task-related and individual-related aspects.

In conclusion, our results indicate clear differences in how pre-service and in-service teachers perceive the difficulty of tasks. Experience plays a key role in shaping these perceptions, and understanding these dynamics is crucial for improving teacher professional development and ensuring that teachers are better equipped to assess and address students PD face in solving mathematical tasks.

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The effects of classroom quality on students' lesson-specific mastery experiences on easy, medium difficulty, and hard tasks

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The importance of classroom quality on students' learning outcomes is well documented, but the impact of classroom quality on students' mastery experiences, especially in the context of tasks of different levels of difficulty, has received less attention. Investigating this issue is important to understand the role of different domains of classroom quality on student outcomes. In this article, we present our preliminary analysis of how quality of student-teacher interactions predicted students' mastery experiences across a sequence of four lessons. Students reported their end-of-lesson mastery experiences. These reports were linked with observed scores for certain classroom quality domains: emotional support, classroom organisation, instructional support, and student engagement. Across two types of models, which examined within- vs. between-teacher variability, a similar pattern was replicated, highlighting the significance of instructional support for students' mastery experiences.

Keywords: Mastery experiences, classroom quality, teacher-student interactions, level of difficulty, multilevel structural equation model.

Introduction

Classroom quality is important to student outcomes. Many studies demonstrated that high quality classroom interactions are related to positive student outcomes such as social competence, engagement, and learning (see, e.g., Allen et al., 2013; Patall et al., 2018). In mathematics, Zhu and Kaiser (2022) found that teachers' social-emotional support and instructional quality were associated with student outcomes such as students' self-efficacy and interest. König et al. (2021) found that teachers' competence was associated with their instructional quality, while instructional quality (cognitive activation) was associated with students' mathematics outcomes. Expanding our knowledge regarding the role of classroom quality on students' perceived mastery experiences can help unwrap the role of different dimensions of classroom quality on important student outcomes. Furthermore, domains of classroom quality might be differentially related to students' experiences of mastery on tasks of different levels of difficulty, which can have implications for teachers' instructional strategies.

We define *mastery experiences* as students' perceptions of succeeding on tasks, e.g., on learning mathematics during the previous lesson. Mastery experiences are important in at least two ways. First, they are a proxy for students' actual performance experiences, and thus an indication of what they have learnt. Second, as appraised performance experiences they are powerful predictors of students' self-efficacy for learning mathematics (Byars-Winston et al., 2017), which we know is related to their engagement, perseverance, socio-affective states, and mathematics performance (Street et al., 2024).

In this article we focus on students' mastery experiences of learning mathematics for different levels of task difficulty. The ICMI Study Volume on Task Design (Watson & Ohtani, 2015) reflects the importance of tasks as elements of mathematics instruction with important influence on students' opportunities to learn mathematics. The tasks used in the classroom not only convey messages about what doing mathematics entails (e.g., Zaslavsky, 2005), but they also become the objects of students' mathematical activity and shape their experience.

We know that students' (negative) attitudes towards particular mathematical topics are related to their perceptions of task difficulty (Mielicki et al., 2021). Furthermore, we know that difficult tasks are related to steeper changes (growth) in students' mathematics self-efficacy over time (Street et al., 2022), and that the relationship between self-efficacy and performance differs according to perceived task difficulty (Reinhard & Dickhäuser, 2011). These studies highlight the benefit of differentiating between types of outcomes in mathematics, e.g., according to task difficulty. However, we know little about how various classroom factors, such as classroom quality domains, can (differentially) support students' experiences of mastery on tasks of different levels of difficulty. This kind of knowledge might inform teachers' decisions on both task selection (e.g., difficulty level) and type of teacher support (e.g., emotional vs instructional support).

Teachers can support students' mastery experiences both through the types of tasks made available to students and the types of teacher support provided for different tasks. In this study, we focus on classroom quality in relation to classroom interactions, i.e., teacher-student interactions. We draw on the Teaching Through Interactions theoretical framework (TTI; Hamre et al., 2013) where quality in teacher-student interactions is conceptualised through three broad domains, namely emotional support, classroom organisation, and instructional support. Student engagement is included as a separate domain, which can be seen as both a predictor and an outcome of classroom quality. The TTI is operationalised through the Classroom Assessment Scoring System (CLASS; Pianta et al., 2012), in which the three domains are operationalised through 12 dimensions (e.g., positive climate, behavioural management, quality of feedback) and 43 indicators (e.g., relationships, clear expectations, building on student responses). Each indicator is exemplified with several behavioural markers, which are directly observable.

One affordance of classroom quality observation manuals, such as CLASS, is the operationalisation of latent constructs into observable classroom interactions. This enables a deconstruction of the role of different kinds of classroom quality for different kinds of student outcomes. König et al. (2021) conceptualised classroom quality according to the domains of student support, classroom management, and cognitive activation. Of these, only cognitive activation predicted lower secondary students' mathematics performance (König et al., 2021). Patall et al. (2018) found that when students perceived their (science) classwork to be more difficult than usual, this was associated with decreased perceived competence and increased dis-engagement. However, these effects were minimised when the students perceived their teachers to be supportive of their autonomy. In these studies, classroom quality was measured through a novel observation manual (König et al., 2021) or through student reports (Patall et al., 2018). In our study we use an established and standardised observation manual to investigate how quality of student-teacher interaction predicts students' mastery experiences at the end of each lesson during a sequence of four lessons when a new mathematics topic was introduced.

The research question guiding our study was: What are the within- and between-teacher effects of domains of classroom quality (emotional support, classroom organisation, instructional support, and student engagement) on end-of-lesson student mastery experiences for different levels of task difficulty? We present in these proceedings preliminary analyses to this ongoing study.

Methods

170 Norwegian students and their nine mathematics teachers took part in the study across three to four mathematics lessons, where the students were introduced to a new topic in the first lesson of the sequence. Students reported on 597 end-of-lesson mastery experiences for easy, medium difficulty, and hard tasks respectively. In the analysis we report herein, student reports were linked with 34 observations of classroom quality in the second half of each lesson. In future analyses, we intend to include beginning-of-lesson self-efficacy and overall classroom quality as predictors of end-of-lesson mastery experiences.

We administered nine questionnaire items to measure students' mastery experiences at the end of each lesson, three per difficulty level. Students responded on 11-point scales (0 = not at all true; 10 = completely true) to how successful they had been with learning mathematics during the previous lesson, for different levels of perceived task difficulty (example items: *During today's lesson in mathematics, I managed to solve all the tasks that appeared easy to me*; medium difficulty task: *All the tasks that appeared of medium difficulty to me*; hard task: *All the tasks that appeared difficult to me*). Individual differences in changes in mastery experiences are presented in Figure 1.

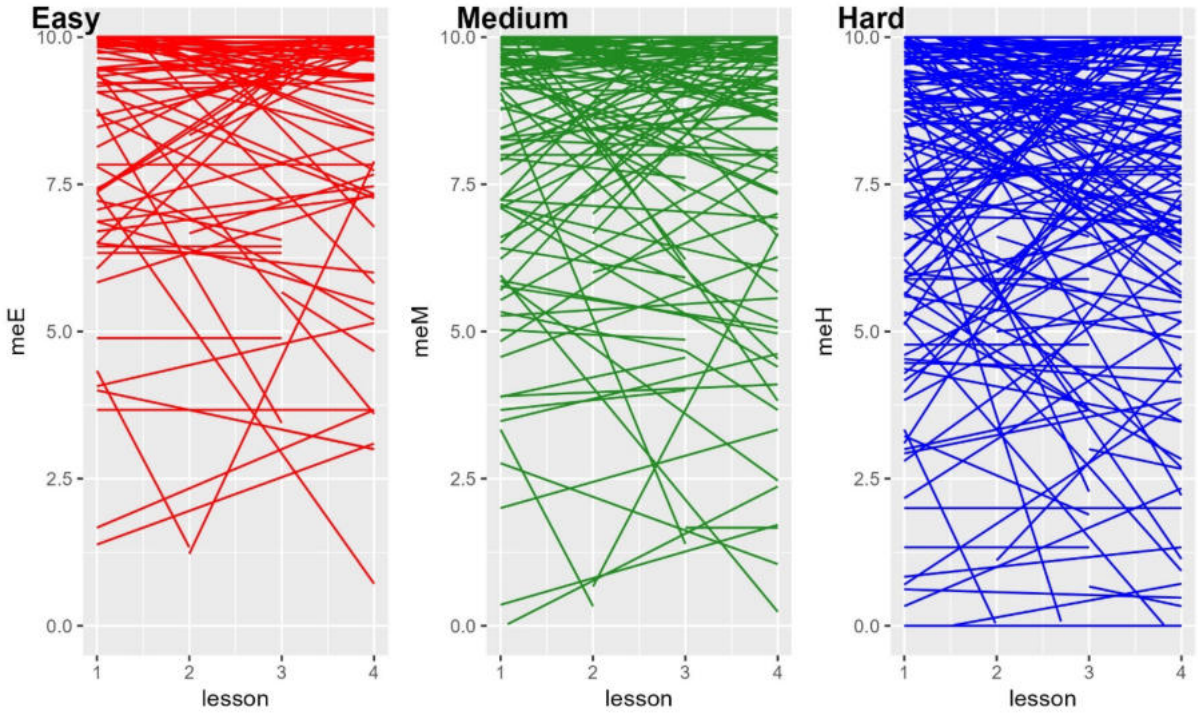

Figure 1: Individual differences in starting point and change in students' mastery experiences for perceived easy, medium difficulty, and hard tasks (predicted linear trajectories).

We used CLASS (Pianta et al., 2012) to measure classroom quality. One certified observer conducted in-vivo scoring according to the CLASS manual, including numeric quality scoring on a seven-point scale (1 = low quality, 7 = high quality) of each dimension of classroom quality. Scoring with CLASS requires observers to complete a certification course and a test to learn how to reliably assess the available evidence for each of the 12 dimensions considering the presence of (observable) behavioural markers. Scoring is conducted on classroom segments of minimum 15-minute duration. Domain-scores were obtained through averaging the dimension scores for each domain. Observed end-of-lesson quality for each domain, for each classroom (teacher) is presented in Figure 2.

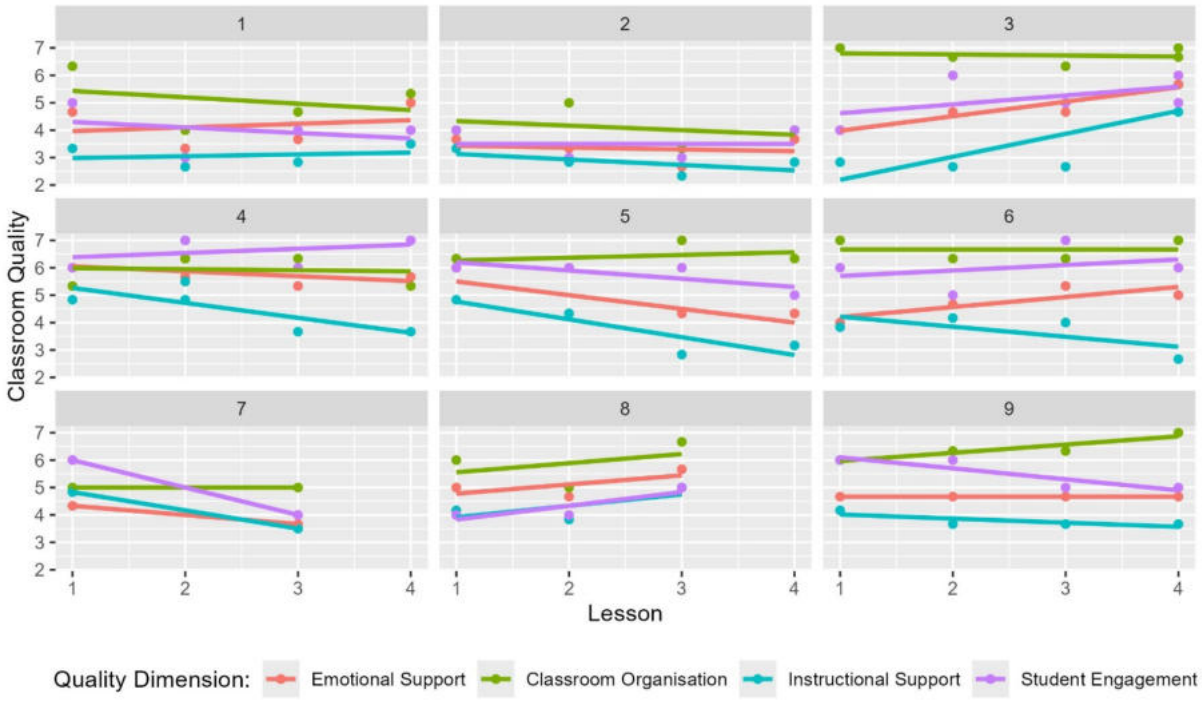

Figure 2: Observed end-of-lesson classroom quality (four dimensions) of nine teachers

We specified cross-classified multilevel models in which lessons (four lessons in a sequence) were nested in both students and in teachers using the Bayesian estimator (Asparouhov & Muthén, 2020; Muthén & Asparouhov, 2012). This way we can assess the effect of quality at the lesson-level, taking account of differences between students and between teachers. An initial multilevel model for change suggested that individual trajectories were linear. As shown in Figure 3 the cross-classified model was specified for time-points (lessons) nested in both students and teachers. This way it was possible to link the observed quality (scored at the classroom level) to each student's lesson-specific report. Models were carried out in Mplus 8.8 (Muthén & Muthén, 2017-20) using the Bayesian estimator. We specified 12 models, three for each quality domain, so that separate models were run for mastery experiences of easy, medium difficulty, and hard tasks.

At the within-level (lessons) we regressed mastery experience for easy, medium difficulty, and hard tasks on lesson-order (0, 1, 2, 3) and observed quality, in separate models (see Figure 3). We specified a random term for lesson, meaning that we allowed individual student trajectories of mastery

experiences to vary across students and for different teachers. This means each student had different starting points and change over time. At the between-level we observed the average mastery and average mastery-on-lesson slopes.

To answer our research question, we ran all models twice: first with within-teacher-centred (centred within cluster, CWC) observed quality as predictor, and then using the grand-mean-centred (GMC; Singer & Willett, 2003) observed quality as predictor. When centring within cluster we subtract the cluster-mean from the individual observation (i.e., $x_{ij} = X_{ij} - \bar{X}_j$). When we grand-mean centre we subtract the grand-mean from the individual observation (i.e., $x_{ij} = X_{ij} - \bar{X}_{..}$). The effect of the within-teacher-centred predictor is interpreted as such: when a teacher deviates from his or her own mean, this has an effect on the outcome. The effect of the grand-mean-centred predictor is interpreted as such: when a teacher deviates from the average of all teachers, this has an effect on the outcome.

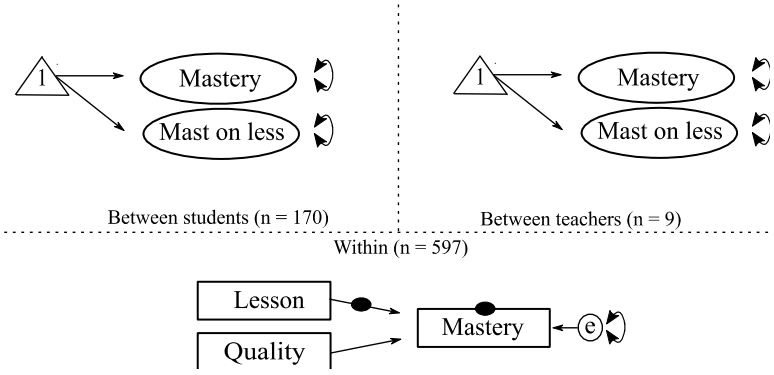

Figure 3: Cross-classified models of students' mastery experiences and observed classroom quality

Results

For the within-teacher effects the quality effects are shown below (Table 1): All four teaching-quality domains predicted students' mastery experiences for easy tasks. Classroom organisation, instructional support, and student engagement predicted students' mastery experiences for medium difficulty tasks. Instructional support predicted students' mastery experiences for hard tasks.

Table 1: Effects of classroom quality on students' mastery experiences: Within-teacher centring

Quality is centred within teachers	Easy		Medium		Hard	
Fixed effects	beta	cred	beta	cred	beta	cred
Emotional support	0.296	*	0.157		0.120	
Classroom organisation	0.268	*	0.245	*	0.124	
Instructional support	0.212	*	0.257	*	0.325	*
Student engagement	0.274	*	0.239	*	0.260	

*95% credible interval

For the grand-mean centred effects (see Table 2), the same pattern was replicated with the addition of a student engagement effect on mastery experiences for hard tasks. In terms of interpretation, the

models where classroom quality is centred *within* teachers indicate that when the quality of instructional support is higher than a teacher's own average across lessons, this has a positive impact on the mastery experiences of the students in her classroom (for tasks of all levels of difficulty). Similarly, the models where classroom quality is centred *across* (or between) teachers indicate an effect on students' mastery experiences when quality of instructional support *or* student engagement is higher than that of the average lesson, of the average teacher.

Table 2: Effects of classroom quality on students' mastery experiences: Grand-mean centring

Quality is grand-mean centred among all teachers	Easy		Medium		Hard	
Fixed effects	beta	cred	beta	cred	beta	cred
Emotional support	0.298	*	0.175		0.157	
Classroom organisation	0.272	*	0.257	*	0.163	
Instructional support	0.233	*	0.287	*	0.365	*
Student engagement	0.281	*	0.258	*	0.310	*

*95% credible interval

Discussion

In our study, cross-classified multilevel models demonstrated variations in classroom quality, both within and across teachers. Furthermore, increased classroom quality was associated with stronger perceptions of mastery experiences, both when classroom quality was compared to a teacher's own average lessons, and when it was compared to average lessons across teachers. While the effects were somewhat stronger when classroom quality was grand-mean centred, this conveys a positive message that classroom quality a) is important, and b) can vary from one part of a lesson to another and from one lesson to another. Knowing that there is individual variability across both students' and teachers' engagement and learning experiences (Li et al., 2024; Malmberg & Hagger, 2009), and across classroom quality (Mashburn et al., 2008), it is important to investigate which kinds of classroom quality factors might be associated with variations in which kinds of student outcomes.

While all the included classroom quality domains predicted students' mastery experiences on easy tasks, fewer predicted their mastery experiences on medium difficulty tasks, and only instructional support (and, partially, student engagement) predicted students' mastery experiences on hard tasks. König et al. (2021) similarly found a direct effect of cognitive activation (comparable to instructional support in our study) only on student performance in mathematics. These studies thus highlight the role of instructional support for both performance and affective outcomes in mathematics. In the TTI framework, instructional support includes teachers' work to facilitate conceptual development and strategic processing of information through, e.g., responsive feedback and instructional dialogues. While such instructional moves can directly influence students' task performance, interactions in other CLASS domains are likely related to students' mastery experiences through more indirect paths (through, e.g., increasing their sense of belonging or the time spent on relevant learning tasks).

Succeeding on harder tasks is obviously harder than succeeding on easier tasks, and while teachers' work to support students emotionally and ensuring learning is well organised has a bearing on students' experiences of mastery for easier tasks, our findings point to the importance of teachers' instructional moves that directly support students' effective learning strategies when it comes to harder tasks. Teachers' work to enhance student engagement, which can be seen as both a predictor and an outcome of classroom quality, is also important for students' mastery experiences.

Although our study included 597 reports of mastery experiences of 170 students, a limitation is that there were only 34 observations of 9 teachers available, and the context is limited to Norwegian classrooms. Future studies would need to include observations of more teachers across more lessons. Studies in other national contexts, e.g., countries with more stratified educational systems, would also be valuable. Furthermore, observation scoring was conducted in-vivo by a single observer, thus observer reliability could not be calculated. However, CLASS is a standardised protocol which requires observers to regularly re-certify to retain their observer status, to limit bias. Finally, the analyses we reported in this paper included quality scores from half the lesson only; we plan to include overall classroom quality in subsequent models we will explore as part of our ongoing analysis.

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ChatGPT is going to replace us! Or not? "Narrative Insight" about secondary-tertiary transition in mathematics

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This study explores the use of Narrative Insight, a personalised GPT model, to analyse undergraduate students' journals in the context of secondary-tertiary transition in mathematics education. The goal is to evaluate the extent to which artificial intelligence can support the researcher in narrative analysis by identifying relevant themes to categorise keywords and highlight emerging themes from student texts. The results show that Narrative Insight can offer new thematic perspectives, although it requires human supervision to ensure accurate interpretations. The model has proven to be a promising tool for automating part of the analytical process for qualitative data, especially in its preliminary stages, without giving up the depth of the research.

Keywords: secondary-tertiary transition, AI, narrative analysis, qualitative analysis.

Introduction

Qualitative research has assumed a central role in the educational sciences, allowing for a deeper understanding of phenomena in the field. In mathematics education, qualitative analysis of narrative data, such as interviews and texts produced by students, has emerged in the past two decades as one of the fundamental methods in affect research (Di Martino & Zan, 2015; Hannula, 2002), because they allow the "inner world" of the student to be explored as authentically and realistically as possible, showing also how social and cultural aspects are grounded in it (Radford, 2008). Unlike quantitative studies, which can often be conducted with relatively large numbers of students using statistical tools and software, qualitative studies require prolonged attention to detail and a considerable amount of time to collect and analyse data (Creswell, 2014), which require intensive coding and interpretation. This often implies the need to reduce the number of participants or sacrifice the depth of the analyses conducted, in favour of research with smaller samples, which allows for detailed exploration of participants' experiences, at the expense of a greater possibility of generalisation of results or class group analysis in its entirety (Jirwe, 2011). Despite the lack of literature on the subject, this issue is a well-known fact within the academic and scientific community, confirmed by the presence of very few works conducting qualitative investigations of large groups and university classrooms (Whitt & Kuh, 2017), and particularly in the mathematics education field.

In this context, generative artificial intelligence (e.g., ChatGPT) may emerges as an innovative resource for qualitative analysis, accelerating processes such as coding and theme identification due to its ability to handle large amounts of textual data. Based on advanced artificial intelligence technologies (Radford et al., 2018), MyGPT can adapt to different needs (Montenegro-Rueda et al., 2023; Gabashvili, 2023), supporting both quantitative and qualitative analysis. The possibility of customization through the "builder" tool makes it a versatile instrument for researchers, allowing them to automate part of the analysis process and provide an initial interpretation of the data.

In this study, we explore the use of MyGPT for analysing journals of students enrolled during mathematics at the degree program of Architecture of the Politecnico di Milano, focusing on the transition from secondary to tertiary mathematics education. Artificial Intelligence has already been used to analyse texts produced by students, mainly to provide didactic and evaluative feedback (Wang et al., 2013), but rarely to understand narrative content. We ask, therefore: to what extent can a MyGPT (i.e., a customised GPT model) be used as a tool to support the qualitative analysis of students' narrative data in the context of secondary-tertiary transition in mathematics education? We will discuss this issue by assessing Schoenfeld's (1998) *robustness* of the model, which we will explain in more detail in the next section.

Methods and Research Context

In the present study, a customised GPT model (MyGPT) was developed to assist researchers in the narrative analysis of texts produced by students, with a focus on topics related to mathematics education in the transition from secondary to tertiary education (STT). This model will be evaluated in terms of *robustness* in Schoenfeld's (1998) sense: how well a model can be applied to different cases, maintaining sufficient flexibility and adaptability without losing its explanatory power. In order to do this, we will carry out a four-step analysis in which we will measure, in particular, the ability to assign keywords to predetermined subject categories (explanatory potential) and the ability to extrapolate categories a priori and subsequently assign keywords to these categories (flexibility, adaptability and variation in explanatory potential). After verifying the robustness of the model, we will go over the analyses carried out in our previous research (Bernardi et al., 2024) using GPT as the only tool. Being a first approach to the topic, the measurement of the model's characteristics was carried out in a qualitative manner, and we chose to use the graphical representation using Venn diagrams to visualise and monitor the evaluations, attributions and "choices" made by the machine, and their variability.

We decided to use the narrative data collected during the mathematics course for the Architectural design degree program. These are collections of journals written by first-year students and had already been analysed by this team in light of the STT themes (Bernardi et al., 2024). Those study involved 180 first-year Architecture students (ages 18–22) from Politecnico di Milano, who provided informed consent to participate. Each student was asked to submit four reflective journals during the semester with no requirements related to structure, length, or content, but suggesting, however, to focus on their experiences, thoughts and feelings related to the course. To foster participation, additional points were awarded for students who completed all four submissions. Only a designated tutor (the first author) accessed these journals, ensuring anonymity when reporting general insights to instructors. In the previous work, after a data selection process, journals related to the entire course written by 54 students were analysed, and from these we identified six a priori categories to which keywords extrapolated from the texts could be assigned, choosing them from the words used most (and least) frequently by the students: Self-Regulation (S-R), Structural Differences (SD), Social Integration (SI), Instrumental vs Relational View of Mathematics (I/R), Identity (Id). Only one word was present across all themes ("test"), making it difficult to categorise; it was therefore chosen to isolate it from the others (see Figure 1).

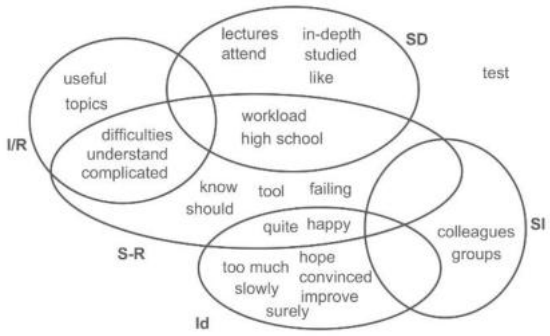

Figure 1: Classification of the selected words (Bernardi et al., 2024)

To observe how the Large Language Model carried out the categorisation of meaningful words used by students; we created a MyGPT called "Narrative Insight" (NarrI). First, we provided NarrI with the texts from the relevant scientific literature on which we based the study and identification of the a priori categories in our previous work, and which dealt with STT via Clark and Lovric's (2008) *rite of passage* theory (Social Integration and Identity categories), the new challenges in terms of study method, autonomy, organisation and self-regulation (Self-Regulation and Structural Differences categories) (De Guzmán et al, 1998; Zimmerman, 2000), and the necessary shift from an instrumental view of mathematics to a relational one (Instrumental/Relational category) (Skemp, 1976). On the basis of these works and the categories determined on them, we submitted to NarrI the identified keywords (which can be seen in Figure 1 already divided into groups with respect to the previous classification) and ask the machine to divide them with respect to our categories; in this way, we wanted to check the explanatory potential of the machine through the understanding and use of the categories and keywords related to the STT theme.

We then asked NarrI to identify a priori categories that, in its opinion, were relevant to STT issues, again based on the relevant literature. We list below the seven categories identified together with NarrI's own brief explanation of them (in *italics*) and our comment to underline the correspondence with the previous categories:

1. Phases of Mathematical Transition: *how students adapt to different learning environments and expectations in mathematics, including shifts in learning style, curriculum complexity and level of independence required*. It therefore contains themes in the Self-Regulation and Structural Differences categories;
2. Mathematical Self-Concept: *students' perceptions of their own mathematical abilities and identity*, which we can correspond to the category Identity;
3. Relational vs. Instrumental Mathematical Views: *whether students view mathematics as connected, relational field or as a set of rules and procedures to follow*, which corresponds exactly to category Instrumental/Relational;
4. Navigating Emotional Barriers: *how students manage feelings like anxiety, fear, or frustration related to learning mathematics*. This theme, which is very present in the literature on STT, had not been taken into account by us neither in the a priori nor in the a posteriori analysis: since the students always expressed *what* triggered certain emotions in them, we were always able to categorise the *cause* of them in the categories we initially chose;

5. Adjusting to Mathematical Discourse: *students' ability to engage with and understand the language and communication styles used in higher-level mathematics*. Even this category had not been perceived as necessary by the team, as it was often attributable — in our opinion — both to the necessary transition to a relational view of mathematics and to the change of cultural institution of reference, which required a more specialised and "high" language;
6. Developing Autonomy in Learning: *the process by which students become more independent and take responsibility for their learning in mathematics*, corresponding to the Self-Regulation category;
7. Social and Academic Integration: *how students balance social and academic demands, and their sense of belonging in the academic environment*, which partially refers to both the Social Integration and Self-Regulation categories.

This request allowed us to assess the machine's ability to extrapolate macro-themes from foundational texts, demonstrating its flexibility and adaptability to the educational context: in fact, despite the fact that we might suppose that a possible user-researcher already has certain categories of reference in mind, they might be interested in identifying others in order to observe the collected data in a different way, either by asking NarrI to identify relevant themes, or by integrating some already highlighted by the user.

In this way, we verified that Narrative Insight was a sufficiently robust model for further investigation, going on to assess its usefulness as a support tool for the researcher who wants to carry out a qualitative analysis of narrative data.

To test the usefulness of NarrI in the analysis of narrative data, we retraced the steps taken in the analysis previously carried out (Bernardi, et al., 2024), using, however, NarrI as the only tool. In this way, we were able to experience the need to automate certain actions of the machine, thus implementing its construction via the building interface. For example, we needed a list of stop-words to be neglected in the counting phase of the frequencies of the terms present in the texts; or again, we set up those variations in gender or numerosity of the terms to be neglected, aggregating all concordances under the masculine singular term.

Finally, we asked NarrI to extract the data from the files, containing the student's narratives and broken down by author, and create a table containing a number associated with each student, the STT categories into which his or her narratives could fit, and textual citations justifying the choice.

Results

The first categorisation carried out by NarrI of the 27 proposed words, and examples of sentences in which they were used by the students, was very similar to the one we carried out: 19 words remained unchanged even partially, while 8 were categorised completely differently (Figure 2).

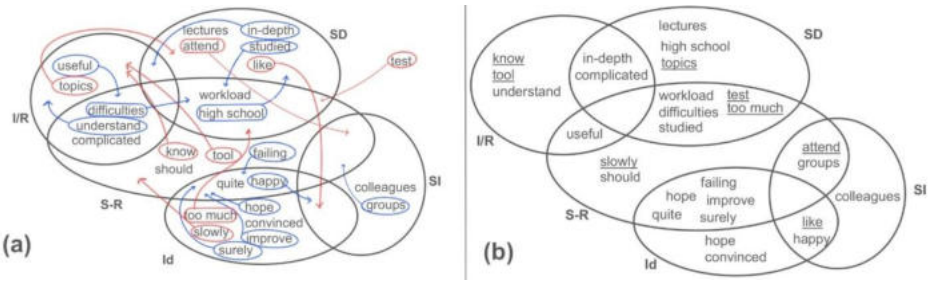

Figure 2: (a) Changes in categories from the previous categorisation: in red completely changes, in blue partial changes; (b) new categorisation of the selected words

To better understand the differences between the two categorisations, we also asked NarrI to explain the reasons for the various assignments. The discrepancies that emerged seem to stem from the interpretative nuances present in the narrative data. For example, words such as "attend", which in our study had been categorised in "Structural Differences" between secondary and tertiary education, were placed by NarrI in the categories "Self-Regulation" and "Social Integration". Our choice was based on the novelty introduced into students' lives by optional participation in academic life and lessons, as these aspects are left to the individual's organisational capacity, autonomy and desire to become involved in the academic environment, NarrI opted for other categories. This suggests that NarrI grasped aspects of student behaviour differently than human researchers, probably also due to different ways of understanding the educational context. This, that may be seen as a limitation, can also be a potentiality: NarrI seems to be able to make an original and motivated classification, just as it might happen to two researchers who independently categorise a set of words on the basis of predetermined and fixed themes, acting as a useful tool for the researcher for a preliminary analysis and a one-way "discussion" with the machine, having a different — and valid — interpretation of the collected narrative data.

Observing from afar the themes that NarrI has identified as fundamental for the analysis of the STT phenomenon, we note, as already pointed out, that these are similar to those of our working group: the cognitive aspect, the social aspect, the affective aspect and the cultural aspect. This is not surprising, as these are macro-topics frequently dealt with both in mathematics education and in the pedagogical field in general. As we get closer, however, we notice that the focus NarrI has placed on these topics is, in some aspects, different from ours, and this is particularly noticeable for categories 4 and 5 highlighted by the machine. The user-researcher asking NarrI to identify categories from scratch may want a different viewpoint on his data, or may be approaching this area of investigation for the first time and seek guidance on where to direct his work.

We report in Figure 3 the categorisation of keywords made according to these categories. Again, we asked NarrI about the rationale behind its choices and concluded that they too were well-founded and consistent with the themes of the STT.

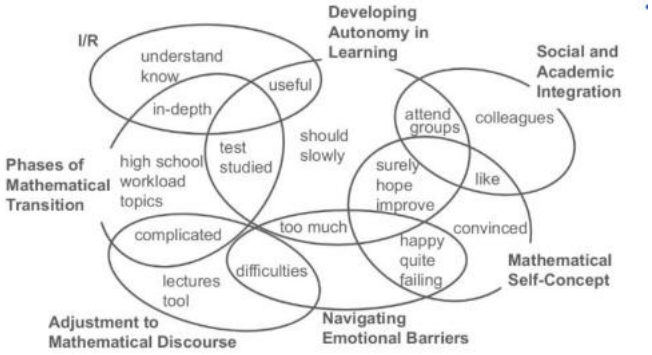

Figure 3: Categorisation of the keywords based on the STT groups selected by NarrI

Finally, we shared the files containing the students' journals with Narrative Insight, asking the machine to identify the most frequent and least frequent words present, without any particular request on the list of stop-words used. We expected to obtain results similar to those identified in our previous research using a free corpus analysis toolkit designed for text analysis, but this was not the case: NarrI was not able to correctly identify the occurrences of the words it considered important. For example, the term "difficulties" (and its variants) was only counted 10 times by NarrI (and 136 by us), and the result did not improve even after several attempts at correction.

We therefore chose to have the machine classify each student individually, analysing all the narratives they had written (four, during the entire duration of the course) providing the categories within which it would place the topics, and examples to justify its choice. The results of this task, which we can consider to be both *counting* (considering each student individually) and *assigning* (of the category according to the theme), behaved as we expected, considering the previous results: it identified and classified the themes dealt with adequately, also presenting contextual examples, but it took more than 5 requests to obtain this classification on all the students considered in our reference sample, in each experiment tried.

Conclusion

This study is among the first to explore the application of a customized generative Artificial Intelligence tool, Narrative Insight (NarrI), for qualitative analysis of student-generated narrative data within the context of the secondary-tertiary transition (STT) in mathematics education. One of the primary challenges in qualitative research, especially with large groups, is managing substantial amounts of data while maintaining analytical depth. By automating parts of the thematic categorisation process, NarrI demonstrates a promising approach to scaling qualitative research without losing insight into individual experiences. This strength is particularly valuable for large datasets, where traditional qualitative methods might struggle to capture complexity. Through the analysis of reflective journals, NarrI enabled us to capture complex, multifaceted insights into students' cognitive, social, and emotional experiences.

In our study, we evaluated the robustness (Schoenfeld, 1998) of NarrI in supporting narrative analysis related to secondary-tertiary transition (STT) in mathematics. This model was customised based initially on academic reference texts, working both with a priori categories derived from previous

work (Bernardi et al., 2024) and determined by the model itself. The results indicate that the use of Artificial Intelligence allows for sufficiently accurate analysis of large amounts of data without sacrificing complexity and precision in the interpretation of central themes. The comparison of manual and automated categorisation revealed a broad correspondence between the two methods, but also the ability of NarrI to detect subtle differences and propose equally valid alternative interpretations, in line with existing literature (Clark & Lovric, 2008; De Guzmán et al., 1998). This flexibility and capacity to suggest alternative perspectives are key strengths of the NarrI model, that seems to have the potential to reduce the manual workload for researchers by speeding up coding and categorisation processes, especially in the preliminary stages of analysis. However, our findings suggest that the use of Artificial Intelligence still requires careful human supervision: discrepancies in frequency and categorisation analysis indicate the need for continuous interactions and a thorough understanding of the domain to ensure accurate interpretations, that surely can be partly considered a limitation of this method although researcher supervision is a fundamental part of any research regardless of the method used (Creswell, 2014; Whitt & Kuh, 2017).

It is necessary to specify that in order to have a more complete and complex evaluation of the tool, it would be necessary to verify specific characteristics recognized by the scientific community as key elements for the reliability of a method or tool, such as the consistency of test scores across repeated applications; since the repetition of the same prompt ChatGPT responds by producing different outputs each time, it is immediately clear that the evaluation of this research/measurement tool is extremely complex. Some inherent features in this type of technology, such as its continuous updating, make it impossible to use the currently existing validation models recognized by the scientific community. For this reason, it was decided here to use the definition of robustness, although not contextual to mathematics education. A challenge to be faced in the future, then, is precisely to adapt or develop an appropriate method of evaluation and validation of research tools using AI, such as NarrI. In addition, it would be interesting to observe how models similar to NarrI might behave in other research contexts and on other topics, maintaining a crucial role for human expertise in guiding and verifying analyses produced by Artificial Intelligence (Montenegro-Rueda et al., 2023; Gabashvili, 2023).

Can artificial intelligence, then, replace us? Not yet, or at least not totally: the limitations outlined in this paper highlight the inevitable and crucial role of the human researcher, particularly in issues that require the analysis and interpretation of narrative data, which are fundamental to areas of research such as affect (Di Martino & Zan, 2015; Hannula 2002).

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Lower secondary pupils' academic motivation in a collaborative problem-solving intervention

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In many countries, there is a need to address the descending trend of secondary pupils' mathematics performance (between cohorts) and pupils' mathematics-related affect that develops negatively throughout the school years (within cohorts). To address these trends, a 1-year collaborative problem-solving intervention was implemented in six mathematics classes in the academic year of 2019-2020. In this paper, the combined effect of the intervention and the COVID-19 pandemic is studied through pupils' academic motivations. The results show that while pupils value learning mathematics, they do not think they need to show competence or avoid showing incompetence in mathematics lessons. Similar results were gained both on a personal and on a classroom level. The implications of these results are discussed.

Keywords: Mathematics-related affect, academic motivation, goal structures, collaborative problem solving, lower secondary school.

Introduction

Previous research has shown the positive effects of problem solving on pupils' performance and affect in mathematics (e.g. Felmer et al., 2016; Liljedahl & Cai, 2021). Despite this, there is a lack of studies on the systematic, daily use of problem solving in Finnish mathematics classrooms. Based on studies on Finnish pupils' mathematics performance and affect, such studies are much needed. Following the international trend, Finnish pupils' mathematics performance has decreased constantly during the past two decades (OECD, 2023). Furthermore, like in many other countries, Finnish pupils' affect in mathematics start declining from a very early age (Niemivirta et al., 2024). Exceptional is, that Finnish pupils' view of mathematics is distinctively negative (Mullis et al., 2020).

This paper reports from an intervention study that had its purpose on increasing pupils' mathematical thinking and learning, as well as engagement in mathematics through collaborative problem solving (CPS). CPS has received increased research attention recently (Liljedahl & Cai, 2021; OECD, 2017). It makes pupils' mathematics learning visible to teacher and peers while developing their social skills and engaging them into learning mathematics (Liljedahl, 2014, 2021).

The intertwined relationships between mathematics-related affect, problem solving and mathematical thinking and learning (e.g. Schoenfeld, 1992; Hannula et al., 2018) are also the starting point for the 1-year CPS intervention reported in this paper. Lower secondary pupils' mathematics-related affect is studied through pupils' personal and classroom goal structures (Midgley et al., 2000) as part of their academic motivation (Kaplan & Maehr, 2007). The paper aims to answer the following research questions:

1. How can pupils' mathematics-related personal and classroom goal structures be characterised in the beginning of the intervention?

2. How can pupils' mathematics-related personal and classroom goal structures be characterised over the span of the 1-year collaborative problem-solving intervention?

Theoretical framework

Affect in mathematics education is a well-established research field that lacks, to this day, consensus on the concepts and their structures used in the field (Haser et al., 2023). Hence, it is important to start by saying that, in this study, affect is understood as an umbrella term for the cognitive, motivational, and emotional aspects of affect (e.g., beliefs, attitudes, and feelings; Hannula, 2012). Furthermore, affect is studied as an individual psychological trait, indicating to the more stable aspects of pupils' mathematics-related affect (Hannula, 2012).

One of the goals for the intervention study reported in this paper, was to engage pupils in learning mathematics (Viitala, 2022). One way of looking at dispositions towards engagement is to study pupils' purpose to learn (mathematics) through pupils' academic motivation (Kaplan & Maehr, 2007). Academic motivation in this study is understood as pupil's mathematical learning goals that direct their behaviour in mathematics lessons (cf. Hannula, 2012). Academic motivation has often been studied through achievement goal theory in complex and social situations (Urdan & Kaplan, 2020), that is, in situations such as CPS (Liljedahl & Cai, 2021). So is done also in this study.

Pupils' academic motivation, and more closely, their achievement goals have generally been studied through mastery goal orientation (developing competence), performance-approach goal orientation (demonstrating competence), and performance-avoid goal orientation (avoiding demonstration of incompetence) (Urdan & Kaplan, 2020). While the first orientation is focused on the task of learning and understanding mathematics, the latter two orientations are focused on the self and how the self is presented amongst peers (Midgley et al., 2000). Previously, mastery goals have generally been associated with positive emotions, interest, and deep learning strategies, performance-approach goals with higher achievement, and performance-avoidance goals with negative emotions, disengagement, and low achievement (Urdan & Kaplan, 2020).

Based on previous research on CPS (e.g. Liljedahl, 2021; Liljedahl & Cai, 2021; Felmer et al., 2016) and results from this intervention (Viitala, 2023, 2024), the expectation would be that pupils' mathematics-related affect develops positively during the intervention, or at least remains on the same level as in the beginning. That would mean a positive development on mastery and performance-approach goals, and a negative development on performance-avoidance goals. However, the intervention was interrupted by the global COVID-19 pandemic which has shown to have a negative effect on pupils' mathematics-related affect (Metsämuuronen & Nousiainen, 2021). The results represent a joint effect of the intervention and the pandemic.

The intervention

The 1-year CPS intervention was implemented in six lower secondary mathematics classrooms during the academic year 2019-2020. During the intervention, pupils worked collaboratively with problem-solving tasks in their daily mathematics activities following a teaching paradigm called Building Thinking Classrooms (BTC; Liljedahl, 2021). The BTC paradigm has previously shown to increase pupils' engagement and social skills in mathematics classrooms, among other things (e.g. Liljedahl,

2014). CPS is defined as "the capacity of an individual to effectively engage in a process whereby two or more agents attempt to solve a problem by sharing the understanding and effort required to come to a solution and pooling their knowledge, skills and efforts to reach that solution" (OECD, 2017, p. 47). In this study, collaboration is part of the environment where problem solving takes place (Liljedahl & Cai, 2021).

Seven months after the beginning of the intervention, the schools were closed due to the global COVID-19 pandemic. During this two-months' time, the pupils studied at home in isolation from their peers. The intervention ended, together with the school year, two weeks after the school closures ended and the pupils returned to schools – continuing to practice social distancing.

Methods

Participants

The data used in this paper are from two grade 8 classes (N=40) and three grade 9 classes (N=43). One grade 8 class dropped out from the intervention halfway through the academic year and its data is not used for this paper. In Finland, grade 8 pupils are 14-15-year-olds and grade 9 pupils are 15-16-year-olds. All but two (N=81) pupils answered the pre-questionnaire. However, due to the on-going pandemic, not all pupils returned to schools for the last two weeks of the school year (e.g. for health reasons). Hence, the post-questionnaire was answered only by N=49 pupils. In the missing answers were also one whole mathematics class.

Data collection

The data were collected using pre- and post-questionnaires. The pre-questionnaire was answered during the first school week (mid-August 2019) and the post-questionnaire during the last two school weeks of the school year (end of May 2020). The questionnaires were answered individually during mathematics lessons. To study pupils' academic motivation, Midgley and colleagues' (2000) Patterns of Adapted Learning Scales (PALS) was chosen because the validated scale includes aspects of personal *and* classroom goal structures. This dual focus was considered as important because the intervention is based strongly on collaboration in mathematics lessons, highlighting both individual and commonly built classroom-level structures.

From PALS, (revised) personal achievement goal orientations and perception of classroom goal structures were included in the study (see Midgley et al., 2000). Both categories included questions regarding mastery goals, performance-approach goals, and performance-avoid goals (cf. Urdan & Kaplan, 2020). Also questions regarding avoiding novelty were added since the intervention implemented working methods that were new to the pupils (e.g. daily problem solving). The statements were adapted so that they were directly connected to mathematics (e.g. "It's important to me that I improve my skills this year." → "It's important to me that I improve my mathematics skills this year."). Data were collected through Likert-scale from 1 = Fully disagree to 5 = Fully agree.

Data analysis

Due to the small sample size, the data are analysed through descriptive statistics and correlations. Reliability of the sum variables is tested using Cronbach's alpha. Significance of the descriptive statistics is tested using paired samples t-test.

Results

Personal achievement goal orientations

Pupils' personal achievement goal orientations were studied through mastery goal orientation (five statements, e.g. "It's important to me that I improve my mathematics skills this year."), performance-approach goal orientation (five statements, e.g. "It's important to me that other students in my class think I am good at mathematics.") and performance-avoid goal orientation (four statements, e.g. "It's important to me that I don't look stupid."). All the sum variables were between highly reliable and acceptable (mastery pre α=.90, post α=.90; performance-approach pre α=.93, post α=.94; performance-avoid pre α=.73, post α=.84). Also, the sum variable of avoiding novelty (five statements, e.g. "I prefer to do work as I have always done it, rather than trying something new.") was reliable (pre α=.83, post α=.85).

Table 1: Descriptive statistics, personal sum variables

	Mean		**Median**		**St. Deviation**	
	Pre	Post	Pre	Post	Pre	Post
Mastery goal orientation	**3.61**	**3.13*** **	3.60	3.20	.852	.974
Performance-approach goal orientation	2.07	1.93	2.00	1.80	.977	.892
Performance-avoid goal orientation	**2.45**	**2.26***	2.5	2.25	.900	.945
Avoiding novelty	3.45	3.46	3.40	3.55	.861	.833
* Difference is fairly significant, $p<.05$ (2-tailed); *** Difference is highly significant, $p<.001$ (2-tailed)						

The descriptive statistics for personal achievement goal orientations and avoiding novelty, together with the significance tests, are presented in Table 1. The results show that pupils think that it is important to learn new skills and understand mathematics (pre: M=3.61, SD=.852; post: M=3.13, SD=.974). However, the decline in the mastery goal orientation over the intervention year is highly significant, remaining only slightly on the positive side ($t(49)=3.516$, $p<.001$). A closer look at the pupils' answers reveals that there are 19 % points increase on the proportion of pupils that disagree with the statements (25 % and 44 % respectfully, sum variable between 1–3). There is a moderate correlation between the pre- and post-test answers ($r(48)=.48$, $p<0.001$).

Performance-approach goal orientation show that it is not important to pupils to show that they are good at mathematics or that mathematics is easy for them (pre: M=2.07, SD=.977; post: M=1.93,

SD=.892). Their view does not change significantly during the intervention. There is a moderate correlation between the pre- and post-test answers ($r(48)=.51$, $p<0.001$).

Performance-avoid goal orientation reveal that pupils do not think it is important not to show lack of competence in mathematics lessons (pre: *M*=2.45, *SD*=.900; post: *M*=2.26, *SD*=.945). At the end of the intervention, their view is fairly significantly lower than in the beginning ($t(49)=2.098$, $p<.05$). There is a moderate correlation between the pre- and post-test answers ($r(48)=.56$, $p<0.001$). Furthermore, there is a strong correlation between the pupils' performance-approach goal orientation and performance-avoid goal orientation within the pre- and the post-tests (pre: $r(79)=.79$, post: $r(48)=.84$). That is, if the pupil does not worry about looking "smart" in mathematics class, they probably do not worry about looking "stupid" either.

Lastly, pupils want to avoid novelty, that is, stick to familiar concepts and class work (pre: *M*=3.45, *SD*=.861; post: *M*=3.46, *SD*=.833). There are no differences between pre- and post-tests, nor is there mentionable correlations.

Perception of classroom goal structures

Pupils' perception of classroom goal structures was studied through mastery goal structure (four statements, e.g. "In our class, how much you improve is really important."), performance-approach goal structure (three statements, e.g. "In our class, getting right answers is very important,") and performance-avoid goal structure (four statements, e.g. "In our class, it's very important not to look dumb."). The reliability of performance-approach and performance-avoid goal structures were between acceptable and highly reliable (performance-approach pre α=.78, post α=.84; performance-avoid pre α=.90, post α=.88). Interestingly, the reliability of mastery goal structure was not acceptable (pre α=.63, post α=.26), and will be studied through separate statements.

Table 2: Descriptive statistics, classroom sum variables

	Mean		**Median**		**St. Deviation**	
	Pre	Post	Pre	Post	Pre	Post
Performance-approach goal structure	**2.59**	**2.29***	2.67	2.00	.827	.950
Performance-avoid goal structure	2.17	2.07	2.00	2.00	.941	.807
* Difference is fairly significant, $p<.05$ (2-tailed)						

The descriptive statistics for classroom goal structures, together with the significance tests, are presented in Table 2. The results on performance-approach goal structure show that, on a classroom level, pupils' do not think that it is important to get the right answers or good grades in mathematics (pre: *M*=2.59, *SD*=.872; post: *M*=2.29, *SD*=.950). The negative development is fairly significant ($t(49)=2.406$, $p<.05$). There is no mentionable correlation between the pre- and post-tests. However, personal and classroom performance-approach structures correlate moderately within pre- and post-test (pre: $r(78)=.58$, post: $r(48)=.58$). Furthermore, the proportion of pupils disagreeing strongly with

the statements (sum variable between 1–2) increases by 24 % points between the pre- and post-tests (28 % and 52 % respectively). There are no differences in the pupils who agree with the statements.

The results on performance-avoid goal structure show that the pupils do not think it is important not to look bad in class or make mistakes in front of everyone (pre: M=2.17, SD=.941; post: M=2.07, SD=.807). There is no significant change or mentionable correlation between pre- and post-test. However, there is a strong correlation between performance-approach and performance-avoid goal structures within pre- and post-tests (pre: $r(78)$=.63, post: $r(48)$=.75), and personal and classroom performance-avoid structures within pre- and post-test (pre: $r(78)$=.83, post: $r(48)$=.68).

Since classroom mastery goal structure did not form an acceptable sum variable, the statements are studied separately. The results show that, on a classroom level, pupils have very neutral views on the importance of trying hard and understanding in mathematics (M=2.84, SD=.934, M=3.18, SD=.984). They do not think that improving in mathematics is important (M=2.73, SD=.976) but strongly agree that it is OK to make mistakes as long as learning is happening (M=4.04, SD=1.038). Moderate correlations are found in the pre-test between understanding and trying hard ($r(77)$=.50, p<.001), understanding and improving ($r(75)$=.44, p<.001) and trying hard and improving ($r(76)$=.60, p<.001). In the post-test, these correlations did not appear anymore.

Summary and discussion

The purpose of this paper was to study lower secondary pupils' mathematics-related affect during a 1-year CPS intervention. Mathematics-related affect was studied through pupils' personal and classroom goal structures (Midgley et al., 2000) as part of their academic motivation (Urdan & Kaplan, 2020). The results are examined in the beginning of the study (research question 1) and over the span of the intervention year (research question 2).

In the beginning, pupils reported having positive mastery goal orientation that has generally been associated with positive emotions and deep learning strategies (Urdan & Kaplan, 2020). Interestingly, trying hard and improving in mathematics were not considered to be that important on a classroom level, though making mistakes were acceptable. This implies to a very individualistic view of learning mathematics: *"It is important to me that I am improving, but I don't really care about the others"*.

In the beginning, pupils reported having negative performance-approach and performance-avoid goal structures both on a personal and classroom level, with a strong within test correlation. The message is clear: *"I don't mind looking smart or stupid in mathematics class, nor do I care if others do"*. In previous research, high performance-approach goals have been associated with high performance, and high performance-avoidance goals with negative emotions, disengagement, and low achievement (Urdan & Kaplan, 2020). The results imply again to a very individualistic view without high academic ambitions, but it is also looking away from negative emotions, disengagement, and low achievement.

The development during the intervention that was interrupted by the COVID-19 pandemic, can be characterised as mixed. On a task level (mastery goal orientation, Midgley et al., 2000), pupils' goals remain on the positive side even though showing highly significant negative development (cf. Metsämuuronen & Nousiainen, 2021). On a "self" level (performance-approach and performance-avoid goal orientations; Midgley et al., 2000), pupils cared less if someone was demonstrating

competence in class or if they themselves were demonstrating incompetence. From the collaborative point of view, this is good news: pupils do not have to stress over their appearance in mathematics lessons and can collaborate with their peers more freely.

In their article on the state of the art-research on achievement goal theory, Urdan and Kaplan (2020) call for research that integrate systems of perceptions, beliefs, goals, and affect. In previous papers, Viitala (2023, 2024) reported from the same intervention but looked at the development of pupils' mathematical views through pupils' view on their ability, effort, enjoyment, difficulty and success in mathematics, and the views' connections to pupils' performance level. The results showed negative development trends (Viitala, 2023) similarly as the development on pupils' personal mastery goal orientations in this study. However, a closer look revealed that the development was mainly due to one performance group: low average performers (Viitala, 2024). For further study, it would be interesting to see if pupils' different performance levels differ also in respect to academic motivation.

This study has many limitations such as low number of participants and a lack of control groups. It is also impossible to differentiate the effect of the intervention and the pandemic from the results. Hence, more research in the form of a new research cohort is needed to understand better the effects of daily collaborative problem solving on pupils' thinking, learning and engagement in mathematics.

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Mindset and Mathematics Achievement: Mediating Effects of Anxiety and Learning Motivation

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Keywords: Growth mindset, mathematics achievement, mathematics anxiety, mathematics learning motivation

Introduction

Students with a growth mindset tend to achieve superior academic outcomes, including higher academic performance. While existing research has largely focused on the influence of mindset within specific countries or regions, there is a notable gap in understanding how these mindsets vary across different cultural contexts.

Methods

The objective was to evaluate the impact of mindset on mathematics achievement and to uncover subtle differences in the effects of mindset across diverse cultural, educational and economic contexts, as well as investigate the underlying reasons for these variations using data from the PISA2022 assessment, with a specific focus on contrasting East Asian countries/economies with OECD countries/economies (excluding Japan and Korea). This study employs the structural equation model established by Dong et al. (2023).

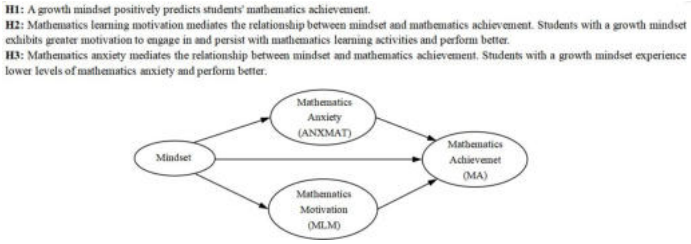

Figure 1 Hypothetical Model

A structural equation model was conducted under the hypothetical model for data analysis through Mplus8.3 and SPSS26 with mathematics achievement as dependent variables. Students' mindset was measured by their levels of agreement with the question on a scale from "1-strongly disagree" to "4-strongly agree", the students who show disagreement hold a growth mindset and who show agreement hold a fixed mindset.

Findings

For East Asian countries/economies, the results indicate that ANXMAT is negatively correlated with MA for both advantaged and disadvantaged students. In contrast, MLM was found to have a significant positive correlation with MA. While a growth mindset(GM) was significantly and negatively correlated with MA among disadvantaged students, no significant correlation was found

among advantaged students. This outcome is inconsistent with H1, which suggests that the benefits of GM may not be equally applicable across different socioeconomic backgrounds.

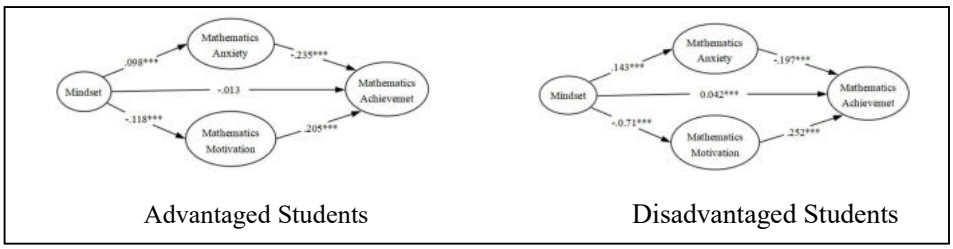

Figure 2 SEM-East Asia

As evidenced in Figure 2, a notable positive indirect effect of GM on MA is observed among the disadvantaged. Similarly, the results for advantaged students corroborate H2 and H3.

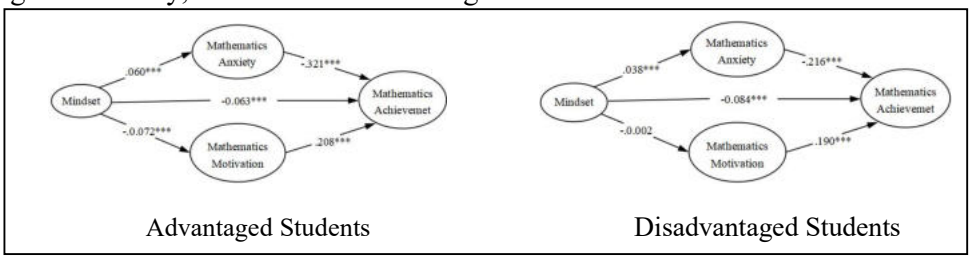

Figure 3 SEM-OECD

For OECD countries/economies, MA is positively correlated with GM for both groups, which is consistent with H1. MLM demonstrated a significant positive correlation with MA for both groups while ANXMAT exhibited a negative correlation with MA for both groups.

Concerning indirect effects, ANXMAT functions as a pivotal mediator in the relationship between mindset and MA for students from disadvantaged backgrounds. In contrast, the mediating role of MLM was found to be non-significant, which contradicts H3. For students from advantaged backgrounds, both ANXMAT and MLM were identified as significant mediators, a positive indirect effect of GM on MA. This finding lends support to H2 and H3.

Discussion and Conclusion

The findings reveal significant regional variations in the influence of mindset on academic performance. The contrasting trend in East Asia and OECD indicates that the relationship between mindset and academic achievement may be associated with culture. The educational environment in East Asia emphasizes hard work rather than strategic thinking, fostering a "false growth mindset" centered on effort rather than on intelligence malleability.

Furthermore, discrepancies in educational policies and the distribution of resources are of paramount importance. OECD countries emphasize educational equity, and superior educational standards. Conversely, East Asia grapples with an inequitable distribution of educational resources, giving rise to discrepancies in both the quality of education and student performance. In this context, even if disadvantaged students develop a growth mindset, their educational system lacks support for their development, potentially increasing anxiety and negatively impacting academic performance.

Theoretical framework linking teacher feedback and students' mathematics self-efficacy, attributions, and mathematics achievement

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Keywords: Self-efficacy, attribution theory, mathematics achievement, feedback.

Background

As shown in the review by Street et al. (2024), the relationship between students' mathematics self-efficacy and their mathematics achievement is well-documented. However, a significant gap remains in our understanding of how students appraise information from various sources to calibrate their self-efficacy. This systematic review will focus on teachers' feedback practices as a primary mechanism influencing students' appraisal of self-efficacy information, with a particular focus on students' attributions as a potential mediating factor. In doing so, this review aims to answer the following research question: What are the relationships between teacher feedback, students' self-efficacy, and students' causal attributions of their performance in the domain of mathematics?

Self-efficacy is an individual's belief in one's capabilities to organize and execute the courses of action required to manage prospective situations (Bandura, 1997, p. 3). Particularly in mathematics, strong self-efficacy empowers students to tackle complex problems, persevere through challenges, and ultimately grasp concepts more deeply (e.g., Martin et al., 2015; Pajares & Graham, 1999). Bandura conceptualizes self-efficacy as a dynamic construct that is continuously updated through self-efficacy information from four key sources: mastery experiences, vicarious experiences, social persuasion, and physiological states. Importantly, the cognitive appraisal of information from these various experiences is what shapes one's self-efficacy. Research has shown that effective feedback, characterized by being specific, timely, and growth-oriented, can significantly impact students' beliefs in their abilities (Shute, 2008; Wiliam, 2011). Conversely, negative feedback can diminish self-efficacy and confidence. Additionally, attributing success to effort and effective strategies, rather than innate ability, reinforces a growth mindset—the belief that performance can be improved through perseverance and dedicated practice (Graham & Williams, 2009).

Method

This systematic review adheres to PRISMA 2020 guidelines (Page et al., 2021), ensuring a rigorous and transparent approach. Two comprehensive searches were conducted across five academic databases—Academic Search Elite, ERIC, PsycINFO, Scopus, and Web of Science—on November 12, 2024, using search terms that captured a broad scope of research, accounting for synonyms, related concepts, and terminology variations. The searches were limited to peer-reviewed empirical reports published in the English language. The search results were screened through two stages. The initial abstract screening included studies that examined relationships between at least two of the three constructs in the domain of mathematics and focused on lower secondary school students. The second full-text screening focused on the conceptualisation of (1) self-efficacy as future-oriented

statements reflecting specific contexts and goal-referenced confidence in one's ability; (2) attributions as domain-specific causal ascriptions; and (3) feedback as contextualized information (e.g., on task, process, self-regulation, attributions) provided by teachers, either orally or in writing. A risk of bias assessment for experimental studies was conducted using the Education Endowment Foundation's Padlock tool (Education Endowment Foundation, 2019). Search #1, which included all three constructs combined with the 'AND' operator in the title field, yielded 156 unique records across the five databases. Search #2, which included pairwise 'AND' combinations in the title field under the same search restrictions, yielded an additional 398 unique records. The two-stage screening yielded a final selection of two qualitative, six observational, and 10 experimental studies.

Findings

The synthesis of the selected studies resulted in two theoretical models—one based on observational studies and the other on experimental studies. Both models included strong associations between teacher feedback and students' self-efficacy, as well as a bidirectional relationship between students' attributions and self-efficacy. The experimental model further identified causal links between teacher feedback and both attributions and self-efficacy. Additionally, both models delineated a pathway in which teacher feedback influenced students' attributions and self-efficacy, shaping their cognition and learning behaviours and, ultimately, their mathematics performance.

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Emotion graphing: Tool to study emotions and foster meta-affect

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Keywords: Emotions, graphing, affect.

Emotions may be defined as "rapidly-changing states of feeling experienced consciously or occurring preconsciously or unconsciously" (DeBellis & Goldin, 2006, pp. 135). The quick, rapidly changing nature of emotions also makes this construct difficult to measure. Understanding emotions is vital however because they occur in the moment and thus are malleable – by teacher, environment, etc. - and so are a way to change deeper-seated affect (Middleton et al., 2017). Attitudes may also be formed from repeated habitual emotions (Grootenboer & Marshman, 2016).

I present a tool called *emotion graphing,* where students are asked to draw a graph of their self-perceived emotions during an experience. I explore the affordances of emotion graphing for studying participants' emotions and fostering a larger theorized structure of *meta-affect*, "the individual's monitoring of affect" (DeBellis & Goldin, 2006, pp. 136). By looking across studies of emotion graphing from undergraduates learning how to prove mathematics to middle school students in a geometry camp, emotion graphing serves a vehicle for more effective communication during interviews as well as a form of experience-based sampling, with pedagogical use as exit tickets.

Methods: Emotion Graphing as an Instrument

Pictorial methods can capture shifts in affect, through drawing maps or pictures (Gómez-Chacón, 2000). Emotion graphs consist of an x-axis representing time and y-axis representing valence of emotion (positive and negative). Participants retrospectively sketch how their emotions changed in intensity over an experience, here a proof construction task (Figure 1). Participants then marked on the horizontal axis and/or annotated their graph with short captions at inflection points where they felt emotions shift. We explore the affordances of emotion graphs as used in two populations: The first study with N=12 undergraduates in a course focused on learning how to prove, over a series of interviews. The second study regards N=35 middle school students (ages 11-14) in a geometry summer camp.

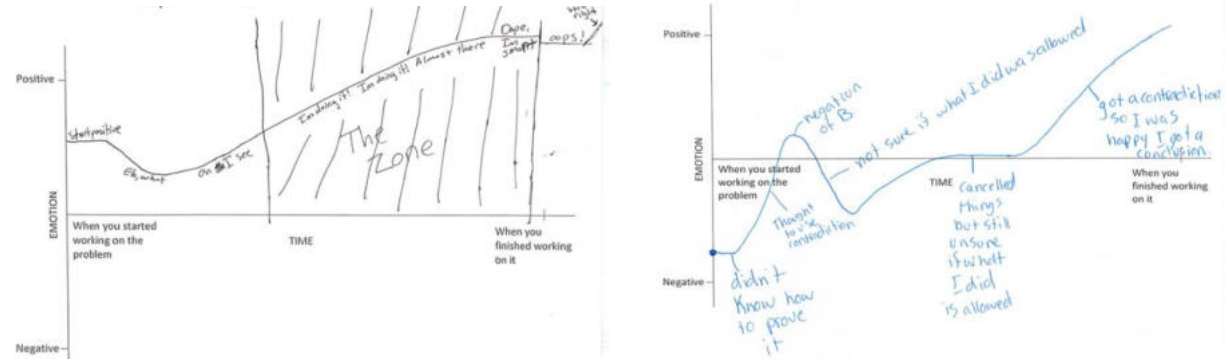

Figure 1: Example emotion graphs from undergraduates working on proof constructions

Discussion

In Satyam et al. (2022), we analyzed three studies with undergraduate students graphing affect and found four clear benefits: (1) temporal ordering of recollections, (2) explicit representation of change over time, (3) reduced dependence on verbal data, and (4) the creation of visual objects (the graph) for discussion. Emotion graphs can be used to describe variations in students' emotions ranging from years to single problem-solving episodes (e.g., Andrà & Scalvini, 2013).

Emotion graphing has shown promise with middle school students as well. In a recent study conducted during a geometry summer camp, we used experience sampling methods to study students' emotional engagement. After every math activity, N=35 students filled out a lesson exit ticket, which included a blank emotion graph and a suggested emotion word bank for them to label their graph with corresponding emotions. In total, we collected approximately 455 emotion graphs over the span of one week. Data analysis is underway, and we anticipate insights about the potential of emotion graphing with middle school students. In a post-survey, students reported emotion graphing helped them "realize where I was confused", "express about the topic", "if I didn't like it I had hope it would change", and "I liked you hearing my opinion."

In summary, an emotion graph tool serves as an instrument for assessing students' recollections of their emotions. This work advances research on emotions by providing students a process through which they can track, monitor, and reflect on their emotions. Through this process, students can identify barriers, strategize changes, and improve their own mathematical learning experience.

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TWG 09: Interaction, Communication and Language in Mathematics Education

Interaction, communication and language(s) in mathematics education: The work of TWG09

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In this paper, we report on the work of TWG09 at CERME 14 while exploring a central tension in our field: whether it is possible to attend simultaneously to the social and cultural dimensions of language and communication and to the mathematical content in focus. TWG09 brings a diverse set of perspectives, methodologies and foci to consider the interplay between mathematics, language and culture. Mathematics is taught and learnt through interaction, using communication and language, and as such is a cultural, social and linguistic activity. The meaning mathematics has for us is shaped by and through communication, and this communication is fundamental to deepening mathematical understandings. Furthermore, ignoring the social and cultural dimensions creates and perpetuates inequities. The presentations within TWG09 illustrate, reveal and challenge this central tension with important implications for the teaching and learning of mathematics at all levels.

Keywords: Language, interaction, communication, discourse.

Introduction

TWG09 brings a diverse set of perspectives, methodologies and foci together to address questions around interaction, communication and language(s) in mathematics education. The growth and maturity of research with such a focus is reflected in the diverse theoretical and methodological approaches within this TWG, but also in the range of special issues and edited books dedicated to this area of research. Accordingly, the presentations at the conference provoked deep discussions around a range of mathematical topics (e.g., angles, equivalent fractions, zero), and mathematical linguistic practices (e.g., explaining, naming, responding), drawing on a range of theoretical and methodological perspectives. This paper introduces and reflects on the diversity of perspectives and insights shared within TWG09 at CERME14.

Can we pay attention to the social and cultural dimensions of language and communication at the same time as the mathematical content in focus?

At CERME14, TWG09 dedicated a session to an Oxford-style debate exploring a central tension in the field: whether it is possible to attend simultaneously to the social and cultural dimensions of language and communication and to the mathematical content in focus. Two members argued in favour of this possibility, while two opposed it. The debate highlighted many aspects of our research focusing on interaction, communication and language, and the tensions that arise in our research designs, analysis and communication of findings to peers, teachers and beyond. We have organised

this introductory paper around the key points that arose during this debate, highlighting how these issues were more widely discussed and debated in the presentations and discussions in the group.

The Interplay of Mathematics, Language, and Culture

Mathematics is both taught and learned through language and social interactions. Mathematical knowledge is socially constructed and mediated through language, communication, and cultural context. If learning mathematics is fundamentally a process of meaning-making, then it can be argued that attending to both language and social and cultural dimensions is not a distraction from content learning but rather an essential component of conceptual development. Yet the question posed focuses on whether these aspects can be attended to *at the same time*. It is not an argument about whether mathematics should be taught without acknowledging these dimensions, which would ignore the fundamental role of communication and cultural context in meaning-making. This is exemplified in much of the work focused on multilingual classrooms, where more than one language is used in mathematics classroom talk (Gregurević & Fetzer, 2025; Revina & Schüler-Meyer, 2025), including presentations at this conference on the interaction between Maltese and English in learning equivalent fractions (Farrugia, 2025), between German and English in learning 0 (Bürgstein & Söbbeke, 2025), and between German and Croatian in learning about doubling and halving (Deegener et al., 2025). Yet much of our research on this interplay foregrounds language and communication and perhaps neglects, to some extent, the mathematical content. This may partly be a consequence of the constraints in reporting the results of any analysis (e.g., length restrictions, journal guidance, reviewer feedback). We are seeing more research that addresses this concern, and this is exemplified by research that focuses on how language influences the meaning developed for key mathematical concepts, such as negative numbers in the paper by Fabian and Kortenkamp (2025) and angles in the paper by Kimber et al. (2025).

Mathematics as a Cultural, Social and Linguistic Activity

Mathematics is not an isolated, culture-free discipline. Mathematical concepts and practices are socially constructed and vary across cultures. Mathematics has often been framed as a universal language that transcends culture. The field of ethnomathematics has demonstrated that different cultures develop distinct numerical systems, measurement techniques, and spatial reasoning frameworks. While there are six universal mathematical activities: counting, locating, measuring, designing, playing, and explaining; how these are expressed varies culturally (Bishop, 1991). Other mathematical activities can vary considerably in different cultural contexts (Barton, 2008). Mathematical reasoning develops within cultural contexts, and acknowledging these differences enriches learning rather than distracting from it. Different languages stress (and ignore) different aspects of mathematical objects and activities in their words for the objects.

The challenge of integrating social, linguistic and mathematical dimensions is reflected in our work developing analytical tools to examine the interplay of these different dimensions. Much of the research in this area focuses on sociomathematical norms (Yackel & Cobb, 1996), and teacher or student talk moves (Erath et al., 2021; Gustafsson, 2024). Aumann and Erath (2025) presented four categories of student initiations which influence the norms of interaction within which they arise. Both Bessenyei (2025) and Julsgård (2025) focused on teacher prompts, with Bessenyei examining

group problem-solving and Julsgård examining one-to-one tutoring sessions. Drageset and Skoglund (2025) focused on how an expert teacher positioned their students in the learning process, for example as thinkers, explainers, and listeners. These ideas are also explored by Jonker et al. (2025) in peer-to-peer dialogue. Finally, Revina and Schüler-Meyer (2025) considered teacher moves in response to students' multilingual contributions during a lesson on solving quadratic equations. Each of these studies illustrates the different aspects of learning mathematics in classroom interaction. The framework presented by Kimber et al. (2025) brings together many of these dimensions, with the aim of making the connections between these different aspects.

Language and Communication Deepen Mathematical Understanding

Mathematics is about thinking, reasoning, and making sense of problems. Language is the medium through which this happens. Language plays a fundamental role in mathematical reasoning and problem-solving. Mathematics is often described as a specialised discourse (Halliday, 1978; Schleppegrell, 2007) characterised by unique syntactic structures and semiotic representations that differ from everyday communication. Various theories from sociology, psychology, and linguistics underpin much of the research within TWG09, where learning is seen as occurring through social interaction and is mediated by language. Mathematical concepts become internalised as students engage in discussions and verbal reasoning. Another key idea underpinning much of the research in this group is that of Halliday (1978) who describes mathematics as a "register of language" with its own specialised vocabulary and grammar. Students must learn to navigate formal mathematical discourse alongside everyday language, as was exemplified by the presentations described above, illustrating the language associated with particular mathematics concepts in different named languages (Bürgstein & Söbbeke, 2025; Deegener et al., 2025; Farrugia, 2025). This issue is also key to learning through one language. Lomas and Wessel (2025) illustrated this in their analysis of students learning the scalar product, while Mähnert and Erath (2025) illustrated this with examples from design research focused on fraction division. Meaney et al. (2025) also explored this in their work with pre-service teachers focusing on how algebraic variables and constants are described.

Interestingly, the growing use of generative AI, such as large-language models (LLMs) or chain of reasoning models, not only gives us further insight into the relationship between language and mathematics but can also play a role in the teaching and learning of mathematics. Eckhardt and Thurm (2025) used an LLM to prompt students' mathematical explanations using ideas from research on prompting and dialogue in mathematics teaching, while Gíslason (2025) asked preservice teachers to 'teach' the LLM how to solve a mathematical logic problem, which required them to craft prompts that directed the AI to the key mathematical ideas. Interactions with these AI models highlight how reasonable and predictable many of the mistakes or misconceptions made by students are.

Students construct mathematical meaning through participation in structured classroom interactions. Ignoring the role of interaction in mathematical thinking undermines students' abilities to articulate, justify, and refine their reasoning. Focusing on the early years, Henschen and Teschner (2025) showed how spontaneous language used in interactions can reflect emerging mathematical understanding, while Carlén et al. (2025) focused on the shift in textbooks from informal language and visuals to more formal mathematical symbols and routines. Both emphasised the importance of

exploratory and informal language in learning. Yet we are still learning more about how language and interaction influence the learning of particular concepts.

Ignoring Social and Cultural Dimensions Creates Inequities

A mathematics curriculum (enacted or prescribed) that ignores linguistic and cultural diversity privileges certain students while disadvantaging others, e.g., multilingual learners, learners with additional learning needs. There is now substantial research showing that multilingual students benefit from leveraging their full linguistic repertoire in problem-solving (Planas & Chronaki, 2021). Many students engage more deeply when mathematics is connected to their lived experiences. Furthermore, mathematics is not just about the key ideas underpinning it—it is also about thinking, reasoning, and problem-solving, all of which occur through language and within cultural contexts. By embracing communication and cultural diversity, we make mathematics more engaging, inclusive, and accessible for all learners (e.g., Lenz et al., 2024; Prediger et al., 2022). Indeed, ignoring social and cultural dimensions creates inequities in mathematical learning, as without attention to these factors, many students are disadvantaged, leading to unequal learning outcomes. One example of this was presented through Ebel et al.'s (2025) analysis of mathematics school-leaving examinations, revealing the language barriers associated with the different tasks included, with consequences for students in these high-stakes assessments. Another example was presented by Ebbelind and Ivanov (2025), who analysed a spatial reasoning activity for young children, and how language mediated the diagnostic purpose of the activity.

These presentations highlighted how the integration of language and mathematics must be done with careful consideration of cognitive demands, ensuring that language and culture support rather than hinder mathematical reasoning. They also illustrate the consequences when certain students are disadvantaged through the linguistic and mathematical opportunities or barriers they experience.

Implications for teaching and learning

Teaching is complex. Learning is complex. One argument for focusing on the social and cultural dimensions of communication sequentially and iteratively with a focus on the mathematical content, is that this demarcation can support both teaching and learning. The complexity of classroom interaction is widely acknowledged, and one of the teacher's roles is to support students to navigate this complexity. This is reflected in the large growth of research focused on teacher development in this area (e.g., Kasari & Meaney, 2024; Planas et al., 2024), and in relation to how we, as researchers, listen to and respect the expertise of teachers (e.g., Drageset & Ell, 2024; Planas et al., 2025). Several studies in the working group focused on teacher education. Rave-Agudelo and Planas (2025) and Kimber et al. (2025) illustrated the complexity of communicating in the context of professional development focused on angles, considering too the role of gestures in that communication, and Revina and Schüler-Meyer (2025) examined the possibilities for teachers to take up students' multilingual ideas to create learning opportunities.

Conclusion

Ultimately, the question is not whether we can pay attention to linguistic, social and cultural dimensions in mathematics education—it is how we can do so effectively, critically, and equitably.

The research presented in TWG09 at CERME14 not only shows how lively this field of research is, but also how we, as a group, push the boundaries for new ideas for language inclusion, language- and culturally-responsive teaching, and for truly multilingual mathematics classrooms with equal opportunities for all. Accordingly, the productive discussions and constructive exchanges in the group also show that the participants in TWG09 have a great interest in familiarising themselves with the various presented studies, identifying new links and building new collaborations on this basis.

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Characterisation of students' initiatives in whole class discussions

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From an interactionist perspective, learning mathematics at school is primarily about participation in the classroom discourse, which is mainly represented by public classroom discussions. The participation can be of a very different nature. In this paper, we focus on students' contributions that arise without any compulsion from the teacher, the so-called students' initiatives. We argue that these initiatives can make a significant contribution to explaining hidden, implicit expectations and norms. For this purpose, we present four categories of students' initiatives we found in the video recordings of 18 mathematics lessons from a classroom at the beginning of the secondary school. The categorisation of the initiatives of five selected pupils shows different patterns of contributing in the form of initiatives and we see how initiatives from the various categories lead in practice to different expectations and norms being explicated to a certain extent.

Keywords: Communication, interaction, classroom discourse, norms.

Introduction

Participating in mathematics classrooms is a crucial part of learning mathematics in school (e.g. Sfard, 2008). The ways in which students participate can vary and do not necessarily have to be characterised by active speech acts (Krummheuer & Brandt, 2001). Nonetheless, active contributions play a special role, as they bring the students' thoughts into the classroom discourse. While in teacher-dominated classrooms the IRE scheme (Mehan, 1979) is very present, we want to focus on those student contributions which are not a direct response to an initiation from the teacher, but the students initiate themselves. These student initiatives represent a special form of participation, as they break the discursive flow (Drageset, 2015) and introduce new topics or aspects into the classroom discourse. An examination of these special types of contributions therefore appears promising for a deeper analysis of students' participation in mathematics classrooms.

Theoretical background

The importance of participation for learning mathematics has already been emphasised in many studies (e.g. Cobb & Bauersfeld, 1995; Sfard, 2008). From an interactionist point of view, individuals construct their own reality through interaction, based on the meaning that objects or situations have for them (Blumer, 1986; Garfinkel, 1967). In these interactions, the interactants display each other their individual interpretation of objects or situations which may lead to an interpretation, that is regarded as shared (Krummheuer, 1983). This "shared" interpretation is sufficiently suitable, so that the interactants can go on with their interaction, which marks the starting point for successful learning processes, since learning, seen as learning-as-participation (Sfard, 2008), takes part in interaction. Such collective negotiations of meanings in the classroom mostly take place in public class discussions. The defining aspect of these whole class discussions is, that they (at least in theory) are more or less accessible for everyone, although the participation in these polyadic interactions can

vary from hearer or recipient to speaker and many other roles, which may also change over time (Krummheuer & Brandt, 2001). Participation in whole class discussions is therefore essential for learning success, whereby active involvement in the form of verbal contributions plays a prominent role, as the word is the realisation of the thought (Vygotskij et al., 1962) and therefore represents a window to one's cognition, a publication of one's individual interpretation of an object or a situation.

By acknowledging the importance of students' contributions in the whole class discussions for the investigation of students' participation, the various ways open up to us, in which they can be shaped. Drageset (2015) differentiates between five superordinate categories of student comments: explanations, partial answers, teacher-led responses, unexplained answers and student initiatives. The first four categories can be seen as categories for utterances that are made as responses within student-teacher interactions and associated practices. The last one, student initiatives, on the other hand, describes comments that are more proactive in nature, as students "break the communicative flow" (Drageset, 2015, p. 35) to contribute own intentions or content. This kind of student's utterances seems therefore very interesting, as, again speaking with Vygotskij et al. (1962), the initiative represents the realisation of the student's thought, which is not forced by questions from the teacher but has developed through the interaction without any social compulsion to contribute anything. Since successful, appropriate participation in (mathematics) classrooms is mostly a question of fulfilling the teacher's expectations respectively following the norms (e.g. Aumann & Erath, 2024; Garfinkel, 1967; Yackel & Cobb, 1996), initiatives are a particularly delicate form of student utterances, because, firstly, there is no direct expectation of the teacher as there is no open question, and, secondly, because initiatives make it possible to make previously implicit aspects explicit and therefore can help to identify (normative) expectations (Aumann & Erath, 2024) or "hidden rules" (Jablonka, 2008) from the teacher. Hence, an investigation of student initiatives seems promising.

Our framework for the examination of student initiatives is based on the presented definition from Drageset (2015). We define students' initiatives as a break in the communicative flow, where students contribute their own intentions or content. This means, that there is no compulsion for the students to say anything, but they do it from an intrinsic motivation, making the initiative itself proactive. According to the distinction between social and sociomathematical norms, a distinction between utterances regarding (social) behaviour and actions and utterances regarding (mathematical) objects and actions seems close to hand. This differentiation between action-related and object-related utterances allows us to categorise the initiatives on a content level and draws parallels with the terms "regulative" and "instructional discourse" (Bernstein, 1996). In addition to this distinction, we want to differentiate concerning the referential context or the discursive orientation of the initiative. While Sfard (2008, p. 58) introduced the terms "proactive" and "reactive" for the categorisation of meta-discursive intentions, we want to stay on a content level. Although, as stated above, all initiatives are proactive by definition regarding their discursive function, they can be proactive or reactive on a content level, as it depends on the referential context and the kind of contribution they make. To delimit it more clearly, we want to distinguish the direction of the initiatives between backward-looking and forward-looking initiatives. The defining characteristic of the first is, that in this case no new own (mathematical) content is contributed, and the contextual reference is clearly set in the past which makes it more repetitive or reactive. In contrast, forward-looking initiatives introduce new

(mathematical) topics or aspects with a focus on the future, which makes them contributing and proactive on a content level.

The questions therefore arise as to what extent the categories presented are useful for describing student initiatives and how they can help to identify expectations in the classroom.

Methods

The data used in this study was obtained from a video study conducted in a fifth-grade classroom at a German upper-secondary school (German: "Gymnasium") in a medium-sized city. The first four weeks of regular mathematics lessons after the transition to secondary school were video recorded, resulting in a data corpus comprising 18 lessons of video material, which were all recorded in their entirety.

For further research, we chose five pupils out of the class, namely Adem, Elijah, Lukas, Merle and Maja. Two of them were also part of the previous video study on the last lessons of the fourth grade in primary school, three other children intuitively caught our attention during the study due to their very different participation patterns in the classroom. Firstly, we reduced the data material by marking every public utterance from the five pupils. These utterances, together with the preceding and following turns, each represent situations of public participation of one of the five students. Of these situations, those that corresponded to our definition of students' initiatives were then selected and, as the categories presented in the previous chapter form a categorial system, sorted using qualitative content analysis based on video-graphed data (Dinkelaker, 2016; Kuckartz, 2018).

Empirical insights

In the following sections, we want to give insight into our categorisation. The distinction between object-related and action-related utterances in one dimension and forward-looking and backward-looking utterances in a second dimension allows the utterance to be categorised in a 2 by 2 table. We therefore present a prime example for each of the four cases. For clarity, each transcript is numbered starting at 1, but the situations are from different lessons at different times.

Action-oriented and backward-looking

In the second lesson, the pupils counted their pets and documented them on a tally sheet. Afterwards, the teacher introduced the bar chart as a form of representation for data. At the end of this lesson, this short sequence took part.

1	Oskar:	What should we write into the bar chart? The pets again?
2	Teacher:	Yes, please note as homework: bar chart and in brackets: pets. Then you know it. [Adem points up] Adem?
3	Adem:	So, should we now turn this table into a bar chart?
4	Teacher:	[points the thumb-up to Adem] Yes, correct.

In line #1 Oskar asks, what he or she has to do. The question is very precise and is about, what data has to be "written" into the bar chart. He directly asks if it's still about the pets, that they talked about earlier in this lesson. The teacher confirms and summarises, that as homework they should make a bar chart and that they should add "pets" so that the pupils don't forget what they have to do (line #2). Now Adem points up, signalling that he wants to make an initiative, as no question is asked.

After getting called up, he summarises and asks in his own words whether what he has understood as homework is what needs to be done. The teacher confirms orally and gesturally. In this short sequence, we can identify Adem's initiative as an *action-oriented* question, as his questions do not aim at the mathematical object of a bar chart, but at the confirmation of his understanding of the homework given by the teacher. This question is by no means unnecessary, as it clarifies for him what he has to do to complete the task in an accepted manner. But in his question, he is also *backward-looking*, as he refers to something already said and does not contribute any new content of his own.

Action-oriented and forward-looking

An example of an also action-oriented but *forward-looking* initiative is the following. In the second lesson, the teacher starts with a so-called "daily exercise". After the correction, the pupils tell the teacher, how many points they achieved. Directly after that, Maja points up.

1	Maja:	[points up]
2	Teacher:	You have a question, Maja?
3	Maja:	Are we allowed to write in notebooks in mathematics?
4	Teacher:	No, because I will give you worksheets from time to time and you may also have to staple out a sheet and hand it to me

The teacher's question in #2, if Maja wants to ask something makes it clear one more time, that Maja's raised hand and the associated wish to make a contribution are to be seen as an initiative, as it seems, that there is no open question, that could motivate Maja's hand raising. Maja's question in #3 confirms this impression, as she asks, whether they are allowed to write in notebooks (instead of folders). She doesn't refer to something said before this sequence but opens a new topic. This new topic is not really about a mathematical object, it is about, how to work in mathematics classrooms properly. Additionally, this new topic seems not to be really relevant for the moment, as there was no prompt to write something in their folders, but it has high relevance for the future, as it makes it explicit, what the teacher expects from the pupils to do, respectively how to work in mathematics classroom. The initiative is therefore *forward-looking* and *action-oriented.*

Object-oriented and forward-looking

In the sixth lesson, the class works on several exercises to determine predecessors and successors of natural numbers. In one task, the number given is zero. After a short discussion of what the predecessor of zero could be, the following sequence took part.

1	Teacher:	So, the correct answer is to draw a dash at this point. There is no predecessor because we are in the number system of natural numbers. We have now justified that, we go on. Task j is still missing. Where did we leave off? With Merle.
2	Merle:	908, 909 and 910. I wrote "not solvable" in the previous task, is that also possible?
3	Teacher:	You can do that too, exactly.

In this short sequence, it is interesting, that Merle gets the word from the teacher to present her solution to the Task j in #1. She does it and adds directly after that her initiative regarding the discussion just finished (#2). That could indicate, that for her the outcome of the discussion was not totally satisfying. However, she presents an alternative solution to the dash, which the teacher highlights as the correct answer. She therefore contributes an own new suggestion and, by asking for

the validity, helps to make alternative possible solutions explicit, which makes her initiative *forward-looking.* The teacher accepts her initiative and confirms the solution as correct (#3). It gets clear, that it is not so much about the way she wrote it, which could make it action-oriented, but more about the object of the denomination of the predecessor of zero, which makes it *object-oriented.*

Object-oriented and backward-looking

This case was the most uncommon in our video study. In the eighth lesson, the power notation is introduced. On the blackboard, there is a table with "big numbers" notated digit for digit and the corresponding power notations. Additionally, there is a number written in power notation, but not sorted by the powers.

1	Teacher:	So, the next one going to the board has to sort it in his head- do you have a question? *[leans over and looks to Lukas]*
2	Lukas:	It says … in the lower line, two times ten plus eight plus nine … nine times ten?
3		*[indistinct chatter]*
4	Teacher:	So, we just found, that I didn't sort it, what must not happen is, that one exponent occurs twice. Let's have a look. *[points on the exponents one by one]* We have the eight, the five, the six, the seven, the four, of what do we have to think here when there is nothing written, Lukas? That bothered you a bit.
5	Lukas:	The three?
6	Teacher:	No, which exponent? Eight times ten to the power of what equals eight? That's how you need to think.

After there is no answer from Lukas and another pupil gives the correct answer to the question in #6, the teacher directly goes on with the next task. Again, the question from Lukas in #2 breaks the interactional flow as it interrupts the explanation from the teacher (#1). Therefore, Lukas seems to have important reasons, why he wants to make a contribution. When given the word, he doesn't refer to the teacher's explanation but starts to bring something else, a number written on the blackboard, into the focus (line #2). He starts to read what is written and refers to the "lower line" (#2), which sets the focus not on the entire term, but on special parts of it. Obviously, he has some problems with it as he makes some pauses and does not formulate a proper question. That gives the impression, that he himself is not quite sure what problem he has with the term, but something is bothering him. Here we can find an indicator of the *backward-looking* character of his initiative. The teacher then starts with an explanation of probably what she thinks his problem could be. With lines #4, #5 and #6 it seems likely, that Lukas was not able to comprehend the term 10^0, respectively the translation process there, if no power of ten is written behind a number. That makes it clear, that his initiative aims at the object level of this term, which makes it an example of an *object-oriented* contribution.

Discussion

As we can see, we are able to classify contributions in the form of initiatives in the four categories presented above. However, the mere categorisation of individual utterances is not expedient here, as there may be good reasons for attempting each category and we do not want to take the liberty of assigning a value here. As stated above, student initiatives offer the possibility to make implicit norms or requirements explicit. By categorising the students' initiatives, we can see what type they are, if they address object-related or action-related discourse and whether they are more proactive or

reactive. Figure 1 shows the frequencies of the students' initiatives from the five chosen pupils in the first 18 lessons for each of the presented categories and the respective total numbers of initiatives.

Table 1: students' initiatives from Adem, Elijah, Lukas, Merle and Maja

	Obj.	Act.
F.	2/15	3/15
B.	1/15	9/15

Adem

	Obj.	Act.
F.	2/4	0/4
B.	0/4	2/4

Elijah

	Obj.	Act.
F.	2/10	6/10
B.	1/10	1/10

Lukas

	Obj.	Act.
F.	5/8	2/8
B.	0/8	0/8

Merle

	Obj.	Act.
F.	1/5	4/5
B.	0/5	0/5

Maja

Table 1 highlights that there are large variations in the total number of initiatives in the observed first lessons in the new school. While Adem started 15 initiatives followed by Lukas with 10, Elijah and Maja have not even started their initiatives half as often. These figures should be treated with caution due to the statistically not particularly reliable scope, but we can still recognise clear trends. The variation in the total numbers could be an indicator that Adem and Lukas see more need for clarification in class, or that they are more socialised to bring their own thoughts into the classroom discourse more often. However, an interesting aspect here is that Lukas and Maja were in the same primary school class, so they were socialised within the same mathematics classrooms and yet Lukas initiates twice as often as Maja. Yet, more interesting than the total numbers is the respective distribution regarding the direction and the type of discourse addressed. When looking at Adem's initiatives, we can see clearly a trend toward backward-looking and action-oriented initiatives. In the presented prime example, we have seen that this is mostly a repetitive question about what was said what is to do. This makes it likely that it may often not be possible for him to follow the instructions in such a way that he knows what to do, which is why such an initiative is necessary. With this kind of initiative, he makes the expectations of the teacher explicit (Aumann & Erath, 2024) and is therefore able to fulfil them, which is a worthwhile goal. At the same time, it may show, that Adem is not as successful as the other pupils to understand (more or less) "hidden rules" (Jablonka, 2008) and therefore this action-oriented discourse seems restricted to him (Bernstein, 1996).

Elijah's initiatives are, as already mentioned, significantly less frequent than Adem's. Again, the numbers are not too reliable, but we can recognise indications of patterns consisting of an accumulation of initiatives in the categories object-oriented/forward-looking and action-oriented/backward-looking. The second case might be, as with Adem, a sign of missing understanding of what was said what is to do, although the trend is not as strong as with Adem. Meanwhile, the object-oriented and forward-looking initiatives show contributing character, as here new (own) content is suggested and brought into the classroom discourse, which can become the object of negotiation processes and can lead to a common interpretation of this content and its meaning (Krummheuer, 1983). For the process of learning, this kind of initiative seems to be very substantial.

Meanwhile, Lukas' and Maja's initiatives show a trend towards a forward-looking action-orientation. The situations are categorised in this way when the type of discourse is about the (social) behaviour but there is no backward reference to something already said. Therefore, they can be interpreted as a

preliminary evaluation of the teacher's expectations as to how to behave. In contrast to Adem's typical pattern, this case seems to be rather a productive and proactive one. Evaluating the expectation helps to make it explicit and to act in an appropriate and accepted way and therefore to participate successfully.

While Maja's initiatives are most often categorised as action-oriented and forward-looking, but there was no situation to categorise as backward-looking, Merle's table shows a similar pattern. She also initiated always forward-looking, but less action-oriented and more object-oriented. So, while Maja seems to try to evaluate the norms and expectations regarding actions and participation, Merle's initiatives are more about the mathematical objects, and she is in this cluster similar to Elijah. These patterns can also be described as forms of creating autonomy (Yackel & Cobb, 1996), which is again a crucial part of learning as participation.

Conclusion

The suggested categories draw an interesting picture of student initiatives, as they show signs of patterns of participation. Although, as stated above, there are many forms of participation (Krummheuer & Brandt, 2001). We discovered that student initiatives are a crucial way for the students to recognise and identify norms and (situational or normative) expectations of the teacher, which is central for appropriate participation in mathematics classrooms (Aumann & Erath, 2024; Bernstein, 1996; Jablonka, 2008). The patterns discovered should be treated with caution but following the work of Drageset (2015) they indeed show, how different the pupils contribute to the classroom discourse in their initiatives and therefore participate in mathematics classrooms. While Adem's pattern shows an accumulation of initiatives, that might be interpreted as attempts to make already given work instructions clear for him, Lukas' and Maja's pattern can be interpreted as a try to evaluate accepted behaviour in advance. This discovery is exciting, as both were in the same primary class. Their similar pattern might be an indication of their similar earlier socialisation. The examination of their participation in the last weeks of primary school will be part of further research. Also, a further investigation of participation patterns in other types of students' utterances as answers in teacher-led discussions might add a different point of view on the pupils' competence to participate in different situations, where different types of discourse and different competencies are addressed and required. Finally, a longer observation period and the resulting larger number of student initiatives would also be advantageous in order to obtain a more reliable picture of the patterns. Nevertheless, the categorisation of student initiatives presented here seems to be useful for examining student attempts to recognise expectations and norms.

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Teacher guidelines during group work

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This research aims to explore the role of teacher-student interactions in the process of group mathematical problem solving, with a particular focus on prompts. If the students make meaningful discoveries and set milestones by themselves, the teacher's prompting can be considered successful. To explore successful teacher prompting, the following research question was formulated: What kind of prompts did the teacher use, and how did they help the students to solve the task during a collaborative problem-solving session? Through the eyes of the researcher (who is the same as the teacher in this research), the teacher's prompts in the classroom showed a variety of diverse types, which helped to improve the efficiency of the problem-solving process.

Keywords: Group work, interaction, prompt, teacher interventions.

Introduction

Group work, collaborative problem-solving in small groups, has become an important tool for teaching mathematics, with many cognitive, metacognitive, and social implications (Hofmann & Mercer, 2016; Smith & Mancy, 2018). At the same time, this approach is challenging for the teacher, who has to monitor the progress of several groups and give prompts to individual groups or the whole classroom when necessary. This paper aims to contribute to understanding the richness and complexity of interactions by examining the impact of consciously designed intentional prompts and spontaneous, unintentional teacher reactions in the classroom and unveiling how teacher interactions activate students' problem-solving skills. This study investigates teacher-student interactions in a mathematics classroom where students in two small groups engage in collaborative problem solving. This simplified scenario helped document the interactions.

Teachers' knowledge, beliefs, goals, and established social and mathematical norms shape classroom interactions in mathematics teaching. Teachers make moment-to-moment decisions during interactions, carefully considering how to respond to students' thinking (Mason, 2015). These decisions involve balancing the discussed mathematical content, the interaction's objectives, and the students' needs. The complexity of managing these factors, combined with the occurrence of unexpected situations, can lead to cognitive overload for teachers, making planning essential to reduce this burden (Kondé et al., 2023). These factors' interplay significantly impact classroom learning's flow and effectiveness (Ingram, 2023).

The present research highlights the importance of the teacher's planning for the lesson, which includes not only selecting heuristic strategies for problem-solving but also designing possible prompts. To explore the problem, the following research question was formulated: What kind of prompts did the teacher use, and how did they help the students to solve the task during a collaborative problem-solving session?

The framework

The focus of the research was to study the role of prompts during the problem-solving process. Prompts are strategic interventions in the problem-solving process by the teacher to help students move forward in the problem-solving process without directly receiving the answers to their questions (Kramarski et al., 2013).

In this study, teacher prompts were analysed using two separate dimensions. The first dimension, derived from Giles and Gilbert (1981), categorises prompts as motivational, process-oriented, or product-oriented. The second dimension, based on Hofmann and Mercer (2016), classifies the prompts according to the teacher's role: authoritative, initiating, or continuing interactive. Each prompt was independently classified along both dimensions, providing a two-dimensional analytical framework.

Motivational prompts are intended to keep the students moving through the activity; for example, they reinforce students' contributions. Process-oriented prompts are intended to bring students' attention to methods or procedures, assuming they know them. An example is when the teacher asks for a drawing representing the problem, provided the students are already familiar with graphical representation as a heuristic strategy. Product-oriented prompts provide students with information that they are unable to proceed without. When the teacher directly corrects students' answers or explains the next step of the problem-solving process, a product-oriented prompt is given. When the teacher draws attention to a new approach, a heuristic strategy the learners may not have used before, it is also considered a product-oriented prompt. Another dimension of teacher interaction refers to the extent of the teacher's role. The intervention can be authoritative, where the teacher takes the lead; initiating, where the teacher leaves the continuation to the students; or continuing interactive (interactive in what follows), where the teacher responds to the students' ideas (Hofmann & Mercer, 2016).

In small group problem solving, teacher interactions should also support effective group work. Research indicates that group discussions are productive when students share ideas, discuss different views, and resolve them to achieve consensus. However, the literature is not in full agreement on what prompts and interactions teachers can use to achieve this aim (Hofmann & Mercer, 2016). Moreover, these characteristics are not commonly found in classrooms, and teachers find managing group discussions difficult. This research also aims to contribute to a better understanding of this phenomenon.

Method

The research is based on a small-group talent management program for 8th-grade students, aged 13-14, in which seven students participated. The program was organised in a school setting, but the sessions lasted 30 minutes instead of the 45 minutes usually spent in school lessons. This paper deals with one session, the first of the semester. The evaluation of the whole program is the subject of further research.

The teacher in the classroom was the author of the present paper. After transcribing the audio recording, the text was segmented, separating the teacher's voice and the in-group conversations. I

analysed the teacher's voices together with an experienced university colleague. Prompts were first identified and then categorised by double consensus. I marked the students with A, B, C, D, E, F, and G. The students worked in two groups, three in the first group (A, B, C) and the rest in the second. For the sake of simplicity, the group number is given next to the pupils' letter; for example, F2 means that student F is from the second group.

The challenge of the session was the following problem (Bolyai team competition for 8th grade students, National level, 2010/10. task):

In an empty pond, we release 50 hungry pike, which soon start to devour each other. A pike is called satiated and will not eat any more fish if it has eaten three other pike (hungry or satiated). How many of the 50 pike can be satiated in this pond in their lifetime?

To make it easier for readers to understand, the following demonstrates how 16 pike can be satiated. Count the pike from 1 to 50 and arrange them in order. The fourth pike eats the three before it. Now, the seventh pike has only three pike in front of him, which he eats. The same procedure is continued, i.e., the 10th pike is again preceded by three pike, etc. This way, the 4th, 7th, 10th, ..., and 49th pike can be satiated, a total of 16. Questions are raised as to whether more pike can be satiated or whether it is possible that fewer pike can be satiated if the rules are followed.

As this session was the first stage of the talent management series and the working method was unfamiliar to the students, I was mainly prepared for prompts to help them collaborate. Students were expected to find it difficult to understand the rules in the text at first reading, so I decided to directly answer questions about understanding the problem (product-oriented prompt). Moreover, concerning heuristic strategies, I prepared to give two prompts: encourage the students to represent the problem graphically (process-type prompt) and, if they get stuck, to understand the problem better with fewer pike (e.g., ten pike, product-oriented prompt).

The two groups sat at two tables, with a microphone recording the communication at each table. They were allowed to draw on a single large drawing board, each student using a different-coloured marker pen.

Results and Discussion

Although the teacher is the same as the author, the singular third person is used for the teacher's voice from now on. Looking at the course of the lesson, first, the moments that triggered the teacher's interventions were examined. The most important ones are also illustrated through examples of how the teacher reacted to the situation and to whom. The selection was based on the interventions that contributed most to the success of the task, and on the need to present as many different types of prompts as possible.

Students first asked questions about the interpretation of the task. The questions were addressed to the teacher, but the answers were addressed to the whole class, as they were important feedback for everyone. In the following discussion, members of both groups participated. The original questioner must have known the information was relevant to both groups, so he asked his question 'out loud'. As earlier mentioned, the teacher was prepared for the need for clarifying interaction.

D2: "During their lifetime" means that not only once [...]
A1: Well, he could die.
E2: If it was eaten, it could not eat […]
D2: No, my question is, if he is satiated once, can he still be hungry afterwards?
Teacher: No. Once he is satiated, he cannot eat any more. So, he is already satisfied in this respect.
A1: The text states that he cannot consume any more fish.

D2 thinks of the everyday experience: after we are satiated, we can be hungry again. The class reacts to D2's suggestion in a lively way, even if they do not initially understand the question. Later, D2 formulates the question accurately. The teacher clarifies the interpretation with an interactive, product-oriented prompt to the whole group, but A1 does the same nearly simultaneously.

After clarifying the questions, the participants of the two groups made drawings according to the teacher's instructions (process-oriented prompt). Figuring out a graphical representation of the problem was an effortless process for students; however, they differed in idiosyncratic characters. The groups' drawing sheets show that the students started drawing separately; however, students in group 2 used the same coding approach. After about 8 minutes of work, the teacher noticed that the students were getting stuck, so she suggested they investigate the problem for ten pike. This initiating, product-oriented prompt was intentional, and the teacher designed it in advance. The aim was to get experience with the problem. Solving the original problem with smaller numbers may be a well-established strategy. However, the fact that the two groups interpreted the teacher intervention differently presented a difficulty and an additional challenge. Group 1 ignored the instruction because they were immersed in their work. Three members of group 2 (D2, F2, G2) worked on the instruction, and the fourth member (E2) went his own way.

Teacher: I would like to ask you a question: Do you consider this 50 to be a lot? Or couldn't you try it on fewer fish first? I would say try it on ten pike and see how it goes.

Group 1 essentially ignored the teacher's suggestion. They did not want to give up their own train of thought, which they had already started, and continued with the 50 pike.

B1: What? To ten?
A1: I understand, but let's look at this first; let's finish it.

Group 2 seemed to have followed the teacher's instructions, but not in the way the teacher intended. They stayed with the 50 pike, but fish were divided into groups of 10, bringing them to a possible solution.

F2: Now, how many is that? [Thinking of the left part of Figure 1, explaining the process from top to bottom.][1] Now, that's ten. And three of them were satiated.

D2 approves the solution, but G2 finds it confusing that some satiated pike will be eaten during the process. D2 and F2 explain it, and in the end, G2 recites the process using the figure.

F2: OK, we can make it four more of these [we can repeat this process four more times], so three will always be satiated, and one will always stay [alive]. […] The solution is 15. [Referring to the full dots in Figure 1]

[1] Brackets refer to the non-verbal manifestation of the student or to the comment made by the author.

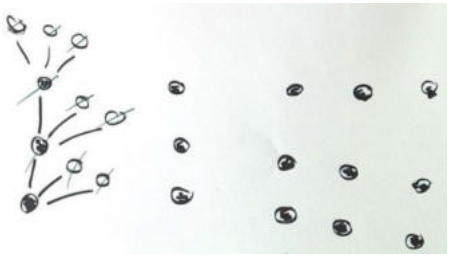

Figure 1: The representation of group 2 for the case 10 pike

It was important for the teacher to have the students explain their thinking to each other to reinforce collaboration and clarify the several individual idiosyncratic approaches. The teacher saw an interesting drawing in group 1, the group that did not follow the teacher's instruction concerning ten pike. The teacher asked the student to interpret his drawing (Figure 2), so it was an interactive, process-oriented prompt for him. The drawing is a perfect representation of the problem. The pupil was thinking in a linear process and marked the hungry and satiated pike differently from the previous drawing. (Read the drawing from top to bottom and from left to right.)

Teacher: This is very interesting. Can you tell me how to interpret this drawing?
B1: The circled ones are the ones who have eaten someone, but then they are eaten afterwards. So maybe they are eaten by being eaten at once.

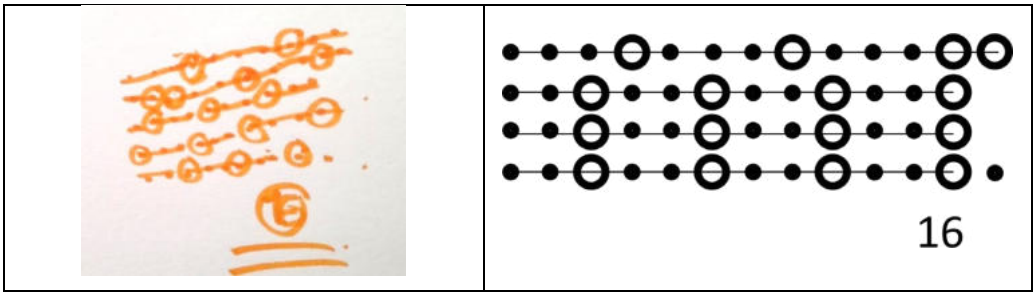

Figure 2: The representation of student B1 (original and edited figure)

While the students were drawing, they sometimes shouted their solutions out loud. The teacher noticed 17 and, as 17 pike could not be satiated, she addressed the group intending to clarify.

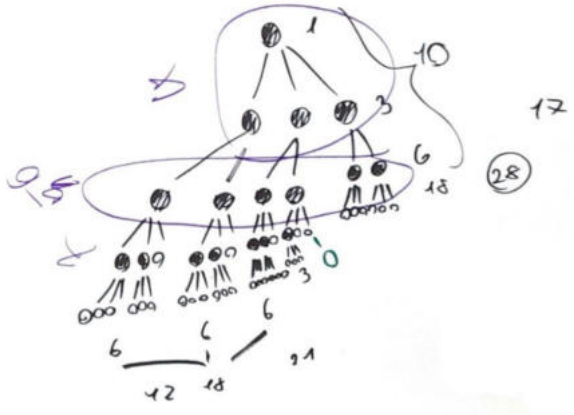

Figure 3: F2's false solution

The interpretation of the student's drawing is as follows. There are 49 circles in the figure representing the pike (Figure 3). The 50th circle was drawn by another student (in a different colour), but this did not play a role in the solution. The full circle represents the satiated pike, and the empty circle is the hungry pike. The pupil drew 17 satiated pike but missed the picture because three pike have eaten only two others and are still shown as satiated.

The following transcript shows an interactive, product-oriented prompt where the teacher explicitly points out the error. This was appropriate because the students, by their own statement, had already found the solution several times.

F2:	17 comes out.
E2:	Already, three people counted it. It is 17.
F2:	However, it seems suspicious.
Teacher:	[Pointing to one of the wrongly satiated circles.] It's not satiated because only two lines are coming from there […]
D2:	But these are no good either.

In the same group, another student (E2, who followed his own path and did not join the group work) reached the 17 pike, but his solution was essentially correct. He just counted the number of pike wrongly, the teacher noticed and, using an interactive, motivating prompt, asked him to explain. Then two of his groupmates (D2 and G2) noticed the mistake immediately and corrected it.

Teacher:	Okay, can you show me how 17 pike can be satiated? Somebody tells me, let us say E2, what is happening, how can this 17 […]
E2:	I started the whole thing by saying that there is a pyramid, the top one is filled, or the last one […] and at the bottom of the pyramid there are 27 [pike], the nine satiated ones ate 27 in total, and then, I don't know, there are nine satiated ones, three satiated ones [one row above the row containing the nine satiated pike], here three more, and one at the top […].
Teacher:	Yeah, and where are the 17 that are satiated? The ones who got satiated?
E2:	With that [thinking of the top one], yes, I think, that's it. But maybe I miscalculated.
D2, G2:	It is 16. [D2 wrote the number 16 on the drawing sheet.]

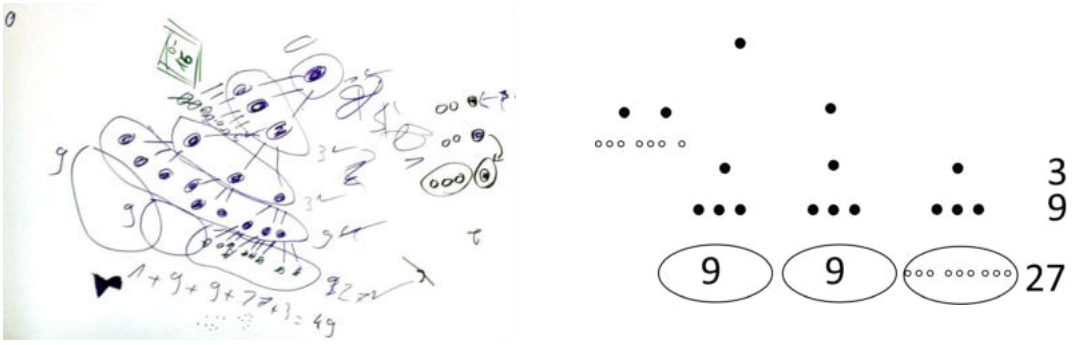

Figure 4: The representation in Transcript 5 (E2's original and edited figure)

In the session, the students came up with some solutions, and it turned out that no solution was found for 17 satiated pike. However, the question remained open whether 17 pike could really not be satiated. Due to time pressure, the teacher took more control using an authoritative, product-oriented prompt and, reverted to the planned heuristic strategy of "solve the problem for a smaller number of

elements," i.e., for ten pike. However, the students did not seem to understand why the teacher insisted on ten pike and kept returning to the original problem during the discussion. They are talking about the left part of Figure 1.

Teacher:	So, for ten pike, let's say three can be satiated. Right? That was on your drawing.
C1:	But then four can be satiated.
Teacher:	Well, how can four be satiated?
C1:	[We continue the drawing with] the little branch on the right, the one that's not there yet, right? [C1 continued the algorithm and mentally added another three circles to the picture.]
Teacher:	But there are only ten pike […] right? So, four can't be [satiated], and then 3, 2, and 1 [can], you could draw that.

Summary

The data quantify the 41 longer and shorter teacher prompts during the session. To address the research question, in terms of the extent of the intervention, more than half of these were interactive, around a third were initiating, and only a sixth were authoritative. The teacher mostly used process-oriented and product-oriented prompts; motivational prompts were only present to a limited extent. Process-oriented prompts, such as encouraging students to use graphical representation, helped them effectively use familiar heuristic strategies. Product-oriented prompts, including clarification of misunderstandings and encouraging the strategy for breaking down problems into smaller cases, directly facilitated students' progress when they got stuck. While product-oriented instructions often successfully guided students, they sometimes caused confusion or partial misunderstanding, suggesting that these instructions need to be carefully aligned with students' current problem-solving state. This phenomenon points to the need for the teacher to be sufficiently flexible and to build a rich schema around the problem (Kondé et al., 2023).

The problem solving was successful, as both groups independently found good solutions to a possible number of satiated pike, supported by explanations. With a little help, they could work out all the solutions and explain why there were no more solutions. The success of the lesson was also due to the inspirational nature of the task and the skill of the students, but it was certainly a good example of how students can do well under more restrained teacher guidance.

This article also highlighted the need to link planned prompts with the dynamic and often unexpected demands of teaching at the moment. I have reported that the teacher may not have achieved the intended effect with the planned prompt. As Mason (2015) discusses, the ability to perceive and respond flexibly in real time is a feature of professional teaching that bridges the gap between preparation and implementation in practice.

Limitations

The author acknowledges her dual role as researcher and teacher in the study. While the dual role provided valuable insight and context, it also introduced potential biases in interpreting interactions and prompts.

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Contrasting meanings of ‘0’ in bilingual primary school settings

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Current findings from mathematics didactics and foreign language research highlight the potential of contrasts for learning processes. The presentation of mathematical concepts in bilingual lessons not only leads to a linguistic contrast, but can also contrast different mathematical conceptual facets. Against this background, the current PhD project aims to investigate and characterise the role of the different languages (German and English) in bilingual lessons for the learning of mathematics in more detail. For this purpose, both languages are considered as complementary components in order to initiate a contrastive approach. The article uses a bilingual learning setting on the topic of ‘0’ to show how children’s interpretations differentiate and how their reflection on different meanings of the number ‘0’ arises.

Keywords: Bilingual education, English (second language), mathematical concepts

Introduction

The commission of the European Union (2003) recommends that member states ensure effective language learning and linguistic diversity from an early age, with the teaching of at least two foreign languages in kindergarten and primary school. In addition to the regular foreign language teaching that takes place in the majority of German primary schools, bilingual teaching is another component of second language learning. The establishment of bilingual learning opportunities corresponds with the recommendation to supplement pure foreign language lessons in elementary school with the preparation of learning content in order to enable a deeper understanding of language functionality (KMK, 2013). In essence, bilingual teaching, such as CLIL, entails the unique combination of a foreign language with subject-specific learning situations within a subject (e.g. mathematics). The aim thereby is not only to acquire language skills during lessons, but also to teach subject-specific content. In light of this, such bilingual teaching and learning processes are also of particular interest for mathematics education research. With regard to mathematics teaching at primary level, this content-related link represents a field that has so far been little researched with regard to “process-oriented studies […] that allow insights into classroom interaction, favourable circumstances of language learning and beneficial conditions of competence development” (Viebrock, 2013, p. 53). While there are already findings from research concerning the development of language learning in bilingual lessons, there is a lack of empirical research on the development of subject-specific mathematical competence in CLIL contexts.

The PhD project aims to investigate the role of the different languages (German and English) in bilingual lessons in primary school for the learning of mathematics. The project is at an early stage of the research process, in which an initial learning setting to the topic of ‘number 0’ has been developed and tested in a bilingual primary school. This article uses documents and a short episode from the current case study to present initial findings on whether the learning setting can cause mathematical reflection on the meaning of the number ‘0’ among the learners.

Language and mathematical learning processes

From a process-oriented perspective, mathematical learning and language are inextricably intertwined. Language takes on two different functions in learning processes (Maier & Schweiger, 1999). On the one hand, language serves as a tool to organise one's own thinking and to open up new possibilities of thinking and knowledge (*cognitive function*). On the other hand, processes of exchange and communication with others are also important in order to expand one's own mathematical knowledge. At this juncture, language assumes its *communicative function* by acting as a medium of communication. The communicative and cognitive functions of language should therefore not be understood as separate entities. Rather, they are inseparably linked. This is corroborated by empirical findings from numerous studies that have identified language competence as one of the most significant predictors of mathematical learning ability (Prediger et al., 2019).

Several studies in mathematics education emphasise that the negotiation of different perspectives and subject-specific insights with the teacher or other learners is fundamental important for the further development of mathematical knowledge (Nührenbörger & Schwarzkopf, 2016; Steinbring, 2009). In particular, young learners, who are not yet autonomous learners, depend on such collective negotiation processes in which all participants stimulate each other (Miller, 1986). For this purpose, the diversity and heterogeneity of learners can be particularly fruitful when dealing with a common learning object. This also applies to language as one of many dimensions of heterogeneity, as different languages can provide different conceptualisations for mathematical concepts (Barton, 2008). Bilingual teaching can exploit this by employing a variety of linguistic terms to represent a mathematical concept in a multifaceted manner. Beyond this, empirical studies highlight that the implementation of the student's first language (i.e. Turkish) in mathematics lessons have shown advantages for mathematics learning. Kuzu (2019) found some evidence that among other things, cognitive multiperspectivity can be productive by linking conceptual nuances that are close to different languages, as in the case of understanding fractions (secondary school). For primary school, Maisano (2019) has shown that the use of Turkish, in addition to German, offers potential for describing and explaining. With the help of bilingual podcasts, Klose (2019) has been able to demonstrate the appropriate use of technical language in the conceptualisation process of primary school children in German and English. Against this background, the PhD project aims to investigate and characterise the role of the different languages (German and English) in bilingual lessons for initiating mathematical learning processes in more detail. The focus is to create contrasts through language diversity in order to initiate productive irritations for mathematical learning processes. Productive irritations have been identified in mathematics education research as valuable starting points for learners' mathematical reflection, negotiation and understanding.

Productive irritations and mathematical learning processes

One purpose of mathematics instruction is to facilitate the progressive development of reasoning and argumentation skills, both within the context of individual learning and through collaborative discourse. In order to achieve this goal, it is crucial that the initiation of argumentation is not solely dependent on interactions with the teacher. Instead, it should be authentic and derived from a subject-specific perspective itself. This is particularly effective when learners challenge their established

expectations when working on mathematical tasks or making mathematical observations. This phenomenon is referred to as *productive irritations*, a concept derived from Piaget's work on cognitive conflicts. It occurs when children develop new ideas when they are confronted with ideas or facts that contradict their initial expectations (Nührenbörger & Schwarzkopf, 2016). Productive irritations can de defined as deviations from a preconceived expectation that require clarification and can be resolved through arguments (Nührenbörger & Schwarzkopf, 2016). In addition, it is conceivable that productive irritations may not only emanate from the subject matter itself, but also from the linguistic tools employed to convey it. In particular, bilingual teaching offers the potential to contrast two languages, thereby facilitating the occurrence of productive irritations. For this, it is important to understand the different roles of the languages in bilingual learning processes.

Bilingual learning and contrasting languages

In light of the various criteria that inform the designation of bilingual programs, such as CLIL, Immersion, etc., Diehr (2012) proposes a typology that differentiates bilingual concepts according to the role the languages play in the learning process. In her tripartite typology, she distinguished between the use of the foreign language in relation to the first language and the associated didactic functions. The first type A is characterised by monolingual teaching, which permits the use of the foreign language solely as a medium of learning, thereby considering the first language as inappropriate as language of instruction and excluding it. Type B is characterised by a limited integration of the first language. Nevertheless, the foreign language continues to be the dominant language in the classroom interaction. In contrast, in type C, both languages are regarded as equally valid. The concept of bilingual teaching in which two languages are used in a meaningful way, extends beyond type B in that is assumes an independent and specific contribution of the language of instruction to the students' learning progress. With regard to type C, Diehr (2012) specifies that it is insufficient to merely consider isolated technical terms from bilingual vocabulary lists. Rather, it is necessary to relate the two languages to each other, in order to teach a *contrastive approach*. This will enable the cultural construction of specialist terminology to be addressed, cultural multiperspectivity to be ensured, and intercultural awareness to be promoted.

Implications for the research project

A comparison of the perspectives derived from foreign language research (Diehr, 2012) with the findings from mathematics didactics research described above (Miller, 1986; Nührenbörger & Schwarzkopf, 2016; Steinbring, 2009) reveals a high degree of correspondence with regard to the productivity of contrasts. In particular, the linking of different perspectives through productive irritations in negotiation processes is seen as a crucial factor in the development of new mathematical knowledge. Bilingual teaching, which uses both the first and the foreign language as complementary elements, thus opens up a rich environment for comparing, contrasting and negotiating certain facets of the subject matter and the language.

For the research project, basic linguistic and subject-specific terms in German and English from the national standards of the mathematical learning in primary schools have been analysed on a subject-specific, linguistic and etymological level to identify potentials from a didactic perspective. This didactic and linguistic analysis entailed an examination, which exhibited both linguistic and

mathematical conceptual differences, including in their fundamental concepts. On this basis, it was possible to work out different mathematical topics and types of bilingual resources ((1) *derivation*, (2) *word-analysis*, and (3) *contextualisation*; Bürgstein et al., 2023), that can be used for the empirical survey in the research project. The bilingual learning setting presented in this paper corresponds to type (3) *contextualisation*. This type is characterised by the fact that there are more terms for a particular concept in one language than in another, depending on different contexts.

The number '0' exemplifies this phenomenon. In German, regardless of the context in which the number is embedded, the same number word is used, namely 'Null'. In contrast, in English different terms are used depending on the context. For example, the term 'love' is only used in the context of tennis, whereas in football 'nil' is used, the term 'oh' is often utilised to encode or nominate something, for example telephone or bus numbers. The term 'zero' is usually used to describe quantities, for counting or calculating (Bürgstein et al., 2023). We assume that these terms, which are used differently depending on the context, can produce a moment of productive irritation and give children an impulse to reflect about different meanings of the number '0'. It is essential that learners understand that numbers can represent not only quantities (cardinal aspect), but also other concepts (ordinal aspect, numbers to calculate, nominate aspect etc.), depending on the context in which they are used (Padberg & Benz, 2021). Establishing relationships between different aspects of numbers is essential for the development of comprehensive knowledge about numbers and their use and representation (Cockburn et al., 2008; Padberg & Benz, 2021). This doesn't mean that children should explicitly name different number aspects, but they should develop an awareness of diverse meanings associated with numbers.

Against the background of theoretical explanations, the linguistic and mathematic-conceptual differences in the context of the number '0', and the few studies on subject-specific mathematical learning in primary bilingual lessons, there is a desideratum to investigate the role of the different languages (German and English) for reflecting about mathematics. This leads to the formulation of the following research question: *How can learners' individual interpretations of the number '0' be characterised in bilingual negotiation processes using language contrasts*?

Method

A bilingual learning setting was developed to collect the empirical data. This was conducted in the form of a case study for an initial pilot study with a group of seven third graders in a bilingual class at an international school in western Germany. The case study method (Lamnek, 2010) makes it possible to identify typical aspects of learners' mathematical interpretations. Before first insights are presented, the learning setting and the analysis method are described. Children with English as their first language (L1, in this article: Ryan & Anna) and children with English as a second language (L2, in this article: Caro) were selected for the case study.

The learning setting is divided into two distinct phases. In *phase one*, two or three children engage in monolingual activities in German or English. The learners are required to collect and compile examples in which they encounter the number '0' in everyday life. To fulfil this task, the pupils can utilise provided materials, illustrations and audio files. Additionally, they can employ pieces of paper to record their ideas and examples, either in written form or as drawings. Subsequently, the learners

are required to sort their examples, to reflect different meanings of '0' and to find appropriate headings for the different meanings. In the *second phase*, two children who have previously worked in different languages in phase one work now together in a bilingual phase. The learners present the results of their work from the first phase. Subsequently they are asked to sort their examples together, giving reasons. Finally, they write down a number word that is particularly suitable for the '0' in the respective examples and find a suitable heading. The moment when the children realise that there are different names for the number '0' in English as opposed to German can be a moment of productive irritation. The expectation to utilise a singular term for the mathematical concept of '0' in each situation may be disrupted during such a joint negotiation process. This can lead learners to clarify the conflict by contemplating the mathematical concept in greater depth and to enter into a mathematical dialogue about different meanings of the number '0'. After the bilingual phase, interviews were conducted with the children.

The data generated in this first pilot study was analysed with the help of the "epistemological triangle" (Steinbring, 2009). The epistemological triangle is a meaningful analytical tool for the present research interest, as it reconstructs the emergence of mathematical conceptual interpretations and their underlying reference contexts. Mathematical "concepts" (e.g. here number '0') require certain "signs" in order to be grasped and encoded. The meaning of the signs must be created by the learners, who use familiar "reference contexts" (Steinbring, 2009). In this article we analyse photos that show which objects the children chose to represent the number '0', how they sorted these objects and named their categories. Therefore, we work out which *signs* the children interpret, which familiar *reference contexts* they use for this and which *conceptual interpretations* can be reconstructed. The analysis offers initial insights into the data, which at this stage of the research project can provide initial indications of meaningful reflections by the children.

First insights

Phase 1: Ryan and Anna (English setting)

The children Ryan and Anna speak Mandarin and English as their first languages. In phase 1 they collected and sorted examples for the number '0' (see Figure 1 a)-d)). The objects and images they put together can be understood as *signs* that represent the number '0' for the children. For the task of sorting these representatives and forming categories, they use everyday experiences as a familiar *reference context* by naming their categories in relation to concrete everyday objects and experiences (e.g. a) body parts, food; b) public transport; c) maths; d) sports; see Figure 1a)-d)).

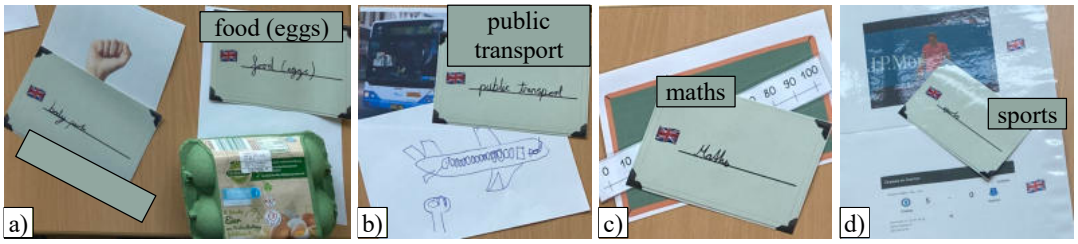

Figure 1 a) – d): Sorting from phase 1 (monolingual English)

At this stage, the categories do not illustrate any mathematical relationship to the representatives. For example, the picture of the bus with the number 308 is an appropriate example for the category public transport, but this category does not describe a mathematical function of the number. Such function could be, for example, that numbers on buses or airplanes are used to precisely number traffic lines or encode flights.

Phase 2: Ryan and Caro (bilingual setting: English and German)

In phase 2 Ryan works together with Caro (Caro worked in the first setting in the German speaking team). They bring together their results, discuss them and develop a new joint sorting. Here, too, the objects and pictures that the children have chosen could be understood as *signs* which represent the number '0'. During the negotiation process between Ryan and Caro, their categorisation evolves into a more mathematics-based perspective (e.g. a) count; b) number; c) calculating; d) sport results; see Figure 2a)-d)), so that it can be assumed that their *reference context* has changed.

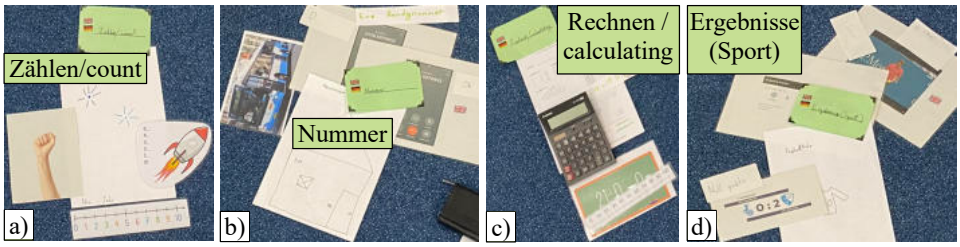

Figure 2 a) – d): Sorting from phase 2 (bilingual: German and English)

The comparison of Figure 1and 2 gives a first indication that the children may have adopted a new perspective. While in Figure 1 the picture of the hand (Figure 1a) was labelled as an example of "body parts", Ryan and Caro now establish a relationship between the picture of the hand and mathematics. Thus, the hand is categorised as a representative of counting ("count") together with other examples such as the number line or the rocket countdown (Figure 2a). While in the first phase the blackboard picture and number line (Figure 1c) were categorised in a rather general way as "mathematics", the children now name a specific function for which the numbers can be used: calculating (Figure 2c). After the children have sorted the examples, they are asked to label the categories with the corresponding number words for '0' in German and English (German: Null; English: e.g. oh, nil, or love). Prompted by the interviewer's question as to whether they had learned anything new about '0' in English and German, the two children enter into an exchange that moves beyond the concrete examples to a meta-level:

1	Caro:	That there can be, erm, are a lot of different zeros in English, so how you can say it differently. And I've never noticed how often you come across the shape or the number zero […]
2	Ryan:	For example, I wonder why you say oh, nil, and love because you can put it all in just one thing […]

Interestingly, Ryan (L1: Engl.) reflects on the efficiency of a single number word for the number '0', while Caro (L1: Ger.) emphasises the variety of number words in English and reflects in this context that she had never before realised how often she encountered '0' as a shape or number. The children's comments seem to be an initial indication that they were encouraged on a conceptual level to think in a more differentiated way about the meaning of the number '0' and the variety of number aspects.

Discussion

In order to investigate to what extent learners' interpretations of and reflections on the number '0' can be initiated by bilingual negotiation processes (see research question), a learning setting was developed in which children engage with contrasting terms for the number '0' in German and English. The analysis of initial data (students documents, video or transcript excerpts) using the epistemological triangle provides cautious indications that the children's *reference contexts* change from more everyday terms to more mathematic-related terms through activities of joint sorting and negotiation. Furthermore, it appears that the particular design of the learning setting can open up *irritations through language contrast* by confronting leaners with phenomena that contradict their original expectations. In the empirical data above, both children seem to discover new and surprising aspects in relation to the denotation and occurrence of the number '0': While Ryan emphasises the efficiency of a single term for the number '0', Caro expresses her surprise about the high frequency of the number '0' ("*And I never noticed how often you come across the shape or the number zero*") and how many different terms are used for it in English. With regard to the research interest, it is less relevant, that the children learn to use different terms for the number '0' (language skills). Rather, the aim is to provide learners with meaningful opportunities for reflection on a mathematical level (mathematical skills). Based on the first data, it can be assumed that the contrastive approach of the languages can create a catalyst for thinking with learners about different mathematical meanings of numbers (numbers for calculating, counting, coding, etc.). This process seems to be important in order to promote not only the linguistic but also the mathematical skills of learners in bilingual lessons. For further research, it must be determined whether these insights can also be reconstructed in other cases and to what extent meaningful mathematical reflection processes can be initiated for other mathematical concepts using language contrasts.

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Mathematical discourse changes during the introduction phase of the early years - A textbook perspective

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The aim of this paper is to investigate the mathematical discourse, as communicated by textbooks, during the introduction phase of early years mathematics education, and to identify how it changes over this short period of time. While textbooks play a central role in communicating this discourse, less is known about how the discourse as communicated by textbooks changes during the introduction phase, i.e., the first months of school. We use commognitive concepts to examine part of the discourse, specifically routines and visual mediators, of the first 100 tasks in three textbooks for Grade 1 in Sweden. Our findings reveal a rapid increase in the routine of re-organization and the use of symbols as visual mediators, implying a swift change from a discourse characterised by colloquial visual mediators and germinal routines towards a more formal mathematical discourse.

Keywords: Mathematics, discourse, commognition, early years, textbook.

Introduction

Research about early years mathematics education is vital for understanding the initial encounters with the school mathematical discourse and its potential consequences. Research in this field has been conducted for several decades but there is still a knowledge gap about the mathematical discourse during the introduction phase of early years education (i.e., the first months). To address this gap, Rezat (2024) suggested that research should focus on the interrelationships between textbooks, mathematical discourse, and other entities such as teachers, students, and the subject of mathematics itself. The present study contributes to this body of research by investigating textbooks and how discourse changes is established during the introduction phase. Mathematical discourse is characterised by four interrelated, what Sfard (2007, 2020) termed, tenets. The tenets are *keywords* (vocabulary and word use), *visual mediators* (images, graphs, symbols, and signs), *narratives* (stories about mathematical objects), and *routines* (patterns of repeated actions). In this paper, we assume the mathematics textbook to be a "second teacher", i.e., an experienced and authoritative interlocutor who requests students to perform specific actions and provides exemplifications of mathematical discourse tenets. The guiding questions are: *What characterises the mathematical discourse as communicated by textbooks during the introduction phase of early years' mathematics education?* and *What changes in the mathematical discourse as communicated by textbooks can be identified during the introduction phase?* To answer the questions, we examined how the routines and visual mediators change in the initiation and closure of the first 100 tasks in three Swedish mathematics textbooks for Grade 1 (seven-year-old students).

Mathematics textbooks and the early years

Research has shown that textbooks hold a prominent role as a curriculum resource in mathematics education because it is commonly viewed as a ubiquitous, trustworthy (and

therefore authoritative), and comprehensive resource for teaching and learning (Rezat, 2024). Textbooks can provide a consistent curriculum, and, when used with the teacher's guide, a complete set of suggested teaching methods, examples, concrete materials, and assessment tools (Van Steenbrugge & Ryve, 2018). Mathematics textbooks pose questions in specific ways that prescribe to students and teachers what the school mathematics discourse is. Hence, by foregrounding some content and activities, textbooks influence teachers' and students' conceptions of what and in what ways mathematics should be taught and learnt (Rezat, 2024). In virtue of this influence, textbooks are often used as a medium for the national curriculum and for fostering educational traditions (Kim & Remillard, 2020; Rezat, 2024; Van Steenbrugge & Ryve, 2018). For example, the central role of the mathematics book often downplays reform-oriented approaches for the benefit of more formal traditions (Nachlieli & Tabach, 2019; Rezat, 2024). For instance, textbook tasks offer opportunities for students to engage in the mathematical discourse in ways that may restrict students' opportunities to participate in mathematical exploration (Christiansen et al., 2023).

In Sweden where the present study was conducted, roughly 90% of students use textbooks for learning (Norberg, 2021). Students are often expected to work silently while the teacher guides students individually (Van Steenbrugge & Ryve, 2018). Many textbooks in Sweden are designed for use from the early years (Norberg, 2021) which position the textbook as a core resource for learning during the introduction phase. An important part of mathematics education in the early years is the development of foundational number sense (Andrews & Sayers, 2015). This refers to the understanding of relationships between numbers, which is essential for learning arithmetical operations, place value, and problem-solving. Foundational number sense includes understanding how mathematical objects are visualised in different ways, such as viewing numerals as sets of objects. Recent research on textbooks and textbook tasks for Grade 1 in Sweden has shown significant differences in the distribution of number sense content depending on the book series (Norberg, 2021). Norberg (2021) mapped resources for communication in number sense-related tasks (subtraction) from the complete set of curriculum programs in Sweden. Her results showed that the most requested answer form for these tasks is in the form of mathematical symbols.

Theory

In our study, learning is viewed as a process of individualising the mathematical discourse according to the theory of commognition (Lavie et al., 2019; Sfard, 2020). This process includes talking, writing, drawing, and thinking as part of communication (Sfard, 2020), regardless of whoever the interlocutor is, e.g., with-self, a classmate, a teacher, or as in our case a textbook. As we study the discourse tenets in textbooks, we interpret the textbook as representing the voice of a more knowledgeable other. The textbook, as the voice of a more knowledgeable other, introduces and maintains rules for participation in the mathematical discourse that apply to meta- and object-levels (Sfard, 2007). Meta-level rules have traditionally been characterised by several key features: direct instruction, emphasis on memorisation, a teacher-centred approach, standardised methods and routines, textbook reliance, and a focus on assessment (Kim & Remillard, 2020). During the introduction phase, object-level rules frequently address various categories of number sense, which often represent the largest distribution of content

areas (Sayers et al., 2021). This is not surprising, since developing foundational number sense is central in early years education.

For the present study, the use of the two discourse tenets routines and visual mediators are key for understanding how interpersonal communication such as communication between teacher, student, and textbook, is performed when verbal language is omitted. Routines are either process-oriented rituals or outcome-oriented explorations (Christiansen et al., 2023; Lavie et al., 2019). One central process of learning is understood as routinisation (Lavie et al., 2019), i.e., adopting previous experiences to a larger extent towards mastering a specific procedure or action. To increase participation in the mathematical discourse, children must master several important, so called *germinal routines* (Lavie et al., 2019). Germinal routines are the most foundational routines in mathematics discourse, e.g., comparing sets and counting. As students develop knowledge of mathematical discourse they encounter new more complex routines, which can require a series of different germinal routines to be performed in a systematic pattern (Lavie et al., 2019). Thus, knowledge of previous germinal discourse routines is part of more demanding mathematical discourse routines. Consequently, germinal routines are crucial for developing knowledge about more complex routines, such as re-organising quantities, and endorsing or rejecting narratives about mathematical objects. For this exploration to occur, students need to be comfortable and autonomous in deciding which routine to perform in a given situation, hence the importance of practicing rituals through tasks (Nachlieli & Tabach, 2019). Visual mediators are "perceptually accessible aspects of material reality […] employed as 'helpers'" (Lavie et al., 2019, p. 15). Examples of visual mediators are symbols, e.g., numerals or mathematical signs, connecting lines, images of objects e.g., cats and balls, and circles. Different visual mediators can play different roles in mathematics. For example, symbols are signifiers of mathematical objects, connecting lines can illustrate relationships, images of objects are often used as (colloquial) representations of real-world mathematics, circling objects can mean classifying by properties. The choice of visual mediators is either determined beforehand by explicit instructions, e.g., "draw three cats", or open to the student to independently choose a visual mediator to provide the solution.

Since textbook tasks play an integral role as a teacher's voice during the introduction phase, they play a significant role in determining the mathematical discourse. A textbook task "serves as a context where the students explore the content, and they indicate what students are required to do" (Kim and Remillard, 2020, p. 31). Mathematical tasks, according to Nachlieli and Tabach (2019), are roughly organised as initiation (requirement of specific actions), procedure (performance of specific actions), and closure (communication of the result of the specific actions). Here we draw on what characterises mathematical discourse as communicated by textbooks. Hence, this study does not focus on how students perform textbook tasks. Rather, our focus is on the voice of the textbook in terms of what actions it asks students to perform (initiation), and how it requires ways of communicating results (closure). To understand how the initiation of the task is communicated by the textbook, we view the routines as the action that the initiation of the task requires the students to perform. To understand how the closure of the task is expected to be communicated by the student, we view what students are asked to produce when mediating their closure to a given task.

Methods

In Sweden most students formally encounter a mathematics textbook for the first time in grade 1, at the age of 7. Following theory and previous research, which suggest that mathematics education is often "task-driven", relies on textbooks, and emphasise certain discourse characteristics (Kim & Remillard, 2020; Van Steenbrugge & Ryve, 2018), we investigated three books available in schools in Sweden to answer our research questions. The mathematics textbooks were selected by their popularity on the market for school textbooks. Since there is no formal national review board of curriculum material in Sweden, the market is open for any textbook author to publish material independent of scientific validation.

All three books were designed for students' individual work, with the possibility to draw and write directly in the book. The data material (n=300) consists of the first 100 tasks about number sense in each of the textbooks. We focused on number sense tasks because number sense is important in the introduction phase (Andrews & Sayers, 2015). The examined tasks covered approximately the first two months of lessons in school year 1. Each task was qualitatively coded for routines (initiation) and visual mediators (closure). We compared our categorisations and applied intercoder reliability (O'Connor & Joffe, 2020). Firstly, we based our coding of routines on key words from Lavie et al. (2019). Our aim was to identify examples of mathematical discourse routines: counting, quantifying sets, comparing sets, and naming a number on a line. Other routines, such as endorsing or rejecting, and estimating were rarely present (below 1%), hence their absence in the findings. The routine re-organising was added abductively to code tasks that require students to re-organise elements in a set which is more complex than germinal routines. Consequently, tasks coded as re-organising required students to perform more than one germinal routine. Secondly, the categories for visual mediators were inductively coded as coloured field, drawn image, drawn line, circle, and written symbol.

How many? Write the number.

Figure 1: Task featuring a routine [counting] and a visual mediator [a written symbol]

Write the dots that the dice might show.

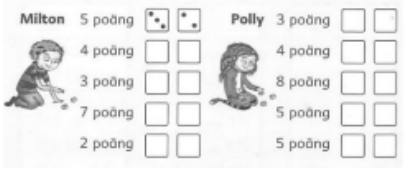


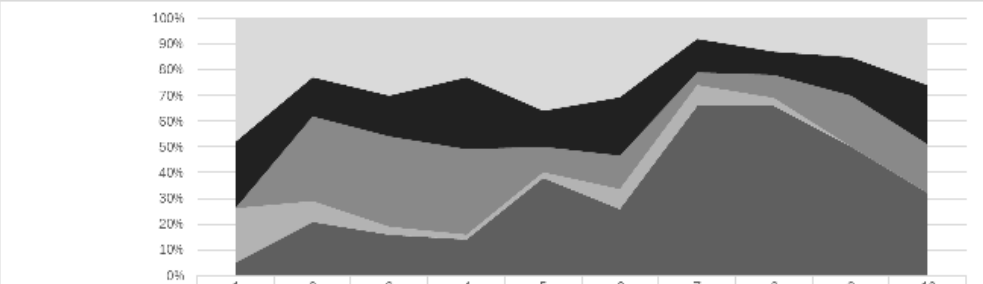
Figure 2: a task featuring a routine [re-organising] and a visual mediator [a drawn image]

There were no other types of visual mediators that the textbook tasks requested. Figure 1 shows an example of the task "How many? Write the number", i.e., requesting a ritual performance that was coded as counting routine, with the visual mediator coded as written symbols. Figure 2 shows a task coded as the routine of re-organising, asking the student to draw the dots that the dices might show, that is, parting the numbers into sets of 2. After coding, the data sets were merged into one large set and divided into 10 sections of 30 tasks, i.e., 10 from each of the three books as there were no significant differences between the distribution of discourses in the

books. This procedure facilitated the analysis and the graphical presentation (see figure 3 and 4 in the Findings section) of gradual change.

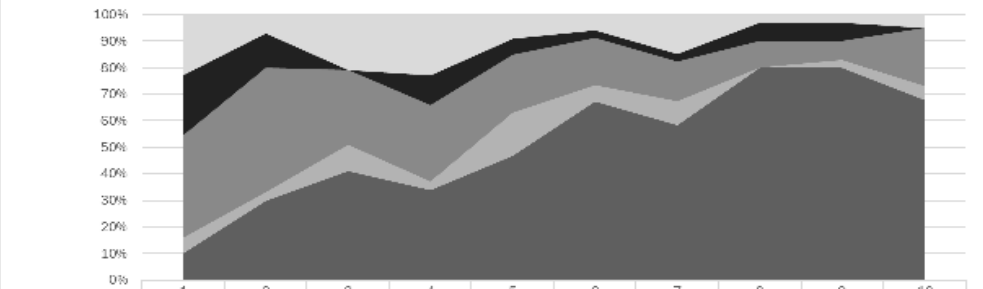
Findings

In this section, we describe the main results related to change over time in the mathematical discourse during the introduction phase, one about routines (task initiation) and one about visual mediators (task closure). In addition, we point towards an interpretation considering routines and visual mediators taken together.



	1	2	3	4	5	6	7	8	9	10
Counting	48%	23%	30%	23%	38%	31%	8%	13%	15%	26%
Quantifying sets	26%	15%	18%	28%	14%	23%	13%	5%	15%	23%
Comparing sets	0%	33%	35%	33%	10%	13%	5%	5%	20%	19%
Naming a number on a line	21%	8%	3%	2%	2%	8%	8%	3%	0%	0%
Re-organising	5%	21%	15%	14%	38%	26%	66%	66%	50%	32%

Figure 3: Frequencies of routines (10 groups of 30 tasks)



	1	2	3	4	5	6	7	8	9	10
Circle	23%	7%	21%	23%	9%	6%	15%	3%	3%	5%
Drawn line	23%	13%	0%	11%	5%	3%	3%	7%	7%	0%
Drawn image	35%	47%	28%	29%	22%	18%	15%	10%	7%	22%
Coloured field	5%	3%	10%	3%	16%	6%	9%	0%	3%	5%
Written symbol	10%	30%	41%	34%	47%	68%	59%	80%	80%	68%

Figure 4: Frequencies of visual mediators (10 groups of 30 tasks)

Firstly, our findings show that all five routine categories were present in our data. In relation to frequencies, Figure 3 shows a gradual change from germinal routines towards more demanding routines. Four of the germinal routines (counting; quantifying and comparing sets; naming a number) decrease from approximately making up 80-90% of the tasks at the beginning to 50-60% at the end. The more demanding routine (re-organising) increases from making up 5-20% of the tasks at the beginning to 30-50% at the end. Secondly, our findings show that all five categories for visual mediators were present in our data. In relation to frequencies, Figure 4 shows a gradual change in visual mediators from colloquial towards more discourse-specific examples. Four visual mediators (coloured field; circle; drawn lines or images) decrease from approximately 80% to 25% and one (written symbol) increase approximately from 20 to 75%.

In summary and taken together, the two findings above indicate how the mathematical discourse rapidly (considering that this is the introduction phase) move from being more playful, colloquial, and informal towards being more formal and disciplinary.

Discussion

This study investigated changes in the mathematical discourse during the introduction phase of early years' mathematics education. Here we interpreted the results as showing how the mathematical discourse rapidly became more formal. Our findings show that the visual mediator *symbols*, as a task closure, increases during the span of the introduction phase, while *circles* and *connecting lines* diminishes. The results show similar patterns for task initiation. The observed connection between the use of symbols and re-organisation routines suggests that the textbook presumes that students' familiarity with symbols is sufficient to perform more advanced routines at this stage. However, introducing more advanced routines and formal symbols early might increase the intellectual load on young learners. This rapid change could pose challenges for students who struggle with the development of number sense (Andrews & Sayers, 2015), when trying to grasp relationships between mathematical objects and their different signifiers, in this case, sets of objects and their corresponding numerals. Our research findings are consistent with those of Norberg (2021), demonstrating the change towards mathematical symbols becoming the predominant closure.

Another notable change in the mathematical discourse that the textbooks communicated was that while tasks involving germinal routines tended to diminish, they did not completely vanish. This persistence is crucial as it facilitates routinisation which enable students to engage in similar mathematical situations (Christiansen et al., 2023; Lavie et al., 2019). However, we noted a gradual reduction in the variety of visual mediators. Consequently, students' experiences with more informal visual mediators, such as coloured fields, and drawings, may become less emphasised, thus introducing a new meta-level rule of mathematics discourse. Although children might enjoy using coloured pencils, our results indicate such practices are not part of the formal methods for communicating mathematical solutions. Our study confirms that the textbook communicates a meta-rule that says that mathematics is about re-organising routines, and a meta-rule that says that mathematics is communicated with symbols. These two meta-rules are foundational aspects of the formal mathematical discourse described by Kim and Remillard (2020) and Norberg (2021). Sayers et al. (2021) showed how the distributions of tasks in textbooks place emphasis on routines in number sense-related tasks to various degrees throughout the first semester of Grade 1. This indicates that, even though the introduction phase involves tasks that aim at using symbols and re-organising quantities, other types of routines and visual mediators might appear in later stages of the school year. However, students' first impression of mathematics might be crucial for their understanding of the mathematical discourse. Viewed as a central source of opportunities for participation in mathematical discourse, textbooks might limit the ways that students can engage with and understand the meta-level rules of mathematics education and we agree with Norberg (2021) who stresses the importance of providing tasks and exercises that offer a variety of mathematical situations to avoid simplified understandings of number sense. The reviewed literature made clear that textbooks, as teaching materials, play a central role in forming teaching practices (Norberg,

2021; Van Steenbrugge & Ryve, 2018). The emphasis shown in our results on certain routines and visual mediators can shape what students and teachers perceive as important in mathematics. This could lead to a narrow understanding of the mathematical discourse due to limited experiences, which could interfere with exploration routines where independent decision-making is crucial (Christiansen et al., 2023; Lavie et al., 2019).

Limitations and future research

This study has limitations. 100 tasks in three mathematics textbooks do not necessarily encompass the full variation of how a textbook in general communicates the mathematical discourse. However, based on findings that mathematical textbooks usually are similar in character (Van Steenbrugge & Ryve, 2018), we argue that the findings of the present study still provide relevant insights about the mathematical discourse during the introduction phase. The textbook, although important, is only one resource used by teachers. Students encounter the mathematical discourse and its changes in other ways too. Furthermore, almost all children who enter Grade 1 in Sweden have already participated in a pre-school mathematical discourse during the pre-school years, where they have encountered several germinal routines and explored different visual mediators. There are however differences between the mathematical discourse of the pre-school and the more formal school mathematical discourse. Future research on mathematical discourse and its changes in the early years could pursue various trajectories. By mapping other commognitive tenets like narratives and vocabulary, or by extending the data collection to a longer period, more extensive quantitative research could further inform the changes in the mathematical discourse during the introduction phase.

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Impacts of multilingualism on mathematical learning in primary school - Focusing on everyday language differences

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The influence of language on mathematical learning is undeniable. At the beginning of primary school, the everyday language appears especially relevant. Moreover, in an increasingly super-diverse society, such as Germany, there are numerous children who speak more than just one language in their daily lives. Therefore, it seems crucial to consider all their multiple language skills. In this study we investigate the impacts of these competences on mathematical learning processes. We focus on differences between everyday language registers of several languages. Initial empirical findings are presented.

Keywords: Mathematical learning processes, language, multilingualism, primary education, everyday language.

Everyday language and mathematical learning in primary school

According to current research, language plays a crucial role in mathematical learning processes (Ingram et al., 2023). If we take a closer look at the learning of mathematics in school, we will observe the use of different linguistic forms (Meyer & Tiedemann, 2017). Depending on the foci of interest in research on the correlation of mathematical learning and language, reference is made to different theoretical frameworks. One internationally strongly recognized approach is Halliday's (1978) concept of linguistic registers, which is based on the assumption that language is used functionally in communicative situations.

In Germany, there are three registers that are repeatedly discussed in the context of mathematics education. These are a technical-mathematical register, an everyday language register and a school academic register (Meyer & Prediger, 2017). The mathematical register consists of technical terms and typical syntactic structures, such as those used in formal proofs or theorems and aims to represent a mathematical structure as precisely and accurately as possible. In contrast, the function of everyday language is to facilitate unhindered conversation between people who are mostly physically in the same place, resulting in short, often grammatically fragmentary sentences supported by gestures and facial expressions. According to Feilke (2012), the school academic register mediates between the other two registers. It is not specific to communication about mathematics, but rather to academic communication in general and to educational situations. Complex and grammatically correct sentences are characteristic for this register. In research, a strong influence of both, technical and educational language on mathematical learning processes is proven (Meyer & Tiedemann, 2017). However, competences in these two registers cannot be expected from students, especially at the beginning of primary school, and must be developed in school first. However, viewed from another perspective, almost all students have everyday language skills when they start school. Everyday language skills therefore form the linguistic basis for learning mathematics in primary school.

This raises the question of the role that everyday language plays in mathematical learning processes. Previous research has already examined the relationship between everyday language and students' conceptual understanding. According to the idea of *Grundvorstellungen*, which is widespread in Germany, the linking of formal mathematical concepts with practical experiences and real situations is essential for the individual conceptual development (Leufer et al., 2019). However, these practical experiences and real situations are often acquired in everyday life and are therefore linked to the everyday language register. Therefore, if the learner's experience is to be captured, the interwoven known linguistic means must also be considered. In this sense, everyday language plays an important role in conceptualisation in primary school (Ruwisch & Weigand, 2023). Although school academic and the mathematical registers become increasingly important in the later schooling, everyday meanings ascribed to mathematical terms can influence students' understanding of mathematical concepts even in higher grades (Antonpoulos et al., 2023).

When focusing on the relationship between students' everyday language skills and mathematical learning processes, it is important to note that in Germany, as in numerous other countries, a significant proportion of children grow up multilingual. These children have knowledge of several everyday languages, and their experiences are frequently related to these languages. Provided that there is a connection between everyday language and mathematical learning (in primary school), it can be assumed that being familiar with everyday language registers of different languages also has an impact on mathematical learning. Indeed, several research projects have demonstrated that the explicit inclusion of students' multilingualism can have a (positive) impact on their mathematical learning (Barwell et al., 2019). Nevertheless, there remain numerous questions regarding the concrete impact of everyday registers of different languages. Therefore, this paper focuses on the influence of multilingual learners' everyday language competence in their various languages on their mathematical learning processes.

Everyday language registers in different languages

In the course of investigating the everyday language registers of several languages and their potential impact on mathematical learning, our research group identified notable differences between some languages. For example, in the context of doubling and halving, the following expressions are often used in everyday German: "doppelt so viele" [*double as many*] and "halb so viele" [*half as many*]. In other (potential) languages of origin of the children in German classes, however, this (mathematical) relationship is formulated quite differently in everyday language. Everyday Croatian, for example tends to use expressions such as: "duplo/dvostruko više" [*double more*] and "duplo/upola manje" [*double/half less*]. Figures 1 and 2 show the use of these linguistic terms in whole sentences with literal translations into English.

German	Croatian
Amy hat 6 Buntstifte.	Amy ima 6 bojica.
Ben hat doppelt so viele Buntstifte.	Ben ima duplo* više bojica.
Amy has 6 coloured pencils.	*Amy has 6 coloured pencils.*
*Ben has **double as many** coloured pencils.*	*Ben has **double more** coloured pencils.*
	**alternatively: dvostruko* [double]

Figure 1: Doubling

German	Croatian
Amy hat 6 Buntstifte.	Amy ima 6 bojica.
Lisa hat halb so viele Buntstifte.	Lisa ima duplo* manje bojica.
Amy has 6 coloured pencils.	*Amy has 6 coloured pencils.*
*Lisa has **half as many** coloured pencils.*	*Lisa has **double less** coloured pencils.*
	**alternatively: upola* [half]

Figure 2: Halving

Looking more closely at the linguistic expressions, we can observe that the word “viele” [*many*] is used in both German contexts. In contrast, different words are used in Croatian: “više” [*more*] and “manje” [*less*]. These Croatian words indicate the direction in which the quantity is changing, whereas the German expression simply refers to the quantity itself. Another observation is that one word can be used in the different contexts in Croatian. The word “duplo” [*double*] actually corresponds to the word “doppelt” [*double*] and “upola” [*half*] corresponds to “halb” [*half*], but the word “duplo” can be used in both contexts. This phenomenon illustrates the connection between doubling and halving. In addition to these differences between the Croatian and the German everyday language, we also found differences between German and other everyday languages in similar content-related contexts (e.g. French, Russian, Arabic). Furthermore, negotiating with various native speakers, we learnt that there is no direct equivalent in Spanish for expressions such as “doppelt so viel” or “halb so viel”. Instead, Spanish speakers would use “el doble de” [*the double of*] or “la mitad de” [*the half of*]. Besides, the direct translations into English, which is used as the reference language in this paper, show that there are differences between English and all the other languages that have been mentioned so far. English native speakers would probably not use “double as many” (equivalent to German) or “double more” (equivalent to Croatian), but rather “twice as many”, which emphasises the multiplication by two.

The notion that there are instances where no direct, literal equivalents exist between different languages is not a novel one (Prediger et al., 2019). Instead, the idea that languages may offer alternative structures and conceptualisations, has been part of the scientific discourse for a long time. This suggests that similar observations about contrasts between languages may also be found in other everyday contexts, which are relevant to mathematical concepts in primary education. Furthermore, it can be speculated, that these differences between different everyday languages may influence mathematical learning processes in primary school. A wide range of influences, both positive and negative, are imaginable. On the one hand, students might be confused by the different linguistic expressions, which could result in mathematically incorrect approaches, ideas and conclusions. For example, students who speak Croatian and German may be confused that the German word “viele” [*many*] is used in contexts with doubles and halves. They might transfer their experience from the Croatian language, where the corresponding word indicates the direction in which the quantity changes. On the other hand, linguistic differences may prompt reflection on mathematical concepts, which could have beneficial impacts on mathematical learning. These effects may be further enhanced by the activation of additional linguistic resources and associated experiences. A number of research projects with multilingual students, mainly in higher grades, have already proven positive effects of language differences on the individual development of mathematical concepts. For example, Prediger and colleagues (2019) were able to demonstrate that language differences between Turkish and German on the topic of fractions can enhance the mathematical learning of Turkish-German bilingual students. However, particularly in primary education and with a focus on everyday language differences, there are numerous mathematical topics and various language combinations that still require research (Deegener, 2024). This paper therefore focuses on the differences between the everyday language registers of several languages, especially on those that seem to be relevant at the beginning of primary school in Germany.

Methods

Our research project consists of two parts. The first part is a theoretical investigation of typical mathematical topics in primary school and the corresponding everyday language expressions in German. In light of these observations, a comparison is made with the corresponding everyday language means of several other languages and differences are identified. The second, empirical part of the project investigates possible effects of linguistic differences on mathematical learning processes in primary school and how they can be used as a resource for learning and teaching.

An initial pre-study data collection was carried out with children of pre-school age (five- and six-year-olds). In June 2024, several interviews were conducted with bilingual children. Each interview involved two children who shared the same languages. In this paper, we refer to an interview with Spanish-German-speaking children. The interviewer had a basic knowledge of Spanish as a foreign language and only used German in the interview. However, at the beginning of the interview, the children were encouraged to utilise both languages. The topic of the interview was doubling and halving. Real objects and a machine (a large box) representing the doubling and halving process were chosen as a basis to encourage exchange about doubling and halving. Concretely, several objects were put into one side of the machine and half or twice as many objects were taken out on the other side. One direction of the machine represented doubling, the other halving (Figures 3 and 4). The children were prompted to describe what would happen to the quantities and to make predictions. The names of the children have been changed in this paper in order to ensure anonymity.

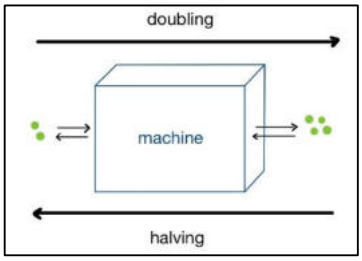

Figure 3: Sketch of machine

Figure 4: Machine in reality

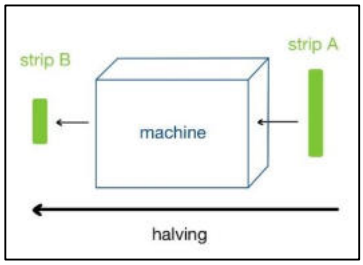

Figure 5: Strips of paper

First analyses of potential impacts

The following sequence has been chosen as representative, in which differences between Spanish and German everyday language seem to influence the focus of the learning process as well as the activation of basic action-related mathematical concepts. References to transcript turns are marked with <Turn x>.

Episode 1

Contextual information: Before the episode begins, the children had already observed and described various doubling and halving processes. Then they were asked to predict what would happen to a strip of paper (strip A, Figure 5) when it was pushed through the machine in the direction of halving.

15	Theo:	ich denke dass es dann halb ist *I think that it will be half then*
16	Interviewer:	was glaubst du? ((holds strip A in front of Lily)) *what do you think?*
17	Lily:	mmh klein ist ((moves her outstretched hands towards each other))

		mmh is small
18	Interviewer:	wollt ihr mir mal ungefähr zeigen was meint ihr wie groß ist es dann wenn ichs jetzt hier rausnehme ((places strip A on the table)) *would you like to show me roughly how big you think it will be when I will take it out here now*
19	Lily:	eins zwei drei ((inaudible)) fünfzehn ((moves her finger along strip A at this time)) *one two three ((inaudible)) fifteen*
20	Interviewer:	und [wenn? ja] *and [when? yes]*
21	Lily:	[und dann] ist glaube ich (.) fünf *[and then] I think it is (.) five*
22	Interviewer:	kannst du mir das ungefähr mit den Fingern [zeigen wie siehts dann aus auf der anderen Seite gleich?] *can you show me roughly with the fingers [what it will look like on the other side?]*
23	Lily:	[eins zwei drei] vier fünf ((moves her finger along strip A at this time until the centre)) bis hier ((points to the centre several times)) bis hier *[one two three] four five (.) until here (.) until here*
24	Theo:	ich denke das wär halb sein (.) ein Teil nicht und den anderen doch *I think it will be half (.) one part is not and the other still*
25	Interviewer:	((addressing Theo)) kannst du mir sagen wie lang das dann ungefähr ist (.) das neue Band *can you tell me roughly how long it is (.) the new strip*
26	Lily:	in die Mitte (.) in die Mitte *in the middle (.) in the middle*
27	Theo:	so ich denke (.) [eins] ((puts his finger on the left end of strip A)) *so I think (.) [one]*
28	Lily:	[hier] ((points to the centre of strip A)) *[here]*
29	Theo:	ich denke bis da ((puts his finger on the centre of strip A)) *I think until there*
30	Interviewer:	Warum genau da? *Why exactly there?*
31	Lily:	Weil das ((points to the centre of strip A)) hier ist die Mitte *Because that (.) here is the middle*

In episode1 it is noticeable that Lily utilises expressions with "Mitte" [*middle*] <Turn 26; 31> and that she repeatedly points to the middle of the strip of paper <Turn 23; 28; 31>. In contrast, Theo utilises terms like “halb” [*half*] <Turn 15; 24> and refers to two parts <Turn 24>. The formulations used with “Mitte” and “halb” suggest different, but both sustainable, basic action-related mathematical ideas about halving lengths. The German term “Mitte” seems to be more in line with the basic action-related idea of cutting or folding the length exactly in the middle. Whereas the German term “halb”, seems to be nearer to the mental representation that halving a length results in two parts of equal length (each half as long as the other). From a German-speaking perspective, the utilisation of “halb” in halving contexts appears to be more natural, whereas the utilisation of “Mitte”, seems not wrong, but rather unusual. Looking more closely at the episode, Lily consistently uses this rather unusual term <Turn 26;31> and there doesn't seem to be any linguistic approximation to Theo. For example, even when she explains Theo’s prediction of how long strip A will be, she sticks to her formulation <Turn 31>. However, if we take a closer look at the Spanish meaning of “Mitte”, these observations appear in a different light, since there are at least three different terms that can be

considered as equivalents of "Mitte": “medio”, “centro” and “mitad”. The term “mitad” has already been mentioned, as it is used in Spanish for the equivalent of “half”. In addition, “mitad” can be used to describe the central point of a path. In the context presented here, with a strip of paper, both uses of “mitad” are appropriate (Figures 6 and 7). As a result, it is plausible that Lily chooses the term “Mitte” due to her familiarity with the use of “mitad” in similar halving contexts in everyday Spanish and “Mitte” appears to be the most suitable translation for her. The phonological similarity between “mitad” and “Mitte” may further influence her selection. This suggests that Spanish everyday language might activate a fundamental action-related concept in her thinking, which seems less obvious from a German-speaking perspective. Moreover, “mitad” seems to be able to transmit both basic action-related ideas presented. Therefore, from Lily's Spanish perspective, it is possible that she uses “Mitte” to represent these two ideas. This interpretation could explain her linguistic focus on the middle and the absence of linguistic approximation to Theo. In conclusion, it seems reasonable to suggest that differences between the everyday registers of different languages may affect which basic action-related ideas are activated and focused.

Figure 6: Overview use of “mitad” I

Figure 7: Overview use of “mitad” II

The next sequence was selected because it suggests that Lily is in fact activating both basic content-related ideas, and also because the interviewer would not have expected this flexible thinking based on her own linguistic knowledge.

Episode 2

Contextual information: Episode 2 is derived from the same interview as Episode 1. There is approximately one minute of an interval between the two. During this period, strip A was put into the machine and strip B was taken out. The interviewer then gave the children both strips.

58	Lily:	((takes strip A and places it next to the strip B)) in die Mitte war das (.) das war hier (.) das war hier (.) ((points to the centre of strip A)) also in die Mitte war das (.) in die Mitte war das *it was in the middle (.) that was here (.) that was here (.) so it was in the middle (.) it was in the middle*
59	Interviewer:	bist du dir sicher? *are you sure?*
60	Lily:	ja guck mal *yes look*
61	Lily:	((points to the centre of strip A)) in die Mitte war das *it was in the middle*
62	Interviewer:	du darfst das hier ((points to strip A)) auch irgendwie anders legen wenn du magst *you may lay this (.) also somehow different if you like*
63	Lily:	((places the two halves of strip A on top of each other)) ((inaudible)) ich gucke ob das richtig ist *I'll check if that's right*
64	Interviewer:	Du schaust ob das richtig ist sagst du? *You check if that's right, you say?*

65	Lily:	ja ((places the folded strip A exactly next to strip B)) *yes*
66	Interviewer:	mmh (.) du darfst es richtig knicken (.) kannst du nicht kaputt machen *mmh (.) you can bend it properly (.) you can't break it*
67	Lily:	((looks at the srips)) jap das Gleiches, ja in die Mitte. du hast recht Theo *yes the same, yes in the middle. you are right Theo*
68	Interviewer:	((laughing))

Lily's confirmation of her prediction in combination with her statement that Theo is right <Turn 67> indicates that for her both basic mathematical ideas agree and that she activates both. However, the interviewer told us afterwards that she was very surprised by Lily's statement and so reacted with an uncertain laugh <Turn 68>. She thought that Lily had believed that her prediction was correct and that Theo's idea contradicted it. So she wondered why Lily was confirming her idea and agreeing with Theo's at the same time. She would probably have had a changed understanding of the situation and acted differently if she had known that the Spanish word "mitad" can be used in these different contexts. However, given her limited knowledge of the Spanish language, she miscalculated the linguistic compatibility of the two ideas, leading probably to an underestimation of Lily's flexibility in understanding.

Discussion

This paper highlights the importance of the everyday language register in mathematical learning processes at the beginning of primary school. For multilingual children, everyday language registers of all their languages appear to be relevant. Accordingly, this paper focuses on everyday language registers of several languages, emphasising their differences and their influence on mathematical learning processes. Initial insights from the first empirical data indicate that such linguistic differences may evoke the activation of specific, additional, or different basic action-related mathematical ideas. Thus, they may influence the focus and the progression of mathematical learning processes. Furthermore, there is an indication that these differences may also lead to a discrepancy in teachers' interpretation of the learning process if they are not informed about the linguistic differences. This could result in the inadvertent omission of additional, sustainable, and basic action-related ideas.

The empirical data presented indicate and specify the impact that multilingual everyday language registers can have on mathematical learning processes in primary schools. Our research aims to differentiate this relationship more specifically. We also try to clarify to which extent multilingual language skills of students at early primary school age can be seen as a challenge or as a resource in super-diverse (German) classrooms. In addition, our pre-study highlights the importance of looking more closely at the role of teachers. They are the ones who guide the children's learning processes and make a significant contribution to the development of basic mathematical skills through their evaluation of statements and actions. It is therefore worth discussing the value of making teachers aware of the differences between the everyday registers of different languages, as it is already the case with the differences between everyday and technical language (Antonopoulus et al., 2023; Meyer & Tiedemann, 2017). In this context, it would be valuable to consider whether teachers need to be sensitised to the linguistic differences between all languages, or whether sensitisation to selected languages would be sufficient.

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Classroom discourse and positioning: A case study of a highly regarded teacher

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The starting point of this study is that some teachers master the art of teaching mathematics and that studies of such practices can contribute to developing our understanding of mathematics teaching. This article reports on a study of a highly regarded teacher in grade two. The aim was to describe how the teacher used interactions during plenary talk and what this could tell about the teacher's positioning. To achieve this, we used frameworks for studying interactions to describe the practice and then positioning theory to understand how this builds a practice. We described seven aspects that together characterize this practice (alleviate, develop ideas, involve students, keep focus, conclude, tell, reduce complexity) and discussed how these related to positions and storylines.

Keywords: Classroom discourse, interactions, positioning theory.

Introduction

Mathematics classrooms are the main arena for studying mathematics teaching and learning, and arguably, the discourse is the most central element of the classroom (Erath et al., 2021; Cobb, 2000) and a large body of research on mathematics classroom discourse, communication, and interactions have emerged (Erath et al., 2021; Xu & Clarke, 2019). However, even though there is a growing consensus that shifting classroom interaction patterns toward more student talk and participation is beneficial (Alrø & Skovsmose, 2004; Kazemi & Hintz, 2014; Stein et al., 2008), change seems to be slow and difficult to sustain (Xu & Clarke, 2019). At the same time, some teachers have changed towards a participatory pattern, where students' mathematics talk and sharing ideas is a vital part of their practice. Studying such practices can add to our understanding of such practices and how to establish them. Thus, our research question is: Which aspects of a proficient teacher's practice can be described by studying the teacher's interactions during plenary discourse, and what can this tell about the teacher's positioning?

Literature review

Concepts and frameworks

Researchers have developed a large number of concepts and frameworks describing teacher interaction during mathematics teaching (e.g., Drageset & Ell, 2024, found 94 concepts in 44 articles). These concepts describe several exciting phenomena. For illustration, we will give three examples of how concepts describing teacher interactions can be combined into more general ideas. One phenomenon is simplification, which is described in different ways and nuances as the Topaze effect (Brousseau & Balacheff, 1997), guided algorithmic reasoning (Lithner, 2008), and hints (Wood et al., 2006). The effect of simplification is that it becomes easier for the student to reach an answer, but it contributes little to helping the student find the answer to the next task. In isolation, this seems like

a contra-productive teacher interaction. However, little is understood about how simplification might be a productive interaction with others, looking at other aims (mastery, motivation) in the long run. Another phenomenon is probing (Warshauer, 2015), which also links to concepts such as inviting (da Ponte & Quaresma, 2016) and eliciting (Fraivillig et al., 1999). These concepts describe how teachers try to help students think and solve problems without the teacher simplifying. A third phenomenon is uptake (Correnti et al., 2015), which describes situations where student ideas or reasoning are taken up for discussion. This requires that the teacher first gets access to the students thinking through interactions such as monitoring (Stein et al., 2008), eliciting (Fraivillig et al., 1999), or elaborating (Henning et al., 2012). By accessing, sharing, and taking student ideas up for discussion, students are positioned as owners of ideas and their thinking as a core element of teaching and learning. As these three examples illustrate, conceptualizing the classroom brings our understanding forward. Some authors have developed frameworks, either created to develop certain aspects of teaching (e.g., talk moves by Kazemi & Hintz, 2023) or created to be capable of describing all kinds of interactions in the classroom (e.g., the IC model by Alrø & Skovsmose, 2004). One such model is the redirecting, progressing, and focusing framework (Drageset, 2014), which describes 13 interactions in three groups. Redirecting describes interactions where the teacher aims to change the student's approach by putting aside, advising a new strategy, or correcting questions. Progressing describes interactions where the teacher aims to move the process forward, often towards an answer or solution, by either demonstrating, simplifying, using closed questions for each step, or using more open questions and probing. Focusing describes interactions where the teacher stops the progress to focus on something important by requesting students to enlighten details, justify or apply a method to similar problems, or by pointing out (through a recapitulation or telling the student to notice something).

Positioning theory

Positioning theory is a way to understand interaction by studying communication acts, positions, and storylines created when people interact (Davies & Harré, 1990; Harré & Moghaddam, 2003). Studying positions can shed light on how discourse and participation impact students' learning opportunities (Hiebert & Grouws, 2007) and can contribute to understanding the challenges teachers face when trying to change their practice. A key feature of positioning theory is that it seeks to understand the dynamic nature of social relationships, describing how people adopt and develop. This immanent nature of positioning theory makes it useful in mathematics education for studying how discourse unfolds and the consequences of that unfolding (Wagner & Herbel-Eisenmann, 2009).

A position is defined by its rights and duties (Harré & Moghaddam, 2003); thus, if you are a teacher, you typically have the right to lead the classroom but also the duty to help students learn. Storylines are loose clusters of narrative conventions (Harré & Moghaddam, 2003). These can emerge as participants interact, but past experience and longer-term storylines might mean that participants are simultaneously in different storylines without realizing it (Wagner & Herbel-Eisenmann, 2009). Communication acts are an integral part of positions and storyline building, but also where it becomes possible to identify positions and storylines.

Taking on a position is also about positioning others, and another layer to positioning theory, willingness, capability, and power (Davies & Harré, 1990), contributes to explaining why there are

differences in taking on positions during an interaction. This is particularly relevant for teacher-student interactions, where long-established dynamics shape discourse patterns. Willingness describes whether or to which extent participants accept a position or position others. Capability describes to which extent a person is capable of taking on a position. Power describes how and to which extent participants are allowed to take on positions (Harré & Moghaddam, 2003). Together, the three aspects of positioning theory (communication acts, positions, and storylines) and the understanding of how willingness, capability, and power affect positions provide a way to connect classroom interactions with longer-term storylines about mathematics teaching and learning.

Drageset and Ell (2024) used positioning theory to review the literature on concepts describing teacher interactions. First, they categorized 94 concepts into 14 communication acts. This was then used to identify five distinct teacher positions within three storylines. The first two positions, *a teacher who tells* and *a teacher who supports*, both describe teachers who position themselves as the ones who know and the students' needs to position themselves as listeners and to accept help. Thus, this creates a storyline of *teachers as providers of insight*. The third and fourth positions, *a teacher who uses student ideas to create learning* and *a teacher who orchestrates*, both describe teachers who position themselves as facilitators for learning and not providers of insight, thus creating the storyline of *teachers as facilitators of learning.* The fifth position, *a teacher who participates*, describes teachers who position themselves on the same level as students and work together to learn, thus creating a storyline of *teachers as participants in learning*. The authors emphasize that the immanent nature of positions means that these might change many times during a lesson.

Method

To answer our question and aim of studying a proficient teacher, we chose a teacher well known for conducting mathematical talk during lessons. This teacher has more than twenty years of experience in teaching mathematics, has a master's degree in mathematics education, is educated and working as a teacher specialist, and teaches colleagues how to teach dialogic and inquiry-based mathematics in the area. The teacher has a solid knowledge of research literature and is teaching a second-grade class (students aged 6-7), where she has been the main teacher since they started in grade one.

We observed all her mathematics teaching for two weeks to study the teacher's practice. The data were nearly ten hours of teaching, where the content of the lessons can be described as numbers, basic calculation, and number sense. One could discuss if this is sufficient to describe a teacher's practice, but it is close to the standards set by the Learners Perspective Study (Clarke et al., 2006), often referred to as best practice. The observation was done using a video camera and a microphone on the teacher. This captured everything the teacher said and what students said to the teacher or in plenary talk, and the video captured all activity and information on the blackboard and smartboard. We found that the interactions during plenary talk were more varied, and since much of the student talk was difficult to grasp outside of plenary, we chose to limit our study to plenary talk.

The analysis was conducted using nVivo in two phases. In the first phase, all teacher interactions in plenary were coded using the redirecting, progressing, and focusing framework (Drageset, 2014). The coding was done on a turn-by-turn basis (Linell, 1998), where each utterance was coded. Such coding of single utterances is not done in vacuums; each utterance must be understood in light of prior

utterances, particularly the last one. Significant adjustments were made to the framework during the coding by adding interactions (see Table 1). This is not surprising, as the original framework was developed based on five quite regular teachers, while this data is from a proficient teacher who knows the research literature on teacher interactions. In the second phase, we grouped the interactions in new ways to shed light on different aspects of the teacher's practice, using thematic analysis (Braun & Clarke, 2017). In this phase, we also used positioning theory as a tool to clarify each of these category groups. These phases were not pre-planned; instead, the first phase was done to explore, and the second phase came as a result of searching for meaning and answers to our research question.

Findings and discussion

During the coding of all teacher interactions using the redirecting, progressing, and focusing framework (Drageset, 2014), we changed the framework when we observed qualitatively different interactions (see Table 1)

Table 1: Overview of interactions observed

Interactions from the redirecting, progressing, and focusing framework (those in brackets were not observed)	Closed question - Correcting questions - Demonstration - Enlighten details - Justification - Open question - Put aside - Recap - Simplification (Advising a new strategy) - (Apply to similar problems) - (Notice) - (Request assessment from other students)
New interactions observed	Appreciate errors - Confirm - Connect - Correcting explanation - Encourage - Repeat - Request alternatives - Thinking time - Uptake

Using the concepts from the redirecting, progressing, and focusing framework (Drageset, 2014) and the additional interactions described during the analysis (Table 1), we developed a set of groups related to how interactions were used. These groups gradually developed into seven aspects of the teacher practice. Each aspect describes a part of the teacher's practice, and together, the seven aspects characterize this teacher's practice.

The first aspect, *alleviate*, describes interactions in which the teacher worked to reduce students' fear of trying and participating. *Alleviating* was particularly evident when the teacher encouraged students to work and share. This included reminders that errors were accepted and part of the learning process, such as saying, *"Yes, yes, but we can think wrongly, can't we? That's totally fine*."

The second aspect, *developing ideas (something to talk about)*, describes interactions where the teacher gives the students time to think alone or talk with peers (turn and talk). Thinking time was often given after open questions (open progress initiatives). We saw how simple interactions such as *thinking for a minute* or *turn*ing *and talk*ing are powerful, as they enable everybody to develop ideas or suggestions, and multiple ideas are necessary to create real discussion.

The third aspect, *involving students*, describes interactions where the teacher invites students to participate in the discourse. One way to do this was to *enlighten detail*, where the teacher requests

students to tell all the steps they used towards a solution, how they think about a problem or concept, or request *justification* of an answer or suggestion. In addition, the teacher did not stop at the first idea but instead requested students to share *alternative ideas*, such as this: *"Okay, did anyone think of another way to reach 45? "*. Sometimes, one or more ideas were taken up for discussion and further development (uptake), which was either done by challenging or directing attention towards an answer or idea. The effect of such interactions is that students are involved in the discourse and that ideas are shared and discussed, giving them opportunities to learn from each other.

Table 2: The seven aspects that characterize the observed practice. Aspects in the top row and the connected interaction concepts in the second row.

Alleviate	Develop ideas (or something to talk about)	Involve students	Keeping focus	Conclude	Telling	Reduce complexity
Encourage Appreciate errors	Open questions Think alone Think together (turn and talk)	Enlighten detail Justify Request alternative ideas Uptake	Remind Point out Repeat Confirm	Recap Connect Put aside	Demonstrate Correcting question Correcting explanation	Closed questions Simplification

The fourth aspect, *keeping focus*, describes interactions where the teacher worked on keeping the students focused on the task or discussion. Interactions such as *remind*ing, *pointing out*, *repeating,* and *confirm*ing were typical of how the teacher worked to keep the focus. While reminding and repeating are used to get the students back on track, pointing out is used to emphasize something important, and confirming can assure them that this is where the focus needs to be. One example of a reminder is when a student asks if just writing the answer is fine, and the teacher says: ”No*, you know that I do not think that is enough; you have to think about how you found the answer”.* This example also illustrates how reminders are used to keep focus on how they should work.

The fifth aspect, *conclude*, describes interactions where the teacher ended the discourse on a task or theme to move onto something new (another discussion, change to working on tasks, or ending the lesson). This was done by either *recap*, *connect*, or *put aside*. When using recap, the teacher summarized the discussion to conclude. When using connect, it was more pointed towards a connection that was important to remember. With put aside, it was also a way of ending a discussion by staying with an established answer or definition. One example of a recap is when the students and teachers discussed which way -7,5 should be rounded, as -7 is larger than -8, and the teacher ended the discussion by saying: *”As you say you should round towards the largest number, that is 8”*.

The sixth aspect, *telling*, describes interactions where the teacher tells or informs students about mathematical ideas or procedures. This was mainly done in two ways, by either *demonstrating* or *correcting*. When demonstrating, the teacher presented ideas or procedures while expecting students to listen, and this is often done in plenary and when introducing new ideas. When correcting, the teacher either corrected by a question or by explaining. An example of a correcting explanation is: *"Or the associative law tells us that we can choose which number we want to start with"*.

The seventh aspect, *reducing complexity*, describes interactions where the teacher made it easier for the students to solve a task or answer a question. This was done by either asking *closed questions* or by *simplification*. When the teacher used closed questions, they were often simple, one-step questions like this: *"Okay, so 34 minus 4, how much is that?"*. The use of simplification typically adds information about how to solve a task.

The seven aspects are presented in Table 2, with related interactions. The first four (alleviate, develop ideas, involve students, and keep focus) describe how the teacher organizes classroom discourses. There is consistent work on preparing the participation in discourses by ensuring the students feel safe and valued (alleviate) and that everyone has something to share (develop ideas). The discourses involve students in various ways by enlightening their thinking, requesting alternatives, focusing on justification, and taking ideas up for discussion. While this goes on, the teacher works on keeping focus. Looking at Drageset & Ell's (2024) five positions, these four fit well into the position of *a teacher who uses student ideas to create learning* as alleviate and develop ideas about creating opportunities for all to participate, involving students, and keeping focus on managing a discourse based on student ideas. This further belongs to the storyline *of teachers as facilitators of learning*, and facilitating is a precise description of these four aspects.

The last three aspects (conclude, telling, reduce complexity) describe different ways the teacher shares insight and knowledge. Concluding and telling belong to the position of *a teacher who tells* since both are about the teacher sharing knowledge. Reducing complexity belongs to the position of *a teacher who supports* since both closed questions and simplification are ways to help students find solutions. Consequently, concluding, telling, and reducing complexity fit into the storyline of *Teachers as providers of insight.*

While we separate into seven aspects, three positions, and two storylines to understand the practice, shifts between aspects happen frequently. This aligns with Wagner & Herbel-Eisenmann (2009), who state that positions and storylines are immanent and that several positions and storylines might exist simultaneously. This shifting of aspects and positions also means that students meet changing expectations during a lesson. *A teacher who tells* will try to position students as listeners, *a teacher who supports* positions students as receptive to help, while *a teacher who uses student ideas to create learning* positions students as owners of ideas and expects them to share these.

Conclusion

This article reported on a study of the practice of a proficient teacher with long experience and solid knowledge of research literature. We aimed to study the interactions and what this told us about teacher positioning. To do this, we used the redirecting, progressing, and focusing framework

(Drageset, 2014) as a starting point and further developed this with nine new concepts describing interactions. We then organized these interactions into seven aspects that characterized the practice. The first four (alleviate, develop ideas, involve students, keep focus) fit well into the position of *a teacher who uses student ideas to create learning* and the storyline of *teachers as facilitators for learning* (as defined by Drageset & Ell, 2024). The last three fit into the positions of *a teacher who tells* (conclude, tell) and *a teacher who supports* (reduces complexity) and the storyline of *teachers as providers of insight* (as defined by Drageset & Ell, 2024).

This article illustrates how interactions and positions are useful tools for characterizing practices and developing concepts that others can use. Concepts still need to be developed to characterize practices, as these can capture practices and nuances not described before. However, there is a growing need to review what we have and organize concepts describing limited interactions into more general ideas or frameworks.

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On the linguistic complexity of mathematics school-leaving exams: Differences between subject areas

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A common pool of mathematics tasks for the Abitur, the final high school exam, is provided by the IQB Institute in Berlin and is used by the 16 German federal states to create their exam questions in the areas of calculus, linear algebra, analytic geometry and stochastics. With the help of natural language processing, we compare lexis, syntax and changes in semiotic modality in these tasks for the different subject areas to evaluate the features of context-based task design that might be typical for those subject areas. Possible implications for the linguistic complexity of the tasks are discussed.

Keywords: Secondary school mathematics, assessment, Natural Language Processing.

Introduction

Given the importance of school-leaving examinations for society as a whole and their impact on school life and access to tertiary education, their tasks have been subject of discussion over the years; see Bruder (2021). In Germany, a common pool for the tasks of the final examination (Abitur) has been introduced in 2017 to centralise the tasks across the federal states. The pool aims to cover the content and competencies that the curriculum develops in the last two years of school. As described by Frenken et al. (2024), the tasks have a comparatively low cognitive potential. Furthermore, the design of the task often involves a context that needs to be understood to make sense of the mathematical activities with the amount of text in the tasks being criticised, see Jahnke et al. (2016). The aim of this project is to study the exam questions in terms of linguistic complexity. We investigate the differences between the subject areas of calculus, linear algebra, analytic geometry and stochastics as curricular differences might have an impact on the linguistic structure of the task text as well.

The design of Abitur tasks in mathematics

After the unsatisfactory results of German students in the PISA and TIMSS assessments in the early 2000s, the idea of centralising the curricula and the final examinations was developed. To achieve better comparability this led to the introduction of *educational standards* by the Standing Conference of the Ministers of Education and Cultural Affairs (KMK) in 2004, for the primary and lower secondary levels, and in 2012 for the upper secondary level and the Abitur examinations. The educational standards define argumentation, problem solving, modelling, the use of representations, the use of symbolic, formal and technical elements and communication as the central competencies in the subject of mathematics at the upper secondary level, which are to be worked on at three different levels of requirement. Abitur exams include at least two of the three subject areas calculus, linear algebra/analytic geometry and stochastics (KMK, 2012). In general, mathematics tasks in the common Abitur pool seem to contribute to a centralisation not only of the examinations but also of the preparations beforehand (IQB, 2024). Nevertheless, this form of assessment does not seem to be living up to expectations. On the one hand, a generally low and diverse level of cognitive potential

for mathematical skills is found in Abitur tasks (Frenken et al., 2024). On the other hand, it remains unclear which task elements influence task difficulty aside from mathematical performance**.**

Mathematical register and linguistic complexity in mathematics tasks

Different contexts require different linguistic resources, a register defined by the subject matter about which the communication takes place (Halliday, 1994). In the school register students are required to read and write texts according to certain conventions (Schleppegrell, 2001). Among others, this register can be characterised by technical lexis and structured clauses with logical relations, for example via the use of noun phrases, nominalisations, relative clauses, prepositions and conjunctions (Schleppegrell, 2001). As register is domain-specific (Halliday, 1994), language features in mathematics differ from those of other domains (e.g. Lee, 2006; Österholm & Bergqvist, 2013). First, in the mathematical register symbolic language and illustrations are used that complement the natural language (O'Halloran, 2015). Natural language plays a significant role in contextualised mathematics tasks, partly due to its use in the description of problem situations. "[However,] it is not clear how, or even if, mathematical texts […] can be described in common linguistic terms" (Österholm & Bergqvist, 2013, p. 762). First approaches to the mathematical register describe texts as syntactically dense and compact (Lee, 2006; Österholm & Bergqvist, 2013). Furthermore, the mathematical lexis consists of a mixture of mathematics-specific vocabulary, everyday language expressions and those to which a different meaning is assigned in mathematics, all of which have different effects on the complexity of mathematical tasks (Lee, 2006; Dyrvold et al., 2015). Following Halliday's (1994) idea of domain-specific language, it remains necessary to specify the 'register' of mathematics tasks in relation to (curricular) characteristics, as those might not only influence the situational contexts of the tasks, but their linguistic structure as well (e.g. Österholm & Bergqvist, 2013).

Students' language skills are closely related to their test performance or the difficulty of the task in mathematics (Peng et al., 2020). In recent years, a variety of studies in German mathematics education have combined several dimensions of language, such as syntax, lexis and structural features like word count, to estimate the linguistic complexity of mathematics tasks, partly using Natural Language Processing (NLP) to investigate several linguistic features at once, as well as larger text samples (e.g. Bednorz et al., 2024; Niederhaus et al., 2016). First, changes in semiotic register could reduce task difficulty, as they provide additional or explanatory information (Martiniello, 2009). Symbolic notation, however, follows its own grammar and therefore could also increase task complexity when there is no redundancy between symbolic sequences and natural language (Österholm, 2006). Gürsoy et al. (2013) show that elements of text length have a positive correlation with test performance, see also Niederhaus et al. (2013) and Österholm & Bergqvist (2013). Linguistic elements close to everyday experiences, such as personalisation (e.g. pronouns), familiar words and shorter sentences, might contribute to a lower task difficulty (e.g. Bednorz et al., 2024). Dyrvold et al. (2015, p. 1) suggest, however, that everyday vocabulary may as well increase the linguistic complexity of mathematics tasks as it is "uncommon" in the mathematical context. Academic language, on the other hand, with referential structures or unfamiliar vocabulary seems to increase task complexity (Schleppegrell, 2001), especially in mathematics (Bednorz et al., 2024; Dyrvold et al., 2015).

In this project we aim to identify the occurrence of factors of linguistic complexity in tasks of different subject areas: *Do school-leaving exam tasks differ in lexical and syntactical complexity or changes of semiotic modality according to their subject area?*

Sample and methods

Based on the studies presented above, we operationalised linguistic complexity through central linguistic features of three dimensions of the mathematical register: *change in semiotic modality, lexis* and *syntax,* see Table 1. For this study, a quantitative approach through Natural Language Processing (NLP) was chosen to identify patterns on a larger scale. The study excluded the morphological level, as well as the semantic level, as those are the ones requiring a manual or more qualitative approach instead of NLP. Furthermore, structural features, such as word count, as well as readability indices and coherence values were later added in the project but are not the focus of this study. The analysis was based on the Abitur tasks for the years 2017 to 2023 provided in the Common Pool of Abitur Tasks for the Federal States. These included 562 Abitur tasks with 2721 informative or instructive task sections. First, the tasks were categorised according to the subject areas they addressed. Pictures were separated from written language, and symbolic sequences were replaced by "XX". All verbal elements per task made up one unit of analysis. NLP was used to automatically measure the linguistic key figures, mostly the German language model of the Python library spaCy (Honnibal et al., 2020). In addition, the corpus SUBTLEX-DE (Brysbaert, 2024) for German subtitles, a corpus established based on German Wikipedia articles (Zenodo, 2024) and a corpus of the Abitur tasks themselves have been consulted to estimate the amount of everyday language, technical/academic language and the language in Abitur tasks. The degree of abstraction of the vocabulary was determined through the German list of extrapolated affective norms (GLEAN), which assigns empirical concreteness values to German lexemes (Lüdtke & Hugentobler, 2023). Finally, we approached syntactic density through indicators such as prepositions, as parts of prepositional phrases, in comparison to conjunctions, or the mean length of sentences, and, on the other hand, syntactic dependency through features such as pronouns or dependency distance in general. An ANOVA with pairwise t-tests and Bonferroni correction was used to determine differences between different subject areas. Where the graphical inspection indicated a violation of requirements, a Kruskal Wallis and Dunn test were applied.

Table 1: Elements of linguistic complexity that were examined, sorted by language dimension

Semiotic modality	Lexis	Syntax
• Symbolic sequences • Reference to pictures	• Everyday language • Non-mathematical technical language • Vocabulary of Abitur tasks • Variance • Concreteness	• Sentence length • Dependency distance • Entropy • Prepositions, pronouns, (coordinating and subordinating) conjunctions

Linguistic analysis

First, tasks from different subject areas show large differences with respect to their reference to semiotic modalities other than natural language, see Table 2. The investigation of the use of symbolic sequences and reference to pictures shows that stochastics tasks make comparatively less use of

symbolic sequences, while for calculus tasks the opposite is true. Meanwhile, tasks in calculus or analytic geometry refer significantly more often to visualisations.

The vocabulary used in stochastics tasks can be identified as closer to everyday language according to the frequency of its words in the subtitle corpus. The opposite is true for calculus and analytic geometry tasks. In addition, words in calculus tasks show a lower frequency in the Wikipedia corpus and therefore seem to be more distant to the academic register. The vocabulary used in stochastics tasks shows the least variable lexis in terms of unique words per task, which is significant in comparison to tasks of analytic geometry, in which the words used in the text also seem to be the most concrete. The most abstract vocabulary can be detected in calculus tasks.

Although there was no difference in sentence length or dependency distance, as can be seen in Table 2, there is a slight difference regarding the entropy values between the subject areas. Stochastics tasks show a lower entropy level which might hint to a lower information density. For words that convey dependencies within or across sentences or which contribute to syntactic compactness the findings are mixed. First, prepositions and pronouns appear in a higher proportion in calculus tasks, whereas the least were found in stochastics tasks, with a generally large difference in use of prepositions.

Table 2: Mean, standard deviation, variance analysis, pairwise comparison and effect size for linguistic characteristics of three language dimensions according to subject area

	Calculus (n = 182)	**Lin. Algebra** (n = 79)	**Stochastics** (n = 191)	**Anal. Geom.** (n = 110)	**Variance analysis**	
Semiotic Modality	*M(SD)*	*M(SD)*	*M(SD)*	*M(SD)*	$F(3,558)$	η^2
Symbols[1]	5.50^{a} (4.09)	4.10^{b} (3.01)	0.82^{c} (1.19)	4.24^{b} (2.79)	83.9****	0.31
Ref. to pictures[1]	1.02^{a} (1.02)	0.25^{b} (0.59)	0.22^{b} (0.50)	0.76^{a} (0.95)	37.2****	0.17
Lexis						
Subtitle[2]	1.17^{c} (0.50)	1.38^{b} (0.48)	1.61^{a} (0.42)	1.24^{bc} (0.40)	32.8****	0.15
Wikipedia[2]	1.71^{b} (0.18)	1.83^{a} (0.24)	1.77^{a} (0.22)	1.78^{ab} (0.24)	6.38***	0.03
Abitur tasks[2]	3.31^{a} (0.27)	3.27^{a} (0.38)	3.00^{b} (0.24)	3.30^{a} (0.24)	50.6****	0.21
Variance[3]	79.0^{ab} (46.0)	78.8^{ab} (47.4)	71.3^{b} (38.4)	89.0^{a} (52.6)	3.60*	0.02
Concreteness[4]	3.70^{c} (0.24)	3.79^{bc} (0.24)	3.84^{b} (0.32)	3.93^{a} (0.20)	19.6****	0.10
Syntax						
Sentence length[4]	15.5 (4.07)	15.8 (4.15)	15.1 (4.06)	15.0 (4.20)	0.89	0.01
Dependency distance[4]	2.94 (0.59)	2.99 (0.56)	3.04 (0.55)	2.93 (0.49)	1.39	0.01
Entropy[4]	2.90^{a} (2.20)	3.04^{a} (2.59)	2.07^{b} (1.44)	3.18^{a} (2.18)	9.56****	0.05
Prepositions[5]	12.4^{a} (2.82)	10.2^{b} (2.93)	8.62^{c} (2.85)	9.48^{bc} (2.61)	59.8****	0.24
Pronouns[5]	4.94^{a} (2.08)	4.96^{ab} (1.86)	4.42^{b} (1.92)	5.02^{ab} (1.63)	3.57*	0.02
Coordinating conj.[5]	1.99^{b} (1.53)	2.63^{a} (1.55)	1.66^{b} (1.73)	2.74^{a} (1.83)	12.6****	0.06
Subordinating conj.[5]	1.05^{b} (0.88)	1.25^{b} (1.34)	1.80^{a} (1.28)	1.25^{b} (0.90)	15.5****	0.08

* $p < .05$; ** $p < .01$; *** $p < .001$; **** $p < .0001$

[1] total number; [2] mean word frequency in corpus; [3] number of unique tokens; [4] mean value; [5] in percentage of task text

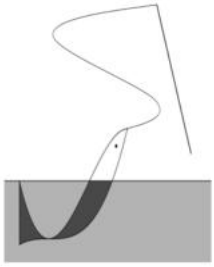
Linear algebra and analytic geometry show higher proportions of coordinating conjunctions, while stochastics tasks were observed with the most subordinating conjunctions.

Discussion

In conclusion, none of the four subject areas shows an overall higher level of linguistic complexity. Instead, each of them can be characterised by different elements of complexity which may also need to be addressed differently in classes preceding the exams. This supports the hypothesis that the domain of a (mathematical) text determines the register used (Halliday, 1994). The different register features may lead to different factors influencing the linguistic complexity of the task.

Calculus tasks, see Figure 1 for an example, contain a comparatively high number of symbols and references to pictures. Although it remains unclear whether those characteristics increase their linguistic complexity or not (Martiniello, 2009; O'Halloran, 2015; Österholm, 2006), tasks of that domain seem more typical for the multimodal mathematical register (Lee, 2006; O'Halloran, 2015). The vocabulary used in the calculus tasks also seems to be typical for the Abitur tasks themselves, which may also stem from the fact that these together with stochastics tasks make up a larger proportion of the sample. For calculus there has been a debate as to whether the context of the Abitur tasks is necessary or if the actual focus still lies on calculating recurrent algorithms instead of adapting to the real-life situations (e.g. Frenken et al., 2024). From a lexical perspective, there is no indication of this in comparison to the other subject areas as the vocabulary in calculus tasks is not significantly less variable than that of other subject areas, which would have indicated a higher proportion of recurring task formulae. However, a further analysis of phrases in calculus tasks, and the register of

Die Abbildung zeigt das Logo eines Geschäfts für Anglerbedarf. Die obere Spitze der Schwanzflosse des Fischs liegt auf der Wasseroberfläche; die Strecke zwischen oberer und unterer Spitze der Schwanzflosse steht senkrecht zur Wasseroberfläche.



Bei Verwendung eines geeigneten Koordinatensystems kann die untere Begrenzungslinie des Fischs mithilfe der in $\mathbb{R}$ definierten Funktion $u: x \mapsto \frac{1}{8}x^3$, die obere Begrenzungslinie mithilfe der in $\mathbb{R}$ definierten Funktion $v: x \mapsto \frac{1}{4}x^2 \cdot (4 - x)$ beschrieben und die Wasseroberfläche durch die Gerade mit der Gleichung $y = \frac{5}{4}$ dargestellt werden.

a Zeigen Sie, dass

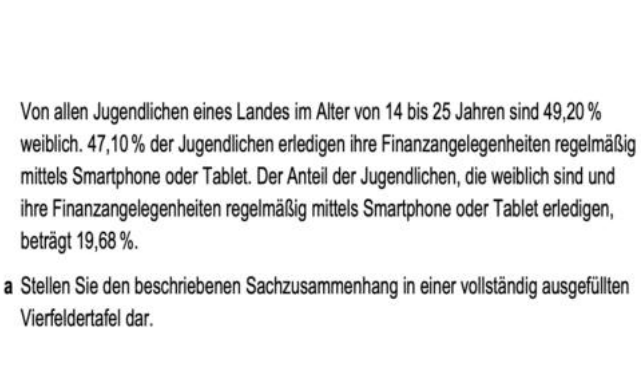
- die Graphen von u und v nur die Punkte $P(0|0)$ und $Q\left(\frac{8}{3}\middle|\frac{64}{27}\right)$ gemeinsam haben;

The illustration shows the logo of a fishing tackle store. The upper tip of the fish's tail fin lies on the surface of the water; the distance between the upper and lower tip of the tail fin is perpendicular to the water surface. [illustration] Using a suitable coordinate system, the lower boundary line of the fish can be determined using the function u: … defined in $\mathbb{R}$, the upper boundary line by using the function v: … defined in $\mathbb{R}$ and the surface of the water is described by the straight line with the equation $y = \frac{5}{4}$.

a Show that

- *the graphs of u and v only have the points P … and Q … in common; […]*

Figure 1: Example of part of a calculus task (introduction, subtask, symbols, illustration, & translation)



Von allen Jugendlichen eines Landes im Alter von 14 bis 25 Jahren sind 49,20 % weiblich. 47,10 % der Jugendlichen erledigen ihre Finanzangelegenheiten regelmäßig mittels Smartphone oder Tablet. Der Anteil der Jugendlichen, die weiblich sind und ihre Finanzangelegenheiten regelmäßig mittels Smartphone oder Tablet erledigen, beträgt 19,68 %.

a Stellen Sie den beschriebenen Sachzusammenhang in einer vollständig ausgefüllten Vierfeldertafel dar.

Of all young people in a country between the ages of 14 and 25, 49.20 % are female. 47.10 % of young people regularly manage their financial affairs using a smartphone or tablet. The proportion of young people who are female and regularly handle their financial affairs using a smartphone or tablet is 19.68 %.

a Describe the factual context in a fully completed 2x2 contingency table.

Figure 2: Example of part of a stochastics task (introduction, subtask, & translation)

mathematics tasks in general, as well as a content analysis, is needed to verify this claim. Calculus tasks have a higher proportion of prepositions and pronouns, as referential structures, compared to tasks from other subject areas, which may increase syntactic complexity, but also contributes to a higher contextualisation and personalisation of the task text which reduces task complexity (Bednorz et al., 2024). On the other hand, calculus tasks seem to consist of a higher number of main clauses than other tasks according to their lower proportion of conjunctions. This may indicate that referential structures are established through symbolic sequences and more dense prepositional phrases.

Stochastics tasks seem to be less variable in terms of their reference to other semiotic modalities. They implement fewer or no symbolic sequences or references with respect to pictures; see Figure 2 for an example. On the other hand, the vocabulary can be characterised as closer to everyday language which may increase its linguistic complexity (Dyrvold et al., 2015) or as well decrease it being closer to the reader's everyday context (Schleppegrell, 2001). According to the elevated use of subordinating conjunctions, hinting to more unpacked syntactic structure, stochastics tasks appear less syntactically dense, also according to the entropy value, indicating a possibly lower task difficulty (Bednorz et al., 2024). However, the comparatively stronger use of subordinating conjunctions may indicate a more complex description of conditions and references in stochastics tasks. This could lead to a higher cognitive load as the solution of stochastics tasks often depends on the identification of such relations; further analysis is required here. Tasks in linear algebra cannot be characterised as (not) significantly complex. Only their use of non-mathematical technical language could be a factor to higher task difficulty (Dyrvold et al., 2015). Linear algebra tasks rarely refer to images and implement symbolic sequences roughly as often as do analytic geometry tasks. Especially without the implementation of supporting visuals the task difficulty might be increased (Martiniello, 2009). Coordinating conjunctions make up a higher proportion in linear algebra tasks. This potentially increases the referential structure but decreases syntactic density (Bednorz et al. 2024).

Abitur analytic geometry tasks show some similarities to linear algebra tasks, for example concerning the use of conjunctions. However, they contain a significantly greater number of references to visualisations, which might increase the task difficulty as this means a further change in semiotic register, but it might as well offer additional information (Martiniello, 2009; O'Halloran, 2015). The task lexis seems to be more variable, which might indicate a higher task difficulty (Bednorz et al., 2024). Furthermore, the vocabulary is more concrete, which suggests a diversity of real-life situations

in analytic geometry tasks, while it remains unclear whether this increases or decreases the task complexity.

Outlook

The concept of linguistic complexity in this study is based on a selection of linguistic parameters. Some of them have not been standardised, which makes the result less comparable and prevents an absolute interpretation of the complexity of Abitur tasks for different subject areas. Any type of comparison, whether it might be with Abitur tasks of a different subject, from a different time span, or with final examination tasks in other countries or tasks from textbooks (see Niederhaus et al., 2016) could usefully supplement the results of this linguistic analysis. This study also sets aside almost all investigations of the pictures and symbolic sequences used in the tasks which makes any estimation of the difficulty in understanding the whole task text obsolete, since the mathematical register is not represented in its entirety (O'Halloran, 2015). Especially calculus and analytic geometry tasks would show a different complexity structure as those make the most use of symbolic sequences and visualisations. Finally, the results of the linguistic analysis should be linked to the students for whom the tasks are designed. The main potential of this research lies in the possibility to compare linguistic elements in mathematics tasks with test performance, see e.g. Dyrvold et al. (2015), and reading behaviour to determine how and whether certain complexity factors influence the solution process.

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On using operators to introduce negative numbers: How do students describe states and transformations?

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The aim of this paper is to focus attention on the use of the operator interpretation as an appropriate language-sensitive approach for introducing negative numbers. We derive the distinction between 'states' and 'transformations' from different theoretical considerations and utilise this as a framework for a qualitative content analysis. The study investigates which lexical means are used by 8th grade learners while giving examples for the use of negative numbers. Nine interviews were conducted and analysed, and the subsequent collection of learners' utterances offers insights into the lexical means enabled by the learners. From these utterances we can draw conclusions regarding their conceptual understanding of negative numbers. The empirical results can also be used for additional research on how building up an operational conception of negative numbers can be enhanced in mathematics lessons.

Keywords: Negative numbers, operator interpretation, numerical extensions, states, transformations.

Introduction

Learners encounter the concept of negative numbers on a theoretical basis usually by the start of secondary education. This can be explained by the fact that the introduction of integers emphasizes that learners become familiar with counting numbers and performing simple additions and subtractions on non-negative integers. Furthermore, these activities do not require learners to have a theoretical concept of negative numbers from preschool and elementary education. Teaching the sequence on numerical extensions in German schools corresponds to what Bruno and Martinón (1999) describe in the Spanish context as the learning progression: namely, by progressing from integers via positive rationals to rational numbers including negative numbers. Subsequently, with up to four years of operating with positive integers only and two more years that include positive rational numbers, that are used to build up mental representations of the concept of numbers, "new numbers are adjoined to the old system" (ibid, p. 792). This perspective can easily lead to students developing a *deficit understanding* of negative numbers as they are subsequently added to the previously known 'normal' numbers. Indeed, more than thirty years ago, Davidson (1987) remarked that introducing negative numbers in this manner "requires students periodically to overthrow their previously well-founded intuitions" (p. 431). Since negative numbers are not "representable concretely as manipulable objects" (ibid), learners experience that the concept of cardinality is *limited* to non-negative integers – a perception by which a deficit understanding of negative numbers is further promoted. This is also indicated by the results of Fabian (2023), who found that 7th grade learners' described negative numbers using expressions such as 'less than nothing' or 'the quantities beyond nothingness', suggesting that their conceptual understanding of negative numbers still seems to rely on cardinal interpretations.

Negative numbers are often introduced by the help of familiar real-word contexts or models, in which the use of negative numbers is the expression of a *state*, e.g., temperatures below 0°C, debts or

elevation. The difficulty of these contexts, as Freudenthal (1983) puts it, is that the negative numbers are "introduced as results of operations, which formerly were impossible" (p. 438), which again reinforces a deficit understanding rather than building up sustainable conceptions. An indication of this can be found in learners' language when reasoning about negative numbers as something that is 'less than nothing' or that indicates 'the quantities beyond nothingness' (Fabian, 2023).

These initial considerations show how *describing* negative numbers in a deficit way and having deficit *conceptions* of them could be linked. Conversely, for the introduction of negative numbers and to develop sustainable conceptions of negative numbers, learners need to get access to the crucial lexical means that can support the process of concept formation. In this paper, we argue that one possible approach for the introduction of non-positive integers, which doesn't rely on a deficit description of negative numbers, can be found in enhancing an *operational conception* of them. With the aim of supporting learners' language development, we investigate which lexical resources learners can activate when talking about negative numbers and how their use of lexical means can be linked to an operational conception of negative numbers.

Theoretical Background

This paper builds on the theoretical connection between language and thinking that has been discussed frequently in mathematics education research (e.g., Pimm, 1987; Sfard, 2008). These findings emphasized that language as a learning medium plays an important role in the process of developing conceptual understanding of a mathematical topic: Language unfolds 'epistemic power' and can make a significant contribution to building up sustainable conceptions as "it shapes the thinking" (Pöhler & Prediger, 2015, p. 1701). As the introduction of negative numbers should be taught in a way that combines lexical and conceptual learning in class, appropriate *lexical means* are required. By the term 'lexical means' we refer to words, sentence components, syntactical constructions, and graphical representations that need to be known – need to be mastered – by learners to, on the one hand, communicate about the concept of negative numbers and, on the other hand, construct the meaning of the concept (Pöhler & Prediger, 2015).

To learn more about how to enhance learners' understanding of negative numbers using lexical means, it's important to determine the mathematical concepts and mental representations that learners need access to. While numbers can be used to describe states in the world that surrounds us, they can also be used to describe processes or actions that transform one number into another, i.e., heating up and cooling down, spending and earning money, or shifting something to the left or right. With the introduction of *negative* numbers, these situations of use are extended to states 'below zero' and transformations that are applicable to *any* states, thus allowing unrestricted backward movement on the number line. Hence, two conceptions that appear to be crucial for the introduction of negative numbers can be found in using them to express *states* and *transformations* (Kirsch, 1977).

In our view, it is particularly the conception of *transformations* that can support learners' understanding of negative numbers and of operations with integers, as a didactical approach proposed by Sfard (1991). Sfard distinguishes between 'structural conceptions' and 'operational conceptions' as two different stages of concept formation learners must go through. What Sfard indicates for mathematical concepts in general is also applicable to negative numbers. Before a person can treat

negative numbers as an (theoretical) object and recognize their meaning as a subset of integers (structural conception), the person needs to become skilful in performing subtractions (operational conception). Therefore, when dealing with negative numbers, an operational conception needs to be developed *first* and only afterwards can a structural conception slowly evolve from the previously established operational conception. With reference to Jourdain (1956), Sfard (1991) denotes negative numbers as "nothing but a type of process" (p. 12).

Considering both lexical and conceptional perspectives, our initial point of research is founded in what Sfard (1991) concludes her paper with: "We ought to ask ourselves what means should be used to ensure that the students go safely through these doubts-about-meaning periods when they still feel certain uneasiness about the object they have to manipulate" (p. 34). Since the operational conception seems to be handled mainly through verbal representations[1] (Sfard, 1991), it is the *lexical means* that become even more important as they can help learners make the operational conception cognitively accessible. The study reported in this paper aimed to investigate the lexical resources that learners can (already) enable and to identify potential connecting points for supporting learners in language. Hence, the research question is: How do learners describe *states* and *transformations* while giving examples for the use and usefulness of negative numbers?

Methods

The study draws on the distinction between *states* and *transformations* that is considered as an important mental representation when introducing negative numbers and that is also made by Sfard (1991) as she addresses 'structural' and 'operational' conceptions in the context of learning and teaching mathematical concepts. With the aim of collecting lexical means that learners use to describe states and transformations in the context of negative numbers, the first author of this paper conducted nine interviews with 8th grade German students (aged 13-14).

In Germany, negative numbers are usually introduced in 7th grade. Therefore, it can be assumed that the interviewees have already had the chance to develop their conceptional understanding of negative numbers and to develop the lexical resources that are necessary for dealing with them. The sample included four students attending a grammar school (*Gymnasium*) and five students attending a comprehensive school. The length of the interviews varied from 8 to 14 minutes. The interviews were semi-structured and consisted of two parts: First, the students were asked to explain what they understand by the term 'negative numbers' and to give examples and possible contexts for the use of these numbers. To get an indication of which lexical means learners can enable, and how they can be linked to an operational conception of negative numbers, the learners' vocabulary and the structure of their utterances were the focus of the study. The participants were asked questions such as *What do you need negative numbers for? In what kind of situations do negative numbers appear to be important?* Second, they were requested to perform different tasks on comparing and operating with

[1] Since "verbal encoding […] must be processed sequentially, so it seems more appropriate for representing computational procedures" (Sfard, 1991, p. 7).

negative numbers. The second part of the interview is not within the scope of this paper, thus, only the analysis of the first part is described below.

All interviews were videotaped, transcribed, and subjected to a qualitative content analysis according to Mayring (2015). The expression of *states* and *transformations* were deductively taken as the two basic codes of the analysis. Within the data, learners' answers that either included describing states explicitly or that indicated a structural conception of negative numbers were assigned to the code *States* (S). Likewise, learners' statements were coded as *Transformations* (T), if they verbalized transformation situations explicitly or if they indicated an operational conception. For example, the statement "If I have minus ten euros in my account now"[2] was coded as S while the statement "For example, if you are at the 4 [*puts a finger at the edge of the table*] and then go six steps minus that is backwards [*gestures a jump to the left with the finger*], you end up at the –2" was coded as T. To investigate the characteristics of the given descriptions further, sub-codes were then developed inductively. Namely the codes S1 (non-exist/non-possess), S2 (lack/need), S3 (negative magnitude or quantity) and S4 (static representation) as sub-codes of S and T1 (return/neutralize), T2 (arithmetic operation), T3 (operator interpretation for comparing numbers) and T4 (dynamic representation) as subcodes of T were added to the inventory of codes. The interviews were coded separately by both authors of this paper. Validity and reliability were established by comparing the two sets of results, clarifying codes and re-coding data until agreement.

(An Excerpt from the) Research Results

The empirically obtained results of the research consist of both the differentiated category system and the collected descriptions – being them state-bound or transformation-bound – that 8th grade learners gave when giving examples for the use of negative numbers. With regard to the paper's research question, the inventory of codes first shows that *both* types of descriptions (states and transformations) can be identified in learners' utterances. The developed sub-codes can then provide an initial indication of what the learners' descriptions of states and transformations are characterized by. By investigating the coded utterances in more detail, additional information on *how* the learners described states and transformations, when giving examples for the use of negative numbers can be collected, e.g.: which *lexical means* they used for describing states and transformations, what kind of *situations* and *contexts* they thought of when giving those descriptions, whether the given examples appear to be *suitable* for either describing states or transformations, at which points the learners seem to show *difficulties* explaining their examples (indicated by e.g., pauses in speaking, interruptions or searching for words) or *change the strategy of explaining* (from being state-bound to being transformation-bound and vice versa). In total, 50 utterances were coded for the first part of the interviews, of which 36 were assigned to the category *States* (S) and 14 to the category *Transformations* (T).

Due to the limited scope of this paper, the full category system will not be discussed in detail. As most of the coded utterances have been assigned to the category S, we will focus on presenting the *state* descriptions we found. Therefore, we will first show an excerpt from the inventory that depicts

[2] All utterances were translated from German into English by the authors for this paper.

an overview of the sub-codes S1 to S4 as well as examples for utterances that have been coded into the respective category (Table 1). Later, we will provide deeper insights into some of the interview transcripts to enrich the reported findings.

Table 1: Sub-codes of the category 'States' with illustrations from transcripts

Code	Example	Number of occurrences
S1 (non-exist/non-possess)	"Minus numbers, because you're just taking something that isn't there."	15
S2 (lack/need)	"And then you just go into debt because you still need three euros, […], that's minus three euros that are missing."	9
S3 (negative magnitude or quantity)	"So, you just have negative money."	11
S4 (static representation)	"For example, if you have a debt of ten euros. That's also minus ten at the bottom when you look at your account balance."	1

When the learners were asked to think of examples for the use of negative numbers, it was noticeable that they described a situation in which negative numbers are being used as an expression of a state rather than a transformation. The created situations were mainly characterized by states of 'non-existing' or 'lacking' with negative numbers used to express something 'deficient'. For example, the utterance "[You need negative numbers] to represent something with numbers that something is not there, but even less" (S1) indicates a deficit conception that the learner might have of negative numbers. Some learners also made use of lexical means that verbalize the (contradictive) state of having a negative quantity or magnitude, e.g.: "And Lisa has absolutely no money right now, so she still has minus twenty money" (S3). It turns out that some learners even tried to give state descriptions for situations that would have been more suitable for describing transformations, illustrated by the following transcript:

Interviewer:	Can you think of another example or situation in which negative numbers are being needed?
Student:	Hm, for example, when you buy something and you – you want to buy something that costs maybe five euros.
Interviewer:	Mhm.
Student:	But you only have two euros for instance.
Interviewer:	Yes.
Student:	And then you just go into debt because you still need three euros, and when you do the maths, that's minus three euros that are missing.
Interviewer:	Mhm. Can you describe again where exactly the negative numbers are important in your example?
Student:	They are important because they are not there, and they are missing. And because they– Yes, I'm not sure [*shakes head*] Äh [*4 sec*] Because if you had five euros now, for example, that would be zero that would be missing.

Because there are three euros missing, that's minus three euros, because they're not there and you need them.

The learner created a situation that could have been perfectly described as a transformation, since 'buying something' can be characterized as a process of action. One operation that would be suitable for that situation is: $2 - 5 = -3$. However, the learner only made use of state descriptions (e.g., 'you only *have* two euros', 'you still *need* three euros') and tried to connect them to grasp the whole transformation that is being processed within the situation. Two observations can be made regarding the learner's reasoning. Firstly, that the learner seems to struggle with describing the role of negative numbers in their example by only referring to states. This is seen by several interruptions in speaking, by many attempts of re-formulating the situation ('they are not there', 'they are missing', 'you need them') and by their self-verbalized insecurity ('I'm not sure'). Secondly, the learner seems to identify the negative number itself ('minus three') with the state of 'not-existing' or 'lacking', which led to incorrect descriptions ('that's minus three euros that are missing'). Those difficulties in explaining the concept of negative numbers, and describing situations of use, were observed with other learners too as the following transcript shows:

Student:	They [negative numbers] indicate the quantities beyond nothingness so after zero.
Interviewer:	Could you explain again what you mean by that?
Student:	There's even less of the nothingness. I have absolutely no idea how I can explain it in any way– so I don't know how I can explain it, that is– hm, it's– with temperatures for example there is zero degrees Celsius, for example, and then it goes up and down. There are still minus degrees then...

Again, a rather deficit conception of negative numbers was being verbalized by the learner who mainly made use of static descriptions. For example, the learner referred to a state of 'not-existing' ('There's even less of the nothingness') and to negative quantities ('quantities beyond nothingness'). The learner seemed to find it challenging to explain their understanding of negative numbers, as the learner interrupted themselves multiple times assumably trying to find the right words and explicitly repeating not to know how to explain any further. Thus, their utterances were being characterized by incomplete sentences and by an unprecise use of lexical means. The only description that could be read as an expression of transformation can be found in 'then it goes up and down'. This process seemed to be cognitively assigned to the context of temperatures for the learner; however, the relation to negative numbers was not made explicit.

We conclude the presentation of the research results by also taking a brief look at learners' utterances that have been coded into the category *Transformations* (T). Although we cannot provide detailed analyses of those utterances at this point, we would at least like to mention the following observations: The coded statements indicate that some learners seem to have a dynamic perspective on negative numbers ("And negative numbers basically run backwards." (T4)) – often verbalized by referring to movements along the number line. Also, transformations were expressed as situations of 'returning' or 'neutralizing' a few times. What can be observed within the respective utterances is that the learners verbally switched to deficit – or even incorrect – state descriptions when they seemed to struggle with further developing their self-created examples. The utterance "Hm maybe, if– that's [negative numbers] good when you get something back from someone, that then just– he has like minus things." (T1) exemplifies that assumption.

Discussion and Conclusion

Even though this paper can only provide a small insight into the interviews with the learners, and into what could be found in their enabled descriptions, a discussion of the research results will be attempted at this point. What can be observed with the findings of our study, namely the differentiated sub-codes of the inventory and the excerpts from the transcripts, is, that the learners seem to have a *structural conception* of negative numbers. According to Sfard (1991), at this (more advanced) stage of concept development the learners already should be able to (mentally) manipulate negative numbers as theoretical mathematical objects. However, even if the learners themselves often give examples in which *transformations* with negative numbers are being processed, they continue to try to describe *states* – which then often appear to be less suitable for the situation or even incorrect. One possible reason for this could be the lack of lexical means that express transformations. It could also be assumed that learners have not had enough opportunity to develop an *operational conception* of negative numbers – which would, as Sfard (1991) argues, be necessary for a structural conception to evolve from. Consequently, the learners seem to build their structural conception on a *deficit understanding* of negative numbers. Given that they identify negative numbers with states of 'lacking', 'non-existing' or 'negative quantities', they meet an obstacle that is both of a cognitive and linguistic nature: The non-existence (of quantities and magnitudes) is not *imaginable* and therefore not *describable*.

Despite its brevity, the analysis presented in this paper exemplified how learners' language might be 'limited' to the description of states, even when thinking of situations in which negative numbers could be used for expressing transformations. The empirical results indicate that the lexical means including deficit descriptions that are currently being used by the learners support a deficit understanding of negative numbers rather than fostering the development of sustainable conceptions. Inspired by the considerations made by Sfard (1991) and encouraged by the more recent findings of this study, we want to draw on the *operator interpretation* as a possible conceptional basis for introducing negative numbers.

This approach is not new, yet it seems to have been neglected in recent years, at least in the German schooling system. For example, from a technical mathematical point of view, Kirsch (1973) explains how negative numbers could become more cognitively accessible by referring to operators. Operators in the meaning of dynamic *state changes* or *action processes* are easier to describe and much easier to imagine than something 'less than nothing'. Learners can find them in natural processes that are invertible, such as adding or removing something or moving forward or backward – conceptions that learners already know from operating with non-negative integers, and which are also sustainable for negative numbers because they keep their mode of effect. Hence, the operator interpretation offers a chance to introduce negative numbers as descriptions of well-known performed actions rather than referring to something that is 'new', 'different' or 'lacking'. As it doesn't require a deficit description of negative numbers but rather builds on well-known mental representations and strategies, focusing on learners' development of an operational conception might be a helpful approach. Once again, referring to Davidson (1987, p. 432): "Unlike non-positive collections of material objects, non-positive actions […] not only exist but are familiar to children from infancy." Whether this approach indeed holds potential for mathematics teaching has yet to be shown. For example, further

considerations could be given to the role that appropriate lexical means – that help to understand negative numbers as expression of transformations – play in getting access to the operator interpretation as a conceptual basis for learners.

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Using Maltese meaning-related vocabulary for the topic '*Equivalent Fractions*' in a bilingual Grade 6 classroom in Malta

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Mathematics education in Malta is frequently conducted through translanguaging with English and Maltese. In this paper I describe a design-based research study carried out in collaboration with a Grade 6 elementary teacher. The topic in question was Equivalent Fractions *and the main thrust of the lessons was to heighten the teacher's awareness of useful Maltese vocabulary for meaning-making, to maximise student verbal participation, and to introduce the innovative idea of using Maltese for written mathematics texts. I provide excerpts of English / Maltese classroom interactions, using the illustrations to reflect on language-responsive pedagogy.*

Keywords: Meaning-related vocabulary, translanguaging pedagogy, design-based research, fractions, elementary mathematics education.

Introduction

In many classrooms in Malta, mathematics education is experienced bilingually through English and Maltese. The presence of English in the education system, and society at large, is the result of 165 years of British colonisation, which ended in 1964, while Maltese is the home language of the majority of Maltese people. In many classrooms, teachers and students communicate verbally, and negotiate mathematical meaning, through both languages, but all written mathematics is in English. In this paper, I describe a collaborative design-based study carried out with an elementary school teacher. The research question addressed was: *What Maltese vocabulary is salient for meaning-making for the curricular topic* Equivalent Fractions*?* The aim of the study was to encourage, and validate, spoken Maltese and to introduce *written* Maltese for mathematics.

The curricular topic *Fractions* and language-responsive teaching

Fractions are among the most difficult concepts that elementary children come across (Charalambous and Pitta-Pantazi, 2005). Van de Walle et al. (2020) highlight that fractions have five interpretations: *part-whole* (e.g. ¾ of a circle or ¾ of a class of 24 children); *division* (3 pizzas shared amongst 4 children); *measurement* (¾ as a number on the number line, the fraction being a measure of the whole number 1); *operator* (¾ of €20) and *ratio* (in a classroom, the ratio of boys to girls is 3:4). The focus of this paper is the *part-whole* interpretation and, more specifically, equivalent fractions. Two fractions are equivalent if they are representations for the same amount or quantity or, when considering fractions as numbers, they represent the same number. Equivalent fractions can be created by multiplying or dividing the numerator and denominator by the same number, but Van de Walle et al. (2020) stress that students need to see the connection between the concept and the procedure. Van de Walle et al. recommend that, since fraction symbolisation is difficult for young students to understand, words should precede symbols in instruction.

Several authors have written about the importance of teachers focusing implicitly on language during mathematics lessons. For example, Gibbons (2015) and Lee (2006) highlight the benefit of explicitly

focusing on mathematical vocabulary and 'ways of saying', while Melanese et al. (2011) and Moschkovich (2013) similarly advise with reference to contexts wherein students are learning through a language which is not their home language. A contemporary notion is *language-responsiveness*. Teaching mathematical content language 'responsively' is defined as "instruction that enhances language for mathematics learning" (Erath et al., 2021, p. 246). More specifically, materials and instructions should: engage students in rich discourse practices (Principle 1), establish various mathematical language routines (P2), connect language varieties and multimodal representations (P3), include students' multilingual resources (P4), use macro-scaffolding to sequence and combine language and mathematics learning opportunities (P5) and compare language pieces (form, function, etc.) to raise students' language awareness (P6). Götz and Baiker (2023) explain that language-responsive lessons cannot follow all six design principles simultaneously and my study touches mainly on principles 1, 3 and 4. Prediger (2019) advises that language-responsiveness should not be restricted to the communicative function of language, but should also take into consideration the epistemic function of language as a tool for thinking and knowledge construction. She points out that meaning-related vocabulary is likely to be a mix of both subject-specific words and 'everyday' words.

My study is not the first design-based one focusing on fractions. Two recent mixed-methods studies were carried out in Germany by Prediger and Wessel (2013) and Schüler-Meyer et al., (2019), both with the aim of aiding students' conceptual understanding. In the former study, German was used to scaffold learning for second language learners. The focus was on student explanations, and the student to teacher ratio was 2:1. In the latter study, Turkish was introduced into lessons with German / Turkish students. The focus was bilingual connections, and each teacher worked with three to five students. Both studies involved large numbers of teachers and students. While my own overall intention to support conceptual understanding is similar, my study features only qualitative data, collected in two natural classroom settings. Furthermore, my study differs in that it was carried out in a post-colonial context, wherein Maltese is an accepted language, and already used orally (to varying degrees) for mathematics, although generally in subordination to English and never for written mathematics.

Theoretical framework

My theoretical framework is underpinned by three key elements. First is the definition of translanguaging itself. The term 'translanguaging' has been used by various writers in different ways and, indeed, the interpretation of the term has changed over time (see MacSwan, 2022). In this paper, I use the perspective adopted by Cummins (2021), namely, a *Crosslinguistic Translanguaging Theory*. In this view, translanguaging is considered to involve shifts between distinct, identifiable languages, which are used according to the social context at hand. (This view contrasts with the view proposed by Ofelia García and her colleagues (e.g. García and Otheguy, 2014), wherein bi-multilinguals have one unitary language system with undifferentiated mental grammars). Second, I hold a non-deficit view of bilingualism. With respect to our contemporary world, where so many people operate through a second language, Cook (2022) challenges traditional views of language by highlighting bilingual advantage over monolingual deficit. Third, I subscribe to the belief that "translanguaging pedagogies represent an epistemic and pedagogical commitment to multilingual

learners by legitimizing and intentionally incorporating their multilingualism and multilingual identities in the classroom" (Shepard-Carey & Tian, 2023, p. 1).

Methods

Shepard-Carey and Tian (2023) believe that the exploration and development of translanguaging pedagogies is best served through researcher-teacher partnerships and collaboration. They propose a research / practice approach of "co-stance, co-design, and co-shifts" (2023, p. 6). Co-stance refers to the dynamic process of dialogue that allows both researcher and teacher to deepen their understanding of the complexity of classroom contexts; co-design refers to the iterative process of planning lessons that build on students' repertoires; and co-shift refers to moments when teacher and researcher engage in self and communal reflections. My study reflects this approach. Furthermore, the method utilised was design-based research, described by McKenney and Reeves (2019) as an approach that involves developing and implementing interventions to address problems in practice. Prediger et al., (2015) point out that design research generally has dual objectives: it focuses on a specific curricular topic while simultaneously considering a wider instructional issue. In my case, the curricular topic was *Fractions*, while the wider consideration was *the use of Maltese for meaning making in mathematics*.

The school was chosen in a purposive manner, since it had a minimal number of non-Maltese students. I wished to avoid a multilingual context which, from my experience, I knew would result the use of English as a lingua franca. Two teachers volunteered their participation. This paper focuses on the Year 6 teacher, Daniela (pseudonym). Her class comprised 18 students, aged 10 to 11 years. Daniela's home language was Maltese, but she was also fluent in English. She explained that she usually started mathematics lessons in English, but if she noticed that the students were not understanding, she repeated in Maltese. Daniela reported that the home language of all but one of the students was Maltese and that she always allowed her students to participate in Maltese, if they so wished. The Saudi Arabian student present in the class could speak Arabic, English and Maltese.

McKenney and Reeves (2019) mention key features of design-based research: It is explorative in nature, has a flexible, iterative structure, and involves ongoing evaluation and retrospection. In line with this, I started the study by conducting a one-hour interview with Daniela, during which we discussed the topic and her views on language for mathematics. I then planned and prepared six lessons. Minor amendments were made following Daniela's feedback. Daniela delivered the lessons, while I observed. Every day, we discussed the lesson for about 15 minutes, and went over the following day's lesson. Finally, another one-hour interview was held, during which we shared our overall reflections. The interviews and lessons were audio-recorded, while the lessons were also video-recorded. As I observed the lessons, I wrote down reflective thoughts for discussion with the teacher and also as a means of immediate analysis of the data being collected. I carried out daily transcription of the interviews, documenting my reflections and any points to be followed up with the teacher. Thus, the analysis and data collection were closely intertwined. For practical reasons, the finer analysis and transcription of the classroom interaction was done at a later date.

Adopting a positive view of bilingualism, I sought to encourage the teacher to use Maltese vocabulary more consciously, and more explicitly, for meaning making. I validated, and encouraged, the teacher's and students' use of Maltese and integrated Maltese text into written work, on the

assumption that this would have pedagogic benefits. The key ideas addressed – comparison of fractions of regions and quantities, equivalence and fractions on the number line – were chosen based on the mandatory syllabus. In the lesson plans, I included vocabulary – both English and Maltese – that the teacher might use. These included the Maltese term for English nouns that Grade 6 students would already be familiar with e.g. *numerator* / **numeratur**, but also expressions like *the same part of the whole* / **l-istess parti mill-ħaġa sħiħa [the same part of the whole thing]**. Thus, as pointed out by Prediger (2019), the meaning-making vocabulary was a mix of both subject-specific words and 'everyday' words. My participating students already had a few years of experience with fraction symbolisation, hence my aim was to focus on strengthening the link between symbols (and procedures) and concepts (Van de Walle et al., 2021).

The lessons were designed in a way so as to target key principles of language-responsiveness. First, verbal participation was maximised through pair work and reporting back to the class (for example, the students would work in pairs to formulate an explanation of some point for a friendly alien named Alfred) (Principle 1). Second, in line with Principle 3, both English and Maltese were associated with diagrams and symbolisation that the students were already familiar with. Furthermore, Maltese was included in the written texts of worksheets, study notes and interactive-panel projections. For example, in one worksheet, I wrote the problem: "*Alfred the Alien says that the fractions 4/12 and 12/36 are equivalent because both include the number 12 (**it-tnejn li huma fihom in-numru 12) [both of them include the number 12]**. Is he right? Explain.*" Third, Maltese was encouraged as part of verbal interaction. The topic *Fractions* was not new to the students, but Daniela acknowledged that pushing students to *explain* ideas would render the lessons more challenging (Principle 4).

Discussion of data

As Daniela delivered the lessons, she used both English and Maltese, although English was more frequent. However, as a result of my lessons plans and our discussions, Daniela now also used certain Maltese words and phrases that were anticipated to support meaning making. She admitted that some expressions 'did not come naturally' at first but that, over the days, she came to appreciate their usefulness. One example is the phrase '**jiswew l-istess**' [**they are worth the same**] with respect to 'equivalent fractions.' There was variation in the students' own use of Maltese and English, with some students choosing to use more English and some more Maltese. However, the students engaged with interest with the increased use of Maltese. When Daniela asked them (not in my presence) their opinion about the Maltese written insertions, they told her that they found this approach helpful.

I will now discuss two stretches of interaction. The first is taken from the third lesson. The expression *Equivalent Fractions* was projected on the interactive panel, and the students were asked how they would explain this expression to Alfred the alien in English.

Patrick:	Equivalent fractions are fractions that are the same/
Eddie:	(*Interrupts his partner*)/ But with different numbers.
Teacher:	Can you specify? "Different numbers" means…?
Patrick:	Like one half and three sixths. Although you think that the numbers are bigger, but, in reality, one half/

Teacher: (*Interrupts*)/ Wait a second, wait, wait. So maybe instead of saying "different numbers" we can say "NUM..."

Patrick: Numerators and denominators. Like one on two and three on six.

Teacher: Like one on… one half and…? Can you give me the fraction?

Patrick: Three sixths.

At this point, the thrust of Daniela's questions / prompts was to encourage the students to express themselves more precisely in terms of pointing to the elements of the fractions (numerator / denominator) and expressing the fraction more formally. As other students took their turn, two pairs reported the procedure for creating equivalent fractions by multiplying or dividing the numerator and denominator. Other pairs focused on the concept, using vocabulary like; *the same, always equal, equal but not same denominator and numerator, have the same amount*. After this first round of explanations, the teacher invited the students to explain the same expression to Alfred in Maltese, thus encouraging and validating their home language for mathematics. Students' explanations included expressions like: **l-istess, jfissru l-istess, għandhom l-istess ammont, għandhom valur l-istess** [**the same, they mean the same, they have the same amount, they have the same value**]. While these expressions mirrored quite closely the students' English expressions, it is interesting to note that the students picked up on Daniela's expression '**jiswew l-istess**' [**they are worth the same / they have the same value**], although here expressing it as **l-istess valur** [**the same value**]. The Maltese reference to the idea of 'same value' appeared to add further depth to the articulation of meaning. A contribution worth commenting on is Fabian's, who read out the explanation he had drafted together with Ian.

Fabian **Mela, Alfred, il-frazzjonijiet** equiv**ilanti** [sic] **għandhom l-istess ammont, pereżempju wieħed fuq tnejn, tnejn fuq erbgħa** […] **sebgħa fuq erbatax u erbatax fuq tmienja u għoxrin.**

[So, Alfred, equivalent fractions have the same amount, for example, one on two, two on four […] **seven on fourteen and fourteen on twenty-eight]**.

Teacher: **Sebgħa minn erbatax**. [**Seven from / out of fourteen**]. (*Teacher looks at researcher as if to validate the expression of the fraction in Maltese*).

MTF: **Iva. Sebgħa MINN erbatax. Tista' tgħid 'fuq', anke bl-Ingliż ngħidu** 'three on, over four', **imma proprja tgħid tlieta MINN erbgħa, jew tlieta minn KULL erbgħa.**

[Yes, seven from/out of fourteen. You CAN say 'on', even in English we say 'three on, over four' **but really, we say three FROM / OUT OF four, or three from/out of EVERY four].**

Fabian took pride in using Maltese and here he coined a mixed-code word *equiv***ilanti** (instead of **ekwivalenti**). He asked Daniela if this word 'made sense' but, not hearing his question due to some classroom distraction, she did not respond. Fabian's word was heard and used by other students. The incorrect articulation of this word was not an issue *per se* but, after the lesson, I did draw Daniela's attention to Fabian's query, not only to validate the student's efforts but also because I believe that if we are to promote Maltese for mathematics, then the words might as well be said correctly. A more significant aspect of the above interaction was Daniela's, and my own, rephrasing of 'seven on fourteen' as **seven FROM / OUT OF fourteen** or **seven from / out of EVERY fourteen.** Here the

Maltese foregrounded the part-whole concept more explicitly than English. The excerpt also shows how bilingual connections can be made collectively, involving students, teacher and researchers.

The second excerpt is taken from the fifth lesson. The lesson started with a projection of the image of Alfred the alien on the interactive panel, together with the phrases *simplest form* and *lowest terms*. The students were asked to work in pairs to formulate an explanation of these terms, and any language could be used. Here the implicit message was that both languages are acceptable, and students could use whatever they preferred. Kyle and Amy reported their respective pair-work as follows:

Kyle: It means that a fraction keeps going down to the smallest fraction, for example, twelve on twenty-four, six all over twelve, three all over six, one half. You need to keep going down until you find the smallest fraction. I kept dividing.

Amy: We know this: 'simplest' means smallest, so when you have to turn them into the simplest form, we have to think of a [multiplication] table that fits in them. **Mbagħad għamilna**, two fourths **għamilnieh** division by two **jiġi** one half; **u** eight tenths **erġajna għamilnieh** division by two, **jiġi** four fifths.

[**Then we did**, two fourths, **we did it** division [we divided it] by two, **it comes** one half; **and** eight tenths **we once again did it** division by two, **it comes** four fifths].

Having listened to seven procedure-oriented explanations, Daniela found an opportunity to focus on *conceptual* meaning. Patrick had suggested the fractions 5/10 and 2/4 and ½, saying that they were "the same fraction with different numbers". Daniela picked up on this cue as follows:

Teacher: (*She indicates the fractions written on the panel. She inserts equal signs: 5/10 = ½ and 2/4 = ½*). So these are equal because they are equivalent fractions, just in their simplest form. So you need to also explain to me something else: that even though the numbers are different ...?

Theo: But they still have the same value.

Teacher: They still have the same value. **Jiswew indaqs. Il-parti mill-kbira, mill-ħaġa shiħa, jiswew indaqs.** Is that understood? **Dik importanti li tgħiduhulu [lil Alfred]**.

[They still have the same value. **They are worth the same. The part from the large [one], from the whole thing, they are worth the same**. Is that understood? **That's important to tell him [Alfred]**.

George: Hello Alfred. If you have a fraction of two fourths and you need to make it simplest form, it's easy! Two fourths, you can turn it into one half.

Teacher: And you need to tell him [Alfred] that the fractions are in the simplest form. They are ...?

George: The same.

Daniela's latching onto Patrick's point illustrates a commitment to language-responsiveness on her part, and the translanguaging within the same stretch created a linguistic connection that served to shift the students' attention away from procedures to a more conceptual interpretation.

Conclusion

I set out to explore what Maltese vocabulary is salient for meaning-making for the topic *Equivalent Fractions*, utilizing the notion of meaning-related vocabulary (Prediger, 2019) in a 'natural' classroom setting within the Maltese bilingual context. The teacher's increased use of purposive

translanguaging to support meaning-making, and the keen engagement with both languages by the teacher and the students, were indications that the pedagogy I designed has potential to be utilised in our bilingual classrooms.

My choices about what constituted meaning-related vocabulary were based on my professional and academic knowledge. The teacher also gave her input based on her knowledge of the subject, her experience teaching Grade 6, and her familiarity with this particular group of students. Certainly, a class teacher plays a crucial role in identifying meaning-related expressions, especially when these utilise vocabulary that is 'everyday' language. Indeed, the study highlighted that it is not always straightforward to distinguish between subject-related and everyday vocabulary. *Multiply* and *denominator* – and their Maltese counterparts – may easily be identified as subject-specific and *you keep going down* might be considered as everyday vocabulary. However, expressions like *have the same amount* and *part of the whole thing* are not so easily categorised. So, while the theoretical distinction is useful, in practice the dichotomy can be blurred. However, this blurring can help to shift our attention to the more important aspect of *meaning-making*. The study highlighted that Maltese not only served its purpose but, at times, expressed meaning in a more explicit manner than English.

I approached this study with a positive disposition toward translanguaging pedagogies. I believed that English and Maltese could be drawn on for pedagogic advantage through flexible translanguaging. While verbal translanguaging is not new in our classrooms, the use of Maltese for *written* mathematics was an innovative experiment that was well received. I believe that this method can offer a practical and helpful compromise between, on one hand, sticking to English-only written work and, on the other, a potential duplicating of all resources in both languages. Possibly, this initial experiment might find relevance in other post-colonial contexts.

The design-based approach allowed for ongoing reflection about translanguaging pedagogy for both myself and the teacher, resulting in professional growth for both. I hope to extend the study by working with other teachers. I believe that further collaborations between researchers and teachers will strengthen our efforts to establish "translanguaging pedagogies as consistent and intentional practices" (Shepard-Carey & Tian, 2023, p. 2).

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The impact of multilingualism on mathematical teaching-learning situations

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In an increasingly multilingual society, the teaching of mathematics in German primary schools is particularly challenging. On the one hand, multilingualism opens up access to different linguistic perspectives, on the other hand, differences in linguistic expression and mathematical terminology can lead to difficulties in mathematical understanding. This is also confirmed by everyday experience. In this study, we investigate the effects of Croatian as a first language on the mathematical learning of primary school pupils in Germany.

Keywords: Language, multilingualism, bilingualism, primary education, Croatian.

Mathematical learning, language, and multilingualism

There is a consensus in mathematics didactics research that language plays an important role in mathematical learning processes. Vygotsky (1978) already described language as a central medium for cognitive development and learning. Since then, a lot of research has been done underlining the close connection between language learning and mathematical learning (Setati, 2005; Moschkovich et.al., 2018; Ingram et al., 2023). Lately, special attention is given to multilingual learners who have competences in their first as well as in their second language (Barwell et al, 2019) In this context, Cummins' "interdependence hypothesis" emphasises the importance of preserving the first language. Following Cummins, the loss of the first language on the one hand can have a negative impact on a child's cognitive and academic potential in the second language (Cummins, 1981). On the other hand, appropriate backing and strengthening of the first language supports academic success in the second language. Eventually, the development of language competence is not only an important learning goal, but also an important prerequisite for the learning of mathematics.

Language is not only a means of communication but also influences our thinking. In this respect, the communicative and the cognitive function of language can be distinguished (Bruner, 1974). Both are closely interwoven and influence each other. In the sense of the cognitive function, language is an important instrument for organising our thinking and structuring our knowledge. It is true that our thinking is not exclusively linguistically bound. But language and its differentiation play an essential role in the differentiation of ideas. The more differentiated linguistic means learners have to describe mathematical concepts, the more differentiated insights into these concepts they will be able to develop. In the process of developing and differentiating language skills, the exchange with others, respectively the communicative function of language, plays an essential role. Communication processes are crucial for developing new perspectives on one's own prior knowledge, they enhance not only our thematic or mathematical learning but also improve our language skills. The cognitive function of language is directly related to its communicative function (Cummins, 1981).

This aspect of (mathematical) concepts and ideas developed with relation to the individual language skills is particularly interesting in a multilingual context. If two (or more) languages are involved in

learning, it is not just a matter translating. Instead, language use is necessarily bound to a specific cognitive system which is dependent on the language structure. Different aspects of reality can therefore be emphasised due to the phrases and vocabulary given by the linguistic structure. In linguistics, this connection is described by the relativity hypothesis - also known as the Sapir-Whorf hypothesis (Whorf, 1940). It assumes that each language has a specific cognitive system that influences perception, restricting access to ideas that are not encoded within it. From a resource-oriented perspective, multilingual learners stand to benefit from drawing on multiple language systems, thereby enriching their cognitive abilities. Like communication and interaction, which enable perspective-taking and knowledge restructuring, multilingual learners can reflect on reality through two or more linguistic perspectives. Studies from secondary education prove empirically, that multilingualism can influence mathematical learning in a positive way. For example, Prediger and colleagues (2019) were able to demonstrate, that differences between Turkish and German on the topic of fractions can enhance the mathematical learning of Turkish-German bilingual students. Moreover, linguistic variation can facilitate a more nuanced comprehension of mathematical concepts (Prediger & Uribe, 2021). However, it is essential to address the differences in terms, ideas, and approaches to subject content associated with multilingualism explicitly in order to fully capitalise on this benefit. Reflection on language and concepts on a meta-level are the key to mathematical learning processes in studies in secondary mathematics classrooms (Barwell et al., 2016). Following Whorf (1940) and recognising the specific nature of the cognitive systems of different languages, it needs to be clarified to what extent the results from research relating to secondary education can be transferred to primary education or other language combinations and to what extent there are generalisable aspects. The first step towards a more comprehensive understanding of multilingual mathematical learning processes is to focus on the influence of learners' different first languages. This study follows this approach and focuses on German-Croatian bilingual primary school students from the second and third generations. As the teacher is a Croatian native speaker, microanalyses of the emerging interactions are possible with special attention to the use of Croatian as the first and shared language as well as the second language German. Data from a pilot study is introduced.

Pilot Study

For the data collection, a teaching-learning situation was designed and implemented that offers a bilingual communication occasion. German and Croatian are chosen for the setting in order to benefit from the teacher's own language skills in these two languages.

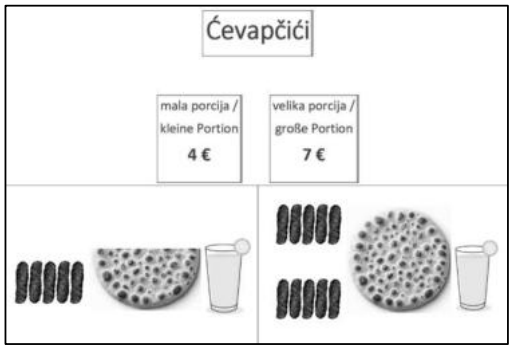

Figure 1: price board and illustration of the portions small portion 4€ / big portion 7€

A real-life context on buying well-known snacks is chosen in order to enable culturally assignment to the Croatian-speaking area and to encourage learners to use their first language when completing the task. The offered material is bilingual (see Figure 1) and the teacher leads the discussion in both languages.

As part of the teaching-learning situation, the children were given the task of explaining whether they thought it was more sensible to buy two small portions or one large portion of the ćevapčići-menu. This task enables an exploration of the mathematical topic of halving and doubling. Negotiating this topic in German or Croatian reveals a range of different formulations and grammatical structures that go hand in hand with conceptual differences. One example is given here for clarification (Table 1).

Table 1: Example German – Croatian

German	"doppelt so groß"	"halb so groß"	**Croatian**	"duplo veće"	"upola manje"
	[*double as big*]	[*half as big*]		[*double bigger*]	[*half smaller*]

If you take a closer look at the linguistic expressions, you will notice that the word "groß'" [*big*] is used in both German contexts. In Croatian, on the other hand, different words are used: "veće" [*bigger*] and "manje" [*smaller*]. These Croatian words indicate the direction in which the size of an object changes, while the German refers to the size itself. Initially, only the price board was presented. After the first justification, the picture of the "portions" including the lemonade was also shown, whereupon a further justification was initiated. Children of different ages and with different durations of school socialization in the German education system were selected. The intention was to clarify and specify challenges and resources associated with multilingualism in mathematical learning and to ascertain whether the observed effects are independent of age and duration of school socialization.

The teaching-learning situations of the pilot study were videotaped and transcribed. English is used as the reference language within the transcripts. The translations into English highlight the linguistic differences between Croatian and German, as well as the potential conceptual differences that may arise from them. For this reason, the verbalisations have been translated word by word into English. They are presented in italics. Non-verbal actions are shown in grey, reference to certain lines of a transcript are given as <5>. The names of the children have been pseudonymized.

Empirical examples

The two examples presented in this paper refer to one interview with a second and one with a fourth grader. In this way, first insights into primary mathematical learning at different learning levels concerning both, mathematical and language skills, may be introduced and compared. The data was collected during the last week of the summer holidays and the first week of lessons in Germany. As the children had been in Croatia and/or Bosnia for between three and four weeks prior to the data collection, it can be assumed that the first language was increasingly used as a means of communication immediately prior to the data collection.

Nina

Nina is the second generation in Germany. She is in the second grade. Nina's first language is Croatian and the first contact to German was at the beginning of kindergarten at the age of 3. She only speaks Croatian at home, as her parents only speak weak German.

The initial question was posed in German "Was wäre schlauer?" [*What would be smarter*?]. After glancing at the price board, Nina immediately answers in German. She decides that it would make more sense to buy the large portion because it is cheaper in total than the two small portions. When the illustration is uncovered by the teacher, Nina sticks to her conclusion. For her, the price seems to be essential for the decision. Consequently, the remaining question for her is that of a fair-minded way to share that large portion. Nina first explains in German that the large portion can certainly be shared. The teacher then asks whether Nina can also formulate her reasoning in Croatian, to which she replies in the affirmative. Figure 2 presents the corresponding transcript in a synoptic layout, allowing for direct comparison between the German and Croatian versions. To preserve this format, the transcript has been presented as a figure. Reading the transcript keep in mind that the Croatian explanation actually sequentially followed the intuitive German first try.

	German		**Croatian**
1G	Also hier ist ein halber Brot *Well, here's half bread* und hier nicht. *and here not*	1C	Znači ovdje je pola kruha. *Well, here's a half bread,* ovdje nije. *heres´s not.*
2G	Und dann können wir *And then we can* das doch halb machen. *do it half* *(draws a horizontal line through the illustration of the bread with the hand)*	2C	Pa mogu sa mojojm prijatljicom *So I can with my friend* ovako odrezati kruh *cut the bread like that* *(draws a horizontal line through the illustration of the bread with the hand)*
2aG	Und dann ja, ist ja auch praktisch. *And then yes, it's also practical.*	2aC	i onda podijelimo. *and then we share*
3G	Und dann hier ist eins für mich *And then here's one for me* *(draws a line with her hand across the top row of ćevapčići)*	3C	Ovo je za moju prijateljicu - gore *This is for my friend,- on top* *(draws a line with her hand across the top row of ćevapčići)*
4G	und das da alles für meine Freundin *and that there is all for my friend* *(moves her hand back and forth over the bottom row)*	4C	a ovo je za mene dolje *and this is for me on the bottom* *(draws a line with her hand across the bottom row of ćevapčići)*
5G	und das hier, das *and this here, that* *(draws a line with her hand horizontally through the centre of the lemonade illustration)*		
6G	das können wir doch zusammen trinken. *that we can drink together.*	6C	a ovaj sok možemo zajedno piti. *and this juice we can drink together.* *(taps the illustration of the lemonade)*

Figure 2: Transcript Nina

Nina divides the bread in half, arranges the ćevapčići into two rows, and shares the lemonade. While there are many parallels between German and Croatian expressions, some statements are even identical upon closer examination <1G-1C, 3G-4C, 4G-3C, 6G-6C>. Non-verbal actions she uses to reinforce her words are often identical <2G-2C, 3G-3C, 4G-4C). This could lead to the conclusion

that Nina mentally follows the same (narrative) path. After providing an explanation in German, she is now transferring this path into Croatian. The non-verbal actions may indicate that Nina wants to make her argument even clearer. In this way, she achieves relief on a verbal level, but still increases the chance of making her arguments explicit (Fetzer, 2011). On the other hand, however, translating word by word seems to reach its limits in lines 2G and 2C. While in German the verb "halbieren" [*to halve*] is used <2G>, the equivalent technical term in Croatian does not seem to be available, at least in this situation. Instead, Nina seems to paraphrase <2C>. Drawing the horizontal corresponds to the verbalised "halb machen" [*do it half*] <2G>. So here <2C> the non-verbal argumentation relieves her of the need to explain things verbally. These gestures substitute words when she lacks the appropriate term at least here in Croatian.

This case study indicates that the provision of multilingual material and the encouragement to formulate explanations in two languages does not necessarily have an impact on mathematical learning. The reconstruction of the case demonstrates that technical terms in the first language cannot be recalled in some instances. Consequently, a multi-perspective view resulting from different formulations of mathematical facts is precluded. The following example contrasts with the previous one, highlighting excerpts that illustrate the effects of multilingualism on mathematical learning.

Toni

Toni is the second generation in Germany. He is in the fourth grade. Toni's first language is Croatian and the first contact to German was at the beginning of nursery at the age of 1 1/2. He speaks Croatian and German at home. His parents speak German very well.

The first question was posed in Croatian and German "Što bi bilo pametnije (Croatian)? Was wäre schlauer (German)?" [*What would be smarter?*]. Toni first begins to answer in Croatian "Ja bih uzeo veliku porciju i nju popoloiviti, da ja i moj prijatelj…" [I *would take a large portion and halve it so that me and my friend…*]. When the teacher mentions that Toni can also answer in German, Toni immediately completes his answer in German "…dass ich und mein Freund die selbe Portion nur für weniger Geld haben" [*that me and my friend have the same portion for less money*].

A grammatical error seems particularly notable here. In German a hypothetical statement could be formulated as follows: "Ich würde eine große Portion nehmen und sie halbieren" [I *would take a large portion and halve it*]. The German grammatical tense "Konjunktiv II" is here made by combining the verb "würde" [*would*] and the infinitive "nehmen" [*take*] or "halbieren" [*halve]*. The Croatian Grammatical tense "Kondicional I" is made by combining the verb "bih" [would] with the past participle of the main verb "uzeo" [*taken*] or "popolovio" [*halved*]. In Croatian you would not use the infinitive in this context. Toni begins with "Ja bih uzeo veliku porciju" [I would take a large portion] and he uses the past participle of the main verb correctly. In the following coordinate clause, however, he does not follow the grammatical rules of Croatian but those of German by using the infinitive form of the verb „popoloviti" [to *halve*]. On closer analysis, in this sentence he conjugates the everyday context correctly [*I would take*], whereas the technical term [*halve*] is correctly translated in terms of meaning but follows the grammatical rules of German. This may indicate that Toni had first mentally formulated the idea of halving in German and that there was an error in the translation into Croatian. The fact that he ends the sentence much more quickly in German than he

begins it seems to confirm this assumption. Toni's arguments between Croatian and German show a number of parallels, both in verbal and non-verbal action. However, it is worth taking a closer look, especially in cases where a literal translation is not possible (Transcript Toni). For Example, the German "halb so groß(e Portion)" [*half as big (portion)*] cannot be translated into Croatian word by word (see table 1). In Croatian you would say "upola manja porcija". The German term "groß" [*big*] could not be used in the Croatian context. The Croatian adverb "upola" is composed of the terms "u" *[in]* and pola *[half]* and it means figuratively as much as *by half*, so the comparative "manja" [*smaller*] is required here. This linguistic difference becomes a topic of conversation when the teacher explicitly asks whether there is a difference between "halb so groß(e Portion)" and "upola manja (porcija)" <1> The following dialogue emerges (Transcript 1).

Transcript 1

1	Teacher:	Wenn du sagst mala porcija je upola manja od velike? *When you say the small portion is half smaller than the big one?* Gibt´s einen Unterschied zu der Aussage? *Is there a difference to the statement?* Die kleine Portion ist halb so groß wie die große? *The small portion is half as big as the big one?*
2	Student:	Ja *Yes*
3	Teacher:	Gibt´s einen Unterschied? *Is there a difference?*
4	Student:	Ja. Im Kroatischen heißt es ähm ja ähm - ich übersetzte es mal *Yes. In Croatian it says ehm, yes ehm - I'll translate it* diese Portion ist halb so klein als diese hier. *this portion is half as small as this here.*
5	Teacher:	Upola manja. *Half smaller.*
6	Student:	Ja. Und hier heißt es halb - doppelt so groß. *Yes, and here it says half - double (twice) as big.*
7	Teacher:	Und wie heißt es im Deutschen mit der Hälfte? Dieses halb? *And how is it said in German with half? This half?*
8	Student:	Die Hälfte? Das ist die Hälfte von dem. *The half? This is the half of that.*
9	Teacher:	Aber ich kann ja auch sagen, es ist halb so groß. *But I can also say, it is half as big.*
10	Student:	Das ist dann aber ein bisschen verwirrend, weil es ähm, weil es halb? *But that's a bit confusing because it ehm, because it half?* Aber so groß ist. *But it's so big.* Das wäre ein bisschen logischer, wenn es heißen würde halb so klein. *It would be a bit more logical if it said half as small.*

Toni seems to be comparing the language systems of Croatian and German here. He translates from Croatian into German and explicitly stresses that he is translating, indicating that he would not normally say "halb so klein" [*half as small*] in German <4>. Although the teacher asks how it is normally expressed in German, Toni does not use the common phrase "halb so groß" [*half as big*] <4>. Instead, he says "die Hälfte von" [*the half of*] <8>. After the teacher remarks that it can also be formulated "halb so groß" [*half as big*] <9>, Toni does not deny. He employs an interrogative tone to emphasise "weil es ähm, weil es halb?" [*beacause it ehm, because it half?*] <10>, subsequently

continuing with “Aber so groß ist” [*But it´s so big*]. The use of the adjective “groß” [*big*] in the context of the adverb “halb” (*half*) seems inappropriate to him. As a result, Toni finds a way of expressing himself that makes more sense to him. In this context the formulation “halb so klein” [*half as small*] he uses is very uncommon in German. However, the adjective “klein” [*small*] he uses would be perfect in Croatian. A possible explanation is that the differing Croatian and German formulations prompt Toni to reflect on his concept of halving, which assumes a reduction from the initial state. It can be deduced that Toni compares his concept with both language formulations and chooses the one he finds "more logical" <10>. The German phrase “halb so groß” [*half as big*] does not seem to be appropriate in this context, as “groß” [*big*] does not linguistically reflect the “becoming smaller” after halving. For Toni this fact should be reflected in the formulation so that it is also linguistically logical (according to his mathematical concept of halving). From a math didactics perspective, his concept is limited, particularly with negative numbers. Nevertheless, this example highlights the positive impact of multilingualism on math learning. Linguistic differences prompted Toni to question and restructure his understanding of halving. His generalization aligns with his knowledge level and should not be viewed as a step backwards. However, the need to reflect on different formulations and the concept of halving doesn't emerge directly from the original task. This was evident with both Nina and Toni. Toni's reflective thinking at the meta-level only occurs through interaction with the teacher. Specifically, it is the teacher's questions about differences between the German and Croatian formulations that initiates reflection. The learner is introduced to the different formulations by the teacher.

Discussion

This paper analyses the relationship between multilingualism and mathematical learning in primary schools. Theory suggests that multilingual learners have significant potential due to their access to two different language systems. To explore whether learners can benefit from this potential in practice, we designed and implemented a teaching-learning situation that allowed bilingual communication. The results of the pilot study confirm that translations and language comparisons can enable a multi-perspective approach to mathematical concepts (Prediger & Uribe, 2021). Using the example of the concept of halving, it was shown that this is already possible in primary school and can have a positive impact on mathematical learning.

Furthermore, it suggests that examining how the teacher manages the learning process could be particularly revealing. At least for the primary education it seems that this factor plays a more decisive role in helping learners to benefit from their first language than the mere provision of bilingual materials or tasks. The question therefore arises as to whether it is sufficient to sensitise teachers to the linguistic differences between the languages or whether basic knowledge of the different languages is required.

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Children's block play language and the manifestation of mathematical content

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This paper explores the mathematical content present in preschool children's block play, focusing on their verbal interactions during play. Starting with Henschen's (2020) research, the study analyses how children's language and negotiating processes are linked to mathematical concepts, particularly in geometry and aspects of spatial orientation. Using thematic analysis of video data, the study identifies categories related to mathematical content, such as size comparisons, spatial orientation, and geometric shapes. It highlights how children use everyday language to discuss and solve construction problems, revealing their understanding of mathematical relationships. The findings suggest that children's block play language reflects mathematical concepts in a way that facilitates various connections.

Keywords: Mathematical language, block play, geometric concepts, kindergarten.

Introduction

Children's block play can be considered an integral part of preschool educational settings (Ramani et al., 2014). At the same time, the importance of children's building activities for the development of mathematical skills is undisputed, not only with regard to geometric learning (Sarama & Clements, 2009). In this context, the importance of learning geometric concepts through guided play is particularly emphasised. Little attention has so far been paid to analysing children's block play concerning children's utterances and negotiation processes. Looking at the verbal interactions and negotiation processes seems interesting, considering that language is both a means of communication and a tool for thinking (Vygotskij, 2017) and can therefore provide insight into the (mathematical) thinking of children while block play.

This paper aims to start here. Selected findings from Henschen's (2020) PhD project, in which children's block play was analysed using a thematic analysis, are presented for the first time in an English version and are extended by an interaction analysis in the sense of a mixed-methods idea. Following Henschen (2020), this paper shows how the mathematical content relates to the children's utterances during block play. While the thematic analysis focuses on the manifest content of the children's utterances, the addition of an interaction analysis, as presented in this article, enables the reconstruction of the mathematical meaning that lies behind it. In this way, it is possible to show which of these manifest mathematical contents are meaningful for the children in their negotiation processes.

Mathematical content and language in block play

An examination of different theoretical and empirical contributions shows that with regard to mathematical content, similar aspects are associated with block play. The activities of locating, measuring, and counting mentioned by Bishop (1988) can be used to categorise and classify these aspects, as Henschen (2020) carries out. For this paper, the focus is on locating, which includes

exploring and conceptualising one's spatial environment with, for example, models, words, or other means (Bishop, 1988). Many of the mathematical facets attributed to block play can in this sense be labelled as locating. So, it seems appropriate to differentiate locating into the three aspects of geometric shapes, symmetry, and spatial orientation (Henschen, 2020). Geometric shapes play an important role in children's building activities, as shown in the analyses by Ginsburg and colleagues (2004). Their analyses highlight aspects such as naming and using geometric shapes, as well as composing shapes during building activities. More specifically, they describe a young boy constructing a square based on his knowledge of geometric relationships (Ginsburg et al., 2004). Corresponding to the aspect of symmetry, the work of Fröbel (1851/1967) is interesting. He discussed the creation of aesthetic forms as part of his play gifts, which often involve symmetrical objects that children can further develop by rearranging building blocks in an orderly or symmetrical way. Among other contributions, the work of Seo and Ginsburg (2004) describes and categorises activities such as telling a location or direction under the broader concept of spatial relations. It stands to reason that spatial relations and also spatial orientation play a major role in building, for example, with building blocks.

Henschen (2020) points out that in the description of building activities used for the above classification, the authors focused more on the actions with the building material and less on the children's verbal utterances. In their analyses, Ginsburg et al. (2004) even speak of spontaneous or intuitive mathematics. In contrast, Bryant and colleagues (2024) suggest that one mechanism for how block play influences mathematical learning is mathematical language. Therefore, they focus on mathematical language use during block play. The authors differentiate between six dimensions of spatial language: "1) spatial locations (e.g., top, bottom), 2) deictic terms (e.g., here, there), 3) spatial dimensions (e.g., short, long), 4) spatial features and properties (e.g., curvy, straight), 5) shapes (e.g., circle, square), and 6) spatial orientations and transformations (e.g., turn, rotate)" (Bryant et al., 2024, p. 3). All dimensions are part of the "Spatial and Quantitative Mathematical Coding system" (Bryant et al., 2024, p. 12) developed by the authors as part of their study. Their coding then follows a quantitative approach that is based on the counting of mathematical language terms.

Mathematical content therefore appears to be present in at least two ways in children's block play. On the one hand, children spontaneously or intuitively use mathematical knowledge in their building activities. On the other hand, children use mathematical language terms while building constructions with blocks. What this focus ignores, however, is the importance of language as a means of communication and thinking (Vygotskij, 2017). In relation to block play, it can be stated that language can help children to reflect and make abstractions and communicate mathematical ideas (Bruce et al., 1992). Given these considerations, it is interesting to analyse the verbal interactions in block play situations in relation to their mathematical content. To do so, two levels can be distinguished: On the one hand, the use of certain words and phrases makes mathematical content explicitly visible on a manifest level, while on the other hand, a mathematical meaning may be present on a latent level, which only becomes manifest in the negotiation process between the interactants (Vogler, 2020).

This paper therefore addresses the following question: Which of the mathematical contents that can be described on a manifest level in block play interactions can be interpreted as meaningful in the children's negotiation processes?

Method and methodology

As this article reproduces findings of Henschen (2020), the method used there is briefly summarised here. The data was collected using videography. Small groups of children (usually 2-4 children between the ages of 4 and 6) were filmed playing with the block play materials available in the kindergarten during free play phases without the guidance of an adult and without a given task. This video data was analysed using a thematic analysis (Kuckartz, 2014). The thematic analysis procedure is shown in Figure 1 in the lower part. Firstly, scenes in which the children engaged in a controversial discussion about the building activities were selected and transcribed. These transcripts were used to develop categories that summarised the overarching themes or content-related aspects of building that were evident in the children's statements. The category names were based on the idea of "natural categories" (Kuckartz, 2014, p. 44) in an attempt to stay as close as possible to the children's expressions.

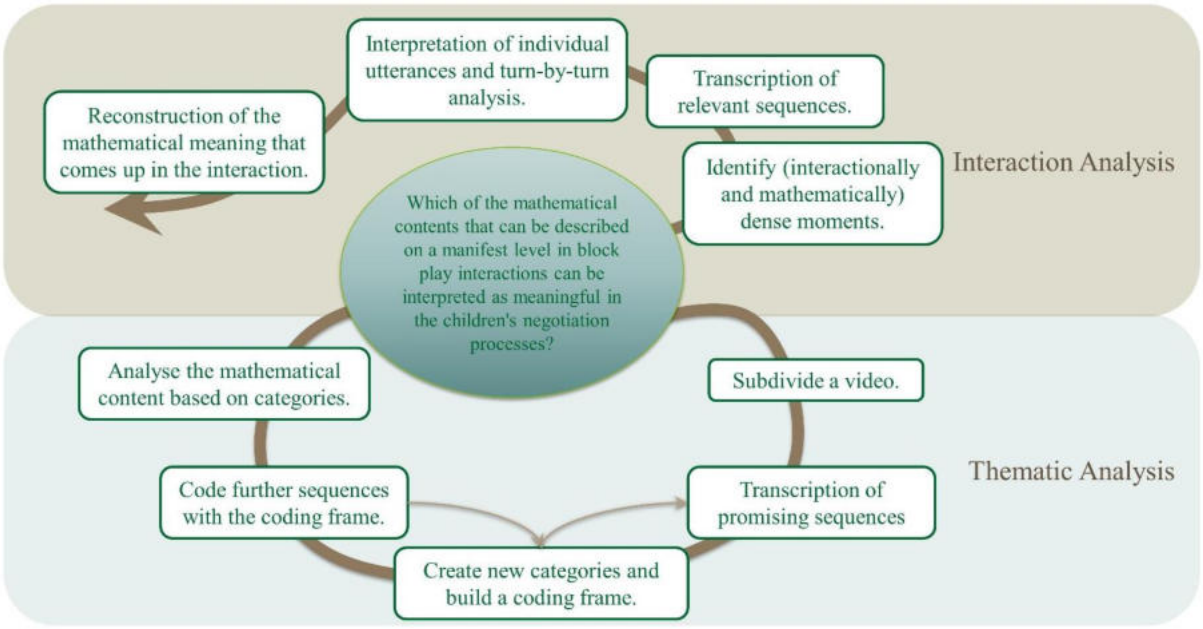

Figure 1: Method of analysis

In the empirical findings, a brief overview of some of the categories developed from the data will be given, and the differentiation and analysis of the (mathematical) content based on categories is shown on the basis of transcripts of two block play situations. These two transcribed block play situations are also adequate sequences for the interaction analysis (Figure 1, upper half), so they are taken as empirical examples for the interaction analysis. Following an interpretative paradigm, a turn-by-turn analysis is conducted to interpret the utterances of the interactants with regard to their interrelatedness (Vogler, 2020). This allows reconstructing which mathematical content appears to be meaningful in the children's negotiation process. The interplay between thematic analysis and interaction analysis

is essential for uncovering how mathematical content, which is outwardly observable, carries meaning for children within the context of their interactions, and for understanding the ways in which this meaning is constructed.

Empirical findings

With the thematic analysis, a total of six categories that can help to describe and systematise mathematical content in children's block play were developed from the data. The categories are small/large, wrong way/right way, slanted/straight, open/closed, fixed/unfixed, and equal/unequal. These categories, which are described in detail in Henschen (2020), are linked to mathematical content in different ways. Based on the scenes that are assigned to each category, various differentiations of content can be analysed and illustrated for each category. This is explained as follows for the categories wrong way/right way, open/closed (Figure 3) and slanted/straight (Figure 5). With thematic analysis it also becomes apparent that the categories and also the facets of each category are interwoven with each other. This is shown below using two empirical examples.

Figure 2: Ladder

Ron:	*points to the bottom end of the ladder*
Max:	da müssen– [we must]
Ron:	mal da ganz runter [all the way down there]
	shows the gap down to the floor (figure 2)
	dann hebts\ gell/ [then it holds\ right/]
Max:	dann müssen wir ganz runter bauen [then we must build all the way down]

With regard to the mathematical content analysed, this interaction can be best described by the following categories: wrong way/right way, open/closed, fixed/unfixed.

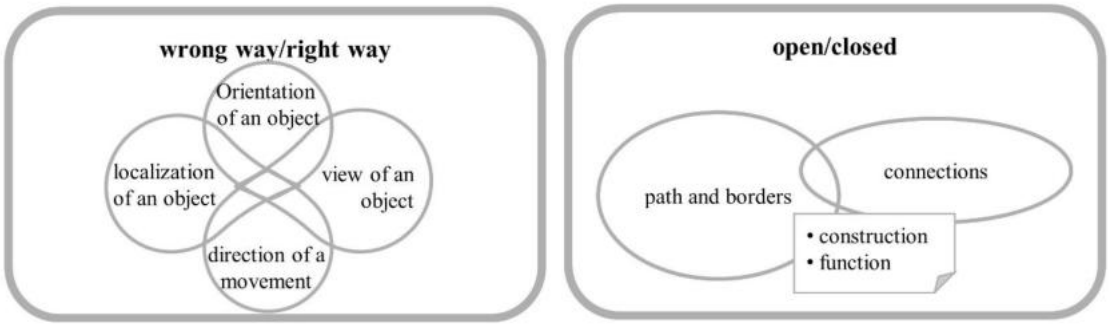

Figure 3: Illustration of the categories wrong way/right way and open/closed

Figure 3 (left side) shows the facets and connections that could be found when differentiating the category wrong way/right way with the thematic analysis. For each of the four illustrated facets, references to certain spatial relationships or concepts could be described on the basis of the children's

interactions. For the two facets of orientation of an object and direction of a movement, this is comprehensible in the above empirical example. The children use terms like “downwards” or “build all the way down,” which, on one hand, refer to the direction of a movement and, on the other hand, describe the orientation of the components that still need to be added.

The category open-closed (Figure 3, right side) references the mathematical topic of topology. As a result of the thematic analysis, a differentiation concerning the construction method and the function of the building emerges. It is, for example, about creating joints between components (connections) or ensuring a path between two locations in the building for imaginary users (path/border). In the empirical example, the statement “build all the way down” illustrates that these two aspects are often intertwined and inseparable. When the focus is on the insertion of a ladder element, it aligns with the concept of making connections. If the focus is on the incomplete connection between the end of the ladder and the floor, it fits the path/borders facet. In this empirical example, the interconnectedness of the two categories, open/closed and wrong way/right way, is striking, as can be seen in Max's statement, “then we must build all the way down”.

To determine which of the contents identified through thematic analysis hold significance in the children's negotiation processes, the empirical example referenced above is examined with a turn-by-turn analysis. In the situation, Ron and Max are building a ladder for their construction. They seem to agree that the ladder needs to be completed down to the ground. Ron first completes the sentence started by Max, “we must”, then Max repeats the whole sentence word for word and adds the verb “build”. This can be interpreted as a taken-as-shared-meaning of the idea “building all the way down”. But why do the children bring this up? Their focus seems to be on stabilising the ladder, “then it holds”. Concerning the mathematical content, it is therefore not necessarily the orientation of the components to be attached that is coming up here, but rather the idea of closing the gap, which leads to a more quantitative consideration in the further course of the block play situation with the question of how much (ladder) is still needed. This suggests that the contents, which are categorised as fixed/unfixed and open/closed, are meaningful for the children in the negotiation process.

The second example illustrated in the following transcript is about a situation in which the children build a construction from building blocks. They name it a corn maze.

Anna:	*places a stone at a right angle to the previous*
	rüber biegen [curve over]
Emma:	*continues the row with a stone straight ahead*
Anna:	rüber [over]
	continues the row with a stone straight ahead, places next stone at a right angle
	dann muss das so abbiegen [then it has to turn like this]
Max:	keine (?) [none]
	continues the row with a stone straight ahead
	so/ [thus]
Anna:	nein [no]
	places a stone at a right angle to the previous
	das muss aber so rüber [but it must turn over like this]
	continues the row with a stone straight ahead
Emma:	*places a stone at a right angle to the previous (figure 4)*
Anna:	warte da (?) so geradeaus [wait there (?) straight ahead like this]

Figure 4: A corn maze

Regarding the analysed mathematical content, this interaction can be well described by the following categories: wrong way/right way, slanted/straight, and open/closed. In relation to the category wrong way/right way (Figure 3, left side), the two facets of orientation of an object and direction of a movement can be identified in this situation. This is shown by the fact that terms such as “curve over”, “turn”, and “straight ahead” are used and corresponding construction actions are carried out that point too the orientation of an object. Simultaneously, these terms serve to indicate the direction of movement, similar to those used in a route description.

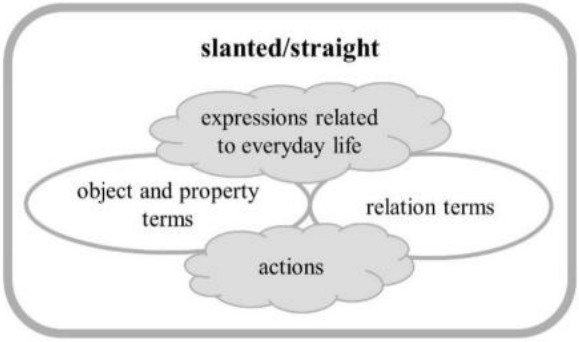

Figure 5: Illustration of the category slanted/straight

With regard to the category slanted/straight, the empirical example shows connections with relation terms. While the addition of blocks in a straight row—geometrically, one could also say at an angle of 180°—is described by the term “straight ahead”, the addition across a corner or perpendicular to the previous block is represented by modelling and pointing as well as by the everyday term “curve over”.

Here, too, it is interesting to reconstruct with turn-by-turn analysis what mathematical content seems to be meaningful in the children's negotiation process. It looks as if Max and Emma mainly carry out actions, i.e., placing bricks, while Anna seems to conduct the collective building process in addition to placing bricks herself. It can be interpreted that in this scene the process of building a (geometric) shape becomes manifest. The obvious mathematical content on the manifest level, which is summarised by the category wrong way/right way, appears meaningful to the children due to the fact that the intended shape enforces a defined arrangement of the building blocks. The fact that creating a certain shape is meaningful for the children in this situation also refers to the category slanted/straight (Figure 5).

Conclusion and Outlook

In both empirical examples, on a manifest level, at first glance, the mathematical content of spatial orientation is evident. In their negotiation processes, however, the children focus on topics that are linked to further mathematical content, such as the gap between the end of the ladder and the floor or the construction of an object in a certain geometric shape. Finally, it can be stated that the mathematical aspects summarised as locating in the theoretical framework and, in particular, the aspects' spatial locations and spatial orientations and transformations from the Spatial Mathematical Coding System of Bryant et al. (2024) can be recognised in the empirical findings presented above.

The diverse links with mathematical concepts and various connections between the mathematical contents and their facets that are evident in the block play interactions and negotiation processes of the children can be seen as a resource. It is evident that children use this resource in their interactions when they solve construction problems together (Vogler et al., 2024). One reason for this could be the effort to achieve an intersubjective understanding of the situation, as can be found among peers in block play (Henschen et al., 2024). For further mathematical learning, also in formal learning contexts, it is a relevant question how professionals can recognise and support this mathematical content, which is still approached with imprecise language by young children.

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Semantic profiles in screening 6-year-olds' spatial sense and reasoning in Sweden: The impossible of not being a teacher

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This study explores the semantic profiles in teacher-student interactions while screening six-year-olds' spatial sense in a Swedish preschool class. The focus is on an activity where students identify and communicate spatial relationships. Using Legitimation Code Theory (LCT), the study examines changes in semantic gravity and density during interactions and the implications that the resulting semantic profiles entail for the teacher's role. While iterative shifts in the teacher's language use from context-dependent and simplified to decontextualised and highly condensed, known as semantic waves, are crucial for students' understanding and mastery of subject-specific language, these are expected to be absent in screening situations. The findings highlight the complexity of maintaining the role of screening administrator for a teacher, who might find it hard to resist the urge to transform a screening session into a learning opportunity.

Keywords: Early childhood mathematics education, LCT, screening, semantic waves.

Introduction

The foundation for students' academic success is widely acknowledged by the research community to be established at an early age (Reikerås & Dahle, 2022), while the subject content of spatial reasoning is considered a crucial part of students' development of mathematical knowledge (Clements & Sarama, 2011). Spatial reasoning skills facilitate, for example, problem-solving and creative thinking in various situations (Lowrie et al., 2019). In Sweden, six-year-olds attend preschool class, the first year of the formal education system, intended to provide a smooth transition between preschool and school. Teaching six-year-old students should "stimulate pupils' general development and learning" (National Agency for Education, 2022, p. 22), and mathematics is an integral part of the curriculum. The curriculum advocates that teaching six-year-old students should provide opportunities for students to use mathematical concepts and reasoning in contextually relevant situations, which places demands on how teachers engage in mathematical discussions with students.

Teacher interaction with students can be viewed in terms of semantic waves (Maton, 2013), reflecting iterative shifts in the teacher's language use from context-dependent and simplified to decontextualised and highly condensed. These shifts are believed to enable students to gradually master the subject-specific language use that is subsequently emphasised and assessed throughout the educational system. Therefore, examining how teachers introduce six-year-olds to one of the subject-specific language uses in more detail is essential.

To ensure individually tailored teaching and learning practice, preschool class teachers in Sweden have to use screening tools since 2019. One of those tools is entitled *Find the Mathematics* and is used to assess six-year-old's "mathematical thinking" (National Agency for Education, 2019). Its primary objective is to identify students at risk of not meeting the learning outcomes, students needing additional support, and students needing further challenges. The screening is conducted during the

autumn term, marking the beginning of the student's first year in school. It consists of four activities: patterns, number sense, measurement, and spatial sense (see Walla, 2024 for a comprehensive overview). Although the Swedish screening tool is helpful for assessment as well as teaching and learning purposes, it remains unclear to the teachers whether they should assess students' current mathematical knowledge or whether they should assess what the students learn during the screening (Walla, 2024). This might create an equity issue in screening and its results for students' further mathematical development. To follow it up, this paper delves into teacher-student interaction in one of the screening activities, *Playground*, and examines the language used in conducting this activity from the point of view of the semantic waves.

Semantic waves are ideal situations of iterative shifts in the teacher's language use from context-dependent and simplified to decontextualised and highly condensed. To describe those shifts, Maton (2013, pp. 11–12) introduced the notions of *semantic gravity* (the degree of context-dependence of meaning) and *semantic density* (the degree of condensation of meaning). Both semantic gravity (SG) and semantic density (SD) may strengthen and weaken in the ongoing classroom interaction; but a semantic wave only emerges when the language use goes from being decontextualised (SG–) and highly condensed (SD+) to being highly situated in the context (SG+) and simplified (SD–) and then necessarily back. However, strengthening and weakening of semantic gravity and semantic density do not necessarily follow this pattern and may form other *semantic profiles*, for example, a *low semantic flatline* (SG+, SD–) in which language use retains the same degree of high semantic gravity and low semantic density over the observed time (for more detail, see *Theoretical background*).

Given the ambiguity of the screening task and, to our knowledge, the lack of studies that examine semantic profiles of language use in screening situations, there is a need to shed light on how teacher-student interaction unfolds in a screening activity, focusing on six-year-olds spatial sense. Thus, this paper aims to explore semantic profiles in the teacher-student interaction in the activity *Playground,* which is a part of the Swedish screening tool *Find the Mathematics.* To achieve the aim, the following research questions are addressed:

- How does the semantic gravity of interaction between the teacher and six-year-olds change in the screening activity?
- How does the semantic density of interaction between the teacher and six-year-olds change in the screening activity?
- What does the semantic profile of the interaction reveal in regard to the teacher's role in the screening activity?

Since the screening activity, *Playground* is purported to assess six-year-olds' spatial sense, which is how they understand shape, size, position, direction, and movement internally in their minds, we need to turn our attention to *spatial reasoning*. Spatial reasoning, defined by Lowrie et al. (2019), involves spatial sense and how spatial information is represented internally and externally. Finally, spatial reasoning involves interpreting and manipulating spatial information and is comprised of the way six-year-olds communicate the information to others. It is, therefore, reasonable to investigate teacher-student interaction as a whole.

Spatial reasoning concerning six-year-old students

Spatial reasoning is conceptualised as understanding and manipulating spatial relationships between objects or shapes. It involves the mental manipulation and visualisation of objects in two or three dimensions (Duff et al., 2021) and to reason about it with others (Lowrie et al., 2019). Spatial reasoning is a fundamental cognitive skill that helps individuals make sense of the physical world, solve problems, and navigate their environment (Duff et al., 2021). Young students rely on their spatial reasoning skills to complete various daily activities. Whether they perform ordinary tasks like playing on the playground or navigating to school or more mentally demanding tasks like perspective-taking, they rely on their ability to manipulate aspects of their environment mentally.

Further, research has shown that improving spatial reasoning skills in children can support learning achievement (Lowrie et al., 2019). In recent years, the connections between spatial reasoning and mathematics ability have become particularly clear, with classroom research demonstrating that (i) spatial reasoning is straightforward to improve and (ii) improving spatial reasoning skills benefits mathematics outcomes (Lowrie et al., 2019; Mulligan et al., 2020).

In the activity *Playground*, the students are asked to imagine, from a perspective from above, what a girl should see if she stood on the ground, which is further explained in the methodology section. This activity means engaging in a cognitive process that involves mentally manipulating and rotating an object or a mental image of an object to understand how it would appear from different angles or orientations. Since the teacher-student interaction while screening provides insight into students' current level of spatial reasoning skills, there is a need for a linguistic lens to understand interactions.

Theoretical background

In mathematics educational research, there has been a social turn (Lerman, 2009) from viewing learning-as-acquisition to learning-as-participation (Sfard, 2009). In this turn to the social, interest emerged in the linguistic side (Lerman, 2009) of learning-as-participation. One theory that focuses on the linguistic side of teaching and learning, specifically on language use as a means of cumulative knowledge-building, is *Legitimation Code Theory*, *LCT* (Martin & Maton, 2013).

Stemming from Systemic Functional Linguistics, SFL, and code theory, LCT uses novel concepts such as semantic gravity and density (see *Introduction* for definitions), which are also the cornerstone of this paper. Maton (2013) introduced these concepts to enable studying knowledge-building as it unfolds in teaching and learning and pointed out the need that classroom practice "traverse[s] the semantic gap" (p. 14) that arises between students' reading of textbooks and students' high-stakes examination, generally in written form, both of which are full of decontextualised (SG–) and highly condensed (SD+) meanings. According to Maton (2013), successful classroom practice must be characterised by two *semantic shifts* in teacher-student interaction, constituting a semantic wave. First, the condensed meanings from the textbooks are unpacked in simpler ideas and placed in the context to facilitate students' internalisation of the ideas, thus strengthening the semantic gravity (SG+) and weakening the semantic density (SD–), which forms a *downward semantic shift*. An upward semantic shift should follow up on this shift, weakening the semantic gravity (SG–) and

strengthening the semantic density (SD+) so the students learn to master the kind of language use that is expected from them in the high-stakes examinations.

In the current study, which involves a screening activity of six-year-olds, there are at least two core inherent properties of the situation that are likely to impact the semantic profile of teacher-student interaction. The first one is related to the screening situation. Screening per se is not meant to develop students' knowledge but rather to assess their present mathematical knowledge level, which is hypothesised to require the teacher's language use to reveal a low flat semantic profile (i.e., SG+, SD–) and to be responsive to the students' language use. In other words, ideally, the teacher's language use is expected to retain the same semantic gravity and density throughout the screening, and at times it might mirror the students' language use if required by the communicative situation. The second inherent property is related to the age of the students. Given that LCT was initially developed to highlight cumulative knowledge-building in secondary schooling, the rationale behind what might be categorised as a highly condensed meaning (SD+) and simplified meaning (SD–) must be adjusted to fit the language use of the targeted age. These issues of language use and its semantic profiles are also related to students' spatial sense and spatial reasoning, which is addressed in the following methodological section.

Methodology

This section describes the screening activity *Playground* through a narrative example of its conduct. It includes data collection, ethical considerations, and data analysis.

Data collection

At the end of the autumn term in December 2022, a female teacher conducted the screening activity *Playground* with three groups of six-year-olds on three consecutive occasions on the same day. Each group consisted of 4 students with the gender distribution of 7 females and 5 males. The screening sessions lasted 20–26 minutes and were video-recorded.

In the initial phase of the activity, the students were instructed to solve a problem in a group and share their thoughts. The teacher showed the students a large picture of the playground and highlighted that it was drawn from a bird's-eye view (see *Figure 1*). The exact wording of the teacher cue was: "Imagine that you are flying above the playground and looking down. It would look like this. What do you see in the picture?" The teacher then proceeded to ensure that all students understood what the picture showed. In the next phase of the activity, the teacher took out the first of six picture cards showing what a girl could see in the playground and asked: "If the girl sees this [showing *Figure 2*] when she looks forward, where is she in the playground? You will have to show where she is on the large picture. Now, take a moment to think for yourself before you talk to each other." In the subsequent phase of the screening activity, the students were prompted to answer the following questions: "Why did you choose that spot? What can the girl see? How do you know that? Can the girl see the boy? Why/why not? Is there anything visible on the picture card that is not seen in the large picture?" The procedure was then repeated for the remaining five picture cards. In the final phase of the activity, the teacher asked questions about the seasons of the year. This phase was excluded from the scope of this paper since it did not concern six-year-olds' spatial reasoning.

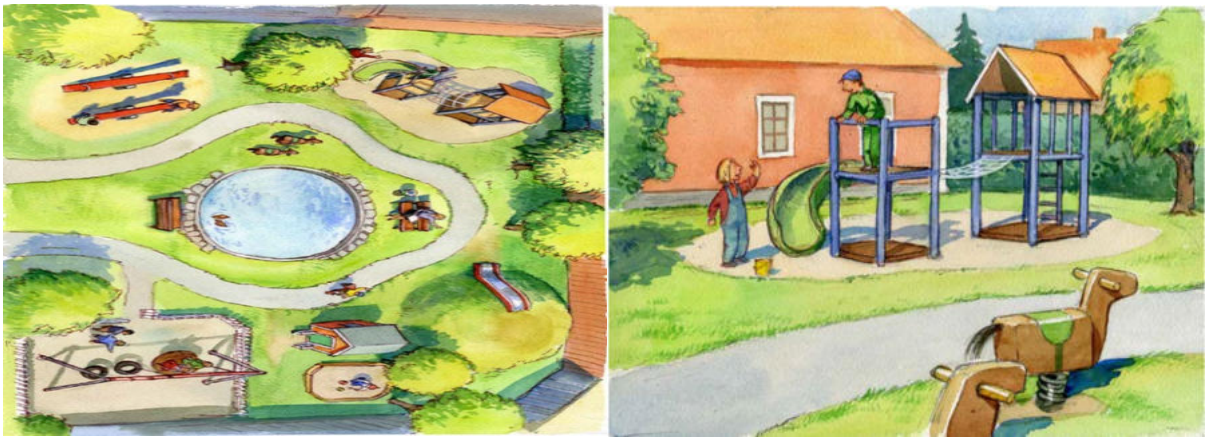

Figure 1 (left). The map of the playground; Figure 2 (right). An example of a picture card (National Agency for Education, 2019)

Ethical considerations

Before the screening activity, the students' informed consent to participation was obtained orally, while the guardians' and teacher's consent was obtained in writing. The participants were also informed that they could withdraw their consent at any time without prejudice. The students were given pseudonyms in the paper to ensure their anonymity and confidentiality. The collected data are handled through secure storage at Linnaeus University's servers, which adhere to GDPR. As such, the study conforms to the ethical regulations for research in Sweden.

Data analysis

To facilitate data analysis, the video-recorded screening sessions were uploaded to NVivo (software for conducting qualitative data analysis) and transcribed verbatim with additional information, such as gestures in the videos or omissions/modifications in the transcript, provided in square brackets. NVivo was used to analyse the video and the transcript using the codes described below. The coding continued until data saturation was achieved (Saunders et al., 2018), which occurred after ten minutes, indicating that any new data began to repeat what had already been found in the previous data.

In our coding, we used the existing codes (SD–, SD+, SG–, SG+) and introduced two new codes (SD0 and SG0); all the codes were used separately for teacher and student utterances. Introducing the new codes was necessary to capture small changes in semantic density and semantic gravity that were typical of the language used in the teacher-student interaction in the given age group. For example, an utterance was coded as SD– when only the placement of an object was expressed, while a longer utterance describing the placement of an object in relation to other objects was coded as SD0. These codes are illustrated in the following extracts: "That one… that and [pointing at the picture]" (SD–) and "The climbing frame, right, and where is it in the playground?" (SD0). Code SD+ was assigned to an utterance which, in addition to expressing the relation of an object to other objects, was also concise, thus bearing highly condensed meaning as illustrated in bold in the following: "No, because she is standing **backwards** [...] so she cannot see the sandpit".

Concerning semantic gravity, codes SG– and SG+ were assigned to the following: “Then if we see some different things that might not be visible on the big picture…” (SG–) and “[pointing at the picture]. They are very big and close...” (SG+). Code SG0 was used to categorise teacher questions when she invited another student to develop or disagree with the previous student, thus making the degree of contextualisation open to interpretation as illustrated: “Yes, what do you think, Roger?”

Results

As hypothesised, SG+ and SD– were the most frequent codes in the teacher's utterances (approximately 57 % and 71 % of all teacher codes for semantic gravity and density respectively). This suggests that the teacher strived to keep a low flat semantic profile of her language use and to assume the role of screening administrator. However, the data also revealed patterns in the teacher-student interaction, which highlighted the complexity of the task.

In a significant part of the coded screening activity (83 %), the teacher’s language use followed a mirroring pattern, that is the teacher’s subsequent utterance maintained the same degree of semantic gravity and density as the previous student’s utterance. This is illustrated in the extract below (codes are provided in the brackets).

Roger:	That one... that [points at the card]? (SG+, SD–)
Teacher:	You were thinking about that... why don't you see those two then... in this picture? (SG+, SD–)
Alva:	Because it's far away... (SG+, SD0)
Teacher:	Far away... because it's outside the picture here or this map or whatever we should call it. (SG+, SD0)

In the exchange with Roger, the mirroring was in line with the expected semantic profile (SG+, SD–), while in her mirroring follow-up of Alva’s utterance, the teacher adjusted semantic density to SD0.

The analysis of changes in semantic profiles demonstrated that semantic density fluctuated as the teacher-student interaction unfolded, as shown in the extract below.

Anton:	Windows... windows on the house. (SG+, SD–)
Teacher:	Yes, exactly... why don't we see that in the picture then? (SG+, SD–)
Anton:	Because it's from above, and that one is from the side, so we can't see it. (SG+, SD+)
Teacher:	So you can't see this one... yes... (SG+, SD–)

In this exchange Anton's language use indicates strengthening of semantic density (SD– to SD+) from one utterance to another, while the teacher language use displays the low flat semantic profile. Notably, in this case, the teacher did not mirror Anton’s strengthened semantic density and remained in the role of screening administrator, thus diminishing the risk of her impact on the other six-year-olds' language use.

However, there are a few instances in the coded data when the teacher deviated from her role as screening administrator and assumed the teacher role. One such instance is evident in the extract below.

Teacher:	OK, that's why she [the girl] stands a bit to the side. You can see the edge of the sand around the seesaw. Did you think about that? [hesitating] (SG+, SD0)
Teacher:	What do you think, Roger, or did you raise your hand? (SG0, SD–) [...]
Roger:	There... it is a bit closer. (SG+, SD–)

Teacher: Yes, but I was thinking about which direction she is looking at. Yeah... but I understand what you mean. Do you think like that, don't you? (SG0, SD0)
Roger: Yes.
Teacher: Yeah... but it's roughly the same place, and I think that fits quite well. Because then she sees... she's close to the seesaw, but she sees the other things even though they're a bit farther away. Is there something we can see that she sees here that we can't see on... on the overview map... like... [turning to] Olivia. (SG+, SD0)

In the first utterance the teacher simply mirrored the previous students' utterance (SG+, SD0) and then invited another student, Roger, to contribute (SG0, SD–). Roger appeared to immediately contextualise the meaning, thus strengthening the semantic gravity (SG+, SD–). However, the next two teacher utterances were characterised by strengthening of semantic density of the interaction (SD– to SD0) as she deviated from her role as screening administrator and assumed the teacher role.

Discussion

When looking at the semantic profiles of the interaction and what they reveal concerning the teacher's role in the screening activity, we can conclude that though it is not the aim of the situation, students explore their shared knowledge and create learning opportunities. As such, the teacher has a complex task in sustaining the role of screening administrator. While the teacher in this study mainly keeps the low semantic profile to assess six-year-olds' understanding and does not engage in teaching when learning opportunities arise, some sections show complexity, as presented in the result. The deviation from the low semantic profile is sometimes evident in (1) mirroring when the teacher is likely to conform to Grice's (1989) maxim of relation of efficient communication; (2) situations when she cannot resist the urge to transform a screening session into a learning opportunity thus assuming the teacher role. Therefore, it remains unclear whether teachers assess students' current mathematical knowledge or whether they assess what the students discern and learn during the screening. For students at risk, this means that they might fly under the radar and be deprived of adequate support at an early stage, which is precarious not only for honing their spatial reasoning skills but also for overall school success.

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Investigating pre-service mathematics teachers' requests for information in one-to-one tutoring

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This paper reports on a study of the requests for information posed by pre-service teachers practicing one-to-one tutoring of university preparatory mathematics students. The objective was to capture the topics of requests and pre-service teachers' stated rationales for posing them. Data was collected using audio recordings of one-to-one tutoring in a whole class lesson setting along with individual interviews, and analysed using inductive and deductive coding. The findings reveal requests centred on the task, mathematics, the student's knowledge or experiences, and the textbook. Rationales behind requests include orientation of the student's knowledge, encouragement of recalling prior knowledge, and guidance of delimited procedures. This study offers a topic perspective on teacher-student talk and provides additional insight into this complex yet important practice.

Keywords: Classroom communication, preservice teachers, tutoring, questioning techniques.

Introduction

It is acknowledged from decades of research that teacher-student talk has a significant influence on students' mathematical learning outcomes (Brodie, 2011). However, pre-service teachers are faced with challenges when learning how to communicate effectively in the mathematics classroom. This has been exemplified by Gaspard and Gainsburg (2020), with results showing discrepancies between pre-service teachers' knowledge about successful questioning practices and their performed practices. To develop pre-service mathematics teachers' skills in classroom interaction, it is important to study their practice of and rationales for requesting student input in mathematics teaching.

Mathematics teachers' verbal communication has largely been studied in whole class settings. This study is an attempt to extend this knowledge base of teacher-student interaction to one-to-one tutoring conversations. These conversations are particularly interesting for interaction and communication research in mathematics education, since they are usually initiated by the student, whose issue frame the conversation. Additionally, unlike whole-class teaching, teachers cannot plan as much for what to say or how to lead the conversation forward. This makes the rationale behind the teacher's communicative moves an interesting object of study, which this paper seeks to address.

In various ways, mathematics education research has identified and classified teachers' requesting information from students in classroom conversations (Boaler & Brodie, 2004; Drageset, 2014; Mehan, 1979; Nathan & Kim, 2009; Sinclair & Coulthard, 1975). These studies have produced knowledge about the structure of question-response interactions, the usage of "higher order" versus "lower order" questions, and the various ways by which teachers ask for student input in classroom conversation. However, one question remains, that is, what are the teachers asking about? This study attempts to contribute to efforts toward a more comprehensive understanding of teacher-student communication in mathematics classrooms by addressing this issue.

The aim of this study is therefore to investigate pre-service teachers' (PSTs') requests for information in mathematics one-to-one tutoring in a way that reveals topics of the requests and rationales behind them. Hence, the study is guided by the following research questions:

RQ1: What do PSTs request information about during one-to-one tutoring in mathematics?

RQ2: What rationales do PSTs provide for their use of requests for information from students?

Background

Seminal literature on classroom discourse (Mehan, 1979; Sinclair & Coulthard, 1975) made an extensive work in classifying the structure of classroom talk between teacher and students. Sinclair and Coulthard (1975) identified three major speech acts performed by the teacher in the classroom: *elicitation* (request of linguistic response, realised by a question), *directive* (request of non-linguistic response) and *informative* (give information or opinion). In later work, Nathan and Kim (2009) added to the elicitation category by also including pauses and provocative statements made by the teacher. The present paper is limited to the study of requests for information. This demarcation includes more requests than the syntactic questions of Sinclair and Coulthard's (1975) elicitation category, while excluding other, more subtle, ways of inviting students to participate in the conversation.

The notion of speech *acts* used by both Mehan (1979) and Sinclair and Coulthard (1975) reveals the assumption that *saying something means doing something*. An act is defined as the smallest unit of analysis of discourse, and may be described as the functional equivalent to the grammatical clause (Mehan, 1979; Sinclair & Coulthard, 1975). Seeking to identify PTSs' requesting in tutoring conversations, the same basic assumption is essential to this paper. One or several acts are made within a *turn*, which Mehan (1979) defines as the period in a conversation during which a single teacher or student speaks, before giving the word to or being interrupted by someone else.

Previous literature on requests, elicitations or questions posed by the teacher in mathematics teaching have resulted in various classification systems. In a literature review of research on mathematics teachers' use of questions, DeJarnette et al. (2020) conclude that classifications are often based on what type of responses the different questions elicit. One common approach is to classify questions by the cognitive demand they place on students for responding, ranging from lower to higher order (e.g. Nathan & Kim, 2009). The same approach has been used when analysing the questioning practices of PSTs in one-to-one mathematics tutoring (e.g., DeJarnette & Hord, 2022; Kilic, 2018). These studies have shown that PSTs usually use lower order questioning that requires short answers, but also that participation in training related to posing mathematical questions can increase the usage of higher order questions. There are also studies identifying teacher questions according to intentions with asking them, as interpreted by the researcher (e.g. Boaler & Brodie, 2004; Drageset, 2014). As a third approach, Gaspard and Gainsburg (2020) classified PSTs' questions in whole class teaching into the main categories predictable and unpredictable questions, referring to if the response anticipated from the students was predictable by the PST or not. The authors also interviewed the PSTs for their rationales for asking different kinds of questions over time. Results include an increasing avoidance of unpredictable questions motivated by their invocation of negative reactions among the students (Gaspard & Gainsburg, 2020).

The current study does not attempt to classify requests according to the cognitive demands placed on students when answering them, the teacher's intention with asking them (as interpreted by the researcher), or whether or not the answer is predictable by the teacher. Instead, the aim is to describe the topics of the requests, along with rationales for posing them as stated by the PSTs. The way in which the study was conducted in order to achieve this aim is outlined in the following section.

Method

Four PSTs in their third or fourth year of study were invited and accepted to assist 35 students enrolled in university preparatory courses during two mathematics lessons. Individual interviews with the PSTs were held after each lesson. Both PSTs and students received written and oral information about the study, and written consent was obtained. Data collection and analysis in response to each research question was conducted in two separate stages, and hence are presented separately below.

To address RQ1, audio recordings of one-to-one tutoring conversations were collected and thereafter analysed using inductive coding. In this paper, one-to-one tutoring (tutoring for short) refers to in-class personalised teaching occurring in a conversation including one tutor and one or a few students. The tutoring conversation aims to guide the student(s) towards understanding of the mathematical content and solution of specific textbook problems. During the lessons in the current study, the students were supposed to work individually or in pairs on tasks from their mathematics textbook. The subject matter of the two lessons were integrals and trigonometry, respectively. However, students also worked on tasks concerning functions, logarithms, equation solving, and differentiation.

The audio recordings were transcribed verbatim. All participants were anonymized in conjunction with the transcription process, assigned pseudonyms (PSTs) or numbers (students). Transcription served as the first phase of analysis, familiarising the researcher with the material, while the subsequent step identified the requests. In each conversation, the PST's verbal contributions were examined. The objective was to identify turns or sub-turns constituting acts of information request. This study uses the concept of turn similarly to Mehan (1979), and acts similarly to Sinclair and Coulthard (1975). Moreover, this study considers request for information to be those acts made by the PST where he or she, in the syntax form of a question or not, asks for student input. The input is requested to be informative, not confirmative, and a linguistic response is expected. Table 1 illustrates the process of identification of information requests. These acts might often appear as questions, but not always – and not all questions were identified as requests for information. Excluded acts were confirmations, directives, explanations, corrections, encouragements, and "hidden telling" (e.g. explanations ending with "right?", or "isn't it?", requesting a confirmation from the student). This first identification yielded 173 PST requests relevant to this study.

Aiming to capture the topic of each request, the third phase of analysis used inductive coding (Braun & Clarke, 2006), focusing on *what* the request centred on, and is illustrated in Table 1. During coding, the context of each request was considered, e.g. the problem the student was working on or the most recent talk. When necessary, the audio files supplemented transcription to facilitate the analysis, providing important information on verbal emphases. Thereafter, codes were formed into broader categories, considering similarities and differences across codes.

Table 1: Illustration of identification and coding of requests for information

Turn	David: *And that equals…? Yes! And, have you used the pq-formula before?*		
Identification of requests	*And that equals…?*	*Yes!*	*And, have you used the pq-formula before?*
	Request 1	Not request	Request 2
Code (topic of request)	What it equals	-	The student's prior experiences

In the one turn presented in Table 1, David performs two different requests with a confirmation in between. The question marks added during transcription, as well as the actual words being pronounced by the PST, provided guidance in the process of identifying requests for information. The confirmation ("Yes!") in the middle of the turn, acknowledging the student's writing in their notebook, along with a brief look at the content of the requesting parts of the turn informed the division of it into two different requests for information. The first request takes the form of an incomplete sentence ending with a raise in intonation, which asks for the result of a multiplication. This information was provided by the context of this turn, i.e. the teacher and student talk preceding this one. This context, along with the explicit use of the word "equals", motivated the code *what it equals*. In the end of the turn, the PST asks for information about the student's possible prior usage of a certain formula. This was coded as requesting information about *the student's prior experiences*.

In order to answer RQ2 and gain insight into the PSTs' rationale behind requests for information in mathematics tutoring, individual interviews were conducted immediately after the lessons. These audio recorded interviews focused on the PST's thoughts on, approaches to, and strategies in one-to-one tutoring. Conducted semi-structurally (Kallio et al., 2016), the PST was first asked to share their recent experience of tutoring in the recorded lessons. Following this, the main question of the interview was "In general, how do you think when you are tutoring students in mathematics?". Further questions varied based on each PST's responses, including examples like "How do you decide when to ask questions and when to show a procedure?", "How do you adjust your communication to individual students?" and "What difference do you think it makes between asking a question you know the answer to and one you don't?".

The interviews were transcribed verbatim and thereafter analysed by identifying parts where the PSTs talked about requests for information in tutoring. This analysis was conducted deductively, based on the codes and categories found in response to RQ1, presented in the result section. Relevant parts of the interview, often constituting PST reasoning in a few sentences, were assigned a suitable category from the RQ1 results. Next, a description of the rationales for each request category was produced, and representative examples were identified. Categories of requests and rationales for them will be presented and exemplified in the following section.

Results

The findings of this study are presented in relation to the two research questions respectively.

RQ1: What do PSTs request information about during one-to-one tutoring in mathematics?

The analysis revealed four categories of information that the PSTs requested during one-to-one tutoring, centring on different elements of the tutoring situation. The requests were task-centred (n = 24), mathematics-centred (n = 90), student-centred (n = 43) and textbook-centred (n = 16). The first two categories were found in tutoring conversations of all four of the PSTs, whilst the two latter were found among three out of four PSTs.

Task-centred requests concern task information and the student's work or thoughts on the task that is currently being worked on. The PSTs ask for what is known or what to seek in the specific task. When requesting information about the student's work or thoughts on the task, the PSTs ask for a description of the work performed by the student so far or how the student think it is appropriate to move forward in the solution. A generic example of this category is "What have you done so far [on the task]?".

Mathematics-centred requests are broader in terms of application compared to the requests centred on the task. They often stem from a task being worked on at the moment but focus on mathematics in general. The PSTs ask for concept explanations, procedure steps, or mathematical relationships. The straightforward question of "what it equals" is frequent in this category. Examples are

Christopher: I know we're supposed to take e to the power of ln 3, and what do I mean by 'to the power of'?

Bella: And what value does it equal then? If we put in 90 degrees?

David: When can the Pythagorean theorem be used?

In the first example, Christopher pauses his explanation of procedure to direct the student's focus to the concept of "to the power of". He requests general rather than task specific information about the meaning of this concept, from the student. The same goes for Bella's request on the value of sin 90° and David who asks for a general prerequisite for using the Pythagorean theorem.

The third category, *student-centred requests*, ask for information about the student in relation to the mathematical context. These requests concern the student's feeling or opinion about a certain solution ("Do you find it weird?" - Christopher), as well as questions explicitly asking if the student understands. Other requests in this category concern the student's work on other tasks, like questions on if, or how, the student has solved problems the PST considers relevant to the one at hand. There are also requests belonging to this category in which the PST asks explicitly for student's prior knowledge or if the student recognises something being shown or told. Some examples are

Annie: Do you recognize this from school?

Christopher: What do you know about calculating the area of a surface?

Requests in this category centres on the student while still asking about mathematics. This emphasis on the student can often be detected by the formulation of "do *you*"/"did *you*" in the request, and distinguishes student-centred requests from the mathematics-centred ones. In mathematics-centred

requests, the request is also directed to the student, which is the one expected to produce a response, but it is the mathematics that is at the centre of the request.

The fourth and last category of requests is *textbook-centred requests*, focusing on the textbook used during the lesson. The PSTs ask for where they find formulas in the book or what the book's solution key says for correct answer. This is exemplified by David below.

David: How did they [the authors of the book] do in the answer?

The use of "they" as referring to the authors of the textbook makes this example textbook-centred.

RQ2: What rationales do PSTs provide for their use of requests for information from students?

In the interviews, the PSTs offered several reasons for the requests being posed during the lesson. Here, those rationales will be presented in relation to the four categories of requests for information.

Task-centred requests were used by all four of the PSTs. Annie, Bella and David all motivate this kind of request being posed in the beginning of the conversation as a way of finding out what kind of assistance the student needs. Asking for what is known or what to seek in a certain mathematics task is described by Bella as a kind of summarising in order to progress on the task.

Bella: I don't like to give them the answer, I'd like to make them think for themselves. And to summarise what we've got, what the task description tells you, and what you know yourself.

In this example, "what we've got" and what the text in the task tells you are information provided by the textbook but requested from the student by Bella. Similarly, David describes how he could proceed from the information gained from task-centred requests by asking the student about a mathematical procedure. This refers to a *mathematics-centred requests*, which were identified among all four of the PSTs. David calls the guidance of a limited procedure a "smaller issue" to face, compared with if the student does not know how to approach the task at all.

The third category of requests, *student-centred*, was used by Annie, Christopher and David. Christopher puts a lot of thought into how he requests and gains information on students' knowledge. In the interview he talks about how he prefers to let the students tell what they know from before.

Christopher: Because they often wonder, I mean, they don't know what they don't know. And if you can find out what they don't know, there's no need to help them all the way.

Christopher describes that when he asks about students' prior knowledge, he tries to avoid questions he already knows the answer to. For example, he would rather ask "What do you know about ln?" (student-centred request) than "What does ln e equal?" (mathematics-centred request). He says that he tries to steer the conversation towards their knowledge, posing a request for which the answer lies in the power of the student. This is rationalised on the basis that there is no clear right or wrong in the student's answer "nothing" to the question of "What do you know about ln?", but that it does provide the PST with information about the student's prior knowledge. Another reflection concerns the closely related matter of the type of requests for which Christopher himself knows the answer to.

Christopher: Why would I ask for something I know the answer to myself? You would never do that otherwise. And it's like, well the teacher has all the answers.

Textbook-centred requests were used by Bella, Christopher and David. In his interview, David refers to the book as a helping hand when he's unsure of something and afraid to admit it to the student. He states that he finds it easier to say he does not remember the formula and look it up in the textbook than to say he doesn't know how to solve the problem.

Discussion

The findings of this study consist of categories of requests for information posed by PSTs in mathematics tutoring, as well as PSTs stated reasons for requesting that information. In summary, the PSTs' requests are centred on the *task*, on *mathematics*, on the *student* and on the *textbook*. Rationales for different kinds of requests include mapping of the student's former work or knowledge, encouragement of the student's independent thinking in place of teacher telling, and provision of guidance for oneself. Categories were formed based on the topic of the request, providing additional insight into classroom interaction in the context of one-to-one tutoring. Along with the description of rationales behind requests, this paper contributes to our understanding of mathematics teachers' questioning by nuancing as well as explaining the topics of the requests being posed.

The categories of requests were found to be unevenly distributed, with mathematics-centred requests being the most prominent (n = 90) and textbook-centred requests the least (n = 16). Considering one rationale for textbook-centred requests being mathematical uncertainty of the PST, this result is not surprising. The findings on rationales for requests relate to previous literature in several ways. For example, Christopher condemns questions that he knows the answer to, such as closed questions asking for mathematical facts. These kinds of questions reflect Gaspard and Gainsburg's (2020) notion of predictable questions. The interview transcripts show how Christopher's rationales for asking student-centred questions instead of mathematics-centred questions builds upon a view of predictable questions as something artificial, while the PSTs in Gaspard and Gainsburg (2020) rationalise the use of predictable questions by reactions from students they have taught. However, DeJarnette et al. (2020) state that teachers often ask questions with a specific response in mind. With data on PSTs' rationales for requesting certain types of information from students, it seems close at hand to compare those to the intention-based classification systems of Boaler and Brodie (2004) and Drageset (2014). In a future study, the relation between rationales behind requests in this type of study and the categories from previous mentioned literature could be analysed.

Regarding data collection methods, the interview was chosen for answering RQ2 on PSTs' rationales behind requests because understanding the rationale behind actions requires asking the actors themselves. Since the interviewer had not yet accessed the audio recordings or conducted an analysis, preparing specific questions about the requests actually posed during tutoring was not possible. This limitation might have reduced the number of rationales identified for each category. However, the narrow time gap between lesson and interview is a strength as it ensures fresh recall of the experience.

The recorded lessons constituted a real-world setting for the students considering teaching method, textbook, and tasks used. What differed was that in addition to the students' ordinary teacher, PSTs participated as tutors, meeting the students for the first time. The PSTs had limited prior knowledge of the students' mathematical performance, requiring them to gather essential information during these lessons. This means that the information that the PSTs considered necessary for one-to-one

tutoring is included in the collected dataset and therefore accessible for study. In addition to the findings on requests for information, however, future studies using a similar approach could address teacher acts of giving information to or requesting action from the students.

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Linguistic demands and more: A framework for analysing mathematics lessons

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We present a framework developed as a tool for analysing rich episodes of mathematics lessons in ways that reflect this richness rather than summarise it through coding. Our project supports language-responsive teaching. This framework was developed to address our concerns that through code-based analysis of lessons we were losing too much of the complexity, particularly relating to mathematical connections and aspects of the interactional context that build, evolve, and adapt during an episode, task or lesson. The framework involves identifying structure within lesson episodes, based on shifts in mathematical focus, and analysing how key dimensions of language-responsive mathematics teaching are at work within this structure. We draw on data from one analysed lesson episode to illustrate the use and potential of the framework, but also to underscore some of the challenges and reflections that the framework has brought to our work within the project.

Keywords: Language-responsive teaching, mathematics classroom discourse, noticing.

Introduction

In 2022, in the context of a broader team including secondary-school mathematics teachers and other researchers and teacher educators, we started a project aimed at examining and discussing language-responsive ways of developing mathematics teaching. We have faced several challenges since then, one of which has been the thinking and use of a framework that allowed us to "see" and investigate some of the features of the mathematics lessons conducted by the teachers engaged in the developmental activities of the project. Videotaped lesson data is always incredibly rich in many respects and decisions need to be made for the prioritisation and analysis of some features over others. Decisions such as splitting up lessons for the analysis and how to do the splitting, for example, are not trivial. As posed by Adler (2017): "A major task for any lesson analysis is how to chunk or divide the lesson into analytic units." (p. 130). Classroom research in mathematics education with a focus on language and communication has provided a range of alternatives for breaking down lessons. Adler (2017) suggests identifying the object of learning and creating related episodes; O'Halloran (2004) interprets mathematics lessons as sequences of curricular content microgenres; and Clarke (2001) proposes the structure consisting of *Lesson*, *Activity*, *Episodes*, *Negotiative event*, *Turn and Utterances.* What comes from the study of the lesson data can be hard to put together again to produce some understanding of how the teaching and learning developed, and more so if the criteria for chunking the lessons and analysing each separate lesson piece are not consistently shared and applied.

In this paper we present a framework that was elaborated as a tool for the analysis of mathematics lessons with experienced teachers and learners of diverse linguistic repertoires. Support for these teachers was planned through participation in a professional development program on language-responsive mathematics teaching, which involved the collaborative design of mathematical tasks on

linear equations, angle properties and probability and the discussion of instances of teaching moves, amongst other key moments of practice. We draw on data from one of the videoed lessons to illustrate the use and potential of the framework, but also to underscore some of the challenges and reflections that the framework has brought to our work within the project.

Background

The authors are members of a research team working on the above-mentioned project on mathematics teaching in linguistically diverse classrooms in England, aiming to support language, communication and mathematics together (for further details of the project see Ingram and Lee, 2023). We view language and communication in mathematics lessons as consisting of words, images, diagrams, and other ways of communicating about mathematical objects, ideas, and practices. As described in Ingram et al. (2024), members of the team have experience with diverse interpretive approaches for researching language and communication in mathematics classrooms, including ethnomethodology, sociocultural theory and systemic functional linguistics. Our collaboration is mediated by this range of theoretical approaches and the belief that the understanding of lesson data can benefit from the adoption of a complementary perspective.

Design of the framework

Our intention was to develop a framework for analysing rich episodes of mathematics lessons that we have identified in our project data, in ways that reflect this richness rather than summarise it through coding. The framework we present (see Figure 2 for an example in use) was developed by the project team. It aims to provide a structure to support analysis of lesson discourse within lesson episodes for the purposes of following the mathematical story of the lesson. We identify episodes as sections of a lesson involving one type of activity in work on a task, e.g., a discussion or students working on a task. The framework uses as a unit of analysis the sub-parts of such an episode, which we call *stanzas*. A stanza could be as small as an utterance or span several minutes of a lesson. We recognise changes between stanzas as a shift in focus, as agreed by more than one member of the team, and adopted the term *stanza* because it has resonances of a smaller part of a whole and having a particular focus.

We name stanzas (column 1, Figure 2) by what we identify as the theme of the interaction in the stanza and give a qualitative characterisation of the mathematics in focus (column 2). Our intention for these columns is to help keep the mathematics foregrounded: how the interaction relates to this mathematics and how this relates to the story of the whole episode. This includes how the teacher or students return to a focus, for example in a developing argument, which we propose to indicate by overlaying arrows on the completed table. The four remaining columns involve analysis of key dimensions of language-responsive teaching (see below), which are a focus of the wider project. These four dimensions were developed by the project team based on prior work identifying design principles and teacher moves that enhance language for mathematics learning (Erath et al., 2021).

Our analysis of lesson episodes focuses on what we noticed during the stanzas, where what we notice individually is influenced both by the theoretical background we each work with and by the noticing of others in the analysis discussions, and may draw on one or more of these dimensions of language-responsive teaching. By focusing on what we noticed in the stanzas in relation to these dimensions, we aim to develop our understanding of the nature of language in classroom discourse and its role in students' learning.

Dimension 1: Linguistic demands of the mathematics. This dimension draws attention to linguistic demands by noting the language teachers or students say or write in relation to key concepts of the lesson, as this reflects the opportunities students have to hear, read, write or speak this language and how language use is modelled (Prediger, 2019). For example, choosing to name or label objects, relationships or processes can support communication about these and can emphasise their importance. Furthermore, choices of names or labels can foreground certain meanings in a particular context (Mason, 2002).

Dimension 2: Student reasoning, explanations and argumentation. Reasoning, explanations and argumentation in mathematics have distinctive features that differ from everyday language or other subjects (Erath et al., 2021; Schleppegrell, 2007). This dimension captures students' explanations, how they share their reasoning or build arguments, and what aspects of these are focused on, developed or evaluated. We note here that explanations or arguments that draw false conclusions may nonetheless contain mathematical features, such as conjectures, connections to previous mathematical tasks or use of mathematical language.

Dimension 3: Listening and feedback. This dimension includes how feedback on explanations draws attention to the mathematics, offers opportunities for students to listen to each other or be supported in this and teachers actively listening for mathematical features of what students say even when it is not obviously mathematical. Listening can serve different roles in mathematics lessons, for example, to evaluate or interpret students' responses, or to make connections with other ideas. Distinctions between these can be indicated by the response that follows.

Dimension 4: Making connections. Making connections is a key part of teaching and learning mathematics, but there are many different types of connection that could be made, for example between ideas, tasks, methods, or processes. In our study we also consider connections that relate to communicating mathematics, including between representations, between families of words and between informal and more technical ways of describing mathematics.

We note that the dimensions are ordered for the purposes of arranging them in the framework table (Figure 2), but we consider them to be intertwined and may occur throughout a stanza. In particular, like Adler and Ronda (2015), we caution that this ordering does not suggest a teaching sequence. This is one design reason for not locating *Listening and feedback* in the final column, which might suggest that it happens 'last' in classroom interaction (as noted by the teachers in the project).

The framework described above is therefore designed to support noticing and a focused analysis of each dimension separately and also consider how they are at work within each stanza and over the course of the episode, including whether and when some dimensions move out of focus.

Method

In the first stage of using the framework, two members of the project team used it to analyse one lesson episode together, then another episode separately. One of these team members and another team member then analysed a further episode separately to assess the potential for using the framework reliably by different researchers. This raised anticipated issues of grain size for the stanzas and time for use. As a group we then analysed two further lesson episodes from different topics, in person and then online (see below for details of episode selection). Our choice to collaborate at this stage was based on our goal of understanding more about the potential of working with the

framework, rather than reliability of coding of our project data. Collaborating enabled us to surface and debate issues the framework raised for us, through which we drew on and examined how our different backgrounds in analysing classroom language and communication may have affected what we noticed in the data through differently developed sensitivities (Mason, 2002) and what we chose to report. For example, one author noticed pauses and intonation and others noted slight shifts in word choices and grammar. These differences raised questions about what level of detail to report in the framework while keeping the mathematics foregrounded.

Episodes were selected for analysis through a two-stage process that is part of the wider project work. In the first stage, lesson episodes were identified by considering the types of activity around a task, for example students working on the task or a discussion. In the second stage, a reduced version of the Global Teaching InSights coding scheme (OECD, 2020) was used to identify episodes that scored more highly in terms of the quantity and quality of interaction between the teacher and the students. We watched the video of the episodes multiple times during analysis, re-watching short sections to confirm our *accounts of* (Mason, 2002) or as our discussion drew attention to different features.

Results: Illustration of our use of the framework to analyse a lesson episode

We now illustrate our use of the framework using data from our analysis of a 5-minute episode from a lesson on angles in parallel lines for students aged 12-13 in England. For practical reasons our analysis took place in a recorded online meeting (approximately 1 hour 40 minutes) after previous work in person on other episodes. The episode involves a whole-class discussion that followed up on a task in which students were shown the diagram in Figure 1 and asked to discuss and "write down as many statements or equations about these angles as possible". In the discussion the teacher selected relationships that students had identified, wrote them on the board and asked for reasons for them.

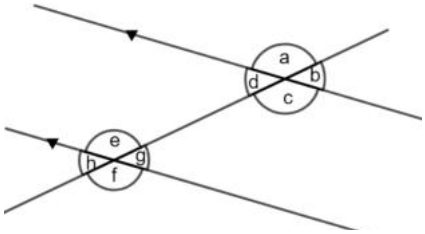

Figure 1: Parallel lines task (reproduced from lesson video)

Figure 2 illustrates our shared analysis of the first three of five stanzas in the episode, presented here in the form produced during our shared analysis session. We now describe some of our deliberations and choices made in this analysis. For reasons of space, we do not describe all details for each stanza.

In our first two meetings, we had first identified a stanza and then analysed each dimension before continuing to identify further stanzas. In contrast, in our analysis of this episode we first focused on identifying stanzas, and then analysed across the dimensions, working stanza by stanza. One challenge in our analysis was the level of detail to include in cells. Figure 2 illustrates our choices, which took into account the goals of our analysis and therefore combine verbatim quotes and summary comments.

In our shared analysis, we quickly reached agreement on the shifts in focus that defined boundaries between stanzas and the core mathematical ideas or activities within the stanzas. For example, we interpreted the two geometric results in stanza 1 as forming one stanza together, rather than separate

Stanza	Core mathematical ideas and activities	Linguistic demands of the mathematics	Student reasoning, explanations and argumentation	Listening and feedback	Making connections
1. Angles prior knowledge (0.00–1.06)	Sum of angles round point and on straight line. Using this as a basis for relationship between angles.	Angles around a point Angles on a straight line (preposition) Degrees, Equals, Equation, point Reason (why, and what you write after the answer), what a reason is in mathematics	Modified from angles in a 'complete circle' Students completing missing words in mathematical statements started by the teacher	Accepting answers as correct but requesting rephrasing and structuring statements into conventional forms Referencing routine of how to give a reason when making an argument	Students communicating using written notation, copied by teacher, but speaking the notation, e.g. equals not = Connecting to prior knowledge by asking for reasons for equations Complete circle given as reason for the equation
2. Straightness of line (1.06–1:28)	Raising assumption that lines are straight in the diagram	Straight, line, straight line Not necessarily, Assuming Parallel lines cut by…			Connecting representations (tracing along line) This diagram being a representation that tells them a line is straight (basis of assumption)
3. Comparing angles (1.28–3:30)	a>d is inequality, a=c is equation. When is it sometimes true, or always true?	Is greater than Equation, Expression, Inequality If… would …always Still the same Opposite angles True Cut line (naming transversal) Same, different Same place' to describe angles on new line In this scenario, case	Is the relationship always true or only sometimes true?	Validates appropriate mathematical responses, repeats appropriate answers	Inequalities are a type of expression, but it's not an equation Speaking 'greater than' connecting with symbol Using dynamic language to explore the static diagram.

Figure 2: Example of the framework in use

stanzas, because they both involve the core mathematical activity of students using prior knowledge in an explanation of a relationship they had identified between angles. We identified the move from stanza 1 to stanza 2 by the teacher drawing attention to an assumption that the transversal line was straight in response a student's use of *straight line* in their explanation in stanza 1. The shift in focus that marks a move from stanza 2 to stanza 3 was indicated by introducing the inequality relationship $a > d$, which holds for the given diagram, and using this as a starting point for the core mathematical activity of contrasting a relationship that is sometimes true with one that is always true, e.g. $a = c$.

In stanza 1 we identified *Linguistic demands* that included statements involving prepositions, such as *angles around a point* and specialised vocabulary such as *degrees*, *equation* and *point*. We included the teacher's clarification that a *reason* is what should be written after stating the relationship. We interpreted this as a linguistic demand because it set out what a reason looks like in mathematics, and because in England there is an expected form for these reasons in geometric arguments. There was an explicit interplay between the dimensions *Student reasoning, explanations and argumentation* and *Listening and feedback* in stanza 1, as *Listening and feedback* was characterised by restructuring students' arguments. For example, the teacher accepted "because it's a complete circle" as a correct reason but then requested "Angles around a WHAT equals 360?" In this way, the teacher's feedback also referenced a routine from general mathematical practice of giving reasons.

Connections that we considered to be made in stanza 1 include students connecting to prior knowledge of geometric results as reasons for their equations. The teacher also connected words with the symbols students used to communicate relationships, for example by speaking "=" as "equals". We discussed whether stanza 1 involved connecting algebraic representations with the diagram. Some students are likely to have made such connections before this episode by forming equations to express relationships between angles in the diagram, but we decided that this connection was not made during stanza 1 itself because neither the teacher nor the students connected the equations with the diagram in speech or through other means such as gestures (Alibali et al., 2014), which, by contrast, the teacher did in stanza 2. This discussion highlights the challenges of using the framework for individual episodes: connections made available previously may later be left tacit and we may only be able to record connections that are articulated in the episode.

Stanza 2 focuses on the assumption that the transversal line is straight (which we consider to develop the statement introducing angles on a straight line in stanza 1 and indicate by a connecting arrow in Figure 2). In stanza 2 we did not identify any student speech, so there are no examples of *Student reasoning* or *Listening and feedback*. *Linguistic demands* include classifying lines, e.g. *straight line*, *parallel lines*, and phrases associated with assumptions, e.g. *not necessarily*, which are important for precise communication within mathematical arguments as well as being everyday expressions. One focus of our discussion was the teacher's use of the diagram to make connections. First, she connected speech with the diagram by asking "How do I know this line is straight?" while tracing along the transversal line. She also connected the assumption that the line is straight with what students could "see from the diagram". We noted that use of the diagram therefore extends beyond a point of reference: its visual properties are used as a basis for a claim for properties of a line. This contrasts with properties made explicit through notation, such as arrows on parallel lines, or stated in the task.

In stanza 3 the mathematical focus is the inequality "$a > d$" and whether it is always true, which the teacher contrasts with the relationship $a = c$ which is always true. In addition to specialised terms,

and giving the name *cut line* to afford communication about the transversal, *Linguistic demands* here particularly include conditional statements such as "If I moved this cut line, would a always be greater than d?" One particular type of connection that we interpret the teacher making is between the related specialised terms *equation*, *inequality* and *expression* as she identifies the *inequality* as a type of *expression* and contrasts *inequality* with *equation*. A further type of connection is between the static diagram and a range of possibilities that it could represent. The teacher makes these connections through dynamic and temporal language, e.g. *moved*, *always*, *still the same*, and drawing an alternative transversal for which $a < d$ could be interpreted visually from the relative size of the angles in the diagram and connecting this with an algebraic representation. As a form of *Listening and feedback*, we considered the teacher's repetition of mathematically appropriate student statements such as "They are the same" to validate these responses. Although opportunities for *Student reasoning, explanation and argumentation* were framed by the teacher questioning whether the result $a > d$ was always or only sometimes true, we did not identify examples of students explaining why this was the case, or being pushed to explain this.

In summary, these descriptions of our analysis of stanzas 1-3 illustrate how our use of the framework supports us to draw out details of how the teacher and students sustain or change the mathematical focus of discussions. For example, we identified a range of topic-specific and mathematical linguistic demands that differed between stanzas, support for more general mathematical practices such as giving reasons and considering whether relationships hold always or only sometimes. We also identified differences in attention to how students make their arguments through the teacher's listening and feedback and the variety of connections articulated in these short sections of discourse.

Conclusion

Data analysis requires the summarisation of data, which reduces the complexity of the situation. The act of videoing or observing a lesson also necessarily results in data reduction. However, classroom interactions are complex, multifaceted and dynamic in ways that intertwine pedagogic decisions, the mathematical content and the interactional context. We lose so much information when we reduce classroom data to a set of codes. This framework was developed to address a concern that we were losing too much in the earlier coding, particularly connections, themes and aspects of the interactional context that build, evolve and adapt during an episode, task or lesson, or even across lessons. Furthermore, many of the coding systems we considered seemed to lose sight of the way in which interactions are intertwined with the mathematical content because they focus on particular pedagogic teacher moves or structures of interaction, rather than how these moves or structures might be influenced by or influence the mathematics in focus. Other systems, such as Adler's MDI framework (Adler & Ronda, 2015), consider these relationships across lessons rather than at the level of interaction. We have illustrated here how the framework has supported our analysis of lesson episodes in our project and now raise the question of whether it could be used beyond our project to capture quality in language-responsive teaching.

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German upper-secondary students' use of discourse practices and lexical means related to the scalar product

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This study examines the role of language in upper-secondary mathematics education in Germany, emphasising the interplay between language, learners' conceptions, and concept formation, particularly regarding the scalar product. The research introduces a further differentiation of the refined model of representations and registers into verbal arrow geometry, verbal vector algebra, and elementary geometry levels. A teaching-learning arrangement was designed to observe which discourse practices and lexical means within these verbal levels students use in learning processes towards understanding the scalar product. The findings suggest a geometry-specific refinement of conceptually rich explanatory forms.

Keywords: Discourse practices, lexical means, scalar product, conceptual understanding.

Context of the study in Germany

In Germany, empirical studies on language in mathematics education predominantly focus on primary and lower secondary education. Language phenomena related to topics in upper secondary education, such as stochastics, analysis, and analytical geometry, have received less research attention so far (exception: Prediger & Sahin-Gür, 2020; Armbrust, 2022). Analytical geometry, one of the major topics in the upper secondary curriculum in Germany, involves algebraizing geometry using vectors. However, research on learners' conceptual development in geometry-specific areas is lacking (Sträßer, 2015, p. 9), although a few recent studies are addressing this gap (Büttner & Erath, 2023 on trigonometry). The interdependence between language, learners' concepts, and concept formation (Vollrath, 1984; Wessel, 2020) highlights the importance of these relationships in mathematical learning. This interdependence suggests a similar need for greater focus on language use and conceptual understanding within the field of analytical geometry. In German schools, analytical geometry is often reduced to procedures, with concepts such as the scalar product being treated as a formula tool. Regarding the scalar product, students often struggle to interpret it geometrically when it is not equal to zero (Frohn, 2020), which suggests a lack of conceptual understanding.

Theoretical background on conceptual understanding, representations and discourse practices

Hiebert and Carpenter (1992) define *understanding* "in terms of the way information is represented and structured. […] mathematics is understood if its mental representation is part of a network of representations" (p. 67). The authors highlight the importance of connecting concepts, facts, and procedures across past and present learning contents (Hiebert & Carpenter, 1992). This idea is also found in the work of Vollrath (1984) with regard on concept formation: a mathematical concept is understood when its *scope* and *content* is clear, and it is embedded in a conceptual *network*. In the cognitive-psychological model of understanding proposed by Drollinger-Vetter (2011), a conceptual

network is further specified. In her model, concepts are deconstructed into their associated representations, which are supported by underlying concept elements that describe partial aspects of the main concept and must be linked and understood. Some concept elements for the scalar product are, e.g., lengths, angles, and vectors, because they represent previously learned content and are integral to the concept itself, which need to be connected for a deeper understanding. On the one hand, learners build their understanding of a concept by consciously compacting several concept elements into the actual concept. On the other hand, conceptual understanding is developed when compacted concepts are unfolded into their underlying elements (Post & Prediger, 2022).

Integrating meaning-related vocabulary, representations, and discourse practices for developing conceptual understanding of the scalar product

The processes of *compacting* and *unfolding* aim to foster learners' conceptual understanding, which in turn requires the use of meaning-related lexical means (Wessel, 2020). While research shows the relevance of these lexical means, the relevant meaning-related vocabulary must still be identified for each mathematical topic. Regarding analytical geometry, research suggests a differentiation of *vector language*, *arrow language*, and *point language* (Artmann & Törner, 1983). The authors show that in proofs and propositions, vector language is predominantly used while in means of illustration and motivation, which support conceptual understanding, arrow and point language dominate (Artmann & Törner, 1983). This study combines arrow and point language under the language of *arrow geometry*, recognizing them as key components of meaning-related vocabulary in analytical geometry.

Given the abstract and primarily relational nature of mathematical objects, understanding a concept in all its facets needs activities of relating concept elements and representations (Drollinger-Vetter, 2011). By connecting *verbal*, *symbolic, graphical* and *concrete* representations, learners develop the mental constructs and relationships that a mathematical concept entails (Duval, 2006). Wessel (2020) developed a *refined model of representations and registers* by further unfolding the verbal representation level into *verbal technical*, *verbal academic*, *verbal everyday* registers. While graphical representations play a significant role in analytical geometry, especially given the focus on language in describing them, this discussion will not delve into the other representational levels (graphical, symbolic-numeric, symbolic-algebraic, and concrete). For the concept *scalar product*, the *verbal academic register* particularly includes lexical means of *verbal arrow geometry*, *verbal elementary geometry,* and *verbal vector algebra*. Arrow geometry involves interpreting vectors as arrows in both graphical and verbal forms. In contrast, vector algebra focuses on formal, abstract descriptions through symbolic and verbal representations. Elementary geometry comprises the foundational elements learned in lower secondary education, present across all representational forms. For example, the relationship between orthogonality and the scalar product can be expressed using formal (verbal vector algebra level) or meaning-related vocabulary (verbal arrow geometry level and verbal elementary geometry level):

- *Example 1* (formal)*:* If two vectors $\vec{a}$ and $\vec{b}$ are orthogonal, their scalar product is zero.
- *Example 2* (meaning-related)*:* The two arrows with a common starting point represent two vectors and form a right angle. In this case, the scalar product is zero.

Even more importantly than the lexical level, activating students in rich discursive practices such as *explaining* is needed for fostering conceptual understanding (Erath et al., 2021). *Explaining* complements *understanding.* Klein (2009) differentiates three types: *explaining-what* (meanings of terms and constitutive properties), *explaining-how* (sequences of actions and constructions), and *explaining-why* (justifications of connections and relationships). They are referred to as *explanatory forms* in the following. In relation to the different explanatory forms in mathematics education, Erath (2017) highlights that oral explanations of meanings and justifications in class 5 mathematics lessons occur less frequently, while procedural explanations, particularly explaining-how (e.g., the steps in a specific calculation) seem to take centre stage.

Research question

Regarding analytical geometry, there is a lack of empirical research on how learners handle these explanatory forms and which lexical means they actually use in these processes. Also, processes of relating representations of elementary geometry, arrow geometry, and vector algebra, are not yet empirically investigated and the potential and hindrances of the interplay of representations, explanatory forms, and specific lexical means in analytical geometry remain mostly unclear. This leads to the central research question: *How do upper-secondary students handle specific explanatory forms related to the scalar product, and which lexical means do they use?*

Methods and research design

The overarching project which aims at designing a language-responsive teaching-learning arrangement towards the *scalar product* follows a didactical design research with its topic-specific features and iterative cycles. The teaching-learning arrangement was designed following a structured specification of the learning content and incorporating the design principles *relating different representations* and *engaging students in rich discourse practices*, which have proven to be fruitful in other studies on language-responsive approaches (Erath et al., 2021).

Design of the teaching-learning arrangement

The designed teaching-learning arrangement consists of five stages and includes various tasks for learners to complete. The scalar product is first introduced through an angle problem in an irregular hexagon, then distinguished from other products. Subsequently, both the arithmetic formula $\vec{a} * \vec{b} = a_1 b_1 + a_2 b_2$ and the cosine formula $\vec{a} * \vec{b} = |\vec{a}| \cdot |\vec{b}| \cdot \cos\alpha$ are presented and demonstrated to be equivalent using concrete vector examples. Next, the effect of positional relationships of arrows with a common starting point for two vectors on the scalar product is explored (see Figure 1 for segment of the task).

Exploring characteristics of the scalar product

a) $\vec{a} = \binom{4}{1}, \vec{b} = \binom{3}{5}, \alpha = 45°$ **b)** $\vec{a} = \binom{5}{2}, \vec{b} = \binom{-2}{5}, \alpha = 90°$ **c)** $\vec{a} = \binom{4}{4}, \vec{b} = \binom{3}{3}, \alpha = 0°$

Use GeoGebra to geometrically represent the vectors and angles from each task.

- What do you notice about the scalar product? Make notes.
- Explain why your observations are valid by examining the magnitude of the cosine on the unit circle.

Figure 1: Segment of stage 3 in the learning pathway

Initially, learners relate the symbolic-numeric representation to its graphical counterpart and then articulate the observed relationships in a written explanation. The fourth step focuses on the equality of scalar products, especially when the scalar product equals the product of vector magnitudes. Figure 2 illustrates the derivation task through the decomposition of vector $\vec{b}$ into two parts.

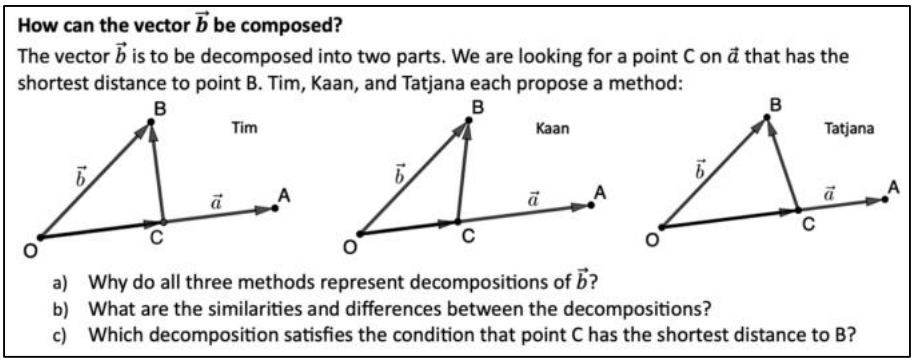

Figure 2: Segment of stage 4 in the learning pathway

Like previous tasks, learners are required to connect their verbal descriptions with the graphical representations. Next, vector $\vec{b}$ is further defined as having an orthogonal component $\vec{b}_{\perp}$ and parallel component $\vec{b}_{\parallel}$ forming the basis for deriving the relationship $\vec{a} * \vec{b} = |\vec{a}| \cdot |\vec{b}_{\parallel}|$. This prepares learners to interpret the scalar product as the area of a rectangle in the final stage of the learning pathway.

Data collection and data analysis

The teaching-learning arrangement was tested in a classroom setting ($N = 25$) over six 45-minute lessons. All plenary discussions, as well as two focus groups (each with three participants) were recorded during the lessons, and all learners' written work was collected. The video material was transcribed and analysed using Mayring's (2015) combined deductive and inductive approach. The analysis involved three steps:

(1) *Theoretical inventory of language demands:* Language requirements were inventoried according to elementary geometry, arrow geometry and vector algebra, with further specification of *explaining* and the construction of examples. For instance, an *explaining-why* example in arrow geometry could be: The scalar product is zero because the two arrows of the vectors are orthogonal to each other.

(2) *Coding (further) discursive practices and lexical means:* Terms like *segment*, *perpendicular*, and *parallel* were assigned to lexical means of elementary geometry; *arrows* and *decomposition* to arrow geometry; and *vector* and *linearly dependent* to vector algebra. Additionally, deictic means and individual terms used by students like *these things* and *overlapping* were also coded. To avoid redundancy familiar concepts like *angle* from lower secondary education were assigned to elementary geometry.

(3) *Analysis of interplay:* The relationship between task, discourse practice, and lexical means was examined to identify peculiarities, similarities, and differences.

Empirical insights into students' explanations regarding the scalar product

In the following, cases are presented in which an inductive refinement of specific explanatory forms emerged during data analysis. These include *explaining-how to visualize a geometric construction* as an important conceptual specification of *explaining*, *deictic means* and *individual terms as support for explaining-why* in the context of arrow geometry, and *describing graphical representations as scaffold for explaining-why.*

Explaining-how as a potentially conceptual specification

In relation to explaining within the context of (arrow) geometry, students tend to explain how to visualize a (given) situation geometrically. An example of this can be seen in the transcript excerpt from Emma and Tina. During the plenary session, vector addition which forms the basis for decomposing a vector into two parts (see task from Figure 2) was to be explained to everyone, while Emma was speaking quietly to her neighbour, Sarah. The teacher labelled the parallel and orthogonal parts from Tim's decomposition as $\vec{c}$ and $\vec{d}$ in the sketch and asked who could explain what vector addition is

Emma:	[*Talking to Sarah:*] I think she [*the teacher*] means that two new vectors are created by this path [*points to the adjacent arrows, which are parallel and orthogonal to the arrow of* $\vec{a}$]. Instead of using vector b, you kind of use two, you know? […]
Teacher:	[*Have Tina go next*]
Tina:	You add vector c and d […]. And that then creates b.

Emma explains to Sarah how vector addition can be visualised using two adjacent arrows "instead of using vector b". Emma argues on a conceptual level within arrow geometry, imagining vector addition as a *path*, which represents her individual use of lexical means. She stays close to the graphical representation of the concept referring to the arrows. Explaining-how to visualize a situation geometrically was found more often in the data. Putting a (given) graphical geometrical relationship into words seems conceptually fruitful. Tina, on the other hand, explains how to perform vector addition, using lexical means from vector algebra. Such rather procedural explanations were also found in the data, but it appears that *explaining-how* extends beyond mere procedures and is a conceptually rich discourse practice, particularly in geometrical contexts.

Individual and *deictic means* as support in explaining-why

An analysis of students' explaining-why statements reveals a frequent reliance on graphical representations for justifications. In the following transcript, Nina and Jana first exchange ideas on how to explain why all options suggested by the three fictional characters qualify as vector decompositions (see task a) from Figure 2).

Nina:	[…] Yes, because everyone draws a different stroke. So/ … You know? Here, this one draws a stroke there [*points to the part of the decomposed vector that is orthogonal to* $\vec{a}$ *in Tim's decomposition*]. This here.
Jana:	Yes, because you always get to the vector through two strokes. Sorry. I mean to the point.

The explanations by Nina and Jana refer to the graphical representation of the three situations presented to them in the task. In the negotiation process between Nina and Jana, it is evident that they use both deictic means like "Here, this one draws a stroke there. This there" and individual lexical

means such as "one draws a stroke there". It appears that the proximity to the graphical representation and the use of deictic means support their explanation process. Since Nina and Jana struggle with explaining-why, they initially rely on individual meaning-related lexical means within everyday language in their oral explanations to each other to bridge linguistic difficulties. Similar simplifications are also found in the data, particularly in relation to *explaining-why*, where students often rely on graphical explanations drawing on arrow geometry.

Describing **as scaffold for** ***explaining-why***

Looking at the transcript of Nina and Jana above, it becomes evident that describing and explaining are closely connected, as Nina, for example describes similarities in the individual decomposition by stating "Yes, because everyone draws a different stroke [*explaining-why*] […] Here, this one draws a stroke there. This there. [*describing*]". This phenomenon is not unique to the case of Nina and Jana. The data analysis suggests a close connection between *describing* and *explaining-why*. A further example can be found in the excerpt from Emma. She relates to the task in which the learners were asked to explore the properties of the scalar product (see Figure 1), particularly the relationship between types of angles and the magnitude of the product.

> Emma: Yeah, exactly, that they are, like, similarly aligned and that it's, well, an acute angle [*writes on her worksheet*]. And I also wrote down that they are not orthogonal or perpendicular to each other, so, yeah, that's why the scalar product is still greater than zero.

Emma refers to the first part of the task, where observations were to be noted. She does not focus on the section explicitly requiring a why-explanation using the cosine on the unit circle. Instead, Emma describes the positions of the arrows of $\vec{a}$ and $\vec{b}$ to her classmate by initially talking about the directions of the arrows and the included angle. In the next part of her statement, she draws on the previously learned relationship between orthogonality and the scalar product as a resource. While doing so, Emma adopts both terms *perpendicular to each other* and *orthogonal*, which likely represent synonyms for her. Only after her description does Emma, on her own initiative, add an explaining-why statement by providing the reasoning that the scalar product in this case is greater than zero. In the descriptive part of her statement, Emma uses lexical means from arrow geometry in combination with graphical representations and only when explaining does she switch to previously learned lexical means from vector algebra. It appears from the data that for the other learners who completed the tasks, describing first and then providing an explanation seems easier than producing explanations right away. A look into German linguistic didactics confirms this assumption, as Feilke and Rezat (2021) state that explaining aims to make concepts understandable, and descriptions are helpful as part of the explanation process.

Conclusion

The transcript analysis highlights the need for theorizing and applying the model of registers and representations by Wessel (2020) within analytical geometry. The study offers a preliminary categorization of verbal elementary geometry, verbal arrow geometry, and verbal vector algebra as relevant categories for distinguishing between formal and meaning-related vocabulary.

First, the findings indicate that students frequently combine explaining-why statements with descriptive lexical means, suggesting a natural progression from description to explanation, where initial descriptions help ground the more abstract justifications that follow. Second, students engage in explaining-how regarding geometric concepts, such as imagining or visualizing geometrical relationships related to the scalar product. This explanatory form potentially aids both in developing procedural fluency in the context of vector algebra and in supporting conceptual understanding in the context of arrow geometry, as students articulate and refine their thoughts on the application of geometric principles. In this context, *explaining-how* is understood as explaining how mathematical objects can be visualised. Finally, the data demonstrate that students use a wide range of lexical means, drawing from elementary geometry, arrow geometry, and vector algebra. Their diverse use of individual lexical means highlights the multifaceted nature of their understanding, suggesting that different representational levels and registers are dynamically integrated into their learning processes. These insights underscore the need for careful consideration in designing teaching-learning arrangements and selecting discourse practices, especially for representation-rich concepts as the scalar product.

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Language-responsive support for connecting understanding division and fraction division: A design research case study

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The division of fractions is often regarded as one of the most procedural and least comprehensible topics in school. The study presented in this paper gives insights into a design research project aiming to provide learning opportunities for developing a conceptual understanding of fraction division. The study introduces a language-responsive teaching-learning arrangement designed to enhance students' conceptual understanding, where the division of natural numbers is used as a starting point. Short snapshots from a case study illustrate how students interact with the tasks and what potential and opportunities can be observed.

Keywords: Fraction division, relating registers, discursive, design research, conceptual understanding.

Introduction

Academic language can be challenging for many students due to language's cognitive and epistemic function (Schleppegrell, 2004). These challenges are highly relevant for learning mathematics. Consequently, research in mathematics education works towards developing and investigating learning arrangements that effectively combine the learning of mathematics with language learning, as highlighted in Prediger and Wessel (2013) and Pöhler and Prediger (2015). The presented paper focuses on fraction division, known to be one of the most challenging concepts in middle school mathematics. Since students often struggle with interpreting the division of fractions (Malle & Huber, 2004), a teaching and learning arrangement was created to develop a deeper conceptual understanding of fraction division. To enhance the ability of describing fraction division and explaining formal tasks and patterns, language-responsive instructional approaches may be effective (summarized in Erath et al., 2021). Since there is a gap in the interpretation of the arithmetic operations, an understanding should be built by reflecting on and expressing ideas as well as using different representations. In the first cycle of a design research project, we discovered that students used both meaning-related and formal-related language to solve division tasks (Mähnert & Erath, 2024). However, students not only used these two language varieties, which are important to enhance mathematical understanding, but they also solved division problems in different ways (e.g. inverse operation, part-whole concept, equivalence of fractions, and repeated addition). Consequently, as we move into the second cycle of this project, we are now exploring how learners develop multiple and linked ideas of the concept of fraction division. Therefore, we focus on linking different representations and encouraging students to articulate their mathematical ideas.

Theoretical background: Design principles and the division concept

Research has shown (Prediger et al., 2022) that academic language proficiency has a significant impact on mathematics achievement, and that language challenges can affect monolingual students too. To overcome these challenges and support conceptual understanding, it is effective to teach

mathematics language responsively, defined as "instruction that enhances language for mathematics learning" (Erath et al., 2021, p. 246). This approach integrates content and language to provide targeted learning opportunities for the academic language needed to articulate mathematical concepts. Previous design research studies already explored instructional approaches integrating content and language to support students (e.g., on fractions Prediger & Wessel, 2013, and percentages Pöhler & Prediger, 2015). However, similar approaches have yet to be explored specifically for dividing fractions. This highlights the need for designing learning resources that enhance students' understanding of fraction division. Our research aims to fill this gap by investigating language-responsive instruction for dividing fractions. For such instructional approaches that enhance language for mathematical learning, six central design principles were extracted from the literature (summarized in Erath et al., 2021) P1: engage students in rich discourse practices, P2: establish various mathematical language routines, P3: connect language varieties and multimodal representations, P4: include students' multilingual resources, P5: use macro-scaffolding to sequence and combine language and mathematics learning, P6: compare language pieces (form, functions, etc.) to raise students' language awareness. Integrating all six design principles for language-responsive instruction may not be practical. Therefore, prioritization should depend on the specific mathematical content and the language skills of the students. Before discussing the advantageous design principles for our mathematical content, it is important to first discuss the concept of division.

In Germany, primary school students learn about whole number division, understanding it as partitive division (e.g., What is the fifth part of 25?) and measurement division (e.g., How many times does 5 fit into 25?). During the first cycle, students used various ways to solve division tasks in the measurement context, which seems to be a good starting point for discussion (Mähnert & Erath, 2024). As a result, the measurement context is incorporated into the second cycle, focusing on linking the division of natural numbers. Before detailing its implementation in teaching, the concept of understanding is addressed.

Understanding is not a static property or state, but a dynamic process of development. Carpenter and Lehrer suggest studying the "mental activity that contributes to the development of understanding" rather than "static attribute[s] of an individual's knowledge" (1999, p. 20). They propose five forms of mental activity from which mathematical understanding emerges: (a) constructing relationships, (b) extending and applying mathematical knowledge, (c) reflecting about experiences, (d) articulating what one knows, and (e) making mathematical knowledge one's own. According to them, understanding involves various mental activities and not all students learn or demonstrate understanding in the same way. Ideas must be connected to other ideas, but these connections can vary.

Therefore, in the teaching-learning arrangement, the concept of division of natural numbers is addressed and then extended to the new number range of fractions in order to connect ideas. Similarly, reflecting on and expressing ideas is crucial for understanding, and each person's journey to understanding is unique. This underscores the importance of creating a teaching-learning arrangement that fosters opportunities for students to articulate their thoughts and ideas. To realize this, design principle P1 was implemented, because conceptual understanding can be achieved when students are engaged in rich discourse practices such as arguing and explaining meaning because this creates

opportunities for language and mathematics learning (Erath et al., 2021). Emphasizing formal definitions and expressions early in the learning process may lead students to focus on superficial descriptions and procedures. So, it is important to engage in more elaborate discourse practices like arguing or explaining that would be needed for productive work in later steps of the knowledge construction process (Erath et al., 2021). Additionally, research has shown (Duval, 2006) that the cognitive development of mathematical concepts is closely linked to the ability to connect concepts in different representations. As a result, design principle P3 was implemented, meaning that activities in mathematics classes involving the connection between different representations were integrated to enhance conceptual understanding. It is recommended to utilize various graphical representations and connect different forms of representations, such as contextual, verbal, and graphical, to enhance learning. The instructional approach of relating representations (symbolic, numerical, verbal, and graphic) and registers (everyday, academic and subject-specific technical language) and its effects have been theorized, empirically investigated, and practically implemented in mathematics (Prediger & Wessel, 2013). It was shown that the activities of relating registers and representations provide the discursive opportunities to explain and argue, hence to construct meanings. Our study focuses on connecting representations and language varieties for division fractions and linking them to division with natural numbers. By combining design principles P1 and P3, we aim to create learning opportunities at both conceptual and discursive levels. The study addresses the following research questions: *How can the designed teaching-learning arrangement with the design principles "engage students in rich discourse practices" and "connect language varieties and multimodal representations" support students' learning processes? What language challenges and opportunities can be identified?*

Case study on explaining the concept of division and fraction division

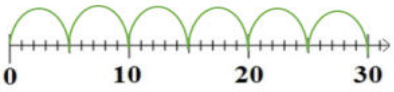
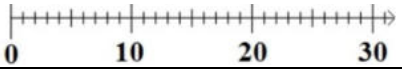
Method: Design Research

Design research (Gravemeijer & Cobb, 2006) was chosen as a methodology to focus on learning as a process in which students are actively engaged in making sense of mathematical ideas. It involves an iterative interaction between designing teaching-learning arrangements, conducting design experiments, and analysing the teaching and/ or learning processes. There are two main goals: creating effective learning opportunities and gaining practical insights into the learning processes related to the content. This paper presents insights from the second cycle of design experiments.

Design of the teaching-learning arrangement

The second cycle emphasizes understanding division in the measurement context. In our case, we are considering the learners' prior knowledge regarding the division of natural numbers. Initially, students are prompted to share their associations with division (see Table 1, Task 1). We then reinforce the basic concept of division with natural numbers using the number line (see Table 1, Task 2), before moving on to division involving fractions (see Table 1, Task 4). Table 1 demonstrates how the design principles P1 and P3 are implemented through these tasks.

Table 1: Example tasks realizing the refined design principles P1 or P3

Design principles	Examples of tasks on division and fraction division
P1: engage students in rich discourse practices (here, explaining division and its connection to the procedure) **P3: connect language varieties and multimodal representations** (here, from symbolic to verbal)	**Task 1 – Associations with division and fraction division** a) What do you associate with the task 25:5? Explain. b) What do you associate with the task $1:\frac{1}{2}$? Explain.
P1: engage students in rich discourse practices (here, explaining the connection between multiplication and division with meaning-related phrases) **P3: connect language varieties and multimodal representations** (here, from graphic to symbolic)	**Task 2 – Jumps on the number line** a) Write down a suitable division task and multiplication task. Then, explain the connection between the division task and the multiplication task. Explain using the picture. 
P1: engage students in rich discourse practices (here, students explain their drawings in Task 4c)) **P3: connect language varieties and multimodal representations** (here, from verbal and symbolic to graphic)	**Task 4 – Represent division on the number line** a) How many times does 4 fit into 24? Draw the jumping arc in the picture that corresponds to the task 24:4. c) How many times does $\frac{1}{2}$ fit into a whole? Could you draw the jumping arc in the picture that corresponds to the task $1:\frac{1}{2}$? Explain how you would draw the jumping arcs. 

Methods of data collection

Our research involved 10 sixth-grade students from two different schools in a city in central Germany. These students had previous experience in working with arithmetic operations involving fractions. They worked in pairs for 180 minutes using the provided learning materials. The first author or a colleague attended all sessions and intervened in the learning process to a minimum. The sessions were videotaped and student's written materials were collected. The overall project aims to design a content- and language integrated teaching design on dividing fractions that combines conceptual understanding with subject- and language-integrated support for learners. The brief illustrative excerpts presented here are intended to provide insights into the individual learning pathway of the learners along the hypothetical learning trajectory. They especially show the role of graphical representations and the discourse practice of explaining. For this, documents and short transcripts of the videotaped episodes of the learning process were contrasted. The representation of the analysis is more illustrative, focusing on the specific task sequences rather than serving as a methodologically rigid tracing of individual learners' individual learning processes across the entire learning pathways. So, the analysis in this paper could take only a local perspective for answering the research question.

The case of Lia and Alina: Challenges and learning opportunities when explaining division and fraction division

This section presents selected snapshots as the first results of an ongoing data analysis. We focus on the case of Lia and Alina, two 12-year-old girls in grade 6 who worked together. Selection criteria was that they showed a high potential for eliciting discourse practices in a first rough analysis of the video material. In the transcripts. ./../... marks a 1/2/3sec. pause.

Snapshots of students' associations with division (Table 1, Task 1)

At the beginning, students were asked what they associated with task $25:5$ to explore the prior experiences they brought from primary school and connections to measurement or partitive division. Alina stated that the result is 5, and Lia agreed.

6	Alina:	So actually, the first thing I see is five, because 25 divided by five is five. Right? Yes.
7	Lia:	Yes.
8	Interviewer:	Can you explain a bit how you know that's five?
9	Alina:	Well, because the 5 fits in the 25 five times. And then you know that if you divide 25 by 5, you have to look how many times the five fits in and that's five times, so I know that 25 divided by 5 is 5.

Alina discusses the result of the task (line 6), but it is unclear if she or Lia fully understand what "25 divided by 5" means. The interviewer's invitation to explain their thinking (line 8) leads to insights. Alina shows her conceptual understanding of measurement by explaining that dividing 25 by 5 means figuring out how many times 5 fit into 25 (line 9). Her language structures indicate her understanding, using phrases like "have to look how many times" and "fit in five times". She also uses "divided by" meaningfully because she refers to its contextual meaning. Next, the students are asked about their associations with the task $1:\frac{1}{2}$, which involves a fraction.

15	Lia:	So, it's also a division task, um, because that's like 1/2 and to make a whole, you need two halves. Yes, so it fits in there twice.
16	Alina:	So, when I saw that, the first thing I had in my head was 0.5, because that's kind of in my head, I don't know, and then I just looked at how a normal division task would be, so 1 divided by 0.5 equals 2.

The students revisit the measurement concept, highlighted by Lia's utterance in line 15, where she explains that $\frac{1}{2}$ fit into a whole two times. This explanation could also reflect the idea of division as the inverse of multiplication, as Lia implies that two halves are needed to obtain a whole, implying that multiplying $\frac{1}{2}$ by 2 results in a whole. Consequently, the context of measurement presents an opportunity to discuss the inverse operation, fostering a deeper understanding and connection of concepts. Furthermore, Alina initially describes having 0.5 in mind and then proceeds with the standard division process, thus establishing a connection between fractions, decimals, and division. Therefore, the combination of representations and the discourse practice of explaining gives insights into the students' understanding of division and reasoning process.

Snapshots of solving division problems with natural numbers (Table 1, Task 2)

The following snapshot refers to a task completed after the initial questions (see Table 1, Task 2) illustrating the relationship between multiplication and division through a graphical representation.

This activity focuses on connecting the graphical and symbolic representation, encouraging students to use language that is related to the meaning of multiplication and division. Initially, students were unsure whether the multiplication task was $6 \cdot 5$ or $5 \cdot 6$, but they finally agreed on the correct answer and changed their initial response. The teacher asked about their reasoning for this change.

92	Interviewer:	Now you have changed the results. Could you explain why you changed it?
93	Alina:	Because we thought, well, if it were five now, then you would mean that there are always five steps and we just think that this rather describes .. so we don't know, but that there are just six .. um that you just need six times the five steps to reach 30 … but actually I don't think it matters, you could also use five times six and then you could … then you would mean something different in principle, but basically neither is wrong.

The linear representation of the number line illustrates both the product and the factors. In line 93, Alina first mentions the five steps, signifying that there are consistently five steps in the graphical representation. She then states "you just need six times the five steps to reach 30", thereby defining the mathematical operation of multiplication. This clarifies that five (jump size) serves as a multiplicand and the six steps (number of jumps) serve as the multiplier. The snapshot highlights the potential benefits of linking language varieties and multimodal representations with the discourse practice of explaining, which helps students to visualize and conceptualize the abstract operation of multiplication in a concrete way. Alina demonstrated her understanding, explaining that the order of numbers does not affect the result. By saying "five times six and then you could (…) then you would mean something different in principle, but basically neither is wrong" (line 93) it can be assumed that only one multiplication task corresponds to the given picture. Her use of the number line and her reflection on the phrases "six times the five steps" and "five times six" indicates that she succeeded in unfolding the concept. Therefore, the combination of graphical and symbolic representations, along with the discourse practice of explaining and the precision of students' language, is crucial for understanding multiplication concepts. However, this can be a challenge for other students. The question arises: how do students handle the situation when they are not dealing with natural numbers, but also incorporating fractions into such tasks? The following snapshot addresses this.

Snapshots of solving division problems with fractions (Table 1, Task 4)

In Task 4, students are required to represent division tasks by drawing jumping arcs on a number line. While part 4a) is straightforward, part 4c) becomes more complex due to the scale of the number line. The consistent graphical representation of the number line remains to encourage students to engage in explanations and discussions. In Task 4c), it is crucial for the students to recognize that they must adjust either the division or the range of the number line. Drawing the jumping arcs in the previous tasks functioned effectively without further discussion. The following snapshot shows a conversation about the definition of a whole in Task 4c).

151	Alina:	I think it's the same task only in reverse and we know twice that ½ fits into a whole. But what is meant by a whole? 30 or 10?
152	Lia:	10
153	Teacher:	What would you guess?
154	Alina and Lia:	10.
155	Alina:	But 30 is also possible. Then there are thirty wholes.
156	Lia:	But 1/2 it is 0.5
157	Alina:	But then it's 1. … No, it could also be 10. It could also be 10.

158	Lia:	Yes, exactly. It could also be 20.
159	Alina:	With twenty it would be 0.10. Nope.
160	Lia:	Yes, of course.
161	Alina:	Eh? With twenty, it would simply be 10.
162	Lia:	So it's ten.
163	Alina:	Yes. 0.5
164	Teacher	What was the reasoning behind the last one?
165	Alina	Because 0.5 times two is 10. But if it were up to 20, for example, then it would be 10 times 2 and then 10 would be half and not 0.5. Although no. 0.5 can also be twenty. That's always half. No, 0.5 is half of a whole.

The students are uncertain whether a whole can be 10, 20, or 30, (lines 154, 155, 158) despite recognizing that $\frac{1}{2}$ fit into a whole twice (line 151). They perceive 0.5 as a flexible value and struggle to apply it as a fixed value representing half of one whole. The confusion may arise from the flexible usage of the terms "a whole" and "a half" in everyday German language. After the students have chosen to answer 10, the teacher asks if they can justify this (line 164). Alina starts by saying that 0.5 multiplied by 2 is 10. Then she tries to argue with a counter-example using 20 to explain that 10 multiplied by 2 equals 20 and therefore 10 is half of 20. While 0.5 as a number is always half of 1, half of another number is simply that number divided by 2. Moreover, they faced difficulties with arithmetic operations. Alina argued that they chose 10 because 0.5 multiplied by 2 equals 10, and did not consider 20 as a whole because "then 10 would be half and not 0.5". The explanations that were given at the beginning of the learning environment (see Table 1, Task 1) for the same task and which took up the conceptual understanding of the measurement in the area of natural numbers could not be incorporated at this point. In the end, the students do not answer the question and the teacher intervenes and explains the task to them. To summarize, the students face challenges when transitioning from verbal to graphical representations, as they have not yet developed a clear understanding of the concepts of wholes and fractions. Although the number line was used the same way in all tasks, it is evident here that the graphical representation alone is insufficient to overcome these misunderstandings. More activities are needed to deepen the work with the number line and help students to grasp the concept of fractions and wholes more clearly.

Conclusion and Outlook

To address the research questions, the learning pathway of Alina and Lia provides insights into the role of language and the development of mathematical understanding of fraction division. While both students demonstrate a strong grasp of division as measurement, they struggle to apply this concept to fractions. In Task 2, Alina reflects on varied phrases "six times the five steps" and "five times the six" to interpret the multiplication task within a graphical representation. This underscores the role of language in developing conceptual understanding, as Alina uses and reflects language that pertains to the meaning of multiplication. While Lia and Alina demonstrate a strong understanding of division as measurement with natural numbers, it becomes more complex with fractions, particularly in defining the whole. These insights highlight the increasing complexity of the conceptual idea of measurement and the need to emphasize phrases related to division more effectively. We think that building upon language varieties acquired in primary school for division can offer appropriate support in deepening students' conceptual understanding. In answering the research question, the design principles "engage students in rich discourse practices" (P1) and "connect language varieties and

multimodal representations" (P3) proved their potential for developing and deepening an understanding of division and fraction division (see Task 2). In summary, the individual learning pathway of the two students shows that linking division and fraction division opens up the potential for building conceptual understanding. However, more research is needed, also to identify supportive linguistic means and phrases. Also, in this project, more data will be analysed to contribute to a better understanding of language learning and conceptual understanding for the case of fraction division.

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The connection between task design and how algebraic variables and constants are described

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In designing mathematical tasks that utilise language(s) as a resource, there is a need to identify which aspects of language are most likely to contribute to developing important mathematical concepts, such as algebraic variables and constants. The aim of this study is to identify how questions in a task about patterns contributed to preservice teachers, as fluent users of the mathematics register, describing variables and constants in specific ways when discussing their solutions. The analysis suggests how the task question could be revised to prompt the use of algebraic rather than arithmetic thinking.

Keywords: Algebra, language as a resource, task design.

Introduction

In an ongoing international project, we have been exploring how to develop mathematical tasks that utilise the epistemic potential of multiple languages to provide "mutually supporting perspectives on a mathematical concept" (Schüler-Meyer et al., 2023, p. 102). However, pursuing this aim has identified many tensions, including that "the use of technical language and how to structure mathematical argumentation seemed to affect how the students interpreted and responded to the requirements of the task" (Ferrari et al., 2023, p. 1589). Thus, rather than continuing to develop new tasks to trial in multilingual classrooms, it became essential to better understand how task questions could contribute to specific aspects of mathematical language being used. In this study we focus on how questions in a task about patterns elicited different ways of describing variables and constants from preservice teachers, who were assumed to be fluent in the mathematics register in Norwegian.

The focus on algebra was chosen because difficulties had been identified in shifting thinking from arithmetic operations with specific numbers to abstract generalisations (Ferretti et al., 2022). This shift is not immediately recognisable in the products of the thinking. For example, Radford (2014) stated that deducing a formula for patterns could require algebraic thinking whereas guessing a formula and trying it out until it worked for a range of specific quantities would indicate arithmetic thinking. It is likely that the language used to describe the deducing would be different to that used for guessing. Therefore, patterns seemed an appropriate mathematical topic for considering how task questions might affect whether algebraic thinking was required and how it was described.

In the table and chairs task, used from the early stages of the project (Schüler-Meyer et al., 2023; Ferrari et al., 2023), the focus was on multilingual students developing rules for patterns in everyday language. To do this requires an understanding of both variables and constants and can be considered a form of functional thinking, an aspect of algebraic thinking, which "involves generalizing relationships between covarying quantities and representing and reasoning with those relationships through natural language, algebraic (symbolic) notation, tables, and graphs" (Blanton et al., 2015, p.

43). Like Radford's (2014) focus on deduction, reasoning about the covarying quantities can affect and be affected by the ways that language is used.

In this study, we aim to identify how variables and constants are described in responses to specific task questions. The tasks were completed by preservice teachers for grades 5–10 in pairs. We presumed that they could complete the tasks appropriately and would have a rich repertoire of mathematical representations and linguistic expressions for describing variables and constants. The research question is: How do task questions contribute to preservice teachers using specific ways of describing variables and constants when orally determining rules for patterns?

Theoretical framing

To respond appropriately to the research question, it was necessary to identify the different ways that variables and constants could be described. Researchers have investigated students' conceptions of letter symbols in mathematics for decades. Küchemann (1978) distinguished between six levels of understanding, where the most complex was "letter as variable", with the requirement that the student had "an awareness that there is some kind of relationship between the letters, as their value changes in a systematic manner" (p. 25). Yet, this covarying property of variables in relationship with each other may not always be in students' minds when working with patterns. Bardini et al. (2005) found that students described a toothpick pattern with the formula $n = 2x + 1$, where x symbolized the number of the figure and n the corresponding number of toothpicks. However, they seemed to treat x as a "temporarily indeterminate number" and n as a "potentially determinate number" that would eventually be assigned a value.

Ely and Adams (2012) also identified issues, describing letter symbols as "unknowns", "variables", "placeholders" and "generalised numbers" and provided examples of how they could appear in what they referred to as typical algebra lessons. The term "unknown" was used to describe fixed amounts that were yet to be determined. Ely and Adams (2012) gave the following problem in which, "the unknown is 'some amount', and the equation is tacit: 'Jack has some amount of apples and then gives 4 of them away. If he now has 3, then how many apples did he start with?'" (p. 21). Variables describe two indeterminate amounts, both of which could assume "any of a (large) set of values" (p. 21) which are always in relationship to each other, and so are covarying. Whereas Blanton et al. (2015) focused on the covarying relationship in describing functional thinking, Ely and Adams (2012) were more focused on the indeterminate amounts, in a similar way to Küchemann (1978). Nevertheless, reasoning about variables includes considering the arithmetic operations which contribute to the covariation. A placeholder is "a letter standing for a number that will be provided in a particular problem or context" (Ely & Adams, 2012, p. 21). Thus, coefficients and constants in equations can be considered placeholders, in that they have specific values in particular examples of that equation. For students trying to make sense of generic equations, such as "y=mx+b", then "m" and "b" can appear to be variables because they can stand for a range of different numbers. However, because they have a specific value in a particular example of this equation and do not vary in relationship to each other, they are different to variables. Determining a pattern requires reasoning about the specific amounts that could fill the coefficient and/or the constant and so can also be considered part of functional thinking. Placeholders could also be what students, in Bardini et al.'s (2005) study, had

considered to be variables when they took on the role of being potentially determinate numbers. The final distinction that Ely and Adams (2012) made was about generalised numbers in which the intention was not to find a specific result but to keep the discussion at the level of general amounts. They gave an example of "a(b + c) = ab + ac" (p. 22). In identifying the specific aspects of patterns, it may be that simplifying the pattern to its most elegant form could involve generalised numbers.

Our focus is not on how letter symbols are used but on how variables and constants in patterns are discussed in everyday language. The distinctions of Ely and Adams (2012) were a starting point for identifying algebraic thinking. Radford's (2014) description of guessing the formula for a pattern and testing it with different numbers can be related to Ely and Adams' (2012) unknowns, which suggests unknowns are more likely connected to arithmetic rather than algebraic thinking. In contrast, being able to reason about a pattern by determining the relationship between the covarying indeterminate amounts would align with Radford's (2014) algebraic thinking. Therefore, identifying how variables and constants were described in responses to specific task questions could provide insights into how to improve algebraic thinking.

Data collection

To determine how the task questions contributed to variables and constants being described in different ways, we analysed the discussions of second-year preservice teachers for grades 5–10. We considered that this group would have an appropriate linguistic repertoire for interpreting and forming patterns, rather than still gaining aspects of it which could be the case with school students. The preservice teachers were participating in a problem-solving course and completing mathematical problems was a normal part of the lesson.

The first and third authors used one workshop in September 2023 to record pairs of preservice teachers working through the task, along with the plenary sessions. In this article, transcripts of the work of five pairs, who gave permission to be recorded, were analysed. The responses to tasks that included comparing other languages are not analysed in this paper.

An earlier version of the task is in Meaney (2024). In the results and discussion section, we provide details of the questions, especially when they deviated from the version in Meaney (2024). In Figure 1, we provide the illustrations for the three patterns that were the focus of the tasks. After working with these, the PSTs were asked to design another pattern with a different table shape.

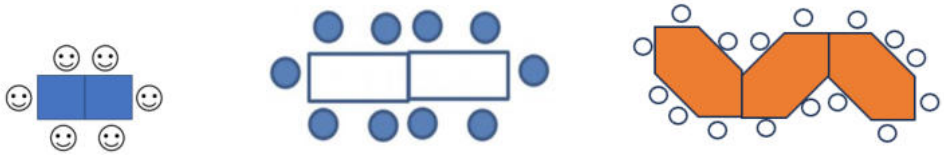

Figure 1: Patterns of tables and chairs used in the task

Data analysis

We began by reading the transcripts carefully, noting where and how the preservice teachers interpreted or described patterns, including variables and/or constants. Table 1 sets out our revised set of categories, adapted from Ely and Adams (2012), to accommodate the mostly natural language,

used by the preservice teachers. To these categories, we added a fifth one to describe how the preservice teachers discussed variables and constants when referring to the structure of patterns, suggesting they were engaged in a meta-level discussion.

Table 1: Categories of aspects of patterns

Category	Description	Language characteristics	Example from the data
Unknown	Amounts that were to be determined.	Use of specific amounts to test the validity of a formula for a pattern.	Eight. What was it here? Twelve? (counts) … Eight, twelve, sixteen.
Variable	Indeterminate covarying amounts	Focus on the relationship between two variables (one or more could be implicit).	The number of tables times two plus two equals the number of people who can sit at the table.
Placeholder	Coefficients or constants.	Focus is on the amounts that should be multiplied or added to the variables.	It will probably be, actually, since it is six, it will then be 6a. But instead of having plus 2, we get ... minus two? But then there will be plus two again, so it will only be 6a.
Generalised number	Focus on simplifying formula.	Identification of unnecessary aspects in the formula.	X one plus two. There's no point in taking times one, then. It is very unnecessary. Tables plus two.
Structure	Meta-level discussion about the structure of a pattern.	Focus on the features, rather than the specifics of the patterns.	Yes, it's the same rule, you just have to change how much it increases by and how much are constants.

Once the different ways of describing variables and constants were classified, we then identified the task questions that seemed to provoke these categories. Although it is not possible to determine a causal relationship between question and ways of describing variables and constants in the responses, the analysis provides insights for further exploration and task adaption.

Results and Discussion

In this section, we present examples from each category and then discuss the task questions which seemed to elicit them.

Unknown

The first task question gave six rules for determining the number of chairs when square tables were placed side by side (see Figure 1) and asked to determine if any represented the pattern. Two of these rules were "Each table you add, gives space for two more people" and "If you multiply the number of tables by two and add two, you get the number of people that can sit down for dinner". This

question seemed to prompt the five pairs to test each rule with specific numbers. Sometimes, they began the calculation with "if" to indicate that there were other choices, rather than just one number.

Pair1PS1: Ja, hvis du har tre bord, så ganger du det med to, da får du seks. Det er jo de seks som sitter på sida her, og så plusser du på de to som sitter på endene til slutt.
Yes, if you have three tables, you multiply that by two, then you get six. It's the six that sits on the side here, and then you add the two that sit on the ends at the end.

The preservice teachers' main focus was not on finding a specific unknown amount, as was the case with Ely and Adams' (2012) original definition of unknown. Instead, the outcomes of the calculations were used to determine the veracity of the rules. The similarities for testing different values for the unknown suggests that arithmetic rather than algebraic thinking was being utilised, similar to Ferretti et al.'s (2022) distinction. Therefore, asking the preservice teachers to evaluate a set of rules, did not on its own seem to result in algebraic reasoning.

As well, in the discussions, very few preservice teachers indicated whether they were talking about tables or chairs. Sometimes, this resulted in the preservice teachers misinterpreting the rule, which is a common issue noted in earlier research (Ely & Adams, 2012). This suggests that the question not only did not contribute to algebraic thinking but also restricted possibilities for determining the relationship between the two variables.

Placeholder

Talk about placeholders appeared when preservice teacher determined the specific coefficients for multiplying the number of tables and the constants, the chairs at the end of the rows of tables, for each pattern. A rule given in the first question about the square-table pattern was, "You take the number of chairs at each table and times it with the number of tables". In discussing this rule, the preservice teachers seemed to focus on the coefficient, as they felt this was not clearly expressed.

Pair4PS1: To av bordene vil jo ha tre. Resten har to.
Two of the tables will have three. The rest has two.

In this excerpt, three refers to the number of chairs at the tables at each end. The rule's imprecision led to this pair abandoning the discussion. In contrast, the same rule seemed to contribute to Pair 5 focussing of determining the constant. Pair5PS1 suggested an alternative rule, "du tar to stoler ved hvert bord, ganger det med antall bord og så plusser på to" (*you take two chairs at each table and times it with the number of tables, and then you add two*"). This alternative rule clarified what the coefficient and constant were for the square-table pattern.

In the task about the rectangular-table pattern, a table was provided to support the preservice teachers to determine the increase in the number of chairs between consecutive figures. This triggered talk about the placeholders, representing the coefficients of the number of tables and the specific constants at the end of the rows of tables. The coefficient as a fixed quantity was also described by preservice teacher 1 in Pair 5 as a specific amount, "antall bord ganger fire pluss to (…) fordi det er jo ikke plutselig at etter fire bord, så er det tre stoler per side" *(number of tables times four plus two (…) because it is not that suddenly after four tables, there are three chairs per side)*.

For the hexagonal-table pattern, the question was "If the table shapes were different, how would the rule change? Use the pattern below to help you explain how the rule could change". The complexity

of the pattern seemed to support the preservice teachers to focus simultaneously on the coefficients and the constants in the rules, as these were not immediately obvious. The example below was typical in that they began by identifying the difference between consecutive figures, to determine the increase, but also identifying the additional chairs at the ends whose sum determined the constant:

Pair2PS2: Ja, her legger du bare til fire. Her legger du også bare til fire, da.
Yes, here you just add four. Here you also only add four, then.
Pair2PS1: [peker på ark] Og så tenker du at disse her bare er på utsida, og disse to er på utsida. Så da er det fire her og. Så det er da fire her også, og fire her også.
[points to sheets] And then you think that these here are only on the outside, and these two are on the outside. So, there are four here and. So, there are four here as well, and four here as well.
Pair2PS2: Ja. Eh ... åtte blir det ... X ganger ...
Yes. Uh ... eight that will be ... X times ...
[Pause i 10 sekunder] [Pause for 10 seconds]
Pair2PS1: X ganger fire pluss fire.
X times four plus four.

Likewise, the question about drawing a pattern with a different shape of table and determining a rule for it, had the potential to support reflection on both the coefficients and the constants, as illustrated by Pair 1 who described a pattern with triangular tables:

Pair1PS1: Det øker med én hver gang.
It increases by one every time
Pair1PS2: Øker med én ja? Men da er det ... bord ganger én pluss to. Det blir det egentlig.
Increases by one, yes? But then it's ... tables times one plus two. It really will be.
Pair1PS1: Ja. Det gjør det egentlig. Bord ganger én pluss to. (Begge ler) Det er bare å plusse to på antall bord. Ja, den er fin.
Yes. It actually does. Table times one plus two. (Both laugh) Just add two to the number of tables. Yes, it's nice.

In these cases, the focus was on the general properties of the pattern, rather than testing its validity, as had been the case when amounts took on the role of unknowns. This seems to align with the points made by Ferretti et al. (2022) who highlighted the importance of students being able to make sense of working algebraically by connecting back to arithmetical operations.

Variable

The questions about the rules for rectangular- and hexagonal-table patterns (see Figure 1) seemed to support the preservice teachers to focus on the relationship between the number of chairs and the number of tables. For Pair 2 in the excerpt in the previous section about the placeholder, the figures on the task sheet supported them to clarify whether they were talking about the tables or the chairs in the pattern, which then helped them to clarify what "X" referred to. This supported the relationship between them, as suggested by Blanton et al. (2015), being considered important, rather than the individual indeterminant amounts.

Pair 5 gave a general description of the linear pattern at the end of the discussion, in which they had used letter symbols to express the constant and the variables.

Pair5PT1: Det blir jo da A som er antall bord ganger x som er ukjent antall stoler som det øker med til vanlig pluss y som er et konstant tall på det ytterste (…) er lik B som er antall personer som sitter rundt bordene. Så A*x+y=B."

It will be A which is the number of tables times x which is the unknown number of chairs it usually increases with, plus y which is a constant number at the end (…) is equal B which is the number of persons who sit around the table.")

Here, the role of each letter was clearly specified, but the choice of the letters (x,y for placeholders) and A, B for variables was unconventional.

Generalised number

There were very few examples where formulas were simplified and these occurred when two pairs independently determined a rule for a pattern about using triangular tables which they had generated. In Table 1, there is an example where Pair 1 simplified the rule. However, these possibilities did not arise because they were prompted by a question, but because the pairs had chosen particular patterns. In having students interpret formulas, it could be possible to include a question that supported them to reflect on the precision of the language, but such task questions would need to be trialled.

Structure

The new category which we added to those of Ely and Adams (2014) was that of structure, which occurred when the preservice teachers talked generally about the pattern formula. Interpreting the given rules for the square table pattern seemed to lead Pairs 3, 4 and 5 to discuss the general structure of the rules, after discovering that they were all valid. Preservice teacher 2 in Pair 3 stated, "Jeg tror det er det samme, bare skrevet på en annen måte, som en av disse her [refererer til de andre reglene]. (*I think it's the same, just written in a different way, like one of these here*)". Although the preservice teachers noticed the similarities, many did not provide details about what was similar.

The other situation in which the structure was discussed was towards the end of the task, after the preservice teachers had produced a rule for the hexagonal-table pattern or made a pattern using a shape of their choice. The example in Table 1 comes from Pair 5. However, such explicit discussions did not seem to be prompted by specific task questions. It may be that a question that focuses more on the similarities and differences between the rules could contribute to students having a structural discussion and leading to understanding about the generalisability of algebraic thinking, which is considered valuable for students' learning (Ferretti et al., 2022).

Conclusion

This study provides some insights into how task questions might contribute to different ways of describing variables and constants in rules for patterns, particularly when using everyday language. Knowing how fluent users of the mathematics register, who are competent in algebra, are prompted to use particular ways to describe variables and constants may support the development of algebra tasks for school students that utilise language as a resource (Schüler-Meyer et al., 2023) and overcome one tension identified in using a previous version of the task with school students (Ferrari et al., 2023).

Questions that required verification of the validity of rules may result in variables and constants being treated as unknowns, as specific amounts are needed to check the veracity of the rules. However, it seems that if the questions prompted a comparison of similarities and differences between rules of the same and different patterns, then variables and constants could be discussed in other ways, which could facilitate algebraic thinking. With this information, it is possible to redesign the task so that the

epistemic potential of language can be maximised (Schüler-Meyer et al., 2023). It also provides opportunities for looking at how other languages might express variables and constants.

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Noticing aspects of the mathematical discourse in a developmental context on teaching angles

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In this report, we examine the noticing of a group of secondary-school mathematics teachers of aspects of the mathematical discourse during the collaborative discussion of a task on teaching angles. Our framework integrates aspects of the mathematical discourse in content-specific teaching that are responsive to students' common challenges in learning angles, as a focus of mathematics teachers' noticing. The data were collected during a one-day workshop. Using hybrid methods of deductive and inductive analysis, we examine what teachers identify, interpret, and decide regarding mathematical vocabulary, mathematical explanations, and graphical examples for teaching angles. A result is that the teachers in the group engaged in noticing mathematical-linguistic explanations with multimodal examples for teaching angles with meaning in dynamic contexts and in relation to non-metric properties.

Keywords: Collaborative discussion, mathematical discourse in teaching, teachers' noticing, teaching angles.

Introduction

In this report, we present a study initiated as part of a research and developmental project with secondary-school mathematics teachers on the mathematical discourse in content-specific teaching. One of the project strategies was the participation of the teachers and of the researchers as facilitators in one-day workshops to discuss professional tasks on aspects of the mathematical discourse in teaching. In Planas and Alfonso (2023), for example, the noticing of two groups of participant teachers was analysed, with a focus on mathematical-linguistic practices of naming and explaining, alongside other foci spontaneously introduced by the teachers. The design and implementation of the workshop in the current study include tasks with written prompts aimed at identifying, interpreting, and deciding on mathematical vocabulary, mathematical explanations, and graphical examples in fictional situations of teaching angles. A notion of supporting students' common challenges in content-specific learning by using aspects of the mathematical discourse in teaching is at the basis of the task design (for details, see Rave-Agudelo & Planas, 2022; 2023). Our study is therefore situated at the intersection of research on mathematics teachers' noticing (van Es & Sherin, 2002; Jacobs et al., 2010) and research on mathematical language and discourse in teaching (Morgan, 2013; Planas, 2021). The study of mathematics teachers' noticing is a relevant line of research, as it is a fundamental aspect of the teaching profession (König et al., 2022). It is also important to ensure that mathematical language and discourse have a place in teacher professional development programmes on mathematics teaching (Planas, 2021). We aimed at examining responses to the following research question: *What does a group of secondary-school mathematics teachers notice regarding aspects of the mathematical discourse when discussing tasks on teaching angles?*

Theoretical framework

Our framework integrates three components: aspects of the mathematical discourse in content-specific teaching, students' common challenges in content-specific mathematics learning, and mathematics teachers' noticing.

The presence of mathematical vocabulary, mathematical explanations, and graphical examples in teaching has the potential to communicate and facilitate the communication of mathematical meanings (Planas, 2021). In our study, these aspects are inspired by the *Mathematics Discourse in Instruction* (MDI) framework (Adler, 2021), particularly regarding the mathematical-linguistic practices of naming and explaining, and the mathematical-graphical practice of exemplifying. We understand *mathematical vocabulary* in relation to mathematical naming, as the set of words or phrases from the mathematical discourse used to name objects, symbols, properties, actions, procedures, and practices in mathematics (Adler, 2021). This vocabulary serves to answer questions such as: *What is this?* (Planas & Alfonso, 2023), *What is it called?* or *What are we talking about?* We understand *mathematical explanations* in relation to mathematical explaining, as sentences incorporating mathematical vocabulary that build mathematical meaning by establishing relationships between objects, symbols, properties, actions, procedures, and practices in mathematics to facilitate students' thinking. These explanations address questions such as: *What is said about something? How does it happen? How is it done?* or *Why is this?* (Planas & Alfonso, 2023). This understanding of mathematical explanation aligns with Leinhardt's (2001) notion of *instructional explanation*, in which the disciplinary nature of the mathematical content is linked to teaching practices that support students' thinking. We understand *graphical examples* in relation to mathematical exemplifying, as graphical representations (pictures, illustrations, photographs, diagrams, etc.) of mathematical examples. A mathematical example is a particular case within a broader equivalence class (idea, concept, technique, ... in mathematics) that supports reasoning and generalisation (Watson & Mason, 2002). A graphical representation is thus a visual mode of communication that conveys complex objects, structures, and processes holistically by integrating visual elements and interpretation rules (Winn, 1987).

A starting point for our investigation is the lack of consensus in defining the concept of angle and its implications for teaching and learning. For our study, common challenges in learning angles were identified in the literature on mathematics education (Rave-Agudelo & Planas, 2022). The studies selected highlighted two common challenges in learning angles that can occur simultaneously: the tendency to communicate meaning about angles in static contexts, without considering dynamic contexts (Mitchelmore & White, 2000), and the tendency to confuse the numerical value of angle amplitude with the angle itself, while overlooking non-metric properties (Tanguay & Venant, 2016).

Regarding mathematics teachers' noticing, van Es and Sherin (2002) conceptualise it by considering the identification of specific aspects of classroom interactions, the connection between these aspects and broad principles of teaching and learning, and the use of contextual knowledge to reason about these interactions. In this initial phase of the study, we adopt the conceptualisation proposed by Jacobs et al. (2010), which involves processes of identifying, interpreting, and deciding on relevant aspects of teaching and learning based on pre-established criteria. While other studies within the broader

project have shown that teachers notice additional aspects beyond those considered in the design of professional development workshops (Planas & Alfonso, 2023), the characterization and description of these processes constitute a crucial step in understanding the complexity of mathematics teachers' noticing. In order to study mathematics teachers' noticing focused on aspects of mathematical discourse in the teaching angles, we adapted Jacobs et al.'s (2010) processes as follows: (i) *Identifying* mathematical vocabulary, mathematical explanations and/or graphical examples of angles in teaching accounts; (ii) *Interpreting* the potential impact of the identified aspects of the mathematical discourse on supporting students with common challenges in their learning of angles; (iii) *Deciding* on proposals of mathematical explanations with mathematical vocabulary and/or graphical examples of angles with the potential to support the students' learning.

Methods

The empirical context of this study was a one-day workshop on aspects of the mathematical discourse in teaching angles. The first author was the teacher educator. Five secondary-school mathematics teachers participated in this workshop. Two participants had a degree in mathematics, and the other three had a degree in engineering. Their professional experience in teaching mathematics ranged from six months to 25 years. They all reported that they had not previously participated in development programmes on aspects of the mathematical discourse in teaching.

In Rave-Agudelo and Planas (2023), we report on the workshop design and the structure, following three stages. The first stage consisted of the presentation and group discussion of two introductory documents. The first document presented and illustrated two students' common challenges in learning angles. The second document illustrated mathematical vocabulary, mathematical explanations and graphical examples for teaching angles. The group discussion of these documents helped introduce the teachers to relevant aspects from the literature on teaching and learning angles. This group discussion also facilitated presentations of the teachers' professional experiences. The second stage involved the teachers' individual responses to two tasks related to teaching angles. Each task provided a fictional account of teaching with three prompting questions. These prompts were invitations to identify, interpret, and decide on mathematical vocabulary, mathematical explanations, and graphical examples to support students' learning. The third stage consisted of a group discussion of the tasks. The teacher educator started the discussion by asking for responses to each of the task prompts. The teachers participated by sharing their answers and focusing on aspects of the mathematical discourse in the tasks that they saw as related to the common challenges in learning. The teacher educator tried not to interrupt the discussion.

In this report, we consider the group discussion of the first task. This task presented a fictional situation in which a student asked the teacher for a mathematical explanation after reading a definition of angle and its classification in a mathematics textbook. In the textbook, the mathematical vocabulary and the mathematical explanations focused on communicating the meaning of angles in static contexts. The graphical examples focused on communicating the metric property. The student's utterance was as follows: '*If an angle is the portion of the plane between two straight lines, why do we take a different portion in the reflex angle? The outer one*'. The everyday meaning of the word

"between" differs from its mathematical meaning, particularly in the phrase "the plane between two straight lines". Figure 1 shows an English version of an extract from the task.

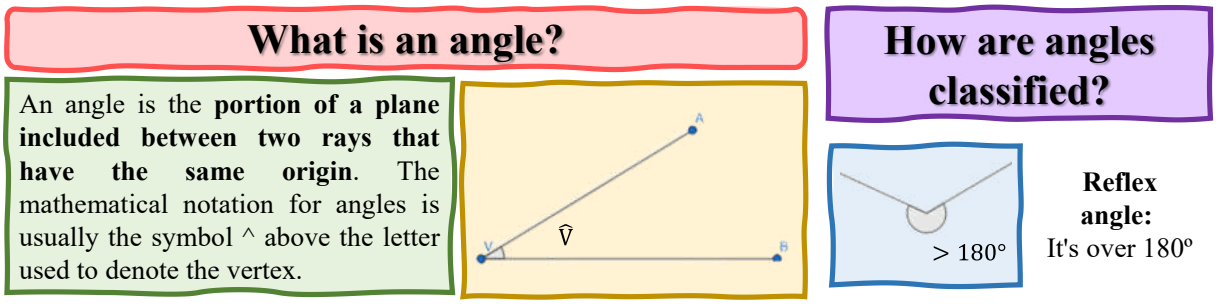

Figure 1: English version of an extract of the task

The prompts provided in the tasks were: (1) What mathematical vocabulary, mathematical explanations, and graphical examples are communicated to the student during teaching? (2) What effects might these aspects of the mathematical discourse have on the students' learning of the concept of angle? (3) What mathematical explanations with mathematical vocabulary and graphical examples, could serve in teaching that is responsive to this student's learning?

The workshop was audio-recorded and transcribed by the first author. The transcript of the group discussion for the first task was divided into three segments, each associated with one prompt. The analysis was then focused on examining evidence of identifying, interpreting, and deciding processes as constituent elements of mathematics teacher' noticing. In this process, we used hybrid methods of deductive and inductive coding analysis (Nowell et al., 2017). A preliminary analysis of the teachers' written responses formed the basis for creating initial codes. The turns in the group discussion were coded deductively, when possible, and this procedure led to the induction of additional codes. The coding was thematically oriented and included processes of noticing (identifying, interpreting, deciding), aspects of the mathematical discourse (mathematical vocabulary, mathematical explanations, graphical examples) and emergent related features. There were variations in the coding process across the task prompts for questions 2 and 3. For the second prompt, for example, the potential effect of the relationship to the students' common challenges in learning and to other aspects of the teachers' professional experience were included. For the third prompt, the potential response to the common challenges in learning and to the other aspects of teaching and learning were added to the created codes. Together with the teachers' turns in the group discussion, the episodes served as a primary unit of analysis. Episodes were created by selecting sets of turns where the teachers' noticing and the aspect of the mathematical discourse in focus were consistent, despite nuances and differences in the features discussed. Although the coding was overall guided by the framework, we remained interpretative and inductive in the analysis of the episodes. For the sake of rigour, we discussed the coding of the turns and the interpretation of the episodes. We present the episodes in an English version.

Analysis and findings

In this section, we present evidence of the participant teachers' noticing of aspects of the mathematical discourse in the teaching of angles.

Identifying aspects of the mathematical discourse in teaching angles

The teachers discussed responses to the first prompt concerning the identification of aspects of the mathematical discourse, but they did not address the identification of mathematical explanations. The extract below illustrates the identification of mathematical vocabulary and graphical examples of angles. In the first part of the discussion, the teacher educator repeated the first prompt, and the teachers identified mathematical vocabulary related to angles. In [2], Senú mentioned the Catalan words for "vertex" and "amplitude". In [3], Tule referred to "plane" and "ray", which Senú repeated in [4]. In [5], Chibcha mentioned "lines" and "origin". In [6], Senú noted that the teacher in the task spent time discussing the classification of angles. In the second part of the discussion, the teacher educator asked about the presence of mathematical explanations and graphical examples, and the teachers focused on graphical examples. In [8], Tule mentioned that there were no graphical examples, but in [9], Nutabe referred to the shapes of angles in his written and graphical response to the task. In [10], Katío referred to the measurements in the graphical examples related to the classification of the angles. In [11], Tule distinguished between graphical representations and graphical examples, associating graphical representations with abstract contexts and graphical examples with real-world contexts. The episode thus illustrates the identification of mathematical vocabulary and multimodal examples (both graphical and real-world context) related to angles.

1	Educator:	What mathematical vocabulary, mathematical explanations, and graphical examples are communicated to the student during teaching?
2	Senú:	She provides many definitions. She talks about the vertex, the angle, and the amplitude…
3	Tule:	About the plane, the ray.
4	Senú:	About the ray…
5	Chibcha:	Straight lines, origin.
6	Senú:	And then she spends a lot of time sorting the angles according to their measure, right?
7	Educator:	What do you think about mathematical explanations and graphical examples?
8	Tule:	There are not any graphical examples, are there?
9	Nutabe:	I have drawn the shape of angles.
10	Katío:	Including the measurements.
11	Tule:	Yes, I mean, these are graphical representations, but when I hear the word example, I think back to when I was a child. I suppose… I need to relate it to something familiar, something tangible that I can physically interact with, right? I mean, for me, those are more than examples; they are graphical representations. But when I think of examples, I picture things like the hands of the clock or the handle of a door...

Interpreting the potential impact of the identified aspects of the mathematical discourse

The teachers interpreted the effects of the identified aspects of the mathematical discourse in relation to common learning challenges, as well as their professional knowledge and teaching experience. In [18], the teacher educator asked for a response to the second prompt. In [19], Senú associated the identified aspects with the communication of meanings for angles in static contexts, without mentioning turns and movements. In [20] and [22], Tule pointed out that "turning" is a missing term that does not appear in the task. In [21], Chibcha, the most experienced teacher, mentioned that the mathematical explanations in the textbook are contradictory. She noticed that these explanations fail

to support the distinction between static contexts, where the angle is described as a region of the plane, and dynamic contexts, where the vertex is emphasised as the centre of rotation. This episode illustrates the interpretation of the potential impact of the mathematical discourse on learning, in relation to the introductory documents discussed in the workshop, to professional knowledge, and to teaching experience.

18	Educator:	Could these things be connected to the common challenges we have been working on? Could you explain how they are connected?
19	Senú:	Yes, sure. She teaches..., in other words, she teaches it in a totally static way, doesn't she? The angle is not presented as a turn; instead, an angle is this, and it is classified in this way, but they don't see that it can be associated with movement.
20	Tule:	The turn is absolutely absent; it does not appear anywhere.
21	Chibcha:	But, you see, it does... That is, in the first explanation, it is the portion of the plane, and in the second, the vertex is part of the angle. Also, she contradicts herself, because the vertex is the centre of the turn, not the ray, but the centre of the turn is important for the angle.
22	Tule:	The turn is absent at any moment.

Deciding on mathematical explanations with multimodal examples

The teachers decided on mathematical explanations with related examples using different modes of communication. The structure of the prompt and the dynamics of the workshop facilitated decision-making regarding these modal aspects of the mathematical discourse in an integrated manner. In [28], the teacher educator asked for an answer, that is, a decision that resolved the doubt raised by the student in the task. In [29], Senú read the explanation in his written answer and stated that moving a ray to one side is not the same as moving it to the other side, which implies a turn. In [30], the educator asked for a graphical example, and in [31], Senú made a hand gesture. In [32], the educator asked if he would present the example with hand gestures, to which Senú responded affirmatively in [33]. In [34], Nutabe presented an example previously discussed that had meaning in static contexts. In [35], Tule presented an example linked to the movement of a clock hand to explain the angle as a rotation and the direction of the rotation. In this episode, mathematical explanations and examples responded to common learning challenges by focusing on communicating dynamic contexts and non-metric properties. Furthermore, the way in which the teachers present an explanation with examples reflects some group decision-making. Although the teachers use words from the mathematical discourse when presenting mathematical explanations, this did not seem to be the focus of their discussion. They also made suggestions that were not related to learning challenges but to other aspects, such as emotions in [34]. Another relevant aspect was the use of examples in modes of communication other than graphical representations, such as hand gestures and manipulatives, suggesting that teachers considered visual representations as complementary to linguistic explanations.

28	Educator:	Can you provide a concrete answer for this student?
29	Senú:	What I wrote is that… I mean, when you go from one ray to the other ray, doing this is not the same as doing that; it is different, hence associated with a turn.
30	Educator:	And how would you present the graphical example?
31	Senú:	Like this, isn't it?
32	Educator:	With hands? With gestures?
33	Senú:	With hand gestures, yes.
34	Nutabe:	I liked the example of the pizza; I think they appeals to emotions, and...

35	Tule:	In addition to the pizza, I used the example of the clock hands. I put one in a different colour, and we played as if we had a time machine. Let's move time forward or backward. Then they see that if you do it on one side, you can do it on the other side too. I am not sure.

Discussion and conclusions

The group discussion of the task demonstrates processes of noticing in collaboration amongst the teachers, focused on aspects of the mathematical discourse in the teaching angles. We suggest that structuring the task around prompts aimed at identifying, interpreting, and deciding facilitated the analysis of data and mediated the production of findings in our approach to understanding *What does a group of secondary-school mathematics teachers notice regarding aspects of the mathematical discourse when discussing tasks on teaching angles?* The teachers identified several aspects of the mathematical discourse in the fictional teaching account. The absence of mathematical explanations in the process of identification can be linked to the complexity of what is meant by mathematical explanation. However, it can also be attributed to the design of prompts that did not sufficiently encourage the teachers to identify explanations as an aspect to be discussed in the workshop. Instead, they may have been involved in other processes of noticing related to mathematics teaching and to mathematical discourse in teaching that we did not capture in our study. Although the processes of interpretation are not related to each of the aspects of the mathematical discourse prompted in the task, the teachers interpreted the potential effects of these discourse aspects on the common learning challenges presented to them, incorporating their professional knowledge and experiences. The group discussion of the introductory documents led to the recognition of common challenges in learning angles. The processes of decision-making included mathematical explanations and multimodal examples that are responsive to common challenges in learning angles. Finally, as stated in our framework, by considering mathematics teachers' noticing through identifying, interpreting, and deciding, we are only beginning to understand the complexity and learning richness of the noticing that can occur during a developmental workshop.

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Teacher responsiveness and didactical decision-making in a multilingual mathematics classroom

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Developing multilanguage-responsive teaching requires teachers to meet the situational demands of harnessing students' multiple languages for learning mathematics. To better support teachers to develop this expertise, it is crucial to understand what prompt them to deploy certain teaching practices in response to students' multilingual contributions. This paper shows how a case teacher's multilanguage-responsive practices link to his perception of situational demands in a multilingual classroom. The study uses the teacher's own video-recorded lesson and conducts a video-stimulated reflection interview that allows the teacher to justify his in-lesson practices. The results present two critical students' contributions and the teacher's responsive decision-making following these contributions – that illustrate how he realizes the demands of multilanguage-responsive teaching.

Keywords: Teacher responsiveness, didactical decision making, multilingual classrooms.

Introduction

Harnessing the epistemic benefits of multiple languages for facilitating conceptual understanding of mathematics (Uribe & Prediger, 2021) requires teachers to master specific situational demands. We call this multilanguage-responsive teaching. It encompasses the teaching practices through which teachers create opportunities for "students' multiple languages to become a resource for conceptual understanding" (Schüler-Meyer et al., in press, p. 6). Multilanguage-responsive teaching has many benefits, ranging from encouraging students to use all their language repertoires for increased participation and agency (Barwell et al., 2016; Norén, 2015), to benefits for enhancing students' mathematical thinking and learning (Uribe & Prediger, 2021). Yet, despite these benefits, we hardly understand how teachers develop multilanguage-responsive teaching practices. To address this issue, we engaged expert teachers in activities to trial specifically designed tasks which have the potential to harness students' multiple languages for conceptual insight, through the principle of contrasting and comparing languages (Schüler-Meyer et al., 2023; Revina & Schüler-Meyer, 2023).

In this paper, we investigate what prompt teachers to deploy certain teaching practices in a multilingual classroom. We focus on analysing how one case teacher Sven picks up students' multilingual contributions and how he responds to the students' utterances. Furthermore, through a post-lesson reflection, we investigate Sven's decision making for specific practices. In this study, we are guided by the following question: "How does a teacher's multilanguage-responsive teaching practices link to his perception of situational demands in a multilingual classroom?"

Responding to situational demands in multilanguage-responsive teaching

Multilanguage-responsive teaching extends approaches of language-responsive teaching aiming at enhancing students' language for enhancing mathematical understanding (Prediger, 2019) by also harnessing multiple languages as resources for mathematics learning. Accordingly, we have adapted demands of language-responsive teaching from prior research on its situational demands (Prediger,

2019) for multilanguage-responsive teaching of mathematics (Schüler-Meyer et al., in press). For this paper, we discuss three relevant situational demands as follows:

- ELICITING LANGUAGES in multilanguage-responsive mathematics teaching is aimed at turning students' languages into an epistemic resource in the mathematics classroom (Barwell, 2020). Eliciting languages should not only engage students in translanguaging but should also create the environments for harnessing the epistemic potential of language reflections.
- SUPPORTING LANGUAGE AND LANGUAGE REFLECTION refers to supporting students in producing utterances in their home languages and supporting the reflection about similarities and differences in how different languages express mathematical concepts.
- IDENTIFYING MATHEMATICALLY RELEVANT LANGUAGE ASPECTS is concerned with recognizing meaning potentials from conceptual connotations expressed in students' languages. Here, teachers must identify what is mathematically important with respect to students (home) language and what is peripheral, for this, topic-specific meaning-related phrases are more powerful than formal technical phrases (Prediger & Uribe, 2021).

For developing teachers' multilanguage-responsive practices for mastering these situational demands, we consider Mason's (2016) perspective on teacher responsiveness: How teachers respond to students' concerns and states significantly impacts both individual student learning and the overall class dynamic. In the context of multilanguage-responsive teaching, this means that teachers' practices in eliciting and incorporating multiple languages influence students' contributions in translanguaging activities. These contributions, then, enable teachers to understand how students interpret concepts in their language(s) and how this interpretation can enhance the class's understanding.

Teachers' responsive practices are shaped through a process of professional judgement: perceiving students' thoughts and ideas, interpreting those thoughts, and making informed instructional decisions to advance students' understanding (Jacobs et al., 2010). In a multilingual classroom, this process involves perceiving and interpreting students' contributions in multiple languages with respect to the demand of enhancing students' mathematical understanding. In other words, teachers need to realize the demands of eliciting language so students' contributions in multiple languages can be exploited. Noticing and interpreting valuable contributions that can create opportunities to harness the potential of students' utterances for meaning making, yet it involves the demands of identifying mathematically relevant language aspects. Ultimately, for making informed instructional decisions to advance students' learning, teachers face the demands of supporting the reflection about how different languages express a mathematical concept and, particularly, how the different connotations can enhance students' understanding of the concept. As teachers often struggle to enact the decision-making phase during the lesson, Mason (2011) suggests providing teachers with opportunities to reflect about the decision making retrospectively, which can help teachers developing a heightened sensitivity, enabling them to adapt their responses rather than relying on habitual reactions.

Methods

The case teacher in the present study, pseudonymised as Sven, is among the teachers who involved in the development of PD activities for multilanguage-responsive teaching. Sven teaches a Year 4 secondary classroom in an international school in the Netherlands, with English as the language of

instruction. He is a native Swedish and fluent English speaker. He has more than 10 years of experience in teaching students with multilingual background. Nearly all of Sven's students were born outside the Netherlands or have had their formal education abroad either in their country of origin or elsewhere, other than the Netherlands. The languages present in the classroom are including Chinese, Hindi, Urdu, Arabic, Turkish, Italian, German, and Dutch.

The lesson we observed in Sven's classroom is on the topic of 'completing a square' method to solve quadratic equations. The completing a square task requires students to identify how 'completing a square' methods is expressed in their home language and reflect upon the meaning – whether the meaning of the expression can be helpful to understand the steps in completing the square method. The task provides students with some examples of how the phrase 'completing a square method' is expressed in other languages, such as in Polish 'do pełnego kwadratu' (to make full square), and later highlights the geometric and algebraic interpretations of completing a square method. The task follows the principle of contrasting and comparing languages (Schüler-Meyer et al., 2023). The observed lesson is Sven's second teaching experiment in the project, and the duration is 50 minutes.

We videorecorded Sven's 'completing a square' lesson and conducted a stimulated-recall reflection interview following the lesson. Video-stimulated recall or reflection interviewing is deemed as appropriate since it allows the subject (Sven as the teacher) to view a video sequence of his teaching practices and is then invited to reflect on his decision-making processes during the recorded event. While the lesson duration is 50 minutes, for the purpose of the video-stimulated reflection interview, we ask Sven to reflect upon the 10 minutes of plenary session at the end of the lesson where comparing and contrasting languages takes place. During the interview, Sven is instructed to stop the video when he wants to remark on a specific situational demand he experienced. The interview evolves around Sven's responsive practices by which he follows up on students' contributions. The lesson and the interview are transcribed and analysed qualitatively.

Results

We present two student contributions to which Sven responded and unfold the potentials for realizing situational demands of harnessing these contributions for promoting deeper understanding of completing a square method in solving quadratic equation. This is followed by an analysis of Sven's practices in terms of how they meet these situational demands and his reflection on them.

Students' contribution 1: Two meanings of 'square' in some languages – as a geometrical shape and as the power of two

During the watching time of video-stimulated interview, Sven pauses the lesson video following the opening of the lesson when he starts the plenary session as below:

> Sven: If I translate completing the square into Swedish, it is *kvadrat komplettering. Kvadrat* [*Swedish words*] means square, both in a geometric sense and the exponent of two. […] If I would translate it [*the Swedish words*] back into English, it would mean completing the square. It has the same meaning. Swedish and English is actually very close. The build-up of sentences and many words have a direct translation. So, Swedish would be a bit a boring example of this. It means basically the same thing as it does in English. We use square for the geometric figure, square, but also for the exponent. Two squared means two to the power of two. But we have found at least one, so far, different meaning for completing the square. [*Sven points*

to a student] Would you like to share that with the whole class? What it means in your language?

In the above episode, Sven starts with a reflection from his own language, Swedish, and shares that the expression for completing a square in Swedish provides him with a reference to both geometric and algebraic meaning. He highlights that he found a different meaning for completing the square, most probably from the time he walks around the class. The first author follows up Sven's remarked practice by asking him to describe what he observed in the paused moment. He describes that he receives an unexpected contribution from a Chinese and a Hindi student before the plenary session begins. Below is an episode from the interview.

Researcher: So, please explain why you think this moment in the lesson is critical.
Sven: In my language, Swedish, it is not that interesting to think about [*translating*] 'completing the square' because it means the same thing as in English. The direct translation is the same. When I walked around the class, I get some feedback from my students. I didn't realize there were so big differences in other languages, such that in Hindi or Chinese. They are very different from the English. That there are different words for the geometric square and the power of two.
Researcher: That part you didn't realize?
Sven: No, I didn't reflect before. So that was new to me as well. That's one of those interesting parts of this.
Researcher: And do you think brought this up [the two meanings of 'square'] to the whole class can potentially open up a discussion? What kind of discussion?
Sven: To explore from different language perspectives. Because as we found out completing the square is translated differently in different languages – could carry more algebraic or more geometric meaning. And that could be interesting to look at, I'm not sure if it helps students' understanding. Difficult to measure but it is at least an interesting point of view to look at.

The episode shows how Sven sees the potential of exploiting students' multiple languages for opening a discussion about what it means by 'completing a square' and how it may help to understand the method both geometrically and algebraically. He notices that in English, the phrase 'completing a square' may bring some confusions, as the word 'square' has dual meanings, as a geometrical shape and as the exponent. He recognizes that contributions from a Chinese- and Hindi-speaking students, that he is unaware of, may offer valuable insights to students learning of the topic. In regard to the situational demands, the episodes illustrate how Sven frames the demands of IDENTIFYING MATHEMATICALLY RELEVANT LANGUAGE aspects, and it is evident that he recognizes the potentials of SUPPORTING LANGUAGE REFLECTION for facilitating students' understanding.

In terms of didactical decision making, Sven follows up the potential of exploiting the meaning of 'square' and 'completing the square' in Chinese and Hindi by providing an example on how he thinks about the problem – doing a back-to-back translation, and then by asking a student to share her thoughts in a plenary session. Sven's actions reflect the three steps for utilizing students' ideas to inform his teaching (Jacobs et al., 2010). In this case, he perceives the students' contribution of two different meanings of the word 'square' in languages other than English (the language of instruction) which may affect how 'completing a square' is interpreted differently. Sven attempts to discuss the meaning of completing an incomplete geometrical squared shape or completing an algebraic form to the power of two, and perhaps to connect the two. Nevertheless, his idea remains superficial which suggests that he struggles to recognize the potentials for meaning making in students' utterances.

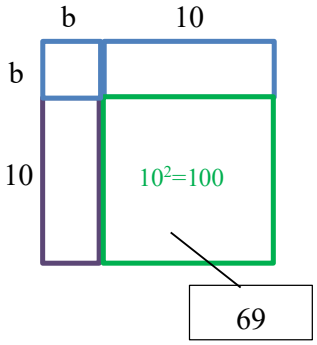
Sven's responsiveness is limited by the unexpected nature of the students' ideas, requiring him to make in the moment adaptions to his teaching practices. In the following episodes, Sven responds to a student's contributions and describes how his perspectives about the situational demands in multilanguage-responsive teaching influence his responsive decision making.

Students' contribution 2: 'Matching the power' may explain the algebraic steps in 'completing a square' method

The interview is continued with the first author and Sven resume the video lesson. Sven pauses the video following the below episode in the lesson.

Sven:	Would you like to share that with the whole class? What it means in your language?
Lin:	Like the name of the method for completing the square, if you translate it to Chinese and translate it back, then it is matching the power. [*Lin wrote on her worksheet:"配方法/ pèi fāngfǎ (matching the power of two)*"]
Sven:	OK. Matching the power. Because you have two different words, for the geometric square and the algebraic expression square [*power of two*]?
Lin:	Yes
Sven:	So, there are two different words in Chinese, two different meanings. […] So, matching the square. In Chinese, the geometric square has different words [*from the algebraic expression square*]. So, if we forget about this figure here [*points to the square drawn on the board/ shown in Figure 1*], but looking at these equations instead, how can we relate to matching the square? Or matching the power. The power here is two. So, I am writing an expression with power of 2. Here is the "green part" [*he highlights* 10^2 *in green*] that I have to do to match these two processes.
Lin:	Yes, like we are trying to find for example $(b + 10)$ square. We are trying to match like the 10 square part of that in $(b + 10)$ square.
Sven:	Yes exactly, I think that is a very algebraic way, we're not looking at any geometric parts here, just algebra. But that's exactly what we're doing right. We're trying to match this equation to this equation [*points to the two equations shown in Figure 1*]. We write it in another form, with a power. Thank you.

The episode shows that Sven intentionally encourages Lin, the Chinese speaking student, to share the expression of 'matching the power', the translation for 'completing a square' in Chinese. Sven uses the visual representation and equations that he provides on the whiteboard, so Lin can reflect on her thought when put the meaning into the 'matching the power'. Lin elaborates her answers and shows what she meant by 'matching' when matching the power in the equations and points to specific part of the equation (and the drawing). In this episode, Sven seems to master the demands of eliciting language and supporting language reflection by connecting registers and representations. Sven's practice of eliciting language has set a crucial step for his students to make sense of what 'completing a square' entails, but still, some potentials remain unexplored, e.g. elaborating in class how Lin's idea could inform the procedure for completing the square. During the interview, Sven shares more about his interpretation of the situational demand in this episode.



Two expressions that teachers wrote on the board:

$$b^2 + 20b = 69$$

$$(b + 10)^2 - 10^2 = 69$$

Figure 1: The geometric figure and algebraic expressions written on the board by the teacher

Researcher: Do you think that the language of 'matching the power' is powerful to discuss about the algebraic meaning? Better than completing the square?

Sven: Yes. It explains more, I think so.

Researcher: In relation to that, why then you decided to focus on the two equations after hearing that the student says, 'matching the power'. Is it like you think, "oh the student is talking about the algebra?"

Sven: Yes. Yes, it is definitely the algebraic interpretation. Yes, because then we see, the point is that they have to write an expression to the power of two so that they can have, take the square root of that expressions with the variable inside the brackets. So that we make that variable as the subject when solving the equation. So, this is a step we have to take, we have to rewrite $b^2 + 20b$ into an expression with variable square, and not the form we started with. So, yes, we have to rewrite it in another form, they have to match. The way she is talking about is the algebraic way. I focus on that because that is the step that we have to take to solve the equation.

Researcher: And, what is your intention when providing the visual representation on the whiteboard [*see Figure 1*]?

Sven: Because I think it helps them to understand what is happening. That b^2 is the area of a square with a side b. So, by putting the algebraic method and the geometric method side by side they see the connection, and they can visualize what is happening. […] That we are actually comparing the area of these squares and rectangles, that we are adding them. So, I think it is easier for the students to understand what the algebra represents.

Researcher: Alright. Do you want to add more comments?

Sven: Well, [*in future lesson*] I would want to try to emphasize their different meanings more, connect it more to the respective ways of looking at it. […] Creating a geometric figure from the algebraic expression. And go all the way, solve the equation. So don't stop at rewriting toward another form. Why are we doing this? They are probably thinking why are we doing this? What is this for? It is to solve algebraic equations. […] And then, start with simple quadratic equations, just to solve $x^2 = 25$ or $x^2 + x = 0$. The simple quadratic equations, start with those. Then move on to the complex quadratic equations where they have to use this method.

The interview highlights that Sven remarks Lin's contribution of 'matching the power' that can be valuable to explain the manipulation algebraic steps in completing the square method to solve a quadratic equation. Following Lin's contribution, Sven interprets what Lin tries to say about 'what to match'. Sven is aware that the expression 'completing a square' is not explicit regarding the procedure to rewrite a given quadratic equation, for instance, $b^2 + 20b = 69$ into a perfect square form, whereas the expression 'matching the power' allows an interpretation to do so. Following Sven's noticing and interpretation of Lin's contribution, he decides to ask for Lin's clarification or

elaboration on what she means by 'matching' in 'matching the power' and how she can relate the two equations he puts in Figure 1 to the idea behind 'matching the power'. It is interesting that Lin can specify that she matches "… the 10^2 part of that in $(b + 10)^2$".

Sven associates Lin's utterance of "matching the power" with the idea of transforming the quadratic expression into a binomial form. In his retrospective, Sven shows his understanding of realizing the demands of supporting language and language reflections by connecting the visual and symbolic representations (shown in Figure 1) which aims to allow students to make connections when trying to understand the algebraic procedures in solving the quadratic equation. Moreover, Sven suggests that alternatively, he would want to start with a simple equation, offering opportunity for students to connect the algebraic and the geometric meanings, and to explore the algebraic manipulation steps in solving quadratic equation more explicitly.

Discussion and conclusion

This paper reports how a teacher realizes the situational demands of multilanguage-responsive teaching by investigating how he responds to two critical contributions from his multilingual students and utilizes the contributions to inform his in-lesson decision making. In his reflections, the teacher highlights some limitation of his current practices and therefore devises some alternative approaches that could have been beneficial to support students' deeper understanding. Nevertheless, as we have discussed elsewhere (Revina & Schüler-Meyer, 2024), it seems demanding for teachers to utilize students' contributions in multiple languages in consistent way. While previous critical inquiries into multilanguage-responsive teaching seem to have enabled teachers to elicit multiple languages in the shown cases, more work needs to be done to better understand how teachers can be supported to continuously develop their capacity to enact responsive instruction (Kavanagh et al., 2020) that utilizes students' multilingual contributions in the classrooms, for promoting mathematical understanding in class. Future work needs to be done to design specific and targeted professional development opportunities for that purpose.

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Fostering mathematical explanations through LLM-based tasks

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Keywords: Artificial intelligence, large language model, explaining, digital escape game.

Introduction and Theoretical Background

Explaining is "[…] an interactive and discursive activity" (Heller, 2016, p. 254) and widely recognized as a crucial learning strategy for enhancing students' outcomes (Dunlosky et al., 2013). In particular, explanations provided by students themselves contribute to the development of deeper understanding (Bisra et al., 2018). Consequently, explaining is frequently observed in classroom discourse. However, active student participation in these discursive activities varies considerably. Factors such as large class sizes, which reduce individual speaking opportunities, and limited discursive competence among students pose significant challenges.

Research Goals

We aim to develop and research tasks that engage students in explanation processes about mathematical concepts by leveraging artificial intelligence (AI), specifically large language models (LLMs) like "ChatGPT". LLMs simulate human-like interactions, allowing questions and statements to be communicated in a natural and conversational manner. The advantage of LLMs in this context is their ability to be preconfigured for specific tasks or purposes through customized GPTs.

In advance, the LLM was configured to act like a learner with difficulties in solving a mathematical task and to ask students, who assume the role of explainers, for assistance. Rather than providing explanations directly, the LLM is designed to initiate deeper levels of explanation from the students by posing understanding-oriented follow-up questions. Moreover, the LLM was configured to reference visualizations embedded in the task, for example, by representing fractions not only in their numerical form but also as parts of a whole in a circle diagram. By doing so, it encouraged learners to engage with different representations in their explanations. In this way, the LLM is overall designed to stimulate rich discourses on mathematical topics with the provision of little help and the omission of own explanations.

Building on the concept of situated learning, we situate the explanation process within a digital mathematical escape game, creating a motivating, immersive, and goal-oriented context (Jiménez et al., 2020). In this environment, students interact with a LLM, which appears as a game character. This design-based research project explores two key questions:

(1) What are relevant design principles for the LLM that enhance the explanation process?

(2) Which mathematical discourse processes are stimulated through interactions between learners and LLM, and how do these interactions influence the quality of students' explanations?

First Results and Outlook

We conducted a first pilot with a self-developed escape game called "The Scavenger Hunt," in which students visit a museum and search for hidden clues. During the game, they encounter Justin (a

configured LLM), who asks for their help on a specific mathematical task he is struggling with. The setting was structured for students to work in pairs, allowing them to collaborate with each other while also interacting with the game character. The learners communicated with Justin via speech-to-text and received his responses in written form. The pilot project was conducted with six pairs of students each from the 6th grade focusing on fraction arithmetic and the 8th grade focusing on linear functions. We collected video recordings of the learning environment as well as screencasts and chat logs. The data is methodologically analyzed using qualitative content analysis.

The situating of the task worked well, as students quickly adapted to the game scenario and engaged with Justin as though interacting with a real human friend. In many cases, interactions with Justin, especially his follow-up questions that consistently pressed for deeper explanations (for example "Why is that…?") encouraged students to discuss their explanatory responses and prompted them to think more deeply about the mathematical topic and their initial explanation, which was typically superficial. As a result, they provided more thorough explanations, and, in some cases, identified additional connections within the task.

However, despite the task focusing on mathematical topics that had been previously covered in class, some students struggled with the explanation process due to considerable knowledge gaps. In these instances, it became evident that a complete omission of explanations by the LLM as a design principle does not work as effectively: Justin's questions aimed at eliciting deeper explanations failed to elicit progress, as the students lacked the necessary vocabulary and comprehension to respond effectively. Moving forward, we plan to explore how the LLM game character can offer subtle scaffolding during the explanation process. For instance, Justin could present suggestions or ideas himself, giving students a pathway to continue their explanations.

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Teaching AI about epistemic logic puzzles: Directing attention

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Keywords: Interaction, preservice teachers, AI, attention, epistemic logic.

Attention, babbling, gargling, and epistemic logic

Mason (2008) explains that teaching is about *directing learner attention* and scaffolding and fading support so that they eventually attend in mathematically productive ways on their own. Mason (2017) contrasts the notions of *babbling* and *gargling* to describe mathematics learners' unsuccessful explanations. Babbling refers to inexpert efforts to express thought. In contrast, gargling refers to learners "throwing technical terms into assignments in the hope that the teacher will recognise them and award marks for them" (Mason, 2017, p. 7). A gargling student is not trying to *mean* anything.

I assigned preservice teachers (PSTs) the homework task of presenting a challenging mathematics problem to the ChatGPT chatbot, leading it to a solution, and, reflecting on the interaction, comparing it with interacting with human students. The rationale is that a session with a chatbot can support preparing for interacting with human students as ChatGPT might make similar mistakes to students in the classroom, including babbling and gargling, and need directing of their (artificial) attention.

One conversation was on an *epistemic logic puzzle*. Epistemic logic puzzles involve reasoning about knowledge, including one's knowledge or the knowledge of other people, especially knowledge of knowledge or knowledge of the lack of knowledge (Hamkins, 2019). The puzzle reads as follows: "At a party, you hear a guest asking about the ages of the host's three children. The host tells the guest that the product of the children's ages is 72, but the sum of their ages is the same as the guest's house number. The guest knows their house number, but after thinking for a while, they ask the host for more information to determine the ages. The host says that their oldest child likes strawberry ice cream. The guest could then determine the ages of the children. Although you do not know the guest's house number, you have received enough information. Assume that age is always given in whole years. How old is the oldest child?" The puzzle requires unusual attention to epistemic clues.

The conversation

In the following, I present segments of the 13 turn-pair interaction between the PST and ChatGPT to explore similarities with human-to-human teaching situations, specifically on directing attention, babbling, and gargling. In the first attempt, it created a list of several (but not all) possibilities to get 72 by multiplying three numbers. Later, in the same turn, it concluded its first solution attempt.

ChatGPT: […] If the sum were 14, there is only one possibility: 2 + 2 + 9. The host's oldest child must be 9 years old because it is the only age that appears in only one combination. Therefore, the answer is that the host's oldest child is 9 years old.

PST: Try to think about the fact that the guest needed another hint, that tells us that although the guest knows the house number, they still needed another hint.

An aspect of ChatGPT's response can be interpreted as babbling if we think that the generation of possibilities of multiplying and considering sums constitutes an engagement with the meaning of the task. But "it is the only age that appears in only one combination" is only superficially related to the

task and could be interpreted as gargling. The PST response seems to be intended to direct attention to a specific (essential) detail, namely the role of the epistemic information in the puzzle. After three more turn pairs, ChatGPT produced a complete and correct list of possibilities, saying that the children are either 3, 3, and 8, or 2, 6, and 6 years old because those numbers uniquely give the same sum. Then, ChatGPT seems to gargle about the age relations of twins and their siblings:

ChatGPT: If the twins are the oldest children, then we know that one twin is older than the other, but we do not know which. However, if the child who is not a twin is the youngest, then the older twin must be the one who likes strawberry ice cream. This is because the host would not have referred to the younger twin as the oldest child if there is an older twin. Therefore, we can conclude in this case that the oldest child is the older twin, and the other twin and the non-twin child are both six years old.

PST: What if I tell you that we always consider twins to be of the same age?

The PST directs the attention to a feature of the task without directly stating what is wrong with the answer. In the tenth round, the teacher candidate summarizes for the artificial student that the possibilities are 3, 3, and 8, and 2, 6, and 6, and directs attention again to the critical clue:

PST: We know from the clue about the ice cream that one child is the oldest, so in which possibility is there definitely an oldest child?

This is already on the verge of being a case of funneling. At last, the PST asks a funneling question:

PST: I think you are overthinking this, how old is the oldest child if there are three children who are 3, 3, and 8 years old?

After this, ChatGPT gave, as a final turn, a reasonably good account of the solution as a whole when asked to do so.

Conclusion

Some of ChatGPT's difficulties in solving the task differed significantly from those of a human student. A human would not need multiple turns to accept that twins are considered the same age or to realize that the clue implies one child must be older than the other two. On the other hand, many human students benefit from having their attention directed to points such as the guest needing another clue to deduce the children's ages. In the conversation with ChatGPT, the PST practiced directing students' attention and getting them to revise their answers while increasing their awareness of babbling and gargling.

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TWG 10: Diversity and Mathematics Education: Social, Cultural and Political Challenges

Report to TWG10: Social, Cultural and Political Aspects of Mathematics Education

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Scope and Focus

Thematic Working Group 10 (TWG 10) has its roots in the *language diversity group/teaching and learning in the multilingual classroom* and progressively transformed to include broader settings, political perspectives, theoretical perspectives. It centres discussions of mathematics education within the realms of the cultural, the social and the political. TWG10 builds on the premise that mathematics education is always more than an encounter between an individual and a mathematical object in a classroom setting. Instead, it views such encounters as shaped and produced by wider cultural and societal contexts that are inherently social and political. At the same time, such encounters are also viewed as contributing and constituting the contexts in which they are embedded in ways that reproduce, challenge or disrupt power relations. TWG10 is characterized by an effort to reflect its own double-role in analysing, shaping and reconfiguring mathematics education practices. The group is specifically interested in research that investigates how diversity and difference affect the possibilities, opportunities, obstacles, privileges and disadvantages associated with mathematics education. This includes issues of gender, race and ethnicity, language, socio-economic status, social class, disability, inclusion/exclusion, life opportunities, aspirations, worldviews and ideologies, school systems, governance structures, space, and settings. Additionally, diversity and difference may occur in relation to who is doing the research and who is being researched, posing methodological issues of an ethical, ontological and aesthetic nature. Diversity and difference are therefore perceived in a broad sense and refer to various ways of doing mathematics education research within the realms of the cultural, the social and the political. As all these multiple diversities and differences intersect, a reflective approach is expected in reporting on implications of any research.

Organization of TWG10's work at CERME14

In the run-up to the conference 25 papers and 5 posters were submitted. 23 papers and one poster were accepted and discussed at the conference, leading to 21 contributions that were included in the proceedings. As at previous CERMEs, the group adopted the principle of *peer presentation*, namely that authors do not present their own paper, but give a short (5 minutes) presentation of a paper submitted by another participant in TWG10. This peer presentation included a description of the

paper's main ideas (as read by the presenter) and the formulation of questions from the presenter's own perspective. This was followed by a 5 minute response from the author(s) and then questions and discussion within the whole TWG. However, the experience of the last conference (CERME13) showed that some authors want to present their own paper for various reasons. We were also discussing, that the separation could create distance between voice of the researcher and their work or denying researchers the opportunity to have their voice heard. For this reason, this time gave the opportunity to deliver one's own presentation, which was at least partially used. The poster was discussed alongside the presentations in order to make it visible to the whole group.

The Papers

In this section, we briefly describe the papers sorted by main topics, starting with papers on the *designing of tasks, reflecting books & curriculum*: *Norén and Wallin* discussed the challenges for second language learners to solve mathematical word problems and how material entanglements play out around mathematical word problems when students talk about them. The paper by *Da Cruz et al.* focused mathematics textbooks in Brazil and Canada, comparing linguistic, cultural, and sociopolitical factors. The study highlights textbooks as pedagogical and cultural artifacts, suggesting that national guidelines may benefit from balanced approaches that include local perspectives. *Jablonka and Nazir* also used a comparative approach but not in analyzing books or tasks, but curriculums. Using Basil Bernstein's notion of framing the Berlin Mathematics Curriculum (Germany) and the Turkish Mathematics Curriculum were discussed. When it comes to curriculums, universal design for learning becomes a more and more common approach, though there is little known about the concept. That is why *Bray, McKenna and Oldham* provide a scoping review that reveals limited but promising research, highlighting a need for further studies to explore UDL as a framework for inclusive and equitable mathematics classrooms. Another paper in this section is about designing a Professional Learning Task (PLT) within the context of preparing mathematics instructors in higher education, focusing on an innovative curriculum that uses technology as a leverage for promoting social justice. *Salazar and da Silva Schneider* discuss elements that integrate mathematical knowledge with its teaching and that aim to challenge mechanisms of oppression affecting traditionally marginalized students in mathematics courses.

Another important topic emerged on *political and environmental crisis*. *López-Serentill et al*. join the analysis of curriculum conceptions and research how sustainability competences have been integrated into the current primary education curriculum in Spain by using the GreenComp European sustainability competence framework. Not only the environmental crisis but other conflicts over inequality are topics for mathematics education. That is why *Neuenfeldt and Dexel* suggest a framework of 'arenas of conflicts over inequality' and transfer it to mathematics educational research. *Fosse and Hauge* present a project on sustainability conducted by pre-service mathematics teachers. Different aspects of the clothing industry within the the subjects of social studies, natural science and mathematics were discussed by the participants and linked to questions of empowerment and disempowerment. A similar project with Norwegian and South African researchers on clean water was carried out and thoroughly researched with reference to video data by *Foyn et al.* The authors report in a very self-reflective way on the difficulties of changing and influencing the positioning of students within such projects.

Empowerment also plays a role in the next notable topic of our working group, the research of student diversity. Drawing on the concept of mathematical empowerment, *Billigen and Söbbeke* analyze the extent to which social participation can be developed in a mathematical negotiation process using a case study. *Wagner et al.* describe the effort to surface experiences and conceptualisations of those experiences drawn from interactions with newly migrated and Indigenous mathematics students. They identify challenges in surfacing student experiences, and in naming their conceptualisation of those experiences as storylines. *Wang and Kollosche* categorise subjective relevance of mathematics students in deriving dimensions of relevance from theory and supplement them with examples from empirical research. *Österling et al.* present a survey study of the mathematics capital of first-semester mathematics students in Swedish universities, using a cluster analysis that shows how mathematics capital is distributed among three clusters of students, distinguished mainly by parents' or significant others' relation to STEM.

One of the most prominent topics at our sessions on student diversity evolves around different conceptions of *gender* and mathematics education. In a quantitative study *Tscholl* examines whether a gender gap in mathematical self-concept can be found among first-year STEM students at the University of Innsbruck and if the accuracy of mathematical self-perception relative to performance differs by gender. She comes to the conclusion that gender-specific filter mechanisms appear to persist when entering STEM degree programs. *Aquilina* takes a closer look at the Italian INVALSI tests aiming to explain how a higher gender gap is constructed. For that reason, she analyzes 657 students answers to the 'Tennis-Tournament'-task. Her results suggest that gender disparities may stem more from situated than from mathematical knowledge. Another approach to determine the relationship between gender and mathematics is made by Brieger, who analyses in detail interviews with mathematicians who describe their mathematics lessons in school and corresponding interpretations by others. In the study female mathematicians describe themselves using mostly masculine-associated wording and women's memories of mathematics lessons in school were being framed in the contexts of competitions and strong will to succeed. The poster in this session by *De Rocco et al.* addressed how mathematics teachers perceive gender difference within a prison school.

As *Teachers* of course play a very important role when it comes to social and political aspects of mathematics education there were different papers prioritizing this group. *Beccuti and Polo* discuss discourses on mathematics and mathematical ability of support teachers in Italy in context of broader systemic issues connected to the support of the teaching and learning of mathematics in Italy. *Guillemette* describes a participatory research project implemented with in-service secondary school mathematics teachers in Montreal, Canada. In the dialogues about mathematics cultural and historical emerge three main thematic topics: Otherness, Hospitality and Reflexivity. In a synopsis of three research projects about socially just approaches to mathematics, *Wright* emphasizes the aspects relational pedagogy, relational authority and relational agency in context of learning and teaching. The paper by *Kitchen et al.* focusses on the development of teacher leaders in hierarchical schools and school systems. They use a socio-political theoretical perspective to explore the opportunities that minoritized teachers have in engaging as teacher leaders in mathematics in their schools and school system, showing that that a ceiling exists for them that limited their growth as leaders.

Next steps

Keeping a critical approach towards 'rigour' during the review process

The number of submissions to CERME has increased quite rapidly during the recent years, which is why the conference organisers have understandably needed to implement measures to ensure as many colleagues as possible could attend the conference. Amid this context, the acceptance rate for CERME papers necessarily gets lower. The question then arises: how could we make informed and fair decisions about submissions to TWG10 in the future? In other words, how can we remain critical and transparent about what sort of work gets rejected from our TWG – especially as such cases are likely to increase?

As with any topic in our TWG, we must also consider the peer review exercise in CERME not as a neutral mechanism, but as a socio-political practice that in many ways represents and renews power relations. What gets accepted and what gets rejected in our TWG is a thorny issue from this point of view. We presume that the upcoming years will need to engage the TWG10 community in tricky conversations about what 'rigour' means within the scope of our TWG. We cannot simply utilise definitions that work elsewhere; ones that emphasise particular worldviews and paradigms beyond our socio-political one. It is likely that CERME will aim to make the peer review process more standardised in the future, and while there is an important message behind such approaches in ensuring all colleagues are treated equally, our TWG may also need to make editorial decisions that help increase the visibility of non-dominant forms of research and research in non-dominant communities'. Representative examples of this might be studies drawing on an activist stance (such as disability studies), or studies drawing on Indigenous approaches (see e.g., the work of Beccutti and Wagner et al. in this year's contributions).

One of the criteria for CERME papers is the fit between the paper and the TWG. We anticipate this criterion will become increasingly important in the future. Many excellent research papers might be excluded from TWG10 if they lack critical perspectives, for example, if they do not adequately contextualise the issues at hand. Yet again, the divisions between what counts as 'critical enough' for our TWG are not simple but most likely require some thorny dialogues. It seems likely that controversial topics will become more frequent in the future, given the many political and social challenges unfolding in different parts of the world.

In discussing these challenges, we believe our responsibility is to keep TWG10 as a space for difficult and important socio-political conversations. Ultimately, the task is about creating space for discussion that is respectful, thoughtful and open to diverse opinions and which welcomes non-dominant scholars and those active in the struggle of the oppressed against oppression. This requires explicitly recognising academic freedom as politically non-neutral whilst upholding ethical responsibility in the struggle for social justice. TWG10 has to remain open to a plurality of perspectives, but it also has to ensure that contributions are transparent, critically contextualised, and aligned with the group's socio-political focus. These tensions are not easy to navigate, but they are central to the ongoing work of TWG10.

Beyond performance in mathematics results: gendered patterns and contextual knowledge in the "Tennis tournament" item

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Standardized assessments in mathematics consistently reveal a so-called gender gap in favor of boys in Italy. The broader study seeks to acknowledge the complexity of this often-oversimplified phenomenon by shifting the focus from quantitative data – the results of standardized assessments – to a qualitative analysis of students' descriptions of their solution processes. This paper examines a grade-8 item from the Italian standardized assessment (INVALSI test) – the "Tennis Tournament" item. The grounded analysis of 657 students' protocols highlighted the gendering of students' solution patterns and the critical role of contextual knowledge in answering correctly. These findings are particularly relevant to the "gender gap discourse", given the widespread belief in the objectivity of standardized test results. Only by focusing on the processes behind students' answers can we meaningfully interpret the data and challenge both the gender gap and its prevailing narrative.

Keywords: Gender gap, mathematics, standardized assessments, contextual knowledge.

Introduction and theoretical background

Over the past decades, national and international standardized tests have gained significant prominence in evaluating educational systems, influencing public opinion, and shaping policies. A recurring outcome of these assessments is the so-called gender gap in mathematics performance observed across most countries. The 2022 PISA (Programme for International Student Assessment) report, which focused on mathematics, highlights that this performance gap remained unchanged over the past decade in 53 of the 64 participating countries or economies with comparable data, including Italy (OECD, 2023). While the average mathematics test scores in Italy aligned with the OECD average, Italy recorded the largest gender gap in favor of boys, with a 21-score point difference compared to the OECD average of 9 points. Globally, boys outperformed girls in mathematics in 40 countries/economies, whereas girls outperformed boys in 17. This international variability underscores the importance of questioning simplistic interpretations of the so-called gender gap and considering the critical role of sociocultural factors in shaping mathematical learning outcomes.

In the Italian context, the INVALSI (Italian National Institute for the Evaluation of the Education and Training System) annually assesses students' competencies in Italian, mathematics, and English across various school grades (2, 5, 8, 10, and 13). Historical data reveal that girls consistently outperform boys in linguistic tests, while a stable and pronounced gender gap in favor of boys persists in mathematics performances (Giberti, 2019). According to the 2023 INVALSI report, male students in the "type" group – an INVALSI category defined by specific socio-demographic characteristics – score between 4.6 and 7.4 percentage points higher than females across school grades, accounting for approximately 50% of the yearly average learning.

The results of standardized assessments are just one indicator of the inherently complex phenomena of educational gaps, yet they are particularly significant because they enable diachronic and

geographical comparisons and because they strongly influence public opinion. Consequently, many studies have attempted to explain the determinants of the gender gap in mathematics through analyses of the overall results of standardized tests. However, Giberti (2019) highlighted a lack of didactical interpretations in the use of such data, which are often analyzed purely statistically. Together with colleagues, she conducted item-level analyses to identify specific characteristics of test items – such as typology, formulation, and context – that could influence the gender gap (Bolondi et al., 2018; Ferretti & Giberti, 2021). Their findings suggest that the "didactic contract" may exert a stronger influence on girls, potentially explaining their response patterns on specific INVALSI items.

I argue that moving beyond the quantitative data from standardized assessments is essential to interpreting the so-called gender gap in final scores and understanding the underlying dynamics. As Di Martino and Baccaglini-Frank (2017) observe:

> Standardized assessments are interested in describing quantitative macro-phenomena rather than in understanding them, also at a micro level. To gain such understanding, using Crespo's words (2000), it is necessary to acquire an interpretative orientation, collecting students' ideas with the purpose of accessing rather than assessing students' mathematical understanding (p. 1).

Cooper and Dunne (1999) supported the need for an interpretative approach, focusing on the analysis of the problem contexts that often reflect implicit class and gender biases. While aiming to enhance relevance and understanding, contextualized tasks frequently rely on cultural capital and prior experiences that are unequally distributed among students, potentially disadvantaging girls and students from lower socioeconomic backgrounds. Similarly, Boaler (2003) highlights how pseudo-realistic word problems can embed biases that influence student performance, impacting particularly on girls, who seem to engage more deeply with contextual variables. Boaler and Sengupta-Irving's (2006) relational approach to learning further stresses that problem-solving cannot be disentangled from the environments and interactions that shape how students engage with tasks. These perspectives reinforce that the context in which mathematical reasoning is assessed is never neutral: it carries implicit messages about whose knowledge and experiences are valued.

These findings stress the need for developing qualitative analysis to meaningfully interpret the so-called gender gap phenomenon and to inform more equitable discussions about mathematics assessment. Questioning the phenomenon and understanding the dynamics behind the performance gap are the first steps toward imagining effective actions to counteract it.

Research aims

In this frame, I am conducting a comprehensive study focused on the qualitative analysis of students' descriptions of their solution processes. The overarching aim of the broader study is to unpack quantitative data to question both the so-called gender gap in mathematics and its underlying narratives. Building on the interpretative orientation advocated by Di Martino and Baccaglini-Frank (2017), I aim to analyze students' problem-solving patterns on INVALSI items that revealed a significant gender gap in the Italian standardized assessment. By doing so, I seek to uncover the underlying dynamics of response construction, moving beyond a surface-level assessment of outcomes. Specifically, the guiding question is: How are gender differences in results reflected by

the gendering of students' solution patterns? Furthermore, I am interested in interpreting the factors affecting these different patterns, particularly their relationship with mathematical competence, as well as other kinds of factors, such as contextual or sociocultural influences.

Once having outlined the methodology adopted in the broader study, this contribution focuses on the results of the analysis of the responses to the "Tennis tournament" item. The goal is twofold: to share the grounded and interpretative approach that guides the study as a whole and to discuss how this case study sheds light on the interplay between the so-called gender gap in the results, the gendered patterns of responses, and contextual factors in standardized assessments.

Research methodology

To select the items for the new data collection, I considered the whole sample of public INVALSI items for grades 2 and 5 (between 2009 and 2022) and for grade 8 (between 2009 and 2017). Then, I conducted a quantitative analysis to estimate the eventual gender gap resulting in each item, which led me to identify an initial sample of 225 items based on the (co)presence of the following criteria:

1. A significant gap in favor of boys in the percentage of correct answers;
2. A significant gap in favor of girls in the percentage of correct answers;
3. An insignificant gender gap in the percentage of correct answers;
4. A significant gender gap in the choice of a specific distractor (for multiple-choice items);
5. An inverted gap between girls and boys among different sub-items of the same item.

I then categorized the selected sample based on: i) item difficulty (measured by the percentage of correct or missing answers, following the INVALSI framework); ii) item typology (e.g. multiple-choice, open-ended or true/false questions); iii) item content according to the National Guidelines for the first cycle (Numbers, Shapes and Figures, Relations and Functions, Data and Previsions.)

Based on a principle of representativeness of the identified categories, I selected a final sample of 21 items: 5 items from grade 2 (the first grade tested in the national assessment) and grade 8 (the final grade of the first school cycle), and 11 items from grade 5 (the transitional grade between primary and secondary school). The "Tennis tournament" item is one of the grade 8 items selected.

New data for the broader study were collected through the voluntary participation of 40 schools across various Italian regions. Since the interest is in students' solution processes, for each item the prompt "Explain how you reasoned to get your answer(s)" was included[1]. Basic instructions for administering the tasks were provided: 1- students' performance would not be assessed; 2- tasks should be completed within a single lesson, with no further constraints; and 3- teachers could clarify item wording if asked. This voluntary participation resulted in the collection – between February and May 2024 – of over 10.000 solution protocols for the 21 selected items.

For data analysis, I adopted the social constructionist paradigm of grounded theory (Charmaz, 2008), an evolution of the original grounded theory method introduced by Glaser and Strauss in the 1960s. This analytical approach explores social reality through an inductively driven systematic praxis,

[1] The INVALSI tests collect just students' answers to the items and not their solution processes.

where theoretical insights emerge from the data rather than being imposed by pre-existing frameworks, acknowledging that researchers actively construct meanings and categories instead of objectively discovering them. Hesse-Biber and Flowers (2019) discuss that social constructionist grounded theory aligns with feminist research principles, which challenge dominant narratives and knowledge structures while emphasizing how socially constructed gendered experiences shape individual perspectives. Grounded theory's iterative process allows for a nuanced exploration of gender as a category of analysis without reinforcing essentialist assumptions. By grounding the analysis in students' explanations, this study avoids a priori assumptions, focusing instead on emerging patterns to guide the interpretation of gendered differences. As the analysis progresses, existing theories in mathematics education will inform the characterization of these strategies. Given the large dataset, the dynamic coding process enables the iterative testing of theoretical hypotheses on unexamined data, refining interpretations until saturation is reached.

The following section presents a case study to illustrate my approach to data analysis and the interpretation and discussion of results.

Case study: the "Tennis tournament" item

The "Tennis tournament" item (Figure 1) was administered in the grade-8 2014 INVALSI test and showed a significant gender gap in favor of boys: 7% in sub-item (a) and 20% in sub-item (b). The INVALSI classification of this item is: *Related area*: Numbers; *Related processes*: solving problems using strategies in different fields; *Keywords*: division, power of a number; *Related goals from National Guidelines*: "recognizes and solves problems in different contexts by evaluating information and their consistency". The discussion of this item is relevant in light of Cooper and Dunne's critique (1999), which highlights the importance of the context to understand the problem-solving processes.

Sixteen players participate in a single-elimination tournament, where each match is one-on-one, and the loser is eliminated.

a. How many games are needed to determine the winner of the tournament?
   A. 8
   B. 15
   C. 16
   D. 32

b. Gabriele won the tournament. How many games did he play?

Answer: ……

Try to solve this problem, and to explain how you reasoned to give your answers (both a. and b.)

Figure 1: Tennis tournament item

Students' protocols related to this item were collected from 15 schools, located in 10 Italian provinces, and divided into two samples: i) 112 boys and 277 girls attending 8th and 9th grades; ii) 92 boys and 176 girls attending 10th and 12th grades (included in the broader study to examine whether advanced mathematical education impacts solution processes). For the division into boys and girls, I followed the binary categorization adopted by INVALSI (which does not make a clear distinction between sex and gender), even though it does not align with my theoretical view on gender. Specifically, I asked

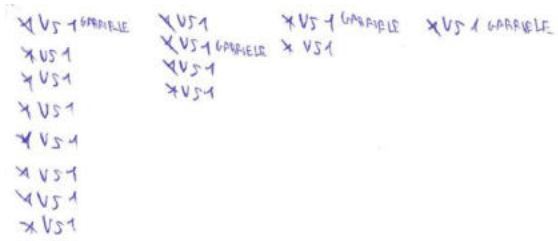
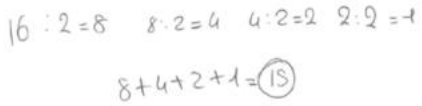
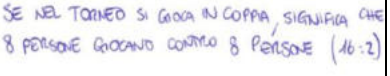
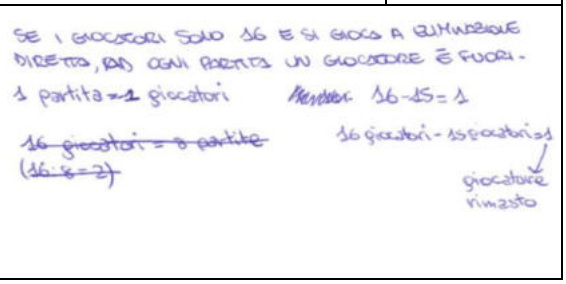
students to write their names on their solution protocols. Since most Italian names are gendered, and with support from teachers when needed, I was able to code the majority of them as male or female.

Results and discussion

The quantitative data analysis showed that the gender gap was higher than in the INVALSI 2014 results, with gaps in favor of boys of 12% and 16% for sub-item (a) and 28% and 20% for sub-item (b) in grades 8-9 and grades 10-12, respectively. This confirmed the item's suitability for examining "gendered" problem-solving patterns, even after 10 years.

Through the inductive analysis of 657 protocols – with particular reference to the students' descriptions of their reasoning – I identified five main problem-solving strategies. Table 1 describes the identified five categories, providing a paradigmatic example for each category and reporting the frequency of adoption of each strategy in my sample, broken down by gender.

Table 1: Main solution strategies adopted by students, with explicit frequencies

<table>
<tr>
<td>1) The "contextualized diagram" strategy: students use or imagine a precise scheme which clearly refers to the scoreboard of single-elimination sports (mainly soccer) and they count the matches from it. In their explanations they often make explicit references to their sports experiences.</td>
<td>Adopted by:
Grades 8-9
M: 55%
F: 26%
Grades 10-12
M: 63%
F: 28%</td>
<td colspan="2"> </td>
</tr>
<tr>
<td>2) The "non-contextualized diagrams" strategy: students draw the total number of players, usually in two different columns/lines, they make couples, they eliminate one line of players and iterate until they have just 1 player left. Then they count the matches.</td>
<td>Adopted by:
Grades 8-9
M: 13%
F: 22%
Grades 10-12
M: 14%
F: 32%</td>
<td colspan="2">  </td>
</tr>
<tr>
<td>3) The "iterative divisions" strategy: students do not use any scheme, but they proceed by dividing by 2 or by making couples and iterate until they arrive to 1.
  </td>
<td>Adopted by:
Grades 8-9
M: 18%
F: 17%
Grades 10-12
M: 29%
F: 24%</td>
<td>4) The "non-iterative divisions" strategy: students do not use any scheme, and they just divide the total number of players by 2, i.e. the number of players in a single match.
  </td>
<td>Adopted by:
Grades 8-9
M: 5%
F: 16%
Grades 10-12
M: 4%
F: 7%</td>
</tr>
<tr>
<td>5) The "N-1" strategy: students do not use any scheme, but they argue, for example, that "If the players are 16 and they play in a single-elimination tournament, at each match one player is eliminated.
$1\ match\ =\ -1\ player$; $16 - 15 = 1$;
$16\ players\ -15\ players\ =\ 1\ winner$"
(translation of the figure on the right).</td>
<td>Adopted by:
Grades 8-9
M: 5%
F: 16%
Grades 10-12
M: 8%
F: 13%</td>
<td colspan="2">  </td>
</tr>
</table>

The mathematical analysis of these categories led me to recognize two opposed categories in terms of their "degree of abstraction". Here, focusing on the 8th and 9th grade samples, I will discuss the most contextualized and the most abstract strategies.

The "contextualized diagram" strategy is the most related to the real context of sports tournaments and the most used by all students. However, a clear "gender gap" emerges: the percentage of boys who drew a diagram resembling sports scoreboards (55%) is more than double the percentage of girls who used this approach (26%). Several students using this strategy recalled their experience with a single-elimination sport to visualize and model the situation proposed, interestingly transferring the context from tennis to other sports (mainly soccer). Once the visualization of the knockout scoreboard was done, students in this category just counted from their scheme of the scoreboard to answer both sub-items. For this strategy, students' contextual knowledge – in this case, their individual familiarity with either watching or playing single-elimination sports – was a determinant factor for the success of their mathematical modeling and, therefore, for their answering. Notably, the "contextualized diagram" strategy was the least fallible, with girls outperforming boys when using it [fallibility rates: 15% and 13% for boys *versus* 8% and 12% for girls for items (a) and (b), respectively].

The "N-1" strategy is the most abstract one, being quite mathematically elegant, and, differently from the "contextualized diagram", not supported by any explicit reference to the real-world experience of sports tournaments. In this case, the percentage of girls using this approach (16%) is more than three times higher than that of boys (5%). The fallibility of this strategy is extremely low for sub-item (a) (~1% for both boys and girls) and extremely high for sub-item (b) (83% for boys and 81% for girls). This was not surprising to me, yet very interesting, because this strategy does not give a clue about how students should count the matches played by the winner and, ultimately, how the tournament works: most of the wrong answers were either "15" or "it's impossible to answer, it depends from the moment in which Gabriele began to play". I coded this latter case as the "King of the Hill" strategy because, in their explanations, students clearly refer to a tournament in which the winning player continues to play until defeat. In this scenario, there is no unique answer to the sub-item (b). As for the "15" answer, they usually combine the argument "I need 15 matches to eliminate 15 players" with "if Gabriele won the tournament, it means that he hasn't been eliminated, so he won all the 15 matches", in a sort of Aristotelian syllogism. In both cases, these wrong answers seem unrelated to a lack of students' mathematical competence or a lack of consistency in their reasoning but rather reflect a different interpretation of the situation. This reveals that the "Tennis tournament" item is based on a clear assumption: everyone knows how a single-elimination tournament works.

Interestingly, similar patterns are observed among 10th and 12th-grade students, indicating that higher mathematical competencies do not reduce the observed gender differences in students' patterns or performance. This suggests that the nature of the item – requiring context-specific knowledge rather than advanced mathematical skills – plays a critical role in shaping outcomes.

In the Italian context, sports - and especially watching sports – are still predominantly male domains. According to the Permanent Sport Observatory (2024), 33% of Italian women report never having practiced sports, compared to 16% of men. In 2022, 63% of men and 44% of women attended at least one live sports event as spectators. Furthermore, 91% of men and 70% of women reported consuming

informational and entertainment content related to the world of sports, ranging from rarely to frequently. The gender gap resulting in the "Tennis tournament" item seems to reflect this contextual bias rather than indicating girls' difficulty in solving mathematical problems or in "evaluating information and their consistency" (National Guidelines' goal related to this item).

The core issue lies in the fact that while the "Tennis Tournament" item aims to measure a form of mathematical reasoning, it also mobilizes a significant amount of contextually situated knowledge, which is itself deeply gendered. In this case, the problem's context privileges students – in Italy, primarily boys – who can draw on their sports-related knowledge and familiarity with tournament structures, mobilizing them as strategies.

Conclusion and further development

The analysis of students' descriptions of their reasoning and the dynamic coding of their solution patterns revealed the significant role of contextual knowledge, such as familiarity with single-elimination sports, in shaping strategy choice and performance. This grounded interpretation suggests that the gender gap observed in the "Tennis Tournament" item is less a matter of strictly mathematical competence and more about access to this contextual – and gendered – knowledge (if the two can truly be separated). As a result, the "Tennis Tournament" item exemplifies how standardized assessments can unintentionally perpetuate the narrative about inequities by privileging contexts that align more closely with traditionally male experiences.

Although this case study focuses on a specific item where context played a central role in its selection, it emphasizes a broader issue: merely measuring the gender gap through selected answers provides limited insight, while analyzing problem-solving processes and looking for gendered patterns can offer generative insights for the interpretation of underlying factors contributing to these disparities. The broader corpus of items included in my research – representing diverse typologies and contexts – aims to generate more nuanced and comprehensive interpretations of gender differences in mathematics.

Administering items like the "Tennis Tournament" does not disrupt the systemic patterns that reinforce gendered identity and (re)produce disparities. As Varenne (2018) argues, researchers, educators, and policymakers must shift the equity lens away from individuals and categories of people and toward the systems that co-produce differences. This requires committing resources to explore the circumstances and contexts that generate disparities between groups rather than reinforcing narratives of underachievement relative to a normative – and mainly male – standard.

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Discourses on mathematics and mathematical ability of support teachers in Italy

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We study essays about mathematics and mathematical ability written by preservice support teachers in Italy, which we analyze discursively in relation to their life experiences and career aims. Overall, while the participants articulate a view of mathematics as a generally worthy, satisfying, and useful form of activity, they nonetheless report a negative personal relationship with it typically after primary school. This is articulated either in connection to a bad experiences with a teacher, as linked to mathematics' challenging growing abstractness, as well as in connection to the participants' own perceived lack of mathematical predisposition or ability. As to the latter, the participants typically depict it as having both an innate and an acquired education-dependent component. After presenting the results of the analysis, we discuss them in view of broader systemic issues connected to the support of the teaching and learning of mathematics in Italy.

Keywords: Special education, Preservice support teachers, Mathematical ability, Discourse analysis.

Introduction

The role that mathematics often has in school and in society as a gatekeeper (Martin et al., 2010) and as a marker of intellectual ability (Nieminen et al., 2024) points to the centrality and problematicity of this subject with respect to phenomena of inclusion and exclusion happening in educational institutions. In particular, discourses surrounding the nature of mathematics and mathematical or general (dis)ability crucially contribute to shape the way in which mathematics is taught and learnt with respect to policies and practices of in(ex)clusion.

As a significant example, the custom of segregating or grouping students according to ability and the connected mathematical pedagogies depend crucially on discourses "about the purposes of education, the potential of students, and the nature of learning" (Boaler, 2020, p. 1). As this example suggests, reaching a full understanding of phenomena involved in inclusive mathematics education involves attending to the related discourses which pervade society and influence the curriculum as well as educational practices. Among these, the discourses articulated by the people involved more directly in the teaching and learning of mathematics appears to be of primary importance. While views on mathematics and mathematical ability have been explored in research with respect to students and teachers of mathematics (cf., e.g., Beccuti et al., 2024; Nieminen et al., 2023; Gildehaus et al., 2024; Kollosche, 2017), it seems that almost no study concentrated specifically on the discourses articulated on these by support teachers. In this paper we thus address the following research question: What discourses do pre-service support teachers articulate around mathematics and mathematical ability in relation to their past life experiences and future career aims?

After addressing this question via a discursive analysis of essays written by support teachers in training in Italy, we reflect on the results in connection to wider systemic issues related to inclusive mathematics education which the analysis will evidence.

Theoretical framework

The philosopher Michel Foucault viewed *discourse* as a primary means through which power and knowledge are woven together, contributing to shape and at the same time being shaped by social structures, cultural norms, and institutions. According to Foucault (1977), discourse is constituted by patterns of knowledge which do not only describe but also systematically construct the shared spaces in which individuals live and thrive as well as concomitantly the characters with which they choose to align their subjectivities to. Understanding discourse is therefore crucial for understanding how knowledge is not only used to describe and represent but also to control, regulate, and administer societal and institutional activities (cf. Kollosche, 2017).

Despite attempts at mitigating the most blatant unjust effects of schooling via the promotion of inclusive policies, schools inherently function directly and indirectly as primary instruments of social gatekeeping through their explicit and implicit role in selection processes justified with the objective of selecting and promoting intelligence, dedication, and merit in view of recruitment in the workforce. Proficiency in mathematics is of foremost importance to these processes, mathematics being a compulsory subject in almost all countries around the world from primary to upper secondary schools. Indeed, the development of mathematical competence or ability is typically predicated to be essential for accessing highly specialized jobs or duties in a society increasingly characterized by technoscientific knowledge. Furthermore, mathematical ability has traditionally been and continues to be narrated to be a proxy for overall intellectual ability and for the capability to solve problems in various non-mathematics related domains (cf. Ernest, 2019). Therefore, discourses on the nature of mathematics and mathematical ability articulated about, around, and in the proximity of those who are predicated as being affected by a disability impairing their school proficiency appear to be paramount for the practices of in(ex)clusion which involve them in schools and elsewhere.

In Foucauldian terms, with respect to phenomena connected to (dis)ability, it is particularly important to attend to how this very same concept and related notions are constructed through discourses (articulated and transmitted explicitly and implicitly inside and outside educational institutions) which impact how different groups of people perceive and address these. Among the affected groups, that of support teacher surely is of primary importance, in this case. It appears thus of foremost relevance to understand and attempt to describe the discourses articulated by support teachers about mathematics and mathematical ability as situated in the cultural and institutional landscape in which their professional activity is incardinated.

The Italian institutional context

Italy represents a particular nationally-situated case where a strong ideological commitment to equity and inclusion appears to pervade the curricular and legislative discursive landscape concerning special education. This commitment started institutionally with the factual abolition of special schools and segregated classes in the 1970s. Nowadays, while few particular instances of special schools still

exist, practically almost all students attend local schools where students are not grouped by ability (D'Alessio, 2010).

In Italy, "normal" secondary teachers are certified as teachers of specific subjects or groups of subjects. While complicated legislative nuances are in place, as a general principle, the completion of a two-year disciplinary-specific master's degree (as opposed to a teaching degree, as in most countries) renders one legally qualified to be a secondary teacher in the corresponding subject(s). In lower secondary schools, the teaching of mathematics is coupled with the teaching of science and typically teachers hold master's degrees in some scientific subject. In upper secondary schools, mathematics is taught alone or coupled with the teaching of physics and typically teachers hold degrees in mathematics, physics, or sometimes in engineering.

Permanent position as support teachers, on the other hand, are accessed after completion of a one-year master's degree. This is an intensive program focusing on special education comprising courses in special psychology, pedagogy, neuropsychiatry, and legislation on special education, as well as laboratories on specific disciplines. While again complex legislative regulations are in place, practically anybody holding a two-year master's degree which is a prerequisite to being a disciplinary teacher (in any subject) can compete for admission to this further one-year master's program. Legally, support teachers are assigned to a class where at least a student with a certified disability is present. Ideally, a support teacher supports the whole class and not specifically the student(s) with disability. Frequently, however, in practice the class proceeds "normally" while the support teacher tutors and interacts primarily or solely with the student(s) with disability (Ianes, 2014).

Experimental context and method

We study 63 pre-service secondary support teachers enrolled in a course within the aforementioned one-year master's program for support teacher qualification in a large research university in the North of Italy. The course is a compulsory laboratory lectured by the first author concentrating on mathematical and logical aspects involved in special education. The participants typically come from a non-scientific course of studies (e.g., literature, history, music, etc.) and their age ranges from about 25 up to 50 years old, with a slight prevalence of female participants. Typically, the participants have had some years of previous experience in school as substitute or non-permanent support teachers.

In the first day of the laboratory the lecturer asked the participants to write an essay entitled "What is to me mathematics" which they wrote individually in the class for about 40 minutes. This group of essays was preliminarily analyzed by means of a thematic analysis inspired by Braun and Clarke (2014) employing the themes developed in the first part of the methodology section of Beccuti and colleagues (2024). These themes are: the challenge of mathematics, the utility of mathematics, the comforting dimension of mathematics, and the relationship to others with respect to mathematics. This preliminary analysis already gave a first picture of how mathematics is viewed by the participants in relation to their of past experiences. However, almost no participant made explicit reference to mathematical ability (possibly in relation to the fact that they never had the occasion to reflect on this issue in their previous educational experiences) and only scattered remark were made concerning their future profession. In view of this, in the second day of the course the lecturer asked the participants to engage in a group discussion concerning mathematics, mathematical ability, and

schooling. Particular topics proposed by the lecturer were if mathematical ability is innate or not and what type of mathematics can and should be taught in relation in particular to disabilities. The participants were encouraged to discuss in groups first and then expose individually their views to the class. Then the lecturer asked the participants to write a follow-up focus essay entitled "What is to me mathematical ability". Again, the participants wrote individually, for about 40 minutes.

Overall, the previously thematically analyzed essays together with the focus essays were further analyzed together by means of a (Foucauldian inspired) discursive analysis, as employed in second part of the methodology of Beccuti et al. (2024). Following these authors, we were not interested in tracing the experience of single individual participants, but in understanding the prevailing *ethos* dominating the chosen group on the selected topics in relation to our research question. In particular, we attended to how the previously selected views of mathematics on utility, challenge, comfort, and relationship to others were articulated with respect to each other and with respect to the participants' articulated notion of mathematical ability. As Walshaw (2016) explained, (Foucauldian-inspired) discourse analyses focus on understanding how the participants' writing reflects their shared understanding of the topics, their positions within the discussion, and the ways their discourse shape their collective knowledge and experience, trying to trace specific *technologies of the self* that the participants express, i.e., ways that articulate and shape how they are perceived by themselves and wish to be perceived by others in the particular and general environments in which they operate. This is in coherence with our theoretical framework, according to which discourse serves as a vehicle for establishing shared norms and truths, determining what can be said and who can speak, what can be seen and what can be done, thus directly influencing how individuals and groups perceive themselves and others with respect to a chosen topic and context (Foucault, 1977). The overall aim of the analysis was to furnish a comprehensive picture of the participants' shared views of mathematics and mathematical ability as stemming from their past and present relationship as well as future association with institutional mathematics in school.

Results

Most of the participants open their initial essays by stating that they deem their past relationship with mathematics as overall not positive or even plain negative. Nevertheless, many participants report a neutral or even good relationship with mathematics in the first part of their schooling experience, often in connection with the positive role of a teacher or in view of its perceived utility in everyday life. Typically, the mathematics of primary or middle school is described as associated with a sense of satisfaction stemming from finding the solution to a problem or from getting some computation right either in the classroom or outside.

> […] I remember that teacher who gave us homework […] [that] I was happy to please.
>
> I could employ it […] for instance at the supermarket where […] I loved to make all ends meet [far quadrare i conti] and have the situation under control

However, the relationship of the participants with mathematics progressively deteriorated, typically from middle or high school. This is explained either with the encounter of a not favored teacher or with the perceived mounting abstractness of the discipline.

> Up until eleven years old, mathematics was a lovely friend, a subject taught by a lovely teacher who could make you passionate; at middle school it became hell, an impossible obstacle […]
>
> […] at high school I lost interest and motivation […] due to a teacher who had little patience.

The participants thus describe a general feeling of discomfort which presents itself each and every time they had to take a mathematical test, often perceived challenge of insurmountable nature almost invariably associated with anxiety and even despair.

> Frustration, inadequacy are just few of the emotions that I associated with this subject […].

Overall, this negative view is justified by many as an general humanistic or artistic disposition which they would possess and to which mathematical achievement seems to be opposed.

> For those who like me have a humanistic disposition [un'indole umanistica], mathematics has always been a too sterile concept without any passion
>
> I feel more affinity with humanistic/artistic subjects where rationality is a bit less important

This is reflected in the view the participants present of mathematical ability, which seemingly some individuals would possess inherently more and others inherently less.

> I always loathed having a more developed logico-mathematical intelligence. I admire who is able in mathematics and I envy them.
>
> Personally, I don't think I have much mathematical ability […]

The participants view mathematical ability as connected to the ease in dealing with numbers and being fast with computations, as well as with to the general ability of solving problems, thinking logically and in general possessing intelligence.

> Mathematical ability represents Intelligence with a capital I […] who possesses mathematical ability is "superior", intelligent, but also mysterious and difficult to accost [inavvicinabile]

As to the problems of the origin and the development of mathematical ability, very few students endorse a completely genetic or "nature" explanation:

> either you have it or you don't
>
> There are people who are more and people who are less apt to do mathematics, so that even reiterate and continuous training could not bring to large improvement, despite the effort and the time employed […]

Noticeably, none of the students however endorse a completely "nurture" explanation of mathematical ability, but more typically state that it is partly innate and partly acquired, without resolving in favor of either option. Overall, the majority of the participants express a positive view of the possibility of nurturing mathematical ability (while overall insisting on the presence of a strong innate component) via proper instruction.

> It is 50 per cent nurtured and 50 per cent innate.
>
> It can be innate but you can also acquire it by experience or in other ways which can be learnt […]

> I think there is […] a completely innate factor [to mathematical ability], but this doesn't mean that it cannot be developed in one's life with exercise.
>
> It is not in my opinion an innate ability but it has to be learnt and exercised for long […] this of course only if cognitive deficit is not present.

Going back to the personal stories of the participants, many affirm that, years after they left compulsory schooling, and especially as a consequence of their recent direct experiences as teachers, they came to terms with their past bad experiences with mathematics. Indeed, eventually they came to appreciate mathematics and its overall utility, to an extent, in everyday life, in modelling scientific phenomena, in computing applications, as well as, more generally, its role in disciplining the mind towards abstraction and exactness (and to its indirect use to access future degrees or professions).

> Afterwards, things changed. I understood the importance of mathematics and of its formulae for describing reality […]

A large majority of the participants reflects on the fact that if they will be called to teach mathematics or to support its teaching, they will in particular attempt to present it in a way that is not too detached from concrete applications as well as

> […] suitable to each student's intellectual and cognitive level.

Indeed, the participants also highlight that care should be used in also conveying overall motivation towards mathematics and stress that teachers should make use of innovative inclusive methodologies (often mentioned in connection to pedagogical theories learnt during their training as support teachers) and methods that may be proper to various levels of intellectual abilities and styles of learning.

> Some pupils can only reach procedural knowledge […] while for others perhaps it is possible to do more by means of innovative methods stressing on everyday practices […]
>
> [the realization that mathematical ability is partly innate] allows me to understand the level to which everyone can reach and that with which I can start with pupils.

Conclusions

As seen, the participants almost unanimously describe mathematics as an important, satisfying, and variously useful form of activity. However, they report either a mostly negative personal experience with mathematics or an initially neutral or positive experience which later deteriorated, typically in connection to a bad experience with a teacher in middle or high school, or else in connection with its growing abstractness. This is articulated as stemming from an overall lack of mathematical ability or an overall non-mathematical disposition of the participants themselves, presented as a general explanation of their failure. Interestingly, the participants see mathematical ability as partly innate and partly acquired, typically favoring a "median" position in the nature-nurture dilemma concerning it. This stance does not appear to derive from previous reflections or dedicated training on the topic, but—we may conjecture—appears to be a discursive point of compromise serving to make sense simultaneously of their overall life story with mathematics, of the progressive commitment to inclusion traditionally ingrained in the working of the Italian system of special education, as well as

of their studies on general pedagogy and cognition. Indeed, this stance on ability may be read as inherently apt to be instrumental to their whole professional endeavor, constituting a particular technology of the self that these participants attempt to enact and which, in a sense, best discursively underlies their specific professional existence as a separated category of teachers.

Generally speaking, the participants appear to subscribe to progressive educational theories of hope (Nieminen et al., 2024), underlying the potential of practices that emphasize diverse mathematical intelligences and contextual learning. This could go in the direction to open up spaces of intellectual and practical challenges to established norms in their future work as support teachers and may help in reframing mathematical ability to include broader or more inclusive definitions of competence and intelligence that challenge the traditional understandings of these notions, as well as, overall, the intrinsically gatekeeping role of mathematical instruction in school and in society. At the same time, nonetheless, the participants' views on mathematics and mathematical ability could fundamentally work in the direction of reproducing systemic disparities, despite the participants' best intentions.

Indeed, the aforementioned partial "innateness argument", coupled with the articulated discursive connection of mathematical ability to general problem solving ability and general intelligence, as well as their overall endorsement of the universal importance of mathematics, could reinforce a view of mathematical ability and mathematical "disposition" as (at least partially) fixed once and for all in an objective sense (both with respect to a universal ideal culture and with respect to an idealized received idea of human cognition). These discursive stances could serve to indirectly reinforce the usual views on the impossibility (or extreme difficulty) of students categorized as disabled to participate in full to school mathematics as well as strengthen the insistence on the connected usual remedial practices focused on a modified, less cognitively challenging, and procedural version of the discipline (cf. Tan et al., 2019).

As said above, the Italian system concerning special education is characterized by a long-established commitment to inclusion which revolves around subject-specialized teachers and generalist support teachers. The system is generally believed to have led to overall improvement impacting positively on the inclusion of special needs students, both in terms of learning and socialization as well as in terms of producing a generally more widespread adoption of diverse and/or innovative teaching methodologies (cf. D'Alessio, 2010). However, as seen, the discourses articulated by support teachers on mathematics and mathematical ability could implicitly foster the institutional concealed reproduction of ableist practices and narratives (somewhat paradoxically conveyed via the words and practices of support teachers themselves). This possibility could evidence a criticality within the Italian system with respect to mathematical instruction, if taken together with the participants' general personal negative disposition towards mathematics and their typical lack of post-secondary education in mathematics. Indeed, as said, for many of the participants the present laboratory is the only post-secondary course specifically devoted to mathematics that they follow before being certified as generalist support teachers. It remains an open question whether perhaps other discursive positionings concerning mathematical (dis)ability might be fostered by alternative and less generalist forms of support teacher training which could be imagined as devoting more time on developing mathematical competences per se as well as reflecting more and more in depth on the nature of mathematics and of mathematical ability.

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Mathematical empowerment and participation in inclusive learning settings

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In inclusive education, all children should work together in their diversity and develop their individual potential. With respect to mathematical difficulties, there are already numerous concepts for inclusive classrooms. In view of the many facets of diversity, however, further concepts and empirical studies are needed, e.g. in the context of mathematical potential or social skills. Social difficulties can overshadow the mathematical potential of children so that they are often unable to show or develop their full individual potential. As part of a larger research project, cooperative learning environments[1] for inclusive teaching have been developed to promote the mathematical potential and social participation of all learners. The article analyzes the extent to which social participation can be developed in a mathematical negotiation process using a case study. On this basis, possibilities of mathematical empowerment in three different domains are discussed.

Keywords: Mathematics education, inclusive education, diversity, empowerment, participation.

Introduction

"*Dustin needs tasks. And I think that's what he's missing in his primary school years and in his biography, that he gets tasks and can show himself*." This quote was made by Dustin's class teacher who was interviewed as a part of this study. The teacher describes Dustin's family background as "difficult". Dustin has a single mother and "difficult circumstances that don't necessarily support him". He was taken in relatively early by his grandparents, where he more or less grew up. In grade 3, custody was withdrawn from the grandparents and transferred back to the mother. This was followed by several changes of residence. During the interview his teacher reflects on what Dustin has already been through and summarizes his situation as follows: "From an early age, Dustin was left to his own devices and had to bite and fight his way through." From the teacher's point of view, this is expressed by the fact that Dustin often attacks other children while playing during breaks, when he doesn't agree with their rules or the way they play. In primary school he was not good at dealing with frustration, for example when losing in competitive games, so that "the whole thing turned into slightly aggressive behavior". Dustin makes threats to classmates and teachers and has to learn to resolve conflicts only verbally and be more considerate to others. In addition to the description of the difficulties in social interaction, the teacher opens up a remarkable perspective in his quote, shown above, to strengthen Dustin: "*Dustin needs tasks* […] *that he can show himself*". From his point of view, such tasks are missing, to help Dustin to show his skills in learning, and in social classroom or school interaction. In this paper the case study 'Dustin' is used to investigate in how far a learner with

[1] In this paper we understand a *learning environment* as a framework for several tasks based on a common mathematical structure (Hirt & Wälti, 2022).

social difficulties can be empowered by a special task design, so that he is not excluded but can participate in joint social and subject-related learning.

Empowerment

The concept of empowerment is characterized by the experience of discrimination and the withholding of participation (Theunissen, 2016). After the concept was introduced into pedagogy, the aim was to empower people through targeted pedagogical efforts so that they gain power and become independent of support. Ernest (2001, p. 123) describes Empowerment as "the gaining of power in particular domains of activity by individuals or groups and the processes of giving power to them, or processes that foster and facilitate their taking of power". Empowerment is aimed, among other things, at a strength that is based on the person and their individual resources, at addressing social disadvantage and discrimination, which should be overcome, and at the professional task of strengthening the vitality of people in a socially marginalized position (Theunissen, 2016, p. 114).

With regard to the empowerment of children in everyday school life, the central question is in what way empowerment can be encouraged and supported by teachers in order to offer help for self-help and at the same time enable the highest possible degree of self-determination (Lindmeier & Meyer, 2020). For this reason, the specific requirements of the learning subject must be taken into account: For the subject of mathematics the achievement of both content-related and process-related mathematical competencies and the understanding of the role of mathematics in everyday life can be understood as contributions to empowerment concerning mathematics. If children and young people are empowered in these, this not only has an impact on their school life, but also on their opportunities to participate in society. Ernest (2001) distinguishes three domains of empowerment concerning mathematics and its uses: (1) The *mathematical empowerment* includes the "Power over the language, symbols, knowledge and skills of mathematics and the ability to confidently apply this in mathematical applications within the context of schooling, and possibly to a lesser extent, outside of this context" (Ernest, 2001, p. 126). (2) When mathematics contributes to participating in a social domain, e.g. work, life and social affairs, it is called *social empowerment*. (3) An *epistemological empowerment* includes the development of personal identity and the growth of confidence in the mathematical and social senses (Ernest, 2001). It becomes clear that these three domains should not be understood separately from each other, but that the aim of mathematics teaching is to empower learners as epistemological agents, by encompassing mathematical, social and epistemological empowerment. For mathematics teaching this means that all children should be empowered to develop their individual mathematical potential using their individual strengths (mathematical empowerment). Therefore, children need needs-based resources in the form of adaptive individual access to the subject of learning combined with challenging mathematical negotiation opportunities. New mathematical knowledge is primarily developed through collaborative, subject-related negotiation processes (Häsel-Weide & Nührenbörger, 2022), so that the ability to learn socially from and with each other always has to be taken into account (social empowerment). The aim should be to offer learners adaptive access to the learning content in a way that they can engage in mathematical activities at their individual level, that challenges them, but does not make them dependent on help. By empowering individual potential, the genuine participation of all children becomes possible and can strengthen confidence in their own subject-specific and social skills (epistemological

empowerment). This orientation corresponds to the idea of inclusion, that aims to recognize and support all learners in their individuality.

Inclusion

In an inclusive education system, all persons should have the chance to develop their individual potential and should be perceived, accepted, and valued in their diversity and individuality. Thus, the goal of inclusive teaching is to support all students in the best possible way (Ainscow et al., 2006). That's why the potential of all students is increasingly focused in academic and educational policy efforts (Dexel et al., 2019). Whereas in Germany, concepts for inclusion, e.g. in dealing with *learning difficulties*, are already established in everyday teaching, many concepts to support *mathematical potential* are designed rather exclusively (e.g. children's university) and offer few possibilities to cooperate, communicate and learn with and from each other (Billigen et al., 2023). Similarly, concepts for dealing with *social difficulties* tend to be exclusive rather than inclusive (e.g. Durlak et al. 2010). However, such exclusive concepts of support do not really correspond to the idea of inclusion. The present study refers to a perspective of inclusion and empowerment as an interplay of joint and individual learning processes. Every learner should develop his or her individual mathematical potential through adaptive mathematical learning opportunities as well as social participation (Krähenmann et al., 2019). In order to promote mathematical potential in inclusive learning settings and at the same time enable the social participation of all children, it is important that learning settings open up an individual access to the subject as well as joint negotiation about mathematical insights. Current studies show that processes of cooperation on a common subject in heterogeneous learning groups are particularly valuable for mathematical learning (Häsel-Weide & Nührenbörger, 2022).

Cooperation and participation

The diversity and heterogeneity of learners can benefit the insights of all so that the joint engagement with a common subject can be fruitful. Therefore, it is necessary that all children *receive the same task*, which is holistic and sufficiently complex, so that it opens up *learning opportunities* for children *on different levels*. By working on a common task, a meaningful discussion about individual strategies and insights can take place. This initiates productive situations of learning with and from each other. In the German didactics of mathematics, the concept of "natural differentiation" is a well-explored concept to reach this (Scherer & Krauthausen, 2010). The children are not restricted by the teacher, but get empowered according to their individual potential concerning mathematics. Thus, this concept offers many advantages for dealing with the diversity of learners. However, looking at inclusive education from a wider perspective that includes particular mathematical difficulties as well as mathematical potential, natural differentiation is not always sufficient to create learning situations for all children. The concept of "parallelization" (Nührenbörger & Pust, 2018) builds on the idea of natural differentiation, but it can offer broader possibilities to realize the demands mentioned above. Due to target-differentiated levels, all learners can be given individual access to the subject of learning. Likewise, this opens up a variety of possibilities for a fruitful communication between the learners. They can contribute their discoveries, which they have obtained at the respective levels, so that social participation is guaranteed for all (Nührenbörger & Pust, 2018).

Methods

The research project aims to promote and empower the individual mathematical potential of all learners even if it is not visible due to social difficulties. To achieve this, parallelized learning environments in combination with natural differentiated tasks have been developed in order to initiate social participation as well as individual learning processes. To answer the research questions, the method *case study* (Lamnek, 2010) is chosen, which allows to identify typical aspects concerning social difficulties, participation and empowerment to reconstruct a holistic picture of the case.

RQ 1: How can the *social participation* in the cooperation between Dustin and Lenny be characterized?

RQ 2: To what extent can initial references to the three *domains of empowerment* be reconstructed?

Before the results are presented, the learning environment and the analysis method is described.

Learning environment "Magic Squares"

The presented learning environment focuses on Magic Squares (MS). MS offer opportunities to discover and explore rich mathematical structures at different levels, which make them suitable for both natural differentiation and parallelization. In a MS the sum of each row, column, and diagonal is equal; it is called the magic number (Figure 1). In MS of odd-order the magic number is related to the odd-order and the mean number (e.g. in a 3x3 MS the magic number is three times the mean number). The current study is an ongoing PhD project, which explores two different parallelized learning environments. In this paper we refer to the learning environment MS, in which so far 31 children (aged 9 to 10 years, grade 4) were videographed working on the different tasks. The researcher also takes the role of the interviewer. The learning environment is divided into three sequences (Figure 2). In the first sequence, children work individually in a naturally differentiating learning environment to become familiar with MS, and to explore first structures. The researcher then forms pairs of students on the basis of the students' outcome from the first sequence. The target-differentiated classification enables adaptive individual promotion and empowerment of the individual potential of all learners. According to the concept of parallelization, some pairs work at level A and other pairs at the more complex level B in Sequence 2. Children working on level A explore relations between mean and magic numbers as well as systematic variations of numbers in 3x3 squares. Children working at level B focus on mathematical structures of different MS of size nxn, so they are challenged to find MS of different sizes (3x3, 4x4, …, nxn) with the magic number 90. Especially the comparison of odd- and even-order squares offers substantial discoveries. In Sequence 3, the students work together in new pairs composed of one learner each from level A and B (Figure 3).

Figure 1: Magic square (3x3)

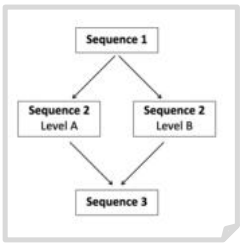

Figure 2: Structure of the learning environment

Figure 3: Task in Sequence 3

Both learners can participate in working on this new research task by applying knowledge gained from previous work on level A or B in sequence 2.

Method of analysis

For the research project, the *participation analysis* (Krummheuer, 2015) represents a substantial analytical tool to reconstruct the autonomy of actions of the learners in the interaction with a social perspective on participation. In the concept of participation analysis, four functions of active speakers can be distinguished with regard to whether and to what extent (a) a *new content* is introduced and (b) this content is introduced with *new formulations* (see Table 1): (1) An "author" expresses his *own new idea* of a content in his *own new words*. (2) A "ghostee" introduces his *own new idea* of a content, but adopts the *formulation of a previous utterance*, whereas (3) a "spokesman" *reproduces the content* of a previous statement in his *own new words*. (4) A "relayer" repeats a *previous content* by using the *same formulation*.

Table 1: Participation analysis: Function of active speakers

Function of active speaker	New Content	Own new formulations
Author	+	+
Ghostee	+	-
Spokesman	-	+
Relayer	-	-

The participation analysis can show the responsibility of each learner for the *development* of the *content* and of the joint *social negotiation*. In the following, a short excerpt from the empirical data is presented, and analyzed using the "participation analysis" (Krummheuer, 2015). Building on this, references to the "domains of empowerment" (Ernest, 2001) are elaborated in the discussion.

Results – Case study 'Dustin'

In Sequence 1, Dustin worked on the task individually and with great interest and made some challenging mathematical discoveries. For this reason, he should work on the more complex level B in Sequence 2, together with his classmate Lenny. Before Sequence 2 begins, Dustin's school break is again characterized by social conflicts. Dustin gets into a fight with other children and, therefore, is late for Sequence 2. He enters the room, sits down besides Lenny, but then very quickly begins to immerse himself in the new discovery task: "*Find magic squares with the magic number 90. How do you do it? Give reasons together.*"

After the boys have found a 4x4 MS with Magic Number 90, they try to use the colored impulse card (Figure 4) to explain how it is possible to find such a 4x4 MS. Dustin thinks about what to write down and presumably refers back to the previous procedure by saying "We have" (turn 1). However, he is interrupted by Lenny (turn 2):

1	Dustin:	What should we write? We have#
2	Lenny:	#Wait, look, that is one, two, [*taps on light yellow fields*] one, two [*taps on light blue fields*].
3	Dustin:	Yes, I wanted to say, [*raises the impulse card (Figure 4)*] here. We write, of each color we counted together and looked, that the result was forty-five. And then when we already had three [*taps from top to bottom on*

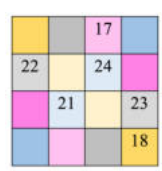

Figure 4: Impulse card

		the first three fields in the left column from their last previous attempt, in which they found a 4x4 MS with Magic Number 90]#
4	Lenny:	#Not forty-five!
5	Dustin:	But it is!
6	Lenny:	The half of the number. Fifty-five ehm forty-five is a half of ninety. And I can already guess that the next one we have to do, if we do a seventy MS next time, I think the#
7	Dustin:	#Thirty-five.
8	Lenny:	We have to calculate thirty-five. I can guess that.
9	Dustin:	That's right, it certainly is.

The participation analysis can now be used to analyze the autonomy of actions and the type of mutual references between the two children on a content-related and social level. For this purpose, the extent to which and by whom a new content is introduced and how this content is continued or rejected is analyzed in each turn:

In turn 2, Lenny is the *author* as he contributes his own idea in his own words. He refers to two fields of the same color (light yellow, light blue) by pointing to them on the impulse card (Figure 4). Dustin responds as well as an *author* by introducing the use of colors to form the sum 45 as a new idea in his own words (turn 3). His wording indicates that he is following the previous intention to write down their done procedure ("We write"). Therefore, he possibly tries to name a strategy of their previous procedure, as he points to three numbers in a column in their found 4x4 MS ("And then when we already had three", turn 3). In general, it would be possible in this way to determine the missing number using the magic number 90. However, Lenny interrupts him again. Lenny then takes on the role of a *ghostee* with linguistic reference to Dustin ("Not forty-five!") by introducing a new content (turn 4). He denies that the sum is 45 and uses Dustin's words for that. At this point a dispute arises (turn 4 & 5) and Lenny seems to have a request for clarification.

The children's statements in the transcript suggest that Lenny and Dustin are arguing on two different levels. Dustin continues to describe the *specific procedure* and states that they first determined the sum of 45 in the two fields of the series (turn 3). Lenny's statements, on the other hand, indicate that he would rather describe a *general structural relationship*: the sum of two fields in a row must always equal half of the magic number (turn 6). As a *spokesman*, Dustin refers to his own statement from turn 3 by reproducing its content in new words. These different levels of argumentation lead to a mathematical controversy between the two boys, which does not cause the breakdown of social cooperation, but leads to a meaningful clarification: In turn 6 Lenny acts as an *author* with reference to the controversy by introducing new content in his own words. To explain his statement, Lenny uses the magic number 70 as another example. Dustin takes up this new idea from Lenny and continues it as an *author* as well by introducing a new content: the sum of the pair of numbers in this case would be "Thirty-five" (turn 7). Dustin is referring directly to Lenny and develops an ambitious new idea further. As a *relayer* in turn 8, Lenny is taking content from his own statement (turn 6) as well as words from Dustin's statement (turn 7). Dustin confirms the joint discovery in turn 9 as a *spokesman* by reflecting the content of turn 6 to 8 in his own words ("That's right, it certainly is"). At this point, it is not yet possible to say whether Dustin has gained any real *structural* insight. However, it becomes clear that the children interact and cooperate with each other honestly on both a mathematical and social level, despite the dispute that arises.

Discussion

The analysis of the case study provides initial indications that both children were given the opportunity to develop mathematical insights through joint negotiation, regardless of social difficulties outside of mathematics lessons. Concerning RQ 1 the numerous roles as "author" can be interpreted as an indication that the boys participate autonomously, because both Lenny and Dustin introduce new content with their own words. This leads to a deep mathematical communication and social participation in which references are made to each other's statements.

Because of the special design of the learning task (individually adaptive access and meaningful cooperative discovery task), both boys worked on a mathematically very demanding task and developed appropriate solution strategies together. The scene presented above indicates initial references to the three domains of empowerment (RQ 2): The boys manage to focus on new mathematical relations in the MS beyond summarizing their approach (*mathematical empowerment;* Ernest, 2001). These new mathematical insights are part of the social discussion. The participation analysis shows that both learners repeatedly make reference to previous statements made by the other, but develop them further in their own words and by introducing new ideas into the discourse. Even when a dispute arises, the discourse does not break off, but leads to new insights. This is remarkable because for Dustin it is often difficult to accept other opinions or rules in social situations (see interview above), which leads to conflicts with his classmates during breaks. The scene presented here provides initial indications that in this mathematical context Dustin is able to withstand both interruptions (turn 2) and counterarguments (turn 4-6) and to use them productively for his own insights (*social empowerment*; Ernest, 2001).

At this moment, further studies with additional case numbers are required to investigate and understand these initial indications in more detail. In addition, the extent to which empowerment on a mathematical and social level can contribute to strengthening children with social difficulties in such a way that the confidence in their own mathematical and social abilities is also strengthened (*epistemological empowerment*; Ernest, 2001) needs to be investigated.

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Creating a task to prepare instructors to teach postsecondary mathematics

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We present considerations for the elaboration of a Professional Learning Task (PLT) within the context of preparing mathematics instructors in higher education, focusing on an innovative curriculum that uses technology as a leverage for promoting social justice. We describe the Brazilian context and how instructors are typically trained to teach within this context. Then, we present our study, grounded in the Professional Learning Opportunities for Teachers (PLOT), exploring ideas on how teaching practices can be racialised and/or gendered. We discuss how a PLT must consider elements that integrate mathematical knowledge with its teaching and that aim to challenge mechanisms of oppression affecting traditionally marginalised students in mathematics courses.

Keywords: Teacher preparation, postsecondary mathematics, racialised mechanisms, gendered mechanisms.

Introduction

The aim of this paper is to present considerations for the creation of a Professional Learning Task (PLT) within the context of a professional development of higher education instructors[1] in engineering programs and other science and technology fields (STEM) in an innovative curriculum that uses technology as a lever for promoting social justice in southern Brazil.

This work is set within two main contexts. The first is the need to prepare instructors to teach in higher education, because postsecondary instructors usually learn to teach through the act of teaching itself, without engaging in the thinking, study, reflection, or discussion of the complex task of teaching (Isaia, 2006). The few initiatives that are beginning to be implemented usually focus on higher education pedagogies, disregarding the important connection between the specific area to be taught and its pedagogy, which has been explored in research on Teacher Education since Shulman's seminal studies (1986). Thus, this paper contributes to the discussion of the training of higher education professionals for teaching, within the contemporary precepts of research in Mathematics Education, taking into account the relationship between the content to be taught and its pedagogy.

The second context relates to the social function of public universities in Brazil. We are looking to policies that increased the number of postsecondary institutions, their distribution across the national territory, and the increase in available seats in all public postsecondary institutions. However, this

[1] In Portuguese, the word *professor(a)* is used for *teacher* and *professor* interchangeably. Some of the references that guide our work are in Portuguese and the term is quite ambiguous, it could mean instructor, or professor, or teacher, or any of those in a broader sense. Whenever broader meaning is implied, we will use the term teacher, following the reference. We will use the term *professor* to indicate a tenured track faculty member, and *instructor* to anyone who teaches any postsecondary mathematics course, as it can vary significantly across the world.

expansion has not been accompanied by appropriate actions and policies to support students to complete their undergraduate courses (Paula, 2017). In other words, there has been some democratisation of access to higher education, with traditionally marginalised students beginning to enrol in greater numbers. However, their enrolment has not been supported by measures to ensure they can succeed and graduate. Given the gatekeeping role that mathematics courses play in courses in STEM fields (Bressoud, Mesa & Rasmussen, 2015), these students should not be blocked by these courses. Moreover, beyond merely overcoming these courses as obstacles, it is important that these students can learn and use mathematics as a powerful tool in building a more just society. Therefore, higher education instructors' training must consider how the content to be taught also interacts with dynamics of society, including power systems and social oppression. In particular, in this work, we intend to advance in this area by focusing on racialised and/or gendered mechanisms occurring through postsecondary mathematics teaching practices as described by Leyva et al. (2020).

We planned a professional development session based on the Professional Learning Opportunities for Teachers (PLOT) model, and, in this initial stage, we present our reflections on the question: *what are important considerations with respect to planning a PLT to support higher education instructors to unpack gendered and racialised mechanisms within mathematics instruction?* We first present an overview of the public universities in Brazil and the training of instructors in this context. Then, we briefly describe the curricular proposal within which this training is embedded. At our university, this proposal is in the initial stages of approval and has been deemed viable in a preliminary instance for nine undergraduate engineering programs; thus, preparing instructors is a necessity. We then present the study itself, including the theoretical perspective, methodology, data, and discussion.

Teacher preparation in higher education contexts

Public universities in Brazil have been undergoing changes that alter their social role and raise questions about the practice of teaching in these universities and even in society (Isaia, 2006). We are concerned with the growing admission of traditionally marginalised students into STEM fields, which are predominantly seen as white and male (Leyva et al., 2020). Teaching at universities should support these students to persist in and complete their courses and to take the lead in processes and mechanisms that challenge social oppression. Thus, the training of higher education instructors, with the aim of addressing specific demands related to this university social role, is pivotal.

Training for instructors in higher education seems to occur alongside teaching, throughout their professional careers, and depends on situations and experiences that are not usually shared with others nor are they the result of research that verifies their effectiveness (Isaia, 2006). Additionally, "the idea that those *who know how to do something know how to teach it*"[2] has supported the logic of recruiting teachers (Cunha, 2009, p. 84), contributing to a culture of overvaluing research over teaching within the teaching-research-extension tripod, which is the core foundation of academic activity in universities in Brazil. Such overvaluing of research is reported in various studies and

[2] Translation by the authors from the original in Portuguese: ideia de que *quem sabe fazer sabe ensinar* deu sustentação à lógica do recrutamento dos docentes. (Emphasis in the original.)

evidenced, for example, in many teacher recruitment and career progression systems (Isaia, 2006). As a result, instructor training initiatives are also undervalued, and postsecondary instructors continue to lack specific training for teaching. When such training does exist, it is often the result of individual initiatives rather than public or institutional policies (Cunha, 2009; Sordi, 2019; Isaia, 2006).

In our institution, there are institutionalised instructors training initiatives, although most of these are of a general pedagogical nature. Such initiatives are beginning to emerge in some higher education institutions: Cunha (2009) describes a group of instructors from various fields seeking pedagogical training, which is also generalist in nature. If certainly a step forward in higher education training, since Shulman's (1986) seminal work, we have recognised that pedagogical training cannot only be conceived as separate from content. Therefore, we propose institutionalised training initiatives for instructors, which considers the specific training for teaching mathematics to promote social justice.

The proposed curriculum

The curriculum we are proposing consists of a sequence of mathematics courses that may appear identical to those in a traditional curriculum (differential and integral calculus, numerical calculus, differential equations, and linear algebra), the pedagogical approach and organisation, however, differ drastically. Essentially, there is a pedagogical shift from algebraic manipulation to understanding the mathematical concepts with the development of technological fluency or, even more ambitious, a technology-based and student-centred curriculum focused on conceptual understanding. We propose using technology as a tool to overcome algebra manipulation as a barrier to access deeper conceptual understanding and refocus the calculus program from merely procedural fluency to critical understanding and problem-solving.

We argue that dedicating a large portion of calculus courses to solving intricate or tedious algebraic calculations can have consequences such as reduction in the time available to explore real-world applications, and limited opportunities to interpret the results. The most dangerous implication, however, is the shift in focus on what is relevant in a calculus course. If the algebraic manipulation receives more emphasis than the interpretation of results and the solving of real-world problems, then it is the algebraic manipulation that will prevail as the more important component of the course.

Conceptual framework

This work considers the PLOT model (Ribeiro & Ponte, 2020), represented in Figure 1, as a theoretical framework for designing teacher professional developmental programs, with the goal of providing teachers with professional learning opportunities (PLOT). In this model, such opportunities are provided through the articulation of the Teacher Educator's Role and Actions (TERA), the Discursive Interactions between Participants (DIP), and the Professional Learning Tasks (PLT). In this study, we focus on the design of a PLT, therefore, TERA and DIP are considered as important planning design elements for the PLT. In particular, when necessary, we use the Mathematical Knowledge for Teaching (MKT) model proposed by Ball, Thames, and Phelps (2008) to interpret the connection between the design of a PLT and the other domains of the PLOT model.

However, because the PLOT model does not explicitly address any kind of power dynamics involved in the classroom and we want to foreground our work in the gendered and racialised relations

intertwined in mathematics postsecondary instruction, we are incorporating into this framework elements to bring such issues to light. To do so, we articulate some descriptions of how women and students of colour can be excluded from participation of mathematical practices and discourse from Leyva et al. (2020) with how micro-affirmations, proposed by Cawley and Wilson (2024), can disrupt some patterns in racialised and gendered mechanisms in postsecondary mathematics education. A few pieces within Brazilian context, such as Silva and Powell (2017), suggest that racialized and genderized experiences, even if not manifested in exactly the same way as in Leyva et al. (2020), are equally prevailing and persistent within mathematics instruction. We are grounding our work in Leyva et al. (2020) to use their more detailed descriptions about how such mechanisms occur, and Cawley and Wilson (2024) to glimpse about how to challenge them.

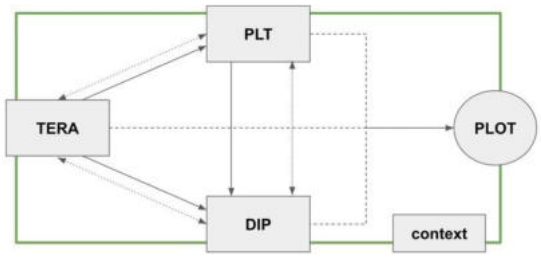

Figure 1: PLOT Model with acronyms translated from Portuguese to English (Ribeiro & Ponte, 2020)

Data collection and analysis

For the mathematical task, we chose the topic of direction fields, as this topic addresses the broader concept of rate of change, which runs throughout the curriculum, from introductory to advanced mathematical courses in engineering programs. We adapted the fall of a body problem, described by the initial value problem $v' = 9.8 - 0.2v$, with $v(t_0) = v_0$, in which the velocity $v = v(t)$ is a function of t, from a traditional differential equations' textbook, raising the level of cognitive demand of the task by shifting the focus of simply solving the equation to connecting different representations related to the concept of rate of change in a differential equation. We incorporated technology accessible to the general population as means to generate one of such representations. Then, we obtained records of practice to include in the PLT by running a workshop to undergraduate students. We journaled our own practice and collected students' solutions of the mathematical task.

In our analysis, we followed a qualitative approach seeking first to capture possible racialised and gendered mechanisms within postsecondary mathematics instruction, that in our case, should be reflected in the PLT. Additionally, how such mechanisms could relate to the PLOT model.

Raising and unpacking potential PLT questions

According to Leyva et al. (2020), Black and Latin* students have fewer opportunities to participate in mathematics classes, instructors often overlook such students in class, not offering support when they need but, specially, not accrediting knowledge whenever they possess it. This kind of practice, fuels and is fuelled by discourses about who knows mathematics. There are not many studies that describe such mechanisms, and even fewer that tries to unveil how to disrupt them. Cawley and Wilson (2024) are one of the few attempts to do so. Drawing from them, showing genuine interest in the students' thought process can be seen as a direct counteraction from what was described by Leyva

et al. (2020). Therefore, it is important to take students thinking into account; in fact, it is particularly important to consider traditionally marginalised students thinking. To achieve such goal, it is, then, necessary to incorporate questions in the PLT that asks instructors, for example, to consider student thinking, to interpret student thinking, to incorporate student thinking in the unfolding of the class.

Using our example, the first question of the mathematical task was filling a table with data from the differential equation $v' = 9.8 - 0.2v$ for some values for v. In a PLT, after initial engagement with the mathematical task itself, instructors can be encouraged to anticipate possible responses from students in a question number (1), creating an opportunity to develop MKT (Ball, Thames, & Phelps, 2008) by connecting the mathematical task to expected student responses. To further this discussion, it is possible, using data collected from the workshop, to incorporate the following question into the PLT: "(2) Considering the following student solutions in Figure 2, how can you interpret them? In other words, do you know how each student filled in their table? What leads you to think that? What questions could you ask to confirm your interpretation?" To create additional opportunities for developing MKT (Ball, Thames, & Phelps, 2008), the following question can be added: "(3) Student C, when asked why they chose these values for the table, answered that they had chosen values close to the target, which in this case meant reaching the ground, with the velocity being zero. Does this answer confirm your assumption? How can this answer modify/refine your assumption? How can this knowledge about the student's thinking inform decisions about teaching this problem/topic?"

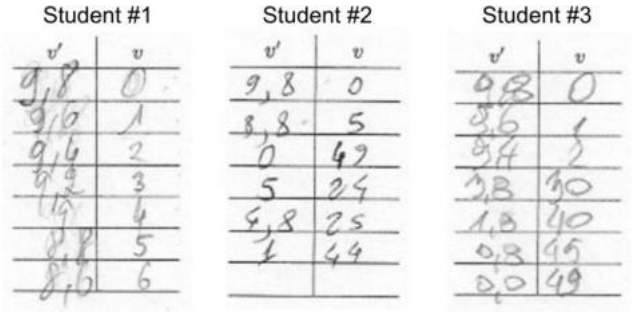

Figure 2: Student's answers.

Moreover, these exact same questions, (1), (2), and (3), when analysed from the perspective of Leyva et al. (2020) and Cawley and Wilson (2024), acquire additional meanings. When directed towards Black and Latin* students, carefully examining the table and listening to the student's reasoning might counteract the action of overlooking their mathematical ability and, consequentially, the distribution of mathematical knowledge altogether. The incorporation of such questions in a PLT needs to be carefully accompanied by TERA to bring such perspectives to light.

Moreover, Leyva et al. (2020) argue that mathematical knowledge is often unequally distributed across race and gender in initial postsecondary mathematics courses. The authors claim that different kinds of instructional events allude exclusionary ideas about who belongs to mathematics and STEM communities. In particular, Cawley and Wilson (2024) assert that leaving part of the work to students with typical assertions found in textbooks ("it is easy to see that", "it is obvious that", or "the proof is left to the reader") can contribute to students feeling a lack of belonging if they cannot see the triviality mentioned by the teacher. Additionally, Leyva et al. (2020) do not detail the event *easy problem*, but

report students who identify differential opportunities for classroom participation and instructor support based on it, which can be associated with student failure to recognise a deemed trivial mathematical fact. To disrupt stereotypical ideas about who can know and do mathematics, Cawley and Wilson (2024) suggest clearly communicating to students they are working in something that is difficult, naming what is difficult about the task and believing students can do and solve the problem.

In our task, we first raised the cognitive demand of the mathematical task, to encourage instructors in to believe students could do it. We did it by shifting the focus of the problem, from solving the differential equation to connecting the different mathematical representations: the algebraic relationship expressed by the equation, the numerical representation given by the table with values of v and v', and the graphical representation given by the slope field. The slope field should be obtained using the *SlopeField* feature of the GeoGebra application. As designed, this task can be carried out on smartphones with internet access. At our university most of the students have access to such phones and some form of internet connection. Additionally, GeoGebra is open, free, and available for the two main smartphone operating systems. Thus, even students from lower-income backgrounds can usually access and complete the task. In the GeoGebra, whenever a slope field graph is generated using only automatic parameters to the *SlopeField* feature, the graph generated may contain an inadequate amount of information and can be displayed in a graphical window that is not very relevant to the problem at hand. Figure 3 shows two slope field graphs for the equation $v' = 9.8 - 0.2v$: one generated using automatic parameters (a), and another with a restricted number of displayed points and a manually adjusted graphical window (b). In this task, technology was used as a tool that enabled the connection of one additional representation (graphic) primarily used to raise the level of cognitive demand of the task. Without the use of the digital resource, the goal of the task could get lost in how to construct such graph (procedurally) rather than reason about it and interpret its meaning.

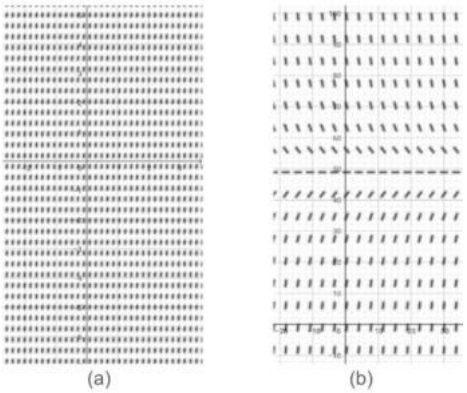

Figure 3: Graphs generated by smartphones using the GeoGebra SlopeField feature.

Then, questions about naming what is difficult might be included in the PLT to afford instructors opportunities to reason about mathematical complexities of the task which will support in naming them to students. In this example, moving from simply solving the equation to connecting the three different representations offer opportunities with respect to: (i) understanding that the table contains values of v and v', but the graph is plotted on a coordinate system with coordinates t and v; also, understanding that v', on the graph, is represented by the slope of each line segment through the

points (t, v); (ii) recognising that points at different values of v, have different slopes, that is, different accelerations v' (in connection with the relationship expressed by the differential equation $v' = 9.8 - 0.2v$ and reflected in the values obtained in the tables); (iii) analogously, understanding that points at the same height on the graph, that is, with the same value of v, have the same slope (same acceleration v'), even if the t values are different. This fact can be interpreted as the temporal evolution of acceleration v' depending solely on the initial velocity v; in other words, two bodies with the same initial velocity, at different times, have the same evolution in acceleration over time; (iv) recognising the limiting velocity or equilibrium solution seen in the asymptote of the graph and, potentially, in the table, when $v' = 0$. In this case, the acceleration is zero, so the velocity remains constant.

While the question "What is difficult about this task? Why?", if added to the PLT, can lead to various interpretations and develop MKT, the teacher educator might plan their TERA to guide the DIP towards the mathematical difficulty of the task with the goal of challenging the neutrality discourse related to the mathematical task that conceals stereotypical views defined by gender and race, and shape mechanisms that create opportunities for unequal participation and support from instructors in the classroom. Therefore, in this problem, it is important for the instructor to recognise the difficulty in obtaining a direction field as a possible qualitative solution to a differential equation. For example, it is challenging to perceive that, based on the representations, whether algebraic, graphical, or numerical, as evidenced in the table, the velocity is not zero as t tends to infinity. It is also difficult to interpret what it means in a direction field for the slopes to be constant for the same v and different t values. And it is particularly important for instructors learning to name what is difficult in the task to name it to students and believe that they students can learn it.

Final thoughts and limitations of this work

A first clear limitation of this work is that the PLT was theoretically analysed, but not empirically tested, therefore, in future work, we plan to carry out workshops with postsecondary mathematics instructors which will give us more data to understand the work of preparing instructors.

One difficulty of this work is the complexities generated by both the level of mathematics explored in the task and the social political dimensions articulated within the context in which it is embedded. To achieve a goal of preparing mathematics instructors with such complex articulation, the collaboration of both authors of this paper was necessary, one with expertise in sociopolitical dimensions of mathematics, but with good transit in postsecondary mathematics, and another with expertise in mathematics, but also with some transit in sociopolitical discussions. Such collaborations are recommended (Bressoud, Mesa & Rasmussen, 2015), but not easily fostered in academic mathematics communities.

Moreover, our analysis shows that the sociopolitical responsibilities are often placed in the hands of instructors, either postsecondary mathematics instructors or teacher educators. At least, it is an essential piece of the PLOT model, because even when designing pedagogical questions to raise racial and gender differentiation in mathematics teaching, much is still left to DIP and how teacher educators are going to moderate them through their own TERA. Such limitation prompted us to include comments on the PLT about possible TERA and intended DIP; however, much is still left to an individual and discretionary space.

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Planning for equity: Does Universal Design for Learning hold promise for mixed-ability mathematics classrooms? A scoping review

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This paper explores the potential of the Universal Design for Learning (UDL) framework to promote equity and inclusion in mathematics classrooms. Disparities in mathematical achievement, often exacerbated by ability grouping, disproportionately affect students from low socio-economic status backgrounds. While mixed-ability classes are recommended to counter this, they present challenges for educators. UDL, with its emphasis on engagement, representation, and action and expression, offers a promising approach to address the diverse needs of students. Through a scoping review of existing literature, this study investigates the evidence supporting UDL's effectiveness in secondary mathematics education. The review reveals limited but promising research, highlighting a need for further studies to explore UDL as a framework for inclusive and equitable mathematics classrooms.

Keywords: Universal Design for Learning (UDL), lower secondary, mixed-ability, disadvantage.

Introduction

It is a well-recognised problem that there are disparities in academic achievement associated with socio-economic status (SES). This is particularly evident in mathematics, with low-SES students in OECD countries "seven times more likely on average than advantaged students to not achieve basic mathematics proficiency" (OECD, 2023, p. 45). However, results from the Programme for International Student Assessment (PISA) consistently show that the strength of the relationship between test scores and socio-economic status (SES) varies markedly between countries (OECD, 2023). This indicates that with an appropriately *equitable* approach – thus, catering suitably for each individual's needs – students can succeed regardless of background (Dietrichson et al., 2017).

While it is important to acknowledge that "gap-gazing" is not always beneficial, particularly through a deficit lens (Gutiérrez, 2008), it is clear from the literature that many factors contribute to discrepancies in achievement. Of particular relevance to this paper is the finding that the achievement gap for students from underrepresented backgrounds tends to widen during the initial years in secondary schooling, a juncture that is understood as being crucial for a student's mathematical journey (McKenna, 2014). This gap is often exacerbated by educational systems that tend to group students by "ability" (or rather, typically, by current achievement level) at this stage, either by school type (e.g., academic or vocational), by syllabus (e.g., "higher" or "lower" level; Advanced Placement), or simply by banding of mathematics classes within a school. Most researchers agree that working-class students are greatly disadvantaged by the incorporation of ability grouping, with students from working-class households disproportionately represented in lower ability groups (Smyth, 2016). Recent research from the OECD (2023) acknowledges that the earlier that students are grouped by ability – contravening *inclusion*, which aims to keep students together irrespective of various characteristics including ability – the stronger the association between SES and performance. An illustration is provided by the situation for Ireland. Irish policies and guidelines maintain a strong focus on inclusion, with policy documents that target both ends of the ability spectrum recommending

the use of mixed-ability classes at lower secondary level, arguing that they are beneficial for all students (Department of Education and Skills, 2020; National Council for Curriculum and Assessment, 2006). Despite this, ability grouping within mathematics, Irish and English is commonplace in Irish schools, with 80% of grade 8 (age ~13-14) mathematics classes grouped by ability, rising to over 90% from grade 9 (OECD, 2013). Aligning with international findings, Ni Shuilleabhain et al. (2021) note that students in Irish schools designated as "disadvantaged" are 40% less likely to study mathematics at higher level than their advantaged peers.

While it must be recognised that upper secondary level offers extra challenges, at lower secondary level there are strong arguments for promoting inclusion through maintaining the open pathways beneficial for less advantaged students by delaying or avoiding grouping. However, if students are not grouped according to (perceived) ability, classrooms will inevitably include a broader range of students, with associated diversity in learning preferences, strengths and barriers. Teaching in such mixed-ability classrooms is not a simple endeavour; Atteberry et al. (2019) note that ability grouping is "often defended based on the contention that detracking and mixed-ability classes are impractical or unworkable" (p. 1). It can be a daunting task for teachers with such mixed-ability classes to create a learning environment that caters to every student in the classroom.

The Universal Design for Learning (UDL) framework, described by Flood and Banks (2021) as "an approach to learning, teaching, and assessment design that is proactive in addressing the varied identities, competencies, learning strengths, and needs of every learner" (p. 2), may have the potential to support teachers and engage students in such mixed-ability contexts. In the present paper, the concept of inclusion connects the ideological notion of equity with the practical considerations of how this might be operationalised in the classroom using the UDL framework (Roos, 2019). The study aims to investigate existing research to identify whether the UDL framework might be an appropriate tool to support secondary mathematics teachers in ensuring equity and inclusion in diverse, mixed-ability classrooms.

Universal Design for Learning (UDL)

Universal Design for Learning (UDL) is a pedagogical framework that was developed by CAST (www.cast.org) in the 1980s. The goal of the approach is to optimise teaching and learning for all students in a way that provides for a diversity of learners, supporting them to develop their own agency. Since the early 2000s, some education systems – for example those in the United States, Australia, South Africa and Ireland – have begun to incorporate UDL as part of their education policy, curriculum development and teacher education (Bray et al., 2024). However, literature to date has focused chiefly on special educational settings and primary or tertiary education, with much less emphasis on secondary mainstream schools (Flood & Banks, 2021).

The UDL framework is divided according to three core principles: *engagement*, *representation* and *action and expression*, which can be used to guide teachers in the creation of environments that are inclusive of all learners. The *engagement* principle aims to provide students with multiple ways to engage in learning, focusing on the "why" of learning; *representation* supports the provision of choice in how students perceive and make meaning of information – the "what" of learning; and the *action*

and expression principle highlights how to provide students with choice and flexibility in how they demonstrate or share what they have learned – the "how" of learning (udlguidelines.cast.org).

Given the complexity of mixed-ability classrooms in mathematics, and the potential of the UDL framework to support teachers in adopting such practices successfully, this research provides a scoping review of extant literature to address the following research questions: RQ1: What (if any) evidence is there for the use of UDL in secondary mathematics classrooms, in particular, for (RQ1a) mixed-ability mainstream mathematics classrooms and (RQ1b) mathematics classrooms that serve low-SES communities? And RQ2: What lessons can be drawn from the current literature?

Methods

A scoping review is a method used to initially assess the potential scope of research literature on a topic and help determine gaps (Grant & Booth, 2009). It differs from a systematic literature review in its smaller scale and scope. This method has proven to be useful in opening up the field and highlighting that future research in this area is justified, and indeed necessary.

EBSCOhost was used as the primary platform for the scoping review, which was conducted across five relevant databases: Academic Search Complete, British Education Index, APA PsycInfo, APA PsycArticles and ERIC. In order to maintain as much breadth as possible, no time constraint was set on the search, and the search terms were expansive, covering the three main categories of UDL, mathematics, and secondary education. The search was further widened by applying the Equivalent Subject expander, but was limited to Full Texts.

Despite the attempts to maintain breadth, only 16 results were returned. These records were imported into the review management software, Covidence (www.covidence.org) and were subject to a first phase of Title and Abstract screening. Exclusion criteria included a sole focus on primary, tertiary or special education, or on the design of digital tools.

The first two authors reviewed the articles, agreeing on eight articles to be included for full review. At the end of that review, two more articles were excluded, leaving six for data extraction. The data extraction template in Covidence was designed to facilitate easy extraction of key study information (Table 1). The papers were subsequently imported into NVivo for coding and analysis.

Table 1: Study Characteristics

Author	Country	Journal	Title	Study Design	Inclusion focus
Dieker et al. (2022)	USA	Middle School Journal	Seven Equity Ideas for Students with Language-Based Needs in Mathematics	Text and opinion	Language (Multilingual/EAL)
Hunt and Andreasen (2011)	USA	Mathematics Teaching in the Middle School	Making the Most of Universal Design for Learning	Text and opinion	Broad inclusion of all learners

McCracken et al. (2024)	USA	TechTrends	Middle School Students' Conceptualizations and Reasoning about the Fairness of Math Tests.	Phenomenological study	Additional/Special Educational Needs; Multilingual/EAL; Ethnicity; Gender
Moleko and Mosimege (2021)	South Africa	Pythagoras	Flexible Teaching of Mathematics Word Problems through Multiple Means of Representation	Qualitative case study; critical emancipatory research.	Language (Multilingual/EAL)
Staats and Laster (2018)	USA	Education Sciences	Extending Universal Design for Learning through Concurrent Enrollment: Algebra Teachers' Perspectives.	Qualitative research: focus group interviews	Multilingual/EAL; Ethnicity; Socio-economic status
Walker and Hunt (2011)	USA	Mathematics Teaching in the Middle School	Social Skill Development through Math.	Text and opinion	Additional/Special Educational Needs

Results

The first clear result from this review is that there is paltry evidence for the use of UDL in secondary mathematics classrooms. Of the six articles, three involved some level of empirical study, and three were classified as text and opinion. The non-empirical articles were all published in practitioner-focused journals, offering practical suggestions for teachers. The three articles that report empirical results are all qualitative in approach, exploring the benefits of UDL within concurrent enrolment programmes, the teaching of word problems, and assessment. Of the six papers, two have a specific focus on inclusion of learners for whom the language of instruction is not their native tongue (Dieker et al., 2022; Moleko & Mosimege, 2021), one prioritises the social skill development of learners with additional educational needs (Walker & Hunt, 2011), and the remaining three papers have a broad equity agenda (Hunt & Andreasen, 2011; McCracken et al., 2024; Staats & Laster, 2018).

Addressing RQ1, it is clear that, if UDL is to be considered as a framework for equity and inclusion in the mathematics classroom, there is a need for further empirical research to be conducted to evaluate its effectiveness. The three articles that include empirical research do show some promise, with each reporting positive outcomes; they are discussed in turn.

Of particular note in relation to RQ1 (a) and (b) is the research outlined by Staats and Laster (2018). The article provides insight into a large-scale, equity-focused study that successfully incorporates the principles of UDL in mathematics education at upper secondary level with a goal of increasing the participation of students (for instance, of low SES) who are underrepresented in third-level education.

The work of McCracken et al. (2024) highlights an approach to assessment design with its foundations in UDL, arguing that “The Universal Design for Learning (UDL) guidelines can inform

assessment development to promote fairness" (p. 1). "Teaching to the test" is a well-known phenomenon, and the constraints imposed by high-stakes examinations is a commonly cited challenge when attempting to change classroom practice. Creating more inclusive and accessible assessments may be one way to address this. By incorporating elements of UDL into assessment design as well as in teaching and learning, the authors aim to ensure that teaching reflects testing practices.

The third paper that incorporates empirical research is that of Moleko and Mosimege (2021), who consider the challenges for both teachers and learners when it comes to mathematical word problems. The focus of this study is on multiple means of representation, and the authors highlight the ways in which teachers can support their students through the provision of "options for language, mathematical expressions and symbols, as well as options for perception" (p. 8).

While it is important to acknowledge the specific barriers that are addressed in some of the papers, there is also a broader consideration given to ensuring that the material is accessible to all learners, that it is engaging and relevant, and that the students can represent their learning in a manner that suits them. Indeed, with the exception of the work of Moleko and Mosimege (2021), each of the papers addresses all three principles of UDL. In order to address RQ2, therefore, the analysis is organised according to each of these principles.

Recommendations for engagement

Five of the six papers provided recommendations on how to engage students in mathematics, with much of the advice focusing on how to optimize relevance, value, and authenticity. Suggestions include asking students to source examples on social media to illustrate a particular concept (Dieker et al., 2022), and ensuring that the context of a mathematics problem is relevant, or at least believable, for the students (Hunt & Andreasen, 2011; McCracken et al., 2024; Staats & Laster, 2018). Walker and Hunt (2011) and Hunt and Andreasen (2011) highlight the importance of offering choice to students on whether to work alone or in a group.

Finally, Staats and Laster (2018) highlight the suitability of mathematical modelling as a pedagogy that aligns well with UDL, encouraging "multiple ways of engaging mathematical scenarios and expressing solutions" (p. 2). According to the authors, teacher respondents noted a change in their students' attitudes towards mathematics through engaging with the modelling process, stating that "modeling was more enjoyable than the more familiar textbook components of the class" (p. 10).

Recommendations for representation

The six papers all provided some guidance on how to design for multiple means of representation, with the work of Moleko and Mosimege (2021) explicitly focusing on this principle. Within the context of mathematical word problems, the authors suggest that teachers should refrain from a mechanistic approach and should rather highlight "patterns in a given word problem, outlining critical features […], as well as the big ideas regarding the concept that is dealt with" (p. 9). Teachers should also clarify mathematical vocabulary and symbols, syntax and structure, and should support learners in understanding how to "decode" text.

In the work of Staats and Laster (2018) there is a clear correspondence between UDL and the steps of mathematical modelling with which students are supported to engage. The authors highlight that

mathematical modelling, by its nature, supports choice in how to interpret information. Throughout this process, learners develop skills in posing questions, making connections, synthesising information, integrating new information with prior knowledge, and generalising. McCracken et al. (2024) highlight the importance of considering the impact on language and cultural differences when designing fair assessments, as these can "influence students' comprehension of information" (p. 13).

The three "text and opinion" articles offer general guidance on how to add "a range of choice to an already well-designed lesson plan" (Dieker et al., 2022, p. 36). The authors caution practitioners to consider the barriers that their students may face, and to take these into consideration when planning, so that all students will be in a position to make sense of the information with which they are required to engage (Dieker et al., 2022; Hunt & Andreasen, 2011; Walker & Hunt, 2011).

Recommendations for action and expression

Provision of choice is a key feature of UDL, and no more so than when designing for how students interact with and demonstrate their learning. The use of choice boards (Dieker et al., 2022) is one concrete way to achieve this, as is providing students with choice regarding the problem(s) that they want to solve (Walker & Hunt, 2011). Mathematical modelling (Staats & Laster, 2018) requires students to use multiple modalities to express and communicate their learning, including text, equations, graphs, and so forth. In the work of Walker and Hunt (2011) and Hunt and Andreasen (2011), capitalising on peer learning is suggested, with the authors postulating that learners could attempt to summarise the solution strategies of others or students could be required to ask for and offer support from their peers before turning to the teacher.

Many of the suggestions for this principle highlight the importance of strategy development within mathematics, highlighting that there is a need to "focus on the processes and organization of problem solving as well as the conceptual underpinnings of procedures" (Walker & Hunt, 2011, p. 299). In order to develop these skills, it is recognised that there is a need for effective, systematic and timely feedback (McCracken et al., 2024; Walker & Hunt, 2011).

Discussion and conclusion

McCracken et al. (2024) differentiate between *equality*, as characterised by transparency and objectivity, and *equity* as requiring adaptation, diversification, and consideration of the individual. If we wish to have an equitable system, then we cannot expect a "one-size-fits-all" approach to provide it. It is clear from the literature that while *equality* may be maintained within a system that relies on (so-called) ability grouping in mathematics, this approach does not provide *equitable* access to the material. Atteberry et al. (2019) contend that by removing ability grouping in mathematics classes, we open courses "to students who were historically denied access based on lower achievement" (p. 8). In their research, high-achieving students were not negatively impacted by the removal of ability grouping, and the average scores of the lower achievers were the same or higher, despite the larger numbers taking the courses. This offers a strong argument for not rationing access to a high quality, challenging curriculum, but rather striving to ensure that it is accessible to a heterogeneous group of learners with diverse strengths and needs. Staats and Laster (2018) advocate for equitable access to

higher-level mathematics syllabi, noting that "inclusive pedagogy does not make a class less challenging or of lower status" (p. 16).

This scoping review sought to address two research questions. RQ1 sought to identify what (if any) evidence there was for the use of UDL in mathematics classrooms, particularly for low-SES or mixed ability. The easy answer is "very little". Only three empirical studies were identified in the review; each was quite niche in its scope. RQ2 asks what lessons can be learned from the literature. Again, any response to this question must be contextualised within the dearth of available research. It must also be acknowledged that, while the articles reviewed for this study offer some guidance, they do not provide adequate discussion of the pedagogical challenges that can face a teacher when trying to cater for mixed-ability mathematics classes. However, it was possible to extract some general (albeit likely obvious) recommendations in relation to RQ2. If we are to design for equity and inclusion, a first step is to know your learners. It is only once you are aware of students' preferences, and the barriers that they face, that you will be in a position to address them. Secondly, it is important to offer choice, while acknowledging the constraints of the learning goals and intentions. Finally, student-centred pedagogies, such as mathematical modelling, align well with UDL and could offer a "way in" for educators to try out the framework using what may be familiar methods.

The research presented in this article is clearly limited by scale and scope. Only one platform was used in the search, and the full-text requirement may have ruled out articles that could have been accessed indirectly. Despite these limitations, the fact remains that there is very little evidence to date for UDL as an approach to support teachers in creating mathematics classroom that can accommodate diverse learners. While the identified articles offer some insight into the implementation of UDL in mathematics, there is a glaring need for more research in mixed-ability settings at lower secondary level. Given the rising profile of UDL in educational policy, curriculum and teacher education, it is imperative that research into the effectiveness of the approach in increasing inclusion, equity and access is prioritised, in order to avoid *hysteron proteron*, or "putting the cart before the horse".

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‘Have-been-doing masculinity’ - emerging gender stereotypes in mathematicians‘ descriptions of their memories of mathematics lessons

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This paper presents the theoretical framework and initial insights from a newly launched research project. The study investigates mathematicians’ recollections of their school experiences, with a particular focus on mathematics lessons and the character traits they identify as instrumental in their development as mathematicians. The data were subsequently analysed with a specific emphasis on the emergence of gender stereotypes. Preliminary findings suggest that male mathematicians tend to attribute their success to innate talent, aligning with traditional notions of natural ability. In contrast, female mathematicians often describe themselves using language typically associated with masculine traits, such as resilience and determination. Additionally, women’s memories of mathematics lessons are frequently framed within contexts of competition and the necessity of possessing a strong will to achieve success.

Keywords: Doing mathematics, STEM careers, STEM education, gender differences.

Introduction

The persistent underrepresentation of women in Science, Technology, Engineering, and Mathematics (STEM) careers is a well-documented phenomenon (World Economic Forum, 2023). In response, contemporary research endeavours to identify the underlying factors contributing to this disparity, particularly those driving women to leave the so-called "leaky pipeline." Rather than framing the inquiry around questions such as "What happened to those girls?" or "Why do women disproportionately opt out of STEM careers?", our approach shifts focus to the narratives of individuals who have successfully established themselves within STEM fields without strengthening the aspect of gender as much. By exploring the common denominators among these successful careers, we aim to adopt a strengths-based perspective, moving away from a deficit-oriented framework that centers on failure and attrition. Our study seeks to understand the formative educational experiences that influenced individuals to "stay in" STEM, with a specific focus on their memories of mathematics lessons during primary school. This emphasis reflects a deliberate departure from the more extensively studied domain of secondary school mathematics and its impact on career trajectories in STEM (e.g., Cerinšek et al., 2013; Nix et al., 2015). The rationale for this focus lies in the potential significance of early mathematical experiences, which may shape children's perceptions of mathematics as a "male-dominated" discipline and establish enduring attitudes towards the subject. Emerging evidence underscores the critical influence of primary school experiences on students' later engagement with mathematics. Marmur and Darragh (2024, p.188), for instance, highlighted the recollections of pre-service teachers regarding their mathematics education, with one student vividly recalling an episode of embarrassment and humiliation in primary school. Similarly,

Sorvo et al. (2017) identified math anxiety as a phenomenon detectable as early as primary school, noting its detrimental effects on students' subsequent mathematical performance. These findings suggest that early educational experiences may play a pivotal role in shaping career choices, particularly with respect to STEM-related pathways. Our ongoing research seeks to investigate the emerging patterns in the memories of individuals who have achieved successful careers in STEM, particularly their recollections of primary school mathematics lessons. This study represents an effort to identify recurring themes and stereotypes within these narratives, which may provide insight into the factors that positively influence persistence in STEM. In this paper, we present preliminary hypotheses derived from the initial findings of a pilot study conducted with mathematicians (representing the "M" in STEM). Although the main data collection phase of the project is still underway, these early results offer valuable perspectives on the role of primary school mathematics education in fostering long-term engagement with STEM disciplines.

Empirical framework: Gender-Gap in STEM-related fields

The persistent gender gap in Science, Technology, Engineering, and Mathematics (STEM) fields remains a pressing issue, with little evidence suggesting significant progress toward equality. Women are significantly underrepresented in STEM careers, comprising nearly half (49.3%) of the workforce in non-STEM occupations but only 29.2% in STEM roles (World Economic Forum, 2023, p. 7). This disparity becomes evident early, as fewer women pursue STEM-related career paths despite increasing educational attainment globally (Delaney & Devereux, 2019). The phenomenon of the "leaky pipeline" further compounds the issue: while women earn 38.5% of STEM degrees, their representation drops to 31.6% in STEM jobs one year after graduation, and only 17.8% and 12.4% in VP and C-suite leadership roles, respectively (World Economic Forum, 2023, pp. 44-45). Critically, such statistics often fail to capture the underlying social dynamics that create and sustain these disparities. Without deeper examination, these numbers risk portraying gender inequality as inevitable or natural, reinforcing perceptions of male dominance in STEM. A more nuanced approach must consider how gender is actively constructed and reproduced within educational and professional contexts. Research highlights secondary school mathematics as a decisive factor influencing STEM career choices (Spreitzer et al., 2024). For example, Nitzan-Tamar and Kohen (2022) argue that the level of mathematics studied and success in science majors like physics and computer science play a crucial role in determining future STEM participation. However, such studies often overlook how educational systems themselves contribute to gendered outcomes. External perceptions of ability, for instance, may be particularly influential for female students, whose choices in STEM can hinge on how their performance is viewed rather than on their actual abilities (Nix et al., 2015). This aligns with findings that women, despite achieving results comparable to men, report significantly lower confidence in their mathematical abilities (Watt et al., 2017). Other studies highlight how classroom dynamics reinforce these patterns. In U.S. secondary schools, for instance, male students completing Algebra II often adjust their self-assessments downward, aligning their confidence with their actual performance, while female students show no comparable shift (Ma & Johnson, 2008). These differences suggest that cultural narratives about gender and mathematics ability continue to influence students' self-perception and subsequent academic choices. By examining these issues through the lens of gender as a social construct, it becomes clear that the underrepresentation of women in STEM

is not merely a product of individual preference or skill. Instead, it reflects systemic biases and institutional practices that perpetuate gendered norms. Addressing these root causes requires moving beyond statistics to understand how gender is enacted and reinforced in educational and professional environments.

Theoretical framework: Gender Stereotypes and 'doing masculinity'

We adopt Greenwald and Banaji's (1995, p. 4) definition of stereotypes as "shared beliefs about traits that are characteristic of members of a social category" and extend this with the insight of Muntoni and Retelsdorf (2020, p. 72), who note that individual traits and characteristics are often overshadowed by those attributed to the broader social category. Viewing gender through the lens of social constructivism ('doing gender'), gender stereotypes can obscure the individual qualities of students in mathematics. In particular, we highlight how the label of being female, or 'femininity,' can overshadow traits such as ambition, rationality, and assertiveness—qualities traditionally viewed as masculine (see table 1). These gender stereotypes can adversely affect young girls' learning of mathematics. For instance, female teachers who harbor insecurities about their own mathematical abilities may unintentionally reinforce internalized gender stereotypes, leading girls to believe that their mathematical skills are inherently inferior due to their gender (Beilock et al., 2010). Additionally, the stereotype of mathematics as a predominantly masculine field (Muntoni & Retelsdorf, 2020, p. 82) can create a self-fulfilling prophecy, where teachers may inadvertently favor boys over girls, offering more encouragement and praise to male students (Becker, 1981; Jungwirth, 2014; Leinhardt et al., 1979). Gender stereotypes continue to hinder women even after school, with job advertisements framed using masculine language—terms like 'competitive' and 'dominant'—resulting in fewer applications from female candidates (Gaucher et al., 2011). Beyond academic success in mathematics, there are various stereotypes associated with gender and masculinity that deserve consideration. For preliminary analysis, we draw on a selected set of deeply ingrained stereotypes. Traditional feminine and masculine traits, as outlined by Murray (2001, p. 248), align with contemporary findings describing women as communal (concern for others, sociability, emotional sensitivity) and men as agentic (assertiveness, independence, instrumental competence) (Hentschel et al., 2019).

Traditional feminine traits		**Traditional masculine traits**	
Gentleness	Sacrifice	Courage	Assertiveness
Modesty	Supportiveness	Strong will	Initiative
Humility	Empathy	Ambition	Rationality
Nurturance	Compassion	Independence	Emotional control
Intuitiveness	Tenderness		
Sensitivity	Unselfishness		

Table 1: Traditional feminine and masculine traits according to Murray (2001, p. 248)

Rather than solely discussing gender stereotypes, some scholars integrate the concept of social processes, using Bakhtin's (1981) distinction between heteroglossia and monoglossia. In monoglossic viewpoints, female traits are inherently linked to the biological sex of women. However, as Francis (2012) argues, heteroglossia is intrinsic to all gender constructions, while monoglossic, binary perspectives tend to "mask and pathologise heteroglossia" (p.1). Disruptions to these binary perspectives, or 'heteroglossic contradictions'—such as girls exhibiting masculine traits—are often subtle, manifesting predominantly at the micro level of individual subjectivity, and are most apparent to those who do not conform to the monoglossic gender order, as is evident within the transgender community (Jaremus, 2020, p. 718). These contradictions are particularly salient in the context of girls and women in mathematics, where "doing mathematics equates to doing masculinity," creating more tension for girls and women who are more invested in their feminine identity than for boys and men (Mendick, 2006, p. 2). Some girls pursuing careers in STEM try to mitigate this perceived 'masculinity' by incorporating traits or interests deemed traditionally feminine (Archer & DeWitt, 2015, cited in Jaremus, 2020, p. 718). Similarly, boys who transgress monoglossic boundaries and are labelled as 'feminine' or as possessing 'female traits' may find themselves excluded from mathematics, as it is often perceived as a masculine domain (Jaremus et al., 2023, pp. 8-9).

Methods, Data collection and limitation of the subjects investigated

For both the pilot study and the main study, we define "having a successful career in STEM" as holding a position in Science, Technology, Engineering, or Mathematics at a level higher than entry-level, with a minimum of five years of professional experience post-graduation. The pilot study focused specifically on the "M" in STEM, restricting the participant pool to individuals in postdoctoral or higher academic roles at German universities. However, this is not the final criterion for a successful career in mathematics, and further data will be collected in the near future. Participants were asked to reflect on their experiences in mathematics lessons during their school years, starting with the earliest memories. Specifically, they were prompted with questions such as, "What comes to mind when you think about your mathematics lessons at school? Please share your experiences." Additionally, they were asked to describe the personality traits they believed had been helpful in learning mathematics, with questions like, "Please describe any attributes or skills that may have been useful in mathematics lessons."

For analysing the pilot study data, we employed a straightforward qualitative content analysis (Mayring, 2014), with a particular focus on emerging gender stereotypes to gain initial insights and structure the interviews. However, the main study, which is planned for the near future, will be analysed using Grounded Theory (Corbin & Strauss, 2015), as it allows for a more inductive approach and provides the flexibility to continuously incorporate and expand upon data collected across various countries, genders, and STEM fields. In this main study, participants will only be asked to provide their school experiences without additional constraints, allowing their responses to emerge more naturally.

First insights

The traits most frequently mentioned by female mathematicians were ambition and assertiveness, which are traditionally categorized as masculine traits according to Murray (2001). This is reflected

in statements such as: *"But I do think it makes a big difference to also learn from others (.) and try to fit in well, and to seek help when it makes sense, right, exactly,"* which demonstrates humility and collaboration but also a strategic approach to learning. Another participant highlighted ambition and competitiveness by recalling a personal anecdote: *"There was another boy who was also very good at math, and we kind of always had a little, well, like a small competition, you know."*

Interestingly, male mathematicians did not attribute any memories or personality traits using descriptions typically associated with femininity. Instead, they tended to emphasize their cognitive abilities and perseverance. For example, one participant stated: *"Uh, I (..) have a very good ability to abstract and can imagine things really well. […] I'm researching how things deform, and for example, I had no problems with things like body constructions or something like that. Because/ well, this object was just there in my head, and I could rotate it however I wanted. I don't think other people have that as much."* This emphasis on spatial visualization skills and abstraction reflects the association of rationality with masculinity. Another participant reinforced this by describing the persistence required for mathematical success: *"Just a little bit, now/ it becomes more apparent at university that you also need to be persistent and have patience."*

Moreover, male mathematicians rarely employed gendered terms when describing their own abilities, instead framing their qualities in terms of capabilities or skills. One participant expressed this by explaining the nature of professional mathematics: *"[…] maybe even starting with the wrong topic, and you just have to stick with it and fight your way through. That's a very important trait; all professional mathematicians I know have phases where they're not sitting in front of a blank page, but in front of pages that end up in the trash because everything they try doesn't work out."* These descriptions underscore the perception of rationality, persistence, and abstraction as essential to success in mathematics—traits that are often aligned with traditional notions of masculinity.

Gender bias in the interpretation of mathematical narratives

Since the use of binary and stereotype-based categorizations in the initial analysis was retrospectively deemed unproductive, a second round of data interpretation was conducted in collaboration with other mathematics education researchers. The primary objective of this re-evaluation was to eliminate potential biases stemming from preconceived notions about gendered traits in mathematical discourse. To achieve this, all data were meticulously cleansed of pronouns and any gender-specific identifiers before the interpretation sessions took place. This ensured that no explicit clues regarding the gender of the narrating participants were available to the researchers during their analysis. By adopting this approach, the study sought to explore whether and how implicit gender associations would still emerge in the interpretative process. Particular attention was given to the pronouns spontaneously used by the mathematics education researchers when referring to the anonymized participants. The results revealed a striking pattern: with only a few exceptions, where researchers explicitly questioned toward the end of a session whether a given speaker might have been a woman, the pronouns *he/him* were consistently used (see Figure 1).

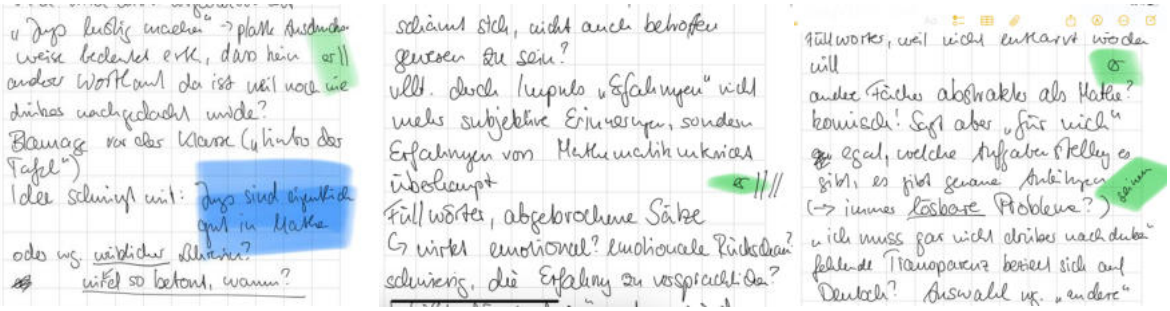

Figure 1: Excerpts from transcripts of the interpretation sessions (in German)

Discussion

Preliminary analysis suggests that both female and male participants framed their school memories and self-descriptions in ways reflecting traditionally masculine traits. This aligns with the concept of *doing mathematics* as *doing masculinity*, where mathematical proficiency is associated with ambition, persistence, and rationality (Mendick, 2006). Male participants tended to present their abilities as inherent skills or talents, avoiding personal or emotional attributions. In contrast, some female participants incorporated elements of compassion or emphasized social learning and collaboration, potentially compensating for the perceived masculinity of their mathematical competence (Archer & DeWitt, 2015). Beyond self-perceptions, interpretative sessions with mathematics education researchers reinforced the implicit association between mathematical expertise and masculinity. Even after gender markers were removed, researchers predominantly used masculine pronouns (*he/him*) when referring to anonymized participants. The occasional questioning of whether a speaker *could* have been female—rather than an even distribution of assumed genders—illustrates the persistence of gendered expectations in academic discourse (Forgasz & Leder, 2017).

These findings align with broader research on gender and competence in STEM fields, which highlights how implicit biases shape perceptions of ability and professional belonging (Carlone & Johnson, 2007; Nosek et al., 2009). The tendency to associate mathematical success with masculinity reinforces systemic barriers for underrepresented groups. While this analysis focuses on gender, it does not yet account for intersecting factors such as race and class. Research on *doing difference* (West & Fenstermaker, 1995) and intersectionality (Yuval-Davis, 2006) suggests these dimensions complicate simplistic gender binaries. Future research will adopt an intersectional approach to explore how multiple identities shape experiences in STEM fields, extending beyond gendered framings.

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Exploring educational resources through sociopolitical lenses: Initial insights from a study of mathematics textbooks

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This paper compares mathematics textbooks from Brazil and Canada to examine how linguistic, cultural, and sociopolitical factors shape educational content and student engagement. Employing a qualitative documentary analysis guided by the Teorema System, the study contrasts Brazil's centralized model via the National Textbook Program (PNLD) with Canada's decentralized approach. The findings reveal that while Brazilian textbooks favor standardization to secure uniform access, this can limit local cultural adaptations; in contrast, Canadian materials integrate diverse cultural and linguistic contexts to resonate with regional realities, albeit with potential disparities in resource allocation. These results underscore textbooks' dual role as pedagogical tools and cultural artifacts and suggest that a hybrid framework combining national consistency with local customization may better promote inclusive and context-sensitive mathematics education.

Keywords: Mathematics textbooks, sociocultural factors, educational policies, comparative analysis.

Initial Considerations

Mathematics textbooks can shape students' learning experiences, presenting knowledge in ways that reflect cultural and political contexts (Fritz et al., 2020). They carry subject-specific language and convey underlying values and priorities (Fan et al., 2013; Gueudet et al., 2021). Planas et al. (2021) describe how textbooks affect second-language learners by linking content with specific forms of language presentation. Gates and Jorgensen (2015) note that textbooks may legitimize societal hierarchies through selective inclusions or omissions.

Research in this area includes studies on how textbooks support modernization ideals in rural settings or frame citizenship and gender (Neto & Valero, 2018; Neto & Pinheiro, 2021; Pacheco & Neto, 2018; Valero et al., 2019). These works show that textbooks can mirror wider social goals beyond disciplinary content (Gueudet & Trouche, 2016; Qi et al., 2025).

This paper examines textbooks from a Brazilian perspective, drawing on a comparison with Canadian materials. The analysis employs the Teorema System (Amaral et al., 2022), comprising (1) contextual mapping, (2) identification of sociocultural and political content, and (3) iterative analysis. This approach addresses textbooks as instruments linked to cultural and ideological dimensions (Fan, 2013). Planas et al. (2021) emphasize that language and culture influence mathematics learning.

Similar findings emerge in comparative studies on how textbooks may integrate or omit cultural content to pursue particular agendas (Cai et al., 2016).

Unveiling the Sociopolitical Dimensions of Mathematics Textbooks

Language and culture are often central in discussions of textbooks, where linguistic differences can help or hinder comprehension (Fritz et al., 2021). Fan et al. (2013) and Gueudet et al. (2021) show that equitable learning depends on recognizing these aspects in teaching materials. Beyond linguistic concerns, textbooks also present perspectives tied to national policies. Jurdak et al. (2018) point out how these materials can endorse certain political or economic viewpoints, illustrated historically by textbooks in China, Saudi Arabia, and other contexts (Kenway & Epstein, 1997).

Some research has noted how materials in rural Brazil may promote specific agendas that guide social and economic development (Neto & Valero, 2018; Pacheco & Neto, 2018). Editors and authors, influenced by governmental frameworks or commercial considerations, also shape textbook design (Gueudet & Trouche, 2016). In settings where centralized systems exist, content choices often follow national standards that aim for uniform coverage (Neto & Pinheiro, 2021). Local adaptation is more frequent in decentralized settings, though variations in funding and oversight can arise (Valero et al., 2019).

Past materials have not always included diverse groups, affecting learners' sense of belonging (Kenway & Epstein, 1997). Some textbooks now feature content on women's contributions or ethnic studies, but digital formats may introduce new differences in access. Considering these factors may inform the design of mathematics textbooks that support a range of student identities.

Methods

This study adopts qualitative documentary analysis based on Bardin (2011) to investigate how Brazilian and Canadian mathematics textbooks address cultural, linguistic, and sociopolitical aspects. The Teorema System (Amaral et al., 2022) guides the process in three phases:

- **Contextual Mapping:** Brazilian textbooks from the National Textbook Program (PNLD) list are extensively used in primary schools. Canadian textbooks employed in the Western and Northern Canadian Protocol (WNCP) illustrate a decentralized environment. The mapping identifies governmental guidelines, cultural settings, and language diversity that shape textbook production.
- **Sociocultural Content Identification**: Informed by Fritz et al. (2020), Planas et al. (2021), and Gates and Jorgensen (2015). Focus on presenting mathematical concepts, cultural representations (e.g., Afro-Brazilian, Indigenous, or multicultural Canadian examples), and language for multilingual contexts.
- **Comparative Analysis**: This approach explores how each system incorporates cultural references, addresses marginalized perspectives, and aligns with policy goals. It draws on

comparative frameworks (Cai et al., 2016; Fan et al., 2013; Qi et al., 2025) to identify similarities and contrasts.

In Brazil, *Matemática: Compreensão e Prática* (Silveira, 2015) and *Projeto Teláris Matemática* (Dante, 2014) were analyzed. *Math Makes Sense* (Pearson Education Canada, 2005–2008) served as the primary reference in Canada. These selections reflect national and provincial curricular frameworks and highlight the differences between standardized and decentralized approaches.

Presentation and Discussion of Data

The findings of the comparative documentary analysis of Brazilian and Canadian mathematics textbooks elucidate the critical cultural, linguistic, and sociopolitical factors that influence mathematical comprehension and engagement. These insights underscore the complex interrelationship between educational content, student experience, and broader societal structures.

Cultural and Linguistic Adaptations

Examining the inclusion of linguistic and cultural dimensions in textbooks clarifies their influence on students' engagement with mathematical content. Fritz et al. (2020) suggest that language in instructional materials shapes students' thought processes and grasp of mathematics. This viewpoint is apparent in the differences between Brazilian and Canadian textbooks. Canadian textbooks integrate cultural scenarios familiar to students, such as hockey or community events, aiming to situate mathematical concepts in relevant settings. *Math Makes Sense,* for instance, provides probability tasks linked to cultural festivals, reflecting Canada's emphasis on multiculturalism and inclusive approaches (Barbeau & Taylor, 2009; Kajander et al., 2018). This practice aligns with Canada's decentralized system, where provinces can adapt resources to local circumstances.

By contrast, Brazilian textbooks (developed under the centralized (PNLD)) tend to feature formal and technical content presentations. Activities typically remain within conventional school contexts, consistent with an emphasis on standardization and nationwide curricular goals (Souza, 2019). While this structure supports uniformity, it may limit the content's cultural adaptability for diverse student backgrounds. Planas et al. (2021) note that bilingual or multilingual environments, such as Canada, benefit from materials designed for varied linguistic needs, whereas Brazil's regional linguistic diversity remains more difficult to address in nationally standardized textbooks.

Sociopolitical Implications

From a sociopolitical viewpoint, textbooks do more than transmit content; they reflect power structures and educational priorities (Jurdak et al., 2018). In Canada, textbooks often emphasize reconciliation with Indigenous communities and multicultural content, aligning with national policies focused on social justice and inclusion (Barbeau & Taylor, 2009). Many activities include data collection linked to daily scenarios, encouraging students to see mathematical ideas within diverse cultural contexts. In Brazil, textbooks have gradually added Afro-Brazilian and Indigenous perspectives, recognizing historical imbalances and advancing cultural representation. Nonetheless, a centralized system constrains local modifications, which may reduce attention to distinct regional

circumstances (Jurdak et al., 2018). These differences show how policy decisions influence textbook design, shaping what is taught and how it resonates with students' cultural backgrounds. In Canada, including Indigenous content is tied to broader reconciliation initiatives, while in Brazil, the national commitment to equity can limit the scope for regional adaptation.

Contributions to Educational Equity: Comparative Study (Brazil vs. Canada)

A comparative analysis of textbooks elucidates distinct pedagogical approaches influenced by cultural contexts and educational policies. Table 1 delineates the key findings, demonstrating how Canadian textbooks emphasize cultural contextualization while Brazilian textbooks adhere to formal standardization.

Table 1: Partial results of the Analysis of Textbooks

Textbook	Example Analyzed	Cultural/Linguistic Aspect	Observation
Math Makes Sense (Canada)	Exercising in sports and hockey	Canadian cultural context: national sports	The exercises incorporate common cultural practices in Canada, such as hockey, providing a closer connection to students' daily lives and facilitating the contextualization of mathematical concepts, as Kajander et.al (2018) discussed.
Mathematics: Ideas and Challenges (Brazil)	Introduction to division	An abstract and formal approach	The presentation of division in Brazilian textbooks follows a technical and formal approach without references to local cultural contexts, reflecting the standardization promoted by the PNLD, as noted by Souza (2019).
Math Makes Sense (Canada)	Probability of cultural festivals	Inclusion of cultural diversity	Cultural festivals are used as the context for probability problems, highlighting the emphasis on multiculturalism and the inclusion of diverse cultural practices, which are in line with Canadian educational policies mentioned by Barbeau and Taylor (2009).
Matemática: Compreensão e Prática (Brazil)	Naming and measuring angles	Use of technical language	The technical language predominates without explicit cultural adaptations, reflecting the centralized curricular uniformity in Brazil, as observed by Gates & Jorgensen (2015).

Textbook	Example Analyzed	Cultural/Linguistic Aspect	Observation
Math Makes Sense (Canada)	Data collection with samples and populations	Application of concepts to Canadian everyday scenarios	The examples of data collection refer to common Canadian everyday scenarios, promoting a practical and meaningful connection to students' realities, as suggested by Kajander et.al (2018) in the educational context of Ontario.
Projeto Teláris Matemática (Brazil)	Ratios and proportions	Traditional Brazilian approach	The presentation of ratios and proportions follows a traditional and abstract line, characteristic of Brazilian textbooks, which priorities strict adherence to national curriculum guidelines, as discussed by Souza (2019).

Brazilian textbooks frequently follow a curriculum-oriented structure, drawing on contexts typical of conventional classrooms. In contrast, Canadian textbooks often integrate diverse everyday scenarios, reflecting the country's ethnic and cultural heterogeneity.

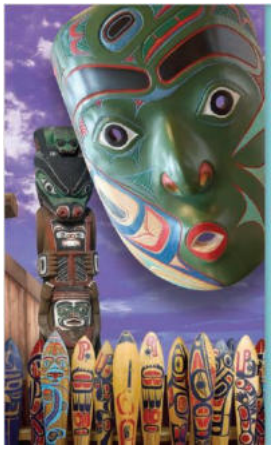

Figure 1: Indigenous Art representations MMS Collection

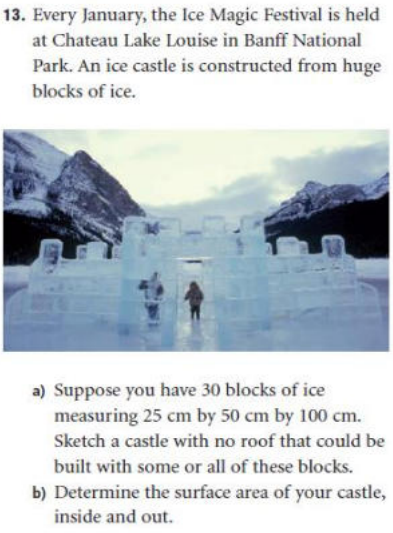

Figure 2: Ice Magic Festival MMS Collection

These materials include illustrations and exercises on contemporary social issues, such as Indigenous reconciliation (Barbeau & Taylor, 2009). For instance, Figure 1 shows how Canadian collections can present geometric transformations directly connected to Indigenous craftsmanship and cultural artifacts, offering students a contextualized perspective on mathematical concepts.

Although Brazil has increased its diversity-related content, many textbooks still adhere to a traditional curriculum. Adapting mathematical tasks for bilingual or multilingual contexts is an important factor. Canadian textbooks often address diverse audiences, including second-language learners, through literal translation and cultural adaptation, making examples more accessible to varied cultural backgrounds. In Brazil, the variety of regional languages, particularly in Indigenous communities, requires specific attention when producing national textbooks.

The federal government's centralized curriculum decisions and distribution policies aim to ensure broad educational access, yet they may limit regional adaptation. By comparison, provincial autonomy in Canada allows educators to tailor resources to local populations, though this process can lead to disparities in funding or materials across different regions (Jurdak et al., 2018). The Figure illustrates how certain Canadian textbooks link local festivals and historical practices to probability, data analysis, or other math topics, indicating the flexibility of decentralized models.

Final Considerations

The findings from this study illustrate how educational policy, cultural representation, and mathematics pedagogy interact in the development and distribution of textbooks. In Brazil, a centralized approach ensures nationwide uniformity, which promotes equity by providing consistent materials to students across different regions. However, this structure may limit the potential for local adaptation and the inclusion of diverse cultural and linguistic contexts.

By contrast, Canada's decentralized model enables provinces to tailor textbooks to their priorities, which can enhance cultural relevance but may produce disparities in content and quality. These contrasting approaches reveal a broader challenge: balancing national standards with local needs. The Brazilian model supports wide access yet may not fully address the country's cultural heterogeneity, whereas Canada's flexible framework could fragment curriculum cohesion. Future research might consider hybrid solutions that combine a core national standard with regional customization, as well as longitudinal studies, to determine how these strategies influence students' mathematical achievement, cultural awareness, and broader educational outcomes. Preliminary evidence indicates that continued examination of how policies shape textbook production and adaptation could shed light on equity in mathematics teaching.

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Empowering and disempowering students in a school project on sustainability

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Four fourth-year preservice teachers with a major in mathematics education, were in their practicum in a Norwegian school. They developed an interdisciplinary project related to clothing and sustainable development involving social studies, natural science, and mathematics. The project started with an assignment where the students wrote a reflective text about their clothing habits to make the project relevant for the students. Then they worked in groups on different aspects of the clothing industry and the consumption of clothes. At the end of the three weeks, the students gave an oral presentation, and the projects were graded. We look at how the preservice teachers facilitated student empowerment. While there are examples where the preservice teachers facilitate student empowerment, we conclude that the efforts may also have caused the opposite, disempowerment. We discuss reasons for this and implications for teachers and teacher education.

Keywords: Consumerism, dilemmas, textiles, preservice teachers.

Introduction

This study was conducted with four preservice teachers, Alex, Billie, Charlie, and Drew, and the practicum teacher Jonny during their practicum in 9th grade in a school in the Western part of Norway. The preservice teachers were all master's students in mathematics education. The practicum lasted three weeks, and they were to teach the subjects of mathematics, social science, and natural science. The Norwegian curriculum (Kunnskapsdepartementet, 2019) has *Respect for nature and environmental awareness* as a core value and *sustainable development* as an interdisciplinary topic in school. This includes understanding dilemmas and that "Our lifestyles and resource consumption have local, regional and global consequences" (p. 17). The preservice teachers had no experience conducting interdisciplinary projects but were challenged to do so in their practicum. They developed a three-week project with the intention to make the students aware of the implications of the clothing industry and their own role as consumers while inviting the students to be active in their own learning. We ask: How do the preservice teachers facilitate student empowerment?

Theoretical framework

In this study we have used Ernest's (2002) concepts for empowerment development in mathematics, where fostering self-belief, taking control, and participating in activities are involved. Ernest (2002, p.1) characterizes three domains of empowerment concerning mathematics and its uses: *mathematical empowerment, social empowerment and epistemological empowerment*. He associates mathematical empowerment with traditional mathematics education focussing on having confidence in knowing how and when to use mathematics in school. Social empowerment is more connected to the use of mathematics outside the classroom, to the power of understanding and being able to exercise critical judgments related to mathematics in society, in particular concerning politics, decision-making and uses in the media. Epistemological empowerment depends on mathematical and social empowerment.

It is about confidence in applying mathematics in new situations and a need to be personally engaged "with mathematics so it becomes an integral part of the learner's personal identity" (Ernest, 2002, p.12). Ernest highlights that teachers should encourage all forms of empowerment.

Skovsmose (1992) underscores the importance of mathematics education in fostering informed citizens who can engage with, and transform, society. He introduces the concept of *mathemacy*, which extends beyond basic calculation to include the application of mathematical techniques and reflective thinking. This aligns with Giroux's (1989) view of literacy as a tool for empowerment. Mathematical literacy, therefore, is essential for social and cultural emancipation. As Skovsmose (2004, p. 1) notes, "Mathematics education can mean disempowerment or empowerment." He emphasises that the interpretation of empowerment varies based on the specific context of the learners and that disempowerment can manifest in all types of mathematics education.

Method

The practicum teacher and the authors of this paper gave the preservice teachers the assignment to develop a three-week project on sustainability in three subjects, mathematics, social science, and natural science. We asked them to start by relating sustainability to the students' own lives. The preservice teachers chose textiles and the clothing industry as the sustainability topic and prepared the project. It started with an assignment where the students wrote a reflective text about their clothing habits to make the project relevant for the students. Then they worked in groups on different aspects of the clothing industry and the consumption of clothes. At the end of the three weeks, the students gave an oral presentation. The student projects were graded by the preservice teachers.

Data material consists of video recordings, audio recordings, and student assignments; all recordings are transcribed and anonymised. We went through the transcribed data and marked situations where the preservice teachers facilitate student participation, where they connect the topic to the students' lives and the preservice teachers reflect on their own teaching related to participation. In the analysis, we use Ernest's (2002) criteria for empowerment to show how the preservice teachers facilitated student empowerment. We do not have data to conclude on whether the students were empowered or disempowered.

Analysis

Now follow some analysed examples of how the preservice teachers facilitated empowerment.

The reflection notes

The first task given to the students is to write a reflection note on their own clothing habits. Mathematics is not applied in this initial part of the project, but we include it because it sets the stage for the three-week project. Some of the students write relatively long texts about how they are conscious about shopping quality clothes and only what they need. They seem to be concerned about sustainable development and act thereafter. Although the task was closely connected to the students' own lives, some expressed that they did not know what to write, and it seems that these students copied each other.

Toward the end of the day, the practicum teacher, Jonny, and the preservice teachers, Alex, Billie, Charlie, and Drew, reflect on the first lesson. Alex brings up problematic aspects of the task which may explain why some students hesitated when working on the task.

Alex: I felt many reflected well on their own habits, and so on, but then I kind of "what is OK to ask?" […] If someone can't afford the clothes they want, it can be hurtful for some.

Drew: That should be paid attention to.

Alex: […] It can't be like, " Yes, very good, you donate to the Salvation Army”". We're not judging people.

Alex suggests that the task can have two negative effects. The first is that it can be hurtful to reflect on clothes for those who feel they cannot afford to buy what they want, a worry that is relevant given that clothes represent status and that students at this particular school come from both wealthy homes and less privileged homes. The other effect Alex suggests is that the task is articulated in a way that some may feel judged because they are not conscious of how their habits impact the environment. Thus, because the task may cause students to feel inferior, the task may disempower some students.

From the preservice teachers' plans, we see that they intended to return to the students' habits related to clothing, asking whether the students think differently about their own habits after having learned more. We don't know whether the preservice teachers actually did this. In any case, this suggests that the preservice teachers expected that after having learnt about how various fabrics pollute the environment, having learnt about working conditions in clothing factories that produce cheap clothes for people in the global north, and having learnt about harmful effects of clothes consumerism, the students would have other thoughts about their own habits and role. The preservice teachers may have had a plan to empower the students to understand the consequences of their own habits, in line with social empowerment, where mathematics played a role in this understanding. They may even have planned to facilitate students' epistemological empowerment as the project, including the mathematical content, could lead to a change in habits and identity as a consequence of students being personally engaged with the project. On the other hand, the preservice teachers may have chosen not to return to this reflexive question because of reasons Alex mentions in the above excerpt. Maybe they were afraid of disempowering students due to causing envy, guilt, shame, or indifference to being part of consumerism.

The group task

After the initial task, the students were divided into groups to work on group projects. The preservice teachers let the students choose a topic related to the textile industry, and their choices spanned from fast fashion to emissions from the transport of textiles. While the preservice teachers gave some short lectures related to sustainability and textiles, the students were also supposed to find information on the Internet. The students were thereby given freedom, trust, and responsibility. Through letting the students take ownership of what information was relevant for their group project, the preservice teachers facilitated all forms of empowerment: mathematical empowerment, as students may feel in control of the mathematical content; social empowerment, as students can become more confident about how mathematics is used outside the classroom; and epistemological empowerment, as the

students may feel empowered to take into account mathematical information, how mathematics is being used in other contexts as well and how mathematics can influence the students' identity.

Developing a checklist

The preservice teachers soon discovered that the students retrieved numbers and statistics uncritically from the Internet, so they arranged two mathematics lessons on being aware of questionable aspects of statistics. After discussing a graph from the media in detail, Charlie led an activity where the students were asked to contribute to making a list of what they should be aware of when evaluating and including statistics in their projects.

Charlie: I thought that we, together, now, could actually make a checklist with what is important to consider when you're reading statistics, like this one. Would anyone like to share with us? What is important to look for when you read diagrams and statistics? We have talked a lot about this now. Will it be easier to contribute when you've discussed it a little in the groups? Then I think we'll do this now, OK? Take some time and talk in the groups, come up with some points which can be useful when reading statistics.

It seems that Charlie intends to let the students take ownership in making the checklist. He asks for suggestions, then repeats his question, probably to clarify in case someone missed what he asked for. When nobody answers, he hints that the students can draw from the discussion they just had. He then changes strategy and makes the students come up with suggestions in their groups first. Charlie thus makes an effort to empower the students to develop the checklist.

A little while later, Charlie continues in plenary.

Charlie: OK? Now I've been around a bit on different groups. There are many who have come up with sensible things. If we're going to make a little list of points that might be good to have when you read diagrams, what points do we want to include on it? Yes?

Student 1: Be critical.

Charlie: Be critical, yes. We can include that. [keyboard crunching] Yes?

Student 2: Check where the information comes from.

Charlie: "Check where the information comes from." Yes? Other things? Yes?

Student 3: Properly understand the numbers.

Charlie: Properly understand the numbers, yes. [keyboard crunching] Do we have more stuff? Yes?

Student 4: To read the headline.

Charlie: Read the headline, yes. Important to see. [crunching] Do we have more stuff? You girls had many points.

Student 5: See what the x-axis and y-axis are saying.

Charlie: Read what the different axes are saying, yes. [crunching] Can we include one more point? There's one more thing I'd like us to include, at least. Yes?

Student 6: It's a bit like what has already been said, but look at how the diagram is made.

Charlie: Absolutely. It's super important, right? Because it says something about the whole impression we get from the chart and the numbers, right? […] you can look at your graph and consider these points we now have made. […] because we require that you use statistics in your project.

This time, Charlie receives suggestions from the students, and all suggestions are typed into a file he makes available for all to use in their projects. He facilitates student empowerment by inviting the students to contribute to the checklist, types their suggestions almost word by word, and connects their suggestions, i.e. the checklist, to their assignment. By applying the students' input, Charlie facilitates the students' gaining of power of what is important to consider when they continue their group work. Six students contributed, but because the students worked on the list in groups, the students who contributed can be seen as representatives of their groups, so even more students have contributed.

Charlie facilitates mathematical empowerment, as the activity is situated in the classroom, connected to the students' projects, and is related to the ability to read mathematical texts and make sense of them. However, we consider it more relevant to connect the activity to social empowerment as the intention of the checklist is for the students to be critical of what the diagrams they find contain, how they are used in the text, and whether they are relevant for their projects. It can be argued that Charlie facilitates students' epistemological empowerment as well because the checklist is intended to be relevant in general, which implies students' future experiences with graphs in the media. While many students might have been empowered by this activity, some might have felt disempowered, for example, if they didn't understand this or the previous task or didn't come up with suggestions in their group.

Evaluation of the group task

When almost three weeks have passed, the groups present their projects, and the preservice teachers grade the projects and the efforts of each student and give feedback to each group and each student. We here present selected feedback to two groups, both of which have chosen fast fashion. Charlie gives feedback to one of the students in the first group:

> Charlie: You show that you are reflective in the presentation. You manage to show that you see connections. And why "fast fashion" exists. You draw in several sides of the issue. What we noticed a bit is that you could use the sources a little more actively, and when you brought this up with the working conditions at the factory, you could have elaborated on this a bit more. We think. But during the process of this work, you show that you take responsibility in the group. You cooperate and help the others quite well. So, we've rated you overall, a 5+.

Charlie is providing positive feedback as he states that the student is reflective, includes several perspectives, and contributes positively to the group, while Charlie also has suggestions for improvement. He facilitates both mathematical and social empowerment implicitly as mathematics has been part of the group project. Because Charlie comments on several reflective and critical aspects of the presentation, it could be argued that he facilitates epistemic empowerment as well. The student received a good grade, 5+ (English B+), which may facilitate empowerment since it can contribute to strengthening the student's confidence. However, good grades may not be empowering in cases when students do not feel they earned the grades, which then would have the opposite effect. As Ernest (2002) emphasises, it is more important that the students feel personally engaged with mathematics, so they have a sense of "self-image as successful in mathematics" (p.12). He also

accentuates that students who have been mathematically empowered often receive personal encouragement from the teacher and have a good relationship with the teacher (p.12).

The following excerpts are from feedback given to another group who also had worked with fast fashion. They opposed to fast fashion and highlighted problematic aspects related to sustainable development.

> Drew: Yes. What we have written about the group together is that you have worked well and efficiently throughout the whole process. You cooperated well and made a very nice presentation. […] You appear unbound by scripts, and we can hear and see that you have worked well together and are collaborating. Your academic content is sound, and you are very familiar with the content, are reflective, and have included criteria from all the school subjects. In a good way. We sense that you have been very united and agreed on what to do in class and at the presentation. And cooperate well.

Billie then gives feedback to each of the students in the group. In the middle of her feedback, she adds:

> Billie: […] And I think it was very nice that you talked about problems with fast fashion […]

The students in this particular group got very good grades. As indicated in these excerpts, they receive positive feedback from the preservice teachers which facilitates certain areas of empowerment, but perhaps not others. The excerpts suggest that attention is given to school-centred values, skills, and knowledge, such as cooperation, answering given questions, presentation skills, argumentation, and reflectivity. What is not emphasized is the understanding of the problems and what to do with them. The only feedback explicitly related to sustainability aspects is the statement Billie showed above. The preservice teachers may thereby have facilitated mathematical and social empowerment as the feedback on academic content could be related to mathematics and how it was applied. However, little emphasis was put on critical aspects of the students' group project, which may have been part of their identity as concerned consumers. The preservice teachers may thus have failed to facilitate epistemological empowerment. The good grades and the positive feedback might therefore have supported mathematical and social empowerment while disempowering the students concerning epistemic empowerment.

Discussion and conclusion

We conclude that the preservice teachers intended to empower the students through the three-week project, but their efforts may also have caused the opposite. The initial task on reflecting on the students' own habits may have empowered the students who were concerned about clothing consumerism, but may have disempowered the students who struggled with what they were to write about, perhaps feeling inadequate about this moral task. Furthermore, the students played a key role in developing the checklist, where Charlie, one of the preservice teachers, often wrote down the input from the students' word by word, showing that Charlie valued the input, which facilitated student empowerment in all Ernest's (2002) domains of empowerment. Yet, this might also have caused disempowerment if there were students who felt they were unable to contribute.

We consider the fact that the student groups could choose topics and find sources of information themselves as a significant example of how the preservice teachers facilitated student empowerment.

The students were given trust and responsibility through the freedom to choose, which is a way to facilitate empowerment, which may have affected in particular those who were concerned about environmental issues. Some of the topics chosen by the students were less consumer and individual focused, such as emissions from transport, and may have been satisfactory for the students who perhaps were less concerned about sustainable development. This may have empowered some students. Furthermore, giving feedback and grades can also be a way to empower students, where, e.g., good grades could inspire the students to continue to critically explore mathematics-based information in the media, while not so good grades could disempower students. The preservice teachers' project plan indicated that they wanted to empower the students to act sustainable. This could be the case for some students who already felt they had a sustainable lifestyle or for those who were inspired to change their habits. However, the three-week project could well be disempowering for those who felt guilt or did not wish to change to a sustainable lifestyle.

We would like to emphasize two important points. One is that even students who received good grades might have been disempowered because what engaged them – moral aspects of consumerism – seemed to be less valued in the preservice teachers' feedback. This is a dilemma of teaching in general. In the real world, the problems of consumerism, working conditions and unfair distribution of goods are far more important than, for example, balancing emphasis on school subjects in argumentation. The preservice teachers seemed to aim for transformative learning, empowering the students, but were trapped in a system where certain aspects of learning are to be valued, and where normative and political viewpoints are not essentially valued when deciding grades. This aligns with Zipin's (2009) exploration of boundaries between lifeworlds and schools, where good pedagogical intentions can sometimes come into conflict with institutional norms and expectations. We expect this is a common paradox regarding normative projects on sustainable development.

Our second point is that empowering students to think and act differently concerning sustainable development is particularly challenging. Sustainability crises can be overwhelming and make students disillusioned and disempowered. Moreover, the crises demand a change in lifestyles, which may be perceived as unpleasant, painful or not justified by many. However, we believe that these are not arguments for avoiding such issues. Rather, these aspects need to be explicitly taken into account. We, the researchers and the practicum teacher, did not prepare the preservice teachers for disempowering aspects of sustainability projects, but now we have learnt that these are essential to include in our teaching and supervision.

In summary, this study delves into the intricate dynamics of fostering student empowerment within the realm of mathematics and sustainable development education. Our findings reveal a nuanced interplay between efforts to empower students and unintentional instances of disempowerment.

Looking ahead, it is imperative for teachers, preservice teachers, and teacher educators to critically reflect on their approaches to projects on sustainability to ensure that empowerment is authentically fostered among students and that attention is paid to handling or avoiding disempowerment concerning those who disagree with normative implications of living sustainable lives. This necessitates a holistic approach that not only encourages autonomy and responsibility but also cultivates critical thinking, empathy, democratic virtues, and active citizenship in addressing complex

societal issues. Sustainable development is a crucial topic in society and therefore relevant for teaching and learning. Teacher training programs therefore need to raise preservice teachers' awareness of dilemmas and paradoxes they will experience in school. From this study, we have become aware of the significance of being explicit about interdisciplinary contexts on sustainability in our plans for teacher training in our mathematics courses. Perhaps we need a new category of empowerment which is related to the context of the applied mathematics in order to emphasise these aspects.

Further research is needed where insights from other academic disciplines are incorporated, such as sustainability education research, transformative education research and feedback in classrooms.

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What does it take to make a change? Pre-service teachers' positionality when working with mathematics inspired by Critical Mathematics Education

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In this paper we explore why our surprise by pre-service teachers' lack of cracking out of their traditional ways of working with mathematics was rather foolish. Norwegian and South African researchers cooperated in developing a full-day session for pre-service teachers in Norway, inspired by Critical Mathematics Education, hoping to challenge the students' traditional ways of working with mathematics. The topic was "Clean water", and the teaching included an open approach to mathematics through this topic, also connecting personal stories from South Africa and issues of global fairness. We hoped to give the students new ways of approaching and working with mathematics that day, but we felt that we could have done better. Analysing video data from one episode through the theoretical lens of Holland et al. (1998) makes us argue that we were rather naive to expect students to rapidly position differently as mathematics learners.

Keywords: Critical mathematics education, pre-service teachers, figured worlds, positionality.

Introduction

"It seems like they just pushed the pause bottom!", one of the researchers exclaimed when examining video data from a teaching session where a group of Norwegian pre-service teachers (PTs) worked with an open mathematics task inspired by critical mathematics education (CME). The PTs had just joined a conversation with a researcher from South Africa, exploring the big difference in access to clean water in the two countries, including political perspectives connected to this. Interestingly, when the conversation ended, the PTs did not seem to connect the emerging issues in the conversation to the mathematics task they were given, even though 'Clean Water' was the topic of the day. Instead, they went back to where they had stopped before the conversation with the South African researcher. We were first surprised by this; it was like this conversation was an isolated happening not connected to the way the students negotiated their approach to work with mathematics. What we saw as a golden opportunity to engage with mathematics differently, was lost. This episode has made us reflect on perspectives connected to the challenge of changing an existing approach to work with mathematics.

Designing and trying out teaching materials that promote new ways of engaging with mathematics teaching and learning is a central aim of the research project 'Mathematical toolkit design North-South' (Kacerja et al., forthcoming). In this project, researchers from Norway and South Africa cooperate in designing research-based toolkits inspired by CME and focusing on teaching mathematics for topics such as sustainable development. We chose to use 'Clean Water', because access to safe water is crucial for human life and is one of the 17 sustainability goals (UN, 2024).

Also, while Norway is at the lowest end of the water risk scale, South Africa is at the highest. We were partly inspired by Skovsmose's (2023) classroom example, *Use of Water*. These research activities align with the new mathematics curriculum in Norway (Ministry of Education and Research, 2019) and the need for South African students to cope with mathematics as a gatekeeper for completing schooling (Julie et al., 2019). The hope is that the toolkits we design and try out with PTs at the University will promote new ways to experience mathematics, and hopefully inspire the PTs to implement such activities in mathematics for school students.

In this paper, we explore why our surprise by the PTs' resistance to accept, what we saw as an invitation to work differently with mathematics, was kind of naïve. We discuss some crucial challenges connected to changing PTs' ways of engaging with mathematics in light of critical mathematics teaching. Our research question is: *How can we understand students' ways of engaging with mathematics in a teaching session inspired by Critical Mathematics Education?*

CME and students' opportunities to participate

Our research project is embedded in a critical mathematics education perspective (Skovsmose, 2023). It is concerned with addressing critical societal issues as part of mathematics while at the same time addressing mathematics critically in all its forms and opening up new possibilities for PTs to explore such focus in engaging with mathematics. A toolkit is a collection of materials we use for teaching a given socio-critical topic in mathematics teacher education, such as literature, different prompts, discussion tasks, etc. (Kacerja et al., 2024). In line with Skovsmose's (2023) presentation of what CME is, we are concerned about the activities that the students will work on, the nature of the problems that are addressed, and how the PTs are invited to interact. In landscapes of investigation (Penteado & Skovsmose, 2022), teachers support students' investigative work by inviting them to wonder about mathematical, sometimes taken from real life. It is up to the students to accept the teacher's invitation and engage with the mathematics, with questions like "what if", and explore the situation at hand from different perspectives.

In previous research, Kacerja et al. (2017) and Kacerja et al. (2021) used the Body Mass Index (BMI) as a context for PTs of mathematics from Norway and South Africa to explore ideas of mathematics in society and point their attention towards being critical of the uses of mathematics. It was found that the PTs accepted the invitation, engaged with the mathematics of the BMI, criticised its uses in several critical contexts, and questioned the people using the BMI and their reasons for doing so. The PTs also discussed if it would be appropriate for them to use such an index to discuss in their mathematics classes. Similarly, Register et al. (2022) found that although PTs could select relevant CME topics, they need more experience with mathematisation, dealing with controversial or challenging discussions, and developing potential students' actions. Several research studies involve sustainability issues. For instance, Helliwell and Ng (2022) discuss how teacher education can support PTs in engaging in climate change issues by developing their practices and offering tasks that align with sustainable futures. Examples of their practices included the PTs watching short videos, engaging in face-to-face discussions, investigating chosen web resources, and designing lessons involving sustainability they would like to implement in future practices. Similarly, Steffensen and

Kasari (2023) described a Norwegian teacher educator inviting her PTs to examine and reflect on important societal issues, such as the Gulf of Mexico oil spills in 2010 and littering in a hiking trail near the students' home. The aim was to bring awareness of the role of mathematics in these issues and to call for action.

Steflitsch (2024) highlighted that students can engage in CME tasks by discussing the mathematics involved, the societal topic, or social aspects in classrooms. As mathematics teachers, it may be considered safer to stick to "the mathematics". However, various examples exist where mathematics teachers engage in the societal issue. For instance, Steffensen (2023) described how PTs expressed they wanted their students to learn mathematics while raising awareness about environmental sustainability. During the practicum, they engaged their 5th-grade students in picking litter, measuring, estimating, and discussing consequences in a modelling activity. Similarly, Steffensen and Kacerja (2021) found that lower secondary school students also accepted the invitation and engaged in critical discussions about climate change. The students found themselves an online carbon footprint calculator, which they started to explore from different sides by making changes in the values of the different variables of the calculator while also reasoning about the effects of those changes in one's life and its significance and feasibility. These examples are inspired by a CME perspective to change the traditional ways of students doing mathematics and engage them with the critical nature of the uses of mathematics in society. In all the examples, PTs and students accepted the invitation to go beyond the traditional mathematics tasks and work with mathematics from a critical perspective, even discussing mathematics' effects on the groups of people it addresses.

Theory – positionality within a figured world

To understand PTs' way of engaging with mathematics in new ways, we will draw on Holland et al.'s (1998) theoretical framework and, in particular, their concept of positionality. Using this framework means that we also need to include the concept of figured worlds. This means that when we try to understand the nature of PTs' positionality as mathematics learners, we need to see the class of PTs in mathematics as a figured world. As Holland et al. (1998) state, a figured world is a "socially and culturally constructed realm of interpretation in which particular characters and actors are recognised, significance is assigned to certain acts, and particular outcomes are valued over others" (p. 52). Moreover, as Holland et al. describe, it is important to underline that the class of PTs in mathematics is a local world with its own valued qualities, and it will be the frame for understanding all the happenings and actions within it. Even though figured worlds are local worlds, they are also fluid worlds, which are influenced by surrounding worlds and its values (Holland et al., 1998). That means the students' previous experiences as mathematics learners will also influence the PT class. Importantly, figured worlds carry a common shared narrative, a storyline, or a standard plot. The standard plot of a figured world works as an unwritten guideline for how to act; like a backdrop for interpretations of the acts that take place within the figured world.

When examining PT's positionality, Holland et al. (1998) emphasise that this is seen in two ways: in relation to the other actors within the figured world and the figured world itself with its standard plot. Moreover, positionality is connected to how the actors see themselves regarding access to power,

privilege, and entitlement, which affect the possible ways of acting or their social position in the figured world. This is not static, the development of social position or disposition relates to voice oneself, to enter spaces, to refrain or to self-censor from activities that happen over the long term and during social interaction. The actors of a figured world will position themselves through indicators of claims that will give access to what to say and which spaces to enter. This means that the way PTs enagage with mathematics in a session needs to be seen in light of what they see as possible acts and how they see themselves entitled to act in a particular way. Actors in a figured world get to know 'their' position as they participate in its everyday practices, which might be out of awareness, more like just mundane happenings (Holland et al., 1998). Understanding the PTs' positionality and their access to change their ways of working with mathematics means we have to pay close attention to what seems to be the standard plot of the class. Importantly for us, using this theoretical lens, is how we can understand individual change or how change can come about. Sometimes, actors might experience happenings that make them conscious of the position they have developed, which has been unmarked so far. They may notice how some of the usual acts or more dramatic activities may affect their entitlement in accordance with their social position. New ways of seeing their social position, or the space they can enter, may lead to reconsidering which acts are possible. This can lead to the possibility of affecting their own behaviour. "Ruptures of the taken-for-granted can remove these aspects of positional identities from automatic performance and recognition to commentary and re-cognition" (p. 140). Thus, positional identity can become "more or less conscious, more or less habitual, moving sometimes out of awareness, (...), and at other times toward consciousness and susceptibility to manipulation" (p. 237). Indeed, this theoretical lens emphasises that by drawing on the different resources at hand, individual's ability to improvise within the frame of the figured world and 'guide their next acts' is a source of redirection through 'serious play' (p. 272).

Methodology

The data for this study was collected as part of a full day of teaching in a mathematics education course for two classes of 4th-year PTs, where we tried out a toolkit about water consumption. Mathematics education researchers from two different Norwegian universities and one university in South Africa collaborated in planning this full-day session. The toolkit included a video as a kick-off activity on water consumption, tasks involving water consumption, literature about risk, sustainability and uncertainty related to CME, ethical issues and teacher perspectives in teaching controversial issues that the PTs were given beforehand. The video was followed by a presentation by one of the South African researchers talking about access to clean water from his context, contrasting the situation in Norway. After some plenary comments, the students were given a worksheet and were divided into groups of 4-5 to discuss various questions on a work-sheet as; how much water is needed to (Task 1) wash your hands, (Task 2) shower, (Task 3) brush teeth, (Task 4) wash clothes.

We collected video and audio recordings of the plenary discussions and nine groups in line with data collection and ethical considerations required by the Norwegian Centre for Research Data (SIKT), and data is anonymised. For this paper, we focus on one group of PTs whose way of engaging with the task was interesting to us. Thus, the example we present here is one episode that unfolded in the class without pretending or aiming to be representative or generalisable.

We looked at the data in teams of three researchers, and the videos were watched individually and in groups. This made us notice and discuss interesting episodes in light of the research project's aim. Also, data were transcribed, and our analysis became an iterative process by going back and forth between video data, transcripts, individual perspectives and perspectives the group discussed. The chosen episode shows a sequence with three PTs, which we saw as a golden opportunity for changing existing approaches to working with mathematics. We closely examined the transcripts and video data to get insight into the PTs' positioning as they work with mathematics. Our focus was on elements such as: The use of language to enter spaces or not; the opportunities the student used or not; the nature of the conversations by focusing on flow, content, contradictions, and easygoingness; the style of their talk; expressions that signalled surprises, strong emotions etc.

Analysis

We are zooming in on a situation with Simon, Adrian, and Thomas, just after the two plenary situations in the beginning of the lesson. The PTs were invited to choose either one of the given tasks or a topic they considered to be appealing and connected to the topic "Clean water".

The young men explained to one of the researchers that they were interested in how much water was wasted in the bathroom. The researcher acknowledges this approach, and the young men proceed to discuss how to estimate how much water is used, starting with hand-wash. This was one of the suggested task on the work-sheet, actually task 1. The young men start a conversation, interacting smoothly, discussing how they can estimate how much water will be used when washing their hands.

Simon:	I can go to the bathroom and take the time.
Adrian:	Now I'm looking for FHI's hand-washing tutorial.
Simon:	Is it ok if we wash our hands for about 30 seconds?
Adrian:	It says to wash the water with soap, and water will take 40-60 seconds.
Thomas:	But the difference between what you should do and what you do…
Adrian:	I don't wash my hands for 1 min
Simon:	No, me neither. No, but according to this one here - you should do it... we can say that maybe you are in contact with water for 20 seconds

They consider information from webpages about how much time they will use and reflect on their experiences. The conversation moves in a nice flow, with no conflicts; they are initiating questions, responding fast, and suggesting various things, but at the same time, they quickly agree, like they are on a well-known path for how to collaborate. Importantly, these three young men have been working together on several occasions over their first years as teacher students, and it seems like there is a pattern among them that Adrian and Simon are positioning themselves by taking up active roles, asking questions, suggesting alternatives and actively accepting a suggestion, with Adrian as the moving force. Thomas is taking up a more passive positioning by being more silent and refraining from the activities Adrian and Simon seem to take up naturally. With the words of Holland et al. (1998), it appears like these three young men are entering spaces based on habitual acts developed over time. They choose the first task suggested on the worksheet, and it seems they are acting along the script of the way of how to work together as students in mathematics, or in line with the standard plot.

Suddenly, Vincent, the researcher from South Africa who presented the perspective of access to clean water in the plenary talk, approaches the boys. He says, "How are you doing?" The young men answer politely, and Vincent starts talking about the South African situation.

Vincent:	The most important for me is the poor [...] When there is no water, they have to buy water to drink. So where do they get the money to buy water? Can you see it?
Adrian:	Yeah, I can see that. How is the situation in South Africa with jobs and such?
Vincent:	Unemployment is close to 48%. That's pretty high. Especially among the youth? So, if you have a high utilisation rate with these problems, it's chaotic.
Adrian:	But what is the biggest jobs in South Africa?
Vincent:	Construction, mining, ... farming. ... They got a lot of money to do it. It disappeared. It's getting better. It was quite low [Vincent explains about corruption and goes into a dialogue with Adrian about hijacking and criminality. Adrian then adds:]
Adrian:	There are two different worlds here in Norway and in South Africa.

These young men seem to be entering this conversation in a habitual way, welcoming Vincent's initiative to discuss with them. As in the first sequence, Adrian takes up an active positioning by joining the conversation with his previous experience. They connect to the conversation by asking questions on what seems perfect in timing and drawing on his experiences about South Africa, which are linked to several topics. Again, we notice the nice flow in the conversation; it is a smooth-running way of moving between the actors in the conversation. It is worth noticing that the conversation seems to influence Adrian; he comments on the large contrast between Norway and South Africa. It is as if he re-reflects about the Norwegian situation in light of this conversation with Vincent.

The conversation between Vincent and the PTs ends here. Vincent leaves, and the young men are looking at each other. After about 2 seconds, Adrian again takes the word.

Adrian:	Number one, then. Shall we think about it? Because it said, you should use 40 to 60 seconds. But large parts had to be scrubbed. You just have to rinse and scrub, then rinse again […] It's maybe 20 seconds out of the 40 seconds you're underwater.
Simon:	But how many seconds does it say you have to ...
Adrian:	40 to 60. But that was the whole process. Yes. 40 to 60.

The young men moved quickly and directly back to their conversation before Vincent appeared, and they didn't comment on the discussion with Vincent in sense. There are no expressions of any feelings or reflections about it. It is as the conversation vanished when Vincent left, and the students had just pushed the pause button for the first conversation among the three of them, which was connected to the given task, during the second conversation involving political issues connected to clean water. This surprised us; it was the topic of the day! When Vincent left, it was like they pushed play to continue. Moreover, in our analysis, we did not see any signs from the young men's talk that they saw the dialogue with Vincent as an opportunity to change either their approach to working with the task or the content of the task. It was just like they stayed on the same path as before, even though it was said that they could freely choose how to work on this topic. Moreover, our analysis made us not see any signs that the PTs seem to be aware of being entitled to or having the power to change directions in their work. We noticed that it was like they went back automatically to what they had started. What we researchers saw as an opportunity to rupture their normal way of working with mathematics through their conversation with Vincent did not affect their way of seeing their entitlement. It did not create a moment of rupture with their typical

approach to mathematics or the standard plot of being a PT in mathematics. Instead, it seemed to be more of a sidestep. They did not improvise or entered the space differently. The conversation with Vincent appeared more like a mundane activity, being polite but unconnected to mathematics. Our analysis made us notice that they kept relating to their traditional positioning as PTs in mathematics through this sequence.

Discussion

In this section, we will revisit the research question: How can we understand students' ways of engaging with mathematics in a teaching session inspired by Critical Mathematics Education? Our analysis has made us notice that the PT's ways of engaging with the mathematics in the sequence explored seem to not carry any sign of change to their traditional way of engaging with mathematics. The PTs seemed to act along a standard plot of how to work with mathematics, without connecting the real-life experiences the conversation with Vincent presents, to the mathematics they chose to focus on and engage with. Drawing on Holland et al.'s (1998) framework, it seems like the conversation with Vincent were not a dramatic enough incident to make any rupture to their habitual ways of working with mathematics, and the PTs seem not to see their social position differently through the sequence; our analysis didn't help us to see any signs indicating that the PT entered the space, their ways of engaging with mathematics, differently. It rather seemed like they automatically stayed on a habitual way of engaging with mathematics, starting with task 1, and continuing on that task even though being involved in a conversation included political issues connected to the topic of the day. This has made us reflect on if the PTs see themselves as entitled to change their already chosen approach to their work. We don't have an answer for that in this paper, but investigating our data from the full-day session may bring us closer in the future.

Even though the literature (Kacerja et al., 2017; Kacerja et al., 2021) points out how PTs and students engaging with CME accepted the invitation to go beyond the traditional mathematics tasks and work with mathematics from a critical perspective, our analysis has made us aware that it sometimes seems to be challenges connected to this. Our analysis makes us aware that it sometimes takes more than what we as teachers think it takes to make PTs go beyond their traditional ways of working with mathematics. Instead of blaming the PTs for not engaging differently than what seems to be their traditional ways of engaging with mathematics, the lens of figured worlds makes us aware that we need to see the happenings in the classroom in light of the broader discourses of education. Figured worlds are fluids worlds, hence we need to include the bigger dynamics of mathematics education, such as what it means to do and learn mathematics, both in the past and the present. Importantly, this makes us as researchers aware of that it takes more to create change, we need to see what we are doing in our classrooms in light of the bigger dynamics of mathematics teaching and learning.

We have learned from the literature (Register et al., 2022) that PTs sometimes need more experience with mathematics dealing with controversial issues to develop actions. To see PTs change in their ways of engaging with mathematics over time are in line with Holland et al. (1998) theories of change. They emphasize that sometimes there might be dramatic incidents that create change, and other times there will be more slow-moving forces that develop change. For the sequence explored in this paper, we will not argue that our aim of promoting students' new ways of engaging with mathematics failed

or that the PTs did not engage in a fruitful way with mathematics. As Holland et al. (1998) emphasise, sometimes we need to see incidents as seeds of change, or as the first steps towards change.

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Bridging the political and the historical in mathematics education: a dialogue with in-service teachers

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In this contribution, I describe a participatory research implemented with four in-service secondary school mathematics teachers in Montreal, Canada, about mathematics cultural and historical dimension. Through the reading of culturally and historically diversified texts, the participants have reflected how mathematics is and has been shaped by the economic, social and symbolic structures within which it is practiced and developed. Anchored in a dialogical approached for the data analyses, the objective was to describe teachers' and researchers' interactions and reflections. Three main thematic emerged from our analyses: Otherness, Hospitality and Reflexivity, from which I developed a reflection on the teachers' practices in relation to social and political issues in/of mathematics education.

Keywords: Social and political dimension of mathematics education, history of mathematics, ethics, dialogism, teachers' practices.

Exploring cultural and historical dimension of mathematics

There is a close relationship between the history of mathematics and mathematics education that has repeatedly been developed and implemented. Especially within research in mathematics education, this relationship has been explored very early, particularly by the International Study Group on the Relations between the History and Pedagogy of Mathematics (HPM Study Group). Affiliated with the International Commission on Mathematical Instruction (ICMI), the HPM Study Group aims to investigate different conceptions and views of mathematics. Its members study different eras, mathematicians, regional and national mathematical schools, mathematical textbooks, and many other topics from the history of mathematics. They seek to connect the history of mathematics as a discipline, its roles in education, and the roles that it continues to have in developing mathematical instruction and the curricula around the globe (see International Mathematical Union, 2021).

In the HPM literature, it is now quite well established that, when learners engage with the cultural or historical mathematical artefacts, especially in the context of the exploration of primary sources, it may provide a sense of "unfamiliarity". These special experiences put the learners on foreign ground, this phenomenon is often referred to as *dépaysement* [disorientation] (Barbin, 1997; Pengelley, 2011). Several studies have shown that the educationally rewarding aspects of such *dépaysement* are that students become more inclined to keep an open mind in relation to mathematics and mathematical concepts (Furinghetti et al., 2006; Guillemette, 2017), and that it may lead to "meta-level" learning (Kjeldsen & Blomhøj, 2012). In this paper, I will explore and question this phenomenon, emphasizing on its social and political aspect.

Toward the social and political dimension of mathematics

In literature, one can find different positions about the idea that the exploration of cultural and historical dimension of mathematics can lead to social and political reflections. It was, for instance, one of the main points of D'Ambrosio's ethnomathematics research program, for whom mathematics has always been shaped by the economic, social and symbolic structures within which it was practiced and developed. He suggests that the original source of abstraction (mathematical and otherwise) is to be found in the material production of life and cultural ways to engage with others. It is within the economic, social and symbolic structures of the society that the needs to be satisfied in educational terms are defined, as well as "ways" of satisfying these needs. These "ways" are in fact "ways of doing things": they appear as "knowledge". This is where the ethnomathematics perspective find, within the social and the historical, its fundamental ontological and epistemological basis. Let's remember that ethnomathematics is a "research program about the history and philosophy of mathematics" (D'Ambrosio, 2006, p. 17).

More recently, researchers focusing on equity and power relations in mathematics education have also been interested in the history of mathematics. They see the encounter with mathematics as something shaped and produced by wider cultural and societal contexts that is inherently social and political and grounded in the history of mathematics and mathematics education. In this sense, the exploration of the history of mathematics could be an occasion to get an awareness of the contexts in which mathematics and mathematics education are embedded in ways that reproduce, challenge or disrupt power relations. For instance, Rochelle Gutiérrez stated that "the importance of the history of mathematics is not just to show that certain racial or cultural groups contributed to the knowledge we have today but to also highlight the ways in which settler colonialism or White supremacy are linked to scientific projects" (2017, p. 16). She emphasizes on the need to acknowledge the settler colonialism and ask whose history and whose language is part of mathematics.

More globally, taking a historical and cultural perspective on mathematics, could foster a repositioning of mathematics education as a social and political project. The political, here, of course, has to be understood as governmentality; that is to say the forces that can determine the behavior or the conduct of the others. Governmentality can find its space of action, of course, in a society, but also in a classroom, in a family or in oneself. The question is to understand better how this link can appear in the context of mathematics education and in what sense. For this contribution, in search bridging the political and the historical in mathematics education, I will take a close but different path of ethnomathematics and equity perspective, the paths of ethics.

Ethics and the experience of otherness

My own research (Guillemette, 2017) on preservice teachers engaging with historical texts has pointed out two interrelated elements: the experience of otherness as well as empathy. The phenomenological description that I provided of the *dépaysement* (see above) of the future teachers show that they make serious efforts to understand different mathematical cultures through history, especially when engaging in the reading of historical texts and exploring historical artefacts. Their difficulties to understand past mathematicians' culture and way of doing mathematics often led to frustration and students can answer *violently*.

As the French phenomenologist Emmanuel Levinas (1971/2010) put it, violence is the thematization of the Other, a reification of the Other, a way to make the Other a Mine. This is where the concepts of otherness and empathy join. With Levinas, empathy can be understood as an effort to establish a nonviolent relation with the Other. Within the experience of otherness, empathy is this modality of being that tends to keep the Other's subjectivity free and alive, to keep it mysterious and indefinite.

Yet, when engaging in the reading of historical texts, we observe that the subjectivity of the mathematicians from the past is arduously preserved by the preservice teachers. They hardly maintain an empathic relation. In the suffering of the *dépaysement*, the answer is often violent, as the author of the historical texts are hardly considered in their own terms, in their mathematical and sociohistorical context. This said, this experience of *dépaysement* is important, because, within these difficulties, I have shown that the students perceived their future pupils as also being confronted with mathematical texts to be interpreted. Aware of the possibility of engaging in violent way, the students can feel *responsability* concerning the mathematical activity which will take place in their classrooms: that of welcoming their learners and their reasoning in a nonviolent way. That is why I suggest that the readings of culturally and historically distant texts in mathematics, through the *dépaysement* that it generates, support the development of empathy toward the classroom and an openness to marginality and diversity in the mathematics classroom, that is to say, in Levinassian terms, a nonviolent mathematical education.

Furthermore, I developed subsequently an ethical perspective on the exploration of cultural and historical dimension of mathematics in the context of teachers' education (Guillemette & Radford, 2022). Based on Mikhaïl Bakhtin (1895-1975) and Emmanuel Levinas (1906-1995) philosophy of otherness, this perspective tries to foster dialogical educational readings of historical texts that questions the self in its theoretical assumptions and offers vicarious aspects around the ways of *being* in mathematics and ways of *doing* mathematics. The encounter with the past is thought as an engagement with a foreign culture where the students reacted to it in terms of an agentic culturally and historically situated positioning in which their subjectivity shows itself polyphonically in the reception of the text and its unavoidable estrangement. Referring to Bakhtin, this dialogical, engaging, polyphonic reaction is called *answerability*. Within this perspective, have been stressed some elements necessary for the preservice teachers to enter into a dialogue with mathematics from another culture, such as the responsive/engaging relations regarding the mathematicians and the setting of a 'dialogical window' in the experience of otherness. An experience of otherness indispensable, I believe, for the reinterpretation and development of the meaning and orientation of mathematics education itself, for teachers' and students' understandings of diversity, for openness to pedagogies recognizing students' and teachers' diverse experiences, needs and perspectives and for inclusive mathematics education.

Toward a dialogue with in-service teachers

In this quest for bridging social and political issues in mathematics education with cultural and historical dimension of mathematics, and having in mind this ethical perspective introduced above, I decided to focus on in-service teachers. The choice of this population was motivated by the fact that cultural, historical, social and political issues are now strongly emphasized in teacher education

programs. In the province of Quebec (Canada), the new *Référentiel de compétences professionnelles: profession enseignante* (Professional Skills Framework for Teaching), in reference to a new cross-disciplinary skills set, explicitly states that teachers must act as ethical professionals with respect to diversity (Gouvernement du Québec, 2020). It is specified that "… on the part of teachers, acting ethically also implies adhering to a certain number of dispositions (attitudes or values) that influence how they carry out their duties" (my translation, *id*., p. 81). It is mentioned that "socioeconomic inequalities challenge schools and teachers in their ability to promote equity and inclusive education by using the levers of intervention available to them in a social justice perspective" (p. 17). These expectations weigh heavily on the shoulders of mathematics teachers.

Furthermore, despite effervescence in the research community and the publication of numerous large-scale studies (see Rubel, 2018), empirical research on the topic remains scarce in the French-speaking world and among in-service teachers (Guillemette & Nicol, 2016). Teachers were identified as being interested in both political and historical issues but reported being poorly trained and lacking in pedagogical resources that could be mobilized in the classroom, citing lack of time and the difficulty of following the curriculum when considering these issues (*ibid*.).

This is why I formulated these research objectives: 1) document the epistemological and pedagogical reflections of in-service secondary school teachers engaging with culturally and historically distant artefacts, 2) describe how pedagogical tools are developed for their classroom through jointly produced activities, and 3) describe their experienced and apprehended constraints and problems. For this contribution, I will focus here particularly on the first objective.

A participatory research

These objectives call for implementing a qualitative study anchored in a comprehensive paradigm. Moreover, these objectives address both the production of research knowledge and the professional development of practicing secondary school teachers in a unified manner. This prompted me to adopt a participatory approach. Four secondary school mathematics teachers were recruited (they had taken part in a professional workshop on reading historical texts led by us two years earlier). Participants were mid-career teachers (10–25 years of experience; two women, two men). A meeting was held to set goals and working arrangements. Three other meetings (150 minutes) took place in the next month. We discussed "historical situations" in order to explore together possibilities for classroom implementation and potential challenges. These "historical situations" (Mesopotamian numeral system, Ptolemy's trigonometry, al-Khwarizmi (4th model for solving 2nd degree equation), Cavalieri (volume of the ball), Arbalestrille (Jacob's staff)) consisted of short excerpts of historical texts derived from HPM literature. There were no historical or pedagogical explanation or assumption but only brief introductions by me. The excerpts were essentially a basis for explorations and discussions, a catalyst to create dialogue. Videotapes of the collaborative meetings, transcription of the audio, and a research diary constituted the data.

A dialogical approach

Aware that diversity and difference may occur in relation to who is doing the research and is being researched, posing methodological and ethical issues, I try to constitute a particular methodological positioning: a dialogical approach (Abtahi & Barwell, 2019; Guillemette, 2017, 2021).

This approach is based on the concept of dialogism developed by Bakhtin. It emphasizes that discourses are inevitably embedded in a dialogue. I'm referring here to dialogue both at the level of language and at the level of ideas and ways of being. For Bakhtin (1981), when we speak, we invest ourselves subjectively: no word can be uttered without an evaluative accent, without an attitude or mode of being regarding what we speak (agreement, sympathy, objection, and so on). The same is true of listeners and their reply. According to Bakhtin, this embodied dialogue constitutes the very fabric of the horizon of meaning of human beings. The "Other" provides, in this context, an additional and external perspective (called exotopic) necessary for the conceptualization of the self as a cohesive and meaningful whole (and vice versa). This dialogical interaction is crucial in the human intersubjective experience and in the formation of a properly ethical consciousness. In this perspective, we account for the multiplicity of the reflections of the participating teachers and researchers in their dialogical interaction.

The conducted dialogical analysis assumed that all discourses, including inner one, are dialogically structured and presupposes an addressee. The main question was therefore to know "from where" the different points of view arose. Specifically, each utterance was considered a "proposition," the next one a "reaction," and the next one a "reaction to a reaction" or "a conclusion". Each utterance was then described, not only in its "literal meaning," but from a "dialogical meaning," i.e., how it appeared in the chain of utterances. Concretely, a spreadsheet editor was used to work on every utterance of significative excerpts of the transcriptions of the video recordings. For each of the utterance was indicated: (1) its "literal meaning," the description of which was guided by a thematic rubric (pedagogical reflection, orientation of co-production, or constraint) and a mode of activity (goal, action, or operation), and (2) was indicated its "dialogical meaning," described from its situation (proposal, reaction, conclusion) and its evaluative orientation (agreement, sympathy, objection).

To gather and make sense of the analysis results overall, a "dialogical narrative" (Guillemette, 2017, 2021) was produced. These short texts will make it possible to present and contrast the participants' (including researchers') various postures, while preserving and promoting their dialogical character.

Some emergent thematic

At this moment, I'm still working on the dialogical narratives. This said, from the first part of analysis and the ongoing elaboration of the dialogical narratives, few interesting ideas and reflections appear.

The first thematic is *Otherness*. By pointing out to the links between past and present modes of doing and being in mathematics, history is perceived as an invitation to consider, in our context, teaching and learning mathematics in accordance with the way in which teachers and students engage in classroom activity. In other words, history reveals the actuality of mathematics education. An invitation to attend to the sensuous manners in which, today, teachers and students bring mathematical ideas to the fore and produce mathematical meanings, sensuous manners which could include

perceptual activity, gestures, kinesthetic actions, language, and the use of artifacts, symbols, graphs or diagrams for instance, all kinds of manners that could be discussed, problematized or reinvented.

It was particularly the case when discussing Ptolemy's trigonometry. In the Almagest (Book 1, Chapters 9 to 11) Ptolemy describes his construction of a table of chords, used as a geometric tool to build is astronomical model. He explains the mathematical construction of his table, quite equivalent to the sine function for angle measurement values α from 0 to 180 degrees (see figure 1).

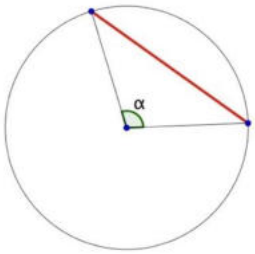

Figure 1: The length of a chord associated with different arcs

Ptolemy trigonometry is understood and practiced in slightly different manners from what we can usually encounter in schools. First, the idea is not to reason on triangles, but on the length of chords associated with given arcs in a circle. In fact, the word "sine" comes from the Latin word "sinus" that means a cavity, a hollow or more broadly a gap or an opening[1]. In a way, the idea of Ptolemy is not to explore mathematically the triangle (or to solve right triangle), but to establish the length of different chords in the unit circle. We reveal here partially lost or occulted senses of mathematics concepts and notions, grounded more concretely in the experience of life and culture, maybe in a more sensuous and direct mode of being in mathematics.

In this experience of otherness in mathematics, when the history enters the game, for the teachers, there was a kind of awareness of the productive aspect of the mathematical activity, including the activity of the classroom. That is to say a kind of a sense of awareness of how mathematics affects us and how we affect mathematics, a productive dialectic between mathematics and the contextualize event of teaching and learning mathematics. Again, here, a sense of responsibility and insight of the sensuous manners in which, today, teachers and students produce mathematical meanings.

A second emerging thematic is *hospitality*. To be a meaningful mediator, cultural and historical elements of mathematics need to appear as something problematic, it must become an enquiry. Such a task, of course, is extremely difficult. With in-service teachers, we can only vaguely perceive its contours. Concretely for the participants, when referring to these elements in the classroom, the gaze turns more unto the teacher, as he becomes a kind of storyteller, this brings special responsibility, different from the usual position of the mathematics teachers. A special attitude toward novelty and that need to be enacted, communicated and shared, novelty of ways of being and doing. New manners that are not the same of that is usually awaited.

[1] Showing an important further cultural circulation, the words *sinus* come from the translation of *jaïb*, which is the Arabic word for cavity or hollow. The word *jaïb* is related phonetically to *jiva*, which is the word for chord in Sanskrit.

The third thematic is more explicitly grounded in the social and the political, the thematic of *reflexivity*. Through our explorations, considering that most of the texts that we read were, in a sense, pedagogical one. Participants were asking: Why mathematics education? Where I want to go with my students? Through these fundamental questions, there was a kind of reconsideration of school mathematical activity. In this perspective, two interrelated ideas emerged. First, the fact that denying learners the opportunity to engage in authentic activity (i.e., posing their own problems, making conjectures, making mistakes, being creatively frustrated, etc.), it is to deny them, in a way mathematics itself, as mathematics, within this cultural and historical perspective appears as something profoundly human, grounded in cultural, scientifical and esthetical context and endeavor. Secondly, mathematics education appears to be linked with social demands that can be critically assessed as mathematics education too has a history and that mathematics education in embedded in it cultural, social and political context. It is interesting to see how the historical, the epistemological, the social and the political intertwine in this precise context.

More generally, my own impression is that there is something there maybe to think about otherness, its valorisation and its defense in a way. Because this experience seems to open the possibility to develop a kind of flexibility for teachers especially. There is an educational reward in terms of social and political awareness for the teachers in the fact of "living a distance". An experience that can help to figure out the mediation of mathematics by the surrounding culture, to perceive learning mathematics as learning a certain manner to be-with-others. A difficulty here is to learn to give ourselves to the otherness of the Other, which is, in a way, the only way to understand ourselves. I would say with Hanna Arendt (1958), to get ready for *action*, that is to say to belong to the world through speaking and acting which take place in the public sphere, which is the very possibility to initiate something new, the possibility to have a beginning. This creation of the new is for Arendt what she calls the miracle of freedom.

Conclusions

In this contribution, I describe some emerging results from a participatory research with in-service teachers on cultural and historical dimension of mathematics. Within an ethical perspective, I brought reflections on the teachers' practices in relation to social and political issues in/of mathematics education. Through these explorations, I discussed diversity and difference in a broad sense and refer to various ways of doing mathematics within the realms of the cultural, the social and the political. I can conclude that the experience of otherness was particularly significant here. These ethical implications should be addressed, explored and developed as it seems to be an interesting way to bridge social and political to cultural and historical issues in mathematics education. Otherwise, on a more research-based perspective, I hope, within this project, having addressed elements of social, cultural or political aspects of mathematics education as well as methodological and epistemological perspectives that could organize such research.

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Curriculum: A restricted or restricting Resource?

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This paper aims at making the discretionary spaces that mathematics curriculum documents communicate to teachers visible by a method of analysis based on Basil Bernstein's notion of framing. The analysis presented is part of an international comparative project. The outcomes show differences in the framing and varying degrees of discretionary spaces granted to teachers for making decisions on how to teach mathematics to their students. Additionally, the outcomes point to the often subtle communication of discretionary spaces including contradictory messages that are conveyed by the issuing school authorities.

Keywords: Mathematics curriculum research, comparative education, turkey, germany.

Introduction

> Competencies of students are at the center of pedagogical processes, meaning the use of capabilities, knowledge and skill for solving problems. This means, that the paths to the achievement of these goals are undetermined. It is the teacher's fundamental responsibility to choose these paths. (Standing Conference of the Ministers of Education [KMK], 2023, p. 5)

Educational policies in Germany emphasize teachers' control over how to teach and aim to reflect this idea in curriculum documents. Hence, large discretionary spaces can be assumed that are necessarily filled out when recontextualizing the curriculum in the classroom. This paper aims to make these discretionary spaces in curriculum documents visible and hence, the mechanisms of control between teacher and the issuing authorities of the curriculum document. However, what this means for the practice of teaching has yet to be explored. This paper is part of a larger study[1] that seeks to explore how variations in realization of curriculum objectives that are shared across educational systems are brought about and how they are executed by schools and teachers. In the comparison, mathematics features as the exemplary school subject, which underwent an international harmonization in light of comparative achievement studies of its curriculum objectives.

By using the notion of *framing* from the complex theorizing apparatus of Basil Bernstein as a methodological tool, two official mathematics curricula are analysed and then compared. The two documents stem from Germany and Turkey, where educational policies and consequently curriculum documents differ significantly from each other. The comparison of these contrasting cases is used to establish a refined understanding of the scope of discretionary spaces in mathematics teaching.

Mathematics Curriculum in Germany and Turkey

We compare the Berlin Mathematics Curriculum [BMC] (Senate Department for Education, Youth and Family, 2023) and the Turkish Mathematics Curriculum [TMC] (Turkish National Ministry of Education [MEB], 2018), as the surrounding project has chosen these two educational systems due

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to their dissimilarities, including e.g. the differences descending from the Turkish centralized approach and the German federal approach to education. The BMC as well as all other Curricula of the federal states of Germany are based on national standards for education (*Bildungsstandards*), which are substantially influenced by the NCTM (KMK, 2004). In the federal state of Berlin, the BMC is legally defined as the basis for school education and published with the status of an administrative regulation by the Senate Department responsible for Education of the state of Berlin (2024, §10 Abs. 1 & §11, Schulgesetz Berlin). The TMC was written and published under coordination of the Turkish Ministry of Education (MEB, 2017). It is influenced by the international OECD framework following the PISA tests and has been revised several times since the AKP-administration came into office in 2002 (Ozturk-Akar, 2022). According to a directive of the MEB from 2019 (MEB, 2019) the TMC serves as the basis for educational activities in school.[2]

Thus, both documents are considered as the basis for mathematics education in their respective context. Despite these contexts differing profoundly from each other, the curricula can be considered similar in terms of their function and objectives, allowing a comparison.

Theoretical Framework

Framing in its original theoretical framework looks at the mechanisms of control in a pedagogic relationship between a transmitter and an acquirer. The educational sociologist Basil Bernstein was able to connect macro and micro level of educational practices by making these mechanisms visible in pedagogic relationships. The framework can only be briefly presented, considering Bernstein's complex underlying understanding of power relations and mechanisms of control in a society.

Power relates to the creation, legitimization, and reproduction of boundaries between categories in pedagogic practice and therefore creates a certain order of things. Control refers to how these power relations are governed in a pedagogic relationship (Bernstein, 2000, pp. 7–12). Control is made visible by the elements of *framing*, consisting of two types of discourses. The instructional discourse [ID] refers to *selection, sequence, pacing,* and *criteria* of a specialized discourse, such as mathematics teaching. The domineering *regulative discourse* [RD] refers to the social relations embedding the ID (Bernstein, 2000, p. 13). The analysis of these five elements can make discretionary spaces visible by assigning Framing values ('weak' or 'strong') to them. While a weak *framing* allows for more discretionary space for interpretation or deviation for the acquirer, a strong *framing* limits this space and leaves the transmitter in more control.[3]

Framing has been used to analyse data from educational settings with different aims (e.g. Al-Ramahi & Davies, 2002; Boelt et al., 2020; Hoadley, 2006; Straehler-Pohl, 2012). Here data usually constitutes of classroom observation and interviews. In this article, the analysed data consists only of written documents. Therefore, it is important to note that the pedagogic relationship is not limited to an educational context but can be found in any field where knowledge is transmitted and acquired

2 The research has been conducted before 1st of September 2024 before the new Turkish Curriculum was put into practice.

3 It is emphasized that the acquirer can only seemingly be in Control, as the acquirer can only be granted Control by the transmitter according to Bernstein (2000).

(Bernstein, 2000). The framework established here understands the official curriculum document that is written by actors on the macro-level of the educational system, disconnected in time and space from educational practices, as the transmitter and the teacher as the acquirer. The document communicates to teachers what legitimate enactment looks like, leaving discretionary spaces to fill.

In this study, the relation between education authorities issuing the curriculum and teachers is considered, as in the studies by Mutemba (2012) and Ferreira & Morais (2010). The idea of a document exercising control, or, *interacting* with its reader disconnected in space and time is found in Wolff (2022), who states that official documents allow their reader to draw conclusions about activities, intentions, and considerations of their authors and publishing entities. Hence, official documents such as mathematics curricula are understood as actively interacting with those who read them, allowing to draw conclusions about control exercised from transmitter to acquirer. Wolff (2022) further argues for the maxim of "order all points" (p. 512) that is based on the idea of the artificial character of a document, revealing its content explicitly and implicitly to its readers. It aims to consider all aspects of the document as possibly meaningful. This approach is adopted in the present study to amplify the possible scope of analysis, as compared to Mutemba (2012) and Ferreira and Morais (2010), who consider only explicit statements on *How* to teach.

Studies that operationalized the concept of *framing* classify the elements of *framing* by assigning values on a scale of 'weak' or 'strong' (Ferreira & Morais, 2010). In our study the international comparison allows us to develop a relative comprehension of 'weak' and 'strong', as the surrounding project understands the comparative approach inter alia as a method to further develop theoretical concepts by involving unfamiliar or unexpected observations (Broadfoot, 1999). Comparing the elements of *framing* on the system level in Turkey and Germany therefore makes it possible to identify and highlight discretionary spaces that otherwise may remain unseen.

Methods

Before analysis both Curriculum documents were translated by the authors, who are German and Turkish native speakers, to ensure a maximum of conservation of meaning and allowing to pivot to the original language when analysing. Layout and structure were equally conserved as close to the originals as possible. The translated samples were chosen according to the surrounding project, which focuses on mathematics of the 4th and 8th grade. Therefore, all continuous text of the mathematics curriculum and any text or tables concerned with the 4th and 8th grade were translated.

For analysis, Wolff's (2022) principle of *order all points* was defined as the consideration of implicit or explicit *Statements* when looking at layout and content. Aspects of order, arrangement and word count are subsumed as layout, semantic meanings are subsumed as content.

For the analysis of content, five elements of *framing* were defined to create a shared understanding:

- *Criteria*: How a teacher or their teaching can be evaluated. It includes any statement about objectives for teachers and their lessons.
- *Selection*: Statements about what mathematics teaching entails and what a teacher does.
- *Sequence*: What comes first, what comes second, etc., when teaching.
- *Pacing*: How much time single elements of teaching are given.

- *Regulative Discourse*: Not primarily concerned with what mathematics teaching looks like, but with social relations and the creation of hierarchies between author and reader.

Statements for the five elements were identified and analysed in each document, allowing a first assessment of the *framing* of each element. While some statements allowed an isolated analysis, many had to be assessed in the broader context of the document and meanings in the original language had to be discussed. The results were compared in a subsequent step and finally discretionary spaces were evaluated according to the findings. Due to the limited format, only a few exemplary findings are presented and cited here.

Findings

The *criteria* in both documents show significant differences. The BMC formulates mathematics teachers' objectives that have a long-lasting influence on pupils but remain underspecified, such as the embedding of mathematics in the pupils' lives (BMC, p. 5), while the TMC describes these objectives more thoroughly, also formulating them for all actors in educational practices. The criteria mentioned allow conclusions on when to evaluate the teaching as well as what points to consider. According to the TMC all actors in education aim to achieve e.g. the following objective:

> Ensuring that students completing primary education, according to their level of development and individuality, become individuals with moral integrity and self-awareness, self-confidence, and self-discipline, possessing basic verbal, numerical, and scientific reasoning skills as well as social skills and aesthetic sensitivity necessary for daily life and effectively using them to become individuals oriented towards a healthy life. (TMC, p. 4)

Despite not only addressing the pupils' mathematical outcomes, the criteria formulated in this statement are directed towards mathematics teachers. Details about when and what to evaluate are becoming clearer when compared to a similar statement of the BMC:

> In mathematics lessons in elementary school, pupils acquire the basic competencies of general mathematics education. This lays the foundations for further learning in secondary education and for a lifelong, successful engagement with demands that involve mathematics. (BMC, p. 5)

Considering the word count, positioning and phrasing used, the BMC shows a much weaker *framing* than the TMC. At the same time, the TMC communicates these objectives as not only the mathematics teachers' responsibility, leaving space for interpretation.

The *selection* is mainly concerned with teaching and assessment to reach pupil outcomes. The TMC elaborates its learning objectives in continuous texts for each grade and provides explanations that accompany each learning objective, which are identified by a code that include the subject (e.g. M for Mathematics) and numbers that identify the grade, learning area, sub-learning area and number of the objective. The document also provides points to consider when implementing the curriculum and elaborates further points and assessment practices in bullet points. All mentioned aspects are provided in detail, often directed towards a teaching entity, as shown in the following example of a learning objective that contains directives providing a limit of teaching and specified activities/tasks:

> M.4.1.6.4. Compares at most three fractions with equal denominators.

a) Models such as length, area, number line are used during comparison activities.

b) Big/small symbols are used when making comparisons. (TMC, p. 47)

The BMC also explicates its learning outcomes in continuous text and tables, however, using far less words and using a more passive language when addressing a teaching entity. Learning objectives are formulated as so-called *learning outcomes* focused on pupils, statements on *how* to teach are rare. Additionally, learning objectives are phrased in a way that allow more content to be included as shown in the following example:

> Arranging fractional numbers on the number line
>
> Comparing and ordering common fractions by direct comparison, finding the common denominator and on the number line (BMC, p. 40)

Despite covering the same topic of arranging fractional numbers, the TMC provides more detailed directives to the teacher, while the BMC only communicates what the pupils should be able to do, at most *suggesting* activities and setting no limits. Overall, the *framing* of the Selection of the TMC is very strong. The *framing* of the Selection of the BMC is weak, except for statements on individualization and pupil orientation as in this example:

> Lessons must enable an individual learning development, which is supported by guidance tools for learning such as learning journals, competency grids or self-assessments (BMC, p. 37)

Here, the BMC provides a similarly strong *framing* to the TMC, as these statements include more details on how to teach and are emphasized by constant mentions, which appears to be in contrast to other aspects of the BMC.

The *sequence* and *pacing* of the TMC appear in a very strong *framing*, as both aspects are communicated in the form of tables and accompanied by explications. In these tables a sequence for the teaching of learning objectives is given, as well as a time distribution in hours and percentages (TMC p. 21). There is no Sequence or Pacing given for other aspects of teaching, such as evaluation practices. The BMC provides a very weak *framing* of both elements, as there is no allocation of time or Sequence to any of the teaching practices. The only Sequence given is concerning pupil outcomes and this Sequence is communicated to the teacher as to be individualized for pupils.

The regulative discourse in both documents often accompanies statements of the elements of the ID, e.g. when referring to pupils' general dispositions to be acquired in the lessons (c.f. criteria above). But it can also be analysed when looking at statements that are exclusively concerned with social relations and hierarchies between transmitter and acquirer. While both documents tend to communicate implicitly what a legitimate teaching practice looks like, the TMC includes statements, where a 'we' is addressed. The 'we' allows the document to formulate the importance of what it says on behalf of the reader, who is not only addressed but seemingly an active part and has responsibility for a collective. Additionally, a few statements in the TMC communicate discretionary spaces directly to the teacher and limit them right after, e.g. when the TMC expects evaluation practices to be the task of teachers and not of the document (TMC, p. 7), followed by a list of seven guiding principles starting with a maximum alignment with "all components of the teaching program" (p. 7). The

Regulative Discourse in both documents can be understood mainly as a mean of legitimization of the documents' messages to the reader. The frequency of these legitimizations led to a 4-way distinction that allows a more structured approach. The four categories are shown in Table 1.

Table 1: Regulative Discourse between school authority and teacher

	unwarranted	warranted
Directive	Very strong Framing	Strong Framing
Suggestion	Very weak Framing	Weak Framing

The categorization of statements is dependent on phrasing (directive or suggestive) and legitimization (warranted or unwarranted). Suggestions leave more decisive space for the reader than Directives. Unwarranted Suggestions are understood as the weakest form of control, while Warranted Suggestions will give a reasoning or legitimacy to follow the Suggestion. An unwarranted Directive can be seen as the highest form of control, and therefore a very strong *framing*, as a warranted Directive leaves the reader in more control by participating in the knowledge behind it. Both documents include few unwarranted suggestions, while unwarranted directives were more prevalent in the TMC. Warranted directives were most common in the TMC, as the document puts emphasis on legitimizing its content through legal and seemingly factual statements, e.g.:

> The learning objectives and explanations in both groups are perfect, up-to-date, valid, and capable of establishing relationships with daily life in the educational process. (TMC, p. 4)

While the BMC contains more warranted suggestions than the TMC, warranted directives can also be found, especially when it comes to pupil orientation and individualization, e.g.:

> Depending on the degree of learning impairment, pupils do not reach the set standards in its entirety or at the specified point in time. This is taken into consideration by individualizing the lessons offered. (BMC, p. 14)

The RD regarding the relation between author and reader as conveyed through the layout differs strongly. While the BMC includes shorter and fewer statements, that are usually presented within a continuous text, the TMC presents a clear structure that makes the information more accessible. Considering all previously mentioned points, the *framing* of the Regulative Discourse of the TMC is considered very strong and the BMC is generally weak but strong when it comes to pupil orientation.

Discussion

The introductory quote to this article from the German Standing Conference of the Ministries of Education speaks of underdetermined paths and the teachers' responsibility to choose these paths. Using the elements of *framing* as means to identify discretionary spaces, this statement is only valid to a limited extent for the Berlin Mathematics Curriculum. While many elements of teaching leave large discretionary spaces for the mathematics teacher, the embedding Regulative Discourse restricts these spaces by focusing on pupil orientation and individualization. To put it into the words of the KMK, some of the possible paths to be chosen when teaching are essentially hidden, while others are promoted. The Turkish Mathematics Curriculum on the other hand, promotes and sometimes even allows only one path and therefore limits the discretionary spaces. This is especially evident when

discretionary spaces are encouraged but at the same time limited, leaving the document in Control over these spaces. The findings hint towards the distribution of power between the issuing authorities and teachers. The effects of this power relation could be meaningful when analysing classroom practice, as the discretionary spaces granted to teachers and schools by curricula constitute possible restrictions that affect the teachers' autonomy for making decisions. For example, the identified limits set for the teaching of the topic of fractions in the TMC, that are not present in the BMC, may grant teachers in Berlin more space to adjust the mathematical content to the students' perceived abilities.

Even when interaction is limited to a document and its reader, Bernstein's (2000) theory of *framing* can be essential regarding a dominant RD. The method of considering the maxim of *order all points* (Wolff, 2022) allowed for the consideration and comparison of elements previously not considered, e.g. the way of structuring the curricula. The established 4-way distinction certainly ignores other possibilities that are neither suggestions nor directives. As this distinction resulted from comparing the two documents and finding differing ways of legitimizations, it should be further refined in future projects. Despite profound differences the comparison of two Mathematics Curricula from different national contexts revealed further findings relevant for future research, such as the addressing of a collective in the TMC, the conspicuous statements regarding individualization in the BMC or the tasks or materials to be used when teaching certain mathematical topics formulated by the issuing authorities. The method is limited when it comes to the interpretation of meaning which is dependent on the researchers' knowledge on the subject, the educational system and language. Additionally, it became clear that learning objectives or outcomes are communicated for pupils and rarely address teachers directly. Considering these challenges, the focus on mathematics has proven to be useful for the comparison, as the documents showed similarities regarding objectives and outcomes and details such as materials, while having highly different approaches vis-à-vis methods and communication. Further research is needed to confirm the validity of the presented method, and the outcomes will need to be confirmed and differentiated in following research for this project.

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Cultivating teacher leaders in hierarchical schools and school systems

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"The Lab" offers professional development in mathematics referred to as the "Institute" for pre-tertiary teachers who work in northern New Mexico. The primary goal of the Institute is to support their learning of specialized mathematical knowledge. A socio-political theoretical perspective is used to explore the opportunities that three minoritized teachers had to engage as teacher leaders in mathematics in their schools and school system after participating in the Institute over many years. Though the teachers characterized themselves as teacher leaders in mathematics, it will be shown that a ceiling existed for them that limited their growth as leaders. This ceiling resulted from the fact that the teachers worked in a hierarchical school system in which they experienced a lack of administrative support. We argue that to cultivate teacher leaders in mathematics, PD facilitators should consider the political context of school districts in which their participants work.

Keywords: Mathematics professional development, politics of education, teacher leaders in mathematics.

Hierarchical school systems are a historical legacy of European colonialism (Williams, 2019). A defining characteristic of hierarchical school systems is the centralization of authority and decision-making in the administrative structure of a school system, which often limits leadership opportunities for teachers (Bimber, 1993). In the U.S. context, hierarchical school systems are a common feature of schools situated in economically underserved areas of the country that largely serve minoritized students (Newmann et al. 1997). Teachers who work in hierarchical and bureaucratic U.S. school systems found in economically underserved communities tend to be isolated from other teachers and administrators, inhibiting the development of teacher leaders in mathematics (Kitchen, 2003). This is problematic because teacher leaders in mathematics are needed to improve the teaching and learning of mathematics in schools situated in economically underserved communities (Kitchen, 2020; Kitchen et al., 2021).

In this paper, a sociopolitical theoretical perspective is used to explore opportunities that three veteran, minoritized[1] teachers had to develop and engage as teacher leaders in mathematics. The three teachers worked in a hierarchical school district that served an economically underserved community in the United States. The three teachers highlighted in this study were unique because of their long-term engagement in professional development (PD) offered through the "Lab." Every academic year since 2009, the Lab has offered PD trainings in mathematics, referred to collectively as the "Institute." Institute participants are elementary- and secondary-level teachers of mathematics, almost all of whom teach in highly diverse, economically underserved communities located in northern New

[1] In the U.S. context, other terms that are commonly used and are essentially synonymous with "minorized" are "culturally and linguistically diverse" (CLD) and "people of color" (See Kitchen et al., 2021).

Mexico, USA. Six full-day trainings during the academic year and a week of full-day summer sessions are offered annually through the Institute. The main goal of the Institute has been to cultivate teachers' mathematical content and pedagogical knowledge—teachers' "specialized mathematical knowledge" (Hill et al., 2005). Throughout, teachers solve problems collaboratively in the Institute and learn about the value of working together, rather than in competition.

In this study, we address the following two research questions:

1. How did teachers who work in a rural, U.S. context characterize themselves as school leaders in mathematics?
2. What opportunities do teachers in a rural, U.S. context, have to engage in mathematics leadership in their schools and district?

We argue that "PD facilitators" who want to cultivate teacher leaders in mathematics should consider the political contexts of the school districts in which their PD participants work. PD facilitators are Mathematics Teacher Educators (MTEs) who lead PD for practicing teachers of mathematics (Karsenty et al., 2023). Facilitators "manage the PD activities, set norms for interactions, support teachers' exchange of experiences and insights, monitor the discussion, and work with teachers toward the goals set for the PD" (p. 28). Before we address our research questions, the research literature on teacher leadership and our theoretical perspective are provided.

Teacher Leadership

Teacher leaders are defined as a teacher or group of teachers who influence colleagues and school leaders to improve learning and achievement (York-Barr & Duke, 2004). Teacher leaders also support their colleagues to improve conditions at their schools and seek to develop a democratic and collaborative school culture (Muijs & Harris, 2006). Teachers become leaders in mathematics as their confidence increases in their specialized mathematical knowledge and instructional abilities (Chval et al., 2010). Teachers build a sense of empowerment as leaders as they develop instructional strategies that lead to increased learning for their students (Godoy-Penteado & Skovsmose, 2009). Teacher leadership includes both formal and informal leadership roles (Liu, 2021). Informal teacher leadership roles can include introducing new teaching techniques to colleagues or mentoring a novice teacher. More formal teacher leadership roles could include taking on the head of a department (Liu, 2021). Among the characteristics of teacher leaders is that they value collegial professional relationships, are student-centric, and are continually engaged in professional development (Hunzicker, 2017). The social, political, and structural contexts of schooling such as how much teacher expertise is valued are among the conditions that support or inhibit the development of teacher leaders (Danielson, 2006). In addition, teacher leadership is influenced by the degree to which school leaders foster teachers taking up leadership (York-Barr & Duke, 2004).

Transforming schools situated in post-colonial countries through teacher leaders

A legacy of colonialism is the historical inequalities found in schools in post-colonial countries (Nyamnjoh, 2012). An outcome is the marginalization of minoritized students who attend schools located in poor communities in the United States, resulting in these students having inequitable access to a challenging education in mathematics (Leonard & Martin, 2013). These schools often lack

resources such as instructional materials, have high teacher turnover rates, and offer second-rate instructional programs (Ingersoll & May, 2012). Educational disparities found in U.S. schools that largely serve diverse, economically underserved communities can also be found in post-colonial countries. As just one example, disparities found in South African schools that serve Blacks are comparable to inequities found in U.S. schools that serve poor, minoritized students (Chirinda et al., 2022). Decentralizing hierarchical school systems can improve access to educational opportunities for students when school leaders distribute leadership opportunities to teacher leaders (Spillane et al., 2003). Distributed leadership models demonstrate the crucial role that teacher leaders can play to positively transform a school's culture (Meyer & Slater-Brown, 2022).

Socio-political theoretical perspective

A socio-political theoretical perspective involves examining educational policies and practices from the viewpoint that differential access to educational opportunities is grounded in differences based on racialized and classed experiences (Battey, 2015). A sociopolitical theoretical perspective places issues of power center stage such as the economic, social, cultural, and political contexts of learning (Tabron et al., 2021). According to Foucault (1977), power and knowledge are in a dialectical relationship that operate to further produce one another. For example, school administrators leverage their knowledge about leadership, teachers, students, and community as a means to help develop a school's learning environment. The administrator's ability to influence the school setting demonstrates the power embodied by the administrator. After initially examining whether teachers identified as leaders in mathematics based on their experiences as participants in the Institute, we use a socio-political lens to study how power, as embodied in hierarchical school systems, limits the development of teacher leaders.

Methods and data sources

We use research strategies used in phenomenology to examine how three minoritized teachers understand, interpret, and qualify their lived experiences (Creswell & Poth, 2018) regarding opportunities provided to them to develop as leaders in mathematics. In-depth interviews were the primary data source for this study. By listening to the teachers, a description of the essence of the research participants' experiences related to leadership was created (Creswell & Poth, 2018). Teachers' narratives were examined through a socio-political lens to interrogate opportunities and barriers that participants had in their schools and district to grow as teacher leaders in mathematics.

Participants

In this study, we employed purposeful sampling. The three participants were unique because they had been part of the Institute for at least four years (few had done this). All three participants worked in the same school district (District M), but not the same school. All three are Hispanic[2] women. One of the participants taught at the middle school level (grades 6-8 in the United States) and the other two taught at the elementary school level. All were veteran teachers who had taught for 15 years or more.

[2] Hispanic is a commonly used term in New Mexico denoting someone of Spanish ancestry.

Data collection

Semi-structured interviews were carried out via Zoom with the three research participants in December 2021. Each interview was carried out by a different author. Each interview was recorded and lasted between 30 and 45 minutes. Eight interview questions were asked, one with a pre-determined probe question. The goal of the interviews was to engage the research participants in a conversation through the use of open-ended questions (Riessman, 1993). At times, an interviewer asked a probe question to help clarify a participant's response and promote conversation, both characteristics of phenomenological studies (Creswell & Poth, 2018).

Data analysis

We used a thematic analysis approach in which the initial goal is to collect the description of experiences and then to inductively group similar experiences (Creswell & Poth, 2018). To address interrater reliability, the four authors conducted a thematic analysis independently, identifying themes found across the interviews. We met once via Zoom and engaged in an iterative process of discussing the themes that emerged across the interviews, reflecting upon, and then clarifying participants' views (Creswell & Poth, 2018). Through this iterative process, we achieved substantial agreement about the themes that emerged from the data. We also analyzed these themes through a sociopolitical lens to consider the themes regarding the political contexts that the participants shared about their work.

Results

The following two themes were identified in response to the first research question: Leadership and teaching confidence and teachers' views of themselves as leaders. The following three themes were identified in response to the second research question: Taking on leadership roles, administrators' reactions to teachers' participation in the Institute, and school leadership and the status quo.

Leadership and teaching confidence

In their narratives, the research participants related growth in their leadership skills with their nascent confidence to teach mathematics for conceptual understanding. For instance, Ava said, "Participating in the Institute has been supportive in terms of me becoming a teacher leader because it helps me to have more confidence in my own mathematical abilities and to explain things to others." This was a common view shared by all the research participants.

Teachers' view of themselves as leaders

Ava described how her colleagues viewed her as a leader simply because she had been a teacher for many years. She explained why she was the only teacher leader in mathematics at her school:

> I don't know of any other teacher leaders presently that are in the school that I'm at other than myself… I have mentored pre-service teachers and first-year teachers, so that is something that I have done. I have partnered with other teachers to give professional development, and plan those activities…

Victoria also saw herself as a teacher leader. She said, "I'm on the team that's working for the after-school math program to put that together for students, that same team is working on the school math intervention plan." She also characterized her role as a teacher leader in mathematics as follows:

> I also feel when my grade level team has questions whether it's about the math standards, the pacing guide, just different aspects when it comes to math, they feel comfortable coming to me, and I can give them suggestions, ideas. I can say, you know, 'I've tried this in the classroom. This is what I found has worked for me.' Or tell them 'you might want to try this if they're struggling…'

In summary, similar to Chval et al. (2010), the participants attributed their emergence as teacher leaders to their increased confidence in their mathematical knowledge and development as teachers of mathematics to their sustained involvement in the Institute. Similar to Hunzicker (2017), the participants' increased confidence led to the development of their identities as leaders. They also saw themselves as teacher leaders because they had mentored teachers and had led PD for their colleagues. The research participants' responses regarding the second research question follow now.

Taking on leadership roles in the Institute and their schools

Ava believed that teachers who had participated in the Institute for many years should have opportunities to assist and mentor other teachers during the Institute: "Several of us who have done this for so long could be put into other roles where we might be able to assist those people in the activities or assist them in some way, other than just being a participant." She believed that PD facilitators such as MSA should work intentionally with school/district leadership so that school/district leaders can continue the work initiated by MSA.

How administrators responded to teachers' participation in the Institute

Ava told us that while "administrators are supportive and see the importance of [the Institute]… they kind of just leave it up to us to attend and implement." She believed that school administrators did not follow up with teachers after they attended the Institute. Analogously, Sofia said: "I just sometimes feel like we learned something and then it's not supported enough to allow the teachers to do what they need to do to branch out to their colleagues in a positive and supportive way." A recurring argument made by Sofia was that teacher leaders were not provided administrative support to develop their colleagues' pedagogical skills. From Sofia's perspective, rather than supporting them to work to improve their colleagues' instruction, administrators exerted their power:

> They are embedding themselves into the Institute as a participant, rather than a support. So, it's almost as if they are learning everything for themselves, all over again, rather than looking at their role as a support system. And that's definitely been a hinder. It's like they're looking at us more as a threat, I guess, that we would know more about different math strategies than they would.

School leadership and the status quo

Sofia talked on several occasions about how her school district administrators were primarily interested in preserving the status quo power structure:

> You know there's a power broker mentality [in the district that was a carry-over from the previous administration]… I think it's getting better, but it does still leave that bad taste in your mouth when you do have certain situations that still keep popping up and it's hard for teachers that, you know, want to gain traction.

Sofia continued by describing how some teachers "have either taken on really strong leadership roles that kind of suppressed the others and so that again stops collaborative work." She explained: "It's one person being overpowering, and it's her way or the highway. This person has the admin's back [because they're 'buddies']. Then it becomes the one-man show."

Ava shared that school administrators placed more attention on getting lesson plans from teachers than on student learning. Ava doubted that her school district's administration was committed to cultivating a culture of learning in the district:

> Their [administrators'] focus is in a different place… When we have our meetings, it's only focused on getting those lesson plans done, which to me is not where the focus should be. It should be on, you know, how are the students thinking, how are they learning, or what are they doing in the classroom… And what are the outcomes of the strategies that we're using?

Analogously, Sofia did not believe that district administrators were committed to building a learning culture:

> But the culture of learning needs to be built and I think admin has to be the backbone of that and allow teachers to either be teacher leaders and not micromanage that and allow it to happen organically. Hopefully, teachers build that relationship of being able to talk about math all day long because that's happening now.

In summary, we explored the distributed opportunities the teachers had to be leaders in mathematics. Similar to Chval et al.'s (2010) study, we found out about the challenges that the research participants dealt with to become teacher leaders such as a lack of administrative support. Ava and Sofia described that their administrators did not support them to improve their colleagues' mathematics instruction. Sofia noted that there was historic precedence in District M concerning the administration brandishing power and perpetuating its power. Moreover, school principals and district administrators rarely afforded teachers leadership opportunities.

Discussion

The three research participants gained confidence in their mathematical knowledge and in their abilities to teach mathematics through their involvement in the Institute (Kitchen et al., 2021). They viewed themselves as teacher leaders in mathematics. Similarly, their colleagues regarded them as teacher leaders in mathematics. However, according to Ava and Sofia, their work as school leaders was not particularly respected or valued by school leaders, resulting in the participants being granted few leadership roles (Liu, 2021). Similar to Newmann et al. (1997), we found that distributed leadership was not a priority in a hierarchical, U.S. school district populated by diverse, economically underserved students. Teacher development as leaders was restricted in District M. From a sociopolitical perspective, the District M administration used its power to prevent teachers from engaging in shared decision-making associated with mathematics programming in the school district. A unique contribution of this study is that it shows how a ceiling may exist on teacher leaders in mathematics who lack distributed leadership opportunities, particularly for teachers who work in hierarchical school systems in which maintenance of the status quo is a priority.

PD facilitators in mathematics need to consider the political context of schooling (Danielson, 2006) if one of their goals is to cultivate teacher leaders among their PD participants. Given the educational inequalities in countries that experienced colonization in particular (Nyamnjoh, 2012), PD facilitators in post-colonial countries need to take into account the political context of schooling as a means to explicitly develop the skills their PD participants need to become teacher leaders in mathematics. Considering South Africa as just one example, PD facilitators should prioritize cultivating teacher leaders who have the skills needed to positively transform mathematics education for Black students. Moreover, as immigration from the Global South to the Global North is fueling growing interest in the Global North in diversity, much can be learned from research conducted in ethnically and racially diverse contexts in which power has operated to exclude diverse perspectives (Pineda & Mishra, 2023). To take on leadership roles in hierarchical school systems, teachers need explicit support from PD facilitators to learn strategies that they can apply to challenge a status quo that limits their opportunities to cultivate their leadership skills.

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Contribution to sustainability competences through mathematics education

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To effectively contribute to a more inclusive and sustainable future through mathematics education, it is essential for mathematics curricula to incorporate sustainability competences. We have analysed the presence in the curriculum of keywords that are related to terms involving the GreenComp sustainability competences, organized into four areas: 1) embodying sustainability values, 2) embracing complexity in sustainability, 3) envisioning sustainable futures and 4) acting for sustainability. The results show that the area most present in the subject of Mathematics is embracing complexity in sustainability, while the least present is embodying sustainability values. We conclude that it is necessary to increase the presence of sustainability competences in the mathematics curriculum and the teacher training programmes.

Keywords: Sustainability, education for sustainable development, sustainability competences, mathematics education, primary education.

Introduction

Since the first international calls on sustainability to address social, environmental and economic crises began to emerge several decades ago (e.g., Orr, 2013;Tilbury, 2011; UNESCO, 2015), Education for Sustainable Development (ESD) has been gradually incorporated into all educational stages, from early childhood education to higher education, to promote the development of sustainability skills (e.g., Bianchi et al., 2022; de Haan, 2010; Rieckmann, 2012; UNESCO, 2017; Wiek et al., 2011). To enhance teacher professional development in sustainability competences, the curriculum must integrate ESD in a cross-cutting manner. Therefore, at present, it does not seem to make much sense to train teachers to teach mathematics if this has no impact on the great challenges and changes of our times. Mathematics teachers should be provided with the necessary knowledge and tools to play an effective role as agents of social change. Preliminary studies have investigated the integration of ESD into the curriculum in general. Guardeño and Monsalve (2024) describe how ESD has evolved in the Spanish educational system through laws from 1985, noting that the latter proposes various updates and brings ESD to light for the first time in the school curriculum, thus responding to the problems facing society involving the Sustainable Development Goals. At the same time, these authors also analyse the curriculum of Ireland, noting the longer history of ESD policies and underscoring the addition of specific subjects, in addition to promoting a cross-cutting approach, as is the case in Spain. More specifically, Tesfamicael and Enge (2024) analysed the presence of sustainability in the Norwegian mathematics curriculum, noting that although the mathematics curriculum does not explicitly incorporate terms such as “sustainability” or “sustainable development”, which could imply that mathematics and sustainability are not connected, its six basic elements

- exploration and problem solving, modelling and applications, reasoning and argumentation, representation and communication, abstraction and generalization, and mathematical fields of knowledge - offer opportunities to integrate sustainability learning. Vásquez et al. (2022) analyse the primary education curricula of four Latin American countries, Colombia, Costa Rica, Chile and Mexico, showing that, in general terms, the teaching-learning process of mathematics is poorly aligned with ESD, with a low presence of key competences for sustainability. Based on these data, the authors conclude that these findings represent a roadmap, both for teacher training institutions and to provide a new educational approach that, through mathematics education, can be used to promote student understanding of the different problems (social, economic and environmental), as well as the measures that must be taken to transform and work for a more sustainable world.

Taking as a starting point these studies, the purpose of this paper is to determine how sustainability competences have been integrated into the current primary education curriculum in Spain (MEFP, 2022), in particular in the section of this document regarding the mathematics knowledge area. To this end, the GreenComp European sustainability competence framework (Bianchi et al., 2022) has been taken as a reference, as it is one of the pillars on which the MEFP itself is based to promote a certain competency model in keeping with the leading supranational recommendations (Saiz, 2022).

GreenComp: a framework for analysing sustainability competences

As stated, this paper relies on the Greencomp report (Bianchi et al., 2022), regarded as one of the foundations of sustainability (Saiz, 2022), as a framework for analysing sustainability competences in the Spanish primary education curriculum. The report GreenComp (Bianchi et al., 2022) is from Science for Policy series of the Joint Research Centre (JRC), the European Commission's science and knowledge service, which aims to provide factual scientific support to the Union's policy-making process. Table 1 shows the two main elements of the framework (areas and competences) and provides a description of each competence.

Table 1: Adaptation of the European GreenComp framework of sustainability competences

Area	Competence	Description
Embodying sustainability values	Valuing sustainability	Reflecting on personal values and their cultural variability. The goal is to raise awareness of the diversity of values and their relevance in ethical and sustainable decision-making.
	Supporting fairness	Supporting equity and justice for current and future generations, learning from past experiences to promote sustainability.
	Promoting nature	Recognizing the interdependence between humans and nature, promoting respect and empathy towards other species.
Embracing comp	Systems thinking	Understanding of the interaction between elements within and between systems, in different scales and subject areas.

	Critical thinking	It involves critically evaluating information and arguments, identifying assumptions, and questioning the status quo.
	Problem framing	Formulating challenges as sustainability issues, considering their difficulty, people, time and place.
Envisioning sustainable futures	Futures literacy	Planning for sustainable futures, developing alternative scenarios and identifying actions to achieve them.
	Adaptability	Promoting the ability to manage transitions and challenges in complex sustainability situations, making decisions in the face of uncertainty and ambiguity, while considering risk.
	Exploratory thinking	Adopting relational thinking by linking various disciplines, using creativity and experimentation with innovative methods
Acting for sustainability	Political agency	Navigating the political system, identifying responsibility and accountability for unsustainable behavior, and demanding effective policies for sustainability.
	Collective action	Promoting the ability to act collaboratively with other agents to promote change.
	Individual initiative	Identifying one's potential for sustainability, recognizing what actions can be taken.

Methods

In keeping with the goal of our study, which, as indicated, consists of determining how sustainability skills have been integrated into the current primary school mathematics curriculum in Spain, we adopted a descriptive qualitative approach (Cohen et al., 2017), relying on the content analysis method using syntactic units as a research technique (Krippendorff, 2013). The Spanish curriculum law includes the principles of basic education (from 6 to 12 years old) and lays out the organization and minimum teaching content of primary education in Spain. In this document, for each area of knowledge of primary education (Understanding Nature, Society and Culture; Art Education; Physical Education; Education on Civic and Ethical Values; Spanish Language and Literature; Foreign Language; Mathematics and Religion), specific competences and basic knowledge are defined.

The keywords were stablished using a set of exclusion keywords related to the four GreenComp competence areas described in Table 1. To do so, the following were considered: a) propose an initial list of keywords generated from the description given in the GreenComp document for each of the 12 competences; b) review of the keywords by three European experts in the field of ESD; c) analyse the curriculum to ensure consistency between it and the keywords; and d) incorporate some keywords that were omitted in the first list and involve the topic in question. The final keywords used in the coding process for each area where:

- Embodying sustainability values: Values (personal, sustainability); empathy / empathize; equity / inequity; justice; nature; ecosystem(s).
- Embracing complexity in sustainability: Systems thinking / systematic perspective / systematic relationships / systematic vision; Critical thinking / critical spirit / critical analysis / critical reflection; Interaction; connection(s); interdisciplinary / interdisciplinarity; complexity; global problems.
- Envisioning sustainable futures: future; plan / planning; scenario; uncertainty; risk(s); adaptability / adaptation; change(s); imagining / imagination; creativity.
- Acting for sustainability: Social justice; collective action; collaboration / collaborate; community; responsibility / responsibly; initiative.

Results

In keeping with the goal of our study, we first analysed the presence of sustainability competences in the curriculum to obtain an initial overview of their presence in the different areas. We then focused our analysis on the area of Mathematics.

We have identified the presence of the terms related to the sustainability competences defined in GreenComp (Figure 1) in all the Spanish primary education curriculum including all the knowledge areas, and we have also identified which of these terms are located specifically in the area of Mathematics (Table 4).

Table 2: Distribution of terms related to sustainability competences

GreenComp areas	Keyword	All the curriculum *f*	Mathematics area *f*
1. Embodying sustainability values	Values (personal, sustainability)	54	1
	Empathy / Empathize	29	1
	Equity / inequity	6	0
	Justice	9	0
	Nature	19	0
	Ecosystem(s)	7	0
	TOTAL	**124**	**2**
2. Embracing complexity in sustainability	Systems thinking / systematic perspective / systematic relationships / systematic vision	7	0
	Critical thinking / critical spirit / critical analysis / critical reflection	38	8
	Interaction	49	1
	Connection(s)	22	15
	Interdisciplinary / Interdisciplinarity	13	2
	Complexity	0	0
	Global problems	1	0
	TOTAL	**130**	**26**
3. Envisioning sustainable futures	Future	10	5
	Plan / planning	0	0
	Scenario	1	1
	Uncertainty	18	7
	Risk(s)	18	1
	Adaptability / adaptation	6	1
	Change(s)	20	1
	Imagining / Imagination	2	0

	Creativity	23	1
	TOTAL	**98**	**17**
4. Acting for sustainability	Social justice	2	0
	Collective action	0	0
	Collaboration / collaborate	18	1
	Community	29	0
	Responsibility / responsibly	36	6
	Initiative	9	3
	TOTAL	**94**	**10**

The data in Table 2 show that the competences in the area "Acting for sustainability" are the least present in the Spanish primary education curriculum. In the Mathematics section, the scarce (or almost zero) presence of terms related to the area "Embodying sustainability values" is striking, while the area with the greatest presence in both the entire Spanish primary education curriculum and in the mathematics section is related to the area "Embracing complexity in sustainability".

Below is an example of an extract from the mathematics section of the curriculum that showcases one of the keywords that have been related to each of the GreenComp areas.

Table 3: Examples of fragments of the primary education Spanish curriculum related to GreenComp areas in the section of mathematics

GreenComp scopes	Term	Fragments
1. Embodying sustainability values	Values (personal, sustainability)	"The goal of this specific competence is for students to work on the values of respect, equality and peaceful conflict resolution, while solving the mathematical challenges proposed, developing skills of effective communication, planning, inquiry, motivation and trust, in order to create healthy relationships and work environments, for example, through participation in diverse teams with assigned roles. This allows building healthy, supportive and committed relationships, strengthening self-confidence and normalizing situations of coexistence in equality" (p. 105)
2. Embracing complexity in sustainability	Connection	"Knowing a variety of strategies makes it possible to confidently deal with challenges and it facilitates engagement. Recognizing the connection between mathematics and other areas, with real life or with one's own experience, increases the mathematical toolkit of students. It is important that students have the opportunity to experience mathematics in different contexts (personal, school, social, scientific, humanistic and environmental) to become accustomed to identifying mathematical aspects in multiple situations". (p. 104)
3. Envisioning sustainable futures	Uncertainty	"Probability as a subjective measure of uncertainty. Recognition of uncertainty in everyday situations and by conducting experiments." (p. 112)
4. Acting for sustainability	Responsibility / responsibly	"Collaborate in the distribution of tasks, assuming and respecting the individual responsibilities assigned and using simple teamwork strategies aimed at achieving shared goals." (p. 113)

Related to the area "Embodying sustainability values", the keywords "values" and "empathy", which are only found once each in the mathematics section of the curriculum, are related to the competence of supporting fairness. The values promoted are respect,

equality and peaceful conflict resolution, which contribute to the support of equity and justice for current and future generations. There are no terms related to the other competences in this field.

Related to the area "Embracing complexity in sustainability", the keywords "interaction", "connections" and "interdisciplinarity" in the mathematics section of the curriculum relate to the competence of systems thinking. The curriculum acknowledges that connections between different mathematical ideas, as well as identifying the mathematics involved in other areas of everyday life, contributes to interpreting diverse situations and contexts, providing a global and interrelated view of the world. Such a vision helps us understand that mathematical ideas are not isolated elements, but are rather related with one another, giving rise to a whole. The keyword "critical thinking" on the one hand, it encourages observation and reflection in the understanding of random phenomena in everyday situations; on the other, it promotes analysis and adaptation of solutions to the proposed context and the implications they would have from various points of view.

Related to the area "Envisioning sustainable futures", the keywords "uncertainty", "risk", "adaptation" and "change" have been identified within the mathematics section of the curriculum. All of them contribute to the development of personal skills that help adapt to new situations, take risks and accept error or changes as part of the learning process, and adapt to situations of uncertainty in everyday situations.

Related to the area "Acting for sustainability", the keywords that are found in the mathematics section of the curriculum are related to the competences of collective action and individual initiative, but none in relation to the political agency competence. On the one hand, the keywords "collaborate", "responsibility" and "responsibly" contribute to collective action by fostering coordination, collaboration and cooperation among peers to solve problems more effectively. On the other, whenever the keyword "initiative" is found in the mathematics curriculum, it is in relation to individual initiative, sensing one's own potential and showing initiative to seek solutions or strategies to the problems posed.

Discussion and conclusions

This paper has analysed how the Spanish primary mathematics curriculum is integrating sustainability competences, since the curriculum is (or should be) a frame of reference that guides the teaching activity of teachers. To carry out the analysis, we used the GreenComp report (Bianchi et al., 2022) as a reference.

From this analytical perspective, we have identified "Embracing complexity in sustainability" as the most present area of the sustainability competences, with the most recurrent terms being associated with "critical thinking" and "connections". This result is not surprising, since, on the one hand, the Spanish curriculum has a specific mathematical competence on "solving problem situations", which seeks to, among other purposes, promote "critical reflection on the adequacy of the solutions to the proposed context and the implications they would have from various points of view (responsible consumption, health, environment, etc.)" (MEFP, 2022, p. 103). On the other hand, there is another specific competence on "recognizing and using connections between different

mathematical ideas, as well as identifying the mathematics involved in other areas or in everyday life" (p. 104). By contrast, the least present area is "Embodying sustainability values", which is surprising, especially if we take into account that in the new mathematics curriculum, specific competences of a socio-affective nature have been added, which describe the skills to be achieved personally, emotionally, and socially.

In light of these findings, on the one hand, we conclude that the area of Mathematics knowledge can contribute to the development of some sustainability skills, due to the very nature of mathematical skills, in line with the data from Tesfamicael and Enge (2024). Thus, there is a practically natural relationship between learning mathematics and sustainability skills related to critical thinking, systems thinking and the ability to establish connections, both between knowledge in the area of mathematics, but also in relation to other areas of knowledge.

The study also exhibits certain deficiencies in the mathematics section of the curriculum, specifically regarding the GreenComp areas "Embodying sustainability values" and "Acting for sustainability". Although we identified the presence of keywords fromboth areas), these are scarce or appear with meanings different from those attributed from the ESD framework. Thus, sustainability appears as a pretext to solve certain mathematical questions. It is presented as context, a backdrop against which to solve a challenge, but the values themselves are not discussed in depth, nor is the students' ability to propose or carry out actions that contribute to the development of sustainability reinforced. It is in this sense that it is necessary to lay out guidelines that allow the establishment of an Agency for Action, whose goal is to promote a greater presence of these sustainability competences in the mathematics curriculum of all educational stages, and teacher training programmes since the decisions of the present will determine the future.

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Mathematics Education and the arenas of new conflicts over inequality – a theoretical framework for research?

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Research on inequality in mathematics education has gained both quality and quantity in the last decades. Since then, new social conflicts and debates have risen. In this paper we therefore present a framework for analysing inequalities and related attitudes developed by the sociologists Steffen Mau and colleagues. We argue that the theoretical assumptions build a useful framework for conceptualising and understanding mathematics educational research on social, political and cultural aspects. To do so we will explain the idea of arenas of new conflicts over inequality. We will then use the theoretical framework to categorise previous mathematics educational research by way of example and re-read it against the background of these new conflicts. Finally, we will analyse data from an own empirical study on practices in early mathematics education to show its heuristic potential.

Keywords: Inequality, research framework, practices in mathematics education.

Introduction

One of the most renowned approaches to conceptualising equity in mathematics education research is known to be Rochelle Gutiérrez's dimensioning of access, achievement, identity, and power (Gutiérrez, 2012). Numerous studies work with this scheme and generate highly relevant findings in the context of diversity and social and political challenges of mathematics education (e.g. McGraw et al., 2023; Mintos et al., 2019). Research also shows that prospective teachers are quite capable of noticing and recognising diversity – but have difficulties when they need to transfer this topic to the teaching and learning of mathematics (Hummel & Bohlmann, 2022). In addition, social conflicts have intensified over the last 15 years and new ones have emerged. The question arises as to whether these conflicts can be identified in the practice of mathematics teaching and in the mathematics educational research. In addition to Gutiérrez's approach, we want to ask how *inequality* can be conceptualised. We propose an approach by the German sociologists Steffen Mau and colleagues (2023), who describe four arenas of new conflicts over inequality, as a heuristic scheme to understand inequality in both mathematics educational practice and research.

The four arenas of new conflicts over inequality

Mau and colleagues (2023) offer a multidimensional approach of conflicts over inequality that enables a profound analysis of the conflict space. They counter the assumption of a two-camp polarisation in German society with a more differentiated picture of four dimensions of inequality conflicts. According to the authors, the core of inequality conflicts are discourses in the context of socially anchored forms of marginalisation and disadvantage on the one hand and preferential treatment on the other. In a society, there are always several conflicts over inequality that occur in parallel and are in some respects interlinked, which is why Mau and colleagues use the term ‘arena’.

Inequality arenas are then rather independent fields of discourse that make it possible to organise the pluralised structure of political disputes and to grasp and analyse dynamics more precisely. According to the authors, the typology of the four arenas is not a new theory of inequality, but a heuristic for structuring the conflict space. It incorporates the work of various sociologists, e.g. Bourdieu's capital theory and the associated symbolic battles for recognition.

The four arenas that Mau et al. (2023) distinguish are labelled *up-down*, *us-them, in-out,* and *today-tomorrow*. Up-down inequalities refer to social status groups and economic redistribution. These are vertical inequalities in terms of material resources, income, and wealth. People have varying degrees of economic capital and the market value of professional qualifications differs, which creates a socio-economic divide within a society. Us-them inequalities relate to societal recognition of diversity, e.g. gender, sexual identity, race, and so on. The focus is on identity policy issues relating to diversity, experiences of discrimination or marginalisation, and recognition of social groups within a society. It is about the prestige of individuals and groups as well as access to resources and positions in the sense of social inequality due to attributed membership of a particular group. As people differentiate between their own group and 'others' by drawing boundaries, the question of what is considered 'normal' and worth recognising is also negotiated here. In-out inequalities refer to the dualism of closure and access, especially with regard to openness towards immigrants. Relevant issues can be categorised according to whether they relate to the territorial border (e.g. controlling immigration) or whether they concern the membership border (e.g. migrants' access to public services). The arena of today-tomorrow inequalities deals with conflicts over ecology and climate protection or change. These are urgent questions of our time that require action today, the results of which will only become visible in the future. Issues such as lifestyle, preservation of property or the fair distribution of burdens arising from climate policy measures are becoming increasingly relevant and are giving rise to dynamic debates.

In their study Mau and colleagues analysed the attitudes of German citizens to issues that can be assigned to the four arenas of conflict (e.g. the immigration of workers from non-EU countries). They summarise that a division in German society cannot be identified, although there are major differences between the arenas. The arena of in-out inequalities is the most polarised; differences in terms of class and education have a particular impact. Our aim is now not do deliver specific information about beliefs on these conflicts, but to use them as a heuristic to understand inequalities in mathematics education.

Social, political and cultural aspects in mathematics education

In order to use the conceptualisation with existing research, we will reanalyse selected studies in the context of social, political and cultural aspects in mathematics education against the background of the four arenas. The first arena to emerge is the 'classic' *up-down* conflicts associated with social stratification. A study conducted by Daniela Steflitsch (2023) is interesting for this context. Steflitsch carried out elements of critical mathematical education in a socially rather homogeneous, privileged classroom. The students discussed the issues raised about the distribution of wealth in a more meritocratic way, rather than focusing on equity, indeed some emphasised the need to make unemployed people more accountable. Using the scheme presented, these attitudes can be interpreted

as an *up-down* conflict in which the privileges of one group in particular are defended. However, it is interesting to note that an *us-them* logic also shines through in the study:

> They distinguished themselves from the 'poor' and the 'rich'. Depending on the focus of argumentation, this meant that they either felt they belonged to the privileged group since they were better off than the lower or bottom percent or that they were in the group of less wealthy and, therefore less privileged people, as they are far apart from the upper or top percent. (Steflitsch, 2023, p. 1848)

Mau et al. (2023) describe this arena as one in which battles for recognition are fought out. While the up-down arena is determined by property and markets, and the in-out arena by birth, the unequal recognition of social groups is also a question of positioning. Only through a certain social position is it possible for students to freely assign themselves to a group and, depending on the argument, not to be considered rich and privileged – although they certainly are in an objective comparison with people affected by poverty. We will take another example to use the *us-them* arena as a heuristic. Hunter and Hunter (2024) conduct a study where they challenge what they call the *story of achievement gap*. In this story aspects of certain groups of students including for example ethnicity are related to who can learn mathematics successfully. "A common response by policy makers and educators alike has been to fix the storied problem of those deemed unable to learn mathematics by putting in place a range of interventions." (Hunter & Hunter, 2024, p. 501-502). However, by taking collectivist approaches to mathematics learning instead of a rather competitive individualistic approach "Pacific understandings of reality, values, and knowledge generation are positioned as the basis of a research paradigm" (Hunter & Hunter, 2024, p. 505). Looking at this approach through the lens of an *us-them* arena shows an interesting shift in discourse. Whereas the story of the achievement gap is clearly suitable to create an 'us' in western society and Indigenous groups are stated as 'them', Hunter and Hunter manage to turn it around and create a new 'us'. How is this possible? Once again, processes of positioning happen, but in this specific case the authors fabricate recognition for a marginalised group. In this context, the third arena of inequality – *in-out* inequalities – deserve special attention. They result from the differences between members and non-members of nation-state societies and the power to define who is a member lies primarily with those who are already there. Interestingly, this is not the case with indigenous peoples. Exclusion was caused by force, which is now tried to be reversed to some extent through empowerment. With the heuristics mentioned, these processes can be understood as renegotiations about who actually belongs to a certain (mathematical) practice and who does not. Mathematics educational research that can be described by the fourth arena of inequality *(today-tomorrow)* has gained much more importance in the last few years. As an example, we highlight the work of Armando Solares-Rojas and colleagues (2022). They provide a methodological design for a didactic intervention with teachers from a school located in a community facing socio-ecological issues caused by massive environmental pollution. In a socio-critical perspective of mathematical modelling they enable "understanding mathematics as a tool that allows modelling phenomena, specifically social-ecological phenomena in a school's communities, to study and analyse them, within the school setting." (Solares-Rojas et al., 2022, p. 207). Solares-Rojas et al. (2022) point out, that the children participating in the project have never known something different then environmental pollution in their community. The modelling skills required today are useful for

the imagination of ‘tomorrow’ and are updated with regard to reflections that are highly relevant for future generations. These examples show, that the arenas can be used to re-interpret mathematics educational research and to broaden the understanding of it. Could it be a tool for analysing data as well?

Empirical validation – three case studies

In this chapter we discuss the heuristic model of the four arenas with empirical data. We try to show the usefulness of it by three different case studies: A classroom discussion in one smaller group, a teacher presentation, and a modelling task. The first two cases are drawn from an ethnographic study conducted in the very first weeks of different primary schools (Dexel et al., 2024).

Case 1 – Why is calculating important?

The following scene has taken place in the first lesson in a Montessori-School in an upper-class neighbourhood.

Mrs R.:	What do you need to write a letter to a friend?
Jakob:	A pen and a sheet of paper.
Mateo:	...and a stamp!
Mrs R.:	Yes, that’s right, but what do you need to be able to write this letter?
Mila:	Maybe we should know the letters?
Mrs R.:	Exactly. First you learn the letters. So it’s important to learn to read and write in order to be able to write a letter. Why is calculating just as important as reading and writing?
Jan:	So that you can recognise the numbers on the houses.
Mrs R.:	Well, what exactly do you mean by numbers?
Jan:	Well, there’s always a number on every house
Mrs R.:	That’s right. These numbers are called house numbers. Which house number is on your house? My house has the number 8.
Mila:	Mine has 11 on it.
Jakob:	Mhh. 27 on my house.
	[…]
Marie:	My parents taught me that. When you see two and four, you don’t need to run any more. So I can remember the number.
Mrs R.:	Oh, so you live in the 24th, but why do you have to learn calculating?
Jan:	Well, so that you know that three plus three equals six.
Mrs R.:	Ok, and what do I need to be able to work that out for?
Frida:	So that you know how much you have to pay.
Marie:	I always get 50 cent pocket money every Monday.
Mrs R.:	What do you do with your pocket money?
Marie:	Well, I go to the shop and see what I want and then I collect the money until I can get it.
Mrs R.:	Yes, what do you do with the money so that you know how much you have? Do you weigh it or what do you do with it?
Marie:	[laughs] No, I count it.

The scene shows a classroom discussion between the teacher, Mrs R., and the smaller group of first-year-learners, in which six children take part. Although the teacher sticks to a teaching by asking, the conversation initially appears cognitively activating. The children are interested, participate actively, and contribute their ideas and experiences about writing letters, house numbers, and the relevance of calculating. Based on the heuristics of the four arenas of new conflicts over inequality, the following can be stated at the same time:

Writing letters is a cultural practice whose importance has diminished nowadays and which has been replaced by digital alternatives (e.g., Email or messaging services). Nevertheless, the children who take part in the discussion are familiar with it and know what is needed to write a letter – in terms of the materials required (sheet of paper, pen, stamp), but also in terms of the relevant knowledge (e.g., letters and house numbers). According to Bourdieu (1979/1987), this knowledge is incorporated, i.e., internalised cultural capital, which is mainly acquired in the family upbringing. The children can now utilise this capital in the classroom, where it allows them to follow and actively participate in the classroom discussion. At the end of the scene, the conversation turns to the topic of pocket money. Marie reports that she is saving her pocket money until she can buy things that she would like to have. This focus on long-term personal goals expresses a sense of distinction and thus a pattern of behaviour that is typical of children from privileged backgrounds. In contrast, children from socially disadvantaged backgrounds often exhibit a short-term habitus. Actions are reviewed in terms of their immediate benefit in order to be able to fulfil needs quickly and with limited resources (Bourdieu, 1979/1987; El-Mafaalani, 2015). Overall, a middle-class orientation can be recognised in this scene. There is talk of letters, house numbers, and pocket money. It is also taken for granted and presented as a matter of course that one should learn to do maths. This reveals a potential conflict in the arena of *up-down* inequalities. Class affiliation, mediated by habitus forms and the extent of cultural capital, significantly determines how successfully children can take part in classroom discussions.

The interaction between the teacher and Marie at the end of the scene is also interesting. Although it is Marie's first ever lesson, she immediately recognises that the teacher is aiming her questions at the practice of counting. She also understands the funny way in which the teacher is framing the conversation and is able to react appropriately. In this interaction, the two create a form of 'us' that probably functions perfectly due to similar habitus and thus creates recognition. At the same time, however, this excludes children who cannot deduce from the interaction that this first lesson is supposed to be about maths. A potential conflict over inequality could arise here. In the scene described, however, such a conflict does not emerge openly, which may be related to the upper-class neighbourhood in which the school is located. Nevertheless, as conflicts in the up-down arena are discussed less frequently by the middle class and also from a greater social distance (Mau et al., 2023) – partly because the members are not affected by poverty and low status – children from socially disadvantaged backgrounds can 'fall by the wayside' as teachers lose sight of the fact that the reality of their lives is different.

Case 2 – Jòzefina, you will try!

The next scene is from a multilingual classroom in a big city in western Germany. The topic of the current series of lessons is arithmetic triangles. The pupils have already worked on one arithmetic triangle together with the teacher on the blackboard and have started on a second one. We could observe that the teacher, Miss M., sometimes switches between German and English in order to address a girl. Most of the following dialogue was held in German. In the transcript we underlined the initially English phrases.

> *Jòzefina (recently immigrated to Germany) seems absent and looks dreamily around the classroom. The teacher discusses an arithmetic triangle on the blackboard with the pupils in plenary.*

Miss M.: Now we have all the inside numbers. Now we can do the maths as normal. Who will do the maths and write the number here on the board. Jòzefina, you will try. Do you want to try to calculate. Four plus five (short break) or five plus four (taps on the five and the four). Five plus four or four plus five (taps on the four and the five again) how much is that?

In this short episode, we see that the teacher addresses Jòzefina, who recently immigrated to Germany, as she looks around the classroom, lost in thought and – at least it can be assumed – finding it difficult to follow the classroom discussion due to the language barrier. Miss M. now actively integrates Jòzefina and switches languages quite naturally during the conversation so that Jòzefina is put in a position to solve the maths problem. This reveals a topic of *in-out* inequalities, namely access and openness towards immigrants. The way Miss M. deals with Jòzefina concerns the question of the membership border and (in this example) migrants' access to education. However, the teacher's actions do not create a conflict in the in-out arena. Rather, she ensures that Jòzefina can actively participate in the lesson and is thus recognised and valued as a member of the class. Therefore, the recognition of diversity, which is a central point of inequality in the *us-them* arena, can also be observed here. An 'us' is created, into which Jòzefina is integrated by her teacher. In comparison to the first scene, a positive resolution of a conflict over inequality is visible here. Although an 'us' is created in both scenes, it is potentially exclusive in the first scene, as it arises from the similar habitus of the teacher and the socially privileged children. This could exclude children from lower status groups, whose realities of life are not taken into account and are not recognised. In contrast, inclusion is created in the second scene when Jòzefina is included in the plenary session by the teacher and diversity is recognised. In this case, a practice of teaching that is aware of inequalities becomes visible (see Hummel & Bohlmann, 2022). While the teacher's practice of teaching illustrates a positive, inclusive resolution of the conflict in the *in-out* arena, it is also possible to ask critical questions at a meso or macro level – for example about access to bilingual classes or schools.

Case 3 – Mathematical modelling with the 'refuelling-problem'

We would now like to turn to a classic modelling task and examine it against the background of the fourth arena, *today-tomorrow* inequalities. The modelling task we want to put up for discussion is an optimisation task that involves finding the best petrol station in border areas with different fuel prices (Blum & Leiss, 2005). It can be found in current task books for mathematics teachers, e.g. in Greefrath (2018). The question is whether a trip to a nearby foreign country for refuelling is financially worthwhile. This depends on many factors, including fuel prices locally and abroad, the distance to petrol stations, fuel consumption, and so on. Although this task offers potential for modelling real-life issues taking into account various factors and promoting process-related skills (e.g. reasoning) as well as content-related skills, it can be described as controversial from the point of view of today-tomorrow inequalities. Considerations on the ecological consequences of such 'petrol tourism' should be included in the deliberations. In this way, the task can be viewed in a larger social context that goes beyond the cost savings for individual people. Furthermore, the task can also be interpreted in a historical context: While the environmental catastrophe was already visible in 2005, such content was still being published completely uncritically as best-practice examples in magazines for teachers. The interpretation as a generational conflict enables a reflective classification of mathematical tasks against the background of social changes. To specify these changes, we can

draw on work by Weiss and Kaenders (2023) who critically reflect on another modelling task that individualises the global social problem of the climate crisis as a problem of one's own diet. Interestingly, both tasks are published in the same teacher magazine. Whereas 2005 the crisis was neglected, tasks are now driven by a zeitgeist that learn students "to cope with social problems by reflecting and changing one's own consumer behaviour." (Weiss & Kaenders, 2023, p. 1865). This is a rich discussion for mathematics lessons in school that is more likely to come up when looking at modelling tasks as symbols of inequality-conflicts.

Discussion

Mau et al. (2023) describe different dimensions of new inequality conflicts, rather than a clear distinction. However, this vagueness also enables a reflexive analysis. Various conflicts over inequality overlap in the aforementioned studies and interactions, and negotiations of difference thus become clear. Through the arenas, an analytical separation of what is at stake here and what is being negotiated can be determined more precisely. It is interesting to note that in our material the conflicts are resolved in one direction, we were unable to recognise any factual conflict (at least not in one and the same scene). In case 1, for example, children from lower status groups are potentially excluded from participating in the classroom discussion (up-down inequality), but an open conflict (e.g. between the teacher and a pupil) is not evident in this scene. It might be worthwhile to take a closer look at open discussions among students in the classroom. All in all, we consider the heuristic of arenas to be a helpful additional instrument to become aware of certain aspects of both mathematics educational research and practice. We do specifically not see it as a replacement for Gutiérrez's (2012) scheme. A future task would be comparing the two approaches to check if there are different outcomes. Two more limitations must be mentioned: If only looked through the lens of arenas, only these conflicts are seen and others are lost from view. The case studies were carried out in German schools, analysed from German researchers and the framework was created in German/western society. It would be interesting to analyse international data in an international research group. Further research and discussions are needed.

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I wish someone could remove all the text from the math book

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It is a challenge for second language learners to solve mathematical word problems in a language they still are learning. Language, as well as norms and contexts cause trouble. In this study three-year study, eighteen middle grade pupils, in 4th - 6th grade, have been intra-acting with materialities, such as strategies to solve mathematical word problems. Elaborating on how relational practices and boundaries within word problems and pupils might be created, attention is given to pupils' and word problems' intra-actions. We draw on Barads relational onto-epistemology and uncover how material entanglements play out around mathematical word problems when pupils talk about them. Strategies like "guess and try", can function as a way to multilingual pupils' agency solving word problems, but "naked" number tasks do not.

Setting the scene

To ensure sustainability regarding pupils, the society and the world, multilingual pupils must feel and experience that they are included, also in mathematics classrooms. This study is part of a three-year project in a multilingual mathematics classroom in a suburban area in Sweden. In the multilingual classroom, second language learners often must learn mathematics in a language they still are learning. It's then challenging to solve mathematical word problems (here Swedish), because language, norms and contexts cause trouble (Barwell, 2003; Caligari et al., 2021; Norén & Caligari, 2020). Word problems regularly produce difficulties for all pupils (Verschaffel et al., 2000), but pupils learning in a second language often struggle more than first language speakers (Moschkovich, 2002).

This study draws on ethnographic methodology (Hammersley & Atkinson, 2019 and Barad's agential realism (2007), which theorizes the entanglement of matter and meaning. Participant observations within a material discursive practice created the data with pupils, teachers, researcher and non-human materialities such as world problems, textbooks, mathematical strategies and experiences. Also, interviews with pupils and informal chats with teachers and pupils become part of the material discursive practice, considering the different aspects of materiality, practices, spaces, bodies and objects.

All pupils engaged within this study are second language learners and speak other mother tongues than Swedish. Some are born in Sweden, and others are more newly arrived. Eighteen middle grade pupils, in their 4th - 6th school year (10-13 years), have been intra-acting (Barad, 2007) with the materialities, such as strategies to solve mathematical word problems. The teaching was organized to grow second language learning pupils' knowledge in solving word problems. The pupils often worked in pairs when solving the problems.

Content and language learning teaching (Swain, 2001) were integrated and developed in the mathematics classroom; thus, mathematical concepts were made accessible. The aim was

furthermore that pupils' skills in the Swedish mathematical academic language would be developed at the same time as strategies to solve mathematical word problems were learned. Strategies taught were for example "guess and try", "draw a picture" and "make a table". When the project was coming to an end, in 6th grade, the pupils were interviewed in pairs, at a time the pupils had completed the National Test in Mathematics (NP) for their age group.

Talking about materialities, school mathematics textbooks that teachers rely on, contain many word problems that cause various problems for pupils s. Swedish pupils used to solve these written mathematical tasks 59% of the lessons and above half of the time they do it individually (Bergqvist et al., 2010. Solving mathematical word problems is a social practice (Barwell, 2018; Caligari et al., 2021) and teachers deal with social contexts when teaching (Chapman, 2006). Barwell (2003) has also shown that middle school multilingual pupils in UK used their narrative experiences to make sense when solving and creating their own word problems.

We elaborate on how relational practices and boundaries within word problems and pupils might be created. We give attention to materiality and account pupils' and word problems' intra-actions, embracing Barads (2007) relational onto-epistemology and ask:

- How do material entanglements play out around mathematical word problems when pupils talk about them and practically try to solve them?

Theoretical framing – creating a way forward

For three years, one researcher spent at least one mathematics lesson per every third week in the classroom with the pupils and one of the teachers, sometime both teachers. Being there, engaging with pupils and the material-discursive practice created experiences to the researcher. The experiences were produced in fieldnotes, photos and videos. Pupils' produced materials and teachers plannings were also collected. Thus, a lot of data was construed. All data generated are not used in this paper, but an ethnographic understanding lies like a grid in the background. The data production was "read together", meaning that all data and experiences are in continuous conversation and relation with each other (Murris & Bozalek, 2019). The interviews draw on produced data in the way that one researcher's and the pupils' materialized experiences can refer back to three years of mathematics classroom work. The 18 interviewees had participated in the activities in this multilingual mathematics classroom for middle school, that means, from 9/10 – 12/13 years of age, in 4th, 5th and 6th grade. All parents were given information about the study and gave written consent for their child to participate. The pupils had given their consent to intra-act with us, and their names are fictive.

The interviews were transcribed and analysed using a Baradian theoretical perspective, meaning that that the data became produced in process and is read together in entanglement, and lens by lens, we were zoomed in and out within materials and productions. In Baradian words, the methodology is diffractive. The reading was made in and out, back and forth, so all the data became entangled with our experiences and the material-discursive practise. We identified agential cuts (think of a cake with mixed ingredients and you cut a very thin piece of the cake). We posed questions concerning for example what pupils gave expressions of

remembering what kind of strategies that became created within productions and experiences. The production of reading the material in entanglement included how to report the analytical process. Thus, we have undertaken a systematically, extensive reading and re-reading of the transcripts, made summaries, and finally we narrated the intertwining intra-actions with the pupils and the word problems. We considered material entanglements by using Baradian (2007) ways of formulating 'material relations' intra-acting with pupils and word problems.

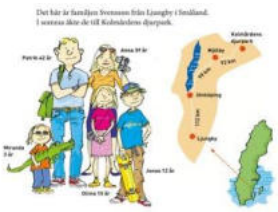
Rather than resembling the position of pupils and mathematical word problem as separate individuations, the Baradian methodology let us view the pupils and the word problems as entangled in intra-actions. This is in contrast to sociocultural perspectives, that since at least 20 years back, have been widespread withing the body of research in multilingual mathematics classrooms (Barwell, 2005; Caligari et al., 2021; Moschkovich, 2002; Norén & Caligari, 2020). Learners are then viewed as participants in social practices as a way of socializing into the mathematics classroom community as well as the language practices in that classroom. Drawing on Barad's onto-epistemological theory emphasizing embodiment and de Freitas and Sinclair's (2013) "relational ontology" (p. 457) the materialities are bound together in processes of becoming.

When interviewing the pupils, the word problems were intertwined within bodily minds and matters, power and experiences in ways we hadn't thought of. These aspects became agentic within intra-action in the material-discursive practise. The sociopolitical perspective being created within the material-discursive practise, engaging in solving mathematical word problems were not just about socialization or mathematics classroom practices, the power also became agentic. As an example, pupils expressed strong feelings about word problems and about the process of solving them. It seemed like the word problems "did" something with the pupils, the word problems became agentic intra-acting with the pupils. Barad (2003, 2007) notes that ongoing relationships and a mutual transformation process grow within the material and the human regardless of human intervention. With the words of Pickering (1995), material agency means that material objects "do things in the world" (p.9). To Barad (2007) all matter – non-human and human – are entangled (intertwined) in mutual material-discursive interconnections. It means that we can't unconnect matter and people. If we do, we lose sight of their reciprocal composition. Engagement and creations within non-human and human materiality reconfigures potentials and patterns of mathematics classroom practices, such as solving word problems. In this paper, the word problems are material objects, and we turn to Barad's (2003, 2007) theoretical toolbox in which "matter and meaning are entangled and interdependent in intra-activities" (Norén, 2018, p. 152). Reading the data production a relation to mathematical power becomes created. This means that the power of mathematics, who's mathematical voice becoming agentic, were produced as a result.

Word problems do things

Reading the materials with a Baradian perspective made us intra-act with the material-discursive practice. Also, in this study the results are being presented entangled with discussion and materials. We start the result-discussion by intra-acting with a word problem and illustrations taken from the textbook Alfa for schoolyear 4:

Figure 1: Illustration by Johan Unenge in the textbook Alfa (2014), printed with permission



This is the Svensson family from Ljungby in Småland. This summer, they went to Kolmården's zoo.
The ages of the family members are written in the figure.

How much older is
a) daddy than mammy?
b) Jonas than Miranda?
Look at the map. The journey started in Ljungby. The first day they went to Jönköping and then to Mjölby. How long was the distance?

This specific illustrated word problem's (Norén & Caligari, 2020) intra-action, with pupils in 4th grade, emerged when the pupils in reaction to the introduction of the task, in the middle of the autumn semester, completely "derailed". There was no indication of any student trying to solve the mathematical word problems. The pupils protested orally, in uproar, shouting to their teachers and fellow classmates. Bodies became agentic intra-acting with the material of word problems. Hands were waving, pupils/bodies were running back and forth in the classroom and jumping up and down intra-acting with chairs. Chairs became slamming and desks were shaking. The room became loudly and unsecure.

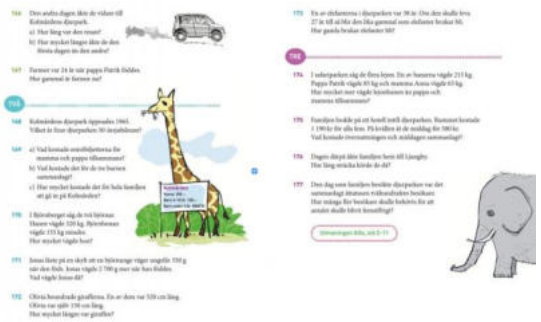
The amount of text was immense (see fig 2) and generated resistance within the pupils towards the word problems. When word problem solving strategies were taught and the strategies intra-acted with the pupils and the word problems, the power of dealing with the word problems became agentic. Engaging within texts and illustrations, the pupils later in the autumn semester solved the word problems (Norén & Caligari, 2020).

When remembering the occasion in an interview, one student said:

> Men till och med i femman så hade jag fortfarande, när jag såg textuppgifter, jag var bara jag orkar inte med det här, jag ska gå ut. Fast till slut så läser jag, jag var bara, det var enkelt, jag gör det nästa gång igen.
>
> *But even in fifth grade I still had, when I saw word problems, I was just – I can't take this, I'm going out. But in the end, I read, I was just, it was easy, I'll do it again next time.*

Still, in 5th grade word problems caused trouble for the pupils. Besides teaching about strategies teachers used second language learning methods, like choosing challenging tasks, scaffolding language and enhancing dialogue (Gibbons, 1998).

Figure 2: page 44 and 45 in the Alfa textbook, illustrations by Unenge (2014), printed with permission



We're not translating the word problems. We show them to illustrate the pupils' and the word problems' intra-actions. The derailment was a consequence of the immense amount of text.

In the interviews, one student referred to strategies that were taught during 5th grade, the focus was on ‘guess and try’, ‘draw a picture’ and ‘draw a table’ (from fieldnotes). Relating to a huge number of written tasks in general, a student, in 6th grade, said[1]:

> För nu har vi typ så här tränat in det och så där, för om vi jämför som i fyran, vi förstod ingenting. Det var jättemycket text och sådant där, och nu är det som, ja, en textuppgift, då tänker jag bara gissa och pröva, jag kan bara använda gissa och pröva eller rita en bild. Förut visste jag inte ens vad jag skulle göra, jag läste texten och räckte upp handen.
>
> *Because now we’ve sort of trained on it and so on, because if we compare like in the fourth grade, we didn't understand anything. There was a lot of text and stuff like that, and now it’s like, well, a text assignment, then I’m just going to guess and try, I can just use guess and try or draw a picture. Before I didn't even know what to do, I read the text and raised my hand.*

This excerpt follows the former one, referring to the teaching in fifth grade, though it here becomes a bit clearer that experiences, ”we´ve sort of trained” and the strategy, ”guess and try” become agentic. When the statements are read together with the illustrations and the task, a discursive practice is created where anxiety, difficulty and powerlessness arise. When strategies such as “guess and try” and “draw a picture” are added to the practice, matter become as trustworthy as language. Thus, agency, power, engagement and confidence in one's own ability to tackle the word problem is instead created.

Other pupils related to the NP, specifically relating to the use of the strategy ‘guess and try’:

> I actually did it on a national exam assignment. It was like this algebra, it was like this x times two equals five something, I don't know what it was, then I guessed and tried first six, seven, then it worked.

Again, in the excerpt above, a pupil intra-acts with the strategy “guess and try” entangled with the word problem. The strategy ‘does something’ to the student. In intra-action the strategy becomes agentic. In the intra-action with the materialities word problems, strategy and experiences of use of the strategy, mathematical engagement become created. The experience of the strategy enables the student to become agentive and to become mathematical within the intra-action. The discourse in which the word problem is produced, and has been produced in, become agentic to. The discourse and the entangled experiences, enables mathematical intra-actions to be created and to become agentic. The strategy “guess and try” when training before NP become agentic and valued, enabling the student to intra-act and to get engaged within the material-discursive practice. The agential cut becomes created within security and confidence in one's own skills and experiences. When one researcher engaged within the discursive-practice and with the pupils talking about it in interviews, the pupils were discussing that the strategies, taught in grade five, could be used when solving word problems in the (NP) for 6th grade.

> Pupil 1: Yes, I remember for sure although I was guessing …
> Interviewer: Guess and…?

[1] We give the original transcript in Swedish just once, because it takes to much space. We have translated the quotes from Swedish to English.

Pupil 1:	Try. I still do, and I passed NP because of that.
Interviewer:	Did you?
Pupil1:	I feel like I passed NP because of that.
Interviewer:	Yes [turning to the second student]. Did you also use 'guess and try' then?
Pupil 2:	No, my brain doesn't work during NP, unfortunately

In reading the interview with the two pupils, mathematical engagement was created for one pupil in intra-action with the strategy "guess and try", while the other pupil expresses "my brain doesn't work". In this dialogue it's noticeable how the NP, and the experiences of it, intra-acts with both pupils. This is read as not just an intra-action with two materialities, the pupil and the word problem, but also within the materialized context, with experiences and imaginations intra-acting with NP. The pupil who could not use the strategy, where the "brain doesn't work", in that intra-action, the mathematical engagement does not become agentic. Instead, it is the negative experiences and imaginations that "kicked in" and become agentic. These experiences are not anchored in beliefs in one's own capacity, power or mathematical independence and engagement. Thus, the strategy "guess and try" does not become agentic in intra-action with the word problem as other experiences "kick in" and become agentic. These experiences are associated with mathematical uncertainty, anxiety and a non-engagement.

In another excerpt, we can again read a pupil's' experiences becoming agentic:

> Thus, I can count in Somali. I also use Swedish too, for example I might have a candy bag, it costs 19, I might have 38, I'm going to buy something for mom, yeast for example. Then I calculate how much I need like this, so yes.

The pupil expresses that s/he can count in Somali and Swedish. When a student's counting in several languages is valued in the classroom, being multilingual becomes an asset. In intra-actions, experiences of tasks, languages and mathematical contexts can become agentic in intra-action with, for example, a word problem. The experience enables mathematical engagement to become agentic. With this way of reading human and non-human materialities, it becomes significant for what is valued in mathematics classrooms. Valuing all resources in teaching, discussing mathematics in everyday life and creating contexts where mathematics can be used and become significant, 'does' something with the pupil's experiences and belief in their own knowing and mathematical engagement.

> When you solve word problems, then you must make long answers like this after and show how you thought about the answer, because if it's a long question, you must have a long answer. So, for example an algorithm, if you're going to solve it, then you just must give the answer, you understand?

The above excerpt shows how intra-action with word problems can evolve when strategies on how to tackle a word problem are lacking. The pupil doesn't express any strategy in the intra-action with the word problem. It is his/her experiences of how word problems "usually" are required to be solved that becomes agentic. The mathematical task becomes the material that has and produces the power, and that determines what mathematical engagement is "requiring". Confidence in one's own knowing and power over the mathematical engagement is not given space in intra-action with the word problem above. In intra-action with tasks

without words, with set-up or algorithms, mathematics becomes mechanical. The task takes on a life of its own and power over what is done. Tasks that include numbers and arithmetic must only be "solved" and given an "answer". The student says this type of tasks as simpler "just to give an answer". The word problem becomes the one having the power. The intra-action with the pupil, the experiences and the word problem, and no intra-action with strategies, disengage the student in the discursive practice.

At the end of every interview, the pupils were asked what they think about working with word problems. Most of the pupils answer that they don't like it, and one of them says:

> Det är det värsta. Jag önskar att alla uppgifter med text kunde tas bort i matematikböckerna och de skulle bara innehålla siffror och tal, det är lättare då.
>
> *I think it's the worst. I wish you could remove all the text from the math book and just have the numbers, it's easier then.*

In the excerpt, numbers and words in the textbook become agentic. The texts become agentic in a negative way and the numbers in a positive way. If the words were removed from the textbook, "everything would be easier". The numbers make the mathematics easier. Mathematics may not become part of the student's every day or future life, or a way to take on challenges and to solve problems that arise. Thus, when the mathematics and the textbook have the power and make decisions, pupils just solve tasks, in the easiest way possible. Materialities are agentic participants in the mathematics classroom.

Conclusion

Reading the data embracing the theory of Barad (2003, 2007), and in contrast to using sociocultural theories, made us intra-act with the productions and the material-discursive practice. Experiences, mathematical engagement, non engagements, language, power, materialities, and discourses became entwined within intra-actions, becoming agentic and made creations. The creations produced the result of the study. Strategies, for example "guess and try", can function as a way for a mathematical engagement to become agentic in intra-action with word problems. Tasks with numbers and arithmetic create mathematical engagement, but not power. The power stays with the task itself and the pupil only must "do" and "answer". Consequently, mathematics teachers can secure future multilingual students' mathematical power in classroom work with word problem solving, by engaging material strategies in their teaching. To end this paper, we assure that material entanglements play out around mathematical word problems when multilingual pupils experience and talk about them. Talking with Barad (2003) all matters matter.

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Patterns of mathematics capital among Swedish higher education mathematics students

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This paper presents a survey study of the mathematics capital of first-semester mathematics students in Swedish universities. A cluster analysis shows how mathematics capital is distributed among three clusters of students, distinguished mainly by parents' or significant others' relation to STEM. In one of the clusters, students had much family support, and talked about mathematics outside school. In the middle cluster, students had acquaintances in STEM but to a less extent engaged in mathematics outside schools. In the third cluster, students liked mathematics but had little mathematics capital. Demographics were similar between the clusters, except for students' age, and parents' level of education. We conclude that for Swedish students with little mathematics capital, the trajectory into higher education mathematics can be longer, but it is a possible trajectory.

Keywords: Mathematics capital, socio-political studies, higher education mathematics, inequalities in education, survey methods, cluster analysis.

Introduction

It is a well-known "fact" in mathematics education research that socio-economic conditions shape students' opportunities to engage in mathematics (e.g., Valero & Meaney, 2014). While this has been well researched in school education, it has been less investigated both for the transition into higher education and in higher education itself. Aspects of knowledge, attitudes (e.g., Biza et al., 2016) and, more recently, identity (e.g., Lahdenperä & Nieminen, 2020) have been highlighted as important for starting and staying in higher education mathematics. Recent socio-political studies of higher university mathematics (e.g., Arideja & Andrews-Larsson, 2017) invite us to explore in which ways enrolling in a mathematics programme is supported by the mathematics specific configurations of material-economic, cultural, and social resources available to students.

Understanding differences in the availability of resources for students' engagement with mathematics is important for research concerned with equitable access to and success in university mathematics. For more than a decade now, researchers in the U.K. have taken a Bourdieusian approach to study the significance of subject-related capital on students' aspirations and engagement in science, technology, engineering and mathematics (STEM) from school to higher education (e.g., Godec et al., 2024). The concepts of science capital (Archer et al., 2015) and mathematics capital (Williams & Choudry, 2016) have been proposed and utilised to identify and explain subject-specific aspects of the social reproduction of inequality with and through education, particularly in societies where STEM skills and knowledge are seen as values connected to power. Thinking further with these scholars, *mathematics capital* can be conceptualised as the interconnected series of "skills and

resources that individuals acquire […] that can generate forms of social advantage in [mathematics] educational settings" (Hernandez-Martinez, 2020, p. 51-52). Mathematics capital is mobilized through the activities and relationships available for individuals belonging to different groups of people in society. As Archer and Mendick argue (2024), making visible how differentially acquired mathematics capital supports and reinforces the development of identities casts light into the connection between social reproduction and aspirations and choices leading to the study of mathematics in higher education.

Although enlightening, studies of mathematics and science capital have so far mostly been produced in the U.K., a society with strong social stratification. Thus, the question of how generative these studies can be in other societies is open. In the Nordic countries, research has shown that despite policies of educational equality and democratisation, education is still an important reproductive force of social differentiation and inequality (Osman et al., 2021), particularly when it comes to participation in higher education. To reduce financial obstacles, Swedish higher education is free of charge and students have access to additional government financial support. Despite these efforts, students from academic family backgrounds dominate the more prestigious university programmes in Sweden (Börjesson & Broady, 2016), whereas students from less educated families more often attend minor, local university colleges (Carlhed, 2016). Also, mathematics and physics are programmes with a documented highly uneven recruitment in terms of gender and social class (Börjesson et al., 2016). It is reasonable to hypothesise that skills and resources connected to mathematics that are available to different types of students may help cast light on such lingering disparities. Thus, the purpose of this study is to understand the distribution of mathematics capital of students who enrol in higher education mathematics in Sweden. We ask: What characterises the mathematics capital of Swedish first year higher education mathematics students?

Methods

This paper presents results from a survey study conducted among first-semester students in higher education mathematics in Sweden. The survey was designed to assess the different dimensions of mathematics capital that students perceive to have accumulated throughout their upbringing and prior educational experiences. The survey drew on the validated instruments in the ASPIRES project (e.g., DeWitt et al. 2016), and connected recontextualisations to Denmark in the SCOPE project (e.g., Pedersen & Nielsen, 2024) and to Finland in the Finsci project (Kaakinen et al., 2023). We selected, translated, and adapted those instruments to suit the Swedish context and the intended target group of mathematics students. We assumed that the categories of science capital outlined by De Witt et al. (2016) would also serve for mathematics capital, and therefore we structured the questions around: who one knows in mathematics/STEM, what support one gets for choosing to study mathematics, what activities and resources related to mathematics were available during one's upbringing, and what significant other's thoughts are about me and mathematics. Additionally, the survey included demographic questions about gender, age, place of origin, area of upbringing, and parents'/guardians' level of education. Selected survey items analysed in this study will be presented in the results section.

The survey was disseminated via e-mail to the entire cohort of approximately 500 students enrolled in either a Bachelor of Mathematics or a Bachelor of Engineering Mathematics programme, offered

at 12 universities in Sweden. The survey was administered at the beginning of their studies, and before their first examinations. We received 150 responses. To assess potential sampling biases, we cross-checked the survey data against Swedish registry data. We found a slight overrepresentation of female students and a marginally lower proportion of older students in our sample compared to the overall population. Given the specific and narrow scope of our study, generalisations are to be made with caution beyond the targeted programmes.

Data analysis

We started with a cluster analysis of responses to the mathematics capital questions to group respondents based on their configurations of mathematics capital. Based on these clusters, we first described the mathematics capital in detail, and thereafter discussed how demographics such as age, gender, or parents' educational background relate to the identified clusters.

A hierarchical cluster analysis was performed, using Ward's method (Jaeger & Banks, 2023) to determine the optimal number of clusters, followed by a *k*-means clustering to assign respondents to clusters. This resulted in three mathematics capital clusters, named McA, McB, and McC. An ANOVA test was run to verify which questions significantly differ between the three clusters. To interpret what characterises each cluster, the positive answers to each mathematics capital question were summarised within each cluster. Finally, cross-tabulations and chi-square tests were used to examine significant differences in demographic variables across the clusters.

Results

The cluster analysis grouped respondents in three clusters, based on the patterns of their responses to questions related to mathematics capital. Among the 150 mathematics students, 39 were assigned to cluster McA, 67 to McB, and 47 to McC, and a visualisation of the patterns is given in Figure 1.

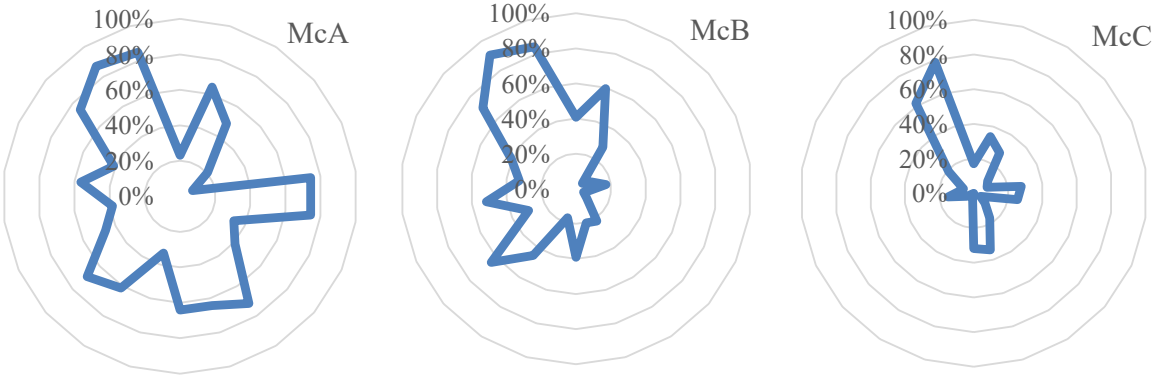

Figure 1: Each graph represents one cluster, and one node in the spider-web graphs represent the proportion of positive responses to a question related to mathematics capital. (More detail below)

Comparing the diagrams in Figure 1 gives that both the size and the patterns differs between clusters. The diagram för McA covers the largest area, showing that many resources related to mathematics capital were available for students in this group. In McB, the size and the pattern is different, with low proportions to the right side of the diagram. The diagram for McC covers the smallest area, and

few resources related to mathematics capital were available to the group of students in this cluster. From this overview, we now explore the characteristics of mathematics capital within each cluster.

Who do you know in STEM?

Figure 2 presents the responses to the question "Who do you know that works in science, technology, medicine or mathematics?", distributed on the three clusters.

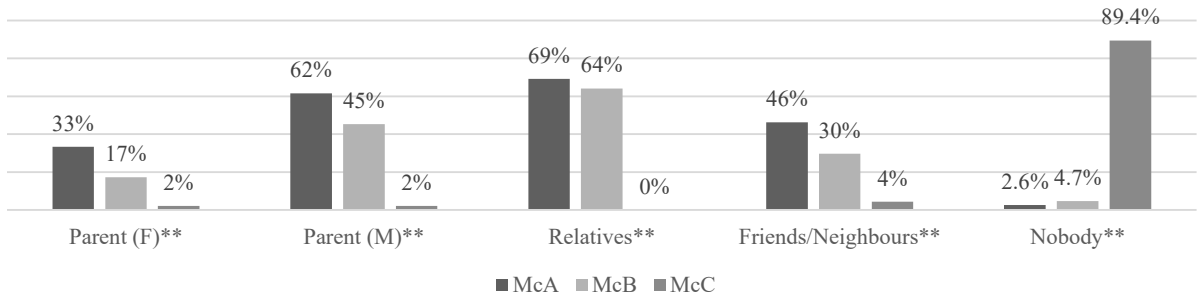

Figure 2: Who you know in STEM?[1]

This question distinguishes cluster McC from McA and McB. The diagrams show that in McC, most students have no acquaintances in STEM; while many in the cluster McA report having parents/guardians, particularly male (Parent (M)), in STEM, and many students in both McA and McB report having relatives or friends in STEM.

Support for choosing mathematics

The question "How important, or not, do you think the following were in supporting your pathways into choosing the programme?" addresses the importance students attribute to support from persons or social contexts, as of the alternatives in Figure 3.

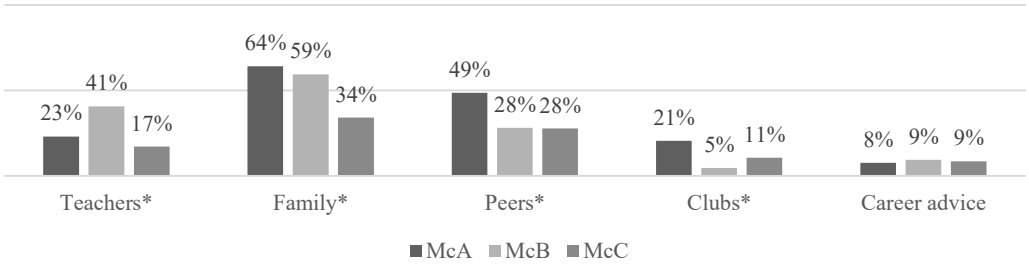

Figure 3: Important support for choosing mathematics

In general, the findings indicate that the importance of support received from teachers, family, peers, clubs differ significantly between clusters, where McA report they received the most support. One

[1] STEM – Science, Technology, Engineering and Mathematics. In all figures, significant differences between clusters according to the ANOVA are indicated by ** for a 99,9% level and * for a 95% level for each item.

exception is career advice, which is attributed little importance for all clusters. The other exception is teachers' support, which is important for 41% in McB, compared to 19% in McA and 17% in McC.

Activities and resources related to mathematics

The mathematics-related activities and resources students had access to outside formal education are part of their mathematics capital. The question asked was "What are the media or social contexts where you took part of mathematics, outside formal education, during your upbringing?"

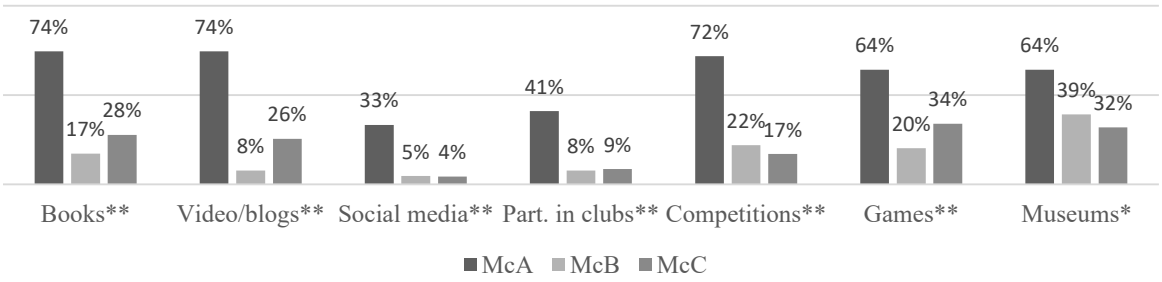

Figure 4: The mathematics related media or resources

We found a significant difference between McA and the other clusters concerning their reported access to and use of media with mathematical contents. In Figure 1, this is what shows at the right side of the diagram, where McB and McC had very low rates. Students in McA report they have engaged with mathematics-related books, YouTube videos or blogs, social media, games, clubs, and competitions. Significant differences between McA and the other clusters were also found concerning who had access to museums or science centres.

Significant others' thoughts about me and mathematics

We also asked about what significant others think about mathematics: "During my upbringing, a significant other thought that…". The following alternatives were provided: "…explained to me that mathematics is useful for my future", "…specifically encouraged me to continue with mathematics after high school", "…thought it was important for me to learn mathematics when I was in school", "…thought mathematics is interesting" and "…thought I was good at mathematics".

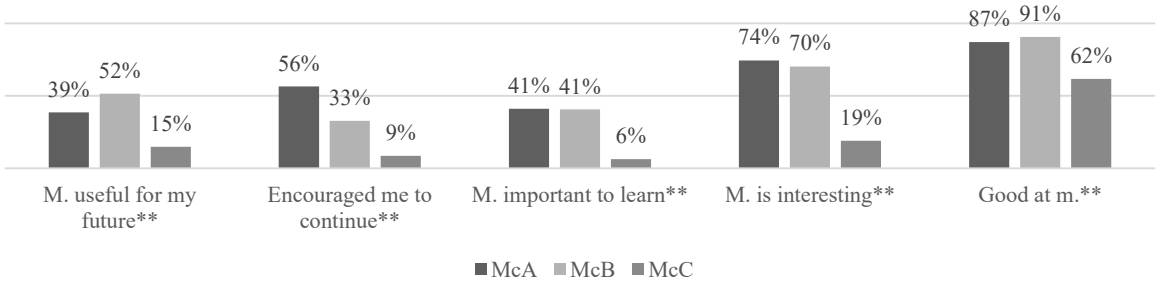

Figure 5: Significant others' thoughts about me and mathematics (M. is used for mathematics)

The majority of students reported they had been told they were good at mathematics by significant others. Nevertheless, this was notably less common in McC. Students in McA or McB were more

likely to have received encouragement to continue to study mathematics after upper secondary school, and to have significant others who think that mathematics is important to learn, who have explained how mathematics is useful for the future, and who think mathematics is interesting. In contrast, students in McC rarely report such strong encouragement related to mathematics.

One question assessed students' own thoughts about mathematics: "It is important to have a basic understanding of mathematics even for people who don't work in mathematics or science", which students in all three groups agree is important. This question, together with the question about being good at mathematics, are the only questions to which a majority of students in McC agree.

Demographic differences between mathematics capital clusters

The demographic variables we included generally show no significant differences between clusters, including gender, area of upbringing (rural or urban), or students' or parents' country of birth. Hence, these variables do not contribute to how mathematics capital clusters were formed. Also, no significant differences between clusters were found in the school achievement, where in all three clusters the majority reported to have good grades in mathematics, and the majority (119 out of 150 students) attended a mathematics-intensive upper secondary programme.

Differences between clusters were found in the educational level of parents. It is lower in McC, where 50% has at least one parent with a university education, compared to 85% in the other two clusters. In addition, there was a significant difference in the age distribution between clusters. Among the 13 students older than 25 years of age, nine were in McC, and four were above 30 years of age when enrolling mathematics higher education studies. As a contrast, in McA, two students were under 18 years old, and hence have had an accelerated educational path.

Discussion

Our analysis allowed to identify three distinct patterns of responses to the five studied dimensions of mathematics capital. All students have acquired sufficient skills in mathematics, evidenced by the fact that they report having good grades in upper secondary mathematics, and have qualified to enrol a mathematics programme at university. Also, the majority of students had been told they were good at mathematics, and they think a basic understanding of mathematics is important for people outside science and mathematics. These findings resonate with earlier research (e.g., Biza et al., 2016; Lahdenperä & Nieminen, 2020) showing the importance of mathematics knowledge/skills, attitudes and identity for entering and staying in higher education mathematics. However, our analysis makes visible the differences between students in relation to other material-economic, cultural, and social resources available for students. We interpret such differences in terms of the availability of resources related to mathematics capital in each cluster. Students in McA have an extensive range of resources contributing to building their mathematics capital, such as parents/guardians —mainly male— with an education in STEM, people around them talking positively about the importance of mathematics, and participation in mathematics-related activities during their upbringing. Students in McA to a high extent engage in extracurricular activities or media related to mathematics, as different from students in McB and McC. Students in McB, the largest cluster, have had access to some resources, but the pattern show how for them, teachers rather than family are an important resource. In contrast, the

students in McC do not report access to many resources, except having been told they are good at mathematics. Against the odds, this group has found a way to enter higher education mathematics.

Earlier research has shown how social factors affect participation in higher education in general (e.g., Osman, et al., 2021). It has also been shown how science capital captures dimensions of significance for participation in STEM-disciplines in particular (Godec et al., 2024). The concept of mathematics capital has a similar relevance for participation in mathematics education (Williams & Choundry, 2016). Mathematics capital grasps the configuration of social, material and cultural resources that differs among social groups and that also have a significance for entering higher education mathematics. In this study, clusters did not show to be significantly different concerning gender and place of origin. Yet, a demographic variable that was significantly different between clusters was parents' educational level, a well-known factor associated to successful participation in education (e.g., Börjerson & Brody, 2016). Also, students' age differed between clusters, where older students were more commonly part of the cluster with less resources related to mathematics capital. Hence, it could be hypothesised that the presence of mathematics capital makes the trajectory for deciding to enter the study of mathematics at university smoother and faster; or, in other terms, that the lack of mathematical capital require for these Swedish students a longer trajectory to make the decision to enrol in mathematics programmes. The results show that students with little mathematics capital manage to enter higher education mathematics in the Swedish system. This does not mean that the support offered to them is sufficient. We see that mathematics capital is unevenly distributed, and we find it particularly concerning that support from teachers is reported to have little importance for students in McC, who might need it the most.

We hope that our results raise awareness about the difference in mathematics capital students bring to higher education. We call for qualitative research to follow up these results to better understand the identified differences in patterns and what it entails to enter higher education mathematics for students with little mathematics capital; what are the obstacles, the expectations for the future, and what makes the students feel they belong (or not) in higher education mathematics.

Acknowledgment

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Gender differences in optimistic, realistic, and pessimistic mathematical self-concepts among first-year STEM students

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This contribution examines (1) whether a gender gap in mathematical self-concept can be found among first-year STEM students at the University of Innsbruck and (2) if the accuracy of mathematical self-perception relative to performance differs by gender. The study analyses samples of 302 and 260 participants using linear mixed-effects models and chi-squared tests. The results show that female STEM freshmen have a higher mathematical self-concept than their male counterparts. Further, women show a larger proportion of optimistic self-concepts – i.e. higher self-concepts than expected based on their mathematical performance – compared to men. Based on theoretical considerations, it is assumed that this is the result of social and criterial comparisons. Overall, gender-specific filter mechanisms appear to persist when entering STEM degree programs, which different stakeholders should consider in the future.

Keywords: Women, self-concept, mathematics, STEM, higher education.

Relevance

As one of the 17 UN Sustainable Development Goals, goal 5 calls for gender equality and the empowerment of all women and girls. This includes ensuring equal opportunities across social, economic, and political domains and ending discrimination against women. Increasing female representation in STEM (Science, Technology, Engineering, and Mathematics) fields is vital to these efforts, as women remain underrepresented (Binder et al., 2021; Goldman & Penner, 2016; Sax et al., 2015). Moreover, with many countries facing a shortage of skilled STEM workers, encouraging women to enter these fields is increasingly important (Bünning et al., 2020).

To determine the causes of female underrepresentation in STEM fields, Kanny et al. (2014) conducted a meta-narrative systematic review of 324 academic publications from the 1970s to the early 2000s. The psychological factor of *self-confidence* emerged as the most commonly cited reason for women not pursuing STEM studies. More precisely, choosing a STEM major is frequently associated with having a strong mathematical self-concept (Goldman & Penner, 2016; Musu-Gillette et al., 2015; Sax et al., 2015).

Austria presents an intriguing case for examining the complex relationship between gender, mathematical self-concept, and STEM enrolment, as this paper will demonstrate. Furthermore, this study introduces a novel perspective by assessing the accuracy of mathematical self-perception relative to actual performance by gender. This approach yields new insights into marginalisation processes at the intersection of mathematics, higher education, and gender.

Theoretical background

Self-concept

Most of the early research on this phenomenon focused on the broad idea of self-concept and frequently lacked a sound theoretical framework. Consequently, Shavelson et al. (1976) developed a well-defined self-concept model that is organised, multifaceted, hierarchical (see Figure 1), stable, developmental, evaluative, and differentiable. They describe self-concept as "a person's perception of himself" (Shavelson et al., 1976, p. 411), shaped by environmental and social influences.

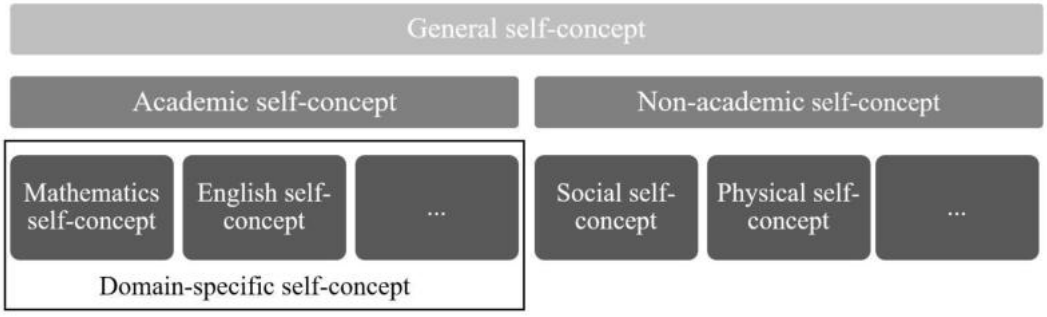

Figure 1: Hierarchical structure of the *self-concept* model according to Shavelson et al. (1976), figure based on Filiz et al. (2020, p. 2)

As Figure 1 shows, the *general self-concept* is divided into an *academic* and *non-academic self-concept*. The academic self-concept is

> generally defined as one's perceived competence in a specific domain (often an academic subject) in a normative way (in comparison to peers or the average person) that includes both cognitive and affective evaluations of the self. (Sax et al., 2015, p. 815)

The perception of one's cognitive and affective competencies in a *specific* academic area, i.e. English, is known as the domain-specific self-concept (Filiz et al., 2020; Sax et al., 2015). The domain-specific self-concept in mathematics is known as the *mathematics* or *mathematical self-concept*.

Mathematical self-concept development

Marsh's (1986) *internal/external frame of reference model* (I/E model) posits that the domain-specific self-concept is shaped by *internal* and *external* comparisons. Internal (*dimensional*) comparisons contrast one's mathematical abilities with those in other domains, often verbal skills, while external (*social*) comparisons involve measuring against peers (Dickhäuser et al., 2005; Marsh, 1986). Recent research expands the model by adding *temporal* and *criterial* dimensions (Dickhäuser et al., 2005; Geisler et al., 2023). The temporal dimension reflects comparisons between past and current performance (Wolff et al., 2019), while criterial comparisons involve self-selected or externally set benchmarks (Geisler et al., 2023). So:

- The better my skills in another domain, the worse I will rate my math skills and vice versa (*dimensional*, formerly *internal*).
- The worse the math skills of my peers, the better I will rate my math skills and vice versa (*social*, formerly *external*).

- The better my past performance in math, the more cautious I will be in assessing my current performance and vice versa (*temporal*).
- The better my performance is assessed by external criteria such as school grades, the better I will assess my performance and vice versa (*criterial*).

Empirical findings on mathematical self-concept and gender

The positive relationship between mathematical self-concept and performance has been repeatedly highlighted by extensive research (Heyder et al., 2019). Further, Goldman and Penner (2016) discovered that STEM career aspirations are more strongly correlated with one's mathematical self-concept than with academic success. Other researchers, such as Musu-Gillette et al. (2015) and Sax et al. (2015), concur with this conclusion.

Unfortunately, there is evidence that girls – even at the same performance level – exhibit a lower mathematical self-concept than boys since elementary school (Fiedler, 2024; Heyder et al., 2019; OECD, 2015). This trend extends into the tertiary education sector: it is typical for female STEM students to have a lower academic self-concept than their male peers (Fiedler, 2024). This applies to both STEM and non-STEM fields, where women exhibit a lower mathematical self-concept according to data from about 1.5 million first-year students from American baccalaureate-granting institutions between 1971 and 2011 (Sax et al., 2015).

Paasch and colleagues' (2017) study provides an Austrian example of such gender differences at secondary school level: The analysis of roughly 76,000 fourth-graders reveals that among students with similar mathematical proficiency, 7% of boys and 13% of girls had a pessimistic mathematics self-concept – a lower self-concept than expected based on performance outcomes (Paasch et al., 2017). Data from 83,000 Austrian eighth-graders show that 16% of the girls but 9% of the boys hold a pessimistic mathematical self-concept, while 21% of the girls but 27% of the boys exhibit an optimistic mathematical self-concept – one that exceeds performance-based expectations (Paasch et al., 2017).

Data and methods

Research questions and hypotheses

While numerous international studies have explored the relationship between gender, mathematical self-concept, and its influence on performance and career aspirations, research on this topic within the Austrian tertiary education sector remains scarce. Austria is intriguing in this context, as it exhibits significant gender disparities in mathematics education. According to the 2022 PISA study, which focused on mathematics, Austria – alongside Italy – recorded the largest performance gap between boys and girls. Moreover, girls in Austria receive on average 46 minutes less mathematics instruction per week than their male peers (Toferer et al., 2023). These distinctive circumstances underscore the need for a focused, country-specific examination of mathematical self-concept and STEM career aspirations as a function of gender in Austria. Additionally, it is still unclear whether the gender-specific patterns of *optimistic* and *pessimistic* mathematics self-concepts can be found in the tertiary education sector. Thus, the research questions are: (1) Is there a gender gap in mathematical self-concept among first-year STEM students at an Austrian university? (2) Is the mathematical self-

concept of STEM freshmen realistic, optimistic, or pessimistic, and are there gender differences in this regard?

Given the research findings of the previous section, it is hypothesised that (1) women exhibit a lower mathematics self-concept and (2) women own a pessimistic mathematics self-concept more often and an optimistic self-concept less often compared to men.

Sample and statistical methods

Given the limited availability of local data in Austria on the relationship between gender, mathematical self-concept, and STEM studies, a quantitative study serves as a valuable first approach to identifying trends in this area. In a subsequent phase, qualitative research methods will be essential to gain deeper insights.

A sample of 100 **s**cience- ($n_f = 39$; $n_m = 61$), 102 **t**echnology- ($n_f = 26$; $n_m = 76$), and 100 **m**athematics-freshmen[1] ($n_f = 65$; $n_m = 35$) at the University of Innsbruck is used to answer the research questions. The sampled first-year STEM students completed an extensive mathematical performance test with about 130 closed-end exercises from the areas of *numbers and functions*, *equations and inequalities*, *vectors and coordinate systems*, *geometry and trigonometry*, *integral and differential calculus*, *stochastics*, and general *mathematical thinking*, an assessment of mathematical self-concept created and validated by the Austrian Federal Ministry of Education[2], and a demographic questionnaire to obtain the necessary data. Mathematical performance data and self-concept scores were further categorised as *high* and *low* based on median splitting. According to Paasch and colleagues (2017), people with an *optimistic* mathematics self-concept fall into the low-performance group but are in the *high* self-concept category. Conversely, a *pessimistic* self-concept applies vice versa. People who fall into the *high* or *low* group for both maths performance and self-concept have a *realistically high* or *realistically low* self-concept respectively.

To answer the first research question, a linear mixed-effects regression model was fitted with *gender*[3] as the main independent input variable. The *standardised self-concept score* is used as the dependent output variable, while the obtained maths performance score (0-100), the last school grade in mathematics (1 – very good – to 4 – sufficient), age categories (15 - <20, 20 - <25, 25 - <30, or >= 30), the degree program (S, T, M), and the nationality (Austria, Germany, Italy, other) serve as control variables.

To address the second research question, nearest-neighbour matching was employed to pair the 130 women from the original sample with the available men, ensuring (mathematical performance) comparability. A chi-squared test was applied to determine whether the proportions of optimistic, pessimistic, and realistic mathematics self-concepts differ significantly between males and females.

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[1] These students voluntarily participated in a mathematics bridging course at the named university. The ***e****ngineering* branch was excluded due to the small sample size. Nevertheless, the term *STEM* is still used throughout the manuscript.

[2] Excerpts of the questionnaire used can be accessed in German at https://p-gw.github.io/oegp-2018/.

[3] The available options for gender identification were *female*, *male*, and *non-binary*. However, the *non-binary* group was excluded from the analysis due to insufficient sample size for a meaningful quantitative study.

Results

Findings for the first research question

The fitted linear mixed-effects model yields an R-squared value of 0.56 for the fixed input variables. The fixed factors that were statistically significant in at least one category are reported alongside their β-coefficients, standard errors, and p-values in Table 1.

Table 1: β-coefficients, standard errors, and p-values of the fixed factors of the fitted linear mixed-effects model

Variable	Estimated β-coefficient	Standard error	p-value
Intercept	-0.976	0.253	< 0.001
Gender = Female	0.196	0.09	0.03
Math. performance test score	0.017	0.004	< 0.001
Last school grade = 2	-0.391	0.102	< 0.001
Last school grade = 3	-1.045	0.131	< 0.001
Last school grade = 4	-1.665	0.193	< 0.001
STEM = M	0.549	0.11	< 0.001
STEM = S	0.375	0.107	< 0.001
Age = 20 - <25	0.189	0.09	0.037
Age = 25 - <30	0.409	0.273	0.136
Age >= 30	-0.028	0.286	0.921

As Table 1 shows, the primary input variable *gender* is significant ($p = 0.03$): Ceteris paribus, the standardised self-concept score increases by roughly one-fifth of a standard deviation ($\beta = 0.196$) when *gender* is changed to female. Furthermore, the self-concept score is significantly influenced by age category, STEM degree program, last school grade in mathematics, and performance on the mathematical assessment. In contrast, nationality does not have a significant effect on mathematical self-concept and is therefore omitted from Table 1.

Findings for the second research question

Despite an equivalent mathematical performance, the proportion of optimistic, realistic, and pessimistic mathematical self-concepts (see Table 2) differs substantially by gender according to a chi-squared test ($p = 0.021$). Subsequent chi-squared tests for each category reveal a significantly (p

= 0.014) higher proportion of females who own an optimistic mathematical self-concept. The odds of having an optimistic mathematical self-concept are 2.292 (95% confidence interval: 1.121 – 4.864) times higher for women than for men.

Table 2: Number of participants by gender (male, female) and degree program who fall within the categories of optimistic, realistically high, realistically low, and pessimistic mathematical self-concept.

	Science		Technology		Mathematics		Total	
	Female	Male	Female	Male	Female	Male	∑ Female	∑ Male
Optimistic	7	4	3	4	20	7	30	15
Realistically high	18	19	2	6	25	14	45	39
Realistically low	8	14	19	31	15	7	42	52
Pessimistic	6	13	2	4	5	7	13	24
∑	39	50	26	45	65	35	130	130

Discussion and outlook

The outcomes of the first and second research questions contradict the stated hypotheses: Women exhibit a higher mathematical self-concept when entering a STEM degree program at the University of Innsbruck, but they overestimate their mathematical abilities more often. This leads to a higher proportion of optimistic self-concepts among females.

Theoretical explanations for these unexpected results are rooted in the *social* and *criterial* comparative frameworks. First, the *social* comparison was found to be highly relevant, especially for females in Austria (OECD, 2013). Second, the *criterial* comparison emerged as the most important frame of reference according to Geisler et al. (2023). Based on these two reference dimensions, it is reasonable to assume that women who choose a STEM degree program come from secondary school classes with a low mathematical performance level. As a result, they had the opportunity to develop a strong mathematical self-concept in *social* comparison, despite their relatively moderate mathematical abilities – also called the *big-fish-little-pond effect* (Geisler et al., 2023). Due to their relative superiority compared to the rest of the class, they probably achieved good grades, which in turn enables a good mathematical self-concept based on the *criterial* dimension. This explanation, which accounts for the results of both the first and second research questions, is underpinned by the results of PISA 2022 (Toferer et al., 2023): Girls in Austria systematically receive less mathematics instruction than boys because they are more likely to attend schools that are less mathematics-intensive and, consequently, classes with a relatively lower level of mathematics performance.

This suggests, however, that women from mathematically strong secondary school classes may be less likely to pursue STEM careers, despite having adequate mathematical ability. This argument

aligns with the research findings of Goldman and Penner (2016), which show that self-concept has a bigger influence on career decisions than performance.

In addition to these psychological explanations, constructs from sociology or social psychology, such as power relations or *brilliance beliefs*, play a significant role when analysing these results. Due to page limitations, these are omitted in this contribution but should be considered in broader discussions.

Apart from the explanation, the key finding of this research is that men and women in Austria still undergo different comparison and filtering mechanisms when it comes to STEM career aspirations. Particularly, women need to have a higher-than-average self-concept in mathematics to choose a STEM degree program. This results in several suggestions for (1) scholars, (2) policymakers, and (3) universities:

(1) It needs to be investigated whether these results can be replicated at other universities and, if so, to identify the filtering mechanisms causing them. Furthermore, research that not only focuses on gaps but also highlights potential is necessary.
(2) The discovered inequities must be taken seriously by policymakers: Laws and other initiatives should address gender inequality and increase societal awareness. To address the lack of STEM workers in Austria, it is crucial to guarantee equitable access to these fields.
(3) Universities need to account for the found gender gaps while recruiting new students and providing remedial instruction for STEM degree programs. Furthermore, it is crucial to support and enhance the accurate self-perceptions of men and women in mathematics.

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Active listening for storylines experienced by Indigenous and newly migrated mathematics students

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From our interactions with Indigenous and newly migrated mathematics students, we describe our team's efforts to surface the students' experiences and conceptualisations of those experiences. Our shifting approaches inform theorisation of positioning, especially in relation to the key concept of storylines in that theory. In particular, we identify challenges in surfacing student experiences, and in naming their conceptualisation of those experiences as storylines.

Keywords: Indigenous knowledge, immigrants, student diversity, culturally relevant education

Storylines and experience

We share here our efforts to find generative ways to interact with Indigenous and newly migrated mathematics students in order to listen to their accounts of their experiences and how they conceptualise those experiences. We have been theorising our work around positioning (e.g., Herbel-Eisenmann et al., 2015) because we are interested in the opportunities for mathematics students who identify with groups who are often marginalised by colonial ideologies of what mathematics should include and value—thus our interest in Indigenous students and students from families who have recently arrived in migration.

People position each other. For example, a teacher may have certain expectations of an Indigenous student that they do not have for other students. These expectations are informed by many things, including previous interactions with Indigenous students, dramatizations of interactions between teachers and Indigenous students in popular culture, or stories that others have shared about their own experiences. People also enact positions. This means that a newly migrated student, for example, could interact/act in a way that aligns with the expectations their teachers have for them. Or they could resist the way their teacher positions them and instead interact/act in a different way. Whether the people involved are aware or not, there are power relations at work in such positioning.

According to theorisations of positioning, in any interaction people position each other within known storylines (Davies & Harré, 1990). In this way, storylines are at once social structures and structures of power that enable people in interaction to make sense of the interaction. Storylines present ideas about which actions are available and which ones are unavailable to the people involved, while positioning is the allocation of power within a structure. Research that focuses on storylines (e.g., our research), addresses the fundamental structures underneath interactions.

We are interested in those structures in mathematics classrooms because they describe what mathematics is for students in those classrooms. Further, we have an interest in all students having access to the powerful social/citizen tools associated with mathematics, and this interest motivates us to promote practices that would support students who have traditionally had less access to these tools.

While most research using theories of positioning focuses on the positioning (e.g., Sengupta-Irving, 2021), the theorisation typically sees a triad at work, including the three elements of (1) positioning, (2) storyline, and (3) speech/communication act (e.g., Harré, 2012). The people involved in an interaction are associated with positions in a story, which guides their choices about how to interact. Storylines are under-researched, both in mathematics education and in other disciplines. There has been some work on storylines in our field—for example, Andersson et al. (2022) and Rodney et al. (2016) have explored storylines related to learning mathematics that appear in news media, while Gerbrandt and Føyn (2023) described the difficulties of recognizing storylines as they emerge during a mathematics education conference. And there is research in our field bearing similarity to storylines using other conceptual tools to address structures of interaction and power in mathematics education.

Our research context

For this reflection on how to listen to student experiences and conceptualisation of those experiences, we draw on our current research project that seeks to document the storylines that underpin interactions involving Indigenous and newly migrated mathematics students in Wabanaki territories. The region commonly referred to as New Brunswick, Canada, was never ceded or surrendered by the Wabanaki peoples who have lived with the land for millennia. The project aims to work with participant students' mathematics teachers to co-develop teaching approaches that would build on strengths identified in the storyline work.

Two major social movements in Canada are relevant. Following the Truth and Reconciliation Commission hearings and published calls for action (e.g., TRC, 2015), there has been new interest and accountability amongst mathematics teachers to develop proactive and responsive pedagogies for Indigenous and settler students. Meanwhile, human migration has been increasing and politicised; mathematics teachers have been looking for guidance in how to teach new immigrants appropriately.

We have interviewed Grade 7 and 8 students to identify storylines. Here we describe our shifting ways of engaging with the students and reflect on the success of these approaches in terms of surfacing storylines that might be useful in the development of strengths-based approaches to teaching mathematics. Our reflection raises questions about how to make a welcoming space for students to share their experiences, and how to connect this to storylines.

What is a storyline?

Storylines provide communication/action repertoires for people in an interaction. People decide what known storyline they associate with the interaction and then use that storyline to position others in the interaction, to choose their own words and actions, and to interpret the words and actions of others. Herbel-Eisenmann et al. (2015) have drawn attention to a limitation of researchers naming storylines, given that researchers rarely participate in the analysed interactions. In our current and previous research projects we have had to decide how to identify storylines and how to describe (or name) them. For example, one of us was a contributor to the work of Andersson et al. (2022) that identified storylines in Norwegian newspapers and other public media regarding mathematics education and either Indigenous or newly migrated students. In that work there was considerable dialogue about

how to name storylines, subsequently leading to the organisation of a PME working group focused on the twin challenge of identifying and naming storylines (Simensen et al., 2022).

In our current project, we continually revisit and reflect upon our reasonings for identifying storylines. We have come to recognise that we are more interested in our research participants' experiences than in finding the best words to name the storylines they fall within. The name of the storyline is not the storyline. Rather, the name indexes a known story (or set of stories). Thus, we have come to see value in the name being brief and evocative: it should connect people to experiences and conceptualisations that they perceive to be common. In order to focus on stories, we have chosen storyline names that include a subject and action—this is a grammatical criterion that distinguishes identifying storylines from identifying themes. Finally, because we are interested in the positioning, we want the name of a storyline to indicate some kind of explanation for the behaviours and actions in the stories associated with the storyline. In brief, the criteria we developed for naming storylines are:

- brevity
- including subject and action (grammatically)
- including an explanation of behaviour

For example, the storyline "Teachers respond to students differently based on the students' reputations" is relatively brief. The subject is *teachers* and the action is *response*, so there is action with clarity about agency. The explanation for action refers to a teacher *judging* students on the basis of their reputations.

At a recent conference presentation, someone asked us how to correctly name a storyline. Our answer was/is that we are not aware of a standardised way to do this. From conversations with Bronwyn Davies, whose collaboration with Rom Harré introduced the concept of story line to positioning theory, we know that she sees the *line* in *storyline* much like she writes about lines of flight in her current work on new materialism (e.g., Davies, 2021): as a point of departure, open to possibility (like a tangent line, shooting off from a circle, in our mathematically-informed conceptualisations). And we know that she sees Harré's conception of the *line* in *storyline* as more rigid, like a boundary, as he has focused on rights and responsibilities in his development of positioning theory (Davies, 2022). We are more inclined to see theory as continually in process, contested (e.g., Bjerre, 2021) and in conversation with applications of the theory in different contexts.

We want our choices about storylines to open possibilities for a productive transformation of mathematics learnings spaces, and we imagine these spaces as welcoming to the particular strengths exercised by Indigenous and newly migrated students. It would be frivolous to try to kill a storyline because storylines are pervasive. For us, the ultimate goal of storyline work is to redeem pathology-based storylines and thus provide bases for alternative ways to deem/identify the people at play in them. We suggest this can be achieved by identifying opportunities to pivot within a given storyline (Gerbrandt and Wagner, 2022). For example, it would be hard to convince people to stop saying that mathematics is full of mystery (because so many mathematically inclined people like to see mathematics this way), but it could be possible for us to work with people to play with the idea of mystery in mathematics, identify anchor points that are important to them, and pivot the story to light new visions for the possibilities for action (e.g., Andersson & Wagner, 2018).

Such pivot work will be important in our collaboration with teachers in the next phase of this research project. But first we need to surface the storylines, name them, and connect them to accounts of experience. We now turn to that process.

Listening through conversation

In the first context of our research project, Dave (the first author) connected with a teacher who, for more than a decade, has worked as a support teacher for the Indigenous students from a local Wabanaki community. Given the history of terrible experiences at the hands of settlers and colonising institutions, it takes time and care to develop relationships of trust with Indigenous students. This teacher's endorsement and collaboration accelerated the students' willingness to speak openly about their experiences (i.e., to trust). In other words, trust can be shared, to some extent, from an already trusted person. Drawing on the teacher's knowledge of the students, we connected with a group of eight students in a series of six biweekly one-hour meetings.

During these meetings, Dave asked questions about what was *happening* in their math classes—sometimes asking students to describe something interesting, or recent, or frustrating. After a student identified something that happened, Dave probed with questions about what is *really happening* when things like that happen, and questions like "Why would [the person] do that?" Those conversations would compare the math-related experiences with other interactions—sports, friendships, etc. The teacher once told Dave that the students looked forward to these discussions and referred to them as "Dave coming to ask us hard questions."

Using the triad of positioning theorisation, the *happening* is a communication act or a series of communication acts, and the description of what is *really happening* is the storyline. After this series of group conversations, we synthesised the stories and dialogue to bring them back to the students. From discussion amongst ourselves about the deep knowledge emerging from the recorded interactions, we synthesised the foci with the following storylines, formulated around the elements in the principles for naming storylines discussed above—brevity, action, explanation:

1. We have to try hard to learn math.
2. Teachers respond to students differently based on the students' reputations.
3. Math is important.
4. The math we do in school is not interesting.
5. We are more worried about failure than attracted by success in math.
6. How I feel about my math performance depends on my expectations for myself.
7. Math teachers don't understand native people [i.e., Indigenous people].
8. Math should be done in silence.
9. Something about parents… <not sure what>

In interviews with the individual students Dave read the storylines one at a time and asked for example stories that illustrate those storylines. For #9 he said that there was a lot of talk about parents but he could not identify a storyline succinctly, and he asked each student if they could say what the storyline might be. One of the students provided a storyline: "Parents expect you to do well." After this, Dave replaced #9 with the student-provided storyline. It became clear that #5 and #6 were not like the

others: the students said they were not sure about them. Emerging from those conversations, Dave learned that it was helpful to ask students if they thought the statement is true, and to ask if they thought the statement is something people say or think. For #5 and #6 almost all the students said the statements were true (indeed, these emerged from their earlier interactions) but that they were not statements that people think or say. This dialogue underscored two things for us. #5 and #6 are not storylines: they are not known stories. And the students' analysis reminds us that storylines do not have to be true to be well known: a storyline's power is independent of its truth.

Following the principle of OCAP (ownership, control, access, possession), "First Nations have control over data collection processes, (…) they own and control how this information can be used" (https://fnigc.ca/ocap-training/). Thus, for the learning from our interaction with the Wabanaki students, they themselves and their community are the most important people to report to. Sacha (second author) identified an elder from the students' community as a person to report to first, and the teacher endorsed this choice. After Dave spent several hours with the elder to report on what happened thus far in the research conversations, the elder joined the next gathering of the student research group. The elder told the students of their own experiences as a student and as an adult dealing with Indigenous and settler peers and with "authorities". The students asked questions and told the elder about some of their experiences in and related to school.

Listening through prompting with identified storylines

The conversations with Wabanaki students informed our approach to newly migrated students. Again, the first challenge with addressing these students was garnering their interest and trust, but we did not have the benefit of a teacher ally with longstanding relationships. We also know that many newcomers are cautious about what they say about their experiences of migration because many of them are in the process of qualifying for residency status. And many of them come from political contexts that prompt them to be extremely cautious of their information getting into the wrong hands.

Dave and Zyrene (Author #4) met with some classes to tell them about the research and invite participation in interviews. In the first session, we used the approach as with the Indigenous students—asking for examples of things that happen in math class and following up with questions about what is really happening—but the students seemed bewildered by this activity. The dynamics of being in a larger group of people and experiencing linguistic barriers probably impacted their ability to answer questions that are unlike the questions they are accustomed to in school or their other cultural contexts. So we switched to an approach informed by the interviews with the individual Indigenous students, in which Dave would read a storyline, ask if it is true, and then ask if they think others believe it to be true. Dave and Zyrene used storylines from the Indigenous students and from the research contexts in a sister project in Norway (Andersson, et al., 2022). For example, Dave would say, "When I do math, I need to use my first language" (Andersson, et al. 2022) and ask students to put up their hands if they thought it was true (fewer than half of the students said it was true). He then asked students to put up their hands if they thought some people *think* it is true (almost all the students thought that some people think it is true). The students were laser focused on these questions, locking eyes with Dave in a way he has not experienced in a classroom before, which indicated to us the students' serious engagement with the questions. The struggle to decide whether to put their hands

up or not was evident on students' faces (even students who had been in Canada less than a month). The students were invested in these storylines, to recognise the importance of these storylines.

Following Zyrene's suggestion that an infographic might help engage students during small group interviews, we designed the image shown in Figure 1. We prompted students to talk about their experiences by pointing to the relevant parts of the infographic, focusing on the storylines and the accounts of "something that happens." Each interview began with example storylines, like we had done in the large group, asking if the students thought the storyline is true. In the intimacy of a small group of students (2–4), we could follow up with more detail, asking who thought it was true, and then asking for examples of things that they have experienced in school. Students talked about their experiences in Canada, sometimes comparing them to experiences in school in other countries.

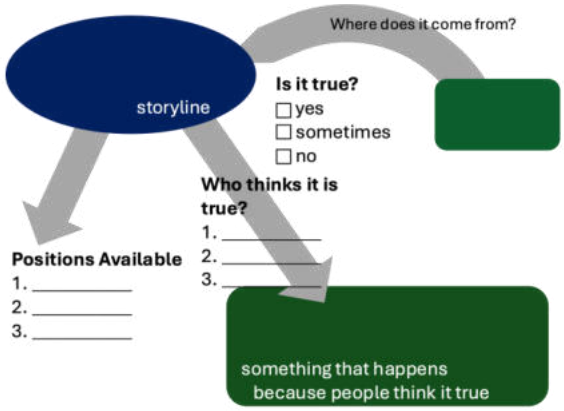

Figure 2: Infographic illustrating the relationships among experiences, storylines and positioning

Then we asked students for examples of interesting or frustrating things that happened in their mathematics learning. From those experiences we pointed to the storyline image and asked what someone must believe to be true for them to behave in the way they did in the interaction. We found that students were readily able to describe experiences and to reflect on reasons and beliefs that underlie the interactions in those experiences—i.e. the storylines. We found, however, that the students did not look at the infographic much. Sometimes we invited students to write on the infographic to fill in some of the details, but none of them engaged in that offer to any significant extent. The graphic organiser might have paradoxically functioned as a barrier, where the bright colours and sharp lines might be associated with professional documents rather than as tools to support idea generation. It might be that students wanted to tell us (humans in interaction with them) their stories, but they were not interested in putting those stories into an abstracted structure.

Discussion

Developing method: In conversations with colleagues about this project, it seems that both the naming of storylines and the methods for surfacing storylines are of particular interest. We suspect that this interest owes in part to the sometimes illusive and intangible nature of storylines. While we might agree that there is a storyline informing our interactions in a given situation, say, between a teacher and a student (e.g., students listen to their teachers), we are also aware that each storyline is rooted within a cultural context. In our interviews with Indigenous and newly migrated students, language,

culture, and identity influenced data collection in visible and invisible ways. For instance, among newly migrated students who were in the early stages of English language learning, there were visible challenges with ensuring they understood the researcher's questions. Though students used digital tools to support multilingual comprehension, the translation apps required loading time and slowed down comprehension. More research is needed to explore the ways that storylines are surfaced across linguistic and cultural divides. We do not know how our attempts to support cross-linguistic communication interacted with students' assessments of the storylines we described.

Surfacing storylines: We sometimes hear educators say "representation matters". Some storylines are visible, recounted in popular culture, such that a majority population could easily recognise themselves holding one of the available positions in the interaction. Saying "representation matters" seems to be a call for available positions in mainstream storylines to start including unexpected or underrepresented voices. This move leaves the dominant stories dominant. So we are motivated to identify different storylines (less recognisable in mainstream/majority/popular culture) from experiences of marginalised populations, which can include positions that are not as universally familiar. Our efforts to surface such stories (described above) is a challenge because those stories are not as familiar to us as mainstream stories. These storylines exist (our findings have surfaced several of them), but they are not as visible and tangible because they come from marginalised groups, thus they are especially important to identify. Indeed, this challenge helped us recognise the distinction between inviting new people into dominant stories and hearing the stories of marginalised people.

We also reflect on the very real positioning that occurs during the research process. What storylines are available to guide the interaction between student participant and adult researcher? How does this positioning influence the resultant data, in turn informing the researcher's understanding of the participant's experiences? We ask ourselves these questions and sit with the unknowing. Even as we sit with the collection of student experiences voiced through stories, we are informed and constrained by the storylines at our fingertips. Assigning a name to a surfaced storyline reflects the preconceptions we carry throughout the process. And the confirmation or refutation of these newly pronounced storylines is again achieved as we participate within the boundary of a storyline.

Pivot points and next steps with teachers: As noted above, we promote the surfacing of storylines not known in mainstream culture. It will be helpful for mathematics teachers to know these, to better understand the experiences of their students. We also have to contend with the mainstream stories, which cannot be killed (because they are in the mainstream), but may, perhaps, be pivoted. This is our next challenge—how to do this in a productive/effective way. We think that the experiences described by marginalised students in relation to those storylines can help us identify what aspects of those stories may be worth keeping and thus be maintained as pivot points to redeem those stories.

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Categorising subjective relevance of mathematics education

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The importance of learning mathematics is rarely doubted. Nevertheless, among students, there is often a lack of concrete ideas about why mathematics education may be relevant to them. These ideas depend on many determinants such as age, culture, social environment, socio-economic variables, interest, self-image, and teaching and teachers. Therefore, they are very individual. Without claiming to be exhaustive, the aim of this paper is to derive dimensions of relevance from theory and, where possible, supplement them with examples from empirical research.

Keywords: Subjective relevance of mathematics education, relevance dimensions of mathematics education.

Introduction

Why is it relevant for me to learn mathematics? What are the benefits of mathematics education for later life? These questions are often posed by students in the mathematics classroom. We propose that these questions are not only an expression of dissatisfaction but also a plea for meaning. Previous research has shown that, when asked for what relevance they see in mathematics education, secondary school students usually merely name the learning of contents for elementary applications, which does not provide relevance to much of what is covered in secondary mathematics education and limits the possible relevance to learning (Kollosche, 2017b; Sachdeva & Eggen, 2023). We assume that such a limited discourse provides hardly any opportunities to experience mathematics as something personally meaningful. Our overall research project is concerned with the questions of how far other discourses about the personal relevance of mathematics education can be identified and how such discourses can be developed among students.

We understand "subjective relevance of mathematics education" as the relevance that students attribute to their engagement in institutionalised mathematics education. We use the phrase "engagement" instead of "learning" because we do not want to narrow the focus down to learning. Instead, students can find mathematics education relevant for different reasons, including the logical nature of mathematics, its gatekeeper function to well-paid jobs, or the self-discipline that mathematics requires. We call such reasons "dimensions" of subjective relevance.

We address the further conceptualisation of subjective relevance of mathematics education elsewhere (Kollosche & Wang, 2024). There, we argue that subjective relevance of mathematics education is important to study for at least three reasons: first, to better understand and improve the performance and motivation of students in the mathematics classroom; second, to better understand and improve the offers of the mathematics classroom in the light of what students find to be subjectively relevant; and third, to better understand mathematics education as a social institution which forms mathematical subjectivities and requires a legitimisation of its existence. To avoid any confusion, we want to stress that it is not our goal to find ways of manipulating students so that they come to believe in the relevance of mathematics education. Here, we want to highlight more specific aspects that are

especially relevant for a critical perspective on the socio-politics of mathematics education: Even if we accepted mathematics education in its current forms, it could be argued that a better knowledge of possible dimensions of subjective relevance would emancipate students, for example, through enabling them to reach better decisions where to invest effort and where not to, or to make demands from other students, teachers, and schools as to what should be offered in the mathematics classroom. Knowledge of the different dimensions in which students come to find mathematics education relevant for them could allow to better create an educational environment which answers to the needs of students. Eventually, discrepancies between what students find to be relevant and what classrooms offer could help to better understand the socio-political nature of the mathematics classroom, for example, concerning the mathematical subjectivities whose development it facilitates or obstructs (e.g., Kollosche, 2018), or concerning the question which social groups are constructed as mathematically able (e.g., Stinson, 2013).

How to study relevance dimensions

The importance of mathematics is demonstrated not only by its global role as a core and compulsory subject (Heymann, 2003), but also by a general consensus among students that it is important to learn mathematics (Kislenko et al., 2007). However, when asked why students think mathematics education is important, they often offer one-dimensional, fragile, and thus easily collapsing ideas about the relevance of mathematics education (Kollosche, 2017b; Sachdeva & Eggen, 2023), such as the idea that you need mathematics in your daily life. As in many other areas, the conviction that mathematics or mathematics education is relevant often reproduces discussions in society without questioning (Ernest, 2004) and is significantly influenced by these discussions. However, based on this basic attitude, questions automatically arise about why mathematics education is important and for what purpose. We assume that the ideas of why mathematics education is personally relevant for someone can vary and that there will be no normative, complete list of relevance dimensions.

In this paper, we want to provide an overview of as many dimensions of the subjective relevance of mathematics education as possible. To achieve this, dimensions of subjective relevance derived from theory and empirical studies are presented. Regarding the theoretical backgrounds of the dimensions we will discuss, we have decided to limit ourselves to selected publications on the educational theory and the sociology of mathematics education instead of conducting a systematic literature review. These two orientations allow us to include both normative and descriptive perspectives on the relevance of mathematics education.

We acknowledge that this choice of literature is very limited and only a first step. With this paper, we do not want to conduct a systematic literature survey, nor do we claim to present an exhaustive categorisation of subjective relevance of mathematics education. Instead, we want to start a discussion about subjective relevance, its importance, its conceptualisation, and its study. Our categorisation can be understood as a first attempt of theory building in that matter. Including more publications might allow us to identify further dimensions later. We also draw insights from empirical studies which have gathered data on why students find mathematics education to be relevant. Interestingly, such findings can challenge the education theory and sociology of mathematics education. An example of this would be the idea that somebody assumes mathematics education to

be personally relevant to be able to help other people, such as helping their own children with their homework (Sachdeva & Eggen, 2023).

As a way of presentation, we present phrases that can follow the beginning of the sentence "Mathematics education is relevant for me because...". The dimensions that we present in this way are no quotes from literature or student data. They rather constitute ideal types of these dimensions in the sense of Weber (1949), that is, "logical constructs" whose "function is the comparison with empirical reality in order to establish its divergences or similarities, to describe them with the most unambiguously intelligible concepts, and to understand and explain them causally" in order to aid sociological analysis (pp. 42–43). However, we will elaborate on how these dimensions are addressed in literature or by students.

Findings

The following paragraphs show our findings. They each start with our formulation of the dimension in question, followed by accounts of where these dimensions have been addressed in literature or by students. In some cases, we also elaborate further on the dimensions, for example, concerning their distinction from other dimensions.

Mathematics education is relevant for me because I need mathematics in daily life. This is a very general statement that is stated very often. From an education theory perspective in the sense of the concept of general education, on which there exist extensive discussions, especially in German-speaking countries, one aim of school, and thus also of mathematics education, is to prepare students for life, as a result of which what they have learnt should be personally applicable and useful in practice (Heymann, 2003; Winter, 1995). However, the prevailing idealistic view of qualifications that should be applicable and usable in everyday private and professional life (Heymann, 2003) reflects the subjective relevance of mathematics education only to a limited extent, as mathematics is more than just an aid in everyday life. Some scholars question the use value of mathematics education altogether (Pais, 2013). When examining empirical studies on this dimension, it becomes obvious that students often mention situations related to shopping, for example, as the following excerpt from an interview study conducted by Kollosche in Germany in 2016 shows: "You need mathematics to avoid being cheated, for example, at the cash desk" (Kollosche, 2019, p. 55, our translation).

Mathematics education is relevant for me because mastering elementary mathematical calculation techniques is fundamental. This statement reflects the importance of elementary mathematical calculation techniques. Furthermore, it emphasises the fact that in terms of content, most of what is to be learnt in mathematics education, especially in German-speaking countries, appears to be of little use to the broad masses, i.e., to those for whom mathematics is no longer an essential part of their lives after completing their education. Following the assumption of Heymann (2003), all the content necessary for daily life has been taught until the 7th school year and, as studies show, everything beyond this is often forgotten anyway (e.g., Department for Business, Innovation and Skills, 2012). Performing elementary mathematical calculation techniques or calculating percentages are, for example, dominant examples of subjective relevance at the content level among students, as the following excerpt from an interview study conducted by Sachdeva and Eggen (2023) in Norway confirms: "especially just plus and minus and division and … multiplication … these are things I find

very useful …" (p. 12). Content that is addressed at higher school levels is rarely described as relevant. Reasons for this could be that the students do not yet know what they can use the content for and that mastery of the content is not necessary in everyday life. In addition, there are a number of technological aids available, particularly in the field of operations, which means that even fewer mathematical skills and knowledge are required of the majority of people (Borovik, 2016). Conversely, these technological aids, especially their (further) development, require more mathematical knowledge from mathematical experts. This leads directly to the relevance of mathematics education in connection with the profession, which is also frequently mentioned by students in several studies (e.g., Dobie, 2019; Sachdeva & Eggen, 2023) and discussed next.

Mathematics education is relevant for me because you need mathematics in almost every job. The perceived relevance of mathematics education in connection with future jobs certainly depends strongly on the students' level of development and how much they have already been involved with different jobs. The proportion of mathematical skills and knowledge, as well as the content required, varies from profession to profession, and therefore, the relevance attributions also differ. From the perspective of education theory, Heymann (2003) argued that the different mathematical requirements of different jobs imply that these mathematical requirements cannot be a part of general education but have to be covered in vocational education. Empirical studies show that in particular children with specific job expectations see the relevance of mathematics education for their future job, as the following quote shows: "I want to be a farmer like my parents and that requires a lot of mathematics, for example, statistics" (Kollosche, 2019, p. 56, our translation).

Mathematics education is relevant for me because mathematical skills are crucial for your opportunities in the labour market. Mathematics is repeatedly ascribed the function of gatekeeper to economic access, full citizenship, and higher education (Stinson, 2004). This effect appears to be strong in countries such as the US, where mathematics is not compulsory throughout the school career but can be dropped at some point. The deselection of advanced mathematics courses by students is usually accompanied by reduced opportunities in the labour market, which means that the gatekeeper function of mathematics often correlates with exclusion, and the subjective relevance of mathematics education takes on a completely different meaning. From a sociological perspective, in addition to qualification, selection and allocation are central functions of school (Kollosche, 2018). Grades as a means of selection usually have a major influence on students' educational paths and, subsequently, on their career paths. In mathematics education, selection often plays a special role in the sense that the mathematics grade in the private sector, for example, has a stronger weighting than other grades. This may be related to the fact that mathematics education is said to demand many qualities that go beyond the content area (e.g., meta-competences or character traits, see below) (ibid.). The usefulness of mathematics education in connection with getting a job is also repeatedly mentioned by students in empirical studies. For example, one student in the study conducted by Dobie (2019) mentioned: "I think it is the most useful because you are more likely to get a job if you are good at math" (Dobie, 2019, p. 40).

Mathematics education is relevant for me because I need mathematics education to pass the tests and to graduate. To a certain extent, this statement is related to the previously discussed dimension of subjective relevance. For many career paths it is a requirement to be graduated or at least to finish the

compulsory school with positive grades. Examinations of all kinds play a particularly central role in this context. This can ultimately lead to students seeing the relevance of mathematics education in passing exams to advance to the next school level, to be able to start an apprenticeship, to obtain a degree, and to gain the right to study or to avoid being assigned to a certain social position. The following quote from an interview study by Sachdeva and Eggen (2023) proves that this relevance attribution exists among students: "I do not know how useful it will be in the future ... I have not experienced a situation where I would need it except for the tests ... so ..." (p. 12).

Mathematics education is relevant for me because I can work completely within mathematics without any reference to another discipline. On the one hand, Winter (1995) demanded that mathematics education has to be application-oriented and has to address real-life problems. On the other hand, reducing mathematics education solely to applications is not representative for the extensive significance of mathematics, which is also considered a particular intellectual experience due to its internal logical order. In this sense, Winter (1995) demands that mathematics education should allow students "to get to know and understand mathematical objects and facts, represented in language, symbols, pictures and formulae, as intellectual creations, as a deductively ordered world of its own kind" (p. 37, our translation). Searching in empirical studies for statements which are linked to this dimension leads, for example, to the following statement of a student from the interview study conducted by Kollosche in Germany in 2016: "I think it's crazy, somehow positively crazy, that you can think up such abstract things in mathematics and then think about them for so long." (Kollosche, 2019, p. 56, our translation). This statement not only expresses the positive experience when doing mathematics, but it can also be something in which a student sees the relevance for themself.

Apart from the predominantly content-related, application-related and career-relevant subjective relevance dimensions discussed previously, the following mathematical meta-competences, by which we mean competences that are independent of the mathematical content, are also particularly important in mathematics education. In fact, these competences are all aspects of mathematical literacy and are often also referred to as key qualifications. However, they still play a subordinate role in education due to the frequent dominance of calculation tasks that often require nothing more than applying a learnt schema, which is also reflected by the students' responses in several interview studies concerning the relevance of mathematics education (e.g., Sachdeva & Eggen, 2023). Nevertheless, precisely these competences make responsible citizens and cannot be outsourced to technology.

Mathematics education is relevant because it helps me in solving problems. The ability to solve problems and, thus, the use of heuristic strategies, illustrate a central competence, which was also formulated by Winter (1995) as one of the three basic experiences that mathematics education should allow. The subjective relevance in this context lies in applying these problem-solving skills to problems encountered outside the mathematics classroom. However, at this point, it is important to mention that there is no evidence that the heuristic strategies taught in mathematics education can be applied outside of mathematics education (Kollosche, 2018).

Mathematics education is relevant because it helps you to learn to think logically and argue coherently. These competences play a central role in mathematics education due to the deductive

structure of mathematics and represent further dimensions of subjective relevance as mathematical meta-skills, which are also important beyond mathematics education, for example, when writing an essay. In addition, mathematics education can contribute to adopting a critical attitude towards reports and statistics circulating in the public domain. This was described by Heymann (2003) as "development of critical thinking" as one of the tasks of general education. On this relevance dimension, there also exist students' thoughts: One student in the interview study conducted by Kollosche (2019), for example, mentions: "I find mathematics totally logical. You don't have to look everything up or just know it, you can work it out for yourself" (p. 56, our translation).

Mathematics education is relevant for me because it helps me to concentrate and work in a disciplined way. In addition to concentration, solving mathematical problems often requires students to be able to stay disciplined. This is also a finding from empirical studies. According to these, students state that mathematics education requires concentration and self-discipline (e.g., Kollosche, 2017a). Therefore, it can be assumed that mathematics education can also train and promote concentration, which can be considered relevant.

However, it is questionable whether mathematics education is partly responsible for promoting character traits (e.g., perseverance, frustration tolerance, ability to concentrate, accuracy) or whether mathematics education rather tests them. At least, there is no evidence that mathematics education actually has an influence on character traits (Kollosche, 2018).

Mathematics education is relevant for me because it allows me to help other people if they have difficulties in mathematics. Although this statement cannot be categorized appropriately in terms of education theory or mathematics education sociology, empirical studies show that mathematics education can also be relevant for someone for this reason. One student, for example, answered the question whether mathematics education will be important for himself/herself later in life in the interview study by Sachdeva and Eggen (2023) in Norway like this: "Yeah … I maybe think that too … like if you will have children then you can help them with their work." (p. 15). This is a dimension that, like many others, is certainly not relevant for everyone, but at least for a few and is probably also strongly dependent on character.

Conclusion

Looking at the subjective relevance dimensions we mentioned in this paper, it is clearly recognisable that relevance attributions, even if mathematics is closely linked to society and its needs, are something personal and, therefore, individual. Thus, there exist neither right nor wrong ideas of relevance. We acknowledged earlier that the above list is certainly incomplete. We are engaged in an empirical study in which we will collect further data about students' subjective relevance and will report on that on a later occasion. This might be the first study where students are directly asked in how far and why mathematics education is relevant for them, while the above-mentioned studies used different questions, for example, on the general importance of mathematics education and the consequences of their answers for the relevance they see in it can only be assumed.

However, our research interest is not only empirical in nature. We also look forward to a further theoretical conceptualisation of subjective relevance, which could begin with a theory-based

organisation of the dimensions we discussed. Candidates for such background theories include Bourdieu's adaption of the Marxist concept of use and exchange value (Williams, 2012), expectation-by-value theory (Eccles et al., 1983), self-determination theory (Ryan & Deci, 1984), and the theory of basic desires (Reiss, 2016).

The most important thing is to offer students dimensions in which mathematics education can be experienced as personally relevant. In this context, it is problematic that the image of mathematics is usually blurred and reduced, not only among students but also in society as a whole. Therefore, it is unsurprising that subjective relevance dimensions which go beyond everyday practical benefits such as shopping are rarely recognised by students. In order to offer students the opportunity to experience other dimensions of relevance, it is necessary to address them in the classroom actively. The subjective relevance dimensions discussed in this paper could provide a basis for this. Findings on the possible further development of ideas of subjective relevance in mathematics education with students are an urgent research desideratum, which we will address in a project in the coming years.

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Socially just relationships between mathematics teachers and learners

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This paper discusses the importance of relationships between teachers and learners of mathematics when considering socially just approaches to mathematics. The need to focus on relationships is justified with reference to mathematics teaching in primary and secondary school contexts. Findings from three research projects facilitated by the author are revisited through a new lens informed by relevant research literature. Three fields of educational research that relate to socially just relationships are explored: relational pedagogy, relational authority and relational agency. Insight generated through this review is used to identify implications for future research and practice.

Keywords: Social justice, relationships, teachers and learners, mathematics.

Introduction

In this paper I discuss the importance of focusing on relationships between teachers and learners when considering socially just approaches to teaching mathematics. I revisit the findings from three research projects I have facilitated, all of which involved working collaboratively with teachers in conducting participatory action research. These include the Teaching Maths for social Justice (TMSJ) project (Wright, 2017), the Visible Maths Pedagogy (VMP) project (Wright, Fejzo, & Carvalho, 2022) and the Primary Maths and Social Justice (PMSJ) project (Wright, Kelly, & Hilton, 2024). I review relevant findings from each project through a new lens, based on relationships between teachers and learners, informed by relevant theories that are presented in the paper.

The common methodological approach involved recognising the complementary expertise of the academic/teacher researchers and reflecting critically on existing practice. Teacher researchers played a key role in the research design, for example by leading the planning and evaluation of research lessons and collecting data from students (through interviews/surveys) to evaluate the impact of their actions. Data were collected from the teacher researchers through recording reflective discussions during research team meetings and interviews. Thematic analyses were carried out on the transcripts.

The three projects referred to above draw on a conceptualisation of socially just mathematics teaching that includes four essential elements:

- Employ collaborative, discursive, problem-solving, and problem-posing pedagogies which promote mathematical sense-making and the engagement of all learners with mathematics.
- Promote mathematical inquiries that resonate with learners' real-life experiences and that help them develop greater understanding of their social, cultural, political, and economic situations.
- Facilitate mathematical investigations that develop learners' individual and collective agency, enabling them to take part in future social action for the public good.
- Challenge common myths surrounding school mathematics, expose processes that lead to the marginalisation of some learners, and open up to scrutiny what it means to be successful. (Wright, 2024, p. 26)

The importance of socially just relationships

The persistent associations between students' socioeconomic status and their mathematical achievement and participation have been well documented (Boaler, Altendorf, & Kent, 2011). School mathematics continues to serve as gatekeeper qualification determining students' future access to higher paid employment and higher status further education courses. School mathematics makes a significant contribution towards what Bourdieu argues is the primary function of the school, that is to reproduce the dominance of the privileged groups within society from one generation to the next (Bourdieu, 1998). Teachers of mathematics play a key role here as, in the students' eyes, they represent the authority of the school. Bourdieu poses the following paradox for any teacher wishing to convince their students of their desire to disrupt this cycle of reproduction within their classroom:

> Either you believe I'm not lying when I tell you education is violence and my teaching is legitimate, so you can't believe me; or you believe I'm lying and my teaching is legitimate, so you still can't believe what I say when I tell you it is violence. (Bourdieu & Passeron, 1990, p. 12)

If teachers are to play a role in challenging inequity within mathematics education, then they need to question their own behaviour and actions within the classroom, and how they are perceived by students, which necessitates fundamentally reappraising the basis of their relationships with learners. They need to become more aware of the discourses they are subject to, many of which are generated by myths associated with mathematics learning, and which strongly influence their practice in the classroom. Bourdieu would argue that, since teachers of mathematics have experienced being relatively successful learners themselves, they are more inclined to accept conventional approaches to teaching mathematics as legitimate and adopt them in their own practice.

One example of a discourse that permeates the teaching profession (and society more generally) is a belief that mathematics is neutral and value free, a claim that has been discredited by critical mathematics educators (Skovsmose, 2021). A tendency to ignore the value-laden nature of mathematics means that teachers (and managers, researchers and policy makers for that matter) are more likely to neglect the importance of relationships in teaching mathematics. This has varying implications for different phases of education.

In the primary phase, at least in the context of Scotland and England with which I am familiar, one teacher normally takes responsibility for the pastoral care and wellbeing of all students in their class (who they teach for most subjects). Primary teachers therefore tend to appreciate the importance for general learning of developing supportive relationships with students based on mutual trust and respect. However, there is often a lack of appreciation of how the principles on which these relationships are built can be applied when teaching mathematics. Alienation and anxiety that many young children experience in learning mathematics can be an important factor in students becoming disengaged from the subject, particularly those from marginalised groups who tend to have less-positive dispositions towards learning. Aldridge and Gifford (2018) attribute such alienation amongst students in primary schools in England to discourses of accountability with students often being tested and assigned levels by teachers on a weekly basis. A resulting tendency to focus on closing the gap in mathematics attainment generates deficit views of learners resulting in damaging strategies such as separating low attaining children from their peers in mathematics lessons (not normally the case

in other subjects), thus denying them access to the full curriculum. Aldridge and Gifford also highlight the increasing prevalence of teaching approaches in mathematics that emphasise answer-getting, rather than sense making. Hence, cultivating positive relationships in the mathematics classroom is key for primary teachers who wish to address social justice and inequity.

In the secondary phase, those mathematics teachers who fail to recognise the value-laden nature of the subject tend to see relationships as unimportant in the mathematics classroom and are often guilty of leaving responsibility for the emotional wellbeing of students to others. In this way, negative attitudes towards the subject are reinforced and manifest themselves through disengagement and negative behaviour, resulting in sanctions that continue the downward spiral. There are other myths surrounding mathematics that damage students' learning. These include the unjustified value that is often attributed to working alone in silence, completing as many questions as possible in the shortest possible time, and avoiding errors at all costs. Whilst these are seen as desirable behaviours by many, they do not reflect what real mathematicians do or effective approaches for enabling students to develop powerful mathematical knowledge and the socio-mathematical agency they need to apply this knowledge collectively to solve real life problems (Wright, Kelly, & Hilton, 2024). Cultivating socially just relationships between mathematics teachers and learners, based on trust and empathy, is necessary if students are to engage in discussion and debate around the nature of mathematics and the best way to achieve success. This reflects the fourth essential element of socially just mathematics teaching highlighted in the Introduction.

Focusing on relationships between teachers and learners of mathematics offers a 'way in' for teachers who may be less willing to make wholesale changes to their teaching repertoire. Since changes identified through raising awareness of relationships can be enacted immediately, it is a potentially powerful strategy for extending the group of teachers who are committed to addressing issues of social justice in their practice.

Much has been written about relationships between teachers and learners in general education, although there is limited literature that relates this to socially just mathematics teaching. In the next sections, I will begin to address this by looking at three fields of educational research that relate to socially just relationships between teachers and learners of mathematics: relational pedagogy, relational authority and relational agency. For each of the three fields, I draw connections to the findings from one of the research projects by viewing the findings through a new lens focusing on relationships between teachers and learners (which was not a primary focus for any of the projects).

Relational pedagogy

Relational pedagogy involves prioritising relationships between teachers and learners over other aspects of education (Aspelin, 2014; Felten & Lambert, 2020). It is often characterised by the desire to establish a cooperative classroom ethos and democratic classroom practices, in which learners take responsibility for their own learning, and which provide opportunities for the growth of mutual trust and understanding between teachers and learners. Noddings (2013) advocates for an ethics of care within education that encourages teachers to consider the need to establish positive and compassionate relationships with students as the basis for pedagogical decisions they take. Watson (2021) expands on what this might look like in the mathematics classroom, suggesting that care in

mathematics education should involve the teacher prioritising listening to (rather than talking to) students, drawing on students' interests, experiences and ideas in developing mathematical meaning, and working alongside students on mathematical tasks.

English et al. (2023, p. 6) articulate five different types of "pedagogical listening" that teachers might engage in during students' productive struggle on open-ended tasks. These strategies, which aim to respect the contributions of learners (as human beings) rather than merely evaluating correctness, are described below. "Empathetic listening" involves the teacher suspending their own judgement of a student's response and focusing instead on what the response says about the student's feelings and how they are making sense of their own world. "Supportive listening" involves encouraging students to listen to each other's mathematical ideas and hence cultivate an inclusive classroom community. "Educative listening" involves teachers posing questions that help students to reflect on what they are struggling with so that they may better understand the nature of the problem. "Self-reflective listening" involves teachers listening carefully for unexpected responses from students that might help them better understand students' learning processes. "Generative listening" involves teachers looking for opportunities to generate new directions in a discussion that will enable better understanding to emerge.

The TMSJ research project, which I conducted with a group of five secondary mathematics teacher researchers from 2013 to 2014, focused on exploring strategies for translating theories relating to social justice into classroom practice (Wright, 2017). The following extract from Brian's (one of the teacher researchers) reflections at the end of the study demonstrates the impact of focusing on social justice in the mathematics classroom on the developing relationships between teachers and learners.

> Through this project … I have come to see the privileged role that mathematics plays in the way individuals are judged and categorized. Due to the project, I ended up discussing these features with students. We also discussed and evaluated the emotional baggage they brought with them to their mathematics lessons. This was particularly constructive with a low achieving (penultimate) set of girls who had already experienced 3 years of secondary school mathematics. Their emotions were overwhelmingly negative: anger, frustration, despair and general unhappiness. By working through these explicitly, and highlighting the pre-eminence of mathematics in society, we were able to find much common ground and thus change the atmosphere in the class. It moved from a teacher attempting to get students to do what they want (e.g, staying on track with the scheme of work), to a teacher helping students to both beat the present (education) system and to be able to engage confidently with a world where much is predicated by mathematics (tax, high paying jobs, shares, etc). A key component of my approach was also to listen a lot (to all sorts of things within and outside the maths curriculum!) and speak encouragements frequently (vital where one of the key barriers was a lack of confidence). (Brian, reflective report)

What comes across clearly is Brian's growing appreciation of the need to listen to students and engage in discussions with them about their feelings towards mathematics and the role it plays in society. This resonates with relational pedagogy's focus on prioritising relationships between teachers and learners and facilitating the development of mutual trust within the classroom.

Relational authority

The relational pedagogy outlined above resonates with Freire's (1972) dialogic approach in which the teacher (or educator) poses questions that engage learners in dialogue that helps them articulate their own perspectives, engage with developing ideas and hence develop a greater appreciation of their own real-life situations. This requires a fundamental shift in the relationship between teachers and learners that challenges conventional assumptions such as the teacher being the sole provider of knowledge. Bernstein (2000, p. 46) argues that a socially just pedagogy involves the cultivation of "personal" authority, in which teachers negotiate classroom rules with students, rather than exerting "positional" authority by resorting to their position of power as a teacher to impose rules.

Bernstein (2000, p. 26) argues that children from more privileged backgrounds are better able to interpret the rules of the game that govern success in the mathematics classroom, including how to "recognise" the mathematics within a task and how to "realise" an appropriate response. Since these rules tend to be less visible when adopting the inquiry-based pedagogies called for as part of the conceptualisation of socially just mathematics teaching highlighted earlier, some researchers have argued that they risk further disadvantaging students from marginalised groups (Lubienski, 2004; Morgan, Hordern, & Hoadley, 2019). However, rather than using this as a justification for adopting more teacher-centred pedagogies (in which the rules for achieving success are more visible), I argue that socially just mathematics teaching should involve teachers discussing and sharing their pedagogic rationale with students so that all learners, including those from marginalised groups, are better prepared to achieve success whilst engaging with inquiry-based or progressive approaches.

The VMP research project, which I conducted with two secondary teacher researchers, Tiago and Alba, from 2017 to 2019, aimed to develop strategies for helping all students to recognise the teacher's intentions when adopting progressive approaches to teaching mathematics (Wright, Fejzo, & Carvalho, 2022). The findings demonstrated how, through holding open and transparent discussions with students about their reasons for adopting particular teaching approaches, teachers can help learners to develop a greater appreciation of their pedagogic rationale and how to respond appropriately to achieve success. The following extracts from interviews with two students, Keira and Neal (both identified as belonging to relatively low socio-economic groups), show how such discussions can lead to an increased understanding of the rationale for agreeing a model solution to a problem (Keira) and for sharing solutions with other students (Neal):

> I learnt, like, more understanding of why we do, like, why we write down stuff in our book. And, like, if you do working out, and we discuss it, then we write it down. I understand why we do that more. (Keira, Interview 1)

> Like, some people could have different, like, reasons why … because, like, not everyone thinks the same. So, like, you could have your own reason, whereas someone else could have their own reason for their own solution. (Neal, Interview 1)

One of the teacher researchers, Alba, reports how students developed a greater appreciation of the strategy she adopted for identifying important information in a question, after she discussed the reasons for doing so with the whole class:

> So, I thought it was a really productive chat/conversation with the class. Like [Student B], for example, he's an EAL [speaker of English as an additional language] student, and for him to be able to say … "pick out relevant information" … one of the reasons he struggles in tests is he doesn't understand what the question's asking him, even though his maths is actually very, very good. So, for him to realise that importance is quite encouraging, to know that, OK, this could be useful to him. (Alba, Research Group Meeting 13)

Through making their pedagogy more visible to learners, the teacher researchers developed greater levels of relational authority and facilitated a more dialogic approach towards learning.

Relational agency

Student agency is often neglected in mathematics classrooms with a tendency to assume that learning mathematical procedures in an abstract way, completely disconnected from real-world problems, is sufficient to provide students with the powerful knowledge they need to become active members of society. However, Manyukhina and Wyse (2019) emphasise that powerful knowledge depends on the agency of the knower. Whilst Bernstein (2000) argues that abstract and specialised knowledge is powerful in that it enables learners to extend their horizons, Muller and Young (2019) contend that powerful knowledge also requires learners to understand 'disciplinary meaning', i.e. how new knowledge is generated within a discipline. This can only be achieved by providing mathematics learners with opportunities to consider new lines of inquiry, make conjectures, pose questions, listen to other people's ideas, consider different solutions, justify and explain their ideas (Mason, Burton, & Stacey, 1985). Such opportunities depend on teachers cultivating relationships in which learners have greater levels of agency over the direction their learning takes. Biesta (2022) warns that, in promoting learner-centred approaches, care must be taken not to lose sight of the purpose of education. He argues that the teacher's role should be to direct students' attention towards the world they live in, enabling them to develop as subjects who can understand and act upon their world.

The PMSJ research project, which I conducted with two co-researchers and six primary teacher researchers from 2021 to 2022, explored how teachers can pursue their interest in social justice through the teaching of mathematics (Wright, Kelly, & Hilton, 2024). One aspect we were particularly interested in was the development of 'socio-mathematical agency', i.e. students' ability to use mathematics effectively in arguing collectively for social change. The following extracts demonstrate how teacher researchers provided greater opportunities for students to use mathematics to explore issues of interest to them and to incorporate mathematical ideas into their arguments for change. This had a significant impact on the teachers' own views of mathematics, particularly their thinking about relationships with learners, as well as enhancing students' socio-mathematical agency.

> And I think, once you start to see the children actually grapple with these issues, and form quite well-articulated opinions … it's really important that we give them a chance to start to understand the problems, and give them a voice to say what they think. (Rose, Year 5 teacher, Interview 2)

> The activities that we did came out of things that the children were interested in at the time … and I think that particularly supports our lower attainers. And they felt like they had a bit of a choice in what we were doing. (Kate, Year 1 teacher, Interview 2)

> So, we, kind of, saw that, when they went off doing their independent task … the children had a better overview at the end and were able to feedback during the plenary, and actually give answers that had reasoning behind them, rather than just: "This is the answer and that's it." … the children have more to say about social justice, or fairness, equality, doing the maths, where it made an impact on their own lives. (Emma, Year 1 teacher, meeting 5)

The findings suggest teachers were successful in cultivating relationships with students that contributed towards generating higher levels of learner agency within the mathematics classroom.

Implications for research and practice

I have highlighted in this paper the need for a greater focus on relationships between teachers and learners of mathematics. Whilst research projects that involve addressing issues of social justice in the mathematics classroom will inevitably consider such relationships, insight generated in this area is often implicit within the findings rather than explicit. I argue therefore that participatory research into mathematics and social justice is needed that foregrounds relationships between teachers and learners. Teachers should be encouraged to critically reflect on existing relationships, and their characteristics, by considering how (and the extent to which) they:

- enhance the contribution that mathematics plays to reproducing inequity
- produce or exacerbate mathematical anxiety and alienation
- promote discourses that misrepresent mathematics as individualistic and value-free
- present mathematics as primarily about answer-getting rather than sense-making

Teachers should be encouraged to identify characteristics of existing relationships that might be considered socially just, by considering how (and the extent to which) they:

- promote a cooperative classroom ethos and democratic practices
- embrace pedagogic listening and enhance dialogic teaching
- make the teacher's pedagogic rationale more visible to learners
- cultivate students' socio-mathematical agency

Through a genuine collaboration with teacher researchers, academic researchers can help identify and articulate what socially just relationships between teachers and learners of mathematics might look like in the classroom and explore strategies that might be employed for their cultivation.

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Gender differences in mathematics education in prison context

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Introduction

In recent years several studies have focused on gender differences in mathematics (Leder & Forgasz, 2008) and international standardized assessments continue to confirm that male students overperform female students in most countries (e.g. OECD, 2023). This gap is closely linked to social and cultural factors, as well as stereotypes influencing metacognitive aspects of mathematics (Leder & Forgasz, 2008), and this has a strong impact also on career choices of females. This study explores gender differences in the teaching and learning of mathematics within the prison context, a setting rarely addressed in educational research and even more in mathematics education. The prison is considered a "total institution" (Goffman, 1961), which disciplines and reshapes bodies and personalities through mechanisms of control (Foucault, 1975), significantly impacting teaching processes. The promotion of cultural and professional growth during detention is a key element in fostering social inclusion.

This work is part of the Italian national project Learning Mathematics in Prison (LeMP) which investigates the conditions and challenges of teaching mathematics within a prison setting, focusing on the insights of the teachers involved. In this preliminary study focused on gender, we will consider data coming from national reports and teachers' perspective to understand which are the peculiarities of teaching mathematics to male and female students in the prison context in Italy. Thus, our research questions are: How do mathematics teachers perceive gender difference within the school prison context? How do they perceive this difference in relation to mathematics learning?

Methods

In the first year of LeMP project four teachers from the three prisons in Italy were involved in semi-structured interviews aimed at exploring their motivations and perceived roles. Interviews were conducted in one-hour online meetings by researchers; in this poster, we focus on the part of the interview aimed at discussing observed differences among students, with a focus on gender. All interviews were audio-recorded and transcribed. The researchers carried out a thematic analysis (Braun, Clarke, 2021) of the collected data, identifying recurring patterns in the teachers' perceptions of gender biases. Excerpts concerning these differences classified and collectively discussed. The analysis was supported by ethnographic tools (Geertz, 1973), such as fieldnotes, which provided contextual depth and enriched the interpretation of findings within the complex dynamics of the prison environment.

Discussion of preliminary results and further steps

In Italy, women represent only 4.3% of the prison population, a stable percentage that renders them a marginal group often overlooked in terms of educational resources and programs. Previous

schooling, future expectations, and gender stereotypes deeply shape their learning paths. Moreover, there is lack, or near absence, of data and studies addressing gender dynamics in this specific context. In the interviews teachers report significant differences in the experience of detention between women and men.

Discussing more specifically issues related to their learning experiences, teachers observed that female students often struggle with focus, influenced by emotional labor, guilt, and family responsibilities. Female students often face psychological and emotional challenges, including depression and guilt related to family responsibilities, which can disrupt attendance and engagement (a teacher said: *"They had an intermittent frequency, because sometimes they were really depressed, so they took therapy.")*.Gender biases among teachers also emerge, as they tend to view female students as more vulnerable, underestimating their abilities (a teacher noted: "*In female sections, women are often mothers, and they suffer due to guilt about being away from their children, which makes them more vulnerable")*. Teachers noted stereotypes about women's mathematical abilities and education, often framing it as preparation for traditionally female-dominated jobs (beautician, seamstress, waitress), and in which mathematics competences are not considered as fundamental. Regarding previous education, the interviews do not provide clear data, reflecting the lack of detailed information in ministerial reports on women and education in prison. Data from interviews with other teachers will be collected next year and, to go further, and get a direct perspective on learning mathematics in prison, in order to include them in designing new educational tools.

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TWG 11: Teaching and Learning of Discrete and Computational Mathematics

Thematic Working Group 11
Teaching and learning of discrete and computational mathematics

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This TWG was newly created for CERME 14, emerging from the encounter of the interests of a previously existing group entitled *Algorithms* (Weber et al., 2022) and a prospective group project on discrete mathematics, based on common observations on recent trends in mathematics linked with computer science.

Indeed, the development of computer science in the last decades has generated changes in the content and practices in mathematics. In particular, it has contributed to the recent emphasis and success of discrete mathematics, a classical topic in mathematics. Also, the development of computer science has led to the growth of computational approaches in mathematics, fostering applications and new ways of thinking and doing mathematics. These evolutions in mathematics, and the related needs for future generations, require and produce new content in school that our TWG intends to address. The topics covered by the TWG are taking a growing place in many current curricula around the world, and have entered into the common core of mathematics as described by international organizations like OECD. For this reason, research developments, and insights are expected by teachers, teacher educators, and policymakers.

To address these challenges at CERME 14, the TWG invited research papers that discuss empirical, theoretical, methodological, or philosophical issues pertaining to the teaching and learning of discrete and computational mathematics, including the following themes: all aspects of discrete mathematics, i.e., both classical subjects, and subjects invoking new needs for computer science like combinatorics, graph theory, discrete geometry, automata, game theory, cryptography, etc.; manipulation and representation of mathematical objects and data in computer science, simulations and discrete modelling, new computational views on classical mathematical objects, computational approaches in mathematics; links between mathematics and computer science, their foundations, concepts, ideas, and methods; algorithms in mathematics, the study and analysis of algorithms; computational thinking, and programming in mathematics; proof, proving, and problem solving in discrete and computational mathematics. Given the current curricular issues, our TWG has interest in these topics in pre-school, primary, and secondary levels, as well as studies at university level, and the implications for teacher education.

We were pleased that the 25 accepted papers and posters presented at CERME 14 covered a wide diversity of the targeted topics. We organized them following four themes.

The first theme was "Implementing Computational Thinking in mathematics", a topic gaining prominence in educational curricula across various countries (Ye et al., 2023). However, participants noted that CT is still a somewhat ambiguous concept, with overlapping definitions and differing interpretations across mathematics and computer science. While CT is often associated with computer science, its core elements, such as algorithmic thinking, problem decomposition, and debugging, have deep roots in mathematical practice. This blurring of disciplinary boundaries sparked ongoing debate throughout the conference, particularly regarding the extent to which CT should be integrated into mathematics education and how it should be differentiated from both mathematical and computer science content.

A key focus of the sessions was on practical implementations of CT in mathematics classrooms. Several presentations illustrated how CT is introduced at different educational levels using both unplugged (offline, physical) and plugged (digital, screen-based) approaches. Throughout the discussions, the interplay between CT and traditional mathematical concepts was a recurring theme. Presenters and participants explored how CT practices, such as writing and testing algorithms, using simulations, or modelling real-world problems, could deepen students' understanding of core mathematical ideas like functions, variables, and spatial reasoning.

Despite the enthusiasm for CT's potential, the conference also highlighted several challenges. A major concern was the risk of conceptual confusion, especially when overlapping terms like "variable" or "algorithm" have different meanings in mathematics and computer science contexts. Educators must also grapple with questions about sequencing: should students first engage with problem solving and then implement solutions in code, or vice versa? How to integrate CT meaningfully into existing curricula without overwhelming teachers or diluting mathematical content? There was general consensus that CT should be seen not merely as a content area but as a way of thinking and working that can enrich mathematics education when used thoughtfully and deliberately.

Finally, the conference encouraged a broader philosophical reflection on the role of CT in education, and on its conceptual consistency. There was agreement that CT offers opportunities for students to engage in hypothesis testing, error analysis, and iterative refinement, practices that align closely with mathematical reasoning. However, participants also acknowledged the need for clearer identification of the concept, curricular scaffolding, and teacher support to realize these possibilities fully.

The second theme was the "Links between mathematics and informatics". In this theme, theoretical and experimental research contributed to discussing the relation between the two disciplines. This included reflections on the place and role of algorithmic thinking or computational thinking in mathematical topics like problem solving in arithmetic, or in combinatorics (e.g. Josefus problem), or in creating algebraic expressions. The extension of problem solving models from mathematics to computer sciences was also discussed. Some papers discussed the relation between programming and mathematics, questioning in particular the way programming is integrated in mathematics textbooks, or the development of abstraction skills in programming tasks (using augmented reality). Another promising topic discussed was the potential of interactive theorem provers in mathematics education, and the design principles for

related learning environments. More generally, a paper questioned how the conceptual knowledge from computer science can enlighten some questions in mathematics education. The diversity of the contributions permitted rich discussions, both on practical and theoretical aspects regarding interrelations between computer science, programming, and mathematics.

The third and fourth themes were connected to discrete mathematics (Sandefur et al., 2022). The third theme explored the "nature and diversity of discrete mathematics", questioning specific topics like graph theory, or tiling problems, but also the way discrete mathematics can connect or is connected with other mathematical topics. The covered topics of the individual contributions were completely different.

The first topic was devoted to an analysis of the surface-level isomorphism strategy that helped students to solve the Catalan number task efficiently. Presentation of a three-step deterministic model allowing identifying proving processes and objectifying students' activity in problem solving was the second topic. The third topic covered experiences with the discovery learning method of graph theory, showing that students remember mistakes they made while proving. The last topic was devoted to the analysis of Hungarian curriculum, which has a long tradition of weaving discrete mathematics topics through the whole curriculum.

Through the general discussions, there were many questions and answers about what discrete mathematics is. When talking about the characteristics of discrete mathematics, participants mentioned the accessibility to students, the development of problem-solving skills, and its connection to computer science. This led to discussions about the reasons and interests for teaching discrete mathematics, e.g., development of the skill to organize things, needs for computer science education, development of many problem-solving skills...

The fourth theme was covered by a group of communications specifically dedicated to "combinatorics", which seems to be the most developed part of discrete mathematics in curricula in many countries. Combinatorics, very common in curricula, has been much investigated in the past, and contributions covered various questions in this area, such as students' recurrent difficulties and errors, specific strategies and approaches in combinatorics, and teaching and learning issues, including the use of programming/algorithmic skills in combinatorics.

A key tension identified throughout the sessions was between teaching efficient combinatorial formulas and fostering deeper structural understanding. Instruction that explicitly highlighted combinatorial structures greatly enhanced students' ability to transfer knowledge. Successful strategies included systematic listings, identifying bijections, and student-generated problems. We had some discussions about the term "isomorphism" in combinatorics, which presupposes the preservation of some organizational principle, rather than just a cardinal equality, but which needs to be clarified.

Research underscored differences in how students generalized numerical patterns — some naturally extended counting processes, while others depended exclusively on numeric or algebraic patterns, potentially leading to superficial generalizations. Effective strategies recommended were partial enumeration, checking smaller cases, parity verification, and algorithm-based enumerations.

The final discussion connected the two main topics in the TWG. There was a shared vision that solving a combinatorial problem is equivalent to understanding an algorithm that would enumerate

its solutions—bridging proof, pattern, and program. Enumeration problems provide a natural playground for CT practices, mainly algorithmisation and abstraction. Conversely, CT tools give combinatorics an experimental dimension; when closed formulas are unknown or unwieldy, an executable algorithm is the solution. Participants called for curricular materials that intertwine these perspectives from an early age.

Only one theme was under-represented in the TWG: the teacher education issues. In the call for papers, we explicitly asked for contributions related to teacher education, as it is essential for any innovation in curricula. For many countries, discrete and computational mathematics is an exotic subject, for others it belongs to the standard curriculum for historic reasons or traditions. In our view, it is necessary to bring these different perspectives together. However, substantial contributions that belong *both* to the area of teacher education *and* our subject matter focus area are rare. While the subject matter is being addressed in some countries, due to its novelty, these programs may not be ready for research yet. Also, if there is research related to teacher education in this area, it can easily be reported in one of the TWGs that are specific to research in teachers' professional development that have a longer tradition than our TWG, which met first at CERME 14. Nevertheless, all participants in the group agreed on the fact that there is a need for good educational programs that help teachers to exploit the full potential of this non-standard topic in mathematics teaching. So, for the upcoming CERME conferences, we will continue to offer room to discuss teachers' professional development and curricular issues, and specifically encourage this kind of contribution. It might also be worthwhile to consider a joint session with TWG18, TWG27, or TWG15 in particular. Our analysis of the contributions to CERME 14 has shown that there were contributions to these groups that could have been included in TWG11 as well.

Apart from that, we were very happy to see that in the discussion, participants, whatever their principal interest, were equally involved in our two main topics, computational mathematics and discrete mathematics. The collective discussions and work showed rich connections between discrete and computational mathematics, and confirmed that it is interesting to join and make them interact. Many perspectives are opened, and we hope that the TWG will be able to develop long-term collaborations on these "hot topics" for mathematics education.

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Shut-the-Box: Fostering computational thinking through the mathematical modelling of games

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This study investigates the presence of computational thinking (CT) aspects in a mathematical modelling project with secondary school students, focusing on the game Shut-The-Box. Participant observations were conducted and relevant scenes were coded for various CT aspects. The findings reveal that multiple CT aspects emerge during the modelling process, particularly in relation to the need for simulation when addressing the game's stochastic nature. As simulations require precise formulations of problems and solutions, and Shut-The-Box facilitates easy unplugged simulation and data generation via playing the game, we suggest that such games may serve as suitable contexts for fostering CT through mathematical modelling.

Keywords: Computational thinking, mathematical modelling, simulation, mathematics education.

Introduction

The relationship between mathematics and computational thinking (CT) has been an active area of research for several years now. While the research has mostly focused on the areas of algebra and geometry, research on integrating CT into the mathematics curriculum in the area of probability and statistics is lacking (Ye et al., 2023). Seeing as most CT frameworks consider using or creating simulations in one way or the other as part of CT (Shute et al., 2017; Weintrop et al., 2016), using stochastic experiments and simulations seems to be a fitting vehicle to enrich mathematics education with computational thinking.

As CT is seen as a problem-solving process (Wing, 2010) and with the presence of modelling in existing CT frameworks (Kallia et al., 2021; Shute et al., 2017; Weintrop et al., 2016), the relationship of CT and mathematical modelling is an interesting, yet mostly unexplored area of research. Ang (2021) explored this relationship from a theoretical point of view by highlighting different mathematical modelling problems and approaches and how different aspects of CT can be found within the modelling process. Ang (2021) argues that the different aspects like abstraction or pattern recognition would normally be seen as mathematical problem solving and concluded that the use of computational tools is what differentiates CT from mathematical problem solving.

Regarding existing empiric research, Villa-Ochoa et al. (2022) observed future teachers in a bachelor's degree program during the development of mathematical modelling projects to identify aspects of CT during the process. Kar Teck et al. (2023) on the other hand investigated the development of CT through a mathematical modelling task in seven school students by combining a rating of the students CT ability on a scale from 0 to 2 with data from interviews and tests deployed before and after the modelling task.

To add to the empirical body of literature regarding CT and mathematical modelling, we investigated the aspects of CT in a mathematical modelling project with secondary school students with three different problems (Bayer et al., 2024). To highlight the connection between CT and the area of

probability, we focus solely on the group that worked on optimal strategies for the game Shut-The-Box in this paper.

Theoretical Background

The concept of computational thinking (CT) and what it encompasses is heavily discussed in the literature. An often cited definition describes CT as "the thought processes involved in formulating problems and their solutions so that the solutions are represented in a form that can be effectively carried out by an information-processing agent" (Cuny et al., 2010, as cited in Wing, 2010, p. 1).

To further identify these cognitive processes and aspects that are connected to or needed for CT, multiple frameworks have been proposed to identify the aspects underlying CT (e.g., Weintrop et al., 2016). Following a review of such frameworks, Shute et al. (2017) developed a framework focused on the foundational understanding of CT in K-12 subjects. The framework consists of six facets of CT, namely *decomposition, algorithms, debugging, iteration* and *generalisation*. The facet of abstraction is further subdivided into the categories *data collection and analysis*, *pattern recognition* and *modelling*, while the facet of algorithms has the subcategories *algorithm design*, *parallelism*, *efficiency* and *automation*.

As CT is often seen as a skillset that is transferable to other disciplines (Shute et al., 2017), there have also been efforts to connect CT to mathematical thinking (MT) and integrate CT into mathematics education. Kallia et al. (2021) considered both CT and MT as problem-solving processes as *contextualisation activities*, that is, the connection of real-world contexts to mathematical and computational concepts. They identified four distinct cognitive activities in this process: (1) translating from a context into a model, (2) working within the model, (3) translating the result from the model back into the context and (4) verifying that the solution solves the problem. This is depicted in Figure 1.

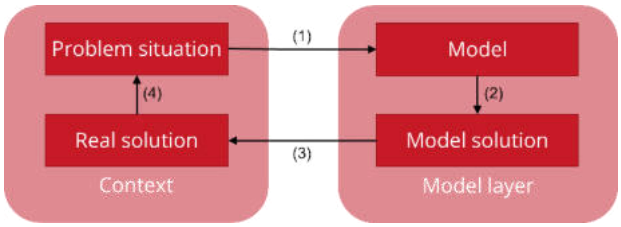

Figure 1: Contextualisation activities with the model layer being of either mathematical or computational nature. Adapted from Kallia et al. (2021).

Kallia et al. (2021) considered the integration of CT in mathematics education by viewing it as using mathematics as a context for CT. The combination of the two processes resulted in the process modell, that is depicted in Figure 2. While the model in Figure 1 for MT as a contextualisation activity is similar to different mathematical modelling cycles (e.g., Blum & Leiß, 2007), the model in Figure 2 is structurally similar to the mathematical modelling cycle under the influence of digital tools with added computer model proposed by Greefrath (2011).

Following a literature-informed Delphi study, Kallia et al. (2021) furthermore identified multiple aspects of CT present in mathematics courses, namely *data analysis*, *data representation*, *abstraction*,

decomposition, *algorithmic thinking*, *pattern recognition*, *automation*, *modelling*, *generalisation* and *evaluation*. Those aspects mostly coincide with the facets identified by Shute et al. (2017).

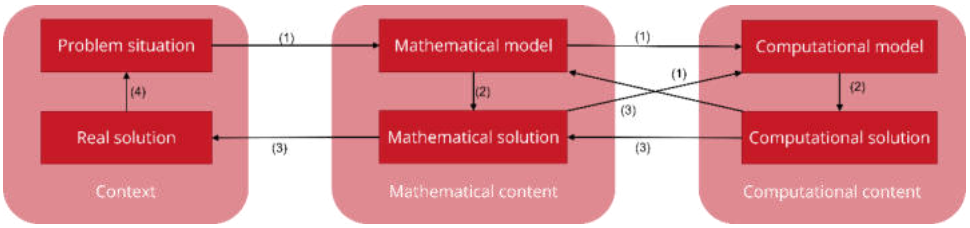

Figure 2: CT in the context of mathematical thinking. Adapted from Kallia et al. (2021).

Methods

Since mathematical modelling and CT both describe problem solving processes and, regarding the cycles of Greefrath (2011) and Kallia et al. (2021), follow similar steps, we looked at common aspects of CT during mathematical modelling activities of secondary school students in a larger study reported in Bayer et al. (2024). For this paper we want to take a closer look at one of the modelling projects – Shut-The-Box – and the possible implications of the stochastic nature of the game for the interplay of CT and mathematical modelling. The research question for this paper is therefore given as: *What aspects of CT are present in the mathematical modelling of the game Shut-The-Box with secondary school students and how do those relate to the stochastic nature of the game?*

As outlined in Bayer et al. (2024), we observed mathematical modelling activities during a modelling project with fourteen students from grade 9 to 11 at a secondary school in Germany, which was part of a project week at the school. The modelling project was offered by a teacher supported by two researchers and a trainee teacher as supervisors. The groups worked for four days on their respective projects and presented the results on the fifth day during a school festival.

The observed group consisted of four male students, with one from grade 11, one from grade 10 and two from grade 9. Their modelling problem was to find an optimal strategy to play the boardgame Shut-The-Box. In the version the students investigated, there are ten tiles numbered from 1 to 10. The player rolls two six-sided dice and adds the numbers together. The player then flips any number of the remaining tiles (a combination of tiles or a single one), whose sum matches that of the dice. The game ends, when no tiles can be flipped to match the rolled sum. For multiple players different evaluation schemes can be applied to determine the winner, for example, adding the remaining tiles as penalty points and trying to score as low as possible or playing as many rounds as possible.

The modelling activity of the group was observed by one of the researchers and relevant scenes were summarized descriptively into an observation protocol. The protocol was than analysed using a deductive qualitative content analysis approach based on Mayring (2014) and coded by three researchers for emerging or underlying CT aspects.

The codes were based on the aspects and cognitive activities given by Kallia et al. (2021): *data analysis*, *data representation*, *abstraction*, *decomposition*, *algorithmic thinking*, *pattern recognition*, *automation*, *modelling*, *generalisation,* and *evaluation*. Following a first scan through the material of all three groups of the modelling project, the codes *debugging* and *data collection* were added, which

are also present in the facets of Shute et al. (2017). As Kallia et al. (2021) didn't define their aspects, the codes are mostly based on the understanding of Shute et al. (2017) of those aspects with two exceptions: *modelling* and *generalisation*. As the activity itself was a modelling activity, the aspect of modelling was heavily discussed between the researchers as this aspect seemed to be present in most of the scenes due to modelling being a complex process itself. Thus, we applied a similar constraint as Villa-Ochoa et al. (2022) and modelling refers only to the formulation of a (mathematical or computational) model, such as a simulation, in our coding scheme.

Shute et al. (2017) understand generalisation as the ability to transfer CT skills to other domains. For this research however, we understand generalisation as a way of generalising from special cases to the general problem and generalising solutions to be transferable to other problems and contexts.

We also looked for occurrences of the main idea of CT as given by Cuny et al. (2010, as cited in Wing, 2010), that is, scenes, where the formulation of a problem or solution with an external agent in mind could be observed and coded them respectively.

Results

The group began by playing the game to explore the context and came up with a draft for four initial strategies, that they considered to be advantageous. Each of the four students adopted one of them to play against each other. If a student can't follow his rule, even if there would be a legal move left, he loses. The strategies of the students were the following:

1. Student A can only flip combinations of tiles, for example, if an 8 is rolled, only 3 and 5 or 1,3,4 can be flipped, but not 8 itself.
2. Student B can only flip a single tile, that is, the tile with the sum as number.
3. Student C tries to keep the lowest tiles (1,2,3) as long as possible.
4. Student D alternates between using combinations of tiles and flipping over a single tile

Over the course of ten games, they noted for each strategy how many rounds they could play and then played another ten games to note the sum of remaining tiles for each strategy, revealing that they had not chosen an evaluation scheme for multiple players yet. According to this data, student B could play the most rounds on average, while student D had the lowest mean sum of remaining tiles. They chose the strategy of the latter as their candidate for an optimal strategy and played a single round with that strategy against the teacher. They lost that round and deemed the strategy to be not optimal. The teacher reminded them that a single game might not be good evidence to discard their strategy.

While the formulation of the strategies can be seen as formulations of *algorithms*, the students also *collected data* on the strategies by playing multiple rounds and then *analysed the data* for picking a favourable strategy. They *evaluated* the chosen strategy in the real context against an opponent with an unknown strategy, albeit that they used only one game to declare that their strategy is not optimal. This could also hint to a missing understanding of the underlying randomness of testing a strategy.

A fifth strategy was then developed by the students around the "usefulness" of each tile. It couldn't be observed if this strategy was already based on the calculations of probabilities or a result of the experience of playing the game. In any case, CT is present here in the form of *pattern recognition* as

they either analysed the structure of the probability distribution to identify less used tiles in combinations or recognized the "flexibility" of some numbers during play. This fifth strategy revolves around trying to flip the tiles 10, 9 and 8 as soon as possible and keeping the tiles 2,3,4,5 as long as possible resulting in a weighting for possible combinations.

While evaluating this strategy against the trainee teacher, the student playing the game was told by a second student which tiles to flip after each dice roll based on the strategy. This can be seen as a form of CT in terms of the definition by Cuny et al. (2010, as cited in Wing, 2010) with the playing student as the *external agent*, who is guided by the second student step-by-step through the strategy.

After input from the trainee teacher, the group started working on a tree diagram to analyse the optimal choice for each role. Their goal was to play as long as possible, that is, taking as many turns as possible and they approached the problem in the fashion of a greedy algorithm. The first dice roll was used to calculate the probability of losing with the next roll, for example, rolling a 3 first, flipping the tiles 1 and 2 and then rolling a 2 would result in a loss. The combination of tiles with the least probability to lose with the next roll was considered optimal and can be seen in Table 1.

Table 1: Optimal choices for the first round in Shut-The-Box

Dice Roll	2	3	4	5	6	7	8	9	10	11	12
Optimal Choice	2	3	4	5	6	7	8	9	10	1+10	5+7

They continued this scheme for the second round and calculated the loss probabilities for each combination of possible optimal choices in the first round and dice rolls in the second, which resulted in the strategy described in the matrix in Figure 3. The first number in the row lists the dice roll in the first round followed by the optimal choices of tiles to flip sorted by possible rolls in the first round. For example, rolling a 2 in the first round and rolling a 12 in the second round would result in flipping the tiles 9 and 3 as an optimal choice.

The students and the trainee teacher discussed if it would be plausible to calculate the third round, but after a short consideration, they decided that after the first two turns they would recede to one of the heuristic strategies identified earlier, namely the strategy of preferably flipping the less used tiles while holding on to the important tiles, that is, those used in more combinations.

Possible strategies for evaluation were discussed by the students with one of the researchers, who suggested to use a random number generator (RNG) to model the random opponent's behaviour. The trainee teacher wrote down all possible tile combinations for all possible dice rolls for the students. The students then simulated a random opponent by using a RNG via smartphone and this list of possible combinations to determine a random flip of tiles. They played 100 games against this simulated random opponent and won 80 games with their strategy.

The process of developing their final strategy addresses many different CT aspects. The most prominent one being *decomposition*, as the students develop their strategy by first looking at only the first round of the game. Ignoring non-optimal combinations for later rounds can be seen as a form of

abstraction, as this reduces the complexity of the calculations by ignoring certain paths in the probability tree. Finally, they *evaluated* the optimality of their strategy against a random opponent. The random opponent was defined by a RNG and list of possible combinations for each turn and can be viewed as an unplugged computational *model* as the behaviour of the random opponent was precisely defined, which would make later automation possible.

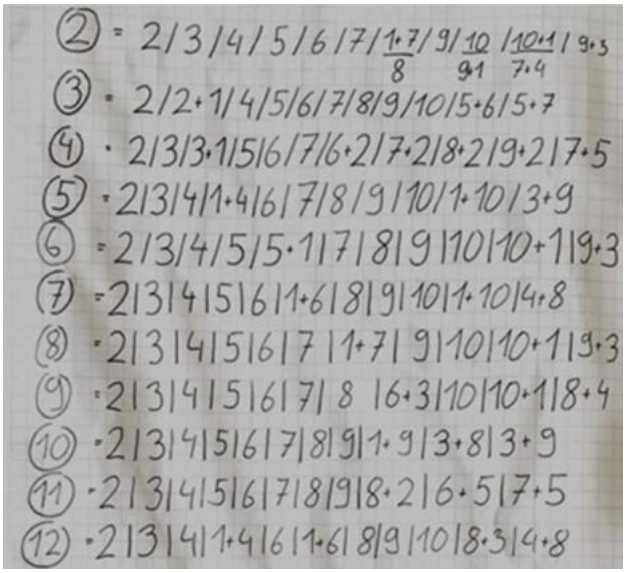

Figure 3: Optimal choices for the second round of Shut-The-Box.

Discussion and Conclusions

Regarding the presence of CT aspects when modelling the game Shut-The-Box, the results show that the CT aspects *data collection, data analysis, pattern recognition, abstraction, decomposition, algorithms* and *evaluation* were present in the modelling project. CT in the sense of Cuny et al. (2010, as cited in Wing, 2010) was also identified when designing a random opponent or evaluating the fifth strategy.

Comparing this to the considerations of Ang (2021), the game requires to work "from processes to model". Problems requiring this approach are described by multiple processes or in the sense of Shut-The-Box by game rules. Ang (2021) suggests, that constructing a simulation can help solving such problems as they allow for an exploration of the context to get a better understanding of the problem. Simulations can also help to discover patterns as a basis for possible solutions and allows to evaluate solutions by simulating the problem multiple times and applying the law of large numbers to get approximate probabilities.

Although the students didn't construct a simulation of the game (for example by implementing it as a program), them playing the game allowed for the same exploration phase to get a better understanding of the problem and collecting data to discover patterns. They also played multiple times to evaluate solutions in the sense of approximating win rates. This becomes especially evident when they evaluated their final strategy by simulating a random opponent and playing the game a hundred times. We argue that this practice of experimentation is a preliminary step towards Monte-

Carlo simulations and thus simulating the game itself. As mentioned, the group still used simulation to model a random opponent by using a random number generator and a list of feasible plays each turn, although this was partly suggested by one of the researchers. In essence, their approach as well as the observed CT aspects overlap with Ang's considerations for a "from processes to model" approach. While Ang (2021) sees the main difference between mathematical problem solving and CT in the usage of digital tools, we present an opposing view. We argue that the essence of CT is the definition given by Cuny et al. (2010, as cited in Wing, 2010) and that digital tools merely allow for an external agent that doesn't interpret the "input" using context knowledge. In contrast to the research of Villa-Ochoa et al. (2022), this project was not conducted with a focus on the application of digital tools. Besides the usage of calculators for the probabilities and a random number generator for the evaluation of the strategy, the group worked completely unplugged while still showing signs of CT in the sense of Cuny et al. (2010, as cited in Wing, 2010). This indicates that games like Shut-The-Box, or more generally problems that lend themselves to a "from processes to model" approach (Ang, 2021), are applicable contexts to address CT in mathematical modelling projects.

Besides the connection of the evaluation strategy of the group and Monte-Carlo-Simulations, , the possibility (and potential need) for collecting and analysing data and constructing a simulation in a stochastic context like Shut-The-Box presents a chance to foster CT in the area of probability and for CT to support students in dealing with similar problems. The usage of simulations encompasses the essence of CT in such problems, as it is necessary to formulate the problem itself but also the solution for evaluation in a way that it can be systematically carried out multiple types. The students lacked a clear, "implementation-ready" definition of their heuristic strategy for turns three and forward, but their experimentation practices and the definition of the random opponent can be seen as preliminary steps towards simulations of the game.

Games like Shut-The-Box allow for a low thresh-hold experience without need for digital tools as experimenting and collecting data is done by playing the game itself and formulating strategies for the players. The stochastic nature of the game carries a need for data collection and analysis to recognise patterns for finding strategies or to evaluate strategies. To apply the law of large numbers for evaluation of strategies, those strategies have to be formulated in such a way that players can precisely execute them over and over again. Therefore, such games also offer a high ceiling by providing motivation for simulating and automating the game and the solution.

We aimed to provide further empirical insight on the relationship between mathematical modelling and CT with a focus on stochastic contexts. The results of this paper however are limited to the observations of a single modelling project. Further research should identify how mathematical modelling can be used to foster CT and how a focus on CT might influence the modelling process. Furthermore, future research should also focus on the simulations of stochastic problems in the classroom as a vehicle for CT in mathematics education.

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Arrow cards as representation when programming educational robots – Algorithmitizing as an interplay between encoding and decoding

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Programming educational robots in primary school is one way to promote computational thinking (CT) from an early age. One of the central CT activities is algorithmitizing, i.e., creating a step-by-step sequence of elementary instructions to solve a problem, in this case to navigate the robot. This paper presents the results of a design research study investigating how arrow cards, as a representation of the elementary instructions, can support algorithmitizing in the context of educational robots. The empirical results indicate that when using arrow cards, algorithmitizing consists of an interplay of encoding and decoding elementary instructions. The change of presentation to the arrow cards can, on the one hand, demand more precision but, on the other hand, also pose challenges. Implications for teaching practice are discussed, and an outlook on ongoing follow-up studies is provided.

Keywords: Educational robots, computational thinking, programming, primary school, design research.

Introduction

Computational thinking (CT) is "the thought processes involved in formulating a problem and expressing its solution(s) in such a way that a computer – human or machine – can effectively carry out" (Wing, 2017, p. 8). Wing (2017) states that CT is an essential skill in the 21st century, on par with reading, writing, and arithmetic. Hence, CT is increasingly considered as an important learning content in school that should be learned as early as possible, including in maths lessons (Wang et al., 2022).

One way to foster CT in primary schools is through programming educational robots. The term CT originally goes back to Papert (1980), who first used the term CT in the context of programming educational robots. He developed the programming language LOGO in order to enhance mathematical learning, enabling learners to explore abstract mathematical concepts through programming. Recent approaches adapt Papert's ideas by programming educational robots for navigating paths to be followed, which requires algorithmitizing the intended motion sequences (del Olmo-Muñoz et al., 2024; Seckel et al., 2023).

In a design research project, a teaching-learning arrangement was developed in which the process of algorithmitizing was to be supported by planning the path with arrow cards (Figure 1). The arrow cards symbolize the possible buttons of the used robot, in particular forward and backward movements of 15 cm, as well as left and right turns of 90°. Arranging the sequence of elementary instructions with the arrow cards can be seen as a representation of the algorithm. This article is dedicated to understanding how the algorithmitizing processes – initiated by using this representation – proceed and the implications for instructional support.

Theoretical background

Algorithmitizing as one CT-activity

Computational thinking generally describes a way of thinking that allows a problem to be described in a way that a computer could solve it (Wing, 2017). Various conceptualizations exist regarding the components of CT (Selby & Woollard, 2013). This paper distinguishes between the following CT activities and differentiates them more precisely in the context of programming plane figures:

- "Reusing and Modifying: Reuse and modify existing code as needed
- Decomposing: Decompose a problem into subproblems
- Testing and evaluating: Testing and targeted adaptation
- Algorithmitizing: Plan a step-by-step approach
- Abstracting: See general properties in special cases
- Generalizing: Inferring a common aspect from different special cases" (Büscher, 2025, p. 368)

This paper focuses on the CT-activity of algorithmitizing. In the context of educational robots, this means creating a sequence of elementary instructions for navigation along a specified path.

Potentials of educational robots in mathematics education

Some studies deal with the question of how the use of educational robots can promote computational teaching. Seckel et al. (2023) conducted a systematic literature review of 25 articles to provide didactic orientations for integrating CT into mathematics lessons using educational robots (such as the Blue-Bot). Among other orientations, they recommend that tasks involving educational robots should establish clear connections to the mathematical content and to be adaptive to the learners' needs. This raises the need for further research on possible pathways and obstacles of learners' ways of algorithmitizing, enabling teachers to effectively respond to the learners. Hong (2023) also conducts a literature review to investigate the overall effect of educational robots on learners' computational thinking. In the study, Hong (2023) analysed 27 experimental and quasi-experimental research papers from the last decade. One result is that educational robots have a significant positive effect on learners' computational thinking development, with the most significant effect at the early childhood level. This speaks in favour of using educational robots as early as possible.

Other studies delve deeper into how the use of educational robots can affect the teaching and learning of mathematics. Francis et al. (2022) found in an intervention study that learners' spatial reasoning improved after participating in a robotics-based intervention. Forsström & Kaufmann (2018, p. 18) show in their literature review that "including programming in mathematics education could improve student motivation to learn mathematics and improve student performance in mathematics". Del Olmo-Muñoz et al. (2024) conducted a mixed-methods study to investigate metacognitive control in learners' problem-solving processes when programming educational robots. Learners were asked to navigate to one of two targets, with the closer target being more difficult to reach because more turns were required. The results of the qualitative sub-study with 82 learners show that younger learners do not base their decisions on the difficulty of the task. In particular, limitations in the learners' ability to express themselves could be observed here. In the subsequent quantitative sub-study, 117 learners

worked with a floor-robot simulator. Their results show the learners' need for metacognitive ability when choosing which path an educational robot should take.

The conditions under which educational robots effectively yield beneficial outcomes remain an open question. Among other things, the development of suitable teaching-learning arrangements as well as more precise empirical insights are needed into how exactly the use of educational robots can promote computational thinking among learners. This article is therefore concerned with a qualitative investigation of initiated computational thinking at the micro level and investigates the following research question: How can a teaching-learning arrangement support learners' activities of algorithmitizing and what are possible learning pathways and obstacles for learners?

Method

The results presented here come from a larger design research project. The aim of the project is (1) to develop a teaching-learning arrangement to promote computational thinking and spatial reasoning using educational robots and (2) to identify typical learning pathways and hurdles as well as conditions for success of the design principles used. Therefore, the project uses a design research approach with a focus on learning processes (Prediger et al, 2015). This article focuses on the applied design principle "using arrow cards to represent the algorithm" and analyses the initiated learning processes with regard to learners' algorithmitizing.

So far, 12 learners (grade 1 and 2) have taken part in design experiments in teams of two. Each team participated in two sessions with a duration of 45 minutes each. No learning robots were previously introduced in the classroom. But some participating learners had previously attended a workshop. The design experiments took place during school. They were videotaped and fully transcribed. The qualitative data analysis was carried out in an approach based on grounded theory (Glaser & Strauss, 2017) to identify ways how learners use the representation when algorithmitizing.

Design

One aim of the teaching-learning arrangement was to promote algorithmitizing in the context of educational robots. Specifically, the learners' task was to create a sequence of elementary instructions to make the robot move along a given path. The robot could be programmed by the learners by subsequently pressing the buttons for moving or turning. The whole sequence of moving and turning could then be executed by pressing the start button. This can be challenging to the learners, because there is a risk of steps being forgotten. It is also more difficult to modify the algorithm if an error occurs because the algorithm is not recorded anywhere. Therefore, one design principle on which the conception of the teaching-learning arrangement was based is "using arrow cards to represent the algorithm" to support algorithmitizing. The idea behind this design principle was to fix the sequence of elementary instructions and thus make them accessible for later modification. It was also intended to reduce the complexity of the programming process, as the planning of the path was decoupled from the implementation on the robot.

One task in the teaching-learning arrangement was to move the robot along the specified path to the target field (Figure 1). The labelling of the fields with A-E (rows) and 1-5 (columns) was only added for this article in order to be able to describe the children's gestures in a comprehensible way. During

the survey, the fields were not labelled for the children. In accordance with the design principle described above, the first step was to lay out the sequence of movements using the arrow cards (Figure 1). Only in the second step were they to transfer the planned route to the robot. If an error occurs somewhere and the robot leaves the path during execution, the learners were encouraged to find the error location and modify the algorithm laid out with arrow cards. This is repeated until the learners arrive at a correct algorithm and the robot moves as desired after execution.

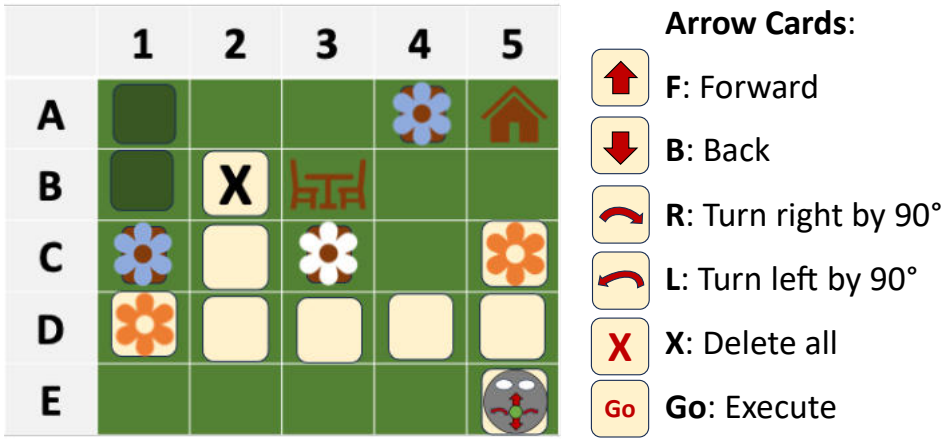

Figure 1: Reconstruction of the map with robot and path (left) and cards (right)

Empirical insights: interplay of encoding and decoding arrow cards

The empirical analysis revealed two main ways how learners used the representation of arrow cards for the algorithm: *encoding* and *decoding*. While *encoding*, the learners lay out a sequence of arrow cards in order to represent a possible programmed pathway. Errors can occur here, and the encoded pathway does not always match the intended pathway. While *decoding*, learners translate a pathway encoded through arrow cards into a movement sequence of the robot. The learners' processes of algorithmitizing consisted of frequent changes between these two ways of using representations.

In this paper, the exemplary case of Elena and Malte is presented to illustrate the possible complexity of an algorithmitizing process when the arrow cards are used as a further representation. Elena had no previous experience with educational robots. Malte had attended a workshop in a different context. The transcripts are a literal translation from German ('#' indicates an interruption).

201 Elena: Forward once. [points to D5]
202 Malte: [picks up F-card]
203 Teacher: Exactly, # lay the cards over there, where Elena is sitting. [M. lays card]
204 Elena: # Left. [takes L-card; lays it left to the F-card] then left. [points to D4] then forward [points to D3]

Encoded pathway: L, F

In Turn 201, Elena suggests „forward once" and points to the field D5. She uses it to describe the route to be taken and points to the target position of the planned movement. Malte then picks up a F-card (Turn 202) and places it in front of Elena on the teacher's instructions (Turn 203). Elena says "left" and takes an L-card. She places it to the left of the F-card, presumably because that is the

direction in which the robot should move from her perspective. Up to this point, *encoding* has been the main feature, as they describe the route to be taken and try to express this through the cards.

205 Malte: And then again [picks up a F-card and wants to lay it right to the other cards].
206 Elena: No, here. # [points to the left side of the cards laid]
207 Malte: # Then it's here and then it turns like this. [imitates the robot's movements on the garden mat with hands; one field forward, then turn 90 degrees to the left] [Elena switches the order of the cards laid]
208 Elena: Wait, delete first. [picks up Delete-card and lays it at the beginning]
209 Teacher: [smirks]
210 Malte: And then it goes forward like this [moves the robot one field forward]
Encoded pathway: X, F, L, F

Here you can first see the disagreement as to where the next forward card should be placed. Malte is in favor of placing the card on the right (Turn 205), Elena on the left (Turn 206). This shows an obstacle when *encoding*, because the learners disagree on whether the cards should be placed from right to left or left to right. Malte then describes the path laid out so far, clearly indicating the robot's position and its subsequent movements according to the cards. He therefore *decodes* the cards that have been laid so far. He does this partly verbally and partly through gestures by simulating the movement on the mat with his hand (Turn 207). Elena then swaps the two cards into the correct order. Malte then places another F-card and manually moves the BlueBot one field forward accordingly (Turn 210). Both the *decoding* of the path taken so far and the subsequent *encoding* of the next step are therefore supported by gestures for visualization or by manual movement of the robot. In order to focus more on spatial reasoning, the teacher asks them to avoid moving the robot manually (Turn 211).

211 Teacher: Try doing this without moving it. [Puts robot back to the beginning.]
212 Malte: [acknowledges].
213 Elena: So, [points to the fields D5, D4, D3 and remains at D3] Then we need forward once again.
214 Malte: [picks up a F-card]
215 Elena: So, so, so, so [brushes her finger across fields E5, D5, D4, D3, D2]; Yes, this forward. [adds F-card from Malte] . [points to C2] Then [3 sec] right, I think. No, left.
Encoded pathway: X, F, L, F, F

Elena taps on the fields D5, D4, D3 one after the other, presumably mentally retracing the path she has already taken with the cards. This would be a new *decoding* of the cards laid so far. However, she then seems to interpret the L-card as a movement to the left, which reveals a possible misconception during decoding. Upon reaching D3, she realizes that another forward movement is needed (Turn 213). Malte picks up a F-card (Turn 214), but says nothing. However, the card still indicates *encoding* for the next step. Elena runs her finger along the path again, field by field, and comments on each step, saying “so” (Turn 215), until she reaches D2. She seems to see this as confirmation that another forward movement is necessary, as she takes Malte's F-card, and adds it to the sequence (Turn 215). Again, this illustrates the connection between *decoding* the previous planned path and then *encoding* the next step. However, the misconception that the L-card is also associated with a movement from D5 to D4 seems to persist. She then taps on C2, which in her opinion is probably the next field to be reached, and initially says: “then right”, but also indicates

uncertainty with “I think”. And then corrects to “no, left”, but does not explain this further (Turn 215). This shows difficulties when *encoding*.

Malte then uses gestures again to *decode* the cards laid so far and to visualize the associated movements:

216 Malte: Like this, [imitates forward movement of robot over garden mat with hand] then he turns like this [turns hand 90° on the spot] and then once forward [points to D4] then again [points to D3] then he has to go forward again. [takes a new F-card and wants to place it] #
217 Elena: # [holds hand over D2] it is like this, isn’t it. No, right [rejects F- card from Malte, reaches for a R-card and puts it on] It goes like this.
218 Malte: [puts F-card back]
219 Teacher: Wait, Malte, you have to talk to each other. Look, Malte just wanted to put another forward. You didn't want that, Elena?
220 Elena: [makes forward movement with hand over path on the map and lands on the left red flower] He then walks over the flowers like this.
221 Teacher: He then walks over the flowers?
222 Elena: Yes.
223 Malte: No. #
224 Elena: # It stands like this [points to D2 and makes a turning movement to the right] # then it turns around and then it can go forward [lands with hand on target field]

Encoded pathway: X, F, L, F, F, R

Malte (correctly) assumes that the robot is still on D3 according to the cards laid so far (Turn 216). Here, he decodes the path laid so far by describing the movements verbally on the one hand and visualizing them with gestures on the other (Turn 216). Based on this, he realizes that the robot then has to “go forward again” and picks up a F-card (Turn 216). Elena is of the opinion that the robot is already on D2 based on the cards played so far (Turns 217, 220, 224). Thus, they disagree on *decoding* the path laid so far. In Elena's case, this could possibly be due to the misconception that the L-card already includes a forward movement. The different *decoding* then leads to them having different suggestions when *encoding* the next step. In this interplay of *decoding* and *encoding*, both use gestures to visualize the intended movements.

225 Malte: # So, when he moves forward like this [makes a movement with his hand over the garden mat from E5 to D5] then he turns around like this [shows 90 degrees turn to the left on D5] then once forwards [shows forward movement on D4] and then again [shows forward movement on D3].
226 Elena: This is that. [points to two laid F-cards] And then remains [hand on D2; puts away R-card]
227 Teacher: Does it then stand, does it stand after the, after the second forward here? [points to D2] or here? [points to D3]
228 Elena: There [points to D3] then forward [points to the F-card] and then this one [points to D2 and then points to R-card in her hand]
229 Teacher: So three times forward? [Malte adds F-card]
230 Elena: Yes. [places R-card at the end]
(As the process continues, they also add two missing F-cards and Go-card at the end)

Encoded pathway: X, F, L, F, F, F, R, F, F, Go

Here Elena seems to find Malte's visualization convincing, she corrects her assessment and is now also of the opinion that another F-card is necessary. However, it remains unclear whether she was

only persuaded by the situation or whether the cause was really clarified (that a turn movement is not also a forward movement). At least this has not yet been explicitly clarified.

Discussion and Conclusion

Educational robots can be an effective tool for fostering computational thinking at an early stage in primary school (Hong, 2023; Papert, 1980). Such programming activities can also improve students' mathematical performance (Forsström & Kaufmann, 2018). However, more detailed insights are needed into how computational thinking can be effectively supported and how learning processes actually take place in the context of educational robots.

A larger design research project is therefore investigating, among other things, which design principles can be used to develop a teaching-learning arrangement that specifically promotes computational thinking, and how the learning processes initiated in this way proceed. The aim is to further develop the teaching-learning arrangement for classroom practice and to gain deeper insights into typical learning processes and obstacles.

This article focused on the design principle of 'using arrow cards to represent the algorithm' to promote algorithmitizing in the context of educational robots and examined the learning processes initiated. In this context, algorithmitizing primarily refers to creating a step-by-step sequence of elementary instructions to navigate the robot along a given path.

The empirical results show that learners do not simply plan the route once in advance, then encode it with arrow cards, and finally transfer it to the educational robot. Instead, there appears to be a complex interplay between *encoding* the next planned step or steps and then *decoding* the previously planned path to reorient themselves spatially before (re-)encoding the subsequent steps. Overall, arrow cards, as a representation of the algorithm, seem to support the algorithmitizing process to a certain extent because the algorithm can be visualised with it.

However, it has also been shown that difficulties in correctly interpreting elementary instructions can arise during both encoding and decoding. One possible misunderstanding seems to be that an L-card also means a further step forward. In the future, it will be worthwhile to conduct a more detailed inventory so that teachers can be made aware of the difficulties that may arise. Other learners also showed a similar interplay between coding and decoding, along with the associated potentials and difficulties. This should be investigated in more detail in follow-up studies.

Studies already point to the potential of using educational robots to support mathematical learning (Francis et al., 2022). Although this was not the focus of this paper, there is still some potential for using arrow cards as a presentation to evoke spatial orientation. The processes of encoding and decoding seem to require spatial orientation, which can, in turn, present certain challenges. The empirical examples shown here illustrate that learners use many gestures, which seems to support both encoding and decoding. This could point to the embodied nature of mathematics learning (Núñez et al., 1999), which could provide a useful perspective for understanding learning processes with educational robots. The possibilities and limitations of using educational robots should also be examined more closely in this regard in the future.

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Algorithmic thinking in problem solving between Mathematics and Computer Science: a case study in modular arithmetic

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Using modular arithmetic as a case study, this research aims to explore, with an interdisciplinary view between Mathematics and Informatics, the ways in which students use algorithms, strategies and heuristics in problem solving activities. In particular, an analytical tool has been developed to analyse the different approaches used by students when confronted with a task at the boundary between Mathematics and Informatics. Our work has provided insights on how such a tool allows a possible characterisation of different problem solving approaches, which may be more oriented towards Mathematics or closer to Informatics, and could be a useful tool in the analysis of tasks to be proposed to students, in order to highlight the characteristics considered useful for the development of algorithmic thinking.

Keywords: Algorithmic thinking, interdisciplinarity, modular arithmetic.

Introduction

When considering Mathematics and Informatics (or Computer Science), we are dealing with two disciplines whose development followed different historical paths. On one hand, Mathematics developed over the past centuries, building upon practical needs before aiming at providing a generalisation of such issues. On the other hand, Informatics is a relatively young discipline that has become more and more relevant in nowadays life. Both disciplines, however, in their history, generated a debate about their philosophical and epistemological framing, giving rise to different points of view (Tall, 2002; Eden, 2007). Looking at the two disciplines' forms of thought we observe how literature of research provides different characterizations of the so called 'mathematical thinking' - MT - (e.g., Schoenfeld, 1992) and 'computational thinking' - CT - (e.g., Wing, 2006) as typical forms of thinking of each one of the disciplines but, in both cases, without reaching an unambiguous and agreed definition. However, nowadays, the two expressions are recognized by the scientific community together with their relevance in education (OECD, 2023). For what concerns the specific case of computational thinking, in spite of this international and widely shared recognition as a key form of thought, there still is an ongoing debate on how to teach it in school: on the one hand, computer scientists seek Informatics as a discipline that should have an autonomous role within school curriculum (e.g., Caspersen et al., 2019). On the other hand, scholars with different backgrounds (e.g., Li et al., 2020), consider that computational thinking should be included within the broader set of STEM disciplines already provided for in school curricula (e.g., Mathematics) spreading it with an interdisciplinary approach. In this frame, literature shows how the two forms of thinking are not disjoint, with different contributions studying the interrelations between Mathematics and Informatics (Kallia et al., 2021) and, more in depth, between mathematical thinking and computational thinking highlighting how, for instance, the activity of resolution of problems represents a common ground between the two disciplines (Kallia et al., 2021).

With a specific focus on the latter, in this paper we would like to reflect on the concepts of algorithm (Korfhage, 2003), strategy and heuristic (Schoenfeld, 1985) and explore the ways in which they are involved in problem solving activities that straddle the boundary between Mathematics and Informatics. These concepts intersect the boundary between the two disciplines and are intrinsically linked to a form of thinking, so-called algorithmic thinking - AT - in the sense proposed by Futschek (2006) - which in turn straddles the concept of mathematical thinking and computational thinking, being part of both (Stephens, 2018). Due to the links that these topics have with the two disciplines, we propose to adopt an interdisciplinary perspective between Mathematics and Informatics, i.e., we want to adopt an approach that enhances the viewpoints of the two disciplines and, at the same time, compares them, highlighting their points of contact and, possibly, those of divergence. In this sense, in order to frame our work and valorise the elements typical of each discipline and those common to both, we refer to the concepts of boundary objects, boundary crossings (Akkerman & Bakker, 2011) and the family resemblance approach (Erduran & Dagher, 2014), which combine the need to highlight disciplinary identities with the need to cross the boundaries between them (Satanassi et al., 2023). In particular, in this paper we will take an interdisciplinary look at the process of problem solving in the case of a course in modular arithmetic, with the aim of investigating the ways in which students use algorithms, strategies and heuristics, and observing whether and how different uses may refer to approaches that are specific to Mathematics or to Informatics. The chosen topic lends itself to an interdisciplinary approach and, in particular, by delving into its applications with students, it is possible to work on problem-situations on the border between two different disciplines therefore stimulating students to design, apply and describe different solving procedures.

Theoretical framework

To frame our work, we took inspiration from the model for mathematical problem solving proposed by Schoenfeld (1985) who, underlining the intrinsic complexity of the process of resolution of a problem, characterised the elements occurring in the process such as the concept of strategy and heuristic. The solver, taking into account his or her resources, must choose a strategy that will enable him or her to produce a solution, making appropriate use of heuristics, i.e., rules of thumb, to better understand and solve the problem. Therefore, according to the author, the problem solving process directly involves the solver and is influenced by several factors: the resources available to the solver (i.e., the tools available to him/her in terms of knowledge, procedures and skills), the solver's control (i.e., the way in which he/she uses his/her knowledge, applies heuristics and makes operational decisions) and the solver's beliefs system. As far as the Informatics context is concerned, the problem solving activity of the computer scientist is also based on the concepts of, for example, strategies and heuristics, but it concerns the resolution of problems of a computational type (Papadimitriou, 1993), i.e., characterised by a specific functional connotation: a computational problem is given by a set of instances I associated with a question Q about them, and solving it means finding an automatic method, called an algorithm, that allows the question to be answered correctly for all instances of the problem. With reference to the definition proposed by Korfhage (2003), an algorithm in Informatics is defined as a method for solving a problem, and its implementation in a language understandable by a computational agent leads to the creation of a programme. The definition of an algorithm determines a finite sequence of steps for any input and must produce the answer for the input (or a

declaration of the impossibility to get an answer for that input). This definition, technical and proper to Informatics, has a different meaning from that used in the mathematical context, where it has historically been reduced to a synonym for "procedure" (Fant, 1993). Similarly, in Informatics, the concepts of strategy and heuristic are used not only in the description of the activity of the problem solver, but also to refer to the methods used by algorithms to find a solution to a problem, that is, to find the answer to any instance of the problem.

We can now summarise the research question as it follows: taking an interdisciplinary view across Mathematics and Informatics, which are the heuristics, algorithms and strategies that a group of secondary school students develop in a problem-solving activity?

Methodology

To tackle our research question, we conducted a case study involving a group of Italian secondary school students attending an extra-curricular project on modular arithmetic. In particular, the group consisted of 26 students of the first year of upper secondary school (grade 9, 14-15 years old) attending a scientific specialty with particular attention to the teaching of mathematics (i.e., “Mathematical Lyceum” - a national experimental project in some Italian upper secondary school). The group of students belonged to a class that did not include Informatics as a curricular topic but, instead, in the context of the Mathematical Lyceum, had additional hours in mathematics compared with other school specialties. In some of such additional hours the students took part in a course on modular arithmetic designed by us and aimed at introducing students to the fundamental concepts of modular arithmetic and exploring its applications leading students to work in a context (that of discrete mathematics) which intersects Mathematics and Informatics. The activities designed for the course were all workshop-based and involved the use of artefacts – in the sense intended according to the theory of instrumental genesis (Rabardel, 2003), in which an artefact is linked to the patterns of use that users assign to it rather than an instrument on itself. Moreover, the activities aimed at getting students to work on a series of tasks through the following structure: working in random groups and through the use of artefacts, the students were requested to solve some tasks dealing with modular arithmetic and whose solutions were later shared and commented in a collective discussion led by the teacher and in which they were requested to formalise, justify and, possibly, generalise the solving procedures they had developed. After the completion of some of the activities the students were proposed a series of tasks to be completed autonomously at home as homework and discussed and revised collectively at school at the beginning of the next session of the course. The research presented in this paper will focus on one of the tasks proposed across the whole course, in particular one of the tasks that was proposed during one of the group activities proposed in class – here in an English translation.

> **Task**: You want to compare the residue class modulo 6 of a string of length $136=17\cdot 8$ and a string of length $10=5\cdot 2$, but among the strings you have available, there is no string length 136. Given the lengths of the strings you have available, is it still possible to perform the comparison? If so, what do you observe? To answer the question, you can help yourself with the hexagon and the strings you have available.

The students were given a regular hexagon with pegs at the corners, numbered from 0 to 5, and strings to be tied to the pegs, whose lengths were an integer multiple of the side of the hexagon. Using the polygon, the pupils were able to find the corresponding modulo 6 rest class for each of the lengths of strings available to them by rolling the strings around the given hexagon. However, the use of artefacts is not strictly necessary to solve the task: the student can apply the algorithm for calculating the residue class modulo m to the numbers 136 and 10 to find the residue classes of the two numbers and, by comparing them, finding that they are equal. We chose to focus this paper on this task because it provides students with the opportunity to approach it from different perspectives that are, for example, the execution of known procedures to perform operations or developing approaches exploiting the artefacts that students were given for the activity. Students' answers were collected in written form. Once the protocols were collected, we proceeded with a qualitative content analysis with inductive category formation (Schreier, 2012). We analysed the protocols generating a list of codes that was updated as all the solutions were analysed – rapidly converging toward a saturation – and grouping them in more general categories. In particular, for what concerns the study presented in this paper, we chose to focus qualitatively on the labels and the categories that resulted from this inductive process rather than on the quantitative analysis of the indicators, to detect either the use of existing or the creation of new algorithms, to analyse the development of heuristics and solving strategies, and to explore the expressions used by students to present algorithms and strategies.

Results

The labels that emerged from the data analysis process are grouped into three categories that, in our view, represent the possible dimensions in which students move when solving a problem (see Table 1). In particular, we distinguished: i) the design dimension (P), which gathers all the labels highlighting the aspects related to the development of a method for solving a problem; ii) the application dimension (A), which gathers all the labels useful for highlighting the aspects related to the practical execution of a procedure; and iii) the description dimension (D), which gathers all the codes related to the choices made by the students in terms of the structure and form of the answer given and the language used to describe and justify the choices made.

Table 1: codes and dimensions emerged from protocols' analysis

P-dimension	A-dimension	D-dimension
P1: artefact P2: use P3: combination P4: distinction P5: knowledge P6: generalisation	A1: example A2: instance A3: passage	D1: natural D2: symbolic D3: programming D4: structure D5: interpreter D6: command D7: action D8: reasoning

In this first part of the section, we propose a brief description of the categories and codes that emerged from our analysis, together with a brief description of the approach they describe. With regard to the P dimension, the following codes were identified: P1 – Artefact: identifies a student's development of an argument based on the use of an artefact. P2 - Use: students refer to an already known algorithm, using it exactly as it was proposed, without combining it with other algorithms they know. We also use this label when students mention a simple learned algorithm, treating it almost as a property or definition now acquired. P3 - Combination: the learner, proposing a new procedure, combines some already known algorithms. We also use this label when the combination designed by the student, although not explained in the answer, emerges from the steps taken and when the student combines simple learned algorithms, treating them almost as properties or definitions now acquired. P4 - Distinction: the students recognise that it is necessary to distinguish special cases for which the procedure they have designed behaves differently from the general one. P5 - Knowledge: this code identifies the situations in which the students' reasoning shows that they reflect critically on the situation they are dealing with and rely on their knowledge to decide what is the best way to solve the problem. P6 - Generalisation: the student's reasoning generalises the task proposed by considering the specific case suggested by the question as an application of a more general procedure.

For dimension A the following codes were identified: A1 - Example: students exemplify their strategy, i.e., they apply the whole strategy to a concrete example. A2 - Instance: students apply the strategy they have worked out to the problem situation of the question. A3 - Passage: students exemplify a single step of the solving strategy, i.e. they use a concrete example to explain a single step of their strategy.

For dimension D, the following codes were identified: D1 - Natural: pupils use natural language, i.e. colloquial language. D2 - Symbolic: students use the typical elements of mathematical formalism, such as the congruence symbol modulo m, the residual class symbol modulo m, etc. We also use this label when students reduce the whole solving strategy to a formula. D3 - Programming: students use elements characteristic of the formal language of programming, such as "if...then". We also use this label for those answers where students reduce the whole solving strategy to a program. This is a use of formal programming language taken to the extreme, leading the student to express the solution in a language that can be understood by a computer. D4 - Structure: students structure their procedure, for instance by ordering a sequence of steps. These steps can be indicated by a series of numbers or a bullet list, but they can also be punctuated using expressions such as "first, then and finally", or by the use of braces and indentation. D5 - Interpreter: students include explicit references to the performer of the procedure they are devising or to their knowledge. D6 - Command: when explaining the procedure they have devised, students give instructions in the form of orders to be carried out. D7 - Action: when explaining their procedure, students describe actions they have taken or need to take to answer the question. D8 - Reasoning: when explaining their solving strategy, students explain the reasoning they have done.

Such system of dimensions can be used to analyse students' responses to a task, and to highlight different ways of using algorithms, strategies and heuristics. To illustrate this potential, we focus, as an example, on the analyses carried out on the task that is the subject of this paper. The task is instantiated in its formulation, so that the learners' possible solving approaches are almost all based

on the analysis of the single instance of the problem [A2]. This approach is a typical heuristic of the problem-solving process in Mathematics, but is not common in computing, where the computational problem, by definition, involves a (typically infinite) set of instances. The work that a student carries out on the specific instance proposed by the problem is, in some approaches, strongly anchored in the artefacts made available for the workshop [P1]: in these cases, the student acts on his/her own to solve the proposed problem, manipulating the artefacts available to him/her. Such an approach is therefore tied to the student's experience and is not effective in solving the proposed problem, because in some cases it is difficult for the student to adequately motivate the results of the work done concretely with the artefact. On the other hand, in some answers, the work done on the specific instance does not contain explicit references to the artefacts used. In these cases, answers can be found that correspond to approaches characterised by abstraction and generalisation: the learner recognises that the instantiated problem is a particular case of a more general problem and therefore tries to generalise the results found by working on the single instance [P6]. Such an approach is very effective because it combines the mathematical heuristics of working on the specific instance [A2] with heuristics based on generalisation attempts [P6]. The latter is referred to in the literature as one of the components of AT (Stephens, 2018) and is common to both Mathematics and Informatics. Finally, there are students who show an approach that is completely detached from the specific case proposed in the problem: in this case, the answers are formulated by referring to a general problem of which the proposed task is one of the possible instances [P6]. Answers of this type thus reveal an approach closer to that of Informatics, which typically deals with problems of a general nature. However, a priori, given the nature of the activity in which the task was proposed, we cannot rule out the possibility that, in order to arrive at the formulation of the answer that we analysed, the students started from the artefact and, by turning it into a tool (in the sense intended by Rabardel, 2003), arrived at the formulation of a generalisation of the proposed problem. In general, we observe that the analyses carried out on the task through the categories that emerged from the students' data allow us to describe and characterise solving approaches of different types, ranging from purely mathematical approaches to approaches that have heuristics (e.g. generalisation) in common with Informatics.

Discussion and conclusions

The process of solving a task is an interweaving of algorithmic and strategic aspects that lie on the boundary between Mathematics and Informatics. Therefore, the role of algorithms, strategies and heuristics in problem solving activities has been explored on the common ground of the two disciplines, focusing in particular on the approaches' characteristic of algorithmic thinking. In particular, the latter, which is an integral part of both mathematical and computational thinking, can be considered as a boundary object, and therefore we can say that our study is part of the strand of research that investigates the relationships between the ways of thinking of the two disciplines and seeks to clarify the ambiguities that still exist today regarding these relationships. The analyses carried out lead to the formulation of an analytical tool that allows us to characterise the possible roles that algorithms, strategies and heuristics play within the problem solving process suggesting that, in order to design and implement activities aimed at developing algorithmic thinking, it is important to understand the plurality of meanings that revolve around these concepts, which turn out to be boundary objects between Mathematics and Informatics. For what concerns the case study

presented in this paper, the categories that emerged from the data allowed us to identify and distinguish the different approaches that students use in problem solving activities. On the one hand, we have identified mathematical approaches that stimulate components of algorithmic thinking that are common to both disciplines (e.g., generalisation, differentiation of particular cases), and on the other hand, we have identified approaches that are very close to Informatics in terms of describing and designing algorithms.

From a first observation, even considering the activity proposed to the students in its entirety, a possible task effect seems to emerge: on the one hand, there are the tasks that we have defined as "calculation" tasks, i.e., tasks that propose to the student an instanciated problem (i.e., they are formulated by referring to a specific situation that could be understood as an instance of a more general problem) and lead the students to work on all three dimensions identified by our coding frame; on the other hand, there are "algorithmic computational" tasks, i.e., tasks that ask for the identification of a solving procedure, that lead learners to work primarily on the description and design dimensions of our coding frame, that encourage the creation of an algorithm and that, while not allowing heuristics specific to algorithmic design and application to emerge, allow students to use algorithmic description techniques specific to Informatics.

Although the proposed observations can be considered as preliminary and related to our case study, and therefore with a possible effect of the proposed tasks, we can say that this research can provide some general indications to be studied in further research, such as a quantitative analysis of distribution of labels identifying the categories, the study and classification of students' heuristics and their ability to work on the concepts of algorithm, strategy and heuristics, and to create interdisciplinary activities that favour the development of algorithmic thinking.

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Deterministic model for objectifying the students' proving process in Discrete Mathematics

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This paper presents a deterministic model for describing and objectifying the students' mathematical activity, targeting the field of discrete mathematics. This model allows to identify their proving processes in problem-solving, involving the manipulation of tangible objects, based on the audiovisual data collected. The model is described in three steps. The first one involves selecting observables relating to the actions and strategies linked to the proving process identified in the mathematical analysis that we wish to identify in the students' mathematical activity. The second one concerns the retranscription and encoding of their activity in relation to the choice of actions performed. Finally, the last one involves analysing the retranscriptions of their activity and comparing them with the strategies identified in the mathematical analysis, to find similarities in the proving processes used to solve the problem.

Keywords: Deterministic model, proving process, problem-solving, discrete mathematics education.

Background and objective of the research

The field of Discrete Mathematics is important for teaching and learning of the reasoning and the proving process. As Gravier and Ouvrier-Buffet (2022) point out, many international works on problem solving and proof use this field (e.g., Da Ronch, 2022; Grenier & Payan, 1999; Hanna & de Villiers, 2012; Hart & Sandefur, 2018; Stylianides, 2016). Discrete Mathematics encourage the students to develop an authentic mathematical activity (Lampert, 1990) similar to that mathematician, particularly in terms of the skills and know-how associated with this activity. Moreover, this field makes it easily to illustrate problems using tangible objects (see Grenier & Payan, 1999; Da Ronch, 2022). But how can we identify in students' mathematical activity and their proving processes used in problem solving when they manipulate these objects? One way of proceeding is to carry out an analysis of the traces using audiovisual data collected with the camera assistance. The audiovisual data allows to researchers to access more easily and more exhaustively to the traces of activities produced by students (Jewitt, 2012; Da Ronch, 2022). This also means that students don't have to select the traces they wish to leave behind a researcher or a teacher when it comes to submitting written work. Written traces lead to a loss of information and make the retranscription of the activity less objective. In a way, they also encourage interpretation biases in the analysis of these written traces by the researcher. The aim of this paper is therefore to propose a deterministic model for retranscribing the students' mathematical activity on discrete mathematics problems and to provide tools for analysing their proving processes. This model is a theoretical model for research in (discrete) mathematics education. It is therefore not intended to be put into practice by teachers, but rather to be used by researchers. On this occasion, we will first give some theoretical elements which will then enable us to describe the main steps in this model, particularly about the choice of observables, the retranscription method and the analysis of the latter. Several examples illustrating this model are given in Da Ronch's work (2022).

Theoretical background and research questions

Many models of problem-solving processes exist (Rott et al., 2021). The authors describe and compare existing models to characterise their potential and limitations for analysing students' problem-solving processes. Among other things, they point out that most of the models proposed are normative. They affirm that these phase models were not designed for the analysis of empirical data or to describe externally observed processes (e.g., videotaped problem-solving processes of students), which highlights the authors' need to build a descriptive model. The model developed by Rott et al. (2021) can be used to code episodes, i.e. to code "A period of time during which an individual or a problem solving group is engaged in one large task […] or a closely related body of tasks in the service of the same goal […]" (Schoenfeld, 1985, p. 292). These episodes are then coded according to a pre-defined typology (deductive approach), which is then enriched by data analysis (inductive approach). However, episodes are coded at a macroscopic level of student activity (e.g., "reading the problem; implementing a plan; verifying a solution", etc.) (Rott et al., 2021, p. 743). This model therefore does not allow us to retranscribe, at a microscopic level, the activity of the students in order to give a retranscription as close as possible to the reality observed. Moreover, this microscopic retranscription of the pupils' activity would make it possible to reduce the interpretation bias that can be induced by coding into episodes and coding the associated episode types.

If we want to identify students' mathematical activity in problem-solving, and their proving processes, one way of doing this is to identify a set of elementary actions considered relevant to the given problem. We are therefore at a microscopic level. Moreover, these actions are directed towards the environment of the situation, but do not all have the same function. The notion of environment named *milieu* in French didactic was developed by Brousseau (1997), Margolinas (1995) and Bloch (2006) in terms of structuring the environment by levels. Other researchers have proposed a characterisation of the environment in which students operate according to three spaces: micro-space, meso-space and macro-space (e.g., Berthelot & Salin, 1992). These three spaces simply allow to classify the environment according to the value of the variable concerning the size of the space. However, the characterisations given in the literature do not allow us to categorise the actions directed towards the environment of the situation in detail, especially as these actions do not all have the same functions (Da Ronch, 2022). It therefore seems more appropriate to categorise our environment according to its functionalities. This will enable us to propose a typology of actions according to the zone of interaction with the environment and their assigned function. For our research, the environment is therefore characterised in *functional zones*, while considering the social dimension of mathematical activity. The *Work Zone* ($\mathcal{W}_Z$) is the main area of the environment where the subject act to solve the problem. Therefore, the actions in this zone are linked to actions about problem-solving. The *Tooling Zone* ($\mathcal{T}_Z$) is the zone of the environment which provides to the subject(s), the objects they need to develop their actions in problem-solving. The *Information Zone* ($\mathcal{I}_Z$) is the zone of the environment which communicates information about the problem: rules of the game, instructions, examples, *etc.* (Da Ronch, 2022; 2024).

In relation to the objective formulated in the first section of the paper and the characterisation given to our environment, several questions can be formulated, namely*: what choices of observables should we make? How can we retranscribe the students' mathematical activity? And how should it be*

analysed? In the following section, we describe the steps in our model that will enable us to answer to these questions. These steps will make it possible to objectify students' mathematical activity, and particularly their proving processes based on audiovisual data collected.

Method for processing and analysing audiovisual data: a three-step model

In this section we present our model called **ORA** (**O** for observable, **R** for retranscription and **A** for analysis). The different steps of this theoretical model will be illustrated by an example based on a problem in discrete mathematics. This model has already been applied to other problems (e.g., Da Ronch, 2022) during research aimed at retranscribing and finely analysing the mathematical problem-solving activity of individuals, out of the classroom (in a context of popularisation of mathematics), with dozens of participants. In each case, coding was carried out by a single researcher based on the choice of observables made through mathematical analysis of the problem (deductive approach).

Choice of observables: typology, actions and strategies for identifying proving processes

Firstly, we have to identify elementary actions noted a_i. The mathematical analysis carried out on the problem allows us to retain only a subset $\mathbb{A}$ of this a_i-actions. Thus, these elementary actions are chosen for epistemological reasons in relation to the knowledge we have on the problem, or for didactic reasons. Furthermore, the subset $\mathbb{A}$ of these actions is finite since the corpus of audiovisual data to be processed and analysed is also finite. The characterisation of the environment by *functional zones* makes it possible to propose a typology of actions. We can therefore distinguish 4 types of specific action. The $\mathcal{W}$*-actions* which are actions that modify $\mathcal{W}_{\mathcal{Z}}$. *The* $\mathcal{T}$*-actions* that interact with $\mathcal{T}_{\mathcal{Z}}$. This may involve selecting or depositing objects in $\mathcal{T}_{\mathcal{Z}}$. *The* $\mathcal{I}$*-actions* that allow access to $\mathcal{I}_{\mathcal{Z}}$. Finally, the $\mathcal{SO}$*-actions* which are actions arising from social interactions involving exchanges between peers or to oneself. They can be actions in oral language or actions that involve gestures (Da Ronch, 2022; 2024).

To identify traces of students' mathematical activity, we need to identify actions based on the knowledge or skills we have about solving of the problem. One way of doing this is to objectify their solving strategies, which require the mobilisation of knowledge, skills or know-how, by means of ordered actions carried out on objects in order to solve a given problem **P**. These strategies will be defined in *extension* or in *comprehension* (Da Ronch, 2022). When we have access to all the strategies and can describe them as a sequence of a_i-actions, on instantiated o_j-objects, we will define them in extension. A *strategy* can therefore be defined as a *word* $\boldsymbol{\omega}$ which takes the form of a sequence $a_i o_j$. On the other hand, when it is not possible to describe all the strategies because there are too many for instance, we will describe these strategies in comprehension, *i.e.,* using a set of actions on generic objects. The example below illustrates a sequence of ordered actions (additions) performed on objects (dominoes) to prove the non-existence of tiling in this configuration (Figure 1). The proof of non-existence consists of reasoning by necessary conditions until an absurdity is reached (two non-adjacent squares that cannot be tiled by a domino). The case study provides a complete proof of the non-existence of tiling.

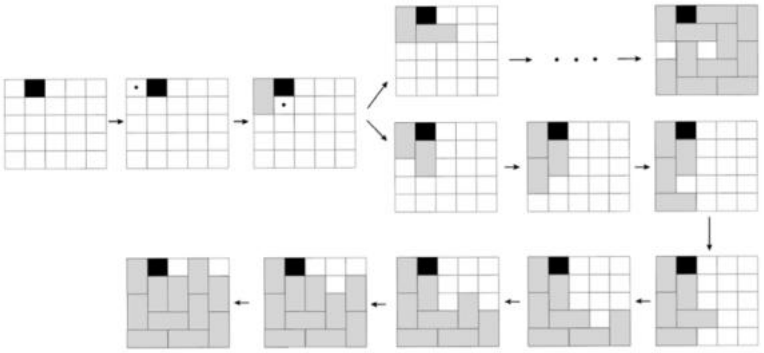


Figure 1: Strategy for proving the non-existence of this tiling

Retranscription of the experience: encoding of elementary actions and retranscription method

To retranscribe the actions that describe the strategies used by the students during the proving processes and the actions linked to the formulation of knowledge, we need to give an encoding. To make our retranscription intelligible, we bijectively encode the actions a_i of $\mathbb{A}$ in an intelligible symbolic format taken from a finite set $\mathbb{L}$ of symbols (Table 1). These symbols are built as close as possible to the observed reality.

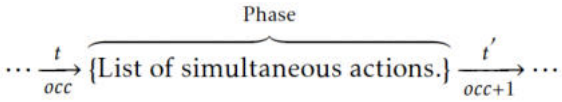
Table 1: Examples of encoding

Type of actions	a_i-action of $\mathbb{A}$	Associated symbol of $\mathbb{L}$
$\mathcal{W}$-action	To add	+
$\mathcal{W}$-action	To remove	–
$\mathcal{T}$-action	To search	
$\mathcal{SO}$-action	To point	
$\mathcal{SO}$-action	To chat	

Now that we have encoded the a_i-actions, we propose a retranscription method. This method is completely deterministic, since it is not open to interpretation and could be carried out by a computer. The actions ($a_i \in \mathbb{A}$) carried out on objects ($o_i \in \mathbb{O}$ with $\mathbb{O}$ that could be empty) by the observed student ($s_k \in \mathbb{S}$) characterise the student's local activity. We define this local activity as a *micro-activity* characterised by a triplet $(a_i, o_j, s_k) \in \mathbb{A} \times \mathbb{O} \times \mathbb{S}$. Between each micro-activity, the time *t* and the discretised occurrence ($occ \in \mathbb{N}$) are specified by an occurrence and a time marker (Da Ronch, 2022).

$$\cdots \xrightarrow[occ]{t} (a_i, o_j, s_k) \xrightarrow[occ+1]{t'} (a_p, o_m, s_k) \xrightarrow[occ+2]{t''} \cdots$$

Between each marker, there may be one or several micro-activities starting at the same time or for which we are unable to give an order, since we do not know precisely when they were carried out. We therefore assume that they all start at the same time. We refer to all micro-activities between two markers as *phases*. Here a phase is not an episode in the sense of Schoenfeld (1985).



Below is an example of a retranscription of the micro-activities of two students trying to solve the problem proposed in the previous section (the dialogue transcription is in Da Ronch, 2022, p. 325).

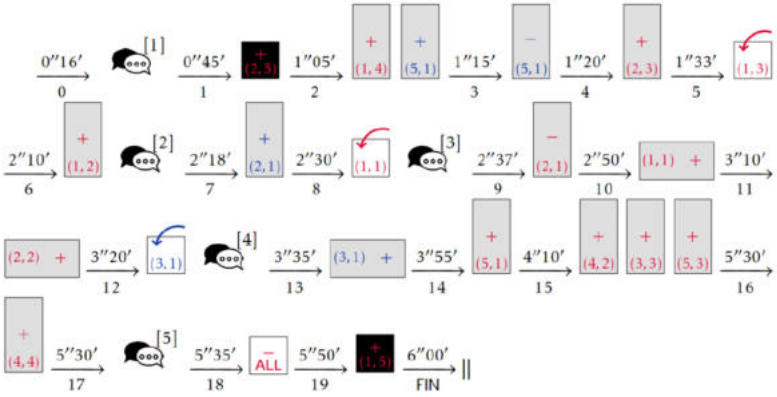


Figure 2: Example of a retranscription of the activity of two students s_1 and s_2

In this extract, two students (s_1 and s_2) are trying to tile a 5×5 grid with dominoes. This grid has a hole (black unit square) and this hole is positioned by the students. In the retranscription, the students are identified by the red and blue colours corresponding to s_1 and s_2 respectively. In addition, when a student puts (+) or removes (-) a domino on the grid, we represent this domino in the colour corresponding to the student, with the plus (+) or the minus (-) symbol and coordinates. To quickly identify the position of the dominoes on the 5×5 grid, we take the bottom left square of the grid as the origin (1,1). For example, on the first occurrence, s_1 puts the unomino (hole) in position (2,5). Then, both students simultaneously put two dominoes vertically in positions (1,4) and (5,1). This means, for example, that the south square of the domino placed by student s_1 on the grid is in position (1,4).

We then propose to enhance our transcription with a *summary*, *i.e.,* an "enhanced" photograph of the Work Zone ($\mathcal{W}_Z$) at a given moment, providing a static view of the situation at a given time. This summary allows us to retrace the students' reality at a *t* time. It also allows us to know exactly the number of $\mathcal{W}$-actions performed by the student knowing the number of $\mathcal{W}$-actions performed by the other students. This gives the number of $\mathcal{W}$-actions carried out between two effectives $\mathcal{W}$-actions and also the distribution of these actions: did s_1 perform more, fewer or as many $\mathcal{W}$-actions as s_2, for example? In the summary below, taken from the retranscription, we can see the reality of the students at a given moment in the retranscription (Figure 3). The symbols of type $\boldsymbol{a}|_{\boldsymbol{b}}$, where $\boldsymbol{a}$ and $\boldsymbol{b}$ are integers, represent the number of $\mathcal{W}$-actions of s_1 (respectively s_2) knowing the number of $\mathcal{W}$-actions of s_2 (respectively s_1). For example, during its first $\mathcal{W}$-action, s_1 puts the unit square in position (2,5), knowing that s_2 has not yet performed a $\mathcal{W}$-action (Figure 2). We thus have coded $\mathbf{1}|_{\mathbf{0}}$.

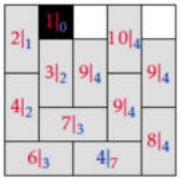


Figure 3: Example of a summary linked to the retranscription (see Figure 2)

The summary reproduces the Work Zone in which the students find themselves at a given moment, which is a perfectly complementary to the retranscription.

Analysis method of retranscription for objectifying students' proving processes

To analyse the retranscription (Figure 2) to objectify the students' proving processes, we need to remove some unnecessary information. This could be, for example, some dominoes which have been added and then removed, or some showing actions which are not significant in identifying a strategy linked to the proving process. We are therefore looking for a subword in this retranscription to highlight a strategy linked to a proving process identified in the mathematical analysis (Figure 1). This subword is shown in Figure 4.

Figure 4: Subword from the retranscription sequence (see Figure 2)

This subword shows a similarity with a part of the strategy for proving the non-existence of tiling in this configuration (Figure 1). However, it does not describe the entire strategy given by the mathematical analysis. In fact, the students dealt with only one of the two cases to prove the non-existence of tiling. Moreover, it is reasonable to say that it is only s_1 who has developed a solving strategy leading to a partial proving process of non-existence. Indeed, the subword described shows that s_1 has carried out almost all the actions needed to develop this partial strategy. In addition, the transcription of the dialogue shows that s_1 reasons well, using reasoning based on necessary conditions ("we have to", see Da Ronch, 2022, p. 325) until it is no longer possible to add dominoes. Furthermore, the end of the retranscription shows that s_1 removes all the dominoes and changes the position of the hole (black unit square). This suggests that he has not realised that his strategy for proving the non-existence of tiling was incomplete. Finally, this subword shows a correct proving process but incomplete that implicitly uses a proof by non-contradiction and a reasoning by necessary conditions without an exhaustive study of the cases. Even if this proof is partial, s_1 has nevertheless developed a mathematical activity. Of course, this example of transcription is given for a particular instance; we would need to have the full transcription to be able to identify other strategies that would also be indicative of a mathematical activity.

Results and discussion

We have presented a deterministic model described in three steps: observable, retranscription and analysis. This model makes it possible to describe and objectify students' strategies in detail, and particularly their proving processes in problem-solving. We illustrated this on a tiling problem in Discrete Mathematics. This enabled us to show, on an example, that this theoretical model was

operational for describing and recognising significant elements of the proving process in problem solving. (for other examples, see Da Ronch, 2022). Moreover, in this model, we can clearly locate the interpretation part. It is precisely at the level of the choice of observables (choice of actions and strategies that seem relevant to what we want to observe). The remainder is completely deterministic, excepted sometimes at the level of retranscription analysis, since the strategies identified may be more or less close to those identified in the mathematical analysis. With this model, a single coder is sufficient, as there can be no interpretation in the coding of an elementary action, unlike episode coding (Schoenfeld, 1985), which requires at least double coding to measure reliability. This model also shows its relevance and effectiveness when it comes to solving problems using tangible objects, whose actions are similar to those carried out in "paper and pencil". Hence, Discrete Mathematics seems to be a suitable field, since a lot of mathematical objects (graphs, tiles, polyominos, *etc.*) can easily be represented in the form of tangible objects (Da Ronch, 2022; Gravier & Ouvrier-Buffet, 2022). In addition, it is probable that such problems, represented in a numerical format, facilitate this retranscription but open up the question of computer transposition. It would be appropriate to consider this type of processing and analysis in an automated way using digital tools, Artificial Intelligence and image recognition, for instance. This would make it possible to search large volumes of data, for a wide variety of users and over a very long period. By the way, recent researches is being carried out by the TWEAK team (LIRIS in Lyon, France), with the kTBS4LA (kernel for Trace Based Systems for Learning Analytics) platform. Based on *Learning Analytics*, kTBS4LA enables observable traces to be collected, stored, manipulated and analysed in order to calculate indicators of individual activity (Casado et al., 2017; Lefevre, 2018). Our model can also be used to perform quantitative analyses by mining data collected on a specific action for instance. It also has its limitations, particularly when the execution time of the actions is different, making it possible the reading and the writing of the retranscription complex. This model also raises questions about its operationality towards complex abstract objects that cannot take on a tangible format, giving an encoding more complex. Finally, it is important to specify that this theoretical model is not intended to be used by teachers, but only to provide researchers with the necessary tools to identify the traces of mathematical activity in problem solving.

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Investigating the use of programming in the process of creating algebraic expressions in 6th grade

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This study was aimed at investigating students' comprehension of algebraic expressions during the transition period from arithmetic to algebra with instruction based on programming in a Scratch environment. The teaching experiment was conducted with 15 students in the 6th grade of a public middle school in Eskişehir. Data was collected from the students' Scratch project files, audio recordings, observations and clinical interviews. The data was analysed using thematic analysis. As a result of the study, it was observed that the students could easily make the transition and transform between verbal algebraic expressions and Scratch code sequences and could clearly see the dependent-independent variable relationship between the variables in the code sequences.

Keywords: Early algebra, Scratch, programming, teaching experiment.

Introduction

Algebra has historically been recognized as the most challenging area in school mathematics (NCTM, 2008; Grønmo, 2018). Research suggests that to overcome students' difficulties in learning algebra and to teach algebra more effectively, we should move away from traditional methods and provide a teaching environment and methods where students can discover and make sense of the nature of algebra (Blanton et al., 2019; Ellis & Özgür, 2024). This can be achieved by understanding the concepts correctly in the transition from arithmetic to algebra and by creating learning environments that enable students to see and understand mathematical relationships by thinking about mathematical structures (Blanton, 2008). Kieran (2007) states that performing generative and transformational activities in algebraic situations and using different representation methods is a link that enables the learning of algebraic expressions and operations. According to Papert (1993), when students create meaningful and concrete products such as a model, a computer program or a work of art in the process of learning abstract concepts, their learning becomes more permanent. For example, it was observed that students concretized mathematical concepts by drawing geometric shapes using the Logo programming language developed by Papert (Boyle,1999).

In recent years, in support of Papert's view, there are many studies showings that activities such as game development and animation creation using the Scratch 3.0 programming language developed by MIT contribute to the development of problem and equation-solving skills and a better understanding of mathematical processes by enabling students to think in more detail due to the nature of programming. Creating educational games using Scratch has been shown to positively affect students' equation generation performance (Chiang & Qin, 2018), and students' cognitive skills such as algorithmic thinking, creativity, collaboration, and critical thinking have positively changed after programming training with Scratch (Jiang & Li, 2021).

These results encourage us to investigate the effect of teaching algebra with programming on concretizing abstract algebra concepts and learning them more effectively. This study is part of a

master's thesis project that aims to investigate the process of making sense of algebraic expressions in students in the transition period from arithmetic to algebra with an instruction based on the Scratch 3.0 programming language. In the mathematics curriculum that was being implemented in Türkiye at the time this research was conducted, algebraic learning outcomes started with the algebraic expressions' unit in the 6th grade for the first time. In the Ministry of National Education (MoNE) textbooks, traditional methods are used for teaching algebraic expressions.

This study seeks to answer the following research question: How do 6th-grade elementary school students translate verbal algebraic expressions into code sequences in Scratch and vice versa? How do the students' cognitive processes required for transitioning between these two representations occur?

Method

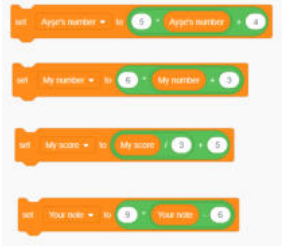
This study is a small part of a project, which is a teaching experiment conducted over eight weeks at a public school in Türkiye. The teaching experiment aimed to help students acquire algebraic expression skills through the Scratch programming language and consisted of 14 activities. This study presents data obtained from one of these activities.

The teaching experiment was carried out in three phases: the first phase involved introducing Scratch; the second phase focused on forming algebraic expressions through activities based on Scratch; and the third phase included problem-solving activities related to algebraic expressions using Scratch. In this study, the findings from the first activity of the second phase of the teaching experiment, titled 'Let's Write a Calculating Code Sequence,' will be examined.

The teaching experiment process was designed and sequenced based on the learning outcomes specified for the algebra domain in the Ministry of National Education's (MoNE) mathematics curriculum. The skills corresponding to these learning outcomes in the Scratch environment were defined, and the activities were designed to incorporate these processes and skills. This alignment was conducted both to design the activities and to analyze the data obtained from these activities. Examples of the activity titled 'Let's Write a Calculating Code Sequence,' prepared based on these objectives, along with the themes used in data analysis, are presented in Table 1 for further clarity.

Table 1: The "Let's Write a Calculating Code Sequence " activity

MoNE Learning Outcomes	Corresponding Scratch Achievement	Let's Write a Calculating Code Sequence Activity	Themes for Data Analysis
Writes an algebraic expression corresponding to a verbally given situation and a verbal situation corresponding to a given algebraic expression.	The ability to represent a verbally given expression in the Scratch environment	"Write the code sequence that calculates 4 more than the score variable."	Verbal algebraic expression to code translation

<table>
<tr>
<td>Calculates the value of an algebraic expression for different natural number values that the variable can take.</td>
<td>The ability to calculate the value of an algebraic expression in the Scratch environment for different values of the variable.</td>
<td>
<table>
<tr><th>Score</th><th>Results</th></tr>
<tr><td>6</td><td></td></tr>
<tr><td>42</td><td></td></tr>
<tr><td>423</td><td></td></tr>
</table>
</td>
<td>Evaluating dependent variables for various independent variable values</td>
</tr>
<tr>
<td>Explains the meaning of simple algebraic expressions.</td>
<td>The ability to write the verbal algebraic expression corresponding to a Scratch code block.</td>
<td>  </td>
<td>Interpreting code sequences as verbal algebraic expressions</td>
</tr>
</table>

The sample of this study consists of 13 sixth-grade students. The participants were selected based on the following criteria: they were required to volunteer for the study, have no prior experience with the unit on algebraic expressions, either at school or through any external course, and have successfully completed the first phase of the teaching experiment.

To ensure the reliability of the data collected during the teaching experiment, the implementation processes were recorded both via video and through the teacher's use of a lapel microphone. Additionally, a researcher's notebook containing observational notes was maintained, and screen recordings of the students' work on computers were gathered. Furthermore, clinical interviews were conducted with the students.

Activity: Let's Write a Calculating Code Sequence

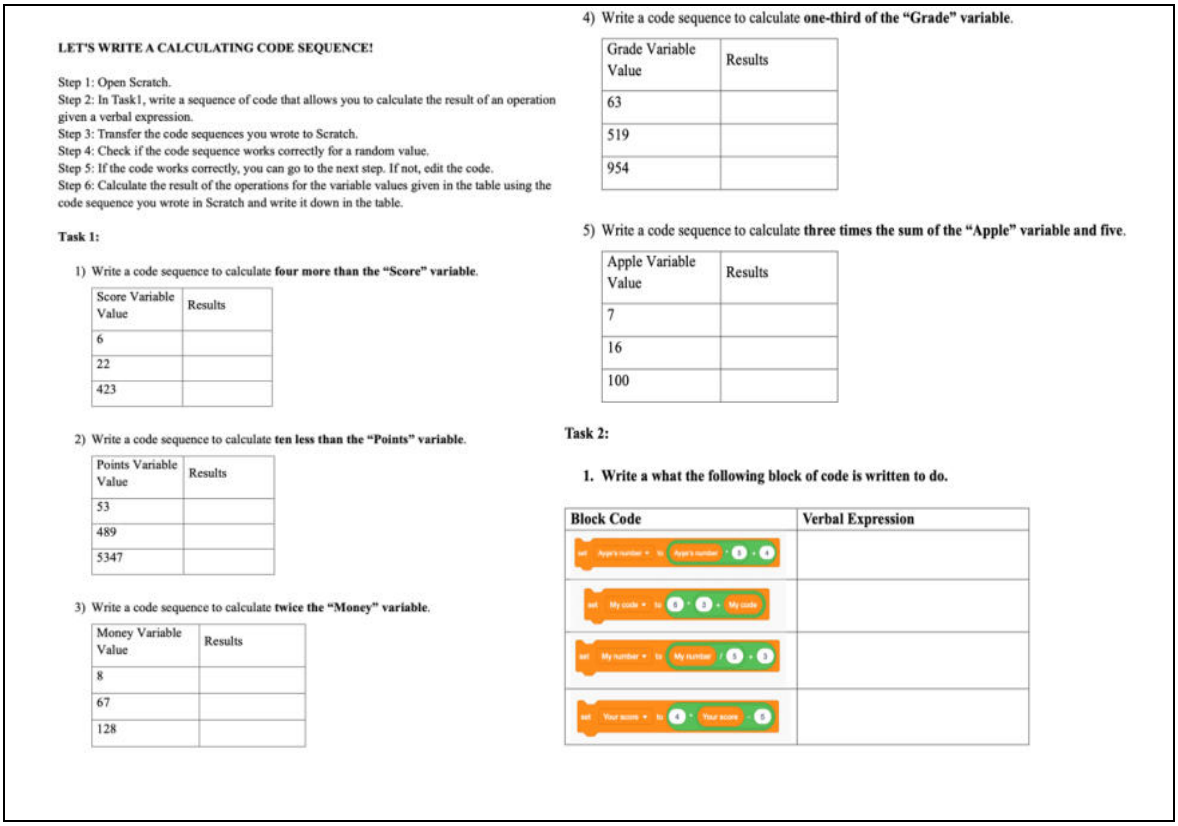
The "Let's Write a Calculating Code Sequence" activity consists of three parts, as exemplified in Figure 1. For this purpose, students were given an activity worksheet.

In the first part of the activity, students are required to define variables, define a second variable that changes based on the value of the first variable, and transfer the algebraic relationship between these two variables into the Scratch environment using operator code blocks. In the second part, students must be able to write a code sequence that calculates and displays the value of the dependent variable for every value given to the independent variable. For example, in the first task of the first part, students are expected to write a code sequence that "calculates 4 more than the score variable" and reach the correct result for different values of the score variable.

In the second part, the values that the independent variable can take are provided by the teacher in a table. Using these values, students will find the value of the dependent variable. The aim here is for students to check their code sequences against the given values and the obtained results. Having all students work with the same values and obtained results. Having all students work with the same values helps facilitate a natural control process within the classroom.

In the third part, students are asked to write the algebraic expressions corresponding to the code sequences provided on the activity sheet, which are different from those in the first part, in verbal

form. In the second task of the third part, for example, the given code sequence and the verbal expression that students are expected to write corresponds to “three more than my number multiplied the six.”



LET'S WRITE A CALCULATING CODE SEQUENCE!

Step 1: Open Scratch.
Step 2: In Task1, write a sequence of code that allows you to calculate the result of an operation given a verbal expression.
Step 3: Transfer the code sequences you wrote to Scratch.
Step 4: Check if the code sequence works correctly for a random value.
Step 5: If the code works correctly, you can go to the next step. If not, edit the code.
Step 6: Calculate the result of the operations for the variable values given in the table using the code sequence you wrote in Scratch and write it down in the table.

Task 1:

1) Write a code sequence to calculate **four more than the “Score” variable**.

Score Variable Value	Results
6	
22	
423	

2) Write a code sequence to calculate **ten less than the “Points” variable**.

Points Variable Value	Results
53	
489	
5347	

3) Write a code sequence to calculate **twice the “Money” variable**.

Money Variable Value	Results
8	
67	
128	

4) Write a code sequence to calculate **one-third of the “Grade” variable**.

Grade Variable Value	Results
63	
519	
954	

5) Write a code sequence to calculate **three times the sum of the “Apple” variable and five**.

Apple Variable Value	Results
7	
16	
100	

Task 2:

1. **Write a what the following block of code is written to do.**

Block Code	Verbal Expression

Figure 1: Activity worksheet

During the stage where students transfer the verbal expression into a code sequence, the control process in Scratch can be done by assigning values to the variables. However, the teacher needs to check the verbal expression written for the corresponding code sequences. After all students correctly complete the tasks and save their Scratch projects as required, the activity is concluded.

After the data obtained in the study was transcribed, the data was analysed using thematic analysis, one of the qualitative research methods. The themes to be used in the data analysis (Table 1) consist of the skills that students are expected to realize while preparing the activities.

Findings

Since the students had difficulty in using expressions such as more, less, multiple, one in ..., the activity was started by reminding the students which operations these expressions correspond to. While transferring the expression “4 more than the score variable” to the code sequence, students first created a score variable. Then, they wrote a code sequence that performs the score+4 operation using the addition operation block from the operator’s menu. All 13 students participating in the activity started coding by creating the variable “score”. 9 of the students used the “What is the score?” block

from the sensing code blocks and defined the answer to this question as the “score” variable. Then, they defined the variable “score” as “score+4” using operator blocks. When the code sequence is executed and the “score” variable is defined as 5, the “score” variable takes the value 9 on the screen.

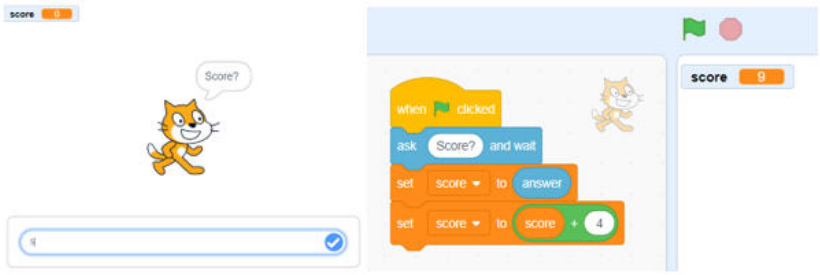


Figure 2: Code sequence using only variable code blocks

4 students first used the “make the score variable 0” block and then used the “make the score variable score+4” block. Here, the “score” variable is given a number value instead of “0” in the “make score variable 0” block in the code sequence and the code sequence is executed. In this case, when the “score” variable takes the value 5, the “score” variable takes the value 9.

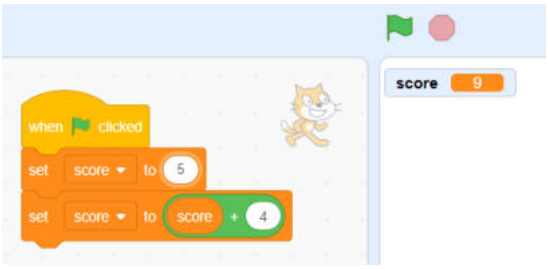


Figure 3: Code sequence using only variable code blocks

Evaluating dependent variables for various independent variable value

After completing the code sequences, the students entered the values given for the variable “score” in the activity sheet into the code sequence and ran it. In this case, two of the students thought that something was wrong and asked the teacher for help.

Yağmur: Now we want to calculate 4 more than the score. If we say score 6 in the code string, we get 10. But the result appears as score again. In this case, 4 more than the score is still the score. Isn't that illogical?
Teacher: You made a good point. What do you guys think?
Yağmur: Then we need to create a new variable and bind the result of the operation to it.

After this conversation, students reorganized their code sequences. They created a “sum” variable next to the “score” variable and defined the “sum” variable as “score+4”.

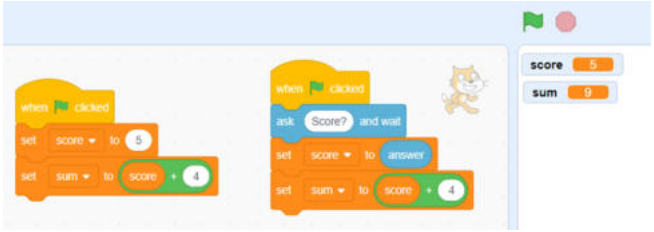


Figure 4: Revised code sequences

All the students reached the result by entering the values given in the table into the Scratch code sequence. In this first question, it was observed that the students calculated the values in the Scratch and then checked them on paper and pencil. When asked why, the answer given is as follows:

Nesli: I check it, the paper gives a guaranteed result.

Then, when it was observed that the same students calculated the values in the Scratch code sequence and did not calculate them on paper and pencil, when asked why they did not continue to check, the student's answer was as follows:

Nesli: Scratch already calculates correctly; it takes less time.

Interpreting code sequences as verbal algebraic expressions

In this section, students were given 4 code sequences. Only 1 student was able to write a correct verbal algebraic expression in all code sequences. In code sequence 1, except for one student (Zeynep), the remaining 10 students wrote a correct sentence. Zeynep wrote the sentence “4 times the sum of 5 and the number of Ayşe” instead of “4 more than 5 times the number of Ayşe”.

Teacher: Which of these is addition and which is multiplication?
Zeynep: The first is multiplication and the second is addition.
Teacher: So, what does she add and multiply Ayşe's number by?
Zeynep: (Looking more carefully) First she multiplies by 5 and then she adds by 4.
Teacher: How does it work in your sentence?
Zeynep: I added by 5 and then multiplied by 4.

After the end of the fourth code sequence, since it was observed that the students made too many mistakes based on operation priority, the activity was not terminated, and the students were asked to transfer the code sequences to the Scratch. The aim was to make the students realize that the nesting of code blocks in Scratch is related to the operation priority. Students were first asked to try the fourth code sequence. First, a code sequence was created as '4 times of 5 minus' and then '5 minus of 4 times'. Then, when the value of the variable was 9, the results were compared and discussed in class.

Ozan: Teacher, this result is wrong, but I think the code is correct.
Teacher: For example, is 5 minus 4 times of 9 equal to 4 times of 5 minus 4 times of 9?
Ozan: It seems equal... Let me try.
Serdar: No, it's not, one is 16 and the other is 31.
Teacher: This should give you a clue.
Yağmur: Teacher, the order of these jellybeans affects the result of the operation! *(At this stage, students named the code blocks symbolizing the variable as jellybeans.)*
Teacher: I thought there was a name for this... What was it?
Rain: Priority of operation, priority of operation!
Ozan: Then these jellybeans act like parentheses. Each jellybean is a parenthesis.

Results and Discussion

Many studies in the literature on algebra instruction have stated that students who correctly learn arithmetic and arithmetic rules tend to have higher success in algebra (Kieran, 2004; Blanton, 2008; Demonty et al., 2018). One of the findings of this study also indicates that, for students to develop algebraic thinking and perform algebraic operations, they must first possess a foundational arithmetic competence. During the instructional experiment, it was observed that students lacked proficiency in

the mathematical language related to arithmetic operations. It was found that students did not understand that phrases like "4 less" correspond to subtraction, "5 more" to addition, "3 times" to multiplication, and "half of" to division. However, by the end of the instructional process, the students' command of the language had improved, and they demonstrated more success both in planning code sequences and in expressing algebraic situations.

Another finding of the study is that Scratch's approach of not processing code blocks randomly or from left to right, but rather treating each block like a parenthesis, paying attention to operation precedence, indicates that Scratch can be an effective tool for teaching order of operations. Students became aware of this situation after encountering problems when first introduced to algebraic expressions that included constant terms.

Research acknowledges that recognizing and expressing relationships between quantities is a crucial step in transitioning to algebra. When writing code sequences, students must analyse the given problem, accurately express the variables, and identify the relationships between them. In this study, it was observed that this process encouraged students to think more deeply about the problem elements, as they made observations about dependent and independent variables during the activities. In fact, when faced with a code sequence where the dependent variable was not provided, students identified, named, and explained the relationship between the dependent and independent variables on their own. This finding suggests that the Scratch can be an effective instructional tool for understanding and establishing relationships between quantities. Calao et al. (2015) suggested that Scratch could be an effective tool for teaching relationships between quantities, and the results of this study support that claim.

In this study, the choice of tools was made to support students in learning programming and mathematics simultaneously. Scratch was chosen for teaching programming because it dynamically and visually represents variables and algorithms, allowing students to interact directly with the code. An alternative tool, such as spreadsheets (e.g. Excel), can also display variables and calculations. However, Scratch provides an environment where only semantic errors can occur, making it easier for students to learn by trial and error. It also reduces the cognitive load on students by removing the syntactic complexity of traditional programming. Furthermore, the visualisation capabilities of Scratch help to create a ‘mental space where the code lives’. Future studies could explore whether spreadsheets can provide a similar learning environment.

Although the data obtained in this study indicate that programming with Scratch can be an effective instructional tool in teaching algebra to students in the early algebra phase, further research is needed to examine the impact of this instructional process on students' success in algebra and the retention of their learning. The results of the study suggest that Scratch programming can be an effective tool for teaching algebraic expressions and introducing algebra. However, further research is needed to prevent misconceptions and learning difficulties.

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Superfluous or meaningful integration of programming in mathematics? An analysis of lower secondary textbooks

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Programming has been integrated in mathematics education in many countries, however, its relation to mathematics in education is not necessarily agreed upon. In this work, we investigate the relations between mathematics and programming in textbooks from two of the largest publishers in Norway. This analysis is inspired by, and compared to, a former study of textbooks in Sweden. Compared to the results of the Swedish study, our results indicate that in Norway there are relatively many examples of programming that have a potential for students to explore and deepening their mathematical understanding. At the same time, an introduction of basic programming is lacking. Our measures of inter-rater reliability also rises questions regarding the applicability of the theoretical framework. The need for more research in several directions is emphasized.

Keywords: Programming, secondary school mathematics, textbooks, mathematical exploration.

Introduction

In recent years, programming has been integrated into the mathematics curriculum in many countries (Oda et al., 2021). The importance and demand for programming skills and skills in information and communication technology in the digital society is high, and is expected to grow (Balanskat & Engelhardt, 2015). The inclusion of programming in school curricula over the last decade can be seen as a second wave of programming in schools. The idea of using programming as a tool for learning mathematics originates back to Papert (1980). He developed the programming language LOGO, which he argued could change the way students think of mathematics, as well as positively affect communication and collaboration between students and teachers. Programming is today introduced in school curricula in different ways. In some countries, a separate subject of programming has been created, while in other countries programming is an integrated part of other subjects. Introducing programming in school is often argued to help students learn computational thinking (Kallia et al., 2021), however, incorporating computational thinking into mathematics is not straightforward (Nouri et al., 2020).

One issue, highlighted in former studies, is that programming can delineate the mathematics towards topics that are programmable and away from other kinds of mathematics (du Boulay, 1980). Holo et al., (2023) find in their literature review that mathematics is mostly applied as a context for learning programming, and that the broad attention of exploring mathematics through programming is yet to come. Kaufmann & Stenseth (2021, pp. 16-18), argue that programming can be included into mathematics education, however, on the condition that the focus "must not be entirely on using programming". This argument is supported by Lie et al. (2017), who indicate that student may learn

about mathematical concepts, as well as practising mathematical competencies as reasoning, communication, modelling and problem solving, when engaged in programming activities.

In their analysis of Swedish and Norwegian curricula, Vinnervik & Bungum (2022) find that programming is broadly referred to as part of the digital competence, and more specifically a tool for learning content within subjects. In both Sweden and Norway mathematics is positioned with the responsibility to provide for basic programming skills. According to the Swedish mathematics curriculum (The Swedish National Agency for Education, 2024, pp. 56), one of the aims of digital tools and programming is "to explore problems and mathematical concepts, perform calculations, and present and interpret data." Vinnervik & Bungum (2022) found in the Norwegian mathematics curriculum connections between modelling and applications and programming, and between exploration and problem solving and algorithmic thinking.

In addition to mathematics, programming is integrated in the Swedish technology subject. Although programming is included in the subjects arts and crafts, music and science in Norway, programming is here intended as a tool to learn subject matter, keeping mathematics responsible for introducing basic programming concepts (Vinnervik & Bungum, 2022). In both the Norwegian and Swedish curricula, basic programming concepts are introduced in early primary school. While the Swedish curriculum explicitly states that the students shall learn visual programming languages in primary school and text-based languages in lower-secondary school (The Swedish National Agency for Education, 2024), the Norwegian curriculum does not specify this (Norwegian Ministry of Education and Research, 2019). However, the most Norwegian textbooks series take the same approach.

Kilhamn et al. (2021) created a theoretical framework for analysing lessons created by teachers in school. They later adapted this framework to analyse mathematics textbooks in the lower secondary school of Sweden (Nyman et al., 2024). In their study, they find that most of the tasks in the Swedish textbooks have a primary goal of teaching programming, and mathematics is mainly a context for the programming.

In the current study, we reproduce part of the study of Nyman et al. (2024), but apply it in the Norwegian context. We adapt their analytical framework to six lower-secondary mathematics textbooks from Norway, to compare and identify possible similarities and/or differences between these two Scandinavian countries. By investigating the textbooks, we may gain insight into how the students in the two countries meet programming in mathematics. Although analysing textbooks does not tell us what students learn, mathematics has a long tradition for being driven by the textbooks (Remillard, 2005), and thus analysing the textbooks can give us an indication of the learning opportunities given to the students. In the recent literature review, Holo et al. (2023) list understanding the potential relation between specific mathematical topics and programming as one of the two major suggestions for future research. The current paper contributes to this understanding of the potential relation between specific mathematical topics and programming.

Based on the framework and the data material in this analysis, we ask two of the same research questions (RQ) as Nyman et al. (2024), adapted to the Norwegian context:

RQ1: What is the relation between mathematics and programming in Norwegian textbooks?

RQ2: What mathematical content is visible in connection to programming in Norwegian mathematics textbooks?

These research questions are studied and discussed with respect to the findings in the Swedish context.

Method

In the current study we collect data from the 8th -10th grade (lower secondary school) mathematics textbooks from two of the largest publishers in Norway[1] (6 books in total). Both textbook series contain visual/block-based programming (Scratch), but they mainly focus on text-based programming (Python). In the analysis, only units that contain programming is included. A textbook unit is included if programming is explicitly mentioned or shows computer code, e.g., a task explicitly requesting the student to solve it using programming. As in Nyman et al. (2024, pp. 6) a textbook unit is defined as "a section in the textbook with a new heading and a coherent content containing, for example, an introduction, worked examples and tasks". In the Norwegian textbooks analysed in this work, the textbook units are often one subchapter. The size of a unit varied from 1 to 16 pages, with a median of 3 pages and an average of 4 pages. In total 96 units were analysed, 36, 27 and 33 in 8th, 9th and 10th grade, respectively.

To examine the first research question, the relations framework was applied. This "includes four types of relations between mathematics (M) and programming (P) in a growing hierarchy from MP1 to MP4" (Nyman et al., 2024, pp. 7). The relations framework was originally a framework formulated in Swedish and applied on teaching lessons (Kilhamn et al., 2021). Nyman et al. (2024) translated the framework to English and adapted it slightly to apply it to textbook units. We based our framework on these two frameworks to be able to compare our results with Nyman et al. (2024), and ended up with the following definitions:

MP1: *Programming without connection to mathematics*
Textbook units that merely consists of programming.

MP2: *Mathematics as a context for programming*
Textbook units where mathematics is mainly a context for learning programming. No new mathematics learning is intended, but programming can be used to repeat or consolidate math skills.

MP3: *Programming as a tool for streamlining calculations within mathematics*
Textbook units where the content is clearly mathematics. A computer is programmed and used in an efficient way to perform calculations.

MP4: *Programming as a tool for exploring mathematics*
Textbook units where programming is used to explore mathematical concepts or relationships. The programming adds new ways of approaching mathematics and is used to deepen the mathematical understanding at a level that is relevant for the specified grade.

[1] The Maximum series of Gyldendal and the Matemagisk series of Aschehoug.

For examples of content of the four categories, we refer to Kilhamn et al. (2021) and Nyman et al. (2024). MP1 and MP2 are described as "poor" categories for learning mathematics while MP3 and MP4 represent productive ways of learning mathematics. This does, however, not mean that all units should be MP3 and MP4. Learning mathematics through programming in an MP4 task will typically require that the students know some basic programming in the first place. Therefore, MP1 and MP2 examples and tasks might be needed as well. In the analysis, a unit is put in the highest possible category, meaning that, for example, a unit comprising at least one example of content that fits into MP4 is categorized as MP4.

The textbooks were initially analysed by three of the authors independently, without any prior discussion on how the units should be coded other than reading the relations framework. In the final unit coding presented in the results, we picked the category of the majority of the coders. For two of the units, all three coders coded different categories (MP2, MP3, MP4). For these two units, the coders discussed the units and reached an agreement. The coding framework has to some extent an ordinal relationship between the categories, from least mathematics to most mathematics. Thus, assuming ordinal data, Krippendorff's alpha for inter-rater reliability for the initial coding is 0.61. However, the categories may also be considered nominal, in which case the Krippendorff's alpha is calculated to 0.46.

To answer the second research question the mathematical content of the textbook units was analysed. Each textbook unit would typically only contain a small section about programming. Thus, to categorize the content of the units we only looked at the content of the tasks, examples, explanations, etc. that contained programming. We used the same content topic categories as Nyman et al. (2024) which were: number sense, algebra, geometry, statistics, probability, and problem solving. Both within a single task, or between several tasks in a unit there can be several different topics, thus, each unit can be categorized with multiple mathematical content topics.

Results

Each category, from MP1 up to MP4, constitute approximately 3%, 43%, 24% and 30% of the textbook units, respectively. Compared with the results from Sweden (Nyman et al., 2024), there are a relatively higher ratio of MP3 and MP4 units in the Norwegian textbooks, and a higher ratio of MP1 and MP2 units in the Swedish textbooks, see Figure 1.

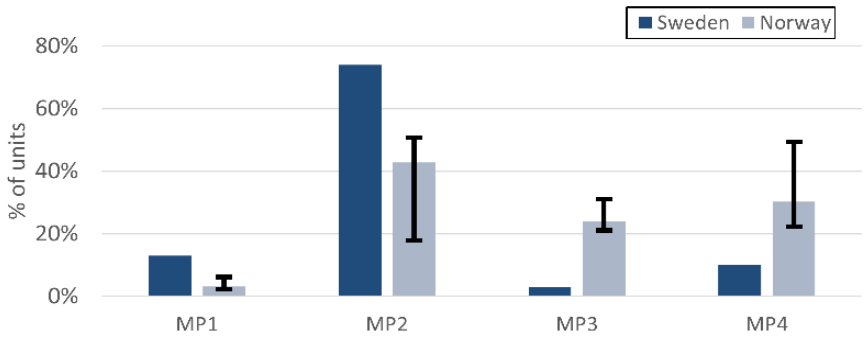

Figure 1. The percentage distribution of relations between mathematics and programming in Norway and Sweden. The Swedish data are from Nyman et al. (2024)

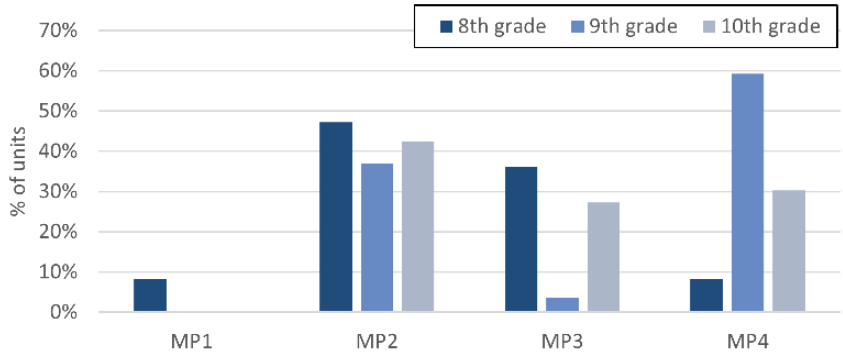

Figure 2. The percentage distribution of the relations between mathematics and programming in each age group

In addition to the distribution of relations in textbook units in Norway and Sweden, the variation between the coders is also illustrated for Norway in Figure 1. This shows the outcome of the coding if the minimum and maximum category was picked, assuming the range from MP1 to MP4 as lowest to highest. Using e.g. the minimum, would put 51 % and 22 % of the units in the MP2 and MP4 category, respectively.

The relations between mathematics and programming vary between the three grades, as shown in Figure 2. The percentage of units in the two highest categories combined are 44%, 63% and 57% for 8th to 10th grade, respectively. 9th grade has by far the highest share of MP4 units, while 10th grade almost have the same share of MP3 and MP4 units. Only 8% of the textbook units in 8th grade belong to the category of exploring mathematics through programming (MP4). We find some introduction to text programming in one of the 8th grade textbooks, but our first impression is that, overall, little attention is given in the textbooks to introducing the fundamental concepts of programming, such as variables, if-statements and loops.

The complete distribution of content in context of relations framework is shown in Table 1. For each mathematical content category, we find that over 50 % of the units were either an MP3 or an MP4 unit.

Table 1. The distribution of mathematical content with respect to the relations between mathematics and programming

Category	Number sense	Algebra	Geometry	Statistics	Probability	Problem solving
MP4	9	16	6	6	8	5
MP3	9	14	1	1	0	5
MP2	15	14	6	4	3	3
Total	33	44	13	11	11	13

Discussion

In the six textbooks analysed we barely found MP1 units (3 of 96), all of them in 8th grade. The high share of MP2 units compared to MP1, indicates that in Norwegian textbooks, the authors try to combine the basic instruction in programming with a mathematical context. One reason for the apparent lack of emphasis on basic programming concepts alone in secondary school might be the

attention to programming concepts in primary school (grade 1-7) (Norwegian Ministry of Education and Research, 2019). In this way, the transition towards a higher share of MP3/MP4 tasks in 9th and 10th grade, compared to 8th grade, may follow a larger trend, where programming to a larger extent is used as a vehicle to learning mathematics. Further, the apparently small focus on introducing fundamental programming concepts can perhaps be due to that the largest publishers in Norway also have digital programming resources, and hence, comprehensive programming material in the written textbooks might not be necessary, if the online material is used in combination with the textbooks. In the Swedish textbooks, the units (except from in one book) were 1-3 pages (Nyman et al., 2024). In the Norwegian textbooks 52 of 96 units consisted of 1-3 pages, and 77 of the units consisted of 1-5 pages. Thus, the Norwegian textbooks have longer units than the Swedish textbooks. This can be important since the units are categorized according to the highest category within the unit, thus, units with more pages will more easily end up in higher categories. This also means that a unit categorized as, e.g., MP4 may have many tasks on MP1 and MP2 that focus on learning programming.

The inclusion of three specialised programming textbooks in the analysis by Nyman et al. (2024) can be another important reason for the high share of MP1 and MP2 units in Sweden. Furthermore, with 2 and 6 MP3 and MP4 units, respectively, their conclusion that the textbooks units with programming provide small opportunities for exploring mathematical content and deepening the mathematical understanding, is reasonable. Our study shows that in Norwegian textbooks, due to the larger fraction of MP3 and MP4 units, there are possibilities of exploring mathematics through programming, given that the learner has some prior basic programming skills.

Some of the differences between Norway and Sweden may be explained by the subtle differences in the curriculum aims in the two countries. Nyman et al. (2024) problematize that although programming is included in the algebraic core content of the Swedish curriculum, the findings of MP3 and MP4 units in the textbooks were mainly found under the content topics of probability and problem solving. However, the curriculum aims do not explicitly link programming to mathematics (The Swedish National Agency for Education, 2024), so it is left to others, e.g., textbook authors, schools and teachers, to operationalize them within mathematical topics. The Norwegian curriculum more explicitly relates programming to mathematics, as in 10th grade, where the aim is "to explore mathematical properties and relationships by using programming" (Norwegian Ministry of Education and Research, 2019). However, the implementation of formal curriculum into teaching materials must be further investigated to fully understand the impact the curricula may have in this context.

Assuming nominal relations categories, the Krippendorff's alpha of 0.46 indicates that the framework by Kilhamn et al. (2021) can be difficult to use consistently. Since the units often included more than one task or example, and the task with the highest score determined the overall unit score, we expect a higher level of agreement. Thus, it should be emphasized that these results come with uncertainties. Nevertheless, our analysis shows a relatively high number of units within MP3 and MP4, and the divergence from Sweden maintains although the minimum is used as selection criteria in the coding, as shown in Figure 1. We propose more elaborate descriptions of the apparently fluent transitions from mathematics only as a context for programming towards using programming in new ways to deepen students' mathematical understanding, as the data has been differently interpreted by the

coders. We also propose a more thoroughly examination and possibly further development of the relations framework.

We performed the mathematical content classification with the purpose of investigating the connection between the relation between mathematics and programming, and mathematical content. Thus, we accepted multiple categories of content topics in the same unit, in contrast to the analysis of Nyman et al. (2024), where each unit is only associated with one mathematical content. The Swedish study shows that number sense, algebra, geometry and problem solving have been largely used as a context of mathematics, constituting 51 of the 54 MP2 units. Looking at the MP3 and MP4 units combined (totally 8), these were found in probability (4), problem solving (3) and number sense (1). Our results show that there might be an opportunity to use programming for exploring within all content topics, which perhaps is not very surprising as the 10th grade competence aim with programming is openly formulated with respect to mathematical content. Further, programming is in 9th grade explicitly connected to statistics and probability (Norwegian Ministry of Education and Research, 2019). Thus, it is not surprising that we found several units with programming connected to statistics and probability. In 14 of 22 (6 of 11 in statistics and 8 of 11 in probability) of these units programming is used to explore mathematics (MP4). Additionally, we found one MP3 unit. Several of the units that were coded as MP4 would also fulfil the requirements of MP3. In one unit, programming is for example used as a simulation tool, but the interpretation of the simulation results moves the unit from the MP3 to the MP4 category. We suggest more thoroughly investigation on possible linkages between MP3 and MP4 content, both within statistics and probability, but also in the other content topics.

The relations framework has now been used in both a Swedish and a Norwegian context. While the Norwegian and Swedish curricula may at first glance look similar, we find that the textbooks have a different realization of programming in mathematics education where the Norwegian textbooks provide larger opportunities of learning mathematical content through programming. We find this for all content areas in the Norwegian textbooks and conclude that programming can be used as a tool for exploring mathematics in all six content topics investigated.

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Towards design principles for learning environments based on theorem provers

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Undergraduate students are faced with significant challenges in learning and producing mathematical proofs. Recent research suggests that theorem provers can support this transition by enhancing students' proof abilities. This paper derives design principles for digital learning environments based on the open-source theorem prover Lean, using a design-based research approach by integrating theories from computer science and mathematics education, as well as empirical results from a study conducted on our prototype.

Keywords: Mathematical logic, undergraduate education, computer assisted instruction.

Introduction

One of the challenges students face in in university mathematics is understanding and producing mathematical proofs. Recently, theorem provers were shown to help students to develop their proof abilities (Thoma & Iannone, 2022). However, implementing them is not straightforward, especially because students have to learn how to use theorem provers. In our project, we are developing a learning environment, which aims at supporting mathematics students' learning of proofing by embedding the theorem prover Lean, in the framework of design-based research. In this paper, we derive design principles for such a learning environment by combining theory from computer science and mathematics education as well as results from a study based on our first prototype of the learning environment. Our paper is therefore a theoretical discussion paper. First, we will present theory about proof in undergraduate mathematics, theorem provers and the challenges they pose for students. Then we will draw on other work in design-based research to establish initial design principles for learning environments that include interactive theorem provers. Our aim for this contribution is then to expand the design principles with the help of the theoretical background from Bayman & Mayer (1988) and Moore (1994) together with the empirical results we collected from the initial learning environment.

Theoretical Framework

Proof in undergraduate mathematics

Undergraduate mathematics regards a narrower definition of proof than high school mathematics. At the university level mathematics is built from axioms and definitions, that can be applied to proof theorems. Students' challenges when producing proofs can be clustered into three categories (Moore 1994): (a) mathematical language and notation, (b) concept understanding, and (c) getting started on a proof, referred to here as strategic challenges. Concept understanding includes difficulties stating definitions, lacking an intuitive understanding and failing to generate examples, as well as struggling to use one's concept images for a proof or structuring a proof based on a definition.

Theorem provers

Theorem provers are software in which mathematics can be implemented in a computer-formalized language. In particular, theorem provers have a symbolic library covering a vast number of mathematical topics, which makes it possible to check the correctness of statements and in particular proofs. While theorem provers have existed for decades, they have been gaining traction in mathematics research in recent years, including work by Fields-medallists (Commelin, 2022; Tao, in press). Theorem provers were developed for research, but their features make them promising tools for teaching as well. In particular, these tools provide interactive feedback as one produces a mathematical proof and a comprehensive overview of the mathematical concepts relevant to the topic being worked on (Massot, 2021). Bartzia et al. (2022) give an overview of different theorem provers and list their educational potential. Thoma & Iannone (2022) have reported a study in which university students used more accurate and correct mathematical language, and were better at organizing and structuring proofs after engaging with the theorem prover Lean.

The interface of the open-source theorem prover Lean (https://leanprover-community.github.io/) is divided into two columns. In the left column, the user provides their input, while the right column provides feedback from the tool. Figure 1 illustrates a proof of the commutativity of the addition over the natural numbers using Lean. The example is a proof by induction over the variable ‘b’, where the user has completed the base case and is currently working on the induction step. The Lean feedback, in the right column, includes the proof state and error messages if any mistakes occur. The dark orange area of the proof state lists all the given statements, in this case, ‘a’ and ‘d’ being natural numbers along with the induction hypothesis named ‘hd’. In the light orange area, the current goal, which is the statement to be proven, is displayed. The red area shows an error message by specifying the location and type of error. Both the given statements and the goal change dynamically as the proof is typed in.

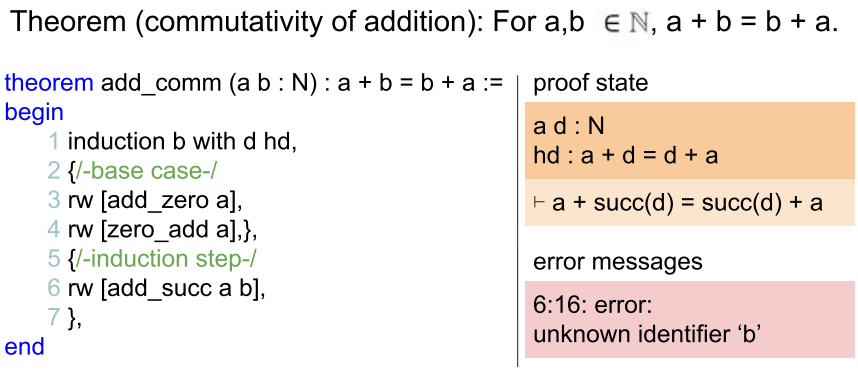

Figure 1: A proof in the theorem prover Lean

To use Lean, students must first learn the Lean language and how the program works. Hoyles (2018) discusses the dual role of digital tools as both a learning subject and a scaffold for learning mathematics, emphasizing the initial investment in learning how to use such a digital tool. Theorem provers were not developed as educational tools. They are therefore not intuitive to use without prior knowledge, and require a considerable amount of work to learn how to use them (Avigad, 2019).

The challenges of students learning Lean

To understand how students learn Lean, we use a perspective from computer science education. Bayman, & Mayer (1988) structure the knowledge needed to master a programming language into three consecutive levels: syntactic knowledge (e.g., proper use of brackets and punctuation, writing commands), conceptual knowledge (e.g., understanding variables, scopes, loops) and strategic knowledge (planning, writing and debugging code).

Iannone and Thoma (2023) have used the model of Bayman, & Mayer's (1988) to categorize students' perceptions of challenges, relying on self-reports obtained from questionnaires and interviews conducted after an 'Introduction to Proof' module that incorporated Lean. Their findings highlight students' struggles, particularly at the syntactical and conceptual levels. In particular, the challenges on these levels are intertwined with the mathematical challenges of the same nature (mathematical syntax and conceptual knowledge), as these issues are not context-specific.

Our learning environment

As learning how to work with a new digital tool requires an initial investment (Hoyles, 2018), we decided to focus the learning environment on a mathematical topic and proof strategies that students were already familiar with. The target audience of our learning environment are second year undergraduate students studying to become mathematics teachers, so we chose the natural numbers and how their properties can be derived from the Peano axioms.

As we are not aware of any existing design principles for the implementation of theorem provers, the initial design of our own learning environment was guided by design principles from other areas of mathematics education. Hein (2021), who researched on pupils verbalizing proof, concluded that students should be offered a linguistic scaffold that helps them in learning how to express mathematical proof. Wessel (2014), who studied non-native speakers learn mathematics alongside learning a language, concluded that students need exercises that link the different representations of the mathematical objects at hand.

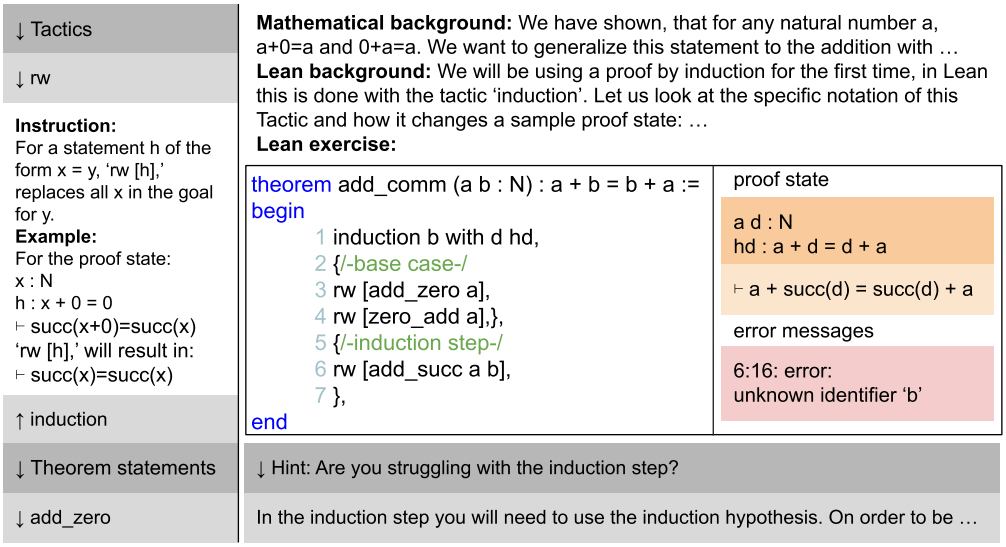

Figure 2: Digital component of the Lean-based environment

Figures 2 and 3 present a schematic depiction of the learning environment (available in German here: http://irenegarnelo.github.io/nat_zahlen_lean/), which is comprised of a digital component and written exercise sheets.

The digital component is a browser-application as depicted in figure 2. It is inspired by the 'Natural Number game' (Buzzard & Pedramfar, 2021). It incorporates exercises to be solved in Lean, featuring the aforementioned input and output columns at the centre of the figure. Additionally, the environment includes texts above the Lean exercises, introducing the mathematical and Lean-specific contents of the exercise. The column on the left contains the descriptions of all tactics and all theorems that can be applied and is meant to provide the linguistic scaffold described by Hein (2021). The environment also incorporates expandable hints to further aid students.

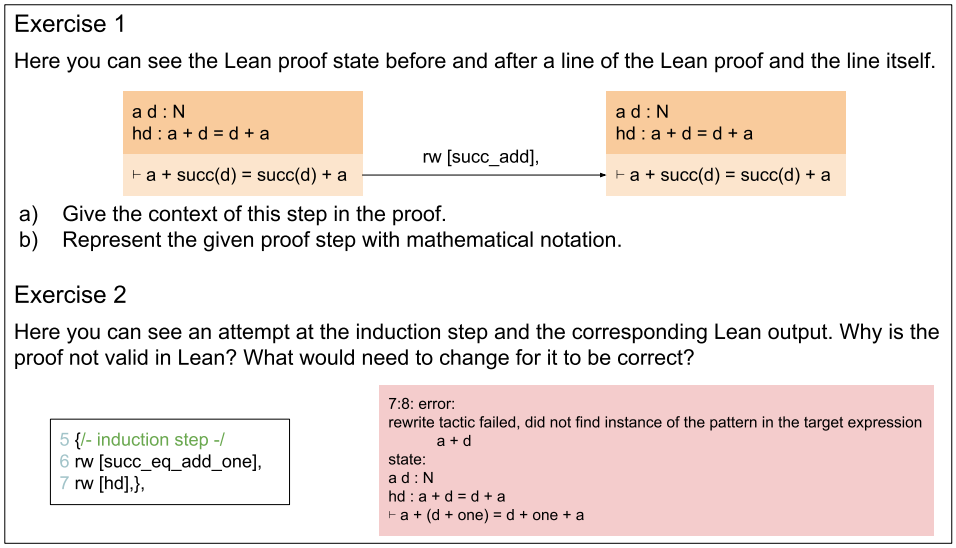

Figure 3: Written exercise sheet from the Lean-based learning environment

The written exercises, as illustrated in figure 3, aim to assist students in learning how to work with Lean, to reflect on their usage of the tool and in particular, to consider the different representations of proofs, following the design principle from Wessel (2014). The written exercises are provided on digital pdf-documents that are linked in the learning environment.

The environment was designed to be used over two 90-minute sessions, in which students are first familiarized with computer-assisted proving, theorem provers, Lean, and the learning environment through a live demonstration. The majority of the time is dedicated to students working in pairs with the learning environment. The working phase concludes with students discussing their experiences and opinions in pairs during a reflection phase.

Empirical results regarding our learning environment

In (Garnelo and Liebendörfer, in press) we presented a study conducted on the learning environment, which was implemented in the lecture 'Mathematics with a Computer' conducted at our university for preservice high-school mathematics teachers in their second year. Our goal in particular was to uncover challenges that need to be further considered in the design of the environment. We documented the work of a subgroup of students and analysed it with regards to the research question:

"At which level of programming knowledge (Bayman & Mayer, 1988) do students face challenges when working with the learning environment and how do the challenges relate to those identified in the process of learning how to proof highlighted by Moore (1994)?". The results that are relevant to the design principles we derive for this work are summarized here.

The challenges encountered when working with Lean occurred at all three levels of knowledge described by Bayman & Mayer (1988). Initially, the majority of problems were related to syntax, which gradually decreased until conceptual and then strategic problems predominated.

Students had trouble with the Lean output. Oftentimes they only looked at whether there was an error message or if the proof state had changed without actually reading them and other times they attempted to read it but fail to understand it. Students reported having trouble with the error messages in particular.

Students oftentimes skipped the written exercises. They were mostly focused on the Lean task, and once it provided the "Proof complete!" message, they moved onto the next Lean task and did not solve the exercises on the sheets. Oftentimes they did not even open the sheets, despite them being linked directly within the environment.

Developing new design principles

Providing a scaffold for the different levels of knowledge

When working with Lean students are learning multiple skills, both on an interdisciplinary level (Hoyles, 2018) and on the level of Lean itself (Bayman & Mayer, 1988). We need to choose the order in which we want students to acquire these new skills. In the first iteration, guided by Hoyles (2018), we had decided to focus on learning Lean as a tool as opposed to adding new mathematical content. (Bayman & Mayer, 1988) present a further subdivision of the skills required within learning how to use a programming language. In particular, they argued that these skills are developed in a specific order (syntax, then concepts, then strategies). In our study, we confirmed that working with Lean also leads to these results. Therefore, we propose designing a scaffold along the learning environment, which provides the levels (syntactic, conceptual and/or strategic) that the student has not acquired yet, so that students are able to focus on the level of knowledge they are currently learning. To this end, the concept of the exercises was changed. While in the initial learning environment, students always had to provide a full proof, we propose that the first exercises consist of reading, completing or correcting given proof attempts in order to focus on understanding and using the Lean syntax initially, before working on tasks related to conceptual and then strategic knowledge. Similarly, once students have mastered the Lean syntax, we recommend exercises where the strategic knowledge needed is given (e.g. in the form of a proof structure), so that students can focus on their conceptual knowledge.

Increasing students' attention to the Lean output

As seen in figure 2, the digital component of the learning environment provides a variety of elements, that students could turn their focus to. When designing such an environment we must decide which of these elements we want students to focus on and how we can achieve this. In our initial design**,** we intended to make use of the advantage of Lean providing interactive feedback as mentioned by

Avigad (2019). To this end, we set the focus on the Lean components of the environment, namely the user input and Lean output, which was achieved by making the left column with all the details on theorems and tactics, as well as the hints collapsible. The only component in the left in the foreground other than the Lean interface, was the introductory text, which was intended to be focused on by the students primarily before work on the task started. Becker et al. (2019) highlight the challenges that students face when confronted with diagnostic messages generated by compilers and interpreters, such as syntax error messages. In Lean, in addition to the error messages, the tool also provides the proof state as an output. In our study**,** we found that Lean is not an exception to the findings of Becker et al. (2019) and that the results also extended to the proof state. We propose introducing instructions and exercises targeted at increasing students' attention to the Lean output. To this end, the instructions both in class and in the digital learning environment on how to understand the output were expanded. A cheat sheet with an overview of how to interpret these outputs was created. Additionally, students received exercises whose sole focus was on reading and understanding the Lean output.

Guiding students' attention to the connection between different representations of proof

Students are faced with the challenge of learning a new language in which to express mathematics as they work on the proofs. This raises the question of whether it would be beneficial for the students to focus on how the proofs they are working on are expressed in a traditional mathematical context or with Lean. As suggested by Wessel (2014), in our first iteration we focused on students connecting these different representations of proof by introducing written exercises. These exercises were posed as a reflection after completing Lean proofs. Lutfianto et al., (2013) showed that pupils prematurely consider a task complete when they reach a mathematical solution, disregarding the intended reflection after obtaining their result. In their case, the reflection consisted in relating the mathematical results back to the context in which they were posed. In our study we also found that students disregarded the exercises that we provided as a reflection after the proof was achieved. We propose that the exercises need to be interweaved better with the Lean-tasks, as opposed to being offered as a final step after proof completion in order to guide students to the connection of different proof representations. To this end the exercises are given in the text introduction, preceding the proofs within the digital environment and are posed in a way, in which they helped the student resolve the Lean proof. For example, an exercise of this type could consist of providing the proof or a proof step in traditional mathematical language for students to translate into its Lean-representation. Similarly, one can set the focus on translating Lean language into regular mathematical language by providing students with the goal states that one will achieve throughout the proof, for them to use as a guideline to develop their mathematical strategy for the proof.

Discussion

In our work we focused on how Lean, or more generally interactive theorem provers, can be incorporated into learning environments for undergraduate students. Interactive theorem provers are complex, but they are have gained increased attention both for mathematical research and education. There are many ways in which a theorem prover could be incorporated into a classroom, which is why design principles are needed. In our work we have started to derive possible design principles from theory in mathematics and computer science education as well as from our empirical results.

The first design principle we derived was designing a scaffold which offloads the higher level of Lean knowledge, so that students can focus on attaining the lower levels of Lean knowledge first. To this end we propose introducing exercises in which students focus on parts of a proof instead of the full proof. The second design principle we derived is to increase students' attention to the Lean output by introducing instructions and exercises targeted at working with the Lean output. The final design principle we derived was to interweave the written exercises with the Lean-tasks in order to guide students' attention to the connection of the different representations of proof. To this end we propose embedding the written exercises within the introductory text preceding Lean tasks and to conceive them in a way that supports students during their Lean task as opposed to serving as a reflection conducted after the Lean task.

Apart from some small structural changes, the design principles described above translate mainly into developing new types of exercises to support the learning journey and in which order these exercises could be posed in order to accompany the different levels of knowledge. The design principles derived in this paper were applied in the development of a new version of the learning environment (available in German here: https://ireneupb.github.io/lean_umgebung/), which we hope to use to get insights on how our design principles affect the learning with the environment.

Further research in the field of theorem provers should study the effect of interactive theorem provers on learning proof. As discussed in Hoyles (2018), digital tools, and in our case Lean, play two roles in the classroom. In particular mastering Lean requires an initial effort, that is exclusively directed at Lean in the role of learning subject. We have analysed the challenges students encounter when learning how to use Lean and tried to mitigate some of them with our design principles. Nevertheless, learning the language will always require some additional effort, even with improved versions of the learning environment. By showing whether mastering challenges with a theorem prover supports learning the analogous challenges in mathematical proof, future research will show whether the initial learning effort is worthwhile.

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Programming in the primary mathematics curriculum in Norway

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This study examines the integration of programming into primary school mathematics in Norway, which the new LK20 curriculum emphasises. Through a qualitative case study across four schools, we analysed teachers' reflections through focus group interviews over three years. Findings indicate initial enthusiasm by teachers and their students. However, programming tends to be treated more as an add-on to mathematics rather than being integrated into the mathematics curriculum. Despite faithful implementation, teachers highlight challenges and a need for more knowledge. We recommend enhanced professional development, further research and ongoing curriculum development to support the integration of programming into the mathematics curriculum.

Keywords: curriculum implementation, programming, primary school mathematics.

Introduction and theoretical framework

The integration of programming into compulsory school curricula has gained significant traction, driven by a recognition of the importance of digital literacy (Bocconi et al., 2018). In Norway, these trends can be seen in the 2020 national curriculum (LK20), mandating the inclusion of programming in Mathematics, Science, Music, and Arts and crafts (Vinnervik & Bungum, 2022). There are varying approaches to integrating programming into school curricula, from having it as a separate subject to integrating it into mathematics and other school subjects (Bocconi et al., 2018). In Finland, Sweden, and Norway, programming is integrated into existing subjects (Vinnervik & Bungum, 2022).

Programming is the process that goes from identifying a problem and proposing solutions to translating these into a computer program, which is then refined and improved through testing and debugging (Sevik, 2016). Computational thinking (CT) is a broader notion including "formulating problems in a way that enables us to use a computer and other tools to help solve them" along with aspects such as data practices and modelling and simulation practices (Bocconi et al., 2016, p. 21). A growing body of research investigates how CT can be integrated with mathematics (Ye et al., 2023).

The LK20 mathematics curriculum for primary school has one competence aim each year, addressing programming. In years 1–3, students create and follow rules and step-by-step instructions during play and games, in years 4–6, they create (year 4) and program (year 5) algorithms using variables, conditions, loops, and functions (year 6). In years 6–7, they apply programming to explore geometric patterns (year 6) and tables and datasets (year 7) (Ministry of Education and Research, 2019). In the same curriculum, the process of breaking down problems and solving them systematically is presented as an aspect of CT, thus linking CT to problem-solving. However, in this study, we focus more narrowly on programming, as this was most prevalent in the interviews. Kaufmann and Stenseth (2023) pointed out that even though programming has been given a prominent position in the LK20 mathematics curriculum, its role in mathematics remains unclear.

For implementing LK20, the Directorate of Education in 2018 commissioned 13 science centres to develop an educational program to strengthen teachers' and students' foundational programming

skills with courses for teachers and schools. They also offer programming teaching resources that they link to the competence aims in LK20 explicitly addressing programming. Using programming as a tool for learning mathematics seems not to have been emphasised (Superbit, n.d.).

There are few studies addressing the preparedness of teachers for the integration of programming into the mathematics curriculum. Kravik et al. (2022) interviewed eight Norwegian teachers in grades 5–7 to investigate their understanding and teaching approaches to programming and computational thinking and their professional development needs. While teachers were generally positive about incorporating programming into mathematics and science, they had a narrow understanding of programming and limited teaching methods. Similarly, Kaufmann et al. (2023) found that prospective mathematics teachers, despite considering themselves "digital natives," had minimal experience with programming. They stressed the need for further research on how programming can contribute to deepening students' mathematical understanding. In Sweden, it was found that the adoption of programming in mathematics education was unclear and challenging for teachers, often focusing on reinforcing previously learned mathematics rather than developing new skills (Stigberg & Stigberg, 2020). Previous studies have thus shown challenges concerning the preparedness of teachers to integrate programming into mathematics education. However, how teachers adapt their understanding and approaches over time remains a crucial yet uncertain aspect of the LK20 reform.

Following Remillard & Heck (2014), we distinguish between the official and the operational curriculum where the official curriculum consists of the designated curriculum, curricular aims and objectives**,** and the content of consequential assessments, while the operational curriculum consists of the teacher-intended curriculum, the enacted curriculum and student outcome (Remillard & Heck, 2014, p. 709). The teacher-intended curriculum refers to the ways the teachers interpret the official curriculum and additional resources, as well as the decisions they make to transform it into practical instructional strategies (Remillard & Heck, 2014). It is the teacher-intended curriculum, rather than official documents themselves, that shapes how teachers facilitate student learning, and several factors are known to influence the teacher-intended, and hence the operational curriculum, including teachers' practices, beliefs, knowledge, access to resources and support and understanding of students need (Remillard & Heck, 2014, p. 714). In addition, the enacted curriculum and student outcomes contribute to additional influencing factors of the operational curriculum, such as contextual opportunities and constraints and students' attitudes and motivation. Given this, our research question is as follows: *What are primary school teachers' reflections about programming in the operational mathematics curriculum and what factors influence it?*

Methods

We use data from a qualitative multi-case study with a longitudinal design, 'EvaFag2025', investigating the implementation of the LK20 National curriculum in Norway (Burner et al., 2022) focusing on Mathematics, English, Music, and the Social Sciences. We report from the first three data collections. Four primary schools representing variations in urban or rural settings, economy and school size were included. Nine mathematics teachers (respectively 2, 3, 3 and 1 teachers from School 1, 2, 3 and 4) took part in focus group interviews lasting 60–105 minutes in 2022, 2023 and 2024. The teachers represented a variation in age and experience and were teaching across grades 1–7.

Among several aspects of the LK20 implementation, the teachers were asked open questions on how they integrate programming into mathematics, if and how they worked with it and about support and hinders in that work. For instance, an initial question could be: "Programming and computational thinking has entered the curriculum. How did you work with this?". Follow-up questions were often more specific, like: "Do you think that programming in mathematics affects students' learning in mathematics in any way?". In subsequent years, we typically started by briefly referring to the previous year and asking for updates.

For the present paper, all interview extracts related to programming in mathematics were identified and coded thematically (Braun & Clarke, 2006). Codes sharing a common meaningful unit were grouped into subcategories, which were then merged into broader categories shaped by our theoretical lenses. The resulting categories are presented and explained in the following section.

Findings and discussion

We identified four main categories and several subcategories (in italics) related to programming in the operational mathematics curriculum. The categories, while distinct, are inherently interconnected, each aligning with specific aspects of the curriculum enactment process.

Category 1: Teachers' knowledge for teaching programming in mathematics

Category 1 includes teachers reflections on their knowledge for teaching. This is known to be a factor influencing the teacher-intended curriculum (Remilard & Heck, 2014). Category 1 contains two subcategories: *Sources of knowledge about programming in mathematics* (coded 11 instances) and *Gaps in knowledge* (9 instances).

The interviewed teachers refer to three *sources of knowledge about programming in mathematics*: courses, self-study, and support from a resource person within the school staff. In the first year, some of the teachers reported attending programming courses at the Science centres. These seem to have played a role in introducing teachers to concepts and tools for teaching programming. However, much seems to rely on self-study as a significant and implicit aspect of professional development.

Teachers also mentioned the importance of resource persons within their schools, and one of the interviewed teachers had such a role. The municipality organises courses for resource persons who then share with colleagues. Other schools report on different practices, less organised at the start, counting more on interested individuals. This changed in 2023 when a group of teachers at one of the schools, focusing on digital aspects of teaching, was "organising a [course] in programming … over two afternoons [...] focusing on mathematics and science" (Teacher 1, School 2, 2023).

Despite teachers' efforts to integrate programming into mathematics and their participation in various courses, several of the teachers still *identify gaps in their knowledge and areas where they need additional support*. One teacher expressed that while they can manage technical aspects, they struggle with explaining and facilitating deeper learning of concepts like "conditions, loops and variables", stating: "I don't feel that I learn that from being able to program them myself" (Teacher 2, School 1, 2022). This reflects a need for professional development that also addresses didactical aspects of teaching programming in mathematics. This sentiment seems to have persisted and developed over time and was expressed by a teacher in a later interview: "… we're not really programmers…"

(Teacher 1, School 4, 2024). This teacher pointed to a lack of competence, rather than a lack of resources, as the root of the problem, indicating a desire for a more comprehensive and detailed understanding of programming, as well as clarity in integrating it into the curriculum, particularly regarding its connection with mathematics. On the other hand, another teacher (Teacher 1, School 1, 2024) emphasised that they do not need to know everything since students themselves should explore and find out. Kravik et al. (2023) also reported a lack of knowledge for teaching programming. After implementing and experiencing the curriculum over three years, our findings suggest that there is still a lack of knowledge and teacher uncertainty about the connections between mathematics and programming.

Category 2: Programming in the teacher-intended mathematics curriculum

Category 2 encompasses teachers' interpretations of programming within the mathematics curriculum, organised into three subcategories: *The importance of programming* (7), *Placement of programming during the school year and at levels* (7), and *Integration of programming and mathematics* (24).

Concerning the *importance of programming,* teachers generally support integrating programming into the mathematics curriculum, recognising its growing importance. One teacher remarked: "The world is heading in that direction, and programming and digital skills are currently much in focus» (Teacher 2, School 1, 2024). Despite the initial positivity, there are indications of a slight decline in motivation over time. One teacher mentioned Micro:bit and not having used it much (Teacher 2, School 3, 2024).

Concerning *placement of programming during the school year and at levels* in line with the designated curriculum, the teachers see programming as relevant across all seven grade levels of elementary school. However, there are challenges in maintaining continuity. One teacher noted: "You kind of start over again each time ..., and you might not fully manage to connect it to the goals you're supposed to link it to" (Teacher 1, School 4, 2024). The teacher highlights challenges in establishing a natural and coherent place for programming in the curriculum and in connecting it to the mathematical part of the learning objectives in the designated curriculum. On a practical level, another teacher struggles to find a natural place for programming based on the progression of the learning in the subject, instead relying on practical considerations, such as doing programming before Christmas.

Concerning the *integration of programming and mathematics*, teachers recognise the potential but do not always emphasise the integration of programming in the traditional mathematical curriculum: "There's a lot of mathematics in it ... It's not like I go around thinking, 'Oh no, now they have to see the connection to programming as well'" (Teacher 3, School 2, 2024). However, some teachers do integrate programming with traditional mathematics content, such as using the terms right, left, forward, and backwards in early grades or exploring areas and rectangles with robots: "We've used the robots to work with area and rectangles, in relation to centimetres with rulers" (Teacher 1, School 1, 2022). However, overall, there were few reflections on the integration of mathematics and programming, and it seems that programming has an unclear role in the traditional mathematics curriculum, as also remarked by Kaufmann and Stenseth (2023). The limited application of

programming in mathematics learning in the primary grades may to some extent, be a consequence of a slow, spiralling progression for programming, which contrasts with a departure from a spiral progression for traditional mathematical topics (Ministry of Education, 2019). This may also be a possible explanation for the lack of reflections on programming in problem-solving, even if this aspect is used as an argument for CT in compulsory education and mentioned in connection with problem-solving in the curriculum documents (Ministry of Education, 2019; Directorate of Education, 2019).

Category 3: Teaching programming in mathematics

Category 3 includes teachers' reflections on how they teach programming in mathematics in practice. This category is divided into three subcategories, focusing on specific aspects of enacting programming: *Tools and curricular resources* (35), *Teaching approaches* (13), and *Interdisciplinary dimensions of teaching programming* (7). All teachers place a significant emphasis on *tools and curricular resources*, a focus that persists across the first, second, and third interviews, as one teacher noted: "What I think about programming is that you need quite a few tools ... you need quite a bit of equipment to get things up and running" (Teacher 3, School 2, 2022). Tools like Micro:bit, BitBot, Arduino, Sphero Ball, Lego Robots, and Edison Robots are mentioned, in addition to block programming platforms like MakeCode and Scratch. Robots are generally associated with programming in mathematics.

Concerning resources for teaching programming, the teachers commonly use ready-made curricular resources such as the adopted textbooks or Superbit (n.d.) (with Micro:bit) that seem to have influenced several of the teachers. There is a danger that an extensive focus on a variety of gadgets, tools and elementary coding tasks may divert the focus from understanding underlying principles and in-depth learning.

In the *teaching approaches,* teachers emphasise starting with simple, hands-on programming activities without computers, focusing on planning and precision, aligning with the lower-grade competence aims in the LK20 mathematics curriculum and the adopted textbooks. Progressing from these unplugged activities to digital programming is seen as beneficial for both younger and older students. One teacher further acknowledges a contrast between traditional mathematics teaching and including robots, "…they have tasks where they need to do a certain amount in the math book, but it's a bit different when they have a robot or something in front of them because then they can actually see it moving physically" (Teacher 1, School 1, 2024).

Teachers see programming as facilitating exploration, as one teacher observed: "It doesn't take long before [the students] are out there trying something" (Teacher 1, School 4, 2022). However, there is a developing concern that while students are actively engaged in hands-on activities, this may not always lead to in-depth learning, thus recognising its limits and stagnation after a while. Also, Kravik et al. (2022) found that the teaching approaches in 2021 were limited, a challenge that seems to persist, with some teachers expressing uncertainty about how best to facilitate exploration. The Nordic approach of implementing programming in compulsory education (Bocconi et al., 2018) integrates programming into various school subjects rather than treating it as a standalone subject. As a result, programming is inherently taught *interdisciplinary,* particularly in mathematics and science

(Vinnervik & Bungum, 2022), though it also extends to creative activities in Music and Arts and crafts. Some of the mathematics teachers reported on interdisciplinary teaching efforts or teaching programming in subjects other than mathematics, mostly connected to Science.

Category 4: Programming and students

Category 4 includes teachers' reflections on the effects of programming on students, resulting in two subcategories: *Students' level of engagement and motivation* (9), and *Challenges and opportunities they experience* (8). The teachers report a high *level of engagement among students* when implementing programming tasks: "[The students] were very inspired […] When we had an activity week, they decided for programming" (Teacher 1, School 2, 2022). The teacher further described a week of programming activities, where students built cars, held competitions, and invited parents to participate.

Teachers also highlighted how programming can be particularly motivating and provide a sense of achievement for students struggling in other academic areas. One teacher observed that "students who might struggle academically in other contexts find joy in this" (Teacher 2, School 3, 2022). Most of the resources used, like Micro:bit, Bit:bot and Scratch, are rooted in international trends and designed to make knowledge accessible and engaging for young students. However, by the last interview, it was clear that the initial motivation might not be sustained over time. One teacher remarked: "[The motivation] is probably very, very high at the start, and then […] it might kind of stall a bit" (Teacher 1, School 4, 2024). This evolution in student motivation suggests that there may be a need to move beyond making programming fun and engaging. Future efforts should also focus on sustaining long-term engagement and fostering deeper understanding.

Teachers reflect on *achievements, opportunities and challenges for students*, pointing out the success many students experience in programming: "there were loops and variables, and they created a lot.... the ones growing up now are so good at these things. They pick things up so quickly" (Teacher 1, School 2, 2024). However, as one teacher acknowledges, some students struggle with programming or lack the necessary patience. Interestingly, it was also observed that students who otherwise struggle can, in some cases, find a sense of achievement in programming. These observations indicate a need for further research on how programming, as part of the mathematics curriculum, may cater to diverse learner profiles.

Contrasts in teachers' perspectives over time and between schools

The transcripts do not reveal strong longitudinal trends; however, the data suggest a gradual waning of initial enthusiasm, an evolving awareness of challenges such as facilitating in-depth learning related to programming, and an increasing desire for pedagogical knowledge among some teachers. In the first interview, teachers spoke more about knowledge and its sources, while in the final interview, one teacher, in particular, shifted focus to the types of knowledge needed, emphasising challenges and the integration of mathematics and programming. No clear differences were identified that could be attributed to the schools' characteristics, such as rural or urban location, size, or budget. Interestingly, teachers at schools with smaller budgets reported no lack of equipment for teaching programming in mathematics.

Conclusion and implications

The interviewed teachers report multiple sources of knowledge about programming, with the Superbit (n.d.) initiative as the most prevalent. Even though teachers seem to have a good basic knowledge of programming, some report a desire for deeper technical and pedagogical knowledge. Teachers view programming as a natural part of the mathematics curriculum, though it appears more as an add-on or as isolated blocks, with few examples of synergies between learning to program and learning mathematics. They often use ready-made curricular resources utilising robots, Micro:bit, and block programming. Exploration is reported as an important teaching approach, although some report lacking approaches for advancing learning beyond the fundamentals. The Nordic model of integrating programming across multiple subjects is seen as beneficial by many teachers, although mathematics is often not a central focus in their implementation of programming. Teachers report engagement, even among struggling students, although this engagement is not universal or sustained over time.

In the official LK20 curriculum, there is only a singular competence aim related to programming for each grade. The Super:bit educational program links these competence aims (for grades 5, 6 and 7) to proposed activities. While it is helpful for teachers to connect activities to the programming competence aim, this is not enough to be able to see further connections between programming to mathematics. Although there is a growing body of research on CT-based mathematics instruction (Ye et al., 2023), more research and curriculum development are needed. In the Norwegian context, one should further explore and map out connections and opportunities for using programming as a tool for learning mathematics. Furthermore, research should also focus on sustaining long-term engagement and fostering deeper understanding while catering to diverse learner profiles.

In the Nordic approach to introducing computational thinking (Bocconi et al., 2018), much of the responsibility falls on teachers. In the authors' view, insufficient efforts have been made to adequately support and prepare teachers. Professional development initiatives need to provide, as they become available, research-based examples of how programming can be integrated into the teaching of various mathematical topics, highlighting both the connections and distinctions between mathematical and computational thinking as well as facilitating sustained engagement and in-depth learning for all students. The aim should be to equip teachers with enough understanding of programming's potential in mathematics to enable them to develop their own teaching resources and strategies. Teachers should also be prepared to teach programming as an integrated component of mathematics, avoiding its perception as a mere add-on. Integrated approaches should enhance mathematics learning while also fostering the development of computational thinking.

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Relationship of computational thinking and spatial reasoning when programming educational robots

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Many studies see potential in the use of educational robots in primary school. On the one hand, the potential is seen to promote students' Computational Thinking (CT) as early as possible. At the same time, a connection between CT and spatial reasoning is often assumed. On the other hand, further research is needed into how exactly CT and spatial reasoning are used in programming processes. In a qualitative study with sixteen second grade students from Germany, this is examined in more detail. This article uses a best case to show how CT activities and aspects of spatial reasoning can be linked in a successful programming process. Consequences for teaching practice are discussed and an outlook on the next steps is given.

Keywords: Educational robots, programming, computational thinking, spatial reasoning, primary school.

Introduction

As early as 1980, Papert had the idea of promoting mathematical skills by programming robots. In this context, he used the term Computational Thinking (CT) to capture the cognitive processes involved (Papert, 1980). Today, CT is seen as an increasingly important skill for responsible movement in today's digitalised world. Wing (2006) even equates the relevance of CT with the basic skills of reading, writing and arithmetic and describes it as a fundamental skill for everyone - regardless of whether they are computer scientists or not.

Teaching spatial reasoning (SR) as interdisciplinary task is also one of the main objectives of general schooling. The sub-component of spatial orientation in particular plays a central role in everyday life. For example, it is constantly necessary to distinguish between right and left - whether driving a car, describing a route or following instructions on which side the doors open on a train. For spatial orientation, however, it is not only important to be able to distinguish between right and left, but also to be able to put oneself in the spatial situation, either mentally or physically, to be able to orient oneself. Only then can children describe routes flexibly by including the directions forwards and backwards in addition to right and left. Among others, the study by Uttal et al. (2013) show that SR can be trained.

Several studies have examined the beneficial impacts of SR and robotic interventions in educational settings (Francis et al., 2022). However, not much research has been done on this current topic. Francis et al. (2022) emphasise that future studies should investigate how exactly the programming of moving robots supports the learning of mathematics. This article therefore examines the extent to which CT and SR demands are placed on children when programming educational robots as the Blue-Bot.

Theoretical background

Framework for the capture of computational thinking

According to Wing (2006), CT is the reformulation and representation of a seemingly difficult problem in a way that makes it manageable and solvable. Wing emphasises: "Thinking like a computer scientist means more than being able to program a computer. It requires thinking at multiple levels of abstraction" (2006, p. 34). That means, students should not only use computational tools but also construct them themselves, which requires many competences. However, there are different views on which activities belong to CT (Brennan & Resnick, 2012; Selby & Woollard, 2013; Angeli et al., 2016; Büscher, 2025). Büscher (2025) subdivided CT in six activities: Reusing and Modifying, Decomposing, Testing and Evaluating, Algorithmitising, Abstracting and Generalising. As there are only three of these activities observed in the case to which this study refers to in the empirical part, only these are explained in Table 1.

Table 1: Important CT activities for this paper (following Büscher, 2025)

Decomposing	Learners break down a complex problem into smaller subproblems.
Algorithmitising	Learners plan a series of several steps.
Testing and Evaluating	Learners test their codes and change them and investigate the effect.

Framework for capturing spatial reasoning

SR includes "the capacity to relate and navigate the world and the ability to imagine objects and their movement" (Francis et al., 2022, p. 928). There are different models of SR, but this paper uses the model of Davis et al. (2015). Davis et al. (2015) identify six different sub-processes that take place during SR and thus, at least in part, during spatial orientation. They break these down into sub-processes, as shown in Table 2. To avoid the impression of conceptual scaling, Davis et al. (2015) presented the categories in their work in a circular model. For reasons of space, this paper uses a tabular representation, as this is sufficient in terms of content, but it should be noted here that the order of the aspects does not matter. In addition, Davis et al. (2015) emphasise that the individual sub-processes often do not take place independently of each other, but that there is an interplay between them. According to Davis et al., both "'mental' understanding" (2015, p. 141) and "'physical' transformations" (2015, p. 141) are necessary in SR, whereby these merge into one another and cannot be clearly assigned to the different aspects. The fact that this definition and categorisation includes both mental and physical activities, which is rarely the case (Francis et al., 2022), is important in the context of this paper as it is relies primarily on the physical reasoning tools that the children use, because only these can be observed.

Table 2: Aspects of SR (following Davis et al., 2015)

Mental understanding	Sensating	Perspective-taking, Visualising, Propriocepting, Imagining, Tactilising
	Interpreting	Diagramming, Modelling, Symmetrising, Comparing, Relating
	(De)constructing	De/re/composing, Un/re/packing, Re/arranging, Sectioning, Fitting
Physical trans-formations	Altering	Dilating/Contracting, Distorting/Morphing, Scaling, Folding, Shearing
	Moving	Sliding, Rotating, Reflecting, Balancing
	Situating	Dimension shifting, Locating, Orienting, Pathfinding, Intersecting

Related research

There are already several studies that deal with programming in elementary school. In their meta-analysis, Lye and Koh (2014) found that programming in the primary school context is primarily taught using the programming languages LOGO and Scratch, but educational robots like the Blue-Bot are also used, even at lower school levels. In the study by Caballero-Gonzalez et al. (2019), for example, first graders were asked to steer an educational robot to a specific target field on a floor mat divided into squares. They were able to show in their pre-post study that the exercises with the educational robots led to an increase in learning, for example in the activities *Testing and Evaluating* and *Algorithmitising*. Thus, they investigated the influence of the use of learning robots on learners' CT activities.

The study by Francis et al. (2022) examines the malleability of SR in robotic interventions. Francis et al. (2022) therefore also transferred the categories for SR according to Davis et al. (2015) to the robotic intervention and tested them there. They found that the *Visualising* and *Imagining* subcomponents of the *Sensating* supercategory were identified in all items of the spatial imagination test and in all robotics tasks. Additionally, elements of the *Interpreting*, *(De)constructing* and *Situating* categories could also be found in the tests and robotics tasks. The elements of *Altering* and *Moving* were also integrated in the robotics tasks. However, it was not possible to investigate whether there were positive effects in the children's spatial imagination regarding these categories, as the spatial imagination test did not cover these elements. In their previous study, in which nine to ten-year-olds were asked to program EV3 robots with LEGO Mindstorms, Francis et al. (2016) were able to observe almost all the elements of SR described by Davis et al. (2015). Only the sub-elements of *Altering* were not used by the children. However, the extent to which this robotics task shapes SR was not investigated here, as the change in orientation skills was not examined using a pre-post-test.

Further research is needed on how CT and SR are related when programming educational robots. This paper therefore addresses the following research question:

RQ: What are the demands for programming educational robots regarding CT and SR and how are they connected?

Methods

For exploring the demands, an explorative case study was carried out (Yin, 2002). It is important that the selected cases are comparable on the one hand, but diverse enough to cover as many different outcomes as possible on the other. Therefore, the sample consisted of 16 learners from Germany aged between 7 and 9 who had no previous experience of CT in the classroom and had not worked with robots. The sessions were videotaped, and relevant scenes were transcribed. A qualitative content analysis according to Kuckartz (2019) was used to evaluate the data, as it enables a structured and rule-based evaluation of the data, makes the interpretation of the data verifiable and allows the results to be transferred to other subjects. To illustrate the results, a child who reached the goal the fastest was selected below to better understand her successful process. We decided to take a closer look at Lisa's case because her example clearly shows how many CT activities and how much SR are required for this process to run successfully. This also illustrates the complexity of the programming process. In other cases where the process is not effective, only a few aspects of the CT and SR that Lisa uses are observed. Using Lisa as an example, we can see what the programming process ideally looks like, which allows us to highlight key aspects of CT and SR in programming.

Empirical insights into the case of Lisa: systematic decomposing and use of diverse aspects of spatial reasoning

In the task focused on in this paper, the learners were asked to move the robot from starting field B4 to target field A1 (Figure 1). However, the exact path was left up to them to keep the task as open as possible and to encourage the children to act independently. In contrast to most of the other learners in the survey, Lisa succeeds in programming a path from the start field to the target field on her first attempt. She programs the path marked in grey (Figure 1).

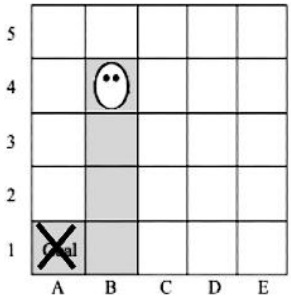


Figure 1: Path of Lisa

118.1	Lisa	So. You could once, erm twice like this. [presses “right” twice] and then it would look like this [puts her hand over the robot and turns it 180° to the right].
118.2	Lisa	Then you could one, [points to B3] two, [points to B2] three [points to B1] fields forward. One, [presses “forward”] two, [presses “forward”] three [presses “forward”].
118.3	Lisa	Erm, then this is how it stands. [puts her hand over B1 in the same way as over B4 in 118-1] … Then to the right again [points to B1] and then he goes there [points to A1]. Right [presses “right”] and then one forward [presses “forward”].
119	Teacher	Okay, try!
120	Lisa	[presses “go”] This could work. [robot moves the intended path] It worked out!

Lisa says: “you could once, erm twice like this” (#118.1) and presses the button for right-turn twice. She then holds her hand over the robot and turns it 180° to the right. In this way, she visualises the position of the robot including its orientation according to the previous programming. In terms of CT, this shows *Algorithmitising*, as she enters the first two elementary instructions. In terms of SR, it shows *Rotating* (*Moving*) when she plans and implements the rotation of the robot. *Locating* (*Situating*) also occurs when she describes how the robot would stand after the movement. This occurs together with *Visualising* (*Sensating*) when she gesturally represents the movement with her hand. It is possible that the visualisation helps here for locating and thus for further planning.

Lisa then points to the fields B3, B2 and B1 one after the other and counts from one to three. She then presses the forward-button three times and counts from one to three simultaneously (#118.2). Regarding CT, *Algorithmitising* is again evident here, as she enters the next three elementary instructions. In terms of SR, Lisa uses *Sliding* (*Moving*) to plan and implement the forward movement by pointing her hand to the fields along which the robot should move straight ahead. Here, Lisa does not use *Moving* to determine the current orientation of the robot, but to plan the forward movement.

Lisa then visualises the position of the robot again, including its orientation according to the previous programming, by aligning her hand in the same way over the field in which the robot would stand after all the previous programming, as she had done before the forward movement over the corresponding field (#118.3). She says: “Then to the right again” (#118.3) and points to field B1 again. However, before she presses the right button on the robot, she says “and then it goes there” (#118.3) and points to the target field A1. Only then does she enter the right-button and the forward-button, saying “right” (#118.3) and “and then one forward” (#118.3) at the same time as pressing the buttons. Regarding CT, this is therefore *Algorithmitising*, as she enters the last two elementary instructions together. Lisa may not need to move her hand here, as she knows that the robot has not been rotated, and the orientation of the robot has therefore not changed. To plan the entire path, she again uses *Locating* (*Situating*) when she describes how the robot would stand after the movement, which occurs together with *Visualising* (*Sensating*) by gesturally representing the orientation of robot with her hand.

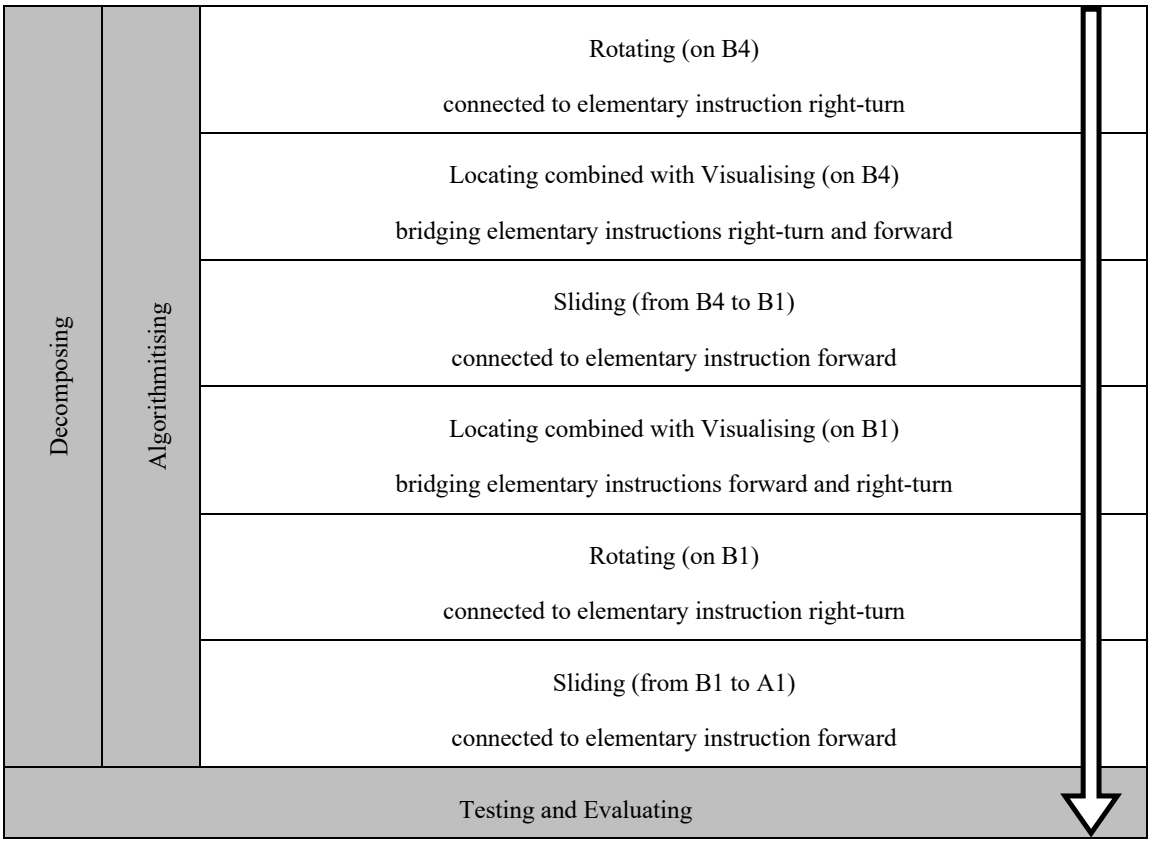
Lisa always proceeds in the same way. First, she plans the turning movements in the sense of *Rotating* (*Moving*), then the forward movements in the sense of *Sliding* (*Moving*), then the next turning movement in the sense of *Rotating* (*Moving*) and finally the last forward movement in the sense of *Sliding* (*Moving*) again. After each of these blocks - except for the last one - Lisa shows *Locating* (*Situating*) combined with *Visualising* (*Sensating*), supported by matching gestures. At the same time, Lisa breaks down the entire path into movements of one kind in one direction according to systematic *Decomposing*. This enables her to divide the planning into smaller blocks for a better overview to successively algorithmicise the path.

After the robot has travelled the path according to Lisa's instructions (#120) and arrived at the target field, she says: “And it worked.” (#120). After testing the planning, Lisa briefly evaluates it. *Testing and Evaluating* is thus shown here as a CT activity.

As can be seen from this excerpt, Lisa is able to plan the route and program the robot correctly with the help of various CT activities and spatial orientation aspects. Overall, Lisa shows the CT activities

summarised in Table 3 (grey) and, in addition to *Orienting* (*Situating*), the aspects of SR (white), whereby *Decomposing* is also part of the spatial orientation processes and here specifically *(De)constructing*. This case study illustrates how complex the process is when both areas of challenge are successfully mastered and how they can be connected.

Table 3: Activities of CT and aspects of SR shown by Lisa



Summary and Discussion

The process summarised in Table 3 shows how complex the programming of the robot is and how many sub-steps it can require, involving CT activities and aspects of SR. Lisa's example represents well that almost all categories of the Davis et al. (2015) model are applied when programming the robot. This applies to other robotic interventions according to the study by Francis et al. (2022). Only the *Altering* component, which was also the only one not founded by Francis et al. (2016) in the programming tasks in their study, did not appear in any of the tandems in this sample, including Lisa. Regarding the sub-processes, not all possible sub-components that are essential for programming the robot are covered. The other interviews have to be examined more closely for these categories.

In addition to *Testing and Evaluating*, Lisa also shows the two activities *Decomposing* and *Algorithmitising*, whereby the first is the central building block in her approach, as it enables structured planning, and particularly the use of aids for planning specific movements. Caballero-Gonzalez et al. (2019) also observed a learning growth in the *Testing and Evaluating* and *Algorithmitising* activities in their exercises with the robots.

It has been shown that, in addition to CT activities, SR can also be important for a successful programming process. The consequence of this for teaching practice is that CT activities as well as SR should be stimulated and supported in a more targeted way. However, to provide students with structured support in the future depending on their individual learning level, further research needs to be carried out with larger samples to determine which demands are placed on children with regard to CT and SR when programming educational robots, in particular investigating which orientation aids are useful in which case.

Conclusion

Studies indicate great potential for the use of educational robots in mathematics classrooms (Zhong & Xia, 2018). The use of educational robots can, for example, promote CT (Hong, 2023) or SR (González-Calero et al., 2019). However, more detailed insights into the actual learning processes are needed to describe the associated demands more precisely and to develop support approaches.

As part of this study, it was found that CT activities as well as SR is repeatedly demonstrated when programming educational robots. When programming educational robots, CT and SR seem to go hand in hand (as in Table 3). To prove whether CT activities must always be carried out before and during the application of SR for a successful process, it is meaningful to analyse more cases. Regarding the groups in this study, it can be stated that those in which both CT and SR activities were used by the students often were more successful than other groups. However, to be able to make a generalised statement, a specific look at other examples would be necessary, but this is beyond the scope of this paper.

The case presented here was an example of a very successful process. This paper has not yet investigated possible difficulties in the use of CT activities and SR. This should be researched further in the future. But the results here can provide a starting point for what the processes could look like in a best-case scenario. Support options for weaker learners can be derived from this. For example, the planning of the route with elementary instructions could be decoupled from the direct implementation on the robot to reduce complexity. And gestures could also help with spatial orientation. This potential should also be investigated in more detail in the future.

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Dragon flights, an unusual introduction of tournament

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Discovery learning is considered as an important way of learning and doing mathematics. The Pósa method is a Hungarian instructional method of mathematical guided discovery through specifically designed problem threads. This paper presents how this method makes it possible for students to discover tournaments. A tournament is a directed graph with exactly one directed edge between each two vertices. In relation to tournaments, we can present many interesting problems to students that cater to their interests, allowing them to encounter fascinating applications. Besides introducing tournaments, the key didactic goal of the problem thread is to highlight a typical misapplication of mathematical induction. This is achieved by the teacher deliberately setting a trap for the students, into which most of them fall, resulting in incorrect solutions to the problem. This creates a memorable moment for them, significantly reducing the chances of them repeating this mistake in the future.

Keywords: Discovery learning, Pósa method, tournaments, mathematical induction.

Introduction

The Pósa method, which developed on the foundation of Hungary's long-standing problem-solving traditions in mathematics education (Gosztonyi, 2016), embodies a significant advancement of mathematical didactics (Győri & Juhász, 2017) incorporating inquiry-based elements (Artigue & Blomhøj, 2013). In the late 1980s, Lajos Pósa began to work with mathematically talented children during weekend camps, and over the years, he designed a specialized method for nurturing talents. To date, more than 400 of these weekend camps have been organized for students in grades 7-12.

The teaching method, devised specifically for talented students, was examined by Juhász and colleagues through a high school experiment to see how it fares among students who are not exceptionally talented (Urbanski et al., 2022). It is worth mentioning that the Pósa method shares many similarities with the Hejny method (Hejný, 2012), while also exhibiting notable differences. A significant portion of Pósa's problems is too difficult for high school students following the standard curriculum, but many of them are accessible, exciting for these students, and can endow them with skills beneficial for their later life. In normal high schools, it is more challenging to implement the parallel running of threads, which is a key feature of the Pósa method. But the experiment demonstrated that this approach also works well. Within the constraints of the usual school environment, there is less time for discovery, so the method can be adapted with appropriate modifications for use in high schools.

The tournament is a special concept in graph theory, not part of the high school curriculum, but it arises naturally in interesting situations for students. The topic is timely for talented students in grades 9-10 and in “normal” high school in grades 11-12. It is important that students must already be familiar with basic graph theory concepts and perspectives.

Problem thread (Gosztonyi & Varga, 2023) - in the Pósa method, we refer to a longer sequence of problems over time this way - is aimed at introducing and thoroughly exploring tournaments, but we also use it to create important didactic situations (Artigue et al., 2014).

Problems and their solutions in this problem thread

The tasks of the problem thread are embedded in a story, which goes as follows:

In Fairyland, dragons transport passengers between any two cities in both directions. However, one day the dragons rebel because the inhabitants of Fairyland do not appreciate them enough. In their anger, they cancel many routes, but still ensure sufficient service, meaning that between any two cities, traffic continues in exactly one direction. This rebellion changes the previously idyllic situation, raising various questions.

P1. Does there always exist a city in Fairyland that can be reached from any other city with at most one transfer, in other words with at most two connecting flights?

At the beginning of the problem thread, we use this story, and we do not yet talk about tournaments, edges, vertices, or paths, but rather about cities, dragon flights, and transfers. As we progress through the thread, this context gradually fades into the background, and we begin to introduce and use graph theory terminology. Exactly when and how this transition occurs depends on the group; we make the switch when everyone feels comfortable using these concepts.

S1. During the solution of the task, students can apply the previously learned method of 'starting from an extreme' (Bóra & Juhász, 2022), as choosing the city where the most dragon flights arrive will be appropriate. The proof is straightforward, so we will not elaborate on it here.

Following this, students are given the opportunity to ask interesting and intriguing questions. In the Pósa method, great emphasis is placed on encouraging students to ask good questions.

For the sake of simpler terminology, let us call a city where one can reach from anywhere with at most one transfer a good city from now on.

The following questions tend to arise:

Q1. Is there always a second good city?

Q2. What can we state about the minimum number of good cities?

Q3. What happens if there are infinitely many cities?

The first question always comes up. It is very easy to answer in this form, so we ask it right away as a lightning round question. Here, too, the idea of examining an extreme situation helps, because if there is a city (C) that only has incoming flights, it is clear that there cannot be two suitable cities, as it is impossible to reach another city from C with just one transfer (in fact, it is impossible to reach it at all). We have now reached an important moment. For a significant number of students, it seems that this thread has come to an end, as there is always one city, but not necessarily two.

In discovery learning, it is worth emphasizing that mathematics is a science that seeks the boundaries between what is possible and impossible (Lützen, 2022, pp. 263-264), and often these

boundaries are more subtle than we initially imagine. This moment is not the first of its kind in our students' experience, so there is a good chance that the following question will arise:

Q1. If there is no city with only arriving dragon flights, does there always exist at least two cities that can be reached from anywhere with at most one transfer?

We set this as a task, meaning this question also becomes P2. The solution of this problem is not as simple as it was for P1.

S2. Let A be a good city. (We know that at least one good city exists.) This city can be reached from anywhere with at most one transfer. Let K be the set of cities that can be reached directly from A. According to the problem's condition, this set is not empty. In the subgraph induced by K, there is also a good city, let us call it B. In this case, B is a good city in the larger graph as well. This is because B can be reached from any vertex in K with at most 2 flights. B can be reached directly from A. From all other vertices outside K, there is a flight going to A, so we can reach B with two flights through A.

After this, interested students can work on question Q2. If we examine the solution to the previous problem more thoroughly and pay attention to the details, it turns out that there are always at least 3 good cities. It is not difficult to construct a general example where there are exactly 3 good cities. With this, we have solved this problem, as we have shown that there are always at least 3 good cities, but 4 is not necessarily guaranteed.

At this point, we ask the students to return to the original story and pose questions that go a bit beyond the initial problem.

A relatively natural question arises:
Q4. Is there always a journey that goes through every city exactly once?

Some of the students already know that in graph theory, this is called a Hamiltonian path, but from now on, we will also introduce this concept in the camp.

We usually receive two different solutions from our students:

S4a. Let us prove the statement using mathematical induction. It is clearly true for two cities. Then, let our induction hypothesis be that the statement holds for n cities. Take a graph with n+1 vertices and remove P from it. In this graph, there is a Hamiltonian path, let it be $S \to T$. If there is an edge from P to S, the path can simply be extended with P at its beginning. If there are two consecutive cities in this path such that an edge leads to P from the first, and another edge from P to the second, then P can be inserted into the path, and we are done again (see Figure 1). If this is not the case, then there is certainly an edge from T to P, thus the $S \to T$ path can be extended with P at its end.

S4b. We can apply a special form of mathematical induction, where we use the statement for all previous values of *n*. Again, choose a vertex, let this be P. Divide the remaining vertices into two sets: the first set contains the vertices that have an edge going to P, and the second set contains those to which there is an edge from P. Both sets have at most *n* vertices, so the induction hypothesis holds, meaning there is a Hamiltonian path in each set. Using this, there is also a Hamiltonian path in the entire tournament (see Figure 2).

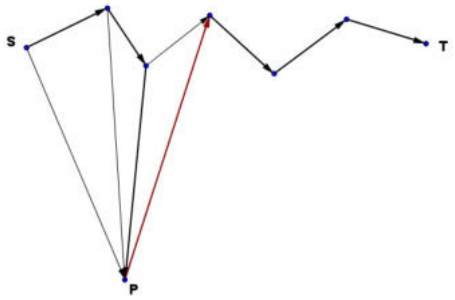

Figure 1: Proof of S4a

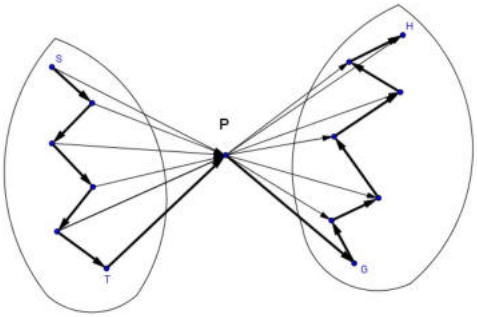

Figure 2: Proof of S4b

The following question arises naturally: instead of a journey that goes through every city, is there also a round trip that goes through every city? Our observations show that students immediately respond that there is not, since if there is a city with only outgoing edges or only incoming edges, a round trip is not possible. However, based on previous experiences, they do not stop there; the next question that arises is: what if we do not allow this, meaning there are flights departing from and arriving at every city? At this point, we introduce the concept of a Hamiltonian cycle for those who are unfamiliar with it. The question then becomes:

Q5. (P5.) Assuming that in a tournament every vertex has both an outgoing and an incoming edge, is it true that there is a Hamiltonian cycle in the tournament?

This question arises for tournaments with at least 3 vertices, and students quickly see that the first step of mathematical induction holds, as the condition is only satisfied for 3 vertices if the tournament is a directed cycle. In this case, there is indeed a Hamiltonian cycle in the tournament. Students can generally be divided into two groups based on how their reasoning continues from here. One group consists of those who thoroughly understood the previous solution (S4a.) and recognize that they can apply the same idea here. Their proof is as follows:

S5. Remove P from the tournament with n+1 vertices. In the remaining graph with n vertices, there is a Hamiltonian cycle due to the induction hypothesis. Given the conditions of the problem, there are edges going both into and out of P, so there will certainly be two consecutive vertices in the Hamiltonian cycle where one has an edge going to P, and from P, there is an edge going to the

other. This means that P can be inserted into the Hamiltonian cycle of the n-vertex graph, so there is a Hamiltonian cycle in this case as well.

The second group consists of students who found the previous reasoning difficult and don't feel a strong sense of ownership over it, so they do not think that the previous method would help here.

This second group can also be divided into two parts: one part includes students who, after this, cannot make meaningful progress on the task, with no advancement toward solving it. The other part includes students who begin examining small cases, and a significant portion of them realize they can find a counterexample, meaning the statement is not true. At this point, we ask them not to share this discovery with the others. (Of course, there is a third group as well, students who understand the solution to the first problem, recognize that this is a similar situation, but also notice that the previously used method does not work properly here. Our experience shows that there are relatively few such students.)

Figure 3: Possible proof of Q5

The most exciting moment in this thread is when students learn that the proof for S5 is incorrect. Not only is the reasoning incorrect, but the statement itself is also false. (We only tell them the first claim.) There is, in fact, a tournament with at least 3 vertices where every vertex has both outgoing and incoming edges, but there is no Hamiltonian cycle. We do not reveal the error to the students; it is their task to find it. This is a task even for those who solved the problem correctly, they also need to find the mistake in the reasoning. For those who believed the statement was true, it is also their task to determine whether only the proof is wrong or if the statement itself is false. After this, many students find the counterexample, and a possible example is shown in Figure 4.

However, it is harder to spot the error in the flawed proof, even when the counterexample is known. The mistake lies in assuming that if the condition holds for a tournament with $n+1$ vertices, it also holds when one vertex is removed. It is possible to find a tournament where every vertex has both outgoing and incoming edges, but if you remove any vertex, this condition no longer holds (the tournament in Figure 4 is an example for this).

We deliberately make this mistake in S4a and S4b as well when we discuss the problem in plenary, increasing the chances that the students will also make this mistake. Later, we explain that, formally, the proof was not correct there either, but it can be easily fixed, because when we remove

a vertex, the original condition is still obviously satisfied. Thus, with this correction, those proofs are correct, because we only need to check that the reduced graph is still a tournament.

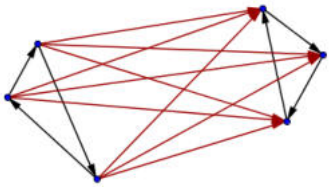

Figure 4: Tournament without Hamiltonian cycle

In this case, the teacher deliberately led the students into a trap. The intention, of course, is not to make the students feel bad. The teacher explicitly points out that this was intentional and done for their benefit. This situation vividly illustrates the importance of the step that was overlooked in the flawed proof. While it often doesn't cause problems, as they have seen before, in certain cases it does. This approach is not unique in the Pósa method. It is crucial that students feel free to think openly about the problems and understand that it is okay to make mistakes. Errors and mistakes can be used as springboards for explorations. (Borasi, 1987) They should also know that sometimes more can be learned from an incorrect solution than from a correct one, because it draws attention to a detail that might not even be mentioned during the discussion of the correct solution.

After the insightful discussion, we ask the students to try to find a condition – preferably one that is relatively easy to verify or at least elegantly formulated – that guarantees the existence of a Hamiltonian cycle in the tournament.

This is a difficult task, and due to its unusual nature, those who typically excel at problem-solving may not necessarily be the ones to find the right condition. The simply stated condition (though not easy to verify) is that it should not be possible to partition the vertices of the tournament into two disjoint sets such that all edges between the two sets have the same direction. We call such a partition a 'malicious division.' In the tournament shown in Figure 4, the vertices can be divided into two 3-element sets, and the direction of the red edges is the same, meaning this is a malicious division of the tournament.

P6. Prove that if there is no malicious division in a tournament, then it has a Hamiltonian cycle.

We will not go into the solution to this question here. The solution is not too difficult, but given the previous tasks, it cannot yet be considered a practice problem.

From the very first opportunity students have to ask questions, it frequently arises whether we can make any claims if we replace a finite tournament with an infinite one.

Q7. Is there always a good city in an infinite tournament?

S7. This task is very useful and, among other things, once again demonstrates how difficult it is to understand the concept of infinity. The answer is no, it's possible to find a counterexample. The solution is particularly simple in the sense that every student has 'seen' this structure before, but it is quite challenging to recognize that this structure can be thought of as an infinite tournament.

Consider the vertices of the graph as the natural numbers, with edges pointing to the greater number between any two vertices. After presenting this example, it's worth asking students to think about why this is a good counterexample and why the proof found for the finite case does not work here. This helps them gain a deeper understanding of infinity and the previous proof.

After solving P6 and answering Q7 the following question often arises:

Q8. If there is no malicious division in an infinite tournament, is there a good city?

Surprisingly, even this seemingly strong condition is not enough, as a counterexample can be found in this case as well. Those who have progressed this far in the thread often succeed in finding the counterexample, as they now understand the situation and can create a counterexample by slightly modifying the one seen in the previous solution.

Students who like this topic raise other interesting questions as well. These questions often arise:
Q9. For every n, does there exist a tournament where every city is a good city?
It is easy to see that for 2 and 4, such a tournament does not exist. For odd values of n, it is not too difficult to construct one. For even values of n greater than 4, the task becomes quite challenging.

Q10. Let P(n) be the probability that in an n-vertex tournament, every city is a good city. (The direction of each edge in the complete graph is determined by the flip of a fair coin.) As n approaches infinity, does the sequence P(n) have a limit, and if so, what is it?

This question also comes up regularly, although only one or two students in each group usually articulate it. Surprisingly, the limit is 1, meaning that in large tournaments, it is almost certain that every city will be a good city (Saile & Suksompong, 2020).

Conclusion

Many years of experience show that this thread is very popular among students. After mastering the basic concepts of graph theory, they have no difficulty handling the concept of tournament. Many students immediately see that tournament perfectly describes a single-round robin tournament in many sports, making the concept seem useful to them. Through problems related to good cities, students become familiar with the fundamental properties of tournaments, so when they encounter significantly more difficult questions related to Hamiltonian paths and Hamiltonian cycles, they are already comfortable using the concept. Years of experience show that those who fall into the trap when trying to prove the existence of a Hamiltonian cycle have a memorable experience, and they remember both the statement, the counterexample, and the faulty reasoning years later. These students are careful to ensure that all logical elements of their proof by mathematical induction are correct. Those who do not immediately recognize the error gain an even deeper understanding of how mathematical induction works.

Acknowledgment

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Identifying isomorphism in combinatorial tasks

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The goal of this paper is to describe how 6th-grade pupils (aged 11-12) solved combinatorial tasks with a focus on how they identified isomorphisms between C(5,2) and C(5,3) type problems. Through video-recorded sessions where pupils solved a series of isomorphic tasks, the research showed that pupils could recognise isomorphism between tasks even when this isomorphism was not easily apparent. This awareness emerged gradually. Key factors included group discussions, moderated discussions, an iterative solving process, and a series of isomorphic tasks. The study introduces an approach to support isomorphism perception as a strategy, suggesting that fostering this skill can enhance mathematical reasoning.

Keywords: Combinatorics, isomorphism, lower secondary school pupils.

Introduction

Despite some reports of success indicating that even young children can exhibit strong combinatorial thinking (English, 1991; Maher et al., 2011), most of the research suggests that pupils generally face significant challenges in the domain of combinatorics (Hadar & Hadass, 1981; Fischbein & Gazit, 1988; Batanero et al., 1997; Eizenberg & Zaslavsky, 2004; Maher et al., 2011; Lockwood & Gibson, 2016). Although problems presented in these studies are often isomorphic, pupils frequently fail to recognise this and instead apply different strategies to solve them (Batanero et al., 1997; English, 2005; Janáčková & Janáček, 2006; Maher et al., 2011). This research aims to trace how lower secondary school pupils aged 11–12 acquire and expand their mathematical reasoning skills by solving C(5,2) type problems. While isomorphism is apparent in some word problems, it remains hidden or difficult to recognise among others. This paper reports how pupils identified isomorphism between problem types C(5,2) and C(5,3).

Literature review

Pupils at all levels face problems even when solving elementary combinatorial problems (Hadar & Hadass, 1981; Fischbein & Gazit, 1988; Batanero et al., 1997; Eizenberg & Zaslavsky, 2004; Maher et al., 2011; Lockwood & Gibson, 2016). Among them is an inability to see two isomorphic combinatorial problems as equivalent (Batanero et al., 1997; English, 2005; Janáčková & Janáček, 2006; Maher et al., 2011).

In mathematical cognition, Greer and Harel (1998) perceive isomorphism as elements within mental representations that individuals construct while assimilating and addressing specific situations. In this context, they introduce three fundamental models:

1. Surface-level isomorphism involves mapping a given problem onto an easier one, employing surface considerations, such as using "friendly" numbers or specific formulas, without necessarily grasping the deeper mathematical structure.

2. Deep isomorphism emphasises a higher quality of mapping formation. Under this model, an individual with advanced reasoning forms mappings beyond surface-level considerations.

3. Mediated isomorphism introduces S as a mediator in constructing an isomorphism between two situations, S1 and S2. This model involves seeing S1 and S2 as specific cases of S, establishing an isomorphism between them through the intermediary S.

An example of isomorphism from the combinatorics field is found in Martino and Maher's (1999) research. Brandon, the 4th grader, solved the tower problem tasking pupils to construct as many towers as possible of height four using cubes in two available colours, and the pizza problem tasking pupils to enumerate all possible pizzas allowing the customer to choose from four ingredients. He proceeded to establish a one-to-one correspondence between each pizza and its analogous tower of height four (Figure 1), explaining how the red cubes in each tower correlated with the "zeros" and the yellow cubes with the "ones" on his pizza chart. He matched each of the sixteen towers to the corresponding sixteen pizzas.

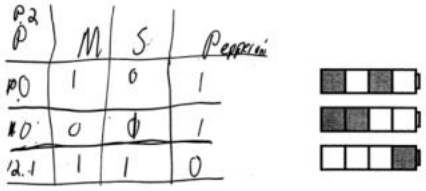


Figure 1: Brandon's one-to-one correspondence between towers and pizzas (Martino & Maher, 1999)

Research questions and methods

This study investigates the effects of engaging 6th-grade pupils in solving multiple tasks of type C(5,2). It addresses the following research questions:

1. Can pupils identify isomorphism between two tasks even when they are not immediately apparent due to differences in numbers or objects?

2. What processes and supports help pupils recognize isomorphisms among combinatorial tasks?

Pupils from 6th grade were selected in the first half of the school year to increase the likelihood that they had not yet encountered similar combinatorial tasks in their mathematics classes. There were no other characteristics required among pupils to be selected. The researcher had not encountered selected pupils before. Pupils split into groups of two to four according to their time possibilities. The experiment was conducted at their school after regular teaching hours. It involved three to four 90-minute sessions during which the pupils progressively solved 24 tasks. Pupils always received a paper with one task, solved it and shared the results and thoughts. Among tasks, 22 were part of one group of isomorphic tasks, while the remaining two belonged to another group. All of them led to the same result - 10. The experimenter did not reveal or mention the possibility of isomorphism. Since a pupil showed some isomorphism between tasks, the experimenter sometimes drew the pupil's focus to possible isomorphism among other tasks. Pupil's discussions and individual explanations of the solutions were video recorded. Data analysis was done by the author alone using the method of decomposing the solving process of each word problem into three parts:

1) Understanding the problem, 2) Grasping the task, and 3) Problem-solving strategy[1]. This paper focuses on analysing the part with a group of three boys — Evan, William, and Jay. During the first session, the boys solved five isomorphic tasks, including Towers, Stones and Cups relevant to this paper (Table 1). Two days later, during the second session, they solved another six tasks, including the Lines and Triangles tasks, which are being focused on for this paper (Table 1).

Table 1: Series of isomorphic tasks relevant to this paper

Towers (2nd task)	We build towers from 5 cubes as shown in the picture. Two cubes are coloured blue, the rest grey. How many different towers can be built?
Stones (4th task)	In the table, the number 102 is formed using 3 stones. How many different numbers can be created using 3 stones?
Cups (5th task)	Tom has two identical cup stickers and five differently coloured cups on which he can place them. He does not want to place more than one sticker on the cup. How many ways can he place the two stickers on the cups?
Lines (9th task)	A regular pentagon *ABCDE* is given. How many different lines are formed by its vertices?
Triangles (10th task)	A regular pentagon *ABCDE* is given. How many different triangles are formed by its vertices?

Results

The first five tasks analyses are described in detail in Kuřík Sukniak (2024), for this paper, a summary of tasks Towers, Stones and Cups follows. When solving the Towers task, boys manipulated with cubes. Jay and William recorded towers on paper as horizontal rectangles. Evan stored individual cases in his mind. Jay combined several methods to find all the cases. William used a "looking for what is missing"[2] strategy, reaching 9 cases. Evan used a two-step organisational principle to identify all the cases and his insight into symmetry led him to find the 10th missing case of William's solution. In the Stones task, Jay used the table to record individual cases. He gradually shifted the stones and found 11 cases, missing the 1-1-1 combination and repeating 2 cases. William grasped the task using numerical values and his strategy was "looking for what's missing". He found 10 cases, counting the case 300 twice and overlooking the case 102, which was the number from the assignment thought. Evan quickly grasped the task without making notes, found the missing 1-1-1 case from Jay's solution, and noticed symmetry in the arrangement of stones. In the Cups task, William recognised an isomorphism between this and the Towers task and solved it using this conceptual isomorphism. Jay identified an isomorphism through the coding system he used when solving the Desk task. Evan probably suspected the isomorphism at a conceptual level and verified it processually.

1 A method I have used several times under the guidance of Professor Hejný.

2 It involves identifying possibilities by focusing on potential cases that have not yet been explicitly listed, relying on intuition or observation rather than a formal or structured system of enumeration.

The following part focuses on an episode when the group of boys approached the Lines and the Triangles tasks, decomposing their solving process as mentioned in the section above. Jay: 1) Jay misunderstood the Line task. 2) He drew a pentagon on a whiteboard, and roughly labelled the vertices. He assigned each vertex two half-lines and labelled these with numbers (Figure 2a). 3) He wrote a pair of numbers for each vertex in turn (1 and 2, 3 and 4 ... 9 and 10).

Other students had no comments. I decided to refine the task. I drew the five vertices of the pentagon, labelled them A through E, and commented that every time two vertices are connected, they form a line. I drew line AB as an example (Figure 2b). Then I asked again how many lines like this were there.

Jay found the task similar and said that the solution was still 10. Evan asked if lines AB and BA were the same. I confirmed. I asked again about Jay's solution. Jay explained in his own words that each vertex is always formed by a pair of lines (half-lines determined by adjacent vertices) and gave a concrete example – vertex C is formed by lines BC and DC (Figure 2c).

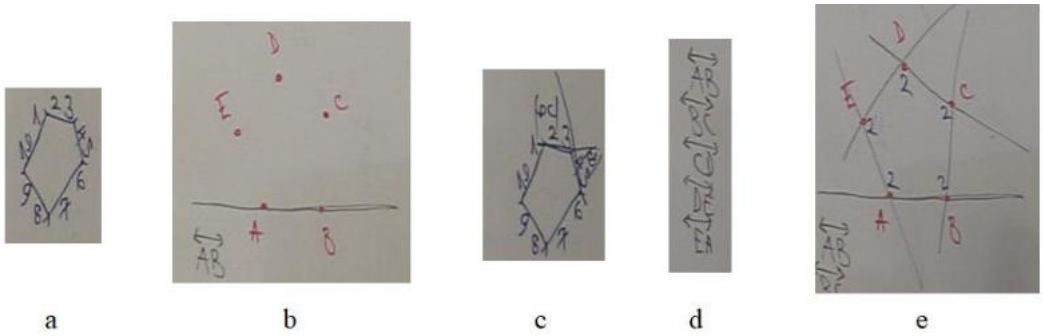


Figure 2: Process of solving the Lines task

Evan: 1) He understood the Lines task after additional input from the experimenter. 2) He used the picture of the five points $A, B, C, D,$ and E. 3) Successively read pairs of adjacent vertices of a pentagon and added the lines BC, CD, DE, and EA to the line AB (Figure 2d).

Jay did not proceed with Evan's alternative solution. The solution offered by Evan made sense to him, but it still seemed to him that five lines were not enough, because each point has two lines in it, there are five points, so there must have been ten lines. However, he could not yet answer where the other five lines are.

William: 1) He didn't understand the Lines task. 2) He numbered the vertices and the sides of the pentagon and the sum of 5+5 gave him the desired ten. He laughed when presenting this idea and said it was just an idea.

Jay verbalised the idea that each vertex was formed by two lines again. He drew twos next to each vertex (Figure 2e) and argued that there must have been ten lines but he didn't know where they were. I suggested moving on to the next task with the possibility of returning to the task with the lines if someone thinks of something.

The boys started to solve the Triangles task. None of them wrote or drew the triangles on their paper. Jay came to the whiteboard to present his solution.

Jay: 1) He understood the Triangles task. 2) He used the triplet of letters (vertices) to list triangles. 3) Even though he said there was no system in his solution the first five triangles were based on the principle of rotation: ABC, BCD, CDE, DEA, EAB. Then Jay stopped and looked for another principle. He chose two adjacent points and looked for a third one. He had already assigned point C to AB (1st solution), so he added point E to AB and wrote ABE. Neither he nor the others realised this was the same triangle already written down the last time (EAB). Then he added D to AB as the next point and wrote ABD. He moved to points BC and looked for a third point. He found E and wrote BCE. He seemed to find the remaining two triangles CEA and DBE using the "looking for what is missing" strategy.

Then William offered him one more triangle DAC and Jay listed it on a whiteboard.

William and Evan: 1) They understood the Triangles task. 2) They used triplet letters (vertices) to list triangles. 3) They probably used the "looking for what is missing" strategy.

The boys found it suspicious that the result was now eleven. In previous tasks, the result was always ten. The list of triangles passed their eyes, but they could not see whether it was complete. I asked if there was some system in the notation that would allow us to be confident, we had all the triangles. Jay said there was no system, he was looking for what had not yet been written down.

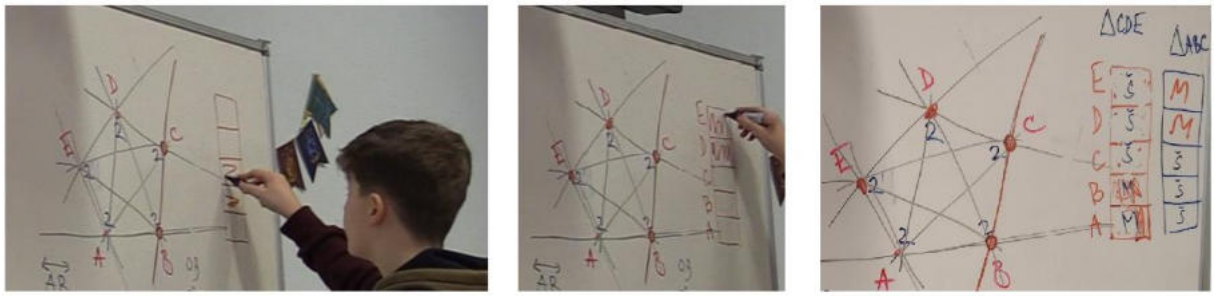
After a while, Evan came up with an idea and pointed to point A, on the whiteboard, which he connected to points B, C, D, E. He counted 1,2,3,4. Then he pointed to point B which he connected to three points C, D, E, and counted 5, 6, 7. Then he pointed to point C and connected it with points D and E. He counted 8, 9. Finally, he connected D to E and said 10. I asked if the boys understood what Evan said. Jay said he did not understand it much. I added that I did not get how it was related to the triangles. Evan himself said he did not understand it either and went to sit down. I asked Evan if he could describe what he showed us, but he could no longer answer.

I navigated the conversation to the Triangles task, but the group did not get further. The task remained with 11 solutions without certainty whether it was everything or whether there was something twice.

I then asked the boys to look again at the Lines task. Now they could see other vertex connections. Jay went to the whiteboard and added lines DB and AC. Jay stated that this was all, but immediately Evan dictated line EB and William line DA, and again Evan line CE. They were happy that they already found 10 lines.

Pupils have already found isomorphisms between some of the previous tasks. I asked whether the Lines or the Triangles task also seemed the same as some of the tasks they had already solved. William speculated the Lines task seemed the same as the Cups or Towers task. He pointed out that all three tasks had numbers 2 and 5. However, he was not sure if this connection was enough. Then Evan came to the whiteboard and revealed the repeating triangle ABE. The number of solutions of the Triangles task was since this point also ten. That made the boys happy. I turned their attention again to the similarities of the tasks. I took William's hypothesis that the Lines task seemed the same as the Towers task and, as in previous cases, challenged the boys to defend this potential similarity. I chose a particular line BC, drew a tower of five cubes next to it, and asked William if he could somehow explain the connection. William assign vertices to the cubes: the cube on the 1st floor

represented vertex A, the cube on the 2nd floor represented vertex B, etc. He coloured the cubes on the 2nd and 3rd floors (Figure 3a). He added another example and associated the DE line to the tower with the coloured 4th and 5th floors (Figure 3b). Evan and Jay confirmed that this made sense.



a b c

Figure 3: Process of uncovering isomorphism

I asked if they could connect the Triangles task to some of the tasks they have already solved. William suggested the task was the same as the Stones task because there were three stones and now, we had three letters (pointing to the names of triangles). He tried to find a specific connection but was unsuccessful. About two minutes later William suggested the Triangles task is the same as the Towers task. He assigned a tower to triangle ABC, where he coloured the first three floors, and then created a tower for triangle CDE, where he coloured the top three floors. Jay argued that for the triangle CDE floors 3, 4, and 5 should be empty and floors 1 and 2 should be colored. I recalled that in the original task with the towers, there were two blue and three grey cubes, but since they didn't want to colour everything, they coloured only the two blue cubes and left the grey cubes uncoloured. I asked if they could connect the triangles and towers of the 3 grey and 2 blue cubes. I wrote Š in the blank boxes (3rd, 4th, and 5th floor) as grey (šedá) and M in the coloured ones (1st and 2nd floor) as blue (modrá) and asked which triangle this tower corresponded to. Jay answered CDE (Figure 3c). I asked if we could rewrite each triangle into the towers using this system. William was not sure if this would always work. Jay and Evan, both thought it would always work this way. William came to the board and drew into the tower what a triangle ABC looks like (Figure 3c). I was ready to move on to the next task and if necessary, return to this one, when William declared another thought. He described that blue cubes from towers represent the lines and grey cubes represent triangles.

In the described session, the pupils identified isomorphisms between given tasks. After moderated discussion and further exploration, the pupils started drawing parallels between the Lines and Towers tasks, later between the Triangles and Towers tasks, and finally between the Lines and the Triangles tasks. The process by which pupils recognised isomorphisms included re-examining previous tasks, collaborative discussion, and identifying shared characteristics. William, for example, began to suspect similarities based on the recurring numbers 2 and 5 and later successfully made connections between the Lines and the Triangles tasks by relating them to the Towers task. The pupils needed multiple steps and cues to finally recognise the isomorphism, such as when Evan re-examined the Triangles task and corrected the initial error of counting eleven triangles, realising that one triangle was counted twice, or as Jay after reviewing the Lines task again realised that also non-adjacent vertices of pentagon can form a line. The pupils frequently discussed their thought processes with

each other. The experimenter asked for explanations of pupils' problem-solving processes, moderated the sharing of ideas to help focus on the tasks, and gave explanations of the assignments when needed. In the *Lines* task, for example, the researcher clarified task instructions and provided concrete examples to help pupils resolve misunderstandings. While the researcher avoided explicitly revealing the possibility of isomorphisms, she occasionally prompted pupils to consider connections between tasks[3]. The researcher, for example, encouraged William to defend his hypothesis that the *Lines* task was similar to the *Towers* task, leading him to map vertexes to cubes.

Discussion and conclusion

This experiment explored how 6th-grade pupils identified isomorphisms between C(5,2) and C(5,3) task types. A unique feature of this experiment was that all tasks shared the same numerical result—10, which noticeably influenced pupils' reasoning. Findings revealed pupils could recognise similar mathematical structures in the Lines, Towers, and Triangles tasks but required collaborative problem-solving, moderated discussion, and task re-examination. It aligns with previous research by Batanero et al. (1997), Janáčková and Janáček (2006), and Maher et al. (2011), suggesting pupils often miss deeper structures when solving by themselves without collaboration. Pupils compared characteristics from earlier tasks to identify isomorphisms. For example, William connected the Lines and the Towers tasks by mapping similarities after realising both involved combinations of two elements. Martino and Maher (1999) and Maher et al. (2011) described a similar iterative problem-solving approach, where pupils used analogous reasoning to solve combinatorial problems. Tasks Lines and Triangles were recognised as isomorphic thanks to the Towers task that functioned here as a mediator. According to the theory of Greer and Harel (1998), mediated isomorphism describes a mediator as some general case. However, the Tower task does not fit this category, which could lead to the refinement of this theory. This study highlights that students do not always immediately recognise isomorphisms but can develop this awareness over time through repeated exposure to similar tasks. The paper illustrates how the iterative nature of problem-solving, combined with moderated discussions, gradually led to students recognising isomorphic structures.

The small sample size and dependence on specific types of tasks limit the generalisability of this study. Future research focuses on the perception of isomorphism among other tasks still of this type and observes other pupils in this age group. Further research could observe how different types of combinatorial tasks or age groups affect the recognition of isomorphisms.

In conclusion, promoting isomorphism recognition through moderated discussion and peer collaboration improves pupils' combinatorial reasoning. Developing these skills can contribute to mathematical thinking in a broader context.

3 The researcher asked for these possible connections even when the two tasks were not isomorphic.

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Generalizing in combinatorial problem solving: Processes-based or numerical-based approaches

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In this paper, we present two cases in which undergraduate students engaged in generalizing activity as they solved combinatorial problems. We note a contrast in what the students generalized (numerical regularity versus process), and we frame these distinctions in terms of Harel's (2008) result pattern generalization versus processes pattern generalization, and in terms of Lockwood's (2013) model of students' combinatorial thinking. We intend these cases to facilitate discussion about what kinds of patterning and generalizing activity is desirable in combinatorics – whether numerical-based pattern generalizations are sufficient, or whether we desire students to engage in patterning and generalizing activity that is based in processes or other aspects of combinatorial thinking. We also connect these ideas to technological advances in generating sets of outcomes with the computer.

Keywords: Combinatorics, Discrete mathematics, Generalization, Counting

Introduction and Motivation

Generalization is a fundamental aspect of mathematics (e.g., Ellis 2007), and students often engage in searching for and identifying patterns as they seek to solve problems involving combinatorial enumeration (Maher et al., 2011; Reed & Lockwood, 2021). In combinatorics, students might employ qualitatively different generalizations, and the content of the generalizations may have an impact on the students' combinatorial thinking and activity. In this paper, we intend to raise the question for readers and participants about what the goal of generalizing in combinatorics can and should be. On the one hand, we see a case in which a student successfully solves a problem in combinatorics, engaging in activity that generalizes a counting process. On the other hand, we see a case in which a student (also successfully) generalizes to solve a problem, but the generalizing activity is centred around numerical regularity. We want to raise the question of what the aims of engagement with generalization and patterning should be in combinatorics and discrete mathematics education. We seek to answer the research question: *When solving counting problems, what kinds of generalizations do students make and use, and what does this imply for the teaching and learning of combinatorics?*

Relevant Literature and Theoretical Perspectives

A Model of Students' Combinatorial Thinking

We frame our findings in this paper on Lockwood's (2013) model of combinatorial thinking. When solving counting problems, students might attend to *formulas/expressions, counting processes,* and *sets of outcomes*. Formulas/expressions are expressions that involve numbers, operations, or variables that yield the cardinality of some outcome set. Counting processes are procedures of enumeration (the procedures students engage in as they solve the problem), and set of outcomes are the set of elements being enumerated by a counting problem. We focus on the ways that students drew upon

(or did not draw upon) these components while solving counting problems. Aligned with Lockwood's model is the notion of a set-oriented perspective (Lockwood, 2014), which is a way of thinking about the mental act of solving counting problems; this construct builds upon Lockwood's model. Other studies have made the case for the value of focusing on sets of outcomes and a set-oriented perspective and have leveraged the model as a way of framing students' combinatorial thinking (e.g. Medova et al., 2020; Soto et al., 2022; Wasserman & Galarza, 2019).

Generalizing with Numerical and Process-Oriented Foci

Students often engage in patterning and generalizing activity as they solve counting problems. In their longitudinal study, Maher et al. (2011) report on students' noticing of patterns across problems and representations as they solve many types of combinatorial problems over the years. In light of such work, we highlight a distinction reflected in students' engagement with counting problems that relates to the focus of their combinatorial activity. In solving counting problems, one may emphasize the cardinality of the sets, and may further emphasize or explore the numerical properties, relationships, or patterns in those cardinalities. For example, Lockwood et al., (2015) reported students creating lists of outcomes among several small versions of a problem and noting numerical patterns among them. This suggests what we call a *numerical focus* as they engaged with generalizing activity. In contrast, students may focus not on numerical patterns, but rather on the counting processes that they employ. For instance, students in Reed and Lockwood (2021) generalized based on processes they employed rather than on numerical patterns and regularity. This distinction between focusing on numerical patterns versus *regularity in processes* is one way we analyzed and interpreted our data.

Related to this distinction, we also consider a distinction between Result Pattern Generalization (RPG) and Process Pattern Generalization (PPG) (Harel, 2008). Harel describes RPG as a way of thinking in which one attends solely to regularities in the result. For example, determining the number of options for flipping a coin n times, someone may notice that the first few cases are 2, 4, 8, 16, etc., and they may suggest the number of options for n flips is 2^n because they notice the numerical pattern of powers of two. RPG is often the type of generalization in which the identification of a pattern is based on empirical evidence alone. In contrast, PPG entails attending to regularity in a process – in our case, a counting process. To extend the above example of coin flips, someone might use PPG to determine that there is a process of doubling to justify the expression of 2^n, noting that by appending an H or a T to the previous sequence, one can double the previous total. Here, the generalization is based not strictly on a numerical pattern, but on regularity observed in the counting process.

Methods

The data presented in this paper comes from a series of task-based clinical interviews (Hunting, 1997) conducted with nine students at a large university in the United States. The interviews were part of a study exploring students' uses of equivalence in counting. We present results from interviews with Ewan and Brian (pseudonyms), two undergraduate students who had taken or were taking discrete mathematics at the time of the interview. They thus had exposure to counting problems in these discrete mathematics courses, although we do not have further information on the specific instruction they received in these courses. In the United States, combinatorics is not consistently part of the high school curriculum. Ewan and Brian majored in computer science and mathematics, respectively.

The students solved various counting problems on an iPad and described their work. During and after their problem-solving activity, the interviewer frequently asked questions to clarify aspects of their thinking and activity. If the students were unsure of a solution, or if the students were unable to proceed after some initial work on the problem, the interviewer often encouraged them to try a smaller-sized example of the same problem (we call such problems smaller-sized problems or smaller-sized instantiations of a problem). In an effort to provide opportunities for students to engage with sets of outcomes, the interviewer would at times suggest exploration of a pre-generated outcome set (of a smaller instantiation). For instance, at times the interviewer provided a pre-generated set of arrangements of 1, 2, 3, and 4, and the student could use the GoodNotes app on an iPad to select and drag outcomes on the screen, creating their own organizations of the provided outcome set.

To identify the ways in which students engaged with patterning activity, the second author reviewed all of the interviews and recorded episodes in which students specifically engaged in patterning activity in their problem-solving process. This researcher articulated broad themes and categories for these kinds of engagements, drawing upon a numerical versus process-oriented distinction. The first author then also reviewed these episodes and themes. Together, via discussion, the first and second authors chose representative episodes to highlight illustrative distinctions among students' engagement with processes and numerical regularity in their solving of counting problems.

For this paper, because of space constraints, we limit ourselves to presenting just two contrasting cases of students' generalizing and patterning activity. We seek to offer points of contrast in students' generalizing activity to facilitate discussion of what kinds of generalizing might be beneficial for students and useful to emphasize in research and instruction in combinatorics education.

Results

We provide two contrasting cases in which students produced patterns as a result of their work with sets of outcomes. We aim to demonstrate a qualitative difference in the kinds of patterning students use in solving these combinatorial problems, and to raise questions about possible limitations and affordances of different approaches.

First, we examine the case of Ewan in solving the Committees problem: *There are 8 females and 10 males. How many different committees of 6 people can you form if there needs to be more women than men on the committee?* On this problem, Ewan began by noting, "Is this addition or multiplication?" Ewan wrote out some expressions – writing specifically C(8,4), C(8,5), and C(8,6) and C(10,2), C(10,1), and C(10,0), as seen in the top of Figure 1; he stated that these expressions represent choosing 4, 5, or 6 from the 8 women and 2, 1, or 0 from the 10 men.

However, Ewan did not write operations and was unsure of whether he should add or multiply to combine these terms. In thinking more, Ewan did see that the "cases corresponded" in some way, so that C(8,4) and C(10,2) would be connected. That is, he understood these expressions in terms of what they did combinatorially – such as selecting 4 females and selecting 2 males. The rest of Figure 1 indicates that he wrote some specific outcomes in reasoning about this process and these expressions. As was typical in our interventions, we then asked Ewan to consider a smaller-sized

instantiation of the Committees problem: *There are 3 females and 3 males. How many different committees of 3 people can you form if there needs to be more women than men on the committee?*

Figure 1: Ewan's initial work on the original Committees problem

Ewan noted immediately that there were two cases, either picking 3 or 2 women. He solved this correctly by writing C(3,3)+C(3,2)*C(3,1), as seen in Figure 2a. He recognized that C(3,3) was the case of selecting 3 women, and as he multiplied the C(3,2)*C(3,1) he noted there were "three possible choices for a different male" that could go with the choice of two females. We interpret that he could justify both cases (of 3 females and 2 females) and could justify the multiplication in the second case.

Figure 2a and b: Ewan's solutions for the two smaller instances of 3 and 4 females

We then increased the total number of females to 4: *There are 4 females and 3 males. How many different committees of 3 people can you form if there needs to be more women than men on the committee?* In solving this problem, Ewan recognized that there were again two cases, that he needed 3 or 2 females (and can't have a committee with just one female). He wrote C(4,3) as the case of 3 females, and the exchange below shows his reasoning on the second case.

> Ewan: In this case [C(4,2)*C(3,1)], uh, I have males left over to choose. Right? So, I need to account for all of the different groupings of males that I have leftover to choose, for each one of these scenarios. Right? […] And we're adding here because, in this case, just like I said, this didn't affect the males in any way. But we also don't need any of the males. So, this is kind of like, almost like a separate problem. … If you pick 3 females, the committee's already formed…the males have no dependence on that group.

We thus note that he was making sense of each formula/expression in terms of the counting process it was representing. Specifically, C(4,3) meant choosing 3 women from 4, and that was the case where the committee was all women. The other possible case was when there were 2 women and 1 man, and he understood that as a separate case. He added the terms together because they represented "separate problems," which we interpret as different cases for the number of women on the committee. We interpret that he understood the processes, and he understood the expressions/formulas as representing those processes.

Finally, we asked him to consider the original problem in light of his work. He thought about it, and we see again that he was extending to the larger original problem by generalizing the process, not just the numerical pattern. He said, "All of the situations where I have no males is going to be when I have 8 females and I'm choosing 6 of them (writes C(8,6)), and that's going to be added to something." Ewan established that there was first the problem with all eight females, and he needed to choose 6 of them, and he wrote C(8,6). He noted that was one case, and he needed to consider another case.

Ewan:	[…] I have eight females, I'm choosing six of them [C(8,6)]. Right. So that is going to be added to something. And I think that does scale just fine. […] I mean, it's going to be kind of a similar thing, right? So, it's going to be like what was it? How did they do it? Ten choose or sorry, no, I did the females first. That's right. So it was like plus eight choose five times ten choose, one [writes C(8,5)*C(10,1)]. But that's not all in this case. So that's where it doesn't scale, at least like at least not immediately. It doesn't scale because there's also one more thing where we need four females. Right?
Interviewer:	Right, exactly.
Ewan:	Is that just added or is this like a sum overall, does it is it just like eight choose, four and 10 choose, two? So this is some sum over all? Actually, that that does make sense to me. That does make sense to me. That feels this [8C4*10C2] is a fundamentally different problem than this [C(8,5)*C(10,1)] in my mind. So, I think that addition, since there is not any interdependency between the things at play here [C(8,4)*C(10,2)] and the things at play here [C(8,5)*C(10,1)], but we need to account for both of them. So, yeah, yeah. I think it's just a sum of all of them. Right.

In this case, we see that Ewan was able to arrive at a solution for the main problem, understanding and interpreting the processes he was laying out and the expressions that reflect the process. First, he noted that there was the case of six women, which he could find by using the process of choosing 6 of 8 women, which was reflected in the expression C(8,6). Then, he realized that there was a case of five women and one man, which he saw as reflected in the expression C(8,5)*C(10,1). Then, he noted a distinction between this larger problem and the previous 3 and 4 person problems, namely that an additional case was needed (where four females are on the committee). He calculated that by noting it would be C(8,4)*C(10,2), and then he reflected on how he was using addition in the problem. He correctly operated on the binomial coefficients he had written down initially, coming to make sense of how to combine them in a way that made sense combinatorially. Notably, he had not been initially sure how to combine the terms to reflect a desirable process, but after his work and connection back to previous smaller-sized problems, he was able to successfully solve the original problem.

We interpret that he extended the counting processes he had used on the problems, and he had a solid understanding of the choosing processes and the different cases in forming the committees. Further, he demonstrated understanding of how the processes he used were represented by the expressions and formulas. So, the binomial coefficients and expressions represented the counting processes he engaged in. Thus, Ewan' work on this problem is an example both of a generalization based on processes (PPG) and also one that leveraged multiple components of Lockwood' (2013) model – namely processes and formulas/expressions.

Second, we contrast with the case of Brian, who solved the Decreasing Sequences problem: *How many ways are there to form decreasing sequences of 3 numbers from the numbers 0-9?* Brian began by considering cases for which number could be first, noting that the sequence could not start with

a 0 or a 1. Brian then engaged in some listing of partial outcomes, noting that, for example, if the sequence started with 4, they had 3 options for what the second digit could be (1, 2, or 3), and then for each of those they had 1, 2 and 3 choices for what was next, respectively (Figure 3a). As Brian write down these numbers, he said, "So we're going to keep going, and see if I start noticing any patterns," and, later, "let's see if we can start extrapolating." We thus infer that Brian's overall approach was to try and identify some regularity in patterns from which they could extrapolate something more general. Brian reflected back on the case so far and in Figure 3b wrote the 1, 3, and 6 corresponding with each row; he then said, "those are triangle numbers" and wrote the triangle of dots in the upper right of the figure. Brian noted that they did not know the formula for triangular numbers but proceeded to try to identify patterns.

Brian then had a shift in his work. As he completed values for 5 and 6, rather than continuing with the process of thinking about options, he noted a numerical pattern and added the numbers to each case. He said, "For five, there should be, um, 6 plus 4 is 10. For six, there should be, 10." He wrote out the sums he observed in each one, as seen in Figure 3c. Ultimately, he summed these numbers are came to the correct answer of 120.

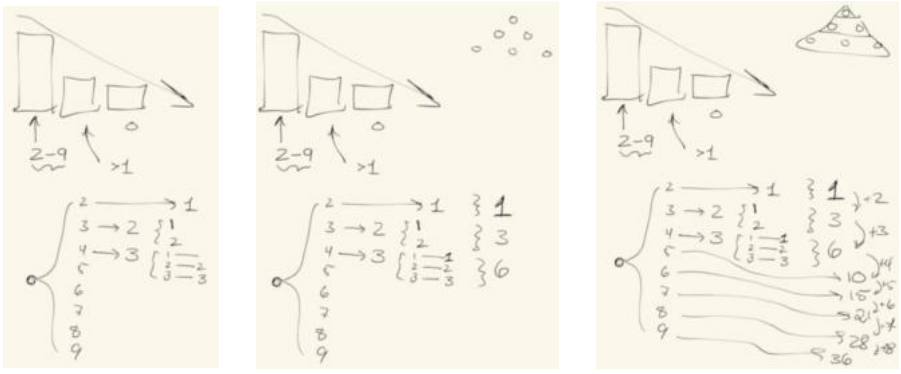

Figure 3 a, b, and c: Brian's work on the Decreasing Number Task, highlighting triangular numbers

The interviewer asked Brian how he went from choices to thinking about triangular numbers (and sums of triangular numbers), and he said "I think I noticed that you're adding a one bigger number every time, and that is the growth pattern here [of the triangular numbers]." Upon reflection, Brian also noted that "I guess there's also a recursive way of thinking about it." He noted that, in the case of starting with 4, the 3 were "new" while the 1 and 2 were repeated from the previous situation. This also helped Brian reason about why the triangular number pattern made sense.

In Brian's case, we appreciate the effective patterning in which he engaged, and we see an efficient and effective use of an appropriate numerical pattern. But, in contrast with the previous case, it raises some questions for us about the overall use of patterning and what was used to develop and justify the patterns. We explore these questions in the Discussion and Conclusion Section.

Discussion and Conclusion

In this report, we see two students engage successfully with patterns to solve combinatorial problems. In both cases, students engaged with multiple smaller-sized problems to arrive at an answer for a larger problem. We claim, though, that there is a distinction between the cases – namely, what

students generalized and what the content of the generalizations were. Ewan engaged in Process Pattern Generalization (Harel, 2008) and carried forward the counting process he had observed in the smaller cases to the bigger case. The numbers and results were not the focus, but rather different aspects of the counting process itself were generalized. Indeed, Ewan focused on the choosing of the people to be on the committee (as seen in the binomial coefficients), the separation into cases, and the use of multiplication to combine male and female committees. In terms of Lockwood's (2013) model, the processes Ewan articulated were closely related to the expressions he wrote, and it seemed clear that his expressions were underpinned by the counting processes, which were the content of his generalizations. We claim that Brian engaged in Results Pattern Generalization (Harel, 2008), as the generalizing activity was based on numerical patterns. Indeed, Brian generated numerical results in observing that larger numbers were added to each case of a starting number, and he leveraged his familiarity with Triangular Numbers to carry that pattern forward.

We intentionally provide two cases in which the students arrived at correct answers, because we do not wish to paint RPG as strictly ineffective or inferior to PPG. Clearly proficiency with numerical patterns may be efficient and effective. These are both correct, but we highlight differences in the nature of what is involved in the patterning. And we raise the question which is more desirable in students' engagement with counting and why. What are potential implications for the teaching and learning of combinatorics as it relates to fostering patterning and generalizing in combinatorics?

Although we do not focus on computing in our data, this discussion (and in particular the phenomenon of numerical patterning we highlight in the case of Brian) is also potentially related to computational solutions and to the role and function the computer might play in generating numerical answers and entire sets of outcomes. Researchers have shown students using the computer to generate large sets of outcomes and to notice numerical patterns (e.g., De Chenne & Lockwood, 2022). Moreover, modern computationally-oriented data analysis methods similarly find and leverage patterns in data that are useful but not always clearly explainable. We consider such activity as providing an important opportunity for researchers and teachers to elaborate appropriate curricular roles that both numerically-based RPG and process-based PPG play in curricula orienting students towards computational mathematics. More research is needed to explore the kinds of generalizing activity such approaches may afford or inhibit, particularly given this distinction between RPG and PPG.

Ultimately, we value patterning and generalizing in students' combinatorial problem solving, and we acknowledge that students tend to leverage patterns, numerical or otherwise, in their combinatorial work. We think this phenomenon opens up important questions for us as discrete mathematics educators – what kinds of engagement with patterning and generalization do we want to foster in students, and how might we practically support hem in such desirable thinking and activity? We hope to engage with such discussions with the field, with the overall aim of continuing to inform and improve the teaching and learning of combinatorics and discrete mathematics.

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Integrating computational thinking in the learning of functions in the 8th grade

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As computational thinking is being included in the mathematics curricula in different countries there is a need to design tasks that support an integrated approach to learning computational thinking and mathematics. Recognizing some common obstacles in the learning of functions, this exploratory study intends to analyse what practices of computational thinking 8th grade students develop in solving a task on affine functions and how this task supports them in distinguishing linear and nonlinear affine functions. A qualitative methodology was adopted, within a teaching experiment, with data collected through observation, and the analysis of students' written work on the task. Findings suggest that the task provided opportunities to enact practices of computational thinking and simultaneously to support first development of students' knowledge about nonlinear affine functions, providing a starting point for discussion on computational thinking and mathematical functions integration.

Keywords: Computational thinking, task design, functions, proportional reasoning, middle school.

Introduction and background

Acknowledging the significance of fostering students' computational thinking (CT), many countries have incorporated CT into their curricula in various ways in recent years, either through dedicated subjects or across multiple disciplines (Bråting & Kilhamn, 2021). In line with this trend, Portugal has introduced CT into its basic education mathematics curriculum (Canavarro et al., 2021). CT may be characterized as a way of thinking in face of problematic situations that involves the enactment of practices such as those outlined by different authors (Kallia et al., 2021), and in the Portuguese mathematics curriculum for basic education: (i) abstraction, which involves extracting the essential information from a problem; (ii) decomposition, which is based on simplifying a problem by breaking it down into more manageable parts; (iii) pattern recognition, which implies recognizing or identifying patterns and regularities when solving one problem and applying them to similar problems; (iv) analysis and definition of algorithms, which involve solving a problem through a step-by-step procedure, often resorting to technology; and (v) development of debugging and process optimization habits, which encourages testing to identify and correct errors and subsequently refine a given solution.

One central issue that emerges when planning the inclusion of CT in mathematics classroom concerns the pedagogical approaches (plugged and unplugged) that best promote its development (Caeli & Yadav, 2020). Plugged approaches are mostly focused on programming and are recognized as a strong pedagogical approach to developing CT, given the importance of the cognitive aspects behind programming (Kallia et al., 2021). However, also several potentialities have been highlighted for unplugged approaches, namely, helping students to construct and outline their reasoning (Caeli & Yadav, 2020), and helping to ease fears of programming among teachers and students, and meaningfully engaging the latter with the big ideas of computer science (Bell & Vahrenhold, 2018). Kotsopoulos et al. (2017) suggest that students should start with less demanding experiences in terms

of understanding (*unplugged* and *tinkering*) and progress to more cognitively demanding ones *(making* and *remixing*). Those authors also note that *unplugged* experience should be the initial approach to promote computational thinking, as it introduces concepts without using computers. This approach requires less cognitive load and does not require knowledge of programming languages. In the *tinkering* experience students modify existing objects, analysing the implications of these changes on the results and correcting possible errors, with a focus on applying concepts and simulation. In the *making* experience, students solve problems by planning and selecting tools, as well as establishing connections between concepts. Finally, in the *remixing* experience, students modify an object to adapt it for a different purpose.

Effectively merging computational thinking with the mathematical topics in the curriculum remains a significant undertaking (Israel & Lash, 2020). Still there is a need to develop studies in less studied mathematical topics, such as functions, to fully understand in which domains of mathematics it is suitable to include computational thinking (Ye et al., 2023). In the case of functions learning, several challenges have been identified within the literature, namely the conversion between different representations of functions, the development of knowledge that becomes connected over time and that is not only context-specific (Wilkie & Ayalon, 2018), and more recently, with introduction of computational thinking, the different meanings variables may assume in the context of algebra and in programming (Bråting & Kilhamn, 2021). Furthermore, it is important to consider the well-documented and widespread students' tendency to make use of proportional reasoning in situations where it does not apply (Van Dooren et al., 2005) and how that may affect the learning of functions when they move from studying linear functions to nonlinear affine functions, an important topic in mathematics curriculum. Hence, there is a need to understand how to integrate computational thinking practices in the learning of functions, namely by conceiving tasks that combine unplugged and plugged activities. This exploratory study, which comes from a teaching experiment aimed at integrating computational thinking into the learning of affine functions in the 8th grade, intends to answer the following two research questions: RQa) What practices of computational thinking do students develop in solving a task on affine functions?; RQb) How did this task support students in distinguishing linear and nonlinear affine functions?

Methods

Task design

The task was the first one in a teaching experiment on the topic of affine functions carried out with two 8th grade classes. In its design, the specific objectives for learning affine functions outlined in the mathematics curriculum (Canavarro et al., 2021), and the characteristics of tasks to promote conceptual mathematics understanding by Swan (2014), namely representing and translating mathematical concepts in their various representations and identifying and analysing structure within situations, including studying relationships between variables, were considered. This lesson was the starting point to support students in building the notion of nonlinear affine functions ($f(x) = ax + b$, with $a, b \in \mathbb{R}$, and $a, b \neq 0$) against linear functions ($f(x) = ax$, with $a \in \mathbb{R}$), they had already studied in the previous school year, in different representations. Targeting the learning goal of interpreting and modelling real-life situations, the task's context presents different plans of gas consumption in

Portugal and it is divided into four parts in connection with the experiences in computational thinking described in Kotsopoulos et al. (2017). The use of Scratch was promoted in this task since it is suggested as a resource in the Portuguese mathematics curriculum and the teacher-researcher (first author) already had successfully explored it previously with these classes. In other tasks in the teaching experiment was also used Geogebra, namely for the graphical representations of functions.

The first part of the task corresponds to the *unplugged* experience, aims students to analyse a situation that can be modelled by a linear function, through out three items where students are asked to: i) explore an example of a direct proportionality situation using specific values of gas consumption; ii) construct a flowchart that identifies the inputs/outputs related to the described situation and outlines the reasoning from the previous item in a more abstract context; and iii) to advance to symbolic language relating objects (x) and images (y) to write an algebraic expression representing the linear function.

In the second part of the task (Figure 1), that corresponds to the *tinkering* experience, students are expected to interpret situations modelled by a nonlinear affine function, make predictions, and correct errors. In the first two items, students should interpret a pre-written code, contextualize some of its instructions according to the given situation, and make a prediction about an output (image) resulting from a specific input (object). In a third item, students are prompted to compare codes and identify errors, using, for example, the rules of mathematical operation priority to justify their reasoning. Finally, they are asked to translate the situation, relate objects (x) and images (y), and arrive at the algebraic expression of a nonlinear affine function.

2. Due to convenience, safety and price, in recent years piped natural gas has reached more and more homes. In the natural gas bill, there is a variable component, associated with gas consumption in cubic meters, and a fixed component, relating to several fees, namely the fixed term, access to the network and occupation of the underground.
For the gas company to automatically issue gas invoices, it is necessary to enter the number of cubic meters that were consumed in a month, real or estimated. The following program was built in Scratch to depict this situation:

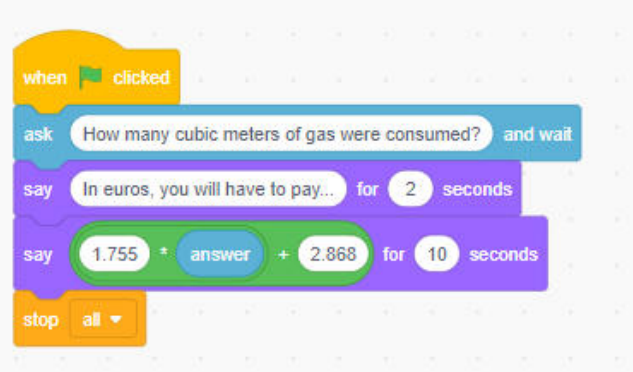

Figure 1: Statement of the task (Part 2)

The third part of the task, the *making* experience, aims to address more problematic situations involving the nonlinear affine function and to establish connections between concepts, through out three items where students are expected to: i) to determine an object corresponding to a given image, using, for example, reasoning with inverse operations or solving equations which were studied in previous years; ii) generalize the reasoning from the former part by completing lines of a code

presented in the task; and iii) build the program in Scratch to validate the obtained result. Thus, both plugged and unplugged approaches are expected from students.

Finally, in the *remixing* part of the task, students are required to build, in a less guided manner, a program in Scratch to model a real-life situation involving percentages, which is associated with a linear function.

Data collection and analysis[1]

This study followed a qualitative methodology within a teaching experiment carried out by the teacher-researcher (the first author). From the two classes that participated in the teaching experiment, eight pairs (four from each class) were selected to represent the heterogeneity of students. To ensure the collection of comprehensive data, these were chosen to optimize the benefits of their interaction during task-solving. For the teaching experiment different data collection methods were employed, including participant observation, with video recordings and the collection of students' written work. In this paper we focus on the analysis of these students' written work which represents the range of solutions from the target groups.

The analysis framework focuses on the five practices of computational thinking mentioned before (Canavarro et al., 2021) and on the evidence of students' use of proportional reasoning or non-proportional reasoning in the context of affine functions. For this study we rely only on the analysis of students' solutions to the items of parts 2 and 3 of the task (the *tinkering* and *making* experiences), where students were expected to understand the difference between a linear and nonlinear affine function.

Findings

The *tinkering* part of the task

The first item in this part of the task asked students for the meaning of the two values (1.755 and 2.868) displayed in the coded representation (Figure 1), in the context of the problem. All students correctly interpreted these values and exhibited the intended *Abstraction*. Differently to most groups, that mentioned the two values as the "price of gas by m^3" and "the value of the fees", one pair interpretated them strictly from the mathematical point of view ("1.755 is the fixed component and 2.868 is the variable component") in connection to what was presented in the task statement.

In the second item of this part, most students were able to determine the values of the input and output values for a gas consumption of 10 m^3, showing that have understood that coded representation translated the situation depicted in verbal language in the statement (Figure 1). In their initial solution (17.55), however, one pair overlooked the fixed rate, focusing solely on the variable component of the gas price. After testing the value in Scratch, they were able to identify and correct the error (*Depuration*), as depicted in Figure 2.

[1] Data presented in this paper is part of a research published by the authors in the paper: https://doi.org/10.48489/quadrante.37041.

Figure 2: Solution to item 2 from Part 2 by one pair of students

In item 3 (Figure 3) all pairs provided a correct answer, accurately identifying and explaining the differences between the two code blocks (*Depuration*). One pair first converted the codes into symbolic language and then justified their response using natural language in the context of the situation (*Abstraction*), showing proficiency in converting different representations (Figure 4).

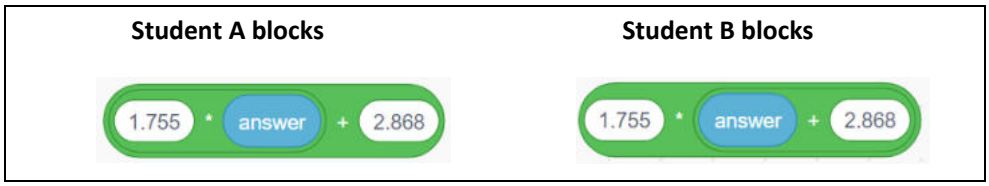

Figure 3: Blocks of code for students to choose (item 3 from Part 2)

This pair of students explained that the correct block of code is the first one: "The idea is to do the price of one cubic meter x (times) the consumed gas + (plus) the fee and not the consumed gas plus the fee x (times) the price of one cubic meter" (Figure 4).

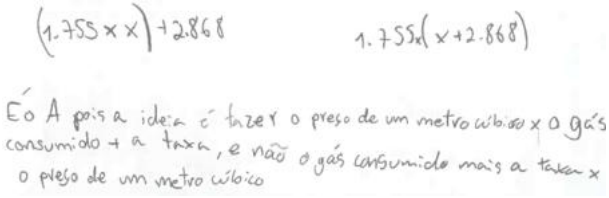

Figure 4: Solution to item 3 from Part 2 by a pair of students

In the last item of this part of the task, students were asked to make a flow chart reflecting the situation depicted by the program (Figure 1), using the variable x to represent the amount of natural gas consumed in a month. All students successfully elaborated a flowchart, defining an algorithm where most of them correctly identified the input data, procedures, and output data (*Algorithm analysis and definition*). This is evident from the geometric shapes used in the flowchart construction, where they use parallelograms for input or output data and rectangles for procedures (Figure 5).

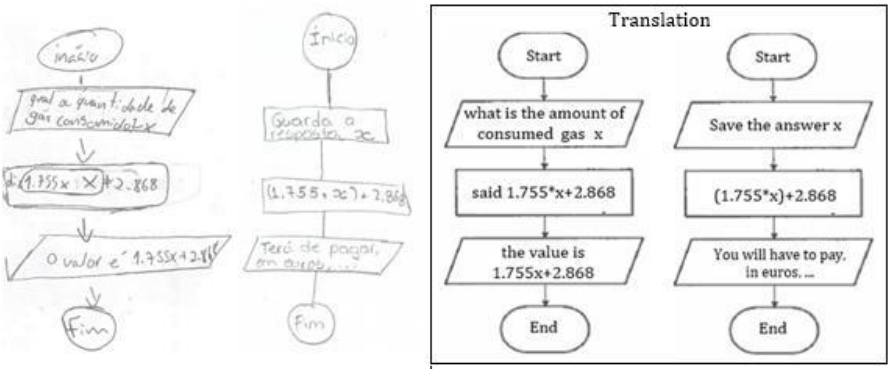

Figure 5: Flowchart by two pair of students

In the solution on the left, one pair of students (Figure 5, left) identifies the input data in the flowchart as a question, using the context of the described situation, and associates it with the variable *x*. For the procedure, it indicates the calculations to be performed in a generalized way, using the variable *x*, and they mention the output data as a response associated with the result of the previously performed procedure. In the solution by another pair (Figure 5, right), there is a reference to the need to store the value of the variable *x*, although this is not identified in the flowchart as an input data.

The *making* part of the task

In the first item of this part, where students were asked to determine the amount of natural gas for a given monthly value (39.95 euros), the pairs that arrived at the correct solution used the inverse operations reasoning. However, several students solved the question incorrectly, some of them by using the direct proportionality reasoning without considering the fixed rates. In these cases, students mainly recurred to the simple rule of three, a procedure quite common in mathematics classrooms.

In the second item, where students were required to fill the blanks in the Scratch blocks (Figure 6), most of the solutions presented by them are generalisations of the correct or incorrect reasoning used in the previous item, allowing for the identification of input and output data and the calculations to be performed (*Algorithm analysis and definition*). Contrary to what was expected, one pair (Figure 6), who had used the simple rule of three without considering the fixed rate in the previous question, was not able to directly translate the rule of three reasoning into code. Instead, these students used the context of the described situation to correctly solve this question, although they did not define the meaning of the variable *x* in their solution, which would prevent the program from working correctly.

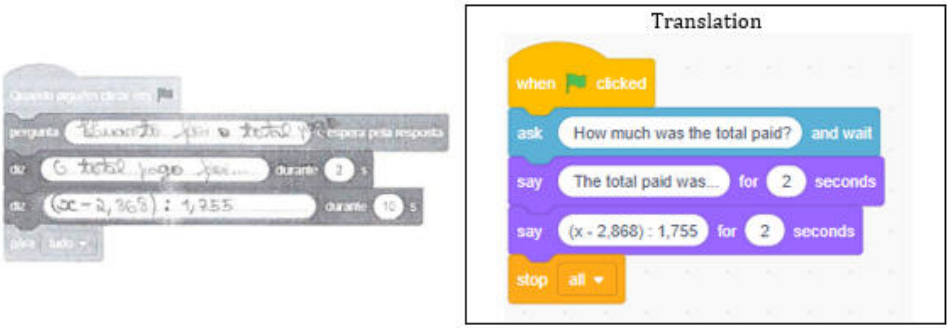

Figure 6: Solution of item 2 from Part 3 by one pair of students

In the third item, students were asked to access Scratch website and to create the above program. Most students managed to test their reasoning by building the program according to the blocks they completed (*Debugging and process optimization habits*). Interestingly, the same pair of students, when constructing their program, noticed something was wrong and managed to solve this by creating the variable *x* to store the value of the response.

Conclusion

Addressing the need to understand how to integrate computational thinking practices in the learning of functions, this exploratory study shows that the chosen approach for the design of the first task in the teaching experiment has allowed students to reveal consistently several practices of computational thinking. Thus, concerning RQa), besides the main practices envisioned for each item of the task, mostly abstraction, pattern recognition and algorithm analysis and definition, students often recurred to debugging for adjusting their solutions. In what regards RQb), we observed that in the *Tinkering* part of the task, students managed to analyse a new way of representing a function, different from the usual representations, a *coded representation*, contributing to the understanding of the concept of a nonlinear affine function. In *Making* part of the task, most students, starting from the analysis of a concrete example of the described situation, managed to advance their reasoning to reach a contextualized generalization of the situation. That was achieved by constructing a program to determine an object given an image in a nonlinear affine function and performed testing and validation of their reasoning. Thus, during parts 2 and 3 of the task, analysed in this paper, students moved between unplugged and plugged experiences which seem to have supported their work.

As documented in the literature, we also observed some instances of overgeneralisation of the proportional model (Van Dooren et al., 2005). However, computational practices practices assisted some students in reflecting on their solutions and made them progress to a non-proportional model, as the one depicted in the *making* part of task. In sum, it seems that the task provided students a good understanding of the two variables involved in the situation (amount of gas consumed and what to pay) and about the relationships between those variables, a key issue in leaning of functions (Wilkie & Ayalon, 2018).

Further analysis of the students' learning during the teaching experiment will allow us to have a deeper understanding of the potentialities of the adopted approach and adequacy of tasks' structure to promote students' knowledge of affine functions. One key point will be to study how students deal with the different meaning variables may have in algebra context and in programming environment (Bråting & Kilhamn, 2021), in the domain of functions. Nevertheless, these findings already provide a platform to discuss the tasks' nature that may support the integration of computational thinking in the learning of mathematics, in regular classrooms, in a connected way.

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Use of re-coding strategy while solving problems related to Catalan numbers by undergraduate student

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This study explores the use of isomorphism, specifically re-coding, in solving combinatorial problems involving Catalan numbers. According to previous research, isomorphism, or recognizing equivalent problems with similar structures, can improve problem-solving success. Through a lesson study with undergraduate students, we examined how one student used re-coding during a diagnostic interview. Despite his high programming skills but low combinatorial proficiency, Alex effectively translated Dyck words and paths into familiar representations like parentheses. These findings highlight that re-coding can simplify problem-solving by reducing cognitive load and suggest that promoting isomorphic strategies can enhance students' problem-solving efficiency in combinatorics.

Keywords: Mathematics education, problem-solving, combinatorics education, problem-solving strategies, undergraduate students.

1 Introduction

Problem-solving in combinatorics presents a challenge for students, often due to the absence of clear verification methods and multiple correct strategies. Studies (e.g., Batanero et al., 1997) show that students face uncertainty when deciding on a solution method, and incorrect estimations frequently arise in counting problems (Fischbein & Grossman, 1997). This reflects the need for effective problem-solving strategies, especially those that facilitate structured approaches to complex combinatorial tasks. One promising strategy for addressing combinatorial problems is the recognition of *isomorphism*, the one-to-one correspondence between equivalent problems that share the same underlying mathematical structure despite differing superficially (Siegler, 1977 in Zenkl and Vondrová, 2023). This problem-solving approach allows students to transform a problem into another equivalent situation that they are more familiar with, which can lead to a correct and more efficient solution (Lamanna et al., 2022). The ability to recognize and apply isomorphism is central to developing a deep understanding of combinatorial concepts.

Research supports the effectiveness of isomorphism as a strategy for solving combinatorial problems (Maher et al., 2010). Lamanna et al. (2022) observed that students who utilized isomorphism were more successful in solving combinatorial problems. In the study by Lockwood and Gibson (2015) students who began solving tasks by enumerating or referencing an isomorphic problem demonstrated a statistically significant improvement in their problem-solving success. Despite its proven benefits, students rarely employ such strategies unless

explicitly guided (Lamanna et al., 2022). This highlights the need for teaching methods that emphasize isomorphism as a problem-solving tool in combinatorics.

Zenkl and Vondrová (2023) identified three distinct ways in which students employed the strategy based on isomorphism: (1) **Re-coding**: Students replaced letters from the problem with numbers or other symbols, or used different letters (e.g., substituting letters from the beginning of the alphabet) to simplify the task and facilitate understanding; (2) **Solving an isomorphic combinatorial task**: Students drew on their knowledge of a previously encountered isomorphic combinatorial problem, either due to familiarity with the solution or a more comprehensible context, to successfully solve the current problem; and (3) **Solving an isomorphic problem from a different mathematical domain**: Students applied concepts or skills from a different area of mathematics beyond combinatorics to address the problem, leveraging their broader mathematical knowledge for the solution.

In this paper, we focus on the application of isomorphism as a problem-solving strategy in problems leading to Catalan numbers, defined as $C_n = \binom{2n}{n} - \binom{2n}{n-1}, n \in \mathbb{N}_0$ (Stanley, 2015). They appear in a variety of combinatorial problems, such as counting valid parenthetical expressions, grid paths not crossing the diagonal, mountains consisting of slashes and backslashes non-descending under the given level, non-crossing pairings of vertices placed on the circle, or polygon triangulations. Catalan numbers provide students with opportunities to explore recursive structures, patterns, and connections between problems (Cofman, 1997).

2 Material and Methods

The findings reported in this paper are from a broader study focused on the affordances of Catalan numbers in undergraduate courses of combinatorics, particularly in courses for pre-service mathematics teachers. The study is designed as a lesson study (Lewis et al., 2019) when two mathematics educators from two different universities (authors) designed the series of lessons in four steps: study – plan – teach – reflect. Besides future mathematics teachers, the designed lessons are also used for students of applied statistics and applied informatics at the university in Nitra. More than 20 task-based interviews (Maher & Sigley, 2020) with students were conducted; most of the sessions and the lessons were (audio- or video-) recorded.

The designed activity consisted of three lessons, 90 minutes each. The first lesson was focused on isomorphisms between eleven combinatorial situations leading to Catalan numbers. As homework after this lesson, students were asked to find as many bijections between the given combinatorial situations as they were able to. The list of the situations, as well as the first categorization of students' strategies to find the bijection, can be found in Medová and Jančařík (2024). During the second lesson the students were divided into groups of 2-4 students. Each group re-discovered one of the three various formulas for Catalan numbers and their proofs in thinking-classroom settings (Liljedahl, 2020). In the non-compulsory homework, after the second lesson, students were asked to write a procedure to list all Dyck's words or valid parenthetical expressions for given n. During the third lesson, students presented their solution strategies and proofs from lesson two to the whole classroom. The lesson three was led by

students and facilitated by the authors. Several instances of students using isomorphism were observed at various stages of the lessons.

In order to deeper understand how students use and switch between different representations and terminology when reasoning about Catalan numbers, and what challenges arise from this recoding during problem-solving we analyzed one of the interviews. As Lockwood and De Chenne (2021) suggested that computer programming might support solving combinatorial problems, we chose a person submitting the non-compulsory coding assignment and, based on our observations from lesson three tending to use a re-coding strategy.

The diagnostic interview was conducted one week after the third lesson, in front of the whiteboard, because the students in the course were used to work with vertical non-permanent surfaces as suggested by Liljedahl (2019), and video recorded. The student provided informed consent with the interview and its use in scientific publications as well as for further educational purposes. The interviewee, Alex (pseudonym), studied applied informatics in his second semester. Based on the fact he got an A mark in the programming course in the previous semester, we can assume his programming competencies were very high. On the other hand, according to the pre-test given at the beginning of the course (the same test as in Melušová & Vidermanová, 2015), his proficiency in solving combinatorial problems was quite low, he solved correctly only two problems and used enumerative strategies.

3 Findings

During the 60 minutes of the diagnostic interview, we identified four instances where Alex used isomorphism between the combinatorial situations. We will now describe the stances and then provide a short analysis. The session started by revising the coding homework. Then, he explained the proofs of the three formulas re-discovered during the second lesson. In the last phase, we focused on isomorphism between the given combinatorial situations.

3.1 Coding

The first instance where the re-coding was observed was the solution of coding homework. The assignment was as follows: “Write a code producing a list of valid parenthetical expressions or Dyck’s words. Input is n (number of characters X in the Dyck’s word), output is the list of the words.” A Dyck word is a string consisting of n X's and n Y's such that no initial segment of the string has more Y's than X's (Stanley, 2015).

The code provided by Alex mixed the use of English and Slovak language. The name of the procedure (GenerateDyckWords) is in English, and the instructions for the user are in Slovak. It can be seen that he does not distinguish between valid parenthetical expressions and Dyck’s words. Although the procedure is called GenerateDyckWords in the source code, it prints the parenthetical expressions. Also, the instructions for the user are written confusingly. The input is supposed to be n, the number of X characters in Dyck’s word, and the user dialogue says "Number of parentheses:". The number of elements in the working set is printed with the comment "Number of words:". It doesn't seem that Alex cared what two characters the Dyck words consist of, whether it's AB, XY or (). In fact, English Wikipedia uses square brackets

when defining Dyck words for the entry Dyck language. Alex claimed that while coding, he also looked up information about Dyck words on the Internet.

When asked why he preferred parentheses to any other character, he answered, “I don't know, but I'll think about it...” (app. 2 minutes later). “Maybe because the parentheses are simpler? Faster to understand? I can see right away if they [generated strings] are okay.”

```
public static void main(String[] args) {
    Scanner scanner = new Scanner(System.in);
    System.out.println("Pocet zatvoriek:");
    int n = scanner.nextInt();
    List<String> result = generateDyckWords(n);
    for (String s : result) {
        System.out.println(s);
    }
    System.out.println("Pocet slov: " + result.size());
}

3 usages
public static List<String> generateDyckWords(int n) {
    List<String> result = new ArrayList<>();
    if (n == 0) {
        result.add("");
    } else {
        for (int i = 0; i < n; i++) {
            List<String> left = generateDyckWords(i);
            List<String> right = generateDyckWords( n: n - i - 1);
            for (String s1 : left) {
                for (String s2 : right) {
                    result.add("(" + s1 + ")" + s2);
                }
            }
        }
    }
    return result;
}
```

Figure 1: The code provided by Alex as the solution of the coding assignment

Note: “Pocet zatvoriek:” means “Number of parentheses:”, “Pocet slov:” means “Number of words:”

3.2 Proving the formula for mountain ranges

The unconscious re-coding can also be observed when Alex proved this solution to the problem about Dyck paths as mountain ranges. Dyck's path is a series of up-and-down steps. The path will begin and end on the same level, and as the path moves from left to right, it will rise and fall, never dipping below the height it began on (Stanley, 2015).

While pointing at the mountain ranges (Figure 2), he described “the way it's divided up is that there's one group where there are three **brackets** down, two brackets down, one bracket down, zero brackets down” and he fluently continued with the description of the bijection used in the proof (Figure 3) “from one form to another, this is adjusted by taking one **step down**. The **stick** is transferred upwards. And when we go the other way around, we take those chopsticks from the bottom”.

Throughout the whole proof, he interchangeably used the terms “parentheses”, “step up”, and “stick” again, in a very unconscious way.

Figure 2: Alex pointing on the whiteboard while describing grouping of Dyck paths

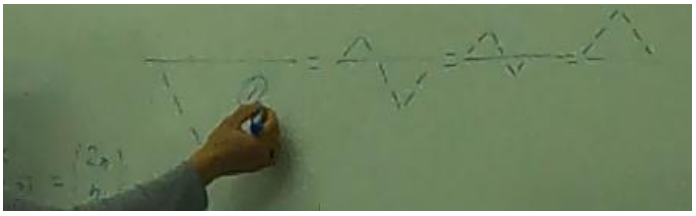

Figure 3: Alex pointing on the whiteboard while describing the bijection

3.3 Proving the formula for Dyck words

While proving the formula for Dyck words using the cyclic lemma, Alex again uses the two words "parentheses" instead of characters, even when reasoning about the Dyck word consisting of characters X and Y.

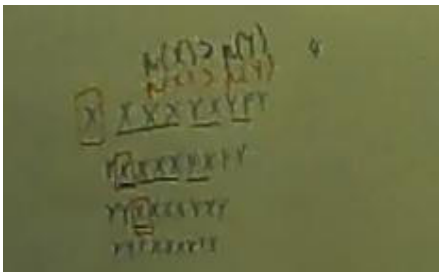

Figure 4: Whiteboard while explaining the cyclic-lemma proof

Alex:	This one doesn't fit because I'm starting with the left parenthesis [pointing to Y in row 4].
Interviewer:	Why won't any new word I get by cyclic rotation satisfy this condition?
Alex:	Because there's always one extra parenthesis, and there's no parenthesis to close that word.
Interviewer:	But that's a different condition we're talking about.

3.4 Explaining the bijections

The last stance when Alex used the re-coding was when he was asked to explain the bijection between the non-crossing pairing of $2n$ vertices and the Dyck words.

Interviewer:	Why does it always have to work like this? Why does the condition of not having multiple ends before the **beginning** always fit?
Alex:	Well, because those **parentheses** can't be like that.
Interviewer:	Well, but does it fit with this picture?
Alex:	(silence for the 40s)

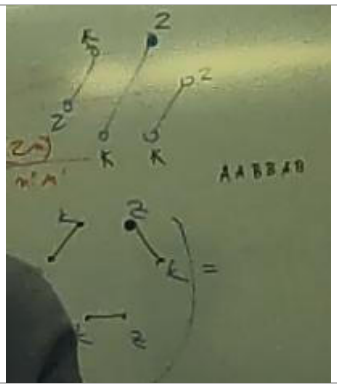

Figure 5: Whiteboard while reasoning about the bijection

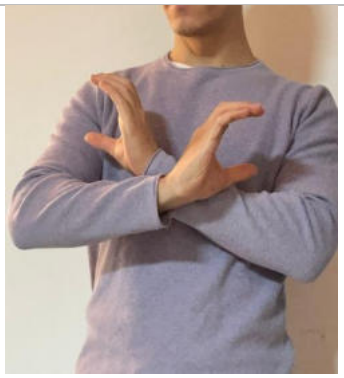

Figure 6: Alex showing the gesture "parentheses can't be like this"

Note: character Z states for "začiatok", i.e. beginning, the character; K for „koniec", i.e. end.

This vignette demonstrates the unconsciousness of the usage of the re-coding strategy. After the direct question of the interviewer "Well, but does it fit with this picture?", Alex was looking at the whiteboard for more than 40 s. It seemed that he was really confused as there were no parentheses on the whiteboard. Also, the word proposed by teachers consisted of letters A and B and not Z/B and K/E, as used for the diagrams of the graph. Then the interviewer recoded the given Dyck work into the ZK notation used for graphs and asked.

Interviewer:	Why does the word I get from such a listing have to be Dyck's? Why can't I get the word KKZZKZ [using English notation EEBBEB]?
Alex:	Because we coded it that way, and it's right.
Interviewer:	Why can't I get this out of a picture like this? ZKKZKZ [BEEBEB]
Alex:	Because we go like this (writes a circle around the vertices of the graph with his finger) and write down the letters (looks at non-Dyck's word ZKKZKZ/BEEBEB). I don't know why you went like that. How did you get this?
Interviewer:	It's not from this picture. I'm asking why I can't get a word like that from the matching graph. Or vice versa: why can't I make a pairing for this word?

Alex looked at the whiteboard and then translated Dyck word into the form of parenthetical expression and underlined the parentheses that made the expression incorrect and looked at the interviewer questioningly.

Interviewer:	I know it's not Dyck's word. The question is why there can't be a non-crossing pairing to such a word.
Alex:	Because the **parentheses** can't be like this (gesture: he crossed his arms in front of his chest, hands were curved into the shape of parentheses, Figure 6). There the edges cannot cross and the beginning must be less than the end and in such a word the end would be before the beginning.

In this turn, Alex used several situations simultaneously: parenthetical expressions, pairings, and Dyck words. They appeared to be three representations of the same concept for him. He seemed to think in the language of parentheses and speak according to the situation. When the question was more cognitively demanding, he switched the code to the language of parentheses. During the whole interview, he did not mention that he switched the representations or used the isomorphism.

4 Discussion and conclusions

The findings from this study highlight the use of isomorphism as a strategy in combinatorial problem-solving, particularly through re-coding. Alex demonstrated four instances of re-coding during the interview, where he unconsciously translated Dyck words and paths into parenthetical expressions or other familiar representations. All the situations where the re-coding strategy was used were used the isomorphism based on substitution of the characters (Medová & Jančařík, 2024) which can be according to Greer and Harel (1998) considered as surface-level isomorphism. The re-coding allowed Alex to rely on a more intuitive understanding of parentheses, facilitating his ability to assess the correctness of the strings without performing step-by-step verifications. The re-coding of the combinatorial tasks into a form where the rules were more immediately recognizable (e.g., parentheses) enabled Alex to redirect his cognitive efforts toward more complex aspects of the problem.

Additionally, the findings suggest that the ability to switch between different representations of combinatorial problems—such as Dyck words, mountain ranges, and pairings—can be a powerful tool for students. The language of parentheses appeared to be a natural and efficient medium for Alex to process and solve problems, indicating that encouraging students to recognize isomorphic relationships and consciously choose familiar representations may enhance problem-solving efficiency. These results point to the potential for re-coding strategies and isomorphism to reduce cognitive load and improve students' success in solving combinatorial problems, particularly in tasks involving Catalan numbers.

Acknowledgement

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Assessing Abstraction Skills Among Secondary School Students: A Study of a Block-based Programming Environment (BBPE) Enhanced by Augmented Reality (AR)

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In the 21st century, educators are increasingly focused on promoting Computational Thinking (CT), crucial for careers in STEM (Science, Technology, Engineering, and Mathematics). A key CT skill is abstraction, which is vital for solving complex problems. This case study explores the development and assessment of abstraction skills in novice 16-year-old programmers in secondary school using a Block-based Programming Environment (BBPE) with Augmented Reality (AR). Data from screen recordings, audio, and interviews show that while students show proficiency in pattern recognition, eliminating, and focusing, they frequently lack higher-order abstraction skills like generalization, decomposition, and debugging. Although most students attain an abstraction level that enables them to organize their thoughts and create logical plans, they often find it challenging to generalize their solutions to more complex problems.

Keywords: Secondary mathematics education, CT, abstraction, block-based programming, AR.

Introduction

In the 21st century, the emphasis on STEM literacy and digital skills has increased due to the importance of computing technologies. Computational Thinking (CT) is crucial for problem-solving in STEM careers (Çakiroğlu & Çevik, 2022), with Wing (2010) defining it as the process of formulating problems so that solutions can be executed by an information-processing agent. Abstraction, a key CT skill, is fundamental in computer science, involving the identification and elimination of unnecessary data (Wing, 2006). Students with weak abstraction skills struggle with complex problems (Kramer, 2007), highlighting the need to teach this skill early (Gibson, 2012).

New technologies such as Block-based Programming Environments (BBPEs) offer strategies for teaching abstraction. BBPEs help students understand basic programming through building blocks (Weintrop & Wilensky, 2017). While popular tools like the Scratch BBPE are widely used, Weintrop & Wilensky (2017) examined that students encountered confusion and difficulties due to the challenge of identifying errors within the complex code. To address this concern, the role of immediate feedback is crucial in BBPEs. Immediate feedback in BBPEs, such as that provided by the <colette> app using Augmented Reality (AR) (Colette-Project, 2022; Milicic et al., 2021), helps students better understand and debug code (Cheli et al., 2018). This study investigates how <colette/> tasks using a BBPE enhanced with AR can improve abstraction skills in novice secondary school programmers.

Theoretical Framework

CT is a concept pioneered by Seymour Papert in 1980. Papert (1980) emphasized two key facets of computation: employing it to generate knowledge and using computers to augment thinking processes for accessing knowledge. The theory from Papert (1980) was then formally introduced by Wing (2006) as the concept of CT. Wing (2006) suggested that CT is the process of using abstraction when solving sizeable complex problems. It includes the skills of "using, modifying, and influencing a large complex system without understanding its every detail" (p. 33).

Various researchers have since refined the concept of abstraction within CT. Csizmadia et al. (2015) described it as eliminating irrelevant information, while Barr & Stephenson (2011) and Lee et al., (2011) emphasized its role in simplifying and generalizing problems. Çakiroğlu & Çevik (2022) expanded on this by identifying five sub-skills of abstraction: pattern recognition, generalization, focusing, eliminating, and decomposition. These sub-skills collectively help reduce problem complexity, making solutions more applicable across different contexts.

Abstraction skills in K-12 education can be developed BBPEs, which simplify programming by representing it as building blocks, making it easier for students to understand and reducing syntax errors. Studies have shown that BBPEs can enhance students' abstraction skills, though further research is needed to identify which specific sub-skills are most effectively developed. For example Rose et al. (2020) found that a BBPE game could improve procedural abstraction but did not specify which sub-skills were involved.

Frameworks to assess abstraction skills in BBPE have also been proposed. For example, Cetin & Dubinsky (2017) proposed a framework which emphasizes the development of mental structures through reflective abstraction to understand CT concepts. This framework, however, primarily focuses on developing students' mental structures through reflective abstraction and does not offer detailed strategies for cognitively assessing the level of abstraction that students attain. Çakiroğlu & Çevik (2022) adapted the RBC Abstraction Model, which outlines three stages—Recognizing, Building with, and Constructing (RBC)—through which students progress in their abstraction skills. They suggested that these stages in BBPEs can be observed across three dimensions: interface use, problem-solving processes, and block structure. In these dimensions, students abstract key components, prioritize efficient solutions, and selectively use code blocks to solve problems, thereby developing their abstraction capabilities.

With respect to levels of abstraction, Ye & Salvendy (1996) proposed a framework for understanding levels of abstraction in computer programming. The framework includes five levels: physical, logical, functional, conceptual, and objective, with each level representing a different depth of understanding in programming. The physical level involves basic syntactic knowledge, which in the context of BBPE relates to how students interact with programming blocks to create code. The logical level goes deeper, involving an understanding of the semantics and purposes of different blocks and how they function when the program is run. At the functional level, students begin to develop a step-by-step plan or algorithm to solve specific problems using the programming blocks. The conceptual level requires understanding the generalized functioning of blocks and recognizing patterns that can be applied to broader contexts. The highest level, the objective level, relates to the comprehensive

analysis of system requirements, but this level is less applicable to BBPE, which focuses on basic programming skills rather than advanced system design.

Augmented Reality (AR) enhances 3D visualization's ability to represent spatial structures more effectively than 2D by integrating real and virtual elements, creating realistic and engaging learning experiences (Ke & Hsu, 2015). Chung and Hsiao (2020) found that AR significantly enhances CT skills, with participants using AR support producing higher-quality programs with fewer errors and edits than those using non-AR methods. While research indicates that AR tools can improve CT skills overall, studies specifically examining their impact on improving abstraction remain scarce, underscoring the need for further investigation into AR's potential to develop abstraction skills in educational settings.

The study utilized the <colette/> app, developed by seven European institutions, to teach CT by enabling students to use their own devices, such as smartphones or tablets, during regular lessons (Milicic et al., 2021). Teachers can select and customize tasks, such as building cubes, where students code structures in a 3D map using Google Blocky. The app also incorporates AR to help students visualize and test their coded structures, enhancing their ability to identify and correct errors in real-time.

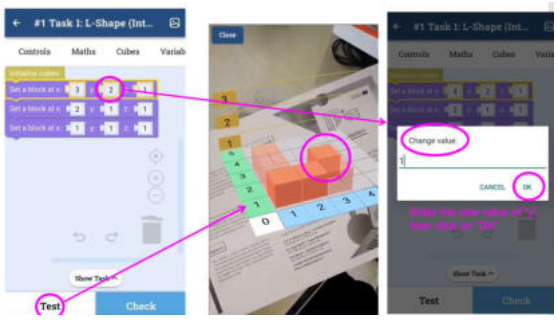

Figure 1: A Screenshot of the task in the <colette/> app in the AR Mode

In summary, this study on BBP tasks in the <colette/> app using AR applies the framework by Çakiroğlu & Çevik (2022), focusing on the dimensions of interface use, problem-solving processes, and block structure. We assess students' abstraction skills, including the sub-skills of pattern recognition, generalization, focusing, eliminating, and decomposition, while considering the levels of abstraction outlined by Ye and Salvendy (1996).

The current study seeks to answer the following research questions:

1. How are abstraction skills developed in the <colette/> Block-based Programming Environment (BBPE) using Augmented Reality (AR) among novice programmers secondary school students?
2. To what extent do students demonstrate different aspects of abstraction skills and levels of abstraction while using the <colette/> BBPE enhanced by AR?

Method

This study was conducted as an explanatory case study to explain how abstraction skills are embedded in BBPE enhanced by AR. An explanatory case study aims to elucidate the underlying causes of a phenomenon, processes, or relationships within a specific research situation (Denscombe, 2021).

In this study, students used the <colette/> app to learn programming with AR. Seven cube-building tasks were created, covering various BBP concepts across four abstraction levels.

The research was conducted at Amsterdam International Community School (AICS) and involved 16 10th-grade novice programmers working in pairs using tablets. The study aimed to observe and analyze students' abstraction skills through task performance and post-task interviews. The study ensured informed consent and anonymous data handling.

Data collection in this study involved screen and audio recordings to analyze students' interactions with the <colette/> app, focusing on their use of interface, problem-solving, and block structures. Semi-structured interviews with six students offered deeper understanding of their reasoning and cognitive processes while performing tasks. Quantitative data further supported the analysis.

To investigate abstraction skills, we used the Çakiroğlu & Çevik (2022) framework within the context of the <colette/> BBPE, with the dimensions interface structure, problem-solving process, and block structure, focusing on pattern recognition, generalization, focusing, eliminating, decomposition, and debugging.

For the first research question, we created an observation form to identify abstraction sub-skills. One of the researchers who is an expert teacher in <colette/> reviewed this form, which included indicators for abstraction in the interface, problem-solving, and block structures. Scores were categorized as “Insufficient”, “Sufficient”, or “Advanced”, and inter-rater reliability was assessed using Cohen's Kappa, comparing a researcher’s and a computer science teacher’s observations (0.79 with 94.98% agreement).

For the second research question, we conducted thematic analysis of interviews to understand students’ behaviors related to abstraction sub-skills. We triangulated data from observations and interviews to evaluate students' abstraction levels according to Ye & Salvendy (1996).

Results

To answer the research questions, we examined the data across three dimensions: interface structure, problem-solving process, and block structure.

The observation result of eight groups of students assessed on their proficiency in abstraction within the context of interface structure in the BBPE indicates that all groups have achieved an advanced level of performance across all four sub-skills. The proficiency levels of students’ problem solving process in the five abstraction sub-skills pattern recognition, generalization, focusing, eliminating, decomposition, and in addition debugging are depicted in Figure 2. The data indicates that students excel in pattern recognition, focusing, and eliminating. However, generalization, decomposition, and debugging emerge as areas of concern.

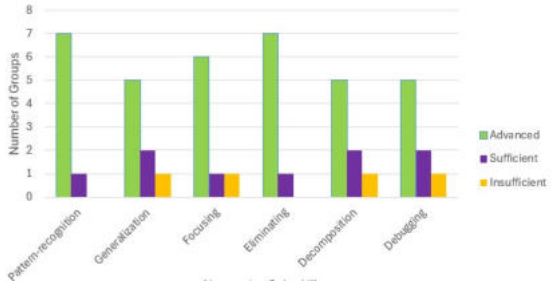

Figure 2: Frequencies of groups performed different abstraction sub-skills in problem solving process

The analysis of students' performance in abstraction sub-skills within block structures presented in Figure 3. The data indicates that a significant majority of groups achieved an advanced level in pattern recognition, focusing, and eliminating.

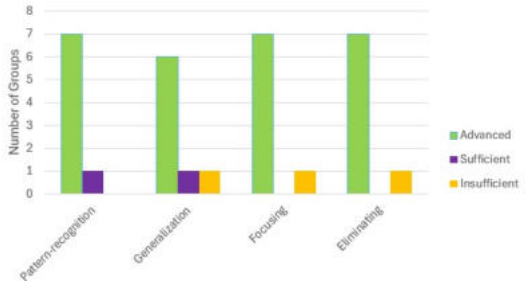

Figure 3: Frequencies of students performed different abstraction sub-skills in the block structures

To further explore the reached levels of abstraction, we analyzed task performance, classroom observations, screen recordings, and interview data. We evaluated students' abstraction level based on two criteria: the correctness of the code and the approach taken to develop the code. A comprehensive analysis of the performance of eight student groups across seven tasks and a qualitative assessment of each group's abstraction level triangulating the different data sources revealed that a majority of four groups achieved a functional level of abstraction, while one group remained at the logical level, and three groups achieved the conceptual level.

A comparison of Groups 2 and 6 highlights differences in task completion and abstraction levels. Both groups completed only the introductory tasks but struggled with the more complex Tasks 3 to 7. Group 2 failed to decompose the solution and attempted to use the same strategy from Task 2 to solve the more complex tasks. A student from group 2 commented on task 4:

> Student 1: … we know this is just 3 different L-shapes, but this is too big. (…) I know we can make it simpler with a repeat block, but we don't know how to do it…

Group 6, despite completing the same number of tasks, showed a deeper understanding of abstraction principles, though they struggled with debugging. Their ability to plan logical sequences placed them at the functional level of abstraction. Reflecting on Task 4, two students from Group 6 commented:.

> Student 2: …the goal of the task was, I think was to use the four loop and understand how to change the dimensions of x, y, z coordinate…
> Student 3: …we just confused what was wrong with our code. We sure it was correct but when we checked it, it always failure…

In contrast, Groups 1, 4, and 8 completed nearly all tasks correctly and effectively , demonstrating abstraction at the conceptual level. A Group 4 student explained their approach to Task 7:

> Student 4: … so the pyramid, it was a three-dimensional structure and we had to use the four loop to do the base and then we have to start slowly decreasing the x and y coordinate… … With that you could make it smaller.

Discussion & Conclusion

Çakıroğlu et al. (2021) emphasize the importance of abstraction skills in problem-solving, highlighting that students who excel in abstraction can solve problems more efficiently. Various approaches, such as game-based learning and robotics, have been explored to enhance these skills. However, previous studies lacked a focus on specific sub-skills of abstraction were embedded. Our study addressed this gap by focusing on what specific abstraction sub-skills are addressed in a BBPE.

Initially, our study focused explicitly on the abstraction sub-skills suggested by Çakiroğlu & Çevik (2022): pattern recognition, generalization, focusing, eliminating, and decomposition. However, during the observation, it became evident that AR in the <colette/> app played a significant role in debugging, assisting students in identifying and fixing errors. Consequently, this study focused on the six sub-skills of abstraction: pattern recognition, generalization, focusing, eliminating, decomposition, and debugging.

In relation to our research questions, we found that while students demonstrated advanced skills in using the interface, and applying pattern recognition, focusing, and eliminating in problem-solving process, they struggled with generalization, decomposition, and debugging, particularly in complex tasks. The study revealed that AR in the <colette/> app supports the development of abstraction skills by guiding students through simple debugging tasks, such as identifying errors in code blocks and providing visual cues for correction. AR feature in the <colette/> app helped students recognize patterns in their mistakes and improve problem-solving. However, the AR support was less effective when the students faced duplicate code in more complex scenarios, suggesting that students struggled to apply higher-order abstraction skills, such as debugging, without additional guidance. This insight suggests areas for improvement in AR-enhanced BBPEs to better support advanced skills.

Furthermore, the decomposition and generalization skills of the students were relatively weak, with some students still struggling to break down complex problems and apply the same code blocks to different tasks. The nature of the tasks, including their complexity and the students' lack of experience with diverse building blocks, hindered their ability to effectively decompose and generalize solutions. Designing tasks with more in-depth scaffolding could address these challenges by providing clearer guidance, breaking problems into manageable steps, and offering examples that show how to apply solutions across contexts. This approach would support the development of higher-order abstraction skills.

The study validated and applied the framework by Çakiroğlu & Çevik (2022) for assessing abstraction skills on the dimensions interface structure, problem-solving process, and block structure. However, we found that the framework's broad categories of "insufficient," "sufficient," and "advanced" were not nuanced enough to capture the diversity in student performance. The incorporation of Ye and Salvendy’s (1996) five levels of abstraction provided a more detailed evaluation.

Most students reached the functional level of abstraction, showing the ability to organize and plan solutions but struggling with generalization in complex tasks. A few reached the conceptual level, demonstrating the ability to break down and generalize solutions, while one group remained at the logical level, understanding block purposes but failing to provide complete solutions. These findings provided deeper insights into the development of abstraction skills when using a BBPE with AR.

The study highlighted the value of immediate feedback, such as AR features, in enhancing debugging skills but noted limitations in handling duplicate code. This limitation led to students expecting to have the correct solution, but their solution was not acknowledged by the system as correct. Their inability to effectively debug and complete the assigned tasks resulted in them remaining at a functional level, rather than progressing to higher levels of abstraction. Future studies should investigate enhanced AR interactions, emphasizing scaffolding that supports generalization, decomposition, and advanced debugging mechanisms. By enhancing the AR feature of BBPE, researchers could investigate how such advancements might facilitate the development of higher level of abstraction, enabling students to move beyond the functional level to achieve a higher level of abstraction in programming tasks. Furthermore, although the study's small sample size and short duration limit generalizability, it offers valuable insights into assessing abstraction skills in AR-enhanced BBPEs.

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“Same, same but different?” – Extending a mathematical problem-solving model to computer science problems

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The long tradition of problem-solving in mathematics education research drew the attention of other disciplines to the emerging models for characterising problem-solving processes. One such discipline is computer science, which relies to a similar extent on its own models as it does rely on Polya's pioneering four-phase model from 1945. However, these applications of Polya's model are neither based on empirical evidence of the suitability of this mathematical model for computer science problem-solving processes nor do they consider further developments since then. The study presented here focuses precisely on this empirical legitimisation ensuring that the models used are up to date and providing first indications that an interdisciplinary model might indeed be appropriate.

Keywords: Problem-solving, Polya, Computer science education, Secondary education.

Problem-solving in mathematics and computer science education

Problem-solving in mathematics education has been a tradition for more than a century. It has been an integral part of school curricula for mathematics around the globe for many years (OECD, 2020). Despite or precisely because of its long-known relevance for mathematics education, the research around mathematical problem-solving has constantly evolved (Liljedahl & Cai, 2021), starting with George Polya’s (1945) famous normative four-phase model, consisting of (i) *Understanding the problem*, (ii) *Devising a plan*, (iii) *Carrying out the plan* and (iv) *Looking back*. Polya marked the beginning of a new era by re-introducing heuristics to the science of problem-solving, generally applicable questions to guide problem-solving processes and a variety of practical examples for teachers and scientists alike. With Schoenfeld (1985) the understanding of mathematical problem-solving receives a more goal-driven character, leading to a new *Exploration* phase, refined definitions of Polya’s existing phases, and a partially circular structure. Fostering detours and improving the quantifiability of problem-solving processes, Rott et al. (2021) present a descriptive five-phase model incorporating the fundamental ideas of problem-solving by Polya (1945) and the advances by Schoenfeld (1985). These and many more studies are proof not only of the intense problem-solving research conducted in mathematics education – currently focusing on teachers’ professional development (Chanudet & Favier, 2023) and interdisciplinary chances (Tran & Nguyen, 2023) – but also of the development since Polya (1945) published *How to solve it*.

In computer science (CS) education, however, problem-solving research is still at its beginnings with only a few forays (Romeike, 2008; Schubert & Schwill, 2011) and no fixed place for problem-solving in school curricula, since CS communities in many countries are still discussing what these curricula should look like (Webb et al., 2017). Nonetheless, in discussions about the CS curricula creation and reform, problem-solving is deemed to be an integral part (European Commission: European Education and Culture Executive Agency, 2022), which highlights the importance of understanding CS problem-solving processes to contribute strategies and material for teaching problem-solving in

future CS classes at school. A closer look at existing CS problem-solving research reveals that a substantial part is guided by Polya's (1945) model, although it is primarily mathematical (Nance, 2016; Oehler & Ludwig, 2024; Thompson, 1996). One possible reason for a mathematical problem-solving model being used in CS could be the increasing research on interdisciplinary skill development in recent years (Tran & Nguyen, 2023) which motivates the investigation of problem-solving processes under the influence of each other's disciplines. Yet, neither evidence nor research justifying the actual applicability of a mathematical model in CS nor in-depth considerations of the described advances in mathematical problem-solving can be found in the CS studies using Polya.

Research question

Considering the historical and current state of problem-solving research in mathematics and CS education, a need for empirical research on existing and future interdisciplinary usage (between mathematics and CS) of mathematical problem-solving models emerges. The pilot study presented in this article is tackling this need by investigating the following research question:

Is it possible to extend a recent mathematical problem-solving model and the corresponding coding manual in a way that meets the requirements of problems from both disciplines?

The answer to this research question holds the potential to provide the theoretical, model-based foundation for the implicit assumption of previous CS problem-solving studies that Polya's model is – to a certain degree – suitable for CS problem-solving processes. The desirable result of this study would be a refined model with a coding manual suitable for problem-solving processes of both disciplines. To decide whether mathematical and CS problem-solving processes might be too far apart for a reasonable interdisciplinary model the following preliminary criteria have been developed:

(i) *Can each phase of the chosen mathematical model be identified in most of the investigated CS problem-solving processes?* This criterion assumes that for an interdisciplinary model, each phase should have relevance in problem-solving processes of both subjects.

(ii) *Can a sufficiently large percentage of each process be coded with the derived model?* Since the extended model should serve both disciplines evenly, this *sufficiently large percentage* should at least be over 50 percent and in most cases significantly higher. Equivalent to criterion 3, the exact threshold depends on the problems, problem solvers and the surrounding setting.

(iii) *Does the unresolved percentage contain less than three new definable phases?* Three, since over the last 75 years most Polya-related models did not exceed five phases (Rott et al., 2021) and extending an existing model by over half might rather call for a completely new model.

(iv) *Can the potentially newly defined phases be found in mathematical processes?* In line with the conditions of criterion 2, newly identified phases should also serve both disciplines. This means that in suitable and comparable settings (e.g. mathematical problems with the use of technology) the newly identified phases can also be found in mathematical problem-solving processes. Findings under this criterion can further motivate research in mathematical problem-solving.

Methods

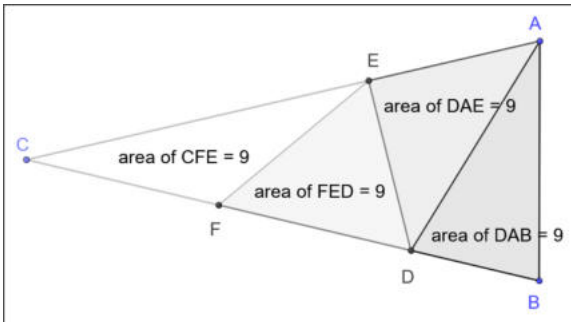
Study design: The core methodological output of this study – of which only the piloting results are presented here – is the development of an interdisciplinary problem-solving model with a corresponding coding manual. This study uses a recent mathematical problem-solving model (described in the next section), serving as the foundation for the deductive development of this manual. The next development step is an inductive extension by analysing problem-solving processes from four different settings (see section "Settings & Problems"). In total, eight pairs of upper secondary students will receive one problem from all settings with each pair solving the problems in a different order – according to the Latin-Square-Design (Field, 2002). The students are selected from the German enrichment program *Digitechnikum* (Wetzel & Ludwig, 2020) to ensure a minimum of comparable prior CS knowledge. The pilot study comprised the deductive derivation and a first glance into one process from each setting, to clarify whether an inductive extension with an interdisciplinary background could lead to a reasonable model considering the developed criteria. Four upper-secondary students (3 male, 1 female) from the *Digitechnikum* with comparable knowledge in CS were grouped in pairs. Each pair was asked to solve two of the four problems while being filmed with no content-related interventions. These processes were then coded using the deductively derived coding manual described in the next paragraph.

Model & Manual: As it is required to consider both Polya's original model and the major developments in recent problem-solving research after Polya, the decision for an initial model, from which an interdisciplinary one could be derived, falls on the descriptive five-phase model of Rott et al. (2021). This model is especially suited for several reasons. Firstly, it evolved directly from Polya (1945) via Schoenfeld (1985) comprising the five phases described in Schoenfeld (1985): *Analysis* [A], *Planning* [P], *Exploration* [E], *Implementation* [I] and *Verification* [V]. Secondly, Rott et al. (2021) describe four additional phases not specific to problem-solving – *Reading* [R] (already in Schoenfeld, 1985), *Writing* [W], *Organisation* [O] and *Digression* [D] – allowing a mathematical problem-solving process to be sufficiently characterised. Thirdly, this model allows for transitions from each phase to another accounting for any possible detour with the potential to reveal process patterns among different tasks, problem solvers or – as in this case – settings. The manual was newly developed deductively by searching for definitions and indicators for each phase in Polya (1945), Schoenfeld (1985) and Rott et al. (2021). Indicators which have changed over time are represented by their most recent version. The resulting manual was first tested in Oehler & Ludwig (2024) and further improved before the usage presented in this paper (see section "Results").

Episodes: Schoenfeld defines an episode semantically as "[…] a period of time during which an individual or a problem-solving group is engaged in one large task […]" (Schoenfeld, 1985, p. 292) and Rott et al. (2021) highlight that "the idea of episodes as macroscopic chunks implies a certain length, thus individual statements do not comprise an episode […]" (Rott et al., 2021, p. 743). In this pilot study each process was first pre-partitioned into semantic episodes, followed by assigning phases to these episodes. However, in contrast to Rott et al. (2021), the coders were advised to not only consider macroscopic chunks but also shorter, maybe simultaneous episodes, since no respective knowledge about CS problem-solving process characteristics is known, yet. This slight change makes it possible to later uncover new dependencies between different phases. Simultaneously, such

empirically observed characteristics of mathematical problem-solving processes like *Planning* and *Implementation* often appearing together or *Verification* only being expected at the end of a process (Rott et al., 2021, Schoenfeld, 1985) have not been given to the coders as guidelines to allow for a more unbiased look at CS processes.

Settings & Problems: Four settings were defined to cover a large field of problem application in both disciplines (M-P, M-C, CS-P, CS-C): Problems from mathematics (M) and computer science (CS) were selected to be either solved by using pen and paper (P) or the computer (C). The latter required the task to be accomplished in a digital environment e.g. GeoGebra in mathematics or the Java Editor in CS. For this pilot study, one problem for each setting was developed of which two are shown in Figure 1, while the remaining two can be viewed in Oehler & Ludwig (2024). In settings M-C and CS-C, students were given a computer with the respective digital environment running and explicit advice beforehand that they were not restricted in using the computer or the internet, respectively. This comment was not given to students in setting M-P and CS-P, which will become relevant when evaluating and discussing the results. To increase the chances of observing proper problem-solving processes, the problems were especially difficult – knowingly at the risk of preventing acceptable solutions in time. In M-C (see Figure 1, upper problem), the students mainly struggled with identifying tools in GeoGebra which would give them more freedom in testing different approaches for constructing the given figure. One student checked an earlier task in GeoGebra using the smartphone. In CS-P (Figure 1, lower problem), the students successfully identified other alignments for the given example and developed an algorithm. However, upon *Verification* they found that their algorithm lacked the optimisation aspects required in the task and did not finish the next version in time.



M-C: Special mosaic windows: A glazier would like to produce mosaic windows in the form of isosceles triangles (here triangle ABC) with the pattern shown in the illustration, whereby each of the partial triangles (these do not have to be isosceles) should have the same area. Construct such an isosceles triangle window in GeoGebra, which retains its properties even if it is stretched in width or length. Can you find a procedure for any number of equilateral triangles?

CS-P: Sequence alignment: One area of bioinformatics is dedicated to the comparison of known DNA sections in of species, which are assumed to have the same function. During evolution, changes have occurred in these sequences due to mutations (insertions, deletions, substitutions), so that these sequences are no longer identical. The aim is now to fit these sequences, considering the different mutation types, so that as many parts of the sequences as possible match – this is known as *sequence alignment*. In the example to the right there are two deletions in S1 (or insertions in S2) which are displayed by so-called gaps "_"), as well as a substitution in the second position. How many other alignments for S1 and S2 can you find with the same number of gaps and substitutions? Develop an algorithm (step-by-step or as pseudocode) that outputs as many alignments with as many matching characters as possible and the least number of gaps necessary between two sequences.

Sequences

```
S1: GAGTT
S2: GGGACTT
```

One possible alignment

```
S1: GAG__TT
S2: GGGACTT
```

Figure 1: Problems for settings M-C and CS-P developed and translated by the first author

Results

The average process length was 52:05 min (SD = 05:11 min) and no process was finished successfully – an expected result given the complexity of the problems. Nonetheless, each team fully engaged in the problem situation and tested, as well as verified different approaches. The pre-partitioning in semantic episodes was conducted as consensus coding among two raters for the first two videos and completed by the first author for the remaining two videos. This led to an average number of episodes of 37 (SD = 11.34) and a mean duration of 01:35 min (SD = 16.58 sec) with both processes using pen and paper having visibly fewer and longer episodes than those using the computer. These episodes were then coded using the nine phases from Rott et al. (2021) plus a tenth phase *Unknown* [U] (none of the nine known phases matches an episode) by two additional raters, as well as the first author. The IRR using Fleiss' Kappa is displayed in Table 1 for all phases specific to problem-solving and cumulated for each of the four settings (cumulated values also consider the phases R, W, O and D). A clear improvement from the first analysis in Oehler and Ludwig (2024) is visible, with four out of five problem-solving specific phases now having *substantial to almost perfect agreement* ($\kappa > .60$) after Landis and Koch (1997). Only *Exploration* falls short with *fair to moderate agreement* ($.21 \leq \kappa \leq .61$). The *poor to slight agreement* ($\kappa < .21$) values, especially for *Writing,* root mostly from only a few episodes being coded as such by one rater. The same applies to *Unknown* in process M-C – yet this process plus process CS-C reveal an interesting novelty. In Oehler and Ludwig (2024) – where episodes were defined as 10 consecutive seconds – for process CS-C, all episodes coded as *Unknown* were independently characterised by both raters as *Research* (with $\kappa = .87$), which led to a preliminary definition of this phase as *the active search for helpful information that can be found neither in the task nor in one's memories. The search is carried out in an external, knowledge-encompassing medium (e.g. formulary, Internet, etc.) by setting search goals and parameters in advance.*

Table 1: IRR using Fleiss' Kappa (κ). All mean values were separately calculated also considering the not problem-solving specific phases *Reading*, *Writing*, *Organisation* and *Digression*

Phase → Setting ↓	A	E	P	I	V	U	Mean (setting)
M	*.63*	*.25*	*.61*	*.82*	*.67*	*< .01*	*.65*
CS	*.68*	*.58*	*.63*	*.65*	*.79*	*1.0*	*.72*
P	*.74*	*.43*	*.42*	*.57*	*.65*	*NA*	*.60*
C	*.59*	*.32*	*.78*	*.83*	*.79*	*.85*	*.73*
Mean (total)	*.66*	*.38*	*.62*	*.73*	*.71*	*.85*	*.69*

In this second analysis, the semantic pre-partitioning yielded episodes at the same positions and both raters rated them independently as *Research* – this time with *perfect agreement*. Moreover, the only *Unknown* episode in process M-C was also characterised as *Research* by one rater and subsequently confirmed in a consensus coding. To complete this finding, no other potentially new phase was

characterised in the context of the *Unknown* phase. Since the overall IRR greatly improved to a point where no phase has a mean agreement less than *fair to moderate* with the majority showing even *substantial to perfect agreement*, a consensus phase coding was conducted for each process. This, in return, allowed for the investigation of problem-solving-specific phase transitions displayed in Figure 2. A transition describes the change from two consecutive phases (e.g.: E followed by Z followed by I: E→I with Z being skipped) or one phase starting and ending within the scope of another phase (e.g.: U while I: I→U→I). This type of visualisation can uncover relationships and indicate dependencies among certain phases. Looking at *Planning* and *Implementation*, which was reported by Schoenfeld (1985) to often appear together, the bidirectional transitions in three out of four processes seem to confirm this tendency. Interestingly, in all processes, bidirectional transitions between *Exploration* and *Implementation* can be observed, as well. In the same way, the absence of transitions holds information about possible process directions, such as *Verification* never following *Analysis* indicating that a certain amount of insight and progress via *Exploration*, *Planning* and *Implementation* must be achieved before *Validation* is possible.

Focusing on the new phase *Research* in M-C and CS-C, there seems to be a tendency for it to appear in interplay with *Implementation*, although more data is required to support this observation. Furthermore, is that all processes start with *Analysis* and end either with *Implementation* or *Verification*. Concerning the four criteria the following preliminary results can be observed: (i) Each original phase of Rott et al. (2021) could be observed in all processes apart from *Verification* in CS-C. (ii) After the consensus coding, no episode was left unassigned, and only in CS-C *Unknown* (later characterised as *Research*) comprised six out of 49 episodes. (iii) Considering all processes, all episodes coded as *Unknown* were – in six out of seven cases independently – characterised as *Research* and no other new phase was identified. (iv) Only one episode in M-C was coded as *Research* in the consensus coding, with M-P not having explicit research opportunities during the process.

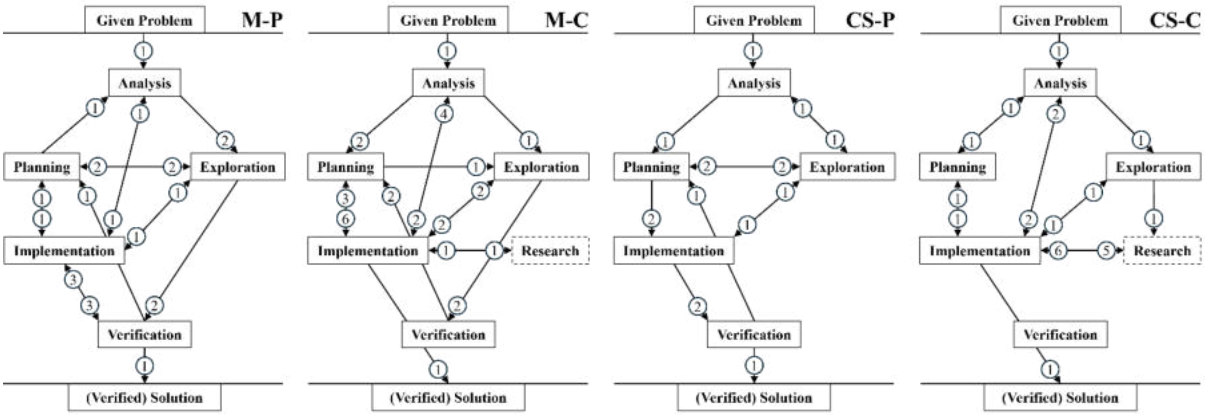


Figure 2: Visualisations of quantified phase transitions for all four settings (excluding phases not specific for problem-solving) with the number of occurrences in circles, following Rott et al. (2021)

Conclusion

This study was motivated by the numerous occurrences of George Polya's (1945) problem-solving model in CS education studies (Nance, 2016; Thompson, 1996), neglecting the profound research and developments in mathematics education since then concerning his and many other problem-

solving models (Liljedahl & Cai, 2021). In times when mathematics and CS education share increasing fields of application and the boundaries between these disciplines become more and more blurred (Romeike, 2008), a solid empirical foundation about the usability of mathematical problem-solving models in CS education is required. Hence, this research project investigates whether it is *possible to extend a recent mathematical problem-solving model and the corresponding coding manual in a way that it meets the requirements of problems from both disciplines.* Four criteria have been defined to eventually answer this question. During this pilot study, first preliminary results concerning these criteria could be collected: Criteria (i)-(iii) were met, which means that (i) each phase of the descriptive problem-solving model by Rott et al. (2021) based on Polya (1945) and Schoenfeld (1985) could be found, (ii) each process was sufficiently codable using this model and (iii) not more than three new phases were identified analysing the episodes coded as *Unknown*. Still, the latter resulted in the definition of one potential new interdisciplinary problem-solving phase – *Research*. In Oehler and Ludwig (2024) *Research* was one of the few reliably coded phases ($\kappa = .87$) and independently characterised as such. This impression could now be confirmed using semantic episodes (with perfect agreement). Criterion (iv) was missing data to get a solid impression of whether *Research* does appear in mathematical problem-solving processes, as well. In addition, students in M-P and CS-P were not explicitly given the opportunity to research, which requires investigating, whether the occurrence of *Research* is rooted in CS problems or the work with computers, in general. In addition, the complexity of the problems prevents a more realistic view of the *Verification* phase and *Exploration* still requires sharpening in the manual. Nonetheless, having three out of four criteria being met – even with little data – the authors are confident that an extended interdisciplinary problem-solving model for mathematics and CS education is indeed possible.

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On the use of Conceptual Knowledge from Computer Science in Mathematics Education

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In its need to describe formal system so precisely that they can be implemented on machines, computer science has developed many ideas that can also help to describe and understand mathematics. In fact, some of these ideas just refine and clarify older mathematical notions. Computer realizations of concepts prove (similar to the role of models in model theory) that concepts are free of self-contradictions. The clarification of objects and their relations has, moreover, the potential to improve students' learning. The paper will investigate formal calculi, type theory, lambda calculus and the Curry-Howard correspondence and explain their contribution to math education.

Keywords: Algebra education, computer science, lambda calculus, reification, type theory.

Introduction

The relationship between mathematics and computer science (CS) is multi-faceted, multidimensional and time-dependent (Modeste, 2015). As an offspring of math and engineering it inherits methods and knowledge from mathematics, and it gives tools to mathematics as is highlighted in the movement for computational thinking (Wing, 2006; Broley et al., 2023). A great deal of current research on this investigates algorithms and procedural tasks that can be carried out by machines, i.e. how actual computation (ranging in scope from using dynamic geometry to programs written by the learner) is useful in learning mathematics. The present paper discusses a relationship that is less prominent: Computer science can provide conceptual knowledge that explains and supports certain aspects of the learning of mathematics. Only very few publications have addressed this explicitly (e.g., Kortenkamp, 2006). There is a partial overlap with a prior more foundational paper (Oldenburg, 2024) but here the aim is to collect ideas that prepare the ground for actual teaching interventions.

Among the tasks of mathematics education is the clarification of the ontological status of mathematical objects and the understanding of how symbols acquire meaning (e.g., Queiroz, 2008). Computer science faces some similar problems, although a bit more difficult due to the fact that computer science needs to put its theories to test by implementing them on computers.

Methods

In its quest to define consistent and meaningful formal systems, CS has developed a bunch of concepts. If they can be successfully realized in soft- and hardware, then one may conclude that they are self-consistent (in analogy to the role of models in logic, see e.g., Mendelson, 2015). Extending this line of argument, one has the following methods on how to transfer conceptual knowledge from CS to math and math education my means of analogical reasoning (Bartha, 2010):

1) If a method of CS allows unambiguous definition of concepts, and if it allows interpretation by humans, then one expects that the method can clarify concepts for humans, too.

2) If a distinction between realizations of mathematical concepts is necessary in CS, one has reason to expect that this distinction is useful for understanding mathematics.

It is important to note that these are heuristic rules, they provide evidence, but no proof. Nevertheless, the research question of this paper is: Do these principles yield plausible results in a number of application cases? This question will be investigated in five topics. The choice of these topics was guided by the wish to present ideas that are hardly discussed in the didactical literature, and to span a range from more concrete to rather abstract and philosophical bridges between the domains.

Topic 1: Make properties of calculi explicit

Mathematics develops and justifies calculi (e.g., Baader & Nipkow, 1998). They are, in a sense, even more fundamental than algorithms: One can view functional algorithms as functional (lambda) calculus together with an evaluation strategy (Barendregt, 1984). Calculi are used in two different ways: They may either be used for semi-automatic reasoning in a given pre-existing theory, then one seeks for calculi that are correct and, if possible, complete (i.e. give the full theory), or they may be used in a creative sense to create a theory, e.g. when a group is defined from generators and relations.

In school, students learn a lot of calculi, e.g. the calculus of equivalent transformations of algebraic expressions, the calculus of transforming equations, forming the derivative or the anti-derivative. It might be beneficial for students to develop some conceptual knowledge about calculi and their properties. For example, equivalence of expressions and equations can be defined exactly the same way (as yielding the same (truth) value under any assignment of values to the variables), and rules of the equivalence calculus are correct but not complete. The derivative calculus is correct (by the usual proofs) and complete on the class of elementary school functions, while the school calculus of anti-derivates is obviously not complete. A complete calculus exists, but is beyond scope in school. An explicit treatment of calculi could also trigger epistemological and ontological considerations: What are the objects that are differentiated? Functions (e.g. as subsets of $\mathbb{R} \times \mathbb{R}$ with some properties) or expressions (i.e. strings of symbols formulated according to algebraic grammar)? Experience shows that the vast majority of teacher students in their third year at university have no idea about this and tend to get confused when asked about it. Most of them cannot even name and explain a difference between functions and expressions. But after explaining that the symbolic text of an expression can be "compiled" to a set-theoretic function, they usually appreciate the diagram in Figure 1.

Confluence is another property of calculi that might be useful to find orientation in the world of symbolic algebra. Moreover, specifying calculus rules often requires concepts that are rarely dealt explicit with in math lessons, e.g. the order of variables and monomials.

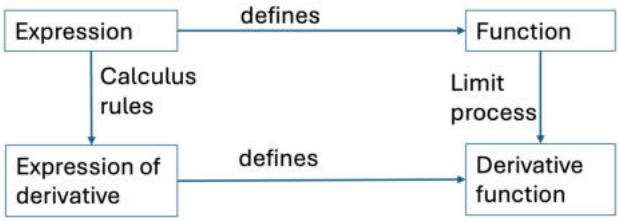

Figure 1: Relation of calculus and objects exemplified for the derivative

Topic 2: The distinction between program-time and run-time

Perhaps every algebra teacher in school knows situation in which students are confused when the values of variables may be changed. This problem even went into a cartoon where a pupil objects the solution $x = 5$ of an equation by arguing that the day before the teacher had said that $x = 3$. Understanding variable conventions correctly seems, however, to be a necessary condition for mastering other parts of elementary algebra (Oldenburg & Stacey, 2024). It is my suspect that the didactical concepts of variables as changing quantities or variables representing many numbers simultaneously are not helpful in this respect (Oldenburg, 2023), but that the distinction between program-time (or design-time) and run-time that students experience from day 1 when starting to program, may help. In fact, this distinction is even more wide-spread: When composing a PowerPoint presentation one can clearly distinguish design-time and presentation-time. This concept of different logical times explains the mentioned algebraic problem easily: One may distinguish between calculus-time where on sets up expressions, transforms them by calculus rules, and chooses (and changes) assignments for variables on one hand, and, on the other hand, calculation-time when an assignment of numbers for every variable is fixed and expressions are evaluated with respect to this assignment. At calculation-time each variable stands for one and only one number, but at calculus-time no value needs to be fixed at all. Changing the value of variables is the same as running a program several times with different inputs.

Topic 3: Type theory and algebra: Enabling the substitution principle

In programming, the data type of objects determine what operations are possible with the objects. Type declarations can ease the understanding of code. In mathematics, type theory has evolved since its incarnation in Russell's approach to avoid paradoxes to a complex system of theories (The Univalent Foundations Program, 2013) that is believed by some mathematicians to be a replacement for set theory and first order logic as foundation of mathematics (Altenkirch, 2023). For the working mathematician and for educational purposes, the less ambiguous simple type theory (Farmer, 2023) provides many benefits, but a discussion of this is far beyond the scope of this paper. Here, I'd like to draw attention to the fact that type declarations can clarify the semantics and support learning. It has been shown (e.g., Hanenberg et al., 2013) that type annotations in programming languages can improve program comprehension and detection of errors. It is plausible that analogous results hold at least in higher mathematics. Consider e.g. the action functional in Lagrangian mechanics which is often abbreviated as $S(q, \dot{q})$ although a correct definition would be $S(q) \coloneqq \int_a^b L(q(t), \dot{q}(t), t)dt$, i.e. the argument of S is a function (i.e. of type $\mathbb{R} \to \mathbb{R}$, rather than type $\mathbb{R}$), so that the type of S is $(\mathbb{R} \to \mathbb{R}) \to \mathbb{R}$. Giving correct type information might help in understanding the theory, although it seems that this has not been investigated. Moreover, one may conjecture that type declarations may support learning of more elementary mathematics as well.

From an ontological point of view, types require less commitment than set theoretic notions: Understanding the full meaning of $x \in \mathbb{R}$ requires to mentally construct the set of all real numbers and saying that x is one of its uncountably many elements. In contrast, the type declaration $x\colon \mathbb{R}$ just says that x is a real number, and to understand this one just needs an idea what a single real number

is, not what the set of all real numbers is. To emphasize this point even more, compare the statements "Snoopy is a dog" and "Snoopy is in the set of all dogs". The latter requires thinking about a complex construct, the set of all dogs (only living or also past and future dogs? Are wolves included?), while the former one only needs the ability to identify Snoopy as a dog. Of course, one has to admit that often the explicit use of sets and classes can be eliminated from propositions ("virtual classes" in the parlance of Quine (1969)), but this eases difficulties that may not be present if set theory would be replaced by type theory from the start.

There is another issue that may pose difficulties for learners and that may be clarified by giving explicit declarations. Leibniz defined identity in terms of substitution: Two expressions have identical meaning if one can be substituted for the other everywhere without changing the truth value of statements (salve veritate) (see, e.g. Mates, 1986). Quine (1960) has noted that there are opaque contexts in language that may violate this principle, e.g. although "Paris" and "the capital of France" refer to the same thing and are thus identical, it may be true that "Peter believes that London is the capital of France" but the substituted version "Peter believes that London is Paris" may be false. In fact, similar contexts exist in algebra: Although $2(x+1) = 2x+2$ holds, one may not substitute one for the other in the proposition "$2(x+1)$ is factorized" because this yields the wrong statement "$2x+2$ is factorized". The problem can be resolved by recognizing, that the type of $2x+2$ is not declared. If one declares $2x+2 : \mathbb{R}$ and $2(x+1) : \mathbb{R}$, then they are equal and can be substituted for each other. However, "factorized" is not a predicate for objects of type $\mathbb{R}$. On the other hand, if the expressions are declared to be of type 'expression', then they differ and one has the property of being factorized, the other not. Hence, type-respecting substitutions heal this problem.

Topic 4: Reification can be formally understood as functional abstraction

The mental construction of mental objects is considered in a large variety of approaches, e.g. reification theory (Sfard, 1991), APOS theory (Dubinsky & Mcdonald, 2001), or procept theory (Tall, 2014). These approaches share a lot but differ in detail. All of them are, however, only described informally and their consistency is implicitly deduced from their plausibility. However, since Church introduced the realizations of natural numbers in pure lambda calculus (Church, 1936), it has been known that objects can be constructed from functions, i.e. from processes. Other examples are the constructions of tuples from procedures (Abelson et al., 1996). Although these examples are loaded with the burden of some technical details that are absent in a natural language explanation, they have the benefit that they can be tested by realizations on computer and that they make the object creation step explicit: Functional abstraction prevents an expression to be evaluated and captures it in a "frozen" form. The same mechanism also enables to deal with infinite sequences, e.g. the sequence of natural numbers or of prime numbers. At first sight, there is little didactical benefit from this insight because it is well known that students need to overcome the urge to carry out calculations and to accept "open expression" (lack of closure obstacle, Thompson & Tall, 1991) – i.e. they need to learn to freeze a calculation to an object. What is, however, interesting is the conclusion that delaying the evaluation and capturing the calculation in unevaluated form is at least at this level not only necessary but sufficient to build new objects. This supports e.g. the sensibility of exercises that present a numerical expression and ask students to explain the structure without calculating it.

Topic 5: The Curry-Howard correspondence bridges "meaning by reference" and "meaning by use"

To understand where the meaning of words and sentences comes from is a major goal of philosophy and several theories have been developed. For the understanding of mathematics two theories (or families of theories) may be considered most influential:

Referential theories: Words acquire meaning because they refer to something. The standard interpretation of predicate logic following the line of Hilbert (Hilbert & Ackermann, 1967), Tarski (1941) and Quine (1981) falls in this class, because interpretations of symbols must be fixed, that assign e.g. a function to each function symbol, and this fixes the symbol's meaning.

Language play theories: The idea that the meaning of words is given by the way they are used in the language game is due to Wittgenstein (1958), but has spread further (e.g., Kripke, 1982).

It is clear that referential theories are more naturally combined with realistic or relativistic convictions on the status of mathematical objects, while language play theory is more in line with constructivist theories. Of course, this dichotomy over-simplifies the philosophy of meaning, but it is quite useful in its application to the learning in general (and of math in particular) and the semantics of programming languages: Assume I shall explain what the German word "Baum" means. One way is to ostentatiously exhibit objects the word "Baum" refers to, i.e., I shall point to trees, or I give examples of sentences where the word is used. Another example bridges math and computer science: What does + mean? The referential explanation is that it refers to the operation of adding numbers. This can be made concrete e.g. in the programming language Scheme, were + refers to a function object that can be applied to numbers, but it can also be stored in data structures or passed to functions. The meaning of + can also be fixed by language play. Then, + does not refer to anything, it is just a symbol that can be used according to some rules, e.g. the rule that $x + 0$ can always be changed to x or that $x + y$ can always be changed to $y + x$. In computer science, this is realized in term rewriting systems (Baader & Nipkow, 1998) such as Mathematica (Wolfram Research, 2024). Lambda calculus can be seen as a term rewriting programming language.

Both theories are in a sense in opposition: For referential theories, objects are fundamental, and language plays the secondary role of describing objects and their relations. In contrast, in language play theories, of course, language is fundamental, and objects emerge from talking as is they were there. This carries over directly to logic: The meaning of $\forall x: p(x)$ in referential theory means that $p(x)$ is true, whatever object x refers to. In language play theory it means that $p(x)$ is true, whatever expression is inserted for in place of x in the formula $p(x)$ (in philosophy, the terms used are usually objectual and substitutional quantification, see Hand, 2007). These two interpretations of quantification may be different: There are only countable many expressions, but maybe more than countably many objects. The mainstream in mathematical logic adopts the referential interpretation and justifies the rules of proof systems from this (Mendelson, 2015). The "world" of referential theories thus includes objects, model theory, denotational semantics-based proof theory as well as referential programming languages like Scheme or Julia. Besides obvious strengths, a clear drawback, especially as seen from an educational perspective is that these theories assume (at least as hypothesis in the sense of model theory or relativistic ontology) that the objects that symbols refer to "exists" in

some form. Language play theory in contrast does need such assumptions. One may first fix conventions of the language game and then objects may emerge from this when one observes that it is a good interpretation of words to assume that they refer to some object. Consider the intersection of sets: from a referential point of view, one may think of A, B as existing sets of existing objects and form $\{x: x \in A \wedge x \in B\}$. From a language play theory, one may know how to speak about having property A or property B and establish rules how to speak about things that have both properties. From this use of language, it may emerge that it is sensible to assume that there are sets of these things. So, language play theory includes processes, term rewriting, much of type theory, and programming languages such as lambda calculus, Pure or Wolfram Mathematica. I should be said that this simple dichotomic picture is adequate only when restricting to the main-stream approaches. However, there is the concept of proof-theoretic semantics which puts semantics in the language world, and models for type theories which puts type theories into the referential world.

With reification theory, we have already met a bridge between these worlds, showing that processes can be turned into objects. This bridge can be extended to span all of the two worlds: The famous Curry-Howard correspondence (Curry & Feys, 1958; Girard et al. 1989; Mimram, 2020) gives an isomorphism between formalized proofs and programs in typed lambda calculus, i.e. programs are proofs, and their types are the propositions proved by them. This surprising result seems not to be recognized much in educational sciences (Oldenburg, 2024). The whole theory is quite involved, but a key idea is that logical modus ponens is the same deriving $f(a): B$ from $a: A, f: A \to B$.

There are several insights gained from this: First of all, the two seemingly distinct worlds described above are tightly linked and therefore consistent. The important role of types that was already pointed out above is further emphasized. Computational thinking (e.g., Wing, 2006) is brought even closer to mathematical thinking because, in the language of ATD (Chevallard, 2019), the connections between the logos of math and computer science extend to the whole praxeologies of both domains. The traditional emphasis of math education on functional thinking is justified. Moreover, it seems plausible that the learning process may start out with language play and referential interpretations may represents a more developed form that should be supported by teaching.

Conclusion

This paper began by questioning whether the principles of transferring conceptual knowledge from computer science to mathematics education can provide valuable insights. The conclusion, as evidenced by the examples provided, is affirmative. The exact way this transfer is accomplished is somewhat different between the five topics, but this is not uncommon with heuristic methods. A key didactical implication is the need to place greater emphasis on precisely defining types, as clarity about types parallels clarity about propositions. It should be noted that the list of five topics discussed is not exhaustive. Additional topics worthy of consideration in this context include generative grammars, the parsing of expressions, and the concept of effective computability—a property that, for instance, the supremum of a bounded subset of $\mathbb{R}$ generally lacks. Clearly, there is an abundance of ideas in this area that remain to be explored.

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Using computational thinking to solve the Josephus problem: Insights from participants with different educational backgrounds

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A study was conducted with the aim of gaining insights into how participants with different educational backgrounds employ computational thinking (CT) to solve a problem that lies at the intersection of mathematics and computer science. Three participants—one with a mathematics and computer science educational background, another one only with a strong mathematics educational background, and a third one with a social sciences education background and self-taught programming knowledge—solved the Josephus problem and reflected on their solution. A conceptual analysis was used to examine their responses, organized around the key CT features: decomposition, pattern recognition, abstraction, and algorithm design. The participants' application of CT varied depending on their educational background and prior knowledge. They approached the problem abstraction differently - the first two participants used mathematical models, while the third one used visual representations. This study draws attention to the flexibility of the CT approach to problem-solving across various educational backgrounds and raises the issue of the nature of the solution.

Keywords: Computational thinking, Josephus problem, Problem-solving

Introduction

Computational thinking (CT) has emerged as a critical competency of the 21st century, like reading and writing. CT skills are essential for solving complex problems in various fields and are now being acknowledged as important for all students, not just those in traditional STEM (science, technology, engineering, mathematics) fields (Grover & Pea, 2013). Wing (2006) defined CT as the ability to formulate problems with computable solutions in a way that can be executed by a computer or human, or a combination of both. The key features of CT include decomposition, pattern recognition, abstraction, and algorithm design.

The Josephus problem (JP) is an old problem in mathematics and computer science. Its solution attracted a lot of attention (Graham et al., 1989; Robinson, 1960; Sullivan & Beatty, 2012). Robinson (1960) mentions the original version of the problem. The JP was chosen because it is non-trivial and can be solved from a mathematical perspective with complex proofs, or from a computer science perspective that makes it accessible to solvers without advanced mathematical knowledge. A reformulated version, used in this study, is the following: N people, labeled 1, 2, 3, …, N stand in a circle. The counting starts from the first person (position 1), and the k-th person is eliminated. The counting continues from the next person (position k+1), and the k-th person is eliminated. The game ends when only one person remains. Given N and the step size k, the goal is to determine the initial position of the last person remaining in the game.

This study aims to gain insights into how three participants with different educational backgrounds employ CT to solve the JP, thus contributing to a broader understanding of the way the CT skills are employed across various educational backgrounds.

The following research question is addressed in this study: **How do participants with different educational backgrounds approach the Josephus problem using CT?**

To answer this question, we analyzed the participants' reflections on their problem-solving process, after they solved the JP. The reflection was structured around the participants' use of the key features of CT: decomposition, pattern recognition, abstraction, and algorithm design.

Theoretical background

Scholars conducting research in problem-solving are recognizing the importance of CT as essential for helping students develop the necessary problem-solving skills to effectively address complex problems (Angeli & Giannakos, 2020; Barr & Stephenson, 2011; Grover & Pea, 2013; Kafai & Proctor, 2022; Li et al., 2020; Wing, 2006, 2008). CT is different than other cognitive approaches to problem-solving, like mathematical thinking or algorithmic thinking (Aho, 2012; Li et al., 2020; Weintrop et al., 2016; Wing, 2006, 2008). Mathematical thinking employs rigorous logical reasoning, abstraction, generalization, and proof to solve well-defined problems within formal mathematical systems. Algorithmic thinking focuses on procedural problem-solving by designing repeatable, step-by-step procedures, and considers the efficiency and complexity of those procedures. CT has been characterized by a list of essential components like abstraction, generalization, decomposition, and algorithmic thinking (Aho, 2012; Angeli et al., 2016; Angeli & Giannakos, 2020; Grover & Pea, 2013; Wing, 2006). Looking for a definition of CT in empirical research, Ezeamuzie & Leung (2022) reviewed 18 studies. They found various approaches to the definition of CT, with most studies defining CT using a list of its main characteristics. The list differs from study to study, indicating that the CT definition remains, for now, an open issue, subjective to researchers' theoretical sensitivity.

Integrating CT into curricula comes with challenges, starting with the lack of consensus regarding a definition of CT, and continuing with preparing our teachers to create and implement pedagogical practices that are tailored specifically to the development of CT skills (Sands et al., 2018). When employed in classrooms, for example in mathematics and science classrooms, CT is conducive to pedagogical practices that enhance the content of those disciplines (Weintrop et al., 2016). Postelnicu & Postelnicu (2020) describe a successful teaching experiment with high school students involved in solving a problem that encourages the development of computational thinking skills. Educators advocate for inclusivity and access (Grover & Pea, 2013) and recommend that CT should not be seen as just another set of problem-solving skills suited for the digital age, and it should be accompanied by a critical understanding of the societal impact of technological advances (Kafai & Proctor, 2022).

Based on the literature review, we adopted the following working definition: *CT is the cognitive process of solving complex problems by formulating them in a way that enables their solution through a series of computational steps or algorithms, which can be executed by a human, a computer, or a combination of both. It involves systematically breaking down problems into manageable components (**decomposition**), identifying patterns or similarities within these components (**pattern recognition**), creating generalizations or models (**abstraction**), and developing step-by-step procedures or rules for solving problems (**algorithm design**).*

Decomposition is the process of breaking down a complex problem into smaller, manageable components. (Barr & Stephenson, 2011; Postelnicu & Postelnicu, 2020). In the case of the JP

problem, one may consider reducing the circle size N or the step k and solving recursively. The original JP problem had N=41 and k=3 (Robinson, 1960). When trying to solve the JP for the general case, one may start with k=2 and various, small values for N. The recursive approach is evident, given the elimination rule given in the problem.

Pattern recognition refers to the ability to identify similarities or recurring themes within problems or their components (Shute et al., 2017). For the JP, this means identifying the recursive relation, defined by the elimination rule from the game.

Abstraction is the process of focusing on the essential aspects of the problem and translating it into a mathematical model or representation (Grover & Pea, 2013, Postelnicu & Postelnicu, 2020). The creation of the model or representation has the benefit of simplifying the problem, reducing its complexity, and helping with the algorithm design. For the JP solution for small values of N and k, the participants used visual representations like a rotary phone dial, a circle or a line of objects, or a deck of cards. For the model with a deck of cards see also Sullivan & Beatty (2012). Depending on the mathematical knowledge, one may find recursive relations to describe the elimination rule of the game, and then a general formula to identify the winning position in the initial circle.

Algorithm design entails creating an algorithm to solve the problem (Postelnicu & Postelnicu, 2020; Wing, 2008). It may include testing and debugging, especially if one uses the algorithm to write a program (code) in some programming language. Solving JP entails a recursive algorithm, given the elimination rule. However, writing a program and executing it are optional, and dependent on one's programming skills and access to appropriate technology.

The CT features may overlap, for example, one may use abstraction to simplify a complex problem by identifying patterns and focusing on relevant details. The abstraction process may lead to a model or representation that will help with the algorithm design.

Methods

Given the nature of the JP, at the intersection between mathematics and computer science, the selection of the participants was based on their educational backgrounds and propensity to successfully solve the JP from a mathematical and/or computer science perspective — Ava, with an educational background allowing her to successfully solve JP from both mathematical and computer science perspectives, and Bob and Cora, with educational backgrounds that restricted them to only one perspective, the mathematical and the computer science perspective, respectively.

Ava is a high school mathematics teacher with a Master's degree in mathematics and strong computer science knowledge. She teaches advanced mathematics and computer science (programming) courses, and values both mathematical and computer science approaches to problem solving. Bob is a university professor with a PhD in mathematics without any computer science training. His research focuses on the foundations of mathematics, the way mathematical knowledge is constructed. For him, games with algorithms describing winning strategies, are not interesting. Cora is a high school social sciences teacher, with a Bachelor's degree in psychology, who loves computer games and has learned programming (Python) on her own. She claims that she has never been good at math, and therefore avoided mathematics since she graduated from high school. The most advanced mathematics course

she took was the equivalent of College Algebra, taken in high school, more than 30 years ago. As a teacher, she has employed Snap! (block-based, graphical programming language) and Python to teach high school students how to design algorithms and write code for games and animations.

The participants were asked to solve the JP and were allotted as much time as needed (from several days to more than one week). The original problem given to them was titled the Sword Problem, and there was no mention of the Josephus problem. When they solved the problem, none of the participants knew that the Sword Problem was the same as the JP. After they solved the problem, a questionnaire containing the working definition of CT (stated above, in italics) and the following question was administered by email: *To what extent does the JP facilitate the development of computational thinking skills? Discuss how solving the problem encourages decomposition, pattern recognition, abstraction, and algorithm design, and provide examples to support your answer.*

The participants' written responses were analyzed using a conceptual analysis. To perform the conceptual analysis (Postelnicu & Postelnicu, 2013; Thompson, 2008), we focused on understanding the constructions elicited by participants. Conceptual analysis involves creating models of understanding that hypothesize the mental actions one must perform to perceive a situation in a particular way. To understand another's understanding the researcher needs to construct a coherent interpretation of their meanings, a compatible fit with what they demonstrated through observable actions, in this case, the participants' responses, structured by the main CT features: decomposition, pattern recognition, abstraction, and algorithm design.

The participants' written answers included both the solutions and the reflections on the JP solutions. Their reflections did not contradict in any way their solutions. Given the space constraints, we analyze here only the participants' reflections on their problem-solving process. We present the analysis for each CT feature and participant, using the excerpts from the participants' reflections (see Table 1, below).

Analysis

Table 1. Excerpts from the participants' answers structured by CT features

CT	Participants' Answers		
	Ava	**Bob**	**Cora**
Decomposition	"JP can be simplified by considering $k = 2$ and taking some samples for N the number of people. If $k = 2$, each round around the circle reduces N by approximately half. […]"	" One will try several concrete cases […] Looking at small steps will teach you what is happening there, and then you can create conjectures for some other cases, and then you may prove them in the general case."	"Trying to keep track of a shrinking circle of people […] I literally was bouncing the nickel around the circle of pennies "killing" every other one until the ultimate final person. […]" (*Researcher's note: Initially, Cora solved JP for k =2).*

Pattern recognition	"[…]Thus, the use of the rule of the game leads to a reduction in the number *N* each round, and after a finite number of complete turns, the diminished *N* decreases to reach 1."	"[…]Next comes pattern recognition, as one needs to know what happens modulo some number."	"I was thinking about a solution to the JP […], just to see if there were any kinds of patterns I could notice to even set up a plan for solving the problem."
Abstraction	"From a mathematical perspective, JP is difficult to approach. The case $k=2$ involves mathematical induction, and use of special functions, like floor function, which are not accessible to most high school students."	"[…] One needs to come up with a formula for the function defined recursively. In the absence of inspiration, one will try several concrete cases and guess a formula, to be then proved by induction."	"[…] I literally took a bunch of pennies […] and tried to just play with the pieces of the problem. The breakthrough […] was reimagining the 'circle of people' as a 'line of people' instead. […]It was much easier to picture it as survivors 'going back to the end of the line.' […] One last final abstraction […] was reimagining the problem as a deck of cards.[…] "
Algorithm design	"To account for the remaining players after each round a possible solution for the algorithm involves two lists. The first list will hold the remaining players after each round, being updated after each round. The other list will have the rule of elimination applied for the new round, and then will replace the first list."	"[…]One needs to come up with a formula […] So, one cannot call the solution to the problem simply computational, given that there is no general rule for the solution of such problems."	"From here – solving the JP was pretty straightforward: […] using python – create a variable that is an empty list; populate that list with a random number of people (the circle or line of people in the JP); copy item 0 to the back end of the list; remove item 0 from the list; repeat until the length of list is 1.[…] Reimagining the problem as a deck of cards; […] 1. Take the top card and place it on the bottom of the deck; 2. Take the top card and remove it from the deck; 3. Repeat until there is only one card remaining."

Decomposition – see Table 1, the row referring to decomposition: Ava managed the complexity of the problem by considering concrete and smaller values for *N* and *k* and using recursive reasoning to decrease the number *N* according to the elimination rule of the game. Likewise, Bob started by solving the problem for concrete, small values of *N* and *k*, and focused on establishing a recurrence relation to describe the rule of the game and solving smaller instances to understand the problem better. To keep track of the way *N* decreased with each iteration of the game, Cora used $k=2$ and tried to reason recursively, for the case $k=2$ and small values of *N*.

Pattern Recognition – see Table 1, the row referring to pattern recognition: Using the rule of the game and recursive reasoning, Ava noticed the way N decreased with each iteration of the game. Bob used modular arithmetic to find a formula for the position of the winning person in the game. Cora sought patterns to establish a plan for solving the problem and after successful abstraction, she would find the recursive pattern of the game.

Abstraction– see Table 1, the row referring to abstraction: Ava found a formula for the initial position of the winner of the game (for the case k= 2) and proved it using mathematical induction. Then, she found a general formula for any N and k. Likewise, Bob used the recursive definition and found a formula for k=2. Cora used representations (e.g., line of pennies, deck of cards). Interestingly, the representation with the deck of cards led her to discover the JP rule for the case k=2, known as the Australian Shuffle (Sullivan & Beatty, 2012).

Algorithm Design– see Table 1, the row referring to algorithm design: Ava designed a recursive algorithm using two lists to manage the players and the rule of elimination. Bob used the algorithmic steps embedded in the rule of elimination to make and test conjectures but remained focused on finding a general formula and proving it. For him, JP needed to have a solution expressed as a general formula, rather than a recursive, algorithmic one that could be executed by a computer. Cora created an algorithm and a program in Python, based on her abstractions from the representations with pennies and the deck of cards.

To summarize, the analysis revealed that the participants approached the JP differently. Ava used both her mathematical and computer science knowledge. She decomposed the problem by reducing N and k, recognized the recursive pattern imposed by the rule of elimination and use it to design an algorithm. She found the general formula and proved it by mathematical induction. Bob relied on his strong mathematical knowledge. He used particular values for N and k and the recursive rule of elimination to make and test conjectures for a general formula. His goal was a general formula, not a recursive algorithm. Cora abstracted the problem through visual representations (a circle of pennies that became a line of pennies, a deck of cards). The abstraction helped her formulate the problem in an algorithmic way and she could write the code (program in Python). She considered the problem solved once the program was executed successfully.

Discussion and conclusion

JP, with its elimination rule, was formulated in a way that supported algorithmic thinking. It was not surprising that Bob and Ava approached the JP based on their mathematical and computer science knowledge, given their educational backgrounds. Both mathematical thinking and CT share algorithmic thinking (Li et al., 2020). The surprise was Cora, who identified herself as a "visual learner" and used representations "to create a mental image or abstraction of the problem." Those powerful abstractions helped her with the complexity of the problem by reducing the number of people N and the step k (decomposition), pattern recognition (she imaged the deck of cards and discovered the rule of the game as the Australian Shuffle), and the algorithm design (the way the game played when she replaced the circle by a line helped with the design of the algorithm with lists). The analysis of these three different problem-solving approaches helps us gain insight into the role of the educational background and prior knowledge in shaping someone's CT skills. The participants

approached the problem abstraction differently - Ava and Bob had strong mathematical backgrounds and used mathematical models, while Cora used visual representations. Moreover, Ava and Bob employed proofs (mathematical induction) to check their solution that included a general formula for any N and k, while Cora had no way of checking the correctness of her solution outside the algorithm she designed. Her solution was viable only for concrete values of N and k.

Historically, computer science evolved as a branch of mathematics. Computer science and the development of CT skills seemed like a natural addition to STEM disciplines. Cora's problem-solving strategy, based on abstractions derived from visual representations, draws attention to the flexibility of CT across various educational backgrounds. As such, the findings of this study suggest that the development of CT skills may expand to non-traditional STEM disciplines like history, literature, and art, as recommended by Grover & Pea (2013). At the same time, the study raises the issue of the kinds of problems that can be solved by employing CT and what constitutes a solution for problems like the JP, in the absence of a mathematical model or proof.

This study had only three participants, only one problem from discrete mathematics, and analyzed only the participants' reflections on their problem-solving processes. Because of these limitations, we cannot make claims of generalization. More research studies should be conducted, with larger numbers of participants from varied educational backgrounds, using a variety of problems requiring the use of CT skills. Future research should focus on the design and implementation of interdisciplinary curricula that foster the development of CT skills.

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Computational problem-solving practices in a Scratch-supported geometry and measurement teaching

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This study investigates the development of a middle school student's computational problem-solving (CT-PS) practices in the Scratch-supported geometry and measurement problem-solving (PS) process. Synthesizing related literature, we identified a conceptual framework with seven computational problem-solving practices. This study adopted the teaching experiment method to examine the development of computational thinking (CT) skills. The data comes from the third teaching sequence and an interview with a selected student. We analysed video recordings of two lessons, final versions of the student's Scratch project, worksheets, field notes, and interview transcripts. The findings indicated a cyclical process between the CT-PS practices. We will discuss the findings in the context of CT and the co-devolution of CT and mathematical thinking.

Keywords: Computational thinking, computational problem-solving practices, geometry and measurement, scratch.

Introduction

Numerous scholars have investigated how computer programming can aid in teaching mathematics since the 1960s. According to Papert (1980), students can use programming to verify their ideas through coding and gain a better understanding of mathematics through debugging. Although there are many studies on CT, it is still unclear how teaching mathematics and CT benefit one another (Ye et al., 2023).

CT can help students develop a deeper understanding of mathematical concepts by requiring them to decompose complex problems into simpler parts and identify patterns and relationships (Brennan & Resnick, 2012; Wing, 2006). This process can enhance students' ability to reason abstractly and quantitatively, which are key components of mathematical thinking (CCSSO, 2010).

Conversely, mathematical thinking can support CT by providing a foundation for understanding algorithms, logic, and PS strategies. Mathematical topics such as algebra, geometry, and calculus provide contexts for developing CT skills, such as pattern recognition, abstraction, and algorithmic design (Grover & Pea, 2013).

Hickmott et al. (2018) emphasized that while the literature emphasizes the reciprocal relationship between CT and mathematical concepts, there is still a lack of research on how CT and mathematics might co-develop (Ye et al., 2023). Lockwood and De Chenne (2020) argue that while programming facilitates learning mathematics in specific contexts, it cannot be claimed to be superior to traditional paper-and-pencil methods across all mathematical domains. Consequently, further research is needed to explore the interactions between mathematical thinking and CT and investigate the advantages and challenges of integrating these approaches. This study examines the development of a middle school

student's CT-PS practices in the Scratch-supported geometry and measurement PS process. In the CT investigation approach of this study, geometry and measurement provide contexts for developing CT skills. Specifically, the study seeks to answer the following research question: How does a middle school student develop computational problem-solving (CT-PS) practices within the context of Scratch-supported geometry and measurement problem solving?

Conceptual Framework: Computational Problem-Solving (CT-PS) Practices

Weintrop et al. (2016) provided a taxonomy to assess computational practices in mathematical and scientific activities comprehensively. They grouped PS practices that are effective for working with computational tools under the term "computational problem-solving practices." Later, various researchers proposed different CT practices (Brennan & Resnick, 2012) and CT-PS practices (Kong, 2019; Ng & Cui, 2021).

To provide a holistic analysis of the CT-PS process, the conceptual framework of the present study includes the following seven practices considering the related literature as reported above: (1) *problem decomposition*, the process of breaking down a complex problem into smaller, more manageable components to simplify it (Waller, 2016), (2) *being incremental and iterative*, an approach to incrementally improve and iteratively enhance solutions through continuous feedback and adjustments (Brennan & Resnick, 2012), (3) *modelling*, creating abstract representations or simulations of real-world problems to analyse, understand, and develop solutions more systematically (Greefrath et al. 2018), (4) *algorithmic thinking*, the process of designing step-by-step procedures or instructions to solve problems efficiently and systematically (Wing, 2006), (5) *abstraction and modularization*, breaking down complex systems into simpler, manageable parts and focusing on essential aspects while concealing unnecessary details (Brennan & Resnick, 2012), (6) *reuse and remixing*, taking existing ideas or components and adapting them to create new solutions or enhancements (Brennan & Resnick, 2012), and (7) *testing and debugging*, systematically examining and fixing problems or errors in a system or code to make sure it works as intended (Brennan & Resnick, 2012). These practices are critical for understanding the cognitive strategies students develop at different stages of the PS process and for improving the efficiency of the process.

Methods

This study adopted the teaching experiment methodology, one of the qualitative research methods (Steffe & Thompson, 2000). The teaching experiment aimed to examine the development of CT skills. It focused on developing CT concepts, CT-PS practices, and CT perspectives. This study is limited to CT-PS practices. The mathematical context is geometry and measurement. The teaching experiment consisted of three teaching sequences. It was implemented in a class of 30 students in the 5th grade of a middle public school in Istanbul, Türkiye, during the academic year 2021-2022. The first author was the mathematics teacher of the class, and the study was conducted in the school's computer laboratory. In each teaching sequence, students sat in pairs and worked on worksheets containing geometry and measurement problems. After solving the problems and writing the algorithms for their solutions, the students coded the algorithm they had written in the Scratch environment.

The study's data consisted of video recordings of two lessons, final versions of the student's Scratch project, worksheets, field notes, and transcripts of the interviews with a selected student. The interview questions with this student were determined by analysing his work in the classroom and observation data during the teaching sequence. The data of each teaching sequence were analysed, and the subsequent teaching sequence was designed. This study focused on Burak's (pseudonym) data from the third teaching sequence (which consists of an 80-minute lesson) to examine the development process of his CT-PS practices in depth.

In the third teaching sequence, students worked on the following problem:

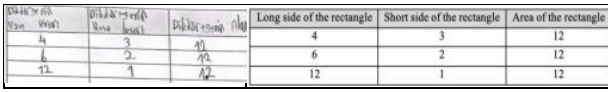
A carpet factory wants to produce a rectangular carpet with an area of 12 square meters. Calculate how many types of rectangular carpets can be produced with an area of 12 square meters. Create a program that calculates the area of a rectangle.

The problem did not explicitly state that the side lengths of the carpet should be integers, which may have influenced how students approached the problem. This limitation was addressed through teacher guidance during the teaching sequence. Grounded in pedagogical principles and supported by existing literature, the teacher led students through a structured process that encouraged them to use models to solve the problem, derive the intended algorithm, and implement it within the Scratch environment. During the data analysis process, the researchers analysed lessons' video recordings, interview transcripts, the student's Scratch project, student worksheets, and field notes of the selected teaching sequence. They evaluated the data within the scope of the CT-PS practices that constitute the conceptual framework. Since the teacher was a participant observer, the second author also codded the data, and possible disagreements were resolved to increase the study's trustworthiness.

The collected data were analysed using thematic analysis (Braun & Clarke, 2006) to examine the student's CT-PS practices. Initially, the researchers open-coded the data from student interviews, Scratch projects, and observation notes. These codes were then grouped into themes based on the CT-PS practices outlined in the conceptual framework (e.g., Brennan & Resnick, 2012; Kong, 2019). For example, instances where the student revised their algorithm after testing it using Scratch were categorized as "incremental and iterative." The first author conducted the analysis manually to ensure a detailed understanding of the data. The second author reviewed the emerging themes and their alignment with the framework for accuracy and consistency, enhancing the reliability of the findings.

Findings

This section first presents the findings on working on the worksheets before using Scratch. Afterward, the findings on programming in the Scratch environment will be presented within the framework of the CT-PS practices, considering the interactions between the practices.



Long side of the rectangle	Short side of the rectangle	Area of the rectangle
4	3	12
6	2	12
12	1	12

Figure 1: Table prepared by Burak to solve the problem

Concerning the process of working on the worksheets, the first finding is related to Burak's difficulty with formulating the problem. During the interview, the student stated that he could not understand

the expression ‘types of rectangles’ in the problem. After the teacher explained it on the board, he realized that he had to create different rectangles and reached the solution by creating rectangle with his friend. This finding showed that the student initially had difficulty understanding the problem and overcame this difficulty with the support of the teacher and his friend.

Another finding is that the student solved the problem by drawing rectangles. During the interview, the student stated that he drew rectangles and found their area as 12 square meters, but he did not use the table (See Figure 1) during the solution. Immediately afterward he stated that the table helped him write the algorithm and that he added whatever he did in the table to the algorithm (See Figure 2). The student expressed the variables as the short side of the rectangle, the long side of the rectangle, and the area of the rectangle in the table on the worksheet and recorded the rectangles he found in the bottom rows (Figure 1).

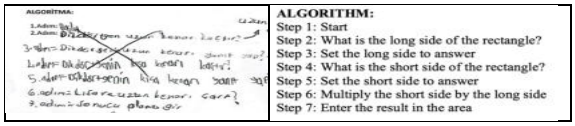


Figure 2: Algorithm written by the student on paper before working with Scratch

Drawing different rectangles helped the student solve and decompose the problem. In addition, recording the sides and area of the rectangles found by drawing as dependent and independent variables in the table (See Figure 1) contributed to the problem's decomposition. At the same time, as the student said, this process helped him write the algorithm.

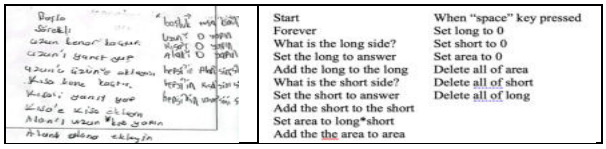


Figure 3: The algorithm written by Burak after completing his Scratch project

The teacher expected the student to solve the problem, decompose it, identify the variables in the problem, and write the process step by step in the form of an algorithm by identifying the relationships between these variables. However, the algorithm he wrote was incomplete (Figure 2). Although the teacher asked the student to write the algorithm in more detail, the student went directly to Scratch without any editing and coded on Scratch based on the incomplete algorithm he wrote (See Figure 2). The video recording shows that the student rewrote the algorithm (See Figure 3) by looking at the code file he created after completing his work in Scratch (Click here to see the Scratch Project).

The student put a cross on the algorithm he first wrote (Figure 2) on the worksheet and wrote the new algorithm he wrote by looking at Scratch (Figure 3). In the interview, the student expressed why he had difficulty in writing algorithms as follows:

Teacher: Why is it difficult to write an algorithm?
Burak: […] What I do in Scratch is easy, but the algorithm is complex and boring.
Teacher: Do you prefer to write algorithms before or after Scratch?
Burak: After.
Teacher: Why?
Burak: In Scratch, we can experiment as we want. If the coding is correct, I write the algorithm from there. If my coding is correct, my algorithm is correct.

Although the excerpt above shows that the student found writing algorithms difficult and boring, he stated that he used the algorithm to explain how he organized the scripts:

Teacher:	How did you arrange the scripts?
Burak:	[…] Even if the algorithm I wrote is incomplete or incorrect, it shows how it should be.
Teacher:	So, you still got help from the algorithm. Is that what you're saying?
Burak:	Yes, I get help from the algorithm, even if it is incomplete or incorrect

Although the student struggled to write an algorithm and avoided it, he completed his work using Scratch. This shows that he preferred producing solutions by testing with Scratch instead of writing the algorithm directly first.

While working on the problem, the student effectively used his prior knowledge from the previous teaching sequences and the information technologies course, as he reported. He also applied the commands such as 'condition, greater than, less than,' which are the learning objectives of the teaching sequence. After completing his Scratch project, he improved it by adding new commands, such as voice and sprite changes in the Scratch environment. During the interview, he said he made these revisions by testing the project. All these processes indicate *being incremental and iterative.*

Figure 4: Scripts to reset variables and lists in the Scratch project created by Burak

Burak stated that he organized the scripts according to the algorithm he wrote. He assigned all the scripts that reset variables and lists to the "when space key pressed" block (Figure 4). By identifying the variables in the problem, he reached a general expression to calculate the area of a rectangle with the models he constructed. Also, in this teaching sequence, he learned the condition command and emphasized that the program should give an error if the long side length of the rectangle is not greater than the short side length. While creating the program, he used the "forever" command to ensure that the sprite in Scratch asks the same questions continuously. He stated that the choice of sprite does not affect the results of the process and that no matter which sprite he uses, it will perform the same operations. Regarding the choice of sprite, he commented, "Even if I put an airplane, it will do the same things." These processes show the development of *abstracting and modularizing*.

Burak said he learned new code blocks such as 'conditions, greater than, and less than' and successfully integrated these codes. In addition, he used the module ("when space key pressed" block), as shown in Figure 4, in all his works similarly. These observations reveal the student's development of *reusing and remixing*.

Burak explained that he preferred to work on Scratch instead of writing algorithms and that he explained this preference by the fact that Scratch offers the opportunity to test how the sequences work by constantly trying them out and editing them when they do not work. In this process, the student tested the scripts. During the interview, Burak mentioned that the code list worked smoothly and that there was no error. However, later in the interview, the code for the sprite change did not

work, and he realized the error and explained how to fix it. He described the errors he made during the lesson and stated that he checked the codes in the part of the program where the error occurred instead of reviewing all the code blocks during the *debugging* process as follows:

Teacher: How do you solve the problem? How do you know where is the error? For example, there was an error with the sprite. Did you look at the sprite or all the codes?
Burak: No, I looked at the sprite. There were no errors in others; I did not bother with them.

From this dialogue and the student's willingness to experiment with Scratch before completing the algorithm or adding new things, we can infer that the student has improved in the testing and debugging process.

Discussion and Conclusion

This study aimed to investigate the development of CT-PS practices of a middle school student in Scratch-supported geometry and measurement PS environment. The findings reveal a cyclical and interconnected development of various CT-PS practices, which provide insights into the integration of CT and mathematical thinking.

The student's preference to test the incomplete algorithm on Scratch and try the new commands on Scratch after completing the project shows the importance of the *testing and debugging*. Since this process helped the student to revise the algorithm written on paper, this finding also points to the development of *algorithmic thinking*. It is also related to *being incremental and iterative* because he switched back and forth between the algorithm on paper and the scripts written in the Scratch environment. Moreover, the student's approach of examining only the code that does not work in the program instead of reviewing all code blocks (Brennan & Resnick, 2012) reflects the development of *testing and debugging*. These findings indicate a cyclical process among the CT-PS practices.

Some computer scientists direct argue that coding might enhance one's understanding of the problem without algorithms beforehand (Knuth, 1997), while others stress the pedagogical importance of writing algorithms (Wing, 2006). In this context, the use of an incomplete algorithm highlights the importance of algorithms.

The student's difficulty understanding the expression 'types of rectangles' and that the student overcame this difficulty with the teacher's support and progressed more easily in problem decomposition reveal that the student had difficulty in the problem formulation phase. Considering this finding, we propose adding *a problem formulation* practice to the CT-PS practices framework, as suggested by Kong (2019).

Considering the findings regarding solving by drawing a model, noticing the variables, making generalizations with the help of a table, and updating the algorithm, we claim that problem-solving, problem decomposition, and modelling stages are preliminary stages for writing an algorithm.

Although the teaching experiment aimed to develop CT rather than mathematical thinking or mathematical learning outcomes, many CT-related concepts, such as the PS process, models used, variables, relationship between variables, and conditional statements, contributed to the development of mathematical concepts and processes. The first implication of how CT supports mathematical thinking is about early algebra. In this regard, the student determined the dependent and independent

variables without knowing the concept of the variable and reached the general expression by *modelling*. This finding indicates that early algebra (Carraher et al., 2008) can develop during CT development. Indeed, Bråting and Kilhamn (2021) recommend further research to examine the relationship between CT and algebraic thinking, particularly regarding the distinction between the concept of a variable in CT and in mathematics.

The second implication of how CT supports mathematical thinking is the relationship between mathematical and *algorithmic thinking*. For example, the student's successful completion of the desired program on Scratch without writing the complete algorithm on paper, his generalizations by drawing rectangles and using tables, and his identification of dependent and independent variables showed that CT deepens mathematical concepts with the development of *algorithmic thinking* (Brennan & Resnick, 2012; Wing, 2006).

Although Bråting and Kilhamn (2021) argue that misconceptions emerge during the development of mathematical concepts in the Scratch environment, studies are showing that it is essential to apply mathematical knowledge to create tasks aimed at developing CT (Grizioti & Kynigos, 2021). In this context, as a future study, we suggest developing mathematical concepts and possible misconceptions while examining the co-development of CT and mathematics. However, it should be noted that this study focuses on an in-depth analysis of the data from a single student, which limits the generalizability of the findings. While this approach provides valuable insights into the development of CT-PS practices, future studies should include larger and more diverse samples to validate and extend these findings.

Acknowledgment

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Investigating students' testing and evaluating as part of computational thinking with the TEC model

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Computational Thinking (CT) allows learners to understand the world around them, which can be considered necessary for general education. Therefore, CT is regarded as an increasingly important learning content. Although extensive work has been done to craft definitions of CT, there is lack of a consensus and little regard for learners' actual processes. This study focuses on Testing and Evaluating (TE) as one CT activity. It presents a model of five Testing and Evaluating Components (TEC) as a framework for capturing learners' TE. This TEC model enables a more differentiated view on learners' TE by identifying and illustrating their differences and similarities. While every pair of learners in this study performed TE, they did so with varying levels of success. Explicating one's observations and comparisons followed by their use to justify the consequences was identified as important predictor of success. Implications for teaching as well as further research are discussed.

Keywords: Computational thinking, evaluative thinking, geometry, design research.

Theoretical Background

Computational Thinking (CT) is mainly traced back to the ideas of Papert (1980), whose school of thought can be described as 'thinking like a computer'. It is often referenced as a vital 21st century skill and should therefore be included in general education. Shute et al. (2017) equate the possession of CT in today's world with the ability to write in the Middle Ages. This assertion draws from the aspect of CT as a form of communication with computers, following Wing's (2017) definition of CT as processes involved in formulation of problems and their solutions such that a (human) computer can carry them out. Following this, CT may be necessary for individuals to participate in an ever more digitalized society. Based on Selby and Woollard (2013), Büscher (2025) specifies the following CT activities to focus on learner's approaches when engaging with mathematical content:

- *Reusing and Modifying*: Reuse and modify existing code as needed
- *Decomposing*: Decompose a problem into subproblems
- *Testing and evaluating*: Testing and targeted adaptation
- *Algorithmitizing*: Plan a step-by-step approach
- *Abstracting*: See general properties in special cases
- *Generalizing*: Inferring a common aspect from different special cases (p. 368)

Testing and Evaluating (TE) as a CT activity enables learners to gather fruitful learning strategies (Papert, 1980) and (employ these to) understand the world around them. *Debugging* could be considered a subset of TE, as it usually only regards the identification and fixing of errors (Shute et al., 2017), not overall evaluative processes. The identification and correction of *programming* errors is considered (software-) debugging in computer science (CS), where Böttcher et al. (2016) see this as a complex and difficult to learn skill, proposing a cyclic debugging process. This process cyclically narrows down where the error occurs and lastly fixes it. Building on this, Michaeli and Romeike

(2019) show a promising approach how software debugging might be taught with the help of teaching a systematic approach; they also note that there are also step–by–step procedures of debugging which were developed for LOGO, Papert's (1980) educational driven programming language. As TE is a broader activity than software debugging, abstracting concepts/ideas from CS (education) research into the latter might aid in closing the research gap for TE, for example by providing starting points.

Despite research efforts into definitions of CT (Grover & Pea, 2013; Selby & Woollard, 2013; Shute et al., 2017; Wing, 2017), not much is known about learners' cognitive processes in CT, even though CT is usually defined as a cognitive or thought/mental process (Selby & Woollard, 2013; Wing, 2017). Additionally, Broley et al. (2017) on CERME10 concluded the necessity to closer examine the experiences of students. Büscher (2024, 2025) presents that students exhibit different manifestations of CT and its activities. She preliminarily characterizes unsystematic, systematic and strategic TE as different approaches (Büscher, 2024). While this enables characterization of pupils' overall approach, albeit presently constrained to geometric programming, the question remains how TE works as a (cognitive) process and differs among learners. Thus, the research question (RQ) is:

RQ: How are TE processes composed and how do they differ between learners?

Methods

This research is part of a larger design research project aiming to understand how CT and mathematics (lessons) can mutually promote learning. By nature, it consists of cyclic phases of design experiments, which inform local theories about the nature of the learning object as well as typical learning progressions and obstacles of learners (Prediger et al., 2015). This study presents the initial local theories regarding TE, stemming from our first cycle. In this design experiment cycle, 26 learners (aged 13–15) were videotaped solving geometric tasks as pairs in a laboratory setting using Turtlecoder (cf. Figure 1), resulting in 638 minutes of recorded activities. Firstly, in a broad analysis, scenes were identified where tests appeared to be conducted by executing the program. These passages were transcribed and qualitatively analysed in a category-forming, inductive process (Kuckartz, 2019), producing five inductive components of TE processes. These categories were subsequently employed for a turn-by-turn analysis coding statements and actions (e.g. implementing an instruction), as recorded graphically in a timeline (Figures 3 & 5).

At first, the students were introduced to the program by drawing a square with a side length of 200 units. The next task involved drawing a larger square of 400 units, which added complexity due to the turtle leaving the screen unless the starting position was adjusted. Following this, they were tasked to draw a rectangle and then a parallelogram with side lengths of 100 and 200 units. Now, pupils were challenged to draw a trapezoid with parallel sides of 100 and 200 units. The final tasks included drawing the House of Saint Nicholas (HSN, Figure 1), a drawing puzzle, and a five-pointed star with 100 units long sides. It should be noted that not all students started the star. In this context, 'drawing' could be described as 'designing an algorithm and implementing its code, which lets the turtle draw'.

Derived from our qualitative analysis, we propose a model of components for the TE process. The next section presents this finding as a theoretical framework. Subsequent empirical insights show the frameworks ability to portray differences in learners' processes and propose potential explanations.

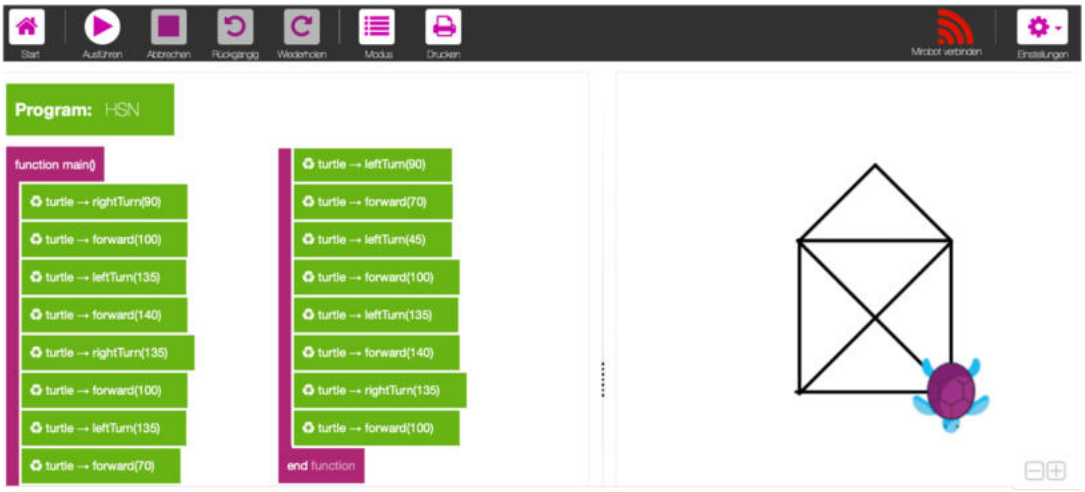

Figure 1: A recreation of Elena and Esma's (Turn 768) HSN program with the Turtlecoder at code-your-life.org/turtlecoder (© Helliwood media & education); the drawing is zoomed in for visualization.

Framework: Testing and Evaluating Components (TEC) Model

This section presents our empirically based model of TE as a key finding of this research. It is segmented into five components which are outlined below. Later empirical insights will demonstrate that these components may overlap, are neither strictly separable nor follow a strictly linear order. We aim to investigate further whether and how a coherent structural model can be devised.

A *cause for the test* is implicitly linked with the goal learners express when testing and evaluating. The reason is rarely made explicit, though the analysis indicates that it often can be inferred from the communication of the partners before they execute the test. In cases where this was not possible, it could sometimes be inferred by the actions the pupils took after executing the test. Such inferences are naturally circumstantial and open to discussion, hence no claim to absolute correctness is made.

The *test* is simply comprised of enacting a solution plan, which in this study can typically be considered as executing the code in the programming environment. However, another form of testing seems to exist, characterized by the fact that it predicts the result of a code without ever executing it. While not further discussed in this paper, this apparent mental simulation of the code shows promising points of contact for further studies examining non-programming tasks engaging with CT-activities.

In the third component one *observes the result or execution* of the solution-plan, in our case it is the step-by-step execution of the code within the learning environment. For example, this is observable, when learners narrate the path their turtle draws or explicitly describe the drawn figure.

It is linked with*, the comparison with one's expectations*, where pupils might verbalize differences in result and expectation or specification. (i.e. sketch from task). Comparing expectations presupposes their existence, which is not always given. Learners (e.g.) describe how the turtle drew 'wrongly'.

As a last component, learners *draw conclusions*, which might stem from their observations and/or comparisons, for example the need to change the parameter of an instruction. It can be seen that in such cases learners often had the next cause for a test to check the new value for the parameter, in this context a length or an angle of rotation. Such cases are called *retests* in this study.

Empirical Insights

The study provides insights into learners' processes of programming geometric shapes in the context of TE as part of CT. For instance, TE proved useful for identifying parameters whose values were not immediately apparent, particularly the lengths of the non-parallel sides of the trapezoids, the 'roof' of HSN, the angle(s) of rotation for these sides, and the lines of the star. We will now portray two compositions through the TEC model that reflect different observed general cases. The first composition (Finn & Felix) is notably structured, possibly cyclic, and demonstrates success as both participants effectively argue their consequences based on explicit observations and comparisons. In contrast, Adam and Anika's engagement with different subproblems often lacks clear argumentation for consequences or elaboration of observations/comparisons, resulting in teacher intervention.

The case of Finn and Felix –justifying consequences with observations and comparisons:

Finn and Felix have already drawn a part of their HSN (Turns 476–704) as can be seen in Figure 2a. Now they are discussing how to approach drawing the second inner diagonal.

711	Finn:	A hundred or so, no clue.
712	Felix:	Hundred, okay, how do we want to do this? [adds instruction leftTurn(100)]
713	Felix:	And then I also do it directly, how much did we have with that one again? [points to the first inner diagonal]
714	Finn:	Ehm, Turtle, so.
715	Felix:	No, I mean the length.
716	Finn:	I think the length was also a hundred.
717	Felix:	Nah, I took hundred forty, because this here triggered me [points to the left upper corner, where the three lines meet].
718	Finn:	Oh, that's right. [Felix adds instruction forward(140); executes program, cf. Figure 2b]

Starting in Turn 711 Finn suggests that they turn their turtle 100°. Felix then asks how they want to do it and implements a 100° turn counterclockwise (Turn 712). Following this, they also add an instruction to move the turtle 140 spaces (Turns 714–718). Felix asking his partner how they want to do it (Turn 712) shows that they intend to evaluate the merit of their approach as their *cause for a test*. If he was certain about the correctness of their idea and its implementation, he wouldn't ask how to do it. This heavily implies that the cause for a test can be temporally prior, with the test not necessarily occurring immediately afterwards. They formulate their plan and execute their code in Turn 718, which is their explicit *test*, the result of which is Figure 2b. They draw their conclusions from this in the subsequent turns:

719	Felix:	So, we have to go diagonal, eh?
720	Finn:	Mhm.
721	Finn:	Ah.
722	Felix:	Wrong direction.
723	Finn:	Then turtle, turn right.
724	Felix:	Yes, only possible.
725	Felix:	Hundred, eh? [changes the last rotation instruction to rightTurn(100)]
726	Finn:	Mhm.
727	Felix:	But from the rotation, it didn't look too bad.
728	Felix:	We try it out. [executes program, cf. Figure 2c]

Finn and Felix verbalize their *observations* in Turns 719 to 722, where Felix' assertion "wrong direction" (Turn 722) is also considered to be a *comparison with expectations*, assuming he

considered their solution to be somewhat correct. Subsequently, they *draw conclusions*, when they change/modify the rotation instruction to turn right instead of left (Turns 723–725). Now a new TE process is in progress, as evident by the (re-)test in Turn 728 (Figure 2c for result). While these two TE processes are undoubtedly connected, they are distinct processes. The evaluation of the new solution is considered a new and distinct reason for testing, which is supported by Felix' assertion to "[…] try it out" (Turn 728). Further, the two learners go through two more TE processes with retests:

729	Felix:	Okay, a bit less, or do we have to increase, nope less.
730	Felix:	Okay .. relaxed.
731	Felix:	Right, eighty-five or so, if not we go back to ninety. [changes last rotation instruction to rightTurn(85); executes program, cf. Figure 2d]
732	Felix:	Ninety like this, or #
733	Finn:	# yes, yes, yes.
734	Felix:	Turtle, we just have to once again. [changes last rotation instruction to rightTurn(90); executes program, cf. Figure 2e]

Four tests are conducted with results seen in Figures 2b–e while Figure 2f shows the diagonal unobscured. This approach is deemed systematic TE following the categories of Büscher (2024).

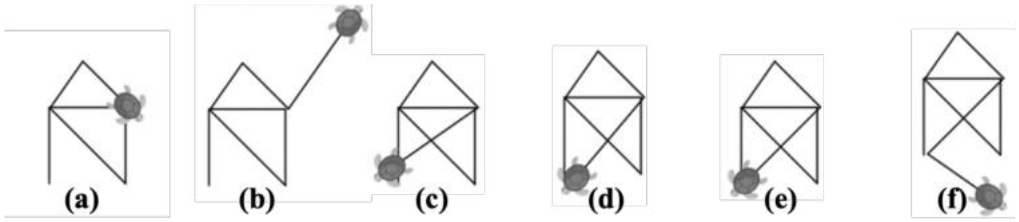

Figure 2: Finn and Felix draw the second diagonal of their HSN (© Helliwood media & education)

Figure 3 illustrates the aforementioned timeline of Finn and Felix' TE (process). It is created using the TEC model to assign explicitly observable statements and actions of one learner to one or more components. If neither action nor statement is observably present, or if they cannot be (clearly) linked to a component, the corresponding space remains empty. One second is used as the smallest measure of time. Here, Finn is represented in blue and Felix in green. Orange denotes overlap between the two, due to simultaneous actions/statements. For example, both use the same execution of the program as a test, lasting 15 seconds in this instance. This timeline illustrates the overlapping of components, for example around 50:36, when *Cause for Test* and *Test* overlap. The inability to sharply separate these subcomponents, partly due to scope for interpretation, is again evident in Felix exclaiming "Wrong Direction" (Turn 722, 50:24–50:25).

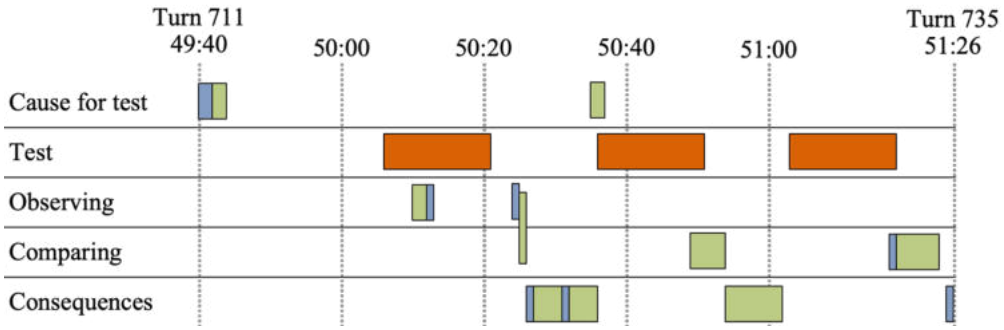

Figure 3: Timeline of Finn (blue) and Felix' (green) TEC (Turns 711–735)

This example demonstrates that the components of TE can overlap, that they cannot always be sharply separated, and that the cause of the test is open to interpretation.

The case of Adam and Anika – no justification of consequences with observations or comparisons

Adam and Anika are also working on their HSN, more precisely they try to construct their first inner diagonal and baseline. Their timeline (Figure 5) visualizes Turns 233 to 247 and was constructed as elaborated for Finn and Felix. Due to space constraints, we limit our explanations to notable sections.

231	Adam:	Yes. [Anika enters instruction rightTurn(45)]
232	Teacher:	Otherwise, I also have a tip for you.
233	Adam:	Yes okay, we can give it a try. [Anika enters instruction forward(100) and executes program, cf. Figure 4a]
234	Anika:	Ah well. [changes to leftTurn(45); executes program, cf. Figure 4b]
235	Adam:	But the line [points to slope] is too short, we have to make something longer there.
236	Anika:	Like this? [changes drawing instruction for the slope to forward(150)]
237	Adam:	Yea, let us try it. [executes program, cf. Figure 4c]

Both refer to different subproblems: While Anika retests her adjusted angle of rotation, Adam considers the slope (Turn 235) from the initial test (Turn 233) to which her retest (Turn 234) does not add new information. This raises the question which test Adam refers to. As he verbalizes a test cause in Turn 233, we consider the first test as his initial point of comparison (see Figure 5, 29:54–30:35).

238	Adam:	And there they are again, make it one hundred and fifty and then #
239	Anika:	# We have to here like this, here like this. [Anika deletes the parameter of leftTurn(45)]

Between 30:19 and 31:16 Figure 5 depicts how both jump between components of the TEC model. They are no longer in sync when they talk over each another and Anika programs while Adam speaks (cf. Turns 238–239, 30:48–30:50). It is also evident that Anika does not wait for her partner's verbal approval before executing the test, as Adams cause for test is observed a second after Anika starts it.

246	Adam:	Yes that is, that could be good, yes one hundred twenty-five is actually quite good .. then ninety, then must one hundred thirty-five go there. [Anika changes rotation to leftTurn(135) and executes the program, cf. Figure 4e]
247	Anika:	Nope, doesn't fit. [changes forward(125) into forward(1130); executes program, result not shown]

Again, Anika implements *consequences*, while Adam verbalizes his *comparison* (Turn 246, 31:35–31:37) and even simultaneously draws her comparison and consequences (Turn 247, 31:46–31:48).

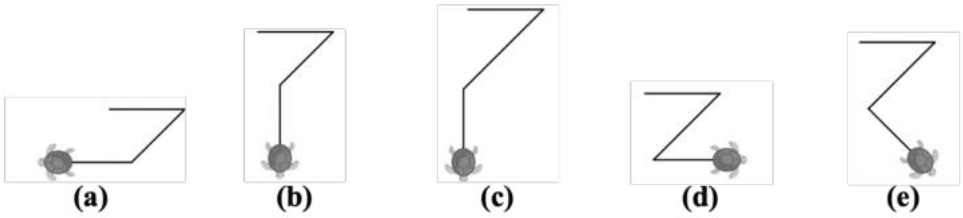

Figure 4: Adam and Anika's first attempt at the HSN (© Helliwood media & education)

This example underscores how TEC can overlap and resist clear separation. It also exemplifies how the TEC model facilitates a nuanced examination of individual processes, even with student pairs.

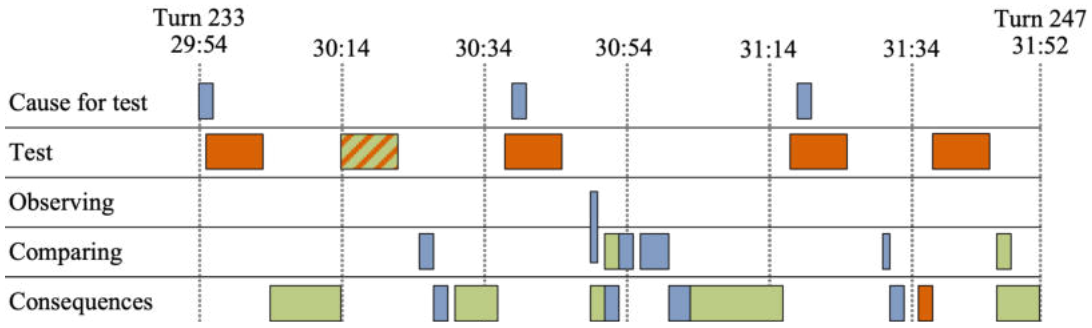

Figure 5: Timeline of Adam (blue) and Anika's (green) TEC (Turns 233–247)

Contrasting the cases

Finn and Felix's TE process(es) lead to them solving their (sub)problem at hand and later successfully constructing a HSN. Adam and Anika did not achieve their desired outcome and later started over with the teacher's help. Using the TEC model makes a major difference tangible: Finn and Felix explicate their observations/comparisons more precisely and use them to argue their consequences. Conversely, Adam and Anika often fail to justify their consequences, also worsening communication.

Various factors, such as communication, influence the success of TE processes. Tasks should thus encourage learners to articulate their observations and comparisons. By providing a structure for arguing consequences, this approach aims to enhance their processes to achieve success similar to Finn and Felix's. Teachers and researchers benefit by being able to better diagnose their pupils' processes. Successful TE processes, as exemplified by Finn and Felix, often appear as systematically structured sequences of TEC. It is unclear how this may contribute to further defining TE or may aid in constructing a coherent structural model of the components – questions we seek to explore.

Conclusion

This study invented, based on empirical observations of learners programming in a geometric context, a five-component model for describing the learners' TE. The TEC model facilitates the analysis and visualization of pupils' TE processes. When applied it demonstrates that its components can overlap, are not always distinctly separated, may appear at different times, and do not follow a strictly linear progression. Given the current small sample size, further research should assess if these components are transferable to other domains and whether a coherent structural model of TEC can be formulated.

Still, the first qualitative insights illustrate that the approach could allow for identifying potential predictors of learners' success. Explicating observations and comparisons, as discussed, seems to be helpful for justifying derived consequences. Therefore, tasks should offer learners opportunities to further these skills. Learning TE on a meta-level might also enable students to acquire learning strategies (Papert, 1980). CT coding interventions can enhance children's executive functions (Montuori et al., 2024). Our study offers initial insights into pupils' TE processes, which could explain these benefits. Teaching software debugging with a systematic process is beneficial (Michaeli & Romeike, 2019). Adapting this CS education research to TE and the TEC model provide a foundation for further research into teaching/learning TE. Thereby, the role of immediate feedback from the learning environment, which allows concentration on observing, merits further investigation.

Acknowledgment

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Analysing post-intervention mistakes in combinatorial problem solving: A study on Czech secondary students

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This study examines the mistakes made by Czech secondary students after participating in a constructivism-based intervention. The intervention aimed to reduce the students' reliance on memorised formulas and promote diverse problem-solving strategies. The study concerns four classes with 88 students who underwent an intervention within design-based research. Following the intervention, students completed a test with combinatorial problems and their solutions were analysed to identify mistakes. The results revealed that while students' overall success rates were relatively low, the intervention resulted in less frequent formula-related mistakes but highlighted persistent issues with systematic listing and partitioning. Notably, a new type of mistake emerged, involving mistakes in counting natural numbers in an interval. The findings suggest that additional focus is needed on systematic problem-solving techniques and unfamiliar problem types to further enhance students' combinatorial reasoning in the intervention.

Keywords: Combinatorics, classification of mistakes, combinatorial model and operation.

Introduction

Combinatorics belongs among school topics which present difficulties to many students due to its complexity and teaching overlying on presenting formulas as the primary solving strategy. Research on combinatorics uncovered types of mistakes students make when solving combinatorial problems, with the ones related to (mis)use of combinatorial formulas being prominent (Fischbein & Gazit, 1988; Maschiach Eizenberg & Zaslavsky, 2004; Wasserman & Galarza, 2019). On the other hand, researchers focused on students' understanding in combinatorics and how to develop it (English, 2005; Maher et al., 2010; Uptegrove, 2015).

In the Czech Republic, students often lack understanding of combinatorial problems and rely too much on formulas (Hejný & Jirotková, 2012). Textbooks do not aid teachers who want alternative methods. The first author developed an intervention based on constructivism, emphasising active learning and various solving strategies (Zenkl & Vondrová, 2023), and implemented it through design-based research. The study examines students' success and errors in combinatorial problems post-intervention.

Theoretical background

This section starts with an insight into the nature of combinatorial problems and students' mistakes when solving them, as both are necessary to design effective teaching strategies. Literature on the teaching of combinatorics follows.

The literature posits that combinatorial problems can be classified according to four aspects (Batanero et al., 1997; Fischbein & Gazit, 1988). The main classification is based on the *combinatorial model* of a) selection (selecting elements from the set), b) distribution (assigning elements to compartments)

and c) partition (decomposing a set into subsets) (Batanero et al., 1997). Problems can involve permutation, arrangement or combination *operations*, with(out) repetition. Problems also differ in *context* (numbers, letters, colours, people, etc.; Fischbein & Gazit, 1988) and in the total *number of configurations* (small, medium and large; Lamanna et al., 2022). Considering students' perspective, combination operations problems fall into two categories: unordered selection of distinguishable objects (I) and ordered sequence of indistinguishable objects (II), with students using a combination approach more frequently and correctly in the former (Lockwood et al., 2018; Wasserman & Galarza, 2019).

Piaget and Inhelder (1951/1975) identified students' mistakes in solving combinatorial problems by analysing systematic and non-systematic approaches from the viewpoint of developmental psychology. Non-systematic methods can lead to failure as they do not guarantee counting all valid options once (English, 2005).

Our study capitalises on the seminal work of Fischbein and Gazit (1988), who identified four types of mistakes, e.g., the use of a wrong combinatorial formula than required or a mere multiplication of the numbers from the assignment. Batanero et al. (1997) extended this list into 14 types of mistakes (Table 1). For example, they distinguished two types of wrong-identifying combinatorial operations: M2 and M3.

Table 1: Errors described by Batanero et al. (1997); the authors' code is in brackets

Code	Mistake	Code	Mistake
M1 (statement)	misinterpretation of the problem statement	M8 (operations)	incorrect arithmetic operations
M2 (order)	confusing if order matters	M9 (formula)	using an incorrect formula of a combinatorial operation that has been correctly identified
M3 (repetition)	confusing if repetition is possible	M10 (parameters)	misunderstanding the meaning of values of parameters in the formula
M4 (objects)	wrong distinguishing of identical/different objects	M11 (treediag)	faulty interpretation of the tree diagram
M5 (exclusion)	excluding some elements to form the configurations	M12 (property)	failing to apply the property of the binomial coefficient
M6 (listing)	non-systematic listing	M13 (cells)	wrong distinguishing of identical/different cells
M7 (intanswer)	mistaken (intuitive) solution without justifying	M14 (partition)	some partitions or types of partitions were forgotten

Batanero et al. (1997) compared the mistakes of students who were taught combinatorics with those of students who were not. Most students without instruction could not list the options systematically (M6), but after the instruction, the main problem lay in deciding if elements could be repeated or if the order mattered (M2 and M3) and there was hardly any M6. Moreover, the students made new types of mistakes, namely, using an incorrect formula of a combinatorial operation that has been correctly identified (M9) or a faulty interpretation of the tree diagram (M11).

Batanero et al. (1997) and others (Hejný & Jirotková, 2012; Maschiach Eizenberg & Zaslavsky, 2004) thus point to teaching overemphasising formulae as one source of students' difficulties. Some propose teaching combinatorics without formulae (Godino et al., 2005) and providing students with opportunities to engage with combinatorial problems, which can lead, by their nature, to rich and challenging mathematical thinking (Lockwood et al., 2020; Uptegrove, 2015). Combinatorial

problems are accessible for secondary students as they "are able to understand what a question is asking, they can begin a solution path by symbolizing outcomes, and, in the process of symbolizing these outcomes, they can engage with and discuss key combinatorial ideas" (Lockwood et al., 2020, p. 5). Research has identified recognising isomorphic problems as a powerful instructional tool (English, 2005; Maher et al., 2010; Wasserman & Galarza; 2019). Building meaning first and then developing the symbolic vocabulary (e.g., factorial, binomial coefficient) supports deep understanding (Uptegrove, 2015). Moreover, instruction based on appropriately selected problems with little teacher intervention and maximum attention to students' reasoning leads to high-level mathematical skills (Maher et al., 2010).

Context and research aim

To enhance Czech students' understanding of combinatorics and prevent their overreliance on formulas, the first author designed a constructivism-based intervention implemented by trained teachers within design-based research. The intervention is grounded in students solving isomorphic combinatorial problems (Maher et al., 2010; Maschiach Eizenberg & Zaslavsky, 2004; Uptegrove, 2015). They are encouraged to develop their own solving strategies and compare them. Among others, the systematic listing strategy is encouraged as even partial listing helps students identify patterns and structure within partial lists of outcomes and reach the correct solution (Lockwood & Gibson, 2016). Teachers allow students to use different approaches (English, 2005) and strive to engage them in mathematically rich discussions (Maher et al., 2010) without presenting combinatorial formulas and mentioning the names of combinatorial operations. Related formulas are to be used only by students who discover them by generalising solutions to isomorphic problems.

The study aims to determine how successful the students are in solving different types of combinatorial problems after the above intervention and what mistakes they still make.

Methods

The study involves 88 students aged 16-19 years from four classes in three secondary grammar schools participating in the intervention delivered by four trained cooperating teachers.

After the intervention, the students were assigned a test of five combinatorial problems of different difficulty (from easy ones such as Code and Rabbit to difficult ones such as Candy and Obraz, see below the problems from one test version) and with a different combinatorial model, combinatorial operation, context and number of configurations (see Table 2). Two equivalent test variants were created to prevent copying, and the order of the problems varied to prevent any undue influence of the place of the problem in the test.

Table 2: Characteristics of the combinatorial problems

Problem	Combinatorial model	Combinatorial operation	Context	Answer
Candy	distribution	combination with repetition	children/equation	20
Code	selection	---	numbers and letters	140
Rabbits	distribution	arrangements with repetition	animals	8
Stamps	partition	combination	slips	20
Obraz	---	---	letters	ZBAOR

Candy/Equation. Choose one of the tasks A, B.

A. Distribute 3 equal candies among 4 children: Adam, Bara, Cecilia and Dan. Identify all the ways this division can be done. (For example, we can give Adam 1 candy, Bara gets nothing, Cecilia gets 2 candies, and Dan gets nothing.)

B. Find all non-negative integer solutions to the equation: $x_1 + x_2 + x_3 + x_4 = 3$. (For example, one solution to the equation is $x_1 = 1, x_2 = 0, x_3 = 2, x_4 = 0$, and the other solution is $x_1 = 0, x_2 = 1, x_3 = 2, x_4 = 0$.)

Combination problems of category II (Lockwood et al., 2018) A and B are isomorphic (Zenkl & Vondrová, 2023) based on the binomial coefficient "6 choose 3". We can code it as *3*-sized subjects of distinct positions (Wasserman & Galarza; 2019) by putting three stickers separating four children, or four adders between three dots representing candies, or the sum on the right side of the equation.

Code. In a three-digit code, the first position is one of the letters A, B, C, or D. The next two positions are any two-digit number from 11 to 45. There are, e.g., codes A22, B45, etc. Determine the number of all the codes thus formed.

There are four possibilities for choosing a letter and 35 possibilities for choosing a number; thus, the result of this selection problem is 140.

Rabbits. A rabbit breeder wants to place 3 rabbits (white, black and spotted) in 2 pens: upper or lower. How many options does he have for housing all these rabbits? Justify which of the results corresponds to the correct solution (the pen can be left empty):
a) $2 + 3$, b) $2 \cdot 3$, c) 3^2, d) 2^3, e) $\frac{3 \cdot 2}{2}$, f) any other number.

Because three distinguishable rabbits are distributed into two distinguishable pens, there are $2^3 = 8$ possibilities for doing so.

Stamps. Mary and Cindy have six post stamps numbered from 1 to 6. They decided to share the stamps, three for each of them. In how many ways can they share the stamps?

A combination problem of category I (Lockwood et al., 2018) which can be solved via the binomial coefficient "6 choose 3" (inspired by Batanero et al., 1997).

Obraz (Picture). The letters of the word OBRAZ can be rearranged in five different positions. It can create, e.g., the "words" BAZOR, AZORB or ZOBRA. Order all these rearrangements of letters alphabetically. What word will be in position 103?

The expected strategy of this problem (De Chenne & Lockwood, 2022) is sub-problems decomposition: 24 of the 120 permutations start with A, the other 24 with B, and so on, i.e., the 97th to 120th permutations start with Z. The first six (97th to 102nd) start with ZA, so the 103rd is the first to start with ZB. The other positions are taken alphabetically by the remaining letters: ZBAOR.

Data and data analysis

Data from 88 students, covering 440 solutions, were analysed. First, the overall success rate was calculated, and incorrect solutions were identified. Mistakes were classified using Table 1, with only

the primary cause of failure coded if the solution included several mistakes. During the coding, a mistake appeared that could not be classified according to Table 1 and coded M15. Indeterminate or crossed-out (unreadable) solutions were coded as "other mistakes", and empty solutions were coded as N/A. Absolute and relative frequencies were then calculated.

Results

Table 3 shows that the average success rates ranged from 23% to 45%. The overall average success rate for all the problems was almost 38%. The students produced 167 correct solutions, 217 solutions with distinguishable mistakes, 24 other mistakes and 32 empty solutions.

Table 3: Students' average success rates for the problems

Obraz	Stamps (partition)	Candy (distribution)	Rabbits (distribution)	Code (selection)
23%	28%	32%	42%	45%

The students solved both problems with combination with(out) repetition with lower success rates than the problem with arrangement with repetition. As expected, they were most successful in solving *Code;* this problem has a familiar context and can be solved by multiplying the number of letters and numbers. The students were the least successful in solving *Obraz*, which was also the most unsolved problem (in 16 cases). The reason might be that *Obraz* was not similar to any of the problems from the intervention. It required finding the configuration in a certain position, not the total number of configurations. The success rate for *Rabbits* was the second highest, and only one student did not attempt to solve it. It is a closed problem, which may have increased its success rate.

The bold frequencies of mistakes in Table 4 warrant our attention. Forgetting some partitions or types of partitions (M14) was the most common for two "distribution" problems: *Candy* and *Rabbits*. Figure 1 illustrates this mistake; the student solving *Candy* forgot configuration 1 1 1 0, which can be permutated; therefore, four configurations are missing.

Table 4: Frequency of mistakes per problem

	M1	M2	M3	M4	M5	M6	M7	M8	M9	M10	M11	M12	M13	M14	M15	other
Candy	3	0	0	0	0	**14**	2	**11**	1	0	1	0	1	**16**	0	1
Code	7	0	1	0	0	0	0	2	0	0	0	0	0	0	**26**	3
Rabbits	3	1	0	10	0	3	8	3	0	1	2	0	0	**14**	0	1
Stamps	1	3	1	0	0	**16**	4	**18**	2	2	0	4	2	0	0	4
Obraz	5	0	0	0	0	7	**13**	2	0	0	0	0	0	2	5	**15**
Total	**19**	4	2	10	0	**40**	**27**	**36**	3	3	3	4	3	**32**	**31**	24

The problems *Code* and *Obraz* cannot be assigned any model/operation, and Batanero et al.'s (1997) study did not address such problems. In many cases, we identified a new type of mistake in the solutions to *Code*, namely, M15: Translation by 1. It consists of the student failing to count the first or the last number from the interval of natural numbers or (in fewer cases) adding one more number. For example, the students usually determined the number between 11 and 45 as $45 - 11 = 34$. The most frequent mistake in *Obraz* was the inability to start solving the problem. The students usually did not write anything or crossed out their previous records.

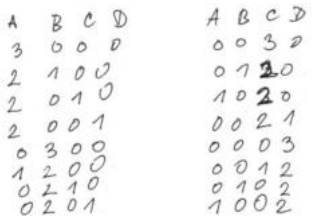

Figure 1: Students' solution to *Candy,* illustrating M14

The most frequent mistake was M6, which appeared in all the problems except *Code*. The students also had difficulties using correct arithmetic operations (M8), considering all partitions or sub-problems (M14) or counting all the numbers of the set (M15). Moreover, many students did not justify their incorrect solution (M7), and approximately 4% of the solutions were mistakes caused by an incorrect interpretation of the problem statement (M1).

On the other hand, there were hardly any mistakes in order and repetition when the operations were identified (M2 and M3), and almost no mistakes related to the incorrect use of formulas (M9 and M10), tree diagrams (M11) or the property of the binomial coefficient (M12). The mistake of excluding some elements to form the configurations (M5) did not appear.

Discussion

The students in our study underwent an intervention on combinatorics, in which problem solving and identification of isomorphic problems were emphasised, and formal terminology and formulas were subdued. The idea was to present combinatorics with understanding and prevent students' overreliance on formulas. The study focused on mistakes students made after this intervention.

We will compare our results with Batanero et al.'s (1997) comprehensive overview of mistakes. The mistakes they identified for the students *before* learning combinatorics (M6 and M14), caused by non-systematic listing or forgetting some types of partitions (in 72 out of 273 incorrect solutions) were made more frequently by our students *after* the intervention. The most common error, M6 (in 40 out of 273 incorrect solutions), was caused by a low ability to enumerate systematically (English, 2005). Thus, the ability to make systematic lists needs more attention in the intervention. Zooming in on this, we find out that almost half of the occurrences (19) of this error are from one group of students taught combinatorics by a beginning teacher just before the final state exam at the faculty.

On the other hand, mistakes related to formulas (M2, M3, M9, M10), which accounted for approximately 60% of all incorrect solutions of Batanero et al.'s students *after* the instruction on combinatorics, rarely occurred among our students. This difference may be partially explained by the nature of the intervention, which encouraged the students to use various solving strategies (including systematic listing and strategy of identifying isomorphism) and subdued formulas as a solving strategy. The deliberate suppression of combinatorial formulas probably prevented students' overreliance on formulas identified in previous research (Batanero et al., 1997).

Our study also revealed differences in the success rates for problems with different characteristics. In agreement with Batanero et al. (1997) and Lamanna et al. (2022), the students were most successful in solving problems with the selection model, followed by problems with the distribution model. The

problem with the arrangement operation had a higher success rate than the problems with combination operations did, which agrees with the results of Fischbein and Gazit (1988).

The problem of *Obraz* is unique, not fitting into combinatorial models and operations. It had the lowest success rate and was often unsolved. Unlike typical combinatorial problems, it does not focus on the number of possibilities. Students had not faced such a problem in the intervention, making it less suitable for the final test. Nonetheless, De Chenne and Lockwood (2022) suggest that such problems enhance the understanding of list formation and organisation. Our future intervention should include problems like Obraz as students struggled to apply their combinatorial knowledge when solving it.

Finally, we added error M15 to the classification of mistakes to Batanero et al.'s list of 14 mistakes. In M15, the student incorrectly determines the number of natural numbers in a given interval. This occurred mainly in *Code,* and, to a lesser extent, in *Obraz*. It did not occur in any other type of problem, which explains why it was not part of the original Batanero et al.'s list.

Conclusions

The main conclusion of our study is that our intervention prevented the students in our sample from over-relying on formulas yet did not provide them with enough opportunity to use the strategy of systematic listing effectively. This aspect needs more attention in the intervention's teaching plan. The main limitation of our study is that we treated all the students together without looking at results from each of the four classes and without connecting them with other data from the intervention, mainly with the data showing the fidelity of the intervention – to what extent it was implemented as designed and how it could have influenced the impact of the intervention. This remains to be done.

Second, we confirmed that the characteristics of combinatorial problems impact their difficulty and added one more mistake type to the existing classification of mistakes. Yet, our results must be considered in light of the small sample in terms of both the number of students' solutions and the number of problems with the same characteristics used in the test.

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Computational thinking in mathematics problem solving: interplay between context, mathematics and computer science

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This research aims to explore the links between the mathematical content and the computer science content in the implementation of a sequence prepared collectively by secondary mathematics teachers and researchers. This contribution reports on a sequence of lessons about approaching the measurement of area of irregular surfaces from a probabilistic perspective at grade 7 (students of 12-13 years). Scratch was used to program a simulation to estimate the area of irregular surfaces.

Keywords: Computational thinking, mathematical problem solving, Scratch.

Strong links between computational thinking (CT) and solving problems in mathematics are recognized (for example, Barcelos et al., 2018). Recently, the Swiss education policy (Conseil d'État, Octobre 2019) considers the teaching and learning of CT as a school priority and sets it as a transversal topic in mathematics as well as in other disciplines. Our research (Batteau & Trgalová, 2023) aims at investigating the links between mathematical and computer science content in the implementation of a sequence of lessons prepared collectively by mathematics teachers and researchers. More precisely, we explore how teachers make visible different transitions between the context of a mathematical problem, the mathematical content and the computer science content. This sequence was based on solving a problem mobilizing mathematical and computer science content.

Kallia & al. (2021) propose a conceptual framework of CT in mathematics, that aims to investigate the interplay between the context, mathematical and computer science content.

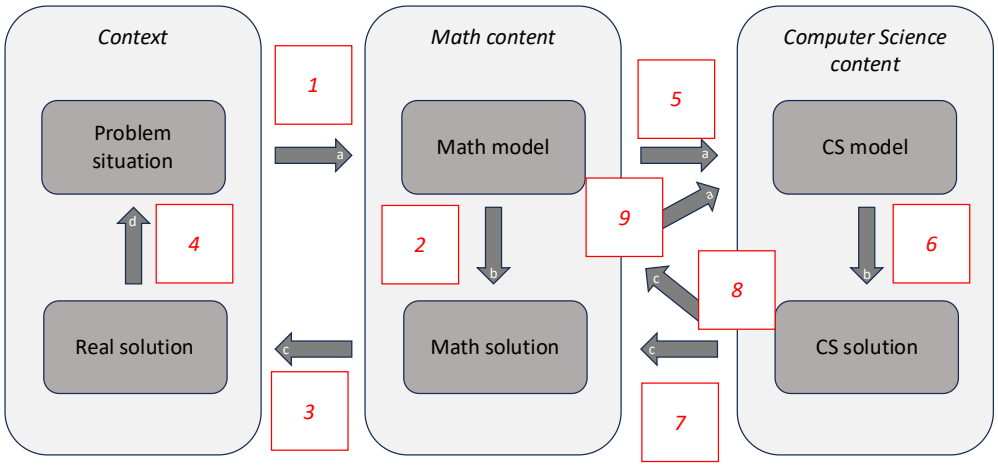

Figure 1: Computational thinking in mathematics based on Kallia et al. (2021, p. 3)

The model (Figure 1) constitutes our conceptual framework. Borrowing the concepts from the didactic engineering research method (Artigue & Perrin-Glorian, 1991), we realized an *a priori* analysis of the task to bring to the fore knowledge and reasoning at stake in the situation (context),

the mathematical and the computer science contents at stake and the transitions between these. Then, we confront this *a priori* analysis with *a posteriori* analysis (based on collected data) aiming at identifying in the students' and teacher's activity the differences between both, that highlight the critical points to be considered in the implementation of a problem-solving lesson involving computational thinking**.** Our aim is to show how the conceptual framework can be exploited to identify critical points in the implementation of a problem-solving lesson involving computational thinking where difficulties linked to transitions between real-life situation, mathematical and computer science content can be expected.

We address the research question: what are the critical points in the different transitions between context, math and computer science contents that need to be handled by the teacher to help students progress in solving the proposed problem?

The main task implemented during the sequence invites the students to measure the area of irregular plane surfaces. Three approaches lead to solving the task: break down the surface into smaller surfaces whose areas can be calculated; tile the surface with unit surfaces (conventional or not); probabilistic approach with simulation using Scratch.

By comparing *a priori* and *a posteriori* analyses, we have identified several critical points situated at these transitions that require specific attention in the design and implementation of mathematical problems, such as making explicit links between mathematical and CS models for example.

An initial attempt was made to use Kallia & al.'s (2021) model as an analytical framework. The aim now is to continue to use this framework, although not exclusively, and if necessary to develop it further. It is this perspective that will guide the rest of our work. This initial use of Kallia et al.'s (2021) model also opens up the prospect of using it in teacher training, or even by the teachers themselves, as a support for designing problems.

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Connections between discrete mathematics and other mathematical topics in Hungarian curricula

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Keywords: Theory of conceptual fields, curriculum, textbooks, discrete mathematics.

Introduction

Combinatorics, discrete mathematics is of great significance in Hungarian mathematical culture. It is marked by names such as Pál Erdős, Pál Turán, Vera T. Sós, Endre Szemerédi, László Lovász. Discrete mathematics plays an important role in Hungarian mathematical research as well as in mathematics curricula of Hungarian public education. Several discrete mathematical topics appear in the curriculum as combinatorics, mathematical logic, sets, number theory and graph theory.

A characteristic of the Hungarian curriculum is its spiral structure. Mathematical topics are built in parallel, and connections between different mathematical domains are encouraged to reinforce deeper understanding of mathematical notions (Gosztonyi, 2020). These connections can be conceived in different ways while implementing the curriculum. In my ongoing PhD research, I investigate opportunities offered by the curriculum to integrate discrete mathematics with other mathematical topics and how this contributes to the emergence and development of mathematical knowledge.

In the first part of my poster, I will give a curriculum analysis. At the level of intended curriculum (Schmidt et al., 1996), I examine what connections are explicitly required by the Hungarian curriculum between discrete and other mathematical topics. However, these connections are not explicit in the text of the curriculum. Textbooks provide more information concerning possible connections. For this reason, my research is currently focused on the potentially implemented curricula (Schmidt et al., 1996), more particularly on textbooks.

The research question discussed on my poster is: What opportunities do Hungarian textbooks in mathematics provide for connections between discrete mathematics and other mathematical topics, in link with the Hungarian curriculum?

Theoretical framework and methods

I rely on Vergnaud's Theory of conceptual fields (2009, 2013). According to him, a concept is formed through a set of different situations, highlighting different aspects of the concept. At the same time, several concepts are encountered in a situation. This approach is very adequate to describe dynamics between different mathematical domains I am investigating. Furthermore, I rely on Vergnaud's distinction between operational and predicative form of knowledge, the first providing means to do and succeed, the second consisting in the linguistic and symbolic expressions of knowledge.

In the case study presented in my poster, I investigate a learning unit in a geometry chapter of a 6th grade textbook. My research question specified for this case: what opportunities are provided by the situation described in this learning unit for the development of students' operational and predicative knowledge both in geometry and in discrete mathematics?

Results

The learning unit in question introduces the chapter on quadrilaterals (Csahóczi et al., 2021). In this situation, students have to construct quadrilaterals from 2, 3, then 4 isosceles right triangles, then from equilateral triangles, and finally from isosceles non-right triangles joined by whole edges.

Reviewing the possibilities for 2 triangles is not difficult. However, in the subsequent problems, using 3 or 4 triangles, it is necessary to systematize the cases to find all solutions. The systematization is determined by the known properties of the triangles used. In my analysis, I identify preliminary (operational and predicative) knowledge necessary to deal with the situation; opportunities to build new operational knowledge and opportunities for transition between operational and predicative form of knowledge, both in the domains of geometry and discrete mathematics. I particularly investigate mutual role of these two domains in the construction of knowledge in the other domain. For example, the symmetries of the triangles can lead to symmetries in a combinatorial sense. Also, the solution process highlights the importance of the parity of the total number of sides of the triangles used. This can provide operational knowledge of parity, which is a typical invariant in discrete mathematical problem solving.

Conclusion

In the presented case, the situation has strong potential to improve knowledge in both domains. Working in both domains can lead to more complex knowledge than if students only apply preliminary knowledge from one of the topics.

In the continuation of my research project, I will analyse further situations, and I will investigate through experimentation how students mobilise operational knowledge for topics that are not discrete mathematical topics but can be approached in a discrete mathematical way, and how these situations contribute to the development of their knowledge concerning both domains.

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Analysis of students' self-written combinatorial word problems

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Keywords: Combinatorics, arrangement, word problems, counting problems.

Theoretical background

The present study attempts to investigate students' understanding at the end of their school career by analysing self-written word problems, which are discussed in the context of the model of combinatorial thinking (Lockwood, 2013). It provides a framework for analysing difficulties in understanding combinatorial problems with three central components: *Formulas/expressions, counting processes,* and *sets of outcomes*. *Formulas/expressions* are mathematical calculations that represent a particular outcome or a number of possible combinations. *Counting processes* describe the mental or practical steps, such as the multiplication principle or the systematic division of a task, that lead to the solution. *Sets of outcomes* include all possible combinations or results of a counting task and help to visualise its structure. These three components are in a dynamic relationship: a counting process can lead to the development of a formula, while a formula can be a counting process. Similarly, counting processes can generate or structure sets of results, and conversely, understanding sets of results can contribute to the selection of an appropriate counting process. Mistakes often occur when students mechanically apply formulas without understanding the underlying counting process, which can lead to logical misunderstandings. The ability to switch flexibly between these components is crucial for understanding and solving combinatorial problems. A lack of understanding of these relationships can lead to results being overlooked or counted twice (Lockwood, 2013).

Research question and methodical approach

This poster aims to answer the following research question: *What understanding do students show of the combinatorial operation arrangement without repetition when formulating their own word problems?* In a test, 161 students (aged 17 to 18) were asked to formulate a word problem for the given equation ($\mathbf{6 \cdot (6-1) \cdot (6-2) = \frac{6!}{(6-3)!} = 120}$) to investigate their understanding of the formula. Thereby they should transfer from a formal representation into a concrete everyday situation. It required them to recognise that the formula calculates the number of ways to select three objects from six options in a specific order (arrangement without repetition). The data was analyzed using qualitative content analysis and a category system was developed ($\boldsymbol{\kappa = 0.72}$ (substantial)). The **deductive categories** were adapted on the basis of the study by Thomas and Pöhler (2024), in which permutation word problems were formulated, and on the basis of implicit combinatorial models (Dubois, 1984): Students can therefore formulate a word problem for arrangement without repetition *(arrangement word problem C1)* based on the combinatorial model *selection (C1.1)* or the *distribution (C1.2)*. The combinatorial model partition is not possible here because there is no division of the basic set *n* into distinguishable classes. It is also possible to formulate the equation according to the *product rule/multiplication principle (C2)*, so that a word problem can be formulated for the equation that is combinatorial, but in which different objects with different cardinality are arranged. Furthermore, students can also write a word problem for *multiplication (C3*) that fits the equation but

is not combinatorial. The **inductive categories** were formed on the basis of the data: *Combinatorial word problems that cannot be clearly assigned to an arrangement (without repetition)* are assigned to category *C4*, in which either a word problem was written in which the elements are partially identical *(C4.1)* or a word problem was formulated in which the elements are partially identical and no selection is made *(C4.2).* There were also *probability word problems (C5)* in which a probability was asked for and not the number of possibilities. There is also the category where *no concrete answer* was given *(C6)*, e.g. the box was left blank, a dash was drawn, or no word problem was written. Finally, there is the category *other (C7)* with answers that do not clearly fit into one of the other.

Results and discussion

Only the most common categories *(C1 and C4)* are presented here. *C1* includes problems where three objects of the same cardinality are selected or distributed from six different objects: *Three marbles are selected from an urn containing six different marbles. How many different combinations can be made? (C1.1)* and *Six athletes are running a race. How many different arrangements are there on the winners' podium? (C1.2).* These problems illustrate two combinatorial models: selection *(C1.1*), because three marbles are *selected* from an urn and distribution *(C1.2),* because athletes are *distributed/placed* on the podium. Applying Lockwood's (2013) model, these word problems demonstrate understanding of the set structure and arrangement process. Students grasp that the number of arrangements decreases step by step, aligning with the formula's logic. Conversely, *C4* problems, such as *Three balls are drawn from a bowl containing three white and three black balls. How many different combinations are possible? (C4.1)* and *Find the number of ways to arrange the letters in the name HANNAH (C4.2),* show challenges in recognising identical elements and applying formulas correctly. In C4.1, students misinterpret the influence of identical elements, while in C4.2, they apply the formula mechanically without understanding the conditions. These difficulties indicate that students do not fully understand the counting process behind the formula and do not always correctly recognise the influence of order. In summary, although students contextualise the formula, they have difficulty interpreting it correctly and recognising the conditions for its application (Lockwood, 2013). Based on this study, there should also be a focus on fostering students' conceptual knowledge (e.g. by formulating word problems) to gain a deeper understanding of combinatorial operations, which ultimately supports students' success in more advanced areas of mathematics and provides opportunities for rich mathematical thinking.

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Underused but powerful: Fostering computational thinking in GSheets on a statistics topic

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Introduction

Since its rejuvenation, computational thinking (CT) has been argued to be an essential skill, such as reading, writing, and arithmetic, which everyone needs to possess (Wing, 2006). However, there are several CT definitions, and they are not yet univocal (see Lodi & Martini, 2021). In this poster, by following Lodi and Martini (2021) suggestions, we defined CT as a way to access knowledge by using problem-solving skills to get solutions with or without the help of an information processing agent.

Realising the importance of CT skills, some European countries and institutions have initiated operationalising CT in compulsory education (Bocconi et al., 2016). Moreover, international organisations such as the Organisation for Economic Cooperation and Development (OECD) and the International Association for the Evaluation of Educational Achievement (IEA) have attempted to make international assessments of CT. Therefore, we are also motivated to contribute to the movements by proposing some CT activities.

CT can be fostered in informatics subjects or related subjects such as mathematics or sciences (Bocconi et al., 2016). Ye et al. (2023) have shown that students' CT skills could co-develop with their mathematics understanding. However, few studies utilised mathematics tools or software in the integration of CT into mathematics lessons (Chan et al., 2022). Therefore, this poster will present the integration of CT into mathematics lessons utilising mathematics software. We raised two research questions: What are the challenges that students face in approaching our designed tasks? And how can computational thinking be supported when engaging in our tasks?

Proposal

Chan et al. (2022), in their literature study, revealed software or tools used in the CT integration into mathematics education, and rarely has mathematics software been utilised, such as spreadsheets. Van Borkulo et al. (2023) utilised spreadsheets in the integrated CT into mathematics lessons and showed students' CT development on computational concepts. However, their study has not provided evidence of how students utilised spreadsheet formulas and fixed them when errors occurred. We then propose the use of Google Sheets (GSheets) to track the inputted and deleted formulas.

This study employed the educational design research (EDR) by McKenney & Reeves (2018); its iterations or cyclical features allowed us to improve our design and understand how our design innovation works. We developed Math+CT lessons (can be accessed at https://bit.ly/CTGSheet) by referring to the Shute et al. (2017) CT framework when incorporating CT components (decomposition, abstraction, algorithm, debugging iteration, generalisation) into the central tendency

tasks. We collected the data from students' work on GSheets and recorded some students' computer screens. The data would be analysed using content analysis following steps by Krippendorff (2004), which consist of making, categorising, and concluding codes that emerge from our data.

Our study has presented the struggles that students face when working on our Math+CT tasks and the CT skills that students have engaged with. Moreover, other topics should be investigated to understand how to optimally utilise GSheets for integrating CT into mathematics lessons.

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TWG 12: History in Mathematics Education

Introduction to TWG 12: History in Mathematics Education

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Introduction

In their introduction to the papers presented at CERME6 in Lyon in 2009, the joint chairs stated that "the working group has no tradition in CERME" (Furinghetti et al., 2010, p. 2679). More than 15 years have passed. This is the ninth consecutive edition of CERME in which this thematic working group has been organized, and the previous statement no longer applies. In fact, although not too large (Figure 1), the presence of this working group at the conference has remained quite stable over time. Table 1 shows the evolution of the number of works published in the proceedings since 2009.

Table 1: Evolution over time of TWG12 papers

CERME6	CERME7	CERME8	CERME9	CERME10	CERME11	CERME12	CERME13	CERME14
13	14	15	16	18	15	13	16	12

One more paper was initially accepted for CERME14. However, the presenting author, who was born in Ecuador, was not granted a visa in time to enter Italy. It is unfortunate that, even in 2025, a lack of visa can prevent researchers from presenting their work at an international congress. Even more unfortunate is the fact that this problem affects researchers from less affluent countries more often. As co-leaders of this thematic working group, there is little we can do. However, we can publicly express our discomfort through these lines.

Figure 1: Participants of TWG12 at CERME14

Overview of the papers presented in Bolzano

The following is the list of possible topics that was provided in the call for papers:

1. Design of teaching/learning materials using history of mathematics, preferably with conclusions based on empirical data.
2. Research on the existing or possible uses of history or epistemology in curricula, textbooks, and/or classrooms practice.
3. Research on the history of mathematics education and/or the relevance of mathematics education in society at the time.
4. Connections and interplay between mathematics education and history of mathematics: theoretical and methodological issues.

This list of topics captures the key areas of research currently being explored within our field at an international level. However, it should be noted that not all topics receive the same level of attention, which is reflected in the uneven distribution of the presented papers across the topics. In particular, the most part of the papers (eight) were related to Topic 3. Of the remaining papers, two addressed Topic 2, while one each addressed Topics 1 and 4 (see Table 2).

Table 2: Distribution of papers among the topics from the call for papers

Presenting author	Topic	Strongly related TWGs	Weakly related TWGs
De Bock	3		
Goemans	3	2	13, 24
Oller-Marcén	3		7
Opsal	3	23	
Pinto	3	1	4
Piña-Aguirre	2	14, 25	24
Rodrigues	3	22	4
Schmidt-Thieme	3	24	
Segev	1		2, 3, 8
Smestad	3	22	
Thomsen	4	17	20
Watford	2	14	8, 17, 22

Most of the papers presented at the conference are potentially related to other thematic working groups (see Table 2). We note that roughly half of all the thematic working groups (14 out of 29) have some relation to the papers presented. Naturally, some of these links are stronger than others, as indicated in Table 2. It is also noteworthy that some papers can potentially be related to many different groups. This is the case with Mark Watford and Kathleen Clark's paper "A transformation of perception in calculus: An exploration of barriers and outcomes through transgressive actions," which is strongly related to TWG14 (University Mathematics Education) as well as TWG8 (Affect and the Teaching and Learning of Mathematics), TWG17 (Theoretical Perspectives and Approaches in Mathematics Education), and TWG22 (Curricular Resources and Task Design in Mathematics Education). Conversely, some papers have a less clear link with other thematic working groups. Some of these papers (such as De Bock and Goemans' paper "Mathematics education in Belgium inspired by Decroly's pedagogy") cannot easily be related to any group.

However, this should not be used as a measure of the paper's interest or quality. In fact, it is beneficial for our group to have both types of paper. Papers with strong links to other thematic working groups demonstrate our belonging to the mathematics education community, while papers with weaker or no links highlight the specificity of our research field.

The workshops

Since CERME12, that took place online, some of our presentations have been given in the form of workshops. This is not uncommon in the HPM (History and Pedagogy of Mathematics) community, and the European Summer University on the History and Epistemology in Mathematics Education (ESU) always includes a good number of workshops. Furthermore, in 2021, the new format was deemed enjoyable, and it was agreed that it should be retained (Chorlay et al., 2022). For this edition, two papers have been adapted for presentation as 90-minute workshops.

In the first workshop, based on the paper by Segev and Elitzur "Using word problems from the 16th century in teaching mathematics", we analyzed some word problems coming from a Jewish math book from the beginning of the 16th century. We attempted to solve them by ourselves and learnt about an experiment conducted with 20 students from an Israeli teachers' college. The results of the experiment indicated an impact on the positive attitudes of the students, as well as some gender differences. The second workshop, based on the paper by Schmidt-Thieme and Hamann "Iconic representations in Renaissance reckoning books – on the way to a table of classification", focused on iconic representations in ancient arithmetic textbooks. We analyzed and classified several such representations excerpted from the *Rechenbuch auff Linien und Ziphren* (1574) by the German Adam Ries. As a result, we reflected on what to understand by 'iconic representation', and we discussed our classification criteria in comparison with those of the paper's authors. The discussion also raised interesting methodological questions and suggested further lines of research.

Another paper, "Papy's minicomputer" by Wendy Goemans and Dirk de Bock, was presented in a hybrid format, combining elements of a traditional presentation with those of a workshop. The authors introduced the so-called Papy's minicomputer, a conceptual, non-mechanical, two-dimensional abacus that was initially intended by its creator for teaching the number system. We were shown how to use it in practice, and we were even able to manipulate some original material from the 1960s.

Group discussions and challenges for the future

CERME conferences usually provide a venue for fruitful conversations, within the thematic working groups, and in more informal situations. During this week of intense work, several discussion topics arose, some of which were addressed as a group during our final session. Here, we briefly report on two topics that were mentioned repeatedly in several of the papers presented: theoretical frameworks in our field and methodological challenges in the history of mathematics education.

Throughout the presentations, various theoretical frameworks and constructs were employed to either provide a foundation for the research or serve as an analytical lens through which to understand a phenomenon or interpret data. These include the theory of didactic situations, the concept of mathematical transgressions, didactic transposition, socioepistemology, the different curriculum levels, the notions of foreground, and Jankvist's hows and whys. Some of these ideas originate from mathematics education and have been adapted to a certain degree for use in a historical context; others are specific to our field. Regardless of their origin or role, they can and should be employed to communicate our findings in a manner that highlights the significant impact that some of our research can have on the broader field of mathematics education. Another issue relates to the multiple methodological challenges that arise in research in the history of mathematics education. Some of these challenges are related to the use of novel technological tools, such as AI, in data analysis, particularly when a quantitative approach is adopted. Others consider the variety of sources that may be relevant to a given research question, or the challenges of making comparisons between different countries. Finally, we consider the possibility (or necessity) of connecting our findings with contemporary mathematics education.

These two issues are related to overarching ideas introduced earlier: the concern for our mutual relationship with mathematics education as a discipline, and for the role, impact and applicability of our research results in this context. In the near future, we need to reflect on these issues, not only to gain visibility, but especially to be able to exchange theoretical approaches and methodologies that could benefit both fields.

Acknowledgment

We thank all the members of the TWG12 for their active participation and the superb working atmosphere throughout the whole conference. See you in Bratislava!

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Mathematics education in Belgium inspired by Decroly's pedagogy

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After WWI, the New School movement flourished, mainly in Europe and the US. The movement proposed a child-centered pedagogy and sought a more just society. In this paper, we focus on the influence of Decroly on mathematics education in Belgium. Before WWII, this influence was limited to primary education (with new curricula starting in 1936). After WWII, mainly under the influence of the École Decroly*, there was an impact on secondary education. In this, the Brussels professor of geometry Paul Libois and his wife Lucie Libois-Fonteyne played a prominent role. The mathematical exhibitions organized by Libois and the students of the* École *drew international attention, for example, from Emma Castelnuovo and Italian teachers. But in the 1960s, Georges Papy's modern mathematics was dominant in Belgium; Libois' ideas could not penetrate widely. In the 1980s, however, ideas similar to Decroly's were revived in the model of realistic mathematics education.*

Keywords: École Decroly*, mathematical exhibitions, New School movement, Ovide Decroly, Paul Libois.*

Introduction

The New School is an international educational movement that emerged around 1890, had its heydays during the Interwar period, but remained influential in many countries after WWII. Other designations are: *Reformpädagogik* (in Germany), *éducation nouvelle* (in France), and Progressive Education (in the US). The movement aimed to "improve education" and it was assumed that therefore children and their (spontaneous) interests should be placed central in the learning process, which implies more attention to the individual learner. Learning also needed to be strongly linked to "real life" (in Decroly's words, "une école pour la vie par la vie" [a school for life through life[1]]). The school should not separate itself from the community in which it is embedded, so community projects and activities were favored. The movement also promoted a meritocratic educational system: social-class differences should not play any role; instead, school should become a vehicle for social progress.

Main protagonists of the movement were John Dewey (1859–1952), Maria Montesorri (1870–1952), Ovide Decroly (1871–1932), Édouard Claparède (1873–1940), Roger Cousinet (1881–1973), Célestin Freinet (1896–1966), among others; the work of 18th century Geneva philosopher Jean-Jacques Rousseau is widely regarded as a common inspiration for the New School reformers. In 1919, schools that endorsed New School ideals united in the New Education Fellowship (NEF) (Clews, 2009). It became a strong network of progressive and experimental education; in several countries national NEF branches were established (see, e.g., Beckers, 2023).

An extensive literature on the New School movement exists today: research on the work of key representatives of the movement (e.g., Depaepe et al., 2003), on national implementations of the reform (e.g., Pozo Andrés, 2021), and comparative studies (e.g., Van Gorp et al., 2017). Mostly,

[1] All translations were made by the authors.

however, research on the New School movement was conducted by general education historians and the influence of the movement on mathematics teaching is discussed marginally at best. At CERME13, following the paper presentation by Carrillo Gallego and Ginés Espín Buendía (2023), the possibilities of research on the influence of the New School movement on aspects of mathematics education were discussed. This discussion prompted the present study.

The purpose of the current study is to examine the influence of Decroly's pedagogy on mathematics curricula and practices in Belgium, both at the primary and at the secondary level. How did Decroly's ideas influence the mathematics curriculum for primary schools implemented in Belgium beginning in 1936? And how did they nourish the debate that began after WWII about reforming mathematics education at the secondary level? To answer these questions, we analyzed pre and post WWII documents, testimonies of those involved, and confronted our findings with the results of other research on the New School movement.

Decroly's views on mathematics education

Given the extensive literature available on Decroly's life, work, and views on child psychology and pedagogy in general (see, e.g., Depaepe et al., 2003; a pedagogical-historiographic "standard article" about Decroly), there would be little point in discussing these issues at length in this paper. Therefore, we will mention only a few key biographical facts and then elaborate on Decroly's views on mathematics education (which are, of course, in line with his general views on education).

Ovide Decroly (Figure 1), born in Ronse in 1871 and died in Uccle (Brussels) in 1932, studied medicine at the University of Ghent, where he graduated in 1896. Through his work with mentally "retarded" children, he became interested in psychology and pedagogy. Soon he became the figurehead of the New School movement in Belgium. In 1907, he founded the *École Decroly* in Uccle (originally named *École de l'Ermitage*), a school for pupils aged 3 to 18 where his pedagogical principles were uncompromisingly put into practice. Decroly was a Freemason, president and co-founder of several pedagogical organizations, including the Belgian NEF branch, and responsible for the chair of child psychology at the *Université Libre de Bruxelles* [Free University of Brussels] (Libois, 1971).

Figure 1: Ovide Decroly at work with children, n.d. (Centre d'Études decrolyennes)

As mentioned, Decroly's views on mathematics education cannot be separated from his broader educational philosophy. A fundamental principle in that philosophy is what he called "globalization" (Decroly, 1928). Children first understand their world in a holistic way. When brought into contact with an object or phenomenon, the child will spontaneously examine it as a whole, in its totality. As unanalyzed totalities, various notions penetrate the child's mind. But gradually, according to their needs and interests, they would continue to analyze, to revisit these notions, and to discover and understand different aspects of the investigated realities, including mathematical ones. The principle of "globalization" has an important didactic implication: to develop children's intellectual faculties at school, it is recommended to start from total realities rather than individual subjects. Already in the early 1900s, Decroly advocated reducing the number of school subjects and avoiding unnecessary subject divisions. "Do everything possible to ensure that the number of subjects is reduced and that the subject matter itself is designed to avoid arbitrary and artificial divisions" (Decroly quoted in Libois, 1962, p. 6). Applied to mathematics, he wanted to prevent young children from learning mathematics as a stand-alone subject; instead, mathematics education should focus on gaining an understanding of mathematical concepts and acquiring mathematical skills from a broader reality. Also, teaching the various sub-disciplines of mathematics (arithmetic, algebra, geometry, trigonometry, etc.) separately was best avoided as much as possible (Libois, 1962).

A second key idea of Decroly was to concentrate the teaching of subject matter around central themes that correspond to dominant interests of the children, the so-called "method of centres of interest". "Interest should be at the heart of everything we [teachers] impart to the children. We have to bow. We have to concern ourselves with their interests, techniques will come later" (Decroly, 1974, p. 14). In mathematics, this means that concepts should be introduced through activities that are meaningful and relevant to the child's everyday life. For example, children might engage in counting, measuring, collecting data or problem solving as part of an interdisciplinary project or activity, such as building something or exploring nature (Decroly & Hamaïde, 1932).

Third, Decroly emphasized the role of children's personal activity in learning, "learning by doing". He recommended to "stimulate a taste for observing, experimenting, questioning facts, expressing thoughts, creating and inventing" (Decroly quoted in Libois, 1962, p. 7). Learning mathematics by doing translates into activities in which children manipulate real objects, construct geometrical shapes, explore patterns and so can discover mathematical principles. It assumes the availability of concrete models and other aids that the child can "see and touch", as well as the necessary time to adapt to each child's individual learning pace. Decroly advocated giving children the time and materials they need: "provide them with the time, space and material opportunities to translate and control through assemblies or objective creations" (Decroly quoted in Libois, 1962, p. 7).

The 1936 curriculum reform at the primary level

In 1936, a new curriculum for public primary education was introduced in Belgium (Ministère de l'Instruction Publique, 1936). The curriculum, signed by liberal minister François Bovesse (1890–1944), was a major innovation compared to the previous one, which dated from 1922. It bore Decroly's stamp.

> It's in the immediate environment that the teaching material is to be found, and it can scarcely be found elsewhere. […] The study of the environment through active observation is at the heart of the program. […] The teacher will draw inspiration from observable things, draw the child's attention to what is around him, interest him in the phenomena unfolding before his eyes and constantly appeal to his immediate experience. (Ministère de l'Instruction Publique, 1936, p. 23)

Catholic schools, which before 1936 applied the same curriculum as public schools, did not follow suit. Catholic leaders could not agree with the purely "scientific" approach of the official curriculum, in which "reality" was reduced to the "concretely observable", and developed their own curriculum.

For the teaching of arithmetic/mathematics, the general tenor of the curriculum was continued and a "realistic" approach was advocated. Calculation should not be a goal in itself, but should be linked to concrete reality. Teaching had to be maximally based on observations and the children's environment. Long and abstract computations had to be avoided. Word problems were not allowed to be abstract, but had to be grafted on the students' activities and interests. In the domain of "measurement", the curricula called for using only the measures that the children would use in everyday life. Yet there were also some elements that were hardly consistent with the general principles. For example, it was stated that learning to calculate quickly and precisely was a main goal of the curriculum, and there was a plea for perfect mastery of the calculation tables and automation of the main arithmetic operations. The authors of the new curriculum recognized that "in the teaching of arithmetic, as in any teaching, there is a routine and mechanical part which must be accepted" (Ministère de l'Instruction Publique, 1936, p. 11).

In a general study of the 1936 curriculum reform in Belgian primary education, Depaepe et al. (1991) have shown that, for various reasons, educational practice did not change fundamentally in those days. This also applied, at least in part, to the teaching of arithmetic/mathematics. The word problems in the new textbooks appeared somewhat more realistic and "taken from life". Regularly, (functional) charts, tables and drawings that came directly from the children's world were used. Also, for the first time, a distinction was made in the difficulty level of the exercises (by designating the difficult exercises with a particular sign). In classroom practice, this probably amounted to tempo differentiation: after classroom instruction, all children made the same basic exercises. Children who were ready could then start on the more difficult exercises.

Most commonly used textbooks continued to pay a lot of attention to long series of bare sums, without "meaning" and unconnected to any applied context (De Bock & Vanpaemel, 2019). Quite deliberately, by the way, as evidenced in the preface of the manual by De Cock and De Vos (1938) for the fifth year:

> Many schoolteachers now reasoned as follows: the main operations have been sufficiently practiced in previous years, and each time the word problems offer, in one form or another, a repetition of the art of cyphering. So it seemed unnecessary to them to add many purely cyphering exercises. In this they wandered. Common experience should have made them sufficiently aware of this. How many word problems had a wrong solution because the ciphering was wrong. (p. 4)

A revised curriculum for public primary education in Belgium was introduced in 1958, but the underlying philosophy did not change substantially (Noël, 2022).

Call for reform at the secondary level after WWII

In Belgium, as in several other European countries, a debate on educational reform flared up toward the end of WWII. The main themes were democratization of education, cultural modernization, and pedagogical innovation (Noël, 2022). A protagonist during that period was the mathematician and communist politician Paul Libois (1901–1991), professor of geometry at the Free University of Brussels (De Bock & Vanpaemel, 2019). As a mathematician, Libois had been educated in part by Frederigo Enriques and Guido Castelnuovo (father of Emma Castelnuovo), figureheads of the Italian school of algebraic geometry. From his Italian masters, he had inherited a predilection for intuitive thinking in mathematics. Libois showed himself to be a fervent supporter of the Decroly's comprehensive child-centered pedagogy. In 1936, he married Lucie Fonteyne (1897–1980), a teacher at the *École Decroly* and later its director.

One of the first post-War initiatives was the founding in January 1945 of the *Comité d'Initiative pour la Rénovation de l'Enseignement en Belgique* (CIREB) [Committee of Initiative for the Renovation of Education in Belgium], a think tank that emerged from resistance circles and strived for a global and democratic reform of the Belgian educational system. The *Comité* was chaired by the physicist Frans Van den Dungen, vice-rector of the Free University of Brussels. Most of the other members of the CIREB were also free-thinking leftist intellectuals affiliated with or graduated from the Free University of Brussels, among them also Paul Libois. In an introductory manifesto *L'École de 6 à 16 ans* [The school for 6- to 16-year-olds] (CIREB, 1945), the *Comité* proposed extending compulsory education, which in Belgium at that time was limited to 14 years, to 16 years. To that end, the manifesto insisted, an entirely new type of school had to be created, a school for all, a synthesis of the existing school systems. "Schools whose task would be not only the teaching of some basic skills (reading, writing, calculating), but the entire formation of the future citizen" (p. 3).

The project of the CIREB was clearly influenced by the views of Ovide Decroly, "the great Belgian psychologist and pedagogue, whose ideas and achievements have largely inspired the current work" (CIREB, 1945, p. 2). In particular, in its call for pedagogical innovation, the "method of centres of interest" was promoted: "All activities are grouped around a central theme, the **Centre of interest**, which corresponds to a dominant interest of the children" (CIREB, 1945, p. 14, bold in original). To help schools implement the proposed reform, between December 1945 and February 1951, the CIREB published the magazine *L'École* which provided several detailed dossiers on cross-curricular themes that could be chosen as "centres of interest" for the various age levels between 6 and 16. Most of these dossiers were prepared by scientists from the Free University of Brussels and dealt with topics such as the "weather", "labor", "housing" and "nutrition".

Following the general principles, the CIREB (1945) also proposed a basic program for each subject. It is a comprehensive program intended for all pupils between the ages of 6 and 16, not divided into ability groups. For mathematics, the CIREB proposed a fairly concise program, organized into five sections: numerical calculations, geometrical figures, transformation of figures, spatial representations, and functions. Interestingly, in geometry, the emphasis was on an intuitive approach to geometrical objects. With the "transformation of figures" section, the program introduced a dynamic element to the teaching of geometry, which could be regarded as an innovation over

traditional approaches. Despite its brevity, it reflects some key ideas of Libois, who certainly had a hand in it.

In May 1945, the CIREB project was presented to the then Belgian Minister of Education, the liberal Auguste Buisseret (1888–1965), but the *Comité* did not wait for political approval of its reform proposals. The CIREB considered that its proposals did not conflict with existing legislation and therefore schools were invited to implement the program directly if they wished. Given the general nature of the program and the lack of precise guidelines, however, it is unlikely that many schools accepted the invitation of the CIREB (Noël, 2022).

In 1948, under the impetus of one of Buisseret's successors, the socialist Camille Huysmans (1871–1968), a major democratic reform of the education system in Belgium was initiated. Huysmans, who himself had worked as a teacher for several years, warmly welcomed the ideas of Decroly and the CIREB. Huysmans' reform led to new curricula for public secondary education, including for mathematics (Bosteels, 1950). The new mathematics program for the first year of secondary education (12-year-olds) mandated a course in intuitive geometry, as advocated by Libois. Catholic schools followed suit. Intuitive geometry was intended as a bridge between the elements of geometrical practice that students had learned in primary school, and the "rational" geometry that started in the second year of secondary education. Intuitive geometry aimed to familiarize students with the properties of basic geometrical shapes on which they could make practical measurements. Much emphasis was placed on actively manipulating real objects to develop students' sense of observation and to induce an "appetite" for learning geometry.

An alternative for Papy's modern mathematics?

In the early 1960s, modern mathematics, say basing mathematics education on set theory and structures, was prepared in Belgium and internationally. After the famous Royaumont Seminar in 1959, a "group of experts", including Paul Libois, met in Zagreb-Dubrovnik to develop a modern program for teaching mathematics in secondary schools. The outcome for geometry was an ambiguous compromise. In the early years (12–15-year-olds), the emphasis was put on an intuitive approach. This attention was largely due to Libois' intervention. For the final years of secondary education (16–18-year-olds), geometry would be replaced with the study of algebraic structures, as advocated by participating mathematicians such as Gustave Choquet and Emil Artin.

The resolutions of the Zagreb-Dubrovnik meeting were to the liking of the teachers of the *École Decroly*. In 1962, a few months after the publication of the official proceedings *Synopses for Modern Secondary School Mathematics* (OEEC, 1961), a pedagogical conference was organized by the Belgian Ministry of Education at the *École*. In her opening speech, Libois-Fonteyne proudly announced that the main conclusions of Zagreb-Dubrovnik corresponded perfectly with Decroly's pedagogical principles, which had already been formulated in 1907 (Libois, 1962). She selected as main conclusions the unity of mathematics, the coordination of the teaching of mathematics with the teaching of other subjects (especially physics) and the personal activity of the student.

Meanwhile, from 1959, Georges Papy (1920–2011), a professor of algebra at the Free University of Brussels, had started experiments with the teaching of "his" modern mathematics at the secondary level (De Bock & Vanpaemel, 2019). Papy's experimental trajectory would expand year after year.

Papy developed a pioneering series of textbooks entitled *Mathématique Moderne* and organized large-scale in-service training for teachers every year. Papy's approach to modern mathematics was innovative, yet from the beginning based on axioms and in many ways opposite to what Libois stood for. Soon Papy's modern mathematics became the leading model for a reform of mathematics education in Belgium. It would lead to a generalized introduction of modern mathematics in secondary schools (from September 1968 on) and, about ten years later, in primary schools.

At that same period, Libois continued working closely together with the *École*'s secondary section, both by advising the team of mathematics teachers and acquainting its members with new mathematical ideas. He also helped the students prepare annual exhibitions on mathematical themes. The alternative "experiments" of Libois continued to attract the attention for several years of a number of Belgian and foreign circles. Until the mid-1970s, Emma Castelnuovo (1913–2014) and many young Italian mathematics teachers came regularly to Brussels to visit the mathematical exhibitions prepared by Libois' students (De Bock & Vanpaemel, 2019). Libois' ideas, however, would not penetrate widely.

In conclusion

Before WW II, Decroly's views on mathematics education were quite influential in Belgian publicly-run primary schools, as they were in secondary schools in the 1950s. From the 1960s this influence waned as Papy's modern mathematics became dominant. However, in the 1980s, after the fall of modern mathematics, realist/constructivist ideas made their entrance (again) in Belgian primary and secondary schools (De Bock et al., 2020a, 2020b). These ideas had much in common with those of Decroly, but were then attributed to the Dutch model of realistic mathematics education.

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Papy's minicomputer

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Georges Papy and Frédérique Papy-Lenger are well known for their involvement in the Belgian and international modern mathematics movement from the late 1950s to the early 1970s. Less well known is a teaching aid invented by Papy and disseminated by Papy-Lenger: Papy's minicomputer. In this paper we present Papy's minicomputer, how it was created and how it works, and we illustrate its reception and use by teachers and teacher trainers in the 1970s. We consult sources authored by the Papys and by contemporaries. Moreover, we discuss recent literature on Papy's minicomputer and link it to the concepts of computational thinking and gamification.

Keywords: Frédérique Papy-Lenger, Georges Papy, minicomputer, modern mathematics.

Introduction

From the late 1950s to the early 1970s, Georges Papy and Frédérique Papy-Lenger were two of the most important Belgian protagonists in the modern mathematics movement. The Papys played a major role in the introduction of modern mathematics into the Belgian school curriculum by conducting classroom experiments, by (co-)organising meetings and training courses for teachers, and as authors of textbooks and as (co-)founders and/or active participants at meetings of national and international organizations. For extensive information on the Papys, their achievements and their detailed role in the Belgian and international modern mathematics movement, see De Bock (2022, 2023) and De Bock and Vanpaemel (2019).

At the end of the 1960s, when modern mathematics was introduced into the secondary school curricula in Belgium, the Papys' attention shifted from secondary to primary education. It is in this context that Papy provided young students with a teaching aid that allowed them to perform large number calculations almost effortlessly: minicomputer. G. Papy (1968) explains how he was inspired by the numbers of Mgr. Georges Lemaître for the construction of his minicomputer[1]. Papy's minicomputer is a non-mechanical two-dimensional abacus based on a binary number system combined with a decimal position system. Paraphrasing Papy, Jean-Claude Matthys stated, "PAPY has invented Minicomputer. He has also composed some of its major masterpieces. FREDERIQUE was the privileged and most inspired interpreter" (Kennes, s.d., no pagination, capitals in the original)[2].

In this paper, we present Papy's minicomputer and how it was used in mathematics education in the 1970s. We also outline how Papy's minicomputer relates to current research in mathematics education. More specifically, Bell and Hickman (2025) promote the teaching of binary numbers

[1] Mgr. Lemaître (1894–1966), a Belgian physicist, posthumously called the father of the Big Bang theory, proposed in 1955 new numbers and rules to perform effortlessly calculations, see e.g. G. Papy (1968).

[2] All translations are made by the authors.

because, according to them, these are the atoms of the digital world. They write mainly about how teaching binary numbers can explain the limitations of technology, but they also refer to how it exercises aspects of computational thinking. Although computational thinking is often associated with solving problems in computer science, it is also intertwined with mathematical thinking. Chan et al. (2023) conducted a literature review on the integration of computational thinking in mathematics education. They identify games as an important tool for computational thinking. Therefore, we mention how Papy's minicomputer was used by Papy-Lenger in gamification in the 1970s and how it can be related to the concept of computational thinking.

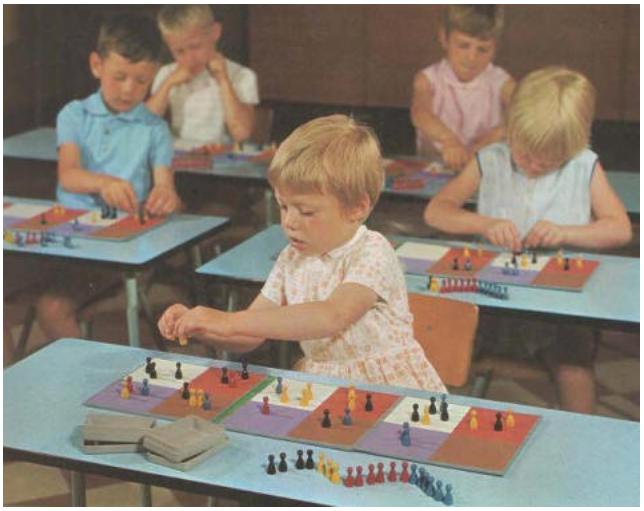

Figure 1: Children at work with Papy's minicomputer (G. Papy, 1968, p. 9)

Theoretical framework

To study the early years of Papy's minicomputer, we first consult documents authored or co-authored by Papy or Papy-Lenger. In addition, we check *Nico*, a journal for teachers which was published by the *Centre Belge de Pédagogie de la Mathématique* (CBPM) and edited by Papy-Lenger (De Bock & Vanpaemel, 2019). The references in these primary sources, as well as additional sources found via the online database GoogleScholar using the keywords 'Papy' and 'minicomputer', are then studied. In doing so, we aim to answer the following research questions: What is Papy's minicomputer? How was Papy's minicomputer received and used?

At the same time, recent literature on Papy's minicomputer is found by searching the online database GoogleScholar. Although scarce, these sources allow us to explore what remains of the potential uses of Papy's minicomputer. We also link Papy's minicomputer to the concepts of computational thinking and gamification. Chan et al. (2023) point out that there is no single definition of computational thinking, but that a definition of computational thinking depends on the subject in which it is integrated. Broadly speaking, computational thinking is the process of thinking required to arrive at a solution to a problem that can then be executed by an information processing agent such as a human, computer or machine. According to Chan et al. (2023), the most assessed computational thinking competency in mathematics education is algorithmic thinking.

Papy's minicomputer

Papy's minicomputer is not an electronic device, but a conceptual, non-mechanical, two-dimensional abacus. F. Papy (1969) describes Papy's minicomputer as a two-dimensional abacus that harmoniously uses the binary number system internally on plates that are organised according to the decimal position system. Papy's minicomputer consists of sequentially arranged square plates divided into four equal, coloured, squares, see Figure 1.

From right to left, the different plates represent the digits of a number, as in a decimal position system. Within each plate the binary number system is used, the white square at the bottom right represents $1 = 2^0$, the red square at the bottom left represents $2 = 2^1$, the purple square at the top right represents $4 = 2^2$, and the brown square at the top left represents $8 = 2^3$. A number is represented by placing a pawn on the correct squares, see Figure 2, which shows the number 697 represented on Papy's minicomputer.

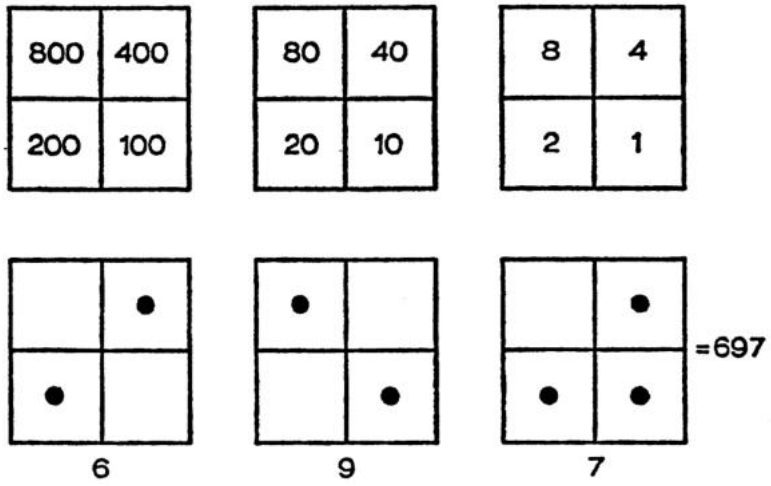

Figure 2: Representation of the number 697 on Papy's minicomputer (F. Papy, 1969, p. 333)

Some may immediately associate the colours of Papy's minicomputer with the corresponding Cuisenaire rods, a teaching aid consisting of blocks of different length and colour, representing the numbers 1 to 10, see for example De Bock and Vanpaemel (2019). This link is justified, as Papy used the same colours in his minicomputer as the corresponding Cuisenaire rods to honour Georges Cuisenaire, see G. Papy (1968, 1975). The first reaction F. Papy (1969) reports on when presenting the results of an experiment that she carried out with Papy's minicomputer in 1967 with 6-year-olds, is "– OH! CE SONT NOS COULEURS!" [Oh! It are our colours!] (p. 333, capitals in original), the child expressing his recognition of the correspondences of the colours with those of the Cuisenaire rods. However, Christiane Vandeputte (personal communication, August 22, 2024), who was for many years a close collaborator of the Papys, stresses that both she and Papy-Lenger observed that the introduction of Papy's minicomputer was easier without the reference to the Cuisenaire rods.

If the binary number system is used purely, then on one square plate subdivided in four equal squares one can represent the numbers 0 to 15, that is, $16 = 2^4$ numbers in total. See for example Figure 3 (left) for a representation of 12 on one plate. As can also be seen from Figure 3 (left), the

representation of a number on Papy's minicomputer is not necessarily unique. Therefore, G. Papy (1968) introduces two definitions. Firstly, every disposition of pawns on Papy's minicomputer represents a number. Secondly, a formation is a disposition that allows the number it represents to be read immediately. For example, the first two lines in Figure 3 (left) are a formation of the number 12, while the other four lines are dispositions but not formations. Figure 2 shows the formation of 697.

Under the title "Secrets of minicomputer" (G. Papy, 1968, p. 53), Papy states that "A disposition is a formation if and only if on all the plates: 1 - never more than one pawn per square, and, 2 - if there is a pawn on the brown, then there is no pawn on the red and no pawn on the purple" (p. 54). G. Papy (1968) continues with "If, for every disposition you are able to find the formation that is equal to it, Then you are able to perform any calculation on MINICOMPUTER" (p. 54, capitals in original). To transform a disposition into a formation, one can apply four reduction rules, see Figure 3 (right). G. Papy (1968) points out that, if a child applies the reduction rules correctly, then it will always have a disposition on its minicomputer that represents the same number, regardless of how many steps forward or backward, even useless steps, the child applies. In this respect, the only really important strategic rule is to apply reduction rule R4 as soon as possible.

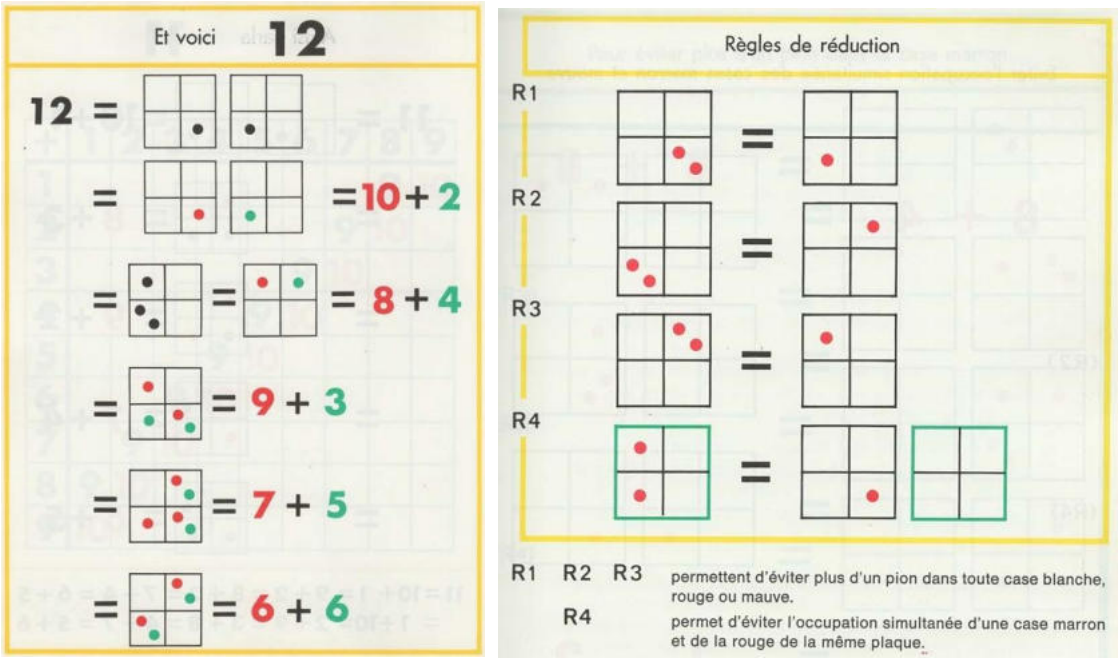

Figure 3: Reduction rules on Papy's minicomputer (G. Papy, 1968, pp. 44 and 55)

Once you know the reduction rules, you can start calculating with Papy's minicomputer. For an explanation on how to add, multiply, subtract and divide two natural numbers using Papy's minicomputer, see e.g. Herbert (1984), G. Papy (1968), Quevedo et al. (2017), Socas Robayna and Espinel Febles (1989), or Van Arsdel and Lasky (1972).

Papy's minicomputer during its first three decades

After F. Papy's (1969) presentation of Papy's minicomputer at the First International Congress on Mathematical Education (ICME-1) in Lyon, France, in August 1969 and on the 48th annual meeting

of the NCTM in Washington, DC, in April 1970, Papy's minicomputer attracted the attention of several teachers, teacher trainers and education specialists. For example, Van Arsdel and Lasky (1972) explain how to calculate using Papy's minicomputer and stress the correspondence of the colours with the Cuisenaire rods. They consider Papy's minicomputer to provide an answer to Jerome Bruner's appeal for "a skill as a tool" (Van Arsdel & Lasky, 1972, p. 451).

Steiner (1974) on his turn, presents Papy's minicomputer as a grouping concept in the sense of Piaget. He applies Erich Wittmann's definition of a grouping, combining a group and a lattice[3] in a hybrid way, see Steiner (1974) and the references therein. Steiner gives some models of a grouping among which one that plays a role in the representation of numbers by means of the Cuisenaire rods, one that is involved in the pedagogical application of Dienes' blocks, one of linear equations that is linked to solving linear equations with Papygrams, and one considering the set of all dispositions of n for $n \in \{2, \dots, 99\}$ on Papy's minicomputer.

Lots of the promotion of, and experiments with Papy's minicomputer took place in the large, but inner circle of Papy. One finds for example a dozen of articles on (experiments with) Papy's minicomputer in different volumes of *Nico*. We mention Dieschbourg (1974), who reports on his work with mentally handicapped children in primary schools and Vandeputte (1974), who discusses her experiences with cerebral paralyzed children. Both authors explain how the ideas of the Papys about Papygrams, arrow representations for relations, sets and Papy's minicomputer for the representation of numbers and calculations are applied to teach mathematics to children with special needs.

At ICME-1, Papy-Lenger also drew the attention of co-workers of Burt Kaufman, director of the Comprehensive School Mathematics Project (CSMP), see De Bock (2022). The New Math project CSMP aimed at modernizing both the content and the teaching methods of mathematics. From 1973 till 1978, Papy-Lenger was director of research of the CSMP (De Bock, 2022). In the final evaluation report of CSMP, Herbert (1984), we read that "the minicomputer, string pictures, and arrow diagrams" (p. i) [4] make up three representational languages that are used frequently throughout the program. Herbert (1984) gives also examples of the CSMP teacher's guide. These include two games invented by Papy-Lenger to get children to do arithmetic in a playful way using Papy's minicomputer. By playing these games, *Minicomputer Golf* and *Tug of War* (rope pulling), individually or in competing teams, children do arithmetic and develop computational skills while playing a game, which is now called gamification (Quevedo et al., 2017).

Herbert (1984) reports on the development of CSMP, its content and organization, training and coordination and on CSMP materials. The latter were in 1974 sent to five academical reviewers, of which three were positively impressed by most of the content and the applied representational languages, except about Papy's minicomputer. As a matter of fact, all five reviewers expressed a negative opinion on the use of Papy's minicomputer, with the worst quotes being "a horrible

[3] A lattice is an ordered set in which there exist specific least upper bounds and greatest lower bounds, see Steiner (1974).

[4] String pictures are Venn diagrams.

aberration" and "a disaster of the first magnitude" (Herbert, 1984, p. 30). In the summary of these external reviews, one reads "All five reviewers are dubious to very negative on the minicomputer's mixture of a binary and decimal base" (Herbert, 1984, p. H-1, underlining in original).

The negative opinion of the five CSMP reviewers is the only documented unfavourable perception of Papy's minicomputer that we could find. However, from the interview with Papy that is available on Kennes (s.d.), we deduce that, in Belgium, critical voices about the price of Papy's minicomputer existed, as well as about the necessity to retrain teachers. Both arguments are rebutted by Papy, one can use paper and pawns to fabricate one's own cheap minicomputer, see also G. Papy (1975), and experiences in CSMP show that only a few hours are needed to initiate teachers to use minicomputer.

Compared to the sources discussed so far, Socas Robayna and Espinel Febles (1989) take the use of Papy's minicomputer to a higher level. On the one hand these authors do not limit themselves to the basic calculations of addition, subtraction, multiplication and division, but also show how to calculate powers, square roots, and logarithms with Papy's minicomputer. On the other hand, they present the corresponding algorithms in the form of flowcharts and make explicit reference to what is now called computational thinking, "developing one's own strategy and stimulating creativity" and "the use of flowcharts is a means of telling a machine how to operate, which allows the development of fundamental programming skills" (Socas Robayna & Espinel Febles, 1989, p. 60).

Recent research on Papy's minicomputer

Recent literature on Papy's minicomputer is scarce and almost exclusively found in languages other than English. An interesting example is provided by Dassie and Costa (2020), who describe the Brazilian production of Papy's minicomputer, and, using press articles and records of Papy's visit to Rio de Janeiro in 1971, link this to a discussion of the modern mathematics movement in Brazil. For additional information on the latter, we also refer to chapters 19 and 20 in De Bock (2023).

From Soares Silva Teixeira (2017) we learn that Papy's minicomputer is still being used in the first cycle in some primary schools in the greater Porto area in Portugal. Two teachers who use Papy's minicomputer describe how their students love this teaching aid and how it helps them to become fluent in mental arithmetic. They emphasize that Papy's minicomputer is very useful in teaching the place value system and that it increases the student's awareness of the structure of numbers (e.g. $7 = 1 + 2 + 4$). During classroom observations Soares Silva Teixeira (2017) witnessed the ease with which the students manipulated Papy's minicomputer, it was not quiet in the classroom, but the students were all actively participating in the mathematical activity. Finally, an intervention study with Papy's minicomputer was carried out with a group of 12 students. Afterwards, in a survey, they reported an overall very positive attitude towards learning with Papy's minicomputer.

Quevedo et al. (2017) first present Papy's minicomputer and its reduction rules and then describe the Minicomputer Golf and Tug of War games we mentioned earlier. Similarly, Mladinić (2020) presents arithmetic problems on Papy's minicomputer. However, the reduction rules are not explained, so part of the original design of Papy's minicomputer and its goal of performing large number calculations almost effortlessly, are lost in this way. Similar remarks apply to the games Minicomputer Golf and Tug of War, which were extensively implemented in the CSMP material (Herbert, 1984). In

retrospect, one can wonder if this was the reason for the negative reception of Papy's minicomputer by the reviewers of the CSMP material. If Papy's minicomputer is deprived of its reduction rules and its goal of obtaining a formation, all that remains is a rather cumbersome notation for a number.

Conclusion

We have presented an overview of how Papy's minicomputer was constructed and how it works. Additionally, we provided some examples of experiments that were carried out with Papy's minicomputer. Due to space limitations, we could not discuss all sources on Papy's minicomputer, so we encourage the interested reader to consult the bibliographies of the references. However, to the best of our knowledge, we believe to have touched upon a broad range of aspects of Papy's minicomputer and to have discussed the key references. It is remarkable to see in how many different countries Papy's minicomputer has appeared as a teaching tool and we were surprised to learn that it is still used nowadays in for example some schools in Portugal, see e.g. (Soares Silva Teixeira, 2017).

We believe it might be interesting to consider the use of Papy's minicomputer as a gamification tool to evoke or train computational thinking. However, one should be careful not to use Papy's minicomputer in a way which was perceived unfavourably in the past. Moreover, keeping in mind Bell and Hickman (2025), if one would apply Papy's minicomputer, then its internal binary number system should be stressed more.

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Proofs in school geometry and international movements in the Brazilian context

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The study researches how geometry proofs were approached in Brazilian educational regulations and textbooks between 1930 and 1970, focusing on 13-year-old students. Comparing two modern periods, the first influenced by CIEM in 1908 and the second by the Modern Mathematics movement in 1959, four representative textbooks of the period are analyzed. The results reveal that, despite the reforms proposing more intuitive approaches, the traces of practices remained centered on deductive geometry with an emphasis on memorizing theorems until the 1950s. Only with the insertion of Modern Mathematics, especially in Sangiorgi's book in the 1960s, there was an attempt to break with traditional practices, introducing more experimental and innovative methods. This situation highlights the resistance and adaptations of Brazilian school culture in the face of international movements for pedagogical renovation.

Keywords: Demonstrations, Intuitive Geometry, Deductive Geometry, Two-column proofs, Secondary education.

Introduction

The study in question aims to analyze how geometry proofs, introduced to 13-year-old students, were approached in educational regulations and textbooks, between the years 1930 and 1970, firstly in the Brazilian context in general and, later, focusing on the state of São Paulo, in the southeast area of the country. Two ideas of international scope circulated in the period examined: the first, represented by the creation of the Commission Internationale de l'Enseignement Mathématique (CIEM) at the IV International Congress of Mathematicians, held in Italy, in 1908; and the second, promoted by the Organisation for European Economic Co-operation (OEEC), Royaumont Seminar, in France, in 1959. In Brazil, these movements were characterized as modern initiatives in mathematics teaching, bringing together representatives from different countries, producing curricular guidelines, and being appropriate to the Brazilian educational context, both in regulations and textbooks. The aim is to answer – how do international movements contribute to the introduction of the study of proofs? What changes can be observed during this period regarding introductory proofs exercises?

The theoretical and methodological contribution of this study is based on research on the international circulation of pedagogical ideas, which since the 19th century has created a vast system of exchanges and mutual observations, structured by pedagogical missions (Matasci, 2016). From the 20th century onwards, international scientific congresses and the creation of commissions to discuss new Mathematics teaching programs enable the continuity of cultural transfers, covering the circulation

of educators, experiences, books and knowledge between different countries. Although the centrality and relevance of European culture is recognized in these two international movements, led by influential mathematicians, the debates and recommendations formulated in these spaces gained global visibility, crossing the Atlantic and becoming important references for Brazilian educational reforms. It is important to highlight, however, that these guidelines were adapted to the local context, considering that, as Chartier (2009) argues, the reception of dominant cultural models does not negate the negotiation with deeply rooted shared traditions, even when new disciplines or norms are introduced.

The present study aims to understand the process of appropriation (Chartier, 2009) of two modern ideas by Brazilian school culture, separated by an interval of 50 years. The focus of the analysis is on the proofs processes in geometry, which throughout this period were introduced in secondary education, starting at 13 years old.

1st modern movement – 1908

Schubring (2003) points out that the official from CIEM guidelines, led by Felix Klein, proposed the inclusion of basic notions of variable quantities and functional dependence on mathematical concepts, in addition to a methodological reorientation with an emphasis on intuition and practical applications. Although Brazil did not participate in the IV Congress of 1908 in Rome, teacher Eugênio de Barros Raja Gabaglia (1862-1919), from Colégio Pedro II (founded in 1837 and a reference in the constitution of Brazilian secondary education), represented the country in the V International Mathematics Congress, in England, in 1912 (Schubring, 2003). However, it's possible that Gabaglia did not bring international discussions and proposals for reforming mathematics teaching to the Brazilian community, as no significant changes were incorporated into the programs at Colégio Pedro II in the 1910s and 1920s (Valente, 2003). It was necessary to wait until 1930s for the proposals of 1st modern movement were inserted in the official program of secondary school through the Francisco Campos Reform, in 1931. This happened through another teacher at Colégio Pedro II, Euclides Roxo (1890-1950), who, for the first time, introduced the subject of mathematics as an integrated unit, bringing together arithmetic, algebra and geometry, aligned with the recommendations that:

> Starting from the living and concrete intuition, the logical feature will grow, little by little, until it gradually reaches formal exposition; [...] knowledge will be acquired, at first through experimentation and sensory perception and then, slowly, through analytical reasoning. Thus, regarding geometry, the formal demonstrative study must be preceded by a propaedeutic course, aimed at intuitive teaching, of an experimental and constructive nature (Bicudo, 1942, p. 157).

The 1931 Reform didn't last long and was replaced in 1942 by the Capanema Reform, keeping the division between intuitive and deductive geometry. In 1951, through the Simões Filho Reform, geometry teaching was reduced to the final school years. The three reforms prescribed deductive geometry, that is, the introduction to the study of proofs in geometry for 13-year-old students, as shown in Table 1. The analysis of this table indicates the sign of deductive geometry, preceded by an intuitive preparation, aligned with the modern movement, in the reforms of 1931 and 1942; however, in 1951, geometry loses space in the program and in the intuitive approach.

Table 1: Geometry in programs prescribed for students aged 11 to 14 (1931-1951)

Reform	Geometry organization	Instructions
1931	*Geometric Initiation Geometry*	- Starting from intuition to **gradually reach** formal exposure: from experimentation and sensory perception to analytical reasoning
1942	*Intuitive Geometry Deductive Geometry*	- Intuitive geometry as a **smooth transaction** between experiences with shapes and deductive conception of Geometry
1951	*Geometry* (13 and 14 years old)	- **Gradually awaken** the feeling of the need for justification and proof

Source: The authors based on the regulations.

2nd modern movement – 1959

According to Kilpatrick (2012), between 1950 and 1970, there were several attempts to change mathematics teaching under the name "Modern Mathematics" (MM) or "New Mathematics", with the aim of aligning school mathematics with scientific mathematics of the 20th century. The creation and elaboration of textbooks by the School Mathematics Study Group (SMSG), in 1958, in the United States of America (USA) and the holding of the Royaumont Seminar, organized by the Organisation for European Economic Co-operation (OEEC) and held in 1959, in France, followed by the publication of the modern program for secondary education, are important milestones for Brazil. The teaching of geometry was the subject of intense debates; at least two trends can be mentioned. The first one, prepared by Birkhoff, proposed the change of Euclid's axioms, preserving Euclidean geometry. The second trend recommended the development of geometry through geometric transformations, that according to our understanding, is related to Klein's ideas.

In 1960, teacher Osvaldo Sangiorgi (1927-2017), author of widely circulated textbooks in Brazil, completed an internship in the USA. Upon returning, in 1961, he created the Mathematics Teaching Study Group (GEEM), in São Paulo, in the southeast area of the country, inspired by Birkhoff. The first collection of textbooks published with a modern approach was prepared by Sangiorgi, having great repercussion, especially in the state of São Paulo (Valente, 2008). On the other hand, the second trend of geometry through geometric transformations (GT) is also gaining space in Brazilian textbooks. The study by Leme da Silva and Jahn (2024) analyzed four textbooks by authors considered leaders in MM in Brazil and concluded that there was a diversity of approaches: two of them included GT in the appendix, one presents a complete and formal treatment of GT (with one of the main authors having made visits to several European countries in the 50s), and a fourth one includes only symmetry and homothety, aiming to integrate it with Euclidean geometry, as these transformations facilitate the study of the notions of congruence and similarity of figures.

Still in 1961, the Brazilian National Education Guidelines and Bases Law (LDB 4,024/61) decentralized national programs, allowing each state to develop its own programs. In this way, the state of São Paulo was a pioneer in incorporating MM ideas into its program published in 1965, which was the subject of discussion within the scope of GEEM. The 1965 program resumed the intuitive study of geometry only in the 1st grade of secondary education and kept deductive geometry in the 3rd grade, for 13-year-old students.

To compare with the normative guidelines, we analyzed four textbooks representative of the period, all aimed at the 3rd year of secondary education (for 13-year-old students), in which the study of proofs in geometry was introduced. The first two books are part of the collections "Curso de Matemática" (Roxo et al., 1936) and "Matemática Ginasial" (Roxo et al., 1944), both prepared by Euclides Roxo, Júlio Cezar de Mello e Souza, and Cecil Thiré, all teachers from Colégio Pedro II, which makes the collection a reference for the period. The other two books make up the collections “Mathematics curso ginasial” (Sangiorgi, 1958) and “Matemática curso ginasial” (Sangiorgi, 1967), both authored by Osvaldo Sangiorgi, equally representative in the 1950s and 60s. In methodological terms, we analyzed the content of the textbooks, focusing on the criteria: how deductive geometry is introduced, which proof methods are used, the language applied, how the proof of base angle of isosceles triangles theorem is carried out and the exercises proposed to students.

Analysis of introductory proofs exercises

We present a study that examines the approach to proofs in two sets of textbooks by different authors, which were designated according to the decades of publication. The first set is made up of textbooks by Roxo et al., which we call Book 01 (1930s) and Book 02 (1940s). These texts allow us to understand how the authors integrated the proofs exercises into the regulations of the 1st modern movement. The second set analyzes Sangiorgi's textbooks, called Book 03 (1950s) and Book 04 (1960s). The first set represents the pre-modern period and the second one allows us to examine the appropriation of 2nd modern movement by the author. Our goal is to identify possible changes in the introduction to deductive geometry and how geometric properties are justified or validated.

Proofs exercises – 1st modern movement – 1930s and 1940s

Books 01 and 02 have many similarities in their approach to geometry, the authors present an introduction to the study of deductive geometry, consisting of an explanatory text containing definitions and principles of a deductive axiomatic system for geometry. The terms "proposition", "axiom", "postulate", "theorem" and "definition" stand out. The exercise of proving is understood as "proving the proposition, showing, through a chain of reasoning, that the thesis necessarily results from the hypothesis" (Roxo et al., 1936, p. 202).

In Book 01, the authors warn that "it is not possible to establish a general method for demonstrating all geometry theorems" (Roxo et al., 1936, p. 206), and promise to present different forms of proof in subsequent chapters, and thus promote deductive thinking. In Book 02, two additional topics on mathematical validations are introduced: "Proof methods", which covers direct and indirect proofs, and "Reduction to absurdity", with examples of proofs. A comparative analysis reveals the primary distinction between the two books lies in the significant expansion in the number of proposed exercises, especially those involving proofs; this difference may stem from Book 01 being seemingly aimed at teachers, while Book 02 more explicitly targets students. In Book 02, the terms “demonstrate”, “prove” and “show” are used as synonyms in the statements of the proof exercises.

Due to space limitations in this article, we focus our detailed analysis on triangles, which correspond to the first topic that approaches proofs in geometry. In the study of triangles, the notion of congruence is introduced, along with the cases of congruence of triangles presented as theorems, all demonstrated

in both books. Other properties of triangles are presented, all involving the application of congruence cases to their respective proofs. Figure 1 illustrates the proof of the property of equality of the base angles in an isosceles triangle in Book 01 (on the left) and 02 (on the right).

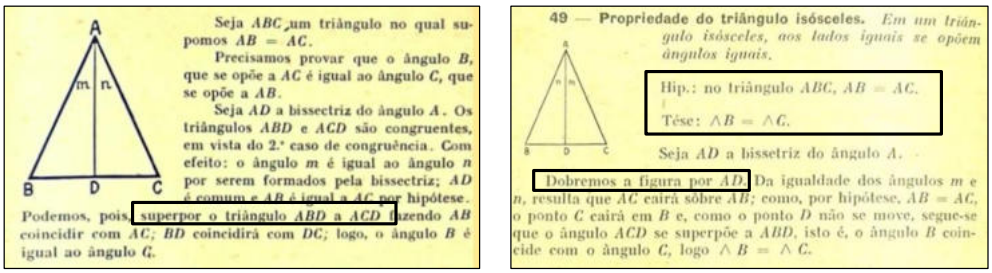


Figure 1: Proofs of property of isosceles triangle (Book 01, p. 235; Book 02, p. 168)

Some differences are noted between the proofs. In Book 01, the authors write in natural language, explaining the starting point and what they want to prove. The proof is justified by mobilizing the 2nd case of congruence, but appeal is also made to the superposition of triangles. In Book 02, the terms hypothesis and thesis, absent in Book 01, are clearly explained, and the proof is inserted in a more concise language, for example, to indicate the angles, the congruence of triangles is not used, the justification is supported by intuition and use of action resources, as in the expression "Let us fold the figure" (Roxo et al., 1944, p. 168). In the exercises in Book 02, before students are asked to do proofs, exercises are proposed such as "Separate the hypothesis and thesis into each of the following propositions, translating them symbolically when possible" (Roxo et al., 1944, p. 138), reinforcing the starting and ending point and the mathematical language.

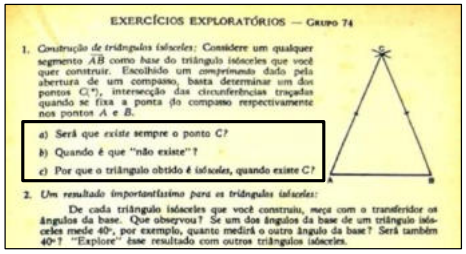
Proofs exercises – pre-modern and MM – 1950s and 1960s

The regulations of the 1950s restrict the teaching of geometry to the final two grades of secondary school. In Book 03, Sangiorgi announces that experimental geometry was created in the primary course and that, for the secondary course, a deductive geometry is reserved. The emphasis on the exposed theorems and exercises for students to do the proofs follows the approach of Book 02, the terms "intuitive and deductive propositions", "axiom or postulate", "theorem and proof" are introduced (Sangiorgi, 1958, p. 85), reducing the appeal to intuitive aspects. However, the mathematical language used in the hypothesis and thesis is even more concise, the proof process is justified by numbered steps. The term "Let's fold the figure" (figure 1, on the right) is replaced by "Let's draw the bisector of the angle of vertex A" (figure 2, on the right), the justifications are done in parentheses and, at the end, there is the acronym c.q.d (as we wanted to demonstrate). The appeal to the movement and superposition of the obtained triangles cannot be further highlighted, the justifications highlight definitions and properties already stated.

Figure 2: Proofs of property of isosceles triangle (Book 03, p.119-120)

We can identify, in Book 03 (1950s), that the evidence is presented using symbolic language, containing more formal aspects in terms of its content. Certainly, such aspects must have contributed to the debate about students' difficulties in understanding geometry proofs getting more space at the National Congress of Mathematics Teaching, in 1957, reporting a practice of "memorizing theorems" used by students to obtain a grade in evaluations (Búrigo, 2015).

Jahn & Leme da Silva (2023) carried out a comparative study of Sangiorgi's Books 03 and 04 and indicated a significant difference between them. Book 04 brought an innovative proposal for the treatment of proofs in geometry, reduced the number of theorems demonstrated, inserted exploratory exercises, as well as new representations used in the proofs. We can realize that, in Book 04, the study of triangles is carried out before the introduction of deductive geometry, in a previous chapter, aimed at intuitive and experimental geometry. There are no proofs of theorems, but a predominance of "exploratory exercises", such as the construction of the isosceles triangle and the exploration of its existence and the measurements of the base angles (cf. figure 3) so that the student "figures out" (or conjectures) experimentally the properties involved.



EXERCÍCIOS EXPLORATÓRIOS — GRUPO 74

1. *Construção de triângulos isósceles:* Considere um qualquer segmento $\overline{AB}$ como base do triângulo isósceles que você quer construir. Escolhido um *comprimento* dado pela abertura de um compasso, basta determinar um dos pontos C_1*), intersecção das circunferências traçadas quando se fixa a ponta do compasso respectivamente nos pontos A e B.

a) Será que *existe* sempre o ponto C?
b) Quando é que "não existe"?
c) Por que o triângulo obtido é isósceles, quando existe C?

2. *Um resultado importantíssimo para os triângulos isósceles:*
De cada triângulo isósceles que você construiu, meça com o transferidor os ângulos da base. Que observou? Se um dos ângulos da base de um triângulo isósceles mede 40°, por exemplo, quanto medirá o outro ângulo da base? Será também 40°? "Explore" esse resultado com outros triângulos isósceles.

Figure 3: Exploratory exercise on isosceles triangles (Book 04, 1967, p. 207)

The demonstration of the base angle theorem of an isosceles triangle illustrates well the changes proposed by Sangiorgi in Book 04. Only after having proposed several exploratory exercises, involving the construction of isosceles triangles, the cases of congruence, and encouraging students to look for a valid property in all of them, is that the proof is presented. The cases of congruence, proven as theorems in Books 01, 02 and 03, are considered postulates in Book 04, after having been studied in exploratory exercises.

In addition to the introductory study, the author uses new methodological strategies for the proofs, such as the proof on the left of figure 4 that separates the triangles, organizes the justifications and at the end writes "how we wanted to demonstrate"; two columns proof (statements and justifications, see figure 4, central) and the proof through drawn schemes (figure 4, right).

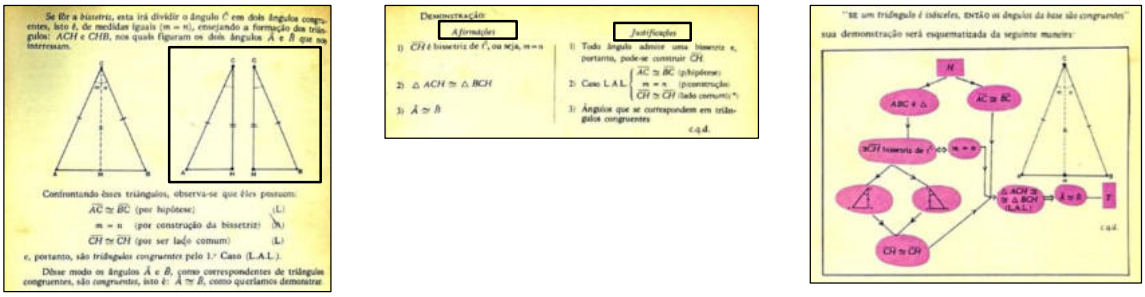


Figure 4: Proofs of properties of isosceles triangles (Book 04, 1967, p. 240-241)

The way in which Sangiorgi (1967) proposes the study of proofs for 13-year-old students is unprecedented in Brazil and, most likely, was conceived based on the contact he established in the USA with study groups, with North American textbooks and with researchers involved in Modern Mathematics. Herbst (2002) reports that, in the USA, two column proofs became a common practice among textbook authors since the beginning of the 20th century, remaining present for a long period in North American textbooks.

Sangiorgi also highlights in Book 04: "Don't 'memorize' proof of theorem! Value yourself, using any of the methods presented. Do it your way, using to these methods and you will be accomplishing in mathematics!" (Sangiorgi, 1967, p. 258). Certainly, the call for the student to do their own proof means explaining that the student can create personal ways of proving theorems, other than those carried out in books or presented by teachers.

Final considerations

Although regulations from 1930 onwards record the appropriation of the first modern movement, highlighting the need for an introductory intuitive study, at the end of the 1950s, secondary education was still dominated by deductive geometry. There is evidence that students memorized theorems, with almost no prior intuitive and experimental study. The analysis of textbooks also reveals a significant number of geometric proofs in Books 02 and 03, in addition to many exercises for students to do proofs. In summary, geometry teaching in the late 1950s and early 1960s was characterized by a strong emphasis on the logical-deductive process, resulting in students memorizing theorems.

The 2nd movement, referring to the ideology of the MM, seems to have had different appropriations. Regarding the introduction of proofs in geometry, they are not highlighted in the 1965 regulations, nor in the need to insert intuitive geometry, nor to emphasize changes in geometry validation processes. On the other hand, Book 04, which announces the modern approach, clearly indicates an attempt to break with sedimented practices of reproducing countless theorems, without prior experimentation, without meaning or understanding by students, and being translated by memorization of theorems.

When we analyze the period in question, we notice the persistence of traditional practices in teaching geometry, in which Euclidean proofs are considered standard of accuracy and, even though they are included in the regulations, they provoke reactions from teachers, book authors and students. It took a 2nd movement, perhaps Sangiorgi's closest contact with North American culture, to present different proposals for teaching proofs, at least in textbooks.

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Mathematics for all, but what mathematics? Comparing the overlap of three coexisting curricula in the mid-20th century Spain

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In Spain, primary and secondary education were not consecutive until the reform that took place in 1970. After the Spanish civil war (1936-1939), education was mandatory (on paper) for children until the age of 12. The 6-10 years period corresponded to primary education but, after that, students from 10 to 12 years old could follow three different itineraries: They could continue and conclude their primary education studies, they could shift to the so-called 'bachillerato universitario' [university baccalaureate] that eventually could lead to higher education, or they could join the so-called 'bachillerato laboral' [working baccalaureate] that was supposed to provide training for different types of jobs. In all cases, mathematics had an important role in the curricula. Here, examining textbooks and official documents, we show similarities and differences in the weight and intentionality given to the mathematics that were taught in these itineraries to students of the same age.

Keywords: Curricula, primary education, middle school, professional education, 20th century.

Introduction

According to Schubring (2015, p. 248): "true universalism can only then become effective when primary and secondary school levels became consecutive". In Spain, despite some previous advances in that direction, this condition for universalism was reached only in 1970. Until that reform, Spanish primary and secondary education overlapped in the 10-12 years period. Moreover, in the case of secondary education, two different itineraries existed, each of them aiming at students with different profiles and eventually leading to different specializations.

Mathematics was present and played a central role in primary education and in both secondary education itineraries. Considering their different goals and orientations, it can be relevant to compare the presence of mathematics in those three curricula, especially in the overlapping grades. Thus, we aim to answer the following research question: What were the differences in the treatment of mathematics according to the itinerary?

We will approach this question by considering different levels of the curriculum and, in particular, the intended and the implemented curriculum (Van den Akker, 2003). From the point of view of the intended curriculum (in official documents), we can focus on the weight of mathematics in the whole program, on the selection and sequencing of contents, on the methodological orientations provided to teachers, etc. On the other hand, from the point of view of the implemented curriculum (particularly in textbooks) we can focus on the treatment of topics and their relative importance, on the number and context of problems, etc. In this paper, we will pay especial attention to the grades corresponding to ages 10-12. This choice is motivated by the fact that this time interval corresponds to the overlapping of curricula taking place in the mandatory schooling period (ages 6-12). As we will see, between ages 6 and 10 there was a common primary education curriculum, and at the age of 10

children had to choose among three itineraries. Since age 12 was the upper bound for mandatory education, we believe that this 10-12 period gains particular interest for comparative purposes.

To carry out this research, we consider different sources of information. Regarding the intended curriculum, we have analyzed some of the official programs and syllabi which were in effect in Spain between 1950 and 1970. Regarding the implemented curriculum, here we have compared textbooks published by the publishing house Bruño, which was directly related to the De La Salle Brothers (officially Institute of the Brothers of the Christian Schools), a religious congregation that played an important role in the Spanish education at that time (Dávila & Naya, 2013).

Some background. Spanish primary and secondary education by 1960

In Figure 1 we provide a slightly simplified overview of the situation of Spanish primary and secondary education by the middle of the 20th century.

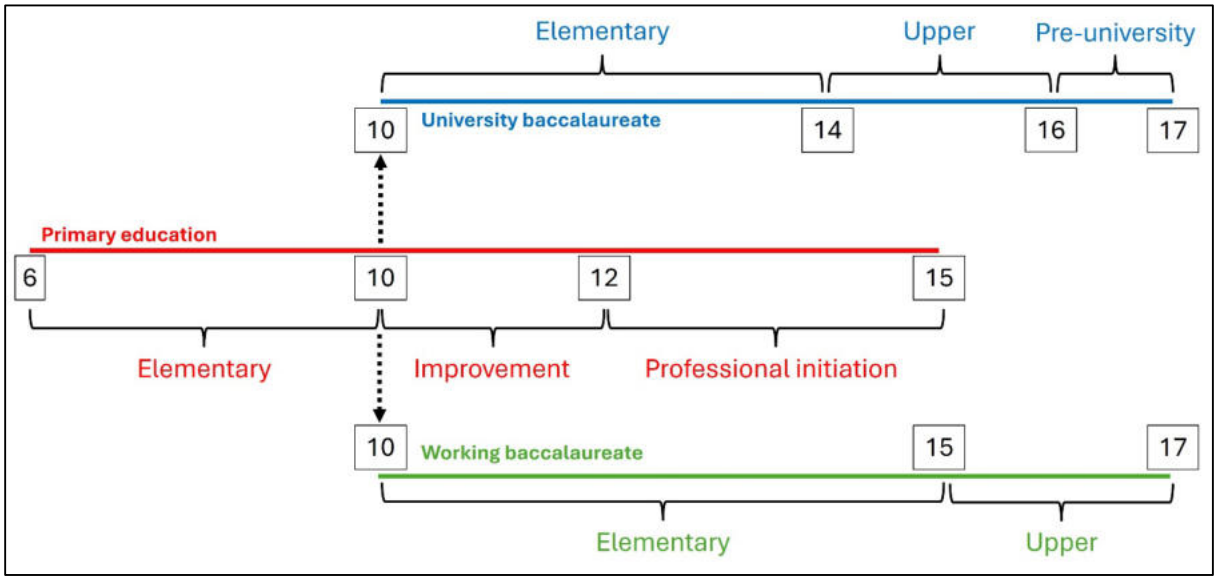

Figure 1: Spanish primary and secondary education by 1960

Education was mandatory for children between 6 and 12 years old. At the age of 10, they could decide whether to continue and conclude their primary education or to switch to any of the two possible itineraries of secondary education: university baccalaureate or working baccalaureate.

The students that continued their primary education could continue it past the age of 12, following the so-called grade of professional initiation that was supposed to provide useful knowledge to carry out certain jobs. After that, they could also enroll in industrial vocational learning util the age of 17 (to become an officer) or 19 (to become a master).

University baccalaureate, as its name suggests, was intended to be the itinerary for those students that wanted and could attend university or to special schools (architecture and engineering). The elementary period was common, and in the upper period the students could choose between a sciences or humanities specialization. Working baccalaureate was intended to provide specialized training for different activities such as agriculture, mining, secretariat, etc. In this baccalaureate there were

common subjects, while others depended on the chosen specialization. After completing their elementary grade, both baccalaureate itineraries could be left to enroll either into the Trade school (to follow studies roughly related to modern economy degrees) or into the Normal school (to become a primary education teacher).

Table 1: Number of students and gross enrollment ratio in Spain (Núñez, 2005)

	1950	1960	1970
Primary education	3,202,915 (66.9%)	4,033,592 (75%)	4,632,253 (74.1%)
Secondary education	214,874 (4.1%)	448,316 (8.77%)	1,286,335 (22.41%)

It is clear, then, that the itinerary followed by the students had a great impact on their future. Furthermore, for many years secondary education was only followed by a minority of students. Table 1 provides some illustrative data about this issue. As we can imagine, the path that a child followed was very often determined by sociological and economic reasons, and not by merit or ability. In fact, even if sometimes there existed mechanisms that connected some of the different itineraries, it was not very usual to move between them so that social mobility was not particularly easy.

Various official syllabi were published between 1950 and 1970. In the case of primary education, the first official syllabi were published in 1953, and they were later reformed in 1965 (Oller-Marcén, 2022). For university baccalaureate, official syllabi with different approaches were published in 1954, 1957 and 1967 (Ausejo, 2010). It seems that the case of working baccalaureate has not been addressed in the literature yet. As far as we know, the first complete official syllabi for this itinerary were published in 1956. Over time, new specializations for the upper period were approved, and their syllabi published (secretariat in 1958, horticulture in 1961, etc.).

The intended curriculum: presence and weight of mathematics

Mathematics was present as a mandatory subject in the nine grades (4 of the elementary period, 2 of the improvement period, and 3 of the professional initiation period) that constituted the primary education itinerary. In the case of university baccalaureate, mathematics was a mandatory subject only during the 4 grades of the common elementary period, while in the 3 remaining grades it was mandatory just for those students following the science option. Finally, in the case of the working baccalaureate, mathematics was a mandatory subject in the 7 grades (5 of the elementary period, and 2 of the upper period). However, the contents of grades 5, 6 and 7 were different according to the chosen specialization.

Regarding the weight of mathematics, in the case of primary education, the official documents from 1953 did not give any number of weekly hours to be devoted to teaching mathematics. According to Galera Pérez (2018) this number traditionally varied among schools, with a likely range of between 3 and 5 hours per week. Later, the 1965 reform established 3 weekly hours in grades 1 and 2, 4 weekly hours in grades 3 to 6, and 6 hours in grades 7 and 8. In the case of university baccalaureate, the official documents of 1953 stipulated 3 weekly hours during all the elementary period, then 6 weekly hours in grade 5, and 3 in grade 6 and pre-university. Finally, in the case of working baccalaureate,

official documents of 1956 stipulated 6 weekly hours of mathematics in grade 1 and 4 weekly hours in grades 2 to 5. During the upper period the number of weekly hours greatly varied according to the specialization.

The intended curriculum: methodological orientations

Official curricula were always accompanied by rather detailed methodological orientations. Sometimes they give indications to the teachers about how to treat certain particular mathematical topics. In other cases, and more interestingly, they provide more general observations about methodological approaches, providing some connections, elaborating on the purpose of teaching mathematics, etc.

In Table 2 we give some excerpts dealing with logical reasoning and abstraction in the different itineraries. We see that, in primary education, the orientations talk about "reasoning well" and there is no further mention of logical reasoning. In fact, the orientations for ages 10-12 explicitly recommend that teachers "avoid difficult proofs". In the case of the working baccalaureate, cultivating logical deductive reasoning is overtly rejected as a teaching goal. These statements contrast with the situation in the university baccalaureate, where logical reasoning is presented as one of the goals of the stage.

Table 2: Logical reasoning and abstraction in the different itineraries

Primary education (MEN, 1953)	**Working baccalaureate (MEN, 1956)**	**University baccalaureate (MEN, 1954)**
"From the earliest years, it is advisable to accustom the child to reason well, step by step" (p. 63).	"The purpose of the teaching of mathematics in the working baccalaureate should not be […] the cultivation of logical deductive reasoning" (p. 561).	"…with it the students must begin logical reasoning, which is one of the purposes of the teaching of mathematics in the baccalaureate" (p. 56).
"Children must acquire the habit of generalization; they must gradually detach themselves from the material and reach the abstract" (p. 63).	"…reasoning should be based on concrete things and images, within the child's reach, using facts and examples from daily life and models made by the students themselves in the handicraft class or in the workshop" (p. 562).	"It must be pointed out to the students that the figures they draw are only approximate, for example, the lines they draw are crude images of those conceived in Geometry […] when introducing the student to the idea of the ratio of two homogeneous quantities, the abstract nature of the ratio must be emphasized" (pp. 55-56).

In the case of abstraction, the orientations of the three itineraries try to avoid an excessive degree of abstraction. However, we see that the working baccalaureate emphasizes the use of concrete things

and images, while the university baccalaureate seems to pursue a higher degree of abstraction and metamathematical reflection when dealing with certain mathematical objects.

The intended curriculum: mathematical contents

In this paper, we are going to restrict ourselves to the case of geometry for students of age 10-11 years. This corresponds to grade 5 in primary education and grade 1 in any of the baccalaureate itineraries. Table 3 shows the contents exactly as described in some of the official published syllabi. We have tried to group the different contents to highlight the existing similarities and differences.

Table 3: Geometry for age 10-11 in the different itineraries

Primary education (MEN, 1953)	**Working baccalaureate (MEN, 1956)**	**University baccalaureate (MEN, 1954)**
Geometry: notion and division. Lines: their classes. Recognition and tracing exercises. Use of the ruler and compass, measurement and operations with segments. Plane angles: classes and properties. Measurement of angles. Exercises. Construction of equal angles. Perpendicular and oblique angles. Use of the bevel and the square in exercises on them.	Elementary movements and constructions in the plane.	Body, surface, line and point. Rectilinear segment and straight line. Measurement of segments. Plane surface. Angles. Perpendicular lines. Mediatrix of a segment and bisector of an angle. Parallel lines.
Polygon in general: classes and properties. Triangles: their classification. Value of its angles. Exercises and problems. Quadrilaterals: their classes. Special study of parallelograms. Exercises and problems. Regular and irregular polygons. Recognition and tracing.	Triangles and polygons.	Triangles: Classification and first properties. Quadrilaterals; parallelograms: Classification and immediate properties. Polygons: Classification. Convex polygon.
The circumference and the circle. Properties. Length of the circumference. Exercises and problems. Relative positions of two circles. Length of the line of the centers.	Empirical measurement of the circumference.	Circumference and circle; first properties about radius, diameter, chords and tangents. Sexagesimal division of the circle. Measurement of angles and arcs.

Similar figures. Proportional lines. Construction of similar polygons.		
Area of plane figures. Area of the triangle, parallelogram and regular polygon. Same for the circle and circular figures. Formulas. To find in them the value of any of its elements.	Area of elementary plane figures. Area of the circle.	Expressions of the areas of polygons and the circle.
	Initiation to the geometry of space with the analysis of the most common corporeal shapes. Volume of the orthohedron.	

We see that the degree of detail is rather different in the three official documents. The primary education syllabus provides a more detailed description of the contents than the other two. For some topics, the differences are quite remarkable, such as in the case of polygons.

When there are more detailed descriptions, other interesting differences arise. Again, for polygons, we see that the primary education syllabus seems to suggest beginning with polygons in general to later particularize to 3 and 4 sides, while the university baccalaureate syllabus seems to suggest the other way round. We also point out that some topics, such as geometrical proportionality or three-dimensional geometry were explicitly mentioned only in some of the itineraries.

The implemented curriculum: the definition or circumference

We present here a small example of how a particular mathematical object was treated slightly differently in two of the itineraries. We focus on the notion of circumference that, as we have seen in Table 3, was part of the geometry contents for age 10-11.

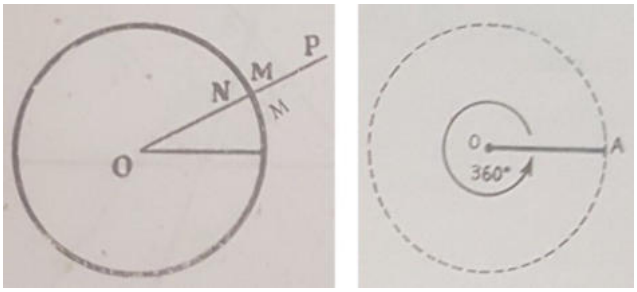

Figure 2: Illustration accompanying the introduction of the notion of circumference in primary education (left) and in university baccalaureate (right)

The following definition has been taken from a book addressed to students in the 10-12 period of primary education (Bruño, 1959, p. 78): “A circumference is a closed plane curve whose points are at the same distance from a point in the same plane”. On the other hand, the following excerpt comes from a book addressed to grade 1 of university baccalaureate (Bruño, 1962, p. 175): “If we rotate a

line segment OA, in a way such that it remains in a plane and the point O is fixed, the point A will follow a closed plane curve such that all of its points are at the same distance from O. This curve is called circumference". In Figure 2, we see the drawings that accompanied these definitions.

Both definitions share a common part, but the one addressed to university baccalaureate students introduced a dynamic approach including the notion of rotation (briefly presented earlier in the book) that makes it richer and more modern than the one for primary education students which is a static more classical definition.

Some final remarks

In Spain, until 1970, students of ages 10-12 had to attend to mandatory education. However, three different itineraries coexisted, each with a different aim. Although mathematics was present in all of them, its weight, contents, and treatment varied. In this paper we have given the first steps towards a systematic comparison of this situation which, as far as we know, has not been addressed in the existing literature.

Mathematics was present practically every year of every possible itinerary. However, its weight was different and somewhat erratic. For example, and for ages 10-12, primary education students received 4 weekly hours of mathematics each year, university baccalaureate students received 3 weekly hours each year, and working baccalaureate students received 6 and 4 weekly hours, respectively. From the point of view of the mathematical contents, even if there were differences (recall Table 2) they might not have been that big. Nevertheless, if we combine the lists of topics with the different workload, it seems rather obvious that the treatment of those topics should differ. This is made even clearer when we have a look at textbooks. Methodological orientations also show the intentionality in giving to each itinerary a more practical, applied or scientific approach.

All the previous observations need to be further explored. To do so we need to globally compare the three itineraries, and not only the overlapping ages 10-12, as well as the different specializations. In doing so, we think that we will contribute to illustrate how mathematics was – and is – concerned not just about educational issues, but also with shaping different types of people (Valero, 2017).

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New Math: A comparison of the NKMM's curriculum with Norwegian and Swedish curricula

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In this paper we compare the content in three curricula, one written by the Nordic Committee for the Modernization of School Mathematics (NKMM) in 1967, one curriculum from Sweden in 1969 and a temporary plan from Norway in 1971. Both the Swedish and the Norwegian plans are said to be based on the NKMM's plan. There are large similarities between these plans, yet there are also some interesting differences: the Norwegian plan goes further than the Swedish plan when it comes to set theory while it has omitted some of the more advanced mathematical topics.

Keywords: Comparative research, curriculum research, educational history, mathematics education.

Introduction

Two of the goals of New Math were to move mathematical concepts from higher grades to lower grades and "to bring school mathematics closer to the academic mathematics of the twentieth century" (Kilpatrick, 2012, p. 563). The foundation of modern academic mathematics was set theory. According to Kilpatrick (2012), one cannot transfer from one country to another how school mathematics should be taught because every country has "a unique school mathematics – a product of its history, culture, and tradition, and conforming to its social, political, and educational systems" (p. 569). De Bock (2023, p. 2) states that New Math appeared differently in the various countries "but in general, the emphasis shifted from developing technical-computational skills to insights into mathematical structures, often but not exclusively pursued by the study of abstract concepts such as sets, relations, algebraic structures, number bases other than 10, etc".

In the Nordic countries, there were several major changes in the school systems that happened around the same time. Norway and Sweden went from a seven-year school (seven to fourteen years) to a nine-year school (seven to sixteen) at the same time as the introduction of a comprehensive school. In Norway, experiments with nine-year *Grunnskolen* started in 1960 to replace seven-year *Folkeskolen* and the schools building on that; *Framhaldsskolen* and *Realskolen*. In 1969, nine-year school was mandatory for all. In Sweden, a reform in 1962 replaced *Folkskolan* and *Realskolan* with *Grundskolan*. It took ten years to implement the reform fully. In addition, the governments both in Norway and Sweden wanted to introduce New Math (Smestad & Opsal, 2023).

At the seminar in Royaumont in 1959, often viewed as the starting point for New Math in Europe, there was a call for regional cooperation in the work to reform mathematics education (NKMM, 1967, p. 6). In 1960, the Nordic Committee for the Modernization of School Mathematics (NKMM) was established by the Nordic Council with members from Denmark, Finland, Norway, and Sweden. The task of NKMM was to develop preliminary plans for mathematics to be tested in schools, and ultimately this work was to result in a report with proposed measures to be handed over to the

authorities in the four countries. The final report included, among other things, a proposal for a curriculum for students from first to ninth grade.

Bjarnadóttir (2023) has studied the work of NKMM and what was the driving forces behind it. “The question is to which degree four independent states with some heritage in common, but different situations in other respect could create a common policy in the field of mathematics education, in particular at the compulsory school level” (Bjarnadóttir, 2023, p. 240). Bjarnadóttir describes the committee’s work as contentious, marked by various interests, perhaps especially among those committee members who were to work with mathematics on the primary level. One of the members of the committee, Norwegian Torgeir Bue, was member of the working team for grade one to six. According to Bjarnadóttir, he wrote an appendix to the main document from this group where sets were not mentioned. Bue left the committee in 1963 (NKMM, 1967).

The secretary for NKMM was Matts Håstad. From the beginning, he was also leading the group responsible for the mathematics curriculum in Sweden in 1969, but left the group “due to conflicts within the group and other people at the National Board of Education” (Prytz, 2018, p. 201). According to Prytz, with references to Kilborn et al. (1977), the use of terms and symbols from set theory was reduced by the new members of the curriculum group.

In Norway, New Math was introduced in the experimental plan of 1971. In the guidelines for the committee working on mathematics, there were clear directions for a transition to New Math as proposed by the NKMM report. Since there were likely several schools and teachers who were not ready for this transition, two proposals for curriculum were made, a traditional one like the experimental plan from 1960 (alternative 1), and one with New Math (alternative 2) (Gjone, Part III, p. 37). During a transition phase, schools could choose between alternative 1 or alternative 2. Sweden was ahead in introducing nine-year compulsory education, as early as 1962 (Prytz, 2015). The 1969 curriculum in Sweden was based on New Math and “included new pedagogical principles on how to introduce and explain concepts” (Prytz, 2021, p. 1038).

In an earlier study we conclude that “While the expansion of obligatory schooling seems to have eased the transition to New Math in Sweden, it also complicated the transition in Norway” (Smestad & Opsal, 2023, p. 172). In both countries, new curricula for obligatory nine-year school provided opportunities to include New Math, but in Norway, complications concerning whether all students should have mathematics in grade nine, slowed down the process. In the end, in Sweden, the teachers were encouraged to choose teaching methods freely from 1973, the national approval of textbook series were removed in 1974, and in the new curriculum of 1980 most of New Math had been removed. In Norway, the final curriculum of 1974 already included far less New Math. Since both the Swedish and Norwegian curricula where New Math was introduced are based on the same plan from the NKMM, it is interesting to see how similar/different the plans are compared to the NKMM plan. Our research question is: How do the Swedish curriculum of 1969 and the Norwegian curriculum of 1971, compare to the curriculum proposal of the NKMM?

Method of analysis

To compare the two curricula with the NKMM proposal, we performed content analysis of the three curricula. Inspired by a framework for analysing mathematics textbooks from Charalombous et al. (2010) and using their concepts, we did a curriculum analysis in two parts: horizontal and vertical. In the horizontal analysis, we established how the plans were developed, by whom and when and in which context. Moreover, we noted the structure of the plans, the number of pages as well as the place the subject of mathematics had in the respective school systems. In the vertical analyses, we focused on one aspect of the curriculum, in our case on the stipulation of content to be covered. The rationale for dividing the analysis in two parts in this way, is that characteristics of the curriculum documents, studied in the horizontal analysis, may well be relevant to interpret the findings of the vertical analysis.

In the vertical analysis, we first created a table with the list of contents from the NKMM's plan (Table 1, column 1). Then we each studied the Norwegian and Swedish curriculum and analysed whether the NKMM contents were represented in these plans, and under which main level (column 2 and 4) and suggested grade levels they were placed (column 3 and 5). We also noted contents that were mentioned in the Norwegian and/or Swedish curriculums that were not mentioned in the NKMM curriculum. When we had finished the analyses individually, we compared the result, discussed and agreed upon those we had analysed differently.

Table 1: Analysis table, the first contents from first grade[1]

NKKM's annual plan	Norwegian, main level (L-M-H)	Norwegian, annual plan Grade	Swedish, main level (L-M-H)	Swedish, annual plan Grade
Grade 1				
1.1 Set, element, member of, subset, empty set	L1	1-1, 2-1, 3-1		1
1.2 Concept of numbers	L1, L2	1-2, 2-2	L	1
1.3 Natural numbers 0-100	L2 (without limit)	1-2, 2-2	L (without limit)	1

[1] In the second and fourth column, the letter refers to main level (L for first to third, M for fourth to sixth, and H for seventh to ninth). The number in the second column and the number after the hyphen in column three refers to the number of the mathematical themes in the Norwegian plan. For example, the number 2-1 in the third column refers to grade 2 and mathematical theme 1.

Result of horizontal analysis

The development processes of the three curricula were different. As mentioned in the introduction, the NKMM was established in wake of the Royaumont seminar to experiment and develop a common curriculum. The committee had members from four countries, and the experiments were international. The work lasted from 1960 to 1967 (Bjarnadóttir, 2023).

As early as 1963, an informal group, with varying members, was established in Sweden by The National Board of Education to follow the development of mathematics education and participate in the discussion about changes in the mathematics subject. In connection with this work, a new group, with some of the same members, was set in 1967 to develop the new curriculum in mathematics. (Kilborn et al., 1977, p. 11). The work stopped for half a year from October 1968 to March 1969 and was only picked up after Håstad and Magne left the group. According to Kilborn et al., Sandström, a chemist, did major edits to the curriculum in the final stages. Magne claimed that "In fourteen days, Sandström rewrote what we had worked on for a couple of years" (Kilborn et al., 1977, p. 23). The curriculum was finished by the end of 1969.

In Norway, a committee began work on a new mathematics curriculum in 1968, where the plan was to be based on NKMM's work. The members of the committee had advanced knowledge of New Math and created a proposal in 1969. Afterwards, the Normal Plan Committee (Normalplanutvalget) took over to adapt the mathematics plan to the other school subjects. There were several changes, both organizational, academic, and methodological. Parts of the changes were done to assure that the curriculum was workable for all students (Gjone, 1985, Part III). An interesting subject-related change was that "set theory" was changed to "basic concepts from set theory" and logic was almost completely omitted. After the Curricula Committee had submitted its proposal, this was to be processed by the Elementary School Council (Folkeskolerådet). The final version of the temporary plan was published in 1971.

The forms of the three curricular documents analysed here are comparable. The NKMM's curriculum proposal in the report first contains the background for the proposal, seven main areas in mathematics that all start with "All students should" (Alla elever bör), and then a description of the content for grades one to nine. In list form, the plan first describes the content for each grade (see Figure 1 grade 1) and then a description of what each theme in the list entails. A total of 27 pages are analysed (NKMM, p. 177-203). The seven main areas in this plan are number area, geometry, measurement, probability, table and diagram, problem solving and the nature of mathematics. The description of the main areas ends with a paragraph that tells what must be fulfilled for this to be carried out.

The 1969 curriculum in Sweden consist of a general part with goals and guidelines, schedules and plans for each level, low (grade one to three), medium (grade four to six), and high (lower secondary), where the subject is included (Skolöverstyrelsen, 1969b). In addition, there are also instructions and comments for the teachers. The structure of the mathematics section of the plan is similar to the structure of the other subjects in the plan and here we have the main elements for low, medium, and high level. A total of 4 pages described mathematics. There is also an appendix for each subject, in which mathematics was treated in 24 pages, with complementary instructions, comments and examples of annual plans in the subject (Skolöverstyrelsen, 1969a).

Årskurs 1

1.1 Mängd, element, tillhör, delmängd, tom mängd
1.2 Antalsbegreppet
1.3 De naturliga talen 0—100
1.4 Större än, mindre än, lika med, inte lika med (symbolerna $>$, $<$, $=$, $\neq$)
1.5 Union, addition (symbolen +). Den kommutativa lagen
1.6 Subtraktion (symbolen −)
1.7 Den associativa lagen för addition
1.8 Verksamhet med material av geometrisk natur
1.9 Enheter

Year 1

1.1 Set, element, belong, subset, empty set
1.2 The concept of quantity
1.3 The natural numbers 0-100
1.4 Greater than, less than, equal to, not equal to, (the symbols >, <, =, ≠)
1.5 Union, addition (the symbol +). The commutative law
1.6 Subtraction (the symbol -)
1.7 The associative law for addition
1.8 Activity with geometric nature material
1.9 Units

Figure 1: Grade one in NKMM's plan (NKMM, 1967, p. 180) to the left, our translation to the right

The Norwegian temporary curriculum from 1971, alternative 2 in mathematics, first gave the goal of the subject, then a list of 10 mathematical themes, what the students should work with at the various levels and finally guiding annual plans (Midlertidig mønsterplan, 1971, pp. 130-137). In addition, there are 30 pages with the headings "working methods" and "symbol list" which we have only included in the analysis for clarification and exemplification. In the NKMM, no details were given on the number of lessons expected for the subject of mathematics. In Sweden, mathematics was given on average 4.4 lessons per week in grades 1–9. Each lesson was 40 minutes, apparently. In Norway, mathematics was given on average 3.5 lessons per week in grades 1–8, with lessons lasting 45 minutes each. Mathematics was an elective subject in the ninth grade with 3 lessons per week. Thus, the Swedish subject had about 15 percent more time than the Norwegian one when the elective subject in Norway is included. Further in this work, we have included the mathematics in the ninth grade in Norway.

Result of vertical analysis

As both the Norwegian and the Swedish plans are based on NKMM, there are many similarities. We will therefore focus on the differences.

Set theory is an important part of New Math and is treated differently in the three plans. The first content mentioned in the NKMM's plan for first grade is "sets, elements, belong, subset, empty set". Furthermore, the students will also be introduced to the commutative and associative laws of addition. Logic is introduced in the third grade. It is also claimed that: "In order for the above program to be implemented successfully, mathematics should be presented in a modern spirit" (NKMM, 1967, p. 179, our translation). In this context, set theory is stressed. We see a greater emphasis on sets in the Norwegian plan than in the Swedish plan – both in the work with geometry, functions, and algebraic expressions. For example, in geometry at the intermediate level in the Norwegian plan, it states "Use of set terminology in the description of points, line segments and polygons" (Midlertidig mønsterplan, 1971, p. 131; our translation), while in the Swedish one it states: "Simple geometric concepts and correlations", without mentioning sets (Skolöverstyrelsen, 1969b, p. 137; our translation).

To exemplify how far the Norwegian plans went in using set notation as a tool in other areas of mathematics, in Figure 2, we give an example of how the plan expects the students to analyse a construction using sets.

In addition, the Norwegian plan has more set algebra, includes number sets (**N**, **Z** and such), and there is more emphasis on the laws. The Swedish plan, unlike the Norwegian one, lacks some logic and set

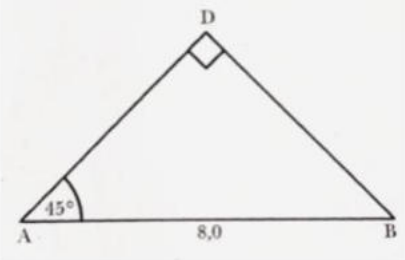
concepts (e.g. open statements, and/or, and intersection) and set symbols. For instance, sets are not included for students at the intermediate level (grade four to six). Also, the Swedish plan lacks some logical concepts such as implication and negation.

Example 1: A triangle ABD is to be constructed.

The side $AB = 8.0\ cm$, $\angle A = 45^\circ$ og $\angle D = 90^\circ$.

Construct the triangle and give a complete solution to the task.

1. *Auxiliary figure*



Measuring unit: cm.

2. *Analysis*

Start with a line *l* and mark $AB = 8.0\ cm$ on *l*.

The point *D* is given by:

1) $\angle A = 45^\circ$
2) $\angle D = 90^\circ$

We use the sets:

1) $V = \{P | \angle(AP, l) = 45^\circ \wedge A \in l\}$
2) $R = \{P | AP \perp BP\}$

$D \in V \wedge D \in R \Leftrightarrow D \in V \cap R$

Then follows the construction.

Figure 2: Example from the Norwegian curriculum (Midlertidig mønsterplan , 1971, p. 159), translation from Opsal and Smestad (2023)

According to the NKMM plan, the first-grade students will work on "Greater than, less than, equal to, not equal to (symbols >, <, =, ≠)" (NKMM, 1967, p. 180). The same wording is also verbatim in the Swedish suggested study plan for the first grade, but here without "not equal to" and the symbol for this. In the Norwegian plan, it says both under level grade one to three and for first grade: "Quantitative concepts in concrete and abstract contexts". In this plan, it includes what is in the NKMM plan but goes beyond this content.

The division algorithm is not mentioned in the Norwegian plan, while in the NKMM, students will work with it in the fourth grade and according to the Swedish plan in the fourth to sixth grade. Neither is the calculator mentioned in the Norwegian plan. According to NKMM, students will use a calculator in the fifth grade. In the Swedish plan, students will use a calculator in lower secondary school and a computer in ninth grade.

Other topics that are treated differently are probability and statistics. In NKMM's plan, probability is one of the seven main areas, and it is one of the points in the list for ninth grade. In the Swedish plan, probability is included, but it is not mentioned in the Norwegian plan. Although statistics is included in all plans, the choice of grades varies. In the NKMM's plan, descriptive statistics first come in the grade eight while in the Norwegian plan this is introduced gradually from fourth grade, and in the Swedish also gradually from second grade.

Vectors and introduction to derivation were two mathematically advanced topics that were included in the NKMM's plan. Derivation is not included in either the Norwegian or Swedish plan. For vectors, Norway and Sweden chose different solutions. While vectors are not included in the Norwegian plan, they are placed in grades 7-8 in the Swedish plan. It is stressed that while most students should be able to add vectors and show that vector addition is commutative, fewer students will be able to subtract vectors or calculate using vectors in coordinate form.

Discussion

To answer our research question: How do the Swedish curriculum of 1969 and the Norwegian curriculum of 1971, compare to the curriculum proposal of the NKMM, we have conducted a horizontal analysis and a vertical analysis of the three plans.

The structure of the NKMM plan is not followed in the Norwegian and Swedish plans. In the NKMM, they first introduce the main area and then what the different classes should contain. It is not divided into levels as we find in both the Norwegian and Swedish plan, where annual plans are just suggestions. The reason for this may be that there is ideally a common structure on how the various subjects should be described in the different curricula, and we see both in the Norwegian and the Swedish plan that the various subjects have the same structure where it is divided by levels and not grades. Even though the structure of the NKMM's plan and the Norwegian and Swedish plan is different, it is still interesting to discuss what is included in all three, and what is emphasized in one or two of the plans and not in the third one something that emerges in the vertical analysis.

Given the much larger amount of classroom time for the subject of mathematics in Sweden, it is not surprising that the Norwegian plan is missing some topics from the NKMM plan, which are included in the Swedish plan. Probability and vectors are two prominent examples of this. It is more surprising that the Norwegian plan goes a lot further in using set terminology in the treatment of other mathematical content, such as geometry. Given that both plans were based on the NKMM plan, this must have been a conscious decision based on different priorities for the subject. Norway introduces set theory as a "tool" in mathematics but does not introduce parts of the more advanced mathematics where this tool is needed. On the other hand, Sweden introduces some of this more advanced mathematics, such as vectors, but not the tool of set theory. It may be that a reason for this was the extensive rewrite of the Swedish plan by a non-mathematician in the final stage.

As Kilpatrick notes, it is not possible to transfer school mathematics materials and practices from country to country, due to each country's unique social, political and educational systems. In our case, several countries with rather similar systems cooperated to create a common curriculum, but still the national curricula ended up differently. Whether this is because of the systems or also individuals' influence, is another story. As De Bock states, there are differences between countries: In Norway, there is more focus on the mathematical structure than in Sweden and less on computational skills – as exemplified by the absence of the division algorithm in the Norwegian plans.

No matter what the background for this difference was, the difference seems to be important when discussing why New Math had such an abrupt end in Norway, at least in the sense that the New Math alternative 2 was abolished already in 1974. The Norwegian way of including set theory must have influenced the whole subject of mathematics more profoundly than the Swedish way, possibly eroding support.

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The use of two-dimensional figures as means of mathematical justification in complex analysis

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To have a better understanding of how different ways of doing mathematics employed by historical subjects are nuanced in contemporary complex analysis scenarios, in this study we present some results of how two participants in a pilot study used two-dimensional figures (without prior instruction on how to use them) as means of mathematical justification when they engaged in a series of tasks based on two problems that Cauchy and Goursat had to face. To carry out the analysis of the mathematical activity used by the participants in the study, we relied on the socioepistemological theory, which allowed us to understand that figures, without homologous representation in an algebraic setting, were used in two different ways: to compare in order to find upper bounds for the modulus of integrals over closed curves, and when the participants felt the need to supplement the algebraic apparatus to give a meaning to purely algebraic expressions.

Keywords: Use of figures, mathematical justification, original sources, complex analysis, history of mathematics.

Introduction

Chorlay et al. (2022) acknowledge that a line of research that has developed in mathematics education over the last 20 years is concerned with how the history of mathematics can become a resource for didactic intervention. Studies such as those of Clark et al. (2022) contribute to this line of research by reporting on three research initiatives that developed during the TRIUMPHS project; a project that aims to teach mathematical topics based on a series of tasks that require university students have to engage with, interpret, and explore the mathematics of primary source excerpts. We envisage that our ongoing doctoral research also aligns with this line of research reported by Chorlay and others. However, before configuring didactic material with the intention of teaching, we decided to explore how the history of complex analysis can enrich didactic research by understanding how the use of two-dimensional figures, that we unveiled in the analysis of original mathematical works, is nuanced in contemporary complex analysis scenarios.

Specifically, in a previous study (Piña-Aguirre & Farfán, 2023a), we conjectured that to develop complex analysis historical subjects had to complement their purely algebraic procedures through the gradual incorporation of two-dimensional figures. In Piña-Aguirre and Farfán (2023b) we illustrate this by means of two problems that historical subjects had to face. On one hand, Cauchy (1825) had to give a meaning to the concept of integral comprised between complex numbers, and to do so he had to complement his purely algebraic procedures with the use of figures accompanied by algebraic expressions. Later on, Goursat (1884) had to use figures, that were not represented by algebraic expressions, in order to find

upper bounds to the modulus of complex integrals over closed paths to prove what we know as Cauchy's integral theorem.

The previous results allowed us to show evidence (Piña-Aguirre et al., 2024) that the use of figures in modern complex analysis textbooks differs from the way in which historical subjects used them to deal with the complex integral concept and to prove Cauchy's integral theorem. Therefore, to give an answer to the question, how do undergraduate mathematics students use figures when facing some of the problems that historical subjects affronted in the development of complex analysis?, we designed a series of tasks (see the next section of the paper) based on original mathematical works, with the particularity that, in order to solve them, students will gradually incorporate the use of figures to their algebraic arguments as evidenced in our historical reconstruction of the development of complex analysis. Nevertheless, before applying the set of tasks with undergraduate mathematics students, we conducted a pilot study to see if the tasks would evoke what we wanted them to evoke, and in this paper we present some results of this pilot study.

The tasks sequence

By relying on Cauchy's memoire (1825), we configured tasks related to the concept of complex integration. Meanwhile, Goursat's (1884) proof of Cauchy's integral theorem allowed us to design tasks that require finding upper bounds for the modulus of integrals around closed curves. Space does not permit us to present the full set of tasks, but we do present one task related to Cauchy's work and one task related to Goursat's work (in which the pilot study participants used two-dimensional figures), accompanied by a brief description of the other tasks that make up the whole set of tasks.

In order to attend to the complex integral concept, we first present a couple of tasks designed to reproduce the type of mathematical activity that was predominant when dealing with complex quantities up to 1825. These tasks require to compute the four algebraic operations with complex numbers, as well as real integrals of real-valued complex functions. In task three, we ask to compute the value of an integral by relying on an excerpt of Cauchy's memoire that deals with a specific definition of real integration. Then, in the next task (which we will call task A), we present the following excerpt from Cauchy's memoir.

> Task A: Therefore, in order to embrace the same definition of integrals between real limits for integrals between [complex numbers] it is appropriate to represent by the notation
>
> $$\int_{x_0+iy_0}^{X+iY} f(z)\,dz$$
>
> the limit [to which is directed] the sum of the products of the form
>
> $$[(x_1 - x_0) + i(y_1 - y_0)]f(x_0 + iy_0)$$
> $$[(x_2 - x_1) + i(y_2 - y_1)]f(x_1 + iy_1)$$
> $$\dots$$
> $$[(X - x_{n-1}) + i(Y - y_{n-1})]f(x_{n-1} + iy_{n-1})$$
>
> When each of the two sequences

$$x_0, x_1, x_2, \dots, X,$$

$$y_0, y_1, y_2, \dots, Y$$

are made up of ever increasing or decreasing terms from the first to the last, infinitely close terms, whose number increases more and more.

We then asked the participants to express, in terms of a limit and a sigma notation, Cauchy's definition of integrals comprised between complex numbers. The following tasks were configured to present a contemporary definition of complex integration via integration paths connecting two different points in the complex plane. Then, we presented tasks related to the concept of simple closed curves. This in turn allows us to present the following task based on Goursat's proof, which we will call task B.

Task B: Consider curve C in Figure 1, oriented counterclockwise, which is made up of sections of the following five curves

$$y_1 = -x + 2; y_2 = x + 2; y_3 = 0; y_4 = -\frac{1}{8}sin\,(31.2x) + x + \frac{1}{2}; y_5 = -\frac{1}{8}sin(25x) - x + \frac{1}{2}$$

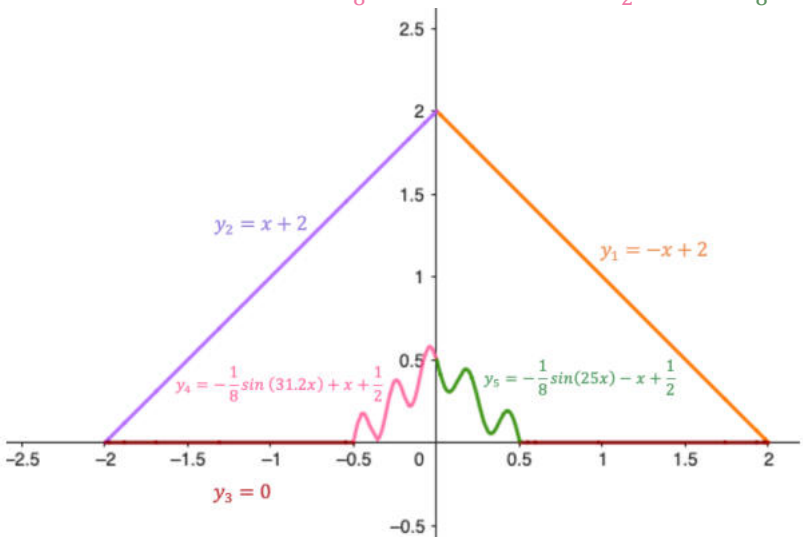

Figure 1: Closed curve C

If it is desired to traverse the entire curve C through 3 or more auxiliary curves C_i dividing the region bounded by the curve C in such a way that the following two conditions are satisfied:

(i) $\int_C f(z)dz$ can be written as the sum of integrals over closed curves γ_i (traversed counterclockwise) composed of sections of curve C and auxiliary curves C_i.

(ii) The distance between any two points in or on the curves γ_i can be bounded by a positive quantity.

Which auxiliary curves would you use?

The final task of the design is identical to task B, the main difference being that the curve C is a freehand drawing similar to an ellipse.

The socioepistemological theory of mathematics education

Since our objective is to understand how the participants of the study use figures to solve the sequence of tasks, we relied on a framework that studies the production of situated mathematical knowledge,

known as the Socioepistemological Theory of Mathematics Education (Cantoral, 2020). According to Torres-Corrales y Montiel (2019), one way in which this theory has analysed how individuals use and do mathematics is through the analytical questionings: *what does he/she do?* and *how does he/she do it?* In this research, we adapted these questions and analysed the mathematical activity of the participants in the pilot study by answering the questions: *what do they do in order to give an answer to the tasks?* and *how do they do what they do?* The second question allowed us to detect the emergence of figures as a means of mathematical justification, which led us to answer the following two questions: *what type of figures are they using?* and *what are the purposes of these figures?*

Following the work of Tournès (2012), we considered that the answers to the question *what type of figures are they using?* could be of two types. If the figures were accompanied by algebraic expressions, we classified them as *quantitative* figures. If the figures were freehand drawings without them being described through algebraic expressions, we catalogued them as *qualitative* figures. Meanwhile, the answers to the question *what are the purposes of these figures?* helped us to identify what these figures enabled the study participants to do.

Method for the recollection of data

Apart from the fact that we wanted to know if the design induced the use of figures, we also considered that it could be an opportunity to get feedback on the design itself (redaction, sequencing, and difficulty of the tasks, etc.). For this reason, the pilot study was carried out with four postgraduate students in mathematics education, with a bachelor's degree in the teaching of mathematics from three different universities in Latin America, who had not previously taken a course in complex analysis, but who were familiar with the four algebraic operations of complex numbers. The reason for looking for participants who had not previously taken a course in complex analysis was that we believed that if the participants had previously worked with the concept of complex integration, the process of discovering how to give a meaning to the complex integral (elicited by the task sequence) would not emerge. In this paper we report some of the responses of the participants called Alejandra and Manuel (both pseudonyms).

We conducted the pilot study using task-based interviews (Maher & Sigley, 2020). One of us (first author of the paper) met with each participant individually in a classroom in the Department of Mathematics Education at Cinvestav. We provided them with pencils, sheets of white paper and an eraser. Each participant was given the following task of the design after completing the previous task. There was no time limit; it was up to each participant to decide how long they wanted to engage with the tasks each day, with the commitment that they would complete all the tasks. The interviews were semi-structured in the sense that we made general comments linking the tasks, any doubts arising from the wording of the tasks were clarified, and if what the task required was uncleared, we rephrased what the task was asking for. As the participants completed the tasks, as they moved from one task to the next, and at the end of the sessions, notes were made about their mathematical work, about errors in wording of the tasks, and about the participants' comments on the tasks. In this sense, we claim that the interviews were also exploratory because they allowed us to provide a basis for a more detailed design.

The mathematical activity of the participants

Our intention with task A was that the participants obtained a symbolic expression that allowed them to attend to the concept of integral comprised between complex numbers. We hoped that the participants extended Cauchy's definition of integral presented in the previous task without the need to rely on the use of figures. However, this is not entirely what happened, as described in Alejandra's mathematical activity below.

What does she do? She obtains a symbolic expression for the concept of complex integration. *How does she do it?* She conceives that the points $x_k + iy_k$ behave in the same way as real numbers, in the sense that they can be arranged in a straight line (see Figure 2). By operating the particular case of $(x_1 + iy_1) - (x_o + iy_o)$, she convinces herself that the general form of these elements is $[(x_{k+1} - x_k) + i\,(y_{k+1} - y_k)]$.

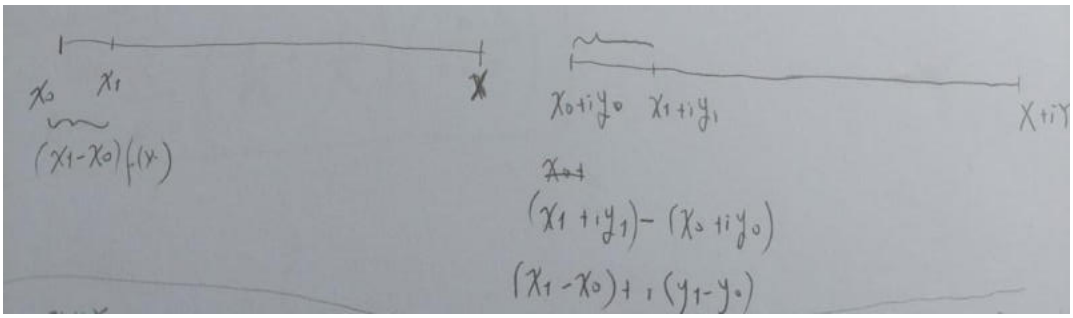

Figure 2: Alejandra's array of complex numbers in a straight line

What type of figure did she use? A qualitative figure. *What are the purposes of the figure?* It allows her to make sense of Cauchy's mathematical activity by imagining that sequences of points in $\mathbb{C}$ behave in a similar way to sequences in $\mathbb{R}$. Unlike Alejandra, Manuel compared the two definitions proposed by Cauchy and realised that what was different in the case of complex integration was that there was an iy accompanying the real expressions denoted by x.

On the other hand, task B was designed with the intention that the participants used figures to divide the inside of curve C. Specifically, Manuel did the following.

What does he do? He obtains the greatest distance in each of the regions delimited by the γ_i curves. *How does he do it?* He divides the region comprised inside the curve C by four γ_i curves. Two of these curves are triangles (yellow triangles in Figure 3) made up of sections of the curves y_1, y_2 and y_3. The other two γ_i curves delimit two regions (whose interior points are marked by green points in Figure 3) that are contained in two triangles (red triangles in Figure 3) that contain sections of the curves y_4 and y_5. Manuel conceives that the four triangles are right-angled triangles.

For the yellow triangles, Manuel states that the greatest distance between any two points within these regions are the lengths of the line segments represented by y_1 and y_2. He argues that these are the greatest distances by comparison. Manuel selects two points inside the yellow triangles and connects them with straight lines, then he uses these straight lines to form right-angled triangles (see white triangle in Figure

3). The two sides of the white triangle are then compared to the sides of the yellow triangle. Since the white sides are always shorter than the yellow sides, because they are contained in the region bounded by the yellow triangles, the greatest distance inside the yellow triangles is the length of the line segments represented by y_1 and y_2.

Then, Manuel states that the greatest distance in the γ_i regions, whose interior points are denoted by green points in Figure 3, are the hypotenuses of the red triangles. Manuel knows that the hypotenuses of the red triangles represent the longest distance between any two points inside or in the red triangles, then the greatest distance between any two points inside the two γ_i regions cannot exceed this hypotenuse, because the γ_i curves are inside the red triangles. That is to say, Manuel obtains upper bounds by comparing the regions delimited by the γ_i curves with the regions delimited by the red triangles.

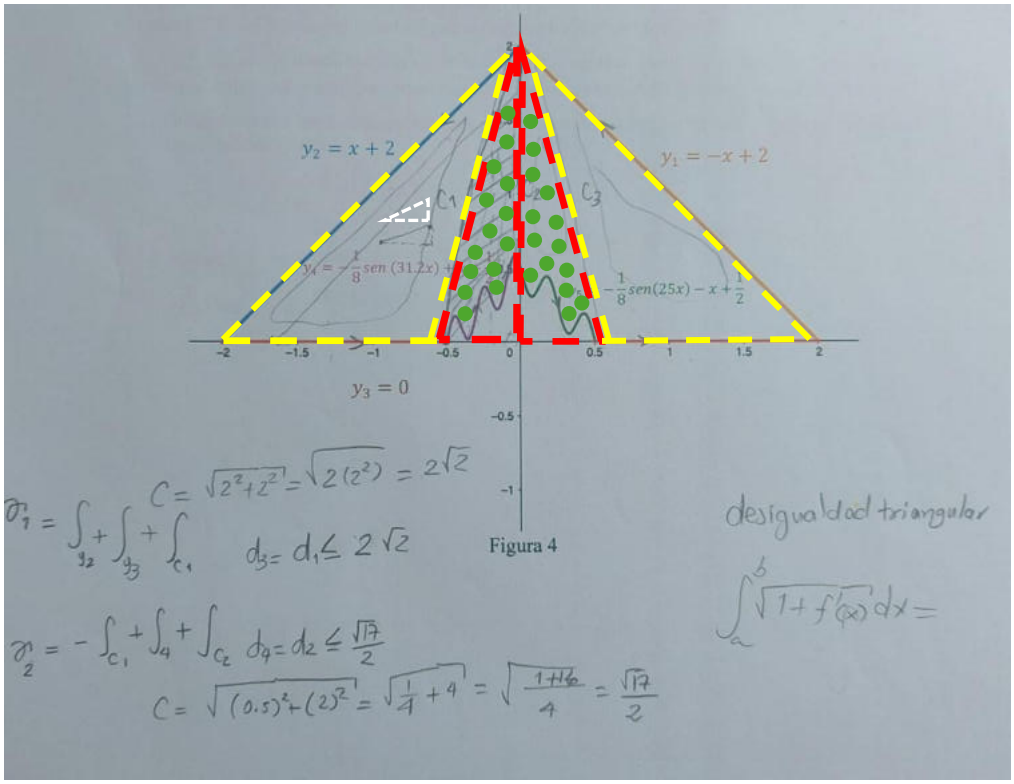

Figura 4

Figure 3: Manuel's use of figures

What type of figure did he use? Although some of the sides or the yellow triangles are represented by algebraic expressions, we argue that Manuel used qualitative figures, because the arguments he used did not involve the algebraic expressions that were already part of the task. *What are the purposes of the figures?* They were used to compare the γ_i figures with known triangular figures. Manuel compares the distance between any two points in or on the γ_i curves with a given distance, which he identifies as the greatest distance in the γ_i region.

It should be noted that in task B Alejandra also used figures to compare. However, Alejandra used rectangles instead of triangles to delineate regions within the curve C.

Conclusions

We argue that the results presented in this paper have allowed us to have a better understanding of how our theoretical results, reported in Piña-Aguirre and Farfán (2023a, 2023b), "live" in contemporary complex analysis scenarios. In the sense that both Alejandra and Manuel used figures, without homologous representation in an algebraic setting and without prior instruction on how to use them, in two different ways when dealing with some of the problems that historical subjects faced in developing complex analysis. Alejandra used them to give a meaning to a purely algebraic expression. Meanwhile, both of the participants used them to compare in order to find upper bounds for the modulus of integrals over closed curves.

In terms of the design, we would like to note that the configuration of the whole set of tasks shares some of the benefits reported by Barnett et al. (2014) when students engage with primary sources. For example, in task A, we hoped to bring the pilot study participants closer to the experience of *mathematical creation* by presenting Cauchy's proposal for complex integration. However, in light of Alejandra's response, we modified this task by removing Cauchy's excerpt in which he gives a meaning to complex integration. We believe that by not presenting the excerpt from Cauchy's memoir, we will give more freedom to the participants of the study to come up with a definition that allows them to deal with this concept. In particular, in Piña-Aguirre and Farfán (2024) we present these modifications to task A (accompanied by the way we designed the whole set of tasks), which in turn allowed us to configure more tasks that will enable us to study in more detail the use of two-dimensional figures. On the other hand, the answers to task B made us realise that all the closed curves that we presented in the set of tasks always delimited convex regions. Therefore, we decided to add a final task involving a non-convex figure, in order to find out what kind of figures the study participants use and how they use them when dealing with non-convex regions.

As our research develops, we hope we can understand in more detail how to transpose scholarly historical knowledge in complex analysis (in this case related to the use of figures) that, as shown in Piña-Aguirre et al. (2024), has not been entirely considered in contemporary complex analysis textbooks.

Acknowledgements

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Shaping geometry in secondary schools in Portugal in the early 20th century

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In this paper, we analyse a textbook from 1906, written by Joaquim d'Azevedo Albuquerque, intended to teach geometry to VI and VII classes of liceus in Portugal. Based on the prescribed curriculum and the one presented to teachers, this article aims to i) compare the prescribed curriculum with that presented to teachers and ii) analyse a proposal for teaching geometry in the final two years of liceus. Using content analysis techniques based on seven categories, we concluded that although the textbook transcribed the programme to be taught in both classes, in the final year of liceus, the author added a chapter on transformation methods, which was not indicated in the official syllabus. Furthermore, the didactic proposal of the school textbook is rather formal, lacking a contextual introduction and providing few application exercises.

Keywords: Curricular development, textbook content, mathematics education, geometry.

Introduction

In developed countries, textbooks are part of the daily routines of students and their parents. They are also an important resource for teachers. Analysing old textbooks reveals, among other things, the objectives to be achieved in past school classes (Nietz, 1941). For example, studying prefaces may allow us to understand how the authors relate to the educational context: the subject matter, teaching values, other educators, etc. Also, how authors choose to present content gives clues to their educational perspectives. Schools reflect and shape society's values, language, processes, etc., and textbooks, in particular, are pervasive cultural artefacts (Burke, 2016). The analysis moves away from Julia's (2001) perspective, for whom the textbook is defined essentially on its use. Textbooks are resources consciously designed to assist instructional objectives both from the student's and the teacher's perspective (Chopin & Bastos, 2009). This broader perspective for the study of school textbooks will be used in this text. Historians struggle to distinguish the different facets of this extremely complex object that is the textbook (Choppin, 2004). Its didactic function intends to convey knowledge, skills, and cultural and social perceptions to younger generations and, as with all curricular productions, mirrors the conflicting forces that shape curricula. Textbooks shape curricula, and we use this term from a wide-ranging perspective, encompassing several levels of decision resulting from pressures of distinct social and cultural forces (Gimeno, 2000). We specifically distinguish between the prescribed and the presented curriculum: the former, the program, usually determined by school authorities and the latter, the one that is printed in textbooks (Gimeno, 2000). Examining the curriculum presented in textbooks cannot fail to consider the programme. The success or failure of "divergent" textbooks may provide insights into the inner workings of curricular change and stability. Furthermore, textbooks are under the technical constraints of their time and participate

in an economic system whose rules and uses, both in production and consumption, necessarily influence their design and material realisation.

This text is part of a broader research program aiming at understanding how school mathematics curricula changed through time as a means to apprehend the broader role of social and cultural forces in forming contemporary curricula. This text focuses on a specific textbook for the upper classes of secondary school that strikes us as distinct from previous and later textbooks for the same classes. *Geometria* is the title of a school book written by Joaquim d'Azevedo Albuquerque and published in 1906, aimed at VI and VII classes of Portuguese *liceus* (current 10th and 11th grades, designed for students with 15-16 years). The author, Joaquim de Azevedo Sousa Vieira da Silva Albuquerque (1839–1912), graduated in Civil Engineering from the Polytechnic Academy of Porto, where he was born. He began his professional career at the *Liceu do Porto*. His intellectual merit earned him, a few years later, a position as the third former student of the Polytechnic Academy of Porto to join its teaching staff [1876/77 academic year]. He was also involved in politics and was a candidate for parliament during the monarchy. He retired in 1910 due to the age limit (Sigarra, 2011). This intellectual was a member of the Lisbon Geographical Society, where he served as its first secretary in Oporto's section. He authored several scientific and educational works in Mathematics, such as "Is the principle of universal gravitation sufficient to explain celestial phenomena?" (1876). He also wrote many mathematics textbooks, some of which became the standard books for teaching arithmetic and geometry in secondary schools. He was an important member of the Porto Instruction Society, where he delivered three lectures between 1884 and 1885 under the theme "The Machine and the World", which introduced a new method of studying kinematics (Guedes, 2004).

This analysis of the textbook aims to i) compare the prescribed curriculum with that presented to teachers and ii) analyse a proposal for teaching geometry in the final two years of *liceus*. To achieve this, we closely examine the *Geometria* textbook, integrating two methodological frameworks: one proposed by Lisa Okeeffe and the other by Jenneke Krüger, combined and applied previously by Almeida and Rodrigues (in press) and Rodrigues (in press). This article focuses on analysing the content, structure, expectations, and language used by Okeeffe (2013) while considering the coherence, visualisation, and context usage more broadly (Krüger, 2010).

The textbook indicates on the first page that it follows the programmes of the 1905 reform for secondary education (Portugal, 1905). The educational system in Portugal includes four years of primary education, followed by secondary education, which consists of a General Course in *liceus* lasting five years (currently covering the 5th to 9th grades), divided into two cycles of three and two years, respectively, and a two-year complementary course. In the final two years, there were two branches: science and humanities. Mathematics was taught only in the science branch, with a workload of four hours per week. Secondary education was aimed at the elite, and illiteracy rates were around 80% of the Portuguese population (Carvalho, 2008).

The 1905 programs (Decreto n.º 3/1905) introduced a key textbook change. In 1895, authorities decided on one, and the same textbook was chosen for all schools (unique textbook). Under the new reform, the unique textbook regime ended, and the production of textbooks was liberated (Carvalho, 2008). The new program also explicitly stated which textbooks were admitted for each grade. For

example, VI and VII classes should have an arithmetic book, an algebra book, a trigonometry book, and a geometry book (Portugal, 1905). Although the first three had similar counterparts in the 1895 reform, as the programs did not change much in these topics, the geometry book did not. Consequently, "older" books on those three topics could still be used, but there was no option for the geometry book, which should contain conic sections, geometric methods, and plane analytical geometry. Previously, Joaquim Albuquerque's textbooks were unique textbooks for the first five years of secondary school but not for the last two years of *liceus*. There is no other textbook for geometry in this specific prescribed curriculum, so Albuquerque's approach to the methods is the only one that is known. Therefore, the 1906 geometry book we are discussing is an attempt to shape the new curricular topics and also an attempt of Albuquerque to extend the range of his collection.

Methods and Theoretical Framework

Methodologically, the text conducts a documentary analysis (McCulloch, 2004) of the curriculum presented to teachers through the *Geometria* textbook without losing sight of the official programme. Document analysis involves representing a document's content in a distinct form to facilitate research and referencing. It is primarily executed through classification, indexing, and transformation procedures to analyse and condense information from documentary sources. Considering the political, social, and economic context of the time, a content analysis (Cohen et al., 2007) was conducted based on examining the school textbook and the official programme. This approach ensures the study's relevance, providing a comprehensive analysis of curriculum documents, school textbooks, and other curricular materials.

Due to the role that textbooks have in the context of teaching and learning, their content and structure are very important for promoting specific visions of curriculum. In this case, a content analysis is performed under two complementary frameworks: one by Lisa Okeeffe (2013) and another by Jenneke Krüger (2010), both used by Almeida and Rodrigues (in press) and Rodrigues (in press).

OKeeffe (2013) established a framework for analysing mathematics textbooks that comprise four primary elements: content, structure, expectation, and language. The content elements include value formation, motivational factors, accessibility, illustrations, and study guides, among others. The structural dimension encompasses how the textbook is organised, the distribution of content, the connections between different subjects, knowledge and information, and the construction process. The expectation dimension is present in all textbooks and can influence how students engage with the topics presented. "For example, if the focus of a mathematics textbook is on repetition and practice, then a student will subconsciously look to replicate a previous method as soon as he/she encounters a question, without attempting to use any problem-solving skills" (Okeefe, 2013, p. 7). The language dimension encompasses features such as the type of discourse (narration, description, etc.), connectors between sentences, and semantic structures; for more refined analyses, the author provides three subtitles: word signifiers, notational signs, and graphic signs (Okeeffe, 2013).

In turn, Krüger (2010) analyses lessons from the early 17th century, using three categories: coherence, visualisation, and use of contexts. Coherence refers to the consistent application of previously taught, aimed at reinforcing skills through practice that can be utilised in more complex situations or diverse contexts. Visualisation pertains to the layout of the textbook, the illustrations, and the contexts

depicted in these illustrations. This category also encompasses the artistic quality of the images and their role in facilitating the content or mathematical problems. Lastly, the contexts are typically drawn from practical or professional situations, relating mathematics to its applications within the students' environment. In this article, Krüger's (2010) framework will be employed to conduct a meta-analysis of the curriculum presented under study.

This text examines the *Geometria* textbook, concentrating on its suitability for the prescribed curriculum. In particular, it analyses two topics: the "Reduction to the absurd method", as Albuquerque chose to explain it without illustrations of the proofs of the theorems, and the "Possibility of solving geometry problems using only a ruler and compass." This presentation, derived from the author, begins the section with a question reinforcing the next content.

Description of the textbook

Considering the *structure*, Albuquerque's textbook (1906) is 156 pages long, with the index at the end. It does not have methodological suggestions, just a description of the methods. It is printed in black ink, with no colour highlights in the titles or figures. The cover states that the school textbook follows the programs published in 1905 and is directed to the VI and VII classes of *liceus*.

The textbook is divided into two main parts, one for each grade. The first part covers the content to be taught in VI class. It begins with the programme's transcription and is followed by four chapters on conic sections: introduction, ellipse, hyperbola, and parabola. These conic sections are those recommended in the official programme, which outlines some properties of these sections and defines the conic sections based on those properties, as illustrated throughout the book's chapters related to geometry for VI class. The second part includes the *contents* for grade VII. This part of the textbook also begins with a transcription of the program, followed by four chapters. The first three explain the methods, while the last focuses on plane geometry. For the VII class, the program prescribes the teaching of elementary contents of plane analytical geometry, which comprise:

> Elementary notions of analytical geometry in the plane – Equations of a point in a rectangular coordinate system. Line equation; angular coefficient. Determine the coordinates of the intersection point of two straight lines; condition for lines to be parallel. Find the equation of the line determined by two given points. Find the acute angle of two straight lines; condition of perpendicularity. Determine the distance between two given points.
>
> Equation of a geometric locus; equation of curvature referred to two perpendicular diameters; equation of the ellipse and hyperbola referred to the axes; equation of the parabola referred to the axis and the tangent at the vertices. Graphical representation of circular functions[1]. (Decree no. 3/1905, p. 489)

The last chapter introduces, in the order in which they appear in the prescribed program, the geometry topics planned for the VII class. Along the chapter, there are sections with exercises, allowing

[1] All translations were made by the authors.

students to practice. The author's choice to dedicate the three previous chapters to exploring the methods stands out since the methods are nothing more than the observation of the programme.

> Observation. - During the review of the doctrines of geometry, attention must be paid to the general methods (analysis, synthesis, reduction to absurdity) and particular methods (geometric places, inversion and algebraic) ((Decree no. 3/1905, p. 489, p. 489)

In the first three chapters of the second part of the textbook, alongside the recommended methods for observing the prescribed programme, the author also includes a section on transformation techniques: multiplication of curves, similar figures, inverse problems, transformation by translation, abstraction or generalisation, algebraic techniques, and the possibility of solving problems using only a ruler and compass. This aligns with the author's pedagogical perspective of teaching different methods, an approach not present in previous or subsequent geometry textbooks.

The figures throughout the textbook either depict a construction or represent graphs of functions, such as the sine and cosecant graphs. The figure is introduced as "The continuous line curve is the graph of the function $y = sin\alpha$, without any other contextualisation of the sine function and introducing $y = cosc\ \alpha$ noting that the ordinates for the same abscissa are the "reciprocal numbers of the corresponding numbers of the function $y = sin\alpha$, since $cosc\alpha = \frac{1}{sin\alpha}$", indicating that it is represented in dashed lines (Albuquerque, 1906, p. 105-106), as Figure 1 shows.

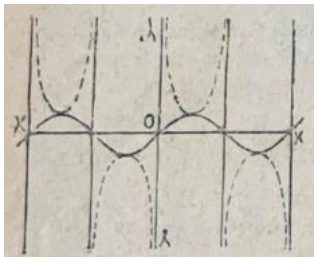

Figure 1: Sine and cosecant graph. (Albuquerque, 1906, p. 150)

There are no motivational elements or indications to guide students' learning. The exercises proposed in the last chapter lack solutions or resolution proposals. The book is highly theoretical, featuring a sequence of methods, definitions, and theorems, and presents the contents in a very formal way. This student-directed organisation fosters a passive *expectation* of learning grounded in a theoretical explanation of the mathematical content to be learned. The student is not encouraged to engage, complete reasoning, or solve exercises when reading the textbook.

For example, in the section where the author introduces the Reduction to Absurd Method, the author begins by explaining what it consists of

> This method consists of demonstrating that the *denial* of the theses, stated as true, evidently leads to absurdity (*ad absurdum,* [sic]), that is, to consequences incompatible with the hypothesis of the proposed proposition or contrary to already established truths, or in themselves absurd (translated from Albuquerque, 1906, p. 53, italic in the original).

The author illustrates the application of the method by proving two theorems: i) If two triangles have equal sides from side to side, the triangles are equal, and ii) The shortest segment that joins a point

outside a straight line to a point on that line is the perpendicular dropped from the point to the straight line. The author demonstrates theorems without any illustration. The lack of illustration makes it difficult to interpret the proof presented by the author, as it stands out in the following example.

> We can always make two equal sides [of the triangles] coincide: let's suppose that the triangles do not overlap (negation of the thesis), that is, that the third vertices do not coincide. Then, these non-coincident vertices form, with each vertex that coincides with direct application, two isosceles triangles with a common base, the segment determined by the non-coincident vertices. The perpendicular raised through the middle of this base must pass through the apex of each of the isosceles triangles, which is absurd (Albuquerque, 1906, p. 53).

Chapter III presents "Transformation methods", a topic not specified in the official program. It includes multiplication of curves, similar figures, contrary problems, transformation by translation, abstraction or generalisation, algebraic, and the possibility of solving problems using only a ruler and compass. The author begins with the question: "How may the possibility or impossibility of a geometrical construction of a problem with the restriction as mentioned earlier be recognised?" (Albuquerque, 1906, p. 110). It then considers the impossibility of solving simple geometry problems using only an unmarked ruler and an Euclidian compass, indicating three famous examples of this impossibility: the duplication of the cube, the trisection of an arbitrary angle and the squaring of the circle. The author answers the question posed at the beginning of the chapter using Petersen's formulation, which ends this section:

> The necessary and sufficient condition for a problem to be solved using a ruler and compass is that the quantities sought are expressible by the quantities given through rational operations or using square roots within a finite number (Albuquerque, 1906, p. 112).

Regarding *expectations*, this book offers a theoretical approach, featuring few exercises, and presents a collection of mathematical methods and concepts. Albuquerque describes mathematics as he perceives it. He employs elaborate formal mathematical *language* to prove results and does not accommodate the textbook for students' independent learning. It's possible to characterise *language* as elaborate by contemporary standards, with little regard for the reader's understanding.

A broader perspective of the textbook

Carrying out a meta-analysis with the categories proposed by Krüger (2010), *coherence*, *visualisation*, and *use of contexts*. It appears that the textbook has a theoretical nature without the *use of contexts* to illustrate what it is intended to teach. Contents are formally exposed using definitions, theorems, corollaries, and demonstrations. Very few examples are given, and the book lacks exercises as a form to allow students to practice.

Regarding *visualisation*, this is a geometry book, so figures appear regularly, especially to illustrate the mathematical content. The book is dense and difficult to read. The contents are organised according to the prescribed program and numbered sequentially throughout the chapters.

Regarding *coherence*, it appears that it exists within the textbook itself, where the teaching of methods for solving geometric problems is privileged. Throughout its 156 pages, the textbook presents a coherent structure, with formal mathematical language constantly used and addressed to the third

person. Although the author did not follow a strict axiomatic approach, arguments for constructing the content almost always follow a rationale supported by logic. The author follows the program's suggestions for revisions of general and particular methods provided for in VII class but introduces a section with a set of methods before geometry in the plane, found in the last part of the textbook.

Final considerations

This article aims to i) compare the prescribed curriculum with that presented to teachers and ii) analyse a proposal for teaching geometry in the final two years of *liceus*.

The fact that there was no unique book during this reform does not allow us to determine whether this would be the only textbook intended for teaching geometry in science courses in the last two years of *liceus*, although we did not find a competing textbook in PORBASE (https://porbase.bnportugal.gov.pt/), the national library database. We also do not know the extent to which teachers using this particular textbook would follow the abstract approach it implied or if they would choose to include illustrative examples and exercises. Also, we do not have data on the number of teachers who used this textbook in their geometry teaching, and we do not know whether they followed the adopted school textbook. However, comparing this textbook with those from previous and subsequent reforms suggests a novel approach to introducing mathematics into the last two years of *liceus* science courses, particularly in teaching plane geometry. According to the 1905 program—or at least Albuquerque's interpretation—geometry takes a secondary role, with the main goal being to develop students' appreciation for specific methods of reasoning.

Drawing a parallel between the prescribed curriculum and that presented to teachers, it is not known what led the author to choose to structure the textbook, including some methods that are not listed in the official program prescribed in the Government Gazette. Due to the lack of a preface, we don't know the sources or the intentions of Joaquim Albuquerque. However, the study of this textbook allows us to look at a set of methods used to prove geometric results with an approach to teaching geometry that is innovative due to the differences found.

The proposal for teaching geometry in the last two years of *liceus* is very formal, and the *Geometria* textbook does not have a didactic nature but is rather a theoretical book due to the lack of exercises, their solutions or resolution proposals, the language used, and the lack of use of contexts, which could make it easier for *liceus'* students to learn plane geometry. However, it is interesting to note that the formal proposal for teaching geometry introduces a series of methods into the curriculum, which allow the teaching of geometry to be systematised. This proposal differs from previous and subsequent proposals for teaching geometry in the last two years of *liceus*.

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Iconic representations in Renaissance reckoning books – on the way to a table of classification

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In this paper, we present a first step of a research project aiming at developing a scheme for the didactical description and analysis of textbooks from the early modern period. The first design feature we take a closer look at are iconic representations and as a first source, we choose a reckoning book by famous German "Rechenmeister" Adam Ries. Within a qualitative approach, we categorise the representations and describe their function in order to find out what we can tell about Ries' didactical intentions that the figures are based on.

Keywords: Textbook research, history of mathematics education, numeracy, Adam Ries

Introduction

When it comes to research on mathematical education in Renaissance Europe, reckoning books written by *Rechenmeister* serve as our main sources for elementary, for the most part arithmetic education. We can rather easily derive from them *what* was to be taught, while the books tell us hardly anything about *how* things were meant to be taught, at least from a strictly methodical perspective. Yet, it is quite clear that these books can tell us something about their authors' didactical principles and intentions. As we see a rather broad agreement on the mathematical subjects of teaching, we need to have a closer look at the differences between the books in presenting their content. Wishing to do that, we found we are missing criteria to describe and compare Renaissance reckoning books in more detail. This paper marks a starting point of a long-term project aiming at developing a scheme for the didactical description of textbooks from the early modern period.

The long-term aim of the project is to find out about didactical concepts respectively communication patterns in mathematical learning situations between around 1500 and 1800. This requires a thorough description of the fundamental database, i.e. persisting reckoning books, in regard to contextual as well as textual aspects. While contextual aspects like "addressers and addressees, […], the physical and social setting of text production and text reception, and the goal(s) of the text" (Jacobs & Jucker, 1995, p. 11) might be extracted directly from the book's surface or introductory parts, an account of textual aspects – i.e. symbolic and iconic representations – requires further differentiated criteria.

For a start, we chose to describe and categorise iconic representations in the 1574 edition of Adam Ries' *Rechenbuch auff Linien und Ziphren* (Ries, 1574; so called *Zweites Rechenbuch*). We use a qualitative approach, characterising pictures and figures first by their components and surface features and secondly by their functions for acting mathematically. From these descriptions, we inductively derive different types of figures and – after considering the text they are embedded in – deduce statements about Ries' didactical intentions. These findings will be used as a base for the analysis of what we find in comparable books, e. g. the one by Johannes Widmann (1526), and will serve as a step on the way to coming up with a table of classification of iconic representations in Renaissance textbooks.

State of research

Considering the methodological perspective, Karp (2014, p. 12, in reference to Schubring) notices a "lack of worked-out theoretical categories for analysis" of sources concerning the history of mathematics education in general. We have reason to believe that this is still true ten years afterwards. Schubring (2022) himself has rather recently devoted a whole book on the question of analysing historical textbooks and ascertains "enormous methodological problems" (p. 3) in this area of research. His main focus, though, is on the "contextual analysis of historical textbooks" (p. 7), referring here to the socio-cultural context of the textbook work. In his own research on Lacroix's textbooks he works out three dimensions of analysis: comparison of consecutive editions of the same textbook, comparison of contemporaneous textbook series and establishing relations between (changes in) textbooks and (changes in) underlying external conditions (e.g. curricula). All of these dimensions of analysis require detailed description of a corpus of textbooks first to allow systematic comparisons at all. However, this problem is not addressed by Schubring. He gives an account of documentation of and research on mercantile arithmetic books – meaning what we call reckoning books in this paper – in different European countries, but does not go into methodical detail here.

Regarding existing research on Renaissance textbooks, especially arithmetic and reckoning books, from Germany, one needs to mention the *Adam-Ries-Bund*. The proceedings of their regular conferences (for an example see Gebhardt, 2023) contain numerous accounts on these, but for the most part this research is limited to the description and analysis of single books and their authors, lacking both a systematic review and a didactical perspective, not considering the teaching context (in the sense of Jacob & Jucker, see above). Holl (2023, pp. 35-42) recently presented an outline of a systematic description of arithmetic books including a categorisation of content-related elements. His work shall serve as a valuable tool when it comes to analysing *what* the author of the textbook wanted his pupils to learn. Didactical features are lacking in his collection, though. Linguistic surveys on reckoning books are rare, as well. Gärtner's (2000) edition of Widmann's book is accompanied by an etymological study on reckoning books being regarded as a specific text type but does not focus on further didactical aspects. Gabriel's (2010) work is the only one known to us to describe Ries' *Zweites Rechenbuch* from a didactical perspective using linguistic methods. He compares Ries' work to Widmann's book focussing content, language issues and notations and therefore analysing the representations that are covered. His work is limited to this particular example; nevertheless, it should be of good guidance for our own purpose in the long-term. Schöneburg (2007) performs a methodical-didactical analysis of the textbook on military mathematics from 1631 by Ambrosius Rhodius, especially regarding addressees, language use and the author's intentions. For this purpose, she analyses the book's content, style, figures and methodical features. The work is limited to a particular example, though, as well.

Analysis of iconic representations in Adam Ries' *Rechenbuch auff Linien und Ziphren*

Types of iconic representations

Before one speaks about iconic representations, it is necessary to make a clear distinction between symbolic representations, i.e. text, and iconic representations, i.e. figures. By figures, we understand

those parts of the written or printed matter that do not only consist of letters, but comprise numbers, lines, operators etc., as well, and that are two-dimensional or offset.

Ries' book contains about 150 figures, and in a first step, we characterised them according to their constitutive parts. We found that the majority of them (about 90%) only consists of numerals/numbers, lines (in different arrangements), dots, brackets, single words (meaning that they are not part of a sentence) and abbreviated notations for units (e.g. currencies). Beyond that, one of these examples also shows a pointing hand and six examples display jettons as part of depictions of a counting board.[1] All of these figures clearly represent mathematical items. Of the remaining figures, one only consists of lines, this one can be easily characterised as an ornament, and ten are made up of pictorial elements, depicting persons in certain contexts from everyday life. While the image on the title page apparently shows a situation inside a reckoning school and by that tells us about the context of use of the whole book, the others each depict the everyday context taken up in the following chapters, in the most part trading scenes (e.g. the trading of cattle). So, this first step, in which we describe iconic representations by their surface features, already allows us to distinguish two categories of figures: those that display **mathematical** items and those that display only **non-mathematical** items. We will call the latter **pictures** and they can again be divided in two sub-categories, **ornaments** and depictions of everyday life situations. As we are interested in Ries' views on the teaching and learning of mathematics, it is obvious that we do not learn enough about the ideas behind the figures of mathematical items from only describing their external characteristics. We need to pose further questions to enable a more sophisticated categorisation of the iconic representations in Ries' book.

In the second step, we strove to ascertain the function of the figures. At this point, we are not yet talking about didactical functions as we seek to deduce these from the more detailed categories later on. The focal question here is: What is it that the figure "does"? As for the ornament, it is clear that it has a purely decorative function and is therefore not of further interest in regard to teaching and learning situations. We will hence exclude this type from our upcoming considerations. The images of everyday life situations illustrate the paragraphs to come; we will therefore call them **illustrations**. Regarding the mathematical items, we find different types. One type of figures **introduces** terminology and notations like numerator and denominator or the names of the place values. Another one **presents** calculations with concrete numbers that are complete in the sense that the given numbers as well as the results are presented. Examples for this type are a multiplication table and images of fully conducted written calculations. These presenting figures differ in how far intermediate results of the calculation process are visible and replicable. Apparently, those presentations may be introductions of calculation methods, as well, but we cannot tell by looking at the figure in isolation. We find that to make further statements and describe the presenting figures more thoroughly, we need to consult further features and integrate the position of the figure within the book. Therefore, the first appearance of a presenting figure is likely to have an introductory function, as well. Furthermore, we

[1] In this book, Adam Ries introduces Arabic numerals and methods of written calculation using the decimal system. Before the introduction of these methods, though, he teaches how to do calculations using the counting board.

find a third type of mathematical figure, which appears numerously. This type consists of numbers and is sometimes complemented by abbreviated units. The numbers are arranged in a certain, often two-dimensional manner, sometimes lines are added. These figures are not self-explanatory when they are looked at in isolation. Again, we need to consult further features, and this time we find that the surrounding text is necessary to make statements on the function of this type of figures. These examples are either part of proofs or part of the presentation of the diverse *regulae* that are typically found in Renaissance reckoning books. The figures do not present the full calculation itself but they provide a scheme how to arrange the given data and intermediate results in such a way that the calculations can easily be carried out in mind. One can say that these figures **structure** the given numbers and data.

Up to this point, we find five types of iconic representations within the second reckoning book by Adam Ries that we can assign all the figures to:

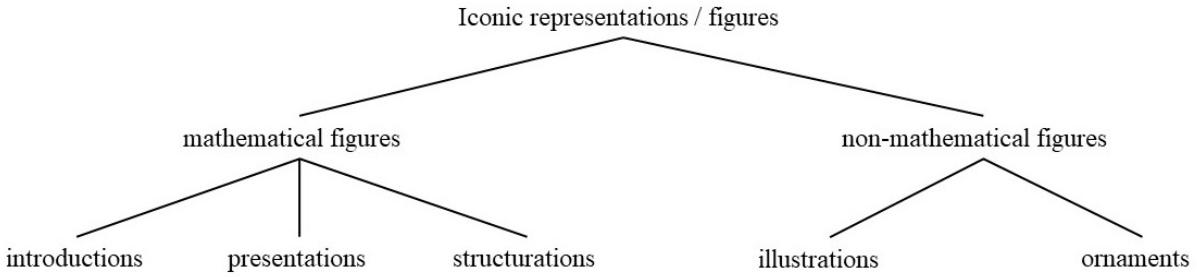


Figure 1: Types of iconic representations in Ries' second reckoning book

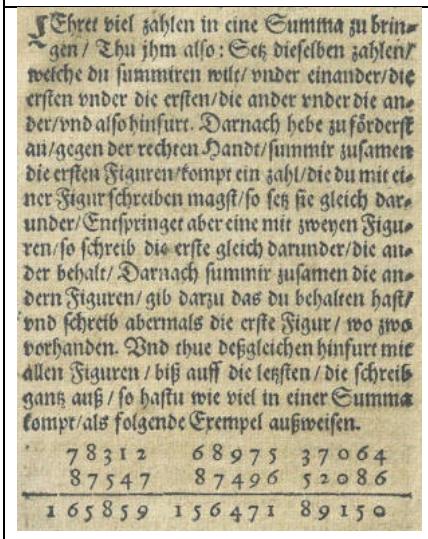
Didactical function of iconic representations

In order to be able to express assumptions about Ries' didactical intentions behind his use of figures, we need to work out their functions in regard to their recipients first. That is, we need to ask next: What is it that the addressee is supposed to do when he or she sees this figure? Starting with the illustrations, being non-mathematical figures they do not aim at a mathematical kind of action. They establish an everyday context for the reader and by doing that they enable the recipient to familiarise with the situation presented. The alleged didactical function of this is identification with the upcoming problems and showing their relevance to practical application. However, we do not know whether Ries himself wanted to include these illustrations at all. Earlier editions do not contain any so it is quite probable that it was the publisher who inserted them of his own accord.

Considering the mathematical figures, we find that it is again not sufficient to look at them separately to work out what mathematical action is supposed to be induced. We need to consider the surrounding text and its relation to the figures. For those figures that introduce terminology and notations it seems clear that they present knowledge that must be grasped and learned by heart. As for presentations and structurations, a closer survey is necessary. In the following, we examine the text-figure relations for two mathematical examples. One of them is the introduction of written addition, which we allocate to presentations; the second one is a part of the paragraph on the *regula falsi*, which can be assigned to the structuring type. We will describe them on a formal (i.e. surface) as well as on a substantive

level. For the latter we compare the content of the accompanying text with the content of the figure itself.

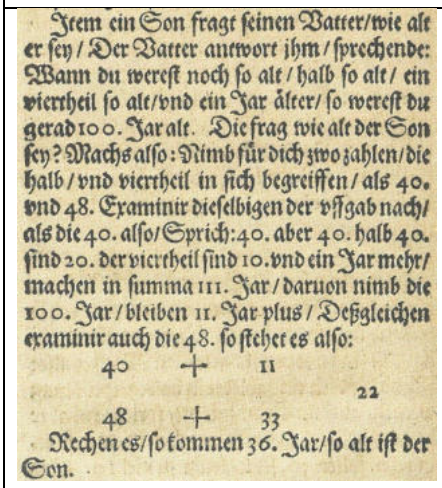
Table 1: Description of text-figure relation in Ries (1574, p. 10r)

Example 1: presentation	Text	Figure
	Stepwise description of the written addition procedure for general numbers: • how to write and arrange the given numbers (one above the other) • how to add up the place numbers • how to write the intermediate sums and how to proceed in case of a carry • how to read the sum	Three examples of fully conducted written addition with concrete numbers: • five-digits summands written one above the other • six-digits sums beneath lines that divide them from the summands
	The text shows direct reference to the figure by saying "as the following examples reveal" (*als folgende Exempel ausweisen*)	

The reference shows that Ries wanted his readers to use both, text as well as figure, so the whole paragraph should be seen as a unit. Consequently, when we wish to work out the figure's function we must analyse its function within this unit.

Ballstaedt (1997, pp. 251-254) names three possible characteristics of text-figure relations that usually appear in combination. Relations may be congruent (meaning that the figure represents the same content as the text), complementary (meaning that the figure fills blank spaces that are left by the text or vice versa, so that comprehension requires to consider both) or elaborative (meaning that the figure presents information beyond the original content of the text or vice versa). The table shows that in the figure information from the text gets lost as generality of application is not visible in it and not every single step described is shown separately. Nevertheless, there are congruent parts, namely the objects mentioned, the notation and the procedure itself as results from intermediate steps are visible. On the other hand, the figure shows concrete examples with numbers that are not mentioned in the text, establishing a clear complementary relation. The function of the figure as part of the text-figure unit is to complement, augment and concretise the text and by this to complete the unit. Ries invites the addressees to use the figure in such a complementary way. They are supposed to reproduce the steps that the text describes for general numbers with the aid of the concrete examples, either mentally or by writing down the calculation stepwise themselves. The didactical function of this intended mathematical action is to reproduce procedural knowledge, possibly in order to gain the ability to applicate this procedural knowledge to other given numbers.

Table 2: Description of text-figure relation in Ries (1574, p. 58v)

Example 2[2]: structuration	Text	Figure
 Item ein Son fragt ſeinen Vatter/wie alt er ſey / Der Vatter antwort jhm / ſprechende: Wann du wereſt noch ſo alt / halb ſo alt / ein viertheil ſo alt/vnd ein Jar älter/ ſo wereſt du gerad 100. Jar alt. Die frag wie alt der Son ſey? Machs alſo: Nimb für dich zwo zahlen/die halb / vnd viertheil in ſich begreiffen / als 40. vnd 48. Examinir dieſelbigen der vffgab nach/ als die 40. alſo/Sprich: 40. aber 40. halb 40. ſind 20. der viertheil ſind 10. vnd ein Jar mehr/ machen in ſumma 111. Jar / daruon nimb die 100. Jar / bleiben 11. Jar plus / Deßgleichen examinir auch die 48. ſo ſtehet es alſo: 40 + 11 22 48 + 33 Rechen es/ſo kommen 36. Jar/ſo alt iſt der Son.	Exemplary task consisting of problem and incomplete solution: • verbal description of situation • question to be answered • detailed description of first step of solution procedure, applied to number 40, giving 11 as positive (*plus*) intermediate result • request to apply first step of procedure to number 48 • solution 36	Numbers and lines arranged in a certain schematic order: • numbers 40 and 11 in a row, symbol + in-between • numbers 48 and 33 in the row below, symbol + in-between • number 22 on the right between the rows
	The text shows direct reference to the figure by saying "so it says:" (*so stehet es also:*)	

Again, the reference suggests that text and figure form a unit and that the recipient is supposed to consider both. Comparing this example with the first one, we see some differences, though, already from a formal typographic perspective. While in the first example the figure follows the text, the number scheme in this example has accompanying text above as well as below it. It is immediately apparent, that the figure is an essential part of the unit completing the text parts and vice versa. The substantive perspective supports this. While in example 1 text and figure might stand for themselves, this is not the case here. The figure is lacking information that is crucial for its understanding, and so is the text, neither of them would be comprehensible if they stood for themselves. There is some congruency, namely the numbers 40, 48 and 11 as well as the word and sign for "plus", but this overlap is very limited. For the rest we can see no content-wise congruencies, the text-figure relation is clearly complementary, and the function of the figure within the unit is to complete it. The mathematical action that the reader is supposed to conduct is to understand the task's solution and in order to do that, it is necessary that they carry out parts of the calculation by themselves with the aid of the number scheme. Apparently, the whole example is missing detailed guidance on the solution procedure, so that the text-figure unit is still incomplete. The necessary instructions can only be found on some prior page, so we have yet another example where not only the accompanying text but the figure's position in the book respectively the paragraph is crucial for its analysis. The didactical

[2] "A son asks his father how old he was. The father answers: If you were twice as old, half as old, a quarter as old and one year older, then you would be 100 years old. The question is how old is the son?"

intention of the text-figure unit is to repeat and practise the *regula* to incorporate procedural knowledge and to gain the ability to apply the rule to other given situations and numbers.

Conclusion and outlook

By describing and categorising the iconic representations in the second reckoning book by Adam Ries, we were able to work out the figures' functions whereupon we could deduce some didactical intentions that Ries built his work on. We found that the mathematical figures' surface characteristics are in line with their functions for the most part. Introductions provide terminology and notations that must be memorised; therefore, they provide learnable knowledge. Presentations and structurations show sections of algorithmic calculations, which are supposed to be retraced and understood in order to enable the addressee to perform the calculations themselves. Hence, these kinds of figures aim at imparting procedural knowledge and abilities. In all cases, it is not intended to lead the reader to a deeper comprehension of the calculation methods.

As this paper is only a first small step in a bigger project there are many steps laying ahead. A first one would be to apply our findings from Ries to other reckoning books as well as textbooks from other subjects (geometry, architecture…) from Renaissance times to see whether the categorisation and the assumed connection to the functions of the figures fit other sources, too. A first glance at the reckoning book by Johannes Widmann shows resemblance concerning the types of figures but reveals that we would have to enhance our categories, as well. It looks like it could be a more promising approach to replace the fixed categories by criteria and to establish a set of characteristics relevant for determining the figures' didactical functions. While keeping the number of criteria to a manageable size, this approach, at the same time, allows combining the criteria in diverse ways enabling a more precise description of the types of figures and leading to more flexibility when selecting categories fitting a research question.

When analysing figures, we also noticed, that in most cases, it does not suffice to look at the figure in isolation. As a second step, the accompanying text respectively the text-figure relation has to be considered, too, as well as the figure's position within the book or within the paragraph. A comparison of Ries and Widmann also suggests that it may be crucial for a didactical analysis to consider whether a figure is part of a sequence of figures, e.g. with increasing difficulty.

So, analysing and categorising further textual aspects and connecting them to contextual aspects has to be done to find out more details about the didactical intentions of the authors.

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Using word problems from the 16th century in teaching mathematics

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Word problems in mathematics is a topic that has received a lot of research interest, mainly because of the difficulties it poses during teaching. In this study, we tried to check what the contribution of bringing such problems from a Jewish math book from the beginning of the 16th century to teaching students could be. The results of the study showed that dealing with these problems and their solution as they appear in the 16th-century book can lead to positive attitudes in the students both in terms of getting to know Jewish mathematicians and in terms of exposure to new ways of solving.

Keywords: History of mathematics, mathematics education research, primary historical sources.

Introduction

The integration of historical content in mathematics teaching has been widely discussed in recent years. For the idea to become a reality, it is necessary, among other things, to introduce the subject into the curriculum of preservice teachers **(**Furinghetti, 2007). In this study, we planned a study unit that exposes the students to the historical content of a specific topic in teaching mathematics: word problems in an arithmetic Hebrew book from the beginning of the 16th century.

Theoretical background

In his 2009 article, Jankvist (2009) proposed a new way to organize the discussion about using the History of Mathematics (HOM) in teaching by answering the questions of "why" and "how" and the interrelation between them.

The "why" refers to the causes of using HOM, and according to Jankvist's opinion, all the arguments for integrating the history of mathematics in teaching can be attributed to one of the following two: arguments referring to history as a tool and as a goal. When we use the history of mathematics to motivate our students, to lower their anxiety, and to improve their cognitive processes, we use the history of mathematics as a tool. When we speak with our students about the evolution of mathematical concepts in time or the development of mathematics in different times and cultures, we refer to mathematics history as a goal.

The "how" refers to the different ways of integrating HOM in teaching. Jankvist classified them into three major categories: Illuminate approaches (short historical notes, mainly about mathematicians, essential events, and famous problems), modules approach (study units are dedicated to the history of mathematics), and history-based approaches (when historical considerations guide how the mathematical subjects are written).

Using primary sources in teaching mathematics

Using primary resources is an obvious choice when we refer to HOM as a goal and want to design an appropriate teaching module. However, the expected difficulties in bringing primary sources to the students must be considered: understanding old texts' language, different terms, and different cultural contexts. However, as Jahnke stated:

> The study of original sources is the most ambitious of ways in which history might be integrated into the teaching of mathematics, but also one of the most rewarding for students both at school and at teacher training institutions. (Jahnke, et al., 2002, p. 291)

Word problems in Elijah Mizrahi's "Book of Number"

The problems were chosen from Elijah Mizrahi's "Book of Number", written in Hebrew in Constantinople at the beginning of the 16th century. This book is particularly relevant because it addresses the didactic aspects of learning mathematics, such as the importance of understanding beyond performing the algorithms for the solution. The book contains various arithmetic topics, such as calculations with integers, simple fractions, and sexagesimal fractions, a chapter on proportions and roots calculations, and a chapter with 100 word problems. In this chapter, each problem is explained in detail; sometimes, more than one solution is given. The problems deal with various subjects from everyday life, such as inheritances, partnerships, shared work, and more, but there are also theoretical problems (Mizrahi, 1534).

Rationale and objectives

In this research, we chose to refer to HOM as a goal and design a teaching module about word problems in Mizrahi's book (16th century). The study unit was designed to give future teachers a broader concept of mathematics and its development process, focusing on Jewish mathematicians. We chose to focus on word problems because this is a mathematical topic relevant to students and their (future) students—a common topic in the grades 7-10 curriculum.

The mathematics department at Herzog College trains mathematics teachers for the 7th - 10th grades. As part of their degree, they study at least one course on the history of mathematics. However, in these courses, there is little reading of original texts and only sometimes discussion about the possibilities of integrating the content into teaching in schools. Beyond the general goal formulated earlier, we set ourselves some more focused goals:

- Learn the solution methods in historical sources and compare them to today's methods.
- Check how these problems can be integrated into teaching in grades 7 to 10.
- Deepen the students' connection to Jewish heritage in a lesser-known field by getting to know prominent figures in the scientific-mathematical context.
- To arouse their awareness and generate a positive attitude toward integrating historical content in mathematics teaching.

Beyond learning the mathematical texts, their content, and the ways of solving the presented problems, the students enrich their background about mathematical knowledge in the early 16th century. For example, they can understand how to do mathematics without negative numbers and some advantages of working without algebra (Heeffer, 2014).

Method

The research was carried out in two classes (20 students) studying a course on different strategies for solving mathematical problems. This course exposes students to diverse and unconventional strategies and solution methods.

In planning the research, we followed the hermeneutic approach.[1] (Glaubitz, 2010). During the study, the students were exposed to other solutions that are common today and methods that were used in the past and "forgotten." They were exposed to non-algebra solutions to problems they are used to solving using algebra. We raised whether there is any added value to these solutions found in historical sources.

In addition, they received some information about Elijah Mizrahi and his book. Some students (mainly the men) were familiar with Mizrahi's halachic writings and did not know he was also a mathematician. Others heard about Elijah Mizrahi for the first time in these lessons.

During the unit's planning, we tried to anticipate difficulties that may arise: The students can read the original book written in Hebrew, but they have to overcome linguistic difficulties originating from the medieval language and the different terminology. They also must be able to understand the cultural environment of the author, Mizrahi, and the stage of development of mathematics at that time.

Participants

The students at Herzog College are primarily religious or traditional Jews with a strong orientation to Jewish tradition. The 20 students who participated in the study are in a program for teaching mathematics in grades 7 to 10 (ages 12-16) and studied in two classes: a women's class (12) and a mixed-gender class (8). Most women's class students were young women who had just graduated from high school and were going on to college, and they were usually without teaching experience. In the mixed group, there were six men and two women, most of whom were older and usually had been working in teaching for several years[2]. So, this was a heterogeneous group that could provide information on different sections of the population.

Research questions

1. What are the students' views about word problems?
2. What are the students' opinions on integrating historical sources in teaching mathematics, particularly word problems?
3. What are the students' opinions on pedagogical aspects of teaching word problems: teaching more than one solution to a problem, solving problems without algebra, and more?

Research design

We chose seven problems from Mizrahi's book: problems that use the rule of three or the false position method, two motion problems (one involving using arithmetic sequences), one inheritance problem, and one dealing with integer division and their remainders.

The subject was taught in two consecutive lessons. The first lesson provided background on Mizrahi's life, work, and cultural environment. Two problems were then presented. First, the students solved the problem the way they wanted (usually using algebra), and then Mizrahi's solution was presented.

[1] This approach concerns local and episodic historical interventions and serves as a means of deepening and reflecting.

[2] Of the 20 students, there were 7 students who worked as teachers and 13 students who were not yet working as teachers.

In the second lesson, the class was divided into groups of 2-3 students, each receiving 1-2 problems from the book. They were required to decipher the problems and Mizrahi's solutions and then present them to the class. We helped the students understand medieval language when necessary. We also had to draw the students' attention to the mathematical content known in Mizrahi's time (without using negative numbers and algebraic tools, but on the other hand, knowledge of the formula for the sum of an arithmetic series).

Table 1: The problems in the research

	The Problem	**Comments on the content and solution method**
1.	If 4 of Constantinople are equal to 6 of Bursa and 9 of Bursa are equal to 3 of Adrianople, how many of Constantinople are 7 of Adrianople?	Rule of three
2.	We added the ninth and tenth to an amount of money, and the result was 50. How much was the amount of money?	False position method, rule of three
3.	A master sent a messenger running 29 miles a day. After ten days, he sent another messenger running 35 miles a day. When will the second messenger catch up with the first one?	Motion problem
4.	Someone swore that each day God doubled his money, he would donate two pešuṭim to the poor. For four consecutive days, God doubled his money, and he gave two pešuṭim each day and ran out of money. How much was his amount of money?	Recursive problem (from the end to the beginning)
5.	If the king sends a messenger to a place, who walks ten miles every day and immediately sends another messenger after him, who walks one mil on the first day, two miles on the second day, three on the third day, four on the fourth day, and so on, when will he catch up with him?	Motion problem, arithmetic sequence.
6.	Someone divides his property among his sons. He instructs that the first will take one zahuv (gold coin) of his money first, then a tenth of the remainder; the second will take two zehuvim of what remains from the first, then he will take a tenth of the remainder; the third will take three zehuvim of what remains from the second, then he will take a tenth of the remainder; so on by this order; the last of his sons will take all that is left from the former ones. We find that all took the same according to this arrangement. How much is the money, and how many are the sons?	Inheritance problem
7.	If one asks: what is the number that when counted by twos - one remains; by threes - 2 remains; by fours - 3 remains; by fives - 4 remains; by sixes - 5 remains; and by sevens - nothing remains.	Integers division and their remainders

After these two lessons, we asked the students to answer a short feedback questionnaire. The questionnaire included 16 questions[3]. There were three open questions and the rest closed questions, in which they had to rate claims/ statements from 1 to 5.

The questionnaire dealt with attitudes and thoughts towards:

a. Word problems in mathematics.
b. Different ways of solving problems.
c. Teaching mathematical content written by the sages of Israel.

Results

Below, we will first present the conclusions regarding all participants. Later, we will focus on the characterizations of different sections of the group.

Most participants expressed positive attitudes towards word problems: They like the topic and think of them as nice puzzles, yet they believe that they are challenging for middle school students and that it is difficult to succeed without algebraic tools. (see Table 4)

Also, on integrating word problems from Jewish mathematicians from the Middle Ages and the importance of knowing Jewish mathematicians, the participants expressed positive opinions despite their difficulty, even though they would not necessarily contribute to raising their students' motivation. The following are some of the reasons they gave for their positive opinion:

Student: As Jews, knowing about Jewish mathematicians is exciting and important.
Student: I think it connects the material taught to us and gives it more meaning when it is connected to Jews and thus also connects to Judaism
Student: I love the Jewish head, and it's nice to see the Jews' wisdom also in real life.

Some students thought it was important to know HOM regardless of the mathematicians' national identities. Other students talked about the connection between the sacred and the profane, especially when the Jewish mathematician is also a halachic, rabbinic authority.

Student: Because it shows that mathematics is not profane and has spiritual and divine creation, and the sages of Israel also dealt with it.
Student: I think there is a mistaken belief that there is a separation between profane and sacred studies.

We also found significant differences between students who are not yet teaching and those already working in teaching. The students already working as teachers tend to attach more importance to content related to Jewish mathematicians (Table 2, Figure 1).

Table 2: Attitudes about integrating HOM in teaching

Statement	Mean (scale 1-5)	Pre-service teachers	Students who are teaching
7. It is important to know word problems that appear in the books of our sages.	3.7	3.4	4.3

[3] The questions were numbered 5-20, while the open questions were no. 9, 19, and 20.

8. I think it is important to know Jewish mathematicians.	4	3.4	4.4
12. Word problems from our sages are more interesting than those in regular textbooks.	3.45	3.2	4.1
15. Word problems from historical sources are too difficult for high school teaching.	2.85	3.2	2.3
17. Teaching word problems from historical sources increases students' motivation to study mathematics	2.85	3	2.7

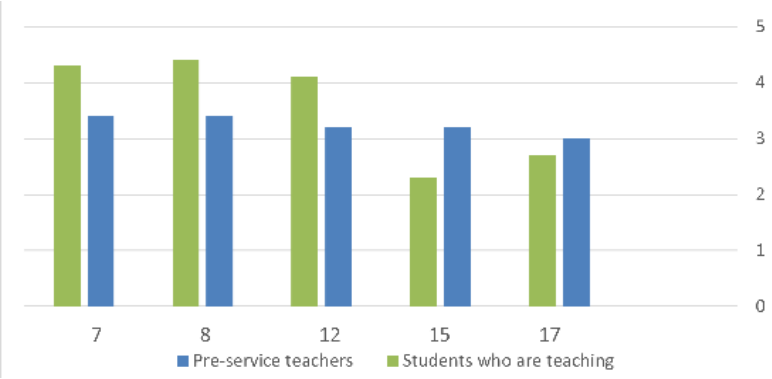

Figure 1: Attitudes about integrating HOM in teaching

Below (Table 3) is a summary of the students' opinions on several didactic issues that emerged during the discussion of Mizrahi's word problems:

Table 3: Attitudes about didactic aspects of teaching word problems

Statement	**Mean** (scale 1-5)
14. It is important to teach solving word problems also without algebra.	4
16. There is no need to teach multiple methods to solve problems.	1.65
18. I like to know different methods of solving problems.	4.1

We see great openness among the students on the subject of various solutions to given problems, including non-algebraic solutions, and also in the answers to the open questions of the questionnaire. Here are some of their answers to the question**,** "Write some new things you learned in these lessons"**:**

Student: I felt I used a way of thinking I usually don't use.
Student: It is incredible how algebra is a narrow way of thinking about problems.
Student: To me, problem-solving in multiple ways develops thinking and creativity.

The attitude to word problems in different genders

Various studies have investigated gender differences in mathematics learning. Most studies dealt with students' cognitive and behavioral differences while studying different mathematical contents, including solving word problems (Zhu, 2007). However, gender differences in teachers' attitudes to

mathematics have been studied less. In this study, although we did not set out to investigate this question, we noticed differences in the students' attitudes towards solving word problems: the men like the subject more than the women; they see the problems as nice puzzles. On the other hand, they estimate that the subject is not too complex for their students. They also think it is not necessarily more challenging to solve problems without algebra than the women who believe there are more difficulties here (Table 4, Figure 2).

Table 4: Attitudes about word problems

Statement	Mean (scale 1-5)	Mean men	Mean women
5. I like to solve word problems.	3.5	4.1	3.2
6. It is difficult to solve word problems without algebra	3.7	3.3	3.9
10. Word problems are a difficult subject for most students.	3.8	3.4	4.0
11. I see word problems as nice puzzles.	3.95	4.4	3.7

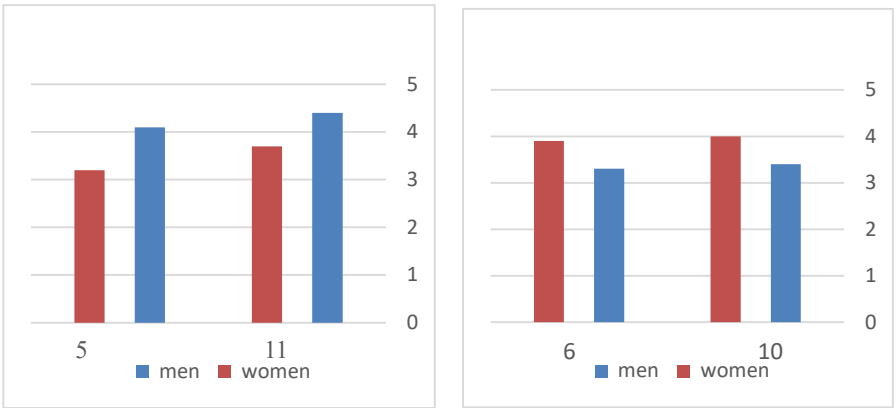

Figure 2: gender differences in attitude to word problems

Discussion and conclusions

The students in this study mainly expressed positive opinions about word problems, similar to the results in Shirvani and Garcia (2014). However, in the same study, gender differences were reported to be insignificant. We can assume that the different results are related to the cultural differences of the study population.

The differences in opinions regarding integration in teaching between students who teach and those who do not can be explained by the fact that only after several years of teaching do they feel the need to do things beyond the daily routine.

The students' positive opinions about word problems or alternative ways of solving, besides algebraic ways, can also result from all the topics taught in this course, which emphasized different, creative, non-standard ways of solving. On the other hand, knowing the solutions of well-known and respected personalities gives additional appreciation to these solutions.

Further and more extensive research can lead to the development of a more comprehensive study unit on word problems from Elijah Mizrahi's book, which is suitable for future teachers. Using word problems from Hebrew books written by prestigious rabbinical figures can also be helpful in curricula aimed at an Orthodox audience that usually avoids non-religious studies.

In his article, Moyon (2019) describes an experiment in which a word problem from Fibonacci's "Liber Abaci," like problem 4 in our study, was presented to secondary school students (aged 14-16). We tried something similar but with prospective teachers in the hope that this activity would change the students' perceptions regarding the importance of integrating the history of mathematics in teaching and that it would continue during the work of these students as teachers.

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Studying curricular reform through a full-text database of Norwegian textbooks: Findings and methodological considerations

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When studying curriculum reforms, it is of interest to investigating when and how key concepts have been included in textbooks. In this paper, we investigate how 390 mathematics textbooks for compulsory school in Norway from 1930 to 1986 treated selected concepts that are usually connected to New Math. We also discuss methodological considerations arising from our study. We use a full-text database of Norwegian books and compare our findings with previous research from Iceland and Norway. We demonstrate that many concepts associated with New Math were included in Norwegian textbooks prior to the 1971 "New Math curriculum", contrasting with the situation in Iceland, and provide a more detailed account of their inclusion. We also discuss how automatic searches can help (and when it does not) in historical research on mathematics education, pointing to some affordances and limitations.

Keywords: Textbook research, mathematics education, history of education.

Background

When new curricula are introduced by governments, policy documents often emphasize the innovations and departures from past practices. However, curricula do not necessarily work according to the plan. Textbooks serve as a bridge between the intended curriculum (the curriculum document) and the implemented curriculum (classroom practice), and can thus be considered part of "the potentially implemented curriculum" (Valverde et al., 2002). Textbooks should therefore be studied as part of a study on the effects of new curricula.

Within the field of history of mathematics education, the reforms under the heading of New Math constitute a particularly interesting case. New Math is often connected to the concept of *set*, but other parts of mathematics, for instance the number line, equations, vectors and statistics, also tended to enter curricula as part of New Math reforms. New Math was based on years of research efforts, was actively promoted through study conferences and international cooperations, and inspired curriculum reforms in many countries (De Bock, 2023). These circumstances created fertile ground for comparative research on curriculum reforms. For instance, Smestad and Opsal (2023) have studied why New Math fared so differently in countries that are often considered to be similar.

Bjarnadóttir (2017) studied textbooks in Iceland before, during and after New Math. Her study shows that set theory concepts were not the only concepts introduced to Icelandic students with New Math. Some of the novel concepts also remained in textbooks after New Math. We replicated Bjarnadóttir's method for the situation in Norway (Opsal & Smestad, 2022). One key difference was that in Iceland, no new textbooks had been published for decades before New Math, while in Norway, several textbook series had been developed in every decade, gradually introducing concepts that in Iceland

co-occurred with set theory. On the other hand, in Iceland there were some topics that only appeared in the 1970s, after New Math had waned there, which in Norway had appeared with New Math.

Both Bjarnadóttir (2017) and Opsal and Smestad (2022) were done by painstakingly studying page by page of textbook after textbook. As we explored the possibilities of expanding our research to include earlier textbooks and more textbook series, we became aware of the possibility of using tools from the National Library of Norway to do searches in a corpus of mathematics textbooks. While corpus-based studies is well known in research on language textbooks to study the vocabulary used (Yang & Coxhead, 2022), we have not found such examples in research on historical mathematics textbooks. This paper therefore has two research questions. The first concerns the content of textbooks, and the second concerns the affordances of full-text searches in databases in historical research on mathematics textbooks.

Our research questions are:

- RQ1: How has the treatment of some key concepts (distributive law, associative laws, the number line, odd and even numbers and equations) changed in Norwegian mathematics textbooks in the period 1930–1986?
- RQ2: What advantages and disadvantages do full-text databases have for historical research on mathematics textbooks?

Norwegian curricula and languages

For context, we will give some background on Norwegian mathematics curricula in the 20th century. At the beginning of the 20th century, curricula from 1889 were used. New curricula appeared in 1922 and 1925, for the countryside and for cities respectively, both for grade 1–7 (ages 7–14). They were replaced in 1939 with new plans, which we will denote N39. In 1960, experiments with nine-year compulsory schooling got its own experimental plan (F60). During the 1960s, there were also experiments with New Math materials produced by the Nordic Committee for the Modernization of Mathematics Education (NKMM). Then, in 1971, a temporary new curriculum, for nine-year compulsory schooling, was published. This included two alternative plans: a traditional Alternative 1, and an Alternative 2 which was close to the NKMM suggestions. However, when a final new curriculum was published in 1974, Alternative 2 was abolished and New Math went back to having a small place in Norwegian curricula. After 1974, there has been new curricula in 1987 and 1997, as well as 2006 and 2020 in the 21st century, but these are not relevant for this article.

As we will study textbooks, we need to give a brief explanation about the official languages of Norway. The Norwegian language has two official written standards, bokmål and nynorsk. From 1974, all students have the right to get textbooks in the standard they use. Sámi is also an official language of Norway.

Methods

To answer RQ1, we will examine when and to what extent the selected concepts are emphasized in textbooks for relevant curricula, supplemented by an analysis of which concepts are closely linked to certain other concepts in the textbooks. RQ2 will be answered by reflecting on the difference between Opsal and Smestad (2022) and this present study.

In Norway, the National Library aims to digitalize all Norwegian books. It also has apps available to do different kinds of analyses, the relevant ones we will describe below. We have developed a corpus that includes all (as far as we have been able to establish) Norwegian mathematics textbooks published from 1930 to 1986. Opsal and Smestad (2025b) describes the development of this corpus in some detail. The corpus included student textbooks, exercise books, teachers' guides etc, in two languages (bokmål and nynorsk) and in all editions (see Opsal & Smestad, 2025b).

For the current project, we have developed subsets of this main corpus for each of the relevant curricula of the period (N39, F60, NKMM, M71, and M74). These subsets include only the first available edition in bokmål of all the student textbooks for each of the curricula. We focused on student textbooks, as those are books used by students where mathematics is explained, as opposed to exercise books (where mathematics is usually not explained) or teacher guides (that students do not use). We chose books in bokmål as the inclusion of both languages would complicate the textual analyses. We chose the first available edition because the inclusion of several editions of some books would skew the results. We ended up with a total of 390 textbooks in the subsets (see Table 1).

Table 1: Number of textbooks included in the subsets

Curriculum	N39	F60	NKMM	M71	M74
Number of books	67	68	10	116	129

The concepts chosen were the distributive law of addition, the associative law of addition, the number line, even/odd numbers and equations. These concepts were chosen since they were shown in Opsal and Smestad (2022) to be part of Norwegian textbooks before 1971, thus making it possible to track them over time. In Iceland, the first three of these concepts entered textbooks from 1966, even and odd numbers had been in textbooks since the 1910s, while equations entered textbooks in the 1970s.

When looking for instances of a concept in a textbook, an operationalization is needed. For instance, the number line can be referred to both as a phrase ("number line") and as an image of a number line. Concepts can also be evoked in other ways, as when the set of even numbers can be written as $\{2n: n \in N\}$. When using automatic text searches, we are limited to searching for particular words or phrases in the textbooks. This limitation needs to be considered when analysing the results, as we will discuss below.

To search for words in textbooks for different curricula, we used the National Library app Word Frequencies (Ordfrekvenser). This app simply counts the occurrences of words throughout the corpus. The words chosen were *distributiv* (distributive), *assosiativ* (associative), *tallinje/tallstråle* (number line), *partall/oddetall* (even/odd number) and *likning/ligning* (equation), and variants of these words, such as tallinjer (number lines).[1] The app includes the possibility of using a grouping variable. In our

[1] The complete list of variants searched for is: distributiv, distributive, assosiativ, assosiative, tallinje, tallinja, tallinjen, tallinjer, tallstråle, tallstrålen, tallstråla, tallstråler, partall, partallene, oddetall, oddetallene, likning, likninga, likningen, likninger, likningene, ligning, ligninga, ligningene, ligninger.

case, we used grade as a grouping variable, to get an overview of which grades words occurred in. The analysis of these tables consisted simply of comparing the tables, individually and together, to point out trends in when and to what degree concepts were stressed.

There is no one-to-one correspondence between a concept and a word. Firstly, a concept may be treated in a textbook without being named (or by using another word). Therefore, we cannot claim that a word search will find all occurrences of a concept. Secondly, the word may be used without referring to that exact concept. However, some mathematical words are exclusively or predominantly used for the concept, and in other cases, each occurrence of a word can be checked. An additional issue is that we have no exact knowledge of the accuracy of the digitalization, so some words may be misspelled leading to the app not finding them. Again, this means that the method may miss some instances of a concept, but probably not lead to extra, erroneous instances. Word searches are therefore more suited to establish that concepts are treated than to establish that they are not.

We do not search for phrases such as "associative law of addition", as there would be too many alternative phrases ("addition's associative property" etc.). The downside of searching just for the word associative, is that we don't know which operation it concerns. To clarify this, we used the app Concordances (Konkordanser), which gives the context of every occurrence of a word in the corpus, by providing extracts of up to 25 words surrounding the search word from the paragraph where the word is used. We used this app in the case of "associative" and analysed the resulting concordances. In many cases, the app returned a text not indicating which operation was discussed. It would be possible to find these paragraphs in the respective books, but that was outside the scope of this article.

Another option for the same goal, would be to use the National Library app Collocations (Kollokasjoner) with the word "associative". This app does a rather more advanced analysis, as it for a given word ("associative", for instance) finds words that are more often included near the given word than it is in a general text. For reasons of space, we did not include this in the present article.

To answer RQ2, we compared the kinds of findings we obtained in the present study with similar findings in Opsal and Smestad (2022). In one instance, we also did a quick look at the textbooks in question to supplement the automatic searches. We noted instances where the automatic analyses of full-text databases gave additional insights or, to the contrary, failed to give insights.

Results

We begin with an overview of the word frequencies (Table 2). Most of the selected words are not used in N39, while the words "even" and "odd" number can be found, but rarely. However, all our selected words were much more frequent in the F60 textbooks. They have stayed in textbooks throughout the period we look at, except "equations", which were not included in NKMM textbooks. When comparing curricula, it is helpful to look at the "words per book" numbers. These show, for instance, that our words (except "equations") were mentioned far more often (per book) in NKMM textbooks than in F60.

The result for "equations" was somewhat surprising, so we had a quick look through the NKMM textbooks. We did not find the word "equations", but we did find tasks such as this "What operations

do you use to find n in these true mathematical statements? […] 5 + n = 6" (*Matematikk for 4. klasse: forsøkstekst. Del I*, 1966, p. 93), showing that the concept of equations was touched upon.

Table 2: Word frequencies (words per book in parenthesis)

	N39	F60	NKMM	M71	M74
distributive		22 (.3)	16 (1.6)	53 (.5)	59 (.5)
associative		46 (.7)	45 (4.5)	69 (.6)	31 (.2)
number line		517 (7.6)	175 (17.5)	491 (4.2)	452 (3.5)
even number	7 (.1)	83 (1.2)	52 (5.2)	55 (.5)	197 (1.5)
odd number	6 (.1)	92 (1.4)	51 (5.1)	50 (.4)	194 (1.5)
equation		1959 (28.8)		750 (6.5)	1622 (12.6)

Table 3: Word frequencies for even and odd number – grouped by grade level (words per book in parenthesis)

	N39	F60	NKMM	M71	M74
1				4 (.3)	10 (1.0)
2	12 (1.2)	9 (2.3)		34 (2.6)	27 (3.0)
3		51 (17.0)		13 (1.3)	23 (2.3)
4		12 (3.0)	80 (26.7)	1 (.2)	36 (4.0)
5	1 (.2)		18 (6.0)	21 (10.5)	28 (2.0)
6		4 (1.3)	5 (1.3)	7 (3.5)	22 (1.6)
7		2 (.2)			180 (15.0)
8		15 (2.1)			32 (2.5)
9		2 (.3)			10 (1.0)
more than one		48 (2.3)		2 (.0)	23 (.8)

We have also run these analyses while grouping for grade level. We find the results for even and odd numbers particularly interesting (Table 3). Even and odd numbers have been included in all the

corpuses, and an analysis by grades can disclose whether their placement in grades have fluctuated. Some textbooks are written for more than one grade, and these are coded as "more than one" in the corpus and in the table.

It is striking that the position of the words even and odd numbers varies so much between curricula. For instance, in the F60 textbooks, there were most mentions in grade 3, while from M74, a large part of the mentions was in grade 7. The striking number of mentions in grade 7 could be investigated further – could it be that even and odd numbers had a particular role in grade 7 in M74 that it did not have in other grades or curricula? A quick use of the Concordances app would give a view of how the words are used in sentences, which could give ideas of what role they play. We will not go into that here.

The contexts of the word "associative" (assosiativ/assosiative) (Table 4) gave some surprises. Although aware that the associative property is relevant for a whole range of binary operations, we imagined that the associative laws for addition and multiplication would be the ones mentioned. However, textbooks also asked whether (or stated that) the associative law holds for subtraction, translation, potentiation[2], addition of vectors and an unknown operation (called "operation quadrilateral"). In M74, however, only associative laws of addition and multiplication are mentioned. The app does not always give enough context – in a large part of the cases, the "associative law" is mentioned without enough information from the app to decide which associative law it is.

Table 4: Contexts (binary operations) of the word "associative"

	N39	F60	NKMM	M71	M74
addition		18	12	36	14
multiplication		13	24	29	9
subtraction		1	1		
division					
translation				2	
potentiation		4			
addition of vectors				1	
the operation quadrilateral			1		
not possible to determine		13	13	18	14

[2] Potentation is the operation of raising something to a power.

Discussion

Broadly speaking, our answers to RQ1 confirm the findings of our manual searches for when concepts were introduced in textbooks in Opsal and Smestad (2022): The concepts distributive law, associative law and number line, which in Iceland entered into textbooks with New Math (Bjarnadóttir, 2017), in Norway were present in textbooks for the F60 curriculum, long before set theory took the stage in what has long been considered the "New Math curriculum" of 1971. The current analysis includes more textbooks, and therefore strengthens the findings in Opsal and Smestad (2022). It also includes the frequency of certain words, and not only first inclusion of a concept. This provides more nuance, as it gives indications of the prevalence of a concept through the grades.

These findings will in turn inform comparisons between Iceland and Norway. As Bjarnadóttir (2006) points out, in Iceland new textbooks with novel ideas were welcome. In Norway, new textbooks were frequent, leading to a more gradual introduction of New Math concepts. When considering the reception of the "New Math" curriculum (M71) in Norway, this fact is relevant. However, other analyses (Opsal & Smestad, 2025a) suggest that the M71 (alternative 2) went further in the incorporation of set concepts than for instance Sweden. Such detailed studies are valuable when studying and comparing curriculum reforms across countries.

Our further analyses, concerning even and odd numbers and the associative laws, rather give inspiration for further detailed analyses than conclusive results. Such future studies could include comparisons with curriculum documents.

Regarding RQ2, we conclude that the use of a corpus, a full-text database and apps has some potential. In our case, the books were already digitalized, but it was still time consuming to create and ensure the quality of a corpus. Once the corpus is established, the analyses can be run quickly, and some hypotheses can be raised which can later be studied in more detail. A traditional textbook analysis of 390 textbooks would be painstaking. Our previous analysis included far fewer textbooks and was content with finding the first inclusion of a concept, instead of all mentions. Thus, the finding above about the apparent differences in use of odd and even numbers in grade 7 in 1960 and 1974 (as judged by word frequencies), would not be found as easily in our manual research.

However, our analyses in the current article are simple. We search only for words, and some concepts are difficult to research in this way. Already, we have not been able to distinguish between the associative law for addition and the associative law for multiplication. Searching for phrases such as "associative law of addition" would risk losing other phrasings (such as "addition's associative law"). The use of other apps gave some more, but still limited, information. Searching for "the role of 0 in multiplication" would probably not give any results, even though we know from Opsal and Smestad (2022) that textbooks do indeed point out what happens when 0 is used in multiplication. Likewise, equations can be found in the NKMM books, even though the word equations is not.

In Opsal and Smestad (2022), we also included implicit inclusion of concepts. The best example is implicitly including the commutative law of multiplication by giving many tasks where the factors are toggled, such as $3 \cdot 6 =$, $6 \cdot 3 =$, $5 \cdot 4 =$, $4 \cdot 5 =$, and so on. There is currently (at the time of writing) no artificial intelligence in the apps to identify such implicit inclusion of concepts.

Automatic searches should be used with care. When specialised mathematics words are found, the corresponding concept is probably included, and this can be checked manually. On the other hand, when words are not found, the concept may still be there. Surprising non-findings should therefore be checked carefully. Used well, we believe such technology will provide good opportunities for textbook research. Simple searches can give preliminary overviews that can spur new studies. Of course, such methods can also be used by researchers who digitalise a small number of books on their own. We are eager to hear of researchers doing such research in this area – as well as to hear of other countries in which the national library or other entities have digitalised all textbooks. This could provide a great arena for detailed, comparative textbook research.

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Historical awareness and foreground in idea generation processes

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Abstract *In this paper we give a suggestion for how historical awareness and the concept of foreground can be a springboard to create a didactical frame, which supports pre-service teachers' possibilities to engage in collaborative teacher communities about idea generation processes of their own teaching and learning modules. First, we emphasize points from literature and the Danish curricula for mathematical education for primary and lower secondary school as well as for pre-service teacher education. On this we base our argument for including history of mathematics in such processes. Then we present and connect the theoretical terms we use in the paper. Finally, we discuss and reflect about how to include history of mathematics to support students to engage in idea generation processes.*

Keywords: Historical awareness, foreground, engagement, collaborative teacher communities, idea generation processes.

Introduction

The purpose for the subject mathematics in the curricula for primary and lower secondary school in Denmark among other things emphasize, that it must "contribute to the pupils' experience and acknowledge of the role of mathematics in a historical, cultural and society context" (our translation, Børne og undervisningsministeriet, 2019, p. 7). To prepare the future teachers to support that, we must include the historical context in the pre-service teacher education as well. The curricula for teacher education part 3—the local subject description—among other things says, that the "students participate in collaborative teacher communities about developing their own teaching in mathematics and contribute to the development of mathematical education in lower and secondary school" (our translation, Læreruddannelsens studieordning [Curriculum for teacher education] 2024/2025, part 3, p. 85). To prepare the future teachers to do that, the pre-service teacher students can be supported in developing, expressing and discussing their ideas, of what they find important in their own education. This can be a way of letting them connect their reflections on their own role as future math teachers and being pre-service teacher students. The teachers of pre-service teachers can orchestrate processes, where the students get the possibilities to express and discuss their ideas of, what they find important. Thereby the students participate in a collaborative students' community, where their ideas contribute to the development of their own local teaching modules. In this paper we use the term idea generation processes as a description of such processes. These idea generation processes of course must be seen in connection with the curriculums for both the teacher education and the national curricula for primary and lower secondary school.

In this paper we focus on a more general view on, how the history of mathematics and different mathematicians' views of mathematics can be a starting point to engage pre-service teachers in common idea generation processes. You might say that we take the perspective of Jahnke's et al.

(2002) description of Freudenthal's (1986) view on using history in mathematical education. Freudenthal among other things:

> took the view that historical interventions in mathematics teaching at schools should contribute more to comprehending the general history (of culture) than to comprehending mathematics. (Jahnke et al., 2022, p. 1430)

Fried (2001) points out, that there is no reason to think, that the techniques and approaches, which we use in nowadays' classrooms, always will be the most powerful ways of using and teaching mathematics in the modern world. To us that is also an argument for why it must be important to let the students meet different perspectives from historical mathematicians. This can give them the experience that mathematics and perspectives on mathematics changes. We consider that the terms *historical awareness* (Jensen, 2010) and *foreground* (Alrø et al., 2009; Skovmose, 2011) can be a springboard to support pre-service teachers' possibilities to engage in common idea generation processes. We unfold these two terms and relate them to Jankvist's (2009) description of *whys* and *hows* in the next section of the paper. The following describes the question we seek to answer from a theoretical perspective:

How can the terms *historical awareness* and *foreground* be a springboard for constructing a didactical setting and a theoretical construct, which support pre-service teacher's possibilities to engage in collaborative teacher communities about idea generation of their own teaching and learning modules?

In the next section we present the theoretical terms we use. Then we describe, how these can be implemented and can interact with each other in an appropriate way to support collaborative idea generation sessions. We have piloted some of the elements in our own teaching of pre-service teachers, but we did not collect data except our own planning of the sessions. Based on this, we keep this as a theoretical paper. In our didactical setting we, among other things, focused on the differences between Poincaré's and the Bourbaki group's view on mathematics and organizing of mathematical working processes. Poincaré's "ideas of teaching and education were closely linked to the roles he attributed to intuition and logic in mathematics" (Jahnke et al., 2022, p. 1422) and Bourbaki's group had a structuralist view on mathematics (Jahnke et al., 2022, p. 1421). Our choice of these different perspectives was inspired by Lindenskov (2009). We also used that text in the didactical setting, where we tried to orchestrate a common idea generation process. This is one perspective of how to use historical mathematicians in idea generation processes. We could also have included other historical mathematicians' views or original sources, which we discuss and reflect on later in the paper.

Theoretical terms

Kjeldsen et al. (2022) introduce Jensen's (2010) terms *historical awareness* from didactic of history and combine it with didactics of mathematics. *Historical awareness* includes what Jensen (2003) coined as a multiple perspective approach. It is both observer and action-oriented and "both the past and the future are present in the present: the past as recollection and interpretations of the past and the future as a set of expectations" (Kjeldsen et al., 2022, p. 46). According to Kjeldsen et al. (2022)

the term *historical awareness* has some resemblances with what Skovsmose (e.g. 2011) call *foreground*. They elaborate as follow:

> In this respect, the notion of historical awareness has some resemblances with Skovsmose's (e.g., 2011) notion of "foreground" within critical mathematics education. Yet, Skovsmose's notion of foreground is embedded in a social context involving students' hope, despair, uncertainty, etc., whereas historical awareness has to do with the students' academic foreground. (Kjeldsen et al., 2022, p. 47)

We interpret this as Kjeldsen et al. (2022) distinguishing between *historical awareness* and *foreground* by emphasizing, that *historical awareness* has an academic *foreground*, and Skovsmose's (e.g. 2011) use of *foreground* is related to students' lives and social contexts. Kjeldsen et al. (2022) found that "the construct of *historical awareness* is well suited to articulate students' academic *foregrounds* in the discipline of mathematics—that is, their potential future lives with mathematics" (p. 65). In this paper we also include the term Skovsmose (e.g. 2011) calls *foreground*. We consider that pre-service teachers' "hope, despair, uncertainty, etc." about being math teachers, both are about an academic *foreground* and a social *foreground*. It is about mathematics itself, mathematics as a subject, and about their personal reasons and visions about being math teachers. That is why we focus on the intersection between, what Jensen (2010) define as *historical awareness*—an academic *foreground*, and what Skovsmose (e.g. 2011) define as *foreground*—a more social *foreground*. We see both *historical awareness* and *foreground* as ways to enlarge and qualify students' possibilities to engage in collaborative idea generation processes of their own pre-service teacher's teaching and learning modules. This, of course, must be within the frame of the educational curricula.

Jankvist (2009) divides the motivations for using history in mathematics educational contexts in two categories: *Whys* and *hows*. The *whys* are argument for using history, which is characterized as history-as-a-goal arguments (meta-issues) and history-as-a-tool arguments (in-issues). Here "the learning of developmental and evolutionary aspects of mathematics either serves a goal in itself or serves to illustrate other historical aspects of the discipline" (Jankvist, 2009, p. 239) and human being is a part of "the developmental and evolutionary aspects of mathematics as a discipline" (p. 239). Using history-as-a-tool "contains the arguments concerning how students learn mathematics" (Jankvist, 2009, p. 237) and can also include, that history has the potential to be a motivation factor for the students and make the mathematics less frightening, by giving the mathematics a more human face (Jankvist, 2009). The *hows* are about approaches to teaching. Jankvist (2009) describe three categories of approaches: illumination, modules and history based (we return to the illumination approach and module approach later). Jankvist (2009) points out, that these distinctions can be used to analyze teaching materials applying history, and that they can be "useful when making decisions about content, form of presentation, and organization concerning the use of history for teachers as well as material designers" (p. 236). We think that the distinction between *whys* and *hows* can also be useful when teachers engage the students in previous described idea generation processes with a starting point in the history of mathematics. Common for both the term *historical awareness* and *foreground* are, that their turning point is about how student's awareness supports their possibilities to act in the present, in the light of both the past and the future. This also relies on their way of

understanding the evolution of mathematics, the subject of mathematics, and the educational systems as a result of human beings taking part in it (see also e.g. Jankvist, 2009).

The *goals* and *tools* argument can be seen as arguments for including history of mathematics in common idea generation processes. It has a concrete role in the design of a didactical setting, which support students' common idea generation processes. Here history-as-a-goal can support, that the students get an understanding of the developmental and evolutionary aspect of mathematics. Furthermore, it can support the pre-service teachers in seeing themselves and their future pupils as a part of this. That is one reason for, why it is important, that they as pre-service teachers participate in collaborative communities, where they discuss and reflect on, what they find can contribute to the development of their own mathematical education and mathematics education in general. In the didactical setting we focus on in this paper, you might say, that using history-as-a-tool on the one hand are about motivating the students to engage in the idea generation processes. On the other hand, they are also about giving the students insights into different views on working with mathematics, give the mathematics a more human face, so to speak. When the students get the possibilities to also engage in discussions about e.g. the mathematical content, they find important to be taught in and how to teach this as future math teachers in primary and lower secondary school, we see this as both *whys* and *hows* perspective.

The didactical setting and theoretical construction

In this section we will try to explain some elements of, how we as teachers for pre-service teachers have tried to form a session, where we could engage the students in an idea generation process for planning some of their own teaching and learning modules at the teacher education. We do not think this as a concept of how to engage students in such processes, but as a concrete example of how we find, that the theoretical term, we have presented in this paper, can interact together in an appropriate way. We finish this section of the paper by giving a visualization of how, we in this example see the theoretical terms are related.

To support the students in sharpening their awareness of their own *foregrounds*, we asked more open questions about e.g. how they see mathematics, how it contributes to their own lives and how they would like to teach mathematics as future teachers. These questions were asked to give the students' a possibility to express and discuss their own connection to math, and their hopes connected to be a math teacher. This is a way to activate their social *foreground* related to their personal academic *foreground*.

Then we used Lindenskov (2009) as a starting point to introduce the students to different historical mathematicians' perspectives of why mathematics is important, their view of mathematics and working processes related to that. Lindenskov (2009) describe Henri Poincaré's (1854-1912) and the group of Nicolas Bourbaki's ideas of mathematics, their perspective of how to work with mathematics and organizations hereof. The idea of presenting different perspectives of historian mathematicians to the students, was to support them in getting an understanding of the developmental and evolutionary aspect of mathematics, and in seeing themselves and their future pupils as a part of this. Here the argument to bring this into play is a *why-goal* perspective. This can be a way to support students in developing a *historical awareness*—a more academic *foreground*. We also found that

Lindenskov's (2009) presentation of these two historical mathematicians' perspectives of mathematics could qualify the students' possibilities to develop their awareness of their own views of mathematics—why it is important from a personal, educational and society perspectives. That was also a *why* perspective related to support the students' awareness of their *foreground*—a more social *foreground* to be a future math teacher. Finally, we found that presenting these two historical mathematicians' perspectives of mathematics to the students, would be a way to motivate the students. It might also have the potential for the students to identify with some of the views. These different views could thereby contribute to give math a more human face, which can be characterized as a *why-tool* argument. Therefore, we found, that this text could be an outset for the further idea generation process.

Poincaré (1854–1912) see mathematics as much more than logic, it is also intuitive and connected to the real world, it can be seen as art, where e.g. numbers and symbols are interrelated in a harmonious way (Lindenskov, 2009). Poincaré's way of working with mathematics was not making a full plan, but starting and seeing where it goes, stopping if the work becomes heavy, because most of the essential ideas comes, when you are not concentrating on the problem (Lindenskov, 2009). Nicolas Bourbaki is a pseudonym for a collective of mathematicians started of six French mathematicians in 1934. The group of Bourbaki found Poincaré's intuitive and non-logical way of looking at mathematics confusing and elitist (Lindenskov, 2009). They wanted more people to be more educated (Lindenskov, 2009). The Bourbaki group was a strong professional and social community, where all the members of the group worked with the tasks, the proofs, and the descriptions, both by writing to each other and by physically meeting (Lindenskov, 2009). Lindenskov (2009) use among others Poincare and Bourbaki to describe, why it might be important to have an aesthetic perspective of mathematics, and how to work with mathematic in an aesthetic way. Based hereof Lindenskov (2009) defines seven keys to open aesthetic aspects of mathematics. We will not go further into the seven keys, neither focus particular on the aesthetic aspects of mathematics. We found, however, that this presentation of different historical mathematicians is a way to support students impression of e.g.: 1) mathematics is created by human beings, and thereby it changes over time, 2) working with mathematics can be seen both from a more intuitive and a more logical perspective or from both perspectives, and 3) you can work and create mathematics, mathematical problems, and mathematical solving using different working strategies. As stated above, we found, that these different historical perspectives could support the students in developing their *historical awareness.* They could e.g. have a *why* discussion related to respectively Poincaré's and the Bourbaki group's view of why mathematical education is important and for who.

We also think it could be a good idea to present the pre-service teachers for the concept of *foreground*. We took our outset in Alrø's et al. (2009) text to introduce the concept of *foreground* to the pre-service teachers. Thereafter we asked them questions divided into the categories: past, present and future. An excerpt of these sounded like: What kind of math did you find important, when you were children? (past); What have been the most important experiences from your own practical experiences of teaching throughout your studies? (present); How will your praxis as a math teacher look like in ten years from now? (future). How do you imagine, we can work with math in the next module/the next year to support you to become math teachers? Try to give some concrete examples (*foreground*

-> present). Here the students could get the possibilities to specify and discuss their *foreground*, their reasons for working with mathematics, and ideals for being math teachers—their social *foreground*. Here they also got the possibilities to use their knowledge of both Poincaré's and the group of Bourbaki's perspectives of mathematics, the different ways they worked with it, and their ideals related to mathematical education. We will characterize our *how* approach to work with history in this case as an illumination approach, because in our didactical setting history was used as a supplement and that agree very well with Jankvist (2009) description: "One way to think of these smaller supplements in the illumination approaches is to think of them as "spices" added to the mathematics education casserole" (p. 246).

These different steps in an idea generation process could lead to more focus on the specific content, that the pre-service teachers found important to work with. This was also related to the curriculum for primary and lower secondary school. Questions that could support this part of the process could e.g. sound like: Which subject areas do you want to work with? Which mathematical didactical perspectives do you want to know more about? Which working methods do you prefer?

These different steps presented above could be characterized as a beginning process of engaging the students in idea generation processes. These could be revisited consecutively throughout their study as pre-service teachers.

A visualization of the interaction between the theoretical terms is presented in Figure 1.

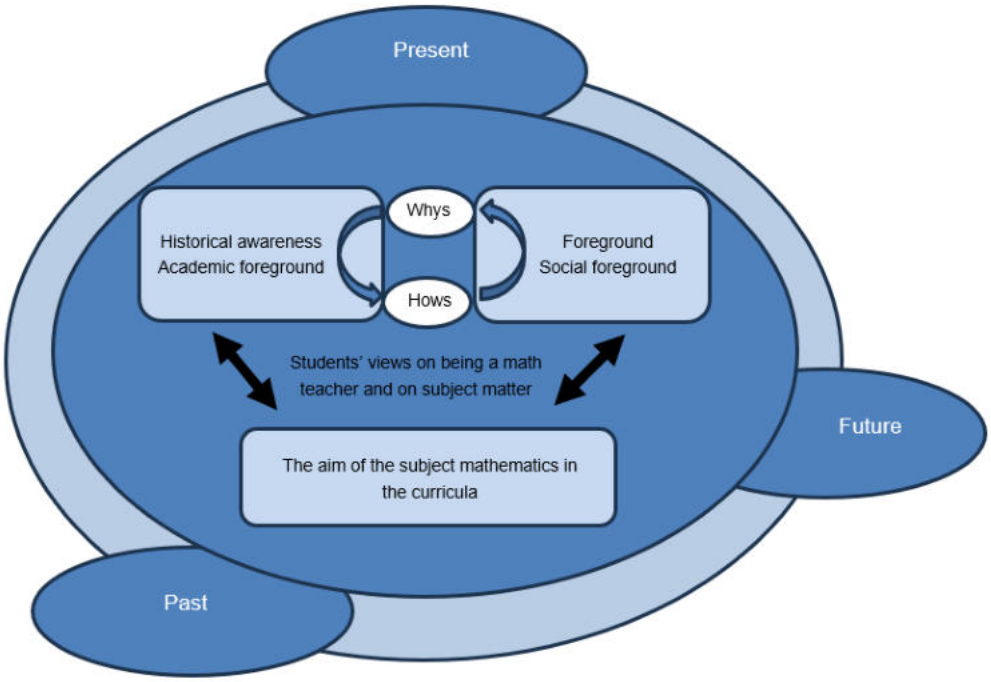

Figure 1: To support idea generation processes—the interaction between the theoretical terms: Historical awareness (Jensen, 2010), foreground (Skovsmose, 2011) and the distinction between whys and hows (Jankvist, 2009)

Reflections

We have presented an example of how teachers for pre-service teachers can focus on the term *historical awareness* and *foreground* as central terms to engage the students in idea generation processes related to their own local teaching and learning processes in mathematics. In this paper we focus on a Danish educational context, but we find that, the terms *historical awareness* and *foreground* can be used broader in other mathematical educational contexts for pre-service teachers. This could also include Jankvist's (2009) description of the *whys* and *hows* for using history in mathematical education. In this paper our concrete historical input were Lindenskov's (2009) description of respectively Poincaré's and the group of Bourbaki's view of mathematics and how to organize working with it. Here we could have used other perspectives or historical mathematicians as well. We could e.g. have used primary historical sources to focus more on supporting the students in developing their *historical awareness* (inspired of Kjeldsen et al., 2022). This could have the benefit which Barnett et al. (2014) emphasizes by referring to Jahnke et al. (2002): "the educational value of original source readings in any discipline lies in their ability to promote "thinking into other persons and into a different world" (Jahnke et al. 2002, p. 299)" (p. 12). Furthermore, they refer to Barbin (1997):

> the educational value of the reorientation (or dépaysement) which reading historical texts can effect in our view of mathematics by "making the familiar unfamiliar" requires a reading which seeks always to understand, rather than judge, the content of those texts (Barbin 1997, p. 24). (Barnett et al., 2014, p. 12).

We think that giving the students the possibilities to engage in situations, which include "reorientation", and the ability to "thinking into other persons and into a different world", have the potential to support pre-service teachers in developing a more reflective awareness of their own *foreground.* In such situations, they might have the possibilities to get insights into other perspectives on mathematics and mathematics education. Kjeldsen et al. (2022) stresses, "that using primary historical sources in mathematics education has potential to develop students' historical awareness" (p. 63). If primary historical sources become a part of such idea generation processes, we think that it would requires another *how* approach, namely the module approach, where more time is used on the historical aspect and the math within the source to support the students in getting insights into the *whys* and thereby supporting them in developing an *historical awareness* to support them in reflecting on their own *foreground*.

In this paper we aimed to focus on how the terms *historical awareness* and *foreground* can be seen as central parts of a didactical setting and a theoretical construct, which has the objective to support pre-service teachers to collaborate in teacher communities about idea generation processes of planning their own teaching and learning modules. This will hopefully also strengthen their abilities to collaborate in such processes in their future life as math teachers.

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A transformation of perception in calculus: An exploration of barriers and outcomes through transgressive actions

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Research on learning undergraduate mathematics via primary historical sources has been investigated for over a decade, and much of the available research has focused on, for example, aspects of students' learning of metadiscursive rules or the reorientation students experience when learning typical mathematical concepts from unfamiliar primary source materials. In our research we employ the lens of transgressions to explore links among transgressive actions, barriers, and outcomes. We present an illustrative example for Simon (a student in a Calculus III with Engineering Applications course), and we highlight the relationship between a barrier and its associated outcomes resulting from engaging in discourse with an instructor and engaging with curriculum materials utilising primary historical sources in the learning of mathematics as transgressive actions.

Keywords: Teaching and learning of analysis and calculus, Undergraduate mathematics education, Primary historical sources, Primary Source Projects, Transgressions.

Introduction

For decades, the role of history of mathematics in the teaching of learning of mathematics has been investigated (Clark et al., 2018). More recently, researchers have paid increasing attention to drawing more explicitly on theoretical frameworks (both within and from outside mathematics education). In doing so, such research has more frequently appeared in the mainstream mathematics education literature, including contributions in a variety of top-tier peer-reviewed journals (e.g., Agterberg et al., 2022; Jankvist et al., 2020).

The focus of our work is to investigate students' learning experiences when primary historical sources (PHS) are used. Researchers have examined students' learning of mathematics via primary sources from different theoretical perspectives, including commognition (Sfard, 2008). For example, Kjeldsen and Blomhøj (2012) illustrated how the use of primary sources promoted students' learning of metadiscursive rules. Additionally, Barnett et al. (2021) used a detailed case study to highlight one student's "figuring out" of metadiscursive rules in the discourse of mathematical analysis through the use and study of curriculum materials drawing upon PHS, called Primary Source Projects (PSPs), in an undergraduate analysis course. Recently, we (Watford et al., 2025) trialled a lens relatively new to mathematics education research (*transgressions*; Kozielecki, 1986) to explore the *transgressive actions* students employ to cross *boundaries* they may have experienced while studying abstract algebra via PSPs[1]. In this paper, we build on our previous work to investigate students' experiences with studying PSPs in a Calculus III course focused on engineering applications. The goal of the investigation was to explore the link between barriers and outcomes related to students' perception

[1] We expand upon these components of transgressions (boundaries, transgressive actions, outcomes) in the "Theoretical background" section.

of mathematics and to determine what actions may have contributed to students' change in perception.

Literature and theoretical background

In the last two decades or so, several experts—particularly within the domain of history and pedagogy of mathematics (the HPM domain)—have discussed claims for the potential effects of students' learning of mathematics via PHS. Pengelley (2011) identified nine reasons motivating the use of original (primary) sources in teaching undergraduate mathematics. Of these, Jankvist (2014) noted several are associated with in-issues (e.g., "participating in the process of doing mathematics") and several with meta-issues (e.g., "understanding origins and discovery"), and most may be subject to interpretation (e.g., "understanding essence"). The most prevalent among Pengelley's reasons is that the use of original sources allows for what is known as *dépaysement* (Barbin, 1997)—which can be thought of as a form of disorientation, cognitive dissonance, or providing multiple points of view for the same idea or concept. The notion of dépaysement was reconceptualised by Jahnke et al. (2000) and became one of three ideas that they proposed as replacement, reorientation, and cultural understanding.

Additionally, Barnett et al. (2014) provided theoretical grounding for teaching undergraduate mathematics via PHS, as well as a discussion of their guided reading approach, which is the primary design tool of the PSPs developed as part of the TRansforming Instruction in Undergraduate Mathematics via Primary Historical Sources (TRIUMPHS) project. The TRIUMPHS project developed, disseminated, and field-tested PSPs, which were designed to teach topics in the undergraduate mathematics curriculum. Each PSP includes author commentary, pertinent primary source excerpts, student tasks, and notes for the instructor. To date, TRIUMPHS has produced some 100 PSPs, which are freely available (via Creative Commons licence) from the grant project website[2].

Within the context of learning mathematics using history and especially learning via PSPs, potential learning outcomes are common among the literature, such as meta-issues (Jankvist, 2014) and instructor perceived benefits (Barnett et al., 2016). Potential barriers to learning mathematics via history and PSPs infrequently appear throughout the research (commonly, a language barrier); however, very few studies have attempted to explore specific barriers and how students overcome barriers to reach outcomes. We found a productive solution in the framing of *transgressions* by Jozef Kozielecki (1986), in which barriers are connected to "new havings or beings" through *transgressive actions*. Such actions are defined as "a purposeful action that leads to an outcome exceeding the boundaries of the individual's past achievements" (p. 89). Although the typical connotation attached to anything transgressive is negative, this is not the case in the context of the theory of transgressions. The conceptualisation of a transgressive action allows for a medium of travel beyond boundaries[3] of limited perception to an expanded perception, which we situate as outcomes often associated with surprise and perceiving what seemed impossible as now possible (Pieronkiewicz, 2015). Additionally, Semadeni's (2015) operationalisation of a boundary, associated with transgressive

[2] The PSPs can be found on the TRIUMPHS website: https://blogs.ursinus.edu/triumphs/

[3] In this paper, we use forms of "barrier" and "boundary" interchangeably instead of the term, "obstacle." We do so to remain consistent with Kozielecki's (1986) use of the same terms.

actions, as a "previously non-traversable limit of own mathematical knowledge or… deep rooted convictions" (p. 27), fits well with our experience with students who study mathematics using PHS.

Initial investigation: Linking transgressive actions, barriers, and outcomes

Possibly due to the original context of transgressions (situated within psychology), the framework has only recently been recognised in the mathematics education research community. We conjecture there are valuable possibilities that the theory affords when studying the mathematics learning process. Recently, we were part of a research team who explored transgressive actions of Abstract Algebra students (also studying via PSPs) based on survey responses; however, the development of the survey instrument was not informed by a transgressions lens. To remedy this limitation, Watford (2022) developed and piloted an interview protocol with the purpose of exploring transgressive actions, outcomes, and barriers identified in Watford et al. (2025). Particularly, Watford's (2022) report demonstrated that engaging in discourse constitutes a transgressive action. In its multiple forms, discursive transgressive actions overcame barriers students encountered in a "Calculus and Its History" course, which was taught primarily with PSPs. Based on that work, we sought to investigate how transgressive actions may have manifested in a different learning context. We draw upon Watford (2022) to situate discourse with the instructor as a transgressive action as well as claim that engagement with the PSP (via reading primary and secondary source material and working through tasks) also constitutes a transgressive action because of the students' purposeful interaction with the PSP and the existence of barriers and outcomes surrounding their interaction.

Methods

The setting for this study is a two-year college located in the western United States. The college boasts three dozen career and technical programs and has 29 academic programs from which students can earn two-year degrees and if desired, they can then transfer to public four-year university or college programs. The course, *Calculus III with Engineering Applications* (or, Calc III-EA), with 12 students enrolled was taught in Spring 2022 (January–May) by Dr. Leaf (all names in this paper are pseudonyms). Dr. Leaf has been affiliated with TRIUMPHS since Spring 2017 as both a site tester and author of PSPs. In the Calc III-EA course, Dr. Leaf implemented two PSPs, approximately one month apart. For the purposes of this paper, we discuss findings for the implementation of the first PSP, "The Radius of Curvature According to Christiaan Huygens" (Lodder, 2022), which students studied for 2.5 class sessions with the focus of instruction being the PSP.

Participants and data sources

Data sources included a 17-item instructor implementation report, in which Dr. Leaf detailed how the PSP was implemented to all students in the course. The report also asked for instructional goals and perceived benefits and obstacles (that Dr. Leaf described for himself, as well as his perceptions of these for students). Interviews with three student participants comprised the second data source. The interviews lasted approximately one hour and were held within a week following the PSP classroom instruction.

Preliminary data analysis

Transcripts were coded using a combination of inductive and deductive coding methods partially informed by previous analysis regarding transgressive actions (e.g., Watford, 2022; Watford et al.,

2025). We searched for instances when a student mentioned they felt limited or unable to do something before performing an action. These actions led to some new insight or state of being. Respectively, clusters of coded data illuminated a *barrier*, *transgressive action*, and an *outcome*, based on Kozielecki's (1986) characterisation of transgressive actions. The implementation report was coded in the same manner. For the purposes of this paper, we highlight the relationship between a barrier and its associated outcomes resulting from engaging in discourse with an instructor and engaging with the PSP as transgressive actions. Engaging in discourse with the instructor fits well with Kozielecki's notion of transgressive action because of its potential to enable students to overcome obstacles leading to outcomes. For the same reason, engagement with the PSP constitutes a transgressive action. The outcomes, which result from these transgressive actions, extend beyond simply learning calculus content. That is, we report on students' ability to connect present mathematical content to previous calculus learning, as well as demonstrating an awareness of the evolutionary aspect of mathematics.

Findings

To depict a student's distinct story while highlighting transgressive components common to all three interviews, we present an especially illustrative example from one participant, Simon, who provided rich responses to interview questions. We begin with a barrier that appeared in each of the three interviews, *Perception of Mathematics*. To account for the unique and personal nature of outcomes, we then discuss a complex assemblage of Simon's outcomes that resulted in his "new having/being" from overcoming the barrier. Finally, we highlight the transgressive actions *Expert/Instructor Discourse* taken together with *Engagement with the PSP* which also appeared in all three interviews. The excerpts on which we focus were prompted by the interview question: How do you feel that your connection to mathematicians and mathematical ideas has changed throughout the course, and what has promoted this change?

Perception of Mathematics as a barrier

Because of the mutually dependent relationship of a barrier to one's new having or being (Semadeni, 2015), we feel it would diminish the importance of the transgressive action if we were to discuss the barrier apart from the outcome. Therefore, we focus on the barrier, *Perception of Mathematics*, in conjunction with its crossing to a complex array of outcomes which generally revolve around meta-issues of mathematics.

A recurring phenomenon in the data was centred around how students' own perception of mathematics defined the scope of their mathematics learning experience. Students' learning experiences were bounded such that, to them, mathematics consisted of a small portion of the vast mathematical landscape. Specifically, students viewed calculus and geometry as two distinct domains with no connection. This concerned Dr. Leaf, who wrote in the implementation report:

> Students often see calculus and Euclidean geometry as two completely separate subjects that have nothing to do with each other beyond that one related rates problem that used similar triangles that one day. I think it is a crime to let them leave the calc[ulus] sequence believing this, and I can't imagine any learning object[ive] more effective than this particular PSP for correcting that misconception, showing how calculus was a method that very directly grew out of Euclidean geometry!

Here, Dr. Leaf demonstrated awareness of a barrier to student learning so endemic to his students that it became a specific learning objective for implementation of the Huygens PSP. Although Dr. Leaf listed other goals more closely related to mathematical content, such as understanding and calculating the radius of curvature, his primary goal was aligned with students understanding certain meta-issues of mathematics.

In what follows, we present the case of Simon, who previously perceived calculus and Euclidean geometry as being two separate mathematical domains. Additionally, Simon viewed calculus as being created by two "insanely intelligent" (i.e., Leibniz and Newton) men who had an "abstract thought that came out of the blue." We show that by Simon's own account of engaging with the PSP and through discourse with the instructor, Simon formed a connection from calculus to Euclidean geometry and now sees mathematics as much more realistic, beautiful, and possible.

Simon revealed his previous perception of mathematics in the following transcript excerpt. He hinted about why he thought calculus and geometry may have been two separate mathematical domains by alluding to Newton and Leibniz as the architects of calculus, implying that Simon may have regarded modern infinitesimal calculus as the authentic calculus. Additionally, Simon imbued the two mathematicians with exceptional qualities such as being "insanely intelligent" people who "randomly think" of great mathematical innovations. This conception of mathematical thought may have limited Simon in viewing others as thinkers, doers, and creators of mathematics, especially himself. In fact, Simon's reference to the previously perceived impossibility of such thought is suggestive of a lack of confidence to think the way Newton and Leibniz thought.

Simon: I remember even in class I said to Dr. Leaf, "that's really wild" because he mentioned before we started the PSP that Leibniz and Newton studied what Huygens was doing… it's just like, yeah, that's infinitesimal calculus. They're … we're creating tangent lines; we're doing this and that, and it was just like, "yep that's exactly it." It was the limit... where we had to actually create the limit. And, it was… one of those mind-blowing moments because … we get it; we're taking a limit, and we get how to do the algebra, but now, that's where that came from! Although Newton and Leibniz were insanely intelligent, … these guys didn't just randomly think of that; this is what they were staring at when they're like, "oh, this is how we're going to now create limits," and it makes it a lot more... realistic and a lot more beautiful.

Mark: How does it make it more realistic to you?

Simon: The way math is always presented is either some crazy calculations that people will magically just know, or it's some very weird abstract thinking that you're just like, "how would somebody think that abstractly, …what would you be thinking about to be able to create the limit as x goes to zero of something? How do you even establish that kind of thought process?" You think it seems impossible, and then you get something like this, and this is how they did it. It's not that it's impossible; it's just that these guys were insanely intelligent, and they… had something to reference so they had something they were trying to solve. They had a problem at hand. It wasn't just abstract thought that came out of the blue.

Not only did Simon have a narrow vision of who was capable of mathematical thought, but he was bound by his previous perception of how innovative mathematical thought occurred. Innovative thinking appeared distant to Simon and out of the scope of his own capabilities. Simon implied that he would never be able to think in such a way, a marker of a boundary (Kozielecki, 1986; Pieronkiewicz, 2015), as he expressed the theoretical impossibility of duplicating a similar thought

process. Simon's narrow perception of (who does) mathematics contrasts greatly with his expanded perception, presented as outcomes.

Outcomes: An attempt to capture the essence of a complex melange

Although we highlighted a clear barrier to Simon's relationship with mathematics via his own perception of mathematics, we found it difficult to clearly define the outcomes associated with the transgressive actions and barriers. Kozielecki (1986) alluded to this obstacle of describing outcomes characteristic of transgressive actions as "more complex and less well-defined" (p. 102) than those of other types of actions. Therefore, we contrast Simon's previous perception of mathematics to Simon's present thinking, which we situate as consisting of outcomes.

In Simon's present thinking, he perceives the possibility of innovative mathematical thought, and he attributed the knowledge of the context of mathematical innovation to be key in establishing the possibility of such thinking. Simon realises "these guys" were not producing random thoughts; instead, he sees the potential connection that Leibniz and Newton could have made while studying Huygens' writing, indicating a more evolutionary aspect of mathematics. Simon's revelation in reference to people "creat[ing] limits" supports this evolutionary perspective. We cannot be certain to whom Simon was referring, but the data suggest a resulting perspective which allows Simon to visualise mathematicians "staring at" and being influenced by Huygens' work. Additionally, Simon can now establish a purpose for innovative mathematical thought saying, "these guys… had something to reference" and acknowledging they were trying to solve a problem.

Simon's revelations of who is capable of mathematical thought and how mathematical innovation occurs is evidence that he was also able to make connections to previous calculus learning. Through his conversation with Dr. Leaf, Simon related what he was doing in the PSP to infinitesimal calculus, and specifically to calculus concepts including limits, tangent lines, and derivatives. Simon even indicated an instant of surprise with the "mind-blowing moment" that he realised what could be a potential origin of infinitesimal calculus. According to Kozielecki (1986), transgressive actions may "culminate in surprise" (p. 102), further supporting the presence of transgressive actions.

Transgressive actions in tandem

Kozielecki (1986) characterised transgressive actions as "every kind of purposeful action that leads to an outcome exceeding the boundaries of the individual's past achievements" (p. 89). In addition, he stated that transgressive actions may often be accompanied by positive emotions, complex outcomes, and surprise. With the evidence presented above, Simon surely engaged in some transgressive action. From his own words and that of the instructor, the actions which brought about the outcomes exceeding the barrier of Simon's own perception of calculus were two-fold. First, *Instructor/Expert Discourse* was necessary for Simon to expand his perception of calculus; without conversing with Dr. Leaf about Newton and Leibniz studying Huygens' work, Simon may not have made a connection from whom he considered the architects of infinitesimal calculus to those that influenced them. However, instructor discourse was not enough to expand Simon's perception of calculus. Simon also needed the transgressive action of *Engagement with the PSP* to add context to what was discussed with the instructor. This included not only reading through the PSP, but also working through the exercises to see primitive notions of limit and derivative comprised engagement with the materials. Although we view the transgressive actions as being distinct (each action appeared

elsewhere in the data without an obvious dependence on another action), we feel the combination of the two was instrumental in assisting Simon over the barrier of his own perception of calculus.

Discussion and future directions

Initial analyses of survey data collected as part of the TRIUMPHS project were focused on students' identification of benefits and obstacles to learning mathematics via PSPs. This initial work fell short of any true explanatory power of what PSPs offered about the learning experience using such an innovation. We found that the theoretical lens of transgressions provided a tool to understand students' experiences with using PSPs in undergraduate mathematics classrooms; moreover, the framework allowed us to link barriers with outcomes through the transgressive action medium, an enlightening relationship we had yet to discover. However, employing this lens with our existing data at the time (survey items that prompted students to discuss benefits and obstacles) was inadequate. A pilot of the interview protocol in Spring 2021 enabled us to explore transgressive actions, outcomes, and barriers, and the current study presents excerpts from our implementation of the refined interview protocol within a content course (Calc III-EA) as opposed to a history course. The transgressions framework provided a valuable lens to describe the link from student barriers to outcomes through transgressive actions, yet these actions remain vague possibly due to the anecdotal nature of interview responses. Therefore, our next step is to more deeply explore transgressive actions through additional in-the-moment data via classroom observations along with other data sources. Our goal is to discover nuances of these transgressive actions which help move students beyond a barrier.

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TWG 13: Early Years Mathematics

Introduction to TWG13: Early years mathematics

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Introduction

In CERME14, presented research about Early Years Mathematics has concerned the mathematics education of children 1 year old (e.g. see the contribution by Björklund & Palmér) up to 7 years old (e.g. see the contribution by Benz et al.). Researching mathematics education during the early years is a complex endeavour and depends on several cultural and institutional constraints. In many European countries, children attend early childhood education, yet the conception of schooling and mathematics in early years varies significantly across different countries as it is testified by the 22 contributions in our group, presented by authors from 14 different countries in the European area (Czech Republic, Denmark. Germany, Greece, Israel, Italy, Latvia, Norway, Portugal, Spain, Sweden, UK) and outside of Europe (Australia, Jamaica, South Africa). Furthermore, transition from childcare to more formal schooling may differ across countries. In TWG13, contributions pay attention to how mathematics is approached more or less explicitly by children, teachers, and other caregivers. The studies within this working group show that mathematical learning occurs in various settings, including kindergarten, preschool, primary school, as in informal environments.

Overview

During CERME14, the 22 contributions (20 research reports and 2 posters) published in the proceedings were presented in seven thematic sessions and were discussed between the participants. A wide range of topics was covered: different areas of mathematical teaching-learning are addressed, and different theoretical and methodological approaches are adopted. In the following, we summarize the wide variety of topics addressed in TWG13 by clustering the contributions presented in CERME14 proceedings.

Numbers and structure

This section examines how young children develop structural awareness in early mathematics, particularly in relation to number concepts. Lindgren and colleagues explore part-whole relationships through a playful distribution task, demonstrating how artefacts become mathematically meaningful when structured with verbal communication, gestures, and finger representations. Sprenger and

colleagues investigate how children describe quantities using ten-frames, identifying four distinct descriptive strategies and highlighting the role of peer interaction in shaping mathematical language and understanding. Elofsson and colleagues focus on children's reasoning with repeating patterns, showing how systematic variation in unit size and repetition—carefully guided by teachers—supports generalization and the development of early multiplicative thinking. Their work underscores the importance of strong pedagogical content knowledge. Together, these studies provide complementary perspectives on structural thinking in early mathematics—through number decomposition and recomposition, visual structuring and peer interaction, and pattern-based generalization—emphasizing the richness of young children's mathematical experiences when instruction is intentional, well-designed, and responsive to their ways of seeing.

Additive problems

Research about additive problems highlighted that tools, gestures, and language support early mathematics learning, hence children often struggle to apply part-whole concepts to arithmetic. Young children verbalise their reasoning inconsistently, and research is needed to understand their cognitive processes. In a study conducted in Germany, Benz and colleagues explored how word problem features—like structure, unknown elements, and visual representations—affect how young children solve addition and subtraction tasks in an online assessment. The results suggest that spatial reasoning consistently emerges as a key factor in problem-solving.

Seo, with colleagues, examined whether spatial abilities—specifically spatial visualisation and visuospatial working memory—help predict how well six- to seven-year-olds in England solve one-step addition word problems. Findings suggest that spatial visualisation plays a role in tackling more complex, dynamic problem types, highlighting a spatial-mathematical link in early arithmetic learning.

Roesch and Hartmann presented a study about children's use of their fingers for addition and subtraction tasks. Although, the sample was small, the results showed that finger use was more common when manipulatives were not provided and particularly beneficial for subtraction tasks.

Multiplicative problems

All studies highlight the role of language, manipulatives, and structured activities in supporting early multiplicative thinking. They collectively suggest that multiplicative reasoning can emerge before formal instruction, but is shaped by age, context, and language. These findings have implications for curriculum design and teacher training in early mathematics education.

Pitta and colleagues demonstrate that structured interventions can support children in kindergarten in transferring multiplicative strategies to novel contexts. Björklund and Palmér identify developmental shifts in toddlers' partitioning strategies, highlighting the emergence of unitizing and equal distribution. Downton and colleagues reveal that linguistic background significantly influences children's interpretation of multiplicative comparison tasks.

Mathematical learning within child and care-givers interactions

Within the 'Early years mathematics' TWG, there are contributions paying attention to how mathematics is approached more or less explicitly by children in informal contexts and in relation with their parents and other caregivers. Graven reports about parents' confidence within the Family Maths Storytime Programme (FMSP), designed for providing opportunities for play-based home numeracy learning among South African children. Voutsina and colleagues present the 'Written Numbers in Everyday Life' project, implemented with families of 3–5-year-old children in the UK and Jamaica. Both contributions provide insights into the wide-ranging opportunities for learning that arise when parents are prompted and supported.

Learning difficulties and disabilities

Research under this theme underlines the importance of family-school partnerships for supporting the early mathematics development of children with intellectual disabilities and specific learning difficulties.

Charitaki and colleagues describe profiles of home numeracy support for preschool children with intellectual disabilities in Greece and underscore the importance of communication and reciprocal support between parents and educators, particularly for families who face systemic barriers that can affect the level of home numeracy support that they can offer. The importance of parent-teacher collaboration is also emphasised by Helmane and colleagues who reveal variations (both similarities and discrepancies) in parents' and teachers' evaluations of the mathematics skills of 4–6-year-old children with and without specific learning difficulties, in Latvia.

The role of play

Two studies, conducted in the context of preschool education in Sweden, contribute to current debates about the role of play in mathematics learning in the preschool years.

Lundvin and colleagues analyse preschool teachers' talk about play and mathematics teaching through the lens of Play-Responsive Teaching theory (PRT). Discrepancies between teachers' talk and PRT, but also teacher's alignment with the position that play and teaching are 'going into one another', emphasise the need for developing shared understandings in relation to play and mathematics teaching within preschool education practice and policy. Comparing instructional sequences that integrate play and mathematics with sequences where mathematics teaching is separated from play, Karlsson highlights the benefits of combining both approaches, for supporting children's exploratory as well as structured learning of mathematics, within a balanced learning environment.

Digital tools and technology

Three papers directed specific attention to digital tools and technology in early mathematics education. Walla and colleagues presented a study on guardians' views on digital tools in Swedish preschools, while Vee and colleagues presented a study on Norwegian preservice teachers' views on the use of digital technology in early childhood institutions. Both of these presentations initiated a discussion on observed differences between countries regarding if and how to use digital tools and

technology and whether and how digital competence is part of mathematics curricula in early childhood education.

Nogueira and colleagues added to the discussion their study on robot programming, concluding that robot programming activities can be used in early childhood education to develop children's itineraries and map-reading skills through activities that not only are highly motivating but also stimulate both cognitive and social skills.

All three papers were further discussed in terms of the risk of digital tools becoming "just play"; raising the question how teachers are supported to use digital tools and technology for mathematics education. A conclusion was that more research is needed and pre- and in-service preschool teachers need professional development regarding digital tools and technology, to be prepared to meet parents, pre-service teachers and children.

Other topics

The topics mentioned above do not comprehend all the numerous issues addressed within the group. Other relevant examples are given by contributions about first graders' mathematical creativity described by Cohen and Levenson in terms of fluency, flexibility, and originality, and younger children's argumentation by Nosrati and colleagues who investigated the factors that may aid or hinder the development of an argument. TWG13 also included contributions about kindergarten teachers, such as their mathematical awareness investigated by Bundgaard, and teachers' level of professional readiness as investigated by Nováková.

Conclusion and future perspectives

The collection of studies presented in TWG13 during CERME14 emphasizes how young children develop mathematical understanding through structured, playful, linguistic and socially rich experiences.

In the last editions of CERME we have registered an increasing number of studies about parental involvement. Home numeracy environments appear to play a relevant role, especially (but not only) for children with learning difficulties. The role of the cultural and linguistic background as embedded in the home, kindergarten, and school environments should remain central in future research about the development of structural thinking, spatial reasoning, and initial arithmetical learning.

The discussion about play-based learning-teaching is ongoing and more theoretical and empirical contributions might fuel it in the future. Digital tools show promise but require clear educational aims and teacher preparation. Future research should explore how to better support teachers and families, integrate play and digital tools effectively, and tailor learning for diverse developmental needs.

Performance on addition and subtraction word problems in an online assessment

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Word problems operationalize different semantic characteristics of addition and subtraction. Often, they are used to assess knowledge of addition and subtraction before written numerals and operation signs are introduced. Earlier research shows that the semantic structure, the unknown set and the use of iconic representations of quantities influence children's performance. To address different requirements of difficulties in an online assessment of children starting school concerning word problems, the problems were varied due to different criteria. However, considering the solution frequency of different problem types, our findings are not completely in line with earlier results. This leads to open questions.

Keywords: Early mathematics education, word problems, arithmetic, assessment.

Theoretical and empirical background

Different addition and subtraction word problems

Informal methods to solve simple arithmetical (word) problems by direct modeling with objects such as counters or fingers constitute an important aspect of children's mathematical competencies before formal schooling. These competencies are seen as "the best predictors of mathematical disabilities/difficulties and later achievement in […] mathematics" (Clements et al., 2020, p. 638). For this reason, in assessments addition and subtraction word problems often are used to assess children's knowledge of addition and subtraction before written numerals and operation signs are introduced. Moreover, word problems operationalize different semantic characteristics of addition and subtraction as operations. This means, different one-step addition and subtraction word problems are also used to assess which semantic characteristics can be understood and solved by children.

Riley et al. (1983) and Carpenter and Moser (1984), distinguish four semantic structures or characteristics of problem types (*change, combine, compare, equalize*) for addition and subtraction. In that vein, two aspects are discerned: The first aspect distinguishes whether the situation is static or dynamic. The second aspect focuses on whether a set is described in a part-part-whole-relation or two disjunct sets are described.

Table 1: Different semantic structures of word problems

	dynamic	static
Part-part-whole	***change***	***combine***
Two disjunct sets	***equalize***	***compare***

Some authors combine the two dynamic situations and, thus, distinguish only three types: combination, change or comparison of sets (Stern 1993, p.7). Furthermore, Clements et al. (2020)

distinguish four types: they merge *change* and *equalize* and they discern change problems into *add to* and *take from* problems.

In addition to the semantic structure of word problems, a further distinction can be made (Verschaffel, 2020; Gabler & Ufer, 2021): additive and subtractive wording. Examples for additive wording in dynamic situations are *she gets* or *she buys*. Examples for subtractive wording in dynamic situations are *she gives away*, *she lost*, *she sells* – the wording describes a dynamic increase or decrease. In *static* situations, the wording does not describe an increase or decrease. In static compare situations, additive and subtractive wording can be expressed by using the terms *more than* and *less than*. In combine situations there are sometimes hints that focus on the whole (*altogether)* or on one part (*out of, of them, of which*). In literature, however, these hints are often not referred to as additive and subtractive wording (Radatz et al., 1996; Gabler & Ufer, 2021). In each of these word problems, three quantities play different roles, any one of which could be the unknown quantity. In dynamic situations the quantities of start, change, and result can be unknown. In compare problems the reference, difference, or compare quantity can be unknown (e.g., Stern, 1993). In combine problems one part or the whole can be unknown. Four different dimensions or "organizing criteria" are therefore used to categorize word problems. The combination of these four dimensions results in a total of 20 possible situation types (Table 2). Table 2 is the extension from two dimensions in Table 1 to four dimensions (Radatz et al., 1996). The four dimensions are:

- Does the situation involve a change (dynamic) or a status (static) (see, two right columns in Table 2)?
- Does the situation involve a part-part-whole relationship or two independent, disjunct sets (see two wide rows in table 2)?
- In which direction does a change or comparison take place? Is the change increasing or decreasing, is a comparison regarding more or less required (see fields marked in light grey, Table 2)? What is given, what is unknown? In other words: Which of the three quantities can be determined by the other two (see fields marked in dark grey, Table 2)?

Table 2: Different problem-types due to semantic structure, wording and unknown quantity

		dynamic						static					
part-part-whole		***change***						***combine***					
	additive/subtractive wording	increase			decrease								
	unknown part	st	ch	res	st	ch	res	whole			part		
two disjunct sets		***equalize***						***compare***					
	additive/subtractive wording	increase			decrease			more			less		
	unknown part	st	ch	res	st	ch	res	ref	dif	co	ref	dif	co

st=start, ch=change, res=result, ref=reference, dif=difference, co=compare

Influence of problem-types on performance

Surveys revealed that the *semantic structure* influences children's performance (Stern, 2009; Riley et al., 1983; Clements et al., 2020). For example, all studies indicated that young children had greater difficulties solving compare tasks (especially when a relation between quantities is described) than solving dynamic tasks like change or equalize tasks. Stern (2009) explained these difficulties (that are independent to the linguistic challenges) by the (missing) idea of a relational understanding of quantities: If the given information does not describe a concrete existing quantity, but the relation between two quantities, the children must activate "a mental image abstracted from concrete things" (Stern, 2009, p. 155; translation by the authors). However, Gabler and Ufer (2021) could not confirm these results for second graders at primary school (average age 7,56 years).

Besides the influence of the semantic structure, studies also revealed that the *unknown quantity* in word problems influences children's performance. In this vein, it was shown that word problems with an unknown reference or start set are more challenging than those with an unknown compare or result set (Riley & Greeno, 1988; Stern, 1993; Gabler & Ufer, 2021). Clements and colleagues describe "Find Results Problems" with an unknown result in their learning trajectory two levels before the level of "Find Change Problems" (Clements et al., 2020, p. 640).

Another factor influencing the difficulty of word problems is the use of *additive or subtractive wording*, especially in combination with the aspect of the unknown quantity: "Word problems in which the directly applicable mathematical operation is inconsistent with the wording are usually harder than consistent word problems" (Ufer & Gabler, 2024, p. 3; Fuson et al., 1996; Lewis & Mayer, 1987). Examples for inconsistent word problems are: "Julie had some cookies. Then she gave 3 cookies away, now she has 2 cookies. How many cookies did she have in the beginning?" (subtractive wording "she gave away", required operations is addition 3+2=5); "Julie has 7 cookies, that is 3 cookies more than Jane has. How many cookies has Jane?" (additive wording "has more than", required operations is subtraction 7-3=4). In Stern's study (1993), the word problems with an unknown reference and an inconsistent formulation were the most difficult.

Also, other aspects, like number range or consistent or inconsistent presentation of quantities can influence children's performance (Fuson et al., 1996). An overview about research concerning different aspects can be found in Daroczy et al. (2015).

Influence of iconic representations on performance

Theoretical considerations and empirical findings indicate that the use of iconic representations for given quantities in oral text problems influences the solution frequency. This is because children do not need to know the quantity only by the number word, they also can see the number of objects corresponding to the number words. This has been confirmed for diagrams (Diezmann & English, 2001), visual presentations of divisions problems (Matalliotaki, 2012) and compare problems in the number range up to 20 (Múñez et al., 2013). Nevertheless, studies investigating performance on written text-problems (especially change problems) also revealed that iconic representation is not always helpful (Van Lieshout & Xenidou-Dervou, 2018; Elia et al., 2007), for example when the whole situation of a *dynamic* problem is represented in one or several pictures. The reported studies

were mostly conducted with children who already received mathematics teaching at school. Thus, it is not clear, whether these results can be transferred to early mathematics education in the absence of formal schooling. However, in an exploratory study in kindergarten, Elia (2020) investigated, the influence of visual representations of quantities in change, combine and equalize problems. The tasks were distinguished in three groups: Tasks with visual representations of two quantities (groups of illustrated objects), tasks with visual representations of one quantity and tasks only with numerals representing the quantity. For the change problems an increase and decrease problem with unknown result was chosen. Moreover, or the combine problems the unknown quantity was once the part and once the whole. Children's performance in problems with two iconic representations was higher for combine and compare problems than for equalize problems. Also, children's performance with two iconic representations was higher in the subtraction combine problems (with the unknown part) than in the subtraction change problems (Elia 2020, p 26).

Despite different results from all studies about representations, it can be concluded: The role of representation seems to influence children's performance on addition and subtraction word problems.

Study

In an online assessment of children's arithmetical performance at the beginning of primary school, the role of situation types and the use of iconic representations of quantities on situation types was varied. The following research questions were raised concerning the word problems:

1. Does the frequency of solving addition and subtraction tasks depend on the situation type?
   Hypothesis: Our findings will confirm the findings of previous studies (e.g. change tasks and combine tasks with unknown results were solved more frequently than compare tasks).
2. To what extent does the use of iconic representations of quantities influence the frequency of solutions within a situation type?
   Hypothesis: The more iconic representations of quantities are given, the higher the correct results.

To standardize software-supported assessment procedure (ILeA Plus), about 2900 children participated in a computerized online interview during the first six weeks of the first school year (age range 5 to 7 years). For this analysis of word problems only children who solved all word problems were considered (N=2275), so that a quantitative comparison over all word problems is possible without missing data. The goal of the online assessment (ILeA Plus) is to assess children's mathematical competencies at the beginning of school. This is done not for labeling children's competencies, but for indicating in which area children should be assessed more differentiated and probably need individual support. As part of the software-supported assessment, numerous aspects of mathematical competencies of each child were documented.

Eleven word problems with different semantic structures, different use of iconic representations of quantities and different unknown quantities were presented acoustically (see table 3, for examples see figure 1). When a new task appeared, the acoustic task was presented automatically. All children could listen to the tasks as often as they wanted by clicking on the speaker symbol. The results of word problems could be selected by clicking on a number field. Only children who showed

knowledge of numerals were considered in the analysis. The number range of all word problems was within the number range of six, to minimize the influence of numbers. At least two tasks per situation type were given. Tasks were constructed for the sematic structures *change combine, compare* (Stern, 1993). We also differentiated for the change problems between *increase* and *decrease* and if the result or change is unknown (Clements et al., 2020, see table 3).

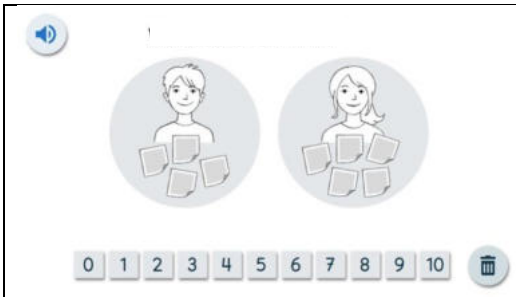
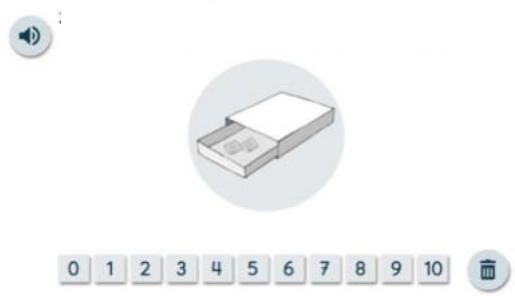
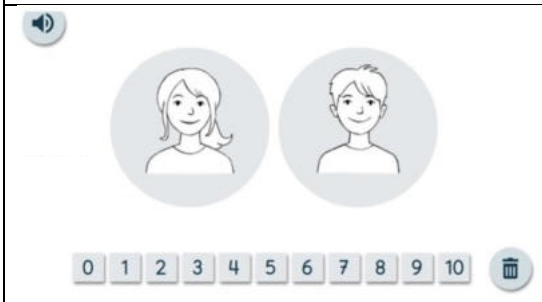
	*Compare*, difference unknown, two numbers represented as quantity Acoustic task: "Ben has 4 stickers. Isa has 5 stickers. How many more stickers has Lisa?" [Wie viele hat Lisa mehr?]
	*Change*, increase, change unknown, one number represented as quantity Acoustic task: "2 stickers are in the box. 5 stickers shall be in the box. How many stickers do you have to add?" [Wie viele Sticker musst du noch dazulegen?]
	*Combine*, whole, no number represented as quantity Acoustic task: "Carla has 2 Stickers. Tom has 4 Stickers. How many stickers do they have together? [Wie viele Sticker haben sie zusammen?]"

Figure 1 Task presentation in the online assessment (Benz et al., 2019)

Results

Tasks of the type *change with unknown result* and *combine unknown whole* were solved more frequently (solution rate more than 50%) than tasks of the type *change with unknown change* and the *compare with unknown difference* (solution rate less than 50%). These two groups of tasks were compared statistically (paired samples t-Test). The difference in students' performance is significant ($t(2274)=5.248$, $p<0.001$) between the group of tasks *change with unknown results* / *combine unknown whole (*M=.63, SD.29) and the group of tasks *change with unknown change* / *compare with unknown difference* (M=.30, SD.39).

At first glance this result is in line with earlier studies, because the tasks *change with unknown result*s / *combine unknown whole* were solved most frequently. However, considering the group of tasks *change with unknown change* / *compare with unknown difference,* the results contradict the results of Stern (2009). Here compare tasks with unknown difference were scored as more difficult tasks than change tasks with unknown change.

The following analysis concerning iconic representations of quantities is based on frequencies of only one item per situation type and representation level (the limitation will be discussed). Therefore, only a descriptive analysis of frequencies was performed. For change tasks with unknown result and combine tasks with unknown whole, the solution frequency was higher the more numbers were represented as quantities. However, the use of iconic representations of numbers in compare tasks with unknown difference and change tasks with unknown change did not lead to a consistent increase of the solution frequency.

Table 3: Frequency of correct solutions

Representation of quantities	*Change* increase, result unknown	*Change* decrease, result unknown	*Change* increase, change unknown	*Combine* whole	*Compare* more, difference unknown
2	72%		40%		28%
1		75%	18%	62%	
0	60%	56 %	29%	53%	35%

Discussion

The first hypothesis can essentially be confirmed. Nevertheless, it is surprising that *change tasks with unknown change* were solved less frequently (except the change tasks with two representations) than *compare tasks with unknown difference.* What aligns with findings is that change tasks are solved correctly more often than comparison tasks. Stern's (1993) study in particular, in which these two types of tasks are directly compared, is repeatedly cited as a reference here. One explanation for this apparent contradiction may be that these two types of tasks have rarely been directly compared with each other in the recent past (for example, Elia (2020) only used change task with unknown result). However, in the learning trajectory of Clements et al. (2020), within the level called “Find change” also compare tasks with unknown difference are mentioned. One possible conclusion from these considerations could be that different word problems should be compared again and, above all, systematically in preschool institutions. Such studies could then examine whether Stern's findings can be confirmed or whether children's understanding of word problems has changed in the past 30 years.

The presence of two iconic representations of numbers does not seem to be helpful for the compare situations and one iconic representation does not seem to be helpful for the change task with unknown change. One possible explanation could be that the unknown quantity cannot be identified easily in the picture – it even leads to wrong solutions (for example, when two numbers were presented as quantities, the presented objects were added).

The second hypothesis could not be completely confirmed. The question remains as to what role iconic representations of numbers in word problems play in performance. Also, in this context the

question remains, what differences can be observed, if the word problems are introduced as enactive representations with concrete objects and actions. However, it became clear that the role of iconic representation cannot be unambiguously clarified. Further research should systematically consider the different semantic types as well as whether the unknown number can be seen in the iconic representation.

Because the items in the online diagnosis were originally not constructed to investigate differences in solution frequencies between task types and representation levels, the analysis shows limitations with only one item per situation type and representation level. Thus, a controlled study is needed in which the unknown part of the problems is considered. Further research questions can be raised, like how the chosen context of word problems or how a presentation of numbers in word problems with concrete objects influences the solution frequency. Finally, another question which also remains is to what extent the digital setting did influence the solution frequencies.

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Cookies on plates – toddlers' partitioning strategies

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This paper presents a study of toddlers (1- to 3-year-olds) development of number knowledge in general and partitioning strategies in particular. We investigate changes in toddlers' ways of experiencing numbers as expressed through actions and utterances when encountering the same task over a prolonged period of time. In this paper, we present the partitioning strategies of these toddlers when engaging in a preschool game, "sharing cookies with Kitty's friends". We identify the strategies used and the meaning of numbers these strategies entail. Further, we present an overview of their progression in number knowledge. The results add to the current knowledge of what it means to develop number knowledge in general and partitioning strategies in particular.

Keywords: Partitioning, toddlers, number knowledge, units, equal distribution.

Background and aim

In situations like play and everyday tasks such as setting the table or sharing objects with peers, young children engage in partitioning activities using different strategies (Björklund, 2010). Partitioning could, in this sense, be considered an intuitive action to solve everyday problems. Partitioning (or "splitting", as Confrey, 1994, terms it) is also known to be a basic feature of mathematical knowledge – one that is foundational for conceptual knowledge of numbers and arithmetic skills, including multiplication, proportional reasoning, and exponentials. Research that focuses on partitioning most often refers to the mathematical content of division and mainly includes school children, even though partitioning is also an aspect of additive tasks. Studies with younger children often emphasize unitizing abilities and are based on theoretical assumptions about how development towards more advanced arithmetical reasoning occurs. Therefore, we consider that there is a need for empirical studies on emerging understanding of partitioning among the youngest learners.

In the project DUTTA, for two years we followed toddlers' (1- to 3-year-olds) ways of reasoning about numerical situations. Play-based interviews were conducted with the toddlers on five occasions. One task in the interview focused on partitioning. In this paper we present the partitioning strategies expressed by the toddlers. The strategies are understood as expressions of the toddlers' ways of experiencing number, in line with the variation theory of learning (Marton, 2015). The research questions are: RQ1) What meanings of numbers are expressed in the strategies enacted by the toddlers in a partitioning task? RQ2) What strategies for partitioning numbers are observed among toddlers over a prolonged period of time?

Conceptual framework and research on partitioning

Partitioning is an aspect of number knowledge that is foundational for multiplicative reasoning and understanding of proportions and fractions. According to Lamon (1996), "Partitioning is an operation that generates quantity; it is an experience-based, intuitive activity that anchors the process of

constructing rational numbers to a child's informal knowledge about fair sharing" (p. 171). Partitioning is, however, a complex skill that involves not only fair sharing but both additive and multiplicative relations. It includes the action of splitting a set of items into smaller units of equal or unequal size (emphasizing an additive part-whole relation), and it also includes switching the focus from individual items to groups of items as composed units that can be compared, or making a one-to-one correspondence with other units, such as when partitioning cookies on a number of plates to make a fair share (equal-sized groups in a multiplicative relation). When partitioning, or exploring different ways to partition a set, a flexible understanding of "unit" is essential because, depending on the number of parts needed, the set sizes (units) will differ. For example, the number six can be split (in an additive sense) into two and four or five and one, but also (in a multiplicative sense) into two threes as well as three twos. Consequently, how children encounter a partitioning task depends on their way of understanding the idea of "unit" and the part-whole relationship of numbers. In agreement with the above-described complexity, Hunting (2003) found in a study of partitioning that children's ability to change focus from counting individual items to identifying the structure of a group was fundamental to the development of their number knowledge.

As partitioning is considered foundational for more advanced mathematical areas, efforts have been made to conceptualize the knowledge that partitioning constitutes. In a study on young students' perceptions and representations of structure, Mulligan et al. (2006) found a high positive correlation between children's structural development (pre-structural, emergent, partial, and structural) and the use of multiplicative structures including unitizing and partitioning.

In a study by Lamon (1996), the partitioning strategies of students (grades 4–8) showed that, at each grade level, a greater percentage of students used more effective partitioning strategies, shifting from the distribution of single units towards the use of composite units. It was also found that the students' strategies were more influenced by the commodity being shared than the numerical portion. Kim (2012) investigated partitioning related to equal sharing situations presented in contextual or symbolic forms of division. The children were found to develop partitioning strategies stepwise with two pre-stages of partitioning for equal sharing: 1) an unequal sharing strategy where children exhaust the whole shared quantity but distribute unequal quantities; and 2) a sorting-out strategy where children distribute equal amounts of shared objects to parts but do not exhaust the whole shared quantity.

To summarize, research on partitioning and structural reasoning provides many examples of strategies, and some studies try to conceptualize learning trajectories (e.g., Kim, 2012) based on empirical observations. There are also studies aiming to describe the conceptual growth behind the development of partitioning strategies, but these concern primary school children (e.g., Castro-Rodríguez et al., 2022; Hurst & Huntley, 2020), leaving a significant gap in the field of knowledge regarding the emerging knowledge of numbers and the idea of partitioning among the youngest learners.

Theoretical framework

The study is based on the variation theory of learning (Marton, 2015), which provides analytic tools to interpret knowledge as different ways of experiencing numbers. According to variation theory, learning the meaning of numbers means discerning necessary aspects of numbers, such as cardinality,

ordinality, part-whole relations, and representations (Björklund et al., 2021). The more aspects discerned, the more advanced the way of experiencing numbers, allowing for more proficient strategies and the use of numbers and numerical features in problem-solving activities. The way a child handles a task is thereby interpreted as an expression of his or her ways of experiencing essential features of the task, which is explained in terms of discerned and undiscerned aspects.

Method

The sample for this paper is 17 toddlers for whom we have interview data from three interview occasions when they were 1, 2, and 3 years old. For the interview to make sense to the toddlers, each interview task was embedded in a meaningful activity or a game. Manipulatives were used partly to engage the toddlers' interest and partly to make possible different kinds of expressions, for example, gestures and actions (Aubrey & Dahl, 2005). Further, the tasks were based on the theoretical framework of variation theory (see above), whereby the use of variation-invariance made it possible to act upon different aspects of numbers in different tasks.

One task was to set the table for Kitty the cat and her friends who are coming to her birthday party. This task is a kind of "distribution problem" where objects (cookies) are to be distributed to a number of recipients (plates). There were always 12 cookies, but as new friends arrived, the number of plates varied. The task included five steps of increasing number of plates. At step 1, there were two plates; at step 2, three plates; at step 3, four plates; at step 4, six plates; and at step 5, six plates in a different setup (Figure 1). For each step, the interviewer first arranged the plates according to the pictures in Figure 1. After that, the 12 identical cookies were placed in front of the child in a pile. Then, the interviewer said, "Can you help Kitty put the food on the plates?" The children were not told to divide the cookies, for example, to distribute them evenly among the plates. Instead, the intention with the task was to explore whether the children would interpret this as a mathematical task, and if so, whether they could discern some structure of numbers and, for example, try to partition the cookies evenly among the plates. The validity of this task is elaborated on in Palmér and Björklund (2024).

The cookies were circular, made of wood, and approximately three centimetres in diameter. The plates were 15 cm in diameter made of plastic.

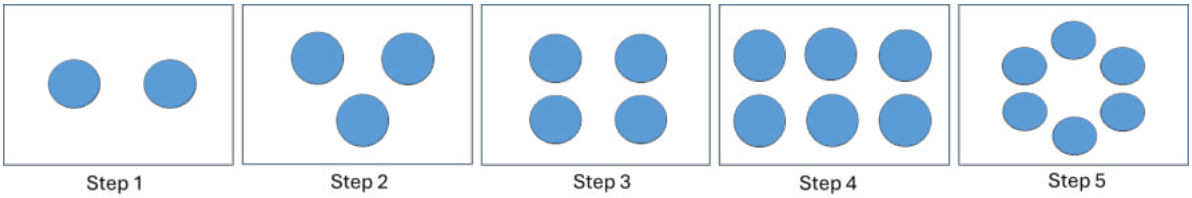

Figure 1: The setting of the plates in the interview task

Toddlers' expressions can be very subtle and demand exclusive knowledge of the individual child's ways of expressing him/herself. Therefore, we involved the toddlers' preschool teachers in conducting the interviews. The preschool teachers were given training in how to conduct interviews with a research purpose. Based on their knowledge of the toddlers, they could be sensitive to the toddlers' responses in the situation (Palmér & Björklund, 2022). The interviews were conducted in a room at the preschools, with one child attending each time. All interviews were video recorded, with

the written consent of the toddlers' legal guardians. All interviews were then coded in NVivo. The analysis of the video recordings of the interviews was inductive, where seven different strategies were identified and then categorized based on the mathematical aspect foregrounded. According to variation theory, a child acts in accordance with how he or she experiences a phenomenon, which opens up for analysing children's strategies as expressions of their knowledge and, in particular, what aspects they discern in a specific situation that guide their actions.

To answer the research questions, we produced an overview of the strategies enacted when the children were 1, 2, and 3 years old. This gives a comprehensive view of the progress on a collective level. To illustrate children's ways of experiencing numbers and numerical features of partitioning, we present one child's progress of partitioning strategies, discussing what it entails for her learning partitioning.

Results

Sometimes, the children were satisfied playing with the toy cat or feeding the cat with cookies. These observations were coded as not attending to any numerical feature of the task and are not included in this overview. We found six strategies for partitioning numbers used by the toddlers. Even though differences in performance can be observed, three different ways of experiencing the numerical feature were identified (see Table 1).

Table 1: Identified strategies and the mathematical aspects foregrounded, PS = Partitioning Strategy

Code	Strategies	Aspects foregrounded
PS1	a) Some cookies on some of the plates	Cookies as quantities (not relating to the fixed set)
	b) All cookies on one plate	
	c) Cookies on some of the plates	
PS2	a) One or more cookie on each plate	Unequal units relating to the fixed set
	b) Approximate equal distribution	
PS3	a) Starts systematic partitioning but loses track on the way	Equal units relating to the fixed set
	b) Systematically partitioning equally onto each plate	

Children who put some of the 12 cookies on some of the plates or all cookies on one plate were interpreted as engaging in completing the task and to some extent attending to numerical features because they were *experiencing cookies as quantities (PS1).* That is, the toddlers were composing units of different kinds, seeing it as possible to combine the single cookies into composed units. However, these units varied in number and did not relate to the fixed set of plates.

Other toddlers were foregrounding the set of plates as an important part of the task and partitioning the 12 cookies onto the plates, although as units of unequal size. A significant difference from PS1 is that these children were *experiencing unequal units relating to the fixed set (PS2)*.

Some of the toddlers made efforts to connect the fixed number of plates to equal-sized sets of cookies. This way of experiencing the numerical feature of the partitioning task was expressed in the toddlers' attempts to partition the cookies in a systematic manner. The toddlers who acted, for example, by putting the same number of cookies onto each plate, were expressing an awareness of units (groups

of cookies) to be connected to other units (the plates). The number of cookies, however, was large (not possible to perceive in an instant), and the toddlers may have made several attempts by trial and error. Nevertheless, this qualitatively distinct way of acting in the partitioning of the cookies is interpreted as the toddler *experiencing equal units relating to the fixed set (PS3).*

The above-described qualitatively different ways of experiencing the numerical features of the partitioning task emerged from the empirical observations of the toddlers' enactment in the interview. The progression in terms of observed strategy use is shown in Figure 2. Among the 1-year-olds, there are only a few observations of toddlers attending to the plates as a significant feature of the task and acting by putting cookies on the plates unevenly distributed. Among the 2-year-olds, we observe a growing awareness of units (cookies) that are to be connected to another kind of unit (plates), both evenly and unevenly distributed. It is among the 3-year-olds where we see a strong tendency among the toddlers to also discern the equal size of the units as a feature to account for in their partitioning.

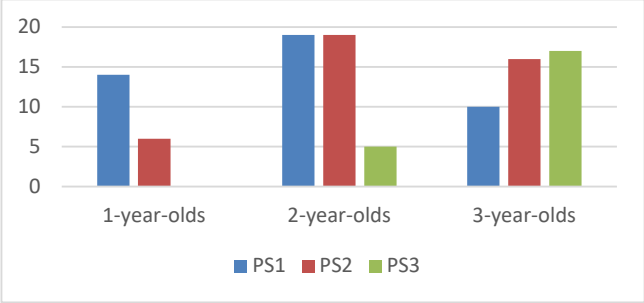

Figure 2: Number of observations across different partitioning strategies at different ages[1]

The progression of the toddler Ester

The toddler Ester has been chosen as a typical example of the progression in partitioning strategies observed among the toddlers in our study.

Table 2 shows that, as a 1-year-old, she makes some attempts to put cookies on the plates. Her strategies include putting only a few cookies on some of the plates and sometimes putting cookies on all of the plates (Excerpt 1).

Table 2: Observed partitioning strategies when Ester was completing the task

	PS1	PS2	PS3
1 year old	x	x	
2 years old	x	x	
3 years old		x	x

Excerpt 1

Ester has two plates in front of her and all 12 cookies on the right side of the plates. She grabs one cookie at a time, putting four on the right plate, then two at a time, and continues one at a time until three are left on the table. Ester then takes one cookie in each hand and moves them to the

[1] The number of observations differs between interview occasions because of varying number of steps that the toddlers were presented with and responded to.

other plate. She moves them back again to the right plate and adds the remaining three cookies to the plate with the other cookies. Ester moves three cookies to the left plate and repeats moving cookies back and forth a few times.

In the excerpt, Ester is fully engaged in the partitioning of cookies onto the plates. However, there are no observations of her trying to be systematic in partitioning the cookies, and therefore our interpretation is that any relationship between the 12 cookies and the number of plates is not discerned; rather, she composes and decomposes units made of the cookies. Her way of experiencing numbers is regarded as seeing the cookies as quantities but these are not relating to the fixed set of plates (PS1).

At 2 years old, Ester seems to attend to number as she more often puts cookies onto each plate (Excerpt 2). This indicates her awareness of some relation between the cookies (as units) that are to be related to the fixed set of plates (PS2). No cookies are left on the table; all of them are used in the partitioning act.

Excerpt 2

Ester has three plates in front of her and 12 cookies to her left. She puts four cookies, one at a time, on the left plate, the rest of the eight cookies on the right plate. Ester suddenly stops, looking at the third (middle) plate for a few seconds, then moves three cookies from the right plate to the empty one, thereafter looking satisfied.

Ester's actions and reactions when completing the task show her intention to compose units of the cookies and these are to be connected to a plate. However, her surprised look when all the cookies have been used to compose units on the two plates while one plate remains empty can be interpreted as indicating that the number of plates is not experienced as a composed unit to which the cookies are to be related.

In the last interview, being 3 years old, Ester acts in more systematic ways to partition the cookies on the fixed set of plates (Excerpt 3).

Excerpt 3

Ester has six plates in an array (level 4) in front of her and 12 cookies to her right. She rapidly puts one cookie at a time onto each of the six plates, then says "no more" and takes all the rest of the cookies in her hands "No more cats; we save the rest."

Ester seems to focus on partitioning one cookie to each cat or plate but without discerning how cookies can be composed into units that in turn can be partitioned as units-to-units. This means she does not yet experience the cookies as something to be composed into units, to be related to other units (plates) in a relationship that may be characterized as multiplicative (units made up of a smaller number of objects are to be connected to a unit of different objects). This indicates that Ester experiences equal units relating to a fixed set (PS3) but does not yet discern the full complexity of this part-whole relationship.

The difference between a child's ways of partitioning the cookies to the varying number of plates can be interpreted as different aspects being discerned; in particular, the relation to the fixed set of plates (constituting an invariant composed whole) is critical, as is how units composed of cookies can be related to the fixed set. When this is not discerned, the child may put a random number of cookies onto the plates and look around for more cookies, surprised that all 12 have already been used. The

advanced way of experiencing numbers in this task is to adhere to the wholeness of the 12 cookies and the wholeness of the set of plates and to understand that there needs to be a relation between the plates and the units composed of the cookies. How this relation looks is for the children to figure out, which some of the 3-year-olds do as they systematically partition the cookies into smaller portions and make changes between number of cookies on each plate to make the units equal in size.

Discussion

Earlier research has often observed strategies that students act out when solving a partitioning task, resulting in descriptions of whether students partition items one by one or in "chunks" (e.g., Lamon, 1996). This presupposes that the children already have an idea of the relationship between units in the task. What our study contributes is empirical observations and interpretations of how children at a very young age experience numerical features (that direct their intentional actions) of a partitioning task. For example, Kim (2012) provides an interesting schema of the development of partitioning strategies among primary school children, but our toddlers also act out one-to-one partitioning (see Ester's attempt in excerpt 3), as Kim observed among primary students. However, a significant difference is that the toddlers may not yet experience the task as constituting a part-whole relationship to be attended to. The primary students likely have more experiences of fair sharing and may even have learnt some strategies, such as procedures connected to certain kinds of tasks. Our investigation of partitioning strategies may therefore provide important insights on the need to look beyond the executed strategies by taking the child's perspective, interpreting *the meaning of the numerical features* embedded in a task.

Some might consider partitioning into equal-sized units evenly distributed across a fixed set to be more advanced than doing so across irregular-sized sets. In a mathematical sense, this may be accurate when highlighting the need to simultaneously consider the number of cookies in each cookie-unit and how many plate-units there are to connect each unit (that is, a multiplicative reasoning of units-within-a-unit related to a different kind of unit). Partitioning in unequal sets only requires relating the number of cookies (12) to be partitioned into parts (in an additive relation), which of course is less complex. Taking the child's perspective, it may, however, be equally relevant to give unequal shares to the birthday guests. This must be considered when trying to implement mathematical reasoning and ideas about partitioning in early childhood education.

Methodologically, it is challenging to do research with the youngest children. Most tasks build on verbal instruction, or a narrative (story) presented to the child, who is expected to respond in some way. Even though the present study takes the approach that any response is of qualitative interest, the children's response is highly dependent on their interest in interacting with the teacher or interviewer at that specific moment and how the child interprets the verbally presented task.

A following step in the analysis process would be to investigate the variety of strategies used by the individual toddlers with differently positioned plates to further explore the essence of the different ways of experiencing numbers' part-whole relationships connected to partitioning.

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Collaborating on researching mathematical awareness in kindergarten class

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In Denmark, mathematical awareness is a competency area in kindergarten class (ages 5-6 years). This study explores the competency area and its development in collaboration with kindergarten class teachers, applying a critical research perspective to identify potentials and challenges across three collaborative processes: pedagogical imagination, practical organisation and explorative reasoning. The study shows that knowledge from both the kindergarten class teachers and the researcher affects the collaboration, but the way it is expressed varies across the three processes.

Keywords: Critical research, kindergarten class, collaboration, mathematical awareness.

Introduction

In Denmark, children start kindergarten class (KC) in the year they turn six. As a transition year, KC has its own pedagogical approach and competency areas, distinct from the rest of compulsory schooling, though it is legally and often physically part of the compulsory school system. In 2014, mathematical aspects were incorporated into KC through the competency area *Mathematical Awareness* (MA). This addition to the curriculum expanded mathematical perspectives in KC, while also distinguishing it from first-grade mathematics. These changes pose didactical challenges in practice for three reasons: (1) MA is a completely new mathematical competency area that is not well-defined in the curriculum or founded in existing practice, (2) the curriculum and supplementary materials provide little concrete guidance on implementing MA, and (3) MA and mathematics are only sparsely covered in KC teacher education, as a bachelor's degree in social education does not qualify teachers to teach mathematics. Additionally, KC's position as a transition year means that MA is not only a mathematics-related competency area but is also expected to foster coherence in students' mathematical learning across different pedagogical contexts. In summary, integrating MA into the curriculum is challenging due to the shift in content and perspective for KC teachers.

A three-year PhD project was launched to investigate and develop MA in collaboration with six KC teachers from two schools. The project follows the critical research (CR) approach (Skovsmose & Borba, 2004). Here, the researcher and practitioners collaborate to explore a practice that does not yet exist but is considered desirable, through three iterative processes: pedagogical imagination, practical organisation and explorative reasoning (Skovsmose et al., 2004). Since collaboration is central to all three processes in CR, this framework positions the KC teachers as active participants in the study of MA. However, the specific circumstances related to MA influence the collaboration. Therefore, this paper addresses the following research question: What challenges and potentials are revealed when conducting critical research on mathematical awareness in collaboration with KC teachers?

This contributes to knowledge on collaboration between a researcher in mathematics didactics and an early-years practitioner with no mathematical background, while also informing the methodological approach of CR when studying an unfamiliar concept like MA.

Background

Collaboration between practice and research has long been a focus in mathematics education research, taking various forms (Robutti et al., 2016). Ponte, Segurado and Oliveira (2003) argue that successful collaboration requires a common goal and meeting participants' diverse needs. While research has explored teachers' collaboration with peers, educators and researchers, theoretical frameworks remain limited (Robutti et al., 2016). In educational design research, close collaboration is emphasised, and the diverse perspectives and expertise of participants are valued (McKenney & Reeves, 2019). Furthermore, practitioners can take on roles as co-designers, implementers or co-explorers (Gundersen, 2021).

CR shares this emphasis on collaboration, focusing on conducting research *with* practitioners rather than *on* them. However, CR does not emphasise the different roles but instead explicates how collaboration occurs through three different processes: pedagogical imagination, practical organisation and explorative reasoning (Skovsmose et al., 2004). In CR, collaboration within pedagogical imagination is shaped by negotiation and deliberation, involving both critical reflection on the *current situation* and envisioning possible alternatives (Skovsmose et al., 2004). Ultimately, this process leads to a description of an *imagined situation*. Wright (2021) highlighted the role of participant agency in this phase, noting that his conceptualisation of the imagined situation prompted critical reflections from teachers. Wright (2021) also found collaboration beneficial in the practical organisation phase, which involves preparing an arranged situation to be carried out in the classroom. Here, the collaboration is pragmatic, focusing on determining and planning the feasibility of an *arranged situation* (Skovsmose et al. 2004). Negotiation is still essential, and structural constraints must be considered. The explorative reasoning phase is an analytic process (Skovsmose et al. 2004). Here, collaboration involves practitioners and the researcher reconsidering the imagined situation alongside an analysis of the arranged situation. Wright (2021) found that this process involved contributions of expertise from both practitioners and the researcher. Andersson (2011) noted that collaboration provided the researcher with valuable insights into pedagogical practices and emphasised that meaningful change requires time and strong participant ownership throughout the process. This reveals elements to consider regarding collaboration in CR.

However, both Andersson (2011) and Wright (2021) demonstrated how CR transformed pedagogical practices in secondary mathematics classrooms. Thus, collaboration in CR has not yet been explored when the participants are a researcher and non-specialist mathematics teachers in an early-years setting. While some early-years studies have involved teachers in designing and implementing interventions (e.g., Sæbbe & Mosvold, 2020), research on collaboration itself remains rare, particularly in the context of developing learning situations during a transitional stage of learning. Therefore, CR serves both as a framework for collaboration in research and as an analytical tool for investigating the collaboration.

The study

Preliminary observations of the current situation revealed that KC teaching closely resembled first-grade instruction, prompting the need for change. This led to the introduction of CR as the research approach, as it is driven by a desire to create change, while also enabling the investigation of a practice

that does not yet exist. Following the three processes, an imagined situation was formulated through *pedagogical imagination*. This process began with the researcher's interpretation of MA, which was discussed during a four-hour workshop with the KC teachers, followed by a preliminary questionnaire. The *practical organisation* of an arranged situation took place during the workshop, focusing on discussing and developing a concrete teaching design. The teaching design was subsequently changed based on the discussion before being executed as the arranged situation. Afterward, a conversation between the KC teachers and the researcher framed the *explorative reasoning*, incorporating both critical reflections on the intervention and forward-looking considerations. Along with the researchers' analysis of video recordings of the arranged situation, this led to a revised understanding of the imagined situation and informed the subsequent intervention. Over two years, two iterations of the three CR processes were completed.

Method

In this paper, purposively selected examples from each process in the second iteration are analysed to reveal potentials and challenges during the collaboration. The analysis focuses on the situations that Skovsmose et al. (2004) characterise as determining the collaboration during the specific process.

The pedagogical imagination is informed by questionnaires from the first iteration, results from the first explorative reasoning and transcripts of conversations with KC teachers following the first arranged situation. The analysis includes situations and statements where the perspectives of the KC teachers and the researcher led to negotiation and deliberation regarding the imagined situation. In the practical organisation, the researcher formulated three guiding principles based on the imagined situation: 1) The mathematical content should be recognisable to the KC teachers, 2) it should be framed by an engaging, realistic context that students can relate to, and 3) students should engage in mathematical inquiry. Subsequently, the researcher designed teaching material centred around a fictional story, where questions within the story engaged students in mathematical inquiry about repeating patterns. The teaching material was explored, discussed and negotiated at a workshop as part of the practical organisation. The practical organisation is documented through video recordings and transcriptions. Situations in the data where structural constraints or other pragmatic considerations are discussed or negotiated are selected as examples of collaboration. The explorative reasoning draws on field notes, transcripts of interviews with the KC teachers and the researcher's observations of the arranged situation. In this process, the concrete experiences of the KC teachers and their reflections in response to the researcher's question, "If you had to do it again, what would you change?" are analysed. Situations from the explorative reasoning where the process resulted in either deliberation or renegotiation of the imagined situation are analysed.

Analysing the collaboration in the three processes

The following examples are from the pedagogical imagination in the second iteration.

As part of the pedagogical imagination, MA had to be established. From the researcher's perspective, this was informed by publications on MA, which, among other things, explicate how MA differs from mathematics in a school setting. In the questionnaire, the KC teachers were asked about their experience teaching MA. KCT4 stated: "I'm new to the world of mathematics". This indicates that

KCT4, to some extent, perceived MA as being equivalent to mathematics as an academic subject. Thus, the pedagogical imagination of MA is disrupted because the KC teachers do not consider MA to be unclear. For them, MA is mathematics as it is taught from first grade onwards, while from the researcher's perspective, MA is something different from school mathematics. At the same time, the statement also reveals KCT4's insecurity about her relationship with mathematics. Some KC teachers describe themselves as novices in mathematics and express feelings of inadequacy in the subject. During the first explorative reasoning, when asked about the mathematical aspects, KCT1 said: "I mixed the tasks as I stood there because I also found it hard to teach". In the teaching situation, KCT1 had difficulty distinguishing between decomposing and composing the number 5 as the sum of two natural numbers. When asked what the mathematical content in the next iteration should be, KCT1 said: "You decide", leaving the decision completely to the researcher. This and other utterances indicate that the KC teachers' main concern regarding MA is their own understanding of the mathematical content, and that they did not want to engage in negotiation about it. This limits the collaborative elements of negotiation and deliberation about the mathematical aspects of the imagination process to the researcher's perspective. Thus, the examples reveal challenges in the pedagogical imagination concerning critical reflection, deliberation and negotiation.

In the first pedagogical imagination, both the researcher and the practitioners contributed an inquiry perspective to teaching, making it a mutually agreed-upon pedagogical approach. In the second pedagogical imagination, this shared understanding of using inquiry as an approach was not re-deliberated. As such, it remained a key element in the imagined situation, demonstrating that when two different professions contribute the same perspective, and this is confirmed during an arranged situation, no re-deliberation is necessary.

The imagined situation that emerges through pedagogical imagination cannot be directly translated into teaching, as the constraints arise from the current situation. I will not delve into the constraints affecting the practical organisation. The following examples are drawn from the practical organisation of teaching material during the second iteration.

As noted in the pedagogical imagination, the KC teachers initially felt insecure about their own mathematical understanding and did not express any opinions on the mathematical content. In the second iteration, the researcher introduced repeating patterns, which KCT1 remarked would be "not just easier for the children to access, but also for us". The suggested content was therefore quickly accepted and recognised by KCT1. During the workshop, there was no didactical discussion on the practical organisation of the content, although the mathematics itself was discussed, such as clarifying what a "unit of repeat" is. While the content seemed familiar, and the KC teachers were engaged, the workshop held greater potential for professional development in the mathematics itself, rather than in evolving the practical organisation of the arranged situation.

The teaching material featured a story about a toy dragon. The KC teacher read a paragraph of the story aloud, ending with an engaging question. The students were then asked to answer the question related to the paragraph. The subsequent paragraph included both a recap of the previous question and an introduction to the next one. At the workshop, the implementation of the designed teaching intervention was negotiated. KCT5 suggested reading the entire story before addressing the questions

one at a time. The researcher supported the KC teacher's reflection, saying: "If it makes sense for you and your class… you can do it… It's not how it's meant to be read… but I also think you can try it". KCT4 then added: "I think someone would only pay attention to the last part" – a point later endorsed by KCT2. In this situation, KCT5 proposed a concrete change, which the researcher was receptive to, while other KC teachers critically reflected on it. Despite these discussions on how to read the story, no compromises were made, thus preserving the KC teacher's agency and expertise in KC practice. Here, the negotiation of the practical organisation regarding a pedagogical aspect is not between the researcher and the KC teachers, but among the KC teachers themselves.

The KC teachers' expertise in KC teaching was also evident in another situation. When asked about challenges in implementing the teaching sequence, KCT1 expressed a concern: "So there is something about following each other's pace and waiting for each other," a concern shared by all KC teachers. The concern was that some students would just sit and wait with nothing to do. This was not something the researcher had considered beforehand. Therefore, a fourth guiding principle was added: 4) The design should be flexible and not too compact, allowing students to work at their own pace while still following the same progression. This led to modifications in the teaching material from a pedagogical rather than a mathematical perspective, such as describing when to start a classroom discussion, even if not all students have finished, and giving quick finishers related tasks.

In these last two examples, the KC teachers apply their specialised pedagogical knowledge regarding KC practice. This sparks deliberation and negotiation of certain pedagogical aspects, which would not have occurred without the KC teachers' involvement in the practical organisation.

The following examples are drawn from explorative reasoning in the second iteration.

Despite recognising repeating patterns, KCT1 still expressed mathematical insecurity when asked about the teaching: "I probably would never, because it's difficult for me, so now, based on myself, …, because I'm in deep water myself regarding all this." In situations like this, the conversation often focused on the KC teachers' learning rather than the students' learning, as when KCT1 said: "I also learned a lot." All KC teachers acknowledged that the students had learned some mathematics, but did not elaborate further on what they had learned. Engaging KC teachers in reflecting on a specific teaching episode highlighted their concerns about their mathematical understanding once again. However, the focus on the KC teachers' learning was, in some ways, an obstacle to the process of reimagining the imagined situation. Additionally, the limited interest in exploring students' mathematical learning impeded the KC teachers' contributions to the process.

Initially, the fictional story was introduced by the researcher as part of the inquiry approach, relating it to the description of MA in the curriculum. This was not questioned by the KC teachers at that time. During this second explorative reasoning, KCT1 emphasised the story's value as an engaging context for the students, stating: "We could have done it in so many ways, where they wouldn't have felt the same ownership or had the same approach…". The main character of the story – the toy dragon – was sitting in the classroom during the teaching. During an observation, the researcher noted that at the end of the lesson, many students gave goodbye hugs to the toy dragon. This indicates that both the researcher and the KC teachers recognised the students' engagement in the story. Consequently, the purpose of the fictional story shifted as critical reflection led the KC teachers to reassess and refine

the idea, thereby enhancing the imagined situation. At the same time, by comparing the researcher's observations with the KC teachers' experiences from the concrete arranged situation, it is possible to also re-confirm elements from the first imagined situation, as seen with the fictional story. During the explorative reasoning, the KC teachers view the mathematical content as difficult but feel confident in the pedagogical approach. For example, KCT1 said: "This is new to me, not to teach this way [the inquiry approach], but the tasks [are different]". This reflection confirms once again the use of inquiry as an approach in the imagined situation, even though it does not address the mathematical content.

Findings

The analysis of the pedagogical imagination shows that negotiating a common understanding of MA is challenged by two main factors. First, differing views of MA and its relation to mathematics make it difficult to engage in creative, critical reflection. While the KC teachers see MA as already well-defined, the researcher seeks to explore its meaning. Thus, for the researcher, it is the understanding of MA that needs to be explored, whereas for the KC teachers, it is the teaching of mathematics that are to be explored. Second, when the KC teachers struggle not only with teaching the content but also with understanding it, engaging in a process of imagining how the current situation could be different regarding the content becomes challenging. However, as this was the second iteration, a potential is revealed in how elements from the first pedagogical imagination do not always need to be re-deliberated if there is already an agreement on them, as in the case of the inquiry approach.

Analysis of the practical organisation process shows that there was no discussion or negotiation regarding constraints or changes to the mathematical content in the teaching material. The KC teachers' focus on their own learning of mathematics, rather than on its teaching, hindered the negotiation of mathematical aspects. However, this created an opportunity for professional development, as engaging with the teaching material supported the KC teachers' mathematical learning. Furthermore, once the mathematical content was clear, the KC teachers began to question the pedagogical aspects of the researcher's teaching material. As a result, their experience and pedagogical knowledge were incorporated, revealing the potential for negotiating the pedagogical aspects of the teaching material through collaboration between practitioners and the researcher.

Explorative reasoning is challenged by the KC teachers' focus on their own mathematical insecurities, hindering their ability to reflect on their students' mathematical learning and limiting their analysis of the intervention, as well as their connection to the imagined situation. However, engaging KC teachers in reflecting on a specific teaching episode enables them to address concerns about their teaching of MA. At the same time, explorative reasoning reveals the potential to reassess and refine the purpose of ideas from the imagined situation, reinforcing the second guiding principle concerning the realistic context and reaffirming the inquiry element.

Potential and challenges in collaboration arise across all three processes, but they differ at each phase. However, collaboration is not simply collaboration; using CR as a methodology enables the exploration of how collaboration varies across the processes when developing a practice that does not yet exist. During the collaboration, the KC teachers are positioned as co-designers when engaging in the practical organisation, and as co-explorers during the pedagogical imagination and explorative reasoning. The challenges revealed show that, in these roles, the KC teachers are sometimes limited

by their mathematical insecurities. Conversely, the potential of collaboration lies in the diverse perspectives that emerge. This is evident when the pedagogical aspects are strengthened by the KC teachers' expertise. Additionally, during the second iteration, the KC teachers felt confident reflecting on another practice and, in relation to this, on their own mathematical understanding, which made them aware of their need for professional development.

Discussion

Ponte et al. (2003) state that two factors define successful collaboration: a common goal and the fulfilment of each participant's needs. In this project, the common goal is the exploration of MA, driven by the need to change the current situation. While KC teachers view MA as equivalent to mathematics and feel like novices in the subject, their challenge lies not in uncertainty about MA itself, but in their own insecurity regarding mathematics. From this perspective, a common goal may seem absent. However, in CR, a common goal can also be a shared desire for change and the recognition that change is necessary. Anderson (2011) highlighted that practitioners initiate change, while Wright (2021) emphasised that collaboration in CR is meaningful because everyone is committed to change. Thus, even in the absence of a specific common goal, the shared aspiration for change acts as the unifying factor. Unlike Anderson (2011) and Wright (2021), who worked with mathematics educators, the KC teachers in this study expressed a need to learn more mathematics, signalling that change was needed. However, addressing this need shifted the focus to the mathematical content itself rather than the exploration of MA as an area of competency.

The status of KC as a transitional year with its unique pedagogical approach emphasises the need to include the KC teachers' perspectives in the various phases. Here, the KC teachers provide pedagogical expertise on what characterises a good classroom environment in their practice. In line with Anderson (2011), this allows the researcher to gain deeper insight into practice, revealing a shared understanding and the potential to accommodate both professional roles during the pedagogical imagination. In this study, engaging in the pedagogical imagination during the second iteration becomes more comprehensive as the KC teachers critically reflect on their experiences from the first iteration, thereby engaging in the renegotiation of the researcher's perspective. Wright (2021) found that collaboration in the practical organisation enables practitioners to share and discuss different perspectives on teaching, as well as take joint responsibility for designing changes. In this study, while discussions on how to read the story did not change the teaching material, they were valuable in fostering critical reflection that might not have otherwise occurred. Moreover, another discussion led to the development of an additional design principle. Thus, collaboration within the practical organisation was not only inspirational but also contributed to expanding and modifying the teaching material, thereby affirming the role of KC teachers as co-designers. As Wright (2021) concluded, the opportunity for both parties to contribute their expertise is distinct in explorative reasoning. In this study, two different professions analysed the same arranged situation and noticed the same thing. Not only did both parties share their expertise, but they also contributed to the collaborative development of the imagined situation, particularly regarding the realistic context. As McKenney et al. (2019) express, this is one of the most valuable elements of collaboration – when different professions contribute their specific knowledge.

The potential for both researchers and practitioners to contribute their expertise in developing a shared understanding of MA is significant and should not be overlooked. These dynamics highlight how different areas of expertise shape the collaboration, demonstrating that both researchers and KC teachers bring unique and valuable resources to all three processes. Although professional development was not the primary aim of the study, it clearly occurred throughout all three phases. When two different professionals discuss and negotiate, mutual learning takes place; thus, it is not only the KC teachers who gain new insights, but also the researcher. However, more than one intervention is necessary for the KC teachers to engage in collaboration as more than implementers. For the KC teachers to actively participate in the development and exploration of MA through engagement in the three processes in CR, it seems they need to be involved in at least three iterations. The project will determine whether this is sufficient.

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Individual Differences in Parental Support for Home-Numeracy in Pre-schoolers with Intellectual Disabilities: A Cluster Analytic Approach

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Mathematics education during the early years involves researching the role of parental practices that could promote numeracy. This study aims to shed light on the individual differences in parental support for home-numeracy in pre-schoolers with Intellectual Disabilities taking into consideration the parent's expectations about his/her child's school readiness. We employed a two-step cluster analysis in a sample of n=374 parents. Statistical analysis yielded seven different profiles for participating parents, including profiles from (1) Moderate Expectations, Few Numeracy-Related Activities to (7) High Expectations, High Numeracy-Related Activities. Also, we examined the effects of activities that parents employ to assist their children during this critical developmental period and are related to direct literacy, fine motor, and general skills unrelated to mathematics or literacy that parents employ to assist their children during this critical developmental period.

Keywords: Home-numeracy, intellectual disabilities, early childhood, parental practices,

Introduction

A significant number of research efforts emphasize the importance of home-numeracy experiences for kindergarten readiness and the acquisition of early numeracy skills (Charitaki & Alevriadou, 2024; LeFevre et al., 2009). During early childhood, numerical concepts are taught through playing. Parents can foster a wide range of activities that can promote their children's numerical skills through a direct [Books with Numbers (e.g. "Using number activity books", "Reading number storybooks"), Counting Activities (e.g. "Using number or arithmetic flashcards", "Identifying names of written numbers")] or an indirect [Applications (e.g. "Measuring ingredients when cooking", "Playing with calculators"), and Games (e.g. "Playing board games with die or spinner", "Playing card games")] manner. Through their choices, parents create a home numeracy environment (Anders et al., 2012; Cannon & Ginsburg, 2008; Charitaki & Alevriadou, 2024; Elliott et al., 2021; LeFevre et al., 2009).

Readiness for the transition to kindergarten over years has gained much of research attention. Findings in the field suggest that the relationships between parental expectations, support, and kindergarten readiness are characterized by significant variability (Slicker et al., 2021). The phenomenon is characterized by such complicity that the best way to be illuminated is through a person-centred approach. Existing applications of person-centred approaches in populations of typically developing suggest that parental expectations demonstrate significant variability which may be attributed to family background factors (Barbarin et al., 2008; Miller & Kehl, 2019). Parental beliefs/expectations regarding their children's academic achievements (numeracy and literacy) are supposed to mediate the choices they make to create their home-numeracy environment (Bradley &

Corwyn, 2002; Charitaki & Alevriadou, 2024; Sy & Schulenberg, 2005) and their children's mathematical attainments (Puccioni, 2015).

Furthermore, existing findings in the field suggest significant variability in terms of early numeracy acquisition of young children with Intellectual Disabilities (Charitaki et al., 2024). Parental aspects seem to have a significant effect on this variability and the use of specific-oriented approaches to domains such as parental beliefs, attitudes, and expectations is considered to have a significant role in terms of enrichment of the home-numeracy environment (Sénéchal & LeFevre, 2002; Skwarchuk et al., 2014). Beliefs and expectations are psychological aspects of parents' personalities that formulate their actions sometimes consciously and some others unconsciously (Grusec & Danyliuk, 2007). Consequently, it is important to take into serious consideration parental beliefs. The population of children with Intellectual Disabilities is characterized both by uniqueness and variability. These parameters that should not be disregarded.

One study that has attempted to shed light on the phenomenon and explain existing variability is Slicker's et al. (2021) study. They employed a sample of 12.670 parents of typically developing children and revealed distinct parental profiles, entitled: (1) high expectations, fewest home activities; (2) high expectations, high activities; (3) very high expectations, moderate home activities; and (4) very high expectations, most home activities. This study suggested clearly that there is a clear effect of parental expectations on their choices to create a home numeracy environment. Though, parents of children with Intellectual Disabilities cope with significant restrictions in terms of functioning and choices to create their home numeracy environment (Greenway & Eaton-Thomas, 2020). Their psychological state of being is formulated in a complicated manner. Their children's functioning is a significant factor that affects their expectations for their children's academic achievements.

Moreover, existing theories related to children with Intellectual Disabilities numeracy acquisition (Burack, Hodapp & Zigler, 1998) suggest that in many cases they tend to follow the developmental pathways of typically developing children but in a slower rate and reach a lower level of achievements. They could be compared to their typically developing matched peers with the same mental age. Consequently, parental support for home-numeracy in pre-schoolers with Intellectual Disabilities could be equally of more complicated compared to analogous support in typically developing children. The significant shortage of person-centered studies in the case of children with Intellectual Disabilities stresses the need for research findings in the field.

The aim of this study

This study aims to gain insights into the individual differences in parental support for home numeracy in pre-schoolers with Intellectual Disabilities and seek evidence to build their specific profiles. Moreover, we intend to research the profiles' association with parental expectations and other parental support regarding direct literacy, fine motor, and general skills unrelated to mathematics or literacy. Consequently, this study aims to answer the following research questions:

1. What are the profiles of parental support for home numeracy in pre-schoolers with Intellectual Disabilities?

2. Are there qualitative differences within the groups of parents' distinct profiles related to parental expectations and other parental support regarding direct literacy, fine motor, and general skills unrelated to mathematics or literacy, and other demographic characteristics?

Method

A total number of 374 parents were enrolled in this exploratory study. The mean age for the mothers was formed at m=37.56 and s.d.=1.75 years, while the mean age for the fathers was formed at m=44.79 and s.d.=1.95 years. All of them had a child with Intellectual Disabilities (IQ scores ranged from 56 to 66) aged between 4 and 7 years old (m=5.33 and s.d.=.498). The data collection process was implemented through two different stages. The first one was implemented during the spring semester of 2022, while, the second one was during the fall semester of 2022 and the first term of 2023. All parents gave their informed consent, which was ensured by providing them with clear written and verbal information about the research and the planned use of data. Confidentiality and protection of data were also ensured for the archiving of recorded and written data.

Measures

Parents Questionnaire

Parents' Questionnaire (LeFevre et al., 2009) is a research tool designed to assess five district domains, including benchmarks (8 items) (e.g. "Count to 100."), direct literacy activities (3 items) (e.g. "Identifying names of written alphabet letters."), fine motor skills (10 items) (e.g. "Putting pegs in a board or shapes into holes, playing with puzzles."), general skills unrelated to numeracy (7 items) (e.g. "Playing store") and numeracy-related activities (20 items) (e.g. "Using number or arithmetic flashcards." or "Measuring ingredients when cooking."). Activities for fine motor skills were assessed through two separate subscales namely: Tasks / Daily life skills (tying shoes, buttoning buttons, tracing, puzzles), and Playing (Movement songs, musical instruments, clay, blocks). General activities unrelated to numeracy were assessed through two separate subscales namely: Role Playing, TV shows or educational software, and Arts and Crafts: Paper or Legos. Finally, the part of numeracy-related activities consisted of four subscales entitled: Books with Numbers, Counting Activities, Applications, and Games. The scale's scoring is made through a 5-point Likert scale. In the benchmark's case the Likert scale that is being used (1=not important at all to 5=very important). In the remaining domains, the Likert scale is being used (0=if the activity was not implemented at all to 4=almost daily).

Caregiver's / Parent's Attitudes Toward Mathematics and Reading

Caregiver's/Parent's Attitudes Toward Mathematics and Literacy (LeFevre et al., 2009) scale is a research tool, which consists of 9 items. The scale's scoring is made through a 5-point Likert scale (1=strongly disagree to 5=strongly agree). Specifically, a high score indicates positive attitudes toward mathematics.

Results

For the study, we employed segmentation analysis. More specifically, we used a two-step cluster technique, which suggested a seven-cluster solution for the description of the profiles of parental

support for Home - Numeracy in pre-schoolers with Intellectual Disabilities. Schwarz's Bayesian Criterion (BIC) was estimated and used to draw a decision regarding the appropriate number of clusters that should be used. We made separate assessments of the fit of all possible cluster solutions from one to seven-cluster solutions. The seven-cluster solution offered the most optimal and parsimonious fit (BIC was minimized at the value of 672.134). Moreover, the silhouette measure of cohesion and separation showed better cluster quality (>.7) in the case of the seven-cluster solution. A comprehensive interpretation of the suggested profiles is provided in Figure 1 and Figure 2.

The cluster analysis revealed seven distinct profiles of parental support for home numeracy among preschoolers with Intellectual Disabilities, each characterized by different levels of engagement, expectations, and educational focus.

Cluster 1, labelled the *Poor-quality Home Environment*, included 60 parents, this group demonstrated limited engagement in numeracy activities. Applications were used occasionally, while counting activities, books with numbers, and games were rare. Despite expressing moderate academic expectations, these parents prioritized non-academic activities such as arts and crafts. Cluster 2, the *Uninvolved/Neglectful Home Environment*, included 61 parents who also reported moderate expectations, but engaged even less in numeracy activities than Cluster 1. Applications and books with numbers were used infrequently, and games and counting activities were largely absent. Their overall activity pattern suggests minimal structured support and a lack of consistent educational engagement. Cluster 3, the *Socially Oriented Home Environment*, consisted of 20 parents in this cluster, numeracy engagement was almost exclusively limited to games. Other structured numeracy activities were seldom used. These parents held low academic expectations and focused primarily on social and play-based activities such as arts and crafts, movement games, and fine motor skill development.

Cluster 4, the *Frustrated Home Environment,* included 70 parents who reported moderate use of applications and games, but limited use of books with numbers and counting activities. Although their expectations remained low, some positive orientation toward numeracy was observed. Non-academic activities such as arts and crafts and limited use of literacy and multimedia tools were also present. Cluster 5, the Authoritative Home Environment, was the largest with 93 participants, this cluster emphasized structured numeracy support, particularly through counting activities and books with numbers. Applications and games were used less frequently. These parents expressed moderate expectations and engaged in a range of academic and creative activities, including music, movement, and literacy-related practices. Cluster 6, the *Demanding Home Environment*, comprised of 30 parents exhibited the highest engagement across all numeracy-related activities. Their home environments incorporated counting, books, games, and applications. High academic expectations were paired with consistent use of literacy, multimedia, and play-based activities, with less emphasis on arts and crafts. Finally, Cluster 7. the *Highly Demanding Home Environment*, included 40 parents, this cluster created rich and structured numeracy environments. All types of numeracy activities were employed frequently, along with strong support in literacy and creative domains. These parents maintained high academic expectations and provided comprehensive educational experiences across domains (Figure 2).

Characteristics	Cluster 1 (n=60)	Cluster 2 (n=62)	Cluster 3 (n=20)	Cluster 4 (n=70)	Cluster 5 (n=92)	Cluster 6 (n=30)	Cluster 7 (n=40)
Frequencies							
Gender							
Males (fathers)	30	31	10	35	47	15	20
Females (mothers)	30	30	10	35	46	15	20
Age Group							
< 30	4	2	0	1	1	13	10
31 – 40	30	44	9	59	83	6	29
41 – 50	27	11	9	10	9	11	11
>51	3	4	2	0	0	0	0
Educational Level							
Primary School	32	23	11	7	4	0	3
Gymnasium	25	34	4	19	5	2	4
High School	3	0	1	25	57	17	0
Technical School	0	4	0	16	27	2	26
University Bachelor	0	0	4	3	0	9	7
Family Income (monthly)							
< 650 €	36	34	0	0	0	0	0
651 – 1000 €	17	13	11	27	32	3	0
1001 – 1500 €	7	9	4	14	9	13	6
1501 – 2000 €	0	5	5	29	52	10	29
2001 – 2500 €	0	0	0	0	1	4	5
Severity of Child's Disability (n=187)							
IQ score: 60-66	2	4	5	32	41	13	16
IQ score: 50-56	28	27	5	3	5	2	4

Figure 1: Sample Characteristics within Clusters.

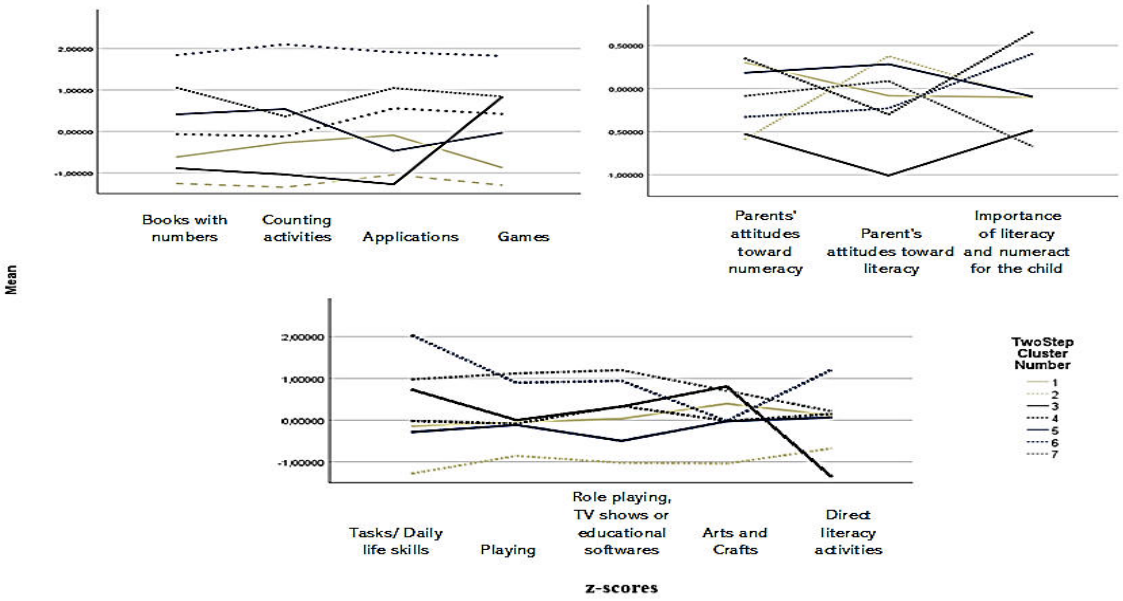

Figure 2: Profiles regarding parental support profiles in pre-schoolers with ID (seven clusters).

Internal Validity

Sensitivity analysis suggested that 87% of parents remained in the same cluster across clustering techniques. Moreover, the cluster solution was re-estimated using a random sub-sample comprising 70% of the entire sample and the solution indicated stability of the cluster solution.

Discussion

Early years of mathematics focused on the home-numeracy environment seem to have a significant impact on children's school readiness and later achievement when entering primary school (Charitaki & Alevriadou, 2024; LeFevre et al., 2009). Existing intra-individual differences across families of children with Intellectual Disabilities justify the need for person-oriented findings regarding the type

of parental support in terms of home numeracy. The present study identified a diverse range of parental support profiles for home numeracy among preschoolers with Intellectual Disabilities, reflecting substantial variability in both the quantity and quality of engagement. These profiles ranged from minimal, unstructured involvement with moderate or low academic expectations to highly enriched and structured numeracy environments aligned with high parental aspirations. Rather than listing each cluster individually, we emphasize the overall pattern: parents differed not only in the types of activities they used—such as counting, reading number books, or playing games—but also in their underlying expectations and priorities, which were often shaped by broader contextual factors, including socioeconomic status.

These profiles revealed a spectrum of engagement, from low-activity environments with modest academic expectations to highly enriched, structured home settings characterized by frequent numeracy activities and strong educational aspirations. Parental behaviour was not only defined by the type of numeracy activity, for example counting, using books with numbers, or playing math-related games, but also by parents' underlying expectations for their children's learning. Importantly, these expectations and support strategies were found to be deeply influenced by contextual and socioeconomic factors.

In line with prior research (Slicker et al., 2021; Bradley & Corwyn, 2002), our results confirm that parental beliefs and expectations are closely tied to the learning environment parents construct at home. Moreover, these beliefs are shaped by multiple factors, including the child's developmental challenges, the level of support and guidance families receive, and critically their socioeconomic background (Sénéchal & LeFevre, 2002; Skwarchuk et al., 2014). Families facing economic hardship and lower educational attainment often experience reduced access to resources and information, which in turn limits their ability to foster numeracy development effectively (Bradley & Corwyn, 2002; Charitaki & Alevriadou, 2024; Sy & Schulenberg, 2005).

Our findings reinforce the well-established association between lower socioeconomic status and reduced academic performance in early numeracy (Kluczniok, 2017; LeFevre et al., 2010; Xu, 2023), which is the case in pre-schoolers with intellectual disabilities (Szumski & Karwowski, 2012). Clusters characterized by minimal parental involvement and low activity levels were predominantly composed of families with limited income and education (e.g., Clusters 1 and 2). In contrast, clusters demonstrating structured and consistent numeracy practices (e.g., Clusters 5 through 7) were primarily composed of families with average to high socioeconomic status. This gradient underscores how economic and educational capital translate into differences in home learning quality, further compounding disparities for children with ID (LeFevre et al., 2010; Ramani & Siegler, 2011).

These findings are of considerable practical significance. Families from disadvantaged backgrounds, especially those who hold lower expectations for their children's academic potential, may be at significant risk of under-supporting their early numeracy development. These families often face systemic barriers to accessing specialized therapies or interventions. Therefore, tailored support strategies are essential. Strengthening family-school partnerships that promote reciprocal communication and shared responsibility between parents and educators could be beneficial for young children with intellectual disabilities. Additionally, low-cost, engaging activities such as

numeracy games can be an accessible and enjoyable way to stimulate early skills across all socioeconomic status groups.

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Supporting collective mathematical creativity in first grade

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This paper describes observations of collective mathematical creativity among first-grade students. Three measures of mathematical creativity were analyzed: fluency, flexibility, and originality. Four groups of students engaged with four open-ended activities suitable for young children, designed to occasion mathematical creativity. Two groups received social support, and two groups received arithmetic support from the first researcher. The results show that these interventions fostered active listening, idea-sharing, mutual support, and diverse responses among students, helping them to exhibit mathematical creativity. Three episodes from two activities are presented here to exemplify how the different supports promoted students' fluency and flexibility as well as a moment of mathematical insight by a student.

Keywords: Mathematical creativity, social support, arithmetic support, first grade.

Introduction

The OECD has identified various skills that are foundational for students in the education system, including collaboration, interpersonal communication, problem-solving and creativity (Chu et al., 2016). Specifically, it is important to nurture mathematical creativity among students (Leikin et al., 2011). Regarding collective creativity, a review of the research literature reveals that most studies on that subject focus on older students (e.g., Molad et al., 2020), and there is a lack of studies involving first-grade students who are just beginning their journey in elementary school. This paper explores the possibility of promoting collective mathematical creativity among first-grade students, thereby highlighting the development of 21st-century skills of creativity and collaboration among young students.

Theoretical Background

What is creativity? According to Torrance (1988), creativity is defined as the process of sensing problems, deficiencies, gaps in knowledge, missing elements, or something that is not in harmony. This process requires individuals to evaluate, test, revise, and re-examine their work to present a final result. Torrance's tests for identifying creativity were based on four dimensions: fluency, flexibility, originality and elaboration. In mathematical education, creativity is typically characterized by three of the four dimensions: fluency, flexibility, and originality. The fluency dimension refers to the ability to generate many possible solutions and ideas to a given problem and is usually measured by the number of correct distinct solutions to a problem (Levenson et al., 2018). The flexibility dimension examines the different strategies students use to solve a problem, and is measured by sorting the responses into categories, where the number of categories represents the flexibility score. Originality refers to the provision of unexpected and unusual answers (Levenson et al., 2018; Leikin, 2009).

Mathematical creativity can be nurtured through specific activities known to foster creative thinking, including tasks that allow for multiple strategies, multiple outcomes, investigation tasks and sorting tasks (Klein & Leikin, 2020). These tasks may also involve deriving general formulas and recognizing

mathematical patterns. For example, open-ended tasks encourage divergent thinking and mathematical exploration (Tabach & Levenson, 2018). An open-ended task is a task with many solutions, allowing students to propose a wide variety of mathematical ideas (Molad et al., 2020).

Additionally, collective learning can promote and develop creativity. Sengil-Akar and Yetkin-Ozdemir (2020) demonstrate that collective creativity can thrive when students are given open-ended, real-world tasks that encourage exploration, discussion, and refinement of ideas. Furthermore, Levenson (2011) highlights a connection between the work environment, the actions of the teacher, and the promotion of creativity among fifth and sixth graders. The teacher's role is to legitimize innovative ideas presented by students, thereby encouraging them to present new ideas (Levenson, 2011). In a study by Gajda et al. (2017), it was found that the teacher's emotional support for his students is important for establishing creative learning environments. A creativity-supportive environment is characterized by an atmosphere of mutual respect between teachers and students and incorporates open dialogue and collaborative activities (Gajda et al., 2017).

There are two types of intervention in the study. The first type is when the researcher offers social support, therefore we call it social intervention. The second type is when the researcher offers arithmetic support, hence we call it arithmetic intervention. The current study explored the impact of these two types of researcher interventions on students' mathematical creativity, while working in groups on tasks designed to occasion creativity. In the social intervention, the researcher guided students to help, listen to, and encourage each other during the activity. In the arithmetic intervention, the researcher corrected mistakes, provided mathematical hints, and asked arithmetic questions related to the tasks. The students' mathematical creativity was assessed based on the three commonly accepted dimensions in mathematical education research: fluency, flexibility and originality. Specifically, we ask, (1) What is the level of mathematical creativity–in terms of fluency, flexibility and originality–among groups of first grade students receiving either type of intervention? (2) In what ways can each type of intervention support collective mathematical creativity?

Methods

The study involved 12 first-grade students, aged 7, attending elementary school. Ethical approval was obtained from the Chief Scientist of the Ministry of Education, written parental consent was obtained, along with the option to withdraw at any time, and all names in this paper are pseudonyms.

The students were divided into four heterogenous groups of three students each, such that each group included one academically advanced student, one average student, and one low achieving student, as based on the teacher's assessment and academic achievements in arithmetic. The research was conducted towards the end of the school year. The mathematics lessons up to that point covered recognizing natural numbers up to 100, as well as addition and subtraction, including chain exercises with two or more operations. In the final months, students also began learning the fundamentals of multiplication and division up to 20.

Group meetings were held at the school, with each group having four sessions, spaced one week apart. The researcher met with each group in a quiet room, separate from the rest of the class. During each meeting, the students worked on a single task, following the sequence described later. All sessions were video-recorded and transcribed by the first author. The research tools included four tasks along with interventions, as well as observations. This paper will focus on two tasks.

The first task, 'Questions from a Picture', is rooted in the visual domain, where students are asked to look at a picture and generate as many mathematical questions as possible. Before beginning the task, the students were given a demonstration task to introduce the concept of formulating questions based on an image. The researcher presented the students with an illustration of two trees, one bearing three oranges and the other four lemons, and asked them to formulate relevant arithmetic questions (see Figure 1).

Once it was clear that the students grasped the concept of generating questions, the researcher introduced the image for the first task and encouraged the students to "be teachers" by generating as many questions as possible based on that picture (see Figure 2).

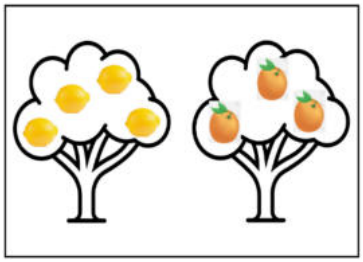

Figure 1: Demonstration Task

Figure 1: First Task

The second task, implemented one week after the first, titled 'Exercises for the Number 12', challenged students to generate as many arithmetic exercises as possible that result in the solution 12. To clarify the assignment, the task was preceded by a preparatory activity. The researcher wrote the number 3 on a sheet of paper, circled it, and asked the students to create exercises where the answer would be 3. Following this introduction, the students were encouraged to "be teachers" by inventing as many exercises as possible for their classmates, all leading to the answer of 12. It is important to note that both activities offer many possible solutions. In both tasks, the researcher handled the actual writing of the answers, as the students were in first grade, not yet fluent in writing.

The students' responses were analyzed using creativity measures, with each task evaluated separately for fluency, flexibility, and originality. For the first task, fluency was measured by counting the number of relevant mathematical questions based on the given picture. For the second task, fluency was measured by counting the number of arithmetic exercises that led to the solution of 12. For both tasks, if the same question or exercise was repeated more than once, it was only counted as one. Incorrect exercises, such as 10+3=12, were not counted. Each question or exercise earned the group one point. Flexibility was assessed by examining the variety of methods proposed by the students. The first author categorized the students' responses, which were then validated by the second author, with discussions held as necessary.

As demonstrated in previous studies (Levenson et al., 2018), the level of flexibility was set by categorizing the students' responses for each task (exemplified below, Table 1).

Table 1: Examples for questions and exercises by category (flexibility)

Task 1 - 'Questions from a Picture'		Task 2 - 'Exercises for the Number 12'	
Category	Example	Category	Example
Count	How many pancakes are on the tray?	Addition	3 + 9

Combining groups	How many apples and strawberries are there in total?	Subtraction	17 - 5
Subtraction	There were 15 balloons, and 4 of them popped. How many are left?	Multiplication	2 x 6
Comparison	How many balloons are there of each color, and which color has more?	Chain exercise	10 x 2 – 8
Inference	How old is Itai? (can be inferred from the number of candles on the cake)	Verbal problem	13 donuts minus one donut.
Measurement	What is the length of the table?	-	-

The number of categories indicates the level of flexibility. Originality in each activity was determined by identifying unexpected or novel solutions (Leikin et al., 2011). Due to the small number of participants, in this paper, we refer only to fluency and flexibility, as summarized in Table 2.

Results

This section begins by presenting the fluency and flexibility scores each group achieved in the selected two tasks. Three episodes were selected from the tasks to illustrate how the different types of interventions – social and arithmetic support – impacted on students' flexibility and fluency. Table 2 presents the measures of fluency and flexibility for each group in the two tasks, 'Questions from a Picture' and 'Exercises for the Number 12'.

Table 2: Measures of fluency and flexibility according to task and group

Group	Questions from a Picture		Exercises for the Number 12	
	Fluency	Flexibility	Fluency	Flexibility
Social Support A	9	3	27	4
Social Support B	11	3	26	4
Arithmetic Support A	13	4	22	5
Arithmetic Support B	14	5	19	5

In general, when it came to fluency, students found it more challenging to generate verbal questions than to create arithmetic exercises. Additionally, in the 'Questions from a Picture' task, the groups that received arithmetic support demonstrated higher fluency in their responses compared to those that received social intervention. Conversely, in the 'Exercises for the Number 12' task, fluency was higher among the groups that received social intervention. For both tasks, flexibility was slightly higher among those who received arithmetic support.

Arithmetic support leads to flexibility

During the 'Questions from a Picture' activity, the students from Arithmetic Support Group B primarily asked questions involving counting items. The researcher noticed a lack of variety in the types of questions being asked, so to encourage greater flexibility, she asked the students if they could think of other types of questions, not just those involving counting that begin with the word "how many". Following the researcher's prompt a student named Carmi asked a question that involved making an inference from the picture.

Carmi: How many children will come based on the number of chairs?

Carmi wanted to know how many children would attend the party based on the number of chairs arranged around the table.

Researcher: Wow, what a great question. So, if we count the chairs, we can guess how many children are expected to arrive. Well done.

It seems likely that Nurit was paying attention, because following Carmi and the researcher's comment, Nurit also proposed an inferential question based on the picture.

Nurit: How many hamburgers will be left at the end of the party?
Researcher**:** How can we know that?

Nurit's question directly followed Carmi's, referring to the hamburgers on the tray. Carmi's response to Nurit's question showed that he was paying attention to her.

Carmi**:** Because there are 12 children, and if they all want some...

It appears that Nurit was thinking about the sequence of events in the picture—specifically, what the children would do after taking their seats at the party.

It should be noted that flexibility in the 'Questions from a Picture' activity was measured by the variety of types of arithmetic questions the students asked based on their observations of the picture. Inference questions require understanding beyond what is immediately visible, as opposed to direct questions that could be asked simply by looking at the picture. This case illustrates how the researcher's intervention encouraged students to increase their cognitive flexibility, enhance their creativity, and collaborate with one another.

Arithmetic support leads to fluency and flexibility

At the beginning of the 'Exercises for the Number 12' activity, the researcher instructed the students to come up with as many exercises as possible that resulted in the answer 12. The students in Group A with arithmetic support, Ela, Miki, and Ronit, initially proposed one multiplication exercise and three different subtraction exercises, all of which were correct solutions for the task. They then attempted to create word problems with the answer of 12, but their questions were unclear and incorrect. For example, Ronit asked, "12 bananas times one apple". Possibly, they were recalling the image from the previous picture activity, leading them to try to invent similar questions. Miki stated, "I have nothing", but the researcher persisted and reflected to the students that while they had used multiplication and subtraction, they had not yet tried addition.

Researcher: I saw that you did multiplication and subtraction, but I don't see addition.

Following the researcher's prompt, the children changed direction and turned to addition:

Ela: 10 plus 2.
Researcher: Wow. What else?
Ela: 9 plus 3.
Ronit: 8 plus 4.

A pattern can be observed in these three exercises. In each, the first addend decreases by 1, and the second addend increases by 1. In this exchange, it is evident that Ronit, (considered by her teacher to be a weak student), was listening to Ela (considered to be a strong student), which led Ronit to think of an additional solution. The researcher's arithmetic intervention motivated the students to come up with more exercises, thereby increasing the fluency, but not the flexibility, of their responses.

Social support leads to fluency and flexibility

During the 'Exercises for the Number 12' activity, Sagiv and Tamir (Social Support Group B) came up with two subtraction exercises that resulted in 12. Afterward, another student, Miri, repeated an exercise that had already been mentioned, 14 minus 2. Although she did not invent a new exercise, the researcher responded with a 'compliment': "We've already said that, but good job!". Immediately after, Miri began generating a stream of subtraction exercises.

Miri: 15 minus 3.
Researcher: Wow, 15 minus 3. No one has said that yet.
Miri: 16 minus 4.
Researcher: Wow, no one has said that either.
Miri: 17 minus 5.

Miri continued to create more exercises, consistently increasing both the minuend and subtrahend by 1 in each new exercise, maintaining the difference so that the result was always 12. It should be noted that Miri was considered an academically weak student by her teacher. It seems that she reached this mathematical insight during the activity, ultimately inventing 14 different subtraction exercises.

Miri: 21 minus 9.
Researcher: Wow!
Miri: 22 minus 10.
Researcher: Wow, Miri, you're doing great today!
Researcher: What do you think of Miri's answers? (asking the two other students.)
(The other students tap loudly and rapidly on the table signifying enthusiasm.)
Miri: 23 minus 11.
Researcher: Nice. What a creative child.

Throughout the activity, the other two students observed Miri, occasionally encouraging her, but did not participate in increasing the fluency of responses. Possibly, the researcher's interventions, such as asking for feedback on a peer's work and giving compliments during the activity, helped Miri generate a significant flow of exercises. Additionally, the non-verbal support from her peers might have also helped Miri find more correct exercises that met the task's requirements.

Up until that point, Miri had been quite passive. The compliments from the researcher appeared to encourage her to speak and motivated her to find more and more exercises. Miri smiled, clearly pleased with the attention and her success in generating a steady stream of exercises, contributing to the task's overall success. This example illustrates how the researcher's social intervention helped enhance fluency and foster mathematical insight during the "Exercises for the Number 12" activity.

Discussion and Conclusion

Qualitative research on this scale is limited and can only suggest possible directions and trends regarding how interventions can contribute to the development of mathematical creativity. In the current study, we found that the researcher's interventions helped students to express mathematical creativity. Regarding the level of creativity, no major differences were found in the measures of fluency and flexibility between the social intervention group and the arithmetic intervention group. Both tasks, 'Questions from a Picture' and 'Exercises for the Number 12,' are defined as open-ended tasks in which students were required to invent questions and exercises. It is important to note that this is a challenging task for young first-grade students. Yet, it also provides them with an opportunity to express creativity by "creating something". Additionally, first-grade students do not yet have fully developed social skills, and working together is not something that comes naturally to them (Veldman et al., 2020). Therefore, teacher intervention is important (Levenson, 2011).

Regarding the second research question, which explores how each type of intervention supports collective mathematical creativity, the findings from the three episodes described above show that both social and arithmetic interventions contributed to increased fluency and enhanced flexibility. In the third case, we also see mathematical insight following social intervention.

Reflecting on the researcher's approach to intervening during the students' work, the question arises: How feasible is it to separate social and arithmetic interventions? In Levenson (2011) the teacher indeed provided both type of supports. As researchers, we try very hard to adhere to the guidelines, in this case, implementing either social or arithmetic interventions. We had prepared in advance a set of questions for each type of intervention. However, there were situations where it seemed very appropriate to deviate from the planned intervention. Our conclusion is that in a regular classroom setting, it would be unnatural for teachers to separate the types of interventions, as some situations require a combination of social and arithmetic interventions to effectively support student work.

In conclusion, a significant contribution of this study is that first grade children, with support from a knowledgeable adult, can build on each other's ideas, support one another, and provide diverse responses to various tasks. This finding is in line with Levenson's research (Levenson, 2011) regarding the contribution of teacher's support to creativity. Although it requires investment from the teacher, this study showed that the effort is worthwhile. First graders can work together and can be mathematically creative. Teachers can show students how to work together, by providing them activities that encourage creativity and lead to teamwork and dialog. As demonstrated by Gajda et al. (2017), social support is important for creative learning environment. Teachers need to mediate the students' work—both social and academic—thereby "planting the seeds" to teach young children how to work together in the future. Students who learn to work together and collaborate are likely to enhance their creativity, both as individuals and as a group.

The study paves the way for larger and more comprehensive studies that also include control groups without intervention.

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The influence of linguistic background on early representations of multiplicative comparison

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Linguistic aspects involved in young children's interpretation of multiplicative comparison situations are investigated by comparing the answers provided by children aged 5-6 years old from three different countries: Australia, Italy, and Israel. Two tasks with different wording are used to analyse how language differences affect children's performance in terms of number of correct answers and most common wrong answers. The study shows that the number of children who can provide correct answers to multiplicative comparison tasks before their formal introduction change significantly according to the linguistic background. Furthermore, the analysis of wrong answers suggests that several possible linguistic factors may intervene including vocabulary knowledge, familiarity with phrases, and differences in the way that languages convey multiplicative meanings.

Keywords: Kindergarten, language, multiplicative comparison, representations.

Introduction

There is a growing (but still scarce) body of literature about how young children represent intuitively multiplicative semantic structures before their formal introduction in school. Through an exploratory study, the first and third authors of this contribution found that by the age of 6, many children can represent correctly different multiplicative situations including equal groups, rectangular arrays, multiplicative comparison, and allocation situations (Downton & Maffia, 2023). In the context of that study, multiplicative comparison (MC) problems were more difficult for the children than the other multiplicative situations. The authors attributed this to children's lack of familiarity with the phrase 'times-as-many', which Nesher (1988) also found. Evidence of this was that the Italian children (for whom this phrase is very unfamiliar, see the Method section) found MC situations more difficult to interpret than Australian children. This observation may lead to the relevant educational implication that familiarity with the words used to describe MC could be a pre-requisite for accessing MC problems (Vanluydt et al., 2020), but this remains a conjecture.

Seeking to better understand this phenomenon, this contribution presents a study, which replicated interviews similar to those by Downton and Maffia (2023) but compared the Italian and Australian contexts with another linguistic background: the Hebrew language. Our research question is: To what extent do linguistic aspects influence young children's answers to MC word problems?

Multiplicative comparison and language

Multiplicative comparison, commonly referred to as 'times-as-many' in research undertaken in English, is a different application of multiplication from that of Equal Groups or Rectangular Array, as it refers to a comparison or an enlargement that may apply to either discrete objects or to a

continuous medium, rather than an iteration of equal groups (Anghileri & Johnson, 1992; Greer, 1992). In short, MC involves two different groups being compared to each other. For example, *John has 3 times-as-many apples as Mary. If Mary has 4 apples, how many apples does John have?* In this instance the number of apples Mary has is being compared to how many John has. In this instance the multiplicative factor (3) may be considered as the multiplier. In essence the MC relationship includes three elements that relate to each other: one quantity is a unit of measure (Mary's apples-4), another is the measured quantity (John's apples-12) and the third is the coefficient (the multiplied measure, which in this context is 3) (Polotskaisa & Savard, 2020). The same problem may also be viewed in terms of ratio or a many-to-one correspondence (3 apples of John's for every one apple of Mary's) which makes 4 the multiplier. Greer (1992) purported that *multiplicative comparison* or '*n times-as-many as'* to be a preliminary stage to ratio and has the advantage over the *groups of* aspect as it relates directly to the nature of multiplication.

While some studies found that young children (6 and 7 years old) had difficulty interpreting this semantic structure (Anghileri, 1989; Mulligan & Mitchelmore, 1997), Downton and Sullivan (2017) found that a high proportion (64%) of Grade 3 children (8 and 9 years old) successfully solved comparison problems using multiplicative strategies.

From a linguistic perspective, Nesher (1988) maintained that an understanding of the textural structure of word problems is critical to interpreting the underlying mathematical structure of the problem. From her analysis of the differences in the propositional structure of three types of multiplicative word problems, Nesher noted that for MC (scalar problems), both the order of the text and the language of 'times-as-many' must be considered. Further, the language of this phrase may confuse students (Anghileri, 1989; Nesher, 1988). Others suggested that it is not uncommon for students to add the two given numbers rather than multiply to solve one-step MC word problems (González-Calero et al., 2015; Nunes et al., 1993).

Several studies have shown that language development and numeracy development are related, and that language skills may account for variation when predicting early numeracy skills (Purpura & Ganley, 2014). This makes sense as understanding and "doing" mathematics necessitates the use of language, and specifically the use of mathematical terms such as more, less, and number names. Studies have also shown that different languages may assist or deter children from understanding number concepts. For example, Kung et al. (2019) found that structural language differences between Chinese and English provide different supports for learning mathematics in preschool, especially since numeracy-related words are embedded in Chinese natural language. Indeed, studies have shown that one of the challenges of teaching mathematics at all ages is bridging between natural, sometimes called informal, language, and the more formal, or technical, language of mathematics. For example, in Greek, the mathematical term for set can have three different meanings in their natural language (Antonopoulos, 2023).

Related to the linguistic perspective are the cultural differences, which also impact on children's interpretation of the phrase 'n times as many' in relation to MC. For example, in Spanish the translation is 'n times more than' (increasing comparison) and 'n times less' (decreasing comparison), which is the case for other languages (e.g., Hebrew, Czech, and Arabic) (Nesher et al.,

2003). These subtle differences indicate that MC is expressed by a different construction to that used in English. In fact, these alternative phrases would be considered grammatically incorrect or confusing to English speakers (González-Calero et al., 2015). Another issue related to the current study is that the same concept, in this case multiplication, may be expressed verbally using different terminology, giving rise to different meanings. Language such as 4 times 6 does not activate conceptual meaning, while saying instead, 4 groups of 6, infers the equal groups semantic structure of multiplication (Downton & Maffia, 2023). Hence, the available literature suggests that vocabulary and wording can affect young children's understanding of comparison tasks. In this paper we investigate to what extent such linguistic aspects influence children's responses to MC problems. We will ascertain this by comparing answers provided by children from three different countries to two different MC word problems.

Methods

We conducted the same interview in three countries: Australia, Israel, and Italy. The interview consisted of several tasks including different multiplicative semantic structures. In this report we will focus only on the MC tasks presented in Table 1.

Table 1: The two tasks in the three languages involved in the study

	English version	Hebrew version	Italian version
Apples task	Mary has 4 apples and John has **double** the amount. How many apples does John have?	לגל יש 4 תפוחים ולתומר יש כמות לתומר. כמה תפוחים יש **כפולה?**	Maria ha 4 mele e Giovanni ne ha il **doppio**. Quante mele ha Giovanni?
Towers task	Mary made a tower using three blocks. Max made a tower **three times as tall**. How many blocks did Max use?	גל בנתה מגדל באמצעות שלוש **פי** קוביות. דן, בנה מגדל בגובה יותר מגל. בכמה קוביות **שלושה** השתמש דן**?**	Maria ha fatto una torre usando tre blocchi. Max ha fatto una torre **tre volte più alta**. Quanti blocchi ha usato Max?

Both tasks present a MC situation but they differ in several aspects. Different numbers are involved and, while both situations involve discrete quantities, the Towers task can also be interpreted in terms of linear comparison (in terms of the height of the towers). In the Apples tasks, the word 'double' is used in all three languages; in the Towers tasks we did our best to translate the English 'times-as-many' into similar phrases in Hebrew and Italian.

Children in the study were aged between 5 and 6 years old. Convenience sampling was used to select the schools that already had been in contact with the researchers for past projects. Children for whom parent consent was provided were interviewed. The Australian sample consisted of 39 children, the Israeli sample 37 children, and the Italian sample 43 children. Each sample was coded according to the country in which they attended school and where English, Hebrew, and Italian are respectively – the language of teaching and the native language for most (but not all) children.

Australian children were interviewed during their Foundation year, the first year of school. Within Victoria children must turn 5 before April 30 of the year they start school. Multiplication is introduced in Year 1 (Grade 1) with a focus on equal group situations (Victorian Curriculum and Assessment Authority [VCAA], 2024). In Foundation the focus is on quantifying and comparing collections to at least 20, partitioning and combining collections up to 10 using part whole relationships and subitizing skills, and practical situations involving partition division. While multiplication is not formally taught in Foundation, children become familiar with the language of 'doubles' in the context of addition and may experience the language of 'times as many' in measurement contexts when comparing lengths.

Children in Israel were interviewed during the last months of Kindergarten. In Israel, there is a compulsory mathematics curriculum for children aged 4-6 years old (Israel National Mathematics Preschool Curriculum, 2010). While learning multiplication is not an explicit aim for kindergarten, the curriculum states that other numerical skills, such as skip counting and creating equal groups of objects, are a prelude to multiplication which will be learned later. The language of teaching is Hebrew, the first language of all children participating in this study. In Hebrew, the word 'times,' as in 'five times three,' is 'kaful.' However, in a different context, this same term could mean 'double,' or 'twice as many,' as in Esther has double ('kaful') what Ann has. There is no one term for triple or quadruple, and instead, those meanings must be said as 'three times (or four times) as many,' but here, the term for "times" is not the same 'kaful' as used above. Instead, there is another term which literally translates as 'bigger by 3'. Finally, when children are first introduced to multiplication in first or second grade, they learn to count the number of equal groups (harking back to kindergarten) and thus instead of immediately using the expression 'kaful' for 'three time five,' they use another term 'pa'amim,' which loosely translates as 'three repetitions of five'.

Italian children were interviewed during their last months of Kindergarten, which is not compulsory schooling in Italy. There is no subject-related teaching and the national curriculum for Kindergarten does not refer to Mathematics directly (Italian National Standards for K-8, 2012). There are recommendations for teachers to work on the procedures of counting and measuring as well on recognizing geometrical shapes. The language of teaching is Italian even if children from migrant families (mainly second generation) are present in almost all class-groups. The use of multiplicative numeral adjectives as 'doppio', 'triplo', etc. ('double', 'triple' in English) is common in the Italian language. In Italian grammar, the use of 'volte' (meaning 'times' as in 'two times' or 'five times') is considered more common than the corresponding multiplicative adjective only when the number is higher than five. Even in those cases, the phrase 'times-as-many' is still uncommon. The Italian word 'volte' has several meanings. In the most popular Italian dictionaries, the meaning as 'repetition' is never the first or second one presented and when examples using the word are provided, they include phrases such as "l'ho detto tre volte" ("I said that three times" in English), but the equivalent of the phrase 'times-as-many' (which would be 'tante volte quanto') does not appear in any of the dictionaries consulted by the third author.

During the interviews in all countries, a research assistant presented verbally the situation and asked the child to provide an answer. For each task half of the children were requested to represent the situation using manipulatives, while the other children were provided with paper and pencil and

prompted to draw the situation. In this way, we could avoid the effect of the means of representation (being a manipulative or a drawing, cf. Downton & Maffia, 2023).

Results

Results from the interviews are summarized in Table 2 indicating the number of correct answers for each task in the three samples.

Table 2: Correct answers in the three samples (corresponding percentages are indicated in brackets)

	Australian sample	Israeli sample	Italian sample
Apples task	14 (36%)	5 (13%)	3 (7%)
Towers task	16 (41%)	2 (5%)	1 (2%)

It is apparent, there is a strong difference: Australian children were able to answer correctly to both tasks much more frequently than Israeli and Italian children. Furthermore, the Apples task appeared more difficult for the Australian children than the Towers task, while this was not the case for the Israeli and Italian children. This is possibly because discrete quantities are involved and the word 'double' may be less familiar to the Australian children than the phrase 'times-as-tall'. As noted in the Methods section, the Israeli and Italian languages differ from English in how the phrase 'times-as-many' is presented, and those phrases may be less familiar for children aged 5-6 years old than the words equivalent to 'double'. Indeed, we had examples of children asking about the meaning of the words; in those cases, the interviewer asked the children what they thought. In some cases, the children replied that they just did not know the word.

For a deeper interpretation of these results, we analysed the most common answers provided by children when the given answer was incorrect. Table 3 summarizes the percentage of selection for the most common wrong answers to both tasks across the three samples. Again, we can observe different phenomena in the three samples: the most common wrong answer is not always the same. In the case of the Apple task, the answer 6 (corresponding to the sum of the involved numbers) was the most common for Australian children, but Italian and Israeli children opt more often for 4 (the first number mentioned in the task). Several children were able to represent the numbers 4 and 2 involved in the Apple task, but then they were not able to answer the question as expected; some of them merely counted all.

As in the Apples task, the first number involved in the Towers task (3) was a common incorrect answer (especially in the Israeli sample), but it was not the most common. For this task, the sum of the two involved numbers (3+3=6) was the most common answer not only for Australian children, but for Israeli and Italian ones as well. The language in the Hebrew and Italian translations of the expression 'times-as-many' directly implies that the second tower is taller than the first one. The necessity to increase the tower's height may have prompted the children to sum the two given numbers. This interpretation may explain also the presence of answers such as 4 and 5. Indeed, we have examples of children that, when asked to justify their answers, replied "because the tower is big" or "because 3 and another 1". Another explained, "I think it is 4 because it is more than 3".

Table 3: Relative frequency (over the total of each sample) for the most frequent wrong answers

	Apples task (correct answer 4×2=8)			Towers task (correct answer 3×3=9)		
	Australian	Israeli	Italian	Australian	Israeli	Italian
Answer "3"	0%	11%	7%	5%	16%	7%
Answer "4"	18%	19%	26%	8%	11%	9%
Answer "5"	10%	5%	9%	0%	11%	14%
Answer "6"	23%	14%	9%	36%	19%	14%

Discussion and Conclusion

We acknowledge that his task may be difficult for young children of this age, but the aim of this study was to investigate the extent to which linguistic aspects influence young children's answers to MC word problems. We compared the answers provided to two different word problems across three samples of 5-6 years old children from three countries with different languages: English, Hebrew, and Italian. The three languages differ in how multiplicative numeral adjectives are used. Furthermore, the three countries differ in how mathematics is taught to 5-6 years old children. These differences are reflected in the results.

Overall, the Australian sample performed much better than the other two. Furthermore, the analysis of most common incorrect answers provides hints about differences in children's interpretation of MC as presented in the two tasks. In particular, we noticed that Australian children tended to sum the two numbers involved in the tasks; this phenomenon may stem from the fact that addition is already introduced in Foundation year. In the case of Israeli and Italian children, adding was more common in the Towers task. This may be related to the fact that the presence of two numbers is more transparent in that task, whereas in the Apple task the number 2 is represented by the word 'double' in a less obvious way. Another explanation is that in the Italian and Israeli translation of 'times-as-tall' there is an explicit reference to the second tower being taller or bigger, as noted by some of the children when justifying their answers.

The general difficulty for Italian and Israeli students to provide a correct answer to these tasks may be explained by their lack of understanding of some words (e.g., 'double' for some Israeli children) or lack of familiarity with phrases such as 'times-as-many' (uncommon in Italian everyday language). Having an understanding of meaning of terms such as these is critical to interpreting a word problem (Nesher, 1988). We believe that linguistic aspects (as an expression of the more general cultural contexts) can explain most of the observed differences. Indeed, different languages have dissimilar features in addressing mathematical concepts that may, as a consequence, result in being more or less familiar to young children. For instance, in Hebrew there is a suffix that can be added to words to express the idea of two-ness (as being in a pair or couple) in the same way that in Latin the prefix 'bi' was used to make explicit the presence of two elements (as the word

'binoculars' which is composed of 'bini' and 'oculus', the Latin words for 'two' and 'eye'). This kind of suffix is common in Hebrew everyday language, which is not the case for 'bi' in English or Italian. Furthermore, there are everyday words (like 'pair' or 'couple'), which may express similar meanings but may be more or less familiar to young children according to their linguistic and cultural background.

While preparing the tools for this study, the authors had to discuss thoroughly both language and cultural issues. For instance, we discussed which objects could be referred to in the texts of the tasks so that they would be familiar to children of all three samples. We believe that our study can contribute to the body of literature about comparative studies; in particular, some of our observations point to the difficulty of translating tasks with fidelity from one language to another, especially for very young children. This is also an important limitation of our study: comparability of the results from the different samples is partial and highly dependent on the different linguistic, cultural, and social contexts in which the data have been collected.

The interpretation provided for the results suggest that Kindergarten teachers can work on different linguistic aspects to promote children's familiarity with mathematical language, which will appear later in schooling (Downton & Maffia, 2023; Vanluydt et al., 2020). For instance, reading story books may help in presenting different words (both 'two times' and 'double') in contexts that make sense to children.

The analysis presented in this contribution has referred only to correct/incorrect answers to the task and qualitative data have only been used to provide a few examples. A further qualitative analysis may provide additional information about the influence of linguistic background on early representations of multiplicative comparison. In particular, we must recognize that there are cases where the child was able to produce a coherent representation (by drawing or using manipulatives) of the situation even if the provided answer was not correct. Furthermore, we did not distinguish between the answers provided by drawing or manipulatives; we believe that those cases deserve attention in future research.

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From patterns to numbers: Supporting six-year-olds' discernment of patterns and structure

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This study applies variation theory of learning to explore the learning opportunities offered by a teacher during a mathematics teaching activity on repeating patterns, aiming to enhance six-year-olds´ understanding of units of repeat and their ability to generalize this structure. Analyzing a video from an intervention study, we identify three critical aspects: 1) the composition of units of repeat, 2) the repetition and variability in the size of the units of repeat, and 3) how identical units of repeat constitute a whole. These findings have implications for early mathematics education, highlighting the need for systematic pedagogical approaches to strengthen students' structural awareness, and emphasize the importance of strong pedagogical content knowledge among teachers.

Keywords: Repeating patterns, structural awareness, early childhood education, variation theory

Introduction

Over the past decade, young students' understanding of patterns and structures connecting fundamental mathematical concepts such as number, measurement, shape and space has gained increasing recognition (Clements & Sarama, 2009; Junker et al., 2025; Mulligan & Mitchelmore, 2009). A range of studies identify patterning as a predictor of mathematical competence (Rittle-Johnson et al., 2019; Verdine et al., 2013; Zippert et al., 2020), linking it to early algebraic, multiplicative (Mulligan et al., 2020; Papic et al., 2011), and spatial reasoning (Rittle-Johnson et al., 2019). Children's ability to abstract structures and represent repetitions and growing patterns enhance numeracy, as the predictable nature of numerical sequences is crucial for counting and calculation skills (Wijns et al., 2021).

The role of patterning in early childhood mathematics education has been the focus of an emerging research domain well researched (Gripton et al., 2023; Wijns et al., 2021; Papic et al., 2011) and linked to number knowledge, part-whole reasoning, and development of multiplicative concepts. However, many studies, including interventions, have been somewhat limited to pattern replicating tasks with few focused on deeper analysis of how to facilitate children's construction of a generalized composite structure and how to develop relationships with number knowledge (Mulligan et al., 2020). Building on this gap in the field and recent intervention studies showing that teachers' ownership of the foundational ideas of patterning and structural awareness is significant for success (Björklund & Elofsson, 2024; Tsamir et al., 2020), the present study aims to explore the learning opportunities offered by a teacher during a mathematics teaching activity with six-year-olds. The focus is explicitly on repeating patterns and students' learning to explain and articulate emergent generalizations. The

following research question will be addressed: In what ways do systematic variation contribute to six-year-olds' discernment of units of repeat and number structure?

The research question will be addressed through an analysis of a mathematics teaching activity that was video-recorded as part of a larger intervention study involving 11 Swedish preschool class teachers and their students. This initial analysis serves as the starting point for a more systematic examination of data (video recordings) from the project. The analysis, informed by variation theory of learning (Marton, 2015) and theories of mathematical structure (e.g., Venkat et al., 2019), directs attention to the critical aspects of repeating patterns that students need to discern in order to develop structural awareness.

Conceptual framework

Mathematical patterns in early education can be found in three categories: spatial structure patterns, repeating patterns, and growing patterns (Papic et al., 2011). This study focuses on repeating patterns, as they are fundamental for more advanced patterning and number knowledge. Repeating patterns (e.g., ABAB or ABCABC) consist of small subsets referred to as the units of repeat (e.g., ABC where the unit of repeat has the length of three), with a cyclic structure allowing prediction of unknown units (Papic et al., 2011). The complexity of a repeating pattern is related to the arrangement of the single units in the units of repeat. Patterns where the units of repeat have two single units, like AB, are seen as the simplest. As the number of single units increases (e.g., ABC or ABCD) or as individual single units are repeated (e.g., AAB or AABB), the pattern becomes more challenging (Lüken & Sauzet, 2021; Papic et al., 2011). Pattern structure requires generalizing relationships, as seen in repeating patterns like ABAB or growing patterns like 1, 4, 9, 16 (Mulligan et al., 2020).

Utilizing a structural approach in teaching aims to deepen students' understanding of both similarities and differences, as well as to enhance their ability to form units and simultaneously consider parts and wholes (Björklund & Pramling, 2014; Mulligan & Mitchelmore, 2009). Systematically arranging sets of items in examples, along with actions that highlight relationships and properties, helps learners perceive local relations, which is believed to foster numerical thinking (Venkat et al., 2019).

Research on patterning

Studies investigating children's patterning can be traced to the field of early numeracy and the focus on numerical competencies for success in the early years of school. A study by Rittle-Johnson et al. (2019) found that preschoolers' understanding of repeating patterns and spatial skills predicted both concurrent and future mathematics achievement (Zippert et al., 2020). With increasing evidence that patterning is a critical predictor of later mathematical performance, several studies have focused on repeating and growing patterns in preschool and early education. For example, research by Papic and colleagues (2011) found that preschoolers show an understanding of the unit of repeat and spatial structuring, retaining and expanding these skills to growing patterns one year later. Additionally, interventions for kindergarten and first-grade students showed impressive growth in patterning and spatial structuring skills including partitioning, similarity and congruence, unitizing and identifying properties of 2D and 3D shapes (Mulligan et al., 2020).

Björklund and Pramling (2014) investigated how six-year-olds discern patterns in concrete activities and their capacity to differentiate repeating units. In line with Papic et al. (2011), they highlight the need for children to be supported in the process of focusing on structural features to discern units of repeat, rather than just focusing on concrete features like color or size. In a related study Lüken and Kampmann (2018) found that patterning and structural abilities positively influenced arithmetic skills in grade 1. More recent studies show that preschoolers can understand and construct growing patterns (Wijns et al., 2021) and in a study with Norwegian first graders (six-year-olds), group differences were found in terms of decomposition and awareness of mathematical pattern and structure: only students with high number sense used relational strategies and those with low number sense used more variants of recursive patterning strategies (Junker et al., 2025).

These studies indicate that children's understanding of patterns often begins with recognizing simple, concrete properties of the objects forming a pattern, such as color or size. However, with guided support that emphasizes structural features, children become able to recognize the units of repeat and use pre-algebraic reasoning to describe patterns. This shift introduces learners to the structural underpinnings of mathematics, especially important for algebraic and multiplicative reasoning.

Method

The present study was conducted in Swedish preschool classes, where an intervention aimed at enhancing six-year-old students' understanding of number structure was implemented over a school year with 11 teachers and their students. The Awareness of Mathematical Pattern and Structure (AMPS) program developed by Mulligan et al. (2020) for the Australian school context, was adapted to fit the Swedish context and curriculum. Specifically, five pathways were selected based on their relevance to the Swedish educational context. The intervention was implemented as part of a larger research project, SATSA[1]. Teachers participating in the intervention were asked to video-record selected teaching activities during its implementation. In this paper, we analyze one mathematics teaching activity captured in a 17-minute-long video recording provided by one of the teachers. Twelve six-year-old students participated in this activity. The activity is from the first pathway in the intervention, conducted at the beginning of the intervention period. The teaching focuses on the concept of repeating patterns, exploring how patterns are composed by identical units of repeat, which can vary in structure across different patterns. This paper presents an initial analysis, serving as the starting point of a more systematic analysis of the video recordings collected from the participating teachers' classrooms.

Variation theory as a framework for analyzing mathematics teaching activities

In this study, we apply variation theory of learning (Marton, 2015) as a theoretical lens to analyze the learning opportunities offered in the mathematics teaching activity. Variation theory directs attention to how a phenomenon is experienced where different ways of experiencing one and same phenomenon is based on what aspects of that phenomenon are discerned. The surrounding world is assumed to be experienced primarily in a holistic sense and becomes more differentiated (detailed

[1] A research project funded by the Swedish Research Council (grant number 2020-03712)

and nuanced) with increasing experiences. Learning is then described as making distinctions that appear as changes in a person's way of experiencing, due to becoming aware of aspects not previously discerned.

Teaching, according to this framework (see Marton, 2015), should start from the aspects of a learning object (phenomenon) that the learner has not yet discerned. Drawing attention to aspects that were previously not discerned is possible through variation and generalization. Critical (new) aspects of the phenomenon are made possible to discern when contrasted against an invariant background. For example, a foundational aspect of repeating patterns is the units of repeat, identifiable by contrasting patterns with varying units of repeat, whereas the aspect "unit of repeat" will stand out and be discerned not as identical in color or number but in how repeated units constitute a similar kind of structure. Taking these principles as starting point, analyzing the video recording means that episodes where the teacher introduces an aspect through her interactions with artefacts are singled out. These episodes are further scrutinized regarding how the artefact is handled (e.g., whether cubes are combined or separated, how units of repeat are positioned, and which gestures are used to emphasize certain features). This process makes it possible to investigate both the necessary aspects for experiencing a phenomenon (like a pattern) in a certain way (as a structure constituting of units of repeat) and how these aspects can be made possible for young learners to discern.

Learning opportunities offered in a repeating pattern activity

The students and their teacher are seated together in a circle on the floor. The teacher introduces a 'train' made of six multilink cubes arranged in an ABC pattern (see Figure 1a). Building on their previous exploration of AB patterns, the teacher asks, "Does this train show a repeating pattern?". She places the train in front of her, repeats the question, and invites responses. The students offer mixed answers, and the teacher asks one student to respond.

Student1: Uh, almost because it needs to be orange-yellow-orange-yellow-orange-yellow.
Teacher: Yes, that's how it would be if it was an alternating pattern.

In this episode, the student expresses that a part of the pattern should be repeated, indicating that the student has discerned that repetition is a key characteristic of repeating patterns (a basic concept in patterning). However, by suggesting an AB pattern instead of identifying the ABC pattern, this also suggests that the student has not yet discerned that the size of the repeating unit can vary across different patterns, and that units of repeat do not always consist of two single units. The teacher then draws attention to their previous work with AB patterns, using this as a comparison to guide the student to discern another important concept: while the structure of repetition remains invariant, the length and complexity of the units of repeat can vary.

The teacher then asks another student for her thoughts.

Teacher: What do you think, Alice (Student2)? Is it a repeating pattern?

The student splits the train into two identical units of repeat and puts them on the floor (Figure 1b).

Student2: No. Yes look, it has two units.

The teacher rearranges the units of repeat and places them in two rows (Figure 1c).

When the teacher asks Alice (Student2) for her thoughts, she chooses to split the train into two identical units of repeat and places them on the floor. By doing so, she demonstrates an awareness that repeating patterns can be divided into smaller, identical units. This indicates that she has discerned the structure of the train as consisting of two units of repeat. When the teacher rearranges the units of repeat into two rows, it further enables the students to see that they are identical. The act of splitting and comparing the units of repeat in one consecutive row and in two separate rows reinforces the idea that the units of repeat are identical and constitute a larger whole.

Teacher: Look, they are exactly the same. See, the units are the same, and there are three cubes in each (simultaneously moves the units of repeat into one row again, as in Figure 1b). How many cubes are there in total?

When asking for the total number of cubes, the teacher shifts attention *from* the units of repeat *to* the whole. The students count, offering different answers. The teacher asks one student (Student3) to tell the total number of cubes. The student counts aloud, pointing to each cube from left to right.

Student3: One, two, three, four, five, six.
Teacher: Exactly. There are six cubes, just like this one, right? (brings out an AB train made of blue and green cubes that they had examined earlier)

Both the ABC train and the AB train are now in front of the students (Figure 1d).

Teacher: How many units of repeat are there in that one? (pointing at the ABC train).

A student (Student4) who raised her hand hesitates when given the opportunity to answer, and the teacher offers reassurance.

Teacher: In this one, there are three. (dis-assembles the AB train into three units of repeat).
Student4: Three, that one has two (pointing to the units in the ABC train).
Teacher: Yes, this one has only two.

In this episode, the whole is kept invariant (both trains consist of six cubes) and the teacher draws attention to how this whole can be seen as a structure of identical units of repeat. By contrasting trains with different units of repeat (two versus three single units, see Figure 1e), the teacher highlights an important concept, particularly for developing multiplicative reasoning. The simultaneous comparison of the two trains is crucial for making this concept discernable; the size of the repeating units directly affects the number of units of repeat in each train. Although both trains have an equal number of cubes, they differ in the number of repeating units. By asking “How many units of repeat are there in that one?” the teacher directs the students' attention to identify both similarities and differences between the two patterns. By splitting the AB train into three repeating units, the teacher makes the concept more tangible, helping the students to see the relation between the total number of cubes and the number of repeating units. This clarifies what is the same and what is different, helping the students to discern the structure of the patterns.

Another student (Student5) signals for attention, indicating that she wants to share something.

Student5: If you want it to be correct, if they are supposed to be the same, then you can do this. If they should be the same. You can rearrange everything so that it becomes like this (moves an AB unit on top of the ABC train) and the same (moves the other two AB unit and places it on top of the ABC train).
Teacher: Exactly. Three 2’s (pointing to the units of repeat in the AB train) are the same as two 3’s (moves the AB units and arranges the trains again in two separate rows).

Student5: Yes!

In this episode, the teacher guides the students to focus on the units of repeat and their role as parts of a larger whole, also using alignment and row and column structure (collinearity). The interaction begins with a student suggesting that the AB units can be rearranged to match the ABC train, which the teacher affirms. By pointing out that "three 2's are the same as two 3's", the teacher highlights the equivalence in the total number of cubes, even though the patterns differ in structure.

The students are next instructed to build their own identical ABC trains with two units of repeat, adding more units gradually as instructed. The teacher asks questions about the number of repeated units and the total number of cubes in the trains, eventually leading to four units and twelve cubes. The teacher arranges the units of repeat into four columns in front of the students (Figure 1f).

Teacher: You can count like this: 1, 2, **3**, 4, 5, **6**, 7, 8, **9**, 10, 11, **12** (counting all the cubes in the left column first: 1 (blue), 2 (yellow), 3 (red); then the next column, and so on). Or 3, 6, 9, 12 (pointing at the blue cube in each column).

The teacher's instruction to build ABC trains with gradually increasing numbers of units of repeat allow the students to explore the concept of repeating units and the relation to the whole.

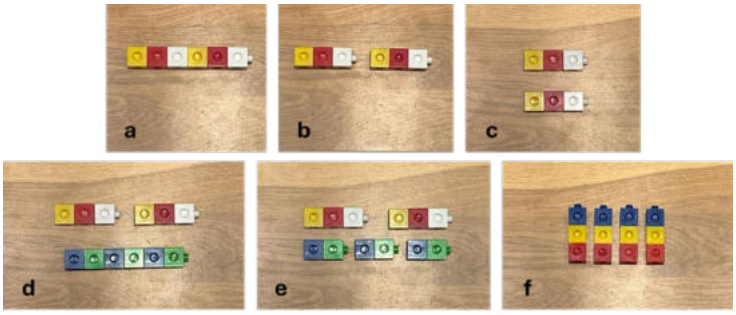

Figure 1 (a-f): Stepwise arrangement of multilink cubes used in the mathematics teaching activity

As the students complete this task, the teacher reinforces their understanding by arranging the repeating units into four columns, making it discernable that they are identical in both design and quantity. This spatial structure helps students see how each unit of repeat is part of a larger whole. During the counting exercise, the teacher first emphasizes the whole by counting all the cubes in sequence using rhythmic counting, and then highlights the units of repeat by skip-counting by 3's. This approach not only helps students recognize the repeating units but also illustrates how these units collectively form the whole and establish a multiplicative counting pattern.

Discussion

In this study, we aimed to explore aspects that students need to discern to enhance their understanding of repeating patterns as composed by units of repeat and to generalize number structure. Our analysis of a mathematics teaching activity in a Swedish preschool class revealed three critical aspects that need to be discerned. The first aspect, *recognizing the composition of units of repeat*, is essential for students to grasp the foundational concept of patterns. The analysis indicated that when the teacher emphasizes the composition of units of repeat (e.g., the cubes connected making up a train), and organizes them with alignment and spatial structure, it helps students understand how single units contribute to the formation of a pattern of composite units of repeat. This aligns with existing

literature highlighting the importance of identifying and distinguishing units within mathematical structures (Björklund & Pramling, 2014; Papic et al., 2011). The second aspect, *the repetition and variability in the size of the units of repeat*, was particularly crucial in this context. As illustrated in the student-teacher interactions, the teacher's guidance prompted students to consider not only the repetition of the units of repeat, but also the complexity of those repetitions. This complexity can significantly affect students' ability to recognize patterns and develop multiplicative reasoning skills, as noted by Rittle-Johnson et al. (2019) and Verdine et al. (2013). By contrasting different patterns, rather than teaching these in increasing order of difficulty and complexity (e.g., AB vs. ABC), the teacher made it possible for students to discern the structural differences in how patterns can be organized, thus enhancing their mathematical reasoning. The third aspect, *how identical units of repeat constitute a whole*, further reinforced students' comprehension of number structure. By asking students to compare the number of repeating units in two different patterns and quantify them, the teacher enabled them to discern the relationship between the units of repeat and the whole. This focus on comparing patterns in this way supports students structural awareness, supporting the development of a deeper understanding of number structure, an essential skill for early arithmetic.

The findings from this study highlight the importance of teachers' strong pedagogical content knowledge in teaching patterns and number, as this supports effective pedagogy and students' mathematical development. Teachers need both content knowledge and effective pedagogical strategies to help students explore these mathematical structures and also connect them to concepts like multiplication, division and fractions (Tsamir et al., 2020). The systematic use of concrete materials in structured ways and drawing attention to the units of repeat is essential in guiding students' development of generalization. This study demonstrates ways in which such systematic teaching can be conducted.

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Becoming confident numeracy players through home play-based early learning activities

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This paper explores how participation in a Family Maths Storytime Programme (FMSP) provided opportunities for play-based home numeracy learning among South African, pre-Year1learners. Based on data from 20 (transcribed and coded) parent interviews, the paper focuses on comments about their children's confidence, which emerged as a theme in all interviews. Through focusing on a single purposefully selected parent interview the paper analyses multi-directional connections between the description of her child's emergent number knowledge, numeracy practices, and numeracy learning dispositions, stimulated by engagement in the FMSP. This analysis highlights how these connections generate confidence in the process of engaging in numeracy practices, and how this confidence simultaneously provides momentum for further numeracy engagement in the home.

Keywords: Family maths, early learning, play-based learning, transition to school.

Introduction and context

Educational inequality in South Africa is extreme. Spaull and Kotze (2015, p. 13) note that "only the top 16% of South African Grade 3 children are performing at an appropriate Grade 3 level" and the "learning gap between the poorest 60% of students and the wealthiest 20% of students is approximately three Grade-levels in Grade 3". Grade R (reception grade, ages 5 to 6 years) is a relatively recent introduction to the South African public formal schooling system (SA DBE, 2011) and is aimed at addressing performance gaps already evident in the early years of schooling. South Africa's participation in international comparative benchmark studies points to the differences in educational and linguistic resources available in learner homes as a key contributing factor to the extreme inequality in mathematical performance (Reddy et al., 2016).

The South African Numeracy Chair (SANC) began in 2011, with the mandate to research and intervene for change in primary mathematics education inequality. As the incumbent Chair, I worked with post-graduate researchers, teacher educators, schools, teachers, learners and communities to establish several projects that included, teacher professional development (PD), after school maths clubs, and family maths events. The Family Maths Storytime Programme (FMSP) was designed to provide home-based support for Grade R children in local communities that were historically marginalised by apartheid. The FMSP and linked research contributes to a gap in research and intervention work with early grade children in relation to use of story- and home-based resources that merge numeracy with early literacy practices.

A range of resources were developed and provided as a stimulus for numeracy (and linked literacy) learning. These included four number story books (with storyboard pages, finger puppets, flash cards of number words and numerals) and dice and cards. The storylines and pictures provided opportunities (first in the number range of 0-5 and then 0-10) for: counting, subitising, counting on, skip counting, establishing number facts, recognising and predicting patterns, number word

recognition and numeral recognition. Linked activities (e.g., using story-linked paper puppets to play number games and retell and enact stories) encouraged learner mathematical talk.

Three sessions were held (spaced at least a week apart) with parents of Grade R learners from two local schools. For succinctness, the term 'parent' includes caregivers who stood in for the role of parent. The language of instruction in both schools was English even though learners spoke mostly Afrikaans or isiXhosa at home. Resources were provided in English and the learners' home language. In each session, I demonstrated reading a number story with the children present, asking them questions along the way (for example, if two frogs jump over how many will be on each lily pad?). I then had a child pretend read and retell the story (using puppets and flashcards on a storyboard) and played number focused puppet, card and dice games—all the while asking a range of *learning focused* questions prompting learners to subitise, count, recognise numerals and number words, predict, follow a pattern, work out one/two more or less, and work out missing parts in part-part-whole games with puppets of story characters. For example, showing them 5 monkey puppets from one of the stories asking 'how many?' — then splitting the puppets between two hands and showing them the puppets in one open hand and asking 'how many are hiding in my closed hand?' (See Jorgensen & Graven, 2022 for an outline of the games).

Literature review and conceptual framing

The FMSP was informed by literature that highlights the role of families in children's mathematical and literacy learning trajectories (Huntsinger et al., 2016) and the importance of play based activities that integrate literacy and numeracy in early learning (De Holton et al., 2001; Purpuro & Napoli, 2015). It integrated numeracy and literacy learning through promoting play-based learning activities linked to number stories. A guiding assumption was that play, in the form of a) representational play (using numeral and word flashcards when retelling the story); b) manipulative play (e.g., using puppets to retell the story on a storyboard page) and; c) rule-bound play (e.g., playing puppet, dice and card games) would provide opportunities for number exploration and concept development and that age-appropriate story scenarios would motivate learning (De Holton et al., 2001).

The FMSP design was guided by socio-cultural theory, viewing learning as occurring within social contexts where language and more knowledgeable others (MKOs) play a critical role in mediating learning within a learner's zone of proximal development (ZPD) (Vygotsky, 1978). Drawing on earlier work (Abtahi et al., 2016) I see the ZPD as multidirectional whereby, within learning activities, the role of MKO can shift between children, adults and the artefacts used in activities. The FMSP prioritised encouraging mathematical talk through having learners pretend read, retell, and enact stories and encouraged parents to ask numeracy focused questions during activities to stimulate talk and mediate for progression. So, for example, when playing with two dice they might ask 'how many dots altogether?' and, if children persist with counting on by touching every dot, they can be encouraged to subitise the number on the dice with the most dots and count on from there.

Bourdieu's (1997) theory of cultural capital and his notion of habitus provided a way of understanding how the noted performance gaps emerge in the early years of schooling. For Bourdieu (1977, p.86), habitus is a "system of internalised structures, schemes of perception, conception, and action common to all members of the same group or class". An implicit sense of habitus begins for learners in the

context of the family. The home habitus becomes reconstituted when new practices emerge and are valued within the home system. The extent to which these new practices can operate as a form of capital outside the home depends on the field one operates in. The field in focus here is the reception year classroom. Cultural capital of wealthier families, with extensive educational resources in the home, translates more easily into capital within the schooling field while learners from families who have been historically marginalised from economic and educational opportunities may not have a feel for the game (Bourdieu, 1977) of schooling (Jorgensen, 2016). Entering the classroom, all children must learn the game of schooling to succeed, involving enacting certain numeracy and literacy practices (e.g., participating in the pedagogic ritual of 'Q&A'—asking and responding to *learning* focused questions.) Supporting learners to become fluent in play-based numeracy learning activities, along with the use of questioning to stimulate mathematical talk and reasoning, was thus considered a productive route to connecting home-based with school-based numeracy learning trajectories.

Methodology of the broader study

Following ethical approval for the study, post-programme interviews were conducted with all 20 participating parents, generating rich data about their experiences of the resources and activities and their observations of their children's learning. Data analysis of the transcribed interviews involved a grounded approach (Glaser & Strauss, 1967) using NVivo to generate coding for thematic narrative synthesis (Elliot-Mainwaring, 2021). I worked with a colleague (Robyn Jorgensen who had attended one of the FMSP sessions) to code the data, as she had some distance from it and was thus more able to generate codes and themes grounded in the transcribed utterances (see Graven & Jorgensen, 2024 for full data coding and general findings). Interviews began with an initial prompt asking parents to share their experience of the FMSP.

Data was coded at the level of whole response rather than at the level of a single idea. This allowed maintenance of the context of utterances and led to responses often having more than one code. Following Robyn's coding, I checked the data against the codes and through discussion a few additional codes were added to data chunks and some codes with highly similar data were merged. Acknowledging that the coding process can yield different results if coded by another coder or at a different time, the value of the coding is not in the specificity of coding (or frequencies for each code) but in the themes generated. The data pointed to ways in which the FMSP activities had influenced children's learning in relation to three main categories, namely, new knowledge, practices, and dispositions, with a fourth category of codes involving the nature of parent learning. The first three categories resonated with Bourdieu's (1977) constructs of capital, practice, and dispositions and these provided a conceptual lens for organising and analysing the full data set.

In this paper I focus on the only code that emerged across *all* twenty interviews, namely learner confidence (or self-confidence), which was placed in the category of new dispositions (see Graven & Jorgensen, 2024). Statements about children's self-assurance of their own knowledge or abilities in relation to engaging in mathematical activities (including reading number words, pretend reading or enacting number stories) were considered indicators of confidence. These indicators were interwoven in data chunks (or segments) across multiple codes and categories. In this paper I explore the question: *How did children's emerging confidence as expressed in parent interviews connect with the*

emergence of new numeracy knowledge, practices and dispositions? As it is beyond the scope of this paper to share coded excerpts from all twenty interviews, I have selected data from a single interview to illuminate and analyse the nature of the interconnectedness between confidence and new knowledge, practices, and dispositions. I have purposefully chosen an interview that has several anecdotes that similarly emerged across multiple interviews, including: playing 'teacher' when sharing activities with neighbours and friends; asking lots of numeracy learning questions; seeing (and engaging with) numbers in the world around them; initiating, leading and willingly engaging in numeracy games. Thus, while the utterances shared in this interview are specific to what Sadie (as parent) noted about Elsa's (her child's) engagement (both pseudonyms) with number ideas and FMSP resources in the home, each of the ideas discussed here recurred across multiple interviews.

Focusing on a single interview allows narrative flow and full data chunks to be shared to support the qualitative analysis of how Elsa's emerging confidence (as noted by Sadie) is interwoven and expressed in her emerging new forms of knowledge, practices and ways of engaging in learning activities. I began with the full interview, deleting non relevant parts. So, for example, I removed initial introductions. I then re-read the interview multiple times with the intention of finding quotes that specifically spoke to Elsa's increased confidence. I noticed, however, that while some quotes were more explicit than others in highlighting Elsa's confidence, the interview tended to shift seamlessly between Elsa's confident ways of *knowing*, *doing* and *engaging with others* around mathematical activities. I thus adopt a narrative style of sharing Sadie's story. What follows is my story of her story (after Sfard & Prusak, 2005). In telling the story, I weave together Sadie's anecdotes about Elsa with my analysis of the interplay between Elsa's self-confidence and her emerging ways of knowing, doing, and engaging with numeracy ideas. To the extent possible within space constraints, I include Sadie's full comments trimming only superfluous utterances. This allows the reader to judge whether my analysis resonates with their reading of the data.

Sadie's story of Elsa's numeracy engagement at home: Findings and discussion

Responding to the invitation to share what she had learned through participation in the FMSP, Sadie began by explaining how Elsa was engaging mathematically with the world around her and seeing the world increasingly through a mathematical lens:

> It kind of opened my brain. When I see my child, even on a Sunday when we go to church she counts the houses. She says, "There are two houses, there's No. 2, there's No. 18." She is getting it right when she sees numbers or words…. When she walks down the steps she jumps, and says, "Count with me, one, two, three. What comes after 20?" And I say "21". Then she counts up to 30. Now she knows 31, 32, …

Sadie noted Elsa's emerging number knowledge stating "she is getting it right" and "now she knows 31,32" while also pointing out that Elsa asks for assistance to extend her knowledge horizons beyond what she already knows (e.g., asking "what comes after 20?"). Sadie continues explaining how Elsa uses questioning to engage herself and others in mathematical activity and conversation.

> She asks lots of questions. She even asks my mother. She says, "Mummy please can you come and sit here, I want to ask you something. Tell me how many plates are in the cupboard, how many

> forks are there?" Then my mother (Elsa's grandmother) says, "Go and count my angel and then come and tell mommy." Then she takes everything out of the drawer, forks, knives and everything so that she can count them all. Then she splits them up. She puts the spoons, forks and the knives separately, then she counts, "10 plus 10 plus 10 Mummy. Give me the answer because I don't know."… Even with her father. "Daddy how many wheels on the car?"

In the second last sentence we see how Elsa asks questions beyond what she knows but are within her reach (or ZPD). Elsa is aware that she does not know how to add 10 and 10 and 10 but expresses an eagerness to learn it. Her questioning and interest thus extends beyond the number range that was in focus in the FMSP (0-10 in the stories and 0-12 for the games with two dice). Sadie then shares how Elsa similarly engages with her older and younger brothers. She shares Elsa's confident declarations about "knowing numbers" and that with this knowledge she can participate in her older brother's homework and help her younger brother to "also" become "clever" with maths.

> Now she wants to use the brother's books. He does his homework at the big table. Then she says "What are you doing? I know the numbers, I know."… She wants to do so much, and you can see she is enjoying it. She is experiencing more about numbers. She tells her [younger] brother she will help him with maths. She says, "Come and sit, you are also going to get clever."

In these anecdotes Sadie expresses how Elsa is confident about having the knowledge capital needed to engage with both brothers mathematically. When asked about the books and resources Sadie explained:

> [Before] she only read for about three minutes, then she was finished with the book. And now with these books you gave us, she knows exactly what is on the pages. She wants to talk about the books… The book my grandmother bought her as a present from the library, she would just look at. Then she would look through the pages, then put it down because she was tired, go and get her [FMSP] book and then say, "Come, mommy, come and sit here, I want to read to you."

Here, we see that Elsa is initiating reading and talking about the FMSP books that support her 'pretend reading'. Sadie contrasts Elsa's engagement with the FMSP books with a book from the library that had her only looking at the pages rather than engaging with the ideas and pretend reading. The pictures, numerals and words (including number words in large font) in the FMSP storybooks are designed to guide pretend reading and storytelling. Sadie continues sharing how Elsa enthusiastically engages with the FMSP dice and card games.

> Then we have to play 'Dingaan's Kraal' and she won't let me go. I have to stay and play with her… She used to throw one dice, but now she has seen me throw two dice, she also wants to play with two dice. And she says the numbers on the two dice… She knows the numbers. She knows that the five has got the two dots at the top, and two dots at the bottom, with one dot in the middle. I think she remembers… She knows the 6, then she counts, "7, 8, 9 and 10." When she gets to 10 she wants to count them on her fingers. And when it is more, I must add my fingers. Like the six and six. But now she wants to learn how to write the word twelve.

Again, we see how Elsa initiates playing mathematical games. Dingaan's Kraal is played as a matching numbers card game and as a 'bonds to 10' card game. Sadie once again explains how while

playing these games Elsa demonstrates and practices her number knowledge while simultaneously extending her learning horizon. Elsa's confidence in being able to subitise and count on, and recognise and write the number words 0-10, provides a basis for wanting to learn to recognise and write the number word twelve. (The FMSP flashcards of numerals and number words were designed to accompany the stories and so only went up to ten while the sum of two die extend to twelve).

Sadie then shares an anecdote about her and Elsa's storybook reading with imaginative role-play, using everyday objects to represent trees, umbrellas, and characters in the stories to enact unfolding number patterns. In the first two FMSP stories in the number range 0-5, five monkeys in a small tree and five children under a small umbrella move one by one to a bigger tree and umbrella respectively. In this imaginative and representational play, Elsa is seen to use comparative mathematical language to speak about what is happening while they are acting out the stories.

> I read to her then she mentions how she reads to me. "I am the mum, you are the child," then she reads to me. With the 'Monkey' book, she wants to sing it for me, "Five little monkeys." She says, "there were five monkeys in a big tree." Sometimes the spoons are the monkeys… And then we take the plastic cups, and they are the trees, and then she takes the spoons and says, "Five monkeys in this tree. And the monkey says, "Oh no this tree is too small," and she jumps the big tree with the spoon. And with the umbrella, she takes a baby blanket for an umbrella. Then I say to her, "Come let's play a game. We take two chairs and make a tent for the big umbrella, then I take the small blanket, and that is the small umbrella. And then we take all the spoons she played with, and they are the children. And I say I am going to pretend I am the child. Then she says, "The child must jump to the big umbrella," and I jump… Then I ask her, "How many more?" and she says, "There are four under this umbrella, and one under that umbrella. More under this one and less."

Elsa's confidence in knowing the unfolding patterns in the storylines enables enacting the stories without the book or the finger puppets. Sadie shares how Elsa has shifted from being relatively shy with children to being eager to read to her friends and teach them counting—again initiating the engagement by calling neighbours to come and play and leading ('teaching them') the activities:

> She did not want to play with the neighbours before, but now she wants to go out and play with the other children, but we don't want her to go because it is not safe in the street. So, when the children ask if she can play, we tell them they must play in the yard. She calls them to come and play…. But she wants to read the book about the monkeys to those children. They are all in Grade R. She says, "Come I am going to read to you. Don't you understand English? I can try and read in Afrikaans"… She never had friends calling, but now she is more interested in teaching them how to count… She was a shy girl but now she has more friends, talks and does stuff.

Here Elsa's confidence in her ability to read her FMSP storybooks and being able to engage in mathematical talk is said to have shifted her from having a shy disposition to one who is willing to lead others and initiate play and learning activities. This confidence is seemingly supported both by her awareness of her emerging numeracy and literacy knowledge, and her developing fluency in doing and talking mathematics. Elsa was given books and resources in both English (the language of instruction in the Grade R class) and Afrikaans (the language spoken at home) allowing her to offer to 'read' in Afrikaans. The excerpt points to the way in which Elsa's strengthened numeracy and

literacy capital, and her ability to engage in mathematical talk and play, generate confidence in her mathematical and reading skills to the point of being ready and willing to 'teach' others. It further highlights how her emergent confidence when enacting her knowledge in various practices simultaneously promotes increased engagement in play-based learning activities, which in turn likely leads to further confidence. This finding resonates with Murray and Harrison's (2011) finding that parents can identify their children's confidence and school readiness. They propose developing early years' home-based learning opportunities to develop resilience and confidence in children to strengthen school literacy and numeracy trajectories.

Conclusion

In the sharing and analysis of *Sadie's story* of Elsa's engagement with mathematical and reading practices at home, we hear of Elsa's emerging self-confidence. This confidence is visible to Sadie in Elsa's appreciation of her own numeracy knowledge (e.g., 'I know numbers') and capacity (and eagerness) to engage with others in mathematical activities and conversations (as evidenced by the many people that Elsa is said to initiate mathematical activities and conversations with.) Through analysis of Sadie's post-FMSP interview, we see multi-directional connections between Elsa's mathematical learning confidence, knowledge, practices and dispositions (ways of engaging).

The FMSP provided resources to develop learners' numeracy and literacy capital at home to strengthen their learning trajectories in their transition to schooling. A play-based approach, which integrated numeracy learning with reading and enacting simple number stories, and playing dice and card games, built a bridge between home play practices and school learning practices (e.g., engaging in Q&A—asking learning type questions to promote conversations that support and extend learning). Resonating with a socio-cultural perspective on learning, confidence emerges as part of the process of learning new knowledge, practices and dispositions and as a product of these processes. Beyond this, confidence, as simultaneously a product of learning and part of the learning process (see, Graven 2004 for a similar finding with in-service teacher learning), gives momentum to Elsa engaging in many more mathematical learning opportunities that transcend engagement with FMSP resources.

The paper highlights how provision of purpose designed numeracy resources and activities, in partnership with parents, can support home based numeracy learning opportunities that provide momentum for learning beyond the resources. As researchers and educators committed to equity in mathematics education, we must search for effective ways to challenge the findings captured in the title of Spaull and Kotze's (2015) article: *Starting behind and staying behind in South Africa: The case of insurmountable learning deficits in mathematics*. Interventions, such as the FMSP, can act for change, challenging the extreme performance gaps that indicate continued marginalisation of most South African learners from successful mathematics learning trajectories.

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Parents' and teachers' view on mathematics skills of children with mathematics learning difficulties in pre-school

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The development of mathematical understanding begins with the birth of a child throughout subsequent stages of learning. Children who experienced a mathematics learning disability at the end of preschool were more likely to continue experiencing this disability throughout primary and secondary school. The article describes and analyses theoretical materials and data about mathematics skills in preschool as an effective indicator of the potential development of mathematics learning difficulties. The research focus on the analysis of parents' and teachers' view of mathematics among 4-6 year old Latvian preschool children. Main findings reveal the relationship between parents' and teachers' evaluation of children' mathematics skills in preschool.

Keywords: Children, mathematics learning difficulties, parents, pre-school, teachers.

Introduction

The acquisition of mathematical knowledge is often challenging for individuals with intellectual disabilities (Chodura et al., 2015; Nuari et al., 2019; Scherer et al., 2017) and those with specific learning disabilities in mathematics (Dekker et al., 2016; Harvey & Miller, 2017; Martin et al., 2017). A specific disability in mathematics, which may be referred to as dyscalculia, is usually diagnosed in the later years of primary school. This approach fails to capitalise on an important period of early brain plasticity during which interventions would be most effective (Ashkenazi & Henik, 2010; Zamarian et al., 2009). The term 'mathematics difficulties' is used to describe both students with low mathematics achievement and students with a diagnosed mathematics disability. Specific learning disabilities are often diagnosed in the later primary grades (Geary et al., 2004; Berch & Mazzocco, 2007). Implementing early diagnosis and targeted remediation can significantly enhance a child's performance in mathematics, as well as prevent mathematics learning failure by targeting basic skills (Desoete et al., 2012; Nelson & Powell, 2018). It is of the utmost importance to identify the early indicators of potential mathematical difficulties to integrate them into appropriate intervention programmes. Consequently, the mathematical achievement of a child in preschool, as assessed by parents and teachers, could serve as an effective indicator of possible learning difficulties in mathematics for children with specific learning disabilities.

Mathematics learning difficulties

As learning disabilities in mathematics are associated with difficulties in understanding numbers and quantities, using number symbols and basic arithmetic operations, the symptoms of this disorder

appear in approximately 6% of children as early as the second year of primary education (Shalev, 2007; Jiménez-Fernández, 2016). In the early years of primary education, students with mathematics difficulties demonstrate lower achievement in counting tasks (Stock et al., 2010) as well as in the accuracy of calculations (Tolar et al., 2016). Furthermore, they experience difficulties in comparing quantities or representing them. Symbolically, they experience difficulties with text-based tasks (Driver & Powell, 2015; Nelson & Powell, 2018). Children with learning disabilities in mathematics typically exhibit greater difficulties in the following mathematical content areas: number sense, basic or fundamental arithmetic operations, and problem solving (Jiménez-Fernández, 2016).

Infants are instrumental in demonstrating an understanding of number that predicts their future mathematical performance in preschool (Starr et al., 2013). Furthermore, early developmental experiences with mathematics influence their preparedness for preschool and school (Nelson & Powell, 2018). Children who experience difficulties in mathematics tend to demonstrate inherent difficulties in a range of informal and formal mathematical abilities from an early age, even before they reach preschool. These difficulties encompass various aspects of numerical literacy, including reading and writing numbers, grasping fundamental counting principles, and learning basic arithmetic facts (Desoete et al., 2012; Geary et al., 2004; Mazzocco & Thompson, 2005). These difficulties in informal and formal mathematics learning place children at an initial disadvantage, as there is a strong correlation between early mathematical competence and the subsequent acquisition of mathematical skills and the rate of growth in mathematical achievement. Consequently, children with mathematics difficulties may experience a widening achievement gap between themselves and their peers, taking to the 'Matthew effects' observed in other domains (Namkung et al., 2019).

The aim of the study is to investigate parents' and teachers' view of mathematics achievement in two subgroups - above and below the 16th percentile level which was determined by the results of the BAASIK Mathematics performance subtest among 4-6 year old Latvian preschool children. The research questions were: RQ1: How do parents and teachers evaluate the mathematics skills of children with and without mathematical difficulties in pre-school (skills level was determined using the results of the e-BAASIK Mathematics performance subtest - above and below the 16th percentile)? RQ2: What is the relationship between parents' and teachers' evaluation of children' mathematics skills in preschool?

Methods

The study, part of the University of Latvia's project "Development of a Unified Set of Methodological Tools for Screening Children's Early Development" on early development screening, aimed to create the e-BAASIK toolkit for assessing developmental risks in children aged 1-6. It focused on 11 mental development domains according to the categories in the International Classification of Diseases, 11th edition (ICD-11). The e-BAASIK construct framework for the age group 3-6 years comprises 11 core scales: large and small motor skills, cognitive abilities, speech and language, adaptive behaviour, disruptive and dissociative behaviour, internalizing problems (anxiety, fear and depression), communication difficulties and autism, attention deficit hyperactivity disorder (ADHD), pre-reading skills, pre-writing skills and pre-primary mathematics skills (Raščevska et al., 2024a; Raščevska et al., 2024b). Data were collected between February 2022 and July 2023 from Latvian preschools in

all regions of Latvia, involving trained professionals and parental consent. The data employed in this study were derived from Math scales (initial mathematical skills) of the e-BAASIK Parent's Survey (PS), Preschool Teacher's Survey (TS) and Child's Ability and Skills test (CAST - a performance test) for the 4-6 age group, comprising the following skills: comprehension of the numerical values 1-5, 1-10; mathematically precise reasoning and fundamental arithmetic operations (see Table 1). Each item of PS and TS was evaluated with 1-4 points, and in the CAST with 0, 1, 2 points.

The sample consisted of children (50% girls, 50% boys) aged from 4 to 6, in age 4 and age 5 group n=280, in age 6 group n=253. For each child, the survey was completed by the parents (either mother or father) who had given their written consent to the study and survey was completed by concrete child's teacher who had participate in project. Descriptive statistics and Spearman's correlation coefficient (v.29, IBM SPSS Statistics) was used to analyse the relationship between the measurements of e-BAASIK Parent Survey basic scales and e-BAASIK Teacher Survey basic scales.

Results

The percentage of children aged 4-6 who achieved the highest rating in the mathematics skills subscale of the e-BAASIK Parent Survey and Teacher Survey is presented in the following table. The children have been grouped according to their risk status and performance on the e-BAASIK test (see Table 1).

A comparison of the maximum scores achieved by children on the mathematics skills items in the parent and teacher surveys, as presented in Table 1, reveals that children who are at risk of developing mathematical difficulties are significantly less likely to have fully mastered the relevant skill than children not at risk of developing mathematical difficulties. A comparison of the performance of children across all age groups (4, 5, and 6 years) in the two groups revealed that those in the NRG group consistently demonstrated superior mathematical abilities compared to their RG counterparts. As an illustration, in the recognition of printed numbers (question 3), 40.4% of the children in the NRG group achieved success, compared to only 17.9% in the RG group, as determined through parent evaluations. Similarly, in both the at-risk and non-at-risk groups, the majority of items were rated higher by parents than by teachers. Children in the at-risk group (aged 4-5 years) exhibited the greatest likelihood of scoring higher on question 1 (Can a child tell which is longer or shorter by comparing 2 pencils or other objects of different lengths?).

The data presented in Table 1 of the remaining articles demonstrate that children at risk exhibited the greatest difficulty in recognising printed digits, counting objects up to 10, understanding which number is larger and which is smaller, arranging a string of numbers in a logical numerical sequence, and finding a larger group of objects among numerically distinct groups of objects (questions 3-7). Conversely, six-year-olds at risk demonstrated the least proficiency in performing addition and subtraction of one and two digits, identifying the largest group of animals by number, and arranging numbers in ascending order (see Table 1). The discrepancy was particularly evident in tasks requiring numerical operations, such as addition (question 8) and subtraction (question 9).

Children in the NRG group demonstrated greater proficiency in complex mathematical tasks, particularly in addition and subtraction, at the age of six. In these operations, 74.8% of children in the

NRG group were able to achieve a correct answer, in contrast to only 40.0% of those in the RG group. This is despite these skills being explicitly outlined in the relevant curriculum. In the 4-year-old group, parents' ratings are generally higher than teachers' ratings on all mathematical skills items (see Table 1). In the sample of five- and six-year-olds, in both the at-risk and the norm groups, teachers were more likely than parents to give higher scores on some items, such as items 3, 6 and 8.

Table 1: Parent's and Teacher's view on children (age 4-6) math skills performance subtest results on e-BAASIK subscale

Questions	Group	Number of children in percentage who gave the highest rating					
		age 4		age 5		age 6	
		P***	T****	P	T	P	T
1. Can a child tell which is longer or shorter by comparing 2 pencils or other objects of different lengths?	RG*	55.4	52.8	75.0	62.7		
	NRG**	81.6	71.0	91.3	86.0		
2. Can the child count orally, calling the numbers in the correct order?	RG	50.0	28.3	85.4	72.5		
	NRG	81.6	61.9	95.2	91.9		
3. Does the child recognise the printed numbers 1 to 9?	RG	17.9	9.4	54.2	45.1		
	NRG	40.4	31.2	74.3	77.5		
4. Can a child determine the number of objects up to 10?	RG	33.9	13.2	73.5	56.9	74.5	75.5
	NRG	68.9	46.3	89.0	87.2	95.2	93.9
5. Does the child understand which number is bigger or smaller?	RG	10.7	1.9	32.7	25.5	59.6	57.1
	NRG	27.2	22.1	65.7	65.8	84.3	83.5
6. Can the child put 4-5 digits in the correct order if the digits are not given in order, e.g. 5, 7, 1, 4?	RG			22.9	20.0	34.0	44.9
	NRG			43.6	50.6	57.6	68.4
7. Can the child find the largest group of animals in the picture, e.g. with 3 cats, 5 dogs, 4 lions and 8 mice?	RG	22.6	13.2			53.3	47.8
	NRG	43.3	32.6			80.1	76.3
8. Can the child add 1 and 2 by performing an addition operation?	RG					40.0	43.5
	NRG					74.8	67.6
9. Does the child know how to subtract 1 and 2 when performing subtraction actions?	RG					33.3	30.4
	NRG					61.7	58.5

*RG - group with math difficulty risk (under 16th percentile according to BSPT results), in the sample of Parent's Survey age 4 and 5: n=56; age 6: n=46; in the sample of Teacher's Survey age 4 and 5: n=53; age 6: n=46; **NRG - group without risk (above 16th percentile according to BSPT results), in the sample of Parent's Survey age 4 and 5: n=224, age 6: n=207; in the sample of Teacher's Survey age 4 and 5: n=227, age 6: n=207; ***P- e-BAASIK Parent's Survey; **** T – e-BAASIK Teacher's Survey

According to parents' ratings, children's mathematical skills improved with age in both the RG and NRG groups. For example, the skill to determine the number of objects up to 10 (question 4) increased

from 33.9% at age 4 to 74.5% at age 6 in the RG group and from 68.9% to 95.2% in the NRG group. The children's performance in mathematics, as assessed by teachers, also improved. For example, the skill to determine the number of objects up to 10 increased significantly from age 4 to 6, with NRG children improving from 46.3% to 93.9% and RG children improving from 13.2% to 75.5%. Despite the fact that teachers' and parents' initial assessments of children's mathematical skills at age 4 differ, teachers' and parents' assessments of 6-year-olds' mathematical skills are, in most cases, the same.

In the majority of tasks, the skills of the children were rated by teachers to be slightly lower than those rated by parents. To illustrate, in task to compare length of objects (question 1), 81.6% of four-year-old NRG children were rated positively by parents and 71.0% by teachers. Similarly, in oral counting task (question 2) 81.6% of four-year-old NRG children were rated positively by parents and 61.9% by teachers. This trend may be indicative of discrepancies in the observational environment, assessment criteria, or the interpretation of mathematical education content.

The tasks that required more complex thinking, such as putting numbers in order (question 6) or performing operations such as addition and subtraction (questions 8 and 9), were more challenging for RG children. There were significant differences between them and their NRG peers in their ability to complete these tasks. The ratings provided by both parents and teachers indicated that RG children exhibited marked difficulties in performing mathematical skills when compared to their NRG counterparts. For instance, the mean scores for skills requiring the ordering of four to five digits (question 6) were significantly lower for RG children, with parents scoring 22.9, while those for NRG children's parents were 43.6. Similarly, the mean scores for RG children's teachers were 20.0, while those for NRG children's teachers were 50.6.

A t-test of paired samples was conducted to ascertain whether there were statistically significant differences between parents' and teachers' mean scores for the scale items in the at-risk and non-at-risk groups. The results indicated that for 4- and 5-year-olds in the at-risk group, there were statistically significant differences between parents' and teachers' scores for scale item 4 only, while the scores for the other scale items were more or less the same. In contrast, the norm group of four-year-olds exhibited statistically significant discrepancies between parents' and teachers' ratings across the majority of scale, with the exception of scale item 5. In the samples of five- and six-year-olds, both in the at-risk and non-at-risk groups, there were minimal differences between teacher and parent ratings, with the exception of scale items 4 and 6.

A statistically significant difference was identified between the parent and teacher risk groups on the mathematics skills summary scale in the 4- and 5-year samples. In the 4-year sample, the mean score for parents was 16.44 (SD=4.4), while the mean score for teachers was 15.02 (SD=3.95, $t=2.40$, $p< .02$). In the 5-year sample, the mean score for parents was 20.00 (SD=3.87), while the mean score for teachers was 18.63 (SD=4.60, $t=2.32$, $p< .02$). The only statistically significant differences between parents' and teachers' scores in the norm group were observed in the 4-year-old samples (M=20.35, SD=3.31; M=19.15, SD=3.64, $t=5.26$, $p< .001$), and not in the samples of 5- and 6-year-olds. No statistically significant differences were identified between the parent- and teacher-rated summative scales of mathematics skills in the sample of six-year-olds, irrespective of whether the children were considered to be at risk or not. In conclusion, the analysis of both the Mathematics Skills Scales items

and the summative scales demonstrates that the differences between parent and teacher ratings are more pronounced at age 4 and in the non-risk groups but decrease or even disappear at ages 5 and 6 in both the risk and non-risk groups.

It is important to note that while the correlations between parents' and teachers' ratings of mathematics skills are statistically significant, they are also moderately close (4 years r= 0.55, 5 years r=0.43, 6 years 0.49, p< .001). This indicates that there are considerable individual variations in parents' and teachers' ratings.

Conclusion

A generalisation of the responses of parents and teachers to questions concerning children's mathematical abilities reveals a similarity in their assessments of performance in situations where no interpretations are necessary or possible. This is exemplified by tasks such as performing simple arithmetic operations, including adding and subtracting one or two digits. Such child performance is readily discernible and assessable for both parents and teachers, as it is mathematically correct, with no inherent interpretations or variations in solution.

In most cases, teachers' assessments diverge slightly from those of parents of children at risk. Teachers tend to rate the mathematical abilities of children at risk slightly lower than parents, whereas parents tend to award higher ratings to the mathematical abilities of children at risk. However, at the age of six, parents' ratings tend to align more closely with those of teachers in relation to children at risk and their mathematical skills. Nevertheless, it can be argued that both teachers and parents are able to assess children's performance accurately when they have a learning disability in mathematics and when they do not have a learning disability in mathematics. The similarities in the assessments are particularly evident in the case of six-year-olds, while the similarities and correspondences are somewhat less pronounced in the case of four-year-olds with and without a learning disability.

The stark contrast between at-risk and non-at-risk groups highlights the crucial significance of prompt recognition and prompt intervention, particularly with respect to fundamental capabilities such as numeracy and calculation. Although both parents and teachers are aware of typical developmental trajectories, teachers tend to be more cautious in identifying potential areas of concern. These findings reinforce the value of a collaborative approach involving teachers and families to support children's early mathematical development. However, discrepancies between parent and teacher ratings could be attributed to differences in observational conditions or interpretation of skills. These data underscore the divergent views of parents and teachers and the necessity for a standardised system to harmonise assessments, thereby promoting consistency and accuracy in identifying developmental needs.

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How teaching actions with artefacts enable children's understanding of the part-whole relationships in an arithmetic task

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This paper originates from a study conducted by preschool teachers and researchers that jointly designed, implemented and assessed activities that focus on numbers and basic arithmetic skills with children 3-5 years old during one school year. The research question is "How can teaching actions and artefacts enable children to discern the part-whole relationship of numbers in a partitioning task?". The data analysed in this paper consist of 33 video recordings of an implemented teaching activity focusing on partitioning and how numbers constitute part-whole relationships. These have been analysed through variation theory of learning. The results support earlier findings that whether the artefact is used in a structured or unstructured way has an impact on the possibility to discern number relations, which can be enhanced by verbal communications, gestures and finger representations.

Keywords: Artefacts, teaching actions, numbers, preschool, variation theory.

Introduction

As part of formal education, preschools in Sweden should provide children with opportunities to develop "an ability to use mathematics to investigate, reflect on and try out different solutions to problems raised by themselves and others" (Swedish National Agency for Education, 2019, p. 15). However, some preschool teachers have been shown a lack of knowledge about mathematics and how to teach mathematics in preschool (Skolinspektionen, 2017). An example of a study addressing this issue is Palmér and Björklund (2023), who argue for the need of applied research in which preschool teachers and researchers work together in order to develop teaching content, activities and artefacts for teaching mathematics.

The study presented in this paper is part of a larger project, which aims to develop teaching about number and basic arithmetic in Swedish preschool for 3–5-year-olds. During one school year (September – May) six preschool teachers and three researchers worked in close collaboration to try out and develop teaching methods and artefacts that specifically enable children's discernment of number structure and the part-whole relationships within and between numbers. Artefacts were introduced and trialled to explore their potential to be used for the project's aims. In this paper, the specific aim is to gain in-depth knowledge of *what teaching practices provide opportunities for basic understanding of numbers and arithmetic skills in preschool*. To fulfil this aim, we seek to answer the research question: *How can teaching actions and artefacts enable children to discern the part-whole relationships of numbers in a partitioning task?*. To guide this exploration, empirically and analytically, we use Variation theory of learning (Marton, 2015).

Theoretical framework

The current project has its foundation in research that takes starting point in learning as change in ways of experiencing phenomena in the surrounding world, which means that numbers are by children experienced in qualitatively different ways that inform how a child handles numbers for example in

a problem-solving situation (see Neuman, 1987; Björklund et al., 2022). Cardinality, ordinality and numbers' part-whole relationships are considered essential aspects to discern, in order to interpret and make use of numbers. Numbers' meaning cannot, however, be "seen" due to their abstract nature (Duval, 2006). Representations are thereby also an essential aspect of numbers, because cardinality, ordinality and the part-whole relationships that are "invisible" features of numbers can be seen through the use of verbal and written symbols, real life situations and concrete materials such as pictures, or manipulatives (Lesh, 1981).

Symbols, materials or manipulatives are artefacts often used in mathematics education to represent some mathematical meaning and thereby aid children's learning. Artefacts can indeed be constructed in ways that more easily bring forth number structure, such as an abacus. Artefacts are however not self-instructive; they afford meaning through the child's use of the artefact and new meanings can be mediated through a teacher's way of using a specific artefact. Thus, it is significant to direct attention to *how* artefacts are used. In preschool settings, unstructured material is often used, such as pebbles or blocks, which can become structuring artefacts depending on how the artefacts are used (Venkat & Askew, 2018).

Variation theory of learning (Marton, 2015) offers a framework for describing and explaining the learning of new concepts in terms of discerned aspects. In line with this theoretical framework, the learner is assumed to have a holistic view of his or her surrounding world. However, this view constitutes of discerned aspects where some are foregrounded in the learner's awareness, and some are left in the background (Gibson & Gibson, 1955). To develop one's way of seeing the surrounding world (learning to see the world in a new more advanced way), such aspects that have not earlier been foregrounded need to be discerned. This is done through the contrasts appearing when an aspect is varied against an invariant background. This is a basic principle of learning in this framework, which is applicable in teaching when the teacher uses artefacts in a systematic way that brings fore necessary aspects through contrast and generalization. Central is also that these contrasts are made simultaneously, to allow the learner to discern significant similarities and differences (Marton, 2015).

Background in research

Children usually learn the counting rhyme in early years (see e.g., Mix 2002; Ahlberg, 1997). But learning to use numbers in arithmetic problem solving is a rather complex skill of making use of counting principles (Gelman & Gallistel, 1978), integrating seriation and classification (Piaget, 1965) and as empirical research has shown also in recent studies, recognizing the mathematical relationships between numbers reflected for example in basic number combinations (Venkat et al., 2019). This means that children experience 5 as a "structural phenomenon", constituting the parts 3 and 2 but also 4 and 1. When this structural aspect of number is not discerned by children, they may rely on single-unit-counting strategies, which are known to be error-prone and not developable strategies in a longer perspective (see Neuman, 1987; Ellemor-Collins & Wright, 2009).

Structural approaches to teaching and learning numbers have been used in mathematics education both with older school children (Schmittau, 2004; Venenciano et al., 2021; Polotskaia & Savard, 2018; Venkat et al., 2021) and younger preschool children (Björklund et al., 2021). These studies provide evidence of success in focusing on number in ways that emphasize part-whole relationships. Intervention studies with younger children have also shown that it is possible to teach numbers' part-

whole relations that support children's advancement of number knowledge and arithmetic skills (Cheng, 2012), even before formal mathematics education takes place (Palmér & Björklund, 2023; Kullberg et al., 2020) through systematically implemented activities. The challenge in teaching in the early years is however to find out and adapt the teaching acts (and artefacts) to the child's way of experiencing the mathematical phenomenon at hand (see Ekdahl, 2019), hence more research is needed regarding design and use of artefacts for number learning in preschool.

Method

The project is a collaboration between researchers and preschool teachers in one Swedish municipality. Teaching activities that have been tried successfully in previous projects are proposed to the participating preschool teachers, with whom the methods and goals are discussed, and the activities are carried out and video recorded for assessment and further development at recurring meetings. The research project thus involves development work in the preschools' practice with a focus on developing teaching. The purpose of the research project is to understand and explain what it is in teaching with a structural approach that contributes to knowledge development in children. The purpose of the development project is to develop and test teaching about numbers, based on an approach that favors children's sense of numbers and basic arithmetic skills in preschool. Ethical permission has been obtained to use video recordings involving children. The video recordings are used both for professional development and for research purposes.

During one school year from September to May, six preschool teachers from four different preschools and their teams participated in the project. Participating children were aged 3-5 years, 110 children in total, but the participation frequency varied during the year. The children were participating in the teaching activities conducted by their teacher. Video recordings of these teaching situations were collected by the teachers themselves.

The analysis for this paper is based on 33 video recordings (5 h and 47 min and 31 s) of teachers teaching one arithmetic task in one of the teaching activities with similar artefacts. The teachers enact the activity with single children or in pairs. These 33 video recordings have first been examined in terms of the teachers' actions with artefacts and how the teaching supports or does not support the child's ability to discern the part-whole relationship in the task. In this paper, three video recordings have been selected for a more thorough analysis of the teachers' use of artefacts to facilitate children's learning. In these recordings, the teachers enact the same activity that includes a partitioning task.

> Instructions for executing the task: Distributing dog biscuits
>
> Tell the children that you have a problem that you need help to solve. You are going to visit a family that has two puppies (show a picture or figures with two dogs). Then tell them that you have bought a package of dog biscuits to give to the puppies. There are 6 dog biscuits in the package (12 if the children find it too easy). Propose the following task: If you are going to distribute the 6 biscuits fairly between the puppies, how many biscuits will each puppy get?
>
> The children are expected to use different strategies to complete the task and distribute the biscuits between the two dogs. The important thing is to stop and examine how the children arrived at their solution and examine the partitioning, i.e. that it is fair (equal number of biscuits for each dog), that all biscuits have been distributed (none were left over), and if the biscuits are differently partitioned, they are still equally many altogether.

Each preschool teacher printed her own pictures of similar dogs and similar dog biscuits as artefacts to use in the teaching activity.

Three different ways of handling the artefact in the teaching were observed. Three video recordings were selected for more in-depth analysis, which are representative for the whole empirical data. These were then analyzed using variation theory principles that emphasize what aspects are foregrounded by the concepts of contrast and generalization.

In the analysis of the video recordings, the teachers' different ways of acting with the artefact bring for the parts and/or the whole to be discerned by the children. These are further interpreted as promoting or limiting children's opportunities to learn about numbers. Here, both how the artefact is structured in the situation and actions that direct attention to essential aspects are important keys to answering the research question. Since a particular interest is directed towards the potential of artefacts to facilitate learning, attention was directed towards how actions to structure the artefact affect what the child can experience and thus what learning opportunities are offered. Thus, in the analysis we look for how numbers' part-whole relationships are made possible to discern for the children through the ways in which the artefact is handled by the teacher. Three themes emerged from this analysis: *structuring with verbal communication*, *structuring with verbal communication and gestures* and *structuring with verbal communication, gestures and finger representations*.

Results

The presented episodes belong to a sequence in the activity of distributing dog biscuits when the preschool teachers introduce the activity to the children. Three different ways of using artefacts, supported by teaching actions that enable number structure to be discerned, emerge in the analysis (see Table 1).

In the introduction of the activity (see Table 1), the task is introduced in teaching activity A by counting the dog biscuits one by one and placing them in a pile. The artefact (dog biscuits) put in a pile then becomes an unstructured artefact. This way of handling the artefact does not make it possible to distinguish the number of biscuits to be distributed. A relationship between parts and whole is not possible to discern. In teaching activity B, the preschool teacher introduces the task by structuring the artefact (the dog biscuits) in two rows. This enables the children to discern that the number of biscuits can be grouped as parts of the larger whole. The preschool teacher further points out the two parts by keeping the groups separate and making a circling gesture over each row. Then, to emphasize that they are to be seen as one whole, they sweep their hand over both parts while simultaneously saying "twelve". In teaching activity C, the preschool teacher points out the two parts and at the same time accentuates the number of the parts by putting six fingers in front of each part with dog biscuits while saying how many there are.

Analysis of the teaching actions and use of artefacts

The artefact used in the analysed teaching activity is unstructured and not self-instructive; it is the actions of the preschool teacher with the artefact that determine whether the artefact becomes structured or unstructured. In the above-described introduction to the partitioning activity, the preschool teachers have used the artefact in three different ways: *structuring with verbal communication*, *structuring with verbal communication and gestures* and *structuring with verbal communication*, *gestures and finger representations*. In the following, these are described in more detail and the learning opportunities they entail are discussed.

Table 1: The preschool teachers' teaching actions when introducing the activity

¤ = dogs, I = biscuits, PT = preschool teacher

Teaching activity A Two dogs and 12 biscuits	**Teaching activity B** Two dogs and 12 biscuits	**Teaching activity C** Two dogs and 12 biscuits
PT puts two dogs on the table in front of the children without saying how many dogs there are. ¤ ¤ PT holds the twelve biscuits in her hand and places one at a time in a pile counting from one to twelve out loud. PT says "Here you get the packet (biscuit). Now you have to distribute them equally, so that they each get the same number of them". I ¤ ¤	PT puts two dogs next to each other on the table in front of the children and says "two". ¤ ¤ PT puts one biscuit at a time above the two dogs. First, six biscuits in one row, then six biscuits in a second row. I I I I I I I I I I I I ¤ ¤ PT asks the children to count the biscuits. The children count the first row and answer "six". PT points to the first biscuit on the first row and confirms that there are six of them. PT points to the first biscuit of the second row and asks the children how many there are. The children count the second row of biscuits. PT points to the first and second row and says, "there were six plus six, how much is that?" One of the children replies "twelve". PT confirms that there are twelve biscuits and swipes her hand above the two rows.	The biscuits are in one row split in two groups. I I I I I I I I I I I I ¤ ¤ PT shows the five fingers on her right hand and only the thumb on her left hand. PT shows the six fingers on the first dog and says, "this dog has six biscuits", then moves her six fingers to the other dog and says, "the other dog has six biscuits". PT swipes her hand above the two dogs with the biscuits and asks, "how many are there altogether?". One of the children replies "seven". PT asks the child to count again. The child counts twelve biscuits and says "twelve". PT repeats "twelve".

When *structuring with verbal communication*, the preschool teacher uses only verbal communication to direct attention to numerical features of the task. For example, by asking the children questions "how many" or by telling them how to handle the biscuits.

The preschool teacher's questions can make the children change the direction of their attention to essential features of the task, but there is limited support for discerning parts and whole or any relationship between them unless the child has already discerned it. By only talking about number of biscuits, the artefact is not structured to visualize such relationships either. Dog biscuits in a pile is an unstructured example and does not provide support for children to discern parts and the whole. This way of handling the artefact and the number relations only supported verbally is shown in teaching activity A.

When *structuring with verbal communication and gestures,* the preschool teacher uses the artefact in ways that point out the parts and the whole by using hands and fingers in interaction with the artefact

and at the same time verbalizes the part-whole relationship. Here, the children are partly supported in their discerning number relations because the artefact is structured in equal straight rows, which enables the children to discern the parts (number of biscuits in each group). The teacher's pointing and circling gestures also direct attention to the parts' relation to the whole. In teaching activity B, the preschool teacher makes visible that there are six in each part and twelve dog biscuits together (verbally communicated) to be distributed. The gestures add a significant component to the verbalized introduction, since it directs attention to what groups that are labeled with a counting word; the part(s) or the whole.

When *structuring with verbal communication, gestures and finger representations*, the preschool teacher, as exemplified in teaching activity C, interacts with the artefact by first arranging the biscuits in groups within a row and simultaneously pointing out and verbalising the parts and the whole, also using the fingers as representations. This addition of the finger pattern enables the children to discern the number structure of equally sized parts that (in combination with the verbal conclusion of the total number of biscuits) constitute the whole, that is, twelve dog biscuits together to be distributed as six in each part. Thus, the relationship between parts and whole is made visible.

Discussion

The activity observed in this study includes dogs and biscuits that are used as artefacts for teaching about number structure. In this paper, we set out to analyse in depth how different ways of handling the artefact offer preschool children opportunities to learn about numbers. To aid our analysis we used variation theory of learning (Marton, 2015). The analysis supports earlier findings (see e.g., Venkat & Askew, 2018) that whether the artefact is used in a structured or unstructured way has an impact on whether it is possible to discern number relations. However, it seems that even if structuring the artefact is significant for enabling children to pay attention to numbers in the task, this can be enhanced by verbal communication, gestures and finger representations. The results also show that an artefact that is unstructured from the beginning can become structured if the preschool teachers' actions together with the artefact are structured so that the mathematical content becomes visible to the child.

The preschool teacher uses representations such as gestures, finger patterns and verbal communication as a way to structure the artefact. If the preschool teacher structures the artefact into parts and at the same time points out the part and whole relationship both verbally and with gestures, several aspects are made visible at the same time and the child is given the opportunity to discern the part-whole relationship and additionally, how many there are in each part (see also Ekdahl, 2019). The finger pattern becomes a semi-abstract representation that clarifies the number for the child and supports the comparison between the number of different groups, that is, connecting the different modes of representation (Lesh, 1981).

Earlier research has shown that systematically implemented activities and teaching focusing on number in a way that emphasises part-whole relationships support children's advancement of number knowledge and arithmetic skills (Palmér & Björklund, 2023; Kullberg et al., 2021; Cheng, 2012). This paper is a first attempt to investigate in depth how this can be accomplished in a preschool activity and how artefacts and actions in teaching with preschool children can support or not support children's progress in numeracy and arithmetic skills.

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Teachers' descriptions of mathematics education in preschool in relation to play-responsive teaching

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This paper addresses the ongoing discussion about what mathematics teaching at preschool level means, and how it ought to be conducted, utilising the perspective of Swedish preschool teachers. The aim is to elaborate on how teachers talk about play, and mathematics teaching, and how this may be understood in the context of play-responsive teaching theory. An analysis of focus group interviews shows that the teachers view play as something self-evident and unquestioned, yet difficult to define. When the teachers talk about mathematics teaching, they distance themselves from how mathematics teaching is performed in school. They also emphasise that teaching requires play, and that learning mathematics often equates to children performing activities that, from the perspective of the teachers, involve mathematics. Thus, how teachers talk about play and teaching may therefore contain opportunities and challenges when seeking to implement play-responsive teaching.

Keywords: Preschool, teaching, mathematics, play-responsive teaching, teachers' descriptions.

Introduction

While researchers agree on the importance of children learning mathematics early, there are diverse views on how young children should learn mathematics, and what teaching mathematics in early education should involve. This issue becomes even more complex if the question of whether mathematics teaching is to be integrated with or separated from play is brought into consideration (Palmér & Björklund, 2016; Björklund et al., 2018). The Swedish National Agency for Education (2018) emphasises that play is fundamental in preschool and that mathematics teaching should be provided. How this teaching should be implemented is the subject of a frequent and ongoing debate (Jonsson et al., 2017; The Swedish Schools Inspectorate, 2016; Vallberg Roth, 2020). According to the Swedish Education Act (2010:800), teaching is processes guided by preschool teachers, aiming for children's development and learning. However, an inspection by the Swedish Schools Inspectorate (2016) showed that many preschools are failing to meet these objectives. These shortcomings are described as a direct result of teachers' uncertainty about what teaching entails and how it should be conducted. However, it has been shown that this can be countered if preschool teachers are made aware of their role in encounters with children (The Swedish Schools Inspectorate, 2016).

Play as a foundation for education means that playfulness should be preserved, while teachers at the same time direct children's attention to specific knowledge content. This is also emphasised as the purpose of play-responsive teaching (PRT) (Pramling et al., 2019). To date, PRT has been studied sparingly in mathematics education. To counteract this, the aim of this study is to elaborate on preschool teachers' talk about play and mathematics teaching: What characterises teachers' talk about

play and mathematics teaching in preschool? Further, we discuss how teachers talk about play and teaching related to PRT.

Mathematics education in preschool

The position of this study is at the intersection of mathematics education and early childhood education, with PRT as a central feature.

Play and mathematics teaching

Although there are different perspectives on how play is to be understood and used as a facilitator for learning, play is most often highly valued in early childhood practices. Sutton-Smith (1997) emphasises that there is a large variety of actions, objects, and forms that are characterised as play, while there are also differences in perspectives on the function of play. Whereas some emphasise that children develop cognitive skills through play, others emphasise play as being a product of children's own amusement, representing the exercising of free choice, power, and identity.

Regardless of view on play, play is a multifaceted phenomenon of which imagination and hypothetical thinking are significant features. These attributes are perhaps best characterised by "as if" and "what if" imagining, which is also recognised in mathematical exploration. For instance, Ginsburg (2006) argues that children commonly use intuitive mathematical knowledge embedded in play activities that provide a strong foundation for cognitive development. According to Vygotsky (2004), play is a creative process in which children may recreate their experiences only limited by their earlier experiences. Following this line of reasoning, it is necessary to provide children with a wider range of experiences as a means for their sense-making. Knowledge development can thus be seen as being guided (for example, in play) by others' experiences, in turn allowing children to "borrow" others' experiences to extend the richness of their own experiences. Play may thereby be seen as offering children relevance and meaning, within which mathematical objects can be integrated (van Oers, 2013a, 2013b). In addition, Brandt (2013) emphasises the importance of teachers as teachers' actions affect the educational potential in preschool activities regardless of the activities involving play or not.

In the last decade, the concept of teaching has been added to the Swedish preschool curriculum. The history of the concept of teaching is thus very brief in the Swedish preschool setting, as compared to other school levels (Swedish National Agency for Education, 2018). How the concept of teaching should be understood in relation to preschool is therefore not obvious (Björklund & Palmér, 2019). In a previous study of Swedish preschool teachers, it was shown that according to the participating teachers, mathematics teaching takes place in activities initiated by the children, rather than by the teachers. Teaching, starting from a specific mathematical concepts or principles was very rare. Furthermore, the material was considered to be an essential part, both as a starting point for teaching and for children's own initiatives. There was, however, an absence of intentional problematisation of the mathematical relationships offered by the teacher. The study also showed that mathematics teaching in the informal situations provided at preschools, confirms rather than challenges children's knowledge (Björklund & Barendregt, 2016). Other studies have shown resistance among Swedish preschool teachers to the use of the term teaching (Jonsson et al., 2017; Vallberg Roth, 2020). Teachers perceive teaching as something that requires more than, or differs from, ordinary preschool

activities. However, there is a wide variation in preschool teachers' opinions on teaching. While some express that teaching does not belong in preschool, others express that teaching is an obvious component. In general, both opinions are practice-based with weak theoretical connections (Vallberg Roth, 2020).

Play-responsive teaching

PRT (Pramling et al., 2019) is a theory and a proposal for teaching in preschool where play and teaching are united in a coherent process. In PRT, play is seen as an activity where children's initiatives, interests and actions are at the centre. Teaching is in PRT based on and adapted to children's play, contrary than using play solely as a tool to achieve predetermined teaching goals. Teachers' participation in play is therefore a prerequisite in PRT. The playfulness can be preserved while specific knowledge content is being attended to. PRT is based on double responsiveness which implies that teachers are sensitive to children's reactions and adapt their actions accordingly. The intention of the theory is not for teachers to consistently participate in children's play or to constantly teach. Rather, the intention is to encourage teachers to sometimes participate in children's play with the aim of teaching (Pramling & Wallerstedt, 2019). The theory characterizes play by the participants' shifting between fantasy and reality. The concept of "as if" refers to a fictional world, i.e. that in which play activity is pretend, and invented based on fantasy. "As is" by contrast, refers to the existing world, i.e. the elements active in play that are considered conventional knowledge, or how something actually is. Play is not considered to be in contradiction to the goal-oriented aspect of preschool; play can be both planned, and have goals (Pramling et al., 2019). Teaching is defined as an activity that is created jointly by the participants, one that assumes an exchange with responsiveness between the participants (Pramling et al., 2019). A prerequisite for teaching is that there is a direction in the interaction, with an aim of making someone aware of, or able to do something they did not know, or were not able to do before. This requires the participants to coordinate their actions, to establish a common shared attention, intersubjectivity (Pramling & Wallerstedt, 2019).

Method

This study is part of a larger research project aimed at implementing play-responsive mathematics teaching in preschools. The empirical material for this study was focus group interviews (Halkier, 2010) conducted with five groups of three to four participants (in total 17 participants). For the purpose of this study, the term *teacher* comprises both preschool teachers (11) and childminders (6). In Sweden, preschool teachers have at least a bachelor level university degree while childminders hold an upper secondary degree. Even though preschool teachers are responsible for the preschool education, it is jointly conducted (Swedish National Agency for Education, 2018) why both professions are included in this study. Each focus group included preschool teachers and childminders well-known to each other. One strength with focus group interviews is its possibility of capturing dynamics between participants, which can provide a deeper understanding of jointly constructed meanings and perceptions instead of individual opinions (Halkier, 2010). Therefore, the analyses focused on the conversations, not individual opinions. Ethical guidelines from the Swedish Research Council (2017) were followed which implies that all teachers were given information about the study and provided their written consent.

The interviews focused on play and mathematics teaching using a semi-structured approach, which implies that questions were predetermined but handled flexibly based on each interview situation. The teachers had no known experience or knowledge about the theory PRT. The interview questions were therefore based on core ideas in PRT, for example play, learning and teaching, but without specifically mentioning the theory. The interviews were audio recorded and were between 37 and 58 minutes in length.

Transcripts of the interviews were thematically analysed meaning that the characteristics of teachers' talk was systematically identified and organised, thereby offering insights into patterns of meaning, that is, themes, across the data set. By focusing on patterns of meaning, it is possible to see and make sense of collective or shared meanings. In line with Braun and Clarke (2012), the analysis was conducted through phases. In the first phase, a general repeated reading of the transcripts was performed to obtain an overall meaning. Content unrelated to play or mathematics teaching was removed. Next, a systematic analysis was conducted through coding implying labelling unique features of the data that were potentially relevant to the research question. A theme was generated when codes shared certain unifying features, thereby reflecting a coherent and meaningful pattern in the data. Here, it became clear how some themes related to play, while others referred to teaching. The themes were then processed, focusing on the uniqueness and essence of each theme. This analysis led to the identification of seven themes that constituted the essence of play and teaching for the participating teachers. Examples of the teachers' descriptions are provided in the results section.

Results

The thematic analysis yielded four themes on play and three themes on mathematics teaching. At first, the themes are presented individually with examples followed by an overall summary of the teachers' talk about play and, respectively, the teachers' talk about mathematics teaching. In the subsequent discussion, these themes are related to PRT.

The teachers' talk about play

In the interviews, the teachers were asked about play; what play is, what play is not, and what the role of play is in a preschool setting. In the teachers' talk about play, four themes became visible: non-questioned play, the joyfulness of play, play belonging to the children, and the hardness of definition.

Non-questioned play

When teachers talk about play, play stands out as being self-evident and unquestioned. For example, one teacher expressed that "play is the most important thing we have during the day". Without further prompting, play is described as having a fundamental role in preschool, according to the teachers.

The joyfulness of play

Teachers' talks also emphasised the joyfulness of play. For the teachers, play is something that is always happy and fun. One teacher expressed this concept by stating, "play must be fun, otherwise it is not play". According to the teachers, joyfulness is a key criterion for determining whether a situation constitutes play or not.

Play belongs to the children

The teachers discussed children as being the ones who rule the play. For example, one teacher stated, “play is controlled by the children, it is free and without demands”. According to the teachers, play is inexplicit, without clear frames, and is something that belongs to the children. The teachers talk about play as being something voluntary and spontaneous based on children's initiative, as the above excerpt shows.

The hardness of definition

Although the three themes mentioned above could themselves be seen as the teachers’ definition of play, the teachers had trouble trying to directly describe what play is and is not. The statements given are very brief and blunt, for example “play is play” and “That was a very difficult question”. The teachers expressed a sense of frustration at not being able to describe what they emphasised as the most important thing in preschool. This was noticeable when they gave examples of non-play situations and changed their minds while talking. For instance, “Clothing is not play, but, when I think, it can actually turn into play”.

Summary: The teachers’ talk about play

The themes about play highlights some tensions. At the same time as play is seen as a self-evident and central part of preschool, there is an uncertainty about how play can be defined. Further, there is an uncertainty regarding the role of teachers in play, as freedom, joy, spontaneity and children’s initiative are focused on in relation to play. A view of play as self-evident may reflect a resistance to ongoing changes in Swedish preschools that may be perceived as a threat to the emphasis on play in preschool. The difficulty in defining play indicates its complex and multifaceted nature and may reveal that play is largely experienced rather than verbally defined.

The teachers’ talk about mathematics teaching

The interviews also dealt with questions about what teaching was and how it was conducted in preschool. In the teachers’ talk, three themes became visible: play makes mathematics teaching fun, doing – with or without the teacher, and distancing from school.

Play makes mathematics teaching fun

In teachers’ talks on mathematics teaching, play is a recurring element. According to the teachers, mathematics teaching in preschools requires play. The teachers talked about how play makes mathematics teaching exciting, fun, and interesting for the children, and makes the children want to engage in what teachers have planned. The teachers observed, “with play, you get the kids’ attention much easier”. For example, one teacher expressed “Play makes teaching mathematics fun and interesting, and then the children want to participate”.

Doing – with or without the teacher

In the teachers’ talk about mathematics teaching, the doing of the children was emphasised. The teachers’ overall talk and the examples provided of teaching mathematics show that children should be allowed to actively participate in the mathematics teaching. Examples of children's doing, or active participation, in mathematics teaching include when children work with clay, climb, paint, use digital

tools, and build. One teacher explained: “Today a child made different balls of clay, then we started talking about bigger and smaller”. Another example of doing being expressed as central in mathematics teaching, also relating to size, was, “We have painted on large and small papers”. According to the teachers, it is important for children to be engaged in activities characterized as “doing”, with room for variation in whether and how the teachers interact, as shown in these examples above.

Distancing from school

When teachers talk about teaching, they often compare teaching in preschool to teaching in school by emphasising the differences. In their talk, they distance themselves from teaching in school, but without explicitly expressing how they perceive teaching in school as being performed. One teacher expressed “In school, there is compulsion in teaching, something every child has to do. At the preschool, we have more freedom and more fun”. Another teacher said, “The school divides play and teaching; in preschool, they go into one another”. As these examples show, according to the teachers interviewed, play, freedom, and fun are key factors that differentiate school and preschool mathematics teaching.

Summary: The teachers’ talk about mathematics teaching

The three themes about mathematics teaching highlight different aspects of preschool education. Play is expressed as important in making mathematics teaching fun, exciting and interesting, and thus helps to get children’s attention and engagement. The importance of children’s active participation, them doing things is emphasised. The teachers describe how the children's involvement in practical activities is central to their learning. This suggests an educational philosophy valuing learning through actions and exploration rather than passive reception of information. The teachers distance the preschool education from the formalized teaching methods in school. In preschool, according to the teachers, play and teaching are integrated in a way that creates a more flexible and joyful learning environment than in school. This distancing from the school can be seen as an effort to preserve the uniqueness of preschool education where play is an important component.

Discussion

The question elaborated upon above pertains to how the interviewed teachers talk about and view play and teaching. In this section, these observations will be discussed in relation to PRT. In PRT, teachers participate in children's play with the aim of teaching. PRT is possible in most situations in preschool, as all situations can become both play and a potential teaching situation (Pramling & Wallerstedt, 2019). Even though play-responsive theory does not provide a fixed definition of what play is, certain characteristics of play are emphasised, for example, *as if* and *as is*. Play is not defined in the teachers’ talks either, and the characteristics of play emphasised by the teachers are not the same as in PRT. For the teachers, the joyfulness of play stands out as a key observation, together with the fact of play belonging to the children. A similar discrepancy was found in connection to teaching. In PRT, teaching is defined as an activity that is jointly created by the participants, assuming an exchange with responsiveness between the parties involved. A prerequisite for teaching is that there is a direction to the interaction with the aim of making someone aware of, or able to do something they did not know, or were not able to do before (Pramling et al., 2019). The teachers interviewed did

not define preschool teaching, but instead talked about what teaching is not, that is, teaching in school. The teachers also talked about how play makes mathematics teaching exciting, fun, and interesting for the children. The role of play is thus very different in PRT compared to how the teachers talk about play and teaching. At the same time, teachers talk about play and teaching going into one another, which corresponds well with PRT. In the teachers' view, teaching does not seem to have the same value as play. Summarising the results shows that if PRT is to be implemented with these teachers in their mathematics teaching practice, an important starting point is to discuss definitions and relations between play and teaching.

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Robot programming and its influence on routes and map-reading skills in early childhood education

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Abstract

In recent years, numerous scientific studies have highlighted the educational benefits of programable toys suitable for kindergarten children. To evaluate the effects of a robotics program on routes and map-reading skills, this study conducted a comparative study involving an experimental group and a control group. A pre- and a posttest were applied to assess those skills, and statistical analysis, including Mann-Whitney U and Wilcoxon signed-rank tests, compared the performances of the two groups. The results indicate a positive effect of the robotics program, leading to the conclusion that robot programming activities can be used in early childhood to develop children's routes and map-reading skills through highly motivating activities and stimulating cognitive and social skills.

Keywords: Robotics, programming, routes, map-reading, early childhood education.

Introduction

The curriculum guidelines for early childhood education in Portugal highlight spatial thinking skills as key competencies for this stage (Silva et al., 2016). However, existing curricula often do not prioritize spatial skills as a curriculum objective, and there is a deficiency in learning activities that foster those abilities for young children (Bruce et al., 2015), especially regarding routes and maps.

Route knowledge refers to recognising your location, navigating through a town, or following a path from one point to another, which implies recognizing landmarks along a route, understanding the sequence of landmarks and turns, and forming a mental map that captures spatial relations between routes and landmarks (Farran, 2012). Routes involve both static and dynamic information, including the order of landmarks, decision points, and associations between landmarks and directional changes (Nys et al., 2015). Alsina (2022) argues that this content is complex as it requires identifying basic spatial notions and relations and developing a mental schema of space and geometric transformations inherent to position changes, such as rotations, reflections, or translations.

Spelke et al. (2011) define maps as two-dimensional and movable pictures representing a three-dimensional and stable environment surrounding the navigator. They argue that an effective use of a map requires either aligning its orientation with the environment or comparing its geometric properties, ignoring differences in scale and orientation. At early stages, map reading poses a challenge as children often struggle with orienting themselves in real space and understanding maps' symbolic and simplified language (Gonzato et al., 2011).

Research shows that robot programming for young children (as young as three years old) offers educational benefits, such as increased motivation, active learning, and early coding skills, as well as

tangible, hands-on experiences that foster cognitive and social skills and promote logical and mathematical thinking (Nogueira et al., 2021). Regarding spatial skills, studies show that robot programming has the potential to enhance the acquisition of basic spatial concepts (Bizarro et al., 2018), to develop spatial awareness and perspective-taking (Shumway et al., 2021), as well as spatial perception, mental rotation, and visual memory (Brainin et al., 2022). In the study of Palmér (2017), kindergarten children showed spatial thinking skills development after a four-month robotics program by mentally comparing paper and grid maps, interpreting symbols, envisioning actions, and conceptualizing relationships between maps and symbols.

Overall, there is a lack of studies focusing on the impact of robotics activities on routes and map-reading skills. Robot programming requires children to mentally anticipate and plan robot movements, which involves understanding spatial relations and navigating environments – skills shared with route planning and map reading. Additionally, it involves cognitive processes such as algorithm design, decomposition, generalization, abstraction, and evaluation (Csizmadia et al., 2015) as well as debugging processes which can enhance cognitive functions like planning, cognitive inhibition, and working memory (Di Lieto et al., 2017), essential for planning and describing routes.

This study's main hypothesis is that robot programming has the potential to enhance the development of routes and map-reading skills. Two main research questions were defined to verify that hypothesis: (1) Does participation in a robotics program influence children's ability to plan and describe a route? (2) What's the impact of a robotics program on children's map-reading skills?

Methods

A quasi-experimental study was carried out to achieve the defined objective, with a standard pretest-intervention-posttest assessment for the experimental group (EG) and the control group (CG). The intervention consisted of a robotics program developed over twelve weekly sessions lasting 45 to 60 minutes, conducted during kindergarten hours. During this period, the children in the CG carried out the activities planned by the responsible educators without any intervention from the research team. They had no interaction with robots in the school context. The pre- and posttest aimed to analyse children's route-planning and map-reading skills and were administered approximately 14 weeks apart. A total of 58 four- to six-year-old kindergarten children from five class groups of three different schools from the same public school cluster in northern Portugal participated in the study. The EG comprised 36 children (15 girls, 42%; 21 boys, 58%) from three class groups. The CG included 22 children (12 girls, 55%; 10 boys, 45%) from two class groups.

The robotics program was implemented over 12 weeks, consisting of a weekly session lasting 45 to 60 minutes during school hours. The sessions incorporated moments of large group, paired, and individual activities, combining structured guidance with opportunities for free exploration. Additionally, children were invited to create a graphical record of each session. The program was designed with a spiral structure and a gradually increasing complexity to adapt to diverse learning rates. It began with simple problem-solving tasks using a Bee-Bot robot, such as navigating paths with minimal direction changes on a grid mat. Then, additional direction changes, unmarked paths, obstacles, and maps were introduced. Using command cards representing a symbolic programming language enriched the children's problem-solving experiences. Some sessions included unplugged

programming activities, in which the children took on the role of the robot and were programmed by each other. In the program's final activity, each group of children told a story using robots.

For the purpose of this study, the following data collection instruments were used: an assessment tool specifically designed for this investigation, serving as both a pretest and posttest and including tasks related to routes and maps (cf. Figure 1), field notes, photographs and video recordings to document children's performance on the tasks. In Task 1, children were asked to conceive, describe, and represent a route to the cheese, avoiding the lava. They should 1A) explain the path they would follow and walk, 1B) represent that path graphically, and 1C) explain the drawn path. In Task 2, the children were given a map of the classroom, which they had to read to locate themselves and find a hidden treasure. Some aids were defined to help children complete the tasks, if necessary. For example, in Task 2, the researcher may need to help the child locate some objects in the room on the map and orient it correctly before asking the child to find their location on the map and locate the treasure.

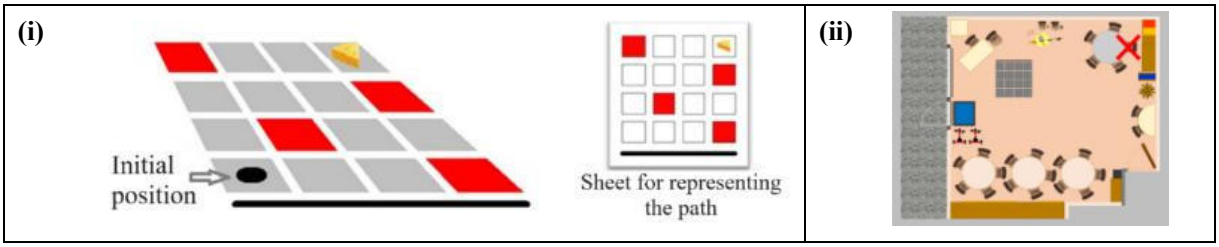

Figure 1: (i) Scheme of the Task 1; (ii) Scheme of the Task 2 of the pre- and posttest

These tasks are part of a broader instrument consisting of seven tasks designed to analyse various aspects of spatial thinking, validated by a panel of experts (Nogueira et al., 2023) and through an application in a pilot sample (Nogueira et al., 2024)[1]. Children's performance on each task was assessed using a Likert scale from 0 to 5. A single researcher conducted all stages of the assessment, took notes during the tasks, photographed key moments of the children's performance, and later analysed the videos along with the other records to assign a score to children's performance in each task. Statistical Package for Social Sciences, version 28.0, was used for statistical analysis. As all the variables displayed non-normal distributions, and given the small size of the sample, non-parametric tests were used: Mann-Whitney U test was applied to both the pretest and posttest scores to determine if there were statistically significant differences between the EG and the CG in each phase. Additionally, the Wilcoxon signed-rank test was used to compare each group's pretest and posttest scores to assess improvements over time. For each analysis, the significance level was set at $p < 0.05$. To assess the effect size for each one of the mentioned tests, we employed $r = Z/\sqrt{N}$, where Z represents the Z-score obtained from Mann-Whitney U and Wilcoxon signed-rank tests, and N represents the total number of observations (N=36+22=58 for Mann-Whitney U tests; N=2x36=72 or N=2x22=44 for Wilcoxon signed-rank tests) (Cohen, 1988; Pallant, 2007).

Results

In this section, we describe the results obtained for each of the EG and CG tasks in the pre- and posttest. The three tasks about routes (1A, 1B, and 1C) were analysed separately.

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As illustrated in Figure 2, for Task 1A, the means obtained by the EG in the pretest and posttest were 1.6 and 2.7, respectively. For the CG, the means were 1.1 in the pretest and 1.3 in the posttest, which seems to indicate a bigger improvement in the EG.

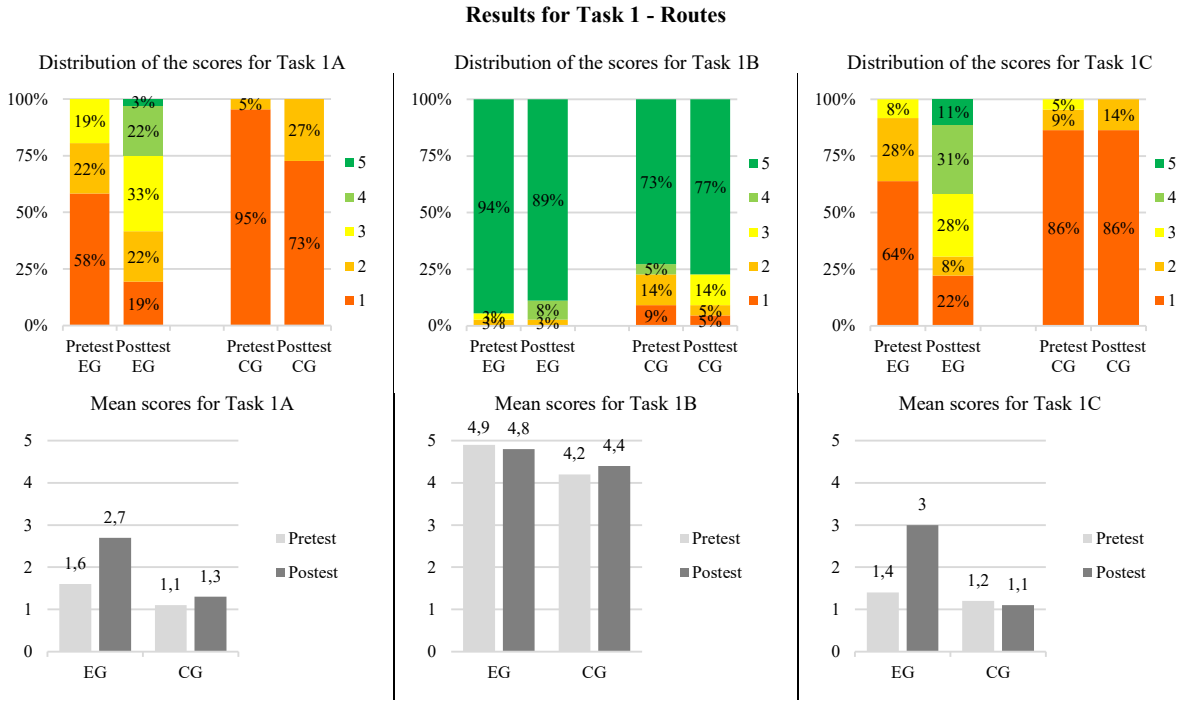

Figure 2: Results for the routes Task (1A, 1B, 1C) from the EG and CG in the pretest and posttest

In the pretest comparison between EG and CG, the results indicate a significant difference between the groups, with a medium effect size (cf. Table 3). In the posttest comparison, the difference was even more pronounced, suggesting the existence of a significant difference between the groups with a large effect size. Comparing the pretest and posttest scores within each group, the EG showed a significant improvement with a large effect size. The CG also showed a significant improvement with a medium effect size. These results demonstrate that both groups significantly improved from pretest to posttest, with the EG showing a larger effect size than the CG.

For Task 1B, both groups showed high scores in both tests (cf. Figure 2). As presented in Table 3, for the pretest comparison between EG and CG, the results indicate a significant difference between the groups, with a medium effect size. In the posttest comparison, the results showed no significant difference between the EG and CG. Comparing the pretest and posttest scores within each group, the EG obtained lower scores in the posttest, although the results indicate no significant change. The CG showed some improvement but no significant change. These findings suggest a significant difference between EG and CG in the pretest but not in the posttest. Furthermore, neither group showed significant improvement from the pretest to the posttest.

For Task 1C, the mean scores obtained by the EG in the pretest and posttest were 1.4 and 3.0, respectively. For the CG, the means were 1.2 in the pretest and 1.1 in the posttest (cf. Figure 2). As shown in Table 3, in the pretest comparison between EG and CG, the results indicate no significant

difference between the groups. In the posttest comparison, the results were extraordinarily different, showing a significant difference between EG and CG with a very large effect size. Comparing the pretest and posttest scores within each group, the EG showed a significant improvement with a large effect size. In contrast, the CG's results indicate no significant change.

Table 1: Statistical tests applied to the results for Task 1 - Routes (1A, 1B and 1C)

				Mann-Whitney U test									Wilcoxon signed-rank tests for each group		
				Pretest					Posttest						
Variable		Group	N	Mean	SD	Z	p-value	r	Mean	SD	Z	p-value	r	Z	p-value	r
Task 1 - Routes	A	EG	36	1.6	0.80	-3.075	0.002*	-0.404	2.7	1.12	-4.597	0.000*	-0.604	4.370	0.000*	0.515
		CG	22	1.1	0.21				1.3	0.46				2.236	0.025*	0.337
	B	EG	36	4.9	0.59	-2.367	0.018*	-0.311	4.8	0.56	-1.336	0.182	-0.175	-0.213	0.832	-0.025
		CG	22	4.2	1.47				4.4	1.18				0.779	0.436	0.117
	C	EG	36	1.4	0.65	-1.783	0.075	-0.234	3.0	1.33	-4.963	0.000*	-0.652	4.615	0.000*	0.544
		CG	22	1.2	0.50				1.1	0.35				-0.333	0.739	-0.05

* $p \leq 0.05$

Concerning Task 2 related to map-reading skills, as indicated in Figure 3, the mean scores obtained by the EG in the pretest and posttest were 2.7 and 3.6, respectively. For the CG, the mean was 3.1 in the pretest and 3.5 in the posttest.

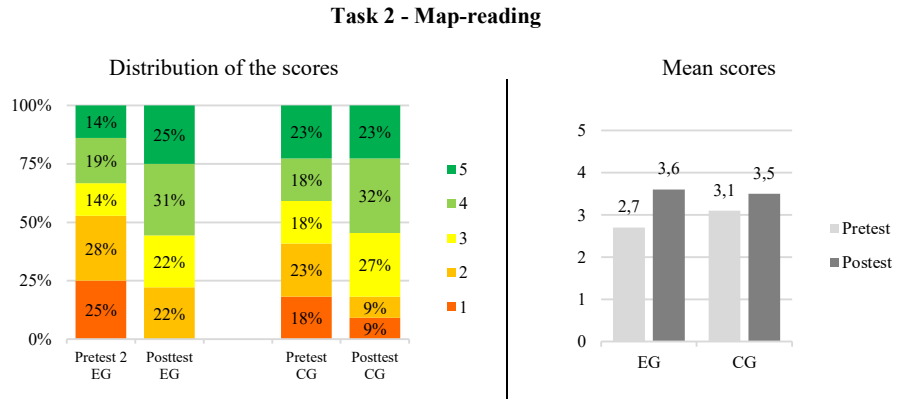

Figure 3: Results for the map-reading task from the EG and CG in the pretest and posttest

Both groups improved their results, but the improvement seems to have been more pronounced in the EG. The results of the statistical tests shown in Table 4 suggest no significant difference between the EG and the CG in the pretest and posttest comparisons. However, comparing the pretest and posttest scores within each group, the EG and the CG showed some improvement. For the EG, results indicate

a medium to large effect size, demonstrating significant improvement from pretest to posttest. As for the CG, results indicated no significant improvement from pretest to posttest.

Table 2: Statistical tests applied to the results for Task 2 - Map-reading

			Mann-Whitney U test									Wilcoxon signed-rank tests for each group		
			Pretest					Posttest						
Variable	Group	N	Mean	SD	Z	p-value	r	Mean	SD	Z	p-value	r	Z	p-value	r
Item 7 - Map--reading	EG	36	2.7	1.41	-0.91	0.363	-0.119	3.6	1.11	-0.132	0.895	-0.017	3.505	0.000*	0.413
	CG	22	3.1	1.46				3.5	1.22				1.589	0.112	0.240

* $p \leq 0.05$

Discussion and Conclusions

Regarding the first research question – Does participation in a robotics program influence children's ability to plan and describe a route? – each of the three tasks was analysed separately. The oral description of a mentally visualized route (Task 1A) or one represented graphically (Task 1C) is a skill shown to be the most challenging for children, and it is also the one where the robotics program seems to have had the most significant impact. In the pretest, no child from either group scored higher than three points, and all children required assistance to complete the tasks. They struggled to verbally and sequentially elaborate on and explain the planned route. This difficulty is understandable, as Alsina (2022) argues that such tasks require mastery of various spatial thinking elements, particularly geometric transformations, and a higher degree of linguistic competence. Additionally, they involve decomposition and sequencing processes inherent to CT (Csizmadia et al., 2015), which are central in robot programming. Also, as argued by Di Lieto et al. (2017), programming robot actions involves mentally anticipating each step, choosing the right command, and continuously updating the program to achieve the goal, which is a process that enhances planning, inhibition, and working memory. This could explain the significant improvement with a large effect size observed in the EG posttest, while the CG generally maintained pretest performance levels. As for the graphic representation of the route observed in Task 1B, results suggest that it was not very challenging for children, with a high percentage completing it without difficulty (cf. Figure 2). Thus, the high pretest performance by both groups (98% of the EG and 73% of the CG scored five points) may have limited the margin for improvement in the posttest.

Considering the second research question – What's the impact of a robotics program on children's map-reading skills? – the results for Task 2 indicate no significant differences between the EG and CG in either the pretest or posttest and an improvement in both groups' performance in the posttest. However, the tests show that the improvement observed in the EG was considerably more substantial than in the CG, suggesting a positive impact of the robotic activities on the development of map-reading skills. Although the robotics program may have positively affected children's map-reading skills, it seems reasonable to conclude that this effect was not very pronounced. Reading and

interpreting a map of a real space is a complex task that requires mastering specific aspects, such as understanding that a map is a top-down view of the space (different from the perspective of someone moving within it), the need to orient the map, and also recognizing reference points. Consequently, as Gonzato et al. (2011) argue, map-reading is a complex challenge for children, who often struggle with spatial orientation and maps' simplified, symbolic language. Therefore, we believe that developing this skill requires activities more suited to its specific nature.

One limitation of this study is the absence of a follow-up to assess the longer-term effects of the robotics program. Implementing such was not viable, as the posttest was conducted at the end of the academic year. Also, being a study developed within the scope of a doctoral thesis, it faced some human resources constraints. The entire development of the robotics program, as well as the administration of the pretests and posttests, was carried out by a single researcher. While this ensured consistency and uniformity in the procedures, it also posed obstacles to data collection. Specifically, it led to the use of a small convenience sample (n=58), which limits the generalizability of the findings and prevented the examination of gender and age-level differences, for example.

Robots stand out as a particularly interesting option among the various technological resources that can be used for educational purposes in early childhood education. In addition to strongly motivating and engaging children, robots promote active and hands-on learning based on problem-solving, which can be leveraged to develop a wide range of skills, from cognitive to social and psychological. This study advances our understanding of the possibilities for exploring robotics to enhance logical-mathematical reasoning, demonstrating that these activities can substantially enhance specific spatial thinking skills, such as routes and map-reading. Programming activities, which require children to anticipate and plan robot movements mentally, have shown to be particularly relevant in developing the ability to plan and describe routes. Thus, this study's results indicate that robotics is a promising option for creating learning environments that engage children and provide rich, problem-solving experiences that support the development of essential daily skills.

Acknowledgment

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Mathematical argumentation in preschool – an exploratory study

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This paper presents results from the first stage of a study on mathematical argumentation among Norwegian preschool children aged 2-5. The aim of the study was to gain insight into situations in which mathematical argumentation might naturally occur among young children, and to better understand what factors may aid or hinder the development of an argument. In observing the youngest children, the question of "which situations can really be classified as mathematical argumentation?" was recurrent. Here two episodes from children's free play are presented, and a discussion of methodological challenges in the process of analysis - with a particular focus on identifying whether 'claims', 'data' and 'warrants' are consistently mathematically anchored.

Keywords: Mathematical argumentation, preschool, free play

Mathematical argumentation

There is a growing body of research on preschool children's mathematical reasoning, justification and argumentation (e.g. Nergård, 2023; Perry & Docket, 2008; Sumpter & Hedefalk, 2015), and researchers have recently turned their attention to preschool as an environment where children develop argumentative skills (e.g. Johansson et al., 2014; Sumpter & Hedefalk, 2018). Some studies have focused on the conversational moves that teachers use to support children`s mathematical argumentation (Björklund et al., 2018), others have explored the preschool children`s use of concrete objects, gestures and verbal language in mathematical explanations (e.g. Johansson et al., 2014; Nergård, 2023; Sumpter & Hedefalk, 2015). Sumpter & Hedefalk (2015) explored children`s use of mathematics in their arguments, and in the context of their finding, they indicate that more research is needed on how we facilitate children`s mathematical argumentation and reasoning. However, in order to facilitate and contribute to mathematical argumentation, mathematical arguments need to be identified.

Identifying mathematical argumentation among younger children can be challenging for several reasons. Young children often lack the formal language and terminology typically associated with mathematical reasoning, and their ideas may be expressed in informal, intuitive ways, making it difficult to discern whether they are engaging in genuine mathematical thinking. The playful, imaginative and frequently shifting nature of early learning environments further complicates the task. The challenge lies in recognizing the underlying mathematical thinking within the informal and sometimes fragmented ways in which young children express their ideas in multimodal ways.

In defining what constitutes a mathematical argument, the literature presents various perspectives that are often superimposed on Toulmin's model for analysing general argumentation (1958). According to Mueller et al. (2012), a mathematical argument is characterized by the intent to convince others of the truth of mathematical ideas. Lithner (2008) offers a more structured framework, introducing the concept of *anchoring* arguments in the relevant mathematical properties of the objects, transformations, and concepts involved. Lithner defines *objects* as fundamental entities, or the

"things" involved in the process, *transformations* as the operations applied to these objects, and *concepts* as the key mathematical ideas emerging from their interaction. Building on Lithner's framework, Sumpter (2016) argues that the presence of these objects, transformations, and concepts allows for the analysis of the *warrant* and *backing* of an argument from the perspective of its mathematical content.

Similarly, Nordin and Boistrup (2018) emphasize the importance of anchoring arguments in these mathematical components when determining whether they are mathematical in nature. Based on the work of Toulmin (1958) and Lithner (2008), they developed a framework for the analysis of mathematical argumentation among school children. In one of very few papers analysing mathematical argumentation among preschool children, Nergård (2023), applied the same framework, consisting of the following four steps: **1)** Identify **claims** in the video recordings of children in play, **2)** Search for **data** which supports the claim, **3)** Search for **warrants** which motivate how the data support the claim, and **4)** Make sure that the argument is **anchored mathematically**. A further two steps were included where applicable, where the author **5)** searched for possible warrants which motivated the warrant (**backing**), and **6)** interpreted the identified elements, taking the particular **context** into account. In these steps Nergård uses a multimodal approach, identifying arguments through a broad range of modes, such as speech, gestures (pointing) and moving artefacts.

It is this framework for analysis that we have applied to the data discussed in this paper. In particular, the first five steps described above were identifiable and applicable.

Methods

Data collection

Video recorded observations of everyday indoor activities including free play, organised activities (such as baking) and daily activities (such as getting dressed or setting tables) were collected over a period of three months in two preschools located in Norway. In each preschool we followed a group of children aged 2 – 5, with 24 and 28 children respectively. Two researchers were present and filmed the children's activities on 12 occasions, for 2-3 hours each time. Where the children directly addressed the researchers with their thoughts, simple follow-up questions were asked, but the researchers would not initiate talk about new or different topics.

This process resulted in a total of 91 clips, where each clip was a self-contained 'scene' in which one or more children were engaged in an activity. The clips varied in length from 30 seconds to 21 minutes. They were named according to the activity taking place and coded as free play (FP), daily routine (DR) or organised activity (OA).

Data analysis

Each of the three authors viewed all clips individually, wrote a small description of what was observed in the form of memos, and coded each clip as red (no mathematical argumentation), yellow (potential mathematical argumentation) or green (clearly containing mathematical argumentation). The framework for identifying mathematical argumentation (Nordin & Boistrup, 2018) outlined in the introduction was used in this process. The clips were also coded according to whether or not there was any mathematical content, how many children were involved, whether they were playing in

parallel, with some interaction, or fully collaboratively, and the available concretes were noted. After having looked through all clips individually, the authors compared the colour codes given. Where there was disagreement or agreement on a yellow code (a total of 9 clips), indicating general uncertainty, arguments for each choice were given, and the criteria for mathematical argumentation revisited, until a consensus of either red or green was reached.

An overview of the categorisations resulting from the above process can be seen in table 1. Of the 6 clips identified as mathematical argumentation, 3 featured an individual child addressing their surroundings, and three featured 2 or more children in joint activity.

Table 1: Overview of clip categorisation

	Number	**Situation**
All clips	91	Free play (36), Daily activity (14), Organised activity (41)
Mathematical content	21	Free play (12), Daily activity (6), Organised activity (3)
Mathematical argumentation	6	Free play (6)

It was relatively easy to identify whether or not a clip contained mathematical content. Examples include the use of shapes, numbers or spatial descriptions. Of the 91 clips, 21 touched on these topics in some way or other. The remaining 70 clips did not contain any clearly identifiable mathematical content, an example being a session where children and adults talked about emotions. However, deciding whether clips containing mathematical content should be categorised as mathematical argumentation or not, was more challenging. In particular we were repeatedly faced with the question of how much we could allow ourselves to infer from the child's actions. That is, how much should suspected intent be given weight in the process of analysis?

Findings

Below we present two examples with two of the youngest children in the dataset. The first illustrates a relatively easy identification of mathematical argumentation (coded green by all authors), while the second highlights potential difficulties in confidently identifying mathematical argumentation (initially coded red by two and green by one author) with children so young.

Example 1: Boy (2 years old) playing with «clip-on»-tiles

In this clip a three-year-old boy plays with clip-on tiles. He bends them around, seemingly with the

aim of completing the circle in order to drive a little toy car around it (See figure 1). The inserted photos are screenshots from relevant time-points in the transcript.

Figure 1

Boy: I don't have a round «runding» in Norwegian, often used by children to denote a circle)
Researcher: You don't have a round? How can you have a round?
Boy: Then I need so many… one, two, three, four, five, six (counting imaginary missing pieces and pointing with his finger to where they would be, see figure 2)

Figure 2

Researcher: Can you see if you can manage?

The boy goes off to find more pieces and attaches them to what he has.

Boy: That was very small (with a tone indicating that what he has done is insufficient)
Researcher: Was it not enough?
Boy: No
Researcher: Why not?
Boy: Can try… no, it doesn't work (he demonstrates how the ends won't meet, see figure 3)

Figure 3

Boy:	It is not a round!
Researcher:	How can you manage?
Boy:	Then I need more of these, all around here (indicating with his hand)
Researcher:	Can you find some?
Boy:	No (he has used all those around him)
Researcher	Are there no more?
Boy:	They are in the little room

They go together to the room next door where the child takes several pieces and brings them back.

Researcher:	Do you think that is enough?
Boy:	Yes, think it is enough… (tries) think we need more… don't think it works… Oh yes! It worked! (clicks the ends of the circle together)

He then picks up the toy car and drives it around what he has built, clearly pleased (see figure 4)

Figure 4

This clip was marked as green by all authors, in accordance with the chosen framework for analysis:

The **mathematical anchoring** is clear from the outset, the boy wishes to close a circle to make a «round», for his little car to drive around (geometrical shape). The boy makes a mathematical **claim**: "I don't have a round". He supports this claim by pulling on the two ends of the track to show that they do not meet to make a full circle (**data**). He indicates why by counting the imaginary missing pieces and pointing to their positions in the unclosed circle (**warrant**). In the final sequence he

concludes and makes another claim: “think it is enough”. He supports this claim by placing more of the pieces he collected into the circle (**data**). He says, “it worked” and illustrates this by clicking the ends of the circle together (**warrant**). He subsequently drives the toy car around the closed track, and this may be interpreted as a statement in support of the warrant (**backing**).

Example 2: Girl builds with wooden blocks

A four-year-old girl is playing with wooden blocks of different colours and sizes. She is building something that appears symmetrical (figure 5a). She then places a small green block on one side (figure 5b) and proceeds to reach for and pick up another green block (figure 6a) before she glances at her creation and puts it back down (figure 6b) while saying: «A small green. A small green». Then she reaches for another green block and places it on her building, seemingly pleased (figure 7 a, b and c).

Figure 5 (a and b)

Figure 6 (a and b)

Figure 7 (a, b and c)

This clip was marked as red by two of the authors, on the basis that the two green blocks look the same and that her rationale for choosing one over the other was not clear. Consequently, there seemed to be an uncertainty about whether or not her building of the tower was mathematically anchored

(through creating a symmetrical pattern), or not. The third author however, noticed a slight difference in size and perhaps shade of green, and argued that the girl makes her **claim** of «small green (needed)» on the basis that her creation was based on symmetry (**mathematical anchoring**), and that the green she had initially picked up would break that symmetry (**data**, towards which she glances), and finally she connects the claim and data with completing symmetry with her new choice (**warrant**).

What made the classification of this clip more complicated, besides the fact that the shades and sizes of the green blocks were only very subtly different, was that the girl, in the second half of the clip, after the sequence described above, continued to build *without* symmetry. This again brought the mathematical anchoring, which is an essential element in the identification of mathematical argumentation (Nordin & Boistrup, 2018) into question.

In this case, based on the observation that the creation remained symmetrical for a good while, and the girl's clear rejection of an item that broke with this symmetry, it was agreed by all authors that the first four steps of the framework were met. The decision also rested on the fact that a child's focus or intention may change rapidly in play, perhaps more so the younger the child, and that it would not be unnatural for her to aim for symmetry initially and then play on without this restriction in place.

Discussion

The examples presented here illustrate the use of Nordin & Boistrup's (2018) framework for identifying mathematical argumentation among preschool children and highlight some of the methodological challenges when studying the youngest children.

Even though a multimodal approach certainly helps (Nergård, 2023), it can still be challenging to identify whether there is a mathematical anchoring in their claims. A certain amount of inference is inevitably left to the researcher, as was found in Sumpter's various studies (Sumpter, 2016; Sumpter & Hedefalk, 2015). However, our view is that this problem is somewhat remediated by analysing such data in teams rather than individually. While existing frameworks for analysing mathematical argumentation among children work relatively well in the more explicit cases, future research may require more detailed guidelines for the interpretation of the different steps of an argument among the youngest children, particularly during free play, where various modalities may look different from that of older children (Johansson et al., 2014).

Also, as a purely qualitative observation it may be worth noting that of the six scenarios which were coded as mathematical argumentation, three were in play with magnetic or 'click-together' shapes, one was with Lego, one was with wooden blocks, and one was in a game of tic-tac-toe.

The fact that mathematical argumentation occurred in free play with shapes of some kind is perhaps not surprising. However, in analysing the data, we found it much easier, and felt it more valid, to infer a child's intentions (e.g. that a missing piece is required to complete a pattern) where the manipulatives showed this clearly, such as with magnetic tiles or the click-in tiles presented in example 1. The more 'degrees of freedom' the manipulatives allowed in their placements, going from the relatively strict possibilities of magnets, through the slightly wider possibilities of Lego, to the even less restricted wooden blocks - with which constructions could be rendered intentionally or unintentionally asymmetrical or irregular more easily - the harder it was to draw conclusions about a

child's intentions. Whether this was because the more restricted manipulatives *invited and resulted in* more mathematical thinking, claims and argumentation from the children, by virtue of leaving very clear visual clues where patterns were not completed, or whether the thinking and argumentation was just *easier to spot* for us as researchers in these cases, is an open question which may be the topic for future discussion and research.

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Subjective perception of the readiness of novice kindergarten teachers to develop children´s early years mathematics

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Abstract

The study reports on research, the aim of which was to find out how novice kindergarten teachers evaluate their own level of professional readiness for the development of children´s mathematical preliteracy at the beginning of their career. The research method of data collection was writing thematic text. The basic method that made it possible to find answers to research questions and formulate findings was the thematic analysis of respondents´ narratives. The results show the respondents´ subjective perception of their own disciplinary mathematical professional competences: in which areas of early mathematics do the respondents consider themselves the most and least competent for the cognitive development of a child in kindergarten.

Keywords: *Kindergarten teachers, professional development, preschool education, early years mathematics.*

Introduction and theoretical framework

Our research focuses on novice preschool teachers. We define them as those at the beginning of their professional careers, typically with up to 3 years of experience. They have relevant higher education qualifications (in our research, a Pre-school Teacher Training bachelor's degree in early childhood education) and pedagogical competence, but they lack practical teaching experience.

Novice teachers are a promising yet vulnerable professional group. They are often associated with (unreasonable?) expectations. It is assumed that they will bring modern approaches and innovations to schools; however, upon entering the profession, they face a range of challenging problems that are difficult to solve. They often grapple with ambivalent emotions; on one hand, looking forward to entering the profession, and on the other hand, experiencing insufficient self-confidence and fears (Darling-Hammond, 2017; Tammets et al., 2019; Stewart & Jansky, 2022). They perceive the start of their teaching careers as a very demanding phase, sometimes referred to as the "practice shock" (Veenman, 1984; Liston et al., 2006).

A teacher's belief in whether and to what extent they can influence a student's performance (Guskey & Passaro, 1994) is based on the psychological construct of self-efficacy developed by Albert Bandura (1997). Beliefs about an individual's ability to organize and carry out a particular activity through which a stated goal is achieved is a determinant of academic success in an educational setting; it is an important motivator. It plays a key role in many aspects of teaching and has a great influence on the learning performance of students and the teaching methods chosen by the teacher (Pajares, 1992).

A teacher's beliefs can be influenced by a variety of factors. Kalaja and Barcelos (2003) state that they depend on the context and experiences of the individual, are dynamic and can be contradictory. Teachers with higher self-efficacy in mathematics pay more attention and focus longer on problem solving, are more patient and exert more effort in solving, and can estimate their own performance more accurately and clearly than others (Klassen & Chiu, 2010; Yavuz et al., 2013). Pendergast et al. (2011) found that after finishing their teaching practice at the end

of their studies, the self-efficacy of student teachers (also kindergarten teachers) decreased. The authors interpreted this decrease as a result of the reality shock that students experienced after the practical experience.

In our study, we note teachers' beliefs in the importance of subjective perceptions of their own preparation of prospective kindergarten teachers in college, and the suitability and appropriateness of various activities for early mathematics learning.

A significant professional competence for (not only novice) teachers is subject-specific competence. Higher education training for teachers significantly develops their knowledge base for teaching, which Shulman (1987) considers a key aspect of the journey toward teaching professionalism (Korthagen et al., 2011). Knowing the necessary amount of basic mathematical knowledge can be useful to them in their profession in developing children's mathematical pre-literacy. By mathematical pre-literacy, we mean a set of gradually developing prerequisites for mathematics in children before they enter school; a complex of abilities, skills, attitudes and values needed for the initiation and successful development of mathematical literacy and its use in different individual and social contexts (Nováková & Novák, 2019). While it is important to respect the unique nature of the professional activities of preschool teachers, given the specifics of the age of the children they work with: the ability to create a multidisciplinary learning environment (Pramling & Pramling Samuelsson, 2011), the ability to integrate play and learning (Pramling Samuelsson & Carlsson, 2008), and to lead a collaborative and supportive interaction with children (Howes et al., 2003).

In preschool teacher education in the Czech Republic, mathematics-related subjects have only a marginal role in the curriculum. In our previous research (Nováková & Novák, 2019), we found that novice teachers generally do not consider further deepening of their mathematical knowledge necessary for their professional development. They believe that they have already acquired the basic mathematical knowledge needed for early mathematical preparation of children in preschool during their primary education. These insights often take the form of implicit, "tacit" knowledge (Švec, 2012). As Žilková et al. (2013) point out, they enter their pre-service higher education and teaching practice with many mathematical misconceptions. These facts led us to attempt to take a more objective look at preservice teachers' beliefs about their professional assumptions based on their knowledge of basic mathematics.

The main aim of the research was to determine how novice, university-educated teachers assess their level of acquired mathematical knowledge as part of professional preparedness for developing children's mathematical pre-literacy. We based our assumption on the fact that this subjective perception is grounded in the individual's reflection on their own situation, influenced by their opinions, knowledge, and experiences.

In line with the research objective, the research question was formulated as follows:

> RQ: Which aspects/areas of children's mathematical pre-literacy do novice teachers consider themselves to be the most prepared for, and which do they consider themselves the least prepared for?

Methods

As a research method for data collection, we used thematic writing focused on the self-reflection of novice teachers. The text they produced is a construction of their own subjectivity. The advantage of free written production is that the mediating effect of the researcher is smaller compared to an interview, and the participant works at their own pace (Elizabeth, 2007).

The research sample consisted of 129 graduates of the Pre-school Teacher Training program at the Faculty of Education, Masaryk University in Brno. The theme of the written self-reflection

was introduced through a prompt corresponding to the research question. The extent was given by one page of text. The writing was free and uncontrolled by the researcher—the flow of writing was not restricted in any way.

The primary method used to formulate findings was thematic analysis of the text (Braun & Clarke, 2006). According to Vaismoradi et al. (2016), thematic analysis is a systematic process that begins with coding and exploring meanings that describe and interpret participants' perspectives. The individual statements were analyzed to find answers to the research question.

The first step in data processing was the creation of codes for individual research areas. We opted for an open coding approach (Švaříček & Šeďová, 2007). For the coding process, we utilized the QDA Miner software (Provalis Research). This software enabled us to determine the frequency of each code's occurrence in the respondents' statements. A total of 18 codes were identified by the researcher across two categories: pre-numerical ideas and pre-geometric ideas. The codes extracted by the researcher from the written statements reflected the participants' subjectively perceived positive (16) or negative (11) aspects of early years mathematics.

Research results

The respondents' statements are highly varied and, in some cases, even contradictory, as is evident from the authentic examples. They reflect the level of theoretically acquired knowledge from previous education as well as the respondents' existing experiences from their own practice. We expressed these frequencies in percentages and illustrated them graphically (Figures 1 and 2). To further illustrate the teachers' opinions, we present examples of their authentic statements, highlighting the corresponding extracted codes.

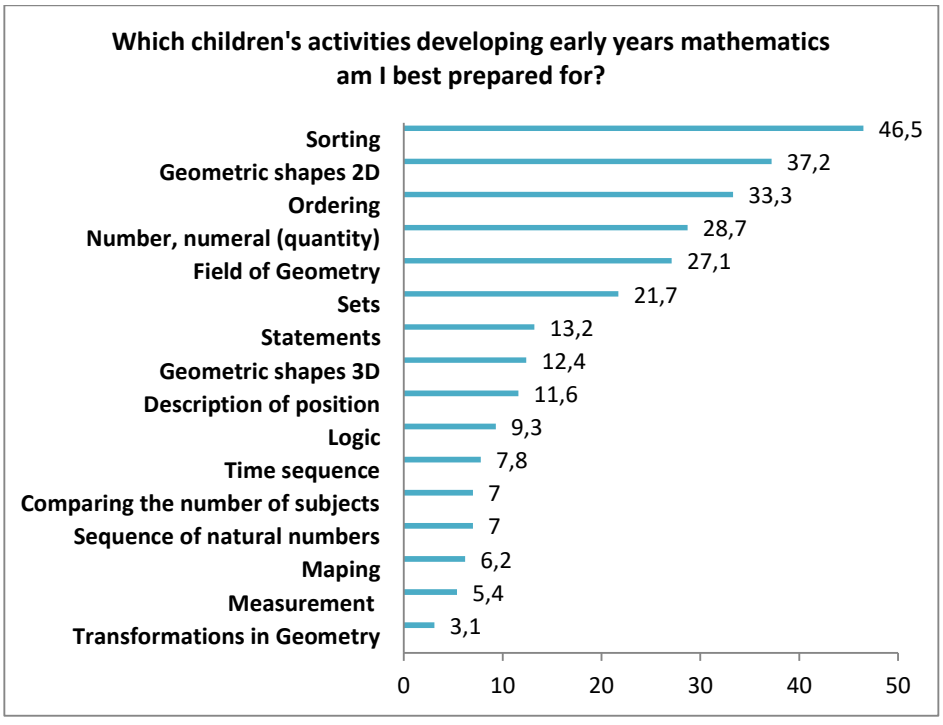

Figure 1

Early years mathematics topics in which respondents consider themselves **best prepared**

The highest frequency is observed for the code *Sorting* (46.5 % of teachers report as an activity for which they feel best prepared to work with children). This code does not appear at all among the activities for which respondents feel least prepared. The interpretation of this finding is clear: sorting objects or toys by visible, sensory attributes is often a deliberately introduced or spontaneous activity not only in preschool classrooms but also with younger children.

> "In my practice, I have had the opportunity to see and apply sorting and ordering and I believe I am well-prepared in this area. I never really realized that the activities I do with children include so many opportunities for developing mathematical pre-literacy."

The high frequency of the code *Ordering* (33.3 %), for which respondents feel well-prepared, is also not surprising. Arranging objects or figures by a given attribute in ascending or descending order is familiar to children – it has its natural reflection in reality (a line of children by size, waiting in a queue, cars stopped at a red traffic light) as well as in many fairy tales (e.g., "The Enormous Turnip," "Snow White and the Seven Dwarfs").

The code *Time sequence* (with a frequency of 7.8 %) also reflects the nature of ordering and relates to mastering orientation in time and distinguishing basic time concepts (day, night; yesterday, tomorrow; before, after). It is also common to sequence events, activities, or happenings throughout a child's day or time spent in preschool, as well as recognizing the sequence of life stages.

> "I discovered that understanding the days of the week, months, seasons, and other temporal sequences, and how things follow one another, also belongs to mathematical pre-literacy."

More than a quarter of the respondents (28.7 %) indicate that they are best prepared for activities represented by the code *Quantity (number, numerals*). They reflect their practical experience, where they use various number models for determining quantity (count of objects) during everyday situations and games. The fact that the Quantity code also appeared among the activities where some students feel least prepared (10.1 %) can be explained by their concerns about the abstract nature of numbers and their symbolic representation by numerals as specific symbols. This is also reflected in an authentic statement from one of them.

> "I feel least prepared in the area of numbers and symbolism. The problem is numerals—I don't know how to better present these symbols to children, other than always using dots or other representations of quantity."

It is not surprising that the highest frequency of strengths within the category of geometric ideas was found in the code *2D Geometric Shapes* (37.2 %). Recognizing and distinguishing basic plane geometric figures—such as circles, squares, rectangles, and triangles—and assigning the correct name to each shape are traditionally among the most common activities in kindergarten settings. However, these activities can sometimes lead to misconceptions (Levenson et al., 2011; Žilková, 2013). For instance, the boundary of a circle (circumference) is often incorrectly referred to as the circle itself, rather than the plane area it encloses. Teachers' statements suggest that they perceive activities involving geometric shapes in a broader context and are capable of incorporating them effectively into their own teaching portfolios.

> "I feel most confident working with geometric shapes, their recognition, differentiation, and sorting. It is easily visualizable, and children find it more relatable than other topics. I already have many activities prepared on this topic."

Similarly, the codes *Field of Geometry*–unspecified (27.1 %), *3D Geometric Shapes* (12.4 %), and *Description of position* (11.6 %) are identified as areas where students feel most prepared.

However, the situation is not entirely clear-cut. These codes also appear among those where students feel least prepared: the codes Description of position (8.5 %), and 3D Geometric Shapes (7.8 %). Perceiving and organizing space, orienting oneself in real space (classroom, playground) in natural situations such as nature walks, sports, dining, etc., and in deliberately created situations, are among the most significant skills developed in early childhood. Therefore, this activity is considered one of the most natural ways to develop spatial imagination, manipulative skills, and creativity.

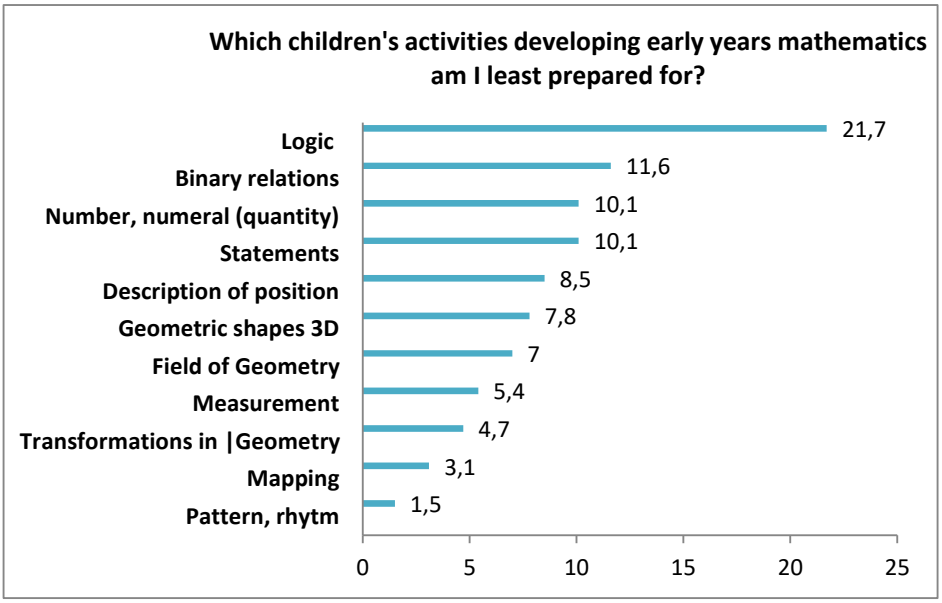

Figure2

Early years mathematics topics in which respondents consider themselves **least prepared**

In the overview of codes where respondents feel the least prepared, the code *Relation* appears frequently (14.3 %). A detailed analysis of the statements reveals that they are referring to the relation of mapping (while relations such as sorting and sequencing are considered their strengths). This reflects some students' limited experience with activities involving mutually exclusive assignments like "how many – so many": how many fingers, so many beads; how many cups, so many spoons.

> "I feel confident with activities where children match objects or connect pictures with lines on worksheets. For example, matching mugs and plates, or pictures of flowers and butterflies."

The perception of logical statements, represented by the codes *Statements, Logic*, is ambiguous. While some teachers consider themselves well-prepared to use them (13.2 % and 9.3 %, respectively), 21.7 % of respondents feel least prepared in this area. Concerns about using statements that are understandable to children but logically accurate and precise in everyday preschool activities seem to be related to teachers' lack of confidence in their own communication skills.

> "I feel less prepared in propositional logic. I am not sure if I can translate this into activities suitable for preschool children."

Transformations in Geometry (4.7 %) and *Measurement* (5.4 %) have a significantly lower representation among the codes where students consider themselves to be most well-prepared. At the same time, both codes are listed among those that students subjectively perceive as their weak areas. The students' statements provide some explanation for this observed fact.

"I feel least confident in the area of measuring geometric shapes because I am not aware of having observed measurement in preschool.

"I struggle with congruence and symmetry. I have not yet been able to fully connect this topic with practice, so I find it difficult to come up with activities for children related to this topic."

Conclusions

In summary, based on the respondents' statements, we can attempt to answer the research question. By comparing the number of positive and negative self-reflections on their own abilities to develop early years mathematics, we infer that novice teachers consider themselves sufficiently competent to effectively contribute to this aspect of children's personal development. Their opinions are highly diverse, as evidenced by the range of topics in which they feel most and least prepared.

The narratives reflect both the knowledge acquired during previous education, particularly from studying mathematical subjects at university, and the initial experiences from their teaching practice in kindergarten. From the individual statements and the identified codes, it is evident that the respondents perceive early math education through a broad range of topics. We consider this to be a good starting point for potentially creating a suitable, comprehensive, and diverse set of activities to develop children's mathematical literacy in their future professional work. Awareness of their weaker areas in their subject-specific (mathematical) preparedness can also serve as a good starting point for focusing on further professional growth.

Regarding the specific areas of early years mathematics, respondents regard pre-numerical concepts (especially sorting and ordering) and geometric concepts (2D and 3D shapes, spatial awareness, and orientation in plane and space) as the most significant and practically necessary. The development of geometric concepts is also commonly featured in didactically targeted activities in preschool education.

Respondents identified logic as a weak point in their preparedness, expressing this concern in various ways. We categorized these under the codes 'logic' and 'statements.' The development of (pre)logical thinking is a crucial component of preparing preschool children for entry into formal education. For beginning teachers, this topic is new, and they find its transformation into specific activities in kindergarten to be challenging.

The Czech Republic, unlike many other European countries, still requires preschool teachers to have qualifications at the level of secondary vocational education (ISCED 2011: level 3), although the proportion of teachers with higher education is currently increasing. The ratio of secondary school-educated to university-educated kindergarten teachers in the Czech Republic was 75.9 % to 22.6 % in 2011, and by 2017, it had changed to 60.5 % to 37.4 % (Loudová-Stralczynská et al., 2022).

International research provides evidence that higher qualifications of teachers lead to better professional knowledge compared to those with lower qualifications; this positively affects the quality of preschool education and children's outcomes (Kelley & Camilli, 2007; Oberhuemer et al., 2010; Saracho & Spodek, 2007). Barnett (2004) analyzed several studies focusing on teacher education and the quality of child care. His analysis shows that the higher the teachers' education, the more responsive they are to children's individuality and the more competent they are in enriching cognitive experiences.

We are aware of the limitations of our research, and its conclusions cannot be generalized for preschool education practice. The respondents in our research are graduates of only one university, and their statements largely reflect the outcomes of their previous education. They

view various areas of early years mathematics from a student's perspective – they have yet to confront their ideas and expectations in their professional practice.

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Dealing with multiplicative problems in pre-primary education: Analyzing children's performance and strategies

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We present findings of the 3rd cycle of an ongoing design research study aimed at developing an instructional program to support pre-primary children's multiplicative reasoning. The 3rd enactment of the program was conducted with a class of 22 Year 1 and Year 2 kindergarten children (mean age 5 years and 7 months) who were pre- and post-tested individually. We focus on two tasks, common at both test moments, which were not addressed during the intervention. The results show that all Year 2, but very few Year 1 children were able to transfer what they had learned during the intervention to novel problem situations. Further, we found emerging inter-individual differences with respect to children's ability to represent mentally the quantities and operations involved in the given problems.

Keywords: multiplicative reasoning, kindergarten, multiplicative problems, performance, strategies.

Introduction

Research evidence indicates that young children have multiplicative reasoning competencies with respect to continuous as well as discrete quantities (Mix et al., 2002). These (limited) competencies can be enhanced when children are exposed to formal and informal multiplicative experiences (Hunting & Davis, 1991; Van den Heuvel-Panhuizen & Elia, 2020). However, early childhood education over-emphasizes counting and additive relations in the context of discrete quantities and pays less attention to multiplicative situations (Hunting & Davis, 1999; Sophian, 2004; Vamvakoussi & Kaldrimidou, 2018). This asymmetry has consequences for children's mathematical development. Indeed, children's performance in simple multiplicative tasks decreases when they are able to represent discrete quantities via counting (Ni & Zhou, 2005). Further, children miss the opportunity to build the conceptual background necessary for understanding important mathematical concepts, notably fractions (Sophian, 2004).

In this paper, we present results from the third cycle of a topic-specific design research study (Gravemeijer & Prediger, 2019) aimed at developing an instructional program to foster kindergarten children's multiplicative reasoning with respect to discerning and expressing multiplicative relations and solve multiplicative problems.

Theoretical background and design of the program

We based our design on three key principles: First, treating discrete and continuous quantities similarly and simultaneously, a choice grounded on different, albeit complementary, reasons: It has been argued that both types of quantity are constructed based on the same mental operation, namely unitizing (Steffe, 2013); that counting and measurement are more similar than typically assumed (Sophian, 2004); and that three fundamental multiplicative operations (namely, equi-partitioning, iteration of a quantity and counting/measuring with different units) are the same in both contexts

(Vamvakoussi & Kaldrimidou, 2018). Second, providing linguistic tools for expressing multiplicative relations, because such tools enable young children to recognize similar relations in different contexts and organize their formal and informal experiences with multiplicative situations (Hunting & Davis, 1991). This claim is supported by recent evidence indicating that simple vocabulary pertaining to multiplicative relations (e.g., "double") at Grade 1 is associated with higher proportional reasoning abilities at Grade 2 (Vanluydt et al., 2021). Third, highlighting the inverse relationship between the operation pertaining to multiplication (iterating a quantity) and those pertaining to division, particularly equipartitioning. This relationship is of great importance for multiplicative reasoning (Greer, 2012). Consistently with this idea, the relationships "A is a multiple of B" and "B is a submultiple of A" were treated as the same relationship seen from two perspectives.

These key principles were implemented in the design of the instructional program which consisted of three story-based activities (A1, A2, and A3) featuring a scenario with a sports team of 10 imaginary creatures. The characters were represented by wooden cylinders of the same base diameter and height ranging from 1 to 10 units in length. Multiple copies of the unit were available to students.

In the first activity (A1), terms for multiples and submultiples were introduced to describe the relationship between the "players" and the "team leader" (whose height was equal to the unit length). The relationship was established through measurement and expressed from two perspectives (e.g., A is triple B/ B is one third of A). The introduction of the terms capitalized on the salient regularity in their production in Greek (Pitta & Vamvakoussi, 2022) which is based on number words and was assumed to allow children to employ aspects of their knowledge of natural numbers productively (Steffe, 2013) to acquire vocabulary necessary to express multiplicative relations. The second activity (A2) targeted proportional problems: A discrete or continuous quantity was assigned to the "team leader" (i.e., the unit) and the quantity corresponding to a "player" was requested based on her height and vice versa. Finally, the third activity (A3) focused on multiplicative change using "fraction machines" (Hunting & Davis, 1991) which, unlike the typical ones, operated in two directions: on one side, the incoming quantity was multiplied by a natural number n, while on the other, by its reciprocal 1/n. This modification allowed for bringing to the children's attention the inversibility of the processes.

In all three activities, manipulatives made of cardboard were used to represent the quantities (disks for discrete, and bars for continuous quantities). Additionally, in A2 and A3 we used a manipulative representation of the ratio between the known and the unknown quantity. Figure 1 illustrates the process of finding a submultiple of a given continuous quantity (one third in the example) using this tool: Children are given options for the unknown quantity and are asked a) to estimate which is the appropriate one (1a), b) to check if the given quantity can be composed by iterating the selected bar 3 times (1b). The tool is then used to illustrate the partitioning of the initial bar (1c), as well as the quotient (1d). In the discrete condition, the available options were presented as linear arrangements of discs glued on paper strips. To find a multiple of a given quantity, the same tool was used in reverse. The use of the tool structures the intended processes similarly for continuous and discrete quantities, highlights the inverse relationship between multiplication and division, and was also assumed to help children keep track of important elements of the problem situation.

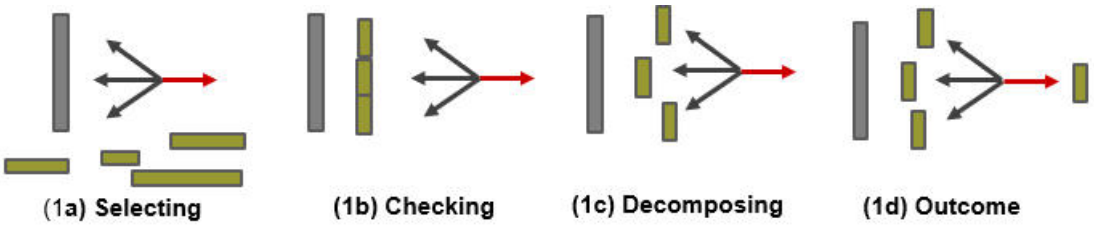

Figure 1: Finding one third of a continuous quantity with a manipulative representation of ratio

The program activities and materials have been designed, evaluated, and redesigned after two enactments with small groups of Year 2 kindergarten children (of a maximum of 4 members in each group) (Pitta, et al., 2021; Pitta & Vamvakoussi, 2022). It should be noted that Greek kindergarten classes typically comprise Year 1 (hereafter, Y1) children who enroll at 4 years of age; and Year 2 (hereafter, Y2) children who enroll at 5 years of age. Findings from the two enactments of the program showed that it was within the range of abilities of Y2 children and substantially enhanced their multiplicative reasoning competencies. In the 3rd cycle of the study, we enacted the program in a more ecologically valid setting: An entire mixed-age class (Y1 and Y2 children), participated in the intervention. We were interested in whether a) the program would have a positive effect on children's multiplicative reasoning competencies in the larger group/mixed age setting; b) the potential learning gains would be transferable to unfamiliar multiplicative situations; c) there would be inter-individual differences regarding potential learning gains. In this paper we focus on one type of difference, namely the extent of children's reliance on manipulative representations during problem-solving. This is because dependence on perceptual input can differentiate between children who solve a problem correctly, with less dependence indicating a higher level of mathematical thinking (Baroody, 2017).

Method

Participants

The participants were a class of 9 Y1 and 13 Y2 kindergarten children from an urban private school in Western Greece (mean age 5 years 7 months, ranging from 4 years 10 months to 6 years 4 months). Nine of the participants were girls (4 Y1). All children were native Greek speakers. They participated voluntarily with their parents' informed consent.

Procedure and research tools

The 3rd enactment started in January and was completed in May of the school year 2022-23. The students participated in the intervention in two groups of 11 mixed-age children; each group participated in nine sessions (approximately 45 minutes each) that took place in their classroom. During the intervention, each problem stemming from the activities was elaborated first for continuous, and immediately after for discrete quantities, to highlight the similarities. For any new problem, the teacher-researcher gave children the opportunity to solve it first on their own; she modelled the intended operations when necessary; she introduced the new terms, used them systematically, and prompted children to use them as well.

The children were pre- and post-tested via individual interviews. The interviews were conducted in two parts and were video recorded.

There were two categories of tasks used for pre- and post-testing. The first comprised problems similar to the instructional tasks, whereas the second comprised two tasks that had not been part of the intervention and tested for transfer of competencies to a new context. In the following, we will focus on these "transfer tasks". The first transfer task targeted "equal groups" and "equal measures" situations, in the discrete and continuous condition, respectively. A total of 18 problems was produced varying the number of equal groups or equal measures (n=2, 3, 4), as well as the type of problems that can be posed based on the same multiplicative situation. Figure 2, presents a model for this task in the discrete condition for n=3.The problems were formulated as follows:

Equi-partitioning (partitive division problem): Three friends (2a) will fair-share these candies (2b). Can you show me how many candies each child will get? How do you know?

Iteration of a quantity (multiplication problem): Three friends (2a) fair-shared a pack of candies. Helen took those two (2c). Can you tell me how many candies were in the pack? How do you know?

Measuring (quotitive division problem): A party of friends fair-shared these candies (2d). I don't know how many children there were (multiple copies of possible recipients available, 2e), but I know that Helen took those two (2c). Can you tell me how many children there were?

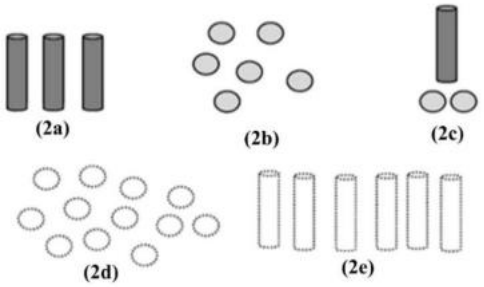

Figure 2: A model for the "equal groups" situation (n=3)

For the second task, the children were given discrete and continuous quantities and were explicitly asked to find a multiple and the corresponding submultiple of this quantity, for n=2,3,4. For example: "I have this chocolate (*paper strip bar*). I will give George one-third of the chocolate. Which of these pieces (*4 options*) do you think George will get? How do you know?" In this task (a total of 12 problems), recognizing and interpreting the terms was a prerequisite for solving the problems.

In all problems, the children were asked to explain their answers to ensure that their choices were not random. The ratio representation tool was available to the children.

Results

Performance

Children's response to each problem was evaluated and scored as 1 (correct) and 0 (incorrect), respectively. Each child's overall performance was calculated as the sum of their correct answers across all problems (a total of 30). Interestingly, among all possible total scores, only few were present in our data. Table 1 presents the frequency of each total score at pre- and post-test. At pre-test, only four children (including one Y1 child) solved two or three problems correctly. All correct answers

were given in the discrete condition, specifically a) in a fair sharing problem with two recipients and, in one case, three recipients, b) recognition of the term "half," and c) recognition of the term "double".

Table 1: Frequency of total scores in the Pre- and Post-test

Pre-test			
Total score	Y2	Y1	Total
3	1	0	1
2	2	1	3
0	10	8	18
Total	13	9	22
Post-test			
Total score	Y2	Y1	Total
30	7	1	8
29	3	0	3
21	2	0	2
20	1	1	2
0	0	7	7
Total	13	9	22

At post-test, 7 out of 9 Y1 children had a total score of zero. Two of them refused to engage with the tasks from the beginning, while the others quitted midway without answering correctly even the simplest of tasks (i.e., when the relation in question was 1:2 / 2:1).

The remaining children (after the intervention) can be broadly divided into two groups: In the first (Group A), 11 children (including one Y1 child) solved at least 29 out of the 30 problems, while in the second (Group B), 4 children (including one Y1 child) solved 20-21 problems. It is worth noting that the incorrect responses were mainly observed in problems with n>2, primarily (but not exclusively) in the discrete condition, and mostly for submultiples.

Strategies

The children's strategies at the post-test were examined based on the extent to which they needed to use the manipulatives given and/or the ratio representation tool during the problem-solving process to represent the involved quantities or relationships. Three categories emerged:

The first category (S1, manipulatives-based strategy) included the cases in which a child used the given manipulatives, including the ratio representation tool, to represent all involved quantities. For example, in a partitive division problem (6 candies for 2 children), Child 1 used the materials representing the candies as well as the dolls representing the recipients; she initially matched each recipient with a pair of candies and then distributed the remaining candies one by one.

The second category (S2, partially mental strategy) included the cases in which a child used the materials, but not for all the involved quantities. For example, in the above problem, Child 2 used the material representing the candies, but not those representing the recipients. More specifically, she placed the 6 candies in a line, stated that she needed to create two groups of candies and pointed to the middle of the line saying, “There are two groups of three on the table, and Helen will take one group, and her sister will take the other one”.

The third category (S3, fully mental strategy) included cases in which a child solved the problem mentally, using the materials only to justify their answer. Such strategies indicate that the child performed mentally the necessary operations. In the following example, Child 3 employs an S3 strategy while solving a multiplication problem (3 candies per child x 3 children):

Child 3: [Looks at the available candies for a while] It's 9 candies [Arranges 9 candies in a line].

Interviewer: How do you know?

Child 3: [Points to a group of three with his fingers] Three, three, three makes nine. It's triple. Three times. Nine.

S3 strategies were also observed in the case of continuous quantities. In the following excerpt, Child 4 estimates in advance which of the options corresponds to half of the given quantity:

Child 4: [Looks at the options for a while and chooses the correct one] This is the one half.

Interviewer: How do you know?

Child 4: If you place it here [points at the bar], it is one of the two pieces. Look. [Opens thumb and index finger to the length of the piece] One, two [measures the chocolate].

An additional category (S0) included cases corresponding to incorrect or unjustified answers. Table 2 presents the frequency of these categories across all problems, for all 15 children at post-test. The dominant strategy was, by far, the S1 strategy; about 15% of the strategies were S2 or S3.

Table 2: Frequency and percent of different strategy categories, in the total of responses

Strategy Category	N (%)
S0 (No /Erroneous Strategy)	41 (9.1)
S1 (Manipulatives-based strategy)	342 (76)
S2 (Partially mental strategy)	37 (8.2)
S3 (Fully Mental strategy)	30 (6.7)
Total	450 (100)

We calculated the number of S2 and S3 strategies per child across all problems and we categorized the children accordingly, as presented in Table 3. Eleven out of 15 children demonstrated at least one S2 or S3 strategy. Four children (including one Y1 child) employed no such strategy; they all belonged to Group B. The remaining Group B children used at most 3 S2 or S3 strategies.

Table 3: Frequency of children groups based on the sum of partially (S2) and fully (S3) mental strategies across all problems at post-test

Sum of S2 and S3 strategies	0	1-3	4-6	7-9	10	14
Frequency	4	5	2	2	1	1

Finally, the sum of S2 and S3 strategies coming from Group A children ranged from 1 to 14. It is also worth noting that most of these strategies appeared in problems pertaining to the relations 1:2/ 2:1. However, 9 children demonstrated at least one such strategy for relations other than 1:2/2:1.

Conclusions and discussion

We presented findings from the third cycle of a topic-specific design research study (Gravemeijer, & Prediger, 2019) aimed at developing an instructional program to foster kindergarten children's multiplicative reasoning. In the previous two cycles, the program was enacted with small groups. In this cycle, an entire, mixed-aged, kindergarten class participated in the program in groups of eleven, which is a more ecologically valid setting in the Greek educational context. We presented findings for two tasks to which the children had not been exposed during the intervention and, thus, constituted a novel context for them. This claim was corroborated by findings indicating that at pre-test very few children were able to tackle a very limited number of the given problems.

Our results showed that, first, all Y2 children and two Y1 children had substantial learning gains: Not only did they recognize the terms for multiples (n=1, 2, 3, 4) and corresponding submultiples, but were also able to construct multiples and submultiples of given discrete as well as continuous quantities. Further, they were able to transfer what they had learned in the context of the intervention to the context of "equal groups" and "equal measures" situations, across all types of related problems (i.e., multiplication, partitive division, and quotitive division problems). The majority showed a remarkably consistent performance across problems; a smaller group of children had not yet stabilized their performance on problems involving relations other than 1:2/2:1 that are known to be the most accessible to children of this age (Hunting & Davis, 1991).

Second, it is evident that these research tasks were too demanding for most Y1 children. The two notable exceptions indicate that this is not merely due to age, highlighting the possibility of substantial inter-individual differences among Y1 children, beyond those between Y1 and Y2 students. We note that, based on our experience of Y1 children's participation in the intervention, we expected more of them to be able to tackle problems at the post-test moment, at least for the second task. Analysis of the intervention data will shed light on Y1 children's ways of engagement with the activities and inform our decision on mixed-age grouping in future enactments.

Third, there are indications of emergent inter-individual differences with respect to the extent manipulatives were necessary for children during the solution process. Mental representation of the problem situation differentiates children who solve correctly a given problem (Baroody, 2017), as is evidenced by the examples we presented. This feature will be further explored in combination with children's problem-solving methods (currently under analysis), with a view to outline a learning trajectory (Clements & Sarama, 2021) pertaining to early multiplicative reasoning.

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Spatial ability and one-step additive word problems in early childhood

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Although the influence of language and numerical abilities on early arithmetic word problem solving has been explored, spatial ability has received limited attention despite evidence suggesting a causal spatial-mathematical link. Using confirmatory factor analysis and structural equation modelling, we investigate the extent to which spatial visualization and visuospatial working memory predict the problem-solving success of 137 six- to seven-year-olds in England. Integrating Thompson's (1993) distinction between quantity and quantitative relations, Davydov's (1982) discrete versus continuous quantities, and Vergnaud's (1982) dynamic versus static relations, we propose a novel framework for categorizing one-step additive word problems. Preliminary analysis indicates that spatial visualization is relevant for solving Dynamic Relation-Relation Unknown problems, aligning with literature suggesting its importance in solving novel and challenging problems.

Keywords: Additive word problems, quantity, quantitative relation, spatial visualization, visuospatial working memory.

Introduction

Early arithmetic word problem solving skills are important, partly because they predict later academic and future career success (Murnane et al., 2001). The influence of language and numerical factors in arithmetic word problem solving has received considerable attention in the fields of educational psychology and mathematics education (Verschaffel et al., 2020). However, one understudied yet promising lens for understanding young children's arithmetic word problem-solving capability is their spatial ability as there is evidence of a causal spatial-mathematical link (Hawes & Ansari, 2020).

While research with older school children suggests that spatial ability may be important for solving certain types of word problems (Gilligan-Lee et al., 2022), there have been few investigations focused on the relationship between spatial ability and arithmetic word problem solving in young children. The present study aims to address this gap in the literature by investigating the extent to which spatial visualization and visuospatial working memory predict one-step additive word problem-solving performance of different problem types in young children. Toward this aim, this paper proposes a novel framework for categorizing one-step additive word problems.

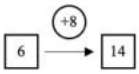
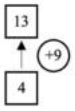
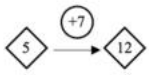
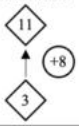
Word Problems

This study focuses on one-step additive word problems. Central to this study's conceptualization of arithmetic word problems is the distinction between a quantity and a quantitative relation (Thompson, 1993). A quantity is an intrinsic attribute of an object that does not depend on anything other than the object itself, such as the set size. A quantitative relation, in contrast, describes the connection between quantities. While both can be represented with a numerical value, when measured, quantitative relation is best represented by integers (Vergnaud, 1982).

Adapting Davydov's (1982) conceptualization, we propose a further subdivision of quantity into two types: *discrete quantity* and *continuous quantity*. Discrete quantities are measured through one-to-one counting, whereas continuous quantities are measured using an imposed unit. For example, the statement, 'Ann has 8 books' presents a discrete quantity given that the number of books Ann has is a specific, fixed quantity, measured through counting. On the other hand, the statement, 'Ann is 8 steps from the wall', presents a continuous quantity given that Ann's position is determined via an imposed measurement unit (i.e., steps).

Adapting Vergnaud's (1982) conceptualization, the quantitative relation can also be further subdivided into two types: *dynamic relation* and *static relation*. A dynamic relation describes the transformation in the state of nature over time (i.e., initial and final quantity). A static relation, in contrast, describes the comparison of disjoint quantities (i.e., referent and compared quantity).

As shown in Figure 1, the two types of quantities—discrete and continuous quantity—and the two types of quantitative relations—dynamic and static relation—form four quadrants. When each quadrant is phrased as a one-step additive word problem, two of the three facts are presented and the problem solver's task is to determine the missing third fact, or the unknown (Verschaffel et al., 2020). Therefore, along with the two types of quantities and quantitative relations, the two types of unknowns (quantity or quantitative relation) form this study's two-by-two-by-two framework for categorizing one-step additive word problems (2 [Quantity] x 2 [Quantitative Relation] x 2 [Unknown]). Word problems can be categorized in different ways, depending on whether one, two, or all three dimensions are used.

Quantity	Unknown	Quantitative Relation			
		Dynamic Relation		Static Relation	
Discrete Quantity	Quantity Unknown	Initial[a] or Final[c] Discrete Quantity Unknown Problems	Ann had 6 marbles.[a] Ann won 8 more marbles.[b] Ann now has 14 marbles.[c] 	Referent[a] or Compared[c] Discrete Quantity Unknown Problems	Ann has 4 marbles.[a] Bob has 9 more marbles than Ann.[b] Bob has 13 marbles.[c] 
	Quantitative Relation Unknown	Dynamic Relation[b] (between Discrete Quantities) Unknown Problems		Static Relation[b] (between Discrete Quantities) Unknown Problems	
Continuous Quantity	Quantity Unknown	Initial[a] or Final[c] Continuous Quantity Unknown Problems	Ann was 5 steps from the wall.[a] Ann walked 7 more steps away from the wall.[b] Ann is now 12 steps from the wall.[c] 	Referent[a] or Compared[c] Continuous Quantity Unknown Problems	Ann is 3 steps from the wall.[a] Bob is 8 steps farther from the wall than Ann.[b] Bob is 11 steps from the wall.[c] 
	Quantitative Relation Unknown	Dynamic Relation[b] (between Continuous Quantities) Unknown Problems		Static Relation[b] (between Continuous Quantities) Unknown Problems	

Note. The dimensions of quantity, quantitative relation and unknown form a two-by-two-by-two framework for one-step additive word problems. The categorization of word problems differs based on whether one, two, or all three dimensions are involved. When each quadrant (Discrete Quantity–Dynamic Relation, Discrete Quantity–Static Relation, Continuous Quantity–Dynamic Relation, and Continuous Quantity–Static Relation) is framed as a one-step additive word problem, two of the three facts are provided, and the missing third fact becomes the unknown to be solved. The unknown is either a quantity[a and c] or a quantitative relation[b].

Figure 1: The two-by-two-by-two framework for one-step additive word problems

Spatial Ability

We define spatial ability as the ability to "generate, manipulate, and reason about spatial relations between and within objects" (Hawes et al., 2022, p. 112). This study concentrates on two spatial skills

that have been most consistently identified in the literature as having a strong association with mathematical thinking: spatial visualization and visuospatial working memory (VSWM). Spatial visualization is the ability to manipulate and transform visual-spatial images (Hawes & Ansari, 2020). Typical measures of spatial visualization include 2-dimensional mental rotation tasks, 3-dimensional mental rotation tasks, mental composition tasks, and mental paper-folding tasks. An extensive body of literature suggests that the relation between spatial visualization and mathematical performance is correlational, possibly causational and even bi-directional (Newcombe et al., 2019).

VSWM is the ability to temporarily store and manipulate visuospatial information (Hawes et al., 2022). There are two types of VSWM: simultaneous and sequential (Allen et al., 2019). In simultaneous VSWM tasks, all information is presented at the same time. In sequential VSWM tasks, the presentation of stimuli is in a sequence. A systematic review on the relationship between VSWM and mathematical performance indicates that sequential VSWM may be particularly promising in studying the spatial-mathematical link (Allen et al., 2019). Yet, studies have only investigated the influence of sequential VSWM on numerical operation (i.e., computation problems) but not on mathematical reasoning (i.e., word problems), which highlights "an evident lack in the literature of such focused work" (Allen et al., 2019, p. 526). Therefore, this study will focus on sequential VSWM and how it relates to one-step arithmetic word problem solving.

Methods

We used data from 137 Year 2 students in England (50.4% female, 49.6% male; age range: 6.52-7.74yrs; mean age: 7.10yrs). A non-probabilistic convenience sampling method was used to recruit participants. A two-step recruitment procedure was used whereby four schools in South East England were recruited first, followed by the pupils. Two schools used the opt-out parental consent approach, and the other two schools used the opt-in parental consent approach. Students were administered a battery of tests one-to-one by the first author, over a span of two testing sessions lasting approximately 30 minutes each, spaced at least an hour apart (but no less than 30 minutes).

Each student was given a total of five tasks. The two spatial tasks were the Wechsler Abbreviated Scale of Intelligence-Second Edition (WASI-II) Block Design subtest, and PathSpan. The Block Design subtest, which requires the participant to use red-and-white cubes to recreate the design shown on the Stimulus Book, within a specified time limit, was used as a measure of spatial visualization (Atit et al., 2022). PathSpan, which is a digitalized version of Corsi Block-Tapping task on iPad, was used as a measure of VSWM (Hawes et al., 2019). Participants are presented with a sequence of dots that light up one at a time in a random pattern, which they must then replicate by touching the dots in the same order they were lit. The Wechsler Individual Achievement Test – Second UK Edition (WIAT-II^{UK}) Listening Comprehension subtest, which involves selecting a picture that matches a word or sentence and generating a word that matches a picture or oral description, was used as a measure of language ability. In this study, listening comprehension was deemed a more relevant measure of language ability than reading comprehension because the one-step additive word problems are read aloud to the participants (Diakidoy et al., 2005). The two mathematical tasks were the word problems task and WIAT-II^{UK} Numerical Operations subtest, which involves identifying and writing numbers, counting, and solving arithmetic computation problems.

The one-step additive word problems were developed through an iterative process for the purposes of this study. The discrete quantity problems were adapted from the literature (Carpenter et al., 1999); the continuous quantity problems were developed based on prior research. All problems have passed three rounds of review and revision, including two rounds of pilot studies, and one round of expert validation with a psychologist who specializes in children's mathematical development and a lecturer in mathematics education. There were 24 problems in total, with three problems in each of the eight categories of one-step additive word problems. Twenty-four unique number facts between 11 and 20 were used in the word problems, selected based on the National Curriculum and consultations with multiple primary school teachers. The number facts were randomized to create four final alternative versions (Forms A-D) of the word problems. Form A was given to 35 participants, while the other Forms were each given to 34 participants. The answers were scored as either correct or incorrect.

Using R and RStudio, confirmatory factor analysis (CFA) was performed on 24 dichotomous word problem items to identify the combination of dimensions with optimal model fit so that the problems can be categorized accordingly. The Weighted Least Squares Mean and Variance adjusted (WLSMV) estimator was employed. Then structural equation models (SEM) were used to answer the substantive question by investigating the relationship between word problem solving performance and spatial visualization and VSWM. Age, gender, language and numerical abilities were included as covariates as they affect word problem-solving success (Daroczy et al., 2015). This paper presents findings from the preliminary analysis of an ongoing study.

Findings

Reliability of Word Problems: CFA

The four versions of the one-step additive word problems were examined for equivalence using Generalized Mantel-Haenszel differential item functioning (DIF) analysis, with test scores as the matching variable. No DIF items were identified with Form D as the reference with a χ^2 value of 3.84 ($p < .05$) as the detection threshold. Therefore, the four test versions were treated as equivalent for subsequent analyses.

The 24 items were tested for internal consistency. The Kuder-Richardson 20 coefficient was 0.93 (95% CI [0.91, 0.95]), indicating strong internal consistency. Most items showed statistically significant tetrachoric correlations, ranging from $r = .14$ to .84. The items were tested for fit. All items had Loevinger's H values greater than .20 (ranging from .27 to .70). However, item 22 did not meet the unidimensionality criterion and was excluded from further analyses.

CFAs were conducted to test the categorization framework of one-step additive word problems (see Figure 1). Eight models, derived from the framework's dimensions—a 1-factor model, three 2-factor models, three 4-factor models, and an 8-factor model—were compared to identify the best-fitting model for investigating the substantive research question. The four-factor model of word problems, wherein quantitative relation and unknown dimensions were used, demonstrated the best fit, as indicated by the model fit indices ($\chi^2/df = 1.14$, $CFI = 0.99$, $TLI = 0.99$, $RMSEA = 0.032$). Therefore, the framework was revised accordingly (i.e., 2 [Quantitative Relation] x 2 [Unknown]) and the four categories of word problems (i.e., Dynamic Relation–Quantity Unknown, Dynamic Relation–

Relation Unknown, Static Relation–Quantity Unknown, and Static Relation–Relation Unknown) were used in subsequent analyses. All factor loadings were significant ($p < .001$), ranging from .70 to .99, indicating that the items reliably measured their respective factors.

Relationship between Word Problem Solving and Spatial Ability: SEM

Table 1 presents the descriptive statistics and correlations among variables. All five variables were significantly associated with each other, except for VSWM and listening comprehension. Performance on word problems had the strongest correlation with numerical operations ($r = .69$), followed by listening comprehension ($r = .46$), spatial visualization ($r = .38$) and VSWM ($r = .36$).

Table 1: Descriptive statistics and correlations among observed variables

	Listening Comprehension	Numerical Operations	Spatial Visualization	Visuospatial Working Memory	Word Problem Solving
Numerical Operations	.36**				
Spatial Visualization	.21*	.47**			
Visuospatial Working Memory	.16	.40**	.46**		
Word Problem Solving	.46**	.69**	.38**	.36**	
Mean (SD)	17.78 (3.89)	12.71 (3.26)	11.65 (6.83)	5.39 (1.22)	9.38 (6.96)

Note. $n = 137$; $^{**}p < .01$, $^{*}p < .05$; The score on Word Problem Solving is the total number of correct responses on items 1 to 24, excluding item 22.

Using the revised four-factor model of word problems (2 [Quantitative Relation] x 2 [Unknown]), SEM was conducted to investigate this study's substantive research question. The two predictors (spatial visualization and VSWM) and the four covariates (age, gender, numerical operations, and listening comprehension) were added to the latent factors using a step-by-step approach to test whether the predictors made unique contributions to word problem-solving performance.

As shown in Table 2, in Model 1, when only gender and age were included as controls, both spatial skills significantly predicted three of the four factors. However, the specific factors predicted by each spatial skill varied. VSWM was positively related to Factor 1, Dynamic Relation–Quantity Unknown problems ($\beta = 0.23$, $p < .05$); Factor 3, Static Relation–Quantity Unknown problems ($\beta = 0.21$, $p < .05$); and Factor 4, Static Relation–Relation Unknown problems ($\beta = 0.27$, $p < .05$); but not Factor 2, Dynamic Relation–Relation Unknown problems ($\beta = 0.19$, $p = .06$). Spatial visualization was positively related to Factor 1, ($\beta = 0.28$, $p < .01$), Factor 2 ($\beta = 0.44$, $p < .001$), and Factor 3 ($\beta = 0.24$, $p < .01$). However, it did not significantly predict Factor 4 ($\beta = 0.13$, $p = .22$).

In Model 2, when numerical operations was also included in the model as a covariate, only spatial visualization significantly predicted Factor 2 ($\beta = 0.25$, $p < .05$). In Model 3, when listening

comprehension was further added to the model as a covariate, spatial visualization no longer significantly predicted Factor 2 ($\beta = 0.22, p = .07$). Numerical operations and listening comprehension significantly predicted all factors.

Table 2: Standardized estimates and model fit statistics for the SEM models

	Model 1				Model 2				Model 3			
	Standardized Estimates (β)											
	Factor 1	Factor 2	Factor 3	Factor 4	Factor 1	Factor 2	Factor 3	Factor 4	Factor 1	Factor 2	Factor 3	Factor 4
Spatial Visualization	0.28**	0.44***	0.24**	0.13	0.05	0.25*	0.03	-0.11	0.003	0.22	0.01	-0.13
Visuospatial Working Memory	0.23*	0.19	0.21*	0.27*	0.10	0.07	0.09	0.12	0.09	0.06	0.09	0.11
Age	0.21*	0.17	0.28**	0.18	0.11	0.08	0.18*	0.06	0.14	0.09	0.20**	0.08
Gender	0.10	0.13	0.01	0.18	0.04	0.10	-0.03	0.13	0.15	0.16*	0.02	0.19*
Numerical Operations					0.62***	0.49***	0.57***	0.61***	0.50**	0.41***	0.47***	0.51***
Listening Comprehension									0.34***	0.23*	0.27**	0.27**
χ^2	333.88				411.31**				393.40*			
df	300				346				338			
χ^2/df	1.11				1.19				1.16			
CFI	0.99				0.95				0.95			
TLI	0.99				0.96				0.96			
$RMSEA$	0.03				0.04				0.04			

Note. ***$p < .001$, **$p < .01$, *$p < .05$; Factor 1: Dynamic Relation–Quantity Unknown; Factor 2: Dynamic Relation–Relation Unknown; Factor 3: Static Relation–Quantity Unknown; Factor 4: Static Relation–Relation Unknown; Acceptable model fit: $\chi^2/df < 2$, $CFI > 0.95$, $TLI > 0.95$, $RMSEA < 0.05$.

Discussion

The role of spatial ability on early years arithmetic word problem solving has received limited attention despite the strong evidence of a causal spatial-mathematical link (Hawes & Ansari, 2020). Adapting Thompson's (1993) distinction on quantity and quantitative relation, as well as Davydov's (1982) conceptualization of discrete versus continuous quantity, and Vergnaud's (1982) conceptualization of dynamic versus static relation, this study proposed a novel framework for

categorizing one-step additive word problems. Using data from Year 2 students in England (6- to 7-year-olds), the framework was tested through CFAs and found a best fit four-factor model (2 [Quantitative Relation] x 2 [Unknown]), suggesting categorization of problems into four types: Dynamic Relation–Quantity Unknown, Dynamic Relation–Relation Unknown, Static Relation–Quantity Unknown, and Static Relation–Relation Unknown problems.

Preliminary analysis found that while spatial ability predicted performance on all four problem types, only spatial visualization continued to predict performance on Dynamic Relation–Relation Unknown problems when numerical ability was included in the model—but not when language ability was also included. This finding supports emerging evidence that spatial visualization is more important for solving novel, difficult problems, while VSWM is more relevant for familiar, procedural ones (Mix, 2019). The finding that young children's spatial ability may be more strongly associated with mathematical reasoning about the relations between quantities rather than the quantities themselves is not surprising given that spatial thinking is fundamentally relational (Hawes & Ansari, 2020). More interestingly, spatial ability may be relevant for mathematical reasoning about the transformation of not only quantities that are naturally measured relative to a reference point, such as position, but also quantities that are measured independent of a reference point, such as set size. This finding confirms studies that indicate that spatial ability is not only related to mathematical domains that are inherently spatial, such as geometry and measurement, but also mathematical domains that do not appear to be inherently spatial, such as numeration and algebra (Casey et al., 2015). Future research should explore the types of visual representations young children produce in solving one-step additive word problems and whether they have a mediation effect between spatial ability and word problem-solving performance, as has been done with older school children (Hegarty & Kozhevnikov, 1999).

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Perceiving and describing structures of kindergarten children

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Successful structuring processes are an important basis for arithmetical learning. This includes describing (individually) perceived structures of a set. Therefore, a learning situation with two groups of children in the last year of kindergarten (age 5) in Switzerland was analysed. In this paper we investigate which structures the kindergarten children perceived and described and whether they oriented themselves towards other participants in their descriptions. The results of the study show that children use four main strategies in their descriptions and that they adopt descriptions from other children as well as from the kindergarten teacher. These takeovers can be seen both positively (children learn how to describe structures) and critically, as they may inhibit their own individual perceptions.

Keywords: Perceiving structures, describing sets, early mathematics education, interaction.

Introduction

There is a broad discussion about teaching and learning in early childhood (Björklund et al., 2020). Numerous studies focus on which materials and structures can support children in their structuring processes, especially regarding patterns and structures. Finger patterns used to support a part-whole understanding (e.g., Kullberg et al., 2020) play just as much a role as well as, for example, dealing with repeating, growing or spatial patterns, e.g. by structuring a set with counters so that you can quickly see how many there are (e.g., Lüken & Kampmann, 2018). This paper analyses a learning situation where kindergarten children in their final year (age 5) are asked to arrange a set in a ten-frame structure (egg cartons) to quickly recognize the number. The analysis focuses on the described structures and participant interactions, discussing their significance for individual learning processes.

Theoretical and empirical background

Structuring processes

Numerous studies indicate a correlation between a structuring perception of sets and both arithmetical and general mathematical learning (e.g., Dornheim, 2008; Kullberg et al., 2020; Lüken & Kampmann, 2018). There is also a correlation between the development of mathematical number concepts and the use of different structuring approaches, which serves as a foundation for arithmetical learning (van Nes, 2009). Additionally, quickly recognizing structured quantities is correlated with later success in arithmetic lessons (Dornheim, 2008). Perceiving sets in structures is also a prerequisite for determining cardinality through structural use: only by perceiving a set in (sub-)structures it is possible to use these structures to determine the cardinality of the set (Sprenger & Benz, 2020a, b). When determining the cardinality of a set, two processes can be distinguished (Steffe & Cobb, 1988; Benz, 2014): the process of perceiving the set and the process of determining cardinality. Benz (2014) developed a theoretical model illustrating the connections between these processes. Initially created through an inductive approach, the model has been empirically evaluated (e.g., Benz, 2014; Sprenger

& Benz, 2020a, b). This model distinguishes three ways of perceiving a set: as individual elements, as a whole, or in (sub-)structures. Each way of perception offers different possibilities for determining cardinality (e.g., Benz, 2014; Sprenger & Benz, 2020a, b). Since perception is an invisible process, it is challenging to analyse and understand. Several research approaches exist to investigate the process of perceiving a set in (sub-)structures (see Sprenger & Benz, 2020b). One way is analysing eye-tracking data to gain insights into the invisible process of perception (Sprenger & Benz, 2020a, b). These analyses have led to the insight that children do not always say what they think. For instance, when children state they have counted the elements of a set, it does not necessarily mean they do not perceive structures (Sprenger & Benz, 2020a). Two other ways for inferring perceptual processes include asking for verbal explanations of perceptions or having children create their own set representations (e.g., Benz, 2014; Gervasoni, 2015; Lüken & Kampmann, 2018). For all three possibilities, interpretations must be cautious to make accurate statements about children's perception of (sub-)structures.

Learning in interaction processes

Mathematical learning in kindergarten is embedded into kindergarten socialization. Interactions between the learners can be seen as a constitutive element for learning (Krummheuer & Brandt, 2001). Such a collaborative learning (Langer-Osuna, 2018) in mathematics education works through interaction between children, where identity (this refers to the way in which children position themselves and others in social interactions), learning and participation are closely linked. Various forms of participation can be distinguished (Krummheuer & Brandt, 2001): A person can adopt another person's wording and apply it to a new content, or they can adopt the idea behind the content and reformulate it. Complete imitation (content and wording) is also possible.

Research interest

Perceiving and describing structures is an important prerequisite for arithmetical learning and it is therefore crucial for children, to learn reliable structures they can use for their calculation strategies (Kullberg et al., 2020). As the ten-frame is also a useful didactical presentation for representing such reliable structures in primary school, this structuring is ideal for investigating how children in kindergarten age perceive and describe structures. This leads to the first research question:

1. How do children in the last year of kindergarten describe sets within a ten-frame-structure?

Talking about structures can be challenging for children, especially those of kindergarten age (Sprenger & Benz, 2020b). This raises the question of whether interaction with others can have an influence on their own descriptions, which leads to the second research question:

2. How do children adopt descriptions of others during a learning situation?

Method

The data presented in this paper are part of the project “MATHEsprechen in kindergarten: Using language repertoires in mathematics productively right from the start”, which investigates the overarching question of the extent to which the use of multilingual resources is stimulated in kindergarten children through mathematical activities with materials in a learning environment. The

mathematical content of the learning environment is about perceiving and using structures in sets. The whole learning environment consists of five parts (three kindergarten parts, two family parts) and was tested in a design experiment in September 2023 in two Swiss kindergarten groups: the first with 10 children (average age: 5 years, 7 months), the second with six children (average age: 5 years, 3 months) in the last year of kindergarten. For the analysis presented in this paper, we examine the learning situation from part 1, conducted in German with both groups. The same teachers (KG1 and KG2) led both situations. KG1 is a person of the research team. Both kindergarten groups had the same learning situation in which they should find a possibility to put a number of eggs in the carton so that one can quickly see, how many eggs there are. The egg cartons for 10 eggs are familiar to the children from their everyday life and resemble a ten-frame, a key visual tool in primary school, whose basic structure (bundle of tens, structure of fives, etc.) can be extended to a field of twenty and hundred. The two learning situations were videotaped with parental consent, transcribed, and coded using MAXQDA, a qualitative data analysis software that helps researchers organize, code, and analyse qualitative data. In a circular, inductive process, four main categories were identified, and participant interactions were coded and analysed.

Results

Strategies for describing structures

Four main strategies were identified when the children placed and described the quantities four, five and six, which are presented below. The adoption of the descriptions of the different participants is described for each group. These relationships are indicated by arrows in the following figures.

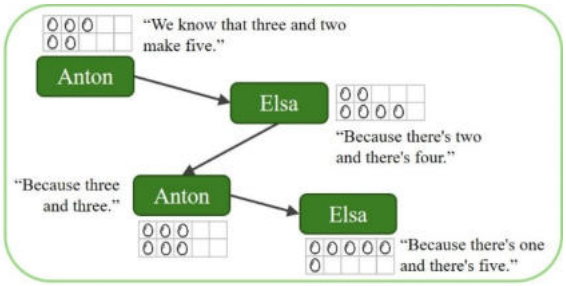

Figure 1: Adoption of the description for strategy "Describing subsets in rows" – (group 1)

In group 1 (green), Anton and Elsa each describe the quantities in two subsets. They name the number of eggs in each row, such as "We know that three and two make five" (Anton) or "Because three and three" (Anton) (see Figure 1). Elsa takes up this approach later and describes two more egg cartons using this strategy by describing the subsets of the two rows. This strategy was also observed in the second group (orange). Ester describes the egg carton with five eggs: "Because here are two and here are three" (see Figure 2). She first names the row with the smaller number of eggs and then the row with the larger number. Lilly adopts this approach immediately afterwards and also names the row with the smaller number first. Ester also keeps her strategy for other arrangements of the five eggs. She always names the smaller number of eggs first, regardless of whether this row is the top or the bottom row (see Figure 2). In an egg carton with one egg in the top row in the middle and three eggs in the bottom row (see Figure 2, right), Lami first describes the arrangement by counting the eggs

one by one. Anna describes the set in two subsets: “Because here are three and there one” and Lami adopts this description and says: “There three and here one.”

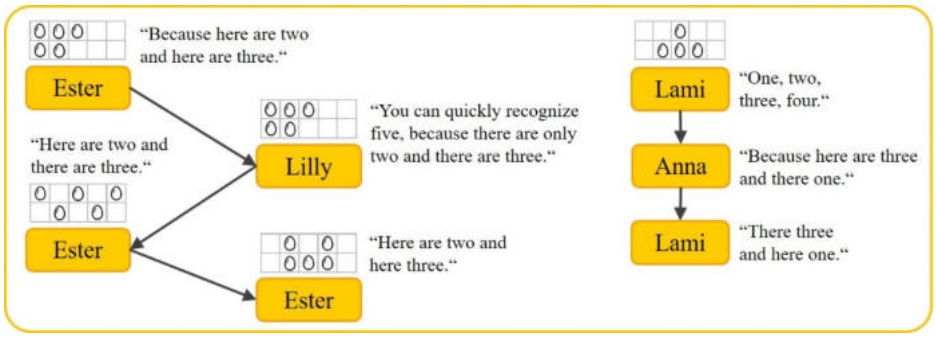

Figure 2: Adoption of the description for strategy “Describing subsets in rows” – (group 2)

Another strategy that children in both groups used to describe the quantities is to name subsets that do not correspond to the rows. In group 1 (see Figure 3) KG1 describes the set with five eggs: “I see four there and one more.” She points to the eggs on the left, which are arranged like a dice-four, then to the third individual egg in the top row. Naomi adopts exactly this description later (see Figure 3).

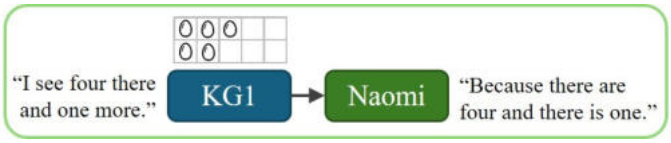

Figure 3: Adoption of the description for strategy “Describing subsets not in rows” – (group 1)

In the second group, Anna also describes this almost identical arrangement: “Because here are four and one” (see Figure 4). Lami takes up such a description in subsets and describes his arrangement: “Here is four and there is two.” He points with his hand to the four eggs on the left and the two eggs on the right (see Figure 4).

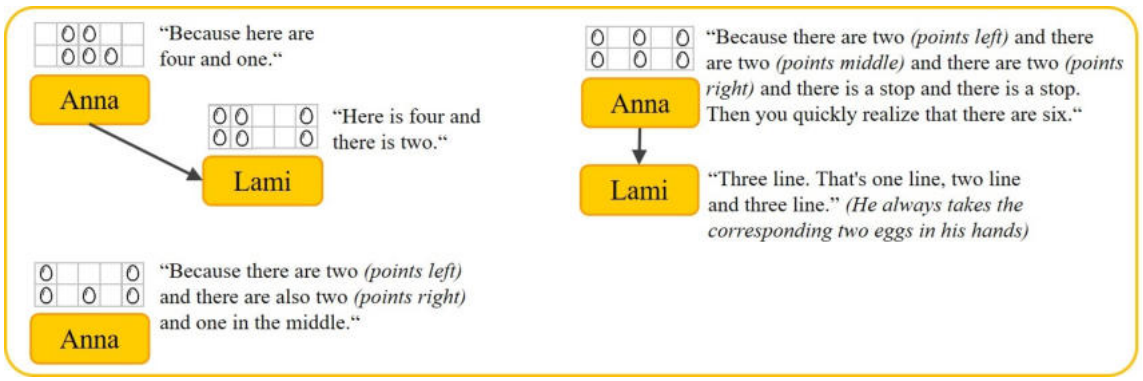

Figure 4: Adoption of the description for strategy “Describing subsets not in rows” – (group 2)

Anna arranges her eggs in three rows of two with the quantity of six eggs (see Figure 4). She points to a pair of two from left to right and explains: “...there are two and there are two and there are two...” Lami has initially sorted his eggs differently in his egg carton. He sees Anna’s arrangement and rearranges his eggs so that they resemble Anna’s arrangement. He also follows her example in the description and talks about three lines. He takes each of the two eggs in his hands and says “... That’s

one line, two line and three line [*sic*].” The arrangement of the five eggs in the egg carton in Figure 3 above was placed by KG2. Anna names three subsets of why she can quickly see that there are five eggs: “Because there are two and there are also two and one in the middle” (see Figure 4, above).

A third strategy, particularly noticeable in group 1, is referring to empty places (see Figure 5 and 6). Right at the beginning of the learning situation, Elsa argues that there is an empty place in the egg carton with three eggs in the top row and two in the bottom row (see Figure 5). She says: “Only one is missing, then it would be six.” She is referring to the empty place under the third egg in the top row, which would create the picture of a dice-six.

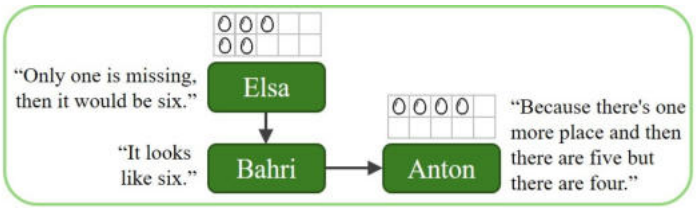

Figure 5: Adoption of the description for strategy “Referring to empty places” – (group 1)

Neither she nor Bahri, who takes up her idea and also says that it looks like a six in the same arrangement, explicitly names the dice pattern. At a later stage, Anton also explains that when four eggs are arranged in a row, it is easy to see that there are four because there is still a free place in the row (see Figure 5).

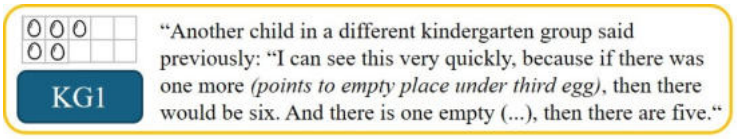

Figure 6: Adoption of the description for strategy “Referring to empty places” – (group 2)

In the other group, KG1 introduced this idea of referring to empty places in the conversation with the children. Throughout the learning situation, no child used this strategy. The fourth strategy observed was the explicit reference to dice patterns (see Figure 7 and 8).

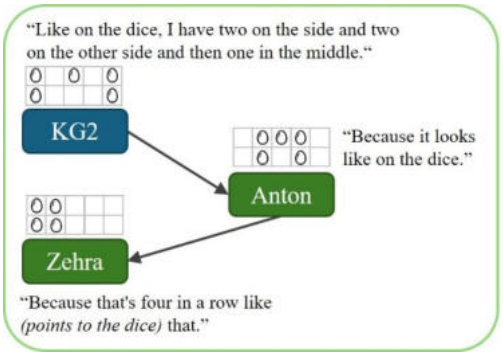

Figure 7: Adoption of the description for strategy “Referring to dice patterns (explicit)” – (group 1)

In the first group of children, KG2 has arranged the quantity of five eggs, which should correspond to the dice pattern of five. She describes: “Like on the dice, I have two on the side and two on the other side and then one in the middle.” Anton also describes a similar arrangement with a reference

to the dice-five (see Figure 7). In the arrangement of the four eggs, Zehra refers to the dice-four as a reason why it is so easy to see that there are four eggs. KG2 adds both to Ester's and to Anna's description of the five eggs (see Figure 8) that it looks like a dice pattern.

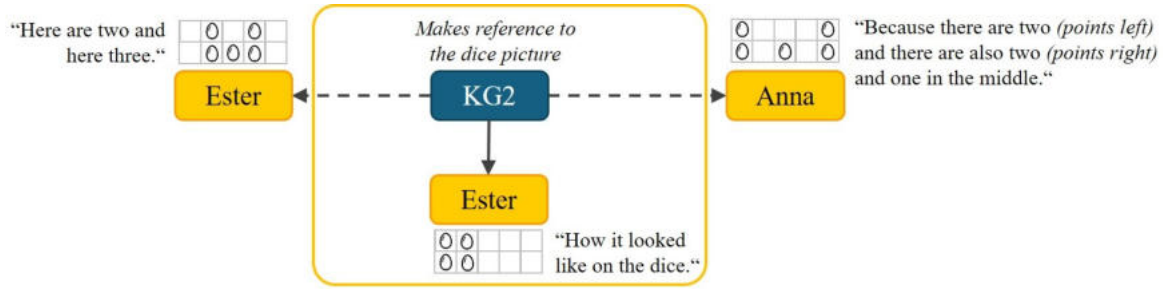

Figure 8: Adoption of the description for strategy "Referring to dice patterns (explicit)" – (group 2)

Only towards the end of the learning situation does Ester describe on her own initiative that the arrangement of the four eggs in the egg carton looks like on the dice. This situation is particularly interesting because a four was rolled with a large foam dice shortly beforehand and the dice pattern of four is still visible to her.

Adopting descriptions in interaction processes

As can already be seen in the previous results, the children refer to each other's descriptions in their descriptions. In the situation shown in Figure 3 there can be seen a complete imitation. Often the children adopt an idea of structuring (e.g. Describing subsets in rows), transfer this to a different arrangement of eggs, slightly (e.g. Figure 1) varying the wording. In Figure 5 Anton's reformulation of the idea to use the empty places differs greatly from Elsa's.

Lami, whose family language is Spanish and who is learning German in kindergarten, shows all forms of participation. In the situation shown in Figure 2, he first describes the arrangement of the eggs in the egg carton by tapping each one individually with his finger and counting them. Anna then explains that she recognises three (bottom row) and one (top row). Lami picks up on this explanation and names the same two subsets, pointing to the corresponding row at the same time (imitation). When laying the number of six eggs (see Figure 4, figures on the right), Lami shows another form of participation: He has initially arranged the eggs differently in his carton and then rearranges them to match Anna's arrangement (three groups of two). He then adopts the idea of Anna's description ("two and two and two") and reformulates it as "Three line [*sic*]. That's one line, two line and three line [*sic*]." He emphasises his explanation by raising each of the two eggs with his hands.

Different observations are made with respect of the role of the two KG in the participation. In group 1, the reference to the dice patterns was made by KG2 and this idea is taken up by Anton and Zehra (see Figure 7). Anton refers to almost the same representation as KG2 (dice-pattern of five), Zehra refers to the dice-pattern of four (see Figure 7) and reformulates the description of KG2. A naming of subsets that do not refer to the rows is initiated by KG1 in group 1 and is adopted by Naomi with the same wording (see Figure 3). In group 2, KG1 introduces a reference to empty places. However, this strategy is not taken up by any of the children (see Figure 6).

Discussion

The children in both kindergarten groups show a variety of strategies for describing quantities. Four main strategies can be identified: (1) Describing subsets in rows (see Figure 1 and 2), (2) Describing subsets not in rows (see Figure 3 and 4), (3) Referring to empty places (see Figure 5 and 6), (4) Referring to dice patterns (explicit) (see Figure 7 and 8). These descriptions allow assumptions to be made about the structuring perception of sets by different children. For example, in the case of Lami, it could be deduced that he does not perceive the set as subsets, but as individual elements (Sprenger & Benz, 2020a, b), since he is counting the eggs (Figure 2). This analysis changes, however, if we consider that Lami has taken over the arrangement of the eggs from Anna and that he initially arranged the four eggs differently in his egg carton. He does not succeed in interpreting a structure in this "foreign" arrangement and possibly uses counting as a familiar strategy (Sprenger & Benz, 2020a) to show that the eggs are the same in both cartons. This gives him the opportunity to "see" her individual perception of a structure and possibly adapt. This can also be seen in the exchange between Ester and Lilly (see Figure 2). After adopting Ester's description of subsets, Lilly ads: "Yes, or you count." As in the scene with Lami, it remains unclear whether we can say anything about Lilly's individual perception at this point, or whether she is merely repeating Ester's description, but counting would actually have been an equivalent or closer strategy for her (Sprenger & Benz, 2020a).

The results show that the children orientate themselves towards each other and adopt both, the idea or the wording of descriptions of structures (Krummheuer & Brandt, 2001). In this setting, children often take one strategy for describing quantities and apply it to different arrangements of eggs, typically adhering closely to the wording of their peers with slight modifications. This demonstrates how children possibly learn through social interaction and how they describe arrangements of quantities. This result must be interpreted with caution, as the children's prior knowledge was not measured in the study. Consequently, no assertion can be made regarding the children's extant structuring skills prior to the study, nor the influence these exerted on the observed interactions. However, the response to impulses from the KG varied between the two groups, indicating that a teacher's introduction of an impulse does not automatically lead to children adopting that description.

Learning from each other by adopting the wording and the strategy of describing quantities seems to play an important role in learning flexible structural descriptions of a set. This adaptation can certainly be viewed critically: It is positive that children learn how structures can be described. On the other hand, it is difficult to evaluate whether the child really perceives the set in this way, or whether individual perception is left "by the wayside". It is therefore important how the conversation is conducted in such a learning situation, what language is used, what questions are asked or what materials are offered. Such variety gives the child the opportunity to show what structures he or she perceives in the set. Careful planning is an important prerequisite, especially as children do not always say what they think (Sprenger & Benz, 2020a). Learning situations in groups seem to be very fruitful here, as children can learn from each other.

Acknowledgment

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Norwegian preservice teachers' views on the use of digital technology in early childhood institutions

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With the inclusion of digital technologies in the Norwegian curriculum as a way that teachers can engage young children, including with mathematics, there is a need for teacher educators to understand how to educate preservice teachers to do this. A survey of 100 Norwegian preservice teachers provided insights into whether seeing digital tools being used on practicum affected their views about using digital technologies to engage children with mathematics. The results show few differences in responses between the groups. If anything, seeing digital technologies being used on practicum seemed to make the preservice teachers more uncertain about their inclusion. This suggests that teacher education courses need to include more input into how digital technologies could be used creatively and in play to engage young children with mathematical ideas.

Keywords: Preservice teachers, early childhood education, digital technologies.

Introduction

In early childhood education and care (ECEC), technology has been employed to enhance teacher-child interactions and is recognised as a valuable tool in teaching and learning (Granone & Reikerås, 2023). In the Norwegian curriculum, known as the Framework Plan (Ministry of Education and Research, 2017), there is an expectation that teachers choose digital technologies to best support children to play, be creative, and engage them with mathematical ideas. Learning is, thus, expected to occur through play, with children's interests being important for play. However, the integration of digital technologies is affected by a number of factors, such as a lack of pedagogical understanding, which is reflected in ECEC teachers' uncertainty about how to utilise digital apps, games, and programs (Pesonen & Valkonen, 2023). The may be part of the reason why this integration has only been partially achieved (Zilka, 2021). Earlier research (Vee, 2023; Vee & Meaney, forthcoming) shows that preservice teachers often only focus on the potential for learning of mathematics when they are considering how to use digital technologies in ECEC and do not seem to consider how it could be done through play. Adding to the complexity of integrating digital technologies, a 'moral panic' to do with screen time presents them as a threat to ECEC practice, which could inhibit their use by teachers (Demeshkant et al., 2023).

Therefore, it is important to identify the factors that support educators to use digital technologies in active and creative ways, based on age-appropriateness, which situates children as active participants in their knowledge development (Undheim et al., 2024). In relation to mathematics education, digital technologies or artefacts have been noted as affecting the interaction process (Granone & Reikerås, 2023), thus, requiring special attention in ECEC courses. Yet, in a literature review by Papavlasopoulou et al. (2024), little research has investigated how ECEC teacher education courses support preservice teachers to integrate digital technologies into their practices, including to support

young children's mathematics education, problem-solving or critical thinking skills. Of the research which has been done, little was undertaken in countries with a play-based curriculum, such Norway (Ministry of Education and Research, 2017).

To contribute to research in this area, there is a need to better understand ECEC preservice teachers' knowledge and attitudes, including to do with digital pedagogical practices (Zilka, 2021). In particular, there is a need to understand whether seeing digital technologies on practicum placements in ECEC affects preservice teachers' beliefs about using digital technologies for engaging children playfully and creatively with mathematical ideas. Although it is commonly considered that practicum experiences can affect preservice teachers' views and attitudes, Österling and Christiansen (2022) found that "research on practicum-as-usual shows minor development of self-efficacy and conceptualizations of mathematics, and sometimes even a negative development of conceptions of reform-oriented mathematics teaching" (p. 9). The studies they reviewed suggested that the differences between preservice teachers were considerably, restricting possibilities that practicum alone might be sufficient to change attitudes or practices.

To determine if seeing digital technologies being used on practicum as part of the teacher education courses affected preservice teachers' views about how they could develop children's mathematical understanding, our research question is: Do preservice teachers who have seen digital tools being used in ECEC have different views on using digital technologies with young children?

Methods

The data were collected through an anonymous online survey from Norwegian ECEC preservice teachers (PTs) in the first two months of 2024. An online survey was chosen because it provided the opportunity to gather data from a range of people in different geographical places, within a limited period of time (Braun et al., 2021). A hundred and twenty-five preservice teachers, from the three campuses of the Western Norway University of Applied Sciences, who participated in a compulsory course on mathematics and language and had undertaken at least one practicum in ECEC completed the survey. Of these 125, 100 provided responses to the questions of interest in this study.

The survey consisted of 22 questions. These were mostly multiple-choice questions and statements requiring Likert scale responses. The PTs could choose between "disagree fully", "disagree partly", "undecided", "agree partly", and "agree fully" and "don't know". This paper focuses on the questions about using digital technologies creatively and in play to engage young children with mathematical ideas. We amalgamated the responses of the first five questions to form two groups of preservice teachers: those who had; and those who had not seen digital technologies on practicum. The questions asked about whether the preservice teachers had seen tablets, computers, mobile phones, digital play, or other digital technologies during practicum. The preservice teachers who gave responses to the survey questions, "I saw the equipment but didn't see it in use", "don't remember", "no, never" were considered to have not seen digital technologies (21 responses). The preservice teachers who stated that they had seen one or more digital technology being used "every day", "once a week", "once a month", or "only once during the practicum" formed the group, who were considered to have seen digital technologies being used (79 responses).

We compared the responses of these two groups of preservice teachers to the questions about the usefulness of apps/robots to provide mathematical learning and play opportunities, about whether digital technologies could increase children's problem-solving competence, and whether it was too early for children to use digital technologies for learning mathematics in ECEC. The last question was about whether digital technologies should be used in ECEC.

The results are presented as graphs that compare the responses between the two groups. The title of each graph includes the survey question. When there were differences in the means in the Likert scale results, we used Fisher's exact test to determine if the differences were significant.

Results

There were very few significant differences between the responses from the two groups for both the general questions about using digital technologies for mathematics and specific questions about the connections between the use of digital technologies creatively and in play to engage young children with mathematical ideas. This suggests that seeing the digital technologies being used on practicum may have little effect on preservice teachers' views.

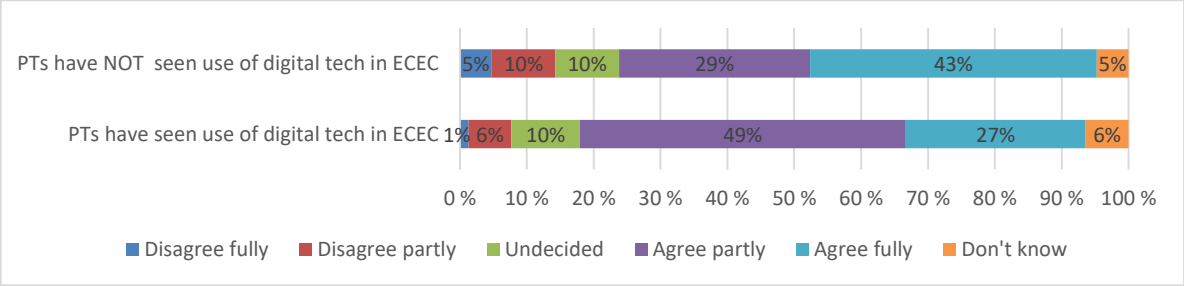

Figure 1: Responses to the statement about the usefulness of apps for providing mathematical learning opportunities

Figure 1 shows the preservice teachers' responses about the use of apps for providing opportunities to engage young children with mathematics. In both groups, the vast majority of preservice teachers were positive about using apps, with 15% of preservice teachers who had not seen digital and seven percent who has seen them being used partly or fully disagreeing about their use for engaging children in mathematics. Fisher's exact test showed no significant difference between the groups for any of the categories or when partly and fully disagree and agree were combined. There were similar percentages of both groups who were "undecided" or who "didn't know" if apps could provide mathematical learning opportunities. This question had the smallest percentages of "undecided" or "didn't know" responses suggesting that more preservice teachers were certain about their responses than they were about the other questions.

In Figure 2, of the group of preservice teachers who had seen digital technologies being used on practicum, almost twice as many were undecided about whether apps could be used to provide learning opportunities through play. Of those who had an opinion, there were no significant differences between the groups about the usefulness of apps for providing opportunities to learn through play. In both groups, there were higher percentages of preservice teachers who did not agree with the statement. This suggests that rather than providing preservice teachers with experiences of

how apps could contribute to children playfully learning through them, practicum experiences may have made them less certain.

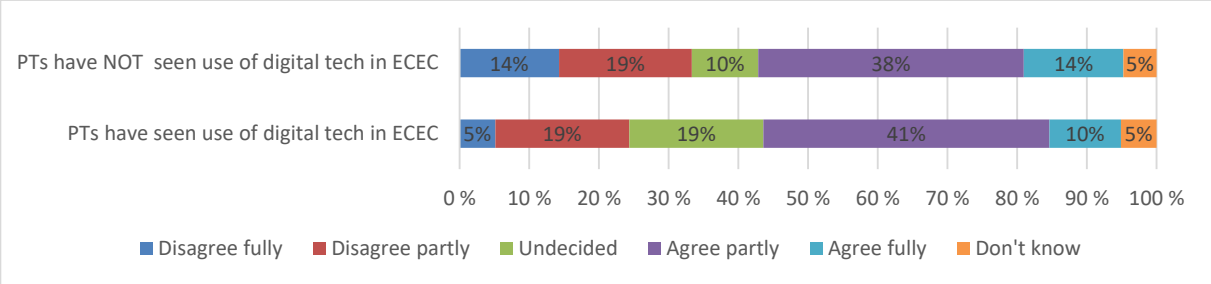

Figure 2: Responses to the statement about the usefulness of apps for providing learning opportunities in play

In regard to the questions about whether robots could provide opportunities for mathematical learning (Figure 3) and learning through play (Figure 4), there were much higher percentages of preservice teachers in both groups who did not know. This suggests that teacher education needs to provide preservice teachers with experiences of how robots could be used in ECEC.

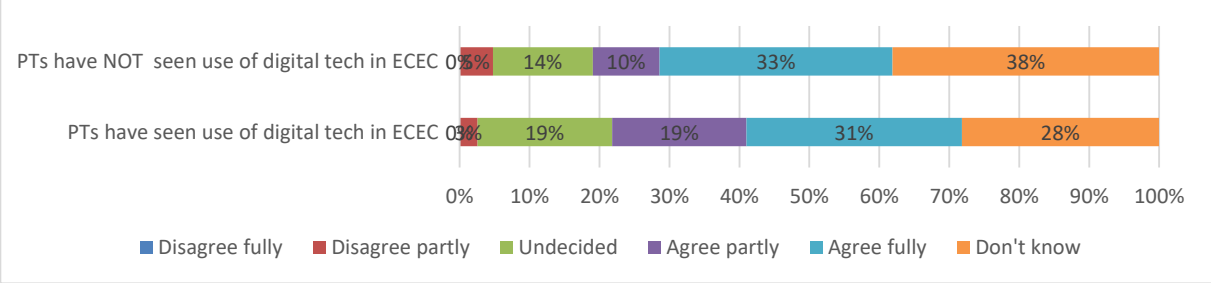

Figure 3: Responses to the statement about the usefulness of robots for providing mathematical learning opportunities

The results in Figure 3 suggest that of those who had an opinion, a much higher proportion of both groups partly or fully agreed with the statement about using robots for supporting children's engagement with mathematics than disagreeing partly or fully. There was no significant difference between the preservice teachers who had seen the use of digital technologies in ECEC and those who had not.

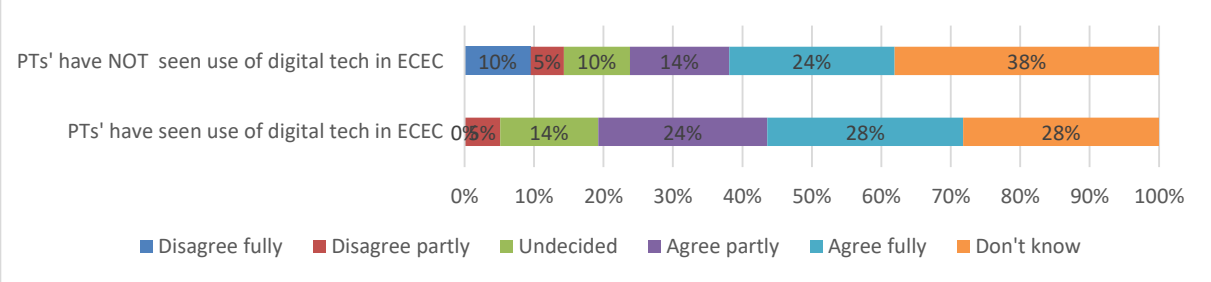

Figure 4: Responses to the statement about the usefulness of robots for providing learning opportunities in play

Both groups of preservice teachers were also generally positive about robots being useful for providing learning opportunities through play (Figure 4), although not to the same degree as they had been with apps. However, those who had not seen digital technologies being used in ECEC were less positive than those who had seen digital tools being used, although the difference was not significant.

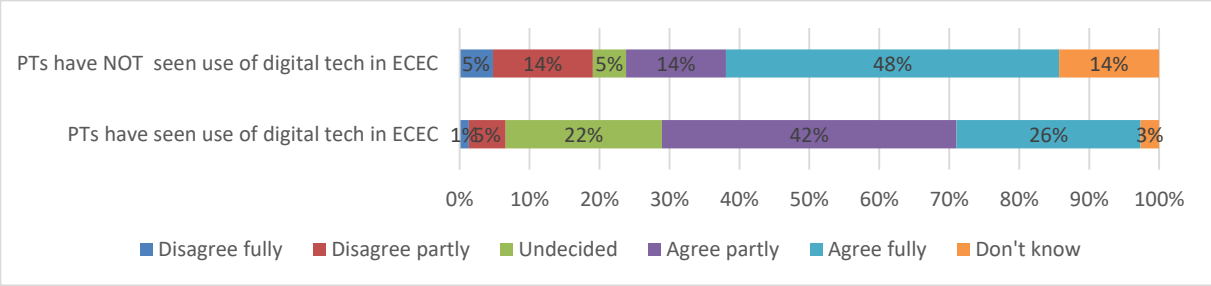

Figure 5: Responses to the statement about the usefulness of digital technology for problem-solving

The preservice teachers in both groups responded positively to the statement that digital technologies could support problem-solving competence (Figure 5). The Fisher's exact test shows a significant difference (P=0.015) in responses to "agree partly" between the preservice teachers who have seen the use of digital technologies and those who have not seen it in use. However, when the respondents for partly agree were combined with fully agreed, there was no significant difference between the groups. This suggests that although positive about the usefulness of digital technologies for developing children's problem-solving competencies, those who had seen digital technologies being used were less certain. This is reinforced by 22% of this group being undecided about this statement, compared to five percent of those who had not seen digital technologies being used.

Figures 6 and 7 show the preservice teachers' responses to general statements about digital technologies being used in ECEC. Although one statement was generally negative about using digital technologies "children were too young to use digital technologies for learning mathematics" and the other more positively worded, "digital tools should be used in ECEC", the responses showed similar results, with no clear trend.

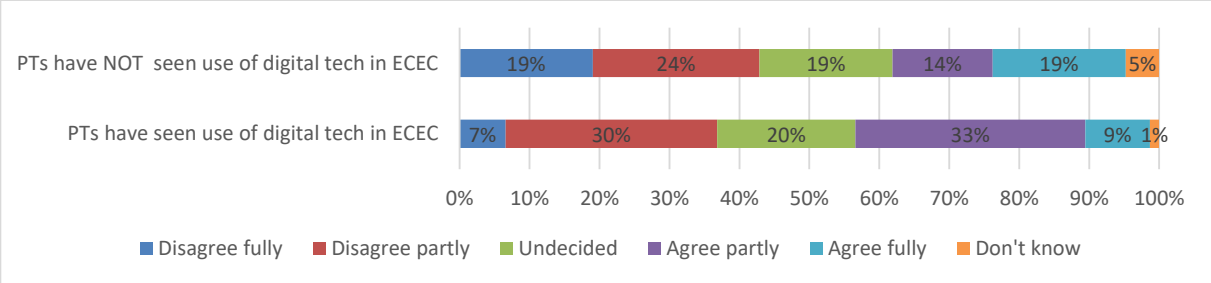

Figure 6: Responses to statement "children were too young to use digital technologies for learning mathematics"

The preservice teachers in both groups did not show a clear indication of agreement or disagreement with the statement about using digital technologies for learning mathematics (Figure 6). Although the preservice teachers in the group who had seen digital technologies were slightly more in agreement with the statement, there were no significant differences in the responses. These responses contrast

with the much more positive responses to the statements about the use of apps and robots for learning mathematics. This suggests that teacher educators may find that in different circumstances, the preservice teachers have different opinions.

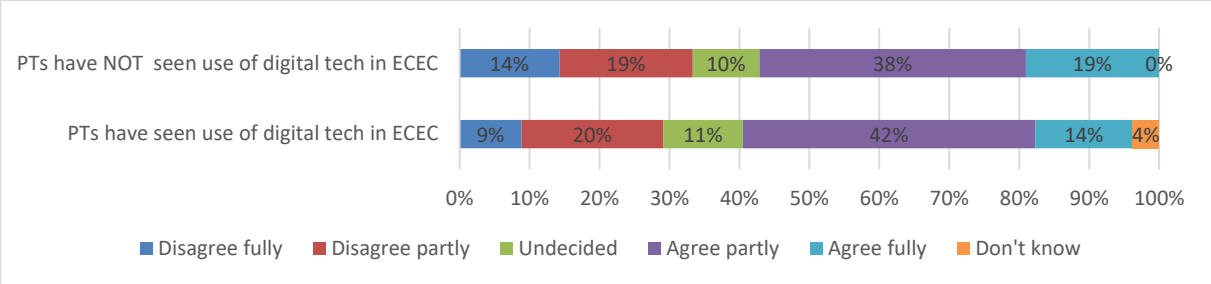

Figure 7: Responses to the statement about the usefulness of digital technologies in ECEC

Although the majority of the preservice teachers in both groups responded positively to the question of whether digital technologies should be used in ECEC (Figure 7), there was still a large number who were negative or undecided about this statement. When the responses "disagree fully" and "disagree partly, "undecided" and "don't know", and "agree partly" and "agree fully" were combined, there were no significant differences between both groups. Given that the Framework Plan (Ministry of Education and Research, 2017) requires the use of digital technology, it may be more difficult for the preservice teachers to disagree with this statement than one about the specific case of young children learning mathematics.

Discussion and conclusion

In this paper, our aim was to determine whether the preservice teachers' views were different if they had completed a practicum in which they saw digital technologies being used in regard to their potential contributions to children's mathematical understanding, problem-solving and play. Similar to the results identified by Österling and Christiansen (2022), it seemed that practicum experience had only a minor effect. In the responses to most of the statements, there were no significant differences in the number of preservice teachers who responded in one way or another. Although there was a difference between those who answered "partly agree" in regard to the statement about digital technologies supporting children's problem solving, this difference disappeared when the preservice teachers who had answered partly agreed were combined with those who fully agreed.

Although we distinguished between those who had seen digital tools being used in ECEC and those who had not, the responses to the survey did not provide sufficient details to know exactly what experiences the preservice teachers had. Future research is needed to know whether the kind of experiences with digital technologies could influence the results. As over 20% of the preservice teachers had not seen any digital technologies being used, there seems to be a need for professional development of current ECEC teachers to ensure that the requirements of the Framework Plan (Ministry of Education and Research, 2017) are met in ways that contribute to children learning through play.

We presume that the preservice teachers who had not seen digital technologies being used in ECEC would have similar views to those held by preservice teachers before they completed practicum. If

this is the case, then it seems that preservice teachers became slightly more uncertain about the use of digital technologies as a result of seeing them being used. Therefore, there also seems to be a need for improved teacher education so that preservice teachers gain more experiences with digital technologies being used creatively and in play to engage young children with mathematical ideas. For example, although an earlier study showed that digital technologies can be an important stimulus for developing problem-solving skills (Granone & Reikerås, 2023), a quarter of the preservice teachers who had seen digital technologies were either undecided or did not know whether they could be used to develop children's problem solving. In the study of Undheim et al. (2024) the preservice teachers identified a lack of personal interest and engagement with the use of digital technologies. However, the results from our study suggest that the preservice teachers held opinions about using digital tools. For example, apps were considered valuable for children's engagement with mathematical ideas. These opinions, aligning as they do with the requirements in the Framework Plan (Ministry of Education and Research, 2017), can be a basis for improved teacher education.

Preservice teachers' responses to statements about digital technologies being used in ECEC highlighted specific areas about which they may need more support. In particular, preservice teachers were less certain about digital technologies contributing to children learning through play and seemed to have little knowledge about how robots could be used in ECEC. Therefore, teacher education courses need to provide more input about the use of digital technologies for preservice teachers that can support learning through play, problem solving and utilising young children's creativity to engage with mathematical ideas. However, as noted by Papavlasopoulou et al. (2024), little research has been done on how to develop and implement courses on using digital technologies in ECEC. Given the uncertainty in preservice teachers' responses, further research is needed.

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Opportunities for learning in parent-child discussions about written numbers in everyday life

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This paper presents findings from the 'Written Numbers in Everyday Life' project, that has been implemented with families of 3–5-year-old children in the UK and Jamaica. The project aims at empowering families to utilise opportunities for talk about written numbers on everyday objects in their environment. We conducted a qualitative thematic analysis of 72 posts (39 from the UK, 33 from Jamaica) that participating families uploaded to an online platform. Three themes emerged in discussions that parents reported that they had with their child: comments with a conceptual focus, comments with a conceptual and mathematical/numerical focus, comments with a conceptual and social purpose focus. The findings provide insights into the wide-ranging opportunities for learning that arise when parents are prompted and supported to recognise and use numerals in the everyday environment, as a learning resource.

Keywords: Written numbers, preschool, home environment, family engagement with learning.

Introduction

Family engagement is an early influencing factor on children's knowledge and dispositions towards mathematics, which, in turn, are linked with early and persistent attainment gaps (James-Brabham et al., 2023). The frequency of intentional parental engagement (Napoli & Purpura, 2018) and parental talk about number (Levine et al., 2010), have been highlighted as important for early mathematics development. Furthermore, positive dispositions, such as curiosity and interest in mathematics ideas are key factors in children's mathematics learning (Ginsburg et al., 2008), and these too are influenced by home experiences (Gervasoni & Perry, 2017). However, research has shown that some parents may be reluctant to engage with mathematics (Gervasoni & Perry, 2017) and parental engagement may be best achieved by incorporating early maths "within parents' overall, existing roles and responsibilities" (Mazzocco & Claessens, 2020, p. 3).

This paper presents findings from the 'Written Numbers in Everyday Life' project, that has been implemented with families of 3–5-year-old children in the UK and Jamaica. The project aims at empowering families to recognise and utilise everyday opportunities for talk about written numbers and their multiple meanings. The study that is reported here is part of the wider project. It is underpinned by the idea that the everyday knowledge and skills that are developed by diverse families' everyday experiences need to be recognised as 'funds of knowledge' (Moll et al., 1992), that can be built upon and used as resources for supporting learning. Thus, we explore the opportunities for learning that can emerge within family-based activities that prompt and support intentional adult-child talk about numerals in everyday environments, within two different cultural contexts. Eason et al. (2022) highlight the need to understand how diverse family contexts and activities can enhance early engagement with mathematics. This study addresses this critical gap in

early mathematics learning by seeking to answer the following research question: What themes emerge in adult-child conversations about the meanings of written numbers in everyday life, as reported by parents in the UK and Jamaica?

Fostering intentional parental talk about numerals in the environment

Written numbers, that is, Hindu-Arabic numerals in digit form (e.g., 8, 63) are cultural artefacts that represent quantitative and non-quantitative ideas (Brizuela, 2004). Research exploring young children's experiences with number in the family context has focused mainly on the impact of mathematically focused family activities on competencies such as counting and computational skills (e.g., Levine, et al., 2010). However, children see "very few examples of the computational use of written numbers before they come to school" as "written numbers, in our society, represent a large variety of numeral and/or quantitative concepts, as well as being exploited for other purposes." (Sinclair & Sinclair, 1984, p. 174). In everyday life, preschool children encounter numerals on everyday objects, such as, food/games packaging, books, the TV, and various kinds of signs that may have quantitative meanings (e,g., representing cardinality) and also numerals that have "social uses" and "nonquantitative purposes" (Tolchinsky, 2003, p. 110), such as, numbers on buses, phone numbers and barcode numbers, which are underpinned by social knowledge and communicate meanings that help us organise our lives. The role of adults (carers, teachers) in supporting children to make connections between their knowledge about numerals at preschool/school and the social knowledge about the use of numerals outside school, is important, for minimizing the risk of school mathematics becoming an alienating experience (Munn & Kleinberg, 2003).

As part of a previous study that adopted a phenomenographic approach (Voutsina & Stott, 2023), we found that, to make sense of the messages/meanings that numerals communicate in their everyday environment, preschool children combine knowledge of the following: Recognition of *a numeral* as a number (and not as a letter, for example), recognition of *the object* on which a numeral appears, recognition of *the context/location/situation* within which they encounter a numeral, and recognition *of other symbols* that may accompany a numeral, denoting its meaning (e.g., £, Kg, ml). Based on this evidence, we designed the "Written Numbers in Everyday Life' programme that aims to foster children's curiosity and awareness about the multiple meanings of written numbers outside the classroom, and support family talk about written numbers in everyday life. The programme involves a sequence of family-based Number Spotting challenges that prompt families to look for, at home or outside, either a specific numeral on different objects and in different contexts/locations that represents different concepts and communicates different meanings (e.g., the written number 4 on different objects representing, age, size, price etc.), or look for different written numbers that represent the same, specific concept (e.g., look for different written numbers that all show size on items of clothing/shoes, or different numbers that show age, or numbers that show quantity etc). In designing the challenges that aim at making children aware of the multiple meanings that numerals have in life outside the classroom, we drew from elements of Variation Theory (Björklund, 2016). We used variation and invariance to contrast examples of numerals, objects, contexts, other symbols and the multiple concepts/meanings that numerals represent, in order to support discernment of each of the aforementioned elements separately, as well as simultaneous discernment of more than one of the above elements and their relationship. All challenges are included in a set of printed family

activity postcards. Each postcard includes the same three prompts that adults are encouraged to use in conversations with their child about the spotted number: *What* does the number tell us? *Who* needs to know this? *Why*?

Implementation and method of data collection

Teachers in participating preschools send two activity postcards to each family every week, for 6-7 weeks. For each classroom, the researchers create a unique link to a Padlet board (Padlet is a software that provides access to shared digital boards where individuals or groups can upload and share visual/textual material). The instructions given to parents are to upload a photograph of the numeral that their child spotted as part of each challenge, showing the object and the location where it was spotted, together with a short comment on what they discussed about the number with their child. The Padlet enables us to collect data that provide insights on how different families engage with the programme, including the contexts, locations, activities and interactions that parents report.

We have introduced the programme to families of 3–5-year-old children in the UK, as well as in Jamaica, as we were interested in examining its transferability to a different cultural context. Jamaica was selected because English is the official language of teaching and the curriculum is still broadly aligned with the English curriculum. For this paper, we focus on the Padlet data that we collected from: three preschools (3–4-year-old children) and one Reception class (4–5-year-old children) in the UK, and two preschools (3–5-year-old children) in Jamaica, during the first year of piloting the use of the Padlet. Ethics permission was granted by the Ethics Committee of the University of Southampton (ERGO 80253) and the University of The West Indies, Mona campus (CREC-MN.099, 2023/2024).

Data analysis

Family engagement with the Padlet was voluntary and varied. Some families who participated in the programme did not post on the Padlet at all or did not post regularly. Parents' comments in posts varied in detail, focus and length. We received a total of 127 Padlet posts from UK families and 105 posts from parents in Jamaica over the course of seven weeks. To address the research question, we focused our analysis on a selected sample of 72 posts (39 from the UK and 33 from Jamaica). These were posts with comments that provided information on the discussion that parents reported that they had with their child about the meaning of the spotted number. Posts with comments related to numeral recognition/identification (e.g., "AC noticed there was a 2 and a 5 on our door!") or other brief comments with no reference to the parent-child discussion, were not included. All posts were anonymous. Therefore, we do not have data on how many different families provided the analysed comments. To analyse the comments, we applied inductive thematic analysis (Braun & Clarke, 2006). Through a "compare-and-contrast" approach (Boyatzis, 1998), we grouped comments under categories based on the aspects of the meaning of numerals that parents focused on in their reported discussions with children.

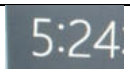
Findings

Three themes emerged: comments with a conceptual focus, comments with a conceptual and mathematical/numerical focus, and comments with a conceptual and social use/purpose focus.

Comments with a conceptual focus

With comments grouped under the conceptual focus category (examples shown in Table 1), parents reported having a discussion with their child about the concept that the spotted numeral represented, such as, the cost of items in the supermarket (e.g., post 1, Table 1), the age or size communicated by numerals on clothing labels (post 2, Table 1), without any further detail about the discussion.

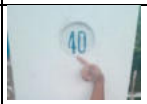
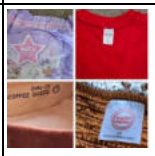
Table 1. Comments focusing on the concept that the numeral represents

1		Whilst out shopping, we discussed the numbers on the shelves and the symbols next to them that show it means the price of the item. Theo was able to identify single numbers, such as 7, 6 etc. (UK)
2		CM saw that there was a number on her sandal. She did not know what the number meant. We had a discussion about shoe sizes. (JM)
3		We found the number 10 at the park and we spoke about how it is used as a date every month! (UK)
4		He visited my work and whilst on the laptop he says mommy I see the number 5. I said, yes, it's the time. (JM)

Comments with a conceptual and mathematical/numerical focus

Some comments indicated parent-child discussions that extended to mathematical/numerical-related notions and skills, such as counting, and/or references to specific attributes of the spotted numerals (e.g., naming of 2- or 3-digit numbers, discussing digital numbers). For example, parents reported prompting their child to count items in a box with the numeral 5 on, or practice verbal counting up to the spotted number (post 1, Table 2), or supporting their child's reading of 2-digit numbers (post 2, Table 2). The third post in Table 2 includes reference to multiplicative ideas ("two of each"), while post 4 reports the parent's use of the opportunity arising from spotting numeral 3 on a digital clock, to explain about analogue and digital representations of time and notions such am/pm, day and night. After a child's query whether 10.2 that they spotted on a thermostat is a 10 or not (post 5, Table 2), the parent uses this as an opportunity to discuss decimal numbers and the symbol for degrees Celsius, making a link with how the temperature that is represented by the numeral is experienced by them at home. Posts 6 and 7 (Table 2) are indicative of discussions extending to notions such as shorter/longer width in the context of spotting numeral 3 on a ruler, and references to comparing higher/lower values and to the notion of weight, based on numerals spotted on children's clothes.

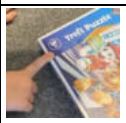
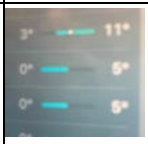
Table 2. Comments that include conceptual and mathematical/numerical focus

1		We found a bath toy in the pharmacy. I asked him to count the items in the package and he did so correctly. I then asked him how many toys are in the package and he said 5. I then asked him to show me the number 5 on the package and he did. We talked about the fact that the number says how many things are in the package. (JM)
2		Caria spotted this number on the column of my coworkers house all on her own and showed it to me. She said "mommy, it's a 4 and 0" I told her the number is 40 and we had a discussion about addresses/street numbers. (JM)
3		We talked about the fact the number 2 appearing twice is because it is saying that two of each item is in the package. (JM)
4		We talked about the time during the day and at night (am and pm), we tell you why sometimes the clock shows the number 15 instead of 3. (UK)
5		“Is that number 10 or not?” We found lots of 10s. This is the only one Jacob questioned if he had found one or not. We had a conversation about decimal place and the symbol of °c relating to temperature. He thought 10.2°C felt cold, which it certainly did. (UK)
6		She looked around the house and selected numbers found on clothing and shoes. She took a few pictures and we talked about the difference in numbers (higher value vs lower value). (UK)
7		“Put it up here and it’s longer, put it down here and it’s shorter.” Margot learning to measure the wall after spotting the number 10. (UK) Akeem and I spoke about the different clothes size, that each represents measurement and that the smaller number is for someone who weighs less. (JM)

Comments with a conceptual and social use/purpose focus

The examples of posts included in Table 3 are indicative of opportunities recognised and used by parents to further extend their discussions of the meaning of numbers to the social use of the spotted numerals, that is, the socially shared rules and conventions that numerals communicate in everyday situations, which inform and shape everyday behaviours, such as, driving behaviour (post 1), identifying locations (post 2, Table 3), protecting information (posts 3 and 5) and deciding what to wear (post 6, Table 3).

Table 3. Comments that include conceptual and social use/purpose focus

1		We talked about how here the speed limit of the road is 50mph but on the bend part of the road there is a lower maximum speed as you need to drive slower on the bend to safely keep control of the car. (UK)
2		We visited a pharmacy. She noticed the numbers hanging above each aisle and started counting. She was able to identify all the numbers. When she finished counting, I led her back to aisle #3 where I explained to her that in a store, things that are similar are placed together so that we can find them easily. "Like the toy aisle; all the toys are in aisle #2 over there". (JM)
3		We found the number "3" on one of my daughter's toys. We realized that numbers are like keys. They help unlock secrets, like combinations to locks. We decided to remember the number “3” and maybe use it as a passcode to open a drawer (3.3.3). (UK)
4		Number 5+ on this picture tells us that these puzzles are for children from 5 year and over otherwise it could be dangerous. Children under that age should not be playing with it. (UK)
5		We went to the ATM to get some cash. I showed him how to enter the PIN, I told him the numbers. He correctly entered the PIN! I told him these numbers stop the bandits from stealing our money. (JM)
6		Harper helped to check the weather on Mam’s iPad. He found the temperature was 5 Degrees C on two days. We talked about checking the weather before we do outside activities so we know what to wear. (UK)

The examples of posts included in Table 3 are indicative of opportunities recognised and used by parents to further extend their discussions of the meaning of numbers to the socially shared rules and conventions that numerals communicate in everyday situations, which inform and shape everyday behaviours, such as, driving behaviour (post 1, Table 3), identifying locations (post 2), protecting information (posts 3 and 5) and deciding what to wear (post 6).

Comments that included two or all three of the identified foci of discussion were categorised under more than one theme. Numerical analysis of the occurrences of the three themes in parental comments (Table 4) shows that overall, a focus on the social use of numerals was the most frequently included aspect of meaning within the analysed sample of posts, as it was identified in 44% of all analysed comments. A focus on the represented concept only, was reported in 28% of comments from the UK and 39% of comments from Jamaica. Discussions extending to a numerical/mathematical focus when talking about the meaning of a numeral were reported slightly more frequently in the UK sample (33%) than the sample of comments from Jamaica (27%).

Table 4: Number of parent comments under each theme of meaning related to spotted numerals

Theme	UK (39 posts)	Jamaica (33 posts)	Total (72 posts)
Conceptual focus only	11 (28%)	13 (39%)	24 (33.3%)
Numerical/Mathematical focus included	13 (33%)	9 (27%)	22 (30.5%)
Social use/purpose focus included	19 (48%)	13 (39%)	32 (44.4%)

Conclusion

Parents' self-reporting posts may have been affected by factors, such as, time commitment and what parents recall or choose to report. Acknowledging the limitation that parental posts do not constitute a direct record of discussions that may or may not have occurred, our aim is not to draw generalised conclusions about parental discussions. Rather, the contribution of this study is in unearthing and delineating the qualitative nature of opportunities for learning that can arise, and parents recognised and reported, within the two cultural contexts of our project. The study, thus, addresses the need for understanding the potential that diverse everyday contexts have for family engagement with number-related ideas (Eason et al., 2022). Our analysis revealed examples of reported parent-child discussions that addressed conceptual, mathematical as well as social aspects of written number meanings, across both cultural contexts. This provides important insights into the multifaceted and wide-ranging opportunities for learning that can emerge in the home environment when parents are prompted and supported to use written numerals that appear on everyday objects, as a learning resource, for drawing children's attention on numbers as attributes of their environment and triggering early discussions about different aspects of their meaning. A focus on the socio-cultural meanings and uses of written numerals in everyday situations provides a fruitful and powerful channel for empowering an inclusive approach to family engagement with various aspects of number. This is because knowledge of the messages that environmental numerals communicate and their social purposes in organising everyday life is knowledge that is socially developed (Moll et al., 1992) and most families have, within their own cultural contexts and environments. We thus concur with Eason et al., (2022) who note that embedding activities within the families' everyday life and diverse contexts can provide the potential for context-based and socio-culturally relevant intervention for all children and families.

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Guardians' views on digital tools in preschool

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This paper presents a discourse analysis of responses to a survey conducted with guardians of children from three Swedish preschools. Two questions are elaborated on: (1) What Discourses occur in guardians' communication about digital tools in preschool? and (2) What are the implications of these Discourses for mathematics education in preschool? Four construed Discourses show diverse views on digital tools, with most guardians taking a positive view as long as these tools are used in such a way that children become producers of content, not consumers. This is in line with research highlighting the importance of focusing on how digital tools are used rather than on for how long. The study shows the importance of preschools clearly communicating when, how, and why digital tools are used in preschool.

Keywords: Digital tools, early childhood education, guardians, questionnaire, discourse analysis.

Introduction

In Sweden, preschool is offered to all children aged 1-5. The national preschool curriculum states that every child should use digital tools in a way that stimulates development and learning. To accomplish this, preschool should give children the opportunity to develop adequate digital skills and an understanding of the digitalization they encounter in everyday life. Further, children should be offered the opportunity to use mathematics to investigate and describe their environment and solve everyday problems (National Agency for Education, 2018).

Digital tools can support mathematics education in preschool. For example, digital tools that require creative solutions and/or reasoning, and that offer children guided discoveries and experiments, promote interaction and communication about shared mathematical ideas (Palmér, 2015; Trigueros et al., 2014). Further, digital tools can combine visual presentations, animations, and sound, thus providing a multimodal environment in which children can see, hear, and act simultaneously. They can also offer multilingual options, which can promote mathematics learning in multicultural preschools, where digital representations have proven to be easier for young children to manipulate and understand than comparable physical materials (Trigueros et al., 2014). Nikolopoulou (2020) has shown that digital mathematics teaching activities may include several mathematical topics such as serialization and sorting of objects, time and space concepts, numbers, classification, and matching. In a teaching intervention by Palmér (2017), programming was used to facilitate preschoolers' learning of mathematics. The children developed their ability to mentally compare and connect (real) movements with maps and symbols. Further, the children demonstrated an ability to mentally envision, hold in mind, and conceptualize actions and relationships between paper maps, gridded maps, and symbols. Thus, several studies indicate the broad potential in using digital tools when teaching mathematics in preschool.

In autumn 2023 the Swedish government changed direction radically when they decided that the preschool should essentially be screen free for the children. Since then, notions about "screen free" and "screen time" have been frequently debated, including among researchers from different disciplines (see for example Government Offices of Sweden, 2023). Some adhere to the World Health Organization's (WHO) recommendation that children under the age of 5 should have as little screen time as possible (The Public Health Agency of Sweden, 2024). Others challenge these notions by referring to educational research, emphasizing that digital tools are used differently at home compared to in educational settings, and arguing that digital tool use at school should not be equated with screen time. The debate about screen time focuses on *how long* children spend sitting passively in front of a screen (see for example Nilsen & Kjällander, 2023). However, the argument for limiting passive screen time is often misconstrued as a need to limit the use of digital tools in general, regardless of *how* children are using them. These debates, alongside the ongoing revision of the curriculum, have led to uncertainty in Swedish preschools. Are they to use digital tools, and if so, how and when? To meet this uncertainty, a combined development and research study was initiated in collaboration between researchers and three preschools aiming to develop a scientifically based strategy for a screen-wise preschool. Quite early in the study, the preschool teachers expressed uncertainty about how to respond to guardians, i.e. the person legally responsible for a child's care and well-being, who sometimes questioned and probed the preschools' use of screens based on the debate described above. It was decided to conduct a survey to find out more about the guardians' views on the use of digital tools in preschool, the extent to which they knew what digital tools are used for in preschool, and the extent to which, and how, their children use digital tools at home. This paper reports these results with a special focus on whether and how the guardians' views may influence the use of digital tools in preschool mathematics education. The following two questions will be focused on: (1) What Discourses occur in guardians' communication about digital tools in preschool? and (2) What are the implications of these Discourses for mathematics education in preschool?

Previous research

Today's preschoolers have never lived in a wholly analogue world, and they are using various digital tools at increasingly younger ages. As digital tools have become part of our everyday lives, the concept of screen time has become important for many guardians (Byrd-Bredbenner et al., 2017). Most guardians consider it important to regulate their children's screen time (Livingstone et al., 2015), in line with WHO's recommendations. However, as digital tools are so central to modern family life, it can be difficult to live up to WHO's recommended limitation, and guardians of young children may feel stressed and stigmatized as a result (Sandberg et al., 2024). Furthermore, there is a risk that the recommendations might lead to a focus solely on screen time instead of on how and why digital tools are used. Guidelines could support guardians in exploring digital tools with their children (Sandberg et al., 2024) as the use of these tools at home differs from their use in preschool (Nilsen & Kjällander, 2023). At home, digital tools are often oriented towards consumption, whereas in preschool, guided by trained teachers, children can explore the tools and use them to become producers of content. According to Craft (2012), the shift from being a consumer of digital content to becoming a producer can give children a sense of ownership and control over digital tools, helping them become more critical and reflective about how they use such tools in the future.

The debate about screen time is often dominated by a black-and-white view in which children's use of digital tools is seen as either a threat or an opportunity (Stephen & Edwards, 2018). However, Nilsen and Kjällander (2023) state that it is more important to focus on what digital tools in preschool can contribute instead of arguing that they are superior to analogue tools. Further, these authors emphasize the importance of quality in digital interaction and its educational purpose rather than the amount of screen time. This is in line with international research highlighting that the question should be about how screens are used rather than for how long. It is not the digital tool itself that determines whether its use will have a positive or negative effect; rather, it is how it is used that is important. The use of digital tools in mathematics teaching can have a positive, negative, or neutral effect, depending on the specific digital tool or software, how it is used, and for what purposes (Baki et al., 2011). Guardians in Sweden have expressed a desire to participate in their children's preschool learning (Öhman et al., 2024). To promote their participation, preschool teachers need to communicate with the guardians about what their role might be in this context (Öhman et al., 2024).

Theoretical framework

In this study, discourse analysis in line with Gee (2014a; 2014b) forms both the theoretical framework and the analytical tool (see Method). According to Gee (2014b), language creates meaning in social practices while also deriving its meaning from social practices. Language reflects and creates the existing reality, where discourses can be seen as part of the ongoing processes that continuously create and reshape meaning in and of social practices (Gee, 2014b). Small <u>d</u>iscourses and big <u>D</u>iscourses are distinguished from each other: in <u>d</u>iscourses, the relationship between words and sentences is described, while <u>D</u>iscourses can be seen as ways of using language, influenced by social positions and power relations (Gee, 2014b). In the present analysis, the construed <u>D</u>iscourses are seen as part of the social practices that create and reshape meanings of using digital tools in preschool.

Method

The empirical material in this paper is from a combined development and research study conducted in collaboration between researchers and three preschools, initiated by the principal for these three preschools. A survey was sent by e-mail to all guardians (n=288) in June 2024 by the principal. After two reminders were sent, 104 guardians had answered the survey, resulting in a dropout rate of 64%. The survey included open and closed questions and allows for a quantitative and qualitative analysis.

The analysis for this paper first focused on guardians' answers to the scaled questions: *What is your opinion of digital tools in preschool?* and *Are digital tools an important part of preschool education?* After that, the free text answers to the questions: *What do you think when you read/hear "digital tools in preschool"?* and *Are digital tools an important part of preschool education? were analysed.* Data consisting of written communication and the specifics in our two research questions led to eight of Gee's 28 tools for discourse analysis (Gee, 2014a). In line with Gee, the discourse analysis initially considered the larger context (#26, #23), and after that, how the guardians used communication to build meaning in a specific context (#17, #14). Finally, the analysis focused on the guardians' explicit expressions (#9, #7) and how the language was used related to this context (#2, #1). The following questions were posed in relation to each tool:

- Deixis Tool (#1): What is talked about as given, based on the context?
- Fill-In Tool (#2): What is not said? What knowledge and what assumptions are required for the communication to be clear?
- Doing and Not Just Saying Tool (#7): What are the guardians trying to do (keeping in mind that they may be trying to do more than one thing)?
- Why This Way and Not That Way Tool (#9): Whose voice is heard? What are the guardians' intentions when they talk the way they do?
- Significance Building Tool (#14): What words are used to build or keep the significance of some things and not others in the guardians' communication about digital tools?
- Relationship Building Tool (#17): How are words being used to build, sustain, and change relationships of various sorts among the guardians, other people, and/or institutions?
- Situated Meaning Tool (#23): What situated meanings, related to the context, do words and phrases in the guardians' communication have?
- Figured World Tool (#26): What typical stories or figured worlds are assumed in the guardians' communication?

Above, Gee's tools are presented as a bulleted list. However, the process of discourse analysis is iterative, meaning that the analysis moved back and forth from large to small perspectives. Through this discourse analysis, four Discourses were construed.

Results

According to the survey responses, 21.2% of guardians find that digital tools are used too much in preschool. At the same time, only 13.5% stated that they were aware of which digital tools are used in preschool, which indicates that some of the guardians stating that digital tools are used too much are not sure about which digital tools are being used. On the question of whether digital tools are an important part of preschool, 28.8% of guardians answered "yes", 49.0% answered "partially", and only 15.4% answered "no".

In the following, the four Discourses are presented, each starting with a translated quotation from the data. The Discourses should be understood in light of the results above, showing that there are more positive or partially positive voices than negative ones regarding the use of digital tools in preschool.

Digital tools need to be introduced early

Our children are growing up in a digitalized world and need to be equipped for the future.

In their communication about digital tools in preschool, the guardians describe how these tools should be introduced early so that "children can keep up with the technological development in society" and understand the world around them. Introducing digital tools in preschool can support children in developing these skills.

To strengthen the argument for using digital tools in preschool, the guardians use expressions such as "good and obvious", "a fun way to learn", "a large part of society", "it's not the 1800s", "the present is digital", and "part of the future". The guardians describe how the use of digital tools has changed and increased since they were children – a trend that will continue into the future. If children

do not receive support in using digital tools, they risk falling behind. Therefore, digital tools should be included in all parts of society.

Digital tools imply screen time, which is harmful to children

I think screen time should be limited as much as possible for children.

In their communication about digital tools in preschool, the guardians use the term "screen time" instead of "digital tools". Further, they write about how children under a certain age should avoid screen time since this may have a negative effect on their eyes and brains. When referring to screen time, the guardians use expressions such as "negative", "unnecessary", "overdose of screens", and "used incorrectly". Together, the guardians' expressions contribute to an image of screen time as something considered harmful to young children.

Implicit in the guardians' communication is a view that all people in society get too much screen time. The guardians express a desire for preschool to refrain from using digital tools and take responsibility for the fact that the children get an "overdose of screens". Mentioning an overdose of screens indicates a view that children are using digital tools in places other than preschool. In their communication about digital tools as screen time, the guardians support their argument by referring to studies showing how screen time is harmful to young children. These first two Discourses, *Digital tools need to be introduced early* and *Digital tools imply screen time, which is harmful to children*, contrast with each other and show guardians' different views on *when* to introduce digital tools.

Digital tools need to be used in the right way

It completely depends on how it is used. It can definitely be positive, but of course it has to be in line with the content.

When using digital tools, children should be producers.

In their communication about digital tools in preschool, the guardians describe how such tools can be an important part of the preschool if they are used in a balanced way and combined with other preschool activities. Hence, the use of digital tools does not appear to be an "either/or" issue since the guardians recognize that they can be used in combination with other activities. The guardians express the use of digital tools in preschool as something positive if they are used "in the right way", referring to *how* digital tools should be used by both children and teachers.

When the guardians express that digital tools should be used in the right way by the children, they mean that the children should be allowed to use the tools creatively and as producers of content. When they express that digital tools should be used in the right way by the teachers, they mean that the teachers should use the tools both to promote children's learning and to vary the teaching. Implicit in the guardians' communication about digital tools is that there is a right way and a wrong way to use them, and only if they are used in the right way can they be viewed positively. The guardians express their doubts about the use of digital tools as a form of entertainment or as a babysitter.

Digital tools take the focus away from what is important

I would have preferred a clearer focus on motor skills, creativity, physical activity, handicrafts.

It is a big question; what type of content is used, and why it is done digitally instead of "for real"?

The guardians describe a risk of overusing digital tools as they may replace other activities and ways to teach. They describe how digital tools are sometimes prioritized over "real" activities. They reveal a view of digital tools as something different and disconnected from play, creativity, physical activity, and social interaction. Implicit in their communication is a view that all preschool activities should preferably be carried out without digital tools.

Further, the guardians express that digital tools may affect how the teachers carry out their teaching. For example, instead of singing with the children, a teacher may choose to show a video of someone singing. Overall, the guardians' communication can be understood as expressing a fear that both the content of the preschool education and how the teaching is carried out will be too much imbued with digital tools. Using digital tools in preschool is not seen as "the right way" to teach and it should therefore not be prioritized. Instead, proper teaching is seen as involving free play and other activities that do not involve digital tools. These last two Discourses, *Digital tools need to be used in the right way* and *Digital tools take the focus away from what is important*, reflect guardians' contrasting views on *how* to use digital tools in preschool.

Discussion

The four Discourses construed in this analysis show different views on digital tools in preschool. This section focuses on implications of these Discourses for mathematics education in preschool.

The Discourses *Digital tools need to be introduced early* and *Digital tools imply screen time, which is harmful to children* reflect contrasting ideas related to *when* to introduce digital tools. If these Discourses affect whether digital tools will be introduced in preschool or not, implications for mathematics teaching are that children might not be given the opportunity to benefit from their use. Children will then not benefit from the support that digital tools can give regarding creative solutions, reasoning, interaction, and communication of mathematical ideas (Palmér, 2015; Trigueros et al., 2014). Further, multicultural schools might not be able to offer the multimodal environment that digital tools can provide and that can be effective in promoting mathematics learning.

The Discourses *Digital tools need to be used in the right way* and *Digital tools take the focus away from what is important* reflect contrasting ideas related to *how* digital tools are used in preschool. In the first Discourse, the meaning of digital tools can be related to previous research about screen time and a consumption-based use of screens at home (e.g. Nilsen & Kjällander, 2023), as digital tools are seen as isolated from other activities in preschool. *Digital tools need to be used in the right way,* the meaning of digital tools can be related to previous research about children exploring such tools in several ways and becoming producers of content (e.g. Nilsen & Kjällander, 2023). In this Discourse, digital tools are seen as something that can be used in a balanced way and combined with other preschool activities. These different views may have implications for how digital tools are used in mathematics education in preschool. For children to benefit from the use of digital tools as a way to promote learning in mathematics, they ought to be producers of content rather than consumers. Further, previous research shows how producing content can contribute to children becoming more critical and reflective about how they use digital tools in the future (Craft, 2012).

In this study, we do not claim that the Discourses construed are the only ones occurring in society or among guardians (especially because of the relatively high dropout rate mentioned earlier), but they occur to varying degrees in the preschools investigated here and are therefore something that the teachers at these preschools need to deal with. Also, Discourses never occur in a vacuum, which is why further studies at other preschools and in other contexts where mathematics in preschool is discussed (for example, the government) could be an interesting continuation of this study. Regarding the dropout rate, another methodological reflection concerns which guardians chose to answer the survey. A large non-response rate in digital surveys is common, but those who answer often do so because the topic is important to them (Bryman, 2016). Thus, we assume that guardians with strong positive or negative opinions about digital tools in preschool have answered the survey to a greater extent than guardians with a neutral opinion. If this is the case, the guardians with a strong negative view on digital tools make up an even smaller part of the total than the results show.

To conclude, 77.8% of the guardians answered "yes" or "partially" in response to the question of whether digital tools are an important part of preschool, and only 15.4% answered "no". The largest group of guardians (49%) answered "partially". This can be related to the Discourse *Digital tools need to be used in the right way*, addressing guardians' opinion on *how* digital tools are used. This indicates that the majority of guardians view digital tools as an important part of preschool, as long as children are producers of content, not consumers. This is in line with research highlighting the importance of focusing on how digital tools are used, rather than for how long (Baki et al., 2011; Nilsen & Kjällander, 2023). In line with Öhman et al. (2024), our study shows the importance of preschools clearly communicating how digital tools are used and why. To ensure such communication, it is important to highlight the use of various types of digital tools, including screen-free options such as robots. Without clear communication about how digital tools are used, guardians may base their views about digital tools on experiences of how such tools are used at home and on what is promoted in media.

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Merging or separating mathematics education and play – does it affect learning?

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Keywords: Play-responsive, teaching, preschool, cardinality, MPM

Introduction

Early numeracy is crucial in children's mathematical development. However, few studies have investigated how to teach numeracy in preschool education (Björklund & Palmér, 2019). Additionally, in several European countries there is an ongoing discussion on the relationship between play and education. This study examines the teaching of cardinality through an intervention where mathematics education is merged with play. The aim is to explore how merging mathematics education and play enhances preschool children's learning of cardinality. In this exploration, the Mediating Primary Mathematics (MPM) framework (Askew & Venkat, 2017) is used to compare the quality of instructional sequences that integrate play and mathematics (PRMT) with instructional sequences where mathematics teaching (MT) is separated from play. Through the use of MPM, the learning opportunities in these different instructional sequences are elaborated on. The research question is: What opportunities for learning in mathematics are provided when children participate in a play-responsive teaching (PRT) intervention?

Mediating Primary Mathematics (MPM)

MPM is a theoretical framework developed to analyse and develop mathematics education (Venkat & Askew, 2017). The framework emphasizes that the teacher conveys mathematical content in mathematics teaching through 1) artefacts, 2) inscriptions, and 3) discussions and gestures. The third component is divided into methods for a) generating or validating solutions, b) building mathematical connections, and c) building learning connections through explanations and evaluations. These three aspects of teaching can be categorized into distinct quality levels. High-quality teaching is more profound and more structured, enabling students to learn in the teaching situation. In lower-quality teaching, materials and examples are used in a one-sided or disjointed manner, and children's responses are not considered or developed, which can lead to a reduction or absence of learning opportunities in the situation (Venkat & Askew, 2017).

Methods

The study is conducted through educational design research, which involves a cyclical process of planning, implementation, and evaluation of teaching. Over the course of one year, the researcher and three preschool teachers met every other week to plan and discuss play-responsive activities. These activities were implemented by the teachers with children in various groups. The implementations were video-recorded. In total, four activities were developed and documented through 73 video recordings from the three different preschools. The analysis was conducted using content analysis. In the first step, the focus was on PRT (Pramling et al., 2019) with the codes "as if", "as is", responsiveness, alterity, agency, metacommunication, and intersubjectivity, as well as codes for early

numerical understanding, including cardinality, subitizing, ordinality, wholes and parts, and representations (verbal expressions, gestures, and body language). This coding was performed using NVivo software. The result of this first analysis was three characteristically different sequences in the video documentations: sequences of play (without explicit mathematics), sequences of mathematics teaching (MT, without explicit play), and sequences with merged play and mathematics teaching (PRMT). In the next analysis step, the MT and PRMT sequences were analyzed using the MPM framework. The purpose of this analysis was to delineate the distinct characteristics of cardinality teaching and to examine the mathematical learning opportunities offered to preschool children in these specific sequences.

Result and discussion

The analyses show that MT sequences include more structured artifacts and more connections to mathematical contexts. According to the MPM framework, MT sequences offer richer learning opportunities through structured artifacts, where the teacher systematically emphasizes cardinality and numbers by linking artifacts, contexts, and body language. This structured teaching support can support children's mathematics learning. In MT sequences, the focus is often on one child, providing a more personal learning experience but reducing interaction between children. PRMT encourages an exploratory, playful approach that promotes creativity and agency. Fewer and unstructured artifacts are used in PRMT, sometimes overshadowing the mathematical content. This may be because the teacher needs to be responsive to children's questions, even those not about mathematics, allowing children to influence and make independent decisions during play. PRMT activities often include multiple children, making mathematics relevant to the entire group. The two sequences complement each other by offering different learning opportunities. MT sequences provide clear connections between artifacts and mathematics, improving children's understanding through structured teaching. PRMT sequences encourage exploration and creativity, allowing children to develop their mathematical understanding through playfulness and independence. Together, these sequences can create a balanced learning environment where children benefit from both exploratory play and structured learning with mathematical concepts.

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Finger use for addition and subtraction in preschoolers

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Keywords: Finger use, early arithmetic, embodied cognition.

Introduction

Empirical evidence suggests that numerical cognition has its roots in bodily and sensory experiences, known as embodied numerosity (Moeller et al., 2012). As many children use their fingers to show their age or to do basic arithmetic, finger counting has been claimed a prime example of embodied numerical development. However, its association with early arithmetic can be understood not only by looking at *if* children use their fingers successfully to calculate (Krenger & Thevenot, 2024), but also at *how* they use them (Björklund et al., 2019). In the current study we combined both perspectives by examining for the first time for different task types how many children use their fingers to calculate, (i) in case manipulatives are provided vs. not provided, (ii) in an addition vs. a subtraction task. Moreover, we aimed at investigating (iii) whether finger use is associated with accuracy and (iv) how children use their fingers.

Method

The following analyses are based on data of n=39 German kindergarten children (M=6.2 years, SD=0.3, range 5.7 to 6.8 years), with an even gender distribution. The children were presented with three types of tasks: a verbal addition problem with manipulatives, a symbolic addition problem, and a symbolic subtraction problem. All children were interviewed individually, in a quiet room of the kindergarten. Videos of the interviews were taken with a digital camera on a tripod. In the verbal addition task, the interviewer read aloud the word problem and placed the corresponding number of marbles in front of the child ("Here you have 5 marbles [interviewer puts five green marbles in random order on the desk]. Here I have 4 marbles [interviewer places another four blue marbles on the desk]. How many marbles do we have together?"). For symbolic addition and subtraction, the problems were presented visually (3 + 5, 9 − 5) and were read aloud to children. We analyzed both accuracy and frequency as well as type of finger use. Type of finger use was coded in line with Björklund et al. (2019) as (i) structured (i.e., instantly showing finger patterns without counting single units), (ii) dynamic (i.e., extending fingers successively for counting), and (iii) strategy mix (i.e., aspects of structured and dynamic finger use are combined). We used a Binomial test to examine if children used their fingers significantly more often in case manipulatives were not provided. Furthermore, we used a Chi-Square test to investigate whether finger use is significantly related to accuracy.

Results

First, we examined how many children used their fingers in the verbal addition task with manipulatives vs. the symbolic addition and subtraction tasks. Results revealed that only one child (3%) used the fingers when manipulatives were provided, whereas n=12 children (31%) and n=10

children (26%) used their fingers to solve the addition and subtraction task, respectively. Thus, significantly more children used fingers, when no manipulatives were provided (Binomial test, $p<.001$), while no significant difference was observed for addition vs. subtraction ($p=.30$). Interestingly, most children used fingers for only one task type ($n=6$ for addition only, $n=4$ for subtraction only) and $n=6$ (15.4%) used fingers for both addition and subtraction. Second, we analyzed whether finger use is associated with accuracy. Overall, $n=27$ children solved the symbolic addition task correctly (69%), whereas only $n=10$ children did so for subtraction (26%). As regards finger use, 83.3% of finger users solved the addition task correctly, whereas only 63.0% of non-finger users did so. However, this pattern did not turn out to be significant ($\chi^2(1)=1.62$, $p=.28$). For symbolic subtraction, 50.0% of finger users solved the task correctly, whereas only 17.2% of non-finger users did so. The association between finger use and accuracy turned out significant for subtraction ($\chi^2(1)=4.19$, $p=.04$), indicating that finger users were more successful than non-finger users. In a last step, we examined *how* children used their fingers. For addition, most finger users applied a mix-strategy ($n=5$; four correct), followed by dynamic ($n=4$; three correct) and structured strategy ($n=3$; all correct). For subtraction, most finger users used a dynamic strategy ($n=5$, only one correct), followed by mix ($n=3$; all correct) and structured strategy ($n=2$; one correct).

Discussion

Results revealed that 26% to 31% of children used fingers to solve the subtraction or addition problem when no manipulatives were provided, which is in line with results reported by Björklund et al. (2019) and Krenger and Thevenot (2024). In terms of accuracy, we particularly found a more pronounced advantage in using fingers for subtraction compared to addition. A reason for these differing results might be that children were in general more proficient in addition and finger use might, thus, not have been a major advantage in this case. A closer look at *how* fingers were used for subtraction revealed that dynamic finger use was less successful compared to other strategies. Although our sample size was too small to draw reliable conclusions, this pattern is still in line with results reported by Björklund et al. (2019). However, our conclusions should be substantiated by further studies, as we only looked at one task of each type and a small sample size.

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TWG 14: University Mathematics Education

Introduction to the papers of TWG14: University mathematics education

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This paper summarises in five themes the 25 papers and 9 posters contributed to the TWG14: University Mathematics Education (UME) at CERME14, and presents an overview of the three thematic discussions conducted in the TWG, before concluding with reflections and suggestions for future research.

Keywords: University mathematics learning and teaching, resources, teaching interventions, mathematics for non-mathematicians, mathematicians and mathematical practices.

Introduction

We opened the activities of the TWG14 at CERME14 by honouring recently departed colleagues who contributed significantly to what we today call UME research: Guy Brousseau, Ed Dubinsky, Simon Goodchild, Anna Sierpińska and David Tall. We also revisited the activities of the group since its launch at CERME7 in 2011, concentrating on an overview of achievements between CERME13 (July 2023) and CERME14 (February 2025). This period saw five special issues related to UME – three in the *International Journal of Research in Undergraduate Mathematics Education* (Dreyfus et al., 2023; Ely & Jones, 2023; Geraniou et al., 2023), which celebrated a decade of contribution to UME research (Dreyfus et al. 2025), one in *Educational Studies in Mathematics* (Martino et al., 2023) and one in *ZDM – Mathematics Education* (Engelbrecht & Borba, 2024); specialised conferences – e.g., the fifth ERME topic conference of the *International Network for Didactic Research in University Mathematics* (INDRUM2024); a book (Bridoux et al., 2023); and, a book chapter (Winsløw et al. 2023). This year, TWG14 received 37 paper and 11 poster submissions. The review process resulted in 25 papers and 9 posters being presented at the conference, all of them published in CERME14 proceedings. This year, we noticed a decrease in the number of contributions compared to previous years, with 36 papers and 17 posters in the CERME13 proceedings. Such decrease was expected due to the emergence of the new TWG25 on "Calculus Teaching and Learning" as well as to the new policy on registration numbers. Contributions address at least one of five themes: *teachers and teaching at school and university level*; d*igital resources*; *mathematics and non-mathematics specialists*; *learning and teaching mathematical topics and practices*; and, *mathematicians and mathematical practices*. Activities included a combination of parallel sessions for paper discussion

and plenary sessions for poster teasers and thematic discussions. The three thematic sessions covered: *digital resources; teachers and teaching at school and university;* and *research methodologies and methods.* In what follows, we summarise contributions in each theme, and the thematic discussions, before concluding with reflections and potential ways forward.

Paper and poster contributions

Teachers and teaching at school and university level

Eleven papers and three posters were classified in this group that proved to be the most popular. Gueudet and Doukhan present a literature review about *Didactic Contract(s)*[1] in UM teaching, particularly teaching characteristics and their impact on students' learning. The review identified three categories of studies that focus on changes of didactic contract at the secondary-tertiary transition; limitations in the students' mathematical activity due to didactic contract rules; and, interventions proposing an 'alternative contract'. Still with attention to teaching environment, and through a commognitive (*Theory of Commognition*, *ToC*) and *Communities of Practice* (*CoP*) perspective, Hooper et al. explore how an inter-community team of mathematicians and mathematics education researchers can redesign an introduction to university mathematics course to facilitate shifts in the lecturer's pedagogical and students' mathematical discourses. Witty et al. discuss the same course from the perspective of the lecturer aiming to redesign the course. Teaching actions towards students' enculturation to university mathematics discourse is the focus of Karavi and Schüler-Meyer, who are interested in *Opportunities to Learn* offered in first-year teaching by four lecturers. The paper exemplifies such opportunities in a session about mathematical proof. Teaching practices towards first year students' introduction to fundamental concepts of Analysis is also the topic of Petropoulou et al. The analysis draws on Radford's *dialectical nature of tensions* to highlight how the lecturer's background in both mathematics and mathematics education contributes to the development of an alternative teaching culture. Butler and Buchbinder use the perspective of *teacher noticing* to study the development of a learning assistant in Calculus over two years. They observe that his noticing shifts from describing particular aspects of student work to reflecting on mathematics generally. With attention to teaching preparation, Canogullari and Radmehr draw on the *Anthropological Theory of the Didactic* (*ATD*) to identify conditions and constraints experienced by lecturers when designing, adapting or selecting tasks. Thorup Eich-Høy analyses mathematics lectures as a way to teach mathematics. Using an original approach based on ethnographic methods and psychoanalytic theory, the paper states that lectures can create situations of resonance, embodiment, and durational pleasure in contrast to the common critiques of lectures as outdated or passive.

This year we welcomed a body of contributions related to the role of mathematics courses (and mathematicians) in mathematics teacher education. Pinto investigates the practices of mathematicians and secondary mathematics teachers when comparing and discussing alternative proofs of the same

[1] Due to space limitations, we do not provide explanation and references to theories, e.g. *Didactic Contract*, *Theory of Commognition* (*ToC*), *Community of Practice* (*CoP*), *Anthropological Theory of the Didactic* (*ATD*), *Cultural Historical Activity Theory* (*CHAT*), etc. We would invite readers to find more in the corresponding contributions in the proceedings.

statement. The study highlights the importance of integrating both mathematical and pedagogical practices in secondary mathematics teaching and showcases how contrasting approaches offer further insights for teacher education. Weith et al., with attention to learning materials, examine the design of resources for pre-service teachers that incorporate theoretical considerations and aim to connect university and school mathematics. The paper presents the design process along with provisional findings on how pre-service teachers draw on their mathematical knowledge to inform their teaching decisions. Viirman et al. bring an international perspective on the role of mathematicians in secondary mathematics teacher education by reporting on the design of a large-scale survey on mathematicians' views and practices towards strengthening mathematics education of prospective teachers. Marchionne et al. examine how prospective secondary teachers judge the value of instructional videos and STACK (System for Teaching and Assessment using Computer algebra Kernel) tasks in learning about secondary students' misconceptions about derivatives. Still in a teacher education context, Hoffman observes how changing the response format of an interface task, from email answers to scriptwriting, led preservice secondary teachers to use overarching lines of argumentation that are more conducive to learning. Finally, with attention to learning materials, Morcos and González-Martín identify inconsistencies in how permutations with indistinguishable objects are presented in secondary and tertiary mathematics textbooks.

Digital resources

Two papers and three posters were classified in this group. Within the emerging field of research regarding the use of genAI (Generative Artificial Intelligence) in UME, Iannone et al. propose a potential research agenda. Using a Delphi method, they collected and classified the research questions considered as most urgent by academics. Miskei-Váradi et al. examine the integration of ChatGPT into a second-year undergraduate numerical methods course and assessed students' perceived usefulness of ChaGPT before and after the course. Findings indicate mixed perceptions on the utility of ChatGPT and its limitations in addressing misconceptions, including risks in accepting false proofs. Du et al. explore university students' mathematical digital competencies in communication in computer-supported collaborative learning and propose a frame that addresses collaborative engagement. Regarding other digital resources, Thoma and Iannone explore a lecturer's perspective on using Lean, an Interactive Theorem Prover, and identify adaptations the lecturer made to his course, the constraints he faced, and the lessons learned in terms of teaching proof with Lean, but also teaching proof more generally. Finally, Markefka et al. describe the development and evaluation of STACK tasks for introductory courses in university mathematics in collaboration with lecturers.

Mathematics and non-mathematics specialists

Five papers and two posters were classified in this group. Dahl et al. explore the integration of virtual reality and problem-based learning to foster engineering students' understanding of the vector cross product. Students reported positive learning experiences, although some of them identified usability issues or/and reported motion sickness. Norris and Noyes compare pre-university mathematics curricula followed by university students of geography, psychology and sociology in England in datasets from 2016-17 to 2019-20, concerning over 90,000 students. Analysis indicates significant differences in the opportunities to study mathematics both between and within the three disciplines,

leading to a complex situation when designing mathematical courses for incoming students.

Several studies in this group take an *ATD* perspective. Šipuš et al. analyse mathematical practices in advanced courses for electrical engineers through interviews with teachers and one master student who point out differences between mathematics and engineering practice. Hitier and González-Martín investigate science students' strategies when solving a non-familiar calculus task, with particular attention to *praxeologies* from mechanics. Students who employed mechanics' or mixed *praxeologies* were generally more successful in solving the task. Florensa et al., also from the ATD perspective, characterise the views of mathematics and engineering teachers about mathematics in the first years of mathematical courses in engineering degrees. Preliminary findings indicate differences in the type of tasks mathematicians and engineers would prioritise (e.g., mathematics-oriented vs practice-oriented tasks). Fernández-Ruano et al. discuss active learning teaching by drawing on *Study and Research Paths* (*SRP*s). The study compares the implementation of *SRPs* instigated by the same problem in three different university-level statistical courses and identifies differences in the ways students engage with statistical techniques. Finally, Kristinsdóttir and Gíslason report on an Icelandic project on the secondary-tertiary transition, aimed at increasing the number of STEM graduates. They discuss similarities and differences in expectations between upper-secondary and university-level mathematics teachers regarding students' mathematical preparation for STEM studies.

Learning and teaching of mathematical topics and practices

Four papers and one poster were classified in this group. Wawro et al. present the design, implementation and analysis of a task introducing the notion of 'least squares' in an introductory linear algebra course. Framed within the *Realistic Mathematics Education* framework the designed task contributes towards student development of graphical intuition with planes and vectors. De la Cruz Benito and Andrews-Larson investigate students' proof methods in the context of linear independence after a course using Desmos, observing that a significant portion of students successfully construct deductive proofs, but others still use empirical proofs. Meanwhile, Wallach and Kontorovich propose a conceptualisation of problem solving based on the *ToC*, distinguishing between *solution generation* routines and s*olution conventionalisation* routines. Rogovchenko and Rogovchenko discuss students' difficulties with differential equations in relation to the general solution. They review influential classical textbooks and suggest an introduction to the notion of the general solution of differential equations as a family of functions. Finally, Veuillez-Mainard studies the teaching and learning of polynomials at the secondary-tertiary transition in France by focusing on students' conceptions of polynomial functions and their graphs.

Mathematicians and mathematical practices

Three papers were classified in this group. Olsen et al. investigate the epistemic tensions associated with the role of programming in the research practices of mathematicians. Drawing on interview data from two mathematicians, one with a background in pure mathematics and the other in applied mathematics, the authors use *Cultural Historical Activity Theory* (*CHAT*) to examine the rules and motives shaping their research practices and the role programming plays within those practices. Goor and Pinto study *positioning acts* of mathematicians and secondary school teachers discussing the practices of a teacher presenting a proof in class, observing that the mathematicians can move from

disciplinary authorities to fellow educators. Finally, L'Italien-Bruneau et al. draw on the *ToC* to study the research discourse of doctoral students in mathematics. They draw on interviews with three such students to describe patterns characterising their specialised mathematics research discourse.

Thematic Discussions

Digital resources: Revisiting UM teaching, learning and assessment

In eight TWG14 contributions, digital resources were the focus or the context of the study. These resources included Virtual Reality, genAI, Programming, Interactive Theorem Provers, Videos, Desmos, Geogebra and STACK. Such attention to digital resources reflects broader trends within UME, as seen in IJRUME's recent special issue on digital experiences (Geraniou et al., 2024), and more generally in mathematics education – especially since COVID-19 accelerated the use of digital resources (Engelbrecht & Borba, 2024). In the discussion, we focused on three digital environments: Interactive Theorem Provers, genAI and STACK. For each of these, we considered: (1) challenges and opportunities they present for learning, teaching and the development of students' employability skills; (2) their influence on assessment, particularly in terms of assessment design; and (3) theoretical frameworks and methodologies that may offer deeper insights into the use of these digital tools.

Concerning the first point, we identified several opportunities and challenges from both student and teacher perspectives. For students, we discussed opportunities for practice, providing applications across different disciplines (e.g., computer science and mathematics), and the development of transferable skills linked to employability. However, concerns were also raised about students potentially becoming overly reliant on digital resources (e.g., genAI) and lacking the appropriate critical engagement allowing for an efficient and effective use of those resources. From the teachers' perspective, the discussion was around the need for targeted professional development and the creation of appropriate teaching materials. It was also noted that support for the teaching implementation of specific digital resources is currently limited, and that some of those resources demand a steep learning curve (e.g., Interactive Theorem Provers). Insufficient availability of appropriate and adaptable teaching and learning materials raises concerns about the practical implementations of these tools in teaching, particularly in contexts where instructors' and students' needs may differ considerably. In terms of assessment practices, we considered tensions between checking correctness and redefining assessment criteria, as well as the contrast between evaluating the final product and placing greater emphasis on the learning process. We also highlighted the importance of rethinking assessment methods to offer a variety of methods but, also, to align better with employability skills. Furthermore, we reflected on potential theoretical frameworks to support research in this area, such as *Instrumental Approach*, *CHAT* and *ToC*. We also discussed how genAI could be implemented within our research process, e.g., as a tool, an agent or a collaborator? Finally, mixed-methods approaches were considered well suited to capture resource-use broadly, as they can provide valuable insights into usage patterns, accuracy, speed and feedback provision.

Teachers and teaching at school and university: A cross-community perspective

In recent CERMEs, TWG14 has seen an increase in the number of contributions related to teachers and teaching at school and university level, and it was a popular theme among contributions also at

CERME14 (see earlier sections). This growing interest is also seen at the INDRUM2024 conference and in recent special issues (e.g., Buchbinder et al., 2023). Given the focus in several TWG14 contributions on cross-community collaboration[2], for instance between mathematicians and mathematics educators, for the second thematic discussion we chose to take a cross-community perspective on teaching at school and university level, including university-based preparation of school teachers and university teacher professional development. The discussion was structured in two parts: (1) reflecting on the affordances and challenges of working with multiple communities in teaching and teacher preparation; (2) considering how can we draw on these affordances and overcome the challenges. Among the studies presented in TWG14 this year, the two main affordances we identified were co-learning partnerships, where members from different communities gain from collaborating on planning and conducting teaching, and benefits for students from cross-community teaching collaboration. However, several challenges were also discussed. Institutional constraints, such as university organisation (e.g., mathematicians and mathematics educators working in different departments or even faculties) might, for instance, lead to a lack of awareness of communities' willingness to collaborate. We also noted potential communication difficulties, stemming, for instance, from different perspectives on teaching and learning in different communities. Finally, the dependence of the 'success stories' on specific individuals involved was pointed out, raising the question of how to make initiatives sustainable. One proposed answer in the second part of our discussion was to conduct research at the 'meta level' on the conditions making cross-community teaching collaboration productive and sustainable. Other proposed strategies are making UME research publications accessible to mathematicians (e.g., through implications for practitioners), and collaborative research on teaching with mathematicians (or other specialists) as co-researchers.

Research methodologies and methods: Specificities, challenges and shared practices in UME

Also, we opted for a discussion on research methodologies and methods, which was motivated by the observation of an increased diversity of methods, including the introduction of methods uncommon in the group, such as a Delphi study and the use of AI tools for coding interview data (see the digital resources theme). The discussion was organised around different interconnected aspects related to methodologies and methods. Discussion revolved around the crucial role methods play, not only in the way research develops, but also in how results are shared and disseminated with close attention to the relation between theoretical frameworks, objects of study, and methods.

One of the issues that emerged from our discussion was the sharing of methodological tools (questionnaires, interview guidelines) within the UME research community, which can become a repository, but which is not sufficiently developed yet. Another topic that was discussed was the need for dissemination of task analysis instruments and approaches, and the possibility for such instruments to be shared with mathematicians – another opportunity for cross-community collaboration between mathematicians and mathematics educators. A further aspect that emerged during the discussion was the international movement towards open data, which may raise issues of

[2]In addition to collaboration within the broader mathematics community (mathematicians, mathematics educators, mathematics teachers, etc.), this could include collaboration with other specialists, like engineers or science educators.

anonymity and confidentiality in our field (typically, for classroom videos). The difficulty of sharing methods and instruments when different theories are employed was discussed as well, a concern that led us to express a need for theory networking towards shared methodologies in UME research. We also discussed the changes in methodological approaches in works presented at TWG14 across CERMEs, noticing a significant development of mixed methods research designs in recent years compared to a dominance of qualitative approaches in the past. Still, access to representative samples for quantitative studies remains an ongoing challenge in UME research.

Many of the issues above could apply to other CERME TWGs. However, specifically to TWG14, UME researchers might be UM teachers as well, which raises issues of balancing priorities and controlling potential biases. For example, a UME researcher who is also a UM teacher might need to reflect on or detach from the teacher's position when doing research. Cross-community collaboration and international/comparative studies were seen as means that might be helpful in this regard. In the latter case, the design of methodological tools that can be used in different cultural and institutional contexts might be an additional challenge. Finally, replication studies were mentioned as an opportunity for transferability, testing and evaluation of methods in different contexts.

Reflections and ways forward

Summarising some of the tendencies we have seen at CERME14 TWG14 compared to previous CERMEs, there is, for instance, an increase in studies focusing on mathematicians' involvement in schoolteacher preparation, and (unsurprisingly, given recent developments in technology and society as a whole) studies on genAI and integration of digital resources more widely. The secondary-tertiary transition is a steady area of interest, as is research on teachers and teaching practices, while this year saw a decrease in papers focusing on the teaching and learning of particular content, and on mathematics for non-mathematics specialist students. This can probably be at least partly explained by the already mentioned new TWG25 on Calculus, as well as by the further consolidation of the TWG26 on mathematics in STEM. This year we returned to contributions around mathematicians' research and teaching practices as well as to their preparation (and pedagogical support) for teaching.

Concerning theoretical perspectives, *ATD* and *ToC* are still widely used in studies presented in the TWG14, while we saw a slight increase in ToC papers this year. Regarding methods, we observed the development of mixed methods, and of innovative methods such as the use of genAI for coding data. There is still a definite lack of contributions focusing on issues of equity and inclusion, despite this being a topic of crucial importance for UME. This lack has been discussed in the group over the years, and one tentative explanation is that researchers focusing on these types of issues, even if their research is conducted at the university level, see themselves more as parts of other research communities rather than of the UME community. This is purely speculative, however, and we would be keen on learning more about how we could attract more contributions from these areas.

In the final session, discussion time was devoted to looking ahead, and thoughts were raised that might serve as inspiration for future work. For instance, for quite some time now, a lot of attention has been paid to the utility aspects of higher mathematics, and its role in other disciplines. Perhaps it is time to also attend to the role of pleasure and enjoyment of pure mathematics? And in connection to this, perhaps it would be fruitful to pay more attention to students' engagement in mathematical

activities as enculturation into mathematical practices? We also discussed the dialectic relation between intuition and rigour. This can be understood both in relation to our common topic of study, namely university mathematics, but also in relation to our own practices as UME researchers. Generally, there seemed to be an interest in further work in the networking of theories and methodologies, in order to see further issues and gain deeper insights.

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Learning assistants' development as mathematics teachers: Insights from an undergraduate to graduate teaching journey

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Learning Assistants (LAs) are undergraduate students who, after completing an academic course, return in later semesters to assist in the teaching of that course. Typically, they are present in class, alongside instructors and/or graduate teaching assistants, and help facilitate student discussions. It has been shown that both students and LAs themselves benefit in LA-supported classrooms. However, few studies have addressed the pedagogical development of LAs as teaching paraprofessionals. We contribute to this line of inquiry by utilizing the lens of teacher noticing to examine how the professional noticing of one Calculus LA, Chase, evolved through his first semester as an LA and into his time as a graduate teaching assistant. Our findings provide insight into LAs' role in undergraduate classrooms.

Keywords: Calculus, learning assistants, professional noticing

Introduction

The Learning Assistant (LA) instructional model was developed in the United States in the early 2000s, to support the facilitation of large introductory STEM courses, like Calculus (Otero et al., 2010). LAs are undergraduates who previously completed a certain course and return to support the teaching of the course. Typically, LAs are present in classrooms alongside the instructor or graduate teaching assistant (GTA) and help to facilitate students' individual or group work by asking or responding to questions, providing hints, and facilitating peer-to-peer interactions. Thus, an LA's classroom role is that of a knowledgeable peer, supporting and advancing student learning. Indeed, multiple studies have shown that in LA-supported classrooms students have better learning outcomes and affective improvements, compared to courses without LAs (Barrasso & Spilios, 2021). These positive outcomes may be surprising, considering that LAs (often non-education majors), receive only minimal pedagogical training. Typically, LAs enroll in a pedagogy course where they learn about teaching and reflect on their practice. To strengthen their content knowledge, LAs meet weekly with the course instructor to refresh their memory of the mathematical topics addressed in that week.

This unique nature of LAs' teaching experience within university classroom environments, combined with the recurring reports of students' success in LAs-supported classrooms, has led researchers to investigate LAs' classroom practices and their professional development as teaching paraprofessionals (Nadelson & Fannigan, 2014). Specifically, it has been suggested that since LAs have only recently learned the course content, they may notice and respond to aspects of student thinking that go unnoticed by other instructors, thus supporting students' success (Wendell et al., 2019). However, to the best of our knowledge, no studies examined how LAs' noticing develops. Our study seeks to contribute to this research gap by examining how LAs' professional noticing evolves as they gain teaching experience. This paper is a case study of one undergraduate calculus LA, Chase (pseudonym). Our research question is: *how did Chase's noticing evolve as he gained teaching experience?*

Theoretical Perspectives

While it has been documented that LAs develop as teachers (Barrasso & Spilios, 2021), the mechanisms behind this development remain relatively unexplored. One promising direction to examine this is the lens of noticing. Teacher noticing has been recognized as an important marker of mathematics teacher development, being developed and widely used for studying the professional learning of practicing and future teachers (Dindyal et al., 2021). Since LAs can also be viewed as teachers-in-training, even if they do not intend to pursue a teaching career, this framework can be appropriately applied to the LA context. For example, Top et al., (2018) observed that as LAs gained experience, their pedagogy shifted from correct students' answers toward attending to students' ideas.

We use van Es and Sherin's (2008) definition of noticing as encompassing attention to events, reasoning about them, and connecting them to broader teaching principles. We additionally use reflective noticing (Buchbinder et al., 2021) to frame this study, examining LAs' retrospective attention to their teaching as opposed to their in-the-moment thinking. While the noticing of LAs has received some attention in the literature (Thompson, 2019; Wendell et al., 2019), these studies do not include the subject-specific nature of development as a mathematics teacher and do not consider development over time, thus allowing for a unique contribution into this under-researched area.

Methods

This study is part of a larger project focused, among other things, on improving Calculus I courses in a large, US public university. Students attend a large lecture (~150 students) taught by a faculty instructor three times a week and learn in smaller (~20 students) recitation sessions, led by a GTA and LA, twice a week. The GTA is primarily responsible for the recitation, including tasks such as: for reviewing the lecture material, proctoring quizzes, facilitating whole class discussions, and interacting with students as they solve problems. The LA assists classroom instruction by circulating the room, asking and responding to questions and facilitating student group work.

Of the four LAs involved in teaching Calculus I in Spring 2022, Chase was a senior mathematics major. He had a strong mathematical background and some tutoring experience. After graduating, Chase taught for a year as a substitute mathematics teacher in a local high school. He then returned to the university for a master's degree in mathematics, and, upon receiving it in 2024, went back to teaching high-school mathematics. Thus, Chase's case is interesting because we were able to examine his long-term teaching development as an undergraduate educator following a novice LA experience.

We examine Chase's development as a teacher by analyzing how his noticing evolved from being a novice LA to an experienced GTA. Data consists of three video-recorded stimulated-recall interviews with Chase, which included reflection on video clips from Chase's recent teaching selected by a researcher based on availability and substance. The first interview took place several weeks into Chase's first semester as an LA (Spring 2022), the second was at the end of his semester as an LA (~13 weeks later), and the third interview was at the end of his final semester as a GTA (Spring 2024). In these interviews, Chase was asked to reflect on his teaching experiences using artifacts like sample student work and short 360-degree video clips of his own recent teaching. These types of reflections have been shown to support pre-service teachers' reflective noticing (Buchbinder et al., 2021). The

interviews were transcribed and analyzed using a coding scheme (Figure 1) based on van Es & Sherin (2008). Each utterance received codes in each of the three categories: actor, topic, and stance.

Category: Actor (Who the LA talks about)

Self: LA talks about their teaching moves, how they think about mathematics, interact with students, their opinions on teaching, or their classroom role.
Student-Individual: LA talks about a specific student.
Student- Group: LA r talks about a specific group of students: what they did, how group members interact with each other, or how groups function in class.
Student–Class: LA talks about students at the class level, e.g., about students they teach or students in general.

Category: Topic (What the LA talks about)

Mathematics: LA talks about mathematics, how someone thinks about or does mathematics, or when they talk about someone's difficulties with mathematics.
Pedagogy: LA talks about their teaching or teaching in general: their actions or pedagogical decisions.
Climate: LA talks about social dynamics or social aspects of interactions with others.

Category: Stance (How the LA talks)

Describe: LA details events they see, hear, or experience, but without judging them.
Evaluate: LA explicitly provides a positive or negative appraisal of a situation or opinion. Or LA provides an alternative action they could have done or could do in the future.
Interpret: LA explains why or how something happens, or makes connections to underlying concepts, causes, or implications by explaining the meaning or significance of events.

Figure 1: LA Noticing Coding Scheme

This coding scheme was developed based on eight LA interviews. As the LA program expanded, a substantial corpus of LAs' interviews (34 interviews) was collected, making their manual turn-of-talk coding unfeasible. Consequently, we developed a technologically enhanced method. Using Dunivin's (2024) chain-of-thought prompt sequence, we designed a prompting protocol for ChatGPT (OpenAI, 2024) to code the interviews. Chase's first and second LA interviews were coded by a human researcher and AI. Coding disagreements were resolved by the researcher, resulting in an interrater reliability rate of 68.7%, indicating substantial to very good agreement. Chase's GTA interview was coded using this prompting protocol and codes were reviewed by a researcher for accuracy.

Results

Table 1: Summary of Chase's noticing across time, in percentages

	Topic	Mathematics			Climate			Pedagogy		
Category	Sub-Category	1st LA Interview N= 13 P= 30.2%	2nd LA Interview N= 36, P= 38.7%	GTA Interview N= 6 P= 16.2%	1st LA Interview N= 15 P= 34.9%	2nd LA Interview N= 35 P= 23.7%	GTA Interview N= 17 P= 37.8%	1st LA Interview N= 15 P= 34.9%	2nd LA Interview N= 22 P= 37.6%	GTA Interview N= 14 P= 45.9%
Actor	Self	23.1	22.2	50.0	33.3	22.7	28.6	80.0	82.9	76.5
	Stud - Class	53.8	55.6	16.7	0.0	18.2	7.1	13.3	14.3	11.8
	Stud - Group	23.1	11.1	16.7	40.0	31.8	35.7	6.7	0.0	11.8
	Stud - individual	0.0	11.1	16.7	26.7	27.3	28.6	0.0	2.9	0.0
Stance	Describe	38.5	47.2	33.3	73.3	50.0	50.0	60.0	25.7	64.7
	Evaluate	15.4	19.4	0.0	20.0	31.8	35.7	20.0	40.0	29.4
	Interpret	46.2	33.3	66.7	6.7	18.2	14.3	20.0	34.3	5.9

Table 1 shows the summary of Chase's noticing codes. N is the number of utterances that received that topic code for each interview, and P is the corresponding percentage.

Chase's Mathematics Talk

In his first and second interviews as an LA, Chase attended significantly to mathematics, with 30.2% and 38.7% of his utterances respectively coded with this topic. However, in his interview as a GTA, Chase focused on mathematics significantly less, with just 16.2% of his utterances receiving this code. The categories of actors and stances shed light on this change in Chase's noticing focus.

In his first interview as an LA, Chase primarily talked about how students in general thought about mathematics (53.8% of mathematics codes received a *student–class* actor code), offering reasons for student thinking or details of students' solutions (in mathematics category: 38.5% of stance codes described, and 46.2% were *interpret*). For example, when discussing a lesson where students evaluated the truth value of statements about continuity and limits of functions, Chase explained how students "found justifying the first [statement] tough. Because that's just a definition and they didn't want to just say, this is the definition of continuity.") Hence, Chase identified both the problem the students struggled with and the source of their difficulty. He also reflected on the mathematical work of specific student groups (23.1% of actor code were *student–group*), but not individual students. He reflected on one group's interaction: "I think [they were] saying, 'Oh, but there are some functions where the limit exists here and it's continuous.' That's true, but if you say the statement's true, you're saying it happens all the time." Thus, Chase's reflections in the first interview were characterized by a strong focus on mathematics and his descriptions of students' mathematical thinking were general and interpretative. Chase also reflected on his own conceptions of mathematics (23.1% of code *self*), comparing them to students' thinking. For example, he said: "I've been taking math classes for a couple of years. I haven't thought about what someone who hasn't seen [logical] implications before would think about them. So, it kind of got me into their [students] mindset a lot more." This shows Chase's emergent realization that others may think of mathematics differently from him.

In his second LA interview, Chase retained his mathematical focus but became slightly more attuned to how individuals perform mathematics, offering more descriptions and evaluations of student work (47.2% and 19.4% of stance codes respectively). Reflecting on one student's solution to an optimization problem, Chase said: "Some confusion on how to find the area. He says $ax=y$, which isn't right. But he does have the right perimeter form perimeter formula." This quote differs sharply from those in the first interview: Chase offered a confident judgment about the strengths and weaknesses of this individual student's work, showing that he began attending more to individuals.

As a GTA, Chase attended less to the topic of mathematics overall, but when he did, he focused more on students in general and provided more interpretations (66.7% of stance codes) than in his previous interviews. When reflecting on a lesson on partial fraction decomposition, Chase shared his thoughts about general student thinking about this procedure, saying "I think there is confusion about, the variable. Because it changes: (ax) is a variable in your integral, but $a, b,$ and c are variables in your system of equations. And I think that shift is not obvious at all to students". As a more experienced teacher, this reflection is not about specific students or tasks but is a more general inference about how students learn a certain mathematical topic and the challenges students encounter with it.

Chase climate talk

Chase noticed the classroom climate (i.e., social aspects of the classroom) less in his second LA interview than in his other two interviews (1st-LA: 34.9%, 2nd-LA: 23.7%, GTA: 37.8%). While the frequencies were comparable in his first LA and GTA interviews, his stance shifted from descriptive focusing on groups and himself, to more evaluative and focused on the class overall.

In his first interview, Chase's noticing was highly descriptive (73.3%) and focused on his own classroom role and student engagement. For example, he noticed students' (un)willingness to engage with him and one another and referred to some recitation sections being more "talkative", or "vocal" while others were "quieter". Occasionally, he offered reasons for a group's lack of communication (e.g., "The class is at 8:00 AM, so they could be sleepy. I'm sleepy."). He also reflected on his own social role in the classroom (33.3% of climate utterances received an actor code *self*), describing how he thought students felt "comfortable" talking to him and treated him with a level of authority since "it seems like [students] approach [the GTA and I] as relative equals when they're asking questions".

In his second LA interview, Chase's stance about classroom climate became more evaluative and interpretive (31.8% and 18.2% of stance codes repressively). He reflected on learning to socially "manage groups" and interact with "strong personalities" by "extract[ing]" himself from excessively "talkative" students to attend to others. He reflected on gender roles recalling one "bright" female student who rarely interacted with others because "when you're a girl in a Calculus class and it's all men, that's intimidating. I wouldn't want to talk to the guys next to me." He also appreciated the fact that some female students were more comfortable with a female GTA than with him and reflected generally about the differences in his and the GTA's roles and authority in class. Overall, Chase was becoming more cognizant of social aspects of the learning environment, moving beyond describing groups' participation, and reflecting on his social responsibilities in the classroom.

Chase talked most frequently about social issues in his GTA interview (37.8% of utterances from this interview were coded as *climate*), indicating an increased focus on the new responsibilities of his role as a GTA. His focus remained strong on his own classroom role (28.6% of actor codes), saying that now as GTA he has a "responsibility to the students to say, you're going to learn this stuff, or I'm going to try my best to have you learn it". He contrasted this with his LA-ing experience, saying that now students "view me as a little bit more authoritative." He reflected on gender equity, saying that students "tended to push back a little more" against his female LA's answers than they would his.

Chase's Pedagogy Talk

The proportion of Chase's talk focused on pedagogy steadily increased across his three interviews (from 34.9% to 37.6% to 45.9%). This indicates that, as Chase gained experience, he focused less on the particulars of the mathematical content at hand and more on the intricacies of his teaching practice. Unpacking the actor and stance codes sheds light on this shift in his noticing of pedagogy.

Naturally, Chase's talk about pedagogy was frequently accompanied by an actor code of *self* across all three of his interviews (80%, 82.9%, and 76.5%). When discussing pedagogy, he foregrounded himself as a teacher, talking about his actions, opinions, or teaching moves (e.g., "I was talking to them, and I was trying to hint again that it would be false", "I was really trying not to be too explicit

about things and trying to get them to make the connections in their own head"). Occasionally, Chase reflected on pedagogy with respect to other actors. For example, in his second LA interview, Chase commented that students' "homeworks were generally better prepared because [course instructor] is very strict on having your homework well presented." This quote shows Chase reflecting on how the actions of other actors: students and the instructor, affected the teaching and learning processes.

Chase's pedagogical stance changed over time. In his first interview as an LA, his talk was highly descriptive (60% of his pedagogical utterances). For example, recalling his interaction with one group, he said: "I was trying to get them to land on the counterexamples that do exist. So, at the end there I stressed the if-then, like the "always-ness" of that implication." Chase occasionally evaluated and interpreted these descriptions, often by hedging them (e.g., "I think I could have landed on this explanation a little bit quicker", "I'm not sure I did any here, but I think earlier I tried to ask them for cases where the statement didn't hold"). These examples also show Chase's reflection on, and dissatisfaction with, his teaching of a particular mathematical content. In his first LA interview, Chase's pedagogical reflections were filtered through his background as a mathematics major.

In Chase's second LA interview, he became less descriptive, turning towards evaluation and interpretation (40% and 34.3% respectively, in the pedagogical category). Chase began connecting mathematics more explicitly to student thinking. For example, reflecting on his teaching approach to an optimization problem where students had to maximize area for a given perimeter, he said:

> If I just tell them to put y in terms of x in the perimeter equation, that's not really motivated by anything. Instead, I can tell "We have two variables here and that's confusing, but we can figure out how to remove one of them and replace it with some function of that other variable." I think that that gives them [students] a way better idea of the underlying [mathematical] machinery.

While Chase was still concerned with the mathematical content of teaching, he now frames this with respect to student learning, realizing that relating to students may be better than explaining to them. Chase also reflected on how his teaching practices changed over his first LA semester, explaining:

> I was trying not to be too explicit about things and trying to get [students] to make the connections in their own heads. Because it's easy to just kind of go into math mode and say, okay, do this and this and this, and I know why, but if they don't know why, the process is very obfuscated for them.

Chase unpacked specific actions as he tried to move away from his "math mode" approach, explaining that he wanted to "ask more questions and better questions… about the concepts themselves as opposed to just telling them concepts." This indicates Chase is learning to center student thinking in his teaching, realizing the importance of students performing mathematics themselves. He also shared that he initially had difficulty approaching students, saying "I was really confused [about] how I talk to these students without them asking me questions?". Gradually, he became "better at interacting with people unprompted and saying how's it going? Sometimes I look at their paper and say hey, can you explain this to me?". By the end of his first semester as an LA, Chase realized the importance of student agency in the learning process and developed techniques for drawing out student thinking.

In his interview as a GTA, Chase's talk about his pedagogy became primarily descriptive, similar to his talk as a beginning LA (64.7% of his pedagogical utterances were assigned a stance code of

describe), but these descriptions were less content-oriented compared to his first LA interview. Instead, he offered descriptions of his classroom practice and how he managed the classroom (e.g. "I break them up in groups of four", "I try to give them maybe not an answer, but a task, and then I'll check in later and say, hey, how'd that task go?"). This is consistent with the shift in his role from LA to GTA and the new responsibilities for managing classroom dynamics. He shared:

> I was an LA under […] a great GTA. And I remember when I started TA-ing, I was thinking a lot, okay, how would [that GTA] do this? And so eventually I did some things differently, but it was cool to see, I'm running the classroom in a way that I admired. It was cool to see that growth.

Summary and Discussion

Across all interviews, Chase's noticing of mathematics shifted from describing the particulars of student work and relating them to his preferred methods to reflecting on mathematics generally across his teaching. His pedagogical noticing became more specific as he moved from describing his actions in the first interview to critiquing his actions and describing classroom management techniques in later interviews. Similarly, his initial descriptive talk about levels of student participation turned to reflection on issues of equity and authority in the classroom. Overall, Chase's pedagogical development from novice LA to experienced GTA can be characterized as a journey from focusing on how *he* himself understood and explained mathematics to focusing on a more integrated view of mathematics and pedagogy, (e.g., how to communicate mathematics effectively, accounting for individual students' attributes). Chase's experience as a high-school teacher before becoming a GTA likely affected this journey; however, we do not have data evidence that specifies this impact.

Our study contributes to the under-researched area in tertiary education of understanding the instructional dynamics in LA-supported classrooms by considering the development of LAs as teaching paraprofessionals. Chase's case is special because he returned to the same university for his graduate studies, which allowed us to trace his long-term professional development. However, he is not unique in being so deeply affected by the LA-ing experience to switch his career to teaching (Nadelson & Fannigan, 2014). While Chase's case cannot be generalized, examining his learning trajectory, especially through the noticing lens (Buchbinder et al., 2021; Dindyal et al., 2021), aids our understanding of how LAs can grow pedagogically, and how LA-ing experience can affect their teaching. The shifts in Chase's noticing of mathematical content in his first interview to reflecting on connections between mathematics and student thinking in his second interview exemplify how LAs' professional noticing can become more advanced over time (van Es & Sherin, 2008). The increased focus on student thinking as a recent learner could contribute to students' understanding of mathematics, as suggested by Wendell et al. (2019).

Our study also offers a methodological contribution of applying generative AI to code transcripts of interviews about mathematics with a promising degree of agreement with human coding. This technologically aided approach was appropriate for use with the noticing analytic lens because noticing is a construct with mature theoretical foundations (as opposed to other areas of study, which may require prior development of coding schemes). We plan to extend this AI-enabled approach to interviews with 18 other LAs to compare their trajectory to that of Chase to gain additional insight into LA noticing patterns and professional learning trajectories.

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Ecology of task design, adaptation, and selection: A phenomenological study with Norwegian mathematics lecturers

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This study aims to investigate the ecology of designing, adapting, and/or selecting tasks for teaching mathematics at the university level by identifying the institutional conditions and constraints that influence lecturers' practice. Drawing on the tenets of the Anthropological Theory of the Didactic, in this phenomenological study data is collected through semi-structured interviews with mathematics lecturers working at a public university in Norway. The findings revealed several conditions at the pedagogy and discipline levels as well as several constraints at the school level. Moreover, instances where conditions and constraints are interrelated were identified. Implications for task design at the tertiary level are discussed.

Keywords: Anthropological theory of the didactic, ecology, task design, mathematics lecturers.

Introduction

Research has investigated task design at the university level from various perspectives. Some studies focus on designing tasks for teaching mathematical topics, such as real analysis (e.g., Gravesen et al., 2017) or multivariable calculus (e.g., Viirman & Jacobsson, 2023), while others emphasize developing skills such as mathematical habits of mind (e.g., Breen & O'Shea, 2019). Despite growing interest, there seems to be a lack of research on factors that explain or influence how lecturers design, select, and/or adapt tasks for teaching mathematics. As Bosch and Gascón (2006) articulated, "when the teacher and the students meet around a knowledge to be taught, what can happen is mainly determined by conditions and restrictions that cannot be reduced to those immediately identifiable in the classroom" (pp. 60–61). It is necessary to consider the conditions and constraints (i.e., ecology) outside of the classroom since they determine activities (i.e., praxeologies) inside the classroom (Chevallard, 2022). This calls for enlarging the unit of analysis, extending it in both directions to encompass more specific and more generic levels (Chevallard, 2002). Since the ecology of a system impacts and explains its functioning, as well as that of praxeologies residing within it (Gascón & Nicolás, 2020), investigating the ecology of task design for teaching mathematics at the university level is important for understanding the system around it and its consequences on lecturers' practice. As such, our study intends to investigate the ecology of task design at the university level by uncovering the conditions and constraints lecturers encounter in their task design practice.

The Anthropological Theory of the Didactic

The Anthropological Theory of the Didactic (ATD) supposes that any human activity can be interpreted as a praxeology, which can further be divided into praxis and logos blocks (Chevallard, 2019). Praxis block refers to the type of tasks and the techniques for carrying out the tasks, while the logos block includes the technologies that explain the techniques and also, theories justifying the technologies (Chevallard, 2019). In this study, we conceptualized the task component of the

praxeology as the practice of designing, adapting, and/or selecting tasks for teaching mathematics. More specifically, it was viewed as a paradidactical praxeology explaining teachers' practices outside of the classroom (Winsløw, 2011). Another important notion proposed within the ATD is called the ecology, referring to "the set of prevailing conditions under which this system lives in actual fact" (Chevallard & Bosch, 2020, pp. xviii–xxxvii). Ecology can be seen as a set of conditions and constraints supporting or impeding the enactment of didactic activities within institutions (Bosch-Casabò, 2018). In ATD, condition is viewed as "anything purported to have influence over at least something" which could "be identified to the state of some system, and therefore to the values of a set of variables that govern this system", whereas a constraint is perceived as "a condition which appears to be unmodifiable by occupants—acting as such—of a given institutional position" (Chevallard & Bosch, 2020, pp. xx-xxi). Chevallard and Bosch (2020) emphasize the importance of identifying the conditions and constraints of an institution (e.g., a university) to explain their influence on the praxeologies (e.g., task design) that are part of a didactic system (e.g., students, university lecturers, mathematics courses) within that institution. They suggest the use of the levels of didactic codeterminancy (LDC) to make this ecology explicit. The LDC, thereby, can be used as a tool to explain the ecology of any didactic organization through the generic and specific institutional levels it offers (Chevallard, 2002). As a tool, the LDC provides researchers a broader perspective to question and study the reality at stake (Bosch-Casabò, 2018). We believe an investigation of such an ecology could help identify significant issues occurring at the different levels of the hierarchy that impact lecturers' task design practice. This research can, therefore, help us to propose more specific solutions to the problems corresponding to a particular level.

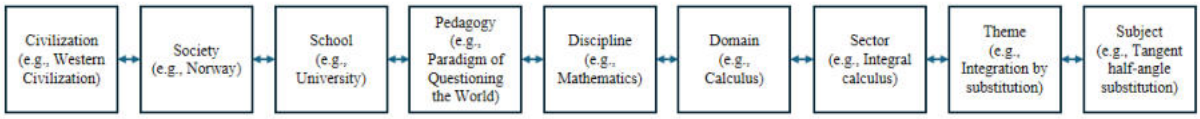

Figure 1: The hierarchy of levels of codetermination with examples, adapted from Winsløw (2011)

In this scale, *society* and *civilization* refer to the highest levels of the hierarchy, with both having their own social norms and values. *School* offers various infrastructures (e.g., buildings, classrooms, and time schedules) for teaching and learning, and *pedagogy* relates to the process of teaching and learning and its organization (Bosch et al., 2021). Align with this view and expanding it further, we perceive conditions and constraints at the school level if they result from or could be addressed by the university stakeholders, such as through setting main policies and regulations. Lower levels structure the *discipline* further into four components (Bosch et al., 2021) (See Figure 1). Against this background, the following research questions are addressed in this study:

> What are the conditions and constraints influencing how mathematics lecturers design, adapt, and/or select tasks for teaching mathematics at the university level? And what consequences do these conditions and constraints have on their task design practice?

Ecology of Mathematics Teaching and Learning at the University Level

Research to date has investigated the ecology of teaching at the university level on various themes. Barquero et al. (2013) studied the ecology of teaching mathematical modeling at the university level

and identified constraints hindering its dissemination across different levels of education. Viirman and Jacobsson (2019) investigated how epistemological differences in the university courses impact the didactical decisions taken by the lecturers and showed that the domain level features influence the decisions made on the pedagogy level. Bosch et al. (2022) designed two study and research paths (SRPs), one in a statistics course at a business administration school and one in an elasticity course for engineering students. One of the constraints they found was the curriculum, which seemed to have an impact on how the questions were generated within the SRPs. Similarly, Hitier and González-Martín (2024) reported the constraints that mechanic lecturers face when working with derivatives in their calculus courses, arguing that lecturers' practices can be accounted for by factors outside of their education and personality. One of the constraints that they found at the school level was the misalignment of mechanics and calculus courses, causing students to attend mechanic courses without having the necessary calculus knowledge and leading mechanics lecturers to adjust their teaching content. Despite the research on the ecology of teaching and learning mathematics at the tertiary level, there seems to be limited understanding of the ecology of task design at this level. Therefore, our study holds a potential for bringing more insights into this understanding, by identifying the issues experienced by lecturers in their task design practice at the university level.

Methods

We employed a phenomenological approach to investigate the ecology of designing, adapting, and/or selecting tasks for teaching mathematics at the tertiary level, considering it a suitable approach for uncovering the experiences of people through in-depth interviews (Creswell & Poth, 2018). Participants were four mathematics lecturers working at a public university in Norway with at least three years of teaching experience. They were selected based on their convenience and willingness to take part in our study. Data was collected by the first author through semi-structured one-on-one interviews as part of her doctoral study. The interview included questions aiming to learn about which principles they consider important in designing, adapting, and/or selecting tasks for teaching mathematics, how they address those principles, and the factors impacting this process. The data we present here focus on the final aspect, aiming to identify the conditions and constraints of task design from the perspective of the ATD. Influenced by the ATD, we conceptualized conditions as situations that lecturers have control over, while we perceived constraints as those beyond their control and cannot be changed immediately. We perceived the LDC in the sense that they have been viewed and defined in the work of Bosch et al. (2021). Their conceptualization of the notions guided our understanding and analysis of our data. The interview transcripts were analysed by the first author at first. Then, the second author analysed 25% of the data to obtain interrater reliability (O'Connor & Joffe, 2020). At the last, we discussed and negotiated our analysis until reaching a full agreement. All names were changed with pseudonyms to maintain participants' confidentiality.

Findings and Discussion

The analysis revealed several conditions and constraints at different institutional levels (see Figure 2). Considering space limitations, here we only present some of the findings that we think are valuable to discuss. Conditions have only been identified at the *pedagogy* and *discipline* levels. One example of a *pedagogy*-level condition was providing individual feedback:

Agnes: We have so many students. We would like to give them […] individual feedback and support them more, but that is impossible. So, we spend a lot of time thinking about tasks that work, that they can do on their own. […] And it [designing tasks that they can work individually] is more important because of lack of resources. If we could follow them closer, I would also make more difficult problems that maybe they do not have a perfect solution.

Agnes expresses the difficulty of offering feedback to each student due to the number of students in her courses, which causes her to design relatively easier tasks that students can work on by themselves. Her concern shows that her task design practice is influenced by the classroom size, something beyond her control. This situation also illustrates how a *pedagogy*-level condition is influenced by a *school*-level constraint. Among others, one suggestion for overcoming such problems could be using peer feedback (Nicol, 2010). Nicol (2010) suggested that feedback should be viewed as a dialogic process, not just a product transferred from teacher to student. This approach allows students to provide feedback to each other and observe peer work, encouraging critical reflection on their own efforts (Nicol, 2010).

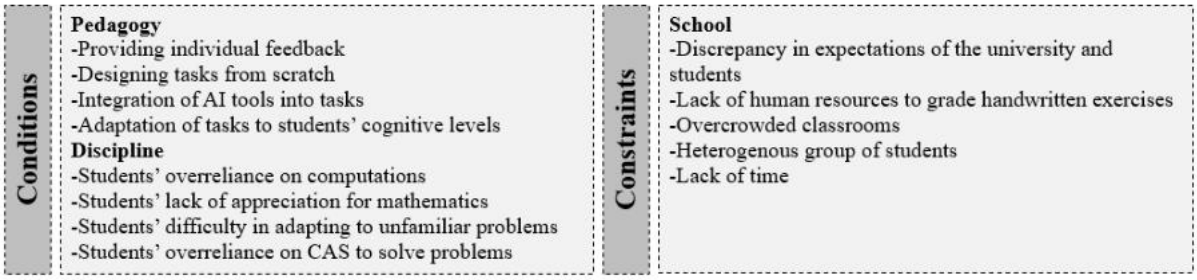

Figure 2: The ecology of task design at the tertiary level

Another condition at the *pedagogical* level was the difficulty of designing tasks from scratch:

Christer: [I use] internet resources or books or things like that because it is really hard to make problems. […] I make some problems myself. I think that sort of making everything yourself and not drawing upon all you know [is difficult].

Christer expressed difficulty in designing tasks from scratch, which often leads him to modify and adapt existing tasks from different resources. He did not mention this difficulty in the context of any constraints influencing his practice; rather, he discussed the challenge of creating problems. Here, it is shown that a *pedagogy*-level condition, although not impacted by a constraint, influences the task design practice of the lecturer. One possible solution to overcome this condition could be to involve different communities in design and implementation of tasks, such as in the form of cooperative engineering (Sensevy et al., 2013). As Sensevy et al. (2013) pointed out, design-based research involving the joint work of teachers and researchers—in this case, mathematics lecturers and educators— enables them to discuss their design and teaching strategies, share their expertise, and translate it into a collaborative knowledge. The collaboration between mathematicians and mathematics educators have been found effective for improving the teaching and learning of university mathematics (e.g., Hernandez-Martinez et al., 2023). The same could also be true for mathematicians. Collaboration among mathematicians could enable them to discuss their design and teaching practices where they could find practical solutions to their common problems.

At the *pedagogy* level, another condition that emerged was the difficulty in integrating artificial intelligence (AI) tools into task design. Important to note is that this challenge could be experienced either at the *discipline* level or at the *pedagogy* level across disciplines. However, from the interview, it seems Christer experienced this at the *discipline* level:

> Christer: If you are going to use this tool [CHATGPT4], it [the task] has to be sort of carefully created, and students have to learn how to [use CHATGPT4], not just use it to get like a copy-paste answer to [their questions] […] but to actually use it to improve their own problem solving and […] construct their own intuition and knowledge […]. But I do not know how to do that.

Christer expressed his thoughts about the need to create well-designed AI-based tasks that would help improve students' problem-solving skills, intuition, and knowledge. He added that he was not sure how to design the kind of tasks that would help him achieve this goal. A recent literature review by MA et al. (2024) argued that there is still a need for research on integration of AI tools into education and identifying effective implementation strategies. In this respect, our findings highlight the importance of investigating this issue at the tertiary level.

At the *discipline* level, several conditions have been mentioned. One issue was students' overreliance on computations and being inattentive to meaningful understanding of mathematics:

> Lukas: They [students] are so focused on solving equations or whatever tasks we give them, but they have no feel really for what it truly means for a function to be continuous or something like that. So, one design principle that we often use is you [students] need to verify something using the definition only. This is very instructive because it means that they really have to think about what the definition says, how you can operate with it, and apply [it] to whatever problem […] at hand.

Lukas perceives that students often prioritize applying procedures over conceptual understanding, which makes him design tasks that require students to use mathematical definitions. This situation demonstrates that a discipline-level condition impacts the lecturer's approach to task design. Our results concur with Breen and O'Shea (2019), who designed a set of tasks for undergraduate level introductory calculus courses, aiming to foster students' habit of mathematical mind and challenge their perception of mathematics as a discipline that solely relies on the application of procedures. Their work proposes six types of tasks that were found to be helpful for this purpose, which are based on example generation, analyzing reasoning, evaluation of mathematical statements, conjecturing and/or generalizing, visualizing, and using definitions.

At the *school* level, one of the constraints was the gap between the expectations of diverse members of the institution:

> Edvard: […] There is often a gap between […] what the students think they should learn and what university or people design the programs think they should learn. And I think it is very important to find a balance there […]. I usually do not like to teach to the test […] [but I give them] exercises [that] will be asked in the exam. We always want them to understand the concepts. So […] a lot of what I do when I choose exercise is sort of concept-driven […]. Sometimes, it is worth doing that [standard procedural tasks], but I do not see sense in […] drilling […] 10 exercises.

Edvard highlighted the difference in the expectations of lecturers, universities, and students, emphasizing the importance of finding a balance to meet those expectations. While he prefers not to

focus solely on preparing students for exams, students typically expect to be well-prepared. As such, he designs tasks in a way that would be similar to the exam so that students' expectations of being ready for the exam would be meet. In alignment with the expectations of the study program of the university and him as a lecturer, he values focusing more on conceptual learning than solely teaching the subject procedurally. A literature review by Struyven et al. (2005) demonstrated that students' preferences of the type assessment determine their learning approaches and vice versa. Therefore, gaining insight into students' assessment and learning preferences is important for designing exam and classroom tasks that meet both students and university expectations.

Another constraint at the *school* level was the lack of tools or human resources for assessing students. For instance, Agnes talked about this constraint in relation to another constraint she experienced:

> Agnes: We do not have the resources to grade handwritten exercises […] This is automatically graded. So that puts […] a limit on what we do. We would like them to also write […] on their own and on a computer. So, like bigger more in-depth problems, and then they could be graded and like getting feedback, but we do not have that. Right now, problems where you need to show reasoning in a better way are not part of this course because we do not have the resources. We cannot grade things [conceptual tasks], so […] more focus [is] on getting to the correct answer.

Agnes lacks the necessary resources for assessing students on more conceptual tasks. Due to this constraint, she is inclined to design more result-oriented tasks. This tendency can also be attributed to the large number of students enrolled to her courses, which affects how she designs tasks and influences her formative assessment of students learning. E-assessment solutions could provide an alternative approach for addressing grading challenges (e.g., Sikurajapathi et al., 2020).

Concluding remarks

This paper investigates the ecology of task design at the tertiary level by identifying the conditions and constraints that lecturers experience when designing, adapting, and/or selecting tasks for teaching mathematics in their courses. As findings showed, there are several conditions at the pedagogy and discipline levels as well as several constraints at the school level of the hierarchy that influence lecturers' practice of task design. One conclusion is that there were instances where a condition was influenced by a few constraints. This situation highlights the complexity of task design, suggesting that effective task design calls for the careful consideration of a range of factors and their potential influence on this practice. Several recommendations are presented in the discussion to address the constraints identified in this study. Finally, the available data provided limited insight into the lower levels of the hierarchy in the ecology of task design. This limitation may be related to the scope of our inquiry, as the study was not explicitly designed to investigate the challenges lecturers faced when designing tasks to teach a particular mathematical topic; thereby, the interview questions did not prompt the participants to delve into this matter in greater depth. Furthermore, care should be taken when interpreting the findings due to small sample from a single Norwegian university.

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Engineering students' experience with the vector cross product in a virtual reality environment

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This paper explores the integration of virtual reality (VR) and problem-based learning (PBL) to enhance engineering students' understanding of the vector cross product in three dimensions. Our design immerses student groups into 3D virtual environments where they collaborate on manipulating vectors representing features of spatial objects. A task is, e.g., that at the outset, the vectors are assigned incorrect values resulting in distorted objects. The setup is tested iteratively at two universities in two different countries. Data are collected through questionnaires. The focus is on students' experiences and perceptions of VR-enhanced learning. Mostly, the students reported positive experiences using the VR environment. It helped them understand the concept of the cross product, see its use in real life, and it was enjoyable. However, some also reported motion-sickness or asked for more control to draw their own vectors.

Keywords: Engineering mathematics, virtual reality, vector cross product, problem-based learning.

Introduction

Mathematics for engineering students and problem-based learning

It is well-known by university teachers that engineering students often find mathematics difficult, and they have little motivation to learn it. This is supported by research showing that engineering students do not develop the mathematics concepts the same way as mathematics students do, and the concepts do not mean the same to them (Maull & Berry, 2000). Bingolbali et al. (2007) found that engineering students see mathematics as a tool and wish to see how mathematics is applied as part of a course. Baillie and Fitzgerald (2000) showed that engineering students who are at risk of demotivation and drop-out, list the level of mathematics and theory as some of the issues. Hence, as argued by Pepin et al. (2021), integrating real-life problems may help engineering students see the use of mathematics.

Christensen et al. (2023) documented a change at Aalborg University (AAU) in Denmark to the first-year mathematics courses in Linear Algebra and Calculus for all students in the STEM-areas by including field specific problems. The study revealed that highlighting the applicability and importance of mathematics in engineering appears to help. However, motivating students to learn mathematics is one thing, understanding mathematics is another. AAU applies problem-based learning (PBL) in projects throughout all curricula and faculties. The AAU PBL principles state that the 'problem' is open and the point of departure of the collaborative project (Askehave et al., 2015). PBL is very beneficial to engineering students (Sukaché et al., 2022), and the usefulness of PBL in mathematics is established at course level using real-life problems (Abou-Hayt et al., 2022).

However, as Dahl (2018) argues, mathematics is an extensive body of abstract knowledge, where the relevance of which to practical, real-world problems is not always immediately evident. This discussion was further developed by Dahl et al. (2023), who argued that both internal and external contextualization can be used in PBL in mathematics projects. In this paper, we therefore apply PBL as the overall theoretical framework for designing the tasks.

Using virtual reality to micro-PBL mathematics concepts

Virtual Reality (VR) is both a learning environment, a motivational tool, and a social medium. As stated above, mathematics education faces challenges due to its abstract nature but also the prevalence of mathematical anxiety (Dowker et al., 2016). However, VR technology offers promising solutions to these issues. VR creates immersive learning environments where students can interact with mathematical concepts. This technology enables visualization of abstract ideas like geometric shapes and functions, enhancing comprehension and problem-solving skills. Thus, VR helps making the inherent abstract world of mathematics concrete. In higher education, VR is already used in medical education (Davidsen et al., 2022). In addition, Timcenko and Thomsen (2023) describe how, using virtual reality, one can expose engineering students to tangible representations of abstract mathematical concepts in support of both students' motivation and their retaining of mathematical knowledge. Moreover, VR can make mathematics engaging by connecting concepts to real-world applications and incorporating them into gaming experiences. It fosters collaborative learning and communication among students and educators, adapting to individual needs. Despite its potential, VR also presents challenges such as accessibility and content quality. Educators need proper training and support to effectively integrate VR into the curriculum and assessment processes.

In conclusion, VR technology offers a transformative approach to mathematics education, addressing the limitations of traditional teaching methods and fostering deeper understanding and engagement. The focus of this study is therefore on how VR can be used to help engineering students understand an otherwise abstract and "invisible" mathematical concept. Hence, we apply the term "micro-PBL" as PBL, in this paper, is used neither to a large project, or a whole course, but to a single concept.

Research question

In this paper, we address the research question: How do engineering students in different countries experience the cross product of two vectors through a virtual reality setup in a micro-PBL setting?

Methods

Which mathematics and in which setting?

This paper is part of the project *myPBL-VRMath* where we focus on visualization and interaction with complex mathematical objects in VR in a PBL environment. The cross product of vectors in $\mathbb{R}^3$ is used as the first example as this is a vector operation essential for understanding direction and magnitude in space. Cross products are used in computer graphics (e.g., computing surface normal), thus representing a crucial tool in 3D graphics visualization, and in physics to describe angular momentum and many other physics phenomena. Given two vectors $u, v \in \mathbb{R}^3$, their cross product $w = u \times v$ is defined as a vector satisfying that w is orthogonal to both u and v; its length satisfies $|w| = |u||v|\ sin\ \theta$ with θ being the angle between u and v; and u, v, and w follows the so-called "right-

hand rule" (Adams & Essex, 2022). One may show that this defines a unique vector, and that the cross product may be computed as $(u_1, u_2, u_3) \times (v_1, v_2, v_3) = (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1)$. Thus, the cross product gives a third vector that is orthogonal to the first two. The length of the cross product is determined by the area of the parallelogram spanned by the input vectors.

We designed a VR PBL environment in which the students are placed in a landscape, Vector Island, where, e.g., they see a sculpture where some facets are wrong because the surface normal is inverted. The students should grab the vectors, using the game controller, and manipulate their positions and magnitudes, observing the instantaneous change in the environment caused by the changed mathematical objects. In this paper, we report a piloting of this VR environment with different types of engineering students in two countries. The testing is done sequentially, in line with principles of design-based research (DBR) where instructional tools are tested, leading to refinements of the design (Sandoval, 2014). Ethical approval has been obtained prior to doing the activities with the students.

Evaluating the students' experiences

We developed a questionnaire to be given after the activities asking about the students' perceptions and experiences. The development was inspired by the Technology Acceptance Model (TAM; Davis, 1985) that explores the acceptance of a new technology using two constructs: perceived usefulness (Q1, Q2, Q5) and perceived ease-of-use (Q3, Q4). Q6 (Recommendation) aligns with TAM's focus on behavioural intention to use technology since recommending the technology to others can reflect a positive experience and intention to use.

- Q1: Usefulness of the activities for own progression of learning
- Q2: Extension of the perspectives over the vector cross product, due to the use of the technology
- Q3: Joy-factors, motivational factors
- Q4: Ease of use of the VR-technology
- Q5: Wish of having VR as a part of several courses during their education
- Q6: If they would recommend other students to try the activities and why
- Q7: Suggestions for improvement of the activities.

The quantitative questions are formulated as 5-point Likert type items (part "a" of questions Q1-Q6). In addition, for each question, we asked the students follow-up questions (parts "b" and "c"), where they should elaborate the answers using own words, thus providing us with qualitative data.

Setup of the tasks

The worked in groups of 3-5 students, where one has a headset, and the others see projections of the environment on their PC flatscreens. The activities are, therefore, designed to encourage dialogues in line with the PBL principles. The student with the goggles sees the 3D world, Vector Island, and can see and manipulate (grab) the vectors that determine the 3D world. Only the students without the goggles can see the underlying formulas needed to compute the cross product, which determine what is rendered in the VR environment. This will make it essential for the students to collaborate as no one has all the information. The students take turns in wearing the goggles.

- If a person in VR changes some of the vectors by grabbing them, the formula and numbers on 2D screens (visible to the other group members) are changed.
- If a change is made from outside the 3D world via an ordinary computer by the other group members, the person in VR sees that immediately.
- The possibility of severe input errors (for instance, caused by a novice VR user performing unexpected moves) is kept to a minimum, i.e., the software checks the input and allows only logical changes so the environment cannot be distorted beyond recovery.

Task 1 – Explore the sun positioning tool

The written material guides the students to: "Enter the virtual world and experiment with the Sun positioning tool. What do you notice? How do the shadows change based on the Sun's position? Try moving yourself slightly (just a little, so you don't get lost!). Observe the statue closely—do you notice anything unusual?" The students should experiment with the tool to move the sun, hence observing a change of the shadows on a sculpture. This task serves as a gentle introduction to the VR environment and using the hardware controllers.

Task 2 – What might be wrong with the triangle and how to fix it?

The written material guides the students to learn that: "An important thing to know is that in Computer Graphics, the objects are made up of triangles. The surface normal of these triangles determine what we can see and how we see it. We can only see the front sides of surfaces, as their normal point towards us. So, how do we (or the computer) calculate the surface normal? That's where our friend, the cross product, comes into play!" The task builds upon Task 1 and the students collaborate on finding a way to fix the sculpture using among other things the right-hand rule.

Task 3 – Fix the problems in the town!

Task 3 is a variation of Task 2 to help the students consolidate their knowledge. However, this is a more open problem where students begin by looking at a picture of a town with many kinds of buildings, mountains at the back, etc. The students should explore and discuss what may be wrong here. This task is designed with inspiration from PBL, asking students to find and solve problems, albeit small (micro) problems. The task is formulated as: "Go down the crossroad. See if you can spot any problems. Observe, talk and act".

Task 4 – Alien attack!

In order to enhance the students' understanding of the cross product, we added a task where the students should not fix something wrong but instead go the opposite direction and create a mistake, a hole, in an otherwise perfect surface. We therefore invented an alien attack, where the students should seek hide from a UFO. The written material stated the following: "Quickly run to safety - The only safe place is under ground. Check the alley between the pizzeria and cafe, to see if there is a way to create a hole to jump in". And later: "Hint: In previous tasks you have learned how to fix the holes in geometry, but now, you need to find a way to make a hole!" See Figure 1 below.

Figure 1: UFO attacking a town on Vector Island

The test sites and the first step of the analysis

Aalborg University as the first testing site – piloting the tool

AAU was chosen as the pilot testing site in the DBR inspired setup as this was where the development of the VR environment took place. This was done before sending the VR tool to other partners. The tool was in a prototype phase, but we needed to catch and fix critical issues, increasing the likelihood of success in broader, out-of-home testing scenarios. The pilot took place during the course "Dynamics and Vibrations" at the 2nd semester of the engineering study programme "Sustainable Design". In the first semester, the students had introductory courses in mathematics, statics and strength of materials. In the course where the test took place, the cross product of vectors was crucial for understanding basic concepts such as angular velocities and angular accelerations of rigid bodies, which are vector quantities in nature. There were more than 10 students present in the class, 7 filled in the consent form, 4 tried the goggles, other observed, several were very engaged in a dialogue, and 4 answered the questionnaire. During this testing, an unexpected bug was detected as well as a need to have written material (mentioned above) with detailed task descriptions for students outside the VR. Those issues were fixed for the second test at Karlstad University (Sweden). However, even this first test on the prototype gave us encouragement that we were on the right track, as can be observed from the selected student comments below:

- I understood it before the VR, but I definitely think that I would have understood it even faster if I had been introduced in VR for it, the first time I had to learn about it.
- I think it is a really nice variation to more traditionally taught ways. It was a cool activity to open up for discussions between students, especially with potential in making small groups.

Karlstad University as the second testing site

At Karlstad University, we tested with 2nd semester civil engineering students, directly after they had finished their first Linear Algebra course. The students participated voluntarily in the activities and no examination or credits were connected to the process. They were first introduced to the study and after that, the students worked together in small groups. We had two groups (4 + 5 students) chosen by taking in consideration their experience of using VR. The activity took about 30-40 min per group. This involved both group cooperation and individual testing for all students in all four tasks. After all activities were finished, the students completed the questionnaire individually. In the rest of this paper, we focus on the analysis of this phase as the VR environment was at a more advanced stage than when piloted at AAU, but we needed the pilot as part of the process. Nine students are not enough to generalise. However, the test serves a purpose of being a step in an iterative DBR-process.

Results of test at Karlstad University

Analysis of quantitative data

Although nine answers are not enough for statistical analysis, a descriptive illustration of their answers should be shown. They clearly lean towards positive attitudes towards our VR tool.

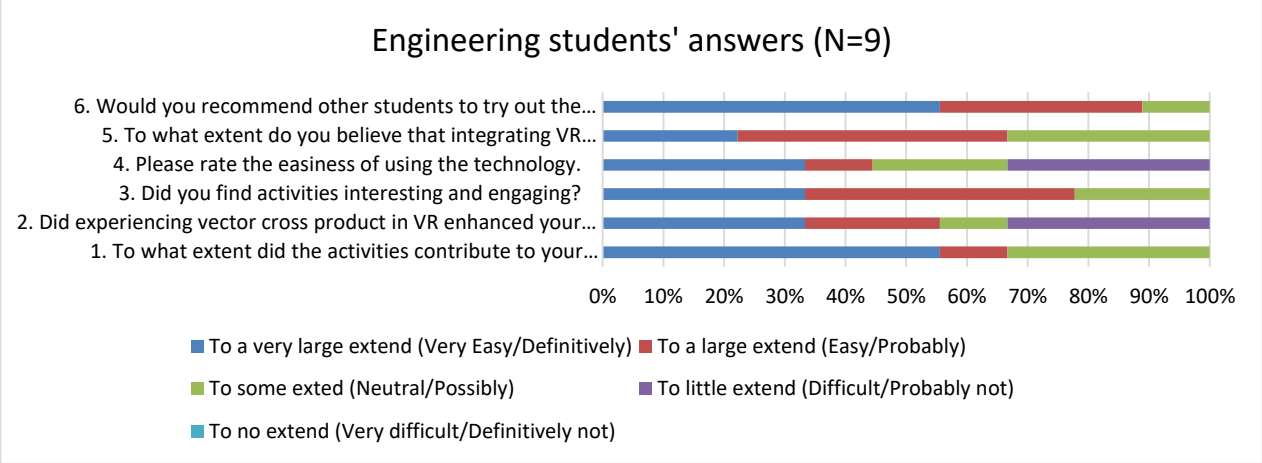

Figure 2: Engineering students' answers at Karlstad University. Questions Q1-Q6 are also seen above

Analysis of qualitative data

We used a data driven approach to analyse the qualitative data since our use of VR is a novel type of activities. As the first step in this analysis, we used ChatGPT to increase the pace of the analysis but also the explore the usability of ChatGPT for such analysis on a relatively small amount of data. Three authors (not from Karlstad University) independently checked ChatGPT's evaluation for mistakes and hallucinations. We found that, mostly, ChatGPT gave valid summaries. However, in Question 1b, asking how the activities (failed to) contribute to their understanding, ChatGPT did not include that a student wrote that the activities were "very funny". Here, "funny" was used in a context of "amusing" not "silly", seen from the context, but ChatGPT might have been confused and then omitted it. Hallucination was seen in the summary of Question 6c, asking if the activities were valuable. The students provided suggestions for significantly more complex mathematics material, while ChatGPT presented it as if this complex material was already present.

In terms of helping students develop understanding of the cross product, the students were positive and experienced the activities as helpful, particularly among students who had not yet fully understood it, while others felt they did not gain any new insights. Particularly the visualisation and interaction with the vectors contributed significantly to their comprehension. This visual approach made complex concepts clearer and more intuitive, which was seen as a significant enhancement compared to traditional methods. Many students also found the interactive and immersive elements of the VR activities enjoyable. The interactivity of the VR environment and the ability to visualize concepts in 3D were strong motivators. In addition, the ability to provide a "big picture" view, showing how mathematical concepts apply in real-life made the abstract mathematics concept more tangible and easier to grasp. Many also stated a wish for more control over the activities to, e.g., draw own vectors, change their sizes, and manipulate them further to see how these changes affect

outcomes. Some students found usability issues and asked for a more intuitive interface. Several students reported experiencing dizziness during use, particularly related to the movement system. Some suggested that teleportation as a movement method could reduce this issue. Some felt that traditional virtual environments or software tools like GeoGebra or Desmos 3D might be more practical and sufficient for certain educational purposes.

Discussion and conclusion

Overall, the results of the testing are promising as students mostly report positive experiences using the VR environment. It helped them understand the concept of cross product, see its use in real life, and it was enjoyable. The observation that students requiring additional time to comprehend mathematical concepts benefit more from VR-based learning environment is unsurprising, given the substantial variation in mathematical proficiency among students. Rather, it underscores the potential of integrating VR technology in mathematics education to assist these students in keeping pace with their quicker-learning peers, ultimately accelerating the overall progress of the class. There are some drawbacks such as motion sickness, and some issues of usability. Motion sickness may be solved by insisting that a student only wears goggles for one task at the time, then rotates. We got ideas for further development such as giving students who may already have a good understanding of the cross product more freedom to play with the vectors. One of the primary limitations of VR integration in education is the significant cost of the equipment. However, since VR is already used in, e.g., medical education and in engineering design, and with anticipated reductions in VR equipment costs in the near future, this research may soon become relevant to a broader range of educational systems. A next step in our project, *myPBL-VRMath*, is to also create VR environments based on rank, eigenvectors and eigenvalues of matrices and explore implementation in courses and with more students as it is a further limitation that our study focused on small groups of students. Building upon the experiences of these two tests, the next step is to do a testing at Azrieli College of Engineering.

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Investigating undergraduates' proof methods in linear algebra

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Research on student reasoning in linear algebra is extensive, yet few studies focus on linearity and proof (Stewart et al., 2019). This study addresses this gap by examining students' proof methods for a mathematical statement on linear dependence. Data from 54 undergraduates in an Applied Linear Algebra I course were collected using an instructional sequence on Desmos. The sequence introduced intuitive and formal definitions of linear dependence and independence through examples and informal reasoning, progressively building toward formal definitions. Students' proof methods were analyzed and categorized using Stylianides' (2008) framework of reasoning and proving. Most students proved the mathematical statement using deductive, rationale-based, or empirical proof methods, each of which includes various subcategories with distinct characteristics, while some exhibited unclear reasoning or did not respond to the question.

Keywords: Undergraduates, linear algebra, linear (in)dependence, reasoning, proof methods

Literature review on the concept of proof and students' reasoning on linear independence

The process of constructing arguments that meet the standards of a proof is often the final stage in a complex and multifaceted activity known as proving. This activity includes several other processes, including working with examples or exploring specific cases, recognizing patterns, and formulating or refining conjectures or other types of mathematical claims. Additionally, it involves developing informal arguments for these claims that may provide insight or ultimately translate into a proof (Stylianides, 2019).

In advanced university mathematics courses, writing proofs is a key goal (Weber, 2001). Proving is an activity that students are expected to engage in, and it is also a common form of instructional explanation that professors provide for their students, as well as the predominant way mathematics is communicated in university classrooms (Mejia-Ramos et al., 2017). As a foundational course, linear algebra provides students across various disciplines with an early introduction to formal definitions and proofs (Carlson et al., 1993; Stewart et al., 2019). As such, it presents an ideal context for investigating students' proof methods. In recent years, student reasoning in linear algebra has been a rich area of research, but relatively few studies have focused on linearity and proof in this area (Stewart et al., 2019). Linear independence is a challenging concept for students, yet mastering it is crucial for understanding other topics in linear algebra (Stewart & Thomas, 2010). Some research done on student reasoning about linear independence indicates that students tend to lean on procedural approaches, such as using pivots to identify the presence of free variables in reduced row echelon forms, for determining if a set is linearly independent (e.g., Payton, 2019; Stewart & Thomas, 2010). Other research suggests that students can also develop many ways to reason about linear independence geometrically in carefully designed learning or problem-solving environments (e.g., Dogan-Dunlap, 2010; Wawro et al., 2012).

In this study, we addressed the question: What are students' proof methods for linear dependence in the context of a Desmos instructional sequence aimed at supporting their proving activity? To explore this, we developed and implemented an instructional sequence on Desmos focused on linear dependence and independence. Desmos is a free web-based platform for interactive mathematics lessons that offers scalability and accessibility, allowing participants in this study to easily access the instructional sequence activities by clicking a link. Additionally, Desmos facilitated the documentation of students' written work, which was used for the analysis in this study. The

development of the instructional sequence was guided by the Authentic Mathematical Proof Activity (AMPA) framework proposed by Melhuish et al. (2022), which includes components of proof comprehension, validation, and construction. This sequence was carefully designed using examples that engaged students in proof activity, aiming to scaffold their understanding of linear dependence and independence through both intuitive and formal definitions while avoiding outside knowledge, such as row reduction techniques. The instructional sequence consisted of seventeen slides: one outlining the learning goals, three focused on proof construction, two introducing definitions, and eleven dedicated to proof comprehension and validation. This study specifically examined a mathematical statement on linear dependence presented toward the end of the instructional sequence, which required participants to prove: *If a set* $S = \{v_1, \ldots, v_p\}$ *in* R^n *contains the zero vector, then the set is linearly dependent.*

Conceptual framework

In conceptualizing students' proof methods, we draw on the mathematical component of Stylianides' (2008) analytical framework of reasoning and proving (Figure 1). This component categorizes supports for claims into two main types: providing a proof and offering a non-proof argument. Providing a proof is further divided into generic example and demonstration. For this study, we only identified demonstrations in our data; we refer to demonstration as a deductive argument in this manuscript. Stylianides defined a *demonstration* as a valid argument based on accepted truths for or against a mathematical claim. An "argument" refers to a connected sequence of statements. "Accepted truths" is a broad term that encompasses axioms, theorems, definitions, modes of reasoning, and representational tools that a particular community may take as shared at a given time. The meaning of 'valid' in the definition of proof should be understood based on current mathematical conventions.

Stylianides' (2008) non-proof argument category includes empirical argument and rationale-based argument. In our analysis, we found the distinction between empirical arguments and rationale-based arguments in students' reasoning. An *empirical argument* offers inconclusive evidence for the truth of a mathematical claim by checking only a subset of all possible cases covered by the claim. A *rationale* is defined as an argument that is neither proof nor empirical. It either does not explicitly reference key accepted truths used in its reasoning within the context of a particular community or uses statements that fall outside the community's set of accepted truths. This construct was both useful and relevant to our work as in our data set, students were instructed to use definitions in their proofs; some drew on other statements as the basis for their argument.

<table>
<tr><td></td><td colspan="4">Reasoning-and-proving</td></tr>
<tr><td rowspan="3">Mathematical Component</td><td colspan="2">Making Mathematical Generalizations</td><td colspan="2">Providing Support to Mathematical Claims</td></tr>
<tr><td>Identifying a Pattern</td><td>Making a Conjecture</td><td>Providing a Proof</td><td>Providing a Non-proof Argument</td></tr>
<tr><td>• Plausible Pattern
• Definite Pattern</td><td>• Conjecture</td><td>• Generic Example
• Demonstration</td><td>• Empirical Argument
• Rationale</td></tr>
<tr><td>Psychological Component</td><td colspan="4">What is the solver's perception of the mathematical nature of a pattern / conjecture / proof / non-proof argument?</td></tr>
<tr><td>Pedagogical Component</td><td colspan="4">How does the mathematical nature of a pattern / conjecture / proof / non-proof argument compare with the solver's perception of this nature?
How can the mathematical nature of a pattern / conjecture / proof / non-proof argument become transparent to the solver?</td></tr>
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Figure 1: Stylianides' (2008) framework

Study context and methods

This research took place within the context of a 4-credit introductory applied linear algebra I class, at a public Southeastern Predominantly White Institution (PWI) university in the United States.

This linear algebra class followed a face-to-face format conducted three days per week throughout the 15-week duration of the Fall 2023 semester. The course enrollment included 64 students, with 54 participating in this study, from a variety of applied majors. Class time frequently featured students working on problem-solving activities in small groups; many problems were taken from the Inquiry-Oriented Linear Algebra (IOLA) project (Wawro et al., 2013). The instructional sequence on Desmos on linear independence and independence was assigned as part of students' class work, and they worked on this activity at home.

Stylianides (2019) noted, proving is a multifaceted and complex process, with the proof serving as the final outcome of this activity. In this study, we focus on the analysis of students' reasoning argument to a mathematical statement and particularly on the final product of individual proving activities. During this process, each participant was first assigned an ID from P001 to P054, and their raw responses to the mathematical statement: *If a set* $S = \{v_1, \ldots, v_p\}$ *in* R^n *contains the zero vector, then the set is linearly dependent* were analyzed. The analysis and categorization of proof methods was conducted using the mathematical component of Stylianides' (2008) analytical framework for reasoning and proving but adapted to the context of this study as follow:

Deductive Proof Method: A student' argument was classified as a deductive proof method if it demonstrated that a set containing the zero vector is linearly dependent by appealing to valid mathematical principles emphasized in the instructional sequence. These arguments were grounded in formally accepted truths, such as the definition of linear dependence and properties of the homogeneous vector equation (HVE) $c_1v_1 + c_2v_2 + \cdots + c_pv_p = \vec{0}$. Within this category, students' responses varied in the formality of their reference to the HVE, which resulted in three subgroups: those who used mathematical notation, those who referred to the HVE verbally, and those who did so implicitly.

Empirical Proof Method: A student's argument was classified as an empirical proof method if they demonstrated that a set containing the zero vector is linearly dependent by relying on specific examples. These arguments typically involved selecting a particular set of vectors, one of which was the zero vector, and constructing a non-trivial solution to the homogeneous vector equation. Within this category, two subgroups were identified based on how students referenced the homogeneous vector equation: one group used mathematical notation explicitly, while the other described the equation's structure and logic through verbal explanation.

Rationale-Based Proof Method: A student's argument was classified as a rationale-based proof method if they demonstrated that a set of vectors containing the zero vector is linearly dependent through reasoning that either did not make explicit reference to accepted mathematical truths emphasized in the instructional sequence or relied on ideas outside that instructional sequence. These arguments did not meet the criteria for deductive or empirical reasoning and were therefore categorized as rationale-based.

Unclear Reasoning or Non-Response*:* Some responses did not fit into any of the categories above. These were labeled as *unclear reasoning* when the explanation lacked interpretability or as *response does not answer question* when the student merely restated the definition of linear dependence or addressed a different mathematical statement without proving the one posed in the prompt.

Findings

The following findings represent how students responded to the mathematical statement: **Prove that if a set $\mathbf{S = \{v_1, \ldots, v_p\}}$ in $\mathbf{R^n}$ contains the zero vector, then the set is linearly dependent.**

Deductive proof methods

Of the 54 participants, 18 constructed arguments using deductive reasoning, invoking general mathematical principles. These students demonstrated understanding of the Homogeneous Vector Equation (HVE) and the formal definition of linear dependence. Their approaches varied in the

specificity and formality of how they expressed this reasoning. I categorized their arguments into three groups based on how they referenced the HVE: 1) explicitly using mathematical notation, 2) explicitly in verbal form, and 3) implicitly in verbal form.

Explicit Reference to the Homogeneous Vector Equation with Mathematical Notation. Seven students made direct and formal use of symbolic notation. These students wrote out the homogenous vector equation ($c_1v_1 + c_2v_2 + \cdots + c_pv_p = \vec{0}$, and clearly stated that a solution exists where at least one scalar coefficient is non-zero. Their arguments were deductive in structure and reflected formal understanding of the mathematical definition of linear dependence. Students in this group explicitly referenced both the equation's structure and its logical implications. For example, P003 stated:

> "The formal definition of linear dependence is that a set of vectors is linearly dependent if there exist weights, not all zero, such that $c_1v_1 + c_2v_2 + \cdots + c_pv_p = \vec{0}$. If a set contains the zero vector, then any real number used as a scalar will still result in the zero vector. Therefore, you can scale every other vector by 0 and scale the zero vector by any non-zero number and get 0 as an answer to the equation every single time."

Explicit Verbal Reference to the Homogeneous Vector Equation. Seven students described the logic and structure of the Homogeneous Vector Equation (HVE) without using mathematical notation. These students used clear verbal reasoning to outline how a linear combination of vectors could result in the zero vector when at least one scalar (typically for the zero vector) is non-zero and the rest are zero. Although these students did not write the equation symbolically, they demonstrated a precise grasp of how the HVE supports a deductive proof. For example, P011 argued:

> "It is always linearly dependent because you can always find at least one nontrivial set of coefficients that satisfies the linear dependence condition. What I mean by this is you can assign values of 0 to the vectors that aren't the zero vector. When you do this, you can assign an infinite number of values to the cp of the zero vector.

Implicit Verbal Reference to the Homogeneous Vector Equation. The remaining four students conveyed deductive reasoning in their arguments without explicitly naming or describing the Homogeneous Vector Equation structure. These students explained that the zero vector can be multiplied by any non-zero scalar while the other vectors are multiplied by zero, leading to a non-trivial solution. Although the homogeneous vector equation was not directly cited, their language reflected a conceptual understanding of its role in proving linear dependence. For example, P010 claimed:

> "If a set contains the zero vector then that vector is free because it can be multiplied by any scalar without changing. This means there must always be a non-trivial solution because it can be multiplied by any scalar while the other vectors are multiplied by 0."

Empirical Proof Methods

Of the 54 participants, 12 students used empirical arguments to show that a set containing the zero vector is linearly dependent. These arguments relied on specific examples, where students selected a set including the zero vector and constructed a non-trivial solution to the corresponding homogeneous vector equation. Within this group, two types of reference to the equation emerged: explicit use of mathematical notation and explicit verbal description.

Explicit Reference to the Homogeneous Vector Equation (with Mathematical Notation). Five students made explicit reference to the homogeneous vector equation using mathematical notation. They wrote out the equation of a form $c_1v_1 + c_2v_2 + \cdots + c_pv_p = \vec{0}$ and provided specific examples showing that not all scalars were zero. Their reasoning was anchored in the formal structure of the equation, even though their proofs were grounded in particular cases rather than general reasoning. For instance, P053 stated:

> " Given $\{v_1, v_2, v_3\}$, where v_3 is the zero vector, we can write the homogeneous vector equation as: $c_1v_1 + c_2v_2 + c_3v_3 = (0, 0, 0)$. We can set c_1 and $c_2$to be equal to 0, and set c_{13} to be any real number, and this would give us a nontrivial solution to the HVE. Similarly, any vector set that contains the zero vector can have a nontrivial solution. By definition, a set of vectors is linearly dependent if is has a nontrivial solution, so any vector that contains a zero vector is linearly dependent."

Notably, one student, P005, in this group went further by assigning specific values to each vector in the set. They stated:

> "Consider a vector set $(1, 1, 1), (3, 2, 1), (0, 0, 0)$. The set is linearly dependent because $0\big((1, 1, 1)\big) + 19(0, 0, 0) = 0(3, 2, 1)$. Since there is a non-trivial solution with not all zero constants, the set is linearly dependent by the formal definition. This same theory applies to any set containing the zero vector because the zero vector can be multiplied by any constant with the other vectors being multiplied by zero."

Explicit Verbal Reference to the Homogeneous Vector Equation. Seven students referred explicitly to the structure of the homogeneous vector equation without using formal notation. These students described the mathematical process of forming a linear combination of vectors with scalar weights and achieving a non-trivial solution. For instance, P027 stated:

> "This is similar to the earlier problem where we were asked if a set was linearly dependent if the solution was $c_1 = 0, c_2 = 0, c_3 = 17, c_4 = 0) = (0, 0, 17, 0)$, given that v_3 was the zero vector. The same solution applies here: assuming that v_3 is the zero vector and $p = 4$, that exact same solution would be true, and since the solution contains at least one nonzero value (17), it is nontrivial and the set is linearly dependent."

This type of reasoning demonstrated awareness of the formal structure of the homogeneous vector equation, specifically, the role of scalar coefficients and their relation to the zero vector, though expressed in verbal or contextual terms rather than symbolic notation.

In this group one student, P046 assigned specific values to the vectors:

> "If a set has solution of only zeros, that doesn't mean the set is necessarily independent. It's only when that is the only solution it can be concluded a set is linearly independent. So, by showcasing there are other solutions to the set in R to the n, it can be proven the set is linearly dependent. For example, the set $(1, 2, 1), (-1, -2, -1), (0, 0, 0)$ contains the solution $(0, 0, 0)$. However, the set also contains the solution (1,1,1). Therefore, the set is linearly dependent."

Rationale-Based Proof Methods

Out of 54 participants, 12 demonstrated that a set of vectors containing the zero vector is linearly dependent by providing arguments that qualify as rationale-based. I identified three nuances in how students employed these arguments: some wrote underdeveloped explanations that omitted explicit references to key accepted truths; others referenced ideas outside the instructional sequence; and a few relied on false assumptions (understood here as statements inconsistent with the accepted truths of the instructional sequence). These issues ultimately prevented the arguments from meeting the criteria for deductive reasoning.

Arguments that were not completely elaborated. Two students showed that the set containing the zero vectors is linearly dependent by communicating the odea of what this will be true but did not really elaborate on their argument. For instance, one student stated:

> "The set would be linearly dependent because the fact that it contains the zero vector indicates it has many non-trivial solutions."

However, the student did not elaborate on the details of those non-trivial solutions or how their existence is known.

Arguments that drew on knowledge external of the instructional sequence. Five students using rationale-based arguments drew on knowledge external to the definitions (and thus external to the Desmos instructional sequence) to prove the that if a set $S = \{v_1, \ldots, v_p\}$ in R^n contains the zero vector, then the set is linearly dependent. For instance, P039 drew on ideas about matrices arguing that:

> "If S contains the zero vector, the matrix would have a line of all 0's, meaning the set is linearly dependent."

Arguments that drew on false assumptions. Five students using rationale-based arguments based their reasoning on false assumptions which are statements that do not belong to the set of accepted truths of the instructional sequence used in this study resulting in an invalid argument. For example, P034 stated that:

> "All linearly dependent sets contain the zero vector, as there is a non-trivial solution to the problem and that can be represented as $0v_1 + 0v_2 \ldots 0v_p = 0$."

Unclear Reasoning

Nine out of 54 students wrote arguments that were classified as demonstrating *unclear reasoning*. These students used idiosyncratic language, making their statements difficult to classify within any of the existing argument types in Stylianides (2008)' framework. For example, P002 stated:

> "If the vs equal zero v's exists and not all that satisfies the equation is a zero."

Response does not Answer the Question

Three students wrote arguments that either stated the definition of linear dependence without proving the statement, or proved a different statement, not the one being addressed. For instance, P035 only stated the formal definition of linear dependence but did not prove the given mathematical statement:

> "The set of vectors $\{v_1, \ldots, v_p\}$ is said to be linearly dependent if there exist weights (real numbers) $c_1, \ldots, c_p$ not all zero, such that $c_1v_1 + c_2v_2 + \cdots + c_pv_p = 0$."

Discussion and conclusion

This study examined undergraduate students' proof methods for a mathematical statement on linear dependence using Stylianides' (2008) reasoning-and-proving framework in an Inquiry-Oriented Linear Algebra (IOLA) class. The findings revealed a range of proof methods. Approximately 33% of participants constructed deductive proofs, drawing on formal definitions and accepted

mathematical structures. This is a promising outcome given that prior studies (e.g., Makowski, 2021) have found deductive proof construction to be rare, even among advanced students. The relatively high proportion of deductive reasoning in this study may be attributed to the design of the instructional sequence, which was grounded in the Authentic Mathematical Proof Activity framework (Melhuish et al., 2022) and emphasized multiple proof-related tasks. Students were guided through comprehension, validation, and construction of arguments, helping them navigate between informal insight and formal expression. Many constructed valid proofs using written language, although few used symbolic notation such as the homogeneous vector equation, consistent with previous findings that students often struggle to incorporate formal mathematical language (Moore, 1994).

Notably, about 22% of students used empirical reasoning, supporting their claims with specific vector examples rather than generalized arguments. While not formally deductive, these responses reflected a strong understanding of the underlying definitions and the logical structure of the proof. Prior work (Weber et al., 2020) suggests that empirical reasoning can serve as a cognitive bridge toward formal proof construction. Students in this category appeared to possess conceptual knowledge but relied on example-based reasoning to express it, which may be understood as part of the natural progression toward deductive thinking. This reinforces the view that proof development is a continuum, not a binary skill (Stylianides, 2008).

An additional 22% of students used rationale-based arguments grounded in informal or intuitive justification. Although these responses did not meet formal proof standards, they suggest that students were engaging meaningfully with the task and attempting to explain why the claim holds. Together, the empirical and rationale-based responses highlight the instructional value of supporting students as they move from informal reasoning to more formal modes of justification. A smaller portion of students, approximately 17 %, wrote arguments that were difficult to interpret due to idiosyncratic language or did not answer the question directly. These responses may still reflect partial understanding, but they point to challenges in translating knowledge into structured argument, an issue also noted by Weber (2001), who found that students often struggle to apply known facts and techniques in proof contexts.

This study has important implications for teaching linear algebra proofs. Instructional sequence approaches that incorporate both informal and formal reasoning, alongside examples and varied proof activities such as comprehension, validation, and construction for enhancing students' engagement in proof activity (Melhuish et al., 2022) may better support their reasoning development and proof construction abilities. Future research should explore students' reasoning processes more deeply through interviews and examine the impact of targeted interventions on proof construction. This research has limitations, including its focus on a single linear algebra classroom, its analysis of only written proofs, and the exclusion of interview data that could have provided deeper insights. Despite these limitations, it offers valuable insights into students' reasoning processes in linear algebra and underscores the importance of supporting their transition from informal to formal proof construction. Its findings contribute to the small but growing body of literature focused on linearity and proof.

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A lukewarm defence of the mathematics lecture

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The lecture is an often-criticised format of teaching that nevertheless prevails in university mathematics classrooms. This paper offers empirically based perspectives on the qualities of a lecture in advanced linear algebra. Using ethnographic thick description based on third person video recordings, the aim is to examine the analytical relevance of psychoanalytic theories when making sense of lectures. I find that this interpretative lens forefronts how the teacher creates a learning environment that affords durational pleasure, thus highlighting affective values of the lecture.

Keywords: Lecture, university mathematics education, embodiment, resonance, erotics.

Introduction

In the beginning of the semester where I conducted my observations, the vice dean sent out a letter addressed to all teaching staff at the Faculty of Science. He wrote:

> The three-phase model at the Faculty of Science has had its time […] We cannot separate well-being and teaching […] Lectures may well be an element that we continue to use, but as a form of teaching, lectures have their natural limitations. (Nielsen, 2024)

The three-phase model, i.e. lectures, homework and tutorials, has been standard in the mathematics department for decades. In the letter, the vice dean seems to distinguish between forward-looking teaching that promotes well-being of the students, and lectures that are implied to be outdated. This plea for student-centred, active learning approaches is consistent with literature on university pedagogy and general didactics, where the legitimacy of lectures is called into question (Jones, 2007). For example, lectures are criticised for their reliance on transmitting knowledge to students who are rendered passive. Regardless, the lecture prevails in mathematics classrooms at universities (Viirman, 2021, pp. 467–468).

Literature on the lecture is of course not monolithic. Based on interviews on first-year mathematics students' reflections on why lectures matter, Melissa Rodd (2003) forefronts the 'spiritual purpose' of the lecture, in that the lecture may inspire wonder, develop identity, and stimulate imagination. She rejects the 'transmission' criticism as a false premise and highlights instead how lectures direct desire for learning. As such, she problematises the juxtaposition of valid participation and social interaction. Rodd asks the question: "if there were no mathematics lecture, what would be lost?" (Rodd, 2003, p. 15), pointing to how the qualities of lectures are difficult to replicate in other communication or teaching formats. Her sentiment is echoed by Steve Jones: "This live experience cannot be replicated through texts or recorded media; the immediacy, the corporality and the subtle dialogue between students and lecturer are irreproducible and ephemeral" (Jones, 2007, p. 402). This calls for qualitative research on lectures that makes sense of them in more ways than their form. As David Pritchard points out in reference to the debate on the legitimacy of lectures, "different disciplines must be treated differently" (Pritchard, 2010, p. 609). The lecture is a diverse practice, not least across disciplines, and according to Pritchard, some of the criticism of lectures appears

superficial when applied to mathematics. He outlines three functions of the mathematics lecture: communicating, modelling and motivating. This paper follows the path of motivation, although motivation is not the affective term, I will be using. Olov Viirman, who has followed the path of modelling in his work on the function of lectures, inquires after arguments about the value of lectures that are based on empirical investigation:

> [A]ny discussion of the role of the lecture in future mathematics teaching at universities, whether among researchers or practicing lecturers, needs to be informed by research on actual lecturing practice. (Viirman, 2021, p. 487)

Within this choir of lecture-apologists, this paper finds its voice. I contribute with empirical research on the qualities of a university mathematics lecture, making sense of the lecturing practice through a theoretically eclectic interpretative lens of embodiment, resonance, and erotics. By trying out these theories, I contribute with ways of understanding university mathematics teaching practice that forefronts alternative perspectives on it. These perspectives might benefit both the university mathematics education research community as well as the local context of my study.

Methods

The lecture analysed in this paper is part of the data for my PhD project. I work ethnographically (Van Maanen, 2011) in the learning environment of the institution where I studied and am currently employed. This makes my study akin to an autoethnography, which requires both intellectual and political considerations about taking 'native' viewpoints for granted and balancing organisational loyalty (Alvesson, 2003, pp. 167–174). As part of my ethnographic work, I have videorecorded 25 lectures given by four different teachers in courses on calculus and linear algebra for first-year students in pure and applied mathematics. The lectures were recorded with a video camera from the back of the class to my side and a microphone placed on the front row pointed at the teacher. The video and audio files have been synchronised and transcribed. While observing the lectures, I took extrospective and introspective notes to support my later writing.

The lecture is the thirteenth and second-to-last lecture in the course *Advanced Linear Algebra* that took place in the spring of 2024. It was transcribed using four tracks of information: The first records timestamps, the second categorises the situation, the third transcribes the speech, and the fourth holds stills from the video recordings, showing the blackboard, the slideshow, or the teacher. A timestamp was made every time the teaching element shifted in a meaningful way. For example, categories are: Going through a proof, giving examples on the board, answering questions from students, and so on.

The transcription serves a heuristic purpose, since I have worked mostly inductively without predefined categories or codes adhering to a specific theory. Instead, in general terms, my empirical approach is exploratory and descriptive, with a mix of hermeneutic, phenomenological and pragmatic viewpoints: Hermeneutic because I interpret the lecture while trying to be conscious about the context of my observations as well as my own prejudice. Phenomenological because I try to understand the social phenomenon of mathematics teaching from a human perspective. And lastly, pragmatic because I work from the assumption that the methods and theoretical notions I employ, produce the reality I describe (cf. Jackson & Mazzei, 2017). My ideal is to first wonder about the data and then

bring in appropriate theories and their vocabularies. In this way, describing a situation is part of analysing and explaining it (Geertz, 1973).

To make the choice of theories transparent, I interweave the sections on theory and results. In doing so, I write in the style of 'confessional tales' and 'impressionist tales' (Van Maanen, 2011), where one's own lived experiences become part of the object of research. I must emphasise that I do not use my own situatedness as a substitute for the student point of view, but for illuminating the research process as an interpretative act. The lecture that I will analyse was in fact a turning point for the theoretical framework I employ, so the interwoven structure reflects the temporal order of events.

Theoretical outset

My ideal may be to wonder first, but I did not set out theoretically emptyheaded when I began my observations. Having just been a student myself, my initial interest was student-centred, and I directed my attention to the learning environment as a site for experiences where students actively construct knowledge. I drew on the theory of embodied cognition (Lakoff & Núñez, 2000), which remains relatively modest in research on the tertiary mathematics education compared to lower levels, not least in combination with design-based methods (Abrahamson et al., 2020).

The theory of embodied cognition holds that human thought is inseparable from our bodies, and that abstractions are metaphorical and based on primary, bodily experiences (Lakoff & Johnson, 1980). This has several implications for mathematics education, some of them being that learning is situated in multimodal experiences (Shapiro & Stolz, 2019), and that a focus on gesture and other extra-linguistic modalities is beneficial for interpreting students' understanding of mathematics (Edwards et al., 2014). With these perspectives in mind, I went into the empirical phase of my research.

The case of the Jordan Decomposition Theorem

As stated, a turning point in my theoretical framework and research questions happened near the end of my empirical work. In the thirteenth lecture on *Advanced Linear Algebra*, the teacher ('Liam') proved the Jordan Decomposition Theorem, a result so major that it was basically the climax of the course. Liam put it like this:

> Liam: All right. Så lad os se på beviset, for vi har faktisk bygget op til det her bevis over de sidste tre, fire forelæsninger. Faktisk igennem hele kurset.
> *All right. So let us look at the proof, because we have actually built up to this proof for the last three, four lectures. Actually, through the entire course.*

I went into this lecture with a brewing sense of ambivalence toward lecturing as a practice. In last week's lecture on nilpotent matrixes, a streak of eight questions from students ended when Liam, in the interest of time, requested the students hold their questions so that they could move on to an example. I was quick to judge this hushing and disengaging in dialogue as bad pedagogical practice, but at the same time, I noticed in myself an excitement that he would keep on lecturing. I also noticed that the students, unsurprisingly, seemed more hesitant about asking questions, and although the lectures had certainly not turned into a completely monologic practice, elements like proofs seemed to be surrendered to Liam's initiative. Now I sat at the back of the class, ever ready to jot down notes on Liam's metaphors, body language, and visualizations, as the thirteenth lecture began.

About fifteen minutes of prefatory ideas, or what Liam called the 'baby version' of the decomposition, led up to the proof of the theorem. The first minute of the lecture was spent by students requesting two recap lectures, of which he agreed to do one before the written exam. Three minutes went with introductory motivations for the theorem. Liam then spent six minutes presenting both the general and the complex case of the decomposition, showing an illustration of the decomposed matrix:

Liam: Denne her blok nederst til højre […] den her matrix har den egenskab, at den ikke har nogen egenværdier. Man kan tænke på det som at vi har alle de her pæne blokke her, her og her, der går ned, og så har vi en matrix, som består af … skrald. Det her, det er en ubehagelig matrix, som ikke har nogen egenværdier. Det kan vi generelt ikke lide. Det er ikke noget, vi er særligt glade for, så det er sådan, det er der, vi prøver sådan at putte alt vores skrald ned i det her højre hjørne.
This block at the bottom to the right […] this matrix has the property that it has no eigenvalues. You can think of it like we have all these nice blocks here, here, and here, going down, and then we have a matrix consisting of…trash. This one, it is an unpleasant matrix that has no eigenvalues. That is something we generally do not like. It is not something we are particularly pleased with, so it is like, it is where we try to put all our trash, down into this right corner.

After this illustration, two and a half minutes went with students' questions. The next minute, Liam presented the theorem in general, non-matrix terms, i.e. the 'actual' Jordan Decomposition Theorem, after which he declared the proof to begin, as quoted at the beginning of this section. He spent five minutes proving the first lemma needed, and then spent one and a half minutes outlining the idea of the proof of the second lemma needed. Finally, seven minutes of uninterrupted proving followed.

I observed the multimodal aspects of the proof: Liam faced the class at a standing desk with his tablet on it, as the slides appeared on a canvas to his left. The proof manifested one line at a time, and from his desk, Liam moved the cursor as a pointer on the slides while talking. Once, after passing his gaze over the students, he turned to the blackboard to elaborate on a detail.

A recurrent metaphor throughout the proof was 'a matrix without eigenvalues is trash,' which Liam introduced in the quote above. He spat out the word 'trash' with a disgusted facial expression every time he said it, while he spoke of the field of complex numbers with corresponding fondness, since the decomposed matrix does not have a trash corner over this field. Most of the proof was about validating that the lowest right corner did indeed consist of trash over a non-complex field. Apart from laughing at Liam's apparent abhorrence of matrixes with no eigenvalues, the students had turned into mute non-data, until Liam concluded the proof with the remark:

Liam: Så nu har I altså set et af de absolut vigtigste resultater inden for lineær algebra: Jordan-dekompositionen. Ja?
So, now you have seen one of the absolute most important results in linear algebra: The Jordan decomposition. Yes?
Student: Du sagde, nulvektoren kunne aldrig være, hvad?
You said the zero vector could never be what?
Liam: En egenvektor.
An eigenvector.
Student: Men er det vektoren, der består af nuller, eller er det den generelle?
But is that the vector consisting of zeroes or is it the general one?
Liam: Det er nulvektoren for vores vektorrum. Nulvektoren for vores vektorrum er aldrig en egenvektor per definition.

It is the zero vector for our vector space. The zero vector for our vector space is never an eigenvector per definition.
[Six seconds of silence]

Liam: Så der har I altså Jordan-dekompositionen, som fortæller at alle matricer kan skrives som denne her blokdiagonale udgave, som jeg nævnte før. Eller som jeg nævnte i starten af forelæsningen.
So, there you have the Jordan decomposition that tells us that all matrices can be written as this block diagonal version, as I mentioned before. Or as I mentioned at the beginning of the lecture.

Later, I sat at my desk to write up the episode, struggling to describe the nuances of the proof with the vocabulary of embodied cognition. I found that I was missing perspectives on time and affect, and so I realized that embodied cognition was necessary but not sufficient for making sense of the lecturing practice I had observed.

Toward a networked framework of mathematical pleasure

I therefore present a tentative networked theoretical framework to forefront alternative perspectives on the case of the Jordan Decomposition Theorem (cf. Bikner-Ahsbahs & Prediger, 2009). To make sense of my observations, I want to take my framework in the direction of psychoanalysis. The versions of psychoanalysis I find particularly relevant for university mathematics education, are the ones that synthesize with critical theory. Here, Herbert Marcuse is an original torchbearer who dealt with erotics and pleasure as societal matters, relevant outside the private sphere.

Marcuse's pleasure principle arises from a critique of Sigmund Freud's belief that gratification is incompatible with progress (Marcuse, 1955). Marcuse disputes that repressing pleasure is necessary in a functioning society. In fact, he builds his Marxian utopia on a freed principle of pleasure, where the resources afforded by the industrial society, if used wisely, create the opportunities to live in accordance with the emancipated human nature, in an erotic relationship to the world.

A contemporary contribution to Marcuse's school of thought is Hartmut Rosa's notion of resonance, which describes a successful way of relating to the world, in that resonance is the counterpart to alienation in the Marxian sense of the word (Rosa, 2019). According to Rosa, a resonant experience will touch you, move you, and even transform you. Resonance requires self-efficacy but is at the same time uncontrollable: You can never guarantee resonance, but some environments might afford resonance more than others. Rosa formulates a critique of the conditions for resonance in our world, which according to him is characterized by acceleration, competition, and a fundamental attitude of manipulation and control of the world. Hence, to Rosa, the experience of being in the world is intimately related to time.

Resonance is a useful phenomenological term for describing learning situations that 'spark' (Rosa & Endres, 2017). Resonance pedagogy deals with the prerequisites for a resonant didactic triad between the subject, teacher, and students: in short, there must be an atmosphere of trust and genuine engagement with the material from both the teacher and the students. Through Marcuse, Rosa also references the erotic relationship to the world as an exemplary attitude in resonance pedagogy, stating that: "Eroticism, to me, is the opposite of trying to manipulate the world" (ibid., p. 35).

Queering the notion of erotics, I finally network KJ Cerankowski's notion of durational pleasures (Cerankowski, 2021). I find their terminology extremely relevant to my observations because it dethrones the orgasm as the pinnacle of pleasure and explores nonsexual pleasures that do not peak by ending. They write: "I want to stake a claim for an erotics of pleasure, of orgasm even, that does not rely on a teleological model of building tension until one arrives at the release" (ibid., p. 136). By removing the end goal, Cerankowski reframes the temporality of pleasure, broadening our understanding of which pleasures may be legitimate and valuable.

Revisiting the case of the Jordan Decomposition Theorem

In the case of the Jordan Decomposition Theorem, it is clear that Liam's lecture does in fact build on a 'teleological model of building tension until one arrives at the release'. In the first quote, he draws attention to this very build-up, which according to him has lasted through the entire course. It is also noticeable that he spends quite a bit of time on prefatory ideas, motivating the need for the theorem, visually illustrating its meaning and laying down all steps necessary before completing the proof in a streamlined manner. However, the release is disrupted when a student immediately asks a clarifying question at the finish line, afterwards prompting Liam to reiterate the 'ta-da', as it were.

Based on the dramaturgy of the lecture, I interpret that Liam intended for the climax to have a certain impact, brought about by half a semester's worth of prelude. The prefatory elements just mentioned can be seen as efforts to create an environment that affords wonder and vicarious feelings of pleasure. Through speech, slides, body language, and chalk, Liam performs multimodal expressions of embodied metaphors that I would argue create the conditions for a resonant didactic triangle. The 'trash corner', for example, is a metaphor whose function is equal parts communicating the idea and evoking a sense of beauty enabled by the 'trash-free' complex matrix, which excites Liam to a contagious degree. The build-up to the proof can then be seen as a way of persuading the students of its relevance, or "to arouse curiosity and a desire to know" (Sinclair & Watson, 2001, p. 40), instead of immediately stating it as a fact. Liam's lecturing reflects an attitude to mathematics not unlike Marcuse's idea of an erotic relationship, where the proof of the Jordan Decomposition Theorem for quite a while gets to be a potential awaiting realisation. Hence, I would argue that Liam's lecture performance is 'awe-inspiring' (Rodd, 2003, p. 17).

The ruining of the climax in turn disrupts who controls the pleasure, a power-smoothing that is made possible by the immediacy of the lecture, where Liam and the students gather in this specific time and place. With Cerankowski's terminology, the pleasure becomes durational in a temporality where the journey is the act itself. The lecture thus operates on two different teleological models at once, building tension and ending 'not with a bang but a whimper'.

Now, a lot of this discussion has hinged on verbal data, and as already mentioned, the students become non-data to me when they listen. However, it would be a fallacy to then conclude that nothing of note happens. For instance, Liam is evidently not just speaking 'at' the students. His gaze before he turns to the blackboard to elaborate on the proof can be interpreted as subtle communication, and while I do not have access to the students' experience, Rodd's (2003) findings suggest that more might go on in a lecture than meets the eye. It is also worthwhile to question whether looking for observable activity as wholesome learning qua activity is the be-all and end-all. As Miya Tokumitsu argues:

> Many critics dismiss lecture attendance as "passive learning," arguing that students in lectures aren't doing anything. Today, declaring something passive completely delegitimizes it […] under capitalism, anything that doesn't immediately appear as productive, even if only in the most superficial way, is dismissed as a waste of time. (Tokumitsu, 2017)

The students in the lecture analysed in this paper are calm and quiet most of the time. Are they learning? Maybe not this moment, here. But later, somewhere else, the desire to know instilled in them by the lecturer carries on. I argue here that learning is mostly unobservable in lectures on a surface level because learning might be displaced in spacetime. A value of the lecture may thus be excitation, understood as "an elevation in energy" (Porsager, 2021, p. 22). Seen through this lens, the apparent passivity of students instead appears as a build-up of desire for knowledge awaiting satisfaction.

Conclusion

Being an analysis of just one lecture out of the 25 I have observed, this paper serves as a pilot study for my doctoral dissertation. It is nonetheless a feature of ethnographic thick description that "small facts speak to large issues" (Geertz, 1973, p. 23). I have demonstrated the analytical relevance of psychoanalytical perspectives when discussing the value of lectures by reading Liam's proof of the Jordan Decomposition Theorem in its erotic dramaturgy, and further speculating about what goes on under the surface. Combining embodiment, resonance, and erotics has enabled me to appreciate the lecture as a learning environment that affords excitation and durational pleasure. Examining whether this perspective is consistent with students' experience is a relevant endeavour for further research.

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Design and implementation of a study and research path for teaching statistics: Impact of the dialectics

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This paper explores the implementation of study and research paths (SRPs) in university-level statistics courses, focusing on the appearance and impact of dialectics within SRPs built around the same generating question at two universities. By comparing these experiments, similar and different factors influencing the development of university-level modelling are identified. The analysis of three SRPs highlights emerging dialectics, and contrasts how students engage with statistical techniques. Future research could further explore why certain dialectics arise, and how they shape the development of SRPs.

Keywords: Anthropological theory of the didactic, dialectics, study and research path, statistics.

Introduction

In recent years, university teaching has changed through the implementation of a common European framework (Bologna Declaration). This has broadened the focus of the study of didactic approaches, going beyond traditional or theory-practice lectures to include approaches such as inquiry-based, project-based, problem-based learning, service learning, etc. Several studies address the relationship that should exist between the learning process and the methodologies applied from a cognitive perspective (Tan et al., 2019). However, few analyse how teaching research methodology connect with disciplinary or epistemological elements, which has changed little since the Bologna process was implemented at university. This can be seen in the fact that the descriptors of subject content have hardly changed (Bosch et al., 2021). The question to ask is the following: "Does a change in teaching approach lead to a change in the conceptualisation of the knowledge to be taught?" It is postulated that the answer is affirmative because of the emergence of new educational purposes in the sense of Gascón and Nicolás (2021).

This paper assumes the postulate of the Anthropological Theory of the Didactic (ATD) that all human activity can be interpreted as a modelling process to be described in terms of praxeologies (Chevallard, 1999). The Bologna process was a shift from the "paradigm of visiting works" to the "paradigm of questioning the world" (Chevallard, 2015). It may, therefore, be inferred that "doing statistics" in the professional world is to implement a modelling process in which the tools that enable solving an initial problematic situation belong to that discipline. From this perspective, engineering degrees will be focused on the teaching of statistics. A learning methodology based on inquiry processes (Artigue & Blomhøj, 2013) in which statistical concepts and techniques contribute to exploring and solving a problem should be used.

Theoretical framework and research question

The ATD proposes a didactic device called a Study and Research Path (SRP) that enables the development of an inquiry process derived from a previous didactic engineering process (Artigue, 1989). An SRP starts with a generating question Q_0, which arises from a problematic situation, and poses a challenge to a study community. Q_0 is expected to be open-ended, and requires searching information in different sources and studying it to develop and validate a final answer. SRPs can be described by the Herbartian schema: $S\ (X;\ Y;\ Q_0) \hookrightarrow A^{\heartsuit}$, where the study community consisting of student(s) X and teacher(s) Y addresses the initial question Q_0 through derived questions Q_i, and intermediate answers $A_j^{\diamondsuit}$ leading to a final answer $A^{\heartsuit}$. The dialectic of questions and answers is the engine for the development of SRPs (Bosch, 2018).

For the *a posteriori* analysis of SRPs, ten dialectics were developed within the ATD to describe the inquiry process (see the latest proposal by Vásquez (2024)): (D1) study and research dialectics: an SRP alternates between study and research moments, evolving through the generation of questions and answers; (D2) individual and collective dialectics: SRPs involve collective responsibility for responses while delegating individual tasks within teams; (D3) dialectics of praxeological analysis-synthesis: students analyse different information sources during the inquiry while teachers make sure Q_0 is answered thoroughly; (D4) dialectics of delimiting and abandoning the topic: mastery emerges when specific knowledge is required to clarify previously misunderstood answers, making it useful for Q_0; (D5) dialectics of the parachutist and the truffle hunter: enquiry alternates between broad brainstorming (parachutist) and focused exploration (truffle hunter) to find relevant knowledge; (D6) dialectics of black boxes and clear boxes: some objects are used without in-depth study (black boxes), leading to grey areas in the inquiry process; (D7) dialectics of media and milieu: the milieu is made up of all the resources, data, tools, and works found in the media (textbooks, the teacher, the Internet, etc.). A rich milieu will allow, for example, the validation of results; (D8) dialectics of reading and writing: information is analysed and integrated into notes and reports, connecting reading with writing; (D9) dialectics of dissemination and reception: the study community justifies and discusses answers in a "common discussion" moderated by the teacher; (D10) dialectics of systems and models: a modelling process refines the initial system and model through intermediate systems to better address Q_0.

The design of an SRP in statistics for engineering degrees, developed in the LABINQUIRY transfer project (Lombard et al., 2024), consists of four phases (Figure 1). In phase 0, the teacher or an external professional presents the generating question Q_0, teams are formed, the SRP methodology is explained, and conditions for delivering the final answer $A^{\heartsuit}$ are set out. In phase 1, brainstorming leads to initial explorations using accessible sources, which are then reorganised and prioritised. Phase 2 involves advanced searches for data, answers, and tools, testing their validity through analyses, experiments, or software, and progress is summarised in an interim report. Phases 1 and 2 may be repeated, if necessary, but within limits. In phase 3, the response to the initial question is presented. Figure 1 shows an outline of the phases of an SRP in statistics.

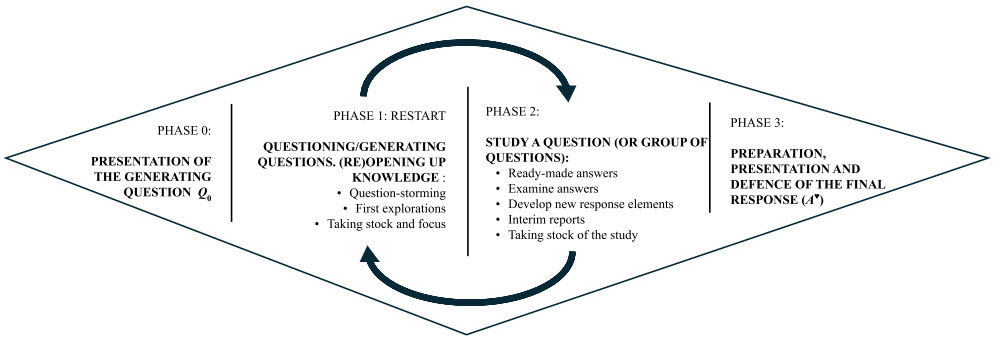

Figure 1: Phases of an SRP in statistics for university degrees

Through the analysis of three implementations of an SRP in statistics, this study aims to identify what elements of the SRP implementation contribute to the emergence and development of different dialectics in statistics at the university level.

Methodology

To address the research question posed, three different implementations of the same SRP, developed during the academic year 2023/24, were used at two different universities (IQS and EUSS) in Barcelona, Spain. The generating question was:

> Q_0: Are the low emission zones (Zonas de Bajas Emisiones, ZBE, after their initials in Spanish) in Catalonia effectively designed, and to what extent does their size affect their efficiency?

A representative of the Department for Transport and Mobility of the Catalan government raised this question. The participating students were organised in teams at the two universities where the SRPs were implemented.

The initial experiment, referred to as SRP-IQS, which was refined over four years, was fully integrated into a second-year statistics course (3 hours a week for 15 weeks) during the academic year 2023-24 in the Chemical Engineering degree at the *Institut Químic de Sarrià* (IQS), Ramon Llull University. The teacher (IQS-T) in charge of the subject is a chemical engineer with extensive experience in both the industry sector and in teaching statistics at the university. In this edition, 43 students organised into seven teams of six members each took part in the experiment.

The experiments conducted for the first time during academic year 2023-2024 in a second-year statistics course at the EUSS School of Engineering attached to the Autonomous University of Barcelona, are referred to as SRP-EUSS. A total of 209 students of statistics for all the engineering degrees (automation and industrial electronic, mechanical, and automotive engineering, engineering in industrial organisation, in renewable energies and energy efficiency) divided into eight groups, participated in the experiment. Two teachers (EUSS-T1 and EUSS-T2) taught four groups each. Both teachers are mathematicians. T1 has extensive experience in teaching mathematics and statistics, while T2, a researcher in didactics of mathematics, has experience in teaching statistics implementing SRPs. The SRP was implemented in the final part of the course in four 2-hour sessions (T1), and six 2-hour sessions (T2).

An analysis was conducted to identify elements and dialectics, and to compare the organisation and nuances of the three implementations. It focused on instructional materials, student team project diaries, and classroom observation notes. Attention was given to statistics topics addressed differently throughout the subject and project. The analysis also included student submissions related to the project. This approach revealed variations in practices and priorities.

Upon completing the SRPs, the students responded to a survey. It concerned the structure of the statistics course, the project in general, its organisation and content, teamwork, and the students' general impression (Fernández-Ruano et al., 2024). Parts of this survey were analysed. From a set of 32 questions (statements), the analysis focused on the four items considered the most relevant to identify the application of the various dialectics outlined and the general impression. They were the following: [Q10] I think the project was related to the content of the subject, [Q12] The project has changed my idea of statistics, [Q25] The descriptive analysis of the variables helped formulate hypotheses, and [Q32] I positively value having carried out this project. As a 5-point rating scale was used, including options from "strongly agree" – 5 to "strongly disagree" – 1, the responses were compared using the Wilcoxon-Mann-Whitney test. A significance level of 10% was applied (Meek et al., 2007).

Results and discussion

This study does not intend to show an in-depth analysis of the different experiments carried out, but a comparative analysis focusing on those dialectics that do not appear in some of the SRPs. They are, therefore, factors that may explain the differences in the students' final assessment. The dialectics (D1, D2, D7, D8) found in all the experiments will not be discussed here, as they are based on a common SRP design.

Experiments at EUSS

The project was implemented *a posteriori* as an application of the content of the course, strengthening statistical and organisational competencies that had not previously been worked on in the course. To implement the SRP, spreadsheets and the statistical software R with its R Commander package, were used earlier than in the previous traditional organisation of the course. This means the students had not used software for descriptive statistics before carrying out the project. The approach and the content used in the IQS implementation were not the same as those at the EUSS implementation. This is mainly because a significant part of the statistics course at EUSS comprises topics on probability, which are not specifically related to data analysis in engineering applications.

At EUSS, in the course taught by T1, the students were limited to applying techniques explicitly covered during the course. In contrast, T2 introduced tools for descriptive statistics and extensions of linear regression techniques that extended beyond the standard course content. As a result, the D4 dialectic cannot be clearly identified in this experiment. In the course taught by T2, regression techniques were developed using software treated as a "black box"; for example, R^2 was referenced and applied without giving the students a clear understanding of its calculation, limitations, or implications, thereby reflecting the D6 dialectic.

An analysis of the studies conducted by the different teams indicates that, at EUSS, T2 provided a broader range of questions (e.g., analysing the number of private cars registered in Barcelona, investigating other regions, and assessing the influence of Barcelona's port area). T1 offered a more narrowly defined scope of study, indicative of a weaker D5 dialectic.

Experiment at IQS

The theory sessions concerned the knowledge the students demanded or needed (through diverse examples that were not related to the project) to advance in the study of Q_0 and derived Q_i, thus combining teamwork and whole-class activities. A detailed account of the design and implementation of those sessions is available in Verbisck et al. (2024). Figure 2 shows a mid-project question-answer (QA) map of the SRP-IQS, constructed in class to show the potential of dialectics D3 and D9. The idea was not for the reader to refer to the exact questions and answers, but to exemplify how the discussions, under the teacher's guidance, resulted in partial answers to Q_0. In Figure 2, dialectics D4 and D6 are identified. The questions in blue are related to the functioning of the ZBE or to general aspects, those in grey to regulations or legislation, the ones in pink to location aspects, purple to seasonal aspects, green to pollutants, yellow to the organisation of the teams, and orange to data (spreadsheet).

During SRP-IQS, tasks characteristic of professional statistics (dialectic D10) appeared. Examples are the delimitation of the system to be modelled (Which radius of influence of the ZBE to consider? Which populations to select?), choosing the variables according to the hypotheses or study to be carried out (Which pollutants to study? During which period?), or all the work of cleaning data, magnitudes, etc. This essential process is absent in traditional statistics courses, where hypotheses and/or data are typically provided in advance, or are directly linked to previously explained techniques.

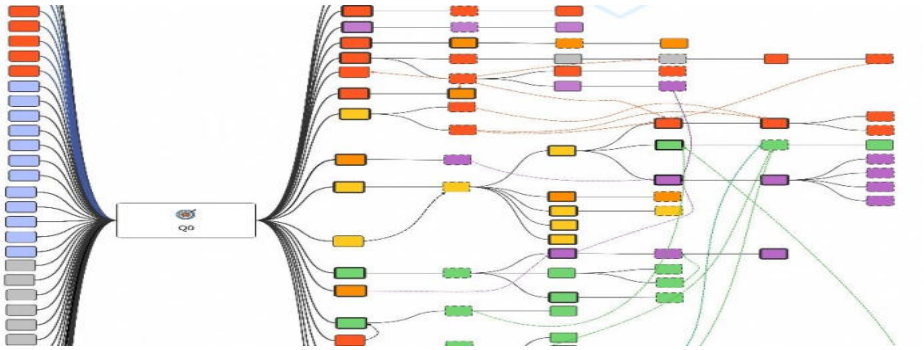

Figure 2: Mid-project QA map of SRP-IQS

The predetermined curriculum was followed (descriptive statistics, probability distributions, inference techniques and parameter estimation, hypothesis testing for one and two samples, analysis of variance, regression models), but some topics were covered in less depth. For example, the study of distributions was reduced due to its theoretical nature, while non-parametric tests were emphasized more, as students explored them through software interaction.

Comparison of group discussions

The moment of collective sharing in the two SRPs at EUSS aligns more closely with a traditional framework of presenting results in which the teacher assumes the role of verifying and validating outcomes (*institutionalisation moment*, as described by Chevallard, 1999). In contrast, in SRP- IQS, this moment of sharing corresponds to the D9 dialectic, as the students actively defend their contributions in front of their peers. They assume knowledge as a response in the strong sense ($A^{\blacktriangledown}$), not as mere reciters of existing works with which they have become familiar, but have not made their own, and have not adapted to the new situation ($A^{\Diamond}$).

This is observed at EUSS, where T1's students limit their world of techniques explained to them in the traditional course and take responsibility for applying them correctly, for example, by replicating the software calculations manually without assessing whether the results answer the questions posed or not. This leads to the fact that the *moment of evaluation* in the case of SRP-EUSS remains the teacher's responsibility. In SRP-IQS, collective sharing aligns more closely with the *technological-theoretical moment* (Chevallard, 1999). Teams debate the most effective strategies, complementary results, or inherent limitations. It is also during this process that, in agreement with the students, the study focuses on the issues that arise from the research. In this sense, the responsibility for validation is shared by the study community.

The epistemological model of statistics challenges the traditional view of the discipline as a hierarchy of techniques, where descriptive and inferential statistics are treated as distinct domains. In SRP-EUSS, this perspective persists, as many students applied inferential tests without first conducting a descriptive exploration that could have guided their work. Conversely, in SRP-IQS, techniques and tests were introduced incrementally, driven by the demands of the investigative process.

Analysis of the student survey

Insight is provided into the students' post-project perspective to highlight some results that expand on the already discussed characteristics of both SRPs. The sample sizes for EUSS-T1, EUSS-T2 and IQS-T are 16, 25, and 32 respectively. When investigating the teacher's distribution of answers in the groups, similar tendencies are observed in the responses to the survey (Figure 3). Regarding Q10, the students whose teacher was EUSS-T1 gave more polarised responses (strong disagreement – 1 and strong agreement – 5 with the statement) in comparison to those of the other groups, but the medians per teacher of 4, 4, and 3.5, respectively, show no significant differences. The same occurs in Q32, where the three medians are 3, and no significant differences are observed (Figure 3).

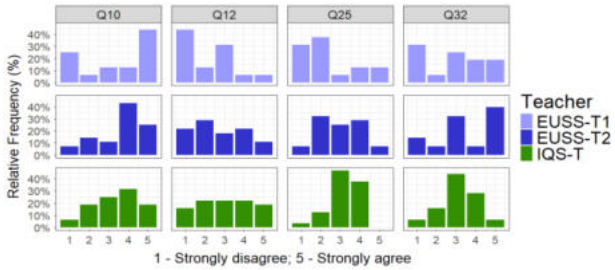

Figure 3: Survey results per groups and per teacher: Q10, Q12, Q25, Q32

Differences are observed in the responses to Q12. The medians shift from 2 for EUSS-T1 to 2.5 for EUSS-T2, and for IQS-T it was 3 (Figure 3). The difference between EUSS-T1 and IQS-T is statistically significant ($p = 0.04$). In Q25, the medians are 2, 3, and 3, and EUSS-T1 has a significantly lower median than EUSS-T2 ($p = 0.09$) and IQS-T ($p = 0.02$).

These results show that a richer modelling process in dialectics leads to greater changes in students' perception. As they shift from procedural learning to deeper engagement, their stance toward the discipline changes—from learning to be examined to learning to do.

Conclusions

The three experiments compared in this paper are part of an ongoing investigation into the implementation, analysis, and evolution of SRPs for statistics for non-statisticians at the university level. The appearance of dialectics in the implementations and their impact on the study and research process were identified. The SRP design does not guarantee the emergence or effective management of dialectics D4, D5, and D6, as these are linked to the teachers' characteristics. In contrast, dialectics D3, D9, and D10 are related to tools for managing discussions and collecting results (e.g., the QA map, class diary), which influence the redistribution of responsibilities within the study community. These elements should be considered in the design of SRPs to ensure the emergence of dialectics D3, D9, and D10. This aligns with the LABINQUIRY project's objective to provide teachers with the necessary resources for inquiry-based activities.

The analysis highlights emerging dialectics and compares student engagement with statistical techniques. Future research should explore the dialectics' drivers and impact on SRP development.

Acknowledgements

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Disciplinary authorities or fellow educators? Exploring positionings of mathematicians when interacting with secondary teachers

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The communication between mathematicians and secondary mathematics teachers have long been recognized as a critical issue in mathematics education. Yet, such communication remains fraught and limited, in part due to profound communal differences and issues of power and authority. This study is part of long-term research project that investigates how communal differences may be leveraged as a valuable resource in cross-community interactions in mathematics education. In this paper, we draw on Positioning Theory to analyse the positioning acts of mathematicians and teachers as they discuss and evaluate instructional moves of a teacher presenting a proof in a videotaped secondary lesson. Our analysis distinct sets of acts suggesting that a positional shift occurred when the mathematicians were situated as fellow educators rather than disciplinary authorities or pedagogical critics. We discuss the potential affordances of different positionings.

Keywords: *Mathematicians, cross-community communication, boundary encounters, positioning.*

Rationale

The field of Mathematics Education is culturally diverse, encompassing various communities such as mathematicians, secondary mathematics teachers, and education researchers. Among these, mathematicians and secondary mathematics (SM) teachers are particularly influential due to their direct impact on student learning and their roles in shaping policies and curricula. In particular, it has been suggested that improving communication and collaboration between mathematicians and SM teachers is vital for addressing persisting challenges that lie at the intersection of these two communities, such as the double discontinuity in teacher preparation (Lai & Baldinger, 2023) or the secondary-tertiary transition (Pinto & Koichu, 2023). However, partnerships between mathematicians and SM teachers remain scarce, among other things, due to a lack of opportunities for interaction, asymmetric relationships, and profound epistemological and sociocultural differences that hinder communication (Fried, 2014; Goos & Bennison, 2018). This study aims to explain how communal tensions and differences between mathematicians and SM teachers shape their communication, and how such tensions and differences could be leveraged to support dialogical learning between these two communities (Akkerman & Bakker, 2011). Our research is conducted within the M-Cubed Lab (Mathematics Teachers, Mathematics, Mathematicians), a setting specifically designed to generate, document and study dialogical learning between mathematicians and practicing SM teachers (Pinto & Cooper, 2022, 2023).

Prior research has shown that engaging mathematicians and SM teachers in dialogue is not sufficient by itself for generating dialogical learning (Pinto & Cooper, 2023). In many cases, the teachers and mathematicians will initially respond to and build on arguments made by their peers, while only relating superficially to the arguments of the other party (Goor et al., 2023). However,

in certain cases, the dynamics of the discussion seem to shift, affording both parties with rich opportunities for dialogical learning (Goor et al., 2023; Pinto & Cooper, 2022, 2023). In this paper we adopt Positioning Theory to explore whether and how such shifts may be explained in terms of different positionings of mathematicians in their interactions with SM teachers.

Theoretical background

A space in which two communities interact can be conceptualized as a case of a boundary encounter, namely, an environment where communal differences and tensions may lead to discontinuity in communication or in action (Akkerman & Bakker, 2011). However, the literature also suggests that boundary encounters may encourage and support exchange of knowledge and perspectives across communities through dialogical learning, or boundary crossing, specifically in relation to mathematicians and SM teachers (e.g., Goos & Bennison, 2018; Goor et al., 2023; Pinto & Cooper, 2022, 2023). For example, Goor et al. (2023) described how a boundary encounter with mathematicians supported SM teachers in recognizing and exploring alternative mathematical interpretations of a question a SM student raised in class and revisit their perspectives on the opportunities for learning this question afforded. Mytlis and Pinto (accepted) examined professional obligations SM teachers expressed in boundary encounters with mathematicians and found that SM teachers revisited their obligation to mathematical precision when considering the pedagogical merits of SM activities that mathematicians deemed mathematically invalid and therefore inappropriate for SM lessons. Pinto and Cooper (2022) showed how mathematicians and SM teachers may join hands and draw on each other's expertise in formulating a refutation to an imprecise mathematical claim of 8th grade students that is appropriate to an 8th grade class.

In all the above studies, the emergence of dialogical learning was not immediate. The researchers describe various barriers in communication, such as tacit use of incommensurable meanings, colliding professional obligations, or different implicit assumptions about students and their motivation for learning. The literature on boundary encounters identifies various means that may encourage and support dialogical learning, such as artifacts that function as *boundary objects* and facilitators' mindful *brokering* (Akkerman & Bakker, 2011). Boundary objects are characterized as possessing interpretative flexibility that enables different communities to collaborate without achieving prior consensus regarding roles and shared goals. Brokering refers to intentional and purposeful actions that help discussants articulate and negotiate the tacit norms and views of their respective communities. Both boundary objects and brokering are essential for addressing the profound different perspectives of mathematicians and SM teachers towards mathematics and its teaching and learning. However, special attention needs also to be given to the troubled history and asymmetric relationships between these two communities that are known to lead to issues of authority and power that may derail communication (Fried, 2014; Goos & Bennison, 2018).

To address this need we turn in this study to Positioning theory (Davies & Harré, 1990). Positioning theory was developed and has been used in numerous works to explain how the roles individuals take or are assigned within social interactions shape communication. It posits that in every interaction, individuals are positioned by others (interactive positioning) and by themselves

(reflexive positioning) in specific dynamic ways, and this positioning guides action by determining, often implicitly, what kinds of actions are considered appropriate within a given context. Two key terms in Positioning Theory that are prevalent for this study are *positioning acts* (hereafter acts) and *storylines*. Acts refer to the specific actions, behaviors, or utterances that individuals perform in social interactions. These acts are not neutral; they are interpreted and understood within a particular social context, where they carry meaning related to the positions of the individuals involved. For example: Making a statement, asking a question or expressing an opinion. Each act can either reinforce, challenge, or transform the positions of the people involved. Storylines are the broader narratives or frameworks within which acts are interpreted. They provide the context that gives meaning to the acts and help establish the expected roles, behaviors, and power dynamics within an interaction. In the context of boundary encounters between mathematicians and SM teachers, Positioning Theory can help explain how the roles of each community are constructed and how they may affect dialogical learning. In this paper we ask: *How are mathematicians positioned when interacting with SM teachers about SM education?*

Methods

Data collection: The M-Cubed Lab

Data for this study were collected within the M-Cubed Lab (Pinto & Cooper, 2022, 2023), an environment designed in accordance with the knowledge accumulated over the years in research on boundary encounters to encourage mathematicians and SM teachers in bringing their experience and expertise to bear in joint explorations of various issues in SM education. In M-Cubed, small groups of practicing SM teachers and mathematicians meet monthly for 3-hour facilitated sessions where they watch segments from videotaped SM lessons, chosen in advance by the facilitators, the authors of this paper, as boundary objects. The participants identify key teaching moves in the videotaped lesson and explore alternative moves that could have been taken. These explorations serve in turn, with the support of the facilitators who act as brokers, as a basis for boarder explorations of different pedagogical and mathematical perspectives. All SM teachers have over ten years of experience, including in high-track classes, while all mathematicians are actively engaged in academic research and have at least six years of university teaching.

The participants in the M-Cubed session discussed in this paper were three mathematicians (denoted as M1-M3), six SM teachers (T1-T6), and the authors of this paper who acted as facilitators (F1-F2). The session revolved around a 12-minutes segment from a 12th-grade intermediate-track videotaped lesson. In this segment, the teacher, referred to here as Ms. N, presented a proof of the derivative of the natural logarithm function $y = lnx$ by deriving both sides of $e^y = x$ (as illustrated in Figure 1) even though this proof is not required by the intermediate-track curriculum. In her presentation, Ms. N repeatedly tried to prompt students to guess her next step, for example by asking "how can we write the expression $y = lnx$ in a different way?". Generally, the students failed to anticipate the next move, and often made false claims, for example proposing that the derivative of e^y is $(y)e^{y-1}$. The discussion of Ms. N's teaching moves that

followed lasted 40 minutes and was video-recorded and fully transcribed. The discussion primarily focused on: critiques of the presentation of the proof; students' prior knowledge; goals and challenges underlying the intermediate track; and similarities between the intermediate track and university mathematics courses for physics students.

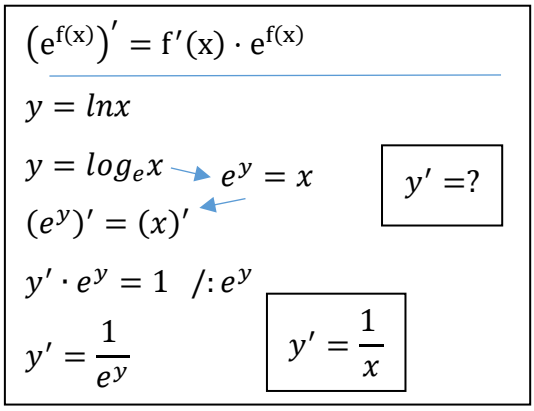

Figure 1: An illustration of how Ms. N wrote the proof on the board during the lesson

Data analysis

Data analysis was conducted in three stages that focused on acts, positioning, and storylines, respectively. In the first phase, we examined participants' utterances and marked repeating acts of the mathematicians or of the teachers at different parts of the meeting. These acts included for example: asking questions; arguing mathematically; making claims about mathematics in general; making claims about the videotaped lesson; making claims about SM education in general; endorsing, challenging or rejecting an observation from a member of the same community, or from the other community. We then refined our initial list of acts to reflect variations of the same act that were observed in the data, particularly when these variations appeared to be communal. For example, the act 'making claims about the videotaped lesson' was refined to 'making critical claims about the videotaped lesson' and 'making supportive claims about the videotaped lesson', with the latter act further refined to additional acts such as 'defending the instructional actions of Mrs. N. The second phase focused on how participants were positioned within the discussion, based on the identified acts. This analysis distinguished between interactive positioning, where participants were positioned by others, and reflexive positioning, where participants positioned themselves. For example, a participant might be positioned as an authority by making a definitive judgment about the lesson or positioned as an advocate by defending the lesson's approach. In the third phase, we tried attributing overarching storylines that could explain the different positionings we identified, drawing on the literature. In this paper we present the first two phases of the analysis, and we briefly refer to our initial findings from the third phase at the Discussion Section.

Findings

Our analysis highlights three distinct positioning of the mathematicians in their interactions with the SM teachers: disciplinary experts, pedagogical critics, and fellow educators.

In the M-Cubed discussion, the mathematicians' acts could be initially characterized primarily as making critical and authoritative statements regarding the proof and its presentation. The mathematicians did not identify an error in the proof, but they considered its presentation flawed. One such flaw, highlighted by M2, related to Mrs. N's use of variables throughout her presentation: "It really bothered me that x and y are sometimes treated as independent variables, and sometimes one is a function of the other, and it is not clear what is a function of what". Specifically, M2 pointed out a step in the presentation where Mrs. N. differentiated the expression $e^y = x$, first by differentiating the left-hand side with respect to y, then differentiating the right-hand side with respect to x, without addressing the relation between x and y.

In the discussion of this issue, the mathematicians talked in one voice, reasoning in definite and decisive terms, without inviting the teachers' perspective. For example:

M2: This thing with [the variable] y, why does it bother me? The nonsense [the students] said, that you derive e^y and get ye^{y-1}, it means that they see e as a symbol, which can either be a variable or not, and y, it's just another symbol that can either be a variable or not.

The teachers, for the most part, did not make claims about the mathematical validity of Mrs. N.'s actions, instead asking the mathematicians questions, inviting them to elaborate their perspective:

T2: I don't understand, it's truly the same function [on the two sides of the equation].
M3: I must say that there are crimes in university physics lessons as well.
T2: But why is it a crime?
M2: Functions of the same variable, that's why they remain equal when you differentiate them.
[…]
T1: I must say […] I still don't understand what is the mathematical problem that [the mathematicians] see, that I'm missing.
M1: It starts with that [a student] asks why not look at $\log_a x$ [instead of $ln\ x$] and it was clear that [Mrs. N.] does not know what would happen.

Furthermore, the teachers turned to the mathematicians to ask for their validation:

T1: What would solve all this is to write *f(x)* instead of *y*?
M2: Yes, this would solve it.

The acts of the mathematicians and SM teachers in these exchanges can be seen as positioning the mathematicians as having an authority with respect to quality and validity of the mathematics. On the other hand, when the mathematicians reasoned about pedagogical implications, often critically, the teachers responded with confidence, explaining and defending the instructional choices of the Mrs. N, and pushing back against the mathematicians' claims. For example, T1 defended Mrs. N.'s use of questions, observing that "her questions were accurate for a group that understands just roughly". T1 also defended Mrs. N.'s decision to still focus on the derivative of $ln\ x$ after a student suggested that it may be simpler to look at the more general case of the derivate of $\log_a x$:

M2: [Mrs. N.] just took the function and pushed them in the direction she wanted.
T1: No that's not accurate. The student asked it wouldn't be simpler [to look at $\log_a x$], and she said no, and it's true, it's not simpler […] Students have difficulty with algebra. Anything you add, another number, another multiple of something, it is complicated for them. And so, pedagogically, at the intermediate track, it is really not simpler to prove [derivate of $\log_a x$]; she is right there!

The teachers also rejected alternative instructional approaches proposed by the mathematicians:

M2: In my fantasy, what I would like [Mrs. N.] to say is: let's look at the equation that defines $lan\, x$, at $e^{lan\, x} = x$, that's our fundamental expression, and we are differentiating it.
T1: No way! At the intermediate track? At the high track, maybe, but at the intermediate track?

The acts of the mathematicians and the SM teachers in these exchanges can be seen as positioning the mathematicians as pedagogical critics, who are invited to make positive or negative assessments regarding instructional decisions and moves and offer alternatives, but whose positions can be questioned, challenged, and even rejected.

A third positioning was evident when M3 asked to present on the board a proof from a university ordinary differential equations course he teaches. He wrote the differential equation $y' = y^2$, and presented a series of manipulations of symbols that without explanation is mathematically senseless, even though it yields a solution to the equation. M3 further observed that such an approach is not uncommon in tertiary mathematics courses for physicists, where solutions are obtained without properly defining variables or procedure. He then suggested that the challenge of teaching proofs to physics students may be analogous the teaching proofs at the intermediate level:

M1: Maybe at the intermediate level you teach [students] to do manipulations like physicists.
M3: That's very bad! I think we're raising [at the university] very bad physicists as a result, and intermediate-level [SM] students deserve better as well.

This analogy shifted the discussion from critiquing Ms. N.'s lesson to collaboratively exploring the question what the purpose of presenting a proof in class is, which is pertinent to both parties. At the tertiary level, the mathematicians debated whether physicists need to see valid proofs:

M3: I taught this [course], you can do all of [the proofs] in a way that makes sense.
M2: It's like units in physics, its work hygiene…
M1: I disagree, you have physicists doing their master's in General Relativity, and they learn to work with this staff, and there is no way that you can explain to them mathematically what they are doing and also have them acquire all the analytical skills in three years.

A similar discussion developed around the reasons for presenting proofs at the intermediate track:

M2: What's the point of showing proofs to intermediate-level students if you're saying they will almost surely not understand them? What's the goal?
T1: Look, in every class, there are two who will understand what [Mrs. N.] was doing. And for these two […], it's worth these ten minutes, so that they will understand that it's not because I say so, but really […] there is a proof for it.
T2: Yes? I don't know, it's a question.

T4: Even when you're not showing all the proofs? […] There is no reason in the world to show a proof that have no value on its own, if all the value of the proof is to show that there is a proof.

Notably, in these two last exchanges, in contrast to the exchanges we analyzed before, neither the mathematicians nor teachers are speaking with one voice and with full authority. Both parties have relevant teaching experience to the question that is being discussed, and diverse perspectives. This enables a cross-community dialogue where the mathematicians are positioned as fellow educators.

Discussion

The literature often underlines the important role of mathematicians in SM education as representatives of the discipline (Fried, 2014; Goos & Bennison, 2018). At the same time, the literature also provides ample evidence that SM teachers are generally unclear about how mathematicians' values, norms, and practices could actually inform their teaching and thus may not see mathematicians' expertise as relevant and applicable for them (Lai & Baldinger, 2021; Mytlis & Pinto, accepted; Pinto & Cooper, 2023). These two facets of the relationships between mathematicians and SM teachers could reflect the storylines that underly the first two positionings we identified in our analysis: mathematicians as disciplinary authorities, and mathematicians as pedagogical critics. Both positionings are characterized in that one of the communities speaks in one voice and with authority, in a way that does not invite members from the other community to suggest alternative perspectives that could be compared, elaborated and refined. Thus, these two positionings may be seen as discouraging dialogical learning.

In contrast, the third positioning we identified – mathematicians as fellow educators – reflects a less common narrative, or storyline, which highlights commonalities between SM teachers and mathematicians and their pedagogical challenges and dilemmas. When mathematicians were positioned this way, the dynamics of the discussion was notably different in that it no longer aligned strictly according to communal divisions. Neither side assumed authority, and instead the two sides supported each other in making explicit and negotiating the meanings of tacit notions such as 'mathematics for physicists' and 'intermediate-level mathematics'. The two sides also shared personal experiences regarding a common practice – proof presentation – and jointly explored their goals, in different contexts, when presenting proofs. This positioning also seemed to have helped mitigate the effect of the deficit narratives that the mathematicians expressed about Mrs. N.'s teaching when positioned as disciplinary authorities. Deficit narratives regarding SM teachers and SM education seem to be prevalent among mathematicians, and they have been shown to hinder communication and collaboration between these two communities (Pinto & Koichu, 2023). Our analysis corroborates this finding by showing that SM teachers initially felt compelled to defend Mrs. N. and rejected pedagogical implications proposed by the mathematicians until the positional shift, that turned the focus to the teaching of the mathematicians and the SM teachers.

One potential implication of this study is that it may be beneficial for lecturers in courses for SM teachers to position themselves as fellow educators rather than disciplinary authorities. This can be done, for example, by sharing personal teaching experiences and highlighting analogies to SM

teaching. Further research is needed for investigating whether doing so could help avoid issues of power and authority, disrupt deficit narratives, and reduce teachers' sense of disconnect between mathematics and pedagogy, towards promoting a more effective learning environment.

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Research in undergraduate mathematics education using the concept of didactic contract: results and contradictions?

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The concept of didactic contract is present in many studies about undergraduate mathematics education. Using a review of the literature, we explore the features of didactic contracts identified by research and their consequences for students. Papers investigating the secondary-tertiary transition observe increased expectations, and students struggling to enter a new contract. Other papers reveal contract rules conveyed by teaching practices and adopted by students, but limiting their activity. Some studies proposing interventions introducing an inquiry-oriented didactic contract observe that the 'usual' contract constitutes an obstacle for students' active involvement. We discuss the apparent contradictions in the results obtained by the different categories of studies.

Keywords: Didactic contract, Expectations, Secondary-tertiary transition, Undergraduate mathematics education

Introduction

The concept of didactic contract is present in many studies about university mathematics education. At the beginning of university in particular, some of the students' difficulties can be interpreted as resulting from a change of didactic contract, since: "Secondary school and university have different didactic contracts" (Gueudet, 2008, p. 245). But what do we actually learn from research about the features of the didactic contract(s) at undergraduate level for mathematics, and about their consequences for students? In the context of a research project aimed at uncovering mismatches between the beliefs and expectations of university teachers and those of their students, we reviewed the research literature that used the concept of didactic contract. In this paper we present this review and its outcomes. Recent studies in university mathematics education stress that the diversity of students and its consequences is an essential direction in current research (Pinto & Koichu, 2023). This diversity challenges the existence of a unique didactic contract at university; we claim that this requires to revisit the use of the concept of didactic contract in university mathematics education, and that a literature review on this theme is needed.

The didactic contract: theory

The didactic contract is one of the main concepts within the Theory of Didactical Situations (TDS, Brousseau, 1997). Many different definitions can be found in the literature, even in Brousseau's own work. In our work we refer to the following definition:

"To each precise notion to be taught, the partners in teaching […] associate expectations, obligations that each undertakes and benefits from, and the means by which they envisage (mutually or separately) satisfying these expectations and obligations as well as the consequences of not satisfying

them. A didactic contract is, in the broad sense, an interpretation of the set of these expectations and obligations, be they compatible, explicit, and agreed to or not." (Brousseau & Warfield, 2014, p.1)

The concept of didactic contract is complex; it has been the subject of debates in the French community of research in didactics of mathematics (Sarrazy, 1995). We do not intend here to propose a comprehensive theoretical discussion, but rather to foreground some issues relevant in our context. A first aspect concerns the implicitness of the didactic contract's rules: most of these rules are implicit, and it is neither possible nor desirable to present them explicitly to the students (González-Martín et al., 2014). This raises a methodological issue for identifying the rules of the didactic contract. A second issue concerns what might be called the "level of generality" of the didactic contract. The definition cited above mentions a didactic contract for "each precise notion"; some studies identify didactic contract rules for specific mathematical situations. Chevallard (1988) claims that entering an institution involves a subjection to an institutional contract, which includes a didactic contract for the mathematical contents living in this institution. Balacheff (1988) proposes to distinguish between the didactic contract, corresponding to interactions between a teacher and their students in a situation, and what he calls "custom," which is a set of implicit but more general rules shared in a given classroom for example. This debate concerns the level of generality of the didactic contract, but also a vision of the contract as a set of already existing, stable rules, versus a dynamic process of negotiating the contract along the interactions between the students and the teacher. The research community has not adopted the concept of custom, while at the same time an increasing number of studies has used the concept of didactic contract to study a broad range of implicit features of the teaching and learning processes. De Vleeschouwer and Gueudet (2011) propose to distinguish between three levels of contract in a given institution: a general level, independent of the knowledge at stake ; the level of mathematics as a discipline; and the level of particular mathematical notions.

Drawing on these theoretical elements, we investigate the following research questions:

Which features of didactic contract(s) at the three different levels are identified by research in mathematics education, with which consequences for students? What categories emerge from a grouping of papers according to these features and consequences?

Review of the literature : methods

To answer our research questions, we conducted a review of the research literature in university mathematics education. We were interested in international research using the concept of the didactic contract to analyse teaching and learning features at the undergraduate level. We focused on research conducted in the last twenty years, i.e. research published between 2005 and 2024. We examined books and book chapters concerning university mathematics education; international journals (IJRUME, ESM, ZDM, IJMEST), and the proceedings of the CERME, EMF and INDRUM conferences. We added to this first selection some of the papers cited in it.

We have included in this review only works where the concept of the didactic contract plays an important role. These comprise either synthesis papers that emphasize the concept of the didactic contract, or empirical studies that use this concept in their data analysis methodology and/or present results in terms of didactic contract. We reviewed a total of 40 papers. In each of the selected papers,

we noted the research questions investigated; we examined the features of the didactic contract(s) identified (contract rules in particular, if such rules are explicitly mentioned), and their consequences for students. Moreover, for each study we identified the level(s) of the contract concerned: general (independent of knowledge), disciplinary (related to mathematics) or content-specific (such as calculus or linear algebra).

Results

Three main categories emerged from our reading of the papers; Table 1 shows these categories and the number of papers in each. The length constraints led us to discuss in this text only some selected papers that represent each category.

Table 1. The three categories, and the number of papers in each

Changes of the contract during the secondary-tertiary transition	The 'usual' university contract, adopted but limiting students' mathematical activity	Alternative didactic contract, accepted or not by students	Other	Total
14	8	14	4	40

Investigating changes of the didactic contract during the secondary-tertiary transition

A first type of studies concerns the secondary-tertiary transition, the change of didactic contract during this transition, and the difficulties of novice students to enter the university contract.

De Vleeschouwer and Gueudet (2011) observe through a university textbooks analysis didactic contract rules concerning the teaching and learning of duality in linear algebra (content level). In particular, since a linear form is sometimes considered as a function and sometimes as an element of a vector space, students should be able to recognise which point of view is relevant and should be flexible in using either view; this flexibility is typical from university expectations, compared with secondary school. Based on this finding, the authors design an intervention in which a systematic work on this aspect of the contract is organised. The analysis of students' answers to a questionnaire shows that this teaching helped them to develop this contract rule. We note that in this work, an intervention is designed to support the adoption by students of some rules of the university contract.

Studies concerning proof (and thus at the discipline level), synthesized by Selden (2012) show that the didactic contract changes between secondary school and university with increasing expectations, especially in terms of rigour, and that novice students have difficulties linked with this new contract. Bloch and Ghedamsi (2005) study the secondary-tertiary transition concerning calculus. They introduce ten "macro-didactic" variables that characterise the "actual didactic contract", for example: "The degree of formalization"; "The quantity of new notions introduced"; "The status of the tasks proposed: technical application or proof of a general result". The analysis of secondary school textbooks and of lists of exercises used at university (in Tunisia) evidences a change in the values of the ten variables and characterises the didactic contract for calculus in secondary school and at university. Even if used here for a calculus course, these variables can characterise the didactic contract at a disciplinary level; they reveal increased expectations at the beginning of university.

Pepin (2014) investigates students' experience during the secondary-tertiary transition, interpreted as their entrance in a new didactic contract. Referring to the three levels introduced by de Vleeschouwer and Gueudet (2011), she studies in particular the general and the disciplinary levels. Drawing on interviews with students over two years, interviews with teachers and classroom observations, she identifies changes experienced by the students. In secondary school, the students are strongly supported by the teachers, while at university they are expected to find their own learning strategies. Combining the didactic contract and a resource perspective, Gueudet and Pepin (2018) further observe that "discussions with the teacher are not easily available resources for students" (Gueudet & Pepin 2018, p.62) is a general contract rule at university; as a consequence, students develop a rule such as "in case of difficulties, I can find help from my peers" (Gueudet & Pepin 2018, p. 69). At the disciplinary level, there is a mismatch between the teacher's views on the homework to be done (including a work on the text of the lecture) and the actual homework done by the students (focused on exercises, and abandoning the search when the solution is not found quickly). Gueudet and Pepin (2018) also observe that the use of online resources can lead some students to incorporate irrelevant contract rules (here at the level of the content, concerning number theory).

Didactic contract rules limiting students' activity

A second type of studies concerns contract rules conveyed by teaching practices at university, of which the teachers may be unaware, and whose consequence is a limitation of students' mathematical activity. Lebaud (2010) investigates how teachers design mathematics exam texts by using interviews and analysing successive versions of the exam text. She finds that teachers' choices are influenced by didactic contract rules, such as: "The values to be calculated when solving an equation with complex numbers are integers" (content level) or "The answer to the first question of an exercise can appear in the text of the second question" (disciplinary level). González-Martín (2013) investigates the teaching and learning of series. He identifies contract rules through a textbooks analysis. Since the teachers align with the textbooks' content, he hypothesises that the contract rules he identified influence the students' activity. By proposing purposefully designed tasks, he confirms that students have integrated two selected implicit rules (content level): "To solve the questions about series that are given, their definition is not necessary," and "Applications of series, inside or outside of mathematics, are not important" (González-Martín, 2013, p.2328). Dorko (2020) analyses the activity of students using an online homework system proposing exercises about sequences. She identifies that one of the students' strategies is to draw on didactic contract rules at the level of the content, like "when a list of sequences is proposed, some are convergent and some are divergent." The results obtained in such studies reveal implicit contract rules that are conveyed by certain features of teaching and of which the teacher is unaware; González-Martín (2013) describes them as differences between the knowledge to be taught and the knowledge actually taught. Some students leverage contract rules to provide expected answers, instead of reasoning mathematically; moreover some rules limit what is actually expected of students.

Introducing an alternative didactic contract: is it possible?

Most intervention studies using the concept of didactic contract concern inquiry-oriented courses, conveying a specific didactic contract, different from the didactic contract in 'traditional' courses.

We use below the term 'usual didactic contract', which appears in some of these articles, but is not clearly defined by the authors. Grønbæk et al. (2009) study a project-based course, where students work in teams. The authors describe rules of the 'traditional' didactic contract at the general level, e.g. the students have to "do the assigned homework"; the teacher has to "institutionalise good solutions and answer students' questions." In the project-based course, these rules are broken and the teams of students organise their work autonomously. Following two teams of students, the authors observe that what they call an "a-didactic contract" develops in the teams, with rules about the important tasks to be performed and how they are shared within the group. This contract has a positive impact on students' representations about the problem-solving activity. Barquero et al. (2022) note similar changes in the didactic contract, in the context of Study and Research Paths (SRPs: courses starting from an open question where teams of students build their own paths in terms of sub-questions and associated answers). Students have more responsibilities, such as for planning their work or validating an answer. Teachers are no longer the sole holders of knowledge in the classroom. The authors interpret this new contract as a paradigm shift, from 'visiting monuments' to 'questioning the world'. Nevertheless this new contract, very different from the 'usual' one, has to be established in the class and its negotiation can be a delicate process. Dawkins et al. (2019) ask whether or not students adopt the new rules of the didactic contract in an inquiry-based course. Interestingly, they note that some students adhere to the new rules, while others reject them, and this is related to their orientation towards learning mathematics: learning orientation (high buy-in) vs performance orientation (low buy-in).

The study by Ellis et al. (2025) does not concern an intervention; but it evidences the possibility of negotiating different contracts. The authors study the link between students' homework and the didactic contract in the context of a national survey in the US involving 500 institutions. They select 5 universities with successful Calculus I programs and, based on a rich set of data (classroom observations and videos, students' productions, interviews with students and teachers), they observe that the didactic contract regarding homework (at the general and disciplinary levels) in these universities differs from the contract in the other universities. Students are expected to struggle with a more complex content; but they do not struggle alone, they are provided with relevant material to support them in their homework; they are also expected to explain their thinking, and they receive feedback. This study evidences the existence of different didactic contracts in different universities, and the possibility of installing a didactic contract fostering a productive homework.

The papers coded "Other" address specific issues like didactic contract rules when using digital resources, like videos or online exercises. The study by Gill and O'Donoghue (2007) concerns the didactic contract in mathematics service courses in Ireland. Interestingly, the authors identify mismatches between students' expectations and the course delivery. Nardi (2008), interviewing lecturers, notes contradictory expectations which lead her to coin the expression "fuzzy didactic contract", and to expect difficulties for students to understand the rules of this contract.

Discussion

We identified three main categories of papers: (1) papers focusing on the change of didactic contract during the secondary-tertiary transition; (2) papers identifying limitations in the students'

mathematical activity resulting from didactic contract rules; (3) intervention studies proposing an "alternative contract." The features of the didactic contract identified by the studies depend on the category. In category (1), the authors note an increase in expectations of students (e.g. in terms of flexibility, rigor, autonomy). This change creates difficulties for students. Moreover, they are sometimes unaware of this change in expectations; making the new rules explicit can support their entrance in the new contract. The studies in category (2) unveil contract rules conveyed by the teaching practices at university, of which the teachers are unaware. These rules can prevent students from developing a rich mathematical activity. Finally, in the context of research-based interventions, studies in category (3) focus on a didactic contract that is different from the 'usual' contract at university. For such studies, the 'usual' contract can constitute an obstacle, since some students reject the breaches in the rules introduced by the new contract of the intervention.

These answers to our research questions raise some issues, since some of their results can seem contradictory. In the context of an 'ordinary' teaching, studies in category 1 evidence significant expectations at university, while category (2) studies identify limitations in what is actually expected at university, at the level of the discipline and of the content. This apparent contradiction could mean that the expectations and obligations of students at university, although higher than those at secondary school, remain below possible expectations for the mathematics taught. A second issue concerns the entrance of students in the didactic contract: while category (1) studies foreground the difficulty for students to enter into the new contract, studies of the second and third categories, on the opposite, produce results showing that students adopt this 'usual' contract. Moreover this adoption has negative consequences: limitation of the students' mathematical activity, or obstacle for their involvement in an inquiry-oriented teaching. Would the students adopt the 'usual' didactic contract only when researchers want them to act differently? An explanation could be that most students, within a few months, actually comply with rules conveyed by the teaching practices shared in a majority of courses (the 'usual' contract), and that breaking these rules is complex. This leads to a third issue: the existence of a 'usual', dominant contract, regardless of the university, the teacher and the students. Intervention studies proposing an inquiry-oriented didactic contract evidence the possibility, under specific conditions, of negotiating alternative contracts. Even in 'ordinary' conditions, students who have several mathematics teachers at university may experience different contracts. The same teacher may also have different kinds of interactions with different groups of students, and this can produce several contracts in the same class. Elucidating all these issues and apparent contradictions requires further research.

Conclusion

While many studies in university mathematics education refer to the concept of didactic contract, there is still a lot to be learnt about didactic contracts at university. Pursuing in our project the aim of determining implicit expectations of teachers and students, we will not restrict ourselves to the didactic contract but will also consider other theories likely to enlighten such implicit expectations, for example the commognitive theory (e.g. Sfard, 2023). Moreover so far the concept of didactic contract is not associated with a solid method allowing to identify its rules; considering other theories can suggest principles for such a method. Understanding the mismatches between students and teachers at university probably requires a combination of several theories.

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Using knowledge from mechanics to solve a calculus task: a study through the lenses of mixed praxeologies

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In this paper, we examine the strategies and reasoning employed by students who have completed college-level calculus and mechanics courses as they undertake an unfamiliar graphing calculus task. Adopting a perspective based on the Anthropological Theory of the Didactic (ATD), we analyse students' techniques and justifications, paying particular attention to their use (or non-use) of praxeologies acquired in mechanics. Our findings reveal that students who employed either a praxeology rooted in mechanics or a mixed praxeology combining elements from calculus and mechanics were generally more successful in solving the task. We conclude by discussing the practical and theoretical implications of these results.

Keywords: Differential calculus, mechanics, Anthropological Theory of the Didactic (ATD), college mathematics, mixed praxeologies.

Introduction and research problem

In her literature review on the interconnections between the learning of physics and mathematics, Marrongelle (2004) concludes that "mathematics understanding plays a role in physics learning, and physics experiences seem to have an effect on learning some mathematics concepts" (p. 260). One such concept is the derivative, which (like velocity and acceleration) is central to dynamics. As Stoffels et al. (2022) highlight, dynamics, is "the most common content […] that connects differential calculus and physics, especially in undergraduate mathematics and physics" (p. 235). However, the derivative has been consistently identified as challenging for students, both in purely mathematical contexts (e.g., Fuentealba et al., 2017) and when applied in a physics context such as dynamics (e.g., Stoffels et al., 2022). Moreover, our literature review (e.g., Hitier & González-Martín, 2022) reveals conflicting perspectives on how physics contexts help students understand the notion of derivative.

To explore how students use a physics context to solve calculus tasks, Marrongelle (2004) developed the *Physics Use Classification Scheme*, comprising four categories (Figure 1). However, Stoffels et al. (2022) critique the ambiguity of the *Language-Mixer* category and note that "there is also no evidence that any of the identified user types is more successful [than the others] in solving differential calculus tasks" (p. 238). Additionally, we identify a possible limitation in Marrongelle's study: the participants were enrolled in an *Integrated Calculus and Physics Class* (ICPC), which could influence their use of physics by, for instance, blurring the boundaries between the two disciplines. To our knowledge, since Marrongelle's (2004) work, no subsequent study has investigated how success in solving tasks might vary across the different categories. This gap highlights the need for further exploration of these categories to better understand how students draw on knowledge from calculus and mechanics to solve tasks.

Physics Use	Description
Contextualizer	Student introduces a physical context as a vehicle to solve calculus tasks. Student show evidence of immersing problems in physical contexts in order to solve them.
Example-User	Student evokes examples from physics to help her/him make sense of calculus concepts and problems. Student talks about physics in a way that is disconnected from the problem at hand, often using physics to justify an answer.
Language-Mixer	Student uses language from both physics and mathematics in her/his discussion of solution methods. This classification is applied to a student when there is not enough evidence to classify the student in another category.
Non-User	Student does not use physics, in any recordable written or verbal way to solve calculus tasks.

Figure 1: *Physics Use Classification Scheme*, excerpt from Marrongelle (2004, p. 263)

Theoretical framework

One of the key tools of the Anthropological Theory of the Didactic (ATD) for studying knowledge is that of *praxeology* (e.g., Chevallard, 2019). A praxeology is composed of two blocks. The *praxis* block, or the "'know-how' component" (p. 87), $\Pi = [T, \tau]$, encompasses the *technique* (τ) used to solve a specific *type of task* (T). This technique is supported and explained by a rationale, known as the *technology* (θ), which is embedded within a broader *theory* (Θ). Together, the *technology* and the *theory* constitute the *logos* block, $\Lambda = [\theta, \Theta]$, of the praxeology.

A foundational principle of ATD is that knowledge is institutionally situated. In other words, praxeologies vary depending on the institution in which they are developed or applied. For example, calculus and mechanics, seen as different institutions, may draw on different praxeologies involving the velocity as a derivative (e.g., Hitier & González-Martín, 2022). Recognising this variability, Chevallard (2019) introduces the construct of *mixed praxeologies* for individuals—"praxeologies made up [by an individual] of constituents taken from different disciplines" (p. 98). However, to our knowledge, mixed praxeologies have been described only theoretically in the literature.

Additionally, we consider a task to be *familiar* in a course if the task is part of the *knowledge to be taught* in that course and can be identified in relevant course materials, such as textbooks. Conversely, a task is classified as *unfamiliar* if it falls outside of these parameters.

In this paper, we explore how students draw on a mechanics context to solve an unfamiliar calculus task, through the lens of ATD. Specifically, we address the following research questions: *From which discipline—calculus or mechanics—do students select praxeologies to solve an unfamiliar calculus task? How successful are students at solving this task based on the praxeology they adopt?*

Methods

This paper draws on data from a larger research project that examined the pairing of a differential calculus course with a mechanics course at a large Canadian college, a post-secondary institution that Quebec students must attend prior to pursuing university studies. The regular syllabi of the two courses were not modified, although the teachers collaborated more closely. Therefore, the calculus and mechanics courses were still distinct and not as fully integrated as in Marrongelle's (2004) ICPC. All students registered in the paired courses were part of the college's Enriched Science program, which offers extracurricular activities such as weekly seminars. Our study involved three cohorts of students: one regular or non-paired (R) and one paired (P_1) in the Fall 2019 term, and one paired (P_2) in the Fall 2021 term. No data were collected in 2020 due to the pandemic.

The participants were asked to answer three questionnaires throughout the term, with the last questionnaire submitted during a calculus class near the end of the term. By the time the final questionnaire was distributed, the students had studied one-dimensional kinematics as part of their mechanics curriculum. At this point in their calculus studies, in addition to the usual differential calculus topics (such as derivatives), they had recently been introduced to antiderivatives and had learned how to determine simple antiderivatives symbolically (e.g., determine an antiderivative for a polynomial function). A total of 85 students responded to the final questionnaire (R: 27, P_1: 30, P_2: 28). In this paper, we focus on the question presented in Figure 2. Our previous research (e.g., Hitier & González-Martín, 2022) indicates that this is an unfamiliar task in calculus, but that similar tasks, such as sketching a position-versus-time graph given a piecewise linear velocity-versus-time graph, are familiar in mechanics.

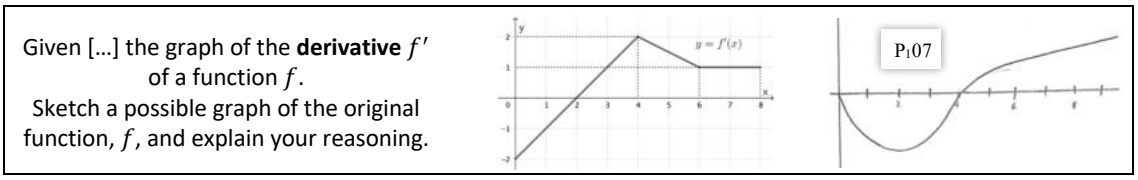


Figure 2: Graphing task (left) and a student's correct sketch of the solution

After an *a priori* analysis of the task, we conducted a praxeological analysis of the students' answers, identifying the techniques and the rationales behind those techniques. We also assessed the graph that each student produced, paying particular attention to their sketching of the four main shapes that comprised it (12 marks in total): decreasing concave up on [0, 2], increasing concave up on [2, 4], increasing concave down on [4, 6], and increasing straight line on [6, 8]. We also assessed three main features (12 marks in total) of the graph: the graph's continuity, its differentiability, and the symmetry of the first parabolic segment on [0, 4] (i.e., $f(0) = f(4)$). Because we asked students to create a sketch and not a precise graph, we did not take into account the use of a scale on the vertical axis and the precision of the values of f with respect to that axis.

A priori analysis of the task

Based on our previous work (e.g., Hitier & González-Martín, 2022), we considered three main approaches in the *a priori* analysis of the task:

1. *AppG*, where the technique consists of identifying relevant features of the graph of the derivative (e.g., its sign), and then attempting to sketch a graph consistent with what can be deduced from the technology. For instance, sketching an increasing graph when the derivative is positive, that is, applying the technology whereby the derivative corresponds to the slope of (the tangent to) the graph of the function, such that the sign of the derivative determines the variation of the graph of the function.
2. *AppE*, where the technique involves determining the equations for the distinct segments of the graph (e.g., $f'(x) = x - 2$) or identifying the form of these equations (e.g., non-constant linear). Subsequently, knowledge related to integration is applied to determine the equations for the corresponding segments in the graph of the function (e.g., $f(x) = \frac{1}{2}x^2 - 2x$) or to establish the form of that equation (e.g., parabola).

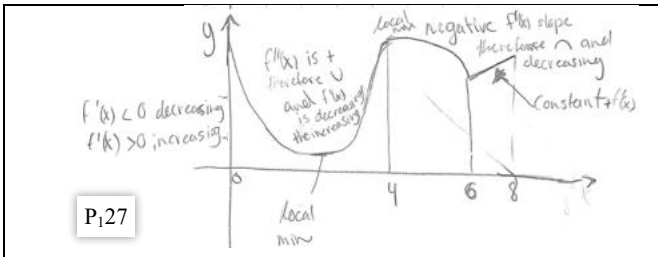
3. *AppM,* where participants clearly draw on their praxeological equipment from mechanics. This could be done explicitly, as demonstrated by student $P_1$24, who explained that they "treated this graph as a velocity-versus-time graph like the ones [they] saw in mechanics, then […] drew the integral of it (position versus time)". In another approach, students could use elements of a praxeology that, according to our previous work, belongs to the mechanics course (for instance, the use of the area under the given graph).

Based on our previous research, we associate the two first approaches with praxeologies from calculus, and the third, by definition, with praxeologies from mechanics. It was anticipated that certain participants' techniques or technologies would be inexplicit or unclear.

Data analysis

Categorising praxeologies

Following the *a priori* analysis of the task and the praxeological analysis of the students' written work, we grouped the students' praxeologies into four categories (see Figure 3):

<table>
<tr>
<td colspan="2"> 
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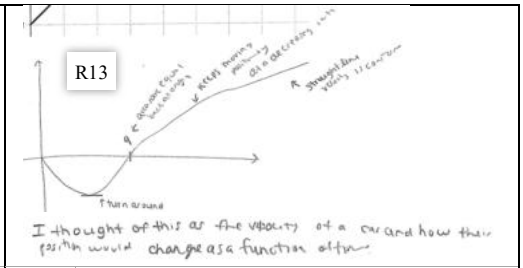
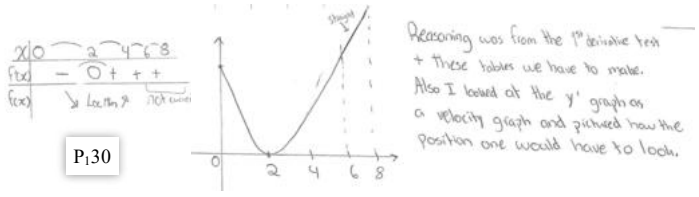
<td colspan="2"> 
</td>
</tr>
<tr>
<td>Cal</td>
<td>Praxeologies like AppG or AppE with no indication that the student drew on their knowledge from mechanics (this would correspond to Marrongelle's (2004) Non-User).</td>
<td>Mec</td>
<td>Praxeologies associated with AppM (this would correspond to Marrongelle's (2004) Contextualizer).</td>
</tr>
<tr>
<td colspan="2"> 
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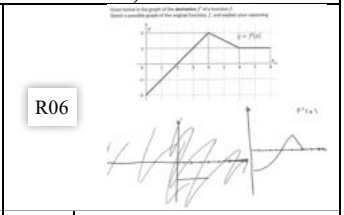
<td colspan="2"> 
</td>
</tr>
<tr>
<td>Mix</td>
<td>Mixed praxeologies where the praxeology used by the student combines elements from AppG or AppE alongside AppM. These students would likely have been classified as Language-Mixers by Marrongelle (2004).</td>
<td>Unc</td>
<td>Praxeologies too unclear to be classified under the previous categories.</td>
</tr>
</table>

Figure 3: Four categories (and examples) for students' praxeologies

As expected, given the purely mathematical context, most students' praxeologies were classified as *Cal* (68% overall; R: 59%, P_1: 77%, P_2: 68%). Within this group, approximately 17% employed *AppE* techniques despite the graphical context (R: 31%, P_1: 13%, P_2: 11%). Notably, four students out of the 85 ($P_1$03, $P_2$06, $P_2$22, $P_2$25) used mechanics vocabulary in their rationale (as in Marrongelle's study), but applied only *AppE* approaches to solve the task. Their explanations described graph behaviour without referencing a physics context, so these praxeologies were classified as *Cal*.

In contrast, almost a quarter of the students (about 24%, R: 22%, P_1: 20%, P_2: 29%) used praxeologies involving mechanics (*Mec* or *Mix*). Among these, 35% employed mixed praxeologies, although the distribution varied across groups (R: 0%, P_1: 17%, P_2: 75%).

Students' use of *mechanics* and *mixed* praxeologies

Some students employed what we call a *geometric-numerical* approach, calculating areas under the graph of velocity ($f'(x)$) to derive position. In the *Mec* category, $P_1$29, R27 and R14 used this approach. We note, however, that for R27, the resulting graph showed the correct increasing and decreasing intervals but appeared almost piecewise linear. In the *Mix* category, $P_2$12 also tried this approach, as part of a mixed praxeology using elements from *AppG*. $P_1$04 (*Mec*) and $P_2$11 (*Mix*) used only the sign of the area to support their rationale; $P_1$04 based their reasoning on the sign and variations of the velocity, while $P_2$11 used the sign of the area to verify part of their work. $P_2$11 exhibited confusion, "because the area under the curve of f' is negative from $x = 0$ to $x = 2$, which means that f should also be negative [although their graph was positive at first]. However, after $x = 2$ the area under the curve is positive but $x = 2$ is a minimum." In their confusion, $P_2$11 conflated displacement with position and considered segmented intervals rather than cumulative areas. The remaining students who used areas under the curve ($P_2$21, $P_2$24, R13) did so mainly to support the fact that $f(0) = f(4)$. The approach of $P_2$21 and $P_2$24 placed them in the *Mix* category, as they drew on mixed praxeologies using elements from *AppG*. The approach of R13 landed them in the *Mec* category, as they "thought of this as the velocity of a car and how its position would change as a function of time".

The rationales offered by participants who used other mechanics-based praxeologies (*Mec* and *Mix*), excluding those involving the area under a curve, displayed considerable variability in the level of detail provided. While some students supplied only brief responses (e.g., $P_1$24, as noted in the *a priori* analysis), others gave more comprehensive explanations. These included interpreting the sign and variation of velocity, as well as other methods such as recognising that linear velocity results in a quadratic position (e.g., $P_2$05). The three students who drew on mixed praxeologies *not* involving area under a curve took the following approaches:

- $P_1$30 used the first derivative test and noted they "also looked at the y' graph as a velocity graph and pictured how the position graph would have to look".
- $P_2$08 alternated in their rationale between mathematical explanations (e.g., "from $x \in [0, 2]$ the slope is up but closing in on 0") and movement-based descriptions ("it then speeds up from after 2s until 4").
- $P_2$34 drew on all three identified approaches: recognising "$f' \rightarrow$ first degree [implies that] $f \rightarrow$second degree" (*AppE*), using the second derivative's sign to infer graph concavity (*AppG*), and describing movement as "the position goes in the opposite direction quickly at first, then slowly" (*AppM*).

Insights gained from student responses

The responses provided new insights into our *a priori* analysis. Initially, we anticipated calculus approaches to be either graphical (*AppG*) or numerical (*AppE*), but we did not expect to find

differences between praxeologies drawing on mechanics. The analysis of students' answers revealed two distinct mechanics-based approaches. The first is a "geometric-numerical" approach based on the area under the graph, which allowed students to identify specific points on their graphs or recognise symmetry in the first parabolic segment; this approach is similar to *AppE*, as it relies on formula application. The second relies on a graphical reasoning, whereby students used features of the velocity graph to sketch a corresponding position graph; this reasoning aligns with *AppG* in *Cal* but often incorporated covariational reasoning—interpreting the slope of the velocity graph as acceleration to infer the concavity of the position graph. This goes beyond rote memorisation of the link between the second derivative's sign and concavity.

Focusing more specifically on the observed mixed praxeologies (approximately 8.2% of participants, R: 0%, P_1: 3.3%, P_2: 21.4%), we note that the students using this approach combined calculus and mechanics in diverse ways. $P_2$12 used mechanics knowledge to refine their graph, employing the area under the curve to determine numerical values for f. $P_2$11, $P_2$24 and $P_1$30 seemed to switch between disciplines as needed. $P_2$11 explicitly noted the interplay of calculus and mechanics, annotating statements like "used calculus initially, but then mechanics to verify that it was right", or "used mechanics mostly, but I understand that this links to calculus". $P_2$24 started with a sign table (*Cal*) but shifted to interpreting $f'(x)$ as a velocity graph, eventually acknowledging, "in the end, the table helps, it's more systematic". $P_2$24 also provided key hints that indicated their shifting between *Cal* and *Mec*, writing that "maybe [they] had a lot more practice turning $v(t)\rightarrow x(t)$ ($f'(x)\rightarrow f(x)$), so [the] physics approach is easy". The other three participants used mixed praxeologies in a different way: their approach largely relied on technologies from mechanics, which they "translated", at least in part, into calculus terms. For instance, $P_2$34 stated they "thought of it as a position-vs-time graph to make it simpler", but their rationale included sentences like "$f'(x)$ is decreasing but the function remains $\oplus$". We see this as an example of thinking in terms of mechanics but writing in terms of calculus, possibly influenced by the question's calculus context.

Beyond analysing the praxeologies used, we also examined the participants' success. Figure 4 displays the median and average score for each group (R, P_1, P_2), categorised by praxeology discipline (*Cal*, *Mix*, *Mec*, or *Unc*), along with the proportion of students in each group using these praxeologies.

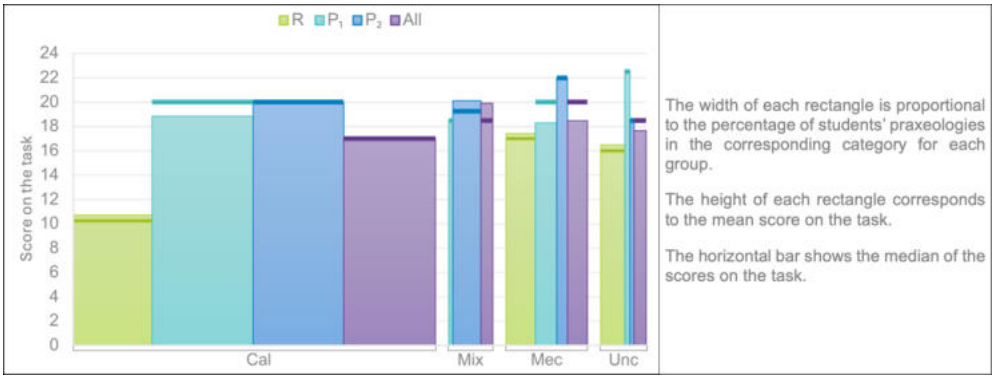


Figure 4: Results of the task for each group depending on the category (average and median)

Differences are more readily discerned among the groups of students than between the disciplines underlying the praxeology:

- Group R achieved overall lower scores than P_1 and P_2, with no mixed praxeologies observed in R. Group R also displayed the greatest disparity in their scores, depending on the type of praxeology used.
- In P_1, scores were consistent across praxeologies, with mixed praxeologies (*Mix*) scoring slightly lower on average. In P_2, students using *Mec* praxeologies outperformed their peers.
- Across all groups, students who used *Cal* praxeologies generally scored lower compared to students who employed *Mec* or *Mix* praxeologies. Additionally, mixed praxeologies yielded the highest average scores, while *Mec* praxeologies produced the highest median scores.

In short, students leveraging mechanics-based praxeologies (*Mec* or *Mix*) generally performed better on this unfamiliar task, underscoring the value of interdisciplinary reasoning. However, the variability in how students integrated mechanics and calculus highlights the importance of enhancing teaching strategies to explicitly connect each discipline—as seems to have been achieved in the paired classes.

Concluding remarks

In this paper, we analyse students' responses to an unfamiliar calculus task resembling a task from their mechanics course. This context allowed us to observe praxeologies drawing on calculus, mechanics, or a combination of the two. When both disciplines were involved, we identified praxeologies that either employed numerical procedures or relied on properties of the given graph and the relationship between the derivative and the original function (or the velocity and the position functions in mechanics). Our findings build on Marrongelle's (2004) work. Students employing praxeologies strictly from calculus were categorised as *Non-Users*, while those using mechanics-based praxeologies were labelled *Contextualizers*. However, in identifying students using mixed praxeologies, we were able to refine the *Language-Mixer* category. Specifically, we observed two distinct ways in which students integrated their knowledge from calculus and mechanics. On the one hand, some students "translated" concepts between calculus and mechanics; while their rationales included calculus-based vocabulary, their reasoning was grounded in mechanics. These students could be considered *Contextualizers*, as their main conceptual framework remained tied to mechanics. On the other hand, certain students seemed to draw flexibly from both disciplines, choosing the most suitable technology or reasoning strategies regardless of their disciplinary origin. Interestingly, we did not classify any participant as an *Example-User*. This may be because our questionnaire offered students limited opportunities to provide examples, unlike Marrongelle's (2004) interview method.

Regarding task performance, students in the non-paired course who explicitly used mechanics knowledge achieved higher scores. However, our results support the conflicting views in the literature (e.g., Hitier & González-Martín, 2022) concerning the role of physics contexts in learning calculus notions. For instance, although $P_2$11 successfully drew on both calculus and mechanics to sketch a reasonable graph, their use of areas under the curve led to confusion. This confusion was traceable to gaps in their understanding of certain mechanics concepts.

Our study also makes a theoretical contribution to the literature. The use of knowledge from one discipline to solve a task in another is often framed as a transfer of knowledge (e.g., Roorda et al., 2015). However, here we adopted Chevallard's (2019) construct of mixed praxeology, which, as far as we know, has never been explored empirically. This enabled us to observe how some students

integrate knowledge from both disciplines to tackle an unfamiliar task, revealing two key patterns: 1) Some students successfully combined knowledge from both disciplines, aligning with literature suggesting that interdisciplinary reasoning fosters a deeper understanding of concepts (e.g., Jones & Watson, 2017); 2) Conversely, other students appeared to use mixed praxeologies as a coping mechanism. For example, $P_1$30's graph resembled a parabola that transitioned into a straight line after 6. This outcome reflected their difficulty in determining the graph's shape on [4, 6].

Finally, we acknowledge limitations in our study. First, although many participants provided written rationales explaining their reasoning, task-based interviews would have provided clarification. For instance, some students may have fully translated mechanics reasoning into calculus terminology, obscuring their reliance on mechanics. Moreover, students from paired courses outperformed those in group R. This may reflect the benefits of pairing, but it could also indicate differences in student profiles; the Enriched Science program could attract students with higher motivation and curiosity, leading them to explore beyond the standard curriculum. Lastly, in contrast to Marrongelle's (2004) broader investigation, our study focused on a single calculus task, which, although graphically contextualised, was relatively accessible to students familiar with similar mechanics tasks.

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Using scriptwriting as a response format for interface tasks: Exemplary analyses in the context of symmetry

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This paper investigates so-called interface tasks as a means to bridge academic mathematics and school mathematics in university courses for pre-service teachers (PSTs). While prior research has focused on the development, implementation, and theoretical foundations of such tasks within a geometry course, less attention has been given to response formats. This study examines the scriptwriting format (as an alternative to the letter/mail format used before), where PSTs are tasked with responding to a student's question about symmetrical figures. I explore the theoretical potential of scriptwriting as a response format and present initial findings from a qualitative analysis of the scripts produced by PSTs.

Keywords: pre-service teachers, interface tasks, scriptwriting, symmetry

Introduction: Interface tasks

The transition from university mathematics to school teaching poses significant challenges for upper secondary pre-service teachers (PSTs), often called the *second discontinuity* of teacher education (Klein, 2016). One way to counteract this problem in university mathematics courses is to use special tasks (e.g., Bauer, 2013; Biza et al., 2007; Eichler & Isaev, 2023; Hoffmann & Biehler, 2023, 2024) that aim to show and reflect on connections between academic mathematics and mathematics in schools. In literature, various theoretical approaches for such task concepts can be found, which have a certain degree of situatedness regarding typical activities of a mathematics teacher in practice as a common design element. In Hoffmann and Biehler (2023, 2024), we provide a theoretical foundation for situated tasks in the context of the framework of content-specific teacher expertise (e.g., Prediger, 2019) and understand so-called *interface tasks* "as learning opportunities whose design is based on identified [jobs] with the following objectives: first, to enhance the probability that PSTs will use their academic CK [content knowledge] and skills in future teaching situations and second, to foster the attitude that their academic CK and skills are relevant by broadening and deepening PSTs' views on future typical requirements." In this, the term "job" is understood as "typical, often complex situational demands of subject-matter teaching […]" in the sense of (Prediger, 2019, p. 370).

While there are some works on the selection and characteristics of suitable situations/contexts/links (Prediger, 2013; Wasserman, 2018; Weber et al., 2023), research on different response formats remains scarce. However, thinking about proper response formats seems to be important in that – at least in university mathematics courses – interface tasks are, to a certain extent, fictitious, usually being completed in written form as homework. For example, there is no real oral discussion with an actual student. This means that no matter how realistic the situational setting of an interface task is, the processing is always somewhat contrived, and suitable response formats are, therefore, essential.

This paper aims to provide a small exploratory contribution to this area of research. A previous study focused on an interface task (Hoffmann & Biehler, 2024) where PSTs were asked to respond to a

high school student's symmetry-related question in an email or letter. For the current study, the response format was modified to *scriptwriting tasks*, which have been successfully applied in other projects (e.g., Zazkis & Kontorovich, 2016).

My primary research interest is to explore how this alternative response format influences typical PSTs' solutions. Before presenting the study, I will introduce the interface task and highlight key findings from the initial research, indicating that changing the response format to scriptwriting could improve the task concerning the intended objectives.

An interface task on 100°-rotationally symmetric figures

The following subtasks are those parts of an interface task we researched as part of Hoffmann and Biehler (2024) that are relevant to the present article.

> Consider the following student statement, "Except for circles or circular disks, there can be no other 100◦-rotationally symmetric figures."
>
> a. Mathematize the statement and clarify the question based on your academic mathematics background.
> b. Respond to the student statement (grade 7) in letter or mail form in a differentiated manner that promotes learning. Feel free to use visualizing elements as well.

The task is part of a sixth-semester course, "Geometry for PSTs". The course is attended by pre-service teachers (PSTs) who will later teach primarily at the upper secondary level. It should be noted that such courses are relatively uncommon in German teacher education, as upper secondary PSTs typically attend mathematics courses together with mathematics majors. Additionally, PSTs participate in courses on mathematics education during their studies. Thus, our interface tasks do not primarily promote pedagogical content knowledge. In particular, the course had no new input on competencies in this area.

The PSTs complete the subtasks presented above after treating the topic of symmetry in the course where the symmetry properties of n-gons were not explicitly discussed. However, it was explained that symmetries of a figure generally form a group. Based on previously proven statements about the concatenation of reflections and rotations, various statements about necessary constellations for the existence of reflectional and rotational symmetries in a symmetry group were shown. This includes a lemma that every α-rotationally symmetric figure is also $(k \cdot \alpha \bmod 2\pi)$-rotationally symmetric for every $k \in \mathbb{Z}$. Thus, an important task in developing a response to the student is to reduce the algebra-oriented university perspective on symmetry didactically into school mathematics.

To complete the interface task, the PSTs must analyze and relate the student's statement to their CK, particularly the course content. We refer to this step as *mathematical clarification*. Building on this, the PSTs must use both their general pedagogical and pedagogical content knowledge, as well as their experience and intuition, to carry out an *elementarization* that is sensitive in terms of content and language to the student's level of understanding (Hoffmann & Biehler, 2024, p. 1449) to answer the

student's question. This step-by-step approach is an established structure of interface tasks, as used, for example, by Biza et al. (2007, p. 303).

In the definition of interface tasks provided above, one central focus is on the *job* that underpins the given situation. For the interface task presented above, this *job* is *examining students' contributions and mistakes and reacting to them in a way conducive to their learning*. Of course, this job can be focused on in university teacher training from various perspectives. The mathematics-focused and idealized approach of *mathematical clarification* and *elementarization* specified by the subtasks is one such perspective ("model job", Hoffmann & Biehler, 2024, p. 1446) that aligns with the course context and content.

In Hoffmann and Biehler (2024), we reconstructed the mathematical arguments of the PSTs and then evaluated them further using qualitative methods. One focus of the analysis was identifying the so-called 'overarching lines of argumentation'. This refers to a holistic description of the mathematical argument underlying a PST's processing of a task part. A corresponding way of categorizing PST responses according to general mathematical approaches can also be found, for example, in Biza et al. (2007, pp. 304-305, 307-308.). In analyzing the answers on a) and b), the correct answers could be assigned to one of two categories: Obviously, the student's statement on the existence of 100°-rotationally symmetric, non-circular figures is incorrect. This can be shown by giving an example (such as a suitable regular n-gon, e.g., n = 18, 36, 72, 360, ...). The two subcategories that work in our analysis for assigning different overarching lines of argumentation are the following: Either the 100° rotational symmetry is justified for a specific n-gon without explaining how this example is found (*Cat. A*), or it is explained how a suitable example of a 100°-rotationally symmetric figure can be constructed systematically (*Cat. B*) (Hoffmann & Biehler, 2024, Sect. 5.2.1).

Regarding subtask 2b, we found that most PSTs' responses fell into Category A: they provided an example of a 100°-rotationally symmetric figure (typically a regular 18-gon or 36-gon). However, they did not explain how this example was found to the student. However, such an explanation would be important from the perspective of the desired "learning conduciveness" of the reaction to the student's contribution. Furthermore, from the perspective of the lecturer and the researcher, it offers an opportunity to observe the PSTs acting mathematically in a school-related situation to diagnose individual strengths and weaknesses and explore the general processing of interface tasks (see also Biza et al., 2007, p. 302).

From a research perspective, this result remains unsatisfactory. It is unclear whether the PSTs themselves did not employ a systematic method to construct a 100°-rotationally symmetric figure—perhaps relying on trial-and-error or internet searches—or whether they had a systematic approach in mind but, for reasons not evident in the data, deemed it irrelevant to include in their letter or email to the student. This article focuses on how a modified response format can help address this open aspect.

Theoretical Background: Scriptwriting tasks in teacher education

In literature, *scriptwriting* in mathematics teacher education describes activities in which fictitious dialogues (often between teacher and student) must be continued from a given beginning of a conversation (e.g., Zaskis & Marmur, 2018). A variety of objectives can be addressed for PSTs'

learning, such as “examining a personal response to a situation, exploring erroneous or incomplete approaches by a student, revisiting and possibly enhancing personal understanding of the mathematics involved, and enriching the repertoire of potential responses to be used in future ‘real’ teaching” (Zaskis & Marmur, 2018, p. 294). In addition to such instructional objectives, scriptwriting is also a helpful research tool because “scripts form a rich data source that can be examined from various perspectives and provide a lens for examining images of teaching and insights into the scriptwriters' understanding of mathematics” (Zaskis & Marmur, 2018, p. 294).

Zaskis and Marmur (2018) used scriptwriting tasks to, among other things, promote the understanding of PSTs on the concept of function and to establish connections between university mathematics and school mathematics. They gained insights into the PSTs’ personal example space on functions and described differences in connectedness and richness. These results formed the basis for follow-up instructions for the PSTs. This underlines the dual usefulness of the scriptwriting format for teaching and research. Campbell and Baldinger (2022) also used scriptwriting tasks in PST training. They could identify and analyze various resources that PSTs used when responding to a student error about the concept of polygons. The resulting scripts helped to reveal “patterns of teaching moves”, the way new mathematical ideas were “introduced”, how the PSTs “represent student’s voice,” and “whether or not the [student’s] error was resolved” (Campbell & Baldinger, 2022, p. 516).

An important commonality in these and other projects (e.g., Buchbinder, 2018; Shure & Rösken-Winter, 2022; Zazkis & Kontorovich, 2016) is that scriptwriting is used as a productive “approximation of practice” for teacher training, which includes “opportunities to rehearse and enact discrete components of complex practice in settings of reduced complexity” (Grossman et al., 2009, p. 283; Zazkis & Marmur, 2018, p. 294). The "reduced complexity" mentioned in the quote fits the definitional idea of grounding interface tasks on *model jobs* (Hoffmann & Biehler, 2024; see above), which, due to their deliberately narrowed perspective (mathematics-focused) compared to the actual job in the classroom, are also of reduced complexity. However, this allows a “focus on individual aspects of teaching” (Campbell & Baldinger, 2022, p. 509), which in the interface task above is the mathematical clarification and subsequent elementarization.

Interface tasks and scriptwriting match well on a theoretical level, which means that the use of scriptwriting activities as a response format seems appropriate in the first instance. I described above the problem that many PSTs used an “overarching line of argumentation” that was not optimal for the fictitious student from the perspective of promoting learning when responding to the student's question about the existence of 100° rotationally symmetrical figures and did not explain how to construct a suitable n-gon systematically. If the PSTs are already aware of such a procedure but do not consider it relevant to the task, this could be due to a possible non-identification with the situation (explain in a letter/email). If the student question is instead used as the starting point for a dialog between teacher and student to be written, the result is a format that is much closer to real classroom interaction (“approximation of practice”). When developing their dialogs, PSTs need to think about how they represent the student's voice and, thus, indirectly think about the learning value of the teacher's response. In addition, new mathematical ideas must be deliberately placed in the dialog for the fictitious student. Both are aspects that, according to Campbell and Baldinger (2022; see above), can be revealed in scripts, i.e., they also play a role during the PSTs’ scripting process.

Overall, the use of scriptwriting activities as a response format for the existing interface task presumably offers advantages for the PSTs in their role as learners as well as for research. In the first step, it can be more validly checked or excluded whether/that individual PSTs are actually unable to construct a 100°-rotationally symmetric n-square systematically, which would illuminate a blind spot in the original study. Furthermore, according to the four features of scriptwriting activities identified by Campbell and Baldinger (2022), there is the potential to gain further insights that can be used to develop and refine design principles for learning opportunities in profession-oriented mathematics courses for PSTs. However, this is not part of this paper.

Research Design

As part of a revision of the interface task introduced at the beginning of this paper, subtask b was modified using scriptwriting, following the considerations in the previous section.

> b. Write a fictitious dialog between a teacher and an interested student at an intermediate level. The dialog starts with the student's contribution and then extends over at least four parts per person.

Following the above analysis of the potential of scriptwriting activities, I expect that the framing of the task by the PSTs as more of a dialog situation will lead to it being more typical to first explain how one arrives at the example that is then further used, rather than it simply falling out of the blue. Accordingly, this paper is intended to contribute to answering the following research question:

How does the interface task's modified response format (scriptwriting) influence the overarching lines of argumentation that can be reconstructed in the PSTs' responses, specifically in terms of their conduciveness to learning?

The formulated dialogs of 19 PSTs are available to answer the research question. These were prepared as part of the homework for the course “Geometry for PSTs” in 2023, and the PSTs have consented to pseudonymized use for research purposes. The dialogs were analyzed using the evaluation method described by Hoffmann and Biehler (2024). The underlying mathematical argumentations were reconstructed and evaluated using qualitative content analysis methods. Initially, the categories from the previous analysis were used, with the option of expanding the respective subcategories inductively if necessary. Of course, this also prepares the way for answering further exploratory research questions on different response formats, which are not part of this paper. Nevertheless, I will make a few comments in the following presentation of the results, which show potential for exciting further analyses.

Selected results exemplified by Nick's dialog

Except for three PSTs, all others (16 PSTs) have formulated an appropriate dialog in terms of content. Only two PSTs gave an example of a 100° rotationally symmetrical figure in their dialog without clarifying how the example was derived (*Cat. A*, see above). Conversely, almost all PSTs (14 of the 16 PSTs) have opted for the second overarching line of argumentation (*Cat. B*, see above), which is that in their dialogs, it is discussed how a suitable example of a 100°-rotationally symmetric figure can be constructed systematically. The following excerpt from the dialog created by PST Nick serves as a typical example of this category:

2 Teacher: […] Do you remember the definition of rotationally symmetric figures?

3	Student:	Yes, well - a figure is rotationally symmetric if we rotate it by an angle and the figure looks the same again, i.e., the rotation does not change the figure.
4	Teacher:	That's right. Then why can a 100° rotation only leave the circle or the circular disk “unchanged” and no other figure?
5	Student:	Yes, that's logical because the rotation angle must be a divisor of 360°. And 100° does not divide 360°. So, only the circle or the circular disk, which is rotationally symmetrical to every rotation angle, can do that.
6	Teacher:	You have already understood the role of the divider very well. However, 100° does not necessarily have to be a divisor of 360°. We have to find a figure that is rotationally symmetric to both 100° and 360°. We would, therefore, have to check whether there is an angle of rotation to which the figure is rotationally symmetric in such a way that if we perform this rotation several times in succession, we get a total rotation of 100°. Then, the figure would not “change”. Do you have any idea how we can achieve this?
7	Student:	That would be the greatest common divisor of 100 and 360, which is 20.
8	Teacher:	Right. What does that mean for your assertion?
9	Student:	Well, then a 20° rotationally symmetric figure would also be 100° rotationally symmetric.
10	Teacher:	Correct. Does such a figure exist? Think about it yourself first, a sketch can also help here - or you can use GeoGebra.
11	Student:	A regular 18-gon! That's right! That means there are other figures besides circles and circular disks that are 100° rotationally symmetric. Thank you!

In Nick's dialog, the teacher first asks about the student's understanding of the concept of rotational symmetry and, building on this asks him to explain how the student comes to his statement. It becomes clear that the student thinks that every angle of rotational symmetry must always be a divisor of 360° [5]. The teacher takes up this contribution appreciatively and then introduces the central argument from a mathematical perspective (rotational symmetry from which 100° and 360° can be generated [6]), which is then completed by the student [9], who names the 18-gon himself [11]. Thus, in accordance with the analysis category, the student's statement is not merely refuted by giving a counterexample. However, the student's misconception is also addressed in a way that is much more conducive to learning, and by building on this, an argument for the correct solution is provided.

Beyond this insight, Nick's dialog can be used to show how the "features of scripts" (Campbell and Baldinger, 2021, p. 516, see above) also represent insightful perspectives for further analysis in the context of the interface task at hand: For instance, we can see relevant mathematics-related teaching moves such as the explication of definitions that are essential for the discussion [2-3]. There are also transcript passages in which the teacher takes up the student's statement to sow the seed for a new mathematical idea (explicitly using the greatest common divisor) [5-7]. In [5], Nick anticipates a very expectable misconception (which still occurs even with PSTs, cf. Hoffmann & Biehler, 2024), namely that non-existence follows because 100 is not a divisor of 360. In addition, the dialogue design also shows (in the sense of Nick's understanding of the underlying mathematics, cf. Zaskis & Marmur, 2018, p. 294, see above) that Nick succeeds in connecting the algebraic perspective on symmetries taken in the university geometry course with the question and elementarizing it appropriately.

Discussion

This paper presented parts of a study in which scriptwriting was used as a format for answering an interface task that had already been developed. A comparison with the analysis of the original

response format (letter/mail) shows a clear shift in the "typical" overarching line of argumentation. Of course, no generalizable statement can be made based on the available data, especially as two different groups with few students are compared. However, both groups had very similar study prerequisites, and apart from the change in the response format, no changes were made to the geometry course's concept. Altogether, the data strongly supports at least stating the hypothesis (in line with the theory-informed assumptions) that the scriptwriting format in the context of the interface task under consideration here leads to many PSTs using overarching lines of argumentation that are more conducive to learning.

Furthermore, the exemplary analysis of Nick's script shows that the categories used in other studies on and with scriptwriting activities (e.g., Campbell & Baldinger, 2022) can also be profitably applied for analysis here. For example, the representation of the student's voice is virtually impossible to evaluate in the original response format. Nick's script also indicates a close connection between the elementarized arguments and his university mathematics background.

For PSTs for whom these links are not so clearly recognizable in the scripts, scriptwriting seems to be both a diagnostic tool and a starting point for individual feedback to the PST. It allows the lecturer and the PST to jointly reflect, for example, on whether there are still gaps regarding the course content or whether specific mathematical concepts and relationships are considered irrelevant to the PST's design of the fictitious dialog. However, valid statements on the instructional potential in the context of our interface tasks require further investigation. Nevertheless, all these considerations point to the initially described research-related and instruction-related potential of using scriptwriting within interface tasks.

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From "chalk and talk" to active learning: inter-community partnerships and shifts in pedagogical and mathematical discourse

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Emergency remote teaching (ERT) during the Covid-19 pandemic exposed lecturers to new pedagogical challenges in which alternative active learning approaches were implemented to manage the absence of "chalk and talk". We explore how an inter-community (Mathematics, Mathematics Education) team can redesign a university mathematics course titled Mathematical Skills (MS) to facilitate shifts in the lecturer's pedagogical discourse and in students' mathematical discourse. Through a commognitive/communities of practice analysis of research design team interviews, we present preliminary findings that evidence shifts: from object-level to meta-level learning features in tasks posed by the lecturer; in the explicitness of lecturer goal setting routines; and, towards tasks that encourage active student participation. We conclude with noting the potential benefits to the students' learning experience emerging from the inter-community partnership.

Keywords: Active learning, Commognition, Communities of Practice, inter-community partnerships.

Introduction

In recent years, research related to Active learning (AL) pedagogies and practice orientated interventions in university mathematics has grown (Biehler et al., 2022). However, the adoption of AL approaches in tertiary mathematics has been gradual, with chalk and talk (C&T) remaining the pedagogical norm amongst mathematics lecturers (Cevikbas & Kaiser, 2023). The Covid-19 lockdowns forced universities to shift teaching online, labelled by Engelbrecht et al., (2023) as emergency remote teaching (ERT). In this new environment, lecturers needed to adapt their teaching approach to keep students engaged (Hooper & Nardi, 2021), as C&T pedagogies were incompatible with ERT. A shift to AL environments with flipped / blended pedagogy elements – including tasks such as problem posing, group work, presentations and time for class discussions – seemed to address some of the challenges of ERT (Hooper & Nardi, 2021), but a reported limitation was the lack of time and resources for effective planning and execution, with much left to trial and error (Engelbrecht et al., 2023). However, conditions are now more conducive to a shift towards AL pedagogies (Engelbrecht et al., 2023), with learned societies such as the Institute for Mathematics and its Applications in the UK funding professional development initiatives for mathematicians on said approaches (Thorogood, 2024). In this paper, we explore how establishing a research design team (RDT) can facilitate a collaborative inter-community partnership (ICP) between mathematicians and mathematics education researchers. The formation of an ICP serves two purposes– the first is to generate innovative pedagogical practice by redesigning a first year (Mathematical Skills, Sets, Numbers and Proof, thereafter MS) course that is infused with AL tasks. The second is to implement this redesign to facilitate shifts in the lecturers' pedagogical discourse, and the students' mathematical discourse. We present preliminary analysis of RDT data, reporting on the shifts in the lecturer's pedagogical discourse and the potential impact these may have on the students' mathematical

discourse. We first outline key elements from the AL and RDT literature before introducing the theoretical framing and methodology of our study.

Literature Review

The relationship between mathematics education researchers and academic mathematicians has been described as fragile (Nardi, 2016). However, the structure and dynamics of this relationship are shifting, as evidenced by the growth in ICPs (Biehler et al., 2022; Nardi, 2016): see, for example, the implementation of a research design cycle towards upscaling an intervention in an abstract algebra course in Biehler et al. (2022). Our study aims to build effective ICPs between mathematicians and mathematics education researchers that are "non-deficit, non-prescriptive, context specific and example centred" (Nardi, 2016, p. 365) and have the potential to spark communications that are "jargon-free, even though their construction is fundamentally driven by mathematics education research findings" (Nardi, 2016, p. 372). By establishing this ICP, we aim to find common ground between the two communities and facilitate a redesign of aforementioned MS course.

Active learning is defined by (Freeman et al., 2014) as a pedagogy that "engages students in the process of learning through activities and/or discussion in class, as opposed to passively listening to an expert" (p. 8413). The broadness of this term is beneficial for adopting a flexible, non-prescriptive approach that may blend various tasks and environments. For example, the diversity of AL approaches is evident in research on flipped/blended classrooms (Cevikbas & Kaiser, 2023). Additionally, research has focused on specific types of AL tasks, such as example generation (Kinnear et al., 2023) or solution sharing (Biehler et al., 2022). The recent growth in AL approaches in university courses may potentially be attributed to exposure to alternative pedagogies during ERT, which highlighted the potential effectiveness of AL to improve student engagement in university mathematics (Engelbrecht & Borba, 2024). Research on AL environments has highlighted benefits regarding student engagement, mathematical understanding and class discussion. However, such environments are time consuming to plan and rely on student buy-in to be successful (Cevikbas & Kaiser, 2023). A critique from Deitrich & Evans (2022) is that the reported benefits have not been consistently demonstrated in practice yet. Additionally, there is a growing body of research exploring specific AL tasks. For example, Kinnear et al. (2023) explored how example generation tasks support students' understanding of challenging concepts such as injective, bijective and surjective functions. Space constraints do not allow us to explore each type of task used in the project; yet, we note that a variety of AL strategies have been included in the redesign of the course to foster student understanding and engagement. Whilst not an exhaustive list, there are four types of tasks regularly incorporated into our course intervention: quizzes, example generation, group work and solution sharing. The ICP that our project is building is underpinned theoretically by networking two theories (Commognition, ToC; Communities of Practice, CoP) which we briefly outline in what follows.

Theoretical Framework

ToC, introduced by Sfard (2008), is a discursive approach to learning, with emphasis placed on the various acts of communication: with oneself, with peers, verbal and non-verbal (e.g. writing on the board). These communicative acts form part of a community's discourse, with mathematics discourse defined by four characteristics: word use, visual mediators, endorsed narratives and routines. Sfard

(2008) notes that mathematical learning can be at object-level (learning about the objects that make up the mathematical discourse, for example, learning the Fundamental Theorem of Arithmetic, what symbols such as ∃ and ∀ mean, and so on) and at metal-level (learning about how discursants are expected to behave within the discourse community in, for example, formulating and substantiating object-level narratives). As students transition to university mathematics courses, they shift from being governed by the rules of school mathematics to new, university mathematics rules. One such rule change students experience is the different expectations for what consistutes an acceptable mathematical argument (or proof). For students, just writing a solution with the correct final answer is insufficient at university level; the lecturer expects evidence of mathematical literacy, developed explanations and logical structure. At university, the rules of the mathematical game change, and engaging with university mathematics involves not merely learning about new mathematical objects (object-level learning) but, crucially, about these new rules (meta-level learning). To facilitate both types of learning, our MS course intervention incorporates variety of AL tasks which are designed to give students the opportunity to explore and engage with these new rules. The ToC was selected to capture shifts in student's mathematical discourse (Sfard, 2008) and the lecturer's mathematical pedagogical discourse (Hooper & Nardi, 2021). Using the ToC as a foundation to inspire our AL tasks, we can see from the examples below how well-designed tasks can facilitate shifts in the student discourse and create opportunities for our lecturer to undergo shifts in their mathematical pedagogical discourse. In this project, we utilise the ToC as a tool to analyse the data and inform the design of our intervention. This will be expanded upon in subsequent sections.

According to Wenger (1998), CoPs are established overtime where a group is defined by the ways members have a shared relationship, beliefs, values, and tasks. In the MS course, we define the community by the beliefs (in this case their view of mathematics, and what it means to them to be a mathematician), the pursuit of shared goals (e.g., gaining skills to be a successful mathematician at university) and the tasks that make up the course (e.g., using proof by induction, deciding if a function is injective or not). Within the CoP, we have two distinct players – the old-timers who are experienced, fully active members of the community (e.g., the lecturer) and the newcomers, those that are yet to engage with the community, its goals and tasks (e.g., the first-year students), but by participating in its practices they aim to become old-timers. In this CoP, Witty is the lecturer of the MS course with the CoP shaped by his previous teaching experiences, his beliefs, the goals he wants students to achieve and tasks. As a researcher in UME, Hooper's role in shaping the CoP is to highlight relevant research literature (e.g., as mentioned in the literature review above) as well as experiences as an old-timer on a similar course. To facilitate the students' transition the new rules of university mathematics, we implement what Wenger (1998) calls an inbound learning trajectory: this involves designing a sequence of tasks and goals for newcomers to engage with during the semester. The concept of a learning trajectory provides a framework for the MS intervention to scaffold tasks that support students from the periphery to the centre of the CoP.

As Biza et al. (2014) note, in CoP there is often a lack of detail regarding the fine-grained elements of the learning trajectory. Networking CoP and ToC provides the nuance needed to trace the aforementioned learning trajectory (CoP) in the terms of what ToC labels as discursive shifts. Defining the MS course community provides us with a boundary to operate within. We root our

definition of the MS community in the meta- and object-level rules we essentially want the students to engage with throughout the semester. So, with the boundary now set, we help the newcomers navigate through it. The construction of a learning trajectory gives a structure for students to engage, peripherally at first, with this new community and its associated meta-rules. The learning trajectory consists of a sequence of tasks, with an underlying goal of shifting students' mathematical discourse. We, therefore, design a learning trajectory which attends to the goals of the community whilst also facilitating students' object-level and meta-level learning. We also use the four characteristics of mathematical discourse both to inform the tasks and to analyse the students shifts along the learning trajectory. By networking these two theories we achieve both a macro and micro level perspective to analyse and inform the redesign of the MS course.

Context, methods, and participants

The ICP is redesigning and implementing an intervention in a 12-week face to face, year 1 MS course with 100% summative coursework. The course offers students an introduction to concepts in Set theory, Number theory, Proofs and mathematical skills. The ICP has two key brokers: Hooper (doctoral researcher supervised by Nardi and Biza, representing the community of mathematics education researchers) and Witty (representing the community of mathematicians). The team held regular bi-weekly meetings lasting 60-90 minutes which were open ended, with discussion informed by goals set and tasks completed by each broker between meetings. Hooper gave a brief overview of ToC and CoP (to ensure transparency and openness on how the analysis of Witty's teaching will take place). These meetings are the focus of the data analysis in this paper. The aim of these meetings is to provide opportunities for discussion, debate and decision making regarding the learning trajectory, AL tasks and the selection of resources. However, the "ultimate substantiator" (Sfard, 2008, p.243), in this case Witty, effectively made the final decision on the content of the learning trajectory. During the design phase, there were two central data collection points: the five RDT meetings (recorded and transcribed by Hooper) and resources uploaded on the shared RDT OneDrive. Hooper conducted a partial analysis of the transcripts using the ToC to identify shifts in the lecturer's pedagogical discourse. Resources shared on the drive included draft session plans which Hooper also analysed. We present three examples in this paper that have emerged from our preliminary data analysis. Example one concerns the lecturer shifting the focus of tasks from object-level learning goals towards meta-level learning goals, with reference to an excerpt on quantifiers. Example two explores a shift in Witty's goal setting routine, where session goals hitherto tacit to students are now made explicit. For example, Witty's session plans began to include a written explanation of what students should achieve from engaging with the various tasks. Example three focuses on a pedagogical shift as Witty seeks to implement tasks to facilitate legitimate student participation in the lectures. In tandem, we try, where possible, to identify potential shifts these changes may have on students' mathematical discourse.

Results

Example One: Shifting task focus from object-level to (also) meta-level learning

The partial analysis of Witty's previous course materials revealed an emphasis on object-level learning, specifically on introducing new definitions (e.g., of a set), theorems (e.g., the Euclidean

algorithm) and notation (e.g., ∃ and ∀). These are important mathematical objects for student participation in the community of university mathematics. However, such participation also requires attending to a new set of meta-level rules which Witty is responsible for introducing. Disproportionate attention to object-level learning may stifle student engagement with these new meta-level rules, potentially hinder students' discursive shifts and limit opportunities for legitimate participation in the MS course. The example Hooper and Witty discussed concerned challenges with making sense of, and using, the notation ∀ (for all) and ∃ (there exists).

> Witty: ...it's weird the notation *for all* and *there exists* then just becomes sort of like I dunno a barrier, it's something they have just seen and are like "I need to comprehend that and I'm trying to do some maths as well".

This highlights challenges students have in engaging with new symbols, deciphering their meaning and use whilst the lecturer has already introduced more mathematical content. This density of content may be overwhelming for new community members. The new rules of the game demand students understand such notation. However, without the opportunity to explore and endorse these new narratives, students are prevented them shifting further than the periphery of the CoP (Wenger, 1998) in the MS course. Therefore – whilst they are engaging with new mathematical content (object-level, quantifiers), their participation in the mathematical discourse of university mathematics is hindered by not being able to utilise notation to solve mathematical problems. Student challenges with quantifiers is highlighted in research, for example, students attempting to solve problems that involve quantified statements without acknowledgement of the quantifiers (Sellers et al., 2021). We discussed this challenge in our RDT meetings as another lecturer (occasionally present at these meetings) stated:

> …my biggest thing when I was an undergraduate especially in analysis was the fact, I literally learnt *for all epsilon greater than 0…* and I didn't, I wasn't registering the for all epsilon I was just registering the sound *for all epsilon greater* and not really understanding what it meant.

Discussing AL tasks that may address problems such as these, the lecturer drew on their experience and suggested utilising tasks such as "fill in the gaps" or "mix and match" style exercises. The aim of these tasks is for students to engage with the new symbols to explore their differences and the implications their use has for the mathematical meaning of a statement – as well as realise how communicating in this new way makes them more central participants in the practices of university mathematics. This example highlights the potential of the intercommunity partnership to identify challenges students may have with these new rules of the game and utilise research and literature to incorporate AL tasks to facilitate meta-level learning where previously the focus had been on object-level learning. By utilising Witty's previous course materials, we leveraged a context driven, jargon free discussion that facilitated reflections, discussions, and modifications which informed the design of the learning trajectory.

Example Two: Shifting from tacit to explicit lecturer's goal setting routines

> Hooper: This [course] is titled maths skills but there is no…this is the point of this module…, this is why we are doing maths skills, these are the things you'll be expected to engage with.

Despite being named Mathematical Skills, it was not explicit to students what these skills were, how they were expected to engage with them, or why they were important for newcomers to the community of university mathematics. If we want students to engage beyond the periphery of the community, it is important they have a clear and explicit trajectory to do so. The comment by Hooper was aimed to spark a discussion on the aims and goals of the course and how this could inform our learning trajectory and build our sequence of tasks. As a result of these discussions, a shift occurred in the lecturer's pedagogical discourse: Witty's goal setting routines shifted from tacit to explicit student goals within the MS course. In practice, this sparked an overhaul in the design of the opening lecture sequence. The above comment likely prompted a re-evaluation of the lecturer's pedagogical choices (e.g., order of the sequence of tasks) and the importance to the students of clarity in terms of what is expected of them. As a result, Witty in Lecture 1 aimed to discuss and explore students' meaning of mathematical skills and their importance.

> Witty: for example, this exercise here leads into them submitting something as well, like having a goal […] by the end of this you're gonna like have solved these problems and written them up.

Additionally, the explicit goal setting extended to Witty's own pedagogical routines as for each AL task included it had an associated goal (or purpose). As Witty notes,

> there are so many things we could do…it's trying to not like, I don't know, just do them for the sake of it. But, also like make sure there is a purpose.

By our fourth meeting, these utterances had become routine and Witty's desire for purpose in each task became regular goal setting. In the above extract, we see the lecturer develop a routine of self-questioning and justification, related to their selection of tasks. As the lecturer engages in this inter-community partnership, they have started adjusting their pedagogical routines. The discussion, sparked by Hooper's CoP informed observations regarding the lack of explicit goals on the MS course, has helped facilitate a shift in the lecturer's goal-setting routines. This shift is crucial for the intervention design, as these goals form the foundation of our learning trajectory that will guide students' engagement throughout the semester. The engagement with this inter-community partnership has the potential to not just produce an intervention to facilitate students' shifts in mathematical discourse but also present opportunities for lecturers to shift their pedagogical discourse as they are exposed to propositions emerging from mathematics education research.

Example Three: Pedagogical tasks to encourage legitimate student participation

As mentioned earlier, one motivation for redesigning the MS course was to help student engagement by shifting from a largely C&T approach to one infused with AL opportunities. Where deemed appropriate, Hooper proposed transformation of previously used teaching materials that were familiar to Witty (e.g., replacing giving examples to students with an example generation task). In response, on one occasion Witty commented that

> Witty: with the quiz they have to think about it, but they are not writing it down and you know maybe that is quite a powerful thing that they need to be doing.

This is a key moment where Witty endorses a change in students' engagement routines to encourage legitimate student participation in lectures. This concept of legitimate participation became a regular theme across our meetings with Witty reinforcing narratives on student participation in the lectures.

> Witty: I guess for a quiz that they might interact with notation, but sometimes it could be we want you to write that down because that's what you're going to be doing in a bit, writing it down. It's good to recognize it, but also like being able to, using it straight away allows you to go "I'm using this correctly."

This provides insight into Witty's aspirations to shift the students' engagement routines from passive observers to legitimate participants in the MS course. The aim is that both object-level and meta-level learning emerges as students are allowed time for exploration and practice when introducing content.

Concluding Remarks

The examples above have highlighted how inter-community partnerships (between mathematics education researchers and mathematicians) can facilitate the design of an AL infused MS course through engagement with research findings, prior experiences and practice. Networking the ToC and CoP provide the theoretical foundations for the RDT, for designing the intervention and for data analysis. The ICP facilitates a shift in the lecturer's pedagogical discourse from C&T to AL pedagogies, exemplified by the three examples above. Example One illustrates a shifting in the focus of tasks set by the lecturer from object-level to (also) meta-level learning. These discussions (in a jargon-free way) provided opportunities for the lecturer to re-evaluate the tasks (and their goals) that have shaped the course previously. Example Two illustrates shifts in the lecturer's goal setting routines, ensuring the aims of the course were now explicit to the students. Example Three illustrates a shift towards designing tasks that encourage legitimate student participation. So far, we have seen how the ICP can facilitate a redesign of a mathematics course and the shifts this presents in the lecturer's pedagogical discourse. Whether and how these changes result in shifts in students' mathematical discourse is the next and imminent phase of our ongoing study.

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Towards a research agenda for generative AI in university mathematics education

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This paper reports preliminary results of a study employing the Delphi method to ascertain the consensus of stakeholders regarding the role and use of generative artificial intelligence (genAI) in teaching and learning undergraduate mathematics. Participants to the study included mathematicians, mathematics educators, and learning designers. The data consists of responses to an online survey which asked participants to state the most urgent research questions related to the use of genAI in university mathematics. We obtained 41 research questions by 34 participants from 23 institutions. We grouped these questions under two broad thematic headings: "questions concerning students' interaction with genAI" and "questions concerning teachers' interaction with genAI". These themes indicate what participants deem a pressing research agenda concerning genAI in university mathematics to be. We conclude with a description of the next steps for this study.

Keywords: University mathematics, genAI and large language models, Delphi method.

Introduction

The debate about the potential of generative artificial intelligence (genAI[1]) and the perceived threat it poses to the teaching and learning of mathematics at university level is pervasive. Opinions are divided (Kramm & McKenna, 2023): many believe that genAI is a threat to the way in which we teach and assess academic performance – and in particular mathematics – and to the delicate relationship between students and teachers (Luo, 2024). Others argue that integration of genAI in teaching and learning at university is unavoidable and indeed desirable (Rane, 2023). In any case, genAI has and will influence teaching and learning processes in higher education at a global scale. Higher education research communities have rapidly responded to these techno-educational shifts. For example, a recent meta-review identified 66 review studies conducted about genAI in higher education alone, exemplifying the urgency of research to help communities of practice to cope with the surge of genAI technologies (Bond et al., 2024).

We see it crucial to produce disciplinary, mathematics-specific research about genAI amid such broader shifts in higher education studies. Research in mathematics education on the affordances and drawbacks of genAI for teaching mathematics has so far spanned from investigating the mathematical

[1]We use the term genAI to indicate all large language models (LLM) such as ChatGPT, Claude, Gemini and others which may be available to students and teachers.

capabilities of a genAI (Green, 2024) to the comparison of the perceptions of teachers and students on the role of genAI in university mathematics (Fuller & Barnes, 2024). A rapid review of the higher education literature beyond mathematics (Lo, 2023) reveals how much of the attention devoted to the use of genAI in higher education concerns issues of misconduct in assessment. However, we could find no coherent research agenda in the work we reviewed, especially with focus on university mathematics, and the current research seems to remain disconnected. We, therefore, have attempted to develop a coherent research agenda regarding the role of genAI in university mathematics using the Delphi method (Hsu & Sandford, 2007). In this paper, we report results from the first phase of this study by focusing on the research questions it generated, and on what these questions tell us about the interests and priorities of stakeholders in university mathematics education.

Data collection: the Delphi method

The Delphi method (Hsu & Sandford, 2007) has recently been used productively to seek consensus on information from stakeholders from a certain community in a relatively new field. The first part of the method requires identifying who the stakeholders are in the chosen community and inviting their contributions. The Delphi method then proceeds in four stages of data collection and analysis. In the first stage, an open-ended questionnaire is sent to potential participants, to gather information about the chosen area. If an extensive review of the literature has been carried out, this stage can be replaced by a structured questionnaire. The second and the third stages of the method consist in asking stakeholders to rank the items obtained from the first stage, for example, via a questionnaire or focus group interviews, to gather consensus about the importance of the items found in stage 1. This iterative process usually produces a high level of consensus that, in the fourth stage, is refined to arrive at a final ranking of the items proposed by the participants.

In our implementation of this method, we followed the study of Kinnear et al. (2024) who, by adopting this method, outlined 55 research questions that represent the interest of mathematicians, mathematics educators, teaching designers and technicians regarding the use, design and research on e-assessment of university mathematics. As in this study, we also choose to gather research questions proposed by stakeholders who, in the case of the use of genAI in teaching and learning university mathematics, were mathematicians teaching university mathematics, mathematics educators researching the teaching and learning of mathematics at university, and/or those academics in positions related to teaching administration such as Directors of Teaching. Firstly, we contacted potential participants thought our extended professional networks; we are part both of Mathematics and Education departments, and our extended network comprises both mathematicians and mathematics educators. After the first contact we employed a snowballing recruitment strategy. We then caried out a rapid review of the literature (Wollscheid & Tripney, 2021) on the use of genAI in university mathematics. In this review we found very few mathematics education studies related to this topic; therefore, we started the Delphi process by distributing an open-ended questionnaire to those who had agreed to take part in the study. The questionnaire comprised some biographical data such as their role in the higher education institution, reason for interest in the study, and the three open-ended prompts: “(1) Please enter details about your first research question. (2) What motivates this question? (3) How would one go about researching this question?”

Participants were encouraged to enter more than one research question with justifications if they wished. These prompts resemble those used by Kinnear et al. (2024), where participants were also asked to propose research questions related to the topic of interest. Ethical approval for the study was sought and obtained at the University of Edinburgh.

We distributed the questionnaire between January and March 2024 to colleagues from 72 institutions in 15 countries. We received responses from 34 individuals from 23 institutions in 12 countries who offered 41 research questions (one verbatim example is in Figure 1). Of those, 21 participants described themselves mainly as mathematicians teaching university mathematics, 9 as mathematics educators and 4 either as learning designers or just interested in the topic.

Question

How can LLMs be used to generate bespoke automated feedback to student-produced text in tertiary mathematics?

What motivates this question?

Recent years have seen substantive efforts committed to generating such automated feedback using established platforms like STACK. Some research suggests that "elaborated feedback" is more useful than simply giving correct/incorrect feedback (Attali & van der Kleij, 2017)* but is it worth the up-front effort for the teacher to devise all this feedback?

How would one go about researching this question?

Initially, a qualitative analysis of LLM generated feedback to students' responses can provide an initial indication of the value and scope of such feedback across different mathematical contexts. Down the line, (quasi)-experimental designs might compare the impact of different feedback types on student learning, behaviour or perceptions.

*(Attali, Y., & van der Kleij, F. (2017). Effects of feedback elaboration and feedback timing during computer-based practice in mathematics problem solving. *Computers & Education, 110*, 154–169. https://doi.org/10.1016/j.compedu.2017.03.012)

Figure 1: Example of one answer in its original form. Here LLM stands for Large Language Models, one of the terms used by participants which we then included in the general 'genAI' category when we made the language of the research questions proposed consistent.

After we collected the data, we worked within our team to make the language of the questions uniform so that they would create a coherent body of research questions for later consideration, and to avoid repetition between questions. We note that this is a departure from the method as described in Hsu and Sandford (2007) but given the variety of educational contexts represented by participants, we deemed it necessary to find a unifying language for the research questions. To be faithful to the intentions of the question setters, we consulted the answers to the second and third questions when editing the text. For example, we decided to use genAI as a collective name for all the large language models mentioned in the questions (LLM, ChatGPT, Claude, Gemini etc.). Once we reached a uniform language, we sought to identify themes among the questions. We held three meetings, each comprising a subset of the authors of this paper. In each of the meetings we collaboratively carried out thematic analysis (Attride-Stirling, 2001) to identify themes and sub-themes, iteratively refining the analysis in each meeting. Through this process, we identified three broad umbrella themes:

'questions regarding students' interaction with genAI', 'questions regarding teachers' interaction with genAI'. A third theme comprising only three questions on the role of genAI in mathematics education research will not be discussed in this paper. The first and the second umbrella themes each contained a number of subthemes, as shown in tables 1 and 2. Overall, it can be said that the larger umbrella themes were rather pedagogical in nature: they concerned the day-to-day practices of teaching and learning of mathematics rather than addressing broader ethical, philosophical or technological conversations as in the bulk of general higher education studies (see Bond et al., 2023). Below we elaborate on the findings of our analysis of the research questions, and what they tell us about the community's perceptions of the role of genAI in university mathematics education.

Theme 1: Questions about students' interaction with genAI

We clustered 17 questions in this umbrella theme (see Table 1). The dominant concerns for participants were how critical students are in respect to genAI's mathematical output (5 questions) and how students use these tools as a resource for their studies (6 questions).

Table 1: List of sub-themes, with examples, within the 'students' interaction with genAI' group. The number in brackets indicate how many questions were found in each subtheme

Students' interaction with genAI (17)	
Sub-theme (number of questions)	**Examples of questions**
genAI as a resource for studying (6)	How do students use genAI to support their independent study? What support do students expect with respect to the use of genAI in mathematics?
Critical evaluation of genAI (5)	How do students evaluate the reliability of mathematical solutions produced by genAI? Does engaging with genAI support students in developing critical evaluation of tasks and their solutions?
Feedback (3)	How do students experience interactions with genAI-based automated systems for formative feedback? How is feedback on a mathematics task generated from a genAI received by students compared to feedback from their teachers?
Student-centred task design (2)	How can students develop a deeper understanding of mathematical concepts by teaching them to genAI? In what areas of undergraduate mathematics is genAI more effective for supporting students' learning? And why?
Support for use of genAI (1)	What support do students expect with respect to the use of genAI in mathematics?

Studies on the accuracy of genAI responses to advanced mathematics questions have shown that, for most topics typically included in a first-year mathematics degree and with the exclusion of proof-related questions, genAI can often provide accurate answers, but they are occasionally subject to hallucinations (Green, 2024). Hallucinations are instances when genAI's output refers to, or directly quotes, material that is incorrect or non-existent (e.g., research papers which seem plausible by their title but have in fact not been written, or a proof following the pattern of a similar proof but in a context where such pattern is not applicable). Therefore, being able to appraise output provided by this tool becomes important for students.

It is interesting to note that participants in our study seem to recognise that genAI is a tool which must be incorporated critically in the teaching/learning interaction and is not something to be banned, as some of the papers in the review by Lo (2023) suggested. This is evidenced by the 11 questions concerning building students' critical skills and orchestrating support for learning which constitute the largest sub-theme in this group. The need to develop students' critical skills to appraise genAI output has been addressed in the literature before (see Drushlyak et al., 2024, for an example of a study in mathematics education concerning proof). Criticality towards the output – if not already developed – needs to be fostered with classroom activities, hence the research question regarding the evaluation of the most effective support that can be offered to students and of the activities that can be devised to this end. Related to the support for students' use of genAI were the two questions directly concerning task design and the three questions about how students respond to automated feedback provided by genAI.

Theme 2: Questions about teachers' interaction with genAI

The sub-themes concerning teachers' interaction with genAI are also revealing. We found 21 questions here (see Table 2). The impact of genAI on assessment design and the threat of assessment misconduct were hardly mentioned by participants, contrary to our expectations. Only five questions were dedicated to assessment, and of those, only one ("How can educators detect AI-generated responses from students?") concerned prevention of assessment misconduct involving genAI. This is contrary to what the general literature on the use of genAI in higher education reports, where the preoccupation with misconduct is pervasive (e.g., Lo, 2023).

Instead, the dominant concern of our participants was with determining how genAI can support the design of tasks aimed at encouraging and supporting students' learning (10 questions). These questions detail the involvement of genAI in practices the literature recognises to be beneficial to students. For example, the question "What are the benefits of asking students to use genAI to generate their own questions on a given topic?' refers to the idea that asking students to generate their own mathematical questions is conducive to conceptual understanding (Voica & Singer, 2013) and the need to find out whether this is the same when the questions are directed to genAI. Another interesting aspect is represented by questions about the use of genAI to support teachers' own practice by rendering some of the tasks related to teaching more effective and less time consuming. For instance, there are questions regarding whether genAI can support lesson planning or writing lecture notes, or even support providing illustrative examples of complex mathematical concepts. Similarly, the assessment sub-theme included questions about how genAI can be used for exam marking or to help

teachers interpret the large datasets that are produced by assessment results. Such questions show again the wish to include genAI in teachers' practice as a supporting and potentially time-saving tool, but to do so responsibly and aware of the capabilities of the tool.

Table 2: List of sub-themes, with examples, within the 'teachers' interaction with genAI' group. The number in brackets indicate how many questions were found in each subtheme

Teachers' interaction with genAI (20)	
Sub-theme (number of questions)	**Examples of questions**
Student-centred task design (10)	What features of genAI-supported tasks have most potential for advancing learning? To what extent can genAI be used to generate examples illustrative of particular topics/concepts?
Assessment (5)	How can genAI be used to support the efficient marking of exam scripts? How can genAI be used to interpret large assessment datasets?
Curriculum design / statistics (2)	How can genAI be used for the teaching and learning of statistical data analysis? What are the implications of recent advances with genAI for priorities in the statistics curriculum?
Feedback (2)	To what extent is genAI-produced feedback reliable? How can genAI be used to support digital scaffolding by providing them with appropriate individualised feedback and hints?
Support for use of genAI (2)	How do we help students learn to make productive use of genAI?

Statistics teaching has a special mention in our dataset. This may be because some of the participants may be involved in teaching statistics, but also it could be the case that our participants believed that genAI is 'better' at solving statistics questions than at providing proofs in pure mathematics (see for example Lo, 2023), at least for now. One surprisingly missing topic regarded the use of genAI for coding. Coding has become a more significant part of university mathematics teaching in the past 10 years (Iannone & Simpson, 2022) where the most used programming languages are Python and R for statistics (Sangwin & O'Toole, 2017). The computer science literature indicates that genAI is very efficient at producing code in these programming languages with a recent study showing that ChatGPT4 achieved an accuracy of 86.2% on the sample used for testing Python coding (Coello et al., 2024). We hypothesise that such questions, which indeed regard university mathematics teaching, are missing from our list of questions due to the composition of our group of participants.

Final remarks and next steps

The categorisation and analysis of the questions presented above shows a picture of engagement with genAI and of preoccupation with how to better shape such engagement, both for students' learning and for teachers' practices. Our participants recognised the potential of genAI tools and the need to explore how they can be productively used by their students. The preoccupations that are manifested by some policy makers and university administrators (see Williams, 2024, for an example in the UK sector) regarding assessment misconduct seem not to be reflected in the priorities of those who teach and research university mathematics. This finding may indicate that stakeholders believe that responsible use of a tool – in this case genAI – achieves better outcomes for teaching and learning than banning or restricting its use. One of the limitations of our study is that the questions posed depended on the expertise and interest of the participants. Although we strived to have a diverse group in our sample, both geographically and in terms of expertise, we have inevitably missed some aspect of the interaction between teaching university mathematics and genAI, as the absence of questions concerning coding suggests. For the next steps of our research, and to produce the research agenda we set out to outline, we will analyse focus group discussions with participants and move to the last step of the Delphi method. This consists in asking participants to discuss the research questions obtained in previous stages and rank them in terms of importance to them and to their community. This will allow us to finalise the research agenda both in terms of what questions will finally be included amongst the ones found, and in terms of outlining what the priorities for the community are. The final research agenda will also contain suggestions on how the research questions may be investigated. Such suggestions will be drawn both form the original questionnaire and from suggestions emerging from the focus group interviews.

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Opportunities to Learn as entry points to university mathematical discourse: The case of proof in introductory lectures

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The transition from school to university mathematics is associated with many student difficulties and dropouts. This study investigates Opportunities to Learn (OTLs) offered in university mathematics lectures. We conceptualise OTLs as the lecturers' attempts to invite students' participation by suggesting entry points in the university mathematical discourse. We explore the OTLs in university mathematical discourse that appear in the overlap of lecturers' mathematising and identifying. For this study, we used data from the first lectures of four first-year courses offered to mathematics bachelor students. We discuss our initial exploration of OTLs through an example from the teaching of proof in an introductory set theory course. By investigating the OTLs, we unfold how they shape expectations towards students' participation in the discourse and the demands for students to meet these expectations. Future research should investigate the impact of OTLs on student participation.

Keywords: Opportunities to learn, university mathematics lectures, discourse, mathematising, identifying.

Introduction

The transition from school to university mathematics, known as the secondary-tertiary transition (STT), presents challenges, often leading to difficulties and dropouts (Di Martino et al., 2023). We view STT not just as an academic shift but as a process of learning the "rules of the game" in university mathematics (Sfard, 2014; Pinto & Koichu, 2023). While resources like textbooks, peers, and bridging courses support this transition, lectures remain a key resource for students. Although lectures typically do not promote active participation, lecturers create entry points for students to participate in the respective mathematical discourse. For instance, Weber (2004) describes a lecturer who began an analysis course by emphasising the logical aspects of proving. This choice stemmed from the assumption that students would struggle with analysis without a foundation in formal proving. By introducing proving through topics like sets and functions, the lecturer provided an entry point for students to grasp proving in analysis. Such opportunities in lectures serve as invitations for students to participate in university mathematical discourse, helping them develop the necessary skills and understanding to navigate this new academic environment.

Nevertheless, whether these entry points in lectures are beneficial for students' learning, or even their STT, remains unclear, preventing us from understanding the contributions of lecturing to a successful STT. Sfard (2014) argues about the great distance between school and university mathematical discourses with respect to reliance on formal rules and the increased level of rigorousness, which would make it doubtful whether entry points in lectures can bridge this distance. At the same time, during STT, the students are expected to reapproach the mathematical practices that they are familiar with from school. From this perspective, providing entry points that explicate how to engage with the presented mathematics could help students to better engage in the university mathematical discourse

during their studies, e.g., during engagement with proving tasks independently. In this paper, we investigate these entry points and how they take shape during lectures.

Theoretical considerations and literature background

Opportunities to Learn as a place of identifying students

From a discursive standpoint, and focusing specifically on university mathematics, the entry points offered in lectures can be conceptualised as Opportunities to Learn (OTLs) (Remillard, 2014). We regard these entry points as invitations to participate in the university mathematical discourse. Building on this perspective, we explore how OTLs emerge in mathematics lectures and are related to lecturers' perspectives on students' participation in mathematical discourse. Chan and Sfard (2020) see OTLs as "circumstances that call for, and support, *a change* in the learner's participation in a discourse, a transformation that would bring him or her closer to the [intended] discourse […]" (p. 3). OTLs can be of two types. Firstly, OTLs call for a change in the "learner's command of the discourse" (p. 4). Here, the lecturers identify students in their talk in a certain way, suggesting "how this participation builds on and is being built by, that of other participants" (ibid, p. 10). Secondly, OTLs invite a change "in the discourse itself" (p. 4). Here, we combine these views and conceptualise OTLs in university mathematics lectures as entry points in lectures aiming at a change in students' participation in the university mathematical discourse. For example, a lecturer might introduce a new mathematical concept by posing a question to raise prior knowledge related to the concept's formal definition. This OTL may allow students to enter the discourse through their prior knowledge.

OTLs are an expression of how lecturers want their students to participate in the university mathematical discourse, so lecturers identify their students in a certain way when explicating the "rules" of the discourse in their teaching. In other words, having access to OTLs is essential for students' participation in the university mathematical discourse as it supports their initiation to the "rules" of the discourse that are otherwise often tacit among expert participants of the discourse. This initiation is particularly important in becoming independent and agentic participants within the mathematical discourse. For example, a lecturer might demonstrate how to construct a mathematical proof by explicitly focusing on a logical style (e.g., Weber, 2004), creating OTL that may allow students to internalise "rules" about how to start and how to structure a proof.

University mathematical discourse in shaping OTLs

Mathematical discourse is distinguishable by a community's word use, visual mediators, endorsed narratives, and routines (Sfard, 2008). Participation in the discourse distinguishes from ritual to explorative (Lavie et al., 2019). Ritual participation in a mathematical discourse appears to be the first necessary step when a person tries to enter a discourse (Heyd-Metzuyanim, 2015). The person focuses on the process and tries to replicate the actions of more experienced participants of the discourse. When the person starts having some experience and can explore independently in the discourse, then the participation becomes more explorative.

University mathematical discourse contains both mathematising and subjectifying talk. Mathematising refers to the talk of mathematical objects. In contrast, subjectifying is the talk about the participants in the discourse. For instance, when lecturers introduce students to mathematical

routines—such as proving or defining—they are not only engaging in mathematising but often also in subjectifying. For instance, they ascribe particular ways of being, which shape participants' identities within the discourse (Heyd-Metzuyanim, 2015). Subjectifying operates subtly, involving statements about who the participants are within the discourse and how participants perceive each other in the discourse (Heyd-Metzuyanim, 2015). Accordingly, an OTL can be conductive to support students' STT when it supports students' subjectifying – their identification of themselves – as members of the university mathematical discourse (Sfard, 2008). Through such social and affective processes of assigning students with identities that are constructed for them by the lecturers, OTLs might contribute to a group's success or failure in learning mathematics (Heyd-Metzuyanim, 2013).

For this study, we focus on identifying a subtype of subjectifying (Heyd-Metzuyanim, 2015) that describes narratives about participants in their talk. Behaviours or ways of being often stay opaque to participants and only become evident when not adhering to them. These characterisations are often implicit, becoming noticeable primarily when participants deviate from them. To uncover such implicit *identifying*, researchers analyse recurrent identifying narratives in the talk of lecturers (Heyd-Metzuyanim, 2013). Overlaps of mathematising and identifying in university mathematical discourse may bring the "researchers closer to understanding how mathematical learning is shaped by students' identities and how students' identities are shaped by mathematical talk" (Heyd-Metzuyanim, 2015, p. 512). Thus, these overlaps highlight how the processes of mathematising and identifying are intertwined in shaping OTLs for students.

Research question

Particularly for the case of proving, OTLs call students to participate in the proving routines when, at the same time, they are trying to familiarise themselves with them (Remillard, 2014). Remillard (2014) identified discursive entry points when students worked on proving tasks. Examples of entry points were the raised questions about managing the proof process and tackling the ambiguity of what needs to be proved. In this study, we ask: *How are students identified in OTLs to allow students' participation in university mathematical discourse, particularly proving, during STT?*

Method

Data collection

This study is exploratory and part of an ongoing project. The project aims to provide understanding and suggest ways to reduce the dropout numbers of first-year mathematics students. We follow a multiple-case study approach where the cases are set by experienced lecturers teaching in four first-year university mathematics courses offered within an applied mathematics bachelor's program in the Netherlands. The four lecturers were chosen as they teach the introductory, proof-oriented courses in the first semester (consisting of two quartiles) of the applied mathematics bachelor and are described as the foundation for the next courses of the curriculum. The selected courses cover introductory topics in the mathematics curriculum.

The primary data set of this study consists of video recordings of the lectures. The courses were video-recorded by the university of the study, and the videos remained online for archive reasons. Each lecture lasted 90 minutes and was transcribed. Captures from the blackboard were also implemented

in the transcripts. A supplementary data set included course notes and other shared material (i.e., homework exercises and extra reading material).

Data Analysis

For this study, we focus on the first lecture of each course as an initial investigation of the data set, as we assumed that OTLs would be particularly prominent here. We aim for a micro-level characterisation of OTLs in the overlap of mathematising and identifying talk. For that, we related them with how they may allow students' participation in the discourse from the lecturer's perspective.

For the first step of the analysis, we started with episode separation. Each episode was related to the presentation of a narrative (e.g., proof, definition, example) or a block of related narratives (e.g., examples). The episodes were coded using in vivo, descriptive, and theoretical codes. The assigned codes aimed to characterise the lecturers' mathematising. At the same time, discourse analysis took place for the characterisation of the identifying. We concentrated on moments that are attached to certain identifications about the person, aiming for the development of a manageable set of identity narratives (Heyd-Metzuyanim, 2015). For instance, "You are in a good shape", "You seem convinced" are examples of moments that were related to certain identifications assigned to the students by the lecturers.

In the second step of the analysis, the episodes were closely examined for the identification of OTLs in lecturers' discourse, doing discourse analysis. In particular, the focus was on the instances, later characterised as OTLs, where the lecturers' identification of students related to their mathematising towards the participation of the students in the discourse. We characterised the OTLs as entry points for explorative or ritual participation in the university mathematical discourse. For instance, phrases like "you immediately see this should be true" were characterised as normative, assigning a membership to the students and considered as an indication of explorative participation from the students to the university mathematics discourse as the lecturer asked for students' independent participation in the discourse, exploring on their own why the narrative is true. We considered these instances as indications of OTLs and compared them among the lecturers (Charmaz, 2014).

Findings

In the following, we present two episodes (consecutive) from the first lecture of the introductory, proof-oriented course in set theory. The course aims to introduce students to formal proof and functions as the foundation for future courses in university studies. The lecturer is an experienced mathematician who taught the course more than five times before the time of the data collection. The following excerpts are from the proof of the first distributive law for sets: Let A, B, C sets. Then the following holds: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. The exploration of OTLs gives insights into the lecturer's attempts to support students' transition from school to university mathematical discourse while suggesting the lecturer's expectations regarding students' participation in the discourse.

Introduction to the theorem and proof

> For example, let's try to do this first distributive law [the lecturer draws Venn diagrams – Figure 1].
> A, B, C, and I intersect A with B, intersection with C…

So, from these Venn diagrams, you immediately see this should be true. But a picture is never a proof. Yeah, it only helps you.
Now, I've seen this picture. And now, I am quite confident that it is true. So, I have to give you a proof.

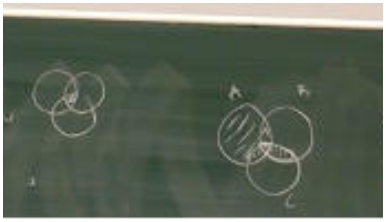

Figure 1: Visual mediator for distributive law

As seen in the excerpt, the lecturer introduces students to the distributive law for sets by presenting the statement of the theorem and subsequently initiates the proof by drawing Venn diagrams. The diagrams offer an initial understanding of why the theorem is true. In the excerpt, we gain initial insights into the lecturer's identification of students as both ritual and explorative participants in the university mathematical discourse, providing OTLs in both cases. On the one hand, the lecturer identifies them as aware of the statement's truth through the presence of the narratives about mathematical proofs (i.e., "you immediately see this should be true", "confident that it is true"), assigning explorative participation for the students as independent participants in the discourse, exploring on their own why the narrative is true. On the other hand, narratives in the lecturer's discourse bring up what counts as valid proof ("a picture is never a proof… it only helps you"), with the lecturer assuming that some students may not see the necessity of providing proof after the lecturer's drawing the diagrams, suggesting a ritual participation for students. The lecturer then continues to the proof of the statement:

Okay, I have to do two sides, I have to prove that A intersects B union C is a subset, the left-hand side is a subset of the right-hand side. And I have to do the same proof but then the other way. Yeah, but let's concentrate on this one.
Proof: let x be an element of the left-hand side.
Then I argue and argue. And I end: so x is on the right-hand side.
This is my goal, yeah. I just have to fill in the details.

In this excerpt, the lecturer starts the formal proof of the distributive law. The lecturer outlines the proof process (i.e., "the left-hand side is a subset of the right-hand side", "I have to do the same proof but then the other way"). Then, the lecturer starts the proof with the main argument, which is to show that an element x from the left-hand side is also on the right-hand side. The lecturer describes the argument as the goal and the process towards this goal as the details in between. In this case, the OTL reflects the outcome of the proving process, suggesting an explorative, outcome-oriented participation in the university mathematical discourse. The lecturer continues with the proof and concludes as follows:

That's a proof of that statement. I take my x in here and prove that it is there.

In the same way, or, similarly, you can do the other statement that if you have an element from A intersect B union A intersect C, that is A intersect B union C.
But that's how you approach this.

In the excerpt, the lecturer concludes the first part of the proof, showing that an element x from the $A \cap (B \cup C)$ is also an element of $(A \cap B) \cup (A \cap C)$. Then, the lecturer identifies that the students have enough information to attempt the proof of the statement that an element x from $(A \cap B) \cup (A \cap C)$ is an element of $A \cap (B \cup C)$, providing them with an OTL to work on their own, following the proof that was presented to them. This OTL is suggesting students' explorative participation as the focus is shifted from the proving process back to the statement that is needed to be proven.

Commentary following the proof

The lecturer continues with a commentary following the proof.

What do you notice? It reads like a novel. Yeah. Yes, you can read this as full sentences. Of course, they are very short and obtain formulas. […] I wrote down some of these identities. Look at the identities in the lecture notes, take one or two of them. And try maybe to draw these Venn diagrams, whether you understand that that is true. Or try to prove them in this way, or read the proof of some of the statements as given in the notes. And really not only try to understand it, but also see how it's formulated. How to make a good text out of it, where all the reasons are in there, look at these words, assume, suppose, let, therefore, then, ands. Yeah.

The lecturer asks the students about the proof on the blackboard, focusing on what they notice. Then, the lecturer mentions that the proof can be read like a novel, has whole sentences, and that the proof's length is short but contains formulas. The lecturer identifies the students as being able to read the proofs and produce similar examples based on the lecturer's presentation of the proof. However, we interpret that at the same time, the lecturer recognises them as newcomers to formal proving because, following this identification, and continues by explicating narratives about proof writing, reading, and understanding ("try to prove them in this way", "draw these Venn diagrams", "read the proof of some of the statements"). In this excerpt, the OTL suggests ritual participation, where the focus is on the process and less on the procedure's outcome. The lecturer adds:

So, don't forget to put in all these words.
That's, that's really important. That's the heart of mathematics is words, not these formulas. But these words give you your assumption, your conclusion, and so on. And that is giving a proof? Explain the reasoning.
And the further you get? Of course. Less detailed you have to be. But right in the beginning, you need to be very detailed because you're building up your knowledge.
Really be prepared. Write on all these.

The final excerpt highlights the lecturer's identification of students as newcomers to proof writing and the suggested ways of producing formal proof. The OTL in this excerpt is related to students' ritual participation in the university mathematical discourse. Within this identification, the lecturer brings up that the "heart of mathematics" is in the words of the proof. The lecturer adds that the students need to be in the beginning "very detailed because you're building up your knowledge".

Then, the lecturer suggests ways to be prepared and write these proofs on their own, allowing an entrance point to the discourse for the students who start engaging with mathematical proofs.

In the example, we unpacked the OTLs in university mathematical discourse that appear in the overlap of mathematising and identifying in the talk around the teaching of proof. The OTLs the lecturer offers to the students are based on the mathematising that is taking place while proving, the students' intended participation in the university mathematics discourse (i.e., "very detailed because you're building up your knowledge"), and the lecturer's identification of students (i.e., "confident that it is true"). The OTLs reveal how students are expected to participate in the discourse. In particular, while the OTLs in the presentation of the theorem and its proof suggested more explorative participation, the OTLs in the commentary following the proof were interpreted as indications of ritual participation. In both cases, the students were interpreted as newcomers in the proving process, suggesting ways of facilitating their independent work with similar proofs. In the example presented here, the lecturer, through the OTLs, seems to emphasise what an independent and agentic participant in the university mathematical discourse is expected to do. Yet, considering that students are not yet independent or agents in the new (for them) university mathematical discourse, it remains questionable whether and how they can benefit from such OTLs.

Discussion

This study is an initial investigation of OTLs offered in university mathematics lectures. We conceptualise OTLs as the lecturers' attempts to suggest a change in students' participation by allowing entry points in the university mathematical discourse. We asked, "*How are students identified in OTLs to allow students' participation in university mathematical discourse, particularly proving, during STT?*". We focused on an example from the teaching in an introductory set theory course, and we provided an exploration of the OTLs that appear in university mathematical discourse in the overlap of the lecturer's mathematising and identifying talk. Thus, the OTLs found in our study may suggest how one should participate in the university mathematical discourse, and depending on the student' success in following the OTLs, their participation in the discourse might develop very differently, potentially reducing the dropout numbers of first-year mathematics students.

The OTLs appeared as entrance points to the university mathematical discourse as they indicated how a student is expected to participate in them. In this case, the OTLs did not focus on proving styles (e.g., Weber, 2004) but more on the ways of studying independently. Similarly to the cases explored by Remillard (2014), the lecturer shared identifying narratives of how students are expected to participate in the discourse. The students are expected to familiarise themselves with the "rules" of the lecturers' discourses, entering the new university mathematical discourse while starting to work independently. Sfard (2014) argued about the importance of students' independent journey in the university mathematical discourse, highlighting it as a difference between school and university mathematical discourses. Yet, considering that students are at the beginning of their university studies, it is unlikely that they will benefit from OTLs that emphasise becoming agentic and independent participants.

STT is a complicated and challenging period for students, and this study is the starting point for further exploration of introductory university mathematics lectures as a resource to support STT. As

Pinto and Koichu (2023) also mention, STT and supporting STT have many interpretations in mathematics education field. Future steps include analysing a larger corpus of data that would allow the refinement of the OTLs for STT. Moreover, students' input would allow a further understanding of the lectures' historical involvement in the STT, enabling characterisation of the OTLs with respect to the shared "rules" but also relating it to their actual agency development.

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The research communication of doctoral students in mathematics

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Doctoral students hold a particular position in relation to mathematics and mathematical learning. This unique position is central in the work presented here as we explore doctoral students' experiences of mathematics through the investigation of their communication of research. Building on the commognitive framework, this paper is a first step in characterising the mathematical research discourse of doctoral students. Our analysis of interviews with three doctoral students in mathematics highlighted three patterns characterising our participants' mathematical discourse. We conclude this paper by discussing how these patterns inform how doctoral students construe their mathematics.

Keywords: Doctoral students, mathematical communication, commognition.

Introduction

Transitions to, across and from university mathematics have been widely examined in recent years (Hochmuth et al., 2021). In the case of the transition *across* university mathematics, two main areas of research were explored: the transition between courses about the same topic (such as analysis) and the transition from general to disciplinary courses (such as engineering) (Hochmuth et al., 2021). Although the transition to higher degrees in mathematics was identified as a possible transition across university mathematics (Hodgson, 2015), doctoral students at the heart of the specific transition from undergraduate to research mathematics received little attention (Edwards, 2010). Yet, as part of this transition, doctoral students play a unique role in mathematics and mathematical learning.

On the one hand, doctoral students can be seen as mathematical students who successfully went through undergraduate mathematics courses and programs. As illustrated by Alcock and Inglis (2008), turning to these students could yield impactful results regarding the "skills needed for success in advanced mathematics" (p. 127). Accordingly, research on doctoral students could inform the sort of habits that foster the learning of mathematics in university courses.

On the other hand, doctoral students can be seen as "becoming mathematicians" (Herzig, 2004, p.171) and, more precisely, as learning the practices of research mathematics. In particular, the process of becoming a mathematician entails learning how to solve "a problem with an 'unknown' solution and no definite 'direction' towards its successful solution" (Geraniou, 2010, p. 288). In other words, doctoral students are not only expected to learn mathematics as part of their program but also to learn how to solve problems for which no solution is known by the mathematical community.

These expectations put doctoral students in a unique position in terms of their experience of mathematics and mathematical learning. Accordingly, in her doctoral project in mathematics education, the first author of this paper seeks to characterise how doctoral students in mathematics experience research and participate in mathematical research activities. Contributing to this broader project, this paper focuses on the mathematical research discourse of doctoral students.

Theoretical background

Building on systematic functional linguistics (SFL) and commognition, Morgan and Sfard (2016) develop a discursive framework that approaches mathematical experiences through the lens of the *ideational* function of mathematical discourses. In SFL, language and communication are described as functional, not representational (Halliday, 1985). In that sense, the *ideational* function of communication pertains to the way we share our experiences of the world by communicating. Similarly, commognition describes communicating and thinking as two sides of the same coin, the former being the public version of the latter. In this sense, Morgan and Sfard (2016) claimed that studying the use of language is a "means of understanding the ways in which participants in communication may construe their 'reality'" (p. 100). Leveraging the work of Morgan and Sfard (2016), we draw on the commognitive framework to explore how doctoral students construe their mathematical reality.

In commognition, the use of language is captured with the fundamental construct called *discourse*. Mathematical discourses correspond to different types of communication and are distinguishable through the *words used* (e.g., the names of mathematical objects and properties), *visual mediators* (e.g., symbols and figures), *narratives* (e.g., definitions and claims), and *routines* (e.g., methods and techniques) (Sfard, 2008). Additionally, discourses define "different communities of communicating actors" (Sfard, 2008, p. 93). Members of a community of discourse, called *insiders*, are said to be able to participate in a given discourse: they are able to communicate with other insiders.

In our work, we view research in mathematics as a form of participation in what we call a *specialised mathematical discourse* (SMD), a discourse corresponding to a specific mathematical field such as dynamical systems, graph theory, or topology. As illustrated in Madsen and Winsløw (2006), pursuing research in mathematics asks for what the authors have called *extensive prerequisites* due to the level of sophistication of the mathematical theories and methods. Commognitively speaking, this means that participation in SMDs relies on discursants' familiarity with distinctive and technical keywords, narratives, visual mediators, and routines.

Additionally, mathematical discourses, specialised or not, are often characterised by what Morgan and Sfard (2016) called the level of *objectification*. Objectification is a process through which participants in a discourse talk about objects and properties instead of people and processes (Sfard, 2008). Objectification occurs through *reification* and *alienation*: in the former, a talk about processes becomes a talk about objects (e.g., talking about fractions as dividing vs fractions as numbers) while, in the latter, all signs of human agency are removed from the discourse. As a result of *objectification*, statements about mathematical objects appear as the "state of the affairs in the world" rather than "a statement about our own commognitive activity" in specialised mathematical discourses (Sfard, 2008, p.107). For instance, a definition appears as a fact about an object instead of a discursive activity.

Building on Morgan and Sfard's (2016) framework, the wider project investigates doctoral students' mathematical discourse as a way to explore their experience of mathematical research. Accordingly, the work presented here is a first step in answering the following question: *How do doctoral students participate in the specialised mathematical discourse of their research?*

Methods

The data for our study were collected from interviews conducted by the first author in the Department of Mathematics at a research-intensive university in New Zealand. These interviews were designed for the first author's doctoral research project. As part of this project, eight doctoral students were invited to a series of individual semi-structured interviews in which they answered questions about their work and progress in research.

As part of the interview, doctoral students were prompted to talk about their research on several occasions. Acting as the interviewer, the first author encouraged the participants to talk about (the state of) their research at the beginning of each interview. Second, she asked participants to provide specific examples from their research to support their answers to interview questions about their participation in research activities. Last, the interviewer occasionally asked clarifying and follow-up questions about the mathematics discussed. For example, when a participant said that "some groups [the mathematical structure] have been explained", the interviewer asked him to detail what he meant by "explained". Additional data was also collected through 30-minute seminars in which students were invited to present their research to other students participating in the study. Accordingly, we view the interviews and the seminars as two settings where doctoral students communicate their research projects and participate in the specialised discourse of their research. However, due to space constraints, we only discuss findings from the analysis of the interviews in the next section.

More particularly, we report on interviews with three doctoral students who provided detailed explanations about the mathematical objects and processes involved in their research. These students are Isaac, Patrick, and Charles (pseudonyms). Isaac and Patrick both work in the field of dynamical systems, and Charles's research is at the intersection of group theory and graph theory.

In total, nine interviews (three for each participant) were analysed. Working with the transcripts of the interviews, we proceeded with a three-stage analysis of the keywords and expressions participants used to talk about and refer to the mathematics of their research. The first stage involved identifying all the instances where participants talked about objects, routines or visual mediators of their SMDs in the interviews. To do so, the first author read the transcripts multiple times and sought clarifications from participants about their use of certain expressions. In the second stage, we looked at the specific words and expressions that participants used. For this paper, we were concerned with the *levels of specialisation* (Morgan & Tang, 2016) and *objectification* of the discourse (Morgan & Sfard, 2016). The *level of specialisation* pertains to the use of specialised and non-specialised keywords and expressions. This categorisation was based on follow-up conversations with the participants and consultations of mathematical references such as books, research papers and conference posters. For example, the names of mathematical objects or properties, such as "wild chaotic attractor" or "fragmentally asymptotically stable", are specialised expressions since they appeared in instances of SMDs found in the references consulted. Conversely, expressions that are not part of the SMDs and could have been replaced with SMD keywords and expressions were classified as *non-specialised*. The *level of objectification* concerns the actors brought into the conversation and what is said about them. A narrative was categorised as *objectified* if it corresponded to a talk involving mathematical objects and properties, and it was categorised as *disobjectified* if it corresponded to a talk involving

people and processes. In the last stage of the analysis, we identified patterns in how our participants used different forms of talk characterised by their level of specialisation or objectification.

Findings

When looking at the levels of specialisation and objectification of our participants' discourse, we found three patterns that characterise their participation in their respective SMDs. We detail and exemplify each of these patterns in the following.

Doctoral students talk about mathematics with non-specialised expressions

The first pattern we identified in our data pertains to the level of specialisation of the discourse generated as part of the interview. In our analysis, we found that our participants talked about specialised mathematical objects with non-specialised expressions. The first example of this pattern occurred in Isaac's interviews when talking about stability and perturbations of a dynamical system:

> So my supervisors talk a lot about stability and perturbations, […] so perturbations, the idea that I have a system, I shake it a little bit and then check again if the system behaves the same or not.

In this excerpt, Isaac elaborated on the object "perturbations", which, in the field of dynamical systems, is a controlled variation of the parameter values in the differential equations characterising a dynamical system. However, Isaac did not talk about this change of parameters when explaining "perturbation" and described it as "shaking [a system] a little bit" (underlined). In particular, "shaking" a system can be seen as capturing the variations of the parameter values, and "a little bit" as capturing the fact that these variations are controlled. Accordingly, we categorised the expression "shake it a little bit" as non-specialised since it builds on the everyday discourse and was not used in the references we consulted.

The second example of the use of non-specialised expressions when talking about mathematics comes from Charles. Here, Charles explains that he builds on the work of other mathematicians to find a specific family of graphs for a proof in his research:

> So, what I am trying to do is to kind of apply a similar idea, so what they did, they were able to build graphs with these groups as local groups, and their stabilisers just explode, they grow super super fast, we call that exponential, you can think of it as a very wild example.

In the excerpt above, Charles offered details about what his discourse community calls "exponential". Even if "exponential" is used in school mathematics (e.g., when talking about functions), Charles used it in a specific way in his SMD: "exponential" is used as a category of groups when associated with the local action of a graph. In short, "exponential" groups correspond to groups for which, when acting on a family of graphs, the size of the stabiliser grows exponentially with the size of the graphs. In the excerpt, Charles talked about this specific type of growth of the size of the stabiliser by first describing the stabiliser as "exploding" and "growing super super fast" and then introducing the specialised keyword "exponential". In particular, these two expressions are not part of Charles' SMD and are closer to the everyday discourse.

The two examples above illustrate how doctoral students used non-specialised expressions when describing or explaining specialised objects or properties from their research. While "perturbations"

and "exponential" could be discussed with specialised words and expressions, Isaac and Charles elaborated on these keywords with non-specialised expressions. More importantly, in the case of Charles, he used expressions similar to "explode" and "grow super super fast" or "wild graphs" again when talking about groups or stabilisers later in his interview. For example, in the following excerpt, Charles talked about how challenging it is to find the family of graphs he needs for his proof:

> […] you just give me a group and I'm supposed to make a graph that has this local action, and on top of that, you are asking me to make this stabiliser explode. That's even harder.

In this excerpt, Charles talked about the stabiliser as "exploding" like in the previous excerpt. However, as opposed to the first excerpt, Charles did not use the specialised word keyword "exponential" as a reformulation of "explode" and only talked about the stabiliser with the non-specialised expression this time. In that sense, this suggests that our participants also use non-specialised expressions outside of explanations of specialised words.

Doctoral students talk about mathematics by talking about people and processes

The second pattern we identified in the data concerns the objectification of the discourse, or more precisely, the *disobjectification* of the discourse. Our data showed that our participants adopted a talk about people and processes when talking about objects and properties. The first example of this pattern comes from Charles' interview, in which he talked finding a family of graphs with specific numerical parameters. When talking about the value of one of these parameters, Charles said:

> Ideally, I would find a family [of graphs] which has the numbers I like.

The numbers that Charles talked about here pertain to the size of the eigenspace for a graph, which is a specific parameter associated with each graph in the family. In particular, Charles wants to find a family of graphs for which this parameter grows linearly with the number of vertices of each graph in the family. In other words, Charles is looking for a family of graphs that will satisfy some numerical conditions. However, when talking about these numerical conditions, Charles did not name the parameters he studied or talk about the numerical value he was looking for. Instead, he talked about "the number [he likes]". An alternative and objectified version could be: "Ideally, I would find a family in which the size of the eigenspace grows linearly with the size of the graph". In a way, Charles referred to the properties of the number by talking about people in the excerpt above.

Similarly, Patrick also talked about people and processes when describing the object at the heart of his research and the sort of work he does:

> We can sort of represent these [heteroclinic cycles] as like a directed graph, and then what you can do is you can do something like stick two of them together, right? And […] this is a heteroclinic network. […] And you take an object like this […], you stick it in a four-dimensional dynamical system, so you just say you have $\mathbb{R}^4$, you stick this [heteroclinic network] in it and then you ask what does this thing's presence in the system do to the nearby points?

In this excerpt, Patrick introduced heteroclinic networks to the interviewer. In Patrick's field of research, heteroclinic networks are the union of heteroclinic cycles and they are said to define dynamical systems as they affect the behaviour of nearby points. In other words, heteroclinic

networks are unions of heteroclinic cycles, and they define the long-term behaviour of nearby trajectories in dynamical systems. Yet, when describing heteroclinic networks to the interviewer, Patrick focussed on the manipulations of mathematical objects and talked about "stick[ing] two [heteroclinic cycles] together" to obtain a heteroclinic network and then "stick[ing]" the resulting object in a four-dimensional system. He then detailed the questions one can "ask" about heteroclinic networks. Accordingly, by talking about people and their manipulations and questions about mathematical objects, Patrick implicitly talked about the properties of heteroclinic networks as the union of heteroclinic cycles and a structure affecting the behaviour of nearby trajectories.

Doctoral students signal their use of specialised expressions with discursive markers

When analysing the level of specialisation of the discourse, we noticed that our participants signalled the use of specialised expressions with a specific kind of markers, that is, markers highlighting the discursive nature of mathematics. For example, in the following excerpt, Patrick detailed some of the functions he works with as part of his research and introduced the names of such functions:

> Then, what we consider is this map or this rule or, as we call it, the flow function [which] is a function that takes where you start and how far forward in time you want to go and spits out where that point will be at the next time.

In this excerpt, Patrick introduced the mathematical object "flow function" as a "map" or a "rule" he studies as part of his research. In particular, this function returns the position of a given point after it moves for a certain amount of time in the dynamical system. When referring to this object, Patrick first used the words "map" and "rule" before introducing the specialised counterpart "flow function". Additionally, we note that just like Charles with "explode" in the first pattern, Patrick also used the non-specialised expressions "map" and "rule" instead of "flow function" later in his interview. However, with this example, we want to illustrate the expression "we call it" with which Patrick introduced the specialised expression "flow function". By highlighting the specialised character of "flow function" with the expression "we call it", Patrick also shed light on the discursive process of defining and naming "flow function" in his discourse community.

Similarly, in the following excerpt, Isaac signalled the particular status of "bifurcations":

> In the world of dynamical systems and in particular, in differential equations, there are different objects that behave differently. So, there are equilibrium points, there are periodic orbits, there are invariant tori but recently people um found out about wild chaotic attractors. The way that these objects are created is through some things that are called bifurcations.

In this excerpt, Isaac introduced bifurcations as something that "creates" the objects that he previously listed. In particular, Isaac detailed that the "things" that created these objects "are called bifurcations". Like in the case of Charles, Isaac used a marker and signalled a specialised expression to the interviewer. Additionally, this marker also highlights the discursive nature of mathematics. As opposed to presenting descriptions of mathematical objects as "a state of the affairs in the world" (Sfard, 2008, p.107), Charles and Isaac both shed light on the processes of naming and defining mathematical objects.

Discussion

In this paper, we investigated doctoral students' participation in research mathematics through the communication of their research projects to a mathematics education researcher. Through our analysis of these interviews, we identified three patterns characterising the discourse of doctoral students when talking about their research in this context. In particular, we found that our doctoral students (1) talked about mathematics with non-specialised expressions, (2) talked about mathematics by talking about people and processes, and (3) signalled their use of specialised expressions with discursive markers.

To our knowledge, the mathematical communication of doctoral students has never been studied before. Our findings are a first step in characterising how doctoral students communicate their research, but also in characterising how doctoral students "construe their [mathematical] 'reality.'" (Morgan & Sfard, 2016, p.100).

The first pattern highlights that doctoral students draw on non-specialised discourses when talking about their research. In this sense, our findings align with the work of Barwell (2013, 2016) who highlighted that learning mathematics might not only be transitioning from what he called the *everyday* or *informal* discourse to the *mathematical* or *formal* discourse. For him, learning mathematics involves a dialectical relationship between various discourses and entails the extension of one's discursive repertoire. Our analysis illustrates that our doctoral students not only drew on their SMDs when talking about their research, but also on the everyday discourse. This suggests that the everyday discourse might be part of the discursive repertoire associated with their research. Additionally, the objectification of discourse is often described as an important stage in learning mathematics: replacing a talk about people and processes with a talk about objects and properties is described as essential for the development of mathematical discourses (Sfard, 2008). Yet, our second pattern indicates that our participants talk about processes and people when describing mathematical objects and properties involved in their research. In addition, our third pattern highlights the discursive processes of naming and defining mathematical objects as part of doing mathematics. In other words, the discourse of our participants sheds light on their involvement and the involvement of other mathematicians in mathematical research.

This suggests that the everyday discourse, but also a talk about people and processes, might be part of the SMDs of our participant. Since commognition claims that communicating is the public version of thinking, it is possible to wonder about the role these ways of talking and thinking play when our participants engage with mathematical research. In other words, when our participants work on their research and try to progress on some of their problems and proofs, what are the contributions, if any, of the everyday discourse and talk about people and processes? More investigations across different contexts, such as conference talks or research meetings, could yield more insights about these contributions. While our preliminary analysis of the seminars suggests that our participants adopted similar patterns when presenting their research to other doctoral students in mathematics, more work is needed to investigate how different discourses and forms of talk play a role in our students' mathematics.

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The role of mathematics in advanced electrical engineering courses: a typical case of numerical integration

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This study, framed within the Anthropological Theory of the Didactic, explores the role of mathematics in advanced courses for electrical engineers. It examines the connection between mathematics and advanced electrical engineering courses through interviews with teachers and a master's student, alongside a didactic and praxeological analysis of teaching and learning about spectrum of laser emission, which relies on integration. The results indicate that mathematics courses mainly provide theoretical foundations, while advanced courses develop techniques specific to electrical engineering. Advanced courses can inform mathematics courses in terms of knowledge selection, adaptation, and organisation, as well as inquiry-based learning. A pedagogy in which the teacher is primarily responsible for knowledge, while the student applies given models, is also a characteristic of courses beyond mathematics.

Keywords: Mathematics for engineers, Numerical integration, Anthropological theory of the didactic, Didactic transposition.

Introduction

Mathematics might be considered a fundamental, but subtly integrated part of many other disciplines. Hence, it is a big challenge to determine the scope and the tools with which one might study the math-related topics in the university education of non-mathematics professionals. There is a growing interest in research of teaching mathematics for students in other disciplines, such as physicists, chemists, engineers, economists etc. (Winsløw et al., 2018, p. 67). Analysing the epistemological relations between mathematical knowledge in mathematics courses and specialised professional courses (Hochmuth & Peters, 2022; Rønning, 2021) enables an understanding of the transitions students undergo, reveals potential gaps, and provides a foundation for didactic interventions.

In this study we wish to explore how mathematical knowledge appear in a more professionally oriented (advanced) course for electrical engineers, with the goal to uncover the requirements of the mathematics discipline that are coming from the subject discipline. This is a continuation to our previous investigations (Bašić & Milin Šipuš, 2024) on the role of the notion of integral in the university education of future mathematicians, economists and physicists, in which we are especially interested in methods and their (dis)advantages in educational research.

Theoretical framework and research questions

Anthropologic Theory of the Didactic (ATD) is a theoretical framework that, in the broadest sense, examines the factors for the diffusion of knowledge through an epistemological, ecological and didactic analysis within a given institution (Chevallard & Bosch, 2020a). The epistemology of knowledge at stake can be explored using the tool and methodology of *didactic transposition*

(Chevallard & Bosch, 2020b). The basic unit for analysing and modelling knowledge is praxeology, which is represented by an ordered quadruple $[T/\tau/\theta/\Theta]$, where: τ is the technique used to solve a task of type T, θ is the technology, the discourse of the praxeology and explains the technique, and Θ is the theory that is a justification of the technology (Chevallard & Bosch, 2020a). The ordered pair $[T/\tau]$ is called a *praxis (block)* of the praxeology and $[\theta/\Theta]$ is called the *logos (block)*. Praxeological analysis enables the understanding of the organisation of knowledge and reveals the purpose of the observed knowledge within the observed institution.

The analysis of the *ecology of knowledge* focuses on *conditions and constraints* set by the various *levels of didactic co-determinacy*: ⋯↔*Schools* ↔ *Pedagogies* ↔ *Disciplines*↔⋯ (Chevallard & Bosch, 2020a). ATD views individuals in relation to the *positions* they hold within the observed *institution* (Chevallard & Bosch, 2020a). In a typical didactic system, the key positions are those of the teacher and the student.

Furthermore, ATD identifies the *paradigm of visiting works* as the dominant educational paradigm, where pedagogy is based on teaching pre-established pieces of knowledge (Chevallard & Bosch, 2020a). The movement towards inquiry-based learning, often seen in research and educational systems, is regarded by ATD as a push for a shift in the educational paradigm—from the *paradigm of visiting works* to the *paradigm of questioning the world* (Barquero et al., 2018; Chevallard & Bosch, 2020a).

In general, we are interested in the epistemology and ecology of praxeologies related to integrals that appear in advanced engineering courses, the role of (both established and new) mathematics praxeologies in such courses and, the relationship between introductory mathematics courses and advanced engineering courses. In this study, we aim to explore potential methods for addressing these broad questions by examining teaching and learning about the spectrum of laser emission. Teachers and experts in the domain of electrical engineering suggested this theme because it includes integration, which poses mathematical challenges to students. Accordingly, our research question is: What insights can praxeological, didactical, and ecological analyses of this theme provide for teaching mathematics to engineering students or for adapting mathematics-related praxeologies to the needs of advanced engineering courses?

Methods

To gain insight into current practices involving the use of integrals in advanced electrical engineering courses, we conducted short semi-structured interviews with two teachers and a student from a faculty of electrical engineering in Croatia. The first interviewee is a professor of Optical Communication Systems with about 20 years of experience teaching electromagnetic courses to students pursuing a specialized study track, typically with about 10 students enrolled. The second professional is a PhD student who also works as a teaching assistant for various courses, such as Antenna Systems. The student participant is in the final year of the Master's programme in Information and Communication Technology, having successfully completed all required courses up to that point.

Our research began by asking teachers to identify content they found particularly interesting as examples of using integrals in electrical engineering. They have selected the theme about the spectrum of laser emission and provided accompanying lecture notes for the students. Our review of

these materials aimed to assess how the mathematical content was addressed in the course. Additionally, interviews with teachers were conducted by the first and second author, focusing on the relevance of the tasks in engineering education and methods for calculating integrals when working with discrete data. We were also interested in determining whether the students were expected to have specific prior knowledge of numerical integration, their use of technology, and any potential difficulties related to mathematics. The interview with a student aimed at the same information but from a student perspective. A praxeological analysis has been performed based on the lecture notes and the interview, while the didactical and ecological considerations are drawn based on the interviews – full transcripts to selected answers and our interpretation is given in the following section.

Analysis

We consider a task in electrical engineering concerning the spectrum of laser emission (Saleh & Teich, 2019). The spectrum consists of many wavelengths and for each wavelength one can measure the corresponding spectral power of the source. The width of the laser spectrum is the distance between wavelengths at which the power is half the maximal power, see Figure 1a.

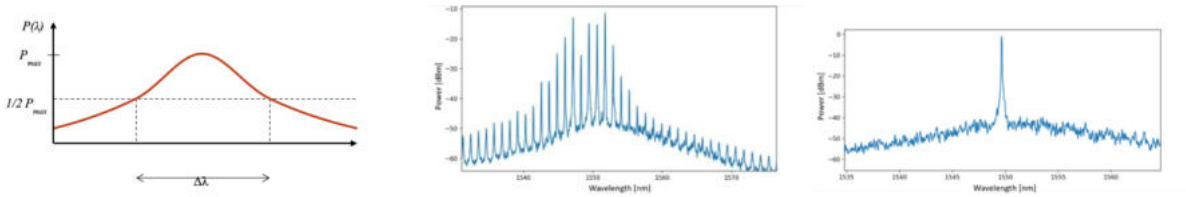

Figure 1: The width of a spectrum (a) and typical spectrum of a multimodal (b) and of a single-modal (c) laser diode

For multimodal (Fabry-Pérot) lasers, the spectrum of possible wavelengths is highly irregular (Figure 1b) and to determine the width, the mean square spectral width is calculated using the formulas:

$$\sigma_\lambda^2 = \frac{\int_0^\infty (\lambda - \bar{\lambda})^2 P(\lambda) d\lambda}{\int_0^\infty P(\lambda) d\lambda}, \qquad \bar{\lambda} = \frac{\int_0^\infty \lambda P(\lambda) d\lambda}{\int_0^\infty P(\lambda) d\lambda},$$

where the second formula describes the average mean of possible wavelengths. The same formulas are used to determine the spectral width of single-mode lasers (Figure 1c) because the spectrum is very narrow, and it is difficult and inaccurate to determine the distance between $1/2\ P_{max}$ points.

In practice, it is common to obtain measurements for a (very dense) set of wavelengths. A typical dataset contains about 20000 entries given as pairs of a wavelength and the corresponding source power. The task for the students is to calculate the spectrum width based on this discrete set of data. The students are not given any further instructions, and the solution of the task should be handed in a form of a report. Since the data is discrete, this is a typical task in numerical integration. In numerical mathematics, numerical integration is usually performed using software which is programmed based on the 'trapezoidal rule' to obtain precise values, but sometimes a less precise 'rectangle rule' is used. The rectangle rule is a very simple approximation technique that is closely connected to the definition of the definite integral using Riemann sums. The integral is approximated by the Riemann sum, i.e.

the ‘area under the graph’ is approximated by a sum of areas of rectangles of certain width of a small interval and height given by the value of the integrated function in one point of that small interval.

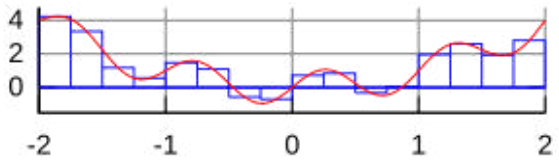

Figure 2: Rectangle rule is based on the approximation of the definite integral by Riemann sum (“Numerical integration,” 2024)

In the interview, the professor (P) explains the task with the specific details about the data provided to the students and the responsibility that is left to them as a small inquiry assignment.

> P: The task is to calculate the mean square spectral width… related to the area under the curve. The definition is classical, and the only thing we do is to apply to a concrete spectrum. This is a fundamental example in our field. The spectrum is given as a discrete sample of data obtained by measurement. The instrument that we use does not give the mean square width, although some other instruments do, so the students need to calculate the integral from the given data numerically. It is left up to students to choose what method and software they will use, but they need to present their work. For the students this is one of the first encounters with real data and for the first time they need to make a calculation using numerical methods.

We observe that the professor expects the students to make connections between the formulas and the data based on their previous mathematical knowledge about the notion of an integral. In the interview, he provides more inputs which explain the students’ praxeologies, i.e. the technology of the ‘area under the curve’ and the ‘rectangle rule’ which the students use as a common technique.

> P: Some students have nice approaches. In general, they interpret the integral as the area under the curve and they use the rectangular approximation, which we also call step or piecewise constant approximation. Trapezoidal approximation would be a step further, but since datasets are large there is no need for it. The students recall how to calculate areas under the graph from their mathematics courses... they don’t have difficulties and provide correct results. Probably the students also exchange basic ideas about integration among themselves. The only problems that occur in some students’ report is related to the fact that the instrument provides data on a logarithmic scale (in decibels), so some students fail to take that into account, but this is a very concrete and manageable problem.

In addition, the professor explains that this task is very special because this specific spectrum can be well approximated by the Gauss curve. It is taught in the lectures that this approximation is related to the numerically calculated integral by a fixed factor (square root of 2), so the professor concludes that this might show students that “their work is meaningful and that the students use this result to check their own work”. Hence, validation of the praxis is also possible in this task using another logos. The professor also comments that the data is such that the issues of numerical approximation (such as floating point error) does not occur in this setting, but this is not discussed with the students, as most of them have not taken the elective course in numerical mathematics and are even unaware of such issues.

The interviewed student (S) recalled immediately the task and described it in the following way.

S: We were given the data about the power of a laser, and we needed to calculate the mean square spectral width. I did not have difficulties because I remember the notion of the integral from my previous courses. An example with integration has been shown in the exercises class and the formulas [of the model] with integrals were provided during the lectures.

We see that the retention of what was the task is very strong, and that the student was interested during different parts of the course. She proceeds by explaining how she solved the task.

S: We have learned in mathematics courses that the integral from somewhere to somewhere is the area under the graph and we saw that we have something similar here…erm… the spectrum of the laser is like a function given by data. So, we didn't need to integrate, I have just simply calculated the data as the small individual areas…and I used the 'for-loop' in Python to sum that up. Of course, there is an error, but as we increase the number of data the error is smaller.

From this quote we see that the student interprets the integral as the area under the graph and uses the technique of approximating that area by 'rectangle rule'. The student also mentioned that they did not take a course on numerical mathematics, and that they "have heard there are numerical methods for integration but have not seen or used them". This indicates that it is possible that the students make connections between the discrete and continuous aspects of the integral based only on intuitive justifications and possibly on the definition of the Riemann integral (theory developed in earlier courses on integrals), and for them numerical integration is an ad hoc technique that they discover while solving this task, not a standard technique from numerical mathematics.

The teaching assistant (TA) brought the focus of the interview to techniques that the students use in a broader sense. He mentions one task with which many students show different techniques than expected regarding taught knowledge. The task reduces to integration in one variable and since integration is not the focus on the exam, the primitive function of the integrated function is provided in the task as a hint. It is noticed that the students do not use that hint and instead plug in the whole integral into a calculator.

TA: For a few years now, I notice that at the exams 80-90% of students rather use the calculator than the antiderivative given in the task to finish the calculation. We suspect they feel safer this way and find it a faster way. I have asked them why they do it but never received a reason. They probably get used to using calculators during their first or second year. Although the mathematics courses in the first year force all the students to integrate well, we might notice that integration becomes less frequent in the higher years. For example, even when the students need to compute Laplace or Fourier transforms, they learn ready-made rules for these transforms, and they stop integrating "by hand" after the second year.

Moreover, the assistant emphasized that the students focus more on the engineering and physics notions while mathematics is an auxiliary tool, and new mathematical notions are not introduced in the course. He confirms that the course on numerical mathematics is elective, and students develop techniques based on intuitive justifications, but believes that for many students the main notions are developed during the course Mathematical Analysis 1 in the first year. He tells the story of his mathematics professor at this course: 'Imagine you are in the woods without a calculator and must estimate the area of an irregular part of wood. You could cut the wood into small rectangles and

measure them with a ruler.' He also discusses students' techniques and mathematical praxeologies, in general.

> TA: Concerning the interpretation of results or making conceptual calculations, such as using that the function is even or has some other symmetry, or a trick using a shift of the graph by $\pi/2$, we see that there are two classes of students. Some students easily do such tricks and others almost never do it, there is no in between. We also see that students often use online graphic calculators, but the professors never teach them to do that – it must be that one student comes to others and shows them what they discovered and that is how some use of software spreads.

Discussion and conclusion

The interviews with the two teachers have suggested the analysis of teaching and learning about the spectrum of laser emission. The analysed theme serves as an example of the interaction between the disciplines of engineering, didactics and mathematics. We see the analysed task as significant in the study programme of future electrical engineers because it is (1) fundamental to the discipline of electrical engineering according to experts, (2) providing learning opportunities for autonomous activities which is shown in the students' freedom to choose techniques for solving the task, (3) providing first encounters with calculations based on real data and establishing connections between theoretical mathematical knowledge and practical domain knowledge, and (4) leads to evolution of the mathematical praxeology of calculating definite integrals.

The extension of integration praxeology is two-folded. First, the task requires students to formulate and justify the technique of calculating the integral of a function given in discrete arguments (table representation). The technique is justified by reinforcing the logos from basic mathematics course: the definition of the Riemann integral and its interpretation as area under the graph. Hence, the praxeology is enhanced in the logos block. Second, the techniques of integration change over the years. At first, in mathematics courses, the main integration techniques are induced by the logos of the integral as an antiderivative and the Newton-Leibniz formula, and through specialised courses numerical integration and the use of software become the more dominant techniques. This evolution of the praxis block is also explained and supported by the episode mentioned by the teaching assistant – the students use calculators to evaluate integrals even when the antiderivative is given as an instruction in the exam task. Exact integration techniques learnt in mathematics courses are not crucial in specialised courses. This aligns with González-Martín's (2021) conclusion that integrals are used to introduce notions in engineering courses, but the techniques in these courses do not heavily rely on calculus techniques (p. 211). Analysing specialised courses from a mathematical perspective can more efficiently guide the didactic transposition of knowledge from mathematics courses. In this case, one may question whether it is necessary to introduce the indefinite integral before the definite. To what extent should numerical mathematics be integrated into the education of engineers, and how should it relate to the knowledge transposed from mathematical analysis? Furthermore, it is crucial to consider how the logos of numerical techniques can be constructed to prevent a disconnect between praxis and logos in engineering praxeologies, Current evidence suggests that while logos is grounded in mathematical analysis, praxis primarily relies on numerical techniques.

From the interviews with the teachers, it is indicated that the students manage in the given conditions, i.e. the economy of the didactic system is evaluated positively from all institutional positions. This is

confirmed also by the interviewed student, not only for herself, but for her colleagues. The ecological analysis spans beyond the local didactic situations to include reasons for its success in: (1) the praxeological equipment that the students attain during their earlier stages of study, and (2) the stability of the implementation of the programme (including the described task) over a period of at least few years. Nonetheless, we may note that the teachers do not formulate all the didactical goals of the task explicitly and the professor clearly emphasizes that the choice of the task is governed by the requirements of the discipline of engineering.

Institutional positions shaped by the specific academic and professional background of teachers who are not mathematicians, enrich the didactic system even beyond the specialised courses these teachers teach, by informing mathematics courses. In the paradigm of visiting works, specialised courses can provide justifications for learning and organising knowledge. In the paradigm of questioning the world, analysing specialised courses, as well as fostering collaboration between mathematics teachers and teachers of specialised courses, can support design of inquiry-based activities (SRPs) even in mathematics courses. Moreover, despite the learning potential of such activities, the pedagogy and ecology of knowledge in specialised courses is within the paradigm of visiting works. The teacher is responsible for providing techniques, assessing their efficiency, and delivering the formula, which can be understood as a model that students must apply (Barquero et al., 2018). This points to the entrenched nature of the pedagogy of visiting works as the dominant institutional pedagogy, imposing constraints on the diffusion of knowledge in non-mathematics disciplines.

Each interviewed participant might be considered to occupy multiple positions. The professor makes academic decisions based both on his position as a teacher and a professional in engineering, while the teaching assistant provided answers in the interviews using his experience as a student and as a teacher. The interviewed student confirms hypothesis about the study programme and the perspective of other students (i.e. from the institutional perspective of a student we confirm that the didactic transposition was successful for some students), but she also explains some of her behaviour based on personal preferences, aware of her differences from other students, and points to her unsatisfied needs for more meaningful connections with other domains of mathematics. This is, of course, one more instance of the personal-institutional dialectic, well-known in ATD, but shows some clear advantages and limitations of the methods in our study – as much as we gain from learning about the personal perspective of a student, we unavoidably lose the power of generalisation of our findings.

To conclude, our study informs us about the role of mathematical tools and notions in electrical engineering that provides opportunities for students' learning on different levels and illustrates a complex relationship between the disciplines that happens in a university study programme for engineers. It might be seen that the strong impact of the discipline is seen in the reduction of the logos block in the discourse from mathematics, which is complemented by the technologies and theories in the discourse of the discipline to justify the established praxis block. We also note the importance of considering different institutional positions, whose disregard, such as considering mathematical notions only from the perspective of the discipline of mathematics and ignoring the epistemology of other disciplines, might lead to methodological limitations. Similarly to some other insights in the discipline of physics (Bašić & Milin Šipuš, 2024), we see that non-mathematics disciplines contain very intricate positions of teachers who, alongside with professional content of their course, teach

domains of mathematics and use their experience as professionals in didactic decisions. The didactic transposition present in their courses shows many positive aspects that certainly would justify the attention of further research in mathematics education.

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Exploring the impact of ChatGPT in teaching numerical methods: Student perceptions, usage and exam performance

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This study examines the integration of ChatGPT in a second-year undergraduate numerical methods course with 32 mathematics major students. We evaluate the perceived usefulness of ChatGPT before and after regular use in a numerical methods course, its correlation with students' willingness to use it in high- and low-stakes environments, and with exam outcomes. Findings indicate mixed perceptions of ChatGPT's usefulness, particularly its limitations in addressing misconceptions and the risk of students accepting false proofs. Correlation data (N = 22) suggest that students who initially viewed ChatGPT as useful were more likely to use it, and frequent users found it more beneficial. A slight positive correlation may exist between ChatGPT usage and exam performance, but its use during the exam correlates negatively with overall performance. Small sample size limits the generalizability of the results.

Keywords: Chat GPT, LLM, attitudes, numerical methods, university mathematics education.

Introduction

The use of technology for educational purposes is as old as education itself. With the emergence of advanced and widely accessible Large Language Models (LLMs) – computer programs that generate human-like text based on training data – we may face a major paradigm shift in education.

The AI Boom triggered by the release of ChatGPT 3.5 by OpenAI in November 2022, and GPT 4.0 in March 2023, was a significant development which have reached a vast number of users. To incorporate ChatGPT into mathematics education, we believe that it must 1) provide reliable mathematical content – so that students may learn valid mathematical content – and 2) learners must perceive them as credible sources of information – which is a critical component of technology acceptance. The fulfilment of these two conditions is not inherently guaranteed. These models may return false or misleading information – a phenomenon called AI hallucination. Hicks et al. (2024, p. 37) explain that ChatGPT is "in an important way indifferent to the truth of their outputs" because they are intended to mimic human language rather than critically evaluate their outputs. Moreover, ChatGPT is often unable to adequately correct misconceptions, as pointed out by Wardat et al. (2023).

Frieder et al. (2023) examined the mathematical capabilities of ChatGPT 3.5 and 4.0 using the GHOSTS and miniGHOSTS datasets, which contain 728 prompts covering various university-level mathematics topics and are designed to benchmark the higher mathematical capabilities of LLMs. Frieder et al. (2023, p. 1) write that "ChatGPT can be used most successfully as a mathematical assistant for querying facts, acting as a mathematical search engine and knowledge base interface."

Furthermore, GPT 4.0 can assist students with undergraduate-level mathematics questions. However, both versions performed poorly on graduate-level mathematical problems.

Kashefi and Mukerji (2023) showed that ChatGPT can generate, debug, and translate code of various advanced algorithms between programming languages, but is prone mistakes like those of a human programmer. Kabir et al. (2023) conducted an analysis with 517 programming questions to evaluate ChatGPT's answers and found that "52% of ChatGPT answers contain incorrect information" (Kabir et al. 2023, p. 1) while the users "overlooked the misinformation in the ChatGPT answers 39% of the time" (Kabir et al., 2023, p. 1) which underlines the importance of user caution when using ChatGPT.

Kelly et al. (2023) conducted a systematic literature review utilising the frequently used extended Technology Acceptance Model (TAM) with 60 articles. TAM explains how users come to accept and use technology, with key constructs such as perceived usefulness (the belief that the technology enhances performance), ease of use (how effortless it is to use), performance expectancy (expected benefits from use), and effort expectancy (anticipated effort required). They conclude that

> perceived usefulness, performance expectancy, attitudes, trust, and effort expectancy significantly and positively predicted behavioural intention, willingness, and use behaviour of AI across multiple industries. (Kelly et al., 2023, p. 1)

Abdaljaleel et al. (2024) administered a ChatGPT-specific modification of TAM called TAME-ChatGPT to 2240 participants from Arabic countries in the context of higher education and found that "ease of use, positive attitude towards technology, social influence, perceived usefulness, behavioural/cognitive influences, low perceived risks, and low anxiety" (Abdaljaleel et al., 2024, p. 1) affected positive attitudes towards the adoption of ChatGPT in higher educational settings.

Given the current known extensive capabilities of ChatGPT as well as its limitations, research was conducted with second-year mathematics major undergraduates at Eötvös Loránd University as part of an introductory numerical methods course, attempting to find answers to the following questions.

- How do students' perception of ChatGPT's perceived usefulness (both before and after the course), their autonomous use of ChatGPT in low-stakes and high-stakes academic situations, and their exam performance (exam scores) correlate?
- How does students' perception of ChatGPT's perceived usefulness for learning numerical methods change because of regular use over the semester?

Based on the literature cited, we believe that answering these questions is critical in establishing the usefulness and applicability of LLMs in university-level mathematics education.

Impact

This study explores how students' perceptions of ChatGPT evolve with experience and how it affects engagement and learning outcomes. Higher levels of acceptance are expected to correlate with higher levels of use, but it is unclear whether more ChatGPT use correlates with better performance, as inaccurate answers might confuse students. However, exposure to erroneous answers may facilitate better long-term knowledge retention and self-correction in some cases (Adams et al., 2014; Große & Renkl, 2007; Paulovics et al., 2023). Establishing such correlation should be a prerequisite to any

argument about the expected benefits of integrating LLMs into the teaching of numerical methods. By examining the correlation between students' perceptions of ChatGPT, and their willingness to use it in varying academic situations, including in examination situation, we can identify some of the potential impact of integrating LLMs on both learning efficacy and assessment practices, thus contributing to their integration in teaching numerical methods, and mathematics in general.

Methods

This study was conducted at Eötvös Loránd University, Hungary (ELTE), which is recognized as the leading institution for mathematics education in Hungary, with 22 male and 10 female second-year mathematics undergraduate students. Participants had previously studied three semesters calculus, two semesters of linear and abstract algebra, and one semester of Python programming. They had not previously studied numerical methods and began their studies in this subject in the spring of the 2023/2024 academic year. Their interests in mathematics are diverse. Not all 32 participants took part in every activity administered during the semester, but all 32 students took the end-of-semester exam.

The course is titled Introduction to Numerical Methods, which includes a 45-minute lecture and a 90-minute lab session for 12 weeks. The course topics include floating point arithmetic, numerical differentiation, interpolation, numerical integration, solving systems of equations numerically, estimating eigenvalues of matrices, and solving ordinary differential equations numerically.

At the end of the semester, students are evaluated based on a complex four-part exam:

- Qualifying Section (50 mins): Three easy, practical programming questions, any non-human assistance allowed. Those who fail this section will fail the whole exam.
- Definitions and Theorems (30 marks),
- Algorithm Comprehension (20 marks): Programming/debugging,
- Proofs (50 marks): State and prove three theorems.

Categorising the research participants based on their prior mathematical and programming skills is not justified due to the high degree of homogeneity in their previously earned grades. Given the sample size and the nature of the research questions, students were not separated into groups of any kind. During the semester, a higher level of diversity in programming skills was revealed than previously suggested by the grades of the Introduction to Scientific Programming (Python) course.

During the 12 weeks of instruction, students completed 5 quizzes (questionnaires) in weeks 2, 3, 5, 10, and 12 to explore how they used ChatGPT for assistance, whether they used it without being prompted to do so, how helpful they found it for solving numerical methods exercises, and whether they find its mathematical claims convincing. They were specifically asked to try and use ChatGPT only before answering Quiz 1 and Quiz 2. It is important to keep in mind that given the same prompt twice, LLMs will most likely return different answers, and thus it is impossible for teachers to plan for every conceivable answer that students may encounter when working with it. Here follow the quizzes and related activities that are discussed later.

Activity before Quiz 1: Students were asked to consider a basic difference scheme and determine which derivative of f it approximates and in what approximation order. Then, they were told to ask ChatGPT for help in any way they like. Finally, they filled out Quiz 1 with the following questions:

1. Did you ask ChatGPT for assistance before being told to?
2. How useful do you feel it was to attempt solving the exercise with the help of ChatGPT? (0 = Not useful at all, 1 = Mostly not useful, 2 = Mostly useful, 3 = Very useful)

Activity before Quiz 2: Students were asked to calculate the Lagrange interpolation polynomial given four input-output pairs and then write a Python code that would find the coefficients of the interpolation polynomial and plot it together with the interpolation points. Then they were told to ask ChatGPT to help them write the code. Finally, students filled out Quiz 2 with the following questions:

1. Did you check the output of the code generated by ChatGPT?
2. Did ChatGPT provide correct code?
3. How useful do you feel it was to attempt solving the exercise with the help of ChatGPT? (0 = Not useful at all, 1 = Mostly not useful, 2 = Mostly useful, 3 = Very useful)

Activity before Quiz 3: Students were asked to prove that if the matrix norm is induced by a vector norm, then the inequality $\|A\| \geq \varrho(A)$ holds, where $\varrho(A)$ denotes the spectral radius of the matrix A.

ChatGPT provided the following answer to this question:

> According to the definition of the induced matrix norm, $\|A\|$ is the maximum rate of proportional increase of a linear transformation performed on a matrix, relative to a given normalized input vector. Formally: $\|A\| = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|}$. The spectral radius is the largest absolute value eigenvalue of the matrix, which means the norm applied to the matrix is the maximum of the absolute value of the matrix's eigenvalues. Therefore, for any induced matrix norm, $\|A\|$ can be greater than or equal to the absolute value of the largest eigenvalue, i.e. $\|A\| \geq \varrho(A)$. Thus, the spectral radius is always less than or equal to the induced matrix norm. Hence, the inequality $\varrho(A) \leq \|A\|$ is valid for any induced matrix norm.

What is your opinion on the solution ChatGPT provided?

a) There are no usable parts in the answer (either in terms of definition or reasoning).
b) There are correct parts, but the reasoning is mostly incorrect.
c) The reasoning is basically good, but some steps are not perfect.
d) The proof is fully valid

Note that ChatGPT's solution was to merely quote the definitions of the matrix norm and spectral radius and then restate the claim to be proven. It did not make any logical steps to connect the two definitions. The actual proof is rather easy compared to how much text ChatGPT returned: it requires the application of equivalent definitions stringed together.

After the exam, we provided an opportunity for the students to fill out a voluntary questionnaire:

1. How much did you use ChatGPT's assistance independently during the semester (outside of the seminars)?
2. How useful did you find ChatGPT's assistance for independent exercise solutions during the

3. semester? (0 = Not useful at all, 1 = Mostly not useful, 2 = Mostly useful, 3 = Very useful)
4. Did you use ChatGPT to solve the test questions in the qualifying section of the test today? If yes, for which questions?

Findings

Results from Quiz (questionnaire) 1 and 2 – completed by 25 students altogether – aimed at measuring students' perceived usefulness of ChatGPT at the beginning of the semester show slightly positive perceptions. Averaging each student's two answers given one week apart yielded data of 25 scores with an average of 1.98 and a standard deviation of 0.57 on a scale from 0 (not useful) to 3 (very useful). The after-exam answers of these same 25 students regarding the perceived usefulness of ChatGPT at the end of the semester show a slight increase in average (2.36) and decrease in standard deviation (0.49), possibly indicating better attitudes towards ChatGPT's usefulness.

Figure 1 summarizes five key correlations (R) and their corresponding p-values between the variables described below, calculated for the 22 students who answered all the questionnaires. All omitted correlations are negligible (|R|<0.3).

- **PPU** (previously perceived usefulness) – the average score for the perceived usefulness of ChatGPT given in response to Question 3 of Quiz 1 after the first ChatGPT session and Quiz 2 after the second ChatGPT session (0-3),
- **P** - the score achieved by the student on the exam (0-100),
- **UDS** - ChatGPT use score during the semester based on the after-exam survey (0-3),
- **PUA** - perceived usefulness of ChatGPT at the end of the semester (0-3),
- **SCR** - the use of ChatGPT during the qualifying part of the exam (0-3) – how many of the three exercises the student asked ChatGPT about.

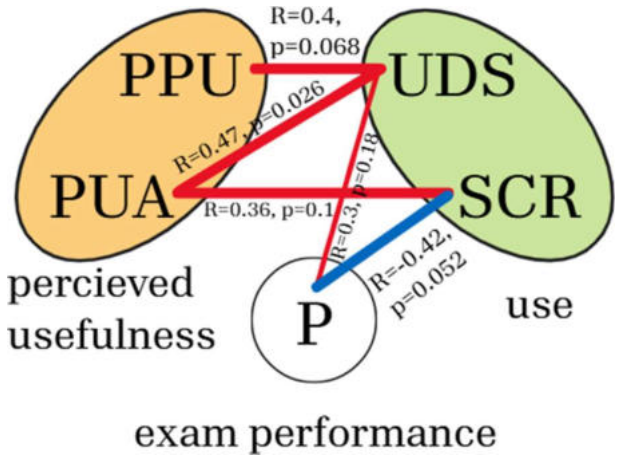

Figure 1: Correlation values and the corresponding p-values between the variables defined above

Based on the data gathered, the previously perceived usefulness of ChatGPT (PPU) may have (p = 0.068) a weak positive correlation (R = 0.4) with individual use during the semester (UDS), and individual use during the semester probably (p = 0.026) has a moderate positive correlation (R = 0.47)

with students' perceived usefulness of ChatGPT after the semester (PUA). Moreover, PUA may have ($p = 0.1$) a weak positive correlation ($R = 0.36$) with students' willingness to try to apply ChatGPT in a high-stakes academic environment, during the qualifying section of the exam. However, this willingness possibly ($p = 0.052$) has a moderate negative correlation ($R = -0.42$) with overall exam performance. Furthermore, students' individual ChatGPT use during the semester might have ($p = 0.18$) a weak positive correlation ($R = 0.30$) with overall exam performance. Finally, there appears to be no noteworthy correlation between students' perception of ChatGPT's usefulness - either before the semester or after the semester - and their exam performance.

The answers for question 3 of Quiz 3, regarding ChatGPT's attempt at solving the theoretical exercise, among the 23 students that answered are as follows:

- 0% (0/23) - There are no usable parts in the answer (either in terms of definition or reasoning).
- 52% (12/23) - There are correct parts, but the reasoning is mostly incorrect.
- 39% (9/23) - The reasoning is fundamentally good, but some steps are not perfect.
- 9% (2/23) - The proof is fully valid.

Given that ChatGPT's answer for this exercise was to quote the definitions of the matrix norm and spectral radius, and then twice restate the claim to be proven, the fact that 11 out of 23 students found this answer either "fundamentally good" or "fully valid" underscores ChatGPT's ability to mislead large fractions of novice students with reasonable-sounding non-answers.

Discussion and conclusions

This study aims to explore the perceived usefulness of ChatGPT in a second-year undergraduate numerical methods course. The findings provide insights into how students interact with and perceive AI-based assistance, highlighting both its potential benefits and limitations.

Results from Quiz 1 and 2, as well as the after-exam survey show mixed perceptions with regards to ChatGPT's usefulness, possibly reflecting the known issues of the current limited mathematical capabilities of ChatGPT when it comes to undergraduate and graduate level mathematics, and its inability to correct misconceptions, as discussed by Frieder et al. (2023), Kashefi and Mukerji (2023), and Kabir et al. (2023). The increase in mean and decrease in the standard deviation between these two measurements may be interpreted as an improved perception of ChatGPT's ability to assist students with numerical methods exercises. Although, this increase may also be the result of several psychological biases, such as the familiarity bias (a preference for what one has been exposed to) or the sunk cost fallacy (continuing an effort because of past investment), as detailed by Greul et al. (2023); rather than ChatGPT's convincing mathematical ability.

In several seminar sessions, students were encouraged to use ChatGPT independently. The after-exam survey revealed many students found ChatGPT useful for solving exercises outside of structured seminar activities. The correlation data suggest that students who found ChatGPT to be useful after a couple of seminar sessions used ChatGPT more frequently during the semester, and that those who used ChatGPT more frequently during the semester perceived it as more useful by the end of the semester. This latter correlation may likewise be due to psychological factors such as the wish to justify effort or mere familiarity bias and is not necessarily caused by ChatGPT's performance.

The data indicates a slight positive correlation between the independent use of ChatGPT and exam performance, suggesting some possible benefits in understanding and application of numerical methods. The numerical values of the correlation coefficient and the p-value are far from showing a significant positive benefit, especially with a moderate-size largely homogeneous pool of participants. Nevertheless, they justify further investigation, which may establish such a benefit more clearly, especially as LLMs are likely to become more powerful in the future.

The notable negative correlation between ChatGPT use during the qualifying section of the exam and the final exam scores ($R = -0.42$) may have at least two explanations. Either ChatGPT reliance for immediate problem-solving without deeper comprehension may hinder performance, or simply those who were less prepared and consequently went on to earn a lower score tended to rely on ChatGPT more heavily in the qualifying section of the exam. It is important to keep in mind, however, that the generalizability of these findings is rather modest due to the small sample size of the study.

In Quiz 3, it is alarming that almost half of the students thought that restating the two definitions and the claim to be proven constitutes an "at least mostly sound." This result is congruent with the findings of Kabir et al. (2023) in the field of programming, and Frieder et al. (2023) in university mathematics, and underscores the possible benefit of including methodology based on students' error correction, such as employed by Adams et al. (2014); Große & Renkl, (2007); and Paulovics et al. (2023).

Integrating ChatGPT into mathematics courses with care and emphasis on its known weaknesses may prove to be beneficial, as it can function as a data-querying tool extremely well, students appear to respond positively to integrating it into courses, and correcting mistakes is also known to increase understanding. This is especially true, as AI tools are likely to get more powerful soon. Several students reported that using ChatGPT is a great tool for them to get their codes to work, suggesting that courses that involve programming might benefit from integrating ChatGPT more.

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Permutations with indistinguishable objects: inconsistencies in the university mathematics training of prospective secondary teachers

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In this paper, we examine how content related to permutations with indistinguishable objects (PIO) is presented in tertiary and secondary mathematics textbooks. Using tools provided by the Anthropological Theory of the Didactic, we observe a compartmentalisation of approaches and missed opportunities to form connections between permutations and combinations in the context of PIO. Our findings highlight important gaps in the rationales presented in tertiary textbooks regarding PIO, as well as inconsistencies between the content studied by prospective teachers and the content presented in secondary mathematics curricula. While the tertiary textbooks studied often rely on a single formula-based technique, the secondary textbooks use a broader range of problem-solving approaches and include tasks with constraints that are absent from the tertiary-level textbooks.

Keywords: University mathematics for prospective teachers, combinatorics, textbook analysis, Anthropological Theory of the Didactic (ATD), Klein's second discontinuity.

Introduction and research problem

Recent years have seen an increase in the number of studies examining the teaching and learning of university mathematics for students outside mathematics programs (Biza et al., 2022). Two major avenues of inquiry can be observed. First, the relevance of mathematical training for various professionals, including prospective teachers, is facing growing scrutiny (Wasserman et al., 2023). In particular, research shows that secondary teachers rarely use their tertiary mathematics knowledge in their teaching, due to a lack of connections between university content and secondary teaching practices, as part of a phenomenon known as *double discontinuity* (Klein, 1908/2016). Second, recent studies highlight the potential of combinatorics to enhance teaching and learning at both secondary and tertiary level. For example, Lockwood et al. (2020) advocate for greater integration of this branch of discrete mathematics into school curricula, noting its potential to foster deep mathematical reasoning and encourage computational thinking, among other benefits. That said, the authors recognise the significant challenges that combinatorics pose for students of all ages and the limited research on teacher training that specifically addresses these challenges.

The first author's Master's thesis explores the intersection of these two topics by identifying consistencies and inconsistencies between the university training of prospective teachers and secondary mathematics. In this paper, we focus on a specific type of task: permutations with indistinguishable objects (PIO). In PIO-related tasks, the goal is to arrange a set of objects in a specific order, where two or more objects are indistinguishable, provided there are at least two distinct types of objects. Lockwood et al. (2018) identify two possible approaches to solving PIO tasks: using permutations or using combinations. They note that students tend to favour permutations, and that textbooks often fail to make explicit connections between these two perspectives. Therefore, while PIO tasks have the potential to draw links between various approaches to combinatorics problems, textbooks often fail to connect related concepts. To better understand this situation, and in line with

our objective to systematically investigate consistencies and inconsistencies between tertiary and secondary combinatorics content, we seek to analyse the mathematical practices related to PIO tasks in university mathematics and compare these practices to those observed in secondary mathematics.

Theoretical framework

We adopt an institutional approach for this study, specifically, the Anthropological Theory of the Didactic (ATD – Chevallard, 1999). ATD situates the teaching and learning of mathematics within the broader context of human activity. Through a process known as *didactic transposition* (Chevallard, 1985), scholarly knowledge is shaped to align with the institutional context in which its teaching and learning take place. *External transposition* describes the process of selecting and organising knowledge to be taught in various didactic institutions, which may result in different reorganisations of the "same" content (Castela, 2017). *Internal transposition* takes place within schools, and occurs when teachers mobilise and transmit knowledge and students acquire it. We are interested in analysing the organisation of PIO-related knowledge in two different institutions: 1) secondary mathematics and 2) tertiary mathematics in teacher-training programs, with the goal of identifying potential gaps. We therefore focus on institutional mathematical practices rather than on teachers' practices. Examining our research problem through the lens of ATD allows us to consider the institutional characteristics that may influence how PIO is addressed in each of these institutions.

ATD provides a tool for analysing this transformation of knowledge: the *praxeology*. Knowledge is mobilised by a task T, which is solved using a specific technique τ. A rationale θ, called a *technology*, explains and justifies the technique, supported by a general theory Θ. This quadruplet $[T/\tau/\theta/\Theta]$ is what is known as a praxeology. It can be broken down into two blocks: the practical block $[T/\tau]$, called the *praxis*, and the theoretical block $[\theta/\Theta]$, called the *logos*. In our case, we aim to analyse and compare the mathematical praxeologies related to PIO in two given institutions. We formulate our research questions as follows: RQ1) *What are the mathematical praxeologies related to PIO in university mathematics courses in a training program for prospective secondary mathematics teachers?* RQ2) *How do these mathematical praxeologies relate to those in secondary teaching?*

Methods

Data collection for the first author's Masters' research is taking place at the Université de Montréal in two phases: 1) textbook analysis and 2) interviews with prospective teachers. This paper focuses on the first phase, as the interviews are still in progress.

Building on previous research (e.g., González-Martín et al., 2013), we analysed textbooks to investigate knowledge to be taught at university and secondary levels. This allowed us to formulate conjectures about the potential difficulties faced by teachers and students using these textbooks, as textbooks provide valuable insights into mathematical practices (e.g., Stylianides, 2014). To identify consistencies and inconsistencies in praxeologies related to PIO content at the university and secondary levels, we examined three secondary textbooks (STs) approved by the Québec government for use in all secondary schools and compared them with two university textbooks (UTs) used in *Discrete Mathematics* and *Probability* courses in a teacher training program. The textbooks analysed are as follows: ST1) Ledoux et al. (2008); ST2) Guay et al. (2008); ST3) Miloudi et al. (2008); UT1)

Ross (2014); and UT2) Rosen (2002). The full bibliographic references for these textbooks are available here. Excerpts from the textbooks were translated for this paper.

We analysed the textbooks and identified the praxeologies related to PIO using open coding methods (Saldaña, 2013), focusing on chapters that explicitly address combinatorics. In the secondary textbooks, this content appears in the chapters on probability; we analysed only the combinatorics aspects of the relevant tasks. Across all textbooks, we identified tasks that involve PIO and examined the recommended techniques for solving them (the praxis block). We also identified content that provided rationales for these techniques (the logos block). The first author conducted the initial analysis and the second author carried out an independent review of the results. Minimal disagreements were resolved collaboratively, resulting in the findings presented in this paper.

Data analysis

PIO in tertiary textbooks (RQ1)

Both UT1 and UT2 dedicate a specific section to PIO, presenting what can be considered "pure" PIO tasks. These tasks involve ordered arrangements of all n objects in a collection, where some objects are indistinguishable. In both textbooks, the first task serves to introduce the concept of PIO. Both textbooks subsequently present the same theorem accompanied by the formula for PIO (UT2, p. 275):

> The number of permutations of n objects, where there are n_1 indistinguishable objects of type 1, n_2 indistinguishable objects of type 2, …, and n_k indistinguishable objects of type k is $\frac{n!}{n_1!n_2!\cdots n_k!}$.

Both university textbooks emphasise the centrality of the formula for PIO tasks.

However, each university textbook supports this theorem with a different logos. In UT1, the combinatorics chapter begins with the multiplication principle, which justifies the formula $n!$ for the permutation of n distinguishable objects. PIO is then introduced as a specific case of permutations: "Example 1.9: How many permutations of the letters in the word PEPPER are possible?" (p. 5). The solution outlines the following steps: 1) calculate 6!, the number of permutations assuming all six letters are distinguishable; 2) identify and compute the cases where indistinguishable letters can be rearranged to form the same "word" (3! 2!); 3) conclude that the number of distinct permutations is 6! / 3! 2!. This example leads directly to the above theorem, accompanied by two solved tasks that use the same technique: applying the PIO formula (τ_{AF_PIO}). However, the "proof" for the PIO formula is based solely on the given example: "Applying the same reasoning as in Example 1.9, we establish the following theorem […]" (p. 5). The reasoning is task-specific and does not lend itself to generalisation. Furthermore, the division principle applied in the final step, which justifies dividing the total number of permutations by the number of distinct permutations that yield the same final "word," is not explicitly stated. Lastly, after introducing PIO, UT1's chapter on combinatorics transitions to combinations without linking either of these concepts to permutations or to PIO.

In contrast, UT2 introduces PIO in a section titled "Generalised permutations and combinations", which follows a subsection on combinations with repetition. The example used is mathematically similar to the task in UT1: "Example 8. How many distinct arrangements of the letters in the word SUCCESS are possible?" (p. 274). However, the proposed solution uses a different technique than

the one presented in UT1: 1) calculate the number of possible ways to arrange the three *S*'s in the seven positions ($\binom{7}{3}$); 2) calculate the number of possible ways to arrange the two *C*'s in the remaining four positions ($\binom{4}{2}$); 3) similarly, calculate the number of arrangements for U and E, using $\binom{2}{1}$ and $\binom{1}{1}$, respectively; 4) use the multiplication principle to find the total number of arrangements: $\binom{7}{3} \cdot \binom{4}{2} \cdot \binom{2}{1} \cdot \binom{1}{1} = 7!\,/\,3!\,2!\,1!\,1!$. This approach justifies the PIO formula while making explicit links to combinations, parameters, and the multiplication principle. However, Example 8 is the only solved PIO task included in UT2, and it is immediately followed by the statement of the theorem, which overshadows the reasoning. As in UT1, despite the initial rigour, the application of τ_{AF_PIO} ultimately emerges as the dominant technique.

While UT1 and UT2 provide distinct approaches—UT1 grounds PIO in permutations and UT2 in combination—the absence of connections between the two technologies perpetuates the compartmentalisation of knowledge. This aligns with Lockwood et al.'s (2018) observation that students may perceive combinatorics techniques as disconnected. Moreover, while an explicit reference to "PIO" could theoretically bridge this gap, UT2 further complicates matters by using different terminology for mathematically identical tasks. Indeed, the subsequent section in UT2, "Distributing objects into boxes", addresses tasks that are mathematically equivalent to PIO. An example is provided of a task that involves organising distinguishable objects in distinguishable boxes, and the conclusion leads to the stating of a theorem for this type of task using the same formula as PIO. Mathematically speaking, distributing n different objects into k distinct boxes with specific numbers of objects in each box is equivalent to PIO.[1] However, the connection between both sections of the textbook is explicit only in a note that follows the resolution of the task. Expanding the discussion from distributing distinguishable objects into distinguishable boxes to distributing distinguishable objects into indistinguishable boxes (Dubois, 1984) could have unified these concepts and clarified PIO as a subcase of distribution problems, but UT2 fails to seize this opportunity.

Both textbooks reveal a compartmentalisation of related praxeologies. In UT1, PIO is isolated within permutations, while in UT2, it is disconnected from related tasks involving object distributions. Neither textbook establishes links between the various possible approaches to PIO tasks. Denied the opportunity to connect the rationales underpinning these practices, students may struggle to integrate their understanding across the two textbooks and courses.

PIO in secondary textbooks (RQ2)

None of the three secondary textbooks has sections dedicated to specific pieces of combinatorics content. Rather, each textbook features a chapter in which combinatorics content is scattered throughout, appearing in various types of tasks. Therefore, we rely on two criteria to identify tasks related to PIO: the presence of an ordered sequence of objects, and the presence of indistinguishable

[1] It can be approached as follows: 1) label each box, assigning identifiers corresponding to the specific number of objects to be placed in each box; 2) arrange the n indistinguishable identifiers (PIO); 3) assign each object a number corresponding to its position in the sequence of identifiers, and place the objects into their corresponding boxes. Thus, the total number of ways to distribute n distinguishable objects into k distinguishable boxes is equal to the number of ways to permute n objects with k distinct types of indistinguishable objects, which aligns with PIO.

objects in the collection. Five tasks—three in ST1 and two in ST2, 6.2% of all combinatorics tasks[2]—meet these criteria and can be classified into four types (see Table 1).[3] In addition to pure PIO tasks (as seen in the tertiary textbooks), these two secondary textbooks also include PIO-related tasks in which an additional constraint must be considered. Moreover, the PIO formula is never explicitly stated. Rather, both ST1 and ST2 rely on a broad range of techniques, which we discuss below.

Table 1: Classification of PIO-related tasks in secondary textbooks

Task classification		Task statement (our translation)
T_{PIO}	"Pure" PIO, i.e., an ordered sequence of indistinguishable objects	ST1: A team of six athletes is competing in a triathlon. Two different athletes must participate in each event. In how many ways can the roles be assigned? ST2: A Christmas garland has ten lightbulb sockets. In how many ways can eight green and two red lightbulbs be arranged?
T_{CP-PIO}	PIO with constraints on the positions of the indistinguishable objects	ST1: In how many ways can three couples sit around a table if the members of each couple must sit across the table from one another?
T_{r-PIO}	A partial PIO: an ordered sequence of certain objects from a collection containing indistinguishable objects	ST2: The numbers 1, 2, 2, 3, 4 are written on individual pieces of paper and placed in a bag. How many distinct numbers can be formed by randomly drawing one, two, three, or four pieces of paper?
$T_{\odot-PIO}$	A circular (as opposed to linear) arrangement of indistinguishable objects	ST1: In how many ways can three boys and three girls be seated around a campfire such that all boys are seated together (assuming rotations of seating arrangements are considered distinct orderings)?

For T_{PIO}, ST1 describes its technique for solving PIO as taking the total number of permutations of all objects and dividing it by the number of rearrangements of indistinguishable objects. That said, as seen in UT1, the rationale does not explicitly address division and equivalence. Concerning $T_{\odot-PIO}$, ST1 uses pictograms to illustrate the six possible outcomes and counts the number of drawings to calculate the total number of possibilities. In doing so, ST1 does not explicitly draw any links between this task and combinations: the number of possible arrangements of three boys and three girls sitting around a fire, with the condition that the three boys must be seated together, is equivalent to selecting one of the six positions around the fire for the "first" boy, i.e. $\binom{6}{1}$, with the other two boys seated immediately after him.[4] Regarding T_{CP-PIO}, ST1 gives the expected final answer, 48, merely providing the following as a justification: "The number of permutations must be multiplied by 2 to account for the fact that two members of the same couple can sit on either side of the table" (p. 359). ST1 does not provide any additional details to reconstruct a technique, but based on the given rationale, we hypothesise that ST1 considers the following approach for this task: 1) choose a pair of

[2] This is a rather small proportion of tasks. We address this situation, as well as the fact that ST3 does not include tasks related to PIO, in the *Discussion and concluding remarks* section.

[3] It could be expected that an analysis of other sections and chapters of STs and of a larger sample of textbooks would lead to even more types of tasks.

[4] In the case of a "linear" PIO (for example, if the people were seated on a bench), this situation would be equivalent to choosing one of four possible places for the "first" boy, leaving room for the other two boys, but this "extra room" is not needed in a circular arrangement. This is also why we consider this task to be different from T_{CP-PIO}: changes occur when considering $T_{\odot-PIO}$ linearly, whereas T_{CP-PIO} can be solved in a linear fashion (with restrictions on the possible arrangements in a sequence rather than around a table) with no mathematical change in the task.

opposite seats for the first couple (three possibilities); 2) from the remaining options, choose a pair of opposite seats (two possibilities) for the second couple; 3) from the remaining options, choose a pair of opposite seats (one possibility) for the third couple; 4) for each couple, multiply by 2 to account for the fact that members of a same couple can switch seats; 5) using the multiplication principle, there are $3 \times 2 \times 1 \times 2^3 = 48$ possible arrangements. We note that this method does not align with other practices in the textbooks that make up our sample, where indistinguishable objects are linked to a division step. Moreover, we believe that a rigorous explanation would indicate that a multiplication by 2 is required *for each couple*, as this would illustrate the use of multiplication for indistinguishable objects and would link this PIO-related task to combinations. However, this detail is absent in ST1.

To solve T_{PIO}, ST2 first gives $\frac{10\times9}{2}$ as the expected answer, but with the sole explanation that "because the number of possible arrangements is determined solely by the colour of the lightbulbs, there are half as many possible arrangements due to repeated permutations" (p. 538). Subsequently, ST2 suggests another technique (Figure 1), asking teachers and students to identify a pattern based on the systematic listing of possible outcomes and the partition into subsets. This approach is absent from all the other textbooks, and the fact this new technique is applied to only one task highlights an inconsistency, both within the secondary textbooks and between ST2 and the university textbooks.

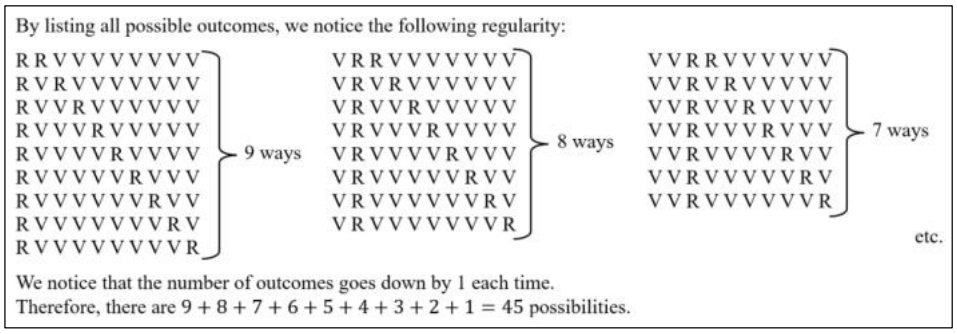

Figure 1: Solving technique suggested in ST2 for T_{PIO} (Guay et al., 2008, p. 585D)

ST2 also includes τ_{r-PIO}, a task related to the permutation of r of the n objects (some but not all objects) from a collection with indistinguishable objects. The suggested technique is to build a tree diagram of all possible outcomes to "make sure that students do not forget possible outcomes or repeat some possibilities" (p. 536). Despite the interesting points about repetition in the diagram, we question whether this technique is practical for a secondary mathematics classroom, as it includes a tree-diagram covering all 60 possibilities.

Discussion and concluding remarks

In this paper, we analyse praxeologies related to PIO in university and secondary mathematics textbooks to identify consistencies and inconsistencies between the training of prospective secondary teachers and secondary mathematics curricula. This analysis is carried out using tools provided by ATD, which yields interesting results concerning the praxis (the tasks and the techniques used to solve these tasks) and the logos (rationales) related to PIO in both institutions.

The two university textbooks provide the same formula for PIO, emphasising the direct application of the formula in solving PIO tasks. This reliance on τ_{AF-PIO} in the tertiary textbooks could help explain prospective teachers' shift towards a formula-based approach to combinatorics during their university training, as observed by Lamanna et al. (2023). On the other hand, we observe differences between the two university textbooks in the way content and rationales are presented: UT1 introduces PIO as a particular case of permutations, while UT2 treats PIO as an application of combinations. UT2 also relies on τ_{AF-PIO} to solve tasks categorised as distributing distinguishable objects into distinguishable boxes. This results in mathematically similar tasks being placed in different sections while establishing little explicit connections between them. Furthermore, neither university textbook provides any rationale for the connections between the permutational and combinational aspects of PIO, which aligns with Lockwood et al.'s (2018) hypothesis that the organisation of textbook content impedes the connection between permutations and combinations in cases where both approaches could be applied. Additionally, our analysis highlights the weakness of rationales provided for PIO content in university textbooks. Although PIO provides opportunities to connect combinatorics concepts, approaches, and broader practices—potentially enriching mathematical thinking—these connections are left implicit and underdeveloped. We conjecture that this could also pose problems for prospective teachers (and potentially for university teachers as well), who may find themselves unable to explain or justify practices related to PIO. In particular, this could prevent prospective teachers from acquiring the necessary tools to address secondary students' difficulties regarding PIO.

When comparing the praxeologies in the university textbooks to those in the secondary textbooks, we observe a number of inconsistencies. In particular, the PIO-related tasks in the secondary textbooks sometimes include constraints that are absent in the university textbooks. The variety of tasks in secondary textbooks is paired with a variety of techniques: formulas, tree diagrams, the listing of outcomes, and the identification of patterns. This suggests that certain aspects of PIO-related knowledge covered at the secondary level might not be addressed in tertiary mathematics education. It also raises questions about whether tertiary mathematics praxeologies related to PIO should be broadened to encompass the variety of tasks covered in secondary mathematics.

Finally, we acknowledge that PIO-related tasks represent only a small percentage of tasks in secondary textbooks. This poses a limitation while also calling for a continuation of this study. Although our analysis of secondary tasks draws on a small sample, there is potential for improvement, especially considering that one of the three textbooks does not include any tasks related to PIO. This also suggests a mismatch between the time allocated to PIO in university education and the time devoted to it in secondary teaching; however, this is difficult to ascertain from a textbook analysis alone, as the amount of in-class time allocated to specific content depends on many other factors. We anticipate that our interviews with prospective teachers will shed further light on this issue, in addition to clarifying the other conjectures that we formulate in this paper.

Acknowledgments

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Mathematical provision across subjects in the pre-university curriculum: Comparing geography, psychology, and sociology undergraduates in England

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Pre-university mathematical provision in England is now distributed across a number of advanced level subjects. However, due to the complexity of subject choices and inconsistency in university entry requirements, it is not easy to determine the potential effect on undergraduate cohorts. Taking geography, psychology, and sociology as illustrative cases, we compare the quantitative curricula followed by undergraduates prior to university using national administrative datasets for secondary and higher education for academic years 2016/17 to 2019/20 (n=90,745). We find variation in prior opportunities to study different types of mathematics both between and within these three disciplines. This study highlights the complexity of the mathematical provision for incoming undergraduates in England, and hence the need for a common mathematical framework across the pre-university curriculum to clarify what students can be expected to know and do at the transition to university.

Keywords: Competences, curriculum, non-mathematics degrees, transition.

Introduction

The question of how to prepare students for the mathematical demands of undergraduate degrees outside of the areas of science, technology, engineering and mathematics (STEM) is of international concern (MacInnes et al., 2016). In Germany, lecturers of non-science subjects agree that students should arrive at university with mathematical prerequisites, which vary widely between subjects (Neumann et al., 2021). Similarly, in the UK, the university disciplines of business, economics, geography, psychology, and sociology all require some prior knowledge of mathematics or statistics (Hodgen et al., 2014). A common issue is undergraduates' difficulty with "applying familiar mathematical concepts in unfamiliar contexts" (Bowyer & Darlington, 2016, p. 2), that is, with transferring the mathematics they know to the new context of their undergraduate discipline.

In England, academic advanced level (A-level) qualifications are taken during the upper secondary (16-18) phase of education, and are the most common pre-university qualifications, although technical and vocational routes are also possible. A feature of the A-level system is that the range of subjects studied is narrow, with most students taking just three subjects, while there are also no restrictions on subject combinations. Students are therefore not confined to specific pathways and can choose a mixture of STEM and non-STEM subjects. A-level Mathematics is optional and has consistently been the most popular A-level subject over the last ten years (Plaister, 2022). Research has used participation and attainment in A-level Mathematics as the primary measures of undergraduates' mathematical backgrounds (Hodgen et al., 2020). However, students could also have opportunities to develop mathematical skills across multiple A-level subjects, depending on their subject choices. For example, a student taking A-levels in Physics, Chemistry, and Computing has clearly studied at least some mathematics. While aspects of mathematics were previously embedded informally within other subjects, reforms to A-level curricula between 2015 and 2018 formalised the

mathematical content of nineteen A-level qualifications. There is now a requirement that quantitative skills form a proportion of the assessment for these nineteen subjects, ranging from 5% for A-level Physical Education to 40% for A-level Physics. Hence, it is now possible to broaden the definition of pre-university mathematical provision to include all 'quantitative' A-level subjects.

This paper uses this broader definition to present an initial exploration of how these curriculum changes affect undergraduates in three higher education disciplines in terms of the mathematics they are expected to have studied prior to university. Geography and psychology are both located at the boundary of the sciences and social sciences, while sociology is a more typical social science. In England, there are no required A-level subjects for entry to university to study geography, psychology or sociology, and all three have the same average entry grades, although grade requirements vary between universities (Universities & Colleges Admissions Service, 2024). While not required, A-level Mathematics is recommended for entry to psychology degrees, but not for degrees in geography or sociology. For all three disciplines, it is also possible, and sometimes even recommended, to take the subject at degree level without having the corresponding A-level. In practice, students enter these disciplines with a variety of different subject combinations, including both quantitative and non-quantitative A-levels. This research therefore seeks to address the questions: what are the mathematical opportunities provided to undergraduate geography, psychology, and sociology students by their pre-university (A-level) curriculum, and how do these vary within and across disciplines?

Conceptual framework

The language of mathematical competences has been used in a number of international contexts to draw attention to "doing" mathematics, rather than simply "knowing" mathematical terms, concepts, methods and topics (Boesen et al., 2014). Frameworks such as those created by the Danish Competencies and the Learning of Mathematics (KOM) project and the Programme for International Student Assessment (PISA) focus on the competences developed within the school subject of mathematics (Niss, 2014). Since our focus is on mathematical competences across subjects, we make use of a framework of nine General Mathematical Competences (Table 1) originally designed to describe the "doing" of mathematics within real, complex situations in a range of different vocational contexts (Advisory Committee on Mathematics Education, 2019), and subsequently adapted for use with academic school subjects.

Table 1: General Mathematical Competence (GMC) framework

<table>
<tr><td>1. Measuring with precision</td><td>6. Understanding data and risk</td></tr>
<tr><td>2. Estimating, calculating and error spotting</td><td rowspan="2">7. Interpreting and representing with mathematical diagrams</td></tr>
<tr><td>3. Working with proportion</td></tr>
<tr><td>4. Using rules and formulae</td><td>8. Communicating using mathematics</td></tr>
<tr><td>5. Processing data</td><td>9. Costing & optimising work processes</td></tr>
</table>

Drawing on Communities of Practice theory, participants in school subjects (and university disciplines), including students and teachers, can be viewed as separate communities of practice (Wenger, 1998). By definition, we therefore expect some level of discontinuity for students moving between the mathematical practices of different subject communities, as well as when moving from school to university. However, there are also ways in which communities connect and communicate with one another, including through concepts or terms that make sense within both practices, known as "boundary objects" (Wenger, 1998, p. 105). The GMCs are intended to function as boundary objects that make sense within different subjects and disciplines, as well as at different difficulty levels and phases of education. Hence, we were previously able to map these GMCs within the intended, written curriculum for quantitative A-levels, providing a common language for discussion of the types of mathematics present in different subjects (Norris & Noyes, 2023). In adapting the framework for use with academic subjects, the number of GMCs was reduced from ten to nine and each GMC was divided into four sub-competences highlighting different aspects of the competence. For example, GMC 1 *measuring with precision* has the four components: (1a) *measuring physical objects and processes*, (1b) *measuring constructs*, (1c) *units of measurement*, and (1d) *accuracy of measurement.* Full details of this expanded framework can be found in Norris and Noyes (2023). While intended to make sense across subject areas, the applications of these GMCs clearly vary between subjects. For example, GMC (1d) *accuracy of measurement* appears in the A-level Geography curriculum as a requirement to "demonstrate the ability to interrogate and critically examine field data in order to comment on its accuracy" (Department for Education, 2014), whereas in the A-level Psychology curriculum the same GMC sub-competence is represented by the requirement to "use an appropriate number of significant figures". Although researching the curriculum cannot tell us what students know or can do in practice, it does provide an indication of the maximum set of mathematical competences that we can expect students to have experienced. That is, where a GMC is not present at all within a students' subject choices, we cannot expect this competence to be well-developed at the transition to university.

Methods

This quantitative analysis uses data from the Higher Education Statistics Agency (HESA) database, with our population consisting of undergraduate students enrolled at universities in England from 2016/17 to 2019/20 to study geography, psychology, or sociology as their first (single honours) or second (joint honours) degree subject. These were linked to data on prior attainment from the National Pupil Database (NPD), with only those students with at least one A-level passing grade retained in the analysis. Subjects achieved at A-level were translated into a binary measure of students' opportunity to study each GMC sub-competence using the results of the curriculum mapping reported in Norris and Noyes (2023). Students are counted as having had the opportunity to access a given GMC sub-competence where that sub-competence is found in the written curriculum for one or more of their A-level subjects. Combining the previous curriculum analysis with national student data from the HESA and NPD thus allows us to investigate the prevalence of GMC sub-competences in the intended pre-university curriculum for different groups of undergraduate students. Since we have whole population data statistical hypothesis testing is not appropriate, so we use frequency analysis only. To prevent the possibility of disclosing individual personal information all counts are rounded to the nearest 5 and percentages are calculated using rounded values.

Findings

Among the population studied (n=90,745), the largest discipline was psychology (50%, n=45,650), followed by sociology (including social policy and anthropology) (37%, n=33,850). Geography (including general and human geography but not physical geography) made up the smallest cohort (12%, n=11,245).

Figure 1 shows the distribution of GMC sub-competences for the undergraduate geography cohort. That is, the proportion that had taken one or more A-level subjects where GMC sub-competences were intended to be taught and assessed. Opportunities to study GMC sub-competences were highly concentrated within A-level Geography, taken by 93% of geography undergraduates. Hence, a core of 15 sub-competences were provided in the pre-university curriculum for almost all geographers, being the same 15 sub-competences previously found to be present within the A-level Geography curriculum (Norris & Noyes, 2023). However, for a small number of undergraduates, these core sub-competences were only present in subjects other than geography, and a few did not have the opportunity to study them at all. Other common quantitative A-levels taken prior to undergraduate geography were Biology (19%), Economics (19%), and Mathematics (18%). Overlap in GMC sub-competences between Geography and other subjects at A-level was somewhat varied. For example, while (8a) *reason with mathematics/ logic* and (8c) *draw conclusions from mathematical information in context* were included in the pre-university provision for similar numbers of undergraduates, the proportion for whom these sub-competences appeared in multiple A-level subjects varied from 16% for (8a) to 67% for (8c). In addition, over half had taken A-levels containing (3a) *working with numerical proportion* (68%) and (2b) *use of number* (58%), which are not present in A-level Geography and so can only have come from other quantitative subjects.

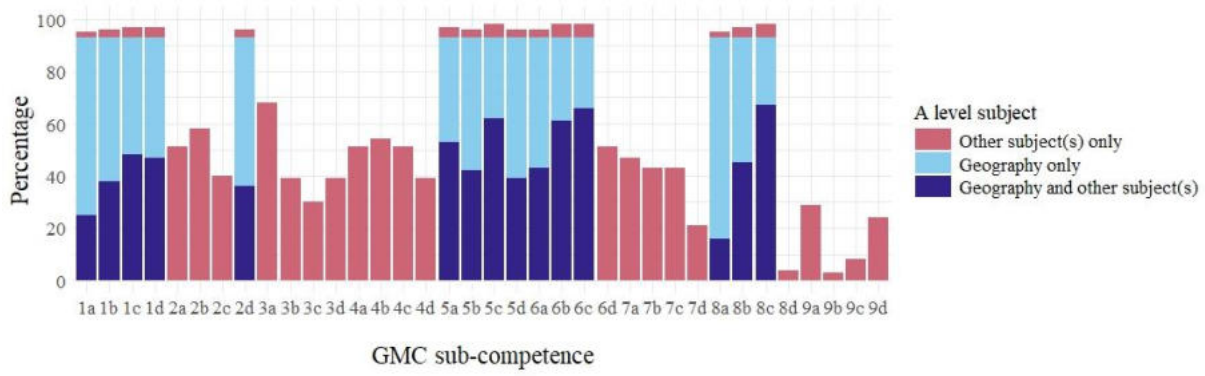

Figure 1: Prevalence of GMC sub-competences in the A-level curriculum followed by geography undergraduates prior to university, by A-level subject

Pre-university provision for the undergraduate psychology cohort similarly included opportunities for most to study a core set of sub-competences (Figure 2), which in this case were the same 18 sub-competences found to be included in the A-level Psychology curriculum (Norris & Noyes, 2023). The proportion of psychology undergraduates who had taken the corresponding subject at A-level was slightly lower than for geography (86%), with a slightly larger proportion of opportunities to study these core sub-competences located in subjects other than A-level Psychology. Of the sub-competences common among psychologists, GMCs (8c) *draw conclusions from mathematical*

information in context and (6c) *establish relationships and make predictions* had the greatest overlap between A-level Psychology and other A-level subjects (54% and 53% respectively), with (2c) *order of magnitude calculations* and (4d) *select appropriate data and units* having the least overlap (both 23%). The opportunity to study (8b) *communicate using mathematics* was offered in A-levels other than Psychology for over half of psychology undergraduates (66%), with their most common other A-level subjects being Biology (27%) and Sociology (26%). Only 13% of psychology undergraduates had taken A-level Mathematics, which therefore provided only a small proportion of opportunities to study GMCs.

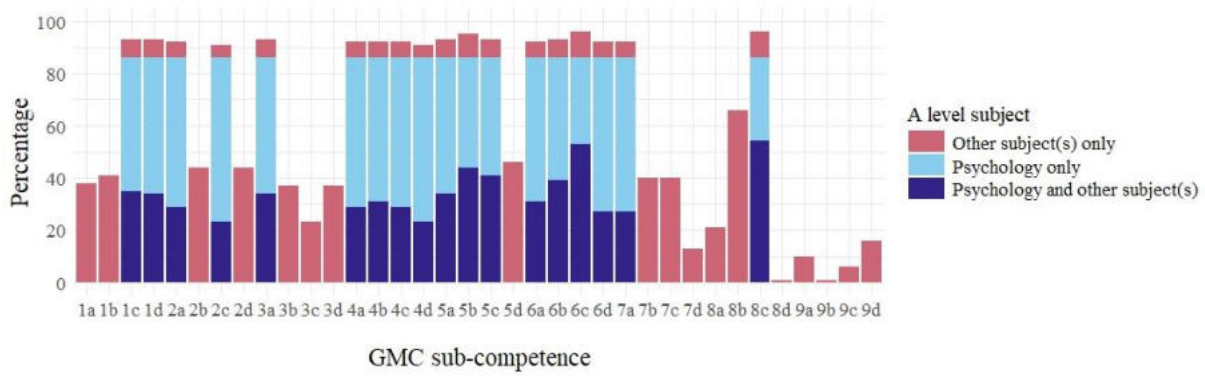

Figure 2: Prevalence of GMC sub-competences in the A-level curriculum followed by psychology undergraduates prior to university, by A-level subject

Most sociology undergraduates had opportunities to study the four core GMC sub-competences found to be present in the A-level Sociology curriculum (Norris & Noyes, 2023), despite the fact that less than half had taken A-level Sociology (49%)(Figure 3). Many undergraduates therefore only had opportunities to access core sub-competences in A-levels other than Sociology, while very few were offered access in multiple A-level subjects. The most common other quantitative subjects taken were A-level Psychology (36%) and Geography (11%), while just 7% had taken A-level Mathematics. Moreover, for the four core sub-competences, between 16% and 31% of sociology undergraduates did not have the chance to study them at all, so they form a much weaker 'core' of mathematical provision than those for geography and psychology.

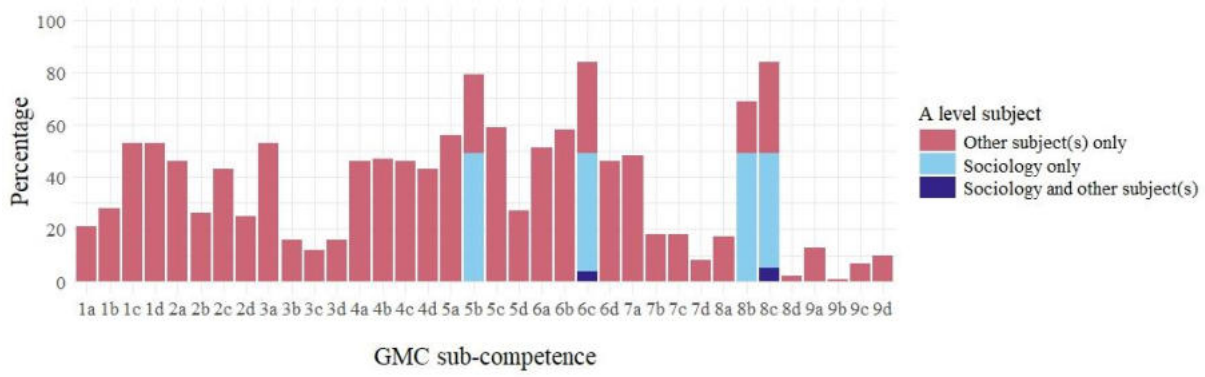

Figure 3: Prevalence of GMC sub-competences in the A-level curriculum followed by sociology undergraduates prior to university, by A-level subject

Discussion

The General Mathematical Competences (GMCs) intended to be taught across the A-level curriculum seem to align reasonably well with the quantitative content of geography, psychology, and sociology degree programmes, which have a common focus on statistics (Hodgen et al., 2014). For geography, psychology, and sociology undergraduate cohorts, core GMCs covered by their pre-university curriculum included some elements of *processing data* (GMC 5) and *understanding data and risk* (GMC 6), which are clearly relevant for further study of statistics. For all three cohorts, this also included aspects of *communicating using mathematics* (GMC 8) that are closely related to statistics, such as data presentation and drawing conclusions from statistical information in context. Core sub-competences prevalent among the pre-university curricula of geography undergraduates had an additional focus on *measuring with precision* (GMC 1), which may link to collecting data through fieldwork (Harris, 2020). In contrast, psychology undergraduates instead had greater opportunities to *use rules and formulae* (GMC 4) in their pre-university qualifications, suggesting that the approach to statistics is perhaps more process-oriented and less practical than for geography. In A-level Psychology, formulae are often used in the context of statistical calculations, although this may not be the case within other A-level subjects. While the English A-level system appears to have offered undergraduate geography and psychology cohorts opportunities to study a wide range of GMCs relevant to their university studies, further work is needed to establish the amount and level of these mathematical competences across subject curricula. In contrast, sociology undergraduates had less opportunity to study a range of data-related competences prior to university, which is perhaps reflected in the low starting point of statistics teaching within sociology degrees (Scott Jones & Goldring, 2017). This suggests that there is perhaps scope to increase the range of mathematical competences provided in the pre-university curriculum for sociology undergraduates, whether in A-level Sociology or in other subjects.

While statistics is an area of common ground, the three disciplines differed in the extent to which their cohorts shared a common set of mathematical opportunities prior to university. Over 90% of geography and psychology undergraduates had previously had opportunities to access a core of GMC sub-competences. Psychology degrees are overseen and accredited by the British Psychological Society, so there is a higher level of consistency in the statistics curriculum across universities than for other university disciplines (Field, 2014), and it therefore makes sense that incoming undergraduates had similarities in the types of mathematics in their pre-university curricula. The consistency among geography undergraduates is perhaps more surprising, as geography degrees are more variable in their quantitative content (Hodgen et al., 2014). This may be because the total geography cohort is relatively small, so universities can select for students with A-level Geography.

The sources of undergraduates' prior mathematical opportunities are also important, as previous exposure to mathematical competences within a particular disciplinary context may aid students' transition to university by making both mathematical concepts and the context of application more familiar (Bowyer & Darlington, 2016). Our findings suggest that sociology undergraduates could face additional challenges in transferring their mathematical experience to university, since a large proportion only had prior opportunities to study core sub-competences in the context of A-levels other than Sociology. Similarly, there are a small number of psychology undergraduates who only had the opportunity to study core sub-competences in other, non-psychology, A-level subjects and therefore

had to transfer these competences across disciplinary boundaries. It is possible that the mathematical transition from school to university may be much smoother for geography undergraduates, the vast majority of whom have already had opportunities to experience mathematical competences within a specifically geographical context. It may therefore not be a matter of great concern that A-level Mathematics is not a major source of access to mathematical competences for any of these cohorts, as a more useful focus could be on encouraging closer connections between university disciplines and their corresponding A-levels.

Conclusion

The contribution of this paper is an increased understanding of who is affected by the curriculum requirement to teach and assess quantitative skills in multiple advanced-level subjects, and therefore how this could contribute to the mathematical preparation of science and social science students for undergraduate study. We find that the current system of advanced-level qualifications in England provides many geography, psychology, and sociology undergraduates with opportunities develop a range of mathematical competences, even without taking A-level Mathematics, by choosing other quantitative A-level subjects. While geography undergraduates primarily gain access to core mathematical competences through A-level Geography, either on its own or alongside other quantitative A-levels, sociologists rely much more on the mathematical competences in subjects other than A-level Sociology. In both cases, there are a number of students who were not provided with these opportunities, either as a result of choosing non-quantitative subjects or because their chosen quantitative subjects provided types of mathematics not relevant to their university disciplines. Filling these 'gaps' would require substantial systemic change, such as moving away from an entirely free-choice system to introduce mandatory subjects of study, or further curriculum reform to embed mathematical competences in subjects that currently have no quantitative requirement. Before such dramatic changes are considered, we argue that there is a need for further development of a common mathematical curriculum framework across advanced level subjects to fully understand the mathematics already provided to students, including not only types of mathematics as presented here but also the amount and level of mathematical content, since there is no existing framework designed for this purpose. These are necessary foundations for informing university lecturers about the pre-university mathematical opportunities available to their incoming cohorts. As well as recognising students' prior mathematics qualifications, we need to find ways both to acknowledge and take advantage of the mathematical opportunities within other subjects to prepare students for studying science and social science subjects at university.

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Between the trivial and the impossible: CHAT as an analytic lens to study the role of programming in mathematical research

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In this study, we explore how Engeström's Cultural Historical Activity Theory (CHAT) can be applied to analyse epistemic tensions related to the role of programming in the practices of mathematicians. Drawing on interview data with two mathematicians from a northern European university, we investigate how our use of CHAT highlights tensions between pure and applied mathematics. The analysis allows us to identify dialectic contradictions between motives and rules governing mathematicians' research practices, demonstrating CHAT's potential for studying epistemic tensions. In future research, we plan to incorporate additional data to study tensions between mathematicians' use of programming in their teaching and in their research.

Keywords: Mathematics, mathematics education, Cultural Historical Activity Theory, programming

Introduction

Over the last decades, there has been an ongoing discussion on the impact of increasing availability and capacity of high-performance computing on mathematics research and education (see, e.g., Buzzard, 2024). However, despite the apparent relevance of computing to mathematics, the epistemic status of programming in mathematicians' practices is unclear. In a survey of 302 Canadian mathematicians, Buteau et al. (2014) found that 43% reported using programming in their research, compared to 18% in their teaching, revealing what we see as a potentially significant gap. In a follow-up study, Broley et al. (2018) found that mathematicians use programming mainly to obtain intuition and numerical results where analytic methods fall short. This is in line with the only other interview study we know of on this topic (Lockwood et al., 2019). Mathematicians' lower use of programming in their teaching is mainly attributed to institutional constraints and the lower epistemic status of programming in teaching (Broley et al., 2018).

Taking a broader view, we note that tensions in the practices of mathematicians have been well-established by philosophers of mathematics and researchers in mathematics education. For instance, Hersh (1991) and Burton (2004) pointed out a gap between the importance mathematicians attribute to intuition, curiosity and experimentation in their research and the more classical and rigid Platonistic image often conveyed to the general public. Hacking (2014) discusses tensions between pure and applied mathematicians from a historical and philosophical perspective, noting that while the distinction between pure and applied mathematics is a recent invention, some mathematicians argue that applied mathematics is not mathematics at all.

In this paper, we give the first report from a PhD project in which we interviewed mathematicians from a northern European university on their research and teaching practices. In this preliminary step, our research aim is to critically evaluate the use of Engeström's Cultural Historical Activity Theory

(CHAT) (Engeström, 2015) as an analytic lens to study tensions related to the use of programming in mathematics. Our choice of CHAT is in line with our view of mathematics as a field whose development is driven by tensions. Specifically, we address the research questions:

RQ 1. What features of mathematicians' programming-related research practices are foregrounded (and not) when we draw on CHAT as an analytic lens?

RQ 2. What additional data sources are needed to substantiate and triangulate findings, and how can we refine our operationalisation of CHAT to promote rigour and transparency in the analysis?

Theoretical perspective

Engeström's Cultural Historical Activity Theory (CHAT) builds on the activity theory of Vygotsky and Leon'tev (see Engeström, 2015) and has been adapted to mathematics education research in different ways (see, e.g., the survey of Núñez, 2009). We adopt the concept of activity as our unit of analysis. We understand activity to be anything humans come together to do collectively that is motive-oriented, enduring over time and situated (e.g., the activity of mathematical research – or teaching - situated at a mathematics department).

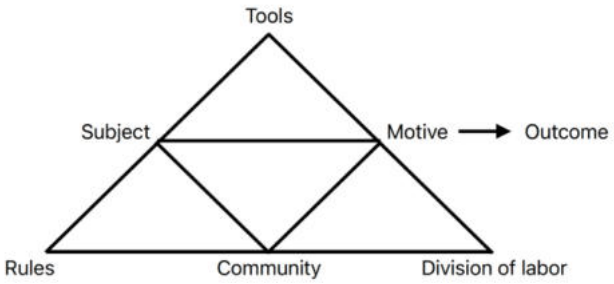

Figure 1: Engeström's extended activity triangle (Engeström, 2015).

We draw on two key constructs of CHAT: the extended activity triangle (Figure 1) and dialectic contradictions. The **extended activity triangle** can be thought of as the anatomy of an *activity*. At the top sits Vygotsky's mediating triangle, which frames the *subject* as using *tools* to pursue a *motive* directed toward a 'tangible' *outcome.* The bottom nodes represent the collective and social nature of the activity, which frames the subject as operating within a community that shares the motive, is regulated by *rules* and norms and is shaped by power structures reflected by the *division of labour*. Following Mwanza and Engeström (2005), we operationalise the extended activity triangle as a checklist of questions we ask the data (see Table 1). The nodes thus provide (overlapping) vantage points that we use to analyse and subsequently synthesise a description of the activity. As the description draws on multiple data sources, it likely contains contradictory information that indicates tensions. Following Engeström (2015), we view the motive as the reason 'why' of the activity and make it the focal point of our analysis as it shapes, and is shaped by, all parts of the activity.

If we think of this operationalisation as a diagnostic tool to examine the well-being of an activity, then **dialectic contradictions** could be thought of as the underlying conditions we seek to diagnose and the underlying causes of the observed tensions. However, dialectic contradictions are not defects

to be removed. Instead, they are intrinsic cultural forces driving the historical development of the activity and have the potential to provide mechanisms for change when understood.

Table 1: Operationalization of the extended activity triangle.

Node	Examples of questions
Subject	What role does the subject identify with as part of the activity?
Motive	Why is the subject pursuing the activity?
Tools	With what means does the subject pursue the activity?
Outcome	What is the 'output' of the activity as perceived by the subject?
Community	Who does the subject see as supporting their pursuit of the activity?
Rules	What rules and norms regulate the subject's pursuit of the activity?
Division of labour	What power structures affects who does what in the activity?

To operationalise dialectic contradictions, we follow Ilyenkov (2009), who defines them as intrinsic properties of the activity that are "the concrete unity of mutually exclusive opposites" (p. 185). We propose the following more explicit operationalisation:

- **Dialectically related observations:** A pair of interrelated observations that provide complementary or contrasting properties of a situated activity that neither can do separately.
- **Dialectic contradiction:** Dialectically related observations that are mutually incompatible and are a cause of tensions that drive change or development in the activity.

Radford (2021), working within mathematics education research, provides an accessible account of material dialectics, where he points to dialectically related observations, such as those between an object's inner nature (or essence) and its material appearance, between what an object is and what it is not, between its history (concrete) and its logic (inner), or between the individual and the collective. Radford takes a less structured view of the concept of activity than Engeström (2015), who, based on the extended activity triangle, classifies contradictions as arising from dialectic observations made within nodes (primary), between nodes (secondary), between old and new versions of an activity system (tertiary) or between activity systems (quaternary). These types of contradictions all concern the inner nature of the activity and resonate with what Radford would call inner contradictions. Here, we restrict our attention to primary and secondary contradictions. A primary contradiction might involve multiple conflicting motives for a single activity. In contrast, a secondary contradiction would be if this conflict of motives leads to, say, a schism within the community in which the activity is situated.

Methods

This study focuses on interviews with two participants, selected from a group of 15 mathematicians interviewed at the department of mathematics at a major northern European university. The broader participant group included researchers from pure and applied mathematics and was purposefully sampled to represent all research groups and capture variation in the use of programming in teaching.

Semi-structured interviews (1.5 to 2 hours each) were audio-recorded and later transcribed and addressed programming in research and teaching, as well as views on institutional context. Interviews were conducted by the first author, a research mathematician, to facilitate communication. Additional participant selection ended when our capacity for analysis was reached. We acknowledge that data saturation was not achieved, as each interview provided new and interesting perspectives.

In this paper, we focus on two participants – given the pseudonyms Mats and Lars – who represent opposite ends of the pure–applied spectrum and offer contrasting perspectives on programming in teaching. To maintain their anonymity, we edited the transcripts to obscure their research fields.

A deductive thematic analysis based on CHAT was initially conducted on the full data set (Braun & Clarke, 2006). During the process, we narrowed our focus of the analysis to the research part of the interviews with Mats and Lars to engage more deeply with the data and better evaluate and refine our operationalisation of CHAT. The analysis produced a set of codes, themes and the set of questions in Table 1. Based on these, we developed two contrasting yet interrelated accounts of mathematical research activity, one for each participant. We then examined the codes, themes and accounts for signs of tensions that met our criteria for dialectic contradictions. The analysis was conducted by the first author and reviewed collaboratively with co-authors and experts in mathematics and mathematics education research.

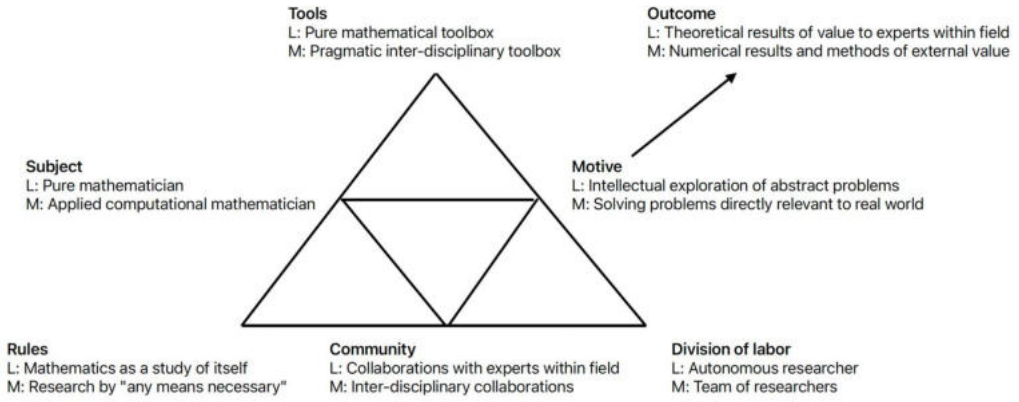

Figure 2: Description of the mathematical research activity in terms of the CHAT triangle. Under each node, Lars' activity corresponds to the top line and Mats' to the bottom line.

Findings from the interviews with Mats and Lars

Here, we present the findings from the analysis of the interviews with Mats and Lars in two parts. First, we provide an account of mathematical research activity based on our analysis of the interview data using CHAT as an analytic framework. Then, we examine signs of dialectic contradictions through the lens of our operationalisation of CHAT. The findings are summarised in Figure 2 below.

The activity from the perspectives of Lars and Mats

Regarding the *subject* of the activity, Lars describes himself as a *classically oriented pure mathematician* who does not involve computers in his research. "I have approached it quite theoretically, meaning there is little in my research that uses programming or that uses any type of

computer-based tool at all." In contrast, Mats identifies as an applied computational mathematician. For him, "it is not possible to imagine doing much without a computer".

Regarding the *motive* of their work, Lars conveyed his passion for mathematics by declaring, "I do love theorems". He describes his motivation as an intellectual exploration of a mathematical landscape in a Platonistic sense: "The mathematically interesting is a very thin membrane that lies between the trivial and the impossible. [...] This is the only place where you can solve problems using mathematics." Mats, in contrast, is driven by solving real-world problems involving real-world data. "Solving that damn problem, that is the important thing." As he sees it, if the link to real-world data is severed, then something is lost: “It becomes like chess. And chess is nice and interesting and everything. But it is just chess."

Regarding *tools*, Lars speculates that "had I been stronger in terms of programming, it is possible that [...] I would have been open to other kinds of solutions and opportunities." However, he does not think he would have wanted to do it any other way. His interest in mathematics is firmly rooted in the use of analytic tools: "[programming] could have helped me, given me some insight, I could have experimented a bit, put together models, and so on. But what I really cared about, ultimately, was theoretical results." Mats, on the other hand, views mathematics and programming as deeply intertwined: "You only get so far by only the tools of mathematics. Simply put, you need to combine the tools of mathematics and programming to have a proper toolbox […] modelling, combined with programming is just insanely powerful." Mats elaborates that analytic tools are necessary to verify computational results because if not, he says, “you could always end up doing things on a computer without having a clue of it is correct or not."

In terms of *rules*, Lars shares a story from the start of his career in which he worked on a project that involved computations on real-world data: "The power series of the solutions were completely incomprehensible, so we had to program a bit […] and compute how these things actually behaved.” So, Lars’ lack of interest in programming is not due to a lack of experience. Instead, he seems to view the use of programming as belonging to a different ‘game’ than the one he is interested in, one with a different set of rules. Mats, in contrast, expresses a rather agnostic view of disciplinary boundaries: "It is solving that damn problem that matters. [...] it is not the discipline as such." As noted earlier, he has little interest in mathematics as an abstract “game of chess”.

Their *communities* also differ. Lars primarily collaborates with other pure mathematicians and PhD students. He estimates that around half of mathematicians in his community know how to program, including some of the best researchers. While he reports that colleagues who focused on numerical approaches broke off and formed a separate community, he now sees a softening of this divide, with computational researchers becoming more engaged in theory. However, Lars indicates that the tensions between pure and applied mathematics persist:

> "There are still communities in pure mathematics who […] look down on purely numerical mathematics. [...] But then you go to the applied computational mathematicians [who are] a bit condescending from the other direction."

Mats positions himself as part of an interdisciplinary community, where most of his colleagues program. When asked about fellow applied mathematics who avoid programming, he says that “I am sure have they have done some programming a long time ago and then put it away. And in that sense, they are not half blind, it is just that they have chosen to close their eyes in that direction."

With respect to the *division of labour*, both Mats and Lars report a strong sense of autonomy. Lars expresses the freedom to explore any ideas he finds interesting, consistent with his view that mathematics is an enterprise of intellectual exploration. As he puts it, “Everything […] is just ideas that are in your head - you want to do this; you want to do that. And then there is something that triggers it, that you follow this trail instead of that." Mats describes having a similar experience in his research group, where this autonomy is extended to PhD students, who are asked whether they prefer analytic work or programming when hired. However, he emphasises the importance of a balance within the group: “It needs the whole package. There is nothing that stands out as more important than anything else.”

Finally, their perceived research outcomes differ in scope and orientation. For Lars, the outcome consists of theoretical papers aimed at fellow experts. If any programming is used, it primarily serves cosmetic purposes. Mats, by contrast, views software implementing numerical methods, usable by the wider community, as a key outcome alongside more traditional research papers.

Dialectic Contradictions

Based on the above descriptions, we indicate two dialectic contradictions in our data: a primary contradiction concerning the participants’ motives for doing mathematics and a secondary contradiction caused by friction between their motives and rules for mathematical activity.

The primary contradiction in motive lies in Lars and Mats’ fundamentally different sense of the purpose of mathematical research. Lars describes his interest in mathematics to be the intellectual exploration of “the thin membrane that lies between the impossible and the trivial”, driven by a curiosity about mathematics itself. Mats, on the other hand, sees mathematics disconnected from real-world data as being just like a game of chess - perhaps interesting, but ultimately just a game. These opposing motives satisfy our criteria for a primary dialectic contradiction: they are (a) complementary and interrelated, (b) mutually incompatible in how they define legitimate mathematical activity: for Lars, the motivation is intrinsic to mathematics, while for Mats it is extrinsic, and (c) a source of tension. Mats’ ‘game of chess’ analogy and Lars’ account of mutual condescension between theoretical and computational mathematicians point to a deeper struggle over epistemically valid ways of pursuing mathematics.

The secondary contradiction arises from the tension between these opposing motives and the corresponding rules of mathematical practice. Lars adheres to the norms of his notion of classical mathematics, where programming has little or no place. His earlier experience with programming was framed as a side-step or a deviation from this norm. Mats, by contrast, insists that his job is to solve that "damn problem" *by any means necessary.* For him, this requires a flexible approach that integrates classical mathematics with modelling and programming. From this perspective, disciplinary boundaries are less important, and success depends on being able to draw on multiple

domains. These conflicting views on rules qualify as a secondary dialectic contradiction as they are (a) complementary and interrelated, (b) mutually incompatible *in light of the contradiction in the motive node*: Lars' motive for doing mathematics is intrinsic to mathematics, while for Mats it is rooted extrinsically, and (c) a source of tension and change, as evidenced by Lars' account of the split between computational and theoretical factions in his field.

Discussion

Above, we used CHAT as the basis for a deductive thematic analysis of interviews with two mathematicians, Lars and Mats, who represent opposite ends of the pure–applied spectrum. As part of this process, we refined our operationalisation of CHAT (see Table 1). We now connect back to the research questions from the introduction and consider implications for our future research.

RQ1: Our operationalisation of CHAT provided an analytic lens with motive as a focal point. In analysing the interview data, this led to us identifying two contradictions. (1) A primary contradiction related to motive: Lars' intrinsically oriented motive of theoretical exploration versus Mats' extrinsic focus on solving real-world problems. (2) A secondary contradiction related to the interaction between friction between motive and the epistemic rules seen as governing mathematical activity: For Lars, programming lies outside the norms of classical mathematics, while for Mats, it is essential for solving real-world problems. These contradictions are not merely a question of incompatible views but reflect a source of tension and conflict and, thus, a force for change in the collective social activity that they are part of. Lars' sense of individual autonomy and disinterest in programming aligns with the norms and division of labour in the community of pure mathematicians that he associates himself with, while Mats' integrated computational approach reflects those of his interdisciplinary community of applied researchers. These findings resonate with findings from the philosophy and history of mathematics that identify such contradictions as driving forces in the historical evolution of mathematics (see, e.g., Hacking, 2014) and mathematics education research where Buteau et al. (2014) connects mathematicians' use of programming to disciplinary norms.

RQ2: Our structured use of CHAT, drawing on Engeström (2014) and Mwanza and Engeström (2005), allowed us to produce a systematic account of Lars and Mats' activities and highlighted dialectic contradictions. However, our focus on two interviews limits our ability to assess collective aspects (rules, community, division of labour). For instance, at the department of Lars and Mats, is a diversity of motives and approaches among mathematicians generally viewed as an asset, a source of conflict, or both? In further work, we, therefore, see it as necessary to include interview data from the broader participant group and other data sources to triangulate findings (such as site visits, background interviews with administrative staff and students or surveys).

While our explicit operationalisation of CHAT, in response to criticism of the use of CHAT by Núñez (2009), promoted transparency and analytic rigour, it may have reduced the flexibility of our analysis. We recognise this tension as similar to the contrast between Engeström's structured framework for CHAT (2015) and Radford's more flexible interpretive approach (2021). So, while Engeström's structured framework was helpful in the early stage of this project, Radford's materialist dialectics may have the potential to allow for more analytic depth in the long run. Going forward, we see a need to balance these perspectives.

The next step of this project is to further refine our use of CHAT, drawing on both Engeström and Radford. We will then apply this evolving framework to the broader set of interview data to shed light on the interplay between mathematicians' teaching, research, and programming practices.

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An alternative approach to teaching the fundamental concepts of Analysis to first-year university students

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Teaching fundamental concepts of Analysis in the first year of university is crucial for students' success in university-level mathematics. This study investigates how a lecturer, with research expertise both in mathematics and mathematics education, introduced these concepts to first-year undergraduate mathematics students. Data collection included observation notes, audio-recordings of lectures and interviews with the lecturer. Through iterative cycles of data analysis, we identified a four-level pattern in the lecturer's teaching approach which we exemplify in the case of the limit of a sequence. We illuminate the meaning of the identified pattern by reflecting on the development of an alternative teaching culture through tensions and attempts of their synthesis. We highlight the role of research experiences in mathematics education in shaping an alternative teaching culture and explore ways to influence the overall teaching culture at the university.

Keywords: university mathematics teaching, concept introduction, tensions, teaching culture.

University mathematics teaching to first-year mathematics students often faces challenges in initiating students to advanced mathematical processes and concepts (Lew et al., 2016; Breen & O'Shea, 2016; Di Martino et al., 2023). While extensive research has focused on teaching proofs, less attention has been given to teaching practices for introducing essential concepts like the concept of limit (Lew et al., 2016) - a fundamental threshold in Analysis which students must grasp to succeed in more advanced mathematical topics (Breen & O'Shea, 2016). In most countries, the expectations at the university level differ from school mathematics which primarily focuses on procedural understanding (Di Martino et al., 2023). In a bachelor's program in Mathematics, students are required to shift their focus towards increased abstraction and rigor (Vollstedt et al., 2014), along with the introduction of advanced mathematical concepts (Hitt & Dufour, 2021). As a result, students may experience situations of a "culture shock" as they are introduced to advanced mathematical language and culture (Burazin & Lovric, 2018). Hence, how key mathematical concepts are introduced is crucial for students' initiation into mathematical culture and success in higher-level mathematics.

In this case study, we investigate the teaching approach of a lecturer during the introduction of the Analysis concepts (limits, continuity and derivatives) in first-year undergraduate mathematics lectures. There is a body of research that points to the mathematical gap between secondary school and university (Di Martino et al., 2023) and values what we can learn from mathematicians' practices. Even more, how their research can influence their teaching in university courses as well as in teacher education programmes (Mali et al., 2021). The lecturer in this study teaches in a mathematics department where the established teaching culture is the presentation the definitions, theorems and proofs. He himself is not only experienced in teaching and research in mathematical Analysis but also has a strong research background in mathematics education (e.g., in teaching in secondary and tertiary

level and teachers' knowledge). This combination of expertise makes him an information-rich case, offering valuable insights into how teaching culture could be shaped by research experience. Therefore, we ask the following research question: How does a researcher in mathematics and in mathematics education approach teaching the fundamental concepts of Analysis to first-year mathematics students?

Theoretical considerations

We adopt a dialectical materialist perspective of culture (Radford, 2021), in which individuals create cultures and cultures create individuals. Radford emphasised the interconnected and dynamic nature of culture. In his view, culture is an integrated system where material and intellectual dimensions interact within a web of societal relationships. These relationships are not static or universally consistent; they shift based on context and circumstances, creating tensions and conflicts that drive cultural change. We also view culture as a polyglot relation of conflict, resistance, and meaning making between individuals and societal practices they engage in. When examining the teaching culture of a lecturer involved in both research and teaching, culture embeds, and is embedded, in the set of practices that influence their approach to teaching and research within the university's affordances and constraints. In this context, we specifically consider the teaching of Analysis and the lecturer's research background, seeking insights into how the dominant university culture could be affected. However, culture is not merely a static set of practices. It is a living, evolving process shaped by the lecturer's interaction with students, colleagues, and the broader academic community. This process is formed through their engagement in societal practices, including how they teach, how they communicate mathematical concepts, and how they integrate research into teaching. In return, this culture influences the mathematician's own thinking behind teaching and research. This thinking includes considerations of how best to enable students to think mathematically, how they work within university affordances and constraints, their own educational history, their views of mathematics and what it means for students to learn mathematics (Potari et al., 2023).

Shedding light on tensions inherent in teachers' doing and thinking has the potential to offer deeper insights, on the one hand, into the source of these tensions, and on the other hand, whether teachers attempt their synthesis (Potari et al., 2023). For example, mathematical rigour and insight are seen as being in tension, and this tension can offer insights into why some teachers prefer giving up some rigour in exchange for insight when teaching mathematical proofs (Sangwin & Kinnear, 2024). The dialectical nature of tensions (Radford, 2021) emphasizes that they are not just obstacles but can also be productive forces that drive deeper engagement and development in the educational context. Culture also serves as a site of tensions and their synthesis towards meaning making, suggesting that the lecturer's teaching practices might involve navigating the differing expectations and challenges that arise during students' enculturation into advanced mathematics and the new university environment. Thus, the teaching culture involves not only the presentation of mathematical content but also facilitating students through this enculturation, so they become successful in higher-level mathematics. The teaching culture would, therefore, be shaped by these broader societal processes, including the difficulties students encounter during their transition, institutional norms and the evolving field of mathematics education itself. Ultimately, we view teaching culture as encompassing teaching practices, teacher thinking and the societal context in which they are situated. We examine

how this teaching culture is developed through tensions handled by drawing on the mathematician's own research background.

Methods

This is a case study focused on teaching the fundamental concepts of Analysis at a university in Greece. The Analysis course under investigation was proof-oriented and its content included limits of sequences and functions, continuous of functions, derivatives and the related theorems. The course was offered to a large cohort of 200 mathematics students in the first semester of their bachelor's programme. The usual way of introducing mathematical concepts in this department was rather traditional, in the form of teaching the formal definitions followed by some examples. The third author was the university lecturer, who held a Ph.D. in mathematical Analysis, and had conducted research in mathematics education as well. He was a well-regarded lecturer among students, with 30 years of teaching experience at the university level.

Data for this study included observation notes and transcripts of audio-recorded lectures covering the concepts of the convergence and the limit of a sequence, as well as the limit, continuity, and derivative of a function. These observations were made by the first and the fourth author. Data also includes transcribed discussions between the lecturer (the third author) and the two researchers (the first and fourth author). Each discussion followed each of the observed lectures, during which, the planning for the next lecture was also addressed. In the retrospective analysis, the second author was involved and further reflections of the lecturer took place that were also part of the data. In this way, the data analysis became systematic. The iterative cycles of analysis began among the three researchers with a grounded study of the data, involving collaboration among the three researchers and the lecturer. Together, they reviewed the transcripts and discussed their remarks, focusing on what the lecturer did and intended during his teaching. In the initial cycles of analysis, a combination of open coding and theoretical coding, informed by extant literature, was used to identify specific teaching practices. For example, uncovering the limitation of students' current content knowledge, coordinating intuitive and formal explanations, providing alternative representations, highlighting a mathematical practice (e.g., abstraction), etc. As the analysis progressed, the researchers identified a four-level pattern in the lecturer's approach to teaching the concepts mentioned above. Following the initial cycles of analysis, the researchers engaged in discussions with the lecturer about his underlying considerations for teaching. They presented him the descriptive names of the identified teaching practices and corresponding examples from the data. The lecturer reflected on how his own background and research expertise influenced his approach to designing and implementing his teaching of the concept of limit. These discussions were audio-recorded and transcribed for identifying tensions.

In the following section, we present examples of observational and interview data concerning the teaching of the limit of a sequence, that were coded as teaching practices and tensions as well. The teaching practices are discussed in the order of the four-level pattern. At the end of the section, we give an account of the identified tensions and their role in the development of an alternative teaching culture comparing to the one described above that had been established for many years in this department.

Findings

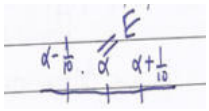
A four-level pattern of a concept introduction was identified by the analysis of our data: *setting a familiar problem unsolvable with known techniques, adopting an intuitive approach to address the problem, decontextualising the solution process to introduce the new concept, making the new concept transparent*. We exemplify this pattern by drawing on data from the teaching of limit of a sequence. At the end of the section, we give an account of the identified tensions and their handling, as they are also findings of the study.

At the first level of concept introduction, a familiar problem was usually introduced and the limitation of students' previous knowledge for solving this problem was established. The selection of an appropriate problem accessible to students was crucial because, in this way, they were encouraged to start thinking about how they could solve it and the lecturer facilitated them in uncovering the limitation of what they knew. The problem he selected to introduce the limit of a sequence was the following: "How we could calculate the area of a circle of radius 1". The students knew from school how to calculate the area of polygons and one student remarked that this method does not work because "the circle does not have a straight section". Thus the limitation of the known measurement process emerged.

At the second level, students were encouraged to adopt an intuitive approach, drawing on their arguably limited knowledge to find a solution method for the problem set. The lecturer asked, "How could we possibly link the area of the circle with the area of regular polygons?" When one student answered, "We could fill the circle with polygons!" the lecturer commented: "Yes, but there [on the circle] will always be something left over." This intuitively obvious comment prompted another student to get the idea, "Let's take polygons inside and outside [the circle]." For the lecturer, this intuitive idea represented students' arrival at a second, crucial for the solution, step: confining the unknown quantity, that is the area of the circle, between two known quantities, that is the area of an inscribed and a circumscribed polygon to the circle. He proceeded to substantiate the student's idea on the board by drawing the simple case of an inscribed and a circumscribed square around the circle, as well as a corresponding number line with the symbols $E1$, E, $E1'$ (representing the areas of the square in the circle, the area of the circle and the area of the square outside the circle, respectively) and the symbolic representation of the inequality $E1<E<E1'$. Then, he asked students another crucial question: "Could we reduce the interval where the number E lies"? This question aimed to lead students pave another crucial step: to think of a way for an increasingly better approximation, arbitrarily close to the intended quantity E. A student immediately responded, "Yes! By increasing the number of the sides of the polygons". Then, the lecturer invited students to "look numerically" at what they had been discussing earlier in a Geogebra environment, where two sliders adjusted the radius of the circle and the number of the sides of the inscribed and circumscribed polygons. After increasing the number of the sides with the slider, he concluded, "The difference in the areas can be made as small as we want and approaches zero." The entire teaching approach intended to ease students' shifts between vital mathematical ideas, such as confinement and approximation through mathematical inquiry and was characteristic for this lecturer's teaching.

At the third level, the solution process was decontextualised from the initial problem and, in this way, students were gradually introduced into mathematical abstraction, as presented in Table 1:

Table 1: A Teaching episode from the introduction to the limit of a sequence

<table>
<tr><td>1</td><td>Lecturer:</td><td colspan="2">The areas are sequences. Let the midpoints of the polygons be considered. We can get as close as we want to the area of the circle.</td></tr>
<tr><td>2</td><td>Lecturer:</td><td colspan="2">If we wanted an approximation to the nearest tenth:
$a-\frac{1}{10}, a+\frac{1}{10}$
[he draws the following figure for the approximation process].
 
If we wanted an approximation to the nearest hundredth:
$a-\frac{1}{100}, a+\frac{1}{100}$. In other words, the distance is the radius.</td></tr>
<tr><td>3</td><td>Lecturer:</td><td colspan="2">How can we express any distance in a general way?</td></tr>
<tr><td>4</td><td>Student:</td><td colspan="2">With a positive number.</td></tr>
<tr><td>5</td><td>Lecturer:</td><td colspan="2">Let's take a positive number and call it ε (epsilon). We said [he divides the board in two parts]:</td></tr>
<tr><td rowspan="4">6</td><td rowspan="4">Lecturer:</td><td>As close as we want,</td><td>For every $\varepsilon > 0$</td></tr>
<tr><td>there exists a polygon</td><td>there exists $n_0 \in N$</td></tr>
<tr><td>such that the distance between its area and the area of the circle is less than ε.</td><td>$|\alpha_n - \alpha| < \varepsilon$</td></tr>
<tr><td>This property is not only true for this particular polygon but also for all polygons with a greater number of sides.</td><td>for every $n \geq n_0$</td></tr>
<tr><td>7</td><td>Lecturer:</td><td colspan="2">When all these conditions are met, we say that the sequence (α_n) converges to the number α. … [the formal definition followed]</td></tr>
</table>

In the above teaching episode, the approximation process was decontextualised from the initial problem of finding the area of a circle to a more objectified form, applicable in other cases as well. Through a detailed translation of the intuitive approach into mathematical terms by the lecturer (turns 3–7), students were gradually introduced to mathematical abstraction at this level. Then, the definition of the limit of a sequence was formulated symbolically, graphically and verbally. The entire process transformed intuition into concrete, interconnected mathematical ideas until the new concept was introduced.

At the fourth level, the introduced concept was made transparent through appropriate examples, counterexamples and non-examples [e.g., the sequence $(-1)^n$]. In parallel, the lecturer addressed potential students' misconceptions (e.g., the convergence implies monotonicity), drawing on his familiarity with such misconceptions from his experience in mathematics education. Furthermore,

the connection of the new concept to expressions from everyday life (e.g., “speed limit”) and to other mathematical concepts (e.g., the density) was discussed, further contributing to the transparency of the limit concept.

Intuition, inquiry as well as pedagogical concerns about how students come to understand mathematics, are all components of this lecturer’s research culture both in mathematics and mathematics education. These components are poles opposed to mathematical rigour, exposition of the advanced content as well as restrictions resulting from the large number of attendees. Attempting to balance these poles creates dialectical tensions of epistemological, pedagogical and institutional nature, identified in this lecturer’s teaching. For instance, we saw above an interplay between intuition and mathematical rigour (epistemological), the inevitable exposition of mathematical content alongside questions calling for students to “do” mathematics (pedagogical) and the large number of students (institutional), which prevents interaction or restricts inquiry in using available technological tools. Thus, the other cultures in which this lecturer participates, like the mathematical, research and university cultures, create tensions in teaching. In his own words:

> “In teaching, you try to arrange ideas in an order so that they come across somewhat naturally. In research, ideas don’t come like that; sometimes a light for something turns on your mind! In teaching, you try for the questions to come as naturally as possible: we have a problem, we try to solve it with what we know, it doesn’t work, we try to think of something else, and that’s how we can connect the knowledge we have and start thinking. Of course, trying to think of something else isn’t easy for students to do in class. For example, I think I had told them to take the regular polygons inside and outside the circle. Then narrowing the interval within which what we are looking for lies—that was relatively easy. Some students mentioned it, like increasing the sides, and that’s how it progresses. But you put things in order. Students should be able to answer smaller questions or at least think about them, even if they don’t give the answer.” [59:31]

We took a step further to ask how these tensions were handled. Our analysis showed flexibility in adapting his teaching to the prevailing conditions in any way possible. For example, the poles of the epistemological tension—intuition (e.g., at the second level) vs. rigour (e.g., at the third level)—were dialectically synthesised in the four-level approach for the introduction of mathematics concepts. This synthesis approach is what develops an alternative teaching culture in this lecturer’s case.

Discussion

In this study, we closely examined the teaching approach of a researcher during the introduction of the concept of limit of a sequence. His goal was to introduce students into how we "do" mathematics. We identified a pattern in his first-year undergraduate mathematics lectures, characterised by four levels: *establishing the limitation of current knowledge for solving a particular problem*, *adopting an intuitive approach to address the problem*, *establishing the new concept*, and finally *making the concept transparent*. This approach provides a nuanced, inquiry alternative to the conventional model, which often relies on a mere exposition of formal definitions.

Reflecting on the development of an alternative teaching culture, we illuminated the meaning of the identified four-level model and its potential for students’ learning. Despite the culture that dominates

university mathematics teaching of a new concept, which includes only the last part of the third level i.e. the presentation of the typical definition, the lecturer in this study forms a teaching culture which includes all four levels and this might be indicative of increased learning opportunities for students. This alternative teaching culture, represented by (but not restricted to) the four-level model, is developed through tensions and attempts of their synthesis. We observed epistemological, pedagogical, and institutional tensions in agreement with previous studies (e.g., Potari et al., 2023). The difference in this study is that the lecturer was aware of the culture of research in mathematics education, the culture of traditional university teaching, and the culture of mathematical research, and he attempted to synthesise his experiences from these existing cultures. In this attempt tensions arose, which he sought to balance, thereby initiating a new, alternative teaching culture.

The findings are particularly relevant when considering the dual roles of research mathematicians who also serve as university mathematics lecturers. Often, there is a perceived disconnect between their roles as researchers and lecturers. However, bridging this gap between mathematical production (research) and teaching could provide students with a more authentic and integrated experience of mathematical inquiry, which is crucial during their initiation into higher-level mathematics. In our case study, the lecturer leveraged his own research expertise both in mathematics and mathematics education to shape the four levels of his teaching practices. By connecting research and teaching, his students may become better equipped to do university mathematics, ultimately being more successful and productive in their mathematical thinking and activity (Lynch & Lockwood, 2019).

The teaching culture that the teacher initiated was further shaped by his own research background in mathematics education. This expertise played a vital role in informing him of students' misconceptions and difficulties in areas such as quantifiers or infinite processes. To address these challenges, he dedicated lecture time to explain these areas through examples and graphical representations. By drawing on his mathematics education expertise, the lecturer created a learning environment which could not only address common misconceptions and difficulties but could also promote a deeper conceptual understanding, essential for student success in university mathematics.

Building on this foundation, the study enhances the understanding of concept introduction practices by increasing awareness of how a lecturer's research background can play a pivotal role in cultivating an inquiry teaching approach. This emphasises that a lecturer's research background both in mathematics and mathematics education serves not just as a source of content knowledge but also as a valuable resource for developing teaching practices towards making abstract concepts more accessible and understandable. Such insights enrich students' learning experiences and ease the mathematical gap identified in the transition literature (Di Martino et al., 2023). Future research will need to investigate the impact of such research-informed teaching practices on students' learning and engagement across diverse student populations and learning environments. While this single case represents a small shift in the existing teaching culture, changing the overall culture of a department is difficult. However, in the context of how the individual shapes the broader culture, it raises the question of how and through which collaborations these research-informed experiences can contribute to shaping the overall teaching culture at the university. In a setting where the context limits significant change, such an approach could still influence how students learn and, consequently, how they develop their teaching as future educators.

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Is this a different proof? Mathematicians' and secondary mathematics teachers' practices when comparing alternative proofs

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There is a growing interest in how teachers in academic mathematics courses can identify and explore connections to their practice. Despite early indications that teachers value examining how their instructors' practices can inform their own, research in this area remains limited. This study addresses this gap by contrasting the practices of a secondary teacher and a mathematician as they analyse the pedagogical affordances of teaching more than one proof in a specific instructional situation. Data were collected in the M-Cubed Lab, an environment designed to study dialogical learning between these two communities. The analysis reveals markedly different practices that lead to opposing pedagogical conclusions, illustrating how exploring mathematicians' mathematical and pedagogical practices in relation to secondary mathematics teaching can be both nuanced and insightful, and underscoring the potentially significant role of such explorations in teacher education.

Keywords: Secondary teacher education, mathematicians, dialogical learning, alternative proofs.

Rationale

More than a century has passed since Felix Klein drew attention to the discontinuity that secondary mathematics (SM) teachers experience when taking academic mathematics (AM) courses, yet the mathematics education community still grapples with delineating the potential affordances of AM courses for SM teaching and how they may be realised in practice. Klein's approach to addressing this discontinuity emphasised connections between specific mathematical concepts and practices in the tertiary and secondary mathematics curricula (Wasserman et al., 2023). Over time, this approach has been implemented, refined, and expanded through numerous capstone courses, teaching modules, and textbooks. Despite these substantial efforts, the double discontinuity persists, and may even be widening (Biza et al., 2022). In response, the past decade has seen growing interest in connections between AM learning experiences and SM teaching that extend beyond curricula to the pedagogical practices of SM teachers (Wasserman et al., 2023). Following an extensive review of the literature, Wasserman et al. (2023) introduced a framework that explicates various distinctions in teacher preparation programs and highlights connections across these distinctions that so far received less attention and remain underdeveloped. One such connection is between the mathematical and pedagogical practices of AM instructors and those of SM teachers. These connections represent how AM instructors may intentionally model certain practices that they deem beneficial for SM teaching. While research on such *modelled-instruction connections* is limited, there are initial indications that they are appreciated by teachers and can influence their practice (Wasserman & McGuffey, 2021).

This study aims to further explore the nature of modelled-instruction connections and their potential affordances for SM teaching by contrasting the practices of mathematicians and experienced SM teachers as they compare alternative proofs in the context of SM multiple-proof tasks. Multiple-proof tasks are tasks that invite students to produce or compare alternative proofs of the same mathematical statement (Dreyfus et al., 2021). Scholars have long advocated for the teaching of alternative proofs,

particularly in teacher preparation programs. However, research suggests that in practice SM teachers rarely present alternative proofs, either because they do not recognise their value, or because they lack knowledge and skills related to the practices of proof and proving (Dreyfus et al., 2021). This makes multiple-proof tasks an intriguing context for investigating modelled-instruction connections.

Theoretical background and research question

The literature suggests that the mathematical and pedagogical practices of AM instructors may play an important role in teacher education by facilitating prospective teachers' learning about teaching through an 'apprenticeship of observation'. Thus, these connections may help future teachers develop productive pedagogical practices (Wasserman et al., 2023). First, prospective teachers may incorporate into their own teaching certain teaching practices and strategies that they experienced and appreciated as learners. Second, prospective teachers may also draw on their instructors' mathematical and pedagogical practices as a resource for informing their own future teaching. Wasserman and McGuffey (2021) introduced the concept of *pedagogical-mathematical practices* to describe practices that are an amalgam of mathematical and pedagogical practices, for example acknowledging and revisiting assumptions and mathematical limitations, or using boundary cases to test and illustrate mathematical ideas. Observing how AM instructors engage in pedagogical-mathematical practices may inform teachers about how to engage with these practices in the future.

Zazkis (2020) observed that teachers' accounts of how they utilise their experiences in AM courses in teaching suggests that such utilizations are often personal and tacit, and thus often remain invisible, not only to outside observers but often also to the teachers themselves. These characteristics, which pose a significant methodological challenge in studying the affordances of AM for SM teaching (Biza et al., 2022), are likely to be even more pronounced in the case of modelled-instruction connections due to the tacit and personal nature of pedagogical and mathematical practices. To address this challenge, Pinto and Cooper (2022, 2023) proposed that boundary encounters between mathematicians and experienced SM teachers may be used as a methodology for generating and studying utilizations of AM in SM teaching. The premise of boundary encounters is that communities are separated and defined by social, cultural, and epistemological discontinuities in action or interaction, which can make the exchange of knowledge across communities difficult (Akkerman & Bakker, 2011). However, the literature on boundary crossing highlights mechanisms that can help boundaries serve as a powerful catalyst for dialogical learning across different communities. This study takes place within the M-Cubed Lab, an environment specifically designed to study boundary encounters between mathematicians and practicing SM teachers (Pinto & Cooper, 2022, 2023). In this setting, this paper investigates the following research question: *How do the practices of mathematicians and secondary mathematics teachers differ when comparing alternative proofs?*

Methods

Data collection: The M-Cubed Lab

Data for this study were collected in the M-Cubed (Mathematicians, Math teachers, Mathematics) Lab. In M-Cubed, small groups of mathematicians and experienced SM teachers convene once a month to view videotaped SM lessons and jointly inquire into mathematical and pedagogical issues

that they recognise therein. This research setting serves as a laboratory for generating and studying the implications of ideas and perspectives from AM in consideration of authentic instructional situations. This paper focuses on two M-Cubed participants: Benjamin, a research mathematician with extensive teaching experience in undergraduate courses, and Daniel, an experienced SM teacher.

Data for this paper comprise a videotaped 2-hour discussion that included Benjamin and Daniel, and an article that Daniel wrote after the meeting for his teacher blog where he reflects on this discussion. The M-Cubed discussion concerned the teaching of multiple-proof tasks, and it followed 15 minutes of individual work where Participants tried to locate different proofs to the Three-Squares Problem:

Given three adjacent squares as illustrated below, show that $\angle\alpha + \angle\beta + \angle\gamma = \frac{\pi}{2}$.

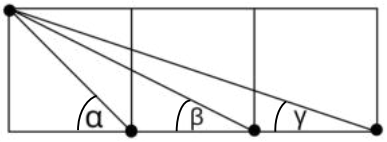

Figure 1: The Three-Squares Problem

The Three-Squares Problem, often attributed to Martin Gardner, has many elementary solutions in various mathematical fields, making it particularly suitable as a multiple-proof task (e.g., Hoseana, 2024). After engaging with the problem individually, participants discussed their proofs in small mixed groups and then watched a 20-minute section from a videotaped lesson in an advanced-track 11th-grade class where six students presented their six alternative proofs of the problem. Three proofs are in planar geometry (presented below), and the other three are in trigonometry, analytic geometry, and complex numbers[1]. Then, the group reflected on the lesson they just watched, comparing how alternative proofs were presented and discussed, and exploring the opportunities for learning that may have been afforded for the students. The M-Cubed meeting was videotaped and fully transcribed. After the meeting, Daniel wrote a ~3000-word article, intended for a blog for SM teachers, reflecting on Benjamin's and the other mathematicians' observations regarding the opportunities for learning that the different proofs afforded, and contrasting them with his own perspective as a SM teacher.

Data analysis

The first stage of the analysis focused on identifying different threads in the data where a thread refers to a sequence of utterances, though not necessarily continuous, that address a common issue and relate to and build on each other. This paper focuses on the thread: "Are the three planar-geometry proofs different and is there merit in teaching more than one of these proofs?" This thread was chosen since it was central in both the M-Cubed session and Daniel's article. Analysis of the practices that Benjamin and Daniel engaged in while addressing this question were informed by Wasserman et al.'s (2023) framework and by Wasserman and McGuffey's (2021) descriptions of pedagogical-mathematical practices. Since such practices are neither strictly mathematical nor purely pedagogical, this paper considers any practice that could reasonably belong to one or both these categories.

Findings

During the M-Cubed meeting, several participants observed that the teacher in the videotaped lesson did not explicitly address what students were supposed to learn from seeing multiple proofs. This

[1] All six proofs are presented in detail here: https://stwis.org/5gylx8.

observation raised the question of what the affordances of multiple-proof tasks are and how teachers can help students realise these affordances, focusing specifically on the opportunities for learning that were afforded by engaging students with the three planar geometry proofs:

Proof A: Add three auxiliary squares and connect the points A and C to the point *G,* as in Figure 2. The triangles ACG and AFE are similar as right angled isosceles and thus $\sphericalangle ACG = \sphericalangle\alpha$. Since $\sphericalangle GCB = \sphericalangle\beta$, we can conclude that $\sphericalangle\alpha + \sphericalangle\beta + \sphericalangle\gamma = \frac{\pi}{2}$.

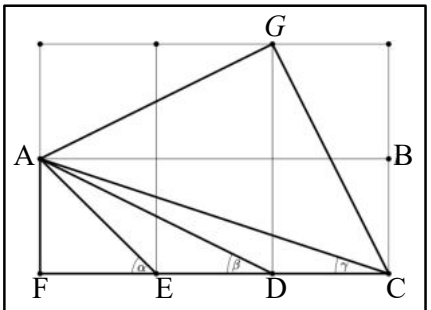

Figure 2: AGC is similar to AFE

Proof B: The triangles AED and CEA are similar since the two triangles share the angle AEC and since $\frac{AE}{ED} = \sqrt{2} = \frac{CE}{EA}$. It follows that $\sphericalangle EAC = \sphericalangle EDA = \sphericalangle\beta$, and since $\sphericalangle\gamma$ and $\sphericalangle CAB$ are alternating angles, $\sphericalangle\alpha + \sphericalangle\beta + \sphericalangle\gamma = \frac{\pi}{2}$.

Proof C: Similarly to Proof 1, add three auxiliary squares and two edges to a point H as illustrated in Figure 3. The new right-angled triangle AHC is similar to with a similarity factor $\sqrt{2}$. It follows that $\sphericalangle ACG = \sphericalangle\beta$ and since $\sphericalangle BCG = \sphericalangle\alpha$ we can conclude that $\sphericalangle\alpha + \sphericalangle\beta + \sphericalangle\gamma = \frac{\pi}{2}$.

Before reading further, the reader is invited to pause and reflect on whether these three proofs are 'different' and on the value of teaching all three proofs in SM classes.

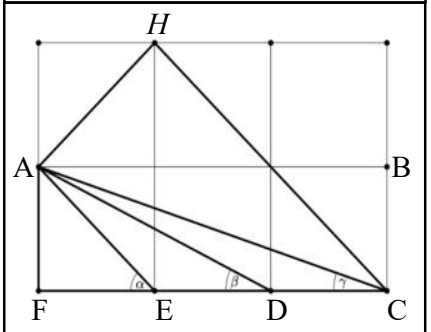

Figure 3: AHC is similar to AFD

Benjamin's analysis of alternative proofs

Benjamin was generally critical of the presentation of alternative proofs in the videotaped SM lesson:

> I don't think we saw different solutions here. I think we saw the same solution several times in different costumes. What could have been valuable was to tell [to the students] that in geometry, the way to show that angles add up to 90° is by embedding them in a right angle. That's what common to all the solutions […] For me, it was boring to see all these solutions […] There is no principal difference, one student looked at this triangle, while another looked at another triangle.

This criticism suggests that Benjamin assess the added value of alternative proofs according to whether the represent fundamentally different *principal ideas*. Since he viewed the three planar-geometry proofs as representing the same principal idea, he saw little added value in teaching more than one proof. However, Benjamin did recognise merit in teaching more than one proof in this case:

> I think it's important to emphasise the difference between a geometric construction, which provides a better explanation but requires innovation, and computational methods […] This is a tactical difference in problem solving, maybe even strategic, that should be pointed out.

When asked to elaborate, Benjamin explained the strategic differences he observed:

> When working with trigonometric functions, or slopes, or complex numbers, you represent the angles as numerical objects in some way, and then … then you compute a sum or a product […] You compute, you just sit down and compute. In geometry, you can't just sit down and compute, you need to find something to compare. It's comparison, not computation.

Benjamin also highlighted a subtle mathematical issue in the computational solutions, which further justified comparing the geometric and computational approaches. He noted that in some of the

computational proofs, the students concluded the sum of the angles without explicitly addressing the fact that this sum was determined up to additions of $\pm 2\pi$. For example, in the trigonometric proof, a student determined that $\beta + \gamma = \arctan(1)$, but did not reason why the sum was not $\pi/4 + 2\pi$. Benjamin noted that the proofs are still valid because all the angles were acute, but this was a geometric argument that needed to be made explicit. He concluded:

> I think it's nice that you don't entirely get away from the geometry. Everyone thinks that you can just compute, compute, compute, but it doesn't really work that way.

In sum, Benjamin's practices of comparing alternative proofs and determining their pedagogical value were focused on the principal ideas each proof illustrated. Specifically, he emphasised the general proving strategies that are exemplified by the different proofs. He considered the three planar-geometry proofs to be essentially the same and saw no value in teaching more than one of them. He did see value in comparing a planar-geometry proof with a computational proof to highlight and contrast two general problem-solving strategies in mathematics and discuss their relative advantages.

Daniel's analysis of alternative proofs

During the M-Cubed session, Daniel opposed the idea that teaching alternative proofs inherently benefits students and agreed with Benjamin that the proofs need to be compared in some way:

> A lesson needs to have content! […] The teacher could have said, wait a minute, let's look at the three [planar geometry] solutions. In all of them we found similar triangles; In all three, our anchor was finding a right angle. This… this would have been a completely different lesson!

However, Daniel did not fully endorse Benjamin's analysis of the proofs and his conclusion, as he observed additional aspects of alternative proofs that he considered valuable for SM students:

> I think, something really nice that you could do is to try to find in each proof the, like, some cues […] What was it in the problem that led us in this particular direction? […] You could ask, now that we've seen a proof [using complex numbers], what in the problem could have made us think in the direction of complex numbers? […] What are the clues at the 'scene of the crime'?

In the Daniel's article, he developed this idea further. He focused on the planar-geometry proofs and explored why Benjamin viewed them as equivalent while the M-Cubed teachers generally considered them different. First, he created a table where he identified the principal ideas in each proof:

Table 1: Daniel's analysis of the principal ideas in the three planar-geometry proofs

Principal idea	Proof A	Proof B	Proof C
Completion to a right angle	V	V	V
Using a known proportion		V	V
Identifying similar triangles	V	V	
Using an auxiliary construction on a grid	V		V

Recapping his analysis, Daniel appeared to agree with Benjamin that all these proofs share a big idea:

> You can see that all solutions use 'completion to a right angle', which they accomplish […] either by using proportion considerations, referring to known proportions, similar triangles, or a grid. Thus, [the] big idea is positioning the three angles at the same vertex to create a new angle that equals their sum, then using a geometric argument to justify that this angle is a right angle.

Yet, Daniel's analysis also highlighted differences between the proofs, particularly the 'cues' in the Three-Squares Problem that could guide SM students towards each proof. Regarding Proof 1, Daniel observed that SM students could recognise that triangle AFE is a right-angled isosceles, which is a well-known auxiliary construct, and use it to identify where it can be placed in the figure to provide useful information. In relation to Proof 2, Daniel suggested that SM students could first look to identify which angles in the figure are or could be equal to α, β and γ, and then 'think backwards':

> We first conjecture which triangles are similar because that would help us, then we identify the proportional edges, then we identify the proportion, and then we prove the similarity.

Regarding Proof 3, Daniel noted that a student could recognise that the triangle AFD has a 1:2 ratio when using the square's edge as a unit of measurement, then replicate this ratio on a grid with the square's diagonal as the unit. Based on this analysis, Daniel concluded that each of the three proofs emphasises a different initial consideration that could guide students towards the proof. He further observed that the different conclusions drawn by the mathematicians and SM teachers might reflect a deeper distinction between AM education and SM education. Specifically, Daniel suggested that the term 'different solutions' might hold different meanings to mathematicians and to SM teachers:

> In [AM], a solution is a proof, and a proof is a sequence of arguments, each one is either known to be true or follows from previous arguments. From this perspective, solutions are equivalent if they represent equivalent lines of reasoning […] In contrast, SM teachers […] may be viewing a solution as be viewing a solution as a process, a sequence of actions that students need to learn. From this perspective, different starting points represent a significant difference [between proofs].

Daniel further suggested that explicating these different meanings could help inform teachers' decisions about teaching alternative proofs. He concluded his article by connecting the different interpretations of 'different solutions' to pedagogical practices of AM instructors and SM teachers:

> In class, mathematicians might ask, "Is there another solution?" while SM teachers might ask, "Did anyone solve it differently?". This difference is not merely semantic but reflects distinct views on the value of [teaching] multiple solutions to mathematical problems.

Daniel further observed that at AM education, the focus is on mathematical products that are independent of the proving process or the identity of the prover, whereas in SM schools the focus is almost entirely on the learners. Hence, for SM teachers, a student's solution is rarely disconnected from the individual solutions process of the student, because "problems, proofs, and [proving], are first and foremost pedagogical tools". Whereas in AM education the 'same' solutions by different proofs are expected to receive the same feedback, the 'same' solutions by difference students may receive different feedback. One student might be praised, while another might receive feedback on subtle errors, all because the students are at different learning stages.

Discussion

The literature suggests that SM teachers may benefit from examining their instructors' pedagogical and mathematical practices (Wasserman et al., 2023). However, the process by which such practices may be transformed into resources that inform SM teaching remains unclear. This study offers further

evidence that this transformation is not straight forward (cf., Pinto & Cooper, 2022, 2023). Daniel did not mimic Benjamin's approach to evaluating the pedagogical productiveness of presenting more than one proof, but instead analysed the proofs differently, reaching an opposing conclusion. While Benjamin argued emphatically that there is no merit in teaching more than one planar-geometry proof while Daniel argued just as strongly that there is. At first glance, this finding might seem discouraging. If this observed difference is, as Daniel suggested, not merely anecdotal but indicative of a deeper distinction between AM education and SM education, then one might question the benefits for SM teachers of examining AM instructors' practices when teaching alternative proofs. However, a closer examination of how Benjamin's observations influenced Daniel's analysis of the three alternative proofs underscores the potential value of modelled-instruction connections.

When comparing the different proofs of the Three-Squares Problem, both Daniel and Benjamin sought to draw insights extending beyond the specific proofs themselves. Notably, the insights Benjamin derived align well with key aspects of proof comprehension that mathematicians expect university students to develop, such as identifying the proof framework, summarising the proof via high-level ideas, and making implicit warrants explicit (Mejia-Ramos et al., 2012). Benjamin's analysis highlighted a single overarching proof strategy common to all three planar-geometry proofs, leading him to conclude that there is no merit in presenting more than one proof.

While Daniel did not endorse Benjamin's conclusion, it served as a catalyst for his own analysis, which was guided by a question more aligned with the pedagogical practices of SM teachers: What cues can an SM teacher highlight in the problem that might direct SM students towards each proof? Daniel's question explicitly intertwines mathematics and pedagogy, as evidenced by his analysis of the three proofs – unlike Benjamin's strictly mathematical approach. Drawing on his knowledge for teaching, Daniel identified three additional heuristics that were differently represented in each proof (see Table 1). In contrast to Benjamin's more general strategy, Daniel's heuristics were specific to SM students and SM planar geometry. In this sense, Daniel's practices and conclusions can be seen as extending and refining Benjamin's rather than conflicting with them.

The question remains: how can the potential affordances of modelled-instruction connections be realised in practice? Analysis of Daniel's approach and practices highlights two significant resources typically not accessible in AM courses. First, the analysis shows how Daniel, an experienced SM teacher, drew extensively on his knowledge for SM teaching when exploring, criticising, and refining Benjamin's practices. This finding adds to growing evidence that SM teaching experience – generally inaccessible in AM courses to both instructors and students – can be a valuable resource for identifying and explicating connections between AM teaching and SM teaching (Even, & Mytlis, 2024; Pinto & Cooper, 2022, 2023). Second, in M-Cubed, Daniel was not only exposed to Benjamin's mathematical practices but was also able to engage in a dialogue with him about his mathematical and pedagogical perspectives. In fact, Daniel noted that what triggered his analysis was the realisation that while Benjamin's exploration of the 'sameness' of the planar-geometry proofs was purely mathematical, the question whether the proofs are the same is not purely mathematical but also pedagogical – meaning that Benjamin's conclusions could be challenged by SM teachers. Creating more opportunities for mathematicians, prospective teachers, and practising teachers to share, explore, and revisit their tacit mathematical and pedagogical perspectives and practices could help

make these resources more accessible in teacher preparation programmes, thereby reducing the perceived disconnect between secondary and tertiary mathematics education.

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On the conceptual understanding of the notion of a general solution

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Many mathematical models for real-life phenomena are based on differential equations. Their importance in applications makes differential equations one of the core subjects in most engineering programs. Empirical studies reveal students' difficulties with the conceptual understanding of the fundamental notions of a differential equation and different types of its solutions. In this paper, we review the presentation of the notion of a general solution as a process in widely recognised classical texts suggesting a pedagogical strategy that can help improve students' understanding of the concept.

Keywords: Differential equations, general solution, conceptual understanding, genesis of differential equations, process-object duality.

Introduction

Differential equations (DEs) are used for modelling a wide spectrum of problems in engineering, physics, economics, biology, medicine, social sciences. The subject is regularly taught to large cohorts of students in mathematics, physics, natural sciences, economics, and engineering. Engineering students traditionally face difficulties with the mathematics subjects, and the course in DEs is not an exception. Mastering different methods of solution is not the only goal of DEs courses – "in the end, it is not the number of tricks that the student has learned, but rather the understanding of the concepts and awareness of the relevance of the techniques, which is important – and, finally, his approach to the new learning which an encounter with a new problem may demand" (Bickley, 1964, p. 383). Rota stressed that by reducing the course difficulty and turning the subject into a "bag of tricks," we impact both students' interest in learning and learning outcomes because "a course taught as a bag of tricks is devoid of educational value" (1997, p. 9). Current educational research on DEs mainly focuses on (i) students' difficulties in understanding DEs and their solutions and (ii) methodological and pedagogical approaches in teaching and learning DEs. In this paper, we are concerned with the important notion of a general solution and teaching strategies that promote its conceptual understanding. The research question we address is: *How presentation of the concept of a general solution through the genesis of differential equations can facilitate students' learning?*

Educational research on students' conceptual understanding of a general solution

The reasons for students' confusion with the notion of "solution to a DE" are well understood.

> Consider the numerous words that are used as prefixes to the word solution – the general solution, particular solution, unique solution, superposed solution, fundamental solution, etc. According to the definition – solution to a DE is a differentiable function that satisfies the DE in some domain, any of the aforementioned would be a solution to the DE. But with the defining process pushed to the background, each of them may stand alone in students' minds. (Raychaudhuri, 2008, p. 175)

The concept of solution as a function has a much broader meaning in DEs compared to the concept of solution as a number for algebraic equations, yet equilibrium solutions to DEs are still numbers (or 'constant functions'). Raychaudhuri (2008) noticed that a small group of high achieving students

demonstrated the reliance on the general solution. When students were asked to validate a given function as a solution to a given DE, they first found the general solution and then verified that it belongs to a given one-parameter family of solutions. Raychaudhuri (2008) argued that the reliance on the generating process could be explained by students' confusion about which solution should be used in the task. If students believe that the general solution should include all solutions to a given DE, they can simply opt for finding it and using it in their reasoning. She contended that although students might have seen examples of DEs possessing solutions which cannot be obtained from the general solution, they may discard them as non-examples of the concept. Singular solutions are often difficult to find, and they are seldom discussed in modern textbooks on the subject. Existence of singular solutions is one of the reasons why the concept of a general solution is often introduced only for linear DEs. The lack of conceptual understanding of the notion of a general solution for nonlinear DEs may result in its wrong use, even for linear DEs with variable coefficients, or in no use at all.

Arslan (2010) found that "the students were successful in algebraic solutions of DEs, but not in conceptualizing DEs and the solution of DEs concepts" and "algebraic solutions of DEs can be found even without a deep understanding and conceptualization of DEs, which is why students do not feel any need to understand DEs and related concepts" (p. 887). Students ignored the presence of a constant in the general solution, substituted the general solution of one DE into the general solution of another DE, viewed it as a slope, and even as an equation of a tangent line. Arslan (2010) concluded that the most striking difficulties were related to the concept of a DE, the general solution, solution, and a particular solution of a DE. Camacho-Machín et al. (2012) reported that "students experience difficulties in identifying the different meanings of the concept of solution to a differential equation" (p. 76). Lacking conceptual understanding of solutions to DEs, students solved a DE and compared the obtained solution with a given function to verify that it satisfies the DE, cf. (Raychaudhuri, 2008). In general, students tend to concentrate attention on specific solution techniques and "often fail to relate them to other concepts or ideas" (Camacho-Machín et al., 2012, p. 76).

Assessment task

We came across students' difficulties with the understanding of the concept of the general solution in a course Mathematics for Mechatronics (MfM) for senior engineering students based on a popular textbook written by Boyce and DiPrima (2012) and taught by the second author (Hernandez-Martinez et al., 2024; Rogovchenko et al., 2020). Several nonstandard tasks were designed for a graded assessment to explore students' understanding of the Existence and Uniqueness Theorems (EUTs) and related notions. A sample problem is presented below.

Problem (a) Verify that

$$y(x) = C_1 + C_2x^2 \qquad (1)$$

is the general solution of a DE

$$xy'' - y' = 0. \qquad (2)$$

(b) Explain why there exists no particular solution of DE (2) satisfying initial conditions $y(0) = 0$, $y'(0) = 1$.

(c) Suggest different initial conditions for this DE so that there will exist exactly one particular solution of a new initial value problem. Explain your choice.

Although the focus in this problem is primarily on the students' understanding of EUTs, we noticed that nearly all students verified that the function (1) satisfies the DE (2), but the overwhelming majority did not provide a valid reasoning to justify that (1) is the general solution (Hernandez-Martinez et al., 2024; Rogovchenko et al., 2020; Rogovchenko & Rogovchenko, 2022b; Treffert et al., 2018). Students learned that solutions to linear DEs exist and are unique on the intervals of continuity of the coefficients. Writing DE (2) in the standard form $y'' - (1/x)y' = 0,$ one concludes that solutions of this DE exist on two disjoint intervals, either for $x < 0$ or for $x > 0$. When the initial conditions are considered at $x = 0,$ the coefficient $(1/x)$ is not defined and thus there exists no particular solution of this DE satisfying initial conditions $y(0) = 0,\ y'(0) = 1$. We purposely asked students to verify that the function (1) is the general solution of (2) to create the situation when the general solution is well defined for all real values of x but initial conditions cannot be met because the coefficient in the DE (2) is discontinuous at zero and the corresponding system of linear algebraic equations for finding the constants C_1 and C_2 is inconsistent. Students correctly addressed parts (b) and (c), verified that the given function (1) satisfies the DE (2), but did not explain why (1) is the general solution.

By the definition, the general solution for a second order DE is a function of the variables x, y and two constants C_1 and C_2 containing all particular solutions of this DE. This can be established either by analysing the structure of the general solution and demonstrating that there are no particular solutions that are not contained in the two-parameter family of functions (1), or by solving the linear DE (2). The latter includes an intelligent guess of a particular solution of the DE (2), substitution reducing the order of the DE, and integration of the obtained first order DE. This technique was not studied in the MfM course; therefore, the task in part (a) became a stumbling block for students, although they possessed sufficient knowledge on linear DEs needed for answering the question.

To address students' difficulties, we analysed relevant literature finding out that Arslan (2010), Camacho-Machín et al. (2012), and Keene et al. (2011) suggested relevant problems, but only one of five problems in the cited papers contributed to the development of the conceptual understanding of the notion of the general solution engaging students into an exploration of an open-ended task (Rogovchenko & Rogovchenko, 2022a). We also posit that the course textbook discusses in sufficient detail only one approach to the notion and advocate complementing it by the definition of the general solution rooted in the genesis of DEs. We believe that the combination of these two process-oriented approaches can facilitate students' conceptual understanding of this fundamental notion.

Process-object duality in mathematics learning

Sfard (1991) argued that abstract notions "can be conceived in two fundamentally different ways: structurally - as objects, and operationally - as processes" (p. 1). Although "we have good reasons to expect that in the process of concept formation, operational conceptions would precede the structural" (p. 10), "the terms "operational" and "structural" refer to inseparable, though dramatically different, facets of the same thing. Thus, we are dealing here with duality rather than dichotomy" (p. 9). According to Sfard, the process of concept formation has three steps: interiorisation, condensation and reification. A learner first performs a process on familiar objects, then starts thinking of the process as a whole, and finally sees a familiar object as a "fully-fledged object." The last step is a rather complex phenomenon often defined as "an ontological shift, a qualitative jump" (p. 30).

In this light, one can define a general solution of a DE as a mathematical *object* (a family of functions depending on n parameters corresponding to the order of a DE and containing all particular solutions of a DE). On the other hand, a general solution can be introduced as an outcome of a *process* of solving a given DE with one of available methods and assuring that it has no other solutions not included in the obtained solution. We argue that a complementary definition of a general solution as a *process* rooted in the *genesis of DEs* can facilitate understanding of this important concept.

The concept of the general solution in the MfM course textbook

For the first time, the general solution is informally introduced in Example 1 (Boyce & DiPrima, 2012, p.11) as the expression that contains all possible solutions of a given DE. It is interpreted geometrically as "an infinite family of curves called integral curves. Each integral curve is associated with a particular value of c and is the graph of the solution corresponding to that value of c. Satisfying an initial condition amounts to identifying the integral curve that passes through the given initial point" (p. 12). This definition includes several important concepts in the Theory of Differential Equations: family of curves, an integral curve, a particular value of an arbitrary constant, an initial condition, a solution to an initial-value problem. Among all these notions, students usually focus their attention on the term '*solution*' and pay much less attention, if at all, to its combination with the terms '*particular*' and '*general*.' Most students encountered the notion of 'solution' earlier in Calculus courses whereas other notions associated with it in a DEs course were not familiar to them. According to Sfard (1991), the notion of '*solution to a differential equation*' is introduced by Boyce and DiPrima using both ways: there are examples which ask to show that a given function is a solution of a given equation (an object) and there are straightforward methods for finding solutions (a process). There is only one problem in Chapter 1 where one has to find a general solution of a very simple first-order linear DE and there are no problems asking students to verify that a given function is a general solution at all. Therefore, the explanation of how one can show that a given family of curves is a *general solution* is not provided, while there are numerous problems asking to verify that a given function is a *solution* of a given DE. This inevitably focuses students' attention on the latter notion. In Chapter 2, a first order linear DE with variable coefficients is solved and its solution, obtained as a result of integration, is explicitly named 'general' (Boyce & DiPrima, 2012, pp. 36-37). Later in this chapter the difference between the general solutions in linear and nonlinear cases is explained, and the justification for the specific use of the term "general solution" is provided.

> For a first order linear equation it is possible to obtain a solution containing one arbitrary constant, from which all possible solutions follow by specifying values for this constant. For nonlinear equations this may not be the case; even though a solution containing an arbitrary constant may be found, there may be other solutions that cannot be obtained by giving values to this constant. ... Thus, we will use the term 'general solution' only when discussing linear equations. (Boyce & DiPrima, 2012, p.74)

The discussion of the terminology for nonlinear DEs remains open because general solutions of nonlinear first order DEs were neither computed directly in examples nor suggested for solution in the exercises. Since students did not see a single example of a general solution to a nonlinear DE, they usually mistakenly associate the concept only with the linear DEs. Although the DE (2) in our assessment task is linear, students did not know how to demonstrate that the function defined by the equation (1) is its general solution. In fact, one can derive this conclusion from the basic properties

of linear DEs. Since the DE (2) is linear, any linear combination of two functions, 1 and x^2, each of which is a solution of this DE, is also a solution of the DE (2). By Theorem 4.1.2 (Boyce & DiPrima, 2012, p. 223), it suffices to check that these two functions are linearly independent and thus form the basis for the solution space for the given equation, so that (1) is the general solution of the DE (2). Linear independence can be easily verified either by the definition or by computing the Wronskian.

The concept of a general solution in terms of the genesis of DEs

A search for a good explanation for the concept of a general solution to a DE brought us to the genesis of DEs thoroughly described in a highly regarded classical book by Ince (1926), where "Dr. Ince has set a new standard for English books on differential equations" (Book review, 1934, p. 357). We also analysed the texts by Forsyth (1885) and Page (1897) that influenced Ince, as well as relevant material in widely recognised books by Boole (1859) and Johnson (1896). Although we tried to make the search as comprehensive as possible using the sources available at https://archive.org/, it is possible that some texts have been missed, especially non-English literature; however, a comprehensive search of all sources is not our focus in this paper. To the best of our knowledge, it was George Boole, an English mathematician widely known as a father of modern symbolic logic, who initiated the discussion of the genesis of DEs. He argued that even though the theory of the genesis of DEs from their primitives is to certain extent explained in treatises on Differential Calculus, there are some points of great importance "which need a distinct elucidation" (Boole, 1859, p. 8).

We explain the idea of the genesis of DEs following Page (1897). Consider an equation of the form

$$\omega(x, y, C) = 0 \qquad (3)$$

relating the variables x, y and a constant C. Differentiate (3) to obtain

$$d\omega \equiv \frac{\partial \omega}{\partial x} dx + \frac{\partial \omega}{\partial y} dy = 0. \qquad (4)$$

Elimination of the constant C after differentiation using equations (3) and (4) leads to a first order DE of the form

$$F\left(x, y, \frac{dy}{dx}\right) = 0. \qquad (5)$$

Equation (3) is called a *complete primitive* (Page, 1897) for a DE (5); it always coincides with a general solution for linear DEs. For nonlinear DEs, singular solutions not described by (3) may exist.

A similar procedure applies to equations involving two, three, or n constants (Ince, 1926, pp. 4-5). The advantage of this explanation is that one starts with a one-parameter family of functions (3) and arrives at a DE (5) satisfied by any function from the family (3). In most cases, the first order DE (5) has no solutions other than those defined by equation (3), and the meaning of a general solution becomes very clear. Boole (1859, p. 15) argued that if one applies the above procedure to a relation containing n distinct constants, there will exist only one DE of the order n which is free of constants. Johnson (1896) emphasised that "any equation containing arbitrary constants is the primitive of a certain differential equation free from these constants" (p. 6), whereas Ince (1926) pointed out the need for "certain conditions of differentiability and continuity of derivatives" (p. 11). Due to its transparency and rigour, Ince's book remained one of the main texts on DEs for over forty years until another influential text by Tenenbaum and Pollard (1963) changed the way the concept of a general

solution is taught today. Focusing attention primarily on teaching students how to use DEs, the authors "have abandoned the use of the terminology "general solution" of a differential equation unless the solution is in fact general, i.e., unless the solution actually contains every solution of the differential equation" (p. xv). This influenced the presentation of the concept in most modern textbooks on DEs and, unfortunately, led to students' difficulties rather than facilitating their understanding of a general solution.

We believe that the idea of the genesis of DEs developed in the texts by Boole (1895), Page (1897), and Ince (1926) can help to resolve the issue with the understanding of the notion of a general solution by complementing the definition of the general solution as an object with the explanation of two processes, the integration of a DE and the derivation of a DE from an algebraic equation relating the variables x, y and the constants C_1, C_2. Indeed, to prove that the function (1) in the assessment problem is a general solution of the DE (2), one differentiates (1) twice (Page, 1897, pp. 2-3) to obtain

$$y'(x) = 2C_2x, \qquad y''(x) = 2C_2. \qquad (6)$$

Replacing $2C_2$ in the first equation in (6) with $y''(x)$ from the second equation, one obtains the DE (2) which, in virtue of linearity, has no singular solutions not described by (1). This simple technique reversing the integration and differentiation not only provides an alternative argumentation that the two-parameter family of functions (1) is a general solution of the given linear DE (2) but, importantly, also applies to many nonlinear DEs. Differentiation is intensively practised in Calculus and is in most cases easier than the integration methods for DEs. In our opinion, this explanation to how a DE can result from a given equation relating x, y and n constants after differentiation and some algebraic manipulations should positively impact students' development of a concept of a general solution of a DE and facilitate learners' ontological shift to its perception as a structural object.

Conclusions

Empirical research reported by Arslan (2010), Camacho-Machín et al. (2012), Raychaudhuri (2008), and the authors (Hernandez-Martinez et al., 2024; Rogovchenko et al., 2020; Rogovchenko & Rogovchenko, 2022b; Treffert et al., 2018) points towards students' difficulties with the conceptual understanding of the notion of a general solution of DE. Assessment tasks may test differentiation skills without contributing to students' conceptual understanding of this notion (Rogovchenko & Rogovchenko, 2022a). The course textbook (Boyce & DiPrima, 2012) and most contemporary texts on the subject do not emphasise the importance of the general solution for DEs other than linear, and students often associate it only with this class of DEs.

We explored how the presentation of the concept evolved in influential textbooks published from the late nineteenth century and came across lucid and rigorous explanations in classical texts by Boole (1859), Page (1897), and Ince (1926). In Calculus courses, solutions to DEs are introduced primarily for the illustration of the differentiation and integration techniques. Therefore, students first approach the notion of a general solution as a *process* (Sfard, 1991). On the other hand, in DE courses, presentation of the notion of the general solution for nonlinear DEs as an *object* receives very little attention, if at all, with the focus shifted to linear DEs. We argue that students can better understand the concept of the general solution if it is based on the genesis of differential equations from equations relating two variables, x and y, and n distinct constants. When students are first presented with an explicit description of a family of solutions containing all solutions (there is no need to verify this)

and then see a DE derived by the repeated differentiation of the original equation and algebraic elimination of all constants, they may better understand the concept. Students know from Calculus that the differentiation of a given equation is technically not as complex as the integration of a DE. Following the idea of the genesis of DEs, students can get a better grip of the concept of a general solution, especially for nonlinear DEs. This approach to the design of conceptual tools for understanding a general solution addresses the claim that

> Teachers at all grade levels tend to view mathematics in terms of "subject matter" (e.g., definitions, theorems, proofs, problems and their solutions), not in terms of the conceptual tools that are necessary to construct such mathematical objects, and that cognitively, pedagogically, and epistemologically both categories are needed. (Harel, 2008, pp. 268-269)

This paper is the first step towards the design of what we believe should become a more effective pedagogy facilitating students' learning and understanding of the concept of a general solution of a DE which we plan to test both in a standard and advanced courses on DEs.

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"How should we solve it?" and "How should we write it down?" Two tacit hurdles in university mathematics discourse

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Learning tertiary mathematics requires students not only to develop new ways to solve mathematical problems but also to communicate their solutions. In this conceptual paper, we use the commognitive framework to construe problem solving as a discursive activity whose metarules are inseparable from the situation and the community within which it unfolds. Then, we distinguish between the processes students go through when generating a solution for themselves (i.e. the solvers' community) and the conventionalization of this solution to make it sharable with a broader community. This distinction raises the question of how solution generation and conventionalization are taught and learned. As a first step towards this question, we present a vignette from student-centred first-semester tutorial in linear algebra. Our analysis delineates didactical routines the instructor implemented to support her students during problem solving in tertiary mathematics.

Keywords: Problem solving, instructional interactions, commognition, linear algebra.

Introduction and research background

The challenges of learning tertiary mathematics have been explored broadly and widely (e.g., Winsløw et al., 2018). Our interest lies with the challenges of problem solving. To justify this interest, let us quote Halmos (1980), who wrote, "what mathematics really consists of is problems and solutions" (p. 519). Similarly, the final report of the Higher Education Curriculum Summit claims, "problem-solving is the most useful skill a student can take with them when they leave university" (Rowlett, 2011, p. 19). A complementary justification is more pragmatic: in the tertiary setting, students' learning of mathematics is mostly occasioned through problems and evaluated through (usually written) solutions students submit for assessment by course instructors. Thus comes the question – how is problem solving learned and taught in tertiary mathematics.

Problem solving processes include linking tentative solutions with a problem situation and appropriate resources (Selden & Selden, 2013). In a classroom, the instructor determines if a tentative solution is acceptable, and this changes according to the level of mathematical rigor required (Stylianides, 2007). Thus, what is an acceptable solution is dependent on the context –the audience, the setting, and the constructed socio-mathematical norms.

Communication of the solution can unfold in different media, including oral and written. Some studies (e.g., Fukawa-Connelly et al., 2017) suggest that oral communication includes mostly informal mathematical content, while written communication is often formal and symbolic. The conversion process from the informal argumentation to a formal written one can be a challenge for undergraduate students, with some of the difficulty stemming from knowing what is acceptable (Selden & Selden, 2013). While a search for literature regarding learning and teaching problem solving yields fewer contemporary publications than may be expected (e.g., Badger et al., 2012),

relevant insights can be borrowed from prolific research on the didactics of proof. Indeed, close relationships between proving and problem solving have been acknowledged (Selden & Selden, 2013). This relationship stems from problem solving generally referring to solving "non-routine" problems necessitating proving (e.g., Selden & Selden, 2013), from considering a proof as a solution to a problem (e.g., Badger et. al., 2012), and from the alignment of the challenges of communicating a solution with the challenges of communicating a proof (e.g., Weber, 2008).

Part of proof research attends to the social process of sanctioning an argument as a proof, foregrounding the key role of communication and audience (e.g., Weber, 2008). For instance, Harel and Sowder (1998) distinguish between the processes where a prover ascertains the validity of a mathematical statement for themselves and where they persuade others in its validity. Famously, Mason et al. (1982) construe proving as an act of convincing not only a friend but also an enemy. Another insight highlights the contextuality of proof. As a process that solves a problem, a proof can be viewed as an event that occurs in a specific time and place (e.g., Goguen, 2001).

The above findings can be complemented by insights from broader problem solving literature. One regards students' challenges with learning to solve non-routine problems, as opposed to exercises that mimic previously demonstrated worked examples (e.g., Selden & Selden, 2013). The literature also points out the difficulties of teaching to solve problems systematically (e.g., Badger et al., 2012). Our aim is to introduce a conceptualization of problem solving that foregrounds some implicit hurdles that students navigate when learning to solve problems in tertiary mathematics. The conceptualization is grounded in the commognitive framework. We put the conceptualization into action in a vignette to cast light on how instructors can respond to students' requests for help with these hurdles.

Commognitive foundations

Commognition is a socio-cultural framework that has been broadly used to make sense of instructional interactions in tertiary mathematics (e.g., Kontorovich et al., 2024). The framework views learning as a process through which learners develop and refine their participation in the target *discourse* to become agentive participants in the corresponding *discourse community* (Sfard, 2008). The framework construes university mathematics as a discourse with characteristic *metarules* that define "patterns in the activity of the discursants" (Sfard, 2008, p. 299).

Language transforms discourses into multilayered structures, enabling discursants to communicate about anything, including the discourse itself. Accordingly, discourse components (e.g., metarules) can be not only enacted implicitly but also articulated, becoming an explicit object of communication (Sfard, 2008). For instance, Kontorovich et al. (2024) explore how metarules of proof can be discussed in a classroom. The researchers contend that ascending to the meta-discursive layer is potentially useful for instructors to spell out how the target discourse compares to what students are accustomed to and for students to specify their challenges with the yet-to-be-familiar realm.

Lavie et al. (2019) position *routines* as a commognitive unit of analysis. Routines are viewed as <*task*, *procedure*>-pairs, where the latter are implemented to achieve the former. Routines are context-sensitive since they are performed in *task situations*. These are defined as "any setting in which a person considers herself bound to act—to do something" (Lavie et al., 2019, p. 159). Sfard (2023)

differentiates between two types of routines. *Algorithmic* routines involve procedures with a well-defined sequence of steps and yield an expected sort of outcome (e.g., solving a quadratic equation). *Heuristics* encompass non-deterministic procedures that "do not specify the basic steps in a unique way. [They] increase chances for a successful performance […] but they do not ensure it" (Sfard, 2023, p. 4). Sfard (2023) contends that most instructional routines are of the second type. Sfard (2023) notes that routines, when taken together, constitute a tightly interconnected *network*. One example of interconnections stems from routines coming in different sizes: from 'extra-large' to 'atomic,' whereas 'smaller' routines serve as building blocks for the 'larger' ones. In this way, routine networks are recursive and have "a fractal-like nature [that] finds its expression in the fact that when you repeatedly zoom into this network, you discover ever new, smaller routines" (Sfard, 2023, p. 5).

Before considering problem solving through the commognitive lens, let us acknowledge that we are not the first to do so. Using the framework, Koichu (2019) conceptualizes problem solving in mathematics instruction as "a socio-culturally shaped process of not-immediate achieving a chosen or imposed goal, in which the involved individuals enact, through private or public exploratory discourse, individual or shared resources that they interpreted, even for a short while, as appropriate for achieving the goal" (p. 7). Koichu builds his conceptualization on cognitive research as well, and acknowledges that "being a discursively oriented, [the conceptualization] is not fully compatible with the 'strong' discursive perspective" (p. 61). We attempt to offer a more compatible version next.

Solution generation and conventionalization in problem-solving task situations

Problem solving can be commognitively construed as a discursive activity that unfolds in a certain task situation where an individual or a group draw on routine networks to produce a *solution*. We borrow Koichu's (2019) definition of solution as "a public narrative, which becomes endorsed, by the problem solvers or the problem proposers, as the one that achieves the goal" (p. 49) or 'a task' in commognitive terms. Consistently, we distinguish between two tasks of problem solving. One is to generate a narrative that the solvers consider as solution to the problem. The other task is to modify this narrative so that the problem proposers consider it a solution as well. That is, the proposed solution must adhere to the conventions of a target discourse community. Modifying an initial *solution-narrative* can go beyond merely "writing it down" and using appropriate notation. Mathematical solutions are comprised of both content and form (Byers & Erlwanger, 1984), which are barely distinguishable from the discursive viewpoint.

The construct of task situation in our conceptualization foregrounds the contextual circumstances of an initial problem-solving task (e.g., Goguen, 2001), since they shape the expectations from the solution-narrative. In tertiary mathematics, course instructors, i.e. the discourse experts, are often the ones to set up mathematical problems for students, discourse newcomers. Then, almost by definition, the proposers and the solvers are not partaking in the same discourse, at least initially. This often results in qualitative differences in metarules each cohort follows to endorse the solution-narratives. University students are not oblivious to this gap. Indeed, variations of "Can I say that on an exam?" and "I know how to solve it, but how can I write this down?" are common in many classrooms.

We propose that, in problem-solving task situations, subjecting the solution-narrative to the metarules of a conventional discourse, in which students are not yet fully versed, can become a sub-task that

students choose to pursue (cf. Koichu, 2019). We introduce the notions of *solution generation* (SG) and *solution conventionalization* (SC) to account for this phenomenon.

With SG, we refer to the set of routines students can implement to bring a solution-narrative into existence. This can involve reformulating the problem, making a sketch, and many other variations of what Pólya (1945) and his followers termed as heuristic strategies. Other algorithmic routines, such as equation solving, can also serve SG purposes. The overall task of SG is for students to transition from an initial task situation where the solution-narrative is conspicuous in its absence to one where the endorsability of some narrative can be considered by the solvers' community.

In some cases, the initially generated solution-narratives can be shared with a broader community as they are, or with minimal changes. In other cases, these narratives need to undergo substantial transformations to align with the metarules of conventional discourse of tertiary mathematics and be endorsed as problem solutions. We use SC to refer to the set of routines the solving students can implement to make these transformations, where the task of SC is to modify the existing solution so that it aligns with the conventions of the setting. These routines can involve switching from the oral to the written medium, which often involves formalization, symbolization, restructuring, adding explanations, and removing redundancies and extraneous steps (Morgan, 2002). These transformations are necessary to make the written solution-narrative be able to 'stand-alone' and to be shareable with a broader community. This conversion process is not necessarily linear as it can involve multiple attempts and cycles (cf. "rewriting drafts" in Vroom & Ellis, 2024).

SG and SC are differentiated by their task, output, and intended audience. At a first glance, it may appear that SG precedes SC since outcomes of the former can be expected to serve as inputs to the latter. Yet, it would be naïve to propose that the network of problem solving routines can be neatly divided into two separate regions. Indeed, the cyclical nature of problem solving has been widely acknowledged (e.g., Koichu, 2019; Selden & Selden, 2013). Then, SC can reveal issues and drawbacks in the initially endorsed solution-narrative, requiring the solvers to go back to the SG drawing board. Furthermore, some routines can have both generational and conventionalization purposes. Thus, it may be more accurate to consider SG and SC as affordances of implementing a certain routine in a specific task situation, rather than associate this routine with a particular type exclusively.

The presented conceptualization aims to provide better understanding of the complexity of students' challenges with problem solving in tertiary mathematics. This conceptualization enables us to ask how SG and SC are taught and learned. This empirical question is at the heart of our broader research agenda. We next present a vignette to make a first empirical step in this direction. It focuses on didactical heuristics that instructors implement to support their students in the development of SG and SC.

Illustrative vignette of teaching SG and SC

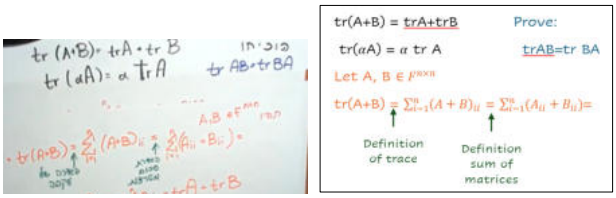
The vignette comes from a course in linear algebra taken by first-year students majoring in mathematics and computer science. The data corpus encompasses video recordings of interactive tutorials, held on an online platform due to Covid-19. The protagonists are six students and Ann

(pseudonym), an experienced instructor who received numerous awards for teaching excellence. We also collected Ann's informal reflections on selected tutorials and consulted with her in the analysis.

The sessions were held twice a week throughout the semester. In the sessions, all students had their cameras and microphones open. Ann used an external camera to project her writing for all students to see. Each session started with students' questions about the homework problems for that week. The students chose homework problems to discuss and shared their preliminary solution ideas.

We reviewed the recordings in a search for episodes where the discussions regarded SG and SC. In most of these episodes, the students sought Ann's help with either SG or SC. Then, our analysis was driven by the question, "*What (didactical) heuristics did Ann implement to assist her students with SG and SC?*" We chose the particular vignette due to the type of help the students sought and Ann's responses to these requests.

This vignette comes from the 8th tutorial (out of 25), whose topic was matrices. It began with proving the properties of the trace of a matrix using formal definitions (see Figures 1a and 1b).



Figures 1a (left) and its translation 1b (right): Screenshot previous to discussion below

After this, Gary asked to discuss the following problem from the homework assignment.

Problem: Given $A, B \in F^{n\times n}$, prove tr(AB)=tr(BA).

11	Gary:	I mostly got stuck with sigmas to prove it, I do not know how that will help.
12	Ann:	Let's try to write, OK? I write trace of AB [writes: "tr(AB)="]. Without justifications, OK? Start writing, OK? What's the trace of AB?
13	Zach:	sigma A B i i
14	Ann:	[writes: $\sum^{n}_{i=1} (AB)_{ii}$] And now, Gary, now what will be the next step?
15	Gary:	[Pause] I don't know. That's exactly where I got stuck.
16	Ann:	So, what is "*i i*" of AB, according to the definition? […] If I tell you that you have two matrices A B, what is *ii* of AB? [2 second pause] How do we find that element? How do you find the *ij* element of matrix multiplication?
17	Gary:	But isn't it dependent on other elements? That is exactly what I tried to see.
18	Ann:	[Writes $(AB)_{ij}$=] So, what is the *ij*-th element of AB according to the definition? Do you remember that?
19	Gary:	The product of the row times the product of the column.
20	Ann:	Correct. And how do we write that in a formal manner? [Pause]
21	Roy:	Sigma
22	Gary:	Sigma A_{ij} times B_{ji} [Ann writes $\sum_k A_{ik}B_{kj}$.]
24	Gary:	Yes. Exactly!
25	Ann:	This is the definition. This [outlining what is on board] is the definition of the *ij* element of matrix multiplication. It's a row times a column. But when we say row times column […]OK? This is the row of A [pointing] times the column of B [pointing] but this [what is written] is the formal definition. So, if we want the *ii* element, we will simply take the formula that is here. […].

The vignette opens with Gary self-reporting that he "got stuck" with the focal problem (see [11]), which we associate with an implicit request for help with SG. (Conceivably, he also needed assistance with SC, yet it seems unlikely that conventionalization concerned him while struggling to produce a solution.) The post-tutorial conversation with Ann suggested that her (didactical) heuristic for getting Gary "unstuck" (task) involved the gradual development of the formulas for $(AB)_{ii}$ and $(BA)_{ii}$ to show their equality (procedure). This is consistent with how Ann led the interaction (e.g., [12], [14]).

We note that Ann's heuristic involves the sigma notation and requires paying close attention to the multiple indexes involved in matrix multiplication. The use of formal definitions and notation is inseparable from SC. In other words, within this heuristic, the generation of a solution heavily relies on systematic implementation of routines with evident conventionalization affordances. For instance, consider [19-20], where Ann nudges Gary to "write in a formal manner" that an element of the product matrix is "the product of the row times the product of the column."

Two characteristics of how Ann implemented her routine are noteworthy. The first one pertains to its implicitness. True, on several occasions, we see Ann making explicit calls to write things down (e.g., [12]) and to use definitions (e.g., [16], [18]). Yet, we view these calls as meta-level utterances preparing the students for her next move, whereas the overall approach, suggesting that the road to SG involves routines with substantial conventionalization affordances, remains mostly unarticulated.

The second characteristic highlights Ann's recurrent attempts to build on Gary's contributions and open space for them. This is evident when Gary remarks that he "got stuck with sigmas" and Ann instantly nudges him to "start writing" them down (see [12]). We view this instructional move as a proposal of an atomic problem-solving routine that closely relates to Gary's one. The exchange also contains several instances where Ann requests Gary to perform a specific routine and then revises her request once she realizes Gary struggles to do so. For instance, in [14-16], she replaces the initial request to unpack $\sum_i (AB)_{ii}$ with the questions "what is *ii* of AB?" and "How do you find the *ij* element of matrix multiplication?" Later, when Gary refers to the *ij* element as "the product of the row times the product of the column," Ann endorses his statement before converting it into the formal $\sum_k A_{ik}B_{kj}$ (see [19-20] and [23]). Overall, these repeated connections and modifications provide opportunities for Gary to contribute to the generation of the solution-narrative that Ann led.

In another interaction, Ann both encourages students to make mathematical contributions to the interaction and explicates the value of SC. She encourages using conventional solution forms for generating different mathematical narratives that are potentially useful to solve a range of problems. In answer to Roy's question, "Do I have to do this move in order to get to the final answer?", Ann responds, "Not for a final answer…on a test or something, but for yourself it's worth it." This is consistent with her task of convincing him that the mathematics community holds on to its conventions not for sheer ritualistic reasons. This is one way in which Ann appears to bring the metarules of Roy's personal discourse closer to the more conventional ones.

Summary and concluding remarks

While learning tertiary mathematics has been studied broadly and widely, we argue that limited attention has been paid so far to students' challenges with problem solving. Coming from the

commognitive standpoint, we construed problem solving as a discursive activity, in which students find themselves in certain task situations. These require them to produce a shareable solution-narrative that will be endorsed by them as solvers and by a broader community of tertiary mathematics discourse, including instructors as the discourse experts. Consistently, we distinguished between routines students implement to bring a solution-narrative into being (SG) and those aimed at aligning the narrative with conventional metarules (SC).

Informed by Koichu (2019), our aim was to propose a conceptualization that is grounded in the discursive perspective on learning (Sfard, 2008). Operationally speaking, one key difference between Koichu's and our proposal regards the interdiscursive gap that we highlight between the community of solvers and a broader discourse community. As a result, the two communities are likely to subject solution-narratives to different metarules. One can argue that it is only natural to expect the solvers to account for the viewpoint of other people (cf. Harel & Sowder, 1998; Mason et al., 1982). Yet, let us recall that students are often newcomers to the discourse of tertiary mathematics, which makes them not yet familiar with the discourse conventions and norms. This interpretation can be viewed as a problem-solving version of the renowned learning paradox (cf. Sfard, 2008).

Research contends that systematic teaching of problem solving can be challenging (Badger et. al., 2012). This is somewhat consistent with the vignettes that we analyzed to learn about the instructor's teaching of SG and SC. In these, the students asked for assistance with either SG or SC, but the instructor did not confine herself to the requested realm. This suggests that SG and SC can be more intertwined in the problem-solving network of a versed discourse participant than newcomers imagine. The analysis also showed that, while implementing different didactical heuristics with a different degree of explicitness, the instructor mobilized both SG and SC to usher her students into the discourse of tertiary mathematics. With this modest first step, we hope to stimulate further research into how SG and SC are taught and learned in tertiary mathematics.

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An inquiry-oriented approach to least squares in linear algebra

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In this paper, we present a research-based task sequence for an inquiry-oriented approach to least squares for introductory linear algebra. The main goal of the task sequence is to engage students in the guided reinvention (Freudenthal, 1991) of the least squares equation and to help students understand why that equation is relevant to solving the least squares problem. The tasks can also introduce vector length, orthogonality, and projection. In this paper, we detail our approach to design research, summarize the task sequence, and share examples of student work.

Keywords: Linear algebra, least squares, inquiry, curriculum design.

Introduction

Introductory linear algebra courses frequently include the topic of least squares approximation. When $A\boldsymbol{x} = \boldsymbol{b}$ has no solution, the set of least squares solutions to $A\boldsymbol{x} = \boldsymbol{b}$ are the vector(s) $\widehat{\boldsymbol{x}}$ that are solution(s) to $A^TA\widehat{\boldsymbol{x}} = A^T\boldsymbol{b}$. One interpretation of the $\widehat{\boldsymbol{x}}$ is that they make $A\widehat{\boldsymbol{x}}$ as close as possible to $\boldsymbol{b}$, which is consistent with a subspace orientation to the problem (i.e., finding the best approximation in a subspace to a vector not in the subspace). How might instructors help students come to develop and understand the equation $A^TA\widehat{\boldsymbol{x}} = A^T\boldsymbol{b}$ and how it is relevant to solving the least squares problem?

This study is part of the Inquiry-Oriented Linear Algebra (IOLA) project (Wawro et al., 2013), which creates research-based curricular materials for the guided reinvention of core concepts within an inquiry-oriented linear algebra class. This design research is driven by making sense of student thinking and the complexity of mathematical ideas, which informs the refinement of the curricular materials. In this paper, we present a research-based task sequence for the guided reinvention of the least squares equation $A^TA\widehat{\boldsymbol{x}} = A^T\boldsymbol{b}$. The tasks can also introduce vector length, orthogonality, and projection. In the results section, we summarize the tasks and share examples of student work.

Theoretical Framing and Literature

Our instructional design work is based in Freudenthal's philosophy of mathematics as a human activity (1991) and the design principles that emerged from his work in Realistic Mathematics Education (RME) (Gravemeijer, 1999). RME design principles include didactical phenomenology, emergent models, and guided reinvention. Didactical phenomenology says "an instructional sequence meant to support the learning of a piece of mathematics should be situated in a context that can be productively organized by students using that piece of mathematics" (Larsen, 2018, p. 25), where 'organize' means to mathematize or make mathematical. The emergent models heuristic describes students' progression from an initial, intuitive understanding to a more formal and mathematized organization of the phenomena (Gravemeijer, 1999). Through engagement in a task sequence over several days, students progress from creating a model of the initial situation through activity, referring

back to this setting while extending, generalizing beyond the initial task setting, and using their new model in future activity by applying it in more formal, mathematically deductive ways towards exploring new phenomena. Finally, the guided reinvention heuristic is a mechanism for students' reinvention of mathematical ideas guided by the task sequence and instructor interactions. The result of this interconnected progression for the student is a new mathematical reality of interwoven mathematical objects and relations. More detail about our use of RME is given in Wawro et al. (2022).

Our instructional design theory also centralizes inquiry-oriented instruction (IOI) (Kuster et al., 2018) to guide student engagement in meaningful mathematics and student collaboration for sensemaking (Laursen & Rasmussen, 2019). IOI also relies on the foundational importance of instructors inquiring into students' mathematics as the instructor generates and builds on student reasoning, helps develops a shared understanding among all students, and formalizes language and notation (Kuster et al., 2018).

In our review of the literature, we found very few empirical studies related to the learning and teaching of least squares in linear algebra. Talbert (2014) presents findings that in an inverted linear algebra lesson (students watching videos outside of class that present material and working on examples in class), students performed well on the final exam solving an application problem using the normal equation $A^T A\hat{\boldsymbol{x}} = A^T \boldsymbol{b}$. Turgut (2013) presents resources designed to teach least squares using Mathematica. The lecture uses commands in Mathematica to plot dots, form matrices from points, transpose matrices, and take matrix inverses all in service of finding the line of best fit through a data set of points. Related to the teaching and learning of least squares is dot product. Donevska-Todorova (2015) identified three dot product definitions aligned with three modes of description (arithmetic, geometric, and abstract-axiomatic) and created an applet to promote students' geometric understanding of the dot product. Dray and Manogue (2006) found projection to be essential in understanding dot product, with the geometric approach benefiting physics and engineering students. In Wawro et al. (2023), we reported on students' use of literal symbols and equations when solving self-generated least squares problems based on the task sequence. We found students used unknowns and parameters in a variety of ways, and we highlighted how their use of dot product equations can be helpful towards supporting their understanding of the least squares equation.

Methods

Our methodological approach for this study was the Design Research Spiral (Wawro et al., 2022), which is composed of five phases: Design, Paired Teaching Experiment, Classroom Teaching Experiment, Online Working Group, and Web (see Figure 1), with revisions occurring between each phase as needed based on ongoing analyses of student thinking. In the Design Phase, we create a first draft of the task sequence. This is influenced by our conceptual analysis of relevant mathematics, our practice- and research-based knowledge of student reasoning in linear algebra, RME, and IOI. In the second phase, we use an adaptation of the paired teaching experiment methodology (Steffe & Thompson, 2000) to investigate pairs of students' interactions with these early task iterations. We use the students' mathematical reasoning during the interviews to inform revisions of the task sequence and instructional implementations. In phase three, we conduct a classroom teaching experiment (Cobb, 2000) to use the task sequence in a research team member's intact classroom setting. During implementation, the teacher-researcher responds to student thinking and revises the task sequence as

needed. These ad hoc preliminary analyses inform further task sequence revisions as well as the instructor implementation notes. In phase four, we work with participating instructors outside of our universities to test the task sequence in their own intact classrooms. We use online working groups (Fortune & Keene, 2021) with modified lesson study to understand instructors' implementation experiences and provide opportunities for reflection and collaboration among instructors. Our goals during this phase are to obtain feedback on tasks, formatting, and sequencing; we also learn how the task sequence played out in various classrooms with various student populations. These insights inform the development and refinement of instructor notes and the tasks themselves. During the Web phase, we add the finalized version of the task sequence and accompanying instructor support to the project website, which facilitates instructors using or adapting the materials for their own class. The task sequence shared in this paper has completed the first four phases of the design research spiral.

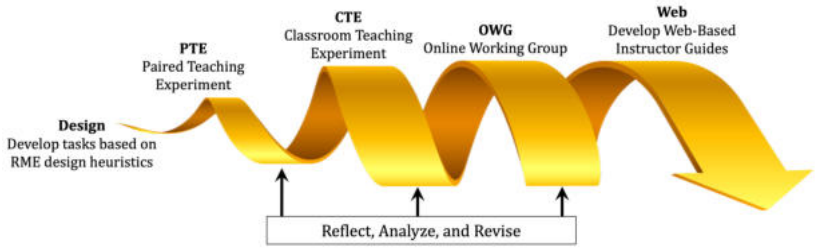


Figure 1: Model of the Design Research Spiral (Wawro et al., 2022)

The examples of student thinking shared in this paper come from a classroom teaching experiment that occurred in an in-person introductory linear algebra class at a large, public, research university in the Mid-Atlantic US in 2022. The course prerequisite was a passing grade in introductory calculus courses on differentiation, integration, sequences, and series. Most of the 27 students were second-year by credit hours and were general engineering majors. The data examples come from students' in-class work or written reflections (short after-class questions graded on effort not correctness).

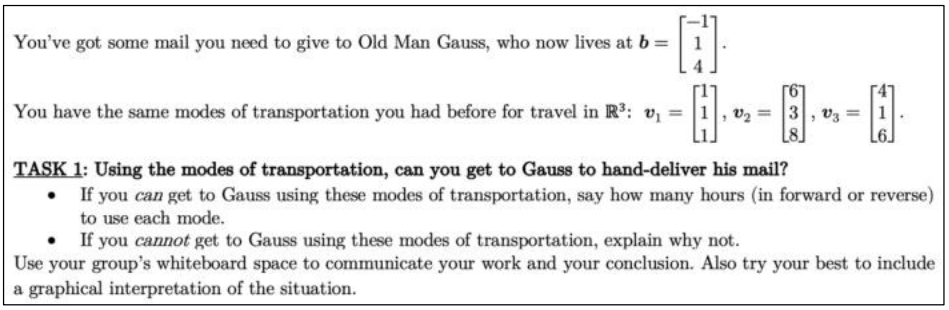
The Least Squares Task Sequence

The main goals of this task sequence are to engage students in the guided reinvention of the least squares equation $A^TA\hat{\boldsymbol{x}} = A^T\boldsymbol{b}$ and to help students understand why that equation is relevant to solving the least squares problem. In Task 1, students try to use three travel vectors in $\mathbb{R}^3$ to get to Gauss to deliver his mail but learn it is not possible. In Task 2, students find locations that they can get to with the travel vectors and determine the distances of those locations from home and from Gauss. In Task 3, students find the location that they can get to that is closest to Gauss and then use a drone to deliver his mail. They determine the location (which will be called $\boldsymbol{p}$), how to get there (called $\hat{\boldsymbol{x}}$), along what vector the drone travels (called $\boldsymbol{e}$), and the length of the drone's trip (which is $\|\boldsymbol{e}\|$). In Task 4, students combine key equations developed during Task 3 to create one general equation to determine $\hat{\boldsymbol{x}}$ no matter what A and $\boldsymbol{b}$ are, which culminates in the equation $A^TA\hat{\boldsymbol{x}} = A^T\boldsymbol{b}$.

Task 1: Delivering Mail to Gauss

The main goal of Task 1 (see Figure 2) is for students to determine if Gauss can be reached using the given travel vectors. It is intended as a review of prior content. Through Task 1, students refresh and solidify their understanding of the equivalence of all linear combinations of a set of vectors, the span

of that set, and a geometric space of all locations that can be reached by that set. They determine that $\boldsymbol{b}$ (the vector denoting Gauss's location) is not a linear combination of $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ (the vectors denoting the modes of transportation), that $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ is linearly dependent and spans a plane in $\mathbb{R}^3$, and $\boldsymbol{b} \notin span\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$; thus, they cannot get to Gauss (see Figure 3 for examples of student work).


You've got some mail you need to give to Old Man Gauss, who now lives at $\boldsymbol{b} = \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix}$.

You have the same modes of transportation you had before for travel in $\mathbb{R}^3$: $\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\boldsymbol{v}_2 = \begin{bmatrix} 6 \\ 3 \\ 8 \end{bmatrix}$, $\boldsymbol{v}_3 = \begin{bmatrix} 4 \\ 1 \\ 6 \end{bmatrix}$.

TASK 1: Using the modes of transportation, can you get to Gauss to hand-deliver his mail?

- If you *can* get to Gauss using these modes of transportation, say how many hours (in forward or reverse) to use each mode.
- If you *cannot* get to Gauss using these modes of transportation, explain why not.

Use your group's whiteboard space to communicate your work and your conclusion. Also try your best to include a graphical interpretation of the situation.

Figure 2: Task 1 - Delivering mail to Gauss

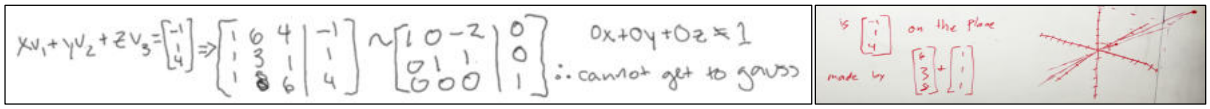


Figure 3: Examples of student work for Task 1

Our rationale for creating this task was to allow students to leverage not only understandings they developed earlier in the course, but also a familiar scenario for the situational activity, an aspect of RME design heuristics. The travel vectors $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ appear in the IOLA "Magic Carpet Ride" task sequence (Wawro et al., 2012), so knowing they are linearly dependent and span a plane is familiar; the new situation is reasoning about and making geometric sense of $x\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + y\begin{bmatrix} 6 \\ 3 \\ 8 \end{bmatrix} + z\begin{bmatrix} 4 \\ 1 \\ 6 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix}$ having no solution. Tasks 2-3 use only $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$, but starting with a linearly dependent set is useful because it sets up exploring least squares problems for linearly independent and dependent sets in Task 4.

Task 2: Getting Close to Gauss

The main goal of Task 2 (see Figure 4) is to leverage students' experience with distance between points and the distance formula as they explore the length of vectors and the distance between vectors, both in the travel scenario and in general. In Task 2, students explore that different locations on the plane (created by the span of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$) are farther or closer to a fixed location off the plane (Gauss, denoted $\boldsymbol{b}$), so it serves as a good foundation before students are asked in Task 3 to find the closest location on the plane to Gauss. As they work on Task 2, some students may know the distance formula from other math courses. An instructor can leverage these student contributions to help formalize with the class that: the length of a vector $\boldsymbol{u}$ in $\mathbb{R}^3$ can be thought of as the distance between the origin and the point defined by the tip of the vector, namely $\|\boldsymbol{u}\| = \sqrt{(u_1 - 0)^2 + (u_2 - 0)^2 + (u_3 - 0)^2} = \sqrt{u_1^2 + u_2^2 + u_3^2}$; the free vector $\boldsymbol{w}$ in $\mathbb{R}^3$ that connects the tips of two fixed vectors $\boldsymbol{u}$ and $\boldsymbol{y}$ can be determined by creating the difference vector $\boldsymbol{w} = \boldsymbol{u} - \boldsymbol{y}$; and the length of $\boldsymbol{w}$ can be calculated as the distance between $\boldsymbol{u}$ and $\boldsymbol{y}$, i.e., $\|\boldsymbol{w}\| = \|\boldsymbol{u} - \boldsymbol{y}\| = \sqrt{(u_1 - y_1)^2 + (u_2 - \mathrm{y})^2 + (u_3 - y_3)^2}$. Students should be encouraged to also model the situation geometrically and to begin build a sense of how changing the location of a point on the plane changes the distance between this point and Gauss.

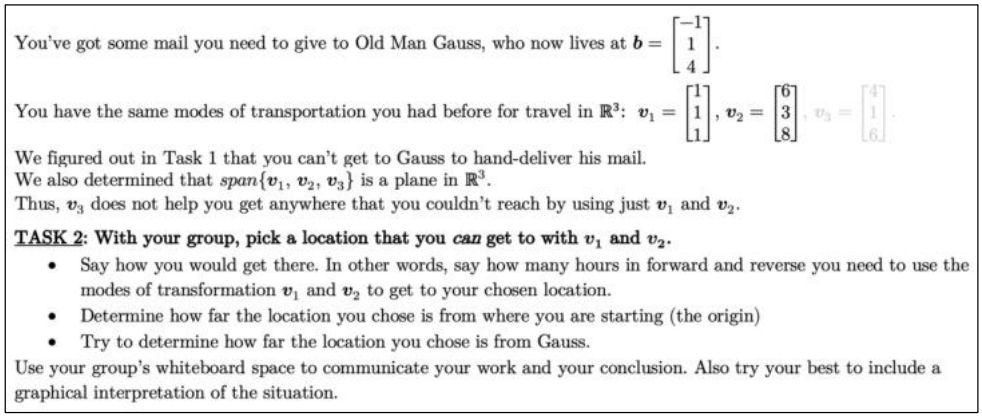


Figure 4: Task 2 - Getting close to Gauss

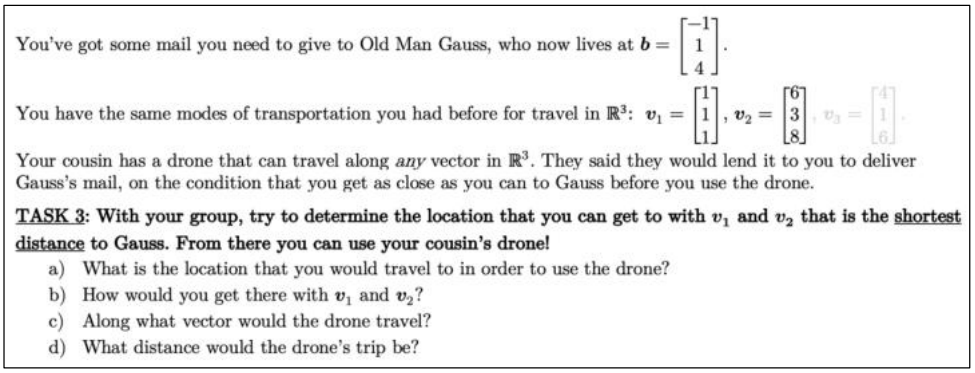
Task 3: Delivering Mail to Gauss with a drone

In Task 2, students determined that the locations they can get to with travel vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ lie on a plane and have different distances to Gauss. In Task 3 (see Figure 5), students determine which location on that plane is *closest* to Gauss so that they can send a drone to Gauss from there. We designed the four questions in Task 3 to bring out four of the main unknown variables in the important relationships at play in least squares approximation. Question (a) asks where you should go to use the drone, the location on the plane closest to Gauss. We encourage instructors to call this location vector $\boldsymbol{p}$ so that they can later tag $\boldsymbol{p}$ as the projection of $\boldsymbol{b}$ onto the plane, if they so wish. Question (b) asks how to get to $\boldsymbol{p}$ with $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$, meaning the scalar values x_1 and x_2 in $x_1\boldsymbol{v}_1 + x_2\boldsymbol{v}_2 = \boldsymbol{p}$. Students will later recognize x_1 and x_2 as the components of $\hat{\boldsymbol{x}}$, the solution to the least squares problem. Question (c) asks for the vector along which the drone should travel. We suggest instructors call this vector $\boldsymbol{e}$ to be consistent with its conventional name, the error vector, if instructors choose to call it that after Task 4. We created the overall problem to be about travel from the plane to Gauss, so the situation matches the vector equation $\boldsymbol{p} + \boldsymbol{e} = \boldsymbol{b}$. Finally, question (d) asks for the distance of the drone's trip, which is $\|\boldsymbol{e}\|$, which could later be labeled in Task 4 as the least squares error.

We recommend students first discuss the main Task 3 prompt in small groups to share any initial thoughts they have about where the location closest to Gauss would be. The purpose is for students to have a chance to suggest that the vector that is *orthogonal* or *perpendicular* to their travel plane will be the one that is the closest to Gauss. In whole class discussion, the instructor can then leverage that intuition to establish that if $\boldsymbol{e}$ is orthogonal to the plane, then $\boldsymbol{e}$ is also orthogonal to $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, and $\boldsymbol{p}$. This is a good time for the class to establish a connection with dot product, that orthogonality implies $\boldsymbol{v}_1 \cdot \boldsymbol{e} = 0$, $\boldsymbol{v}_2 \cdot \boldsymbol{e} = 0$, and $\boldsymbol{p} \cdot \boldsymbol{e} = 0$. The relationships needed to complete Task 3 are now in place for students to work in their small groups to solve for $\boldsymbol{p}$, x_1 and x_2, $\boldsymbol{e}$, and $\|\boldsymbol{e}\|$ (the answers to questions (a)-(d), respectively). These relationships appear in the equations $x_1\boldsymbol{v}_1 + x_2\boldsymbol{v}_2 = \boldsymbol{p}$ (or $A\boldsymbol{x} = \boldsymbol{p}$ for matrix A with columns $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$), $\boldsymbol{p} + \boldsymbol{e} = \boldsymbol{b}$, $\boldsymbol{v}_1 \cdot \boldsymbol{e} = 0$, $\boldsymbol{v}_2 \cdot \boldsymbol{e} = 0$, and $\boldsymbol{p} \cdot \boldsymbol{e} = 0$ and lay the foundation for the guided reinvention of the least squares equation, which occurs in Task 4.

For space, we share one example of student work on Task 3 (see Figure 6). After writing the known relationships, the student wrote $\boldsymbol{v}_1 \cdot \boldsymbol{e} = 0$ and $\boldsymbol{v}_2 \cdot \boldsymbol{e} = 0$ as a system of equations and an augmented matrix. He solved for $\boldsymbol{e}$ as a vector parameterized by t. He then used $\boldsymbol{b} - \boldsymbol{e} = \boldsymbol{p}$ and $\boldsymbol{p} \cdot \boldsymbol{e} = 0$ to

solve for $\boldsymbol{e}$ and then for $\boldsymbol{p}$. He then substituted $\boldsymbol{p}$ into $A\boldsymbol{x} = \boldsymbol{p}$ to solve for x_1 and x_2. He completed the task by finding $\|\boldsymbol{e}\|$. We note that his augmented matrix is a nice aspect for instructors to point out and symbolize as the matrix equation $A^T\boldsymbol{e} = \boldsymbol{0}$, which is central in Task 4.



You've got some mail you need to give to Old Man Gauss, who now lives at $\boldsymbol{b} = \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix}$.

You have the same modes of transportation you had before for travel in $\mathbb{R}^3$: $\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\boldsymbol{v}_2 = \begin{bmatrix} 6 \\ 3 \\ 8 \end{bmatrix}$, $\boldsymbol{v}_3 = \begin{bmatrix} 4 \\ 1 \\ 6 \end{bmatrix}$.

Your cousin has a drone that can travel along *any* vector in $\mathbb{R}^3$. They said they would lend it to you to deliver Gauss's mail, on the condition that you get as close as you can to Gauss before you use the drone.

TASK 3: With your group, try to determine the location that you can get to with $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ that is the shortest distance to Gauss. From there you can use your cousin's drone!

a) What is the location that you would travel to in order to use the drone?
b) How would you get there with $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$?
c) Along what vector would the drone travel?
d) What distance would the drone's trip be?

Figure 5: Task 3 - Delivering mail to Gauss with a drone

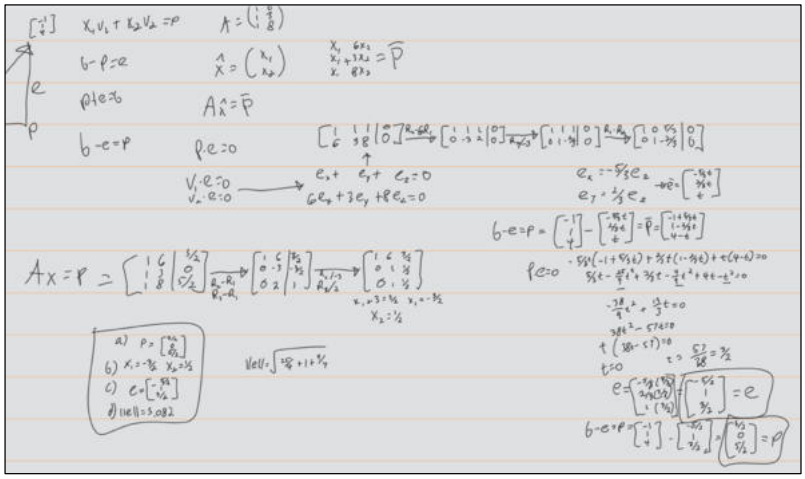


Figure 6: Example of a student's work for Task 3

Task 4: Synthesizing into a general approach

The main goal of Task 4 (Figure 7) is for students to create the least squares equation $A^TA\hat{\boldsymbol{x}} = A^T\boldsymbol{b}$ that allows them to determine $\hat{\boldsymbol{x}}$ no matter what A and $\boldsymbol{b}$ are. As the culmination of the RME-inspired guided reinvention task sequence, this general equation grows out of students' inquiry and more specific work in Tasks 1-3. Task 4 tends to be quick for students in their small groups as they leverage their algebraic substitution skills. For instance, we have seen students substitute $\boldsymbol{e} = \boldsymbol{b} - \boldsymbol{p}$ into $A^T\boldsymbol{e} = \boldsymbol{0}$ yielding $A^T(\boldsymbol{b} - \boldsymbol{p}) = \boldsymbol{0}$, and then substitute $A\hat{\boldsymbol{x}} = \boldsymbol{p}$ in for $\boldsymbol{p}$ yielding $A^T(\boldsymbol{b} - A\hat{\boldsymbol{x}}) = \boldsymbol{0}$. Students might stop there; the instructor can discuss a further simplification to the more common $A^TA\hat{\boldsymbol{x}} = A^T\boldsymbol{b}$**.** The instructor brings together the class's collective work on the entire task sequence by formalizing new terms and definitions; this is in line with an instructor's role in inquiry-oriented instruction to build on student reasoning and formalize language and notation (Kuster et al., 2018). In particular, the instructor can lead a discussion that emphasizes the more general and formal, namely that when no solution to $A\boldsymbol{x} = \boldsymbol{b}$ exists, we can know the best approximation $\boldsymbol{p}$ for $\boldsymbol{b}$ in the subspace spanned by the columns of A by finding what $\hat{\boldsymbol{x}}$ produces the vector $A\hat{\boldsymbol{x}} = \boldsymbol{p}$ in the subspace that is

closest to $\boldsymbol{b}$. The instructor also introduces new terms to complete the guided reinvention, namely: $\hat{\boldsymbol{x}}$ is the least squares solution for $A\boldsymbol{x} = \boldsymbol{b}$, the set of least squares solutions $\hat{\boldsymbol{x}}$ of $A\boldsymbol{x} = \boldsymbol{b}$ coincides with the nonempty set of solutions of $A^T A\hat{\boldsymbol{x}} = A^T\boldsymbol{b}$, $\boldsymbol{p}$ is called the projection of $\boldsymbol{b}$ onto the subspace, and $\|\boldsymbol{e}\| = \|\boldsymbol{b} - A\hat{\boldsymbol{x}}\|$ is called the least squares error. As a follow-up, students can use all three modes of transportation $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ from Task 1 to form A and solve for the least squares solution $\hat{\boldsymbol{x}}$ of the updated $A\boldsymbol{x} = \boldsymbol{b}$, as well as $\boldsymbol{p}$ and $\boldsymbol{e}$. Students will find that $\hat{\boldsymbol{x}}$ is no longer unique but $\boldsymbol{p}$, $\boldsymbol{e}$, and $\|\boldsymbol{e}\|$ are. Instructors could then derive that $\hat{\boldsymbol{x}}$ is unique if and only if the columns of A are linearly independent.

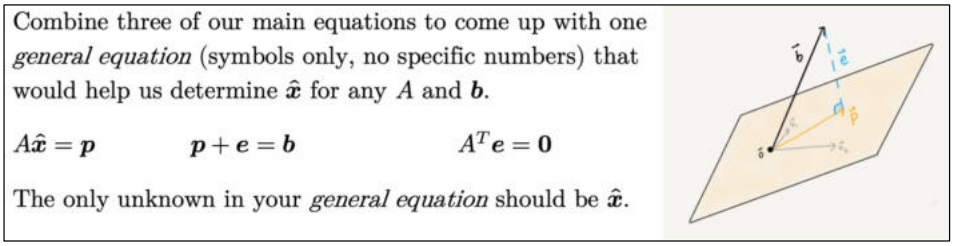


Figure 7: Task 4 - Synthesizing into a general approach

Conclusion

In this paper, we summarized a new task sequence for an inquiry-oriented approach to least squares in linear algebra. We used RME (Freudenthal, 1991), Inquiry-Oriented Instruction (Kuster et al., 2018), and design research (Wawro et al., 2022) to guide our task design and implementation. The sequence uses a subspace orientation to least squares, so that students' graphical intuition with planes and vectors can be leveraged throughout their work on the tasks and beyond. Our work contributes to the limited research on teaching innovations for least squares, as well as on how students in mathematics, engineering, and the sciences make sense of concepts involved in learning least squares in linear algebra. Our future work includes completing the Web phase of the design research spiral, as well as conducting fine-grained analyses from our data of student reasoning about least squares.

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Interweaving university and school mathematics: situated learning opportunities at the transition from university to teaching at school

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The perceived discontinuity between studying university mathematics and teaching mathematics at school is well-known as a challenge in teacher education. While there have been several attempts to resolve this, it remains unclear how teachers can develop the necessary subject matter knowledge for teaching and how they can learn to apply it. Our research project addresses the design and examination of situated learning opportunities for pre-service teachers called 'interweaving problems', which are currently investigated in several cycles of implementation. This paper introduces key design criteria derived from theory, including guiding questions to scaffold the process. We present data from the second cycle and a coding rubric developed to analyse how pre-service teachers draw on university mathematics to inform teaching decisions. We discuss patterns that emerge and possible consequences for the adaptation of specific scaffolding to enhance support.

Keywords: University mathematics, geometry, secondary teacher education, double discontinuity, subject matter noticing.

Introduction

The transition from studying mathematics at university to teaching it at school has been recognised as problematic for a long time, since teachers often fail to see connections between university mathematics and school practice (e.g., Klein, 2016). Our project aims to design learning opportunities that enable pre-service teachers (PSTs) to examine such connections and practice how to make use of them in typical tasks of teaching. The accompanying research is designed to provide detailed insight into how PSTs engage with these learning opportunities. After each semester of implementation, the learning opportunities are adapted based on this research.

Our project works within the context of a geometry course exclusively for pre-service secondary mathematics teachers at the University of Heidelberg. The course, situated in the Master's program, is the last university mathematics class of their education. Each week, PSTs are expected to work on and hand in solutions to a number of exercise problems in groups, which are afterwards discussed in a tutorial. The exercise problems are typically focused on university mathematics, but the goal of our project is to develop problems meant to connect university mathematics and school mathematics. Eleven such problems have been developed and implemented in the course twice, with one cycle of re-design between implementations (Weith et al., 2024), and two more implementations will follow. In this paper, we will outline design criteria for the development of the problems, and present the development of a category-based coding rubric for analysing written solutions to them, which were collected during the second implementation cycle.

Theoretical background

While there is consensus that a profound knowledge of the subject matter is essential for teaching mathematics, studies show repeatedly that PSTs perceive the university mathematics taught in their university education as not particularly relevant to their profession (e.g., Gueudet, 2008; Prediger, 2013; Eichler & Isaev, 2022). There are several theoretical models aiming to characterise the subject matter knowledge teachers need in their profession. Wasserman and colleagues (2023[1]) describe the processes of connecting school mathematics to one's knowledge of university mathematics and letting this inform a teaching decision as *building up from practice* and *stepping down to practice.* Another theoretical approach is that of *school-related content knowledge* (SRCK; Dreher et al., 2018), which is conceptualised in three facets: *bottom-up, top-down,* and *curricular.* While the curricular facet is concerned with the connections within the school curriculum, the other two share similarities with what Wasserman calls *building up* and *stepping down*. SRCK, however, takes a particularly situated approach by giving a number of exemplary questions for each facet that reflect typical professional demands for teachers. Examples include: *Which mathematical ideas might be reflected by a student's remark in class?* (bottom-up perspective); *How can a certain mathematical idea be reduced for teaching purposes in the school context?* (top-down perspective) (ibid., p. 331).

Although such knowledge has been at the focus of a number of successful interventions, there is still little insight into how it is actually acquired, and how it can be made accessible when needed in typical tasks of teaching, such as selecting and adapting learning material or responding to students' questions (e.g., Prediger, 2013; Wasserman et al., 2023). One possible approach is to develop PSTs' subject matter noticing, i.e., their ability to identify and interpret relevant subject matter aspects in a teaching situation to inform teaching decisions (e.g., Sherin et al., 2011). This can be achieved by providing learning opportunities that approximate practice through realistic teaching situations, which is considered beneficial in academic mathematics classes for PSTs (e.g., Prediger, 2013). Providing PSTs with representations of practice is assumed to be especially conducive to the acquisition and restructuring of professional knowledge (Kirwan et al., 2023). It has also been observed that PSTs often perceive the relevance of university mathematics to their profession as very low, which may inhibit their knowledge acquisition and development of subject matter noticing (Sebők, 2023). Exemplifying situations where knowledge of university mathematics can directly benefit teachers may be a way of improving this perception.

Development of interweaving problems as situated learning opportunities

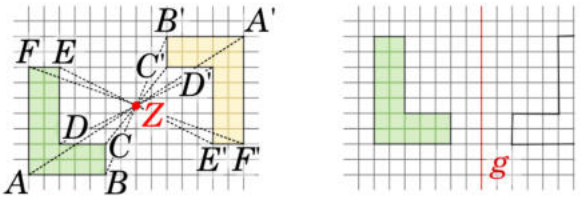
This research project aims to develop learning opportunities where PSTs not only examine meaningful connections between university and school mathematics, but also use this insight to inform teaching decisions. We have come to call this *interweaving* university and school mathematics*,* and the learning opportunities *interweaving problems.* The development process began with the choice of mathematical content; both the university course and the school curriculum were

[1] This reference provides a current and extensive literature survey, and therefore a comprehensive overview of the state of the art.

analysed for pieces of mathematical content that allow for connections that go beyond a superficial recognition of similarities. These were then embedded in text-image vignettes that represent typical situations in a teacher's job where engaging with the mathematics behind the situation can benefit the teacher's instructional decisions. The exemplary questions from the SRCK construct were used as orientation, as they reference exactly such situations. The vignettes were constructed in such a way that answering one or several of the SRCK questions would lead the teacher to a more well-informed judgment or reaction. Lastly, a number of questions was developed for each situation as guidance for the PSTs to productively work on the interweaving problems: asking them to identify and interpret particular aspects of the situation, explicitly connect and compare university and school mathematics, or use their mathematical insight to inform a teaching decision[2]. The finalised interweaving problems consisted of one text-image vignette each along with the corresponding guiding questions, and the PSTs were tasked with solving one problem per weekly exercise sheet.

Figure 1 shows the vignette used in one interweaving problem, presented as the fourth of eleven problems, on the topic of reflections. It is designed so that answering the following SRCK questions can facilitate meaningful engagement: *Which mathematical ideas might be reflected by a student's remark in class?* (bottom-up) and *How can a certain mathematical idea be reduced for teaching purposes in the school context?* (top-down) (Dreher et al., 2018, p. 331)

You are a math teacher teaching a unit on reflections with your 5th grade class. In the past couple of lessons, you have introduced reflections across an axis and reflections across a point.



Baum, M. et al. *Lambacher Schweizer 5 Baden-Württemberg*. Klett, 2014. 70/82. (nachgezeichnet)

One student comes up to you and shows you these two pictures. She says: „I dont't understand why the reflection across a point is called a reflection. When we did the reflection across an axis, it was an L in the beginning, and afterwards the L was the wrong way around. But with the reflection across a point, the green one is an L, and the yellow one is also an L. It's upside down, but it's not really the wrong way around!"

Figure 1: example situation at the basis of one interweaving problem (translated from German)

A possible way to interpret this situation from a subject matter perspective might look like this: The student has recognised that in one reflection, the L shape keeps its *orientation*, while in the other it changes. Both reflections are *motions* (i.e., isometric bijections) of the Euclidean Plane. The reflection across the axis flips orientation, while the point reflection preserves it, as it is actually a rotation by 180° around point Z, or can be seen as the composition of two reflections across perpendicular axes.

[2] In the latest (the second) iteration of the interweaving problems, these guiding questions were quite detailed, as the evaluation of the first iteration indicated that they had been too broad. (Weith et al., 2024)

In this sense, the point reflection is not strictly a reflection. It does resemble a reflection, however, in that it is its own inverse within the group of motions of the Euclidean Plane, which is a property that only reflections across axes and point reflections share. These considerations can then be used to answer the SRCK questions above and thereby inform the teacher's instructional decision in several ways: The teacher might assess that the student has made quite an astute observation, even though she cannot put the correct words to it, and encourage her for this. They might try to help specify her thoughts on orientation by visualising how in the second picture, the L shape would need to be flipped over once to get to the other side, whereas in the first picture, it would need to be flipped over twice (composition of reflections) or not at all and rotated around the centre instead (rotation).

The following guiding questions were provided. They are being grouped here into four thematic sections that we derived from the theoretical considerations outlined above and also used in the data analysis. Please note that this is not a strict separation, as the guiding questions are meant to scaffold a thought process that can in reality be quite fluid and dynamic. (The section headlines were not presented to the PSTs.)

Section 1 examines the relevance of orientation to this situation, and explicitly ties it to the course.

- What do you think the student means by "the wrong way around"? What difference between the reflections is she trying to describe?
- Which motions of the Euclidean Plane can be used to describe the reflection across an axis and the point reflection? How does the difference between the reflections that is relevant here show in their representation as motions of the plane?
- As what other type of motion can the point reflection be described?

Section 2 takes a look back at the student's statement and examines it in light of the previous mathematical considerations. This corresponds to answering the SRCK question *Which mathematical ideas might be reflected by a student's remark in class?*

- Read the student's statement again. How would you assess it after the previous considerations?

Section 3 discovers why the point reflection can still meaningfully be described as a reflection.

- Why are reflections across axes and points both considered reflections in school mathematics? For this, think about how students need to proceed when constructing the reflections.

Section 4 finally works towards a teacher's response to the student, engaging with the SRCK question *How can a certain mathematical idea be reduced for teaching purposes in the school context?*

- Which results of your considerations should be recognisable in your response to the student?
- What would your response to the student be? Choose appropriate examples, representations, etc.

Research interest and research question

As outlined above, the main research interest of the project is to develop and examine PSTs' subject matter noticing, i.e., their ability to engage with the mathematics of a teaching situation and make this directly inform an instructional decision. In this paper, we focus on written solutions to one interweaving problem, where successful subject matter noticing would be expected to appear as meaningful references to mathematical considerations in those sections that work towards a teaching

decision (here, sections 2 and 4). Accordingly, we address the following research question: To what extent and in which ways do PSTs draw on previous considerations on university mathematics when working towards a teaching decision (while engaging with an interweaving problem)?

Data collection and methods

A total of 48 PSTs, all of which gave their consent for analysis of the generated data, worked on one interweaving problem per week in 12 fixed groups of three to five people. Those groups handed in a written solution to each problem, resulting in more than 100 handed-in solutions overall. In this paper, we focus on the 12 solutions to the interweaving problem presented above (see Figure 1) to illustrate the process of analysis and development of a coding manual. The coding was done by the first author, following the guidelines of qualitative content analysis (Mayring, 2021). Coding units were generated as context units by dividing each solution into the four thematic sections outlined above. This resulted in a total of 48 coding units. Each section was then analysed from a specific perspective: Sections 1 and 3, being concerned with examining the mathematics of the situation, were deemed the only ones where the notion of correctness can be applied. The solutions were analysed as to whether the relevant questions (*What is the role of orientation?* and *Why does it still make sense to call both of these motions 'reflections'?* respectively*)* were correctly answered, and if so, how elaborately. Based on this, each coding unit was sorted into one of four categories (see Figure 2, left):

Category	Description
f	Entirely incorrect approach
c-	Correct approach, but elaborated to an inadequate degree
c	Correct approach and elaborated to an adequate degree
c+	Correct approach and elaborated particularly well

Category	Description
0	no discernible reference back to mathematical considerations at all
1	only superficial references
2	mathematically meaningful references are recognizable, but they are not directly used to benefit a teaching decision
3	meaningful references that directly inform and benefit the teaching decision

Figure 2: Coding categories for sections 1 and 3 (left) and sections 2 and 4 (right)

Sections 2 and 4 were examined for the quality of references to mathematical considerations in previous sections and for whether they were used to directly inform a teaching decision. The categories were inferred from the written solutions based on SRCK theory as outlined above and refined inductively during the coding process. Figure 2 shows the categories that resulted from this process; illustrating sample answers will be provided in the following section.

Findings

This section will focus on the coding process of sections 2 and 4, since the most meaningful insight towards answering the research question can be gained there. Sections 1 and 3 were still an integral part of this process because the coding focused on references between the different sections. We would like to provide some insight into the categories by showing and commenting on a number of illustrative excerpts from handed-in solutions. All excerpts were taken from section 4 for better comparability, and all were coded in German and then translated into English by the first author.

C**ategory 0** was only awarded once (group B, section 4), but – differently than expected – not due to a lack of mathematical substance, but to a major misalignment between sections: The mathematical

ideas referenced were entirely different from the ones mentioned in previous sections. Solutions awarded **category 1** typically referenced previous considerations mainly for superficial decisions. The PSTs in the following excerpt (group I), for instance, were only concerned with what language, definitions, and notation to use, while there was no deeper engagement with the mathematics:

> Terms to use: rotation, congruent, (orientation-preserving); Fifth-graders do not know matrix notation yet. […] The definition of motions by their set of fixed points is not useful here, either.

In contrast, **category 2** was mostly characterised by mathematically meaningful and correct references which were, however, not incorporated to directly benefit the teaching decision.

> You can reach the same result [as the point reflection] with several reflections across different axes. Looking at where the two axes intersect, you can see that it is exactly the reflection point.

There clearly is mathematical substance recognisable in this solution (group F). It is mathematically correct and directly draws on previous considerations. However, this is not being used to actually answer the student's question. The explanation neither highlights why the point reflection preserves orientation, nor why it is still called a reflection. The following excerpt from group J, with meaningful references that were beneficially tied into the teaching decision, resulted in **category 3**:

> An L remains an L, but it is upside down. The orientation stays the same. […] That is why it does not look "the wrong way around" to you. You can also picture a point reflection like this. You cut out the L and push it around, without ever lifting it from the table. […] Can you do the same for the reflection across an axis, without the L leaving the table?

The student's question is directly addressed, and the correct mathematical terminology is introduced. Furthermore, the previous mathematical considerations are clearly being used to create a visual argument that highlights the relevant mathematics while being accessible to the student. Another shared characteristic of several solutions sorted into category 3 was a systematic and reflected line of argumentation in the choice of content for the response. For instance, this excerpt from group L:

> Some [motions] preserve orientation, some do not → that is why the student recognises a difference between the reflected L shapes […] We are also leaving out differences regarding the sets of fixed points, as they distract from the essential and might confuse the student. We consider [the choice of a coordinate system, knowledge about matrices and the group of motions, and differences in the sets of fixed points] background knowledge for the teacher, […] but it is not necessary or even possible to communicate these concepts to the student. However, we consider it a good opportunity to revisit reflections in upper secondary and explain the bigger picture with the knowledge acquired in the meantime.

It is recognisable that the group engaged with their previous considerations with the clear goal of achieving an ideal response to the student. They reflected back on all the major ideas from their previous considerations and made choices based on what is best suited to the specific situation. They also incorporated curricular considerations and proposed to expand on the foundational understanding built here a few years later with more elaborate mathematical means, which the students will know by then. What else is notable in this case is that group L was awarded the code c- in section 1, indicating a significantly lacking solution there.

Discussion

This paper aims to outline the design of a new type of exercise problem for the university mathematics education of PSTs, and to present the process of analysing written solutions to such problems. While several studies have used vignettes in PSTs' university mathematics education (e.g., Prediger, 2013), the vignettes developed in our project provide PSTs with explicit scaffolding for interweaving university and school mathematics, a process that has often been identified as difficult (e.g., Sebők, 2023). The scaffolding is designed to enable the integration and restructuring of knowledge related to university and school mathematics (Kirwan et al., 2023), which is critical for mastering the typical tasks of teaching represented in our vignettes. Therefore, the project investigates PSTs' engagement with the vignettes to gain deeper insights into related learning processes. We consider this an inevitable step to improve our understanding of PSTs' learning in this context before existing approaches can be further up-scaled (cf. Wasserman et al., 2023).

Even though it remains to be examined if the developed coding rubric can be applied to the entire data set, findings are promising and the results provide first answers to the research question. Data analysis shows that the solutions vary greatly in quality, which aligns with the assumption that these types of processes are challenging and new to our PSTs. However, a number of patterns, presented here as the different categories, are already recognisable: It is promising that category 0 was only awarded once, and not for lack of mathematical substance, but of alignment to previous sections. This indicates that PSTs mostly manage to draw at least some connections to their mathematical considerations and that the provided scaffolding might support this process. The case of group L seems counterintuitive at first, since they managed to connect university mathematics to their teaching decisions very well although their previous considerations on those mathematics had not been very elaborate. One possible explanation might be that they did have some deeper mathematical understanding of the situation and were able to connect it to their teaching decision (section 4), but they did not consider it relevant when explicitly engaging with the mathematics (section 1). An in-depth analysis of solutions of this type might be warranted in the context of our research interest.

For the purpose of adapting the interweaving problems for the next semester cycle, occurrences of category 2 are of particular interest. They indicate that the PSTs are able to examine the mathematics behind a situation, but then struggle to tie this into their teaching. When more extensive findings from the coding are available, we would expect the prevalence of this category to decline throughout the semester, as applying university mathematics knowledge in a school context is a new activity for our PSTs and we expect them to become more familiar with it over time. Therefore, insight into what characterises solutions from this category is crucial to decide if and how the scaffolding can be adapted in future iterations to ideally support this activity early on in the semester.

Our research complements previous studies (e.g., Sebők, 2023) by expanding the corpus of vignette-based PSTs' university mathematics education, and showing that interweaving mathematics is a challenging endeavour. However, our vignettes are designed in a way that particularly guides interweaving processes, and accordingly, the collected data may provide insight on a more detailed level, which is needed to better understand PSTs' learning in this context and provide suitable learning opportunities. The coding rubric presented in this paper has mainly been derived from the

analysis of one interweaving problem. In a next step, it will be applied to the analysis of the other interweaving problems and adapted, if necessary, also accounting for intercoder reliability.

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Exploring University Students' Mathematical Digital Competency for Communication in Computer-Supported Collaborative Learning

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Keywords: Mathematical digital competency, computer-supported collaborative learning, mathematical communication, digital tools, university mathematics education.

Introduction

The landscape of mathematics education at UK universities has undergone significant transformation in recent decades, driven by advancements in technology. This shift has given rise to the concept of Mathematical Digital Competency (MDC), which highlights the interconnected nature of mathematical learning and digital tool proficiency (Geraniou & Jankvist, 2019). However, the incorporation of technology into mathematics education presents challenges, particularly in selecting tools that align with specific learning objectives. Students' collaboration in mathematics learning adds a layer of both challenges and opportunities, further complicating the integration. Research on Computer-Supported Collaborative Learning (CSCL) highlights the importance of collaboration in enhancing learning outcomes when using digital tools (e.g., Dobie & MacArthur, 2022). Therefore, the focus of this study is on: How do university students employ MDC when communicating mathematically within a CSCL context?

Theoretical Foundation

The present study aims to develop a framework to analyse how students communicate mathematically online, building on existing analytical constructs in both MDC and CSCL. The Theory of Knowledge-Creating Learning (TKCL) (Paavola & Hakkarainen, 2014), which emphasizes the collaborative and individual creation of knowledge through interactions with shared epistemic objects, was chosen for CSCL. The MDC framework, focusing on the interactions between tool mediation and the development of individual mathematical competency, was also considered. We utilize the networking strategy (Bikner-Ahsbahs & Prediger, 2009) that primarily focuses on "comparing & contrasting" these theoretical approaches to identify their compatibility and complementarity, followed by "combining & coordinating" them as an initial step to create a cohesive framework. To enhance complementarity with TKCL, we also consider the two foundational theoretical approaches in developing the MDC framework (Geraniou & Jankvist, 2019), namely the KOM framework (Niss & Højgaard, 2019) for defining mathematical competence through eight interwoven competencies, and the Theory of Instrumental Genesis (Drijvers et al., 2013) which highlights tool mediation in human activity through three dualities. This combination ensures a balanced analysis of both social and individual dimensions in CSCL, allowing for a deeper understanding of how students interact with digital tools in mathematical contexts.

Method and Preliminary Outcomes

This study employs a qualitative research design conducted in the UK. Students' self-organized CSCL groups focusing on mathematics learning were sampled. We select an example group consisting of two 1st-year students: one majoring in Mathematics and the other in Engineering. Data collection methods include learning analytics and observational recordings of their online interactions, in which students work on mathematical problems from the Mathematics Stack Exchange Forum, assisted by ChatGPT. We use reflexive thematic analysis (Braun & Clarke, 2022) to identify key interactions in the students' mathematical dialogues. The analysis is intended to generate themes on the interactions between MDC and CSCL to build a new framework.

In this poster (https://shorturl.at/cj6k0) we present three initial themes contributing to the new framework, focusing on interactions between tool mediation and CSCL progression in mathematical communication. The "***Illustration – Annotation and Setting the Stage***" theme highlights how students collaboratively build a shared understanding of concepts. The "***Verification – Critical Reflection and Confirmation***" theme focuses on critically assessing and confirming the correctness and relevance of their ideas. The "***Inspiration – Elaboration and Iterative Refinement***" theme captures how students seek external ideas and iteratively refine them when encountering challenges. The findings provide tentative suggestions for further improving CSCL activities while considering the concept of MDC.

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Characterising internal discontinuities in mathematics in engineering education through teachers' views

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Keywords: Mathematics for other disciplines, discontinuity, Anthropological Theory of the Didactics

Mathematics education for non-specialists is undoubtedly one of the open fields in undergraduate mathematics education research. Specifically, the mathematical education of engineers presents particularities addressed in previous works, such as the special issue of the IJRUME devoted to this field (Pepin et al., 2021). One of the specific features of engineering education is the distinct role that mathematics plays in both mathematics and engineering courses, which has already been characterised in various engineering degrees. The transition from mathematics courses to engineering ones has been described as the "internal discontinuity" of the triple discontinuity in mathematical education within engineering education (Florensa et al., 2022). Here, we present the first stage of a study to characterise this internal discontinuity through interviews with teachers of mathematics and engineering courses in two institutions.

Study

Our work is framed in the Anthropological Theory of the Didactic (ATD), focusing on the concept of praxeology (Chevallard, 2006). This framework is employed during the interviews to guide interviewed teachers in describing mathematical activity, not in terms of the content to be taught as it is usually done but as types of tasks and their associated techniques. We consider characterising the views of mathematics and engineering teachers to partially describe the prevailing epistemology of two institutions in the ATD sense: those of the mathematical and those of the engineering courses.

Consequently, the research questions we raise in this first phase of the study are: What are the mathematics course teachers' and engineering course teachers' views of mathematical activity in the first years of mathematical training in engineering degrees? To what extent are these two groups' visions different, and how do they help describe the similarities and differences between the prevailing epistemology of these two institutions?

The first phase of this study involved interviewing university teachers from two engineering schools in Barcelona, Spain. Specifically, five teachers who teach mathematics courses (M1 to M5) and four who teach engineering-related courses (E1 to E4) were interviewed. During the interview, two activities used in mathematics courses for engineering students were presented to the teachers (see Figure 1). On the one hand, a classic exercise on function analysis was shown. On the other hand, the generative question of a Study and Research Path (SRP) on population dynamics was presented (Barquero et al., 2011). Interviewees were asked to identify the type of tasks the activities would promote and the associated techniques in their institution. In addition, they were also asked which of the two proposed activities promoted a more helpful activity for engineering education.

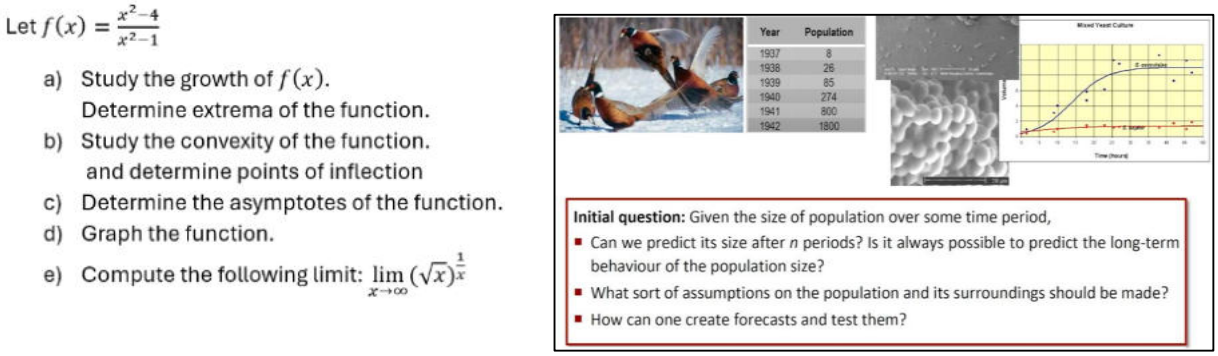

Figure 1: Activities presented in the interview

Findings

Most teachers interviewed identified the first activity as likely to promote manual techniques, as an activity intended to be solved using derivative rules and paper-and-pencil graphing. All the interviewees agreed on the types of tasks this activity would generate among first-year students. In addition, alternative methods, such as numerical calculation software and symbolic computation programs, were also noted among engineering teachers. Regarding the second activity, the diversity of types of tasks increased, including different approaches (statistical, differential equations). While most considered the second activity to be more interesting, some participants thought that the first activity was essential because of its ability to lay the foundations for a future activity.

Acknowledgment

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Fostering dialogue between university researchers and upper secondary school teachers on students' mathematical preparation for university studies in STEM subjects in Iceland

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Keywords: Secondary-tertiary transition, mathematical preparation, organization of studies, interschool communication, teacher expectations of students.

In Iceland, there is an increasing concern about the high dropout rates and low graduation numbers in STEM fields at the university level, with university instructors expressing worries about students' mathematical preparation or lack thereof. Recently, some adjustments have been made to the upper secondary (grades 11–13) curriculum in Iceland, yet teachers at this level report a lack of clear curricular guidance and adequate teaching materials. Brandell et al. (2008) highlight that the challenges students face during the transition from upper secondary school to university often stem from discrepancies between the upper secondary curriculum and the knowledge and skills expected at the university level.

As part of a broader initiative called the SamSTEM project to address these issues, we focus on facilitating collaboration between upper-secondary and university-level mathematics teachers. We explore how these two educational communities perceive the existing problems and consider the potential synergies and conflicts in their expectations.

Our research question is: What are the mutual expectations, as well as the key differences in expectations, between upper secondary and university mathematics teachers regarding students' mathematical preparation for STEM studies at the university level?

Within the SamSTEM project, 25 university instructors from the fields of mathematics, mathematics education, physics, chemistry, statistics, and computer science from three universities (University of Iceland, University of Reykjavík, and University of Akureyri) aim to increase the number of STEM subject graduates. In the school year 2023–2024, members of the SamSTEM project visited 27 out of the 31 upper secondary schools in Iceland that offer natural science strands of studies and held online meetings with the remaining four. Each visit (or online meeting) included a two-hour meeting between the STEM subject teachers at the upper secondary school and 3–6 university instructor members of the SamSTEM project, with the aim to foster a shared understanding of the preparation needed for university-level STEM studies and to map and address the challenges faced by upper secondary teachers.

In Iceland, students' selection of upper secondary school and mathematics courses offered affects their performance in undergraduate mathematics courses at university (Jónsdóttir et al., 2013) and no standardized examination is offered at the end of upper secondary school. Therefore, the SamSTEM project partners brought to the meeting school-specific data regarding how the students at the respective school fared in the university STEM programs in comparison to other schools. The upper secondary school teachers received some slides with preliminary questions to discuss at the meeting,

e.g., regarding the organisation of mathematics courses and selection of teaching materials. In many cases, the meetings also drew in principals, teachers from other subjects, and study counsellors of the schools.

In the first steps of data analysis from the meetings, there was broad agreement on what challenges upper secondary school mathematics teachers in Iceland face. Five of the project participants made a summary of the 12 main resulting challenges and drafted possible suggestions for ways to address these challenges. To check back, the preliminary results were sent to all the participating schools, and they were also presented and discussed at upper secondary school teacher gatherings. After reaching an agreement, the results were presented and discussed at two meetings with university lecturers and two seminars with a mixed group of upper-secondary and university teachers.

In this poster, we present results regarding differences and similarities in teachers' views of students' mathematical preparation for STEM studies at university. We coded and thematically analysed data from the 31 upper secondary school meetings and two university lecturer meetings. Our results indicate that university lecturers' worries align with those of university lecturers in other countries, i.e., they are widely concerned about students' increasing lack of preparation in mathematics (Brandell et al., 2008; Deeken et al., 2020; Wade et al., 2016). They also indicate that upper secondary school teachers would prefer more curricular or university guidance regarding the mathematics course content, although no general desire for standardized national assessment was expressed. Upper secondary school teachers pointed out an increased diversity in the mathematical preparation programs offered by the different upper secondary schools. In many schools, the decisions of school principals affected how teachers could assess their students' knowledge and competencies in mathematics, thus affecting their possibilities to prepare students for university. One of the main hurdles faced by upper secondary teachers was the lack of up-to-date teaching materials in mathematics, forcing them to create their own material on the go.

Link to the poster presented in Bolzano: tiny.cc/CERME14bea

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Learning about students' misconceptions with instructional videos and STACK tasks in university courses for teacher education

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Keywords: Instructional video, STACK, teacher education, student misconception, evaluation.

Theoretical background

The project LLV.HD ("Teaching-learning alliances in mathematics-containing degree programmes - cross-university and digital"; www.uni-kassel.de/go/llv-hd) aims to develop digital tools for teaching and learning mathematics in various study programmes (Eichler et al., 2022). This poster focuses on the teacher education programme at two universities and on the topic of upper secondary students' misconceptions in calculus. For students learning about this, we developed and evaluated two kinds of digital tools: instructional videos and STACK tasks.

Instructional videos can enhance understanding and facilitate learning of mathematics (Kay, 2012). We also use them to support the learning of mathematics education. The tool STACK (System for Teaching and Assessment using Computer algebra Kernel; Sangwin, 2013) supports designing randomised digital mathematical tasks and individualised feedback for different algebraic inputs. Although STACK is widely used to improve school students' learning, it is rarely employed in mathematics education courses for prospective teachers.

The aim of integrating these digital tools into the teacher education programme is to promote prospective teachers' professional knowledge, strengthen their beliefs and foster their motivation. This poster addresses the third aspect. In this regard, we focus on how students judge the utility value of digital tools (Eccles & Wigfield, 2002). Utility value is understood as synonymous to the construct of relevance, which fosters academic achievement (Albrecht & Karabenick, 2018). In our study, digital tools should help "students to construct or otherwise independently identify the value of engaging in academic tasks" (Albrecht & Karabenick, 2018, p. 5). Our research question asks how students estimate the relevance of digital tools for their learning.

The material in this study

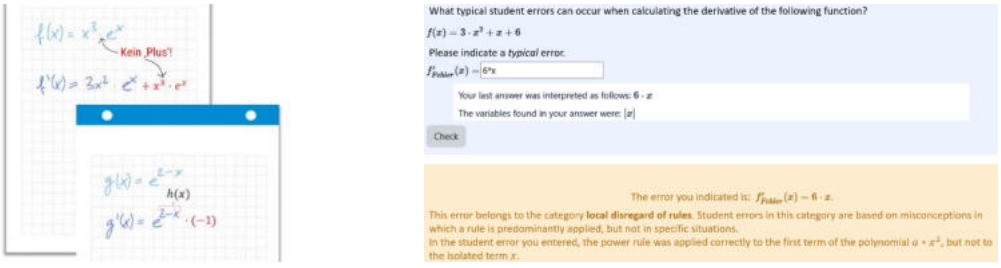

Figure 1: Scene from the instructional video concerning one kind of typical error (left side) and STACK task with the aim of giving this typical error as the solution of the task (right side)

Both the instructional video and the STACK tasks are based on a category system of students' misconceptions concerning procedural tasks when computing derivatives (Eichler et al., 2018). In the instructional video, this category system is explained using real examples of students' mistakes in computing derivatives. While the video's aim was to introduce misconceptions, the STACK tasks aimed to reflect on them and serve as an exercise.

Evaluation and results

The evaluation was conducted through questionnaires administered online immediately after the sessions in the mathematics education courses for prospective teachers, during which students' misconceptions concerning the derivative were introduced. Because this was an initial study, the questionnaires were aimed at all users and included only a few closed and open-ended items. Preliminary results indicate that the majority of students find engaging with the video's topic valuable for their future professional life and consider the instructional video helpful overall. However, while the video is highly accepted, the results regarding the STACK tasks are less positive (Figure 2). The feedback is not considered as helpful and the STACK tasks are perceived as less interesting compared to the instructional video. The video's positive reception suggests it sparks motivation, while the less favourable view of the STACK tasks indicates they may not have the same effect.

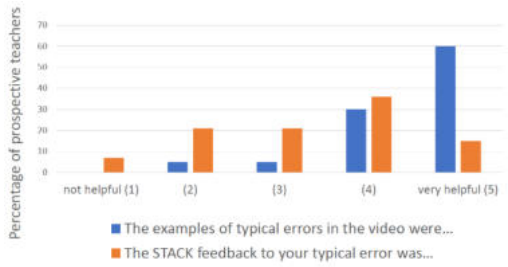

Figure 2: Two evaluation items for the instructional video (blue, $n = 24$) and STACK tasks (orange, $n = 16$)

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Developing and evaluating visual supported STACK tasks for introductory courses in university mathematics

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Keywords: STACK, visualization, task development, evaluation.

Theoretical background

The project MAOAM Visual, funded by the foundation "Stiftung Innovation in der Hochschullehre" and integrated into the Centre for Higher Mathematics Education (https://www.khdm.de/en/), aims to combine the potential of digital tasks using STACK (System for Teaching and Assessment using Computer algebra Kernel; Sangwin, 2013) with the promoting effect of visualisation for learning mathematics in higher education. In particular, it is centred around creating visual tasks that aim to improve university students' conceptual understanding of mathematics.

The digital tool STACK can be seen as an intelligent tutoring system (Hillmayr et al., 2020) as it offers the possibility to design digital mathematical tasks, randomise them and give individualised feedback on different inputs. It is therefore considered to be a promising digital tool for mathematics education that will positively influence school students' learning processes. Visualisation can be regarded as an integral part of mathematics teaching and learning and a powerful instrument for understanding mathematical concepts or procedures (Presmeg, 2014; Arcavi, 2003)

Task Development

The task development is divided into the following four steps.

1) In cooperation with lecturers and tutors, topics where students face conceptual challenges or lack conceptual understanding are identified. Of these topics, only the concepts where another access via visualisation will likely enhance student's understanding are chosen. For example, a geometrical interpretation might shed light on new aspects of Gaussian elimination.

2) A task is then developed around the central concept defined in the previous step. This central concept is divided into sub-concepts. In particular, understanding each sub-concept could prove to be useful in enhancing students' understanding of the central concept. Duval states in (2006) that it is important that a mathematical concept is approached from two distinctly different semiotic representations. Thus, the tasks are designed in such a way that a solution requires a formal and visual approach. Moreover, the tasks aim to include a transition from one representation to the other. Thus, a coordination of two distinct semiotic systems is required. Duval (2002) considers this coordination essential to mathematical understanding.

Upon returning to the concept of Gaussian elimination for a uniquely solvable system of linear equations, one can represent equations of three variables as planes. Consequently, a sensible central concept would be the geometric interpretation of Gaussian elimination. One must first understand how a single elementary row operation changes the planes. Once this is understood, one needs to

translate the concept of a variable vanishing into its geometrical meaning. These two concepts can then be combined to conclude that Gaussian elimination rotates planes along their intersections until each plane is parallel to a different coordinate plane.

3) Task development then subsequently leads to the creation of an appropriate visualisation that can be used to answer concept-related questions. These often-interactive visualisations are usually constructed in GeoGebra (www.geogebra.org). They are designed to be user-friendly for both students and lecturers. Building on Arcavi (2003), the design of the visualisation allows students to discover mathematical properties that might have gone unnoticed when solely employing formal reasoning. Therefore, the visualisations may be implemented in the task itself, or in the feedback, to shed light on possible solutions or errors. At the same time, the visualisations should not present an immediate solution to the problem given in the task. The visualisations are integrated into the tasks as applets. However, the applets can also be used independently, for example, in lectures or tutorials.

In the case of a uniquely solvable system of linear equations of three variables, equations are represented as planes. One of these planes can then be rotated around its intersections via sliders. In matrix notation, this corresponds to the addition of a multiple of one row to a different row when carrying out Gaussian elimination.

4) In the final step, the task is refined to capitalise on STACK's function to give individual feedback as well as its randomisation feature. Latter allows us to randomise the numerically defined parameters and thus the visualisations using them. For Gaussian elimination, the equations, and thus the planes shown in the visualisation, will vary.

In the future, tasks involving visualisations will be available on Andreas Eichler's project page at the University of Kassel, which can be accessed via FB10 on the university's website.

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"The goal is really to have a correct proof that you can also verify in Lean" – Exploring a lecturer's perspective of using Lean in teaching

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Keywords: Teaching with Lean, Interactive Theorem Prover, proof writing, lecturer.

Interactive Theorem Provers (ITPs) have been used in mathematics research for the development and verification processes of proofs and, more recently, they have been used in university mathematics teaching (e.g., Thoma & Iannone, 2022). Lean is an ITP with an extensive verified mathematical library. When writing a proof in Lean, the user receives feedback on the proof code and clear information regarding the context of the proof (see also Yan & Hanna, 2023).

The limited empirical studies investigating the use of Lean have mainly focused on students' use (e.g., Thoma & Iannone, 2022; Hanna et al., 2024). Thoma and Iannone (2022) discuss how students who were experienced Lean users were more precise in the use of symbols when writing proofs in pen and paper, divided the proofs in goals and subgoals and generally wrote in full sentences, unlike most students who had not engaged with Lean. However, students indicated that they found Lean difficult to learn and time consuming, mainly for its syntax (Iannone & Thoma, 2023). Further studies explore the use of Lean in task-based interviews. Thoma and Iannone (2023) investigate the interaction between students and Lean, highlighting differences between students considering "step-by-step verification", mimicking closely a pen and paper proof, or a "multi-step verification" by further utilising coding elements of Lean. Similarly, Hanna et al., (2024) discuss five different ways that students used to prove one logic task using Lean and present students' choices both in terms of the tactics and in terms of the paths they followed to prove the same statement.

However, there is a gap in the empirical studies regarding teaching with Lean. Here, we report initial findings of a study exploring lecturers' perceptions of the pedagogical affordances and constraints of using Lean in an undergraduate mathematics course.

The data come from a larger study examining the introduction of Lean in teaching mathematics at a U.S. university, where Lean has been used in mathematics undergraduate teaching for about 2 years. Previously, a lecturer had introduced Lean and created relevant teaching materials for one course during which the students engaged with both hand-written and Lean proofs. In the academic year that our study took place, two more lecturers used these teaching materials in their own courses. Here, we focus on one of these lecturers whom we call Ethan. The course Ethan taught was designed to closely follow the chapters of a textbook on "The Mechanics of Proof"[1], written by the lecturer who had introduced Lean previously. The content of the course focused on types of proofs, logic, number theory, functions, sets and relations and was designed to capture both hand-written and Lean proofs.

[1] The textbook is available through this link: https://hrmacbeth.github.io/math2001/00_Introduction.html

This was also reflected in the course's assessment. Ethan's course had 21 students from various backgrounds (either majors or minors in mathematics). The classes, 75 minutes each, were offered twice a week. One session was used by the lecturer for Lean demonstrations and the other focused on students working on the homework problems supported by the lecturer and a teaching assistant.

The data collected include course documents (assessment questions and students' responses with hand-written and Lean proofs), students' responses to surveys, task-based interviews with students and interviews with the lecturer. In this poster, we focus on the interview with the lecturer which was transcribed and analysed thematically. The analysis provided insights into Ethan's experience of integrating Lean in his teaching and changing his teaching approach to include pre-recorded materials and less "traditional" lecture time. Having marked students' work, Ethan shared that he had expected to see a difference in students' assessment compared to previous years, but he had "sort of mixed feelings as I was marking the final exams (…) I did look at a few questions on the midterm exam and compared that to previous semesters and it was maybe a little better or about the same. So at least not worse". With the results from students' assessment indicating either no change or a slight improvement he also commented on the constraints of teaching this course with Lean as there were some mathematical topics that could not be taught in this dual manner (pen-and-paper proofs and Lean proofs). Ethan also noted how he changed his own approach to teaching proof and how he would have taught differently this course with or without Lean.

Our poster offers insights into the adaptations Ethan made to his course, the constraints he faced, and the lessons learned in terms of teaching proof with Lean but also teaching proof more generally. In our next steps, we aim to explore the other lecturer's perspectives also in relation to the student data we collected.

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Students' conceptions of polynomial functions and their graphical representations at the secondary-tertiary transition

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Keywords: Polynomial functions, secondary-tertiary transition, graphical representation.

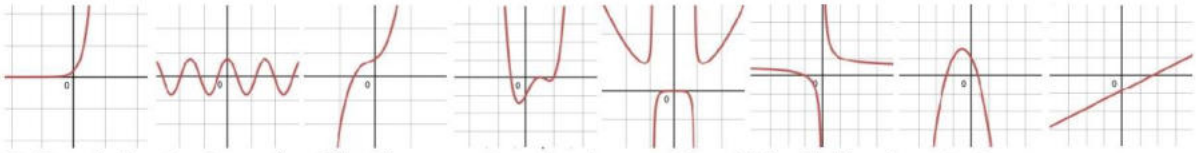
Introduction

Polynomials are ubiquitous objects in mathematics, and in most disciplines using mathematics. Consequently, polynomials hold a fundamental place in the teaching of analysis and algebra. In France, polynomials appear in grade 8 through first-degree equations, but quadratic polynomials are really studied in grade 11. Formal polynomials are introduced in tertiary education, along with their link with polynomial functions. Moreover, in secondary and tertiary education, polynomials are the first functions used in analysis, for tasks such as differentiation. In my PhD, we are interested in the teaching and learning of polynomials at the secondary-tertiary transition in France. We're currently investigating students' conceptions on polynomials; we plan to study teachers' practices around this notion; and the final aim is to develop and experiment activities that could smoothen the transition. Previous research have highlighted various challenges and ruptures happening during the secondary-tertiary transition (Gueudet, 2008). Studies have been conducted to investigate this transition in different domains: analysis (functions, tangent), algebra and linear algebra, probabilities… In the case of polynomials, there are few studies, apart from the PhD research of Pleština (2023). In this poster, we study students' conceptions of polynomial functions and their graphical representations.

Theoretical framework, research question, and methods

To analyse students' conceptions, we use the theory of conceptual fields, introduced by Vergnaud (2009). It emphasises the operational aspect of knowledge by defining operational invariants: theorems-in-action (propositions held as true by students in their activity), concepts-in-action (arguments held as pertinent by students) and action rules (implicit rules that guide the action of the student). In this work, we want to identify some of students' operational invariants, as they inform us on their conception of polynomials. Indeed, a *concept* is characterized in the theory of conceptual fields by three components: the situations which give meaning to the concept (reference); the invariants on which the operationality of the schemes is based (content); the linguistic and non-linguistic forms which allow symbolic representation of the concept, its properties, the situations and processing procedures (signifier). The *conception* of an individual can be defined similarly with these three components. We will draw on Duval's semiotic registers (2017) to describe each component of the conceptions. This leads us to investigate the following research questions: What are the students' conceptions on polynomial functions? In what way do they evolve during the secondary-tertiary transition? To answer these questions, we designed a questionnaire (extract below), focused on three main themes: definition and recognition of a polynomial, polynomial equations and roots, and polynomial functions and their graphs. It was given to 459 students (92 in grade 11, 114 in grade 12, 253 in higher education). An exploratory pre-version of the questionnaire was presented at INDRUM2024 (Veuillez--Mainard, 2024). In this poster, we mainly focus on the third theme.

7. Let P be a polynomial such that for all $x \in \mathbb{R}$, $P(e^x) = 0$. What can be said about P?
8. Plot on the graph below the shapes of the functions $x \mapsto x^n$. Differentiate the cases based on the parity of $n \in \mathbb{N}^*$.
9. Can the following graphs represent polynomial function? Provide a justification for each curve.



10. Does the function $f : x \mapsto (x+3)^2 - 1$ have an optimum (maximum, minimum)? Justify. If so, determine it.

Results

Definitions – Questions 1 and 2 of the questionnaire ask for students' definition of a polynomial. In high school, only polynomial functions are defined (and called polynomials). In higher education, polynomials and polynomial functions are distinguished. Consequently, a large proportion of students define a polynomial as a function (40% in grade 11, 52% in grade 12, 24% in university).

Recognition of graphs – Question 9 (above) was designed to identify operational invariants used by students in the recognition of polynomial graphs. It is an unusual task for students, hence half of them didn't justify their answer for at least one graph. We were able to identify several operational invariants, such as: *a graph of a polynomial is a parabola (or several parabolas)*; *a polynomial function is increasing; a continuous and increasing graph is a polynomial graph*. We also observed that some students use properties of monomial graphs (such as symmetry) to identify polynomial graphs. Moreover, when students provide an algebraic expression of the represented function, they often give the expression of a translated and dilated monomial: $ax^n + b$.

Links with polynomial expressions – In (Veuillez--Mainard, 2024), we presented first results about the recognition of polynomial expressions. We observed disconnections between the recognition of polynomial expressions (e.g. identifying whether e^x is a polynomial expression) and the recognition of polynomial graphs (e.g. identifying the graph of e^x as so, and then identify whether it is a polynomial expression), even in university. This is a manifestation of difficulties in articulating the graphical and algebraic registers. In the poster, we presented more data and illustrated our results with examples of students' answers.

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The roles and responsibilities of mathematicians in secondary mathematics teacher education: An international perspective

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Keywords: Secondary teacher education, mathematicians, the double discontinuity, survey.

The challenges we face as a society require well-educated citizens, and in particular more students going into STEM fields. Since mathematics underpins all STEM disciplines, this requires exceptional secondary mathematics teachers (SMTs). Thus, it is not surprising that the preparation of SMTs has seen an increased research interest in recent years. In this area, one of the main challenges that teacher educators face is connecting SMTs' academic mathematical studies with their teaching in schools (Wasserman et al., 2023). This is an old problem – it has concerned mathematics educators for more than a century – but it remains pertinent and unresolved. In many countries, prospective SMTs are required to complete extensive coursework in mathematics or in some cases attain a bachelor's degree in mathematics, yet many SMTs experience academic mathematics as irrelevant to their future practice (Wasserman et al., 2023). Given that mathematicians are de-facto in charge of the academic mathematics education of SMTs, surprisingly little is known about their views on, and involvement in, SMT education (Wasserman et al., 2023). The literature that does exist addresses issues such as the views of mathematicians on their role in SMT education (Hodge et al., 2010); on what they wish prospective SMTs to take away from their mathematics coursework (e.g., Leikin et al., 2018); on what it is important for prospective SMTs to know about the discipline of mathematics (e.g., Hoffman & Even, 2024); and on how they would design a hypothetical course targeted solely at prospective SMTs (Lai & Ahrens, 2024; Leikin et al., 2018). While the growing body of literature in this area provides many valuable insights, much of the research conducted so far in this area has been small-scale, with few participants ($n \approx 10$) in one country or even at one university. As such, the insights developed so far may be seen as local, and there is a need for research that could support consolidation of these insights and their translation across institutions and countries. To this end, there is a need for a framework for describing similarities and differences between different educational contexts in relation to mathematicians' involvement in the education of SMTs.

In light of this need, the Mathematics Education Committee of the European Mathematics Society (EMS), of which the authors are members, has commissioned a large-scale survey of mathematicians' views on their role, and on the role of academic mathematics, in SMT education, and of local initiatives that different universities have taken to strengthen the mathematics education of prospective SMTs, apart from those already reported in the literature. This survey will take the form of an online questionnaire followed by interviews with selected respondents. The survey will target mathematicians working in mathematics departments across Europe that are involved in SMT education or in efforts to improve it. In designing the questionnaire, we draw on previous large-scale surveys of mathematicians' views (e.g., Pinto & Koichu, 2023), on findings from previous studies,

and also on the literature survey by Wasserman et al. (2023), who identified "three different aspects within the literature in terms of the problems, challenges, and solutions emphasised with respect to university mathematics and teacher preparation" (Wasserman et al., 2023, p. 724): bridging school and university mathematics; distinguishing mathematical concepts from mathematical practices; and recognising the didactical realm and connecting to teaching. Wasserman et al. present a theoretical framework built around these aspects which we will use to frame the survey.

Access and response rates are well-known problems when conducting surveys of this sort, yet past experience suggests that the EMS can provide invaluable help in gaining access to and good will from mathematics departments and mathematicians. Questions will, for instance, concern what kinds of goals the mathematicians have for the learning of prospective SMTs in their courses, whether there are any goals that are specific to prospective SMTs, and what kinds of activities or tasks they use to help the SMTs fulfil these goals. The questionnaire will also include open questions asking for detail on any specific initiatives relating to the mathematical education of SMTs, and we aim to conduct follow-up interviews with mathematicians involved in such initiatives, to elaborate and refine our findings. However, a major obstacle in designing the questionnaire is the great diversity in the organisation of SMT education across Europe, and we made use of the poster as a means of soliciting further information about this diversity, to support us in the design of the questionnaire.

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Research-informed and collaborative reform in university mathematics teaching: The case of secondary to tertiary transition

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Keywords: Secondary-tertiary transition, teaching design, teacher-researcher partnerships.

Mathematics students are known to face many difficulties during their transition from secondary to tertiary education (Gueudet, 2008). During the teaching of an introductory course for first-year mathematics students, the first author (Witty) encountered numerous issues which he associated with this transition, particularly related to students' attempts at mathematical communication, as well as their low engagement levels. For example, it was common to encounter students who would produce solutions that used new mathematical words and symbols, but lacked the rigour he would expect from something 'mathematical'. Gueudet (2008) states that, when students begin university, they notice the new language used by their teachers and imitate it: this "leads them to produce texts looking like mathematics, focusing sometimes more on the form – many symbols, usual language brought to a minimum – than on the mathematical meaning" (p. 244). Similarly, Sfard (2008) notes that "there are important differences between construction and substantiation routines practiced in colloquial and literary mathematical discourses, and these routines change again in the transition from school discourse to the scholarly discourse of mathematicians" (p. 225). As Witty believed it is important to consider how to assist students in overcoming these differences and difficulties in their first year, he became interested in learning more about changes he could make to help. There have been calls for the design and evaluation of interventions that support students during the secondary-tertiary transition. For example, Biehler et al. (2022) state that "there is a growing interest in teaching practices fostering students' active engagement" (p. 10) and that research in mathematics education can support teachers at university-level in this aim. With this in mind, Witty embarked on the design, implementation and evaluation of a new set of teaching sessions created for a course on 'Mathematical Skills' he leads with the aim of alleviating some of the difficulties of the secondary-tertiary transition. Although this intervention was produced in a partnership between him and mathematics education researchers (Hooper, Nardi, Biza), here he reflects on his own experience, mainly of the design part, through a reflexive autoethnographic approach (Ellis et al., 2010).

In this Year 1 course, students are introduced to core definitions and concepts of pure mathematics, such as properties of sets, numbers, and functions, as well as to mathematical proof. When Witty took on this course, he was given materials used previously which lend themselves to a typical 'chalk-and-talk' lecture format. He saw problems in students' engagement and use of mathematical words and symbols whilst teaching this course, and there were additional issues he encountered with his delivery such as an inability to fully convey passion for the new way of mathematical thinking when consigned to a blackboard. He attempted to fix some of these problems in his second year of teaching the course by introducing quizzes into lectures and moving problem solving activity into lecture time. This improved things, but the lack of a systematic approach and understanding of whether he was changing things in a pedagogically sound way meant he did not feel he was reaching the full potential of moving

to a new lecture format. When he was made aware of an opportunity to work with the university's Mathematics Education research group on a project that would involve redesigning a course to include more interactive elements and use of different methods of teaching, this seemed like a good opportunity to improve and expand on his approach to easing the transition to university mathematics.

The approach to planning and designing the new structure for the course was done through regular meetings with Mathematics Education researchers (mostly Hooper) where Witty provided updates on the content and the aims of the sessions, whilst researchers provided research-informed insight on innovative teaching practices, such as inquiry-based group work with flipped classroom elements and use of online tools, such as Padlet, as means for increasing student engagement and contributions during lectures (Hooper & Nardi, 2024). We created a lesson plan template (Figure 1) to detail timings and use of interactivity during a session, along with an explanation by Witty of why he wanted to deliver each part of the session in this way.

Time	Activity	Interactivity	Why
00:00-00:10	Quick recap of statements from pre-lecture video with true/false quiz with some statements.	Wooclap (https://www.wooclap.com)	Need to refresh the idea of the definition from the video as it will be important throughout the lecture. Quick quiz allows immediate interaction and works well for true and false statements!
00:10-00:20	Introduction to truth tables, giving example of not(A) and not(not(A)). Discuss 'and' and then set them the task of completing the truth table.		Good to see an example first of new notation etc. before some doing!
00:20-00:25	Students complete truth tables for 'A and B' and 'A and not(B)'	Working in groups	Students actually work with the truth tables.

Figure 1: Template excerpt for the plan of the beginning of a lecture

The creation of a plan for each session – and its collaborative production with the Mathematics Education research team – is a novelty that we see as of potential pedagogical value to other university mathematics teachers. We see its value in being asked to reflect explicitly on why Witty was choosing a certain method or activity or tool. The poster (https://tinyurl.com/CERME14-Poster-Witty-et-al) presents this design process, highlights of the course implementation and reflections thereof.

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TWG 15: Teaching Mathematics with Technology and Other Resources

Thematic Working Group 15

Teaching mathematics with technology and other resources

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Teaching mathematics with technology embrace both innovative and traditional tools to adopt a broad view of using technology resources in mathematics education such as mathematics-specific learning environments and general digital tools. Technology use is relevant both to teach mathematics in the class and to manage the professional issues from designing curricula to designing learning activities and classrooms in different didactic scenarios, and assessment.

Keywords: Technology, teacher practices, AI, computational thinking, teacher competences

Introduction

TWG15 accepted 19 papers and six posters for presentation at the conference, which reported research from 17 countries (Austria, Canada, Denmark, France, Germany, Greece, Ireland, Israel, Italy, Norway, Portugal, Spain, Sweden, Turkey, UK & United States). In total, 31 participants took part on the group, comprising 15 academics and 16 young researchers. The papers and posters were presented across seven sessions with an additional session dedicated to summarizing the discussions and preparing the final report. The contributions drew upon a wide range of theories – both general educational theories and those specific to mathematics education, both addressing overarching issues as well as those related to teaching with technology. The papers and posters dealt with a wide range of technologies and related issues - from traditional digital learning environments to computational thinking and, most notably, emerging technologies such as artificial intelligence (AI). The contributions touched different *roles of teachers*– teachers as designers, learners, working alone or and as participants in communities. Pre-conference, all papers and posters were organised and grouped into four sub-themes: *New Theories - Future directions* (six papers/posters), *Generative AI* (six), *Computational thinking* (six), and *Technological learning environments* (seven). Each thematic session began with a brief introduction and concluded with a synthesizing summary.

Sub-theme 1: New Theories – Future Directions

In the sub-theme, developed by five papers and one poster, discussions focused on the role of theory in understanding teaching mathematics with technology and other resources. The contributions considered how to adapt theories to encompass emerging technologies (Geraniou et al.; Lyublinskaya & Du) or refine or extend them to address situations involving new configurations of users and/or artefacts (Nøhr et al.; Bretscher et al.; Sayadi & Abboud). Each paper made connections between existing theories in some way. However, in Rocha, comparing theories, to elicit similarities and

differences between them, was the central focus. Drawing on Artigue's (2007) CERME5 plenary, the papers in this session recognise the continuing theoretical diversity inherent in our field, resulting from the need to adapt frameworks in ways that fit the particular needs of the social, cultural and educational contexts that the TWG15 community are living and working in, and at the same time, working on theoretical commonalities, provide a basis for the important work of networking theories.

The instrumental approach provides the theoretical foundation for three papers in this sub-theme. For example, Sayadi and Abboud use instrumental genesis, supplemented by instrumental Approach (Abboud, 2024) and semiotic bundles (Arzarello, 2006), introducing the concept of resonance to account for interactions between different subjects' instruments in the context of geometric constructions. Likewise, Nøhr et al. propose an extension of instrumental genesis to accommodate group cognition, moving away from a theoretical focus on individuals' use of technology to contexts involving multiple users. Geraniou et al. reflect on the strengths and limitations of the Double Instrumental Genesis (Haspekian, 2014) when applied to teachers' use of generative AI, concluding that teachers must continually update their competencies to 'keep up' with the pace of technological change.

The other three papers draw on theories beyond those associated closely with research on digital technologies in mathematics education. From a resource perspective, Bretscher et al. explore how the notion of transparency (Lave & Wenger, 1991) may be used for analysing pre-service teachers' interpretation of other resources, beyond digital technology. Founded in theories beyond mathematics education, Lyublinskaya and Du illustrate how a pedagogical framework for integrating AI literacy can be applied in teaching mathematics. Finally, Rocha presents a 'global model' of teachers' knowledge for teaching mathematics with technology informed by a comparison of the similarities and differences of diverse theoretical frames essentially based on the pedagogical and content knowledge for teaching, with the specific reference to the use of technology by teachers, basing on TPACK (Misha & Koehler, 2006), PTK (Thomas & Hong, 2013) and KTMT (Rocha, 2020).

In summary, the instrumental approach continues to evolve, yet remains central to much of the research on teaching with technology and other resources. In future, networking theories may provide an inclusive approach to managing theoretical diversity, while encouraging continued theoretical growth, adaptation and refinement inspired by new technologies, contexts and concepts from within and beyond mathematics education. These theoretical reflections echo and resonate in subsequent sessions, notably those focusing on generative AI and computational thinking, where emerging technologies exert a disruptive, yet productive, influence on theories in mathematics education.

Sub-theme 2: Generative AI

This session addressed the implications of emerging technologies and AI for mathematics teaching, featuring four papers and two posters. The contributions collectively explored various questions related to teaching mathematics in diverse technological contexts, each with distinct yet overlapping emphases. Notably, generative AI (GenAI) was central to four of the six contributions. The two papers not focused on GenAI were authored by Ovadiya and Marabash. Ovadiya examined challenges

faced by teachers when integrating digital technology into the process of posing mathematical problems. Marabash presented a self-study investigating how mathematical digital competencies manifested in their own experimental approach to posing and solving problems using digital tools.

The remaining contributions addressed different aspects of the intersection between GenAI and mathematics teaching. Yavuz and Dede explored how pre-service teachers utilized ChatGPT as a scaffolding tool to solve a geometry problem within GeoGebra. Segal and Biton investigated whether pre-service teachers' use of ChatGPT for generating mathematically defined problems with specific educational goals could enhance their technological, pedagogical, and content knowledge (TPACK, Misha & Koehler, 2006). Byrne et al.'s poster reported on an intervention in which in-service teachers participated in professional development designed to foster dialogue among teachers, their colleagues, and AI systems. Embid and Perdomo-Díaz's poster examined the ability of various GenAI chatbots—GPT-4, Claude 3.5, Sonnet, Copilot, and Gemini—to generate plausible mathematical problems involving fractions at varying difficulty levels.

Beyond their varied empirical foci, these contributions employed a range of theoretical frameworks and research methodologies, including the zone of proximal development, TPACK, design-based research, and theories of mathematical problem solving. This diversity underscores that GenAI's influence on mathematics education is multifaceted and that our field possesses robust theoretical tools to study and understand these implications.

Mathematics education has a long tradition of adopting and critically engaging with new technologies, continually developing theoretical frameworks to analyse their affordances, applications, and limitations. A key point arising from the discussions was that while existing theories remain valuable for understanding new technologies, there may be a need to network these theories in novel configurations. Despite this point, a central theme of discussion at this session was whether GenAI represents a fundamentally different kind of technology compared to those traditionally used and studied in mathematics education, in relation to different theoriesetical frameworks. The discussions converged to use Double Instrumental Genesis as a foundation for GenAI with a revision to integrate the distinctive characteristics of Gen AI. It is agentic, allowing interactive engagement from both students and teachers; it is dynamic, with unpredictable frequent updates and constantly feature enhancements; and it is probabilistic, often producing varied outputs to identical prompts. Rather than rushing to definitive conclusions about the implications of these features for our field, the discussion focused on critical questions that warrant further inquiry: Should we consider GenAI as agent or artifact, and what are the implications of this choice, in relation to interactions with it, and possible schemes of use? Should they be considered in a traditional way or a completely new way? For what educational purposes might we best leverage GenAI's unique capabilities? How do we balance potential gains in efficiency with the risk of diminishing students' opportunities for deep mathematical thinking? And what new risks and challenges do GenAI introduce to mathematics education? We are confident that research on this type of technology will remain important. The contributions and discussion in this sub-theme illustrated that TWG15 is up for this task.

Sub-theme 3: Computational Thinking

Within the Computational Thinking (CT) theme, we had six contributions, five of them looked into empirical data from small studies with mathematics teachers, whereas one offered a theoretical perspective by exploring how three frameworks offer different perspectives on using computational technology. All the studies highlight the growing interest in integrating computer programming and CT into mathematics education, while also exposing significant challenges.

In more detail, Nordby and Gadanidis focused on the potential role of programming in enhancing teaching and learning of mathematical ideas and relationships in primary school classrooms (grades 1–7). Drawing on diSessa's (2018) framework of computational literacy, they analysed 32 classroom activities using Scratch and found they align with the core principles of computational literacy. Therefore, by highlighting this particular example, they demonstrated how programming can reshape discussions and engagement in primary mathematics, but argued that there is a lack of resources and PD to support teachers in this field. So, they called for more research into how best to support teachers in using computer programming in mathematics classrooms.

Rama et al. presented the case of a grade 8 mathematics teacher, who participated in a lesson study involving four mathematics teachers in total, discussing how the collaboration of these teachers led to the one teacher's "mobilization" of their CT, as they designed a lesson plan involving the use of Scratch. The authors discussed how the teacher's designed lesson encouraged student participation and CT practices, claiming that Scratch played a crucial role in this development. Collaboration with the other teachers was also key as it helped in mobilizing the teacher's knowledge in action, improving the lesson plan and the tasks used and in the effective use of CT practices. As authors concluded, further research could provide insights into how to enhance teachers' abilities at different educational levels to effectively incorporate CT into their teaching and how to more efficiently utilize existing resources (Ponte, 2012).

Borg presented a case study of how a Swedish teacher integrated programming into his mathematics teaching, using the *Structuring features of classroom practice framewor*k (SFCP) (Ruthven, 2009) as an analytical lens for the design decisions and justifications for his teaching approach. The findings highlight how the teacher's background and "craft knowledge" (as a qualified programming teacher) influences the integration of programming in their mathematics classroom. The study shows that the way programming was added to the Swedish curriculum—without removing existing content or clarifying programming goals—has significantly influenced how teachers plan and deliver lessons. This places a heavy burden on teachers, who must teach both programming skills and their application in mathematics, contributing to their criticism of the current approach. Comparatively, countries like England treat programming as a separate subject, and its integration into mathematics depends more on individual schools, highlighting the need for further research into how various curriculum models impact teachers' ability to incorporate programming into mathematics education.

The study by Jemai et al. seeks to offer teacher educators meaningful insights into how preservice teachers (PTs) perceive CT. By examining the notes and reflections of three PTs as they used programming to solve mathematics problems, the authors aimed to uncover PTs' CT perspectives. Using Brennan and Resnick's (2012) CT framework—specifically its focus on perspectives, they

explored what the PTs identified as challenges and how they perceived their value. The results indicate that the PTs viewed themselves as beginners in programming, concentrating on learning to code and creating tools to enhance student learning. They selected familiar math problems and considered programming to be the primary difficulty. Ultimately, they saw their struggles with programming as a valuable part of their learning journey. In summary, the PTs focused mainly on programming, engaged in productive struggle, and aimed to create educational tools for their students—highlighting the need for further research on how teacher educators can help them better connect mathematics and programming.

Kallo also focused on the interaction between CT and problem solving. He emphasizes that introducing CT through problem solving, rather than an extension of 'mathematical thinking' can better harness its potential to develop abilities to engage with complex problems in a digital context.

Last, but not least, Tamborg et al. investigated how three frameworks—the KOM framework (Niss & Højgaard, 2019), CT (Weintrop et al., 2016), and computational literacy (CL) (diSessa, 2018)—offer different perspectives on teaching mathematics using computational technology. KOM sees mathematics as a discipline where technology serves as a supportive tool, relying on teachers' existing mathematical knowledge. CT introduces additional, computer science-related practices that need to be integrated into the curriculum, posing challenges for teachers lacking expertise in this area. CL takes the most transformative stance, suggesting that technology can reshape both the discipline of mathematics and its teaching, requiring teachers to critically evaluate the evolving relationship between tools and mathematics. The paper concludes by emphasizing the need for thoughtful consideration when incorporating computational technology into mathematics education and encourages further research to build on these initial insights.

In the conclusion we highlight that teachers both pre-service and in-service – across different institutional and cultural contexts, struggle with limited resources, lack of clarity in curricula, and insufficient professional development, particularly when programming is introduced without adjusting existing content. Collaboration among educators and supportive frameworks can play a key role in helping teachers meaningfully connect programming with mathematical learning. The perspectives on integration vary—from viewing programming as a supportive tool (KOM), to additional disciplinary knowledge (CT), to a transformative force reshaping mathematics itself (CL). Overall, the research emphasizes the need for clearer guidance, targeted support, and continued investigation into how programming can be effectively and sustainably embedded in mathematics education to support mathematical thinking.

Sub-theme 4: Technological Learning Environments

Within the broader scope of TWG15, this section focuses on the theme of Technological Learning Environments particularly drawn on perspectives pointed out in the seven submissions including five papers by Alan and Akkoç; Cheung and Lyublinskaya; Diamantidis and Kynigos; Läufer and Ludwig; Yiğit Koyunkaya and Bozkurt, and two posters by Lindenbauer and Infanger; Liu et al. All these submissions collectively explore how teachers engage with, design, and integrate digital

technologies in mathematics education. These contributions shed light on the complex interactions between teachers' professional knowledge, technology tools, and classroom practices, revealing critical insights into fostering effective technology-mediated learning environments.

Three contributions focus on pre-service teacher education, highlighting how digital tool use can be developed through structured university-based training. Lindenbauer and Infanger emphasize design challenges in creating GeoGebra-based digital materials (dMs) and propose using an adapted Dynamic Geometry Task Analysis (DGTA) framework to guide future training. Läufer and Ludwig, through the TPACK lens, explore the integration of 3D printing into a mathematics seminar, showing how different phases of 3D modelling correspond to varied knowledge domains. Similarly, Yiğit Koyunkaya and Bozkurt analyze a pre-service teacher's development of Digital Resource Design Capacity (DRDC) during practicum, demonstrating how real teaching experiences support strategic use and critical reflection on technology-use.

In contrast, Liu et al. and Cheung and Lyublinskaya examine in-service teacher practices in different cultural and educational contexts. Liu et al. shed light on Chinese teachers' limited familiarity with tools like GeoGebra and underscore the need for design tasks aligned with teachers' existing practices and available technologies. Cheung and Lyublinskaya trace the evolution of two elementary teachers' orchestration strategies when using the Finch robot, showing how sustained, targeted professional development can foster a shift from teacher-centred to student-centred practices.

Alan and Akkoç as well as Diamantidis and Kynigos contribute a deeper theoretical and conceptual engagement with teacher orchestration and digital resource design. Alan and Akkoç introduce the idea of "linking models" to differentiate didactic goals within orchestration chains, illustrating how identical orchestration types can serve distinct pedagogical purposes. Diamantidis and Kynigos adopt Cultural-Historical Activity Theory perspective to examine tensions in collaborative digital artefact design among teacher educators, offering insights into how institutional and historical factors shape design processes and learning.

Collectively, these contributions highlight not only the complexities of integrating technology into mathematics instruction but also the necessity of context-sensitive, theory-informed, and practice-driven approaches to teacher education. The studies demonstrate a shift from viewing teachers as mere users of digital tools toward recognizing them as designers, facilitators, and reflective practitioners. This evolving perspective underscores the importance of professional development that fosters not only technical proficiency but also pedagogical reasoning and adaptive expertise. As mathematics education continues to evolve in the digital age, supporting teachers' dynamic engagement with technology remains a critical focus for research and practice.

Conclusion: Topics and future issues relevant to the TWG15 community

In addition to the presentations in our group, we created many opportunities for exchange in small groups. After the sessions according to the sub-themes, we reflected on the key points within the topic. At the end of the conference, we discussed and summarised these key points regarding our overall topic of teaching with technology and to explicate the challenges and questions for the future.

We perceived in the contributions and discussions a natural use of theories relevant to our topic as a basis for convergence of different theoretical approaches and networking theoretical frames (Artigue, 2007) to serve as a adequate framing for specific research interests. For example, the synergy of upcoming theories about teacher competences (TPACK, Mishra & Koehler, 2006) and consolidated traditional theories about epistemic processes (such as Instrumental Approach) is convincingly described. The adaptation and expansion of traditional theories was also described very eloquently. Analogous to the adaption of Haspekian (2014) who used Double Instrumental Genesis to describe the epistemic processes using technology on the two levels of students and teachers, Nøhr et al. came up with Group Instrumental Genesis to support understanding and analyzing processes of instrumental genesis within the work of small groups.

The rapid development of emerging technologies was overwhelming – an experience we share with TWG16 about "Learning with technology". Traditional mathematical tools like geometry software or computer algebra were taken into account through contributions to learning environments with GeoGebra in our group, which gave us a good impression of current research in this field. Computational thinking as programming and algorithmic thinking within mathematics teaching is solved in very different ways in the countries and will be an important topic for us in the future. This is also particularly true for generative AI, where we are just at the beginning to understand the challenges popping up with this new unpredictable technology. "Hurry up slowly" (quotation by August emperor) succinctly expresses what the motto for our research must be from now on. These emerging technologies as part of mathematics curricula have to be used as tools in which students' and teachers' creativity, imagination, critical abilities, mathematical and interdisciplinary competencies can be used, supported, and enhanced in education, both individually and in communities of practice. Math education research on teaching with technology is not only evolving with the technological developments, but must foster all fields which are relevant for teachers to design and manage their classrooms with integration technology beneficially. Future issues which were not addressed in our TWG at this time at CERME are still needed for the future. Tools can only be game changers if teachers are prepared for the shift. Therefore, we still need intense research on *professional development* as teachers need ongoing support to learn, adapt and experiment with especially the recent trends. We need research on the *influence of educational policies* on teaching with technology like relieving bureaucratic stress, assessment, curriculum changes, special cultural factors and ethical issues.

Regarding all these diverse challenges in our research field we would highlight the necessity to discuss more about our research interests and research methods which will allow us also further to coordinate our efforts, deepen our collaborations and learn even more from each other.

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Formation of orchestration chains: linking models

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This study elaborates on the notion of orchestration chain (OC), which we previously defined in CERME13 as the orchestrations sequenced to serve a specific didactical goal. For this aim, we designed a case study. The microteaching lesson of a prospective mathematics teacher, together with the lesson preparation process, were examined. Based on the data analysis, two OCs formed by connecting the same types of orchestrations in the same order were selected. We found that two identical OCs may have different didactic goals that link the orchestrations and may be linked to form different models. We called these models as linking models. We believe that the notion of linking models presented in this study will improve the theory of instrumental orchestration and contribute to our understanding of teachers' activities in the classroom.

Keywords: Instrumental orchestrations, orchestration chains, technology integration

Introduction

The emergence of new technological tools offers new potential for mathematics teaching (Lavicza, 2010). However, reaching these potentials is more challenging than it seems (Drijvers & Trouche, 2008). Mathematics teaching practices incorporating technological tools are more complex than initially expected (Drijvers et al., 2009). Recognizing this complexity, researchers have argued that there is a need for a theoretical approach to technology integration that would address technological tools and other resources of the classroom environment together (Monaghan, 2004). The instrumental approach to tool use emerged as a result (Drijvers, 2015).

The instrumental approach focuses primarily on the relationship between an artifact and its user. As long as the user does not know what kind of tasks the artifact can be used for and in what ways, the artifact is meaningless to him. When the user develops mental tools to use the artifact, it becomes meaningful for him (Drijvers & Trouche, 2008). The person should establish an instrumental relationship with the artifact and become able to use it for his purposes. Then, the artifact becomes an instrument for the person (Verillon & Rabardel, 1995). This process is called *instrumental genesis* (Guin & Trouche, 2002). However, teachers must mentor students for instrumental genesis (Drijvers et al., 2010). To describe how a teacher arranges students' use of instruments in the classroom and how they can construct coherent instrument clusters, Trouche (2004) established the theoretical framework known as *instrumental orchestration* (IO).

Theoretical Framework

Instrumental orchestration theory was developed to explain how teachers organize instruments and create coherent instrument clusters (Trouche, 2004). According to this theory, types of orchestrations express teachers' actions. Drijvers et al. (2013) classify orchestrations as individual or whole-class. During whole-class orchestrations, the entire class participates in instructional activities. Some

examples of these types are *technical-demo*, *guide-and-explain*, *link-screen-board*, *discuss-the-screen*, *explain-the-screen*, *spot-and-show*, *sherpa-at-work*, and *board-instruction* orchestrations. The *technical-demo* orchestration involves showing students the instrumented techniques that the teacher will use in the classroom. The teacher guides the procedures during the *guide-and-explain* orchestration by posing questions to the students as they work on their instrumental tasks. The *link-screen-board* orchestration integrates digital technologies with conventional media, including books, paper, and board. The teacher engages the class in discussing what is displayed on the screen during the *discuss-the-screen* orchestration. Explaining to students what is on the screen is part of the *explain-the-screen* orchestration. The teacher examines each student's work individually during the *spot-and-show* orchestration and presents it to the class. The teacher requests students to share their work utilizing individual tools during the sherpa-at-work orchestration with the class. The conventional teaching method, in which the teacher stands on the board, is called the *board-instruction* orchestration. Conversely, individual orchestrations occur when students use their equipment to work alone or in small groups as the instructor moves around the classroom, providing guidance. Individual *technical-demo*, individual *guide-and-explain*, individual *discuss-the-screen*, *link-screen-paper*, and individual *technical-support* orchestrations are individual types of orchestration described in the literature (Drijvers et al., 2013).

Drijvers at al. (2019) states that in instrumental orchestration theory, orchestration types are treated in isolation, and no attention is paid to how they are integrated into teaching. Considering this gap in the literature, two studies focusing on OCs were conducted in the last two years. Ratnayake et al. (2024) showed how orchestration types are linked together through teacher decision-making and defined an OC as "a sequence of IOs together with teacher decision-making links" (p.11). On the other hand, Alan and Akkoç (2023) focused on specific OCs performed at different moments of study and defined OCs as "the orchestrations sequenced to serve a specific goal" (p. 2584). In this paper, we will expand on the concept of an OC and demonstrate how different orchestration types interact to construct OCs that are put together with specific didactic goals of teachers.

In this study, we divided the lesson into moments of study in parallel with Besnier's (2016) research to identify OCs. These moments are *moments of introduction and discovery*, *moments of learning and training*, *synthesis moments*, and *evaluation moments*. Introduction and discovery moments are when a teacher engages students in a task or introduces a topic. Students perform exercises during learning and training moments, work through challenges, and participate in tasks. The practices used in the previous moments of study are combined in synthesis moments. The teacher evaluates students' learning during evaluation moments. We divided the teaching practice into the moments of study in line with these definitions. We identified the types of orchestration used at each moment of the study and analyzed how they were sequenced to align with specific didactic goals.

Methods

We designed a case study with a single case: a prospective mathematics teacher's (Esin) lesson. We considered three criteria in selecting Esin as a participant: Being in the fourth year of the teacher preparation program, using GeoGebra at a sufficient level, and having a positive opinion about technology-enhanced mathematics teaching. She attended a four-year teacher preparation degree at a

state university in Istanbul, Turkiye. She prepared a lesson using the GeoGebra software. Following a three-week preparatory period, Esin taught a forty-five-minute microteaching lesson. Microteaching is a technique used in pre-service training to improve prospective teachers' teaching skills and was part of the course requirement. Esin taught their peers a microteaching lesson on maximum and minimum problems that could be solved using derivatives. Data collection methods were interviews, direct observation, and documentation. PC screens and videos of the microteaching environment were recorded. Other data sources were the lesson plan and teaching materials.

The qualitative data analysis program ATLAS.ti was used to analyze the data. The definitions of orchestration types provided by Drijvers et al. (2013) were used. To determine how the orchestrations were lined up, the microteaching lesson video was divided into sections based on the moments of study (les moments de l'étude) defined by Besnier (2016) based on Chevallard (2002). We identified the OCs in each moment of study in line with the definition of OC (the orchestrations sequenced to serve a specific didactic goal of the prospective teacher) given by Alan and Akkoç (2023). We then analyzed how orchestration types are linked to form certain chains by identifying the instruments used during these orchestration chains and analyzing the didactic goals for which they are employed.

Findings

Our analysis revealed that identical orchestration chains (the same types of orchestrations arranged in the same order) can take on different structures to serve different didactic goals. This section will discuss two moments of study from Esin's micro-teaching lesson—the "learning and training moments II" and the "evaluation moments" indicated as shaded in Figure 1. We identified two distinct OCs for these moments of study, with identical orchestration types but for different goals. After giving a brief overview of Esin's teaching approach, we will present the two linking models emerging from our analysis.

Esin's micro-teaching practice

Esin mainly focused on using the derivative concept to solve maximum and minimum problems and developing an understanding of its geometric meaning. To achieve this goal, she first explained the concept of instantaneous velocity by approximating the number pairs (learning and training moment I). Then, she explained the instantaneous velocity-limit-derivative relationship geometrically (learning and training moment II). Next, she provided a geometric explanation of the relationship between the derivative and the maximum and minimum points (Synthesis-moments II). Lastly, Esin gave the students a real-world problem that involved applying derivatives to find the maximum and minimum points. The students (Esin's peers) then used the board and GeoGebra to solve the problem (Evaluation moments). Figure 1 illustrates the various orchestrations and OCs that Esin used throughout this process.

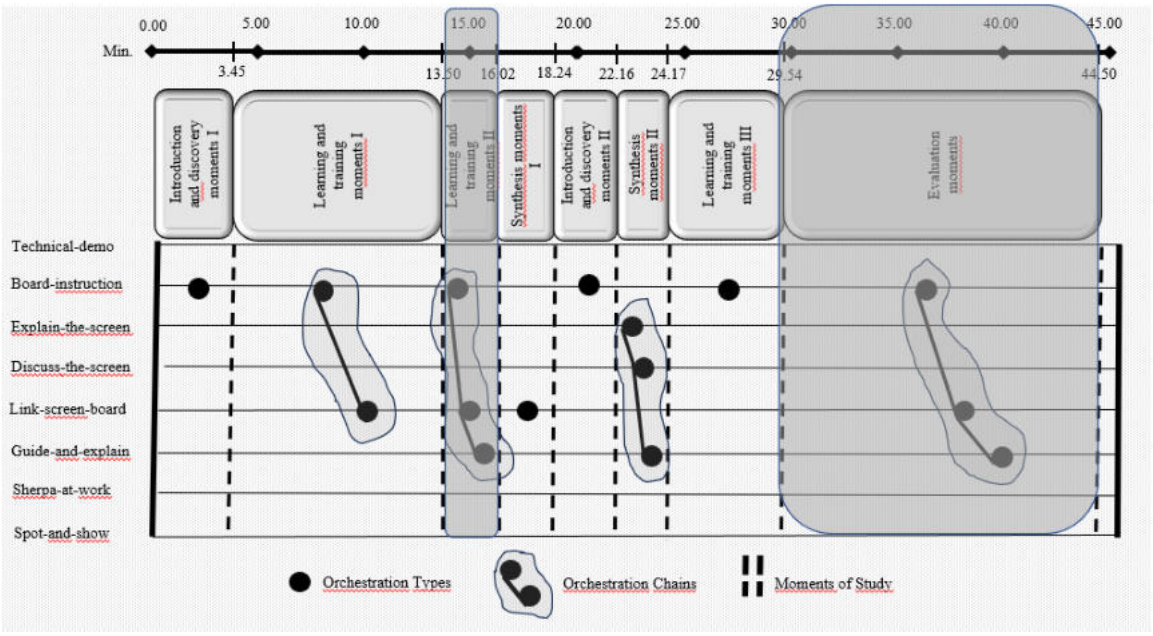

Figure 1: Analysis of Esin's micro-teaching lesson

Linking Model A

During this lesson section, Esin gave a worksheet that included five questions regarding the relationships among instantaneous velocity, limit, and derivative. After completing the first three questions on the board, she verified her answers on the screen using GeoGebra (the learning-and-training-moments I).

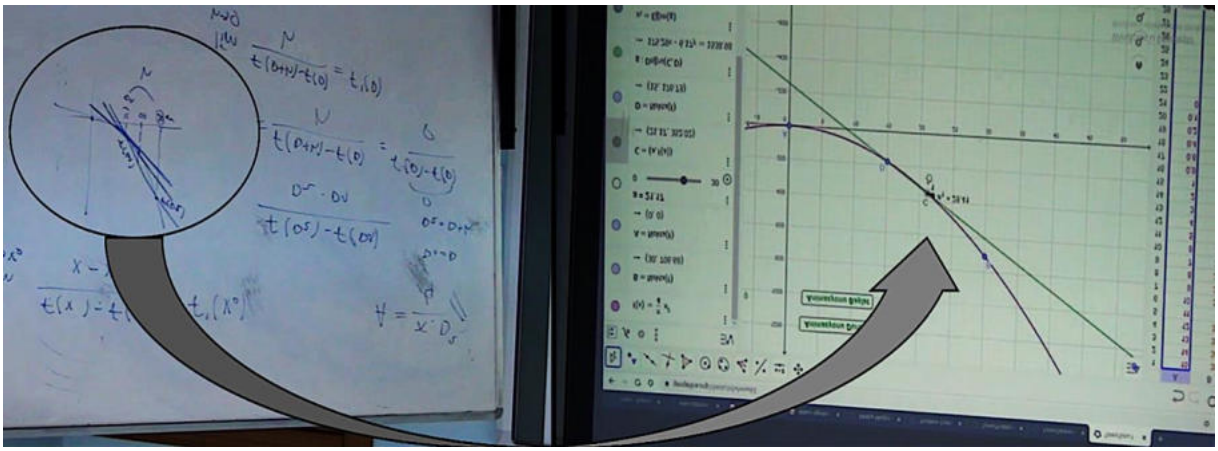

Figure 2. Screenshots from Esin's lesson (derivatives-limit-slope relationships)

For the other questions, Esin began to illustrate on the board how the secant lines approach the tangent or how the average rates of change approach the instantaneous rate of change. Using a step-by-step method, she explained why the result was undefined when h is zero on the board (Board-instruction orchestration). However, when the board was insufficient, she turned to GeoGebra (see Figure 2) and entered $h = 0.001$ to demonstrate that the secant line became the tangent. The result was 23.6, and she said, "So this is the slope of the secant line, the tangent, in that very, very small range" (link-

screen-board orchestration). Using the GeoGebra slide, Esin guided the students with explanations such as "Look, the slope is decreasing, right?". Esin said, "Look, when I get to that point, our secant line is now our tangent line. Our instantaneous rate of change is the slope of our tangent" and ended her explanations on GeoGebra (Guide-and-explain orchestration).

We identified Esin's implementation as a chain that consists of board-instruction, link-screen-board, and guide-and-explain orchestrations. In this process, she utilized GeoGebra because of the board's constraints. Esin overcame the board's constraints with GeoGebra's potential and created an OC with this characteristic feature. She drew attention to the limitation of one instrument and then tried to overcome this limitation with the potential of another instrument (we detailed the formation of this chain in CERME13).

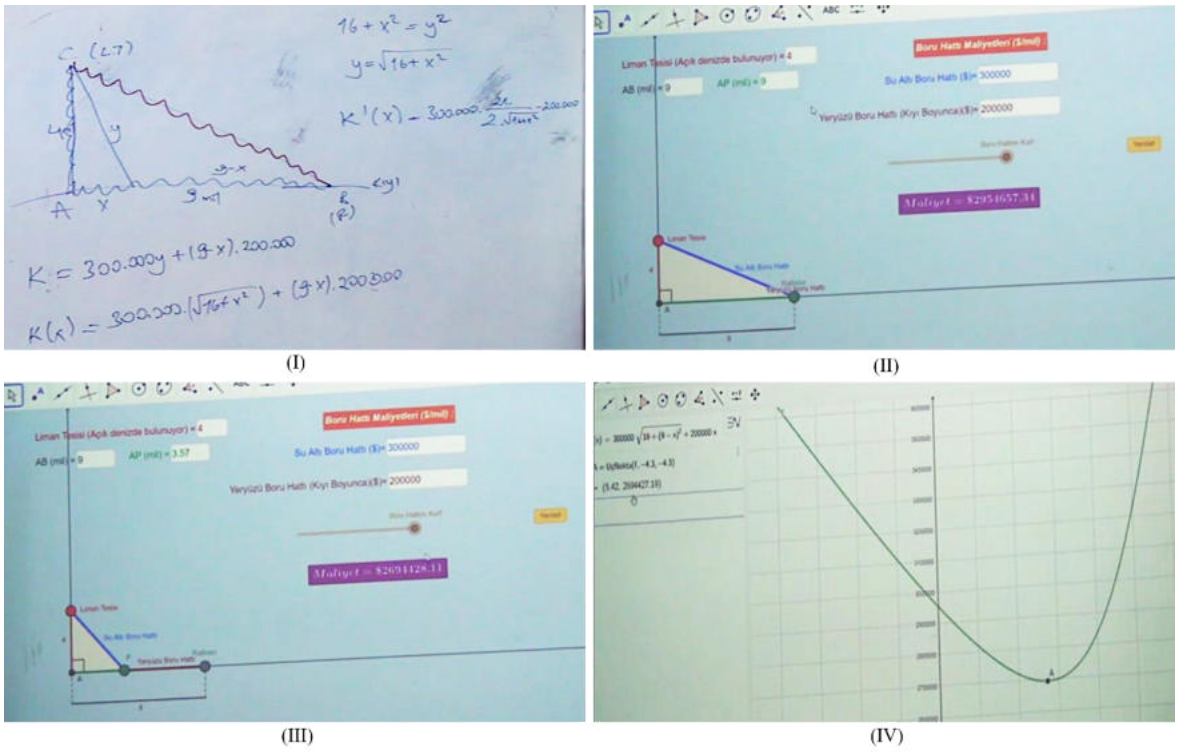

Figure 3. Screenshots from Esin's lesson (a maximum-minimum problem)

Linking Model B

Esin presented the students with a real-world problem to solve in the final section of the microteaching session, requiring them to apply what they had learned. The problem was a maximum-minimum problem about laying a pipeline from an oil facility some distance from the coast to an onshore refinery at the least cost. She told the students that they could work in groups and use calculators. After the students had found their answers, Esin went to the board and constructed the algebraic function to minimize the cost (Board-instruction orchestration). Then, she launched GeoGebra to test the accuracy of the values. The applet Esin prepared was the simulation of the real-world problem (See Figure 3). The model produced and plotted the findings as the variables in the simulation took on various values. Esin looked at cases where the pipeline crosses the simulation's

right triangle's hypotenuse and sides. Esin entered the 3.57 value obtained on the board and the very close 3.56 value into the simulation and showed that the least cost was 3.57 (Link-screen-board orchestration). Finally, Esin moved on to another GeoGebra activity to explain what this means graphically. Esin ended her explanation by showing on the graph that the value of 3.57 she found by experimenting in the simulation was indeed the extremum point of the function graph (Guide-and-explain orchestration).

Esin's practice uses the same types of orchestrations as in the previous section: a chain of board-instruction, link-screen-board, and guide-and-explain orchestrations. However, embracing a different approach, Esin used GeoGebra to show the possibilities of the real-world problem she presented and that the mathematically obtained value is indeed the minimum value. Unlike the other OC, GeoGebra was not strictly necessary here. However, the simulation allowed her to enrich the problem. Esin used simulation to engage with the problem and got results that could not have been obtained by using only the board. In summary, Esin carried out a teaching practice in which the problem's solution started on the board, was enriched with GeoGebra, and then focused on the intended result in GeoGebra. The data indicated that Esin used technology as an enrichment tool. The data showed that Esin used technology as an enrichment tool, prioritizing understanding with GeoGebra over lengthy blackboard procedures.

Discussion

This study aimed to expand upon OCs and explain the different linking models of orchestration types that form these chains. In this regard, we focused on two OCs that consist of the same types of orchestrations arranged in the same order. Our study revealed two linking models in the formation of OCs. We named the first linking model the "Linking Model A," in which the potential of one instrument and the constraint of the other instrument complements each other to form an OC. The second linking model is the "Linking Model B," where the teacher starts on the board, then uses the instrument to enrich the initial problem, and then focuses on the problem's original goal. These two linking models were identified based on the purpose of activating GeoGebra in teaching, aligned with specific didactic goals. Our findings show that the reason for activation shapes the orchestration's structure, making instrument activation and deactivation key determinants of linking models.

Drijvers et al. (2019) suggest elaborating on the relationship between orchestration types and instrumentation schemas. Ratnayake et al. (2024) describe how teacher decisions link orchestration types. In this study, we considered a specific OC as a whole and tried to identify the didactic goals behind it. The linking models we have put forward may have paved the way for us to comment on how orchestration types form chains and why the teacher forms specific chains. In future studies, we expect to increase the explanatory power of IO theory by identifying new OCs and linking models.

The findings of this study suggest that the concept of OCs, taken together with linking models, can help us better understand the patterns teachers use. According to Ruthven (2014), instrumental orchestration theory aims to identify patterns in technology-enhanced teaching practices. However, the patterns constructed by the participants in this study occur in the classroom. Teachers can design broader patterns related to schooling procedures beyond the classroom, which warrants further exploration in future studies. Additionally, GeoGebra (along with AR and 3D) was the only

technological tool used in this study. The orchestration structures (OCs) could become more complex when multiple technological tools are involved. A deeper understanding of OC dynamics requires further case studies. Finally, we suggest integrating them into the professional development of both prospective and in-service teachers.

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Problem-posing based on digitally enhanced experimental mathematics: What does it demand of teachers?

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This is an excerpt from an experienced mathematics-teacher educator's self-study that aims to delineate the expectations of mathematics teachers if the linkage between experimental mathematics, enhanced by digital tools, and theoretical, formal mathematics becomes a part of the school reality. The problems appearing in the paper are authentic and are posed through experimentation in a dynamic digital environment that reveals phenomena otherwise probably undiscoverable. They also present and question situations that necessitate the acceptance of the solution only as a plausible one that is not ascertained from the mathematical-epistemological point of view. The educational implications of the study are discussed from the perspective of teachers, their knowledge, and their competences.

Keywords: Catenary and parabola, computational triangulation, experimental mathematics using digital tools, critical thinking, mathematics teachers' digital competences.

Introduction: Theoretical frameworks of the study

Experimental mathematics in school in the era of advanced digital platforms

Experiment-based conjecture has always been integral to mathematical investigation as an inductive basis for further formal deductive investigation and has been growing with the digital upsurge in mathematics research. In reference to the role of computers, Devlin considers the epistemological implications as follows:

> [E]xperimentation is (...) a significant part of the mathematical enterprise (...). In particular, this (...) gives an epistemological status to assertions that, while supported by (...) experimental results, have not yet been formally proved, and in some cases may never be proved. (Devlin, 2019, p. 65)

Experimental mathematics (EM) characteristics relevant for this paper include gaining insight and intuition, discovering new relationships, visualizing math principles, testing, and, especially, refuting conjectures (Borwein, 2012, pp. 73–74). The intertwining of EM and formal mathematics (FM) as described by Arzarello et al. (2012, p. 132) *inter alia,* enhanced by the "heavy use of new technology", touches upon both epistemological and didactical dimensions. Digital tools, however sophisticated and advanced they may be, do not absolve a user of the need to be alert, critical (Engelbrecht & Borba, 2023), and appropriately skilled and educated in working with them, as the examples in this paper suggest. One source of questionable results of digital experiments is imprecision in the resolution of graphical output. The following configurations, accessible at https://www.geogebra.org/m/ym5vskrc and https://www.geogebra.org/m/qqvrtwzd, provide lucid examples of the need to adopt a critical attitude toward visual computer outputs, even if provided by advanced platforms, and to test them by means of FM. Indeed, Borwein and Delvin (2009) dedicate a special chapter to the need to test computer-generated experimental mathematical results by FM. This is sometimes easier said than done, however, because mathematical problems are not always

explicitly solvable. Thus, special methodologies are needed to interlink EM and FM as a complement to straightforward mathematical testing of experimentally observed results. Denzin (1970, p. 301) worked out the so-called "method of triangulation" in social-sciences research in order to amplify the credibility and validity of research findings by deriving mutual support from various methods. Abramovich (2023) "imports" this methodology from the social sciences to substantiate solutions to mathematical problems that are attained by combining various methods and tools. It is one of the methods that I will apply in this paper to experiment-based problems and attempts to solve them. When an agent as powerful as a digital dynamic platform is added to the classroom reality, mathematics teachers are urged to acquire additional qualifications that pertain to it.

Mathematical knowledge for teaching that involves EM

Mathematical experiments are a rich source of context-based problems that are not always easily solved or even solvable by methods available within the school curriculum, pose a challenge to teachers and impose special demands on them insofar as the teachers adopt mathematical experimentation as part of their classroom mathematical culture. They may face, for example, students' experimentation that yields unexpected results; therefore, they must be equipped with professional pedagogic approaches to the handling of such an occurrence. Kortenkamp (2004, p. 62) is one of a number of mathematics-education scholars who see the requirement of teachers' specific EM–FM-oriented (mathematical) knowledge as "an opportunity to change the focus of teaching from the arithmetic to mathematics in a more global fashion". Therefore, this study proposes a deeper look at this specific strand of mathematical knowledge that is essential for intertwining EM with FM, an expression that I denote below by MK–EM.

Self-study: The choice of a theoretical approach

This study focuses on mathematics teachers' perspectives in coping with EM practices. Therefore, I chose self-study as an appropriate framework for the tracking and analysis of my own actual experience, the challenges I faced at each stage, and the decisions I made. In Hauge's (2021) comprehensive review on self-study by teachers' educators, the interplay between research and practice is emphasized.

> Self-study tends to be methodologically framed through questions, challenges, problems and desires related to areas in practice (...), and where the method most appropriate for examining and collecting data is used according to the study's intent. (...) The researcher must show who he / she is (...) by illuminating thoughts, opinions, interpretations, actions, development, and it (*self-study*) must involve the construction of meaning and knowledge. (Hauge, 2021, ch. 4)

In this spirit, being a mathematician, a mathematics teacher, a teachers' educator, and a researcher, I chose the approach linked to these areas, striving "[to] formalize the work and make it available to the professional community" (Hauge, 2021, ch.4). The study was performed from the teacher's perspective; accordingly, I monitored step-by-step my own experience with digitally empowered EM along a sequence of experiments with reflection properties of the catenary inspired by those of the parabola, performed on the GeoGebra platform. The questions I posed were inspired by the ostensible similarity of the two curves, in the spirit of mathematical problem posing summarized by the "what

if not?" approach, and I tried to answer them. These experiments, I suggest, are illustrative of problems that stem from mathematical experiments, albeit obviously not exhaustively. The experiments, outputs, conjectures, and problems that emerged in the search for formal corroboration or refutation of the conjectures are original and authentic. My methods of solution and my considerations in appraising them emerged "on the fly" and I consider them no less valuable for the purposes of this study than the results obtained. In formulating the research questions, I found it important to refer explicitly to teachers' mathematical knowledge specific to EM (MK–EM) as a basis for the critical thinking needed to appraise the results of the experiments. I adopted some of the components characterized by Geraniou et al. (2024), modifying them to EM, namely:

MDCT2-EM: an awareness of which digital tools to apply in different mathematical situations and contexts and of the capabilities and limitations of the various tools in relation to EM.

MDCT3-EM: "doing mathematics" (problem-solving as defined in MDCT3) within its specific EM adjustment.

Accordingly, the research questions are:

RQ1: How do the characteristics of MDCT2-EM, MDCT3-EM manifest in problem-posing and -solving on the basis of mathematical experiments that make use of digital tools?

RQ2: What occurrences in mathematics teachers' mathematics knowledge (MK-EM) and critical thinking (CT) specific to EM-based problem-posing manifest in *posing problems* based on mathematical experiments and appraising their solutions?

Methods

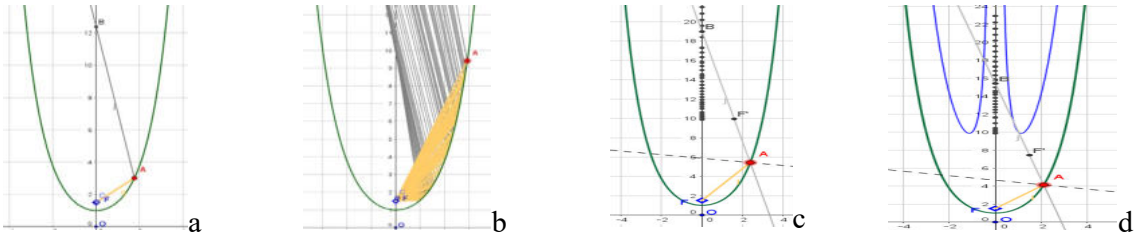
Data collection and analysis

I adopted the approach outlined in LaBoskey (2004) (cited in Hauge, 2021) that self-study uses primarily qualitative methods that are often invoked in educational research. Working out all stages illustrated in the example below, I carefully transcribed all the self-questions and self-answers that arose in the course of this work in real time and a posteriori in a quest for the answers to the research questions. Being an experienced educator of mathematics teachers, I made notes of all occurrences that might have a direct or indirect impact on teachers' competences. Having completed the whole series of problem-posing and -solving, I sorted out my notes in accordance with the characteristics I sought in order to answer the research questions as comprehensively as possible. In the following section, I present an excerpt from this data and its analysis.

The experiment and one of the problems appearing in it

It is hardly surprising that Galileo mistook the catenary ("the chain curve") given by y=cosh(x) (= $0.5(e^{x} + e^{-x})$) for a parabola; they do look similar, for good reason. The approximation of the catenary by first two terms in the Taylor series is $y = 0.5x^2+1$. The next term is $x^4/24$, which means that the difference between the catenary and the parabola is $0.5x^2+1$ is O((x^4)), which yields "pretty close" graphs for small values of x, i.e., around x=0. What does "pretty close" mean for two objects that are not only purely mathematical but also have certain physical properties? A parabola is known to have an important focal property: Any ray emanating from a source located in its focus F, when

mirrored from its inner surface, becomes parallel to its axis. So how do the rays emanating from F behave after being mirrored by the catenary? See Figure 1a for the configuration; the tangent and the normal are not visible. For clarity and due to symmetry, it suffices to consider the reflection only to the right of the y-axis. By applying the Trace On command to the emanating (yellow) and reflected (grey) rays, I follow their behavior as Point A glides along the catenary, see Figure 1b. It suggests, in particular, that there exists a lowest location of Point B where the rays intersect with the y-axis; it is located slightly higher than (0,10), see Figures 1b and Figure 1c.



Figures 1 a, b, c, d: Rays, reflection, intersection point B, its lowest location, graph of $y_B(x_A)$

These conjectures require formal validation or rejection. Therefore, the following problems are formulated and need to be solved**:** Problem (a): Prove or refute the existence of a minimum of $y_B \equiv y(B)$, as a function of x_A. Problem (b): If the minimum exists—what is its value? The solutions by two different methods and applying two different tools are outlined here. In both problems, I express y_B as a function of x_A and look for its minimum. The ray emanating from Focus F(0,1.5) is determined by Points F, $A(x_A, y_A) = A(x_A, \cosh x_A)$. The reflected ray passes through Points A and $F'(x_{F'}, y_{F'})$, the reflection of F with respect to the normal at A: $y = m_{AF'}(x - x_A) + y_A$, or $y = m_{AF'}(x - x_A) + \cosh x_A$; $m_{AF'} = \frac{y_{F'} - y_A}{x_{F'} - x_A}$. One possible way of finding the values of $x_{F'}, y_{F'}$ as a function of x_A is applying equidistance of F and F' from p, the normal to the catenary at A, which yields the location of B: $y_B = y_{F'} - \frac{y_A - y_{F'}}{x_A - x_{F'}} x_{F'}$; $x_{F'} = \frac{-3\sinh x_A + 2x_A + 2\sinh x_A \cosh x_A}{\cosh^2 x_A}$, $y_{F'} = 1.5 + \sinh x_A \frac{-3\sinh x_A + 2x_A + 2\sinh x_A \cosh x_A}{\cosh^2 x_A}$.

An alternative way of constructing Function $y_B(x_A)$ is to consider ΔFAB, in which the normal to the tangent to the catenary at Point A serves as an angle bisector of ∠A. The standard mathematical ways of finding the minimum of y_B as a function of x_A are by differentiating it with respect to x_A, finding the point(s) where the derivative vanishes, and making sure that it is the minimum. In view of the lengthy and intricate expression for y_B, I used the CAS module of GeoGebra for symbolic computations. The expressions for $y_B(x)$ obtained by different approaches did not look equal; an additional computerized procedure ensured their identical equality. In neither approach was CAS able to solve the equation $y'_B = 0$; hence, I turned to Wolfram Mathematica (WM) and obtained a similar result. Unable to complete the formal mathematical procedure of verifying the existence and finding the minimal location of y_B, I turned to "looking under the lamppost," namely, plotting the graph of $y_B(x)$. Figure 1d represents the GeoGebra graph of y(B) as function of x (i.e., of (x_A)), side-by-side with tracked locations of B. Computation of the minimal value of y_B by WM yields the result y(B)= 9.85727330600264, compatible with the location of B as shown in the graphs in Figures 1b–d.

Findings

On the one hand, I cannot claim that Problems (a, b) are explicitly and completely mathematically solved. On the other hand, following two mathematical paths and using two different platforms, I arrived at the numerical value that seems to answer the question of the lowest position of Point B to a precision of 14 decimal digits, although precision alone is not a guarantee of correctness. The plausibility of the positive answer to Problem (a) and the numerical value of the answer to Problem (b) are supported (not verified!) by concurrences of various numeric and graphic results. In Table 1, I present some examples of characteristics appearances discerned on the basis of data.

Table 1: Examples of appearances in the data of characteristics defined in research questions

Characteristics	Examples of appearances and explanations
MDCT2-EM: awareness of which digital tools one should apply	a. The phenomena observed in the experiment were detectable due to special properties of the GeoGebra platform, including construction of dynamic applets subject to the ray-reflection laws, tracing graphic objects and thus discerning phenomena related to their behavior *en masse* (see Figure 1b) across a range of parameter values. The possibility of tracing the location of B on the y-axis (see Figure 1c) provided additional support for the existence of its lowest location and its approximate value. b. Application of CAS and WM for extremum value computation, computing the actual minimum of y(B) by WM, indicates awareness of affordances of these tools specific to EM.
MDCT3-EM: problem-solving based on EM	Discerning the existence of the lowest location of B on the y-axis (Figure 1b–c) led to mathematical problems to be posed and solved. By "doing mathematics" following experiment observations, EM and FM are intertwined as a matter of fact.
CT + MK-EM Critical thinking based on mathematical knowledge	The variable range can be neither exhaustive nor precise in digital representation. Therefore, the observed phenomenon induced a conjecture that had to be cross-checked for validity and evaluated by formal mathematical methods and was dealt with by FM involving MK–EM. FM did not provide an explicit answer; therefore, the "triangulation" was applied to the results. Recognizing that the more diverse indications supportive of the results are accumulated, the more plausible (though not mathematically certain) the results are, is a manifestation of CT.
CT + MDCT2-EM Application of digital tools to resolve the issue originating in critical appraisal of results	Encountering two different expressions for y(B) obtained by two different formal tool-aided solutions (CT), and making sure that they are identical, abate doubt on the way to the solution; this was performed by appropriate use of the digital tools (CAS)— awareness of which digital tools one should apply.

Discussion

This study focused on a specific form of mathematizing—problem-posing based on digitally empowered mathematical experiments that exemplify the EM–FM linkage—from a teacher-centric perspective. The answers to the research questions emerge from my self-study based on a specific set of experiments:

RQ1: By invoking constructs defined by Geraniou et al. (2024) and extending them to EM, I was able to focus on EM-oriented tool affordances and "doing mathematics." The appearance of both characteristics throughout the findings (see Table 1 for examples) suggests their relevance and efficiency for experiment-based problem-posing and -solving. The findings indicate the explicit and essential appearance of MDCT2 extended to EM. As for MDCT3-EM, which refers originally to problem-solving and doing mathematics, in the EM context it extends to problem-posing, which I consider an essential addition to its characteristics.

RQ2: The specific MK required in the EM context, denoted by MK-EM and manifested in different instances in the findings, is on the one hand essential for successful EM-based mathematical activity and, on the other hand, not necessarily included in teachers' MK related to FM. Some examples of occurrences of such knowledge appearing in Table 1 are awareness of the finite variable range in an experiment as against the infinite actual range in FM, the possibility of different analytic attestations of the same result, and the need to search for an *en masse* graphic representation of dynamically obtained behavior of a variating object, such as the one of emanating and reflected rays. The findings also indicate that CT is indispensable in appreciating the outputs of mathematical experiments, more so because some of the outputs are not mathematically dispositive and require critical appraisal of each result separately and of several results as a mutually supportive composite (Engelbrecht & Borba, 2023).

I suggest that self-study was an appropriate method for this type of research. The research aimed to reveal and meticulously record mathematical, computational, and tool-related deliberations of a teacher during encounters with phenomena emerging from an authentic mathematical experiment. The results of the experiment and the types of problems posed, along with their mathematical challenges, were unclear a priori, and the solutions were neither obvious nor premeditated. Thus, notwithstanding my previous experience, the deliberations that comprise the data genuinely reflect, albeit not exhaustively, those that may emerge in similar situations of coping with the unexpected, and do not seem to bias the researcher's opinions.

Data analysis was performed using qualitative research methods. Clearly, with different focuses on EM-FM-in-school-related research, other methods should be considered, depending on research aims, populations, and questions.

Practical and pedagogical implications and future research

The answers to both research questions indicate the important and distinctive modifications of all the characteristics on which they focused for digital–tools-based EM. In reference to the appearance of digital tools in school mathematics, the EM-FM linkage indicates lucid awareness of a distinction between the affordances of tools for EM purposes versus limitations as a source of formal conclusions. The basic understanding that *no output of a mathematical experiment, however*

advanced the tools applied in it may be, is acceptable unless it accords with mathematical theory, leading to a problem that either affirms this accordance or refutes it, is the essence of the twining of EM with FM.

In reference to the expectations of teachers' professional competence, as an educator of teachers, I suggest that further actions are needed in both theoretical and practical- pedagogical respects. In respect to theory, further research seems to be welcome in search of a deeper insight into the adoption of such a mode of teaching school mathematics from the perspectives of teachers, educators, and education systems. A step in this direction is taken by Naftaliev and Barabash (2024). Another important theoretical, pedagogical, and practical-systemic aspect, in my view, is the scope and spirit of school mathematics, as Kortenkamp (2004, p. 62) put it (cited above): "an opportunity to change the focus of teaching from the arithmetic to mathematics in a more global fashion".

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Incorporating programming into mathematics education: The adaptation of a teacher's craft knowledge

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In recent years, programming has gained a more prominent role in mathematics education in various European countries. However, the incorporation of digital tools, such as programming, is not always a straightforward process as it may require, among other things, an adaptation of teachers' craft knowledge. This paper reports on a case study analysing a Swedish teacher's incorporation of programming into his mathematics teaching and the rationales behind such design, using the Structuring Features of Classroom Practice framework as a conceptual lens. The results reveal how the teacher's background shapes the use of programming in his classroom. It also illustrates that the way programming is intended to be incorporated into Swedish school mathematics places high demands on teachers when it comes to teaching students both how to program and how programming can serve as a tool for learning mathematics.

Keywords: Mathematics Education, Programming, Mathematics teaching, Craft knowledge, Structuring features of classroom practice.

Introduction

Programming has long served as an everyday tool for mathematicians. During the 2010s, it was also incorporated into school mathematics in several European countries. Since 2018, students in Swedish compulsory schools are expected to use programming as a tool starting from grade four (age 10), with the objective of creating and utilizing mathematical algorithms. In the non-mandatory upper secondary school (ages 16–19), the extent to which programming is incorporated into mathematics courses varies among different study programmes. When included, programming is intended to be used as a tool for problem-solving, for solving tasks numerically, and/or for data processing. Thus, the Swedish curricula do not associate the use of programming with any specific mathematical content but rather with different practices (Tamborg et al., 2024). Nor do the curricula specify which programming concepts students are expected to master or which specific programming language they should use. In Sweden, where there is no dedicated computing subject, unlike in England (Tamborg et al., 2024), students are expected to learn programming within their mathematics lessons.

In his study of Swedish mathematics teachers' use and perception of how to implement programming in school mathematics, Humble (2022) highlights that many teachers are critical of the way programming is expected to be incorporated into school mathematics. Teachers express how limited time and vague instructions from both school leaders and other stakeholders have negatively affected the incorporation. Kilhamn et al. (2021) illustrate how the arguments for incorporating programming in school mathematics differed among 20 Swedish mathematics teachers. The first category of arguments relates to the view that programming could be a powerful tool on a more general level once the teachers and the students master it. However, learning to master the tool is regarded by many of these teachers as a time-consuming endeavour. The second category of arguments is related to the view that programming in itself is engaging and that it connects to reality. These teachers do not argue

that the use of programming will increase students' understanding of mathematics but rather increase their engagement during mathematical activities. A third category of arguments is related to the view that the use of programming may develop students' computational thinking. The fourth category of arguments relates to the viewpoint that using programming could enhance students' learning of mathematics. These teachers contend that programming can aid students in comprehending mathematical concepts and that programming itself can be regarded as a mathematical activity.

The studies of Humble (2022) and Kilhamn et al. (2021) indicate that mathematics teachers find it challenging to incorporate programming in their mathematics classrooms and hold varying perspectives on the rationale for using programming in school mathematics. It could be presumed that this, in turn, influences how these teachers incorporate the tool into their classrooms.

In this paper, *the Structuring features of classroom practice* (SFCP) framework (Ruthven, 2009) will be operationalised to analyse how structuring features shape a Swedish upper secondary school teacher's incorporation of programming into a mathematics course. Such analysis is of extra interest in the Swedish context where the curricula offers mathematics teachers limited guidance on how to operationalise such incorporation (Tamborg et al., 2024), a situation that appears to frustrate teachers (Humble, 2022). Ruthven (2009) argues that the rationales for how to incorporate digital technology rest on teachers' *craft knowledge*, which encompasses the professional knowledge and thinking used in their classroom teaching. He claims that part of this knowledge has been developed through "(often unconscious, sometimes reflective) adaptation in response to experience" (p. 132). Simsek and Clark-Wilson (2024) argue that the use of the SFCP framework has the potential to make teachers' craft knowledge "more observable and accessible, making it not only apparent to researchers but also accessible to practitioners and thus bridging the gap between research and practice in mathematics education" (p. 22). Given the Swedish context, this paper thus aims to investigate the adaptation and/or development of a teacher's craft knowledge in relation to structuring features of classroom practice when incorporating programming in upper secondary mathematics education.

The structuring features of classroom practice framework

The technological development of the last few decades has meant that for today's mathematics teachers there is a plethora of digital tools available. Teachers are therefore faced with the task of creating a "coherent use of a relatively small selection of [such tools] to form an effective resource system" (Ruthven, 2009, p. 145). The incorporation of new technologies in classrooms is dependent on what Ruthven (2009) refers to as a professional adaptation, and he identifies five structuring features of classroom practice which shape such adaptation. *The curriculum script* is a structuring feature which stems from the teacher's professional knowledge and concerns goals and actions related to the specific mathematical topic, guiding the way the teacher teaches the topic (Ruthven, 2014). A script is in this framework viewed as a psychological construct which relates to an "event-structured organisation of knowledge, forming a loosely ordered model of goals, resources and actions for teaching the topic, incorporating potential emergent issues and alternative courses of action" (p. 386). It thus affects the teacher's lesson design as it interweaves the mathematical ideas to be learned, the tasks to be undertaken, the relevant activity format to be used, and the potential student difficulties to be expected (Ruthven, 2014). A second structuring feature involves the choice of mathematical tools

and the associated curriculum material and how these build a coherent *resource system*. Ruthven's (2009) meaning of the term resource involves "some (physical or virtual) artefact which has either been designed specifically for curricular purposes, or been the subject of educational appropriation for such purposes" (p. 146). The classroom incorporation of new technologies also implies an adaptation of existing activity formats which set the scene for actions and interactions between participants during classroom activities (Ruthven, 2014). Such adaptation of activity formats results in a revised *activity structure* for lessons incorporating new digital technologies. Orchestration types, identified by Drijvers et al. (2010) and originally associated with the Instrumental Orchestration theory, have recently been utilized alongside the SFCP framework (e.g., Simsek & Clark-Wilson, 2024) to describe activity formats. *Time economy* is a structuring feature related to the return (in the form of the pace of the lessons as well as the students' advanced understanding) of the time invested when incorporating a digital tool in mathematics classrooms (Ruthven, 2009). The last structuring feature is the *working environment* which relates to changes in the physical environment as well as to the classroom routines. This implies that the classroom may be organized differently when utilizing digital tools compared to more traditional mathematics lessons.

Method

The findings presented in this paper are part of a case study where the author studied a teacher and his class of Swedish upper secondary school students in the science program, utilizing programming throughout a course in mathematics (Mathematics 3c). In the autumn of 2022, data was collected during five lesson activities where the students were expected to utilize programming to solve mathematical tasks numerically. The first three activities concerned a numerical method for calculating function limits (related to the limit concept, the definition of the derivative, and the derivative of exponential functions), the fourth activity concerned a numerical method for calculating local minimums of functions, and the last activity concerned how to calculate integrals numerically. Since it was expected that students had varying extents of experience using programming in primary and lower secondary school, the school assigned about ten extra hours associated with the previous course to teach students how to use basic programming in mathematics. This is rare in a Swedish context since students are expected to be trained in how to program during regular mathematics lessons. The mathematics teacher in this study had worked as a teacher for 14 years and was also a teacher in physics and programming (specialised in algorithms and computation). He could thus be regarded as a proficient programmer, and he had played an important role at the school in designing the incorporation of programming into the different mathematics courses.

Data collection and data analysis

An audio-recorded and transcribed interview with the teacher, together with field notes taken during all five lesson activities, served as the main data sources when using the SFCP framework to analyse the adaptation of the teacher's craft knowledge. During the semi-structured interview, conducted after the course, the teacher was asked questions concerning his views on programming as a tool in school mathematics and the design of each learning activity. Field notes were taken to portray how the teacher introduced the mathematical content to the students. The researcher also had access to the online programming environment Google Colab used by the students, in which the mathematical

tasks as well as solutions to the tasks (including code) were integrated. After listening to the interview and reading the transcription several times, extracts were classified based on the five components of the SFCP framework, as described and exemplified in the literature presented in the previous section. Parts of the field notes related to the teacher's craft knowledge were also classified according to the components of the framework. Triangulation of data was then conducted, allowing the author to gain a more comprehensive understanding of the teacher's craft knowledge and how his described intentions (related to the components of the SFCP framework) were manifested in the classroom.

Results and analysis

Guided by the structuring features of the SFCP framework, the analysis focuses on the teacher's incorporation of programming in his mathematics classroom and how such incorporation has brought about an adaptation of the teacher's craft knowledge.

Curriculum script

As a foundation for his curriculum script, the teacher in this study expresses a positive attitude towards the incorporation of programming in mathematics education, although he had hoped for more guidance in the curricula on how and when to use programming in the mathematics courses.

> Teacher: My attitude is basically quite positive, although I think it is clear that there has to be a balance between traditional math and programming. But I think, in this digitized society, it is good to have knowledge of computations using programming.

The teacher's design of the learning activities is mainly based on the view that programming offers students opportunities to conduct repeated calculations, which are too time-consuming to conduct using pen and paper, and which could be used to solve mathematical tasks numerically. Based on this view, the primary goals of the learning activities, classified as central parts of the curriculum script, were for students to enhance both their understanding of numerical methods for solving mathematical tasks and their understanding of mathematical concepts through the use of step-by-step instructions.

> Teacher: If you take numerical integration where you sum a bunch of rectangles. Then it may be that you understand what integration or the integral represents more through programming. […] I also think that sometimes the programming can be closer to the definition. If we again take integrals as an example, the definition of an integral strictly mathematically is via this limit value and not via a primitive function.

As part of the curriculum script, the teacher also describes the rationales for the designed activities and how the limited amount of time available to teach students fundamental programming has affected the tasks chosen for the learning activities. These actions are classified as part of the teacher's curriculum script since they are closely associated with the choice and teaching of the mathematical topics. This part of the script is affected by the way programming has been included in the Swedish mathematics curricula. First, the fact that programming was added to the course without any other content being removed forced the teacher to find ways to incorporate the use of the tool and still have time to handle the pre-existing mathematical content in the syllabus. Therefore, the teacher and his colleagues initially identified a number of mathematical tasks related to the mathematical content of the course that were considered well-suited for solving using programming. Some of these tasks were later excluded so that the remaining tasks could be solved using a similar code structure (involving a single loop) together with a limited number of programming concepts (primarily related to variables,

iterations, selection statements, and print statements). Second, since the curriculum for primary and lower secondary school does not explicitly specify which specific programming concepts students are expected to learn, the teacher does not expect new students to master all of the abovementioned concepts. Thus, to prepare the students for the course, the teacher and the school identified the need for additional time associated with the previous course in mathematics to teach students about fundamental programming concepts and the code structure.

With the intention of keeping a clear focus on the mathematical algorithms and not spending an extensive amount of time debugging, the designed task sequences of the learning activities contained examples of code which the students could reuse and modify. This part of the script has evolved from the teacher's prior experiences of using programming solely as a mathematical problem-solving tool.

> Teacher: When [the curriculum] stated that you should use [programming] for problem solving, my interpretation was that then you must be a good enough programmer so that you can write the code from scratch. But then I noticed that quite a few students got stuck in like just… yes but like print [statements] or wrong brackets or something like that. Then they didn't get over that step so they didn't learn anything. And I guess I also think that it's not really a programming course that we're doing, but the focus is on understanding the math behind the algorithms.

In summary, it could be claimed that the teacher's craft knowledge related to the curriculum script has been developed and adapted through the interplay between his own view of programming as a tool in school mathematics, the conditions given by the syllabus, and his own prior experiences.

Resource system

The teacher's resource system comprises digital tools and curriculum material. Classified as central parts of the resource system are the programming language Python and the programming environment Google Colab. The teacher also used the dynamic mathematics software GeoGebra, with which the students were familiar, to make graphical representations of functions (e.g., when introducing the limit concept and the definition of the derivative). This is an example of double instrumentation (Ruthven, 2014) where the teacher manages to handle an existing tool alongside the new one.

The teacher provides specific rationales for the choice of using the programming language Python.

> Teacher: So, I have no formal training in Python, but it felt like a simple programming language for math. There are a lot of built-in math functions from the beginning and relatively little code is required to access the algorithmic thinking compared to like Java or C++ which is a bit more software-related. I also think that Python has a lot of support for making quick visualizations, for example, of processes and so on. It's not something that I work with together with the students, but I demonstrate and it's easy to do. And I have the impression that Python is one of the big languages when it comes to, for example, machine learning, which is something we're aiming for in some way here at the school [in the course Mathematics – specialization].

The teacher's rationale for using Google Colab is that it offers the opportunity to design curriculum materials such as digital worksheets containing task descriptions and windows where students can write and execute their code as well as write their own text. Also classified as part of the resource system is the code structure (mentioned in relation to the curriculum script) which the students had been trained to use and which was applicable to all tasks presented to the students. The code structure included a specific WHILE-loop and an IF-statement used to control when the loop should terminate.

Activity structure

The field notes served as the main data source when analysing the teacher's activity structure during the learning activities, which in many ways resembles traditional teaching in mathematics. As an introduction, the teacher often introduced the mathematical content and methods for solving given tasks using programming in front of the class. Using the typology of orchestration types related to the use of technology in mathematics education, presented by Drijvers et al. (2010), as an analytical construct, this is classified as *technical-demo* orchestration. In this type of orchestration, the teacher demonstrates different techniques required to use the programming environment (e.g., how to construct a given loop). However, the teacher also connected his use of programming to the specific mathematical content, such as relating the generation of function values to the limit value, which is classified as *explain-the-screen* orchestration. There were also examples of *link-screen-board* orchestration, where the teacher stressed how the actions taken when programming could be represented in different media (e.g., algebraically using the whiteboard and graphically using GeoGebra). During the introductions, the teacher asked the students to reflect on new mathematical content as well as on code presented on his screen. The outcome of these reflections, classified as *discuss-the-screen* orchestration, was often a collective discussion about how the use of technology related to mathematics. After the introductions, the students started to solve the tasks presented to them in Google Colab while the teacher moved around the classroom and supported students' work when needed.

Time economy

Several aspects related to the teacher's curriculum script are also classified as time economy features. The additional time associated with the prior course, the choice of tasks to limit the number of programming concepts that the students should master, and the use of a given code structure could all be seen as examples where the teacher tries to change "the 'rate' at which the physical time available for classroom activity can be converted into a 'didactic time' measured in terms of the advance of knowledge" (Ruthven, 2014, p. 386). But the teacher also reflects on such 'rate' in relation to the national exam in mathematics, which do not include tasks involving programming aspects:

> Teacher: In a way, you can say that [the students] would probably have managed the Mathematics 3 course with better results if they hadn't spent so much time on programming and algorithms, if you just look at a national exam. There is no programming there, or at least very seldom. But I try to plant a seed of thought for the future which is probably quite good if you are going to continue with math since you won't get like integrals that can always be solved [algebraically] and so on.

The verbatim extract illustrates how the design of the national exams is associated with the time economy feature since the conversion rate into 'didactic time' is measured not only in relation to the intentions and goals of the syllabus but also in relation to the success rate at the national exams.

Working environment

The analysis of the field notes reveals small adjustments to the working environment in the classroom during the learning activities involving programming. Since the students' textbook did not contain any sections specifically related to the use of programming, the teacher integrated his designed tasks within the programming environment Google Colab (as described earlier). This implies a shift from

solving mathematical tasks in the textbook using pen and paper to a new routine where all students documentation (i.e., the code and the task solutions) is performed within the programming environment using the Chromebook computers provided by the school to each student at the start of their upper secondary school studies.

Discussion

In this paper, the SFCP framework (Ruthven, 2009) has served as a valuable lens to portray and analyse how various features of classroom practices were linked to the adaptation and development of the teacher's craft knowledge during the incorporation of programming in his classroom. In accordance with Simsek and Clark-Wilson (2024), I argue that the results reveal how different structuring features are more prominent than others and to a greater extent connect mathematical and technical aspects. The results particularly highlight how the craft knowledge associated with the curriculum script was adapted in various ways and how such adaptations in turn affected many of the other structuring features. For example, the aim to keep a clear focus on the mathematical algorithms and not spend an extensive amount of time debugging (associated with the curriculum script) led to the development of a code structure related to the resource system. The findings also reveal that the teacher's status as a licensed programming teacher shaped many of his decisions (e.g., focusing on students' use and design of algorithms in order to solve mathematical tasks numerically). The teacher's existing craft knowledge relating to teaching programming thus affected and eased the incorporation of programming in his mathematics classroom. The teacher belongs to the category of teachers who, according to Kilhamn et al. (2021), view programming as a tool that could help promote students' understanding of mathematical concepts (e.g., integrals), and he regards the use of programming as a mathematical activity in itself. The teacher also looks beyond the course and grades on the national exam when he argues that the use of programming may widen students' view of mathematics, provide them with overarching ideas concerning different ways of solving mathematical tasks, and increase their digital competence. For teachers with no or little prior experience of using programming, the rationale and motivation for using programming in school mathematics may differ significantly and be less holistic. Further research is thus needed to explore how mathematics teachers with diverse views on programming and varied backgrounds adapt their craft knowledge when using the tool, and how these adaptations affect students' learning of mathematics.

The results also demonstrate how the adaptation of the teacher's craft knowledge was influenced by the way programming has been introduced into the Swedish curricula. The inclusion of programming in the mathematics curriculum, without removing any existing content and without specifying which programming concepts students should master, affected the teacher's task design in several ways. This was also the reason for the additional time allocated to the prior course. I argue that great demands are placed on Swedish teachers when trying to teach students about programming alongside using the tool for mathematical purposes. This challenge is also one of the reasons why Swedish teachers express their criticism towards how programming is expected to be incorporated into school mathematics (Humble, 2022). Unlike Sweden, England has chosen to introduce a computing subject where students are taught how to program. However, since programming is not mentioned in the English mathematics curriculum, the incorporation of programming into school mathematics rests on the interest of individual schools and teachers (Tamborg et al., 2024). Findings from the ScratchMath

project (Noss et al., 2020) reveal that such incorporation could also be impeded by the high-stakes testing in English school mathematics. The introduction of programming into school curricula varies between countries such as Sweden and England. Therefore, further research is also needed to better understand how various approaches to introducing programming into school curricula set the stage for teachers' incorporation of programming into school mathematics.

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Beyond digital technology: Exploring transparency as a tool for analysing pre-service teachers' interpretation of an ideational resource

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Previous research has used the notion of transparency to understand teachers' use of GeoGebra as a resource for teaching circle theorems. In this paper, we move beyond digital technologies to explore using the notion of transparency to analyse pre-service teachers' interpretation of other resources, specifically an ideational resource called the Mathematics Teaching Framework. We illustrate our theoretical framework using data from two pre-service teachers' written descriptions of their mathematics teaching. We use the findings from this analysis to reflect upon extending theoretical frameworks to other resources, beyond digital technologies, and identify implications and next steps.

Keywords: Transparency, situated abstraction, ideational resource, Mathematics Teaching Framework, pre-service teachers.

Introduction

Developing theoretical frameworks for analysing teachers' use of digital technologies has been a long-term focus of TWG15 (Barzel et al., 2023). A central issue in developing such frameworks is their range of application, that is, to what extent are they particularised for analysing teachers' use of digital technologies and to what extent these frameworks are applicable to 'other resources'. For example, the *instrumental approach* was classically applied in mathematics education to understand teachers' integration of a specific digital artefact, such as a graphic calculator (Guin et al., 2005). Building on the instrumental approach, the *documentational approach to didactics* shifted to analyse teachers' interactions with a *set of resources*, both digital, such as dynamic geometry software, and non-digital, such as textbooks (Trouche et al., 2020). More recently, the definition of resources to which the documentational approach applies has been extended to *meta-resources* (Gueudet et al., 2023), including *ideational resources* (Adler, 2021), which have been "explicitly designed for supporting mathematics teachers' planning of teaching and learning activities" (Gueudet et al., 2023, p. 27). Exploring the range of application of theoretical frameworks for analysing teachers' use of digital technologies can lead to refinement of the theoretical framework itself and/or reflection upon how digital technologies and resources are conceptualised. Such exploration and reflection seem especially important given the rapid evolution and blending of digital technologies with more 'traditional' resources.

Bretscher (2022) used the notion of *transparency* (Lave & Wenger, 1991; Wenger, 1998) to understand teachers' use of GeoGebra as a resource for teaching circle theorems. We build upon this work by exploring transparency as a theoretical tool for analysing pre-service teachers' interpretation of an ideational resource. Specifically, we investigate how a framework for teaching mathematics serves as an ideational resource for developing a shared language about mathematics teaching between pre-service teachers and teacher educators. We introduced the Mathematics Teaching

Framework (MTF), developed by Jill Adler and colleagues at the University of Witwatersrand (Adler, 2021), into university-based sessions of the Post-Graduate Certificate of Education (PGCE) Secondary mathematics programme at our institution with the purpose of making more explicit or *transparent* (Lave & Wenger, 1991) aspects of mathematics teaching. Our aim was to support pre-service teachers focus on mathematical aspects of planning and evaluating lessons by developing a shared language around the aspects of mathematics teaching highlighted by the MTF and valued by the PGCE team of teacher educators. We chose the MTF because the framework was developed for use in professional development with secondary school teachers in a context where mathematics teaching needed significant development starting with explicit attention to core mathematics teaching practices, and their coherence. Hence the framework seemed appropriate in our context of working with pre-service teachers with varied mathematical backgrounds and with little or no experience of teaching mathematics in secondary schools. Central to the MTF is the identification of a *learning goal*, an articulation of the object of learning that is the focus of the planned lesson, i.e. what learners must know and be able to do by the end of the lesson. The MTF highlights three main categories of mathematics teaching practice that serve to mediate this goal in the classroom: *Exemplification* being the teacher's choice of mathematical examples, representations and tasks; *Explanatory communication* being what the teacher says and writes, including how their explanation is justified; and *Learner participation* being how the learners are invited to participate in doing and talking mathematics. Finally, the MTF emphasises that each of the three categories should be *coherent* in focusing learners' attention on the learning goal.

Working with colleagues, we have embedded the MTF into sessions taught at university and into the lesson planning template used by pre-service teachers to support their planning of lessons during their practicum placement in school. From a pragmatic viewpoint, we are interested in whether this innovation has supported our pre-service teachers to develop teaching practices valued by our teacher education team. From a theoretical point of view, we aim to contribute towards understanding how resource use can support the development of a shared language between pre-service teachers and teacher educators. In this paper, we use our investigation as a context for addressing the question: *How does a theoretical framework for analysing teachers' use of digital technologies apply to their interpretation and use of other resources, particularly ideational resources?* In the next section, we explain our theoretical framework, combining the related notions of *transparency* and *situated abstraction* as a means of understanding how pre-service teachers interpret the MTF. We then proceed to operationalise these notions in relation to our research and present the initial findings of our analysis and next steps.

Theoretical background

Adler (2021, p. 1212) refers to the MTF as an *ideational resource* "to foreground it as a reified set of ideas about mathematics and mathematics teaching in general, and ideas in a mathematics lesson in particular". As noted in the introduction, Gueudet et al. (2023) extend their definition of *meta-resources* to include ideational resources, such as the MTF. We draw on two related notions to conceptualise how pre-service teachers use the MTF as an ideational resource to negotiate meaning about mathematics teaching. Firstly, we draw on the notion of *transparency* (Lave & Wenger, 1991; Wenger, 1998) to describe the relationship (Adler, 2021) between a user and a resource in practice.

Transparency describes the extent to which users have access to the communal knowledge and practices embedded in a resource and can interpret and mobilise these in their own practice. Transparency was elaborated by Lave and Wenger (pp. 101-102) through two inter-related notions: *visibility* "in the form of extended access to information" and thus whether the user can "see" or access the ideas embedded in a resource, and *invisibility* "in the form of unproblematic interpretation and integration into activity", and thus whether the user can "see through" these ideas to interpret and embed them into their practice. For the purposes of our study, we are interested in whether pre-service teachers "see" the practices highlighted by the MTF categories, and so can name and talk about them, and whether they can "see through" these ideas, interpreting and embedding them meaningfully into their mathematics teaching.

Secondly, we draw on the notion of *situated abstraction* (Noss & Hoyles, 1996) to provide an over-arching perspective on mathematical knowledge in use in teaching, viewing such knowledge as embedded within situations that occur in the context of teaching and yet simultaneously 'abstract' or generalised, e.g. across lessons or teaching episodes. In the context of digital technologies, combining the two notions of transparency and situated abstraction provided a means of analysing teachers' recognition and interpretation of the constraints and affordances of the software in their mathematics teaching (Bretscher, 2023). Identifying the intertwined processes of recognition and interpretation appears to provide a theoretical means to analyse the non-linear progression of teachers' technology integration. For the purposes of this paper then, combining the two notions of situated abstraction and transparency enables us to conceptualise teachers' progression in a non-linear way in terms of their recognition and interpretation of the MTF as an ideational resource in their mathematics teaching. Building on Bretscher (2023), we use situated abstraction to define a developing user-resource relationship through a continuum of visibility and invisibility in terms of the following four broad categories: (1) *visible-abstract*: a user can access ideas embedded in a resource but their interpretation shows little or no meaning in practice. In our study, this means a pre-service teacher recognises MTF categories but misinterprets or makes little or no connection between them and their teaching; (2) *visible-situated*: a user can access ideas embedded in a resource and their interpretation shows meaning in specific, localised contexts. In our study, this means a pre-service teacher recognises MTF categories, interpreting them coherently in localised ways, e.g. in a specific lesson sequence but not consistently across lesson sequences; (3) *invisible-situated*: a user does not appear to "see through" ideas embedded in a resource, although elements may be used tacitly in specific, localised contexts, e.g. although elements may be tacitly embedded in their teaching, a pre-service teacher shows implicit or no recognition of MTF categories; (4) *invisible-abstract*: a user accesses and "sees through" ideas embedded in a resource and their interpretation consistently shows meaning across contexts. In our study, this means a pre-service teacher interprets MTF categories consistently and coherently across lesson sequences.

Methods

Our overall sample selection for the project aimed to represent all categories of our theoretical framework. We invited pre-service teachers to participate in the project based on teacher educators' perceptions of their mathematics teaching, derived through ongoing interaction with them during the PGCE programme. Of those invited, eight pre-service teachers agreed to participate. For this paper,

our analytic aim was to illustrate the *visible-situated* and *visible-abstract* theoretical categories as well as confirming their descriptive validity for the wider study. For this reason, we selected a subset of our overall sample to ensure that we would be able to distinguish the *visible-situated* and *visible-abstract* theoretical categories and therefore provide an initial insight into how pre-service teachers interpret the MTF. We selected two pre-service teachers, Oliver and Sarah, who represented telling cases (Miles et al., 2020) in terms of their understanding of mathematics pedagogy as perceived by teacher educators involved in the PGCE programme. Oliver was perceived as having good mathematical knowledge, but his understanding of pedagogy was underdeveloped. For example, whilst he was a confident mathematician, lesson observations indicated that he needed to plan his teaching to connect more closely to needs of his students. Sarah was perceived as having a good understanding of mathematics-specific pedagogy in general, but this was not consistent in her teaching. As such, we expected Oliver to exemplify the *visible-abstract* category and Sarah to exemplify the *visible-situated* category. Hence, our sample selection aimed to enable us to take first steps in operationalising the theoretical framework and identify initial findings by comparing these two pre-service teachers. We collected course artefacts that pre-service teachers produce during the PGCE programme as data for this qualitative study. In this report, we focus on analysing a written assignment in which pre-service teachers plan, teach and evaluate a sequence of 3-4 lessons in a specified mathematical domain, drawing on recent and relevant theory and research. The pre-service teachers were recommended to use the MTF as a guide to focus their written presentation of the plan for their lesson sequence, but not required to reference the framework explicitly. Thus, we focussed our analysis on the planning section of the written assignment in which they present a detailed description of the lesson sequence. The written assignment was completed in the first term of the PGCE programme. In this term, teacher educator input had focussed primarily on the Exemplification category of the MTF. Hence, for this report, our analysis focused on how the pre-service teachers interpreted the Exemplification category only.

In our first step of analysis, we broke down the text of the plan for each lesson into *planned teaching episodes*, indicated by a break in the text, such as a new paragraph or a textual indicator such as "this will be followed by…" or "After that, …". We then coded each episode as *visible* if at least one aspect of MTF terminology related to the Exemplification category, i.e. the words *example*, *representation* or *variation*, was used explicitly in the text. We coded the episode as *invisible* if there was no explicit use of MTF terminology. Finally, we coded each episode as either *fully situated*, *partially situated* or *incoherent*. Episodes were coded as *fully situated* if (a) specific examples or representations were named in the text, (b) a rationale for the specific examples or representations was given and (c) the specific examples or representations named in the text were coherent in supporting the learning goal within the context of the lesson. Episodes were coded as *incoherent* where specific examples were named but these appear to distract from the learning goal. Otherwise, episodes were coded as *partially situated*. Each assignment was coded by the first two authors separately and then the results compared to ensure reliability across coders. We illustrate this analytic process with examples of coding in the next section.

Results

We first present examples of planned teaching episodes from Oliver's written assignment to illustrate the application of our coding scheme. In these examples, we use pink-highlighted text to identify explicit use of MTF terminology; red-highlighted text to identify specific, named examples or representations; and green-highlighted text to identify a rationale for the specific examples or representations. We then present the overall results of analysing Oliver's and Sarah's planned lesson sequences.

The learning goal for Oliver's first lesson was that his students should be able to identify and simplify equivalent fractions. His first planned teaching episode was coded *visible* since he made explicit use of MTF terminology including "examples", "exemplification", see Figure 1. In this episode, he named a specific triad of equivalent fractions and illustrated these using a specific representation, which he named as a "bar model". He explains his rationale for introducing the bar model and this seems coherent with the learning goal. Hence Oliver's first planned teaching episode was coded as both *visible* and *fully situated*.

> When considering the actual examples and questions, I was interested in seeing whether the bar model or other exemplifications would be successful in mitigating some of the difficulties of students applying the concept of partitioning to equivalency mentioned in Nunes & Bryant (2009), as was the case in Boylan et al. […]
>
> The exemplification I provided started with simple bar models showing 4/5, 8/10, and 12/15 on top of each other, lined up to show that the shaded parts were the same size (without the numbers), and specifically had the lines that were common in bold. My aim was to cut each part of the 4/5 bar into two to produce the 8/10, and again cut each part into 3 to produce 12/15, asking the question of what fraction each bar represented to students. Then underneath, I would make sure to write explicitly that 4/5 = 8/10 = 12/15" in order to not introduce the same misconception discussed in Muzheve and Capraro.

Figure 1: Oliver, Lesson 1, Planned teaching episode 1 – Coded: Visible, Fully Situated

His third planned teaching episode was coded as *visible* since Oliver again made explicit use of MTF terminology "representation", see Figure 2. He specified introducing a "circle" representation but without providing a clear rationale. This episode was coded as *incoherent* since the introduction of new representations without a clear rationale seemed to distract attention from the learning goal. The incoherence of episode 3 was reinforced by Oliver returning in episode 4 to the original examples of equivalent fractions, see Figure 2. In episode 4, Oliver again names the specific pairs of equivalent fractions and explains his rationale for the choice of numerical example. However, he does not make explicit use of MTF terminology, hence episode 4 was coded *invisible* and *fully situated*.

> Before exploring this formally, I wanted to give the students a chance to practice independently, using again similar representations that involve the same area being partitioned into different denominators, but with different areas [*shapes? Typo?*], such as a circle. While this has issues on traditional freehand drawing as described before, when planned digitally I did not think it would be an issue. I made sure that I would go on to explain how these different representations related to each other, and how we could visualise one as another, using animations. [*Planned teaching episode 3*]

After that, my aim was to allow the students to think about the relationships between the numerators and denominators in the original pairs of fractions, 4/5 = 8/10 and 4/5 = 12/15. My choice of numbers were those that were common multiples but weren't overly simple in order to make sure the students could be convinced that the pattern would be applicable in general. [*Planned teaching episode 4*]

Figure 2: Oliver, Lesson 1, Planned teaching episode 3 – Coded: Visible, Incoherent; and Planned teaching episode 4 – Coded: Invisible, Fully Situated

The sixth and eighth of planned teaching episodes in Oliver's first lesson were both coded as visible, partially-situated, see Figure 3. There is very little detail in the eighth episode, although a form of general pedagogic rationale is provided in relation to tailoring examples to match the "pace of the class". However, the sixth episode provides more detail by naming a specific example and providing a rationale based in general mathematics pedagogy around using open-ended questions to test conceptual knowledge. Currently, the greater mathematics-specific detail in episode 6 is not indicated in our coding. We return to reflect on this later in the discussion section.

For the independent student practice questions, I wanted to provide worksheets that gave some more varied problems that tested their conceptual knowledge and understanding between the numbers in the fractions. Some of these involved discussing with partners questions that were open ended, such as "how many equivalent fractions are equivalent to 2/5" which I planned to discuss as a whole class. [*Planned teaching episode 6*]

I chose to do a matching equivalent activity, where I would write down a lot of fractions on the board and instruct the students to group them into different fractions for after the worksheet, as I would be able to tailor the examples to make them more difficult or more simple matching with the pace of the class. [*Planned teaching episode 8*]

Figure 3: Oliver, Lesson 1, Planned teaching episode 6 – Coded: Visible, Partially-situated; and Planned teaching episode 8 – Coded: Visible, Partially-situated

The results from our coding of Oliver's and Sarah's planned lesson sequence are presented in Tables 1 and 2. Both pre-service teachers have a similar proportion of planned teaching episodes coded as visible and fully situated (for Sarah, 3/11 = 27%; for Oliver 6/18 = 33%). In addition, the majority of their planned teaching episodes were coded as visible in each case, in line with the expectations upon which our sample selection was based. Partially-situated episodes were the modal category for both pre-service teachers. What separates these two cases is that five of Oliver's planned teaching episodes were coded incoherent because he introduced examples or representations that appeared to distract from the learning goal.

Table 1: Summary of the analysis of Oliver's planned teaching episodes

All lessons	Fully situated	Partially Situated	Incoherent	Total
Visible	4	3	5	12
Invisible	2	4	0	6
Total	6	7	5	18

Table 2: Summary of the analysis of Sarah's planned teaching episodes

All lessons	Fully situated	Partially Situated	Incoherent	Total
Visible	1	5	0	6
Invisible	2	3	0	5
Total	3	8	0	11

Discussion and conclusion

Our initial findings indicate that our theoretical framing can be extended productively beyond digital technologies, in terms of identifying and distinguishing pre-service teachers' interpretations of an ideational resource, the Mathematics Teaching Framework, in relation to their teaching. The two pre-service teachers, Oliver and Sarah, were selected with the expectation that they would represent the categories visible-abstract and visible-situated respectively. Our analysis provides support that the teachers can be categorised in this way and that this categorisation is meaningful in terms of describing their interpretation of the MTF. We discuss these findings before indicating implications and next steps for our research.

Our findings show that the MTF Exemplification category was visible to both pre-service teachers in that they both used MTF terminology in describing their teaching in the majority of the episodes analysed. In addition, a similar proportion of the planned teaching episodes for each teacher was visible and fully situated indicating their interpretation of the MTF terminology in the context of these episodes was meaningful, i.e. that their selection of examples was justified and coherent with their learning goal. Incoherent episodes enabled different interpretations of the MTF to be distinguished. In all her planned teaching episodes, Sarah's use of MTF terminology was at least partially situated in her teaching in meaningful ways that were coherent with her learning goal. This is consistent with a categorisation of her interpretation of the MTF as visible-situated. By contrast, in five of Oliver's episodes, his choice of examples or representations seemed to (mis)interpret MTF terminology as meaning that using a greater number of examples and representations improves teaching and learning. In this sense, the MTF seemed too visible for Oliver: he seemed to attribute such importance to using MTF terminology that he overlooked whether its use was coherent with his learning goal. This is consistent with a categorisation of his interpretation of the MTF as visible-abstract, i.e. incoherent with his learning goal and thus lacking in meaning connected to his teaching. Reflecting on our coding of partially-situated episodes leads us to question whether the coding of this category needs further refinement based on the degree of mathematics-specificity in the episode. This finding indicates the significant empirical work required to operationalise and validate a theoretical framework in the context of other resources, beyond digital technology. Nevertheless, our study indicates the productive potential for extending theoretical frameworks to analyse teachers' use of resources beyond digital technologies (Gueudet et al., 2023), encouraging reflection upon how digital technologies are conceptualised in the context of rapidly evolving resources available to teachers. Such theoretical extension could also encourage collaboration with other groups, such as TWG22, in keeping with the CERME spirit. Our next steps are to extend this analysis to the rest of our sample and to apply our coding scheme to the other categories of the MTF. In this way, we aim to identify and distinguish the other categories of the framework and test the reliability of our coding.

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Evolution of Finch robot orchestrations in teaching primary school mathematics

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This study utilised the theoretical framework of instrumental orchestrations to analyse changes in types, frequencies, and sequences of orchestrations of two co-teachers in a 3rd grade Integrated Collaborative Teaching (ICT) classroom in New York City public school. The teachers implemented mathematics activities that used Finch robot and native block-based coding App to address topics in the standards-based curriculum while also attending monthly professional development (PD) sessions to support their implementation of these activities. Analysis of six videos collected over a period of one year indicated that as time progressed, the teachers felt more confident in using Finch to support mathematics instruction. They shifted from mostly teacher-centred orchestrations towards more student-centred ones. At the same time, they started using more types of orchestrations which also led on average to a larger orchestration frequency.

Keywords: Instrumental orchestration, primary mathematics, classroom teaching practice, robot coding integration, technology.

Introduction

The incorporation of robotics and coding into primary school mathematics has attracted considerable interest. Introducing young learners to robot coding as early as kindergarten is increasingly viewed as an effective strategy for developing computational thinking and digital fluency (Angeli & Valanides, 2019). Integrating robot coding into mathematics lessons can make the subject more engaging and relevant, thereby potentially improving students' attitudes towards mathematics (Zhang et al., 2021) and enhancing their content knowledge (Sáez-López et al., 2019; Souza et al., 2019).

We examined how a year-long professional development (PD) program influenced the types, frequencies, and sequences of orchestrations used by two co-teachers in a 3rd-grade Integrated Collaborative Teaching (ICT) classroom in a New York City public school. In ICT classrooms, general and special education teachers jointly instruct classes comprising students with and without disabilities. Initially addressing teachers' needs related to technology skills, PD subsequently refocused to discussions of cognitive demand in mathematics tasks within culturally relevant activities and facilitated the sharing and development of strategies to enhance instruction.

The Finch Robot, with built-in sensors and precise movement capabilities for measuring distances, determining orientation, and performing drawing tasks (https://www.birdbraintechnologies.com/), was used in the study. These features support teaching core mathematical concepts, such as counting, simple addition and subtraction, patterns, measurements, identifying basic 2D shapes, representation of numbers on a number line, fractions, etc., while providing immediate feedback through sound, light, and movement, enhancing student engagement and self-assessment (Kentros et al., 2019). Finch supported culturally relevant mathematics tasks co-designed by teachers and researchers by representing the story in each task, e.g., being a subway train that stops at different stations or a

shopping cart pushed through the store aisle. These tasks made mathematics meaningful to all students, including those with learning disabilities, by linking learning to their cultural and everyday experiences (Lyublinskaya et al., 2024). For example, using forward and backward movement blocks, students were able to link the movement of a robot representing a subway train to addition and subtraction on a number line. They also used sound or light blocks to indicate subway stops, connecting these stops to numbers on the number line.

Theoretical framework

This study applies the Instrumental Orchestration (IO) framework (Trouche, 2004), which examines how teachers intentionally organize and manage instructional environments to foster students' instrumental genesis. Instrumental genesis involves the process by which a digital artefact becomes an instrument through its use in learning activities, shaped by both the tool's design and the users' intentions and practices (Drijvers et al., 2019). Central to this framework are three key components: the didactical configuration, arranging artifacts and participants in the learning environment; the exploitation mode, detailing how these configurations are intended to support specific learning objectives; and the didactical performance, capturing the teacher's adaptive decisions during implementation. These components enable systematic guidance in technology-rich environments.

Initially developed for analysis of Computer Algebra Systems (CAS) integration into mathematics education, the framework applications have since expanded to include diverse digital tools and contexts (Drijvers et al., 2020). This study adds to this body of knowledge by applying this framework to examine robot coding orchestrations in the elementary mathematics classroom.

Instrumental orchestration involves teachers systematically guiding learning processes in technology-rich environments (Drijvers et al., 2020). Teachers' choices of orchestration types depend on their perspectives on teaching, mathematics learning, and practical constraints such as time and technology. Drijvers et al. (2020) identified eight whole-class orchestration types and five individual/group orchestrations. The modified version of this framework, as applied in this study, is detailed in the methods section.

Methods

This year-long study followed mixed methods research design and addressed the following research questions: 1) What are sequences and frequencies of the whole class and individual orchestrations with Finch in teaching mathematics in primary school? 2) In what ways do Finch orchestrations evolve over time?

The two teacher-participants were Maria and Yulia (pseudonyms). Maria, a general education teacher with nine years of experience, holds a bachelor's degree in childhood education. Yulia, a special education teacher with 11 years of experience, holds a master's degree in special education. Their class consisted of 14 students, evenly split by gender, with 43% being students with disabilities.

The teachers implemented 10 activities between October 2023 and May 2024, each lasting around 20-30 minutes, while also participating in grade-based monthly online PD sessions that focused on various aspects of implementation. Data included video recordings of six 35-minute lessons that integrated robot coding activities, e.g., moving Finch along a number line to develop addition and

subtraction strategies, drawing shapes with Finch to understand the difference between squares and rectangles, etc. Qualitative analysis involved using the codebook (Table 1) to identify Finch orchestration types and frequencies of whole-class and individual/group orchestrations with Finch. The study replicated video data analysis procedure suggested by Drijvers et al. (2009). Specifically, each "unit of analysis was an episode, which concerns the whole-class treatment of one task" or individual student/small group treatment of one task. “If this treatment consisted of different orchestration types, the episode was cut into sub-episodes” (Drijvers et al., 2009, p. 3). In addition to the orchestration types identified by Drijvers et al. (2019), the codebook also includes whole-class and individual Link-screen-embodiment orchestrations we identified during the analysis. We also examined the sequence of orchestrations in each lesson.

Table 1: Finch orchestrations codebook

Whole-class orchestration (code)	Colour	Brief explanation	Individual/small groups orchestration (code)	Colour	Brief explanation
Board-instruction (WBI)		A teacher in a whole-class teaching in front of the board. No connections are made to the use of digital technology.	Technical-support (ITS)		The teacher supports the student in technical problems, such as login difficulties, connection issues, or hardware issues.
Technical-demo (WTD)		The demonstration of tool techniques by the teacher.	Technical-demo (ITD)		Individual/small group demonstration of techniques for using the digital content by the teacher.
Explain-the-screen (WES)		Whole-class explanation by the teacher, guided by what happens in the technological environment. The explanation involves mathematical content.			
Guide-and-explain (WGE)		An explanation based on what is in the technological environment provided by the teacher with some,	Guide-and-explain (IGE)		Individual exchange between teacher and (a pair of) student(s) in which the teacher provides guidance and instruction to the student, explains mathematical concepts or methods based on

		often closed, questions for students.			what happens in the technological environment, or raises questions to make the student reflect on his actions and results.
Link-screen-board (WLSB)		The teacher stresses the relationship between what happens in the technological environment and how this is represented in the conventional mathematics of paper, book and board.	Link-screen-paper (ILSP)		The teacher stresses the relationship between what happens in the technological environment and conventional paper-and-pencil and textbook counterparts.
Link-screen-embodiment (WLSE)		The teacher stresses the relationship between what happens in the technological environment and how this is represented through body movements	Link-screen-embodiment (ILSE)		The teacher stresses the relationship between what happens in the technological environment and how this is represented through body movements
Discuss-the-screen (WDS)		A whole-class open discussion about what happens in the technological environment.	Discuss-the-screen (IDS)		The phenomena in the technological environment lead to a discussion between teacher and student(s). The student has considerable impact on the direction and the content of the talk, for example by expressing his difficulties.
Spot-and-show (WSS)		Identification of interesting student work during the lesson, and its deliberate use in a whole-class classroom discussion			
Sherpa-at-work (WSW)		Sherpa student uses the technology to present his or her work, with all students able to follow the actions of both the Sherpa			

		student and the teacher easily.			

Colour codes were assigned to each orchestration type and used in the process of annotated timelining (Lyublinskaya & Du, 2023) to look for changes in types, frequencies and sequences of orchestrations over time. The four shades of green represent student-centred (SC) orchestrations and other colours represent teacher-centred (TC) orchestrations.

Findings and discussion

We present here an analysis of changes in patterns of frequencies and sequences of orchestrations observed over time.

Patterns in frequencies of Finch orchestrations

The frequencies of orchestrations identified in the six videos of lessons are shown in Table 2.

Table 2: Frequencies of orchestrations identified in observed lessons

Week	ITS	WBI	WTD	WLSB	WES	WGE	IGE	WDS	WLSE	WSS	WSW	ILSE	ILSP	IDS
1	4	2	3	1	4	0	1	0	0	0	0	0	0	0
7	2	4	4	4	2	2	0	1	1	0	0	0	0	0
14	2	3	1	2	1	0	1	1	0	0	0	0	1	0
19	3	2	3	0	2	2	1	0	0	0	0	0	0	0
Sub.	11	11	11	7	9	4	3	2	1	0	0	0	1	0
30	5	4	1	3	1	3	1	1	1	3	0	0	0	1
32	2	1	2	2	1	1	3	2	2	1	3	2	0	0
Total	18	16	14	12	11	8	7	5	4	4	3	2	1	1
T/SC	TC	TC	TC	TC	TC	TC	TC	SC	TC	SC	SC	TC	TC	SC

Initially, teacher-centred orchestrations dominated, with frequent whole-class presentations without Finch (WBI = 16), demonstrations of Finch's technical features (WTD = 14), and individual technical support (ITS = 18), reflecting teachers' focus on direct instruction and building technical skills. This aligns with the initial focus of PD on developing teachers' technical skills (coding with Finch). Whole-class discussions (WDS = 5) occurred throughout, but student-centred orchestrations emerged later. The gradual increase in student-centred orchestrations, such as demonstrations of problem solutions (WSS = 4, WSW = 3) and teacher-facilitated individual or group discussions of Finch codes in mathematics (IDS = 1), likely reflects both teachers' growing confidence with Finch and the shift

in PD focus from technical skills to pedagogy, emphasising student coding and connecting that to their mathematical understanding. This changes in PD focus influenced teachers' move towards student-centred approaches.

The findings also indicate a non-linear trend in orchestration frequencies suggesting the influence of factors like lesson topic, student needs, and time constraints on teachers' instructional choices. Nevertheless, the overall trajectory towards increased student-centeredness and diverse orchestrations reflects a positive influence of PD on evolution in confidence in using the Finch for mathematical learning, which is consistent with other studies (Drijvers et al., 2019).

Patterns in sequences of Finch orchestrations

The sequences of Finch orchestrations in each lesson were plotted on a vertical timeline (Figure 1). The evolution of Finch orchestration sequences across the lessons reveals a clear shift towards student-centeredness, as indicated by the ascending green line plot and the increase of green blocks, reflecting more effective integration of the Finch into mathematical practices.

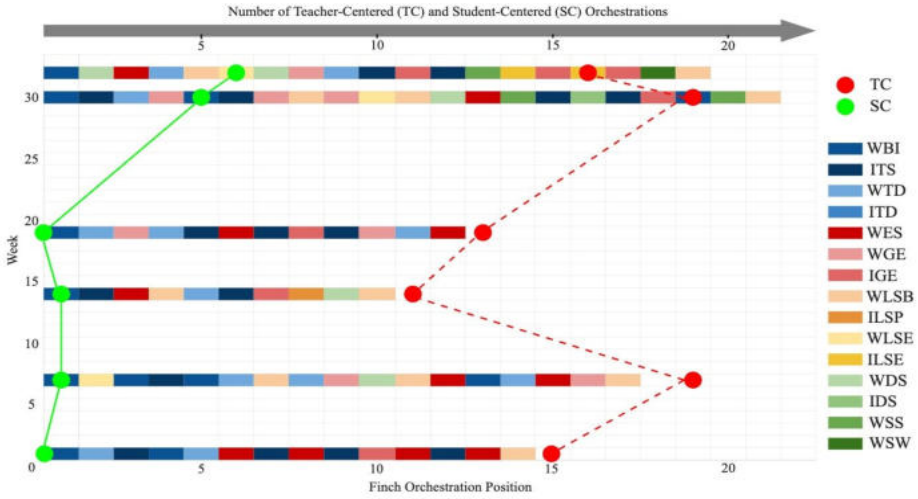

Figure 1: Timeline of Finch orchestration sequences

Analysis indicated that all lessons began with Whole-class board instruction (WBI) without using the Finch to outline the lesson flow and introduce problem scenarios. This was typically followed by technical support (ITS) and technical demonstration (WTD) to introduce Finch, often leading to individual guidance and explanation (IGE). Lessons usually concluded by linking Finch's actions (e.g., movement, lights) with traditional mathematical representations (WLSB), although Week 19 ended prematurely due to time constraints.

In early lessons (Weeks 1 and 7), Finch activities were mainly teacher-led, with a focus on technical demonstrations and explanations (shown in blue or red blocks). Mathematical content was gradually introduced, primarily through whole-class teaching. For instance, in Week 1, Yulia guided the class in counting robot lights to solve an addition problem. By Week 14, explanations and linking activities appeared earlier, reflecting a deliberate effort to connect Finch's functions to mathematics topics. For example, Maria linked robot movements to grid-based subtraction tasks. This phase also introduced

individual student interaction with Finch, enhancing hands-on learning. However, Week 19 highlighted challenges, such as rushed pacing and unfinished tasks, indicating the need to balance teacher guidance with student exploration.

In Week 30, green blocks, signifying student-centred orchestrations, became prominent. Student demonstrations and individual discussions fostered an interactive environment, encouraging students to take ownership of their learning. Examples include Yulia's whole-class presentation of a Finch-drawn shape coded by a student and Maria's individual discussions about shape properties based on student codes. By Week 32, the approach had evolved into a fully student-led model, with open discussions initiated immediately after problem introduction. Combining whole-class and individual activities with embodied demonstrations enriched the learning experience. For instance, Yulia, acting as Finch, used gestures to prompt a discussion on rectangle properties, and guided a student, acting as Finch, to draw a square by physically walking alongside them. Embodied teaching, central to instrumental orchestration, is a crucial role in learning. Drijvers et al. (2019) highlighted embodiment as key in their perspectives on instrumental orchestration, linking to instrumental genesis.

Conclusions

This study adds to the literature by examining the evolution of elementary school teachers' decisions in using robot coding in mathematics instruction. Findings reveal that as teachers engaged in targeted PD, their choices shifted from teacher-centred orchestrations to greater student involvement, reflecting the recognition of robot coding as a tool for supporting mathematical understanding. Embodied teaching provided students with tangible links to abstract concepts. Instructional practices are likely influenced by the ongoing intervention, which included monthly PD sessions that encouraged teachers to reflect on their teaching, build their confidence, and become more willing to let students take initiative and relinquish control over learning. In-class coaching provided support when teachers encountered challenges, reinforcing their growing confidence and suggesting that these changes were facilitated by targeted efforts rather than occurring independently.

Methodologically, this research advances prior studies by combining sequence and frequency analysis with annotated digital timelining, offering deeper insights into the dynamic integration of classroom technology. The study is limited to six lessons from one co-teaching team. Future research should include additional teams and teachers' perspectives to better understand the changes.

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The design of artefacts by mathematics teachers' educators as a window to tension between their actions

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In this research we present the results of two cases' analysis where two pairs of teacher educators on the use of digital tools for Mathematics designed digital artefacts during the preparation of a large-scale training, in Greece. We aim to identify points of disagreements and tensions, whether they were resolved or not. Drawing on activity theory (CHAT), we connect the tensions with the rules of training, the position and obligations of trainees and educators, as well as with the educators' approaches to the broader training framework.

Keywords: Design of digital artefacts, CHAT, tension.

Introduction

The study of mathematics teachers as designers of teaching interventions, with the use of digital or non-digital media has a significant contribution to the understanding of their teaching actions (Remillard, 2018). Furthermore, studies of teachers as designers of digital learning objects for their teaching make clearer the kind of knowledge they draw on as professionals (Laurillard et al., 2018). However, mathematics teachers working in the school can adopt other, hybrid roles, apart from the role of teacher, interacting with the school environment in many ways: as colleagues, as collaborators, as employees, as educators, as trainees, etc. Therefore, we suggest that it is necessary to study their designs considering all these roles and the relevant context. In this research, we will focus on teachers' designs that arise from their roles, as educators of their colleagues.

To conduct the research, we took as our motivation the context of the training of mathematics in-service teachers in the use of digital technologies in teaching, organized by the Greek Ministry of Education, and the large numbers of mathematics' teachers who have been trained in the last fifteen years. This training requires a lot of effort by the trainees and the educators who also are mathematics' teachers, who have been certified as educators for the specific program. The educators' training includes over 400 hours of blended learning courses, at the University, which include courses in Mathematics Education and design with digital media (Zagouras et al., 2022). The study of teachers' designs in the context of this training, with the aim of understanding the type of knowledge they use, the practices they participate in, as professionals and finally to look for and analyse elements of their professional development, should be the subject of an extended research design, likely utilizing a range of methods. As a starting point of our attempt to build this plan we will present the study of two pairs of educators who designed digital artefacts for mathematics to be exploited during the training courses. Through the present study we aim to identify and analyse elements of designers' actions that will help us understand in depth aspects of their practice and to identify contradictions which sometimes may create problems, but on the other hand often led to the development of these practices (Sannino, 2016).

Theoretical Framework

Research around the practices of mathematics teachers in relation to the use of digital media for teaching offers researchers a range of theoretical approaches (Clark-Wilson et al., 2022), while others focus on the knowledge they use (Mishra & Koehler, 2006), how it evolves, in the description of their teaching actions, etc. However, in the present research we wanted to understand holistically and in-depth aspects of teachers' actions who designed digital artefacts for Mathematics in the context of training, having the role of teachers' educator. Beyond the schemes of use of the digital artefacts that led their designers to create them, or capturing the knowledge that they employed in order to design them, we wanted to study other elements, such as time constraints of the training, rules of the training framework, limitations arising from the training curriculum, or design principles followed by designers for other reasons, e.g., due to personal perceptions. In previous research, to holistically understand aspects of the actions of teachers who designed digital artefacts for teaching and to detect evidence of potential changes in these actions (Diamantidis & Kynigos, 2023), we leveraged third-generation cultural-historical activity theory (CHAT) (Engeström, 2001), seeing from the outset the change in teachers' practices as the result of dialectic-type tensions that can appear in any aspect of teacher activity. We did something similar in this research by studying, analysing and interpreting the actions and statements of the teachers, who as educators this time, designed digital artefacts, within the general context in which they acted.

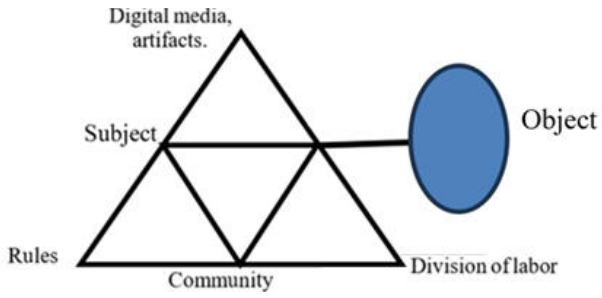

Figure 1: The CHAT 3rd generation activity system (AS)

In our case the "activity" was the design of digital artefacts for training in the use of digital tools in the teaching of Mathematics (design/training) and is described by the "activity system" (AS) and its elements in figure 1 (Engeström, 2001): Subjects are educators/designers (more generally, not just research participants), while rules refer to the context of training and community refers to everyone involved (including schools, mathematics' teachers, trainees and educators of such training, etc). The rationale of employing CHAT is to study the activity by analysing the interactions between the AS's elements. Moreover, it detects possible changes of the activity, where tensions in these interactions occur. The object is described by the objective, the outcome and the motive of the activity (e.g., the digital tool designed for such training and the training activity performed with it). Since different designers/educators were engaged in a similar activity (i.e. design/training), then AS and object can be determined by studying different subjects. Each subject may have different motivations (e.g. different educational goals and design principles) that lead to different outcomes. Thus, the activity is finally described with completeness and with more and more accuracy by the study of more and more subjects and its results (Engeström, 2001). For this specific research, the description of the

activity included elements from the study of the two subjects, while in the future it can be enriched by the study of even more.

Moreover, in the context of CHAT, a possible change in the activity described is favored when tensions are detected, i.e. when inconsistencies are indicated between the elements of the AS (Engeström & Sannino, 2010). For example, in the design/teaching research (Diamantidis & Kynigos, 2023) we identified tensions in the teachers' actions related to their diverse approaches to teaching guidelines for the specific concept that they designed digital artefacts, as formulated by the Ministry of Education. In the present research, we focused on the question: Were there any tensions located at the action of the designers/educators, in these cases we studied and in which interactions between AS's elements were detected? Which were the structural features of these tensions?

Method

For our research, as mentioned we studied the activity of two pairs of designers/educators (DT1 and DT2 was the first, while DT3 and DT4 was the second) who designed digital artefacts for the specific training. These two cases of design took place during the same training, around the same period (2021 and 2022) and were related to a blended model of training (face-to-face and online synchronous and asynchronous activities). The educators aimed to design artefacts which could be exploited during a hands-on session near the end of the training, without having decided in advance what the trainees should do during this session. They also seemed to have at least three common elements, in terms of the AS: the common object "to design a digital artefact for mathematics, to be used at the end of the training", the same digital medium which was "GeoGebra, as an authoring tool", and the same community (consisted of the educators and the participants). All these were reasons for us to present these two cases in the same study. However, all these assumptions had to be confirmed by analysis. Moreover, two more reasons for the selection of these two cases were the centrality of artifact design and the contrasts that emerged, and the long period that each pair worked densely together (more than a year).

The research data consisted of the notes from the face-to-face group conversations, the texts of the remote electronic conversations, the generated digital materials and the transcripts of the semi-structured interviews of those participants who were willing to give. As a result of their long period of collaboration, the pairs of educators developed common codes of communication and thus our methodological approach had elements of ethnography. So, we adopted an approach closer to thematic analysis, focusing more on the meanings and less on detecting patterns in the content; we analysed the data defining the unit of analysis as the "units of meaning" (Veldhuis-Diermanse, 2002). We analysed the units of meaning into two stages. In the first stage we categorized them with a more grounded approach (Kelle, 2005) and in the second we classified each unit of meaning into the elements of AS and thus attempted to match the categories of the first stage to AS's elements.

Results

During the first stage of our analysis six categories emerged, of which we will present those needed for discussion of the results. In our coding, we use the same letter(s) for codes that belong to the same theme, and a different number to indicate a diverse dimension of this theme which often conceals

contradiction, and the possibility of a tension emergence. For example, Codes T3 and T4 correspond to diverse (and somehow contradictory) units of meaning about the same theme; the tradition that the training has created all these years (almost two decades). We use the term "tradition", meaning all those that educators are accustomed to exploit and that attribute some value to their use, whether one thinks it should be continued (T3), or implies that it is not needed at this point (T4). F2 and F3 correspond to descriptions about the functionalities of a tool, i.e.: F2 fits with the approach that there is no point in trying to duplicate the functionalities of a tool, with a different one, because the activity with the new one will have much difference, while F3 stands for an approach like "it's worth designing what is closest to the functionalities of one tool with another and thus simulate them as good as we can". E3 and E4 encode units of meaning about the effort that needed to be put into designing actions and about the person who will perform those actions; the educators who design the training (E4) or those who will participate in it (E3)? D1 and D2 refer to risks designers need to be aware of during the selection of artefacts from other resources, such as training materials, in terms of sustainability; D1 is related to legislation, while D2 is related to variables of the artifact selected that they cannot control, e.g., to prevent it from being inactive. Ph1 and Ph2 refer to issues related to the philosophy of the training, one aspect of which is "the teacher as designer"; Ph1 approach is that the "teacher as designer perspective" does not presuppose technical fluency in mathematics digital environments, while a meaning categorized as Ph2 argues that technical fluency and the designer's perspective strongly correlated. We should say that we faced dilemmas during the coding, since there was no absolute identification of the category and each unit of meaning corresponded, so we made the decision to assign some units of meaning to more than one category. This was not a weak point of our analysis since, we exploited it during the second stage, identifying connections between AS's elements.

Our first case was two educators who discussed and proceeded, based on GeoGebra, in the design of an artefact that aimed to simulate the functionalities of Function Probe (FP), which are no longer available. One of them was to horizontally distort the graph of a function by manipulating a tool. Instead of that tool, they connected the parameters of each available function to a slider, which if someone manipulates, it results in distortion of the graph. Figure 2 shows the specific tool in the case of FP, but also the designers' artefact in GeoGebra. We present two excerpts of their dialogs. The first one took place before designing the specific functionality:

56	DT1:	We should make a medium with GeoGebra, that could be utilized like Function Probe (FP). FP is not available nowadays, but it was a central to training in previous years.
57	DT2:	But is it better to show them how to do with GeoGebra tools, what can be done in FP? Why put effort to build a new tool based on GeoGebra? Let's not mention the following: we won't exactly achieve to "copy" its functions.
58	DT1:	Let's make it look like FP's tool as best we can. So that we can also use the material we have for FP, with modifications, but also for them to see what was there before their training.

The following excerpt was drawn from the data after the functionality's design:

81	DT2:	Do you see? It's not the same! It doesn't feel the same when you move the slider, one is on the plot [FP], the other is a variable off the plot.
82	DT1:	Okay, we'll just show the educators anyway. I think they will understand what we want to tell them. And maybe we'll tell them what you're saying!

83	DT2:	Okay, maybe if we could have touch screen capability and students, or educators could distort the plot with their fingers by touching it we'd be closer to FP's functionality. And maybe a step further!
84	DT1:	This could be a topic of discussion during training. Make a note of it!

From our analysis we categorized lines 56 and 58 to T3, and line 57 to T4, which indicated a tension between DT1 and DT2 around the exploitation of the "training tradition". Moreover, lines 57 and 58 revealed another tension between DT1 and DT2, because we coded them in the categories F2, E3 and F3, E4 respectively.

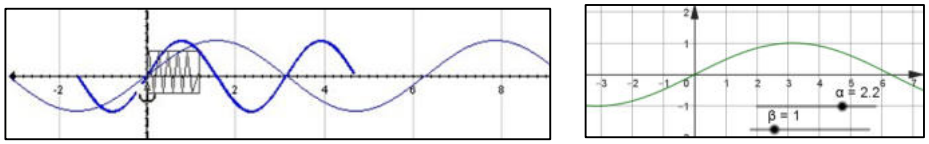

Figure 2: Vertical and horizontal deformation of sinx with FP (upper part) and with FP simulation with GeoGebra (lower part)

At line 81 we found F3, since the inconsistency of the new functionality with the original one is presented at 82 we have F2, T3, as DT1 argued that it makes sense to demonstrate the functionality simulation to introduce the trainees to the FP functionality. At 83 we have F2 to a lesser degree and perhaps T2, as it was discussed how the tradition can be continued (to simulate the particular functionality of FP), but taking into account contemporary possibilities and overcoming the "barrier" of F3; yes, indeed, one cannot copy a functionality of a tool with another, but maybe it makes sense to adapt it using novel technological possibilities. The coding of these units is indicative of the analysis, since there were more units and data related with each code.

In the second case we studied, two educators wanted to leverage a simulation of tossing an ordinary (fair) coin. The special feature of this case is that in the previous phases of training there was no corresponding artefact like the one they wanted to create. Although there are ready-made simulations, they decided to design a new one, with the help of GeoGebra (figure 3): by manipulating the slider, the user sets the number of tosses and sees how many times heads and tails appear. The red points correspond to the head appearances, while the blue ones to tails.

The following excerpt shows part of the designers' dialogue before proceeding with the design:

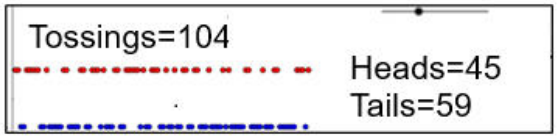

Figure 3: Simulating the flip of an ordinary coin, in GeoGebra

187	DT3:	I suggest that we don't use a ready-made simulation by putting a link, because I don't even know if we can. Does it have any copyrights?
188	DT4:	Maybe we can, but the ready-made one may one day be "offline" and its URL no longer valid. It is also good to design it in GeoGebra, because this is also the philosophy of our training program: "the teacher as designer".
189	DT5:	But we should show them other simulations, because some of the trainees may have difficulty with such designs, which are demanding, in terms of technical knowledge and competencies. We mostly want them to cultivate a designer culture, rather to design everything one can think of!

190 DT6: Okay, but in order to be able to adopt designer culture should they also have some capabilities at a technical level? These go together!

The above excerpt is indicative of the coded meanings that emerged from the analysis we did throughout the design process of this artefact. We categorized line 187 to D1 and 188 to D2, while 188, 189 and 190 discuss the "teacher as designer" issue, while the meaning of lines 189 (Ph1) and 190 (Ph2) corresponds to a tension on the correlation between the technical skills and the acquisition of designer culture by a teacher which would be trained to the use of digital media for mathematics teaching and learning.

Discussion

In the second stage of our analysis, we classified the units of meaning into AS elements, by classifying the corresponding codes (table 1). This classification is the main subject of our discussion. In table 1 it is shown which code corresponds to which elements of the AS. Since all codes of stage 1 were related to the digital medium (GeoGebra as a digital authoring system), the subjects (the teacher educators) and the object (to design digital artefacts, to be used at the end of training) of the activity, we focused on classifying the codes that were also related to the remaining elements of the AS, the community, the rules, and the division of labour.

Table 1: The result of the second stage analysis. The X in each cell means that the code of the same column corresponds to the element of the respective row, i.e. T3-T4 correspond to the community, while E3-E4 correspond to all three elements of the AS, which are listed in the table

AS elements	T3-T4	E3-E4	F2-F3	Ph1-Ph2	D1-D2
Community	X	X	X	X	
Rules		X	X	X	X
Division of labour		X	X	X	X

On the one hand, codes with the same letter but different numbers conceal a contradiction and a possible tension, while on the other hand, the appearance of the same code in more than one AS element of table 1 (i.e., an X in more than one cell of each column), may conceal an interaction between these elements. Since we are looking for tensions between the AS elements, we will focus our discussion primarily on the codes E3-E4, F2-F3 and Ph1-Ph2 of table 1, where there is an X in every cell of the corresponding column. Thus, table 2 shows the tensions that we identified between the AS elements, as we interpreted them based on the corresponding units of meaning.

For example, we noticed that E3-E4 appear in the rules, the community and the division of labour. So, we could say that the process of deciding what and to what extent trainees would participate in the design of artefacts was related to interactions between the educators and themselves, the rules of training and the "position" of the educators compared to the trainees during the training (the division of labour). Another tension also emerged in the discussion about whether it is appropriate or not to replicate the functionalities of digital tools with new ones, as shown in column F2-F3 of table 2. This tension seemed to be resolved by the end of educators' collaboration, while the first one remained

unresolved. Finally, we identified a tension related to the educators' perspective on "the teacher as a designer" concept. It seems that this tension was related to codes Ph1-Ph2, especially when the discussion was about trainees (community) and "to what extent a lack of fluency in creating artefacts is a barrier to acquisition of designer culture?". Moreover, it was also related to D1-D2 when the educators mainly discussed issues of rules that affect the decisions of a teacher as a designer, in general. This tension remained unresolved, as well.

Table 2: Starting from Table 1, we listed the tensions we identified between the elements of the AS

AS elements	E3-E4	F2-F3	Ph1-Ph2
Community	Who is supposed to design?	Replicate functionalities or artefacts or not?	What does 'teacher as designer' mean?
Rules	To what extent should a trainee design artefacts?	Designers/educators argue on transformations that occur when the functionalities of one artefact are duplicated.	Is technical fluency prerequisite or strongly correlated to designer culture acquisition?
Division of labour			

Conclusions

In this study we proposed a method for identifying and analyzing tensions that appeared in the design activity of the educators in the context of our research. Some of these tensions remained pending, while a tension about whether it is acceptable or not to replicate the functionalities of digital tools with new ones, appeared to be smoothing out as educators reached a consensus. However, the normalization of tensions, in the context of CHAT, is a process that is accompanied by significant changes in the subjects' activity, which was not evident from our data. This raises some questions: Did some significant change occur that we did not detect? Has this tension indeed been resolved or is it still pending? Were there factors that favored a temporary "solution" to this tension, which will resurface in the future? Were there factors that allowed the other two tensions to remain unresolved? That is, perhaps training can continue even if the discussion between the educators about the role of a teacher as designer remains open. Perhaps training was designed to allow for such different approaches. On the other hand, a decision had to be made about whether and how they would replicate the functionality of the digital tool to move forward, so perhaps this is why this tension seems to be resolved. Since this is a large-scale training activity, the environment is changing dynamically (e.g., emerging technologies, etc.) and such tensions lurk. By enriching one's data and by studying additional cases one can describe tensions in even more detail, possibly answering some of the previous questions, or new ones that will arise.

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Generative AI in Mathematics Teaching: lessons from spreadsheets through the lens of double instrumental genesis

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Teachers often integrate new digital tools into their practice for pedagogical reasons, even when these tools have not been designed for mathematics education in particular. In this paper, we discuss how lessons learned from the use of spreadsheets in teaching and learning can potentially support us in the use of generative artificial intelligence (GAI) tools in mathematics teaching practice. Through the lens of the double instrumental genesis and two illustrative cases, we look at the didactic use of these two tools and put forward arguments for the GAI tools' genesis and potential challenges and implications for mathematics teachers.

Keywords: Instrumental genesis, double instrumental genesis, spreadsheets, generative artificial intelligence, mathematics teachers.

Introduction

Upon the arrival of a new (digital) technology, the field of mathematics education research often proposes new theoretical constructs and notions specifically targeting that technology. Needless to say, this has led to a plethora of theories, theoretical frameworks and constructs targeting just as many functionalities embedded in various digital tools. However, as illustrated by Jankvist and Misfeldt (2021) in the case of computer algebra systems (CAS) and dynamic geometry environments (DGE), having a look at our field's back catalogue may very likely offer something that can be used as an outset for describing and understanding aspects of our current situation and observations. With the recent introduction of generative artificial intelligence (GAI) we face a similar situation as the field did some decades ago with the introduction of CAS and DGE. Hence, the hypothesis we have is that some of the theories and constructs developed around previous technologies may have something to offer in relation to a (constructive) use of GAI in the teaching and learning of mathematics.

Any digital tool becomes a mathematical instrument in the hands of a learner, through a process of instrumental genesis (e.g., Drijvers et al., 2013). Besides the need for teachers to consider their orchestrations for using digital tools in their practice (instrumental orchestrations, Drijvers et al., 2014), we acknowledge the need for teachers to go through a professional genesis during which the digital tool in question becomes an instrument in their practice. These two different processes are what Haspekian (e.g., 2014) termed as *double instrumental genesis* (DIG). With GAI in mind, this dual process becomes particularly relevant considering also that recently policy and scholarly work puts increasing emphasis in teacher competencies for AI (e.g., UNESCO, 2024). In the context of mathematics education specifically, our previous work suggests that teachers would need to develop their Mathematical Digital Competency (MDC) (Geraniou et al., 2024) when using a digital tool and

in fact to ensure that they continue to develop *mathematical and digital competencies so as to select and implement various tools in their practice in pedagogically productive ways* or in other words develop their Mathematical Digital Competency for Teaching (MDCT) (Geraniou et al., 2024). Considering GAI as a digital tool, we are interested in teachers' mathematics teaching with the use of such. We wonder what may be different, new, challenging and potentially beneficial with GAI, and subsequently what the professional development needs of teachers are to teach mathematics with GAI. We take our departure point in Haspekian's (2014) argumentation about the use of spreadsheets:

> […] the spreadsheet provides a good amplification of the phenomena that play in the development of ICT practices […] Firstly, the spreadsheet is a professional tool without any *a priori* didactical functionality. In this case, the instrumental distance is not negligible and plays a considerable role in the difficulties surrounding the integration of spreadsheets. Secondly, the teacher has to turn this non-educational tool into a didactical instrument through a process of professional genesis, a process made more complex by this instrumental distance. (Haspekian, 2014, p. 253)

Our aim is thus to consider GAI as tools that can be turned into didactical instruments through mathematics teachers' professional genesis and thereby explore, using two illustrative examples, *to what extent the DIG framework suffices to explain teachers' use of GAI tools, although originally developed for spreadsheets.*

Double instrumental genesis - Lessons learned from spreadsheets

The mathematics education research community has increasingly focused on the use of technology in learning and teaching of mathematics. Theoretical frameworks, such as the **instrumental approach** (Guin et al., 2004) which focuses on the concept of instrumental genesis, have emerged and have been widely applied to different studies involving different digital tools (CAS, DGE, etc.). Under the umbrella of the instrumental approach, there are several theoretical frameworks and constructs used in mathematics education, e.g., instrumental genesis, instrumental distance, instrumental orchestration and DIG, which we present below.

The instrumental approach (e.g., Trouche, 2005) focuses on how students interact with digital technologies (DT) and the impact DT have on their mathematics learning. As Trouche (2005) defined it, **instrumental genesis** is "a complex process, linked to characteristics of the artifact (its potentialities and constraints) and to the subject's activity, her/his knowledge and former work methods" (p. 144). There are two main processes involved when a student goes through an instrumental genesis. The first one is termed as *instrumentalization*, which entails the process when the subject shapes and modifies an artifact; and the second one is termed as *instrumentation*, and it involves the process during which an artifact shapes and influences the subject.

Lagrange (2005) highlighted that DT demand new teaching strategies and that "push-button" approaches challenge traditional paper-and-pencil methods. The concept of **instrumental distance** (Haspekian, 2014) considers the gap between the educational potential of paper-pencil mathematics and that offered by specific digital tools. This notion extends beyond the usual concept of "computer transposition" (Balacheff, 1994) to encompass various shifts—cultural, epistemological, or institutional—brought about by incorporating a tool into mathematical practice.

The theory of **instrumental orchestration** (e.g., Drijvers et al., 2014) was developed to analyse teachers' orchestrations when supporting students' learning when interacting with DT. Haspekian (2014), though, considered the necessity for teachers to also professionally develop by going through a process of instrumental genesis, before they are to further develop their teaching practices to pedagogically support their students' learning when interacting with DT. This process was termed as **double instrumental genesis** (DIG) and was inspired when Haspekian looked at the use of Spreadsheets, a tool that is not specifically designed for mathematics teaching and investigated its integration in mathematics teaching practices. Her research led to the idea of the spreadsheet becoming an "instrument for teachers' didactical activity" (Haspekian, 2014, p. 253) through professional instrumental genesis that, in addition to a process where teachers instrumentalize the tool in order to serve didactic objectives, another process of instrumentation takes place to enable the teacher to "incorporate within her (already stable) teaching schemes some new schemes that integrate the use of the tool" (p. 253). Haspekian (2014) presented the case of an experienced mathematics teacher, Ann, who integrated the spreadsheet tool within her teaching practice and who was observed for how she used it for the first time, but also for how her practice evolved as she continued to use it for two years. DIG was used to analyse Ann's actions during the integration in question.

In our experience, the way a teacher uses a tool depends highly on the lesson's objectives and the learning goal and outcomes for her students. Similarly, Ann's activity with the spreadsheet was tailored to the goal of teaching algebraic concepts, such as variables and formulas, as she used the copy function, for example, or reflected upon the numerical feedback of the tool to check the equivalence of two formulae. Such actions reveal the teachers' utilization schemes, which are related to the material aspects of the tool (e.g., Rabardel & Bourmaud, 2003), as opposed to the schemes of instrumented action, which relate to the overall task goals and intentions (e.g., Rabardel, 1995) and their organisational decisions. Ann had to decide how to orchestrate the usage of other tools, such as the data projector to make collective explanations for example. In Ann's case, her professional genesis with spreadsheets was closely intertwined with her personal experience using them, as observations indicated that these aspects influenced each other in a relational way. This interplay became more complex because she wanted her students to use the spreadsheet independently and learn mathematics through the activity, rather than just based on her demonstrations. As Haspekian (2014) continued her argumentation, the students' instrumental genesis in using the tool also became part of Ann's professional genesis with it, leading to further interference.

Ann had to also consider which spreadsheet functionalities her students needed or were ready for, at each stage in their learning journey. For example, Haspekian (2014) mentioned that Ann had decided not to involve or talk about a specific functionality in the first spreadsheet session "as it was above the level of instrumentation she wanted for her pupils at that moment" (p. 257). Such decisions interfere with students' instrumental genesis, but on the other hand showcase that Ann had not "yet turned her personal instrument into a mathematics-teaching instrument" (Haspekian, 2014, p. 258) and revealed how complex these processes are. So, she did use the spreadsheet as a "*calculation-oriented* instrument", but she also had to use it as a "*didactic-oriented* instrument", which evolved her own instrumental genesis with spreadsheets, and further developed her professional genesis (Haspekian, 2014). As Haspekian (2014) summarised, mathematics' teachers professional genesis is

constrained by their own aims and shaped by the way they support their students' instrumentation and instrumentalization processes. Generalising the above, a teacher would have to go through an instrumentalization process, as they would have to instrumentalise the tool to achieve the didactic objectives for their teaching practice, but also an instrumentation process, as they would have to integrate 'new' usage schemes in their existing schemes to ensure a successful integration of the tool in question. As also argued by Haspekian (2014), turning any such tool into a didactical instrument, and therefore go through this DIG process, is a lengthy and complex process and that has implications for teacher training in ways that support teachers through both aspects of tool integration.

Double instrumental genesis in the case of a Large Language Model (LLM)-enabled notebook

In this section, we explore the applicability of the notion of DIG through an example of the Wolfram chat notebooks[1] that integrate GAI, and specifically Large Language Models (LLM) such as OpenAI's and Anthropic's models, that allow for both natural language interaction and coding to interact with mathematical entities and data. Let us consider a scenario where a teacher wants to engage her class in mathematical modelling projects such as "renting vs. buying a house" or "investigating referee bias"[2] while leveraging the chat and LLM functionalities of the notebooks. A teacher would be expected to solve the problem first herself using the AI notebook. We leave aside for now some of the technical challenges that might be encountered.[3] Wanting first to explore the notebook herself, a teacher could simply type "*explore the financial implications of renting a flat versus buying a house over 10 years*" into the notebook. The notebook would respond with a detailed financial comparison between renting and buying. The LLM would automatically define key variables, such as interest rates, property appreciation, and inflation, and would make reasonable assumptions about mortgage terms, and maintenance costs. It would then calculate the total costs for both options over a 10-year period, providing step-by-step explanations along the way. A teacher would be expected to explore further these features and test the current capabilities of AI to gain confidence in effectively integrating this as a tool into her own practice. One noticeable aspect is that through this process a teacher would notice how different GAI is from a calculator, or a spreadsheet. A teacher may feel a lack of autonomy if in the process she does not engage with any mathematical formulas. The AI can do all the work, which may lead her to question whether she would have approached the problem differently on her own. Following her own interactions with the GAI tool, the teacher should wonder how best to approach this issue in a way that supports students develop their problem-solving skills.

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[1] https://reference.wolfram.com/language/tutorial/ChatNotebooks.html

[2] We took inspiration from Wolfram's Computer-Based Maths https://www.computerbasedmath.org/project-repository/

[3] Such challenges could include managing LLM API keys or adjusting model parameters, which, of course teachers should not be expected to have to do. We assume here that such tools could be integrated seamlessly with the school's systems, with costs and infrastructure managed by learning technologists. A big assumption perhaps but this would allow teachers to focus on pedagogy, though ongoing training will still be necessary as these tools evolve rapidly, much like spreadsheets.

Drawing parallels to Ann's journey with spreadsheets, as described by Haspekian (2014), she recognised the importance of actively engaging with the mathematical content to transform digital tools into effective didactic instruments. The teacher using a GAI tool may be worried that the students might bypass some important problem-solving steps, simply inputting questions in natural language and receiving complete solutions. A way forward would be to break the problem down into smaller, more manageable parts that would not only help her students develop a systematic approach, but also require them to engage with the underlying mathematics in each part (e.g., following Polya's problem solving cycle and the well-known DISUM modelling cycle, see Blum and Leiß, 2007). As such, the teacher could structure the notebook into different parts (e.g., understanding the problem, searching and using mathematics and explaining the results) to help the students. A second, equally important way forward would be to realise that LLMs are not aligned with our values and practices 'out-of-the-box'. For example, even if breaking the problem into steps, if asked to identify factors an LLM would respond: "Here are three major cost factors to consider when comparing renting a flat versus buying a house," followed by explicit descriptions of these factors (initial costs, ongoing monthly costs, etc.). Such a response would bypass the students' opportunity to engage in their own thinking. In later steps for calculating ongoing maintenance costs for a home, an LLM would most likely respond directly providing the formula: "You can calculate ongoing maintenance costs using the formula: Annual Maintenance Cost=Home Value*Maintenance Rate".

Therefore, to turn GAI in a true didactical instrument, a teacher would have to (at a very minimum) craft a prompt to ensure the AI's responses would support, rather than undermine, her learning objectives. To achieve this, in the case of the GAI-enabled notebook, the teacher should revise the 'context' of the notebook to include more specific instructions, explicitly directing the LLM to not provide complete solutions and instead guide students to explore the problem. Such an adjustment is crucial in aligning the tool with the pedagogical goals of fostering critical thinking and deeper understanding in mathematics.

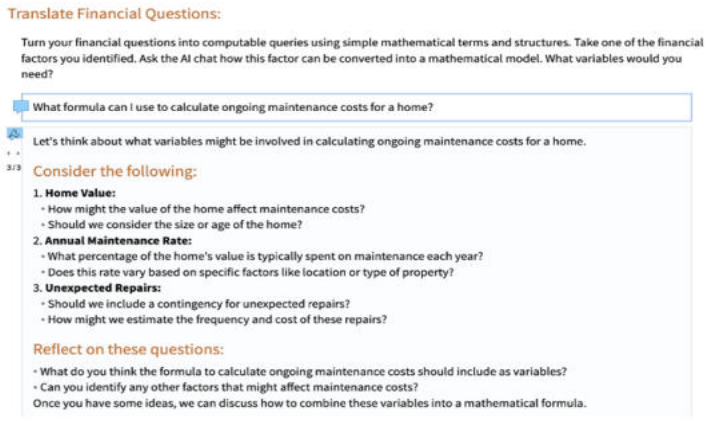

Figure 1: An example of an interaction with an LLM-enabled notebook that is prompted to guide students to specify a formula for home maintenance costs during a mathematical modelling project.

Figure 1 shows an example where the GAI employed by the notebook has been prompted in order to guide the student within the specific context of each step of the mathematical modelling process

instead of providing the answers. Careful crafting of the prompt will shape how the students engage with the tool. If a teacher were to take such actions, then GAI could become an instrument not just for solving problems, but for cultivating problem-solving and modelling skills that were the teacher's intentions in this case for example.

Discussion

We now return to our aim and summarise *to what extent the framework of DIG suffices to explain teachers' use of GAI tools, although originally developed for spreadsheets.* The framework of DIG does offer a useful although limited explanation for teachers' integration of GAI tools in the classroom. While the framework highlights the processes by which teachers transform digital tools into instruments for teaching and learning, the unique features of GAI introduce new challenges and opportunities that require extensions to the original framework. One critical difference between traditional tools like spreadsheets and new AI-based tools is that AI is not a fixed entity. Whereas tools like CAS and DGE have predetermined, static functionalities, AI systems are dynamic, constantly evolving, and capable of regenerating content based on new inputs and interactions. This presents both new opportunities for employing AI but also introduces complexity in the pedagogical process, as teachers must navigate and manage AI's autonomous problem-solving capabilities.

The process of teachers adopting GAI tools involve similar steps to those described in Haspekian's (2014) case study of Ann, who used spreadsheets to teach algebra. Ann underwent a process of DIG, wherein she transformed spreadsheets from mere tools into instruments that supported her teaching objectives and pedagogical strategies. This transformation required her to personally learn the tool, experiment with its functionalities, and make decisions about which features would benefit students' learning and how. The same is true for teachers integrating GAI tools into their lessons. The rapid emergence of AI has generated discussions and scholarly work on AI competencies for teachers and the emergence of frameworks that emphasize the need for teachers to develop both technical and pedagogical skills in working with AI (UNESCO, 2024). However, the process of double instrumental genesis highlights how teachers must develop their own competency with AI tools while simultaneously supporting their students' learning journeys—a dual challenge that is particularly acute with new layers of complexity due to the rapid evolution of GAI and its non-predictable nature and autonomous capabilities.

As argued earlier, one of the core issues with both spreadsheets and GAI tools are that they are not specifically designed for pedagogical use, leaving teachers without clear guidance on how to implement them effectively in the mathematics classroom. While teachers are undergoing such a professional instrumental genesis, they must not only learn the tool's functions but also understand how to guide its outputs in ways that align with their teaching objectives, carefully choose what to present and what to withhold; all of which are decisions that subsequently impact students' learning. In the case of GAI, this is particularly relevant because the AI's ability to autonomously generate answers may tempt teachers to prioritise efficiency over understanding, potentially undermining students' deeper learning, and making the teaching process harder to predict and control. This mirrors Haspekian's (2014) notion of "interference between the personal and professional instrument," where the teacher's use of the tool may unintentionally impact how students engage with it.

The evolving aspect of GAI means that teachers need continuous professional development, not only to refine their MDC (Geraniou et al., 2024), as they did with past tools like CAS, DGE and spreadsheets, but also to stay updated on the constantly changing landscape of AI tools. This implies the need for developing the necessary MDCT (Geraniou et al., 2024). As such, MDCT professional development should emphasize the importance of aligning AI usage with pedagogical values that mathematics education research and practice has demonstrated to be conducive to promoting problem-solving and critical thinking or whichever learning objectives the teacher aims to achieve.

Conclusion

Reflecting on the above points, we can argue that DIG indeed offers a solid foundation for understanding teachers' integration of digital tools. But it may require modification to fully account for the unique challenges posed by GAI. Instrumental distance—the gap between the tool's potential and the teacher's use of it—needs to be expanded to consider the new dimension of alignment that AI introduces. Teachers must develop new competencies to shape AI interactions to support their pedagogical goals, ensuring that AI serves as a means of fostering deep mathematical understanding rather than merely automating problem-solving. In other words, teachers must not only refine their MDC and MDCT but also engage in ongoing professional development to keep pace with the rapid advancements in AI, ensuring that they can effectively integrate these tools into their teaching practices.

Many have expressed both great concerns and high hopes about the impact of chatbots and AI on education (Giannakos et al., 2024). Some may argue that it is best to discard these tools from our education system. But they can be seen as opportunities to enhance learning and focus on mathematical concepts rather than calculation skills, as showcased in the two illustrative cases. We hope that the maths education research community continues to research the new possibilities offered by AI, bearing in mind the lessons learned by past research on digital technologies.

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Preservice teachers and their computational thinking perspectives

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This study aims to provide teacher educators with valuable insights into preservice teachers' (PTs') perspectives on computational thinking (CT). We analyse three PTs' notes from their process and reflections when solving mathematics problems with programming to identify their CT perspectives. We employed Brennan and Resnick's CT framework focusing on CT perspectives and applied Warshauer's concept of productive struggle to gain deeper insight into what the PTs see as struggle and how they valued it. Our findings suggest that the PTs' positioned themselves as novices in programming, focusing on learning programming and making something useful for their students to support their students' learning. They chose well known mathematical problems claiming that programming was the main challenge. Their struggles with programming were valued as part of their learning process.

Keywords: Computational thinking, preservice teachers, productive struggle, programming.

Introduction

In this paper we investigate how preservice teachers (PTs) conceptualize computational thinking (CT) and learning programming in mathematics. Since Wing (2006) promoted CT as a fundamental skill, essential for solving complex problems across disciplines, researchers have emphasized different aspects of CT activities (Nordby et al., 2022a). One aspect is to see CT as a skill that needs a good common definition to accurately evaluate student performance, as mentioned in Shute et al. (2017). They defined CT as the "conceptual foundation required to solve problems effectively and efficiently (…) with solutions that are reusable in different contexts" (Shute et al., 2017, p. 151). Brennan and Resnick (2012) see CT more as a process, incorporating the dimension *CT perspectives* that include the designer's view of the world and themselves in the design process, an understanding of CT we adopt in this paper. Additionally, we apply the concept of productive struggle (Warshauer, 2014), which emphasizes that difficulties encountered while solving mathematical problems can be beneficial and promote learning. Our aim is to provide teacher educators with knowledge about PTs' CT perspectives and how productive struggle is communicated when they solve self-chosen problems in mathematics by using programming. The data analysed are PTs' notes from a mandatory assignment in their final year[1].

Yadav et al. (2014) highlight that it is important to develop the teachers' (and therefore also PTs') understanding of CT in the context of their subject to prevent an overly abstract comprehension that may impede CT integration. However, merging CT with mathematics can be challenging for teachers. Nordby et al. (2022a) note that problem-solving serves as a bridge between CT and mathematics in

[1] Norway has mandatory five years master education for all new teachers in primary to higher secondary school.

schools, providing a rationale for incorporating CT into math education. In a study on primary teachers introducing CT into mathematics (Nordby et al., 2022b), two main approaches were observed: integrating CT into the existing mathematics curriculum, and focusing on programming, sometimes at the expense of mathematics. Teachers play a pivotal role in either facilitating or hindering the integration of CT in mathematics. These researchers advocate for further studies to identify effective strategies for fully integrating CT into mathematics teaching.

Some countries have introduced programming as a new school subject (Nordby et al., 2022a), while others, such as Norway, have integrated programming across multiple subjects, including mathematics. Mathematics is a key subject for incorporating programming in compulsory schooling in Norway (Vinnervik & Bungum, 2021). The Norwegian mathematics curriculum emphasizes that CT should be utilized as a problem-solving method. Consequently, teacher education must align with this curriculum, and CT and programming have become essential components that the PTs must learn. In this context, PTs in their final year of master's studies were given a mandatory assignment to solve a mathematical problem using programming and to document their problem-solving process and reflections. The research question guiding this study is: *What perspectives related to CT and traces of productive struggle can be identified in preservice teachers' notes when solving a mathematical problem through programming?* Analysing and describing PTs' perspectives and struggles, and how they overcame them, can provide teacher educators with valuable insights into the challenges and needs of PTs when applying CT and programming to solve mathematical problems.

Theory

Solving problems and applying CT can often lead to struggles. In mathematics, Warshauer (2014) introduced the concept of productive struggle, describing mathematics as "a dynamic discipline that involves exploring problems, seeking solutions, formulating ideas, making conjectures, and reasoning carefully" (p. 377). Warshauer's productive struggle framework includes 1. getting started, which includes the students' struggle with interpreting the task and how to approach it, 2. carrying out a process, which includes to encounter an impasse, struggle with implementing a process, or being unable to carry out an algorithm, 3. uncertainty in explaining and sense-making, as when they struggle with explaining their work, express uncertainty or are unclear about reasoning their choice of strategi, and the last one, 4. expressing misconceptions and errors related to mathematics topics (p. 385). These aspects of struggle are not only pertinent to problem-solving in mathematics but also extend into the domain of computational thinking. As Herheim and Johnsen-Høines (2020) suggest productive struggle can be linked with programming through computational thinking, which involves systematic steps to solve problems and find solutions, often requiring programming to execute these solutions. Thus, we perceive productive struggle as an integral part of the learning process when practicing CT in conjunction with mathematics. For struggle to be productive, learners must be appropriately challenged. However, their efforts to solve tasks should ultimately lead to a solution, whether achieved independently or with support from others. If struggles lead to students abandoning the problem with a feeling of failure, we see these as non-productive struggles. Brennan and Resnick's (2012) framework for CT encompasses three key dimensions: *computational thinking concepts, computational thinking practices, and computational thinking perspectives. CT concepts* refer to the fundamental ideas in programming, such as sequences, loops, parallelism, events, conditionals,

operators, and data. They demonstrate how these concepts can be applied in Scratch. *CT practices* are described as the skills developed during the programming, such as being incremental and iterative, testing and debugging, reusing and remixing, as well as abstracting and modularization. They explain *CT perspectives* as the perspectives designers form about the world and themselves in relation to technology, identified through; 1) expressing, which includes how learners can express themselves by using programming, 2) connecting, which includes cooperation with others, and 3) questioning, which involves what the students see as opportunities and limitations.

Both the Brennan and Resnick (2012) and Warshauer (2014) frameworks were developed from classroom observations, focusing on learners working through problems, whether in programming or mathematics. Both frameworks emphasize the problem-solving process learners engage in. The CT perspectives focus on learners' reflections on their experiences with CT while working on programming tasks, whereas the productive struggle framework highlights how learners' struggles in problem-solving can foster learning and growth. We see these frameworks as relevant also when PTs are the learners.

Method

The participants in the study were master's students from teacher education programs for grades 1-7 and 5-10, as well as in-service teachers pursuing a master's degree in mathematics education. They had two days of programming instruction, but their all experience with programming from teacher education remained limited. Data was collected from a course titled "Exploratory ways of working in mathematics, problem-solving, and modelling", which all three groups attended together. For simplicity, we will refer to all participants as preservice teachers (PTs).

As part of an assignment in their 5th year, the PTs were asked to select a mathematical problem, solve it using CT and programming, and document the process using screenshots and including reflections on their learning. They were required to explain their choice of the mathematical problem, describe their problem-solving approach using programming, and discuss any coding errors encountered along the way. They worked independently; however, they received some guidance from the teacher educators when requested.

The data set consists of 39 notes from PTs who consented to participate after completing the course. To delve deeper into and further develop our framework inspired by Brennan and Resnick's (2012) CT perspectives and Warshauer's (2014) concept of productive struggle, we decided to examine one randomly selected paper (by drawing lots) from each group. Coincidentally, all three participants reported limited programming experience and chose to use block programming tools like Scratch.

Our preliminary analyses aim to provide insights into the participants' perspectives on CT and mathematics and how they communicate struggle as productive. The process of identifying categories was iterative, starting with theoretical frameworks and then adjusting after the initial readings. Given that our data consists of notes written by PTs rather than observations of school students, we needed to adjust Brennan and Resnick's CT perspectives and Warshauer's productive struggles to fit our context. Due to the limited space, we have chosen here to focus on two of the three elements of the CT perspective: *expressing* and *connecting*.

Expressing: Brennan and Resnick describe "Expressing" in terms of students sharing stories about how they express themselves through programming. We broaden this definition to include: (a) *PTs' notes on how they understand CT,* (b) *their self-reflection on their own learning process*, and (c) *their choice of problem to solve*. We see a connection between (b) and productive struggle, as both are integral to the learning process. As we primarily focus on the reflections in the PTs' notes rather than screenshots due to space limitations, it can be challenging to directly identify their programming struggles. However, our initial examination revealed that PTs openly described their struggles in their learning process. We, therefore, searched for mentions of difficulties or obstacles connected to learning or understanding. Expressing these productive struggles might involve describing uncertainties with implementing a process and detailing the efforts made to solve problems.

Connecting: Since the data consists of individual written notes, there is no observable collaboration between students as in Brennan and Resnick (2012) or interaction with teachers as in Warshauer (2014). Instead, we redefine *Connecting to include input from peers, colleagues, course literature, and other resources* that PTs mentioned as helpful in their process. We also expanded this category to include *connecting mathematics and programming*.

Our analyses in this paper are preliminary. The insights provided can inform us about PTs' reflections and perspectives when learning CT through problem-solving and programming. Incorporating productive struggles helps us identify where PTs comment on their struggles and whether they perceive these challenges as productive in the learning process.

Results

A short summary of the three PTs' assignments

PT1 is in the master's program for grades 1-7. PT1 understands CT as "the process of developing a recipe, the overall thinking and process in developing a systematic approach to problems." They mentioned limited experience with programming and chose to solve the problem: "Create an algorithm that calculates the area of a rectangle with varying length and width" using Scratch. PT1 also tried the program with students during practicum and described the computer as a discussion partner. They expressed enjoyment in mastering and solving mathematical problems. However, PT1 did not explicitly connect programming activity with mathematics.

PT2 is in the master's program for grades 5-10. PT2 described CT as starting with big problems, breaking them into smaller parts, and solving them. They described their programming skills as not good but gained new insights from a student assistant, who suggested viewing the task not just as a programming exercise but as a problem-solving exercise. They initially chose to construct geometrical figures (a rectangle and a right-angled triangle) and used Microbit, a block-based program language like Scratch, to program a BitBot car to trace their perimeter. Due to technical issues with the BitBot car, PT2 shifted focus to creating a game to help students memorize the multiplication table. PT2 noted that the aim was to learn programming rather than create the multiplication table.

PT3 is a teacher pursuing a master's degree in mathematics education. PT3 chose a previously solved problem: visualizing "A bacterial culture doubled every 20 minutes. How many bacteria are there

after 10 hours?" using programming with Scratch. They thoroughly presented the CT practice developed and reflected on different perspectives of CT and mathematics. PT3 detailed description of struggles with programming, such as making the bacteria move and split, providing insights into these challenges. PT3 noted that using programming as a digital tool to learn mathematics offers various opportunities to present models, illustrations, and figures.

Computational thinking perspectives and productive struggles

All three PTs wrote about their perceptions of CT, which we categorized under the CT perspective of Expressing, *(a) notes about how they understand CT*. They are referencing the description of CT from the Norwegian Directorate for Education and Training's website as well as the course literature. For instance, PT1 wrote:

> PT1: Computational thinking I understand as the process of developing a recipe, or in more academic language; the overall thinking and process in developing a systematic approach to problems. […] the process of solving a problem, where the focus is on the development, reflection and execution of knowledge one has or acquires along the way.

PT1's focus is on developing the recipe, the learning gained through systematic approach, the development, reflection, execution and use of knowledge one brings in or acquires during the process. This is a description of CT practice where the practice develops through doing programming.

When analysing *Expressing (b) Self-reflection on learning*, we searched for mentions of programming and/or mathematics, and learning-related terms. PTs primarily reflected on learning programming. For instance, PT2 wrote: "In my case, the goal is not the multiplication table itself but the programming and the journey towards it." This highlights that the aim is learning programming and the process of creating a program, not merely learning mathematics. PT3 also noted that programming is the main challenge, describing the process on a general (1st quote) and specific (2nd quote) level and connecting the struggles to a problem-solving process.

> PT3: (1st quote) Programming is something I find difficult, but from my experience with the previous assignment, it is in the problem-solving process that you really face challenges and learn skills to keep going, even when the task seems impossible.

> PT3: (2nd quote) In this phase of the task, I believe the concepts of decomposition and debugging more clearly describe the process. I continuously fix the program, delete commands, and improve the code. It is a time-consuming process, but I find that I gain a deeper understanding as I continue programming.

What PT3 describes about facing challenges as part of the learning process in 1st quote, is what we identified as a description of being in productive struggle. The 2nd quote begins by describing computational practices such as decomposition and debugging and then shifts to PT3's self-reflection on the learning process. PT3 mentioned that the time-consuming programming efforts led to a deeper understanding. They described a continuous cycle of fixing, deleting, and improving the code, indicating that they encountered struggles and errors that needed correction which we see in connection to productive struggle framework point 2, carried out a process which include encountering impasses (had to delete commands), implementing a process or struggling with carrying out codes. PT3's experience illustrates productive struggle, as it resulted in what they described as deeper understanding, although it was not specified what exactly they understood better.

In *Expressing (c) their choice of problems to solve*, we find that they selected tasks focused on creating something for their students—for instance, a learning program (PT1), a multiplication game (PT2), and a visualization tool (PT3)—rather than solving their own mathematical problems. This choice likely influenced their self-reflections, leading them to emphasize learning programming. Their selections might stem from what they described as a lack of competency in programming, but it can also be seen as an expression of their identities as teachers through the programs they create.

When examining the PTs' reflections on *connecting*, we identified two distinct themes: *connecting with others* and *connecting mathematics and programming*.

In the theme of *connecting with others*, the PTs referenced various sources of interaction and inspiration. PT2 mentioned seeking assistance from a student assistant, while PT1 highlighted interaction with school students and even described the program or PC as a "participant". All PTs drew inspiration from literature. Additionally, PT1 noted connections to wider society.

The shift in perspective, inspired by a student assistant, illustrates how connecting with others can provide new insights and approaches. For PT2, this guidance helped refocus on programming as a tool to solve problems rather than just the tool itself. PT1 noted that the assignment was individual, limiting opportunities to share ideas and reflections. However, inspired by an article from the course literature, PT1 found that the program provided ongoing feedback, acting as a "third participant." PT1, specializing in grades 1 to 7, argued for using Scratch due to its appropriateness for primary school students. PT1 also involved students in practicums to test the program, showing that students were important both as receivers and responders of the programming product.

For all three PTs, their students significantly influenced their choice of programming language, and the mathematical problems selected, even though the primary aim of the assignment was PTs' own problem-solving and learning, in mathematics.

PT1 also connecting to society, stating:

> PT1: Another advantage of working with programming is the practical experience you gain. Society is changing and increasingly moving towards a society that demands knowledge and programming skills. Therefore, these skills can be seen as important for developing into a valuable and good citizen.

This expands the idea of connecting beyond the classroom, linking programming skills to societal needs and emphasizing the role of education in preparing students for a technology-driven world.

Connecting mathematics and programming was not frequently mentioned in the PTs' assignments. However, as current or future teachers, they recognized the potential to link their programming with possible mathematics activities for their students. PT3, for instance, highlighted the opportunity to visualize the mathematical problems they were working on. They also connected this assignment to previous tasks involving mathematical modeling and problem-solving in their reflections.

> PT3: Using programming to solve a mathematical problem gives the impression of combining skills in both modelling, problem solving and CT. ... I model, reflect, argue and prepare solutions using all elements from previous tasks. It is in this assignment that I really get to explore on my own and learn about myself as a problem solver.

PT3 recognized the process of mathematical problem-solving and modeling with CT. While PT1 and PT2 also mentioned this connection, PT3 articulated it more clearly. The linkage between the mathematical problem of exponential growth and programming is evident in this quote from PT3: "This assignment (the note) shows my reflections, processes, and descriptions when I use programming to illustrate a bacterial culture that grows exponentially." Here, PT3 effectively connects a mathematical model with its representation through programming, offering alternative ways of understanding mathematical problems. PT3 noted, "For me, it was about understanding what the different codes and commands did, and a deeper understanding of the process can result in greater cognitive benefits, both in mathematics and in other contexts." This indicates there can be cognitive benefits when integrating mathematical modeling with programming.

Discussions and concluding remarks

As future teachers, the PTs will play a crucial role in introducing CT in mathematics to their students. As Norby et al. (2022b) noted, they can act as gate openers or gate closers for CT. Therefore, it is essential to understand their CT perspective and views on struggles with programming.

We used CT perspectives from Brennan and Resnick (2012) and productive struggles from Warshauer (2014) to develop our analytical framework. With some adaptations, we identified CT perspectives in their notes. The PTs described CT as a method for problem-solving and saw connections between struggling with programming and solving mathematical problems. They viewed problem-solving as a commonality between mathematics and CT. PT3 chose an assignment related to exponential growth, indicating a strong link between programming and mathematics. Creating visual representations of a mathematical model is closely associated with understanding mathematical expressions.

Brennan and Resnick (2012) emphasized that students express themselves through programming, which we adapted to focus on PTs' self-reflections in their notes. PTs expressed their identities as mathematics teachers through the programs they created for students' mathematical learning. Choosing a mathematical problem to solve, using programming, is challenging for novices, so PTs opted for familiar concepts and focused on games or visual representations over new problems. This shows that integrating CT into mathematics may cause the mathematical aspect to shift to the background (Norby et al., 2022b). Yadav et al. (2014) stressed learning CT within the subject context to avoid abstract understanding, which is crucial for integrating CT into mathematics teaching, and this is an area that needs further exploration in teacher education. Designing mathematical programs for didactic purposes is different from solving a mathematical problem for oneself and defends its own place in teacher education. Creating something for students was motivating for the PTs, and such motivation should also be nurtured.

We incorporated the concept of productive struggle into our framework. This decision was based on reflections in the PTs' notes about their struggles and learning. Their documented struggles indicated uncertainty in implementing and carrying out processes (Warshauer, 2014). They showed persistence and engagement in sense-making during programming, demonstrating productive struggle. However, we believe there is further potential in this concept that we have not yet fully explored in our data analysis. We realized that screenshots with PTs' explanations might offer better insights into their sense-making, error searching, and program improvements. Future analyses will include these

screenshots to better capture productive struggle. Additionally, interviews with participants could deepen our insights into productive struggle and enhance our understanding of CT in education.

The research question was: *What perspectives related to CT and traces of productive struggle can be identified in PTs' notes when solving a mathematical problem through programming?* The findings indicate that the CT perspectives chosen by the PTs were primarily related to programming when tasked with solving a mathematical problem through programming. They expressed themselves by creating didactic programs for their students. The process they described we associate with productive struggle. The fact that they themselves go through such a process can help them to support their students through similar processes. To support PTs in using CT to solve mathematical problems, we suggest that a programming course could be introduced before applying CT in mathematics.

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3DMP and TPACK in professional development: Combining actions and knowledge

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3D-printing is more present than ever, with new advancements made constantly. Teachers can use 3D-modelling and print tailored learning objects for their students, simultaneously fostering their own TPACK knowledge, making this context very suitable for student teacher education. Still, research on student teachers, 3D-modelling and printing and TPACK is lacking. Therefore, we present our seminar for mathematics teacher education, and how we use our model to describe the 3D-modelling and printing process in combination with TPACK to gain insight into the knowledge needed for the individual steps of the process. This is needed to efficiently develop teacher training and empower teachers with 3D-printing. The qualitative study is presented, and first indications of the combination of 3DMP and TPACK are shown.

Keywords: 3D-Printing, TPACK, Manipulatives, Interview, 3DMP

3D-printing and education

Gone may be the days when teachers cannot realize their wish for a tailored manipulative (Stigberg et al., 2022) for their classroom. Through 3D-printing, educators are empowered and do not need to rely on commercial solutions for their learning object wishes, but can instead create their own.

Advances in 3D-printing make the technology viable for private and educational use, yet their implementation in schools is hindered still due to a lack of teacher education among other reasons (Ford & Minshall, 2019; Pearson & Dubé, 2022). Therefore, we have created a seminar for teaching student teachers on how to use 3D-printing to create their own manipulatives (Läufer & Ludwig, 2023b; Stigberg et al., 2022), such as the one in figure 1:

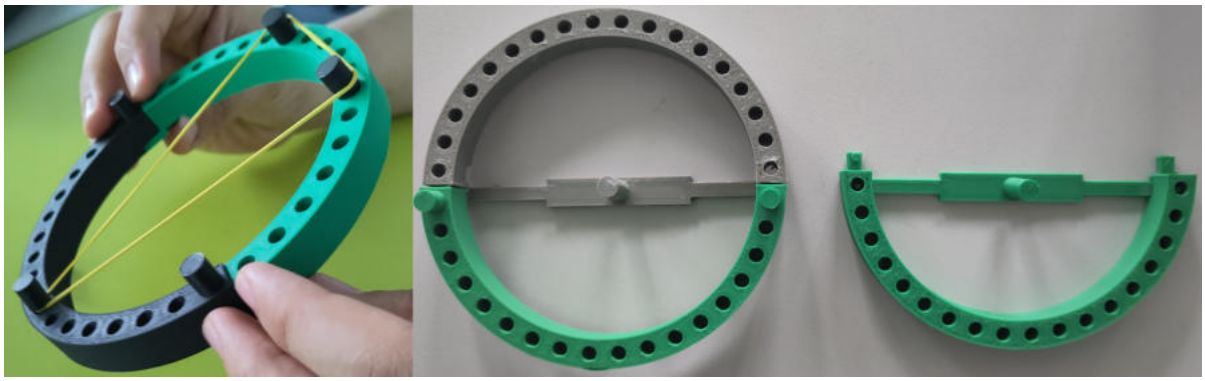

Figure 1: A 3D-printed manipulative made by a student teacher aimed to show Thales' Theorem and the theorem for the peripheral angle and the central angle using a rubber band. The black pieces where the rubber band sits can be inserted into the holes.

The seminar

The target group for our elective seminar is mathematic education students, who already completed the two basic courses, “how to teach algebra” and “how to teach geometry”. We have about 10–12 students every semester, which spans 12–15 weeks. At first, we have an input section (4-5 weeks), outlining the general process, didactics, the tools, and including videos and assignments to learn basic 3D-modelling. Afterward, we have the hands-on section, where the students work on their own projects alone (9-11 weeks). During this phase, we have a first presentation of prototype 1. At the end of the semester, we have presentations of a second prototype (1-2 weeks). The phases are also shown in figure 2. The assignment given to the student teachers is: “Fabricate your own manipulative for a fictional mathematics education lesson, that (1) cannot be bought, (2) is suitable for mathematical education, (3) is complex to some degree, and (4) is 3D-printed.” The example in figure 1 cannot be bought anywhere (criteria 1), but must be fabricated by oneself. It can be used to teach the circumferential angle theorem and more (criteria 2). There is complexity in the design and modelling (criteria 3), and it is 3D-printed (criteria 4). The presentation of the second prototype determines if the students pass or fail. An optional graded 10-page essay can be written for additional credit points.

Theory: Process and knowledge

Research regarding educators (teachers and student teachers alike) 3D-printing and the needed competencies is lacking. The 3D-modeling and printing process (3DMP, (Läufer & Ludwig, 2023a) can be used to describe the steps taken to fabricate a manipulative for mathematical instruction. This model is split into phases (concept, model, prepare, print), tool usage and interim results.

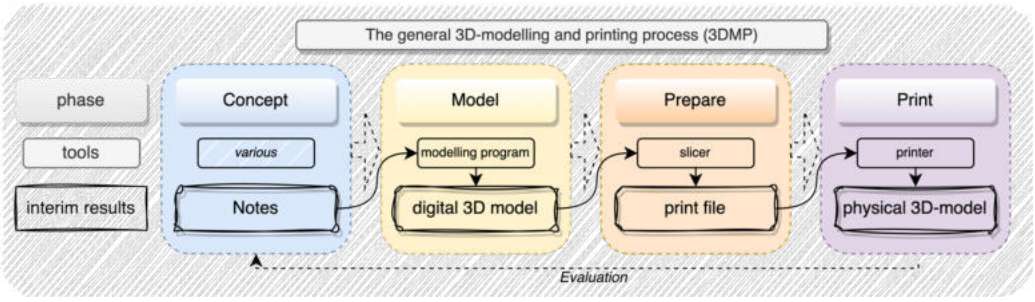

Figure 2: The 3DMP-Process (Läufer & Ludwig, 2023b)

At the beginning, the students have an idea which they want to realize. For that, they need to come up with a **concept** of how they intend to achieve it, answering questions like “What should it look like?”, “What can the students learn?”, “What are the dimensions?” and “What do the students do with it?”. Afterward, they begin **modelling** their concept in a 3D-modelling program. Our student teachers use a programming environment (OpenSCAD) to explicitly require pre-set notes, calculations and thoughts because, in our experience, programming makes it harder to “discover” a manipulative, skipping the concept phase, although this has not been researched yet.

Once the 3D-model is to our expectations, we need to **prepare** the model for printing. This is done in a so-called slicing program, which divides our 3D-model into layers (“slices”; therefore the program is called a “slicer”) and precise directions for our 3D-printer. In the **printing**-phase, the students prepare the printer, the print bed, the filament, and then print their manipulative themselves.

Due to the origin of the process model being a seminar for student teachers, which can pass or fail the seminar, the **evaluation**, which happens anywhere in the process, is only added for completeness but not considered in this paper. Students' evaluation of their manipulatives in a fail/pass seminar does not resemble in-service teacher behavior. Therefore, we focus on the concept, modelling, preparation and printing.

While knowing what the teacher's action are and how to relate them to each other is needed to describe and conceptualize professional development for mathematics (student) teachers, the competencies and knowledge needed during these actions taken is not yet researched. To accommodate for this, we use the TPACK-Framework (Mishra & Koehler, 2006) in this explorative study, which concerns itself with the qualitative description of teacher knowledge and has been used to investigate student teacher tool use (Akkoç & Hangül, 2023; Engelhardt et al., 2023; Rocha, 2023).

Research Question

Teachers want to have good learning materials for their students, while teacher *educators* aim to provide teachers with education resulting in the best possible competencies needed for teaching. Currently, when teaching 3D modelling and printing, we don't know which actions or phase corresponds to which area of TPACK. To fill this gap, especially to accommodate the growing need for teacher education, studies are needed that investigate the connection between these two models. Therefore, we combine the two mentioned theories into the following research question:

What phase of the 3DMP process corresponds with which area of TPACK?

Due to the technical nature of the seminar and the used tools, as well as the mathematical contents that are to be used in and with the manipulative, we expect to see the Modelling, Slicing and Printing phases as mainly technological, and that especially the Concept phase is mathematical and pedagogical.

In the winter semester 2024/2025 this study will be repeated to provide additional data.

Method

To answer this question, we are performing a qualitative study with guided interviews. The first is at the very beginning of the seminar to gain insight on the motivation of the students and if they arrive at the seminar with a goal in mind, which is not included in this paper. We then hold interviews during the semester every time they are to present a prototype (see figure 4). After every prototype, the students have gone through the 3DMP process at least once per prototype. The interview sections are

(a) "Topic and Introduction", which serves as an introduction to the interview
(b) "Concept", where we ask questions on what the students did before 3D-Modelling
(c) "Modelling", where questions regarding the use of the modelling program are asked
(d) "Slicing", where questions regarding the use of the slicing program are asked
(e) "Printing", where questions regarding the printing process are asked
(f) "Feedback", with which we want to see how much influence the instructor and the assistants have on the way the students work on their own manipulatives, and

(g) "Self-locating on TPACK", in where we intend to gain additional insights on how the students view their own work and where they locate each phase of 3DMP. The students have not been taught TPACK but were only shown a very reduced figure of a typical TPACK Venn-diagram.

For this paper, we focus only on (a) – (e). The interviews are not held by the authors (who also grade the students), but by an assistant that has no influence on the grade of the students. The study was piloted and the guided interview, as well as the following methods, were tested (Läufer & Ludwig, 2023b). The interviews are on a voluntary basis.

Overall, they present two prototypes, which do not have to concern the same topic:

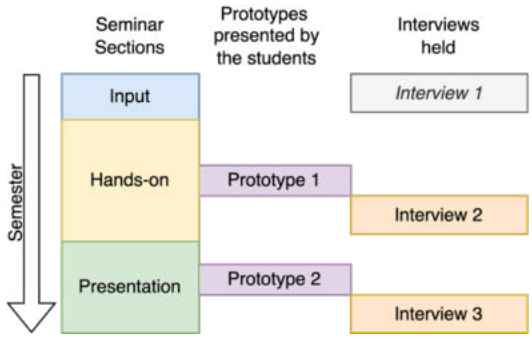

Figure 4: The phases of the seminar, the prototype presentations, and the interviews being held.

After the semester, once the students' grades have been entered, the authors get access to the transcribed, anonymized interviews, ranging from 45 min to 1h 20 min. The answers of the students are then numbered and, if they are very long, split into semantic parts, which are commonly between 1 and 6 sentences. These parts are then rated by 2 raters according to a piloted rating manual for TPACK and 3DMP respectively (see Läufer & Ludwig, 2023b). Due to the *overlap* of the 3DMP ratings and the TPACK ratings, we can then draw conclusions of the connections between the two theoretical models.

First they were rated into the base categories of TPACK (CK, TK, PK) whenever they showed mathematical (CK), pedagogical (PK), or technological (TK) knowledge. We are using the base categories of TPACK to not get lost in details and to avoid an overly complex construct of the 3DMP phases with TPACK, which happened during the piloting phase. In this study, CK is rated when the student uses or mentions explicit mathematical content (i.e. rotation, angles, formulas, …). We rate PK whenever a student references school, education, pedagogy or didactics. TK is rated whenever technology is referenced, like the programs used, research done with computers, and other digital resources.

Afterward, they were rated into the phases of 3DMP every time the actions that the students describe can be attributed to a phase of 3DMP, especially if they reference a tool or an interim result. Every semantic part can be attributed to all phases, if the students talk about their actions in this phase. An example of the rating is shown here (translated from German):

Interviewer: "What notes or preparation did you make during conceptualization?"
Student: "(…) I would use the edge of a semicircle and its diameter, and I would simply construct a semicircle, have it printed, and then I imagined using triangles. I thought

> of printing the triangles separately—some right-angled, some acute-angled, and some obtuse-angled. My first idea was that students, like very young children, would insert the triangles into the semicircle and see which ones fit. (…)"

This is one semantic part that has been agreed on by both raters to be **CK**, **PK** and **Concept**, due to the mathematical shapes (CK), the direct reference to students (PK), and the actions described that can be attributed to the concept phase.

To have methodological solidness, we calculated the inter-rater-reliabilities (IRR) not only in the form of Cohen's Kappa for every category (TPACK) and phase (3DMP), but also calculated Moon's Kappa, which is an abstraction of Fleiss' Kappa (Moons, 2023). Through this, we gain insight not just into the general agreements (Moon's Kappa), but also which agreement is difficult (Cohen's Kappa). The "Self-locating on TPACK" section is only rated during the TPACK rating because the questions are not focused on the ***actions*** that the students have taken, but with their own ***views*** on 3D printing. Therefore, we also don't include it in this paper, and it is not included in table 1.

Results

Every students' 2nd and 3rd interview with the respective IRRs in form of Cohen's and Moon's Kappa values (Moons, 2023) is listed in the following table. The value ranges are 0.2-0.4 (slight), 0.41-0.6 (moderate), 0.61-0.8 (substantial) and 0.81-1(almost perfect) according to Landis & Koch (1977 cited by Moons, 2023).

Table 1: Descriptive Data on the interviews and the agreements of both raters in TPACK and 3DMP. Bold IRR values are >= 0.6. All names are anonymized. *n* refers to the amount of overall ratings over both interviews. The numbers behind the names describe the sum of all semantic parts in interview sections (a)-(e).

Student **(# of semantic parts over both interviews without (f) and (g))**	IRR TPACK				IRR 3DMP				
	Cohen's Kappa			Moon's Kappa	Cohen's Kappa				Moon's Kappa
	CK (n)	PK (n)	TK (n)		Conc. (n)	Mod. (n)	Slice (n)	Print (n)	
Tamino **(111)**	0.35 (15)	**0.67 (12)**	**0.64 (34)**	**0.67**	**0.65 (20)**	0.51 (12)	**0.66 (10)**	0.58 (14)	**0.79**
Sare **(85)**	0.38 (10)	0.44 (9)	0.57 (20)	**0.60**	0.45 (13)	0.45 (6)	**0.82 (5)**	**0.70 (13)**	**0.79**
Jonathan **(99)**	0.49 (25)	0.52 (15)	**0.72 (35)**	**0.63**	0.54 (19)	**0.62 (13)**	**0.76 (8)**	0.46 (14)	**0.72**
Omar **(131)**	**0.75 (27)**	**0.81 (27)**	**0.70 (46)**	**0.75**	**0.75 (29)**	**0.65 (24)**	**0.84 (14)**	**0.69 (18)**	**0.84**
Jürgen **(110)**	0.51 (18)	**0.68 (18)**	0.54 (29)	**0.71**	**0.60 (17)**	**0.70 (16)**	**0.8 (9)**	0.49 (9)	**0.82**

Robin **(122)**	0.31 (21)	**0.60 (27)**	0.55 (27)	0.51	0.45 (24)	**0.69 (17)**	**0.66 (9)**	0.53 (8)	**0.77**
Emily **(93)**	0.48 (14)	0.53 (10)	**0.65 (23)**	**0.70**	0.57 (19)	**0.77 (12)**	**0.86 (9)**	0.49 (10)	**0.80**

For TPACK, the interview with the highest agreement is Omar, while the one with the lowest is Robin. For 3DMP, it is Omar (highest) and Jonathan (lowest). For 6 out of 7 interviews, CK (mathematical knowledge) was the lowest, indicating that this was the least clear to identify. Looking at 3DMP, no such negative trend is visible, but in 6 out of 7 cases, **Slicing** was the one with the highest agreement. For every student, we also generated activity diagrams (Albarracin et al., 2019), as shown here for the students Omar and Robin:

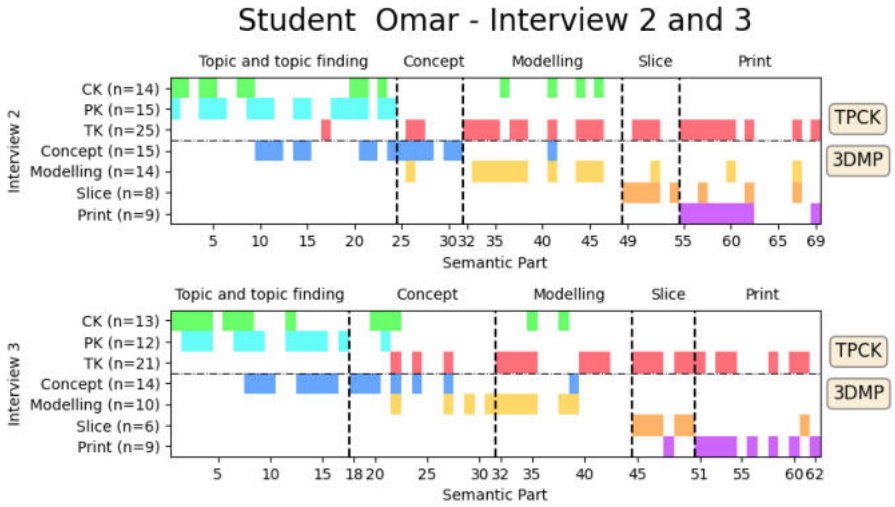

Figure 5: The Activity Diagram of Omar interview 2 and 3, split into TPACK and 3DMP coding. The top sections (vertical splits) divide the interviews into the parts (a)-(e). The x-axis shows the semantic parts (like the one in the interview above, it can span multiple sentences), while the y-axis shows the agreement of both raters for one semantic part. Each row has its own color to help distinguish itself.

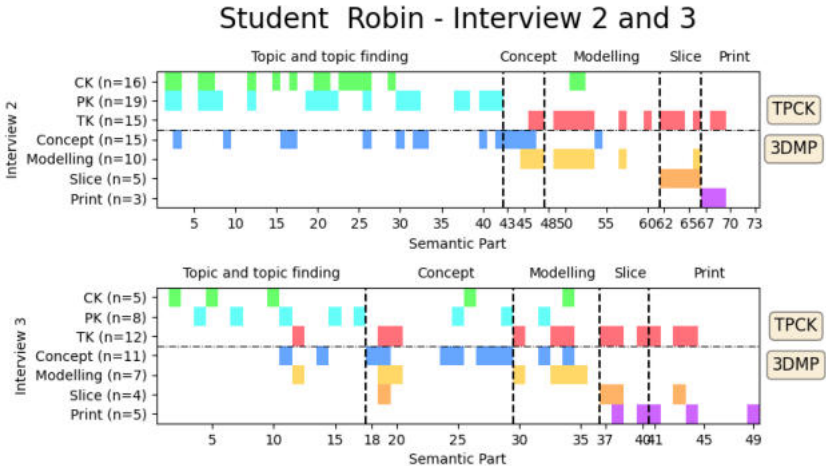

Figure 6: The Activity Diagram of Robin interview 2 and 3, split into TPACK and 3DMP coding.

The diagrams show, that the sections of the interview and the answers of the students are fitting with each other in the sense, that i.e. the student's answers are rated for Modelling inside the Modelling section. Still, the sections are heterogeneous, meaning that the students also mention other phases of 3DMP than the one that's currently the topic of the interview.

In the diagrams (including the 5 not shown in this paper), we can see clear trends that begin to answer the research question:

1) The Concept-phase is not easy to locate in TPACK, since the start of the interview "Topic and topic finding", where the students describe what their manipulative is about, already brings out a lot regarding the concept phase. A small link between CK, PK and TK to the Concept Phase can be seen.
2) The Modelling-phase seems to be linked to CK and TK.
3) Additionally, in all interviews, no semantic part is attributed to Modelling and PK, indicating that the modelling phase may have no pedagogical content.
4) Slicing and Printing are mainly connected to technological knowledge. There is no semantic part that is rated as Printing and CK/PK, while Slicing has a small connection to CK. This is mostly when the students rotate the objects in the slicer or change the size.

Limitations and Discussion

In this paper, we have presented our study on 3D-Modelling and Printing and how we combine our Model describing this process with TPACK to investigate mathematical student teachers' actions through retrospective interviews. Due to the small number of students and the limited space, only basic trends can be surmised, as the detailed investigations of the student's actions are yet to be done.

While the agreement rates, especially with Moon's Kappa, we well and acceptable, the individual categories seem difficult to rate consistently for some students. This can be explained due to the retrospective interviews. The 3DMP model is not yet proven or investigated fully. Still, shows promise in being able to describe the student's actions appropriately for qualitative investigations. This is also since the phases directly relate to the tools and interim results, and the interview was sectioned into (among others) the usage of the tools. The research question is not yet fully answered, as additional interviews should still (and will) be held, and an analysis of additional data taken during the seminar is yet open.

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Pedagogical framework for AI literacy integration: Applications for mathematics teaching and learning

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This theoretical paper proposes pedagogical framework for AI literacy integration into curriculum with focus on K-12 mathematics. This framework was developed based on the guidelines provided by the Five Big Ideas in AI initiative, essential components of human-centred design approach to problem-solving, and fundamental principles for creating optimal learning environments in schools and other settings suggested by the Science of Learning and Development (SoLD) framework. Possible applications of the proposed pedagogical framework to teaching mathematics are illustrated through several examples across different grade bands and curriculum topics.

Keywords: Artificial intelligence literacy, Human-centered design, Five big ideas in AI, Instructional strategies, Integration methods, Mathematics education.

Introduction

In the rapidly evolving landscape of education, the Artificial Intelligence (AI) is reshaping the way we teach, learn, and prepare the next generation for the challenges of the 21st century (Dwivedi et al., 2021; Ng et al., 2023). Growing influence of AI in schools require teachers to have essential knowledge and skills to use, design, and adapt AI literacy curriculum (Kreinsen & Schulz, 2023). It is critical to ensure all students have equal access and resources to achieve success in the era of AI (Pedro et al., 2019).

This paper proposes a pedagogical framework for integrating AI literacy into curriculum with focus on K-12 mathematics. In particular, we suggest strategies for integrating AI literacy across different grade bands and mathematics disciplines, and demonstrate the application of the proposed framework for designing an algebra lesson that integrates AI's concepts of representation and reasoning.

Background

Recent advancement in learning technology and learning sciences helped to expand the range of concepts students can explore with different robotic systems and machine learning platforms (Castro-Alonso et al., 2020). Research also shows that Generative AI could serve as peers, partners, mentees, or tutors to improve cognitive and affective learning outcomes (Mogavi et al., 2024). In particular, Mogavi et al. (2024) demonstrated that ChatGPT helped students to improve their reading, writing, grammar, sentence-building, and conversational language skills; tailored learning experiences for individual students; provided home assistance with assignments; and provided accessible learning resources for students with disabilities.

In the past, students learned about AI in school from their computer science teachers. Thus, the majority of early AI resources and curricula were developed for teachers who had prior training and experience in programming. However, as AI-powered experiences and applications become an

increasingly important part of everyday personal and professional life, being AI literate becomes an important component of K-12 digital literacy and digital citizenship (Bender, 2024). As a result, more recent AI literacy curricula have begun to expand beyond computer science to include interdisciplinary approaches and perspectives (Su et al., 2022). These new AI literacy curricula do not only provide students with a comprehensive understanding of AI, but also include discussions of its impact on society, covering a broad range of topics, from the use of AI in civic engagement and scientific inquiry to the exploration of AI applications and their ethical implications. (UNESCO, 2020).

International AI literacy initiatives have underscored the importance of developing teaching capacities in local regions across the globe (Pedro et al., 2019). Research studies across the world emphasize the need to prepare teachers to integrate AI literacy into their classroom settings, as this can help cultivate students' critical awareness of AI's societal impact and promote responsible use of the technology in different scenarios.

Pedagogical framework for integrating AI literacy

The theoretical foundations for the pedagogical framework for AI literacy integration included Five Big Ideas in AI (AAAI & CSTA, n.d.) that define AI core concepts, essential knowledge, and skills, human-centred design approach (IDEO, 2015) that provides a structure for teaching with AI, and Science of Learning and Development (SoLD) framework (https://soldalliance.org/) that supports inclusiveness and accessibility of a curriculum.

The five big ideas in AI were developed by the AI for K-12 initiative to provide United States national guidelines and resources for K-12 AI education. The five ideas are: perception, representation and reasoning, learning, natural interaction, and social impact, all of which are essential for understanding how AI operates and its impact on society (Touretzky et al., 2022). Perception refers to how computers perceive the world by gathering data from various sensors. Representation and reasoning refer to how agents maintain representations of the world and use them for reasoning. Computers construct representations using data structures, and these representations support reasoning algorithms that derive new information from what is already known. Learning (or machine learning) refers to a way for computers to find patterns in data and explains how computers can learn from data and make predictions. Natural interaction refers to communication between humans and machines that feels intuitive, effortless, and similar to how people interact with each other. Societal impact refers to how AI can impact society in both positive and negative ways.

Human-centred design is an iterative problem-solving approach that prioritizes the needs and experiences of the end-user. It involves deeply understanding those users through empathy, involving them throughout the design process, and generating creative solutions tailored to their specific contexts. This is achieved through cycles of research, ideation, prototyping, testing, and refinement (IDEO, 2015).

Finally, the SoLD framework consolidates evidence from the learning sciences and various educational research fields on proven strategies that foster the necessary relationships and learning experiences to enhance students' health, development, and transferable skills (Darling-Hammond et

al., 2020). The SoLD framework outlines five fundamental principles for creating optimal learning environments in schools and other settings: positive developmental relationships; environments filled with safety and belonging; rich learning experiences and knowledge development; development of skills, habits, and mindsets; and integrated support systems.

The development of the pedagogical framework for integration of AI literacy was guided by Jabareen's (2009) multi-phase iterative process. This included identifying and mapping literature sources; initial categorization of sources to identifying core AI concepts; deconstructing and categorizing these concepts; integrating foundational AI concepts with design thinking and learning science; and synthesis of AI core concepts and core literacy indicators into initial framework. The framework was then refined through iterative cycles of expert validation and pilot testing, ensuring alignment with learning science principles and real-world applications. This process resulted in a practical, evidence-based tool that comprises three fundamental components: *design*, *create,* and *reflect.* Design component focuses on positioning school students as designers and encourages student-led initiatives in student-centred, authentic learning experience. Create component emphasizes development of learning experience for students to become active problem-solvers and apply the knowledge to solving real-world problems. Finally, reflect component fosters development of critical awareness of AI with its societal innovation, while encouraging students to cultivate digital citizenship and actively use AI to solve real-world problems. Table 1 shows school student competencies, learning outcomes, and theoretical basis for these three components:

Table 1: Summary of competencies, learning outcomes, and theoretical basis for the pedagogical framework

	Competencies	Learning outcomes	Theoretical basis
Design component	D1. Applying design thinking process	Students demonstrate proficiency in a deliberate design process, including generating and refining ideas, testing theories, and creating innovative artifacts or solutions to authentic problems.	Human-centred design
	D2. Evaluating AI applications	Students critically evaluate and apply AI technologies, including assessing their effectiveness, accuracy, perspective, and credibility in various applications and contexts.	Five big ideas in AI
Create component	C1. Applying AI to solve real-world problems	Students identify problems that can be effectively addressed with AI technologies, such as data analysis, abstract models, and algorithmic thinking, and develop solutions using a contextualized approach.	Human-centred design Five big ideas in AI
	C2. Decomposing complex problems into manageable parts	Students break down complex problems into manageable parts, extract essential information, and	Human-centred design Five big ideas in AI

		develop descriptive models to understand complex systems and facilitate problem-solving.	
Reflect component	R1. Developing ethical awareness in AI	Students think critically about the ethical implications of AI technologies and evaluate their impact on society, including issues such as bias, discrimination, and privacy.	Five big ideas in AI
	R2. Responsible use of AI technologies with real-world application for the community through collaboration	Students use AI technologies in a responsive and intentional manner, including strategies for managing personal data, maintaining digital privacy and security, and sustaining a culture of innovation and change.	SoLD

In our book (Lyublinskaya & Du, 2025) we further unpacked these competencies for different domains of knowledge, e.g., cognitive, psychomotor, and affective, in order to define performance-based objectives for assessment purposes (not shown in this table). For teachers, the design–create–reflect process naturally leads to effective pedagogical practices such as project-based, inquiry-based, and collaborative learning, discussions and debates, etc. engaging students in real-world practices, brainstorming diverse solutions, and being prompted to think outside the box.

Strategies for AI literacy integration into mathematics education

There is a natural connection between mathematics and Big Ideas in AI, which provides us with many opportunities to effectively integrate AI's concepts into mathematics teaching and learning. Mathematics serves as the language through which AI concepts are formulated, analysed, and implemented. Concepts from linear algebra, calculus, probability theory, and statistics form the foundation of AI's algorithms. Vectors and matrices are used to represent AI features and parameters, while operations such as matrix multiplication, differentiation, and integration enable computations that drive machine learning and decision-making processes.

The following examples illustrate connections between AI concepts and school mathematical practices. These examples suggest strategies for integrating AI literacy into mathematics teaching. We conclude the paper with a lesson plan that demonstrates how to apply the proposed pedagogical framework to integrate AI concepts of representation, and reasoning into an algebra unit on the structure of algebraic expressions.

Example 1. Properties of operations: True or false

In this elementary school lesson, students explore properties of operations by creating and identifying true and false statements. They then train and test AI to recognize these properties, reinforcing their own understanding in the process.

The lesson begins with students exploring properties of multiplication through hands-on activities and discussions. They discover how changing the order of factors doesn't affect the product

(commutative property), how grouping factors in different ways leads to the same result (associative property), and how multiplying a number by a sum is equivalent to multiplying by each addend and then adding the products (distributive property). As they learn these concepts, students individually or in pairs develop verbal statements about multiplication, clearly labelling them as true or false. In the activity that follows, these verbal statements become a training dataset for AI to understand and apply the commutative, associative, and distributive properties of multiplication and to make decisions about the truth of mathematical statements. Students can use Machine Learning for Kids platform (https://machinelearningforkids.co.uk) to set up a text classification model with "true" and "false" labels, and use their true and false statements as training examples. They can also use the trained model to create a simple Scratch game where players input statements and AI classifies them.

Example 2. Statistics in digital photography

In this middle school lesson, students analyse a dataset of brightness values from a digital grayscale photograph (Figure 1), calculating measures of centre and variability. They interpret these statistics to understand the image's overall brightness and contrast, mirroring how AI perceives and processes visual data.

a)

49	57	212	189	34
206	54	70	108	192
222	218	55	61	56
63	89	227	206	28
215	57	65	212	209

b)

c)

Figure 1: Digital (a) and numerical (b) representations of a photograph of stairs (c)

Teachers can introduce students to the concept of thresholding in image processing to show how computers can make binary decisions based on continuous data, which is a fundamental concept in many AI algorithms. Students can choose a threshold value based on their analysis and convert their grayscale image to a binary image (black and white only) based on this threshold.

Example 3. Comparing linear, quadratic, and exponential models

In this high school lesson, students use technology to construct, compare, and analyse linear, quadratic, and exponential functions. They will apply their understanding of these functions to discuss the role of AI in model selection, analysis, and its broader societal implications.

Teachers can provide students with three datasets, one for each function type, are related to various societal issues, e.g. income by education level, health risk factors by age, spread of viral content over time, etc. Without telling students which model is the best fit for each dataset, let students use graphing calculators, online tools, or spreadsheet software to plot data and fit models. For each dataset, ask students to compare and evaluate the fit of each model and the predictive power of each model within a specific context. After students share their models and analysis, teachers can introduce

an AI-powered data analysis tool (e.g., Tableau or Gen AI) and demonstrate how AI can automatically select the best fit model. Let students compare AI models with their models. The discussion could be guided by the following questions: How might AI-driven predictions affect decision-making in societal issues? What are the potential benefits and risks of relying on AI predictions? What is the importance of understanding the underlying mathematics when using AI tools? What are potential biases in AI models and their real-world consequences? While the lesson focuses on constructing and comparing linear, quadratic, and exponential models, this mathematical content is framed within the larger context of AI's growing role in data analysis and its societal implications.

Example 4. Multiplying polynomials

This example demonstrates a high school lesson structured as a design-create-reflect process according to the pedagogical framework for AI literacy. This lesson is part of a larger instructional unit on structure of algebraic expressions which is typically taught in first year algebra course. The key learning activities in this lesson integrate both disciplinary and AI literacy instructional goals. This integration is based on the following key insights from the five big ideas in AI framework (AAAI & CSTA, n.d.):

- AI representations are data structures. The two major types of knowledge representations are symbolic and numerical representations.
- Reasoning methods are algorithms. Representations support reasoning; reasoning methods operate on representations.
- "Knowing" something means the ability to both represent it and reason with it.

The larger performance task for the whole instructional unit places students in the role of transportation coordinators for a school district, who need to optimize school bus routes to minimize transportation costs, fuel consumption, and student travel time. Students are also expected to create detailed schemas and knowledge graphs to represent data in the school bus routing problem, explaining the reasoning behind their chosen representations, and justifying the connections between different data elements.

During the design component of the lesson students identify factors that affect transportation (e.g. travel time, distance, fixed cost, etc.). They then explore how these factors can be represented mathematically using polynomials. Working in small groups, students create mind maps to visualize routing systems and apply polynomial multiplication to calculate combined route costs. This process culminates in the development of draft knowledge graphs for their transportation networks. In the next part of the lesson representing the create component students prototype their AI routing system. Using polynomials to represent different school bus routes, student develop algorithms for route classification. To facilitate prototyping, students can utilize tools like MIT App Inventor (https://appinventor.mit.edu/). Students can also use Scratch (https://scratch.mit.edu/), a block-based programming language to create simplified simulations of transportation systems. The product of this step is a visual representation of their transportation system, highlighting the application of polynomial multiplication in their prototype. Finally, they present their AI routing system and knowledge graphs to the class for feedback. The reflect component is the conclusion of the lesson in a form of a whole class discussion to reflect on student learning emphasize these key points:

- How polynomial terms represent different factors in transportation problems
- The role of polynomial multiplication in combining and comparing routes
- How AI uses structures similar to polynomials for data representation and reasoning
- The connection between mathematical operations and AI decision-making processes

Developing mathematical thinking and problem-solving skills are crucial for understanding and developing AI systems. By integrating mathematical practices, educators can help students not only excel in mathematics but also comprehend the fundamental concepts driving AI technologies and their impact on society. This alignment facilitates a deeper understanding of how mathematical principles underpin AI advancements and how AI systems can be ethically and responsibly developed and deployed.

Conclusion

We need AI-literate teachers who can help all students to prepare for the complex world they are entering, and meet the societal demands for problem-solvers who could use AI responsibly, ultimately widening the AI access for everyone. That means educators need to be familiar with AI technologies and be able to integrate AI literacy into classroom teaching. Moreover, we need teachers who can overcome cultural and financial barriers and limited resources to support the implementation process, especially for students with limited financial resources and access. There is still much work to be done for widespread adoption of AI literacy in classrooms across the world. This includes the development and evaluation of teacher professional development materials that prepare educators with essential knowledge and pedagogical practices necessary to adapt and integrate AI literacy into their teaching (Pedro et al., 2019). Teachers need more resources and support to effectively use AI in order to create student-centred learning environments. The availability of resources, curricular materials, and ongoing professional development opportunities are all core factors that contribute to teachers' self-perceived readiness to integrate AI literacy into their curricula (Milliken et al., 2019).

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Integrating Computational Literacy into Primary Mathematics Education:
Opportunities for developing mathematical ideas and relationships

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In recent years, computer programming has emerged as an important skill in curricula worldwide, including mathematics curricula. In this paper, we explore how computer programming may contribute to mathematical ideas and relationships in primary mathematics classrooms. Using diSessa's principles of computational literacy, we examine 32 classroom activities. The results reveal that only one activity fulfils the principles of computational literacy. By showcasing this activity, we present an example of how computer programming can transform classroom conversations in primary mathematics. However, we argue that more research is needed to identify how (and if) computer programming can be used to develop new ways of learning primary mathematics.

Keywords: computational literacy, computer programming, mathematics, primary school.

Introduction

As far back as the 1970s and 1980s, computer programming was shown to be an effective tool for mathematical education (Feurzeig et al., 1970; Papert, 1980). In his work, Papert (1980) demonstrated how the programming environment, Logo, could be integrated into classroom activities to enhance students' ability to engage with and manipulate mathematical ideas. Today, we see a new and different wave of integrating computer programming into education, based on the assumption that computer programming is necessary to navigate technological environments in our society (Wing, 2006). These new ideas are not rooted in mathematics, but rather seen as fundamental skills and concepts from the field of computer science (Shute et al., 2017). Consequently, the integration of computer programming in education today takes a more general approach that often overlooks the benefits of designing and understanding teaching and learning in specific domains, such as mathematics (Kafai et al., 2020).

At the same time, many jurisdictions have revised curricula, including mathematics curricula, to integrate computer programming across grades (Bocconi et al., 2022). For example, in 2020 and 2021, Ontario, Canada added specific computer programming expectations for grades 1-8 and grade 9 mathematics curricula, respectively (Ontario Ministry of Education, 2020, 2021), with the goal of representing mathematical concepts and relationships dynamically. Another example is Norway, which, in 2020, included computer programming as a central part of mathematics, focusing on variables, algorithms, conditionals, and loops (Directorate of Education, 2020). However, recent research studies show that when primary teachers integrate computer programming into mathematics classroom activities, they struggle to maintain a mathematical focus (Feng & Yang, 2022; Nordby et al., 2022; Nordby et al., 2024).

There is an interest amongst mathematics researchers in how computer programming can be useful and aid in mathematical ideas and relationships. For instance, diSessa (2000, 2018) claims that computer programming can be viewed as a computational literacy because it has infused the field of education (and many other fields of study) changing the structure of "domains of knowledge" (p. 7). diSessa (2018) conceptualizes that new literacies disrupt fields that adopt them through remediation, reformulation, reorganization, and revitalization. In essence, using mathematics education as an example, new literacies change the nature of mathematics education, by disrupting how it is done (taught and learned), who can do it, and when. Noss and Hoyles (1992) note that "there are tasks that children can do with a computer that would be impossible without one" (p. 455). In addition, Borba & Villarreal (2005) suggest that digital media in mathematics education should not be seen simply as tools that serve our purposes, as they may also reorganize our mathematical thinking.

Based in the above-mentioned ideas, more research is needed to highlight the potential for exploring mathematical ideas and relationships when in-service teachers integrate computer programming into their classroom activities. Such understanding will aid in better creating opportunities for using computer programming in learning mathematics. In this context, we are interested in the following research question: *How may the integration of computer programming contribute to the development of mathematical ideas and relationships in primary school classrooms?*

Computational literacy

We recognize that there are different conceptions of computational literacies (Guzdial, 2019; Kafai, Proctor & Lui, 2019). We chose to use diSessa's (2000, 2018) idea of computational literacy as a conceptual framework as it provides 4 explicit categories for identifying the disruptive potentials of computer programming in classroom teaching and learning: remediation, reformulation, reorganization, and revitalization. We illustrate this through a grades 3-4 mathematics and computer programming example (Gadanidis et al, 2024).

Remediation

Computer programming remediates ideas through its ability to dynamically represent concepts and relationships through a new medium. For example, consider the following puzzle:

1. *In the Scratch code (Figure 1a) a bug walks a square path. Run the code at https://scratch.mit.edu/projects/1055180217/editor/*
2. *How does the code work?*
3. *How might we alter the code so that the bug walks a spiral like the one shown in Figure 1a?*

Scratch allows us to remediate this puzzle from a paper-and-pencil activity to a dynamical modelling of a bug "walking" on the computer screen.

Reformulation

In the process of solving this puzzle, students learn about shapes and angles, and encounter a new idea: a variable. A variable is needed so that we can repeat the process walking and turning while changing how far we walk in each stage. The variable "steps" is shown in Figure 1b. This code is available at https://scratch.mit.edu/projects/1055189920/editor.

A traditional approach for engaging grades 3-4 students with a variable may be by solving equations of the form X + 5 = 12 and 2X – 3 = 9. The problem with such approaches is that X in these cases does not vary. X is an unknown constant. X = 5 in the first case and X = 6 in the second case. By dynamically walking a square and then a spiral, Scratch reformulates the mathematics experience so that students may encounter the real meaning of a variable in the spiral walk of a bug. In their work, Kilhamn et al. (2022) show how the variable in the computer program makes it possible to highlight the role of variables as generalized numbers. Hence, reformulation is possible using Scratch as a computer program, due to what diSessa (2018, p.26) calls "cognitive simplicity" that is offered through dynamic modelling of variable.

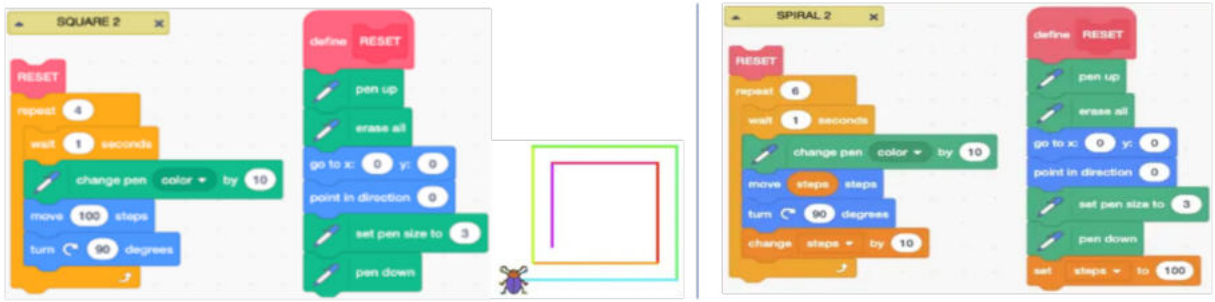

Figure 1: Scratch codes: A bug walks a square path (1a, left) and introducing a variable (1b, right)

Reorganization

diSessa (2018) suggests that computer programming causes a reorganization of the conceptual framework of a field of study, changing what we can do and who can do it (diSessa, 2018). The remediation and reformulation we discussed above reorganizes what mathematics may be done, who can do it and when.

Revitalization

When a field of study makes pervasive use of computer programming, it leads to a revitalization of the field, and how and what teachers teach and how and what students learn (diSessa, 2018). For example, in solving the puzzles in Figure 1a and 1b, students recognize that for the bug to walk a spiral, it needs to take an increasing number of steps. That is, the number of steps need to *vary*. As students in grades 3-4 typically do not know how to define and use a variable in Scratch, the teacher offers them the link to the Scratch code in Figure 1b and poses this new puzzle: *Edit the code to create each of the spirals shown in Figure 2a, 2b and 2c.* This helps mathematics come to life in new ways, offering students access to playful engagement with dynamic representations of the nature and function of variables in a meaningful context.

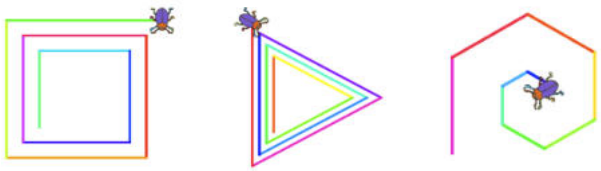

Figure 2a (left), 2b (middle) and 2c (right): Using Scratch to model different spirals.

Methods

The study context is derived from activities conducted in primary mathematics classrooms (grades 1 – 5) in Norway. This study focuses on classroom activities designed by four different primary school teachers from three different schools over a period of one and a half years. In total, this paper draws on data from 27 classroom observations where the teachers integrate computer programming into their primary mathematics teaching practices. Data were collected through video-recorded classroom observations, with audio being recorded via a wireless microphone attached to the teacher.

In our analytical approach, we were interested in identifying how the activities contributed to the development of mathematical ideas and relationships. Drawing on diSessa's (2018) ideas of computational literacy, we used a deductive approach when analysing the data material. First, all the transcribed video-data were analysed using a table in Word capturing all the activities conducted (we view an activity as a range of tasks derived from engagement with similar mathematical concepts). This initial analysis resulted in 32 identified classroom activities (some lessons included more than one activity). Second, we identified the activities where computer programming was used to engage in mathematical ideas and relationships. For instance, if the computer program Scratch was used to engage in mathematical content, the activity was included for further analysis. On the other hand, if the activity was based on students working in programs such as LightBot[1], with no mathematical focus, the activity was removed from further analysis. This step resulted in reducing the activities from 32 to five. In the third and final step, we used the 4 disruptive potentials of computer programming identified by diSessa (2018) to categorize the activities. For instance, we looked for examples of remediation within the activity to determine whether it dynamically represented mathematical concepts and relationships. We also looked for reformulation to ascertain if the activity offered opportunities for students to encounter mathematical ideas in new ways. Ultimately, only one activity fulfilled these criteria. We will present this activity in the result section.

Results

By focusing on how computer programming may contribute to develop mathematical ideas and relationships in primary school classrooms, we will in this section first present three different tasks happening in the classroom activity before we discuss these tasks in terms of the disruptive potentials of computer programming (diSessa, 2018). Moreover, we will also discuss opportunities for potential improvements of the activity.

Task 1

The teacher begins the lesson by introducing the activity as an "exploration of geometrical figures and patterns" (teacher grade 5). Scratch is used as the programming environment, an environment where students can drag and drop different blocks to solve different problems[2]. After a brief introduction to how Scratch works, the teacher asks the students to create a rectangle. In the whole class conversation, the students and the teacher agree that they can use the block called "repeat": "if

[1] https://lightbot.com/
[2] https://scratch.mit.edu/

you want to make a rectangle, you can take one [side] and one [side] and twice – repeat" (student grade 5). The result of their conversation is shown in Figure 3.

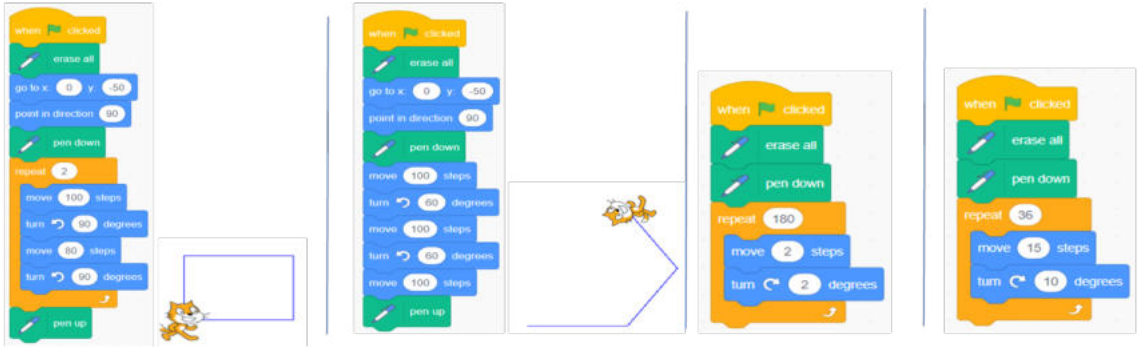

Figure 3 (from left to right): 3a - Code shown on the whiteboard, 3b - Code resulting in the problem, 3c - Code of student 1, 3d - Code of student 2

Task 2

After the whole class conversation about rectangles, the teacher introduces another task, asking the students to create a triangle. When the students start working on creating triangles, a problem arises about what angle to insert in the turning block (see Figure 3b).

Teacher:	what is the problem?
Student:	how many degrees should it turn?
Teacher:	ok, if we have a regular triangle, what are each angle if they are 180 in total [the teacher draws the triangle on the whiteboard]
Student:	60.
Teacher:	yes, but when you are here in the figure in Scratch [points at the right corner of the triangle], you need the opposite of 60 degrees.
Student:	minus 60?
Teacher:	[draws the external angle with dots] How many degrees are here [pointing at the angle with dots].
Student:	120
Teacher:	yes

Task 3

In the third and last task in this lesson, the teacher asks the students to use Scratch to model a circle. While the students are working on the task, the teacher finds that the students are using different sets of numbers in the step and turning block: "you are using different numbers" (teacher grade 5) and asks the students to show their work in front of the class (the students connect their tablet to the whiteboard). The upcoming extract show the whole class conversation:

Student 1:	I took, walk two steps, and turn two degrees and repeat 180 times [see Figure 3c].
Teacher:	great, why 180 and not 170?
Student 1:	because two times 180 is 360.
Teacher:	yes, now we are using mathematics, do you see that? Is there any other way we can create a circle?
Student 2:	[showing his/her work, see Figure 3d]
Teacher:	but 36, repeat 36 times and 10 degrees and move 15 steps? Is that right?
Student 2:	yes, 36 times 10.
Teacher:	yes, because 36 times 10 is 360.

Meaningful moments for developing mathematical ideas and relationships

According to diSessa (2018), computer programming can provide new ways of engaging in mathematical learning. In all the tasks in the activity above, we can identify *remediation*, because Scratch is used to represent mathematical ideas through a new medium (code). For example, in Task 1, students explore the relationship between the side lengths of a rectangle by interacting with codes that dynamically depict a model of the rectangle on the computer screen. We also identify *reformulation,* due to dynamically representing concepts and relationships of angles and how to create different geometrical shapes. This allows for the discovery and discussion of mathematics in new ways that were not previously possible.

Remediation and reformulation combine to *reorganize* the concepts of angles and polygons into a playful investigation with immediate feedback, *reorganizing* mathematics concepts and relationships. This reorganization makes the playful exploration of polygons and their properties more easily accessible than with paper and pencil for younger students. For example, by looking at the conversation in Task 2, we can identify that the students are visually meeting a challenge, when the drawing appears not to be a triangle. The immediate feedback on the students work, *reorganizes* the mathematics, by visually promoting for a discussion of internal and external angles in a triangle in a dynamic representation, in contrast to what can be done with static paper and pencil representations. Thanks to the flexibility of changing the blocks and numbers in the code, students are able to engage with the mathematics from their own starting point by testing and visually seeing the result. An analogy may be the difference between learning about how a bicycle worked by reading a book versus riding a bicycle. The above changes create novel experiences that are mathematically meaningful and deep, *revitalizing* what mathematics is, what it means to do mathematics, and who can do it and when.

Task 2 may be further enhanced by using repeat to automate the repetitive process of drawing polygon sides, as is the case in Task 3. Task 2 may be altered to draw a square rather than a rectangle with different side lengths. This would allow an easier extension to other regular polygons, by altering the repeat number and the angle size. For example, repeat 3 with a turn angle of 120 degrees would result in an equilateral triangle. This automation through code of drawing regular polygons by repeating a number of steps followed by a turn angle helps reduce cognitive load and offers what diSessa (2018) calls *cognitive simplicity.* Additionally, ability to add a wait 1 second code block allows students time to observe the changes as the polygon is drawn.

An extension of this activity could be to include other tasks, such as creating pentagons and hexagons. In that way, by first introducing the square and the repeat, the students can play and test how they can use this starting point to discover other geometrical shapes. In Task 3, the students extend the automation for drawing regular polygons to draw a polygon with many more sides: 180 sides in Figure 3c and 12 sides in Figure 3d. This can help students realize that as we increase the number of sides by reducing the turn angle, we get closer and closer to approximating a circle. In essence, students are taking initial playful steps for investigating ideas of infinity and limit, which are foundational concepts of calculus. That is, as we decrease the turn angle, as the turn angle gets closer to zero, the polygon's "limit" is a circle. However, one constraint with this idea is that the Scratch environment does not allow reducing the step size less than 1 (as the step of 1 = 1 pixel and you

cannot draw ½ a pixel). Nevertheless, introducing the idea will contribute to a conversation about important concepts and relationships in primary mathematics.

Discussion and concluding remarks.

Computational literacy represents a powerful way of seeing the potential of computer programming in primary mathematics education. Findings from this study show that the computer program Scratch, can contribute to the development of mathematical ideas and relationships in primary school classrooms. Using diSessa's (2018) computational literacy principles allows us to discuss and discover how computer programming can be used to identify new ways of engaging with and discussing mathematics. Our findings, in line with Noss and Hoyles (1992), demonstrate that the computer program offers mathematical activities in the classroom that would be difficult to conduct without it. Particularly, how Tasks 1-3 provide students with immediate feedback on their work, leading to discussions about mathematical ideas and relationships of geometric shapes.

However, there are some issues to consider. Our analysis reveals that only one out of 35 activities in our study was considered powerful enough for computer programming to remediate, reformulate, revitalize, and reorganize content in primary mathematics. This finding aligns with both Feng & Yang (2022) and Nordby et al. (2024) who state that teachers struggle to maintain a mathematical focus when computer programming is integrated into classroom activities. Therefore, at its current state, we can argue that computer programming has not contributed to develop new ways of teaching primary mathematics in general. This study contributes to showing how computer programming may develop mathematical ideas and relationship, and that more attention is needed in primary mathematics research to discover ways in which computer programming can support mathematical activities. Developing such knowledge may lead to substantial and powerful changes in how we perceive the teaching and learning of primary mathematics.

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Teachers' challenges integrating technology into teaching problems they have posed: Through the lens of instrumental approaches

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An area of interest for educators today is how technology can support teachers' problem-posing for their teaching needs in mathematics. Teachers often face significant challenges when attempting to incorporate technologies such as applets, and software effectively, in particular for pedagogical practices. By examining these challenges using the theory of "instrumental approaches," that is, the interactions between the user (teacher or student) and technology, we can better understand the complexities involved, which is crucial for addressing specific challenges and improving technology's integration into education. This article highlights three significant challenges identified in a project where teachers used a digital tool to pose problems for teaching purposes.

Keywords: Problem posing, digital literacy, instrumental approach, instrumental orchestration, orchestration types

Introduction

In the last two decades, there is increasing focus on the knowledge teachers develop regarding digital tools (DT) and how DTs can elevate the teacher's role by replacing traditional methods and also to explore concepts, improve techno-mathematical literacy, cultivate knowledge (Jacinto & Carreira, 2023), and promote teachers' problem-posing (PP) processes for teaching needs (Ovadiya, 2023). Cognitive mathematical technology, including digital mediums, motivates learning in diverse ways, sparks curiosity, and opens up new areas for education such as delving into mathematical concepts and operations that were previously beyond reach without technology (e.g., Smith et al., 2017).

Understanding Instrumental Approaches. The concept of "instrumental approaches" assumes that technological tools are useless without specific schemes or cognitive tools to operate them effectively. When that happens, "instrumental genesis" occurs, meaning that the user has adapted their thinking and the tool's function to suit their goals. In education, this involves using technology for effective learning, particularly for complex tasks such as mathematical PP to achieve specific pedagogic goals, which necessitates a profound grasp of the goals of the mathematical task and the skills to adapt the mathematical content to achieve the didactic and pedagogical objectives. Further research in this area is necessary; there is limited research on how teachers pose mathematical problems using DT for specific teaching purposes. This study focuses on understanding the challenges teachers encounter in this area and tries to interpret the teaching process (of the posed problems) through the lens of instrumental approaches by observing the teaching process using the "instrumental orchestration" types specified by Tabach (2013), who consolidated the terms from several studies.

Theoretical background

How Teachers Pose Problems

Research into how teachers pose mathematical problems has explored various aspects, including professional development (PD) workshops to enhance PP practices (e.g., Ovadiya, 2021, 2023).

Studies reveal that teachers struggle to design problems that engage substantive or logical-mathematical ideas rather than just the "calculate" or "find" types (Kozakli Ulger et al., 2022). There are positive indications from studies by Ovadiya (2019, 2021, 2023) that teachers are open to learning how to innovate PP and PD programs that include teacher discussions have led to improved PP skills for specific pedagogical goals, such as understanding graph properties (Ovadiya, 2021) or applying inquiry-based approaches in geometry (Ovadiya, 2023). However, the existing literature often overlooks the practical steps required for effective PP for teaching. This method described herein tries to bridge this gap by describing how teacher educators can leverage PP training to enhance teachers' abilities to pose problems that meet instructional needs.

Teachers' use of DTs in problem posing

Integrating DTs into PP adds complexity to an already challenging task. Since the mid-1990s, research has shifted from focusing on learners to exploring how digital technologies impact teachers (Sinclair et al., 2023). The consensus is that teachers often find problem-posing more challenging than problem-solving, particularly when creating complex problems requiring high-level thinking (Yao et al., 2021). Ovadiya (2019, 2021, 2023) suggests that designing problems with specific pedagogical goals can enhance teachers' techno-mathematical literacy and pedagogical knowledge. Also, when teachers experimented with integrating technology into PP, they discovered – as a result of the technology – properties of concepts, mathematical operations, or geometric theorems that they had not been formerly aware of. In other words, the tools led the teachers to deepen their mathematical knowledge. Indeed, tools such as Desmos, GeoGebra, Phet, Traker (and so on) facilitate the exploration and understanding of mathematical concepts, operations, and phenomena. However, teachers must develop their techno-mathematical skills to use such tools effectively (Ovadiya, 2023).

Problem posing during professional development for teaching it in the classroom

Most studies dealing with how mathematics teachers pose problems, even recent ones, focus on teachers in PD programs or academic courses and not on actual classroom situations. However, successful PP during a training program does not guarantee that the teacher can pose appropriate problems to foster classroom learning. There are many reasons for this, but the main one is the ability (or not) to define the mathematical pedagogical relevance, which differs between the two. In the current study, there was an effort to address this gap as well. Therefore, as part of the course syllabus, the teachers were to design and pose problems that used DT and would abide by the curriculum.

A theoretical framework to understanding the findings: Instrumental approaches

The instrumental approach is based on the idea that the use of instruments by learners (whether physical, mathematical, or technological) does not only involve performing specific actions but also involves integrating these tools into the cognitive structure of the learner. Through this process, the learner appropriates the tool and it is transformed into an instrument that the learner adapts to serve their own purposes and contribute to their mathematical thinking (Revillon & Rabardel, 2004). The instrumental approach focuses on how tools evolve into cognitive learning instruments through user interaction. This concept is also called “instrumental genesis.” This process involves a “dynamic interaction between the learner and the tool, which goes beyond the simple use of the tool and leads to the tool becoming integrated into the cognitive structure of the learner, aiding in the construction

of mathematical knowledge" (Vergnaud, 1994, p. 150). Artigue (2002) emphasized the concept of instrumental genesis, which describes how a tool becomes an instrument for thinking and learning mathematics. She distinguished between "instrumentalization," that is how learners adapt the tool to their needs by modifying and shaping it, and "instrumentation," which is how the tool influences learners' thinking and shapes their mathematical activity. This dual process highlights the dynamic interaction between the user and the tool and reshapes views on how technological and other mediating artifacts may be integrated into mathematical practices.

As part of developing research about teaching and learning using DTs, additional concepts have been developed. For example, "instrumental orchestration" is a term introduced and defined by Trouche (2004) in the context of mathematics education. Trouche's work (2004) emphasizes that the teacher actively mediates the learning process to ensure that the DT is not merely used passively but becomes an instrument that enhances mathematical thinking and engagement. The key elements of instrumental orchestration are didactical configuration (arranging tools, tasks, and the environment for learning activities), exploitation modes (strategies to guide tool use, such as demonstrations, group work, or individual tasks), and orchestration types (context-dependent approaches, including whole-class, small-group, or individual activities). In the years since Trouche, a number of orchestration types have been defined, for example, as presented by Tabach (2013). Recent studies, for example, Artigue et al. (2020), emphasize the need for teachers to adapt their orchestration strategies to integrate DT effectively by developing appropriate skills in classroom management.

Method

Research question: Can the theory of instrumental approach help identify, understand, and interpret the challenges teachers face when using DTs to teach mathematical problems they pose to meet their teaching needs? **Participants**. Twenty in-service high school mathematics teachers participating in a professional development at a teachers' college. They taught grades 7–12, either in Arabic or in Hebrew. Their seniority ranged from 3 to 21 years. All had experience teaching with technological tools, some for 8 to 12 years and some for two.

Procedure. Teachers independently posed mathematics problems using DTs, taught them to their college peers, revised the problem if necessary to better meet their teaching goal, and then taught the problem to their students in school. They then selected an event from the lesson to discuss in the course and then adjusted the problem as necessary. Every stage was fully documented in their journals. Each teacher underwent this procedure six times (total, 120 math problems)**. Data sources.** The teachers' journals, video documentation of their teaching, video and audio recordings of the college lessons and mentoring meetings, and the author's journal. **Data analysis.** Documents underwent inductive analysis driven by the research question: (1) analysis of teachers' journals to identify challenges they recognized and reported; (2) similar analysis of researcher's journal; (3) identification of specific technological challenges; (4) categorization of technological challenges (these include didactical configurations such as understanding the role of the tool(s) as active mediators of learning, using technology to create new ways to think about mathematics and new opportunities for exploration and interaction, or exploitation modes that encourage students to interact with group work or individual problem-solving; (5) associating examples for each challenge; (6)

analysing the challenging events by "orchestration types" and identifying the teacher's instrumental orchestration actions (Technical-demo, Sherpa-at-work, Discuss-the-screen, Work-and-walk-by, Spot-and-show, Not-use-tech, Explain the screen, Link-screen-board. For explanation see table in Tabach, 2013); (6) determining and choosing challenges that reflect the dynamics between the teacher, the acts of teaching, students learning, the math problem, and the DT.

Findings and discussion

The assumption is that a teacher who identifies a challenge in teaching will try to manage it through specific teaching actions. In the present article, the author addresses the challenges teachers identified when teaching the problem they posed using DTs and identifies their orchestration actions to manage that challenge. Below are three examples and their interpretation using instrumental approach theories

Challenge 1. Teaching how to illustrate a mathematical phenomenon graphically

The posing process: Challenges. The teacher posed a problem about local and absolute extrema while finding the shortest distance between the graph of a quadratic function and a specific point. She wrote: "While solving the problem, I encountered a mathematical conflict: differential calculus indicated a specific maximum point, but the graph showed more distant points. This raised the question of how to mathematically explain to students that an internal maximum point is not necessarily an absolute maximum, especially in an open domain, and how DT can mediate it."

The problem. Find the shortest distance between the graph of function $F(x) = \frac{1}{4}x^2$ and point (0,3) (Use digital drawing tools for investigation).

The teaching process: Orchestration types. The teacher opened the lesson by presenting the problem on the general board of the class (Technical-demo). She allowed the students to solve the problem independently in any way they saw fit and emphasized that they were welcome to check on their computers or the application on their cell phones. A discussion about the distance function developed, and a student went to the board to write the solution (Sherpa-at-work). Another student suggested looking at the graph on the board (Discuss-the-screen), noting that the maximum point can be identified without investigation using a derivative, meaning, according to him, that the point on the original graph indicates a maximum distance. Another student suggested both referring to the algorithm for comparing the derivative to zero and drawing the graph of the derivative function, which he did on his cell phone (Work-and-walk-by) to learn about the types of extrema. Another student suggested drawing a graph of the distance function and investigating it (Work-and-walk-by). The teacher encouraged all suggestions and used the ideas to promote discussion (Spot-and-show). A conflict arose over the graph in the applet (distances greater than the maximum distance were found). A student suggested drawing it in another widget; perhaps there would be details about the graph that Desmos and GeoGebra did not reflect. He did this in OneNote and claimed that the maximum point was local (Sherpa-at-work). The teacher and students together investigated the graphs of all the functions suggested for drawing and concluded that the point they found was a local maximum (Discuss-the-screen). (An example is at https://www.geogebra.org/m/e5enuzv4)

The gap. Examining the function's graph in the applet and identifying the maximum distance based on the algorithmic solution raised the question: "… the graph displays greater distances." The students

explored this conflict by plotting distances greater than (d = 3) in the applet, discussed how that was possible, and realized that the algorithmic solution revealed a local maximum, not an absolute one. As their discussion deepened, they concluded that distances larger than that indicated meant that the maximum they had found was local. To confirm this, the students decided to graph the distance function with GeoGebra. They identified two local minima and observed infinitely large distances that they believed represented the maximum. This changed their understanding of "local extrema."

Explanation using instrumental genesis theory. In this lesson, the teacher helped students grasp the concepts of local and absolute maxima by highlighting a conflict between the graph in the application (Discuss-the-screen) and the algebraic solution (Not-use-tech). The lesson included various orchestration activities that facilitated transitions from general discussions to independent or paired explorations. The teacher's approach to leading the lesson to identify the mathematical phenomenon by incorporating drawing and engaging dynamic actions within it led to interaction between the learner and the tool and to the tool becoming integrated into the cognitive structure of the learners, aiding in the construction of mathematical knowledge about "local extrema" as did the diverse orchestration activities and her flexibility in listening to and evaluating each student's ideas, whether expressed verbally or digitally.

Challenge 2. Managing discussions on digital problems to illustrate the parameter phenomenon

The posing process: Challenges. The aim was to pose a problem that incorporates a parameter and then discuss with the students the concept of the graph as a general representation.

The problem. Use a wire of length "*a*" to construct a picture frame with three sides of a rectangle and a semicircle whose diameter is the fourth side of the rectangle. What should the circle's diameter be to obtain the maximum area within the frame? Log in to https://www.geogebra.org/m/djgdvtxn.

The teaching process: Orchestration types. The teacher read the problem aloud (Not-use-tech) and invited the students to work in pairs to send a sketch of the solutions via WhatsApp (Work-and-walk-by). The teacher provided an explanation on the board about GeoGebra, focusing on the data it presents (Technical-demo), and then facilitated a class discussion to find a "target function" while observing the graph (Explain-the-screen). The teacher encouraged questions about the solution without using algorithmic investigation tools (Not-use-tech). Students interacted with the app and asserted that essential features of the graph had changed but were not sure what had happened (Work-and-walk-by). Some students overlooked essential traces in the graph (Discuss-the-screen). The students arrived at a final answer but did not understand the role of the parameter (Not-use-tech). The teacher attempted to revisit the graph to explain the parameter, but the students deemed it unnecessary since they believed the parameter did not represent a central feature (Explain-the-screen, Discuss-the-screen). Students suggested graphing the "target function" graph, and one of them came to the board to do it (Sherpa-at-work).

The gap. After reflecting on the discussion, the teacher felt disappointed that the students concentrated on extracting information from the graph related to size rather than features. One student had suggested drawing the "target function," which was not part of the teacher's original plan. However, this suggestion led to a valuable discussion about the meaning of the parameters in the target function, the graph representing it, and the comparison between the two graphs.

Explanation using instrumental orchestration theory. Here, we can see the reasons for the breakdown in effective orchestration. i) The teacher may not have developed the necessary orchestration schemes to facilitate a class discussion around a general parameter using the DT. Orchestration schemes are ways of structuring tasks to use the tool most effectively and guide the class discussion. If the teacher cannot handle the complexity of general discussions around the parameter, they may be forced to revert to specific cases, which are easier to manage and explain. ii) The teacher's technical fluency with the tool (e.g., handling symbolic or parametric inputs) might not have met the technical-pedagogical goal of discussing the general problem. This mismatch led to simplifying the problem by focusing on specific cases. Nonetheless, the teacher's flexibility in discussing a student's suggestion to draw a different graph led them to a meaningful discussion of the concept of parameter. (See the student's graph: https://www.geogebra.org/classic/agbk5km8).

Challenge 3: Teaching students to create dynamic objects (as "traces") to extend the understanding of mathematical phenomena

The posing process: Challenges. "We never talked in class about the object of the graph of the "target function" in an extrema problem. I expect this to be the most interesting part of the problem by showing the students that when you drag point C, you get the graph of the target function, in this case the graph of the area. … After building the app and talking to my fellow teachers, I realized it was important to teach the students to build the object's 'traces,' as this sharpens understanding of the relationship between the infinite possible triangles and the parabola function. That is, the parabola is a geometric place for all the triangles that meet the conditions of the problem. But how do I teach this without giving explicit instructions on what to do?"

The problem. There is an L-shaped building behind Sammy's school. Sammy wants to build a garden in the shape of a right-angled triangle. Two sides of the triangle will be adjacent to the building. Sammy has a 6-m length of fencing for the hypotenuse of the triangle**.** Log in to https://www.geogebra.org/m/xmnxxhn6 and answer the following question**:** What are the maximum and minimum areas of the garden that Sammy can build? In GeoGebra, we will construct point D together, which describes the target function of the problem. (See https://www.geogebra.org/m/hsm4w92g to see the results after building the trace).

The teaching process: Orchestration types. The teacher presented a problem on the board (Technical-demo) which the students followed on their cellphone app (Link-screen-board). The teacher explained how to understand the graph. The students tried this on their cell phones and those who solved the problem shared their solutions with and without technology (Discuss-the-screen, Not-use-tech). The teacher responded to students who identified an error between the algorithmic solution and the solution they identified, inquiring about the app (Work-and-walk-by). One student led a discussion (Not-with-tech) about target functions and an algorithmic solution to the problem. The teacher told the students to add components and operations to the app to construct traces for the problem to describe a geometric locus of the solution set (Technical-demo). Investigating the app included the traces and an attempt to understand the traces by describing mathematical phenomena. Both independently and by observing what was written on the board, the students understood that the parabola created (through the traces) is a geometric locus that represents the collection of solutions

(Work-and-walk-by, Discuss-the-screen). A discussion and summary around the question “Do we have to understand and draw the locus to solve the problem?” took place that combined the non-technological solutions (Not-use-tech) and ideas from the trace graph (Discuss-the-screen).

The gap. The student's understanding of the problem and solution without the DT preceded their understanding of the graph with the DT. To them, building a target function to describe the infinite areas of the triangles as a parabola using the DT did not resemble the algorithmic solution. The insights from an algorithmic solution differed from those they obtained when required to build an app describing the problem and the solution.

Explanation using instrumental orchestration theories. Building traces in software requires mathematical insight behind the technical operation to understand that they are components in the graph of the solutions of the objective function. The visual graph that emerges dynamically while changing the sizes of the triangles explains the collection of solutions and their relationship. The teacher's orchestration actions led the students, time and time again, to move from an algorithmic to a dynamic solution. This explains the teacher's challenge, which was to teach how to build a digital object that the students do not seem to need because they have settled for the procedural solution. Understanding the digital solution led to the tool becoming integrated into the cognitive structure of the learners, aiding in the construction of mathematical knowledge about how a parabola created (through the traces) is a geometric locus, and this process is reflected as an instrumental genesis.

Summary

The instrumental approach helps the researcher identify and interpret the challenges. The three examples above demonstrate the challenges a teacher may have when teaching a concept or phenomenon by solving a mathematical problem when a gap occurs between how students understand the problem and its solution without technology and understanding using a dynamic tool. To bridge the gap, the orchestration actions moved from a state working without technology to one where they understood the dynamic solution and compared the two solutions simultaneously. The orchestration continued until the dynamic solution of the problem explained the algorithmic solution of the problem and vice versa. In the current study, the researcher implemented orchestration actions defined by other researchers (Tabach, 2013) to clarify the teacher's challenge. It often stemmed from the gap in students' understanding of the concepts and actions required to digitally solve the problems posed by the teacher. Not all orchestration types were suitable for the lesson's analysis. The next step for the researcher is to focus on more precise definitions of orchestration types specific to problem-solving instruction that teachers have posed, especially in challenging situations.

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Teaching computational thinking in a collaborative context: A case study

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This paper aims to address the knowledge of the teaching practice regarding computational thinking (CT) of Lara, a grade 8 mathematics teacher, through her participation in a lesson study. Scratch supports this study as a working tool to promote CT practices. The research is a qualitative and interpretive case study with participant observation design. The participants are four mathematics teachers and the first author, as a facilitator. The results show that the collaboration carried out in this lesson study led to the mobilization of the teacher's CT knowledge regarding planning and leading the research lesson. Her actions and those of the students contributed to effective learning.

Keywords: Computational thinking, Scratch, teaching practice, collaboration, lesson study.

Introduction

Bocconi et al. (2022) published a study in which, among 29 European countries, 18 EU member states and 7 other countries, introduced basic computer science concepts into their mathematics curricula in primary and secondary education. With great emphasis on the development of Computational Thinking (CT) knowledge, the authors suggest Scratch as one of the main ways for students and teachers in this context to explore and develop themselves in this regard. Bocconi et al. (2022) also show great concern in preparing teachers to integrate CT development into their educational practices, pointing out this challenge to ministries and teacher education institutions. Considering the questions raised by Bocconi et al. (2022), it is necessary to take into account the various aspects of mathematics teachers' professional knowledge. When teachers exercise their teaching practice, it is especially relevant to consider their didactic knowledge (Ponte and Oliveira 2002), which a form of knowledge in action (Schön, 1983).

In this study, we use Scratch to promote the development of CT and its integration into mathematics teaching. As a teacher education process that seeks to meet this demand, a lesson study was conducted. The participants were four experienced mathematics teachers, from the grade 8, who sought to include CT teaching in their practice. We consider in particular Lara's participation, as she evidenced her knowledge in action, manifested by her actions and decisions when planning and conducting an exploratory lesson. This communication aims to answer two questions: 1. How was Lara's didactic knowledge mobilized in teaching CT to plan a task in an exploratory lesson and leading that lesson? 2. How did the collaboration between the participants during the lesson study favor Lara in mobilizing the development of CT in her class?

Theoretical framework

Ponte and Oliveira (2002) designate professional knowledge that directly intervenes in teaching practice as didactic knowledge. Knowledge of teaching practice can be evidenced by the manifestation of knowledge in action by teachers, when considering their actions and decisions in the exercise of their profession (Schön, 1983). Knowledge of teaching practice allows teachers to have

confidence in making choices in accordance with their professional identity, establishing their preferences and priorities (Ponte & Oliveira, 2002). Collaborative work may interfere with teachers' choices in planning, designing and conducting tasks (Ponte, 2012). In this sense, Ponte (2012) states that acting collaboratively implies "assuming common goals, in a context of personal trust, in which everyone has something to teach and something to learn, as a pragmatic perspective, as a solution found by a group to solve common problems, which would be difficult to solve individually" (p. 10).

The materials that can be used to design tasks are relevant factors that also influence the choices of teachers when expressing their knowledge in action. Among them, there are a variety of technological resources that support mathematical teaching. Cobb et al. (2017) emphasize that the type of technology is less important than the functionality it provides and, therefore, teachers need to have a deep understanding of how technology and tools can be used to investigate mathematical ideas, generate multiple representations of a mathematical construction, and solve mathematical problems. Only then can these materials used in teaching be integrated into the curriculum in a meaningful way. Teachers need to know criteria for selecting textbooks and teaching materials that include the regular use of tools and technologies, in addition to improving the skills to use them effectively in the classroom to support students in mathematical learning.

Wing (2006) presents CT as a skill with great potential to be explored in mathematics teaching due to its close relationship with mathematical problem-solving. This skill involves five fundamental practices: abstraction, decomposition, pattern recognition, algorithmizing, and debugging (Wing, 2006). For their part, Moreno-León et al. (2017) mention that Scratch allows working on CT practices through its commands, serving as a bridge for the development of this ability. Developing knowledge of CT practices, specifically when using Scratch, has become a relevant topic in mathematics teacher education processes. In this sense, it is important to expand knowledge of teaching practice, which substantiates the teacher's actions in designing or selecting tasks and in their enactment in the classroom, which is the manifestation of their knowledge in action relevant in this study (Schön, 1983). In this communication, we are interested in analyzing the mobilization of didactic knowledge for the design and leading of a task, specifically when using Scratch as a technological tool to support CT learning in the context of teaching mathematics. It's also part of the study to consider the influence of the collaborative context of professional teaching training in which the activities were developed.

Methodology

i) Participants and context

The participants are four lower secondary mathematics teachers from a public school in Lisbon, where the first five work sessions and research lessons also took place, with the remaining sessions being conducted via Zoom. The object of study of this paper is the mobilization of Lara's didactic knowledge. Lara has 26 years of teaching but had no experience with CT, as neither did the other participants in the lesson study. Lara has a master's degree in Mathematics for Teaching. This study takes place in the context of a lesson study with the aim of promoting the development of the teaching practice of the participants. The lesson study took place in the 2023/2024 academic year and followed the steps indicated by Ponte et al. (2016): definition of the study aim, planning the research lesson, teaching the lesson, and post-lesson reflection. Twelve work sessions were held, each lasting around

2 hours. The group planned 3 tasks that were taught by Lara and analyzed by all participants in the post-lesson reflections. This study focused on the third planned task applied to fifteen grade 8 students attending the mathematics class under Lara's responsibility. In the research lesson, the students sat in a U-shape and worked collaboratively for 45 minutes.

The sessions sought to promote the development of CT knowledge to enable the mobilization of didactic knowledge, with greater attention to the aim of the proposed tasks and collaboration among participants (Ponte & Oliveira, 2002). Fujii (2018) states that lesson study supports teachers in developing knowledge on how to teach students not only content but also thought processes so that they become independent thinkers. Paying great attention to the processes of preparation, analysis, and reflection on teaching practice, this author especially values problem solving in the tasks proposed in research lessons. Fujii (2018) emphasizes that promoting teaching through problem solving cultivates students' mental habits in terms of the way they think and their attitudes towards arriving at a mathematical solution. Given that problem formulation and solving is a strong characteristic for the development of CT, it became perfectly compatible with the tasks proposed in the lessons of this formative context. As teaching material for CT development, Scratch was used due to the good results that research, such as that of Moreno-León et al. (2017), recommend in this regard.

ii) Data gathering and analysis

The methodology adopted is a qualitative case study, which follows the interpretative paradigm and the design of a case study, being the first author a facilitator. For data gathering, the following procedures were used: two semi-structured interviews, one before and one after the lesson study; direct observation of the lesson study sessions, including the research lessons; collection of documents of the tasks produced during the lesson study. Information from direct observation was collected through notes in a research journal, audio recordings of all lesson study sessions, and video recordings of the research lessons.

Using content analysis (Bardin, 1979), data were explored inductively, especially in situations that reflected the professional knowledge that Lara mobilized to design and lead the third research lesson, evidencing the manifestation of her knowledge in action (Schön, 1983). Lara was chosen for the case study because she taught the three research lessons, participating intensively in events that were relevant to the analysis of the results that evidenced the manifestation of her knowledge in action. However, in this communication, data analysis takes place within the scope of the planning and design of the third research lesson, as well as the post-lessons reflections and final interviews, as they evidence results of the entire teacher education process promoted by the lesson study.

Based on the video and audio recordings and their transcripts, as well as the lesson plans, episodes were selected focusing on three activities, from which three sub-activities emerged: (i) Planning the research lesson – selection and design of a task (sub-activities: declaration, blocks ordering, representing categories in blocks); (ii) Research lesson (sub-activities: presentation, independent work, whole-class discussion); (iii) Post-lesson reflections (sub-activities: collaboration between teachers before the lesson, collaboration between students and Lara during the lesson, collaboration between teachers after the lesson). This study follows the relevant ethical principles and guidelines, and all names are pseudonyms.

Results

As this is the culmination of the lesson study, we pay special attention to the mobilization of Lara's professional knowledge in the planning, enactment and reflecting of the third task.

i) Planning the research lesson – selection and design of a task

The aim of the task that the group planned was for the students to build a program so that they could calculate their final weighted average in mathematics, according to the real evaluation criteria established for this subject. The following variables of the algebraic formula already known by the students in the paper and pencil solution for this calculation should be created and inserted in the operator blocks: "TestsAverage", "MiniTestsAverage", "TestsCounter", "MiniTestsCounter", and "FinalGrade". Likewise, the grades of the Tests, Mini Tests, Lesson Observation (LO) and Oral Assessment (OA) would determine variables of: "TestsSum"; "MiniTestsSum". The grades of Lesson Observation (LO) and Oral Assessment (OA) always range from 5 to 25. The number of Mini Tests was always twice the number of Tests, requiring specific variables for these parameters. Each assessment was graded from 0 to 100, requiring a calculation to satisfy the 25% proportion that the average of the Tests and Mini Tests made up in the final average. There were two sets of blocks in the statement: in the first cluster, the final weighted average was calculated; next to it, it was converted into levels from 1 to 5.

Using their knowledge in action (Schön, 1983), Lara and the teachers decided that the program should be fully constructed but divided into blocks that followed a similar algorithmic pattern to the blocks immediately above, with the aim of inducing students, through problem-solving (Wing, 2006), to learn how to use the appropriate variables in Scratch. In order to challenge as many students as possible to enjoy every moment of the lesson, those who completed the blurred part of the set of blocks positioned on the left side of the script had to do the same on the right side (figure 1):

> Lara: Because you have groups at different speeds. The fact that we have a plan of a more complex degree of difficulty and a more reasonable plan, so to speak, allows you, because you have the complex one, to then fine-tune it in real time and move on to the next one. Ah, if the whole class needs it, the class needs it. That's it. If only a few need it, then only a few need it. (session 10)

The statement (figure 1) went through three versions, adapted by the facilitator with options to choose from, to meet the objectives of the task determined by the teachers, arriving at the final representation. By using Scratch commands to design the task (Ponte, 2012), Lara had the opportunity to promote the knowledge of CT practices (Moreno-León et al., 2017): i) pattern recognition: the order of the statement was organized so that students recognized that the variables they had to create in the blank blocks ("MiniTestsSum", "MiniTestsCounter", "MiniTestsAverage", and LO) had the same purpose as those immediately above these gaps ("TestsSum", "TestsCounter", "TestsAverage", and OA) and, therefore, the excluded blocks should also be structured like their upper blocks in order to obtain the final weighted average representative of the real evaluation criteria; by using "repeat" to ask the user questions so that they could enter the different evaluation scores; ii) algorithmic: by using "repeat", "question" and the "operators"; organizing these commands with the information that the user provides in "question and wait for answer", so that they are correctly inserted in the "variables" created for the "operators" to calculate their averages; using "if... then... else" to classify the final

average obtained in an organized level from 1 to 5; iii) abstraction: using "says", as it does not interfere with the functioning of the code; focusing on the organization of the algorithm to solve the problem, without worrying about irrelevant information; iv) decomposition: calculating the average independently of the grades of Tests, Mini Tests, Lesson Observation (LO) and Oral Assessment (OA) to obtain the final weighted average of these results; and v) debugging: checking by "clicking on the green flag" to see if the code executed the task correctly.

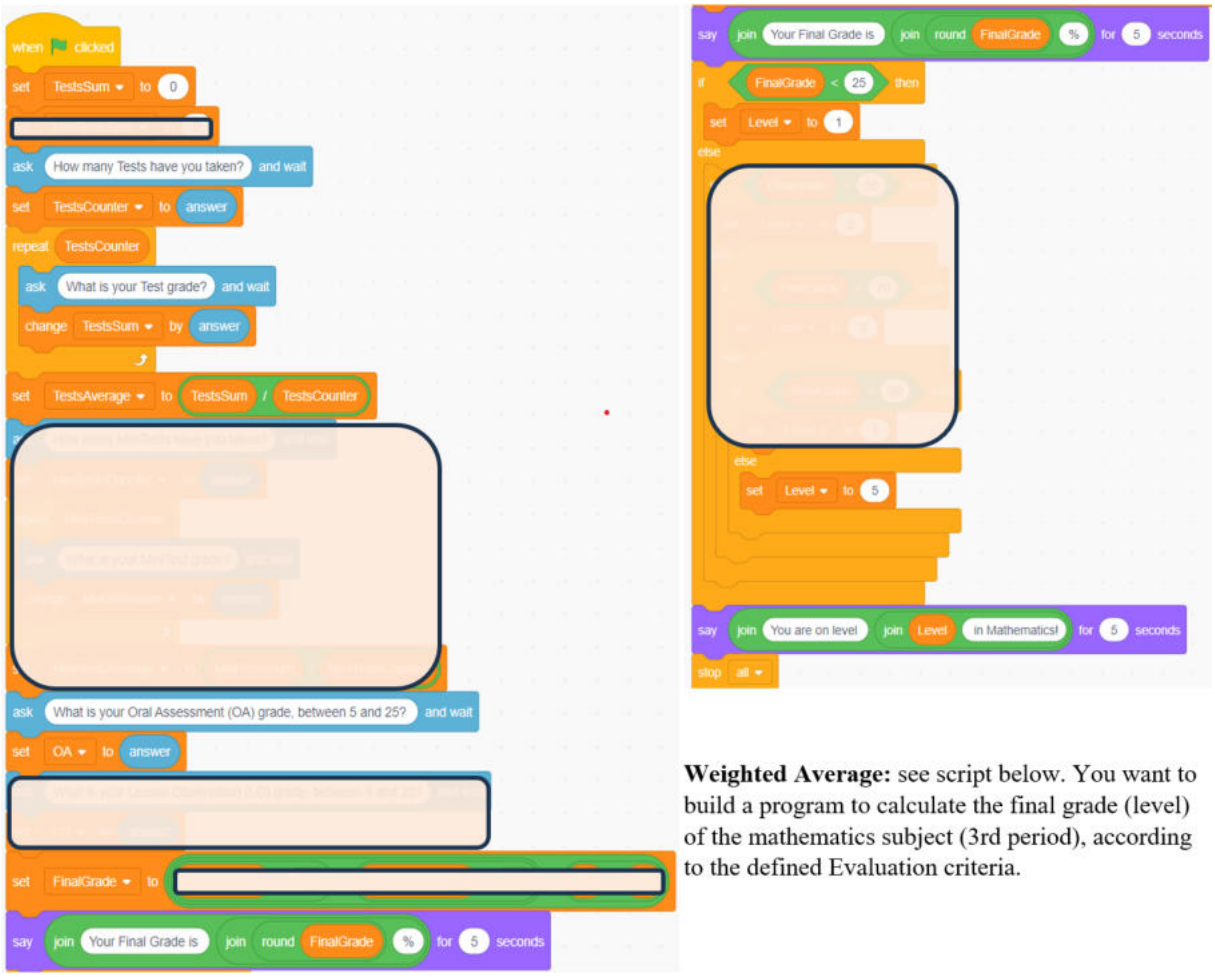

Figure 1: Task for the 3rd research lesson

At this stage, the teachers planned questions (Ponte, 2012) for Lara to lead the students to participate in the lesson in its three stages, as described in the next section.

ii) Research lesson

The research lessons followed the structure: (i) task introduction; (ii) students' independent work; (iii) whole-class discussion (Fujii, 2018). When introducing this planned task, Lara gave each pair of students the statement, explaining the aim of the task. Lara used her professional knowledge (Ponte, 2012) about the dynamics of an exploratory lesson to guide the students through this phase, using the questions provided to participate with information about the calculation of the final math average.

During the students' autonomous work, Lara also used her knowledge in action (Schön, 1983) about leading exploratory lessons (Ponte, 2012), asking them individually: what variables make up the average and how is each of them calculated. These questions, planned in advance, helped the students to identify the variables that should be created to represent all the evaluation criteria necessary to calculate the final grade (Wing, 2006). They also recognized patterns among CT practices (Moreno-León et al., 2017) when they realized that the variables "MiniTestsAverage", "MiniTestsSum" and "MiniTestsCounter", had the same purpose as the variables "TestsAverage", "TestsSum" and "TestsCounter". The same occurred with the variables Lesson Observation (LO) and Oral Assessment (OA). The students also built the blocks containing the operators that calculated the final weighted average, using the variables created to represent all the evaluation criteria used (Wing, 2006).

In the whole-class discussion, led by Lara according to the plan, three students presented different solutions, concluding with a summary by her of the main ideas discussed and the aims achieved. Lara followed the plan by asking challenging questions (Ponte et al., 2016), so that more students would participate in this phase. They analyzed different solutions that address the correct use of pattern recognition applied to the calculation of the weighted average and the mathematical rigor necessary for the algorithm to debug correctly (Wing, 2006):

Lara: So, we've created two routines, one for the Test average and one for the Mini Test average. Would it make sense to make a single routine for the average of the Tests and Mini Tests?
John: I don't think so, but if I had the same number of Tests and Mini Tests, it could be done.
Lara: But for those who did the final operation with all the calculations, was it different from the numerical expression we studied for calculating the average? In other words, what are the priorities in a numerical expression?
Peter: Multiplications and divisions.
Lara: Is the program respecting that?
James: Yes, so I did multiplication first and then addition.

iii) Post-lesson reflection

In the final interview, Lara emphasized how collaboration among teachers contributed to planning the task of supporting the teacher efficiently (Ponte, 2012), helping her to recognize the importance of this context:

Lara: The group made all the difference when it came to choosing the tasks and I'm extremely proud of the work we did because of its coherence. I think it was thought out and consistent. When we had the task-solving sessions, it had a purpose, it was to give us some comfort, ideas, possibilities for what would later be the construction of these tasks. I'd never have achieved this on my own. There was an evolution, even in the codes we used, in the programming routine itself. I couldn't build a set of tasks of this quality, with this consistency. I think things had a logical direction with a purpose, in the right measure, at the right time, and it was something built in a group, which always works out better than one person thinking alone. Even though I like working in groups, sometimes I find that I learn faster on my own. However, I recognize that when we are forced to share, the work is much better.

In her final interview, Maria, who observed the lessons in person, also valued Lara's approach to promote collaboration among students, thus reinforcing the didactic aim of this assignment:

Maria: The questions and the way Lara handled things were a guiding thread for the rest to develop and for the children to complete the tasks they were doing. Any question that came up during the course of the program, the children eventually understood. Those who had not finished the first stage got there. I think that was important and that was what brought all the groups together to get to the final program. Because there were some who had a little thing that they had not done [or] another little thing that they had not finished. And I think that was important.

Discussion

The definition of the learning aim of the task indicates the development of Lara's knowledge about CT practices, as they relate the blocks used in the Scratch categories to abstraction, decomposition, pattern recognition, algorithm, and debugging (Moreno-Leon et al. 2017). In these aspects, the knowledge of teaching practice, referred to by Ponte (2012), was developed in relation to the teaching of CT practices, since Lara and the teachers did not have this competence before the teacher education process. Lara and the participants mobilized the acquired knowledge to develop the aim of the task and integrate the technology in a meaningful way in the teaching of mathematics (Cobb et al., 2017). In turn, Lara's and the teachers' didactic knowledge about task design was mobilized during planning (Fujii, 2018), when they organized the order and appearance of the blocks in the task statement, to boost learning.

The structure of the research lesson (Fujii, 2018) favored student participation so that they could develop CT practices: when the teacher presented the task with an explanation of the procedures involved in the solution; when the students solved the task autonomously; when the teacher guided the classroom discussion on the different solution methods; and, finally, when the teacher summarized the content studied, emphasizing how students constructed their ideas for the different solutions discussed. Different solutions for this task were presented by students in the class discussion (Fujii, 2018), motivated by challenging questions asked at each stage of the lesson, as provided by the respective planning. Lara mobilized her knowledge in action (Schön, 1983) when she planned this lesson structure collaboratively and conducted it rigorously, demonstrating the didactic confidence mentioned by Ponte and Oliveira (2002).

Conclusion

Given the need for teachers to have consistent knowledge of CT practices (Wing, 2006) for their teaching, the different phases of this lesson study contributed to Lara's in-depth study on how to develop this competence (Cobb et al., 2017). Through the tasks solved and analyzed in the problem identification and lesson planning sessions, Lara and the teachers explored Scratch for tools: movement, appearance, sounds, events, control, sensors, operators, variables, and blocks. This allowed Lara to understand how useful these resources are for the development of CT practices, as they are directly related to problem solving. From this developed knowledge (Ponte, 2012), Lara and the teachers planned and Lara conducted a research lesson with the same aim, set up to be engaging for students. This reinforces and extends the perspectives of Moreno-Leon et al. (2017) on the usefulness of Scratch for developing CT.

Lara's and the teachers' reflections helped them to understand that collaboration at all lesson study stages was essential for the success of the research lesson, as it was based on planning aimed at

improving student learning (Ponte, 2012). The collaborative context favored the mobilization of knowledge in action (Schön, 1983), for improving the planning of the task and Lara's actions when leading the lesson and her reflections on effective teaching and learning of CT practices with the support of Scratch. Further studies may lead to understand how to develop the competence of teachers at other school levels to work with CT with their students.

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Teachers' knowledge and technology: the global model

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The teachers' central role in technology integration and the challenges of that integration emphasize the need for a deeper understanding about the teachers' knowledge required to teach with technology. Based on previous work and a systematic literature review, we identified three knowledge models often used: TPACK, KTMT and PTK. The goal of this paper is to discuss the similarities and differences between these knowledge models and present a Global Model. This Global Model is not a new model. On the contrary, it is a model developed based on the existing models and intending to integrate in a single model the knowledge domains considered in the different existing models. The Global Model highlights the common domains considered and the common roots for the three models, but it also makes explicit the differences, mostly related to the understanding of the domains or even to the domains considered, and also to the way how the knowledge's development is conceived.

Keywords: Teachers' knowledge, knowledge model, technology, Global Model.

Introduction

The integration of digital technology has been recognized for its potential to promote mathematical learning (Tabach & Trgalová, 2019). However, achieving that potential has proved to be quite a challenge (Rocha, 2020). Teachers have a central role in technology integration and their professional knowledge is one of the influences over that integration (Clark-Wilson et al., 2020).

The PCK – *Pedagogical Content Knowledge* construct developed by Shulman (1987) is a significative mark in the research on teachers' knowledge. Since then, several authors have developed refinements of PCK (e.g., *Mathematical Knowledge for Teaching* [MKT] by Ball et al., 2008) and even developed new frameworks (e.g., *Knowledge Quartet* by Rowland et al., 2005). The integration of technology as raised new questions and conducted to the development of new knowledge models. These models, with a focus on technology, included a model inspired by the PCK construct (*Technological Pedagogical Content Knowledge* [TPACK] by Misha and Koehler, 2006, 2019), a model intending to promote the inclusion in a knowledge model of aspects related to instrumental genesis (Rabardel, 1995) (*Pedagogical Technology Knowledge* [PTK] by Thomas and Hong, 2013), and a model intending to integrate the research on professional knowledge and the results from studies in technology integration (*Knowledge for Teaching Mathematics with Technology* [KTMT] by Rocha, 2013, 2020).

Although we generally refer to these models as knowledge models, some of them encompass more than just teachers' knowledge within their conception. That is the case of PTK, that includes the teachers' personal orientations, and also the case of KTMT, that globally considers all influences on the teachers, originating from themselves or from the context in which they are inserted.

The goal of this paper is to discuss the similarities and differences between the existing knowledge models with focus on technology. Based on this work, we intend to present a *Global Model*. This

Global Model is not a new model. On the contrary, it is a model developed based on the existing models and intending to integrate in a single model the knowledge domains considered in the different existing models.

Methods

According to Radford (2008, p. 317), "a network of theories can be conceived of as a semiosphere, (…) such a networking space revolves around two different and complementary 'themes' – integration and differentiation". As so, we will base our work on an analysis of the similarities and differences of the existing knowledge models.

Inspired by Jablonka et al. (2018), we started from a literature review (Rocha, 2023), to identify the existing models and the understanding ascribed to each of the knowledge domains considered. We then used all the knowledge domains, their conceptualization, and the relations among the domains to conceive the *Global Model*.

To align with our objective, we adopted the following categories for analysis:

- Domains of knowledge included;
- Other dimensions besides knowledge;
- Characterization of the knowledge development.

In each of these categories, we tried to identify similarities and differences, looking for a way to include them in the visual representation of the *Global Model*.

The knowledge models

According to the systematic literature review conducted in the time frame of 2018 – 2022, using the database Scopus, there are several knowledge models being used in research in Mathematics Education (Rocha, 2023). The results of this study point to the TPACK as the most used model, although there is some diversity in the way how the model is being used. Other models identified in this review were the KTMT and the PTK. As so, in the present work we consider these three models: TPACK, KTMT and PTK.

TPACK

As addressed before, the integration of technology in teachers' practice has raised new challenges. It was clear that teachers need some knowledge about technology, but very soon it was clear that some additional knowledge would be needed. Inspired by the work of Shulman and his PCK construct, Misha and Koehler (2006) developed a new model focused on technology integration. This model emphasizes technology while remaining general, lacking a specific focus on the teaching of mathematics.

Domains of knowledge included

The TPACK model considers the *Content Knowledge* and *Pedagogy Knowledge* domains, assuming PCK as the intersection between these. To these domains the authors add the one on *Technology*, considering then the intersections among each pair of domains: *Content and Technology*, *Pedagogy and Technology*, and the intersection among all the domains – the TPACK. The *Context* is also

considered, although in different ways over time, only recently being assumed as a knowledge domain (Mishra, 2019).

Other dimensions besides knowledge

All the domains in this model are knowledge domains, except for context. In the initial representation of their model, Misha and Koehler (2006) did not include the context, although in the discussion they include some references to it. Later, the model changed to include what the authors call *Contexts*, even if they do not describe what that could be (Harris et al., 2009). Finally, more recently, the model evolved to include *Contextual Knowledge* (Mishra, 2019). This change is clearly based on the recognition by Mishra that *Contexts* is not conceptualized as knowledge, resulting in some inconsistency in a knowledge model. So, the author changed the designation to align it with a knowledge domain. And this new domain of knowledge is assumed as being "everything from a teacher's awareness of available technologies, to the teacher's knowledge of the school, district, state, or national policies they operate within" (Mishra, 2019, p. 76) or, in a shorter way, knowledge about the school reality where the teacher is teaching. The result of this change is that all the domains in TPACK are indeed teachers' knowledge.

Characterization of the knowledge development

The conceptualization of the TPACK model is not so clear about how the knowledge development is conceived. Some authors interpret the model in an integrative perspective and some others in a transformative one (Angeli & Valanides, 2009). As so, it is not clear if knowledge in two base domains (Content, Pedagogy, or Technology) results straightforward in knowledge in the intersection of these domains, but this is the perspective assumed in some studies.

KTMT

The KTMT model recognizes the contribution of the work of authors such as Shulman (1987), and Mishra and Koehler (2006). However, the model intends to go beyond TPACK, integrating in a single model the research conducted so far in teachers' knowledge and in technology integration. With a specific focus on the teaching of Mathematics, this model developed by Rocha (2013, 2020) includes several types of domains in a pyramidal structure.

Domains of knowledge included

KTMT includes three types of knowledge domains: base knowledge, inter-domains knowledge, integrated knowledge.

The base knowledge domains are four: *Mathematics*, *Teaching and Learning*, *Technology*, and *Curriculum and Context*. *Curriculum and context* are regarded as a transversal domain that influences all other domains. It encompasses all influences on teachers' choices, including personal factors (such as teachers' beliefs) and external factors (such as the school context).

Inter-domain knowledge is a distinctive key component of this model that differentiates it from other knowledge models. The KTMT model considers two inter-domain knowledge: the *Mathematics and Technology Knowledge* (MTK), and the *Teaching and Learning and Technology Knowledge* (TLTK). MTK focuses on how technology influences mathematics, enhancing or constraining certain aspects,

and TLTK focuses on how technology affects the teaching and learning process, enhancing or constraining certain approaches.

Integrated Knowledge (IK) is the last type of knowledge in the KTMT model, resulting from the articulation between all the knowledge domains.

Other dimensions besides knowledge

As previously noted, *Curriculum and Context* are viewed as a transversal domain that influences all other domains. It encompasses both external influences on teachers—such as the official curriculum and available school resources—and internal influences, including their beliefs about the potential of technology for teaching mathematics and their self-efficacy. Therefore, this domain extends beyond mere knowledge.

Characterization of the knowledge development

KTMT has a pyramidal structure. Instead of intersections as in TPACK, the inter-domain knowledge is a new knowledge developed from more than one base knowledge and integrating in its characterization results from the research on technology integration. And the *Integrated Knowledge* develops from the base and the inter-domains knowledge in a similar way. This is an on-going process with a continuous development of the knowledge in all the domains of the model.

PTK

According to Thomas and Hong (2013), the PTK model can be useful to promote understanding about how the teachers use technology to teach Mathematics. As so, and as in the case of KTMT, this is a model specific to the teaching of Mathematics. Even so, the authors adopt a very different perspective from the ones in previous models, highlighting the use of technology and how the technological tool can be transformed in different instruments (in line with Rabardel, 1995). However, it is possible to identify in this model, as in the previous ones, the influence of the work of Shulman and his notion of PCK.

Domains of knowledge included

With its roots in the work of Shulman, this model considers as domains the *Mathematical Content Knowledge* and the *Pedagogical Knowledge*. However, instead of PCK, it considers the work of Ball et al. (2008) and their MKT. Besides these, the authors assume as important two domains that are not knowledge domains. That is the case of the *Technology Instrumental Genesis*, assumed as highly relevant, and of *Personal Orientations* also with a strong impact recognized by the authors. All these domains contributing to the PTK.

Other dimensions besides knowledge

As already mentioned, the PTK model includes two domains that are not knowledge domains. This is a difference in comparation to TPACK, but a similarity with KTMT. The inclusion of instrumental genesis, a framework usually not articulated with knowledge models, results in a significative difference of this model when compared with the other two.

Characterization of the knowledge development

The model seems to be based on the MKT model from Ball et al. (2008), to which the technology is added. This resulted in the transformation of the artefact in an instrument, influencing simultaneously the teachers' MKT and the PTK. The *Personal Orientations*, in turn, influence the teachers PTK.

The *Global Model*

The three models are very different in the way they conceived the development of the knowledge. For TPACK we have intersection and, in the integrative perspective, knowledge resulting straightforward from there; for KTMT we have new knowledge developing from the previous level of knowledge, conceived like layers continuously developing; for PTK we have domains influencing other domains.

However, despite the differences, there are also similarities between the three models, with a base on the work of Shuman and his notion of PCK. With nuances, but there are domains of knowledge common to all the models considered. Some kind of context, with differences across the models (and even including it as knowledge in TPACK and as a different type of domain in the other two models), is assumed to be important in all the models.

As a way to highlight the similarities and differences between the models, we start by adopting a similar way of representation. And we adopt boxes and arrows to represent the domains and the relations among them. In the representation of the *Global Model* presented here, we use different colors to represent the relationships between the knowledge domains of each of the knowledge models considered – TPACK, PTK and KTMT – and we resort to partial overlap between the representation of domains in different models when it seems to us there is some relation between the domains of knowledge considered.

The analysis presented here in a very synthetic way, led to the *Global Model* represented in figure 1.

Different colors in the arrows allow the identification of each of the original models. As so, parallel arrows offer the identification of similarities among two of the models. That is the case of TPACK and KTMT, a situation that was possible to anticipated, given the influence of TPACK on KTMT conceptualization.

Partial overlap between the representation of domains is also important, revealing some connections between models. These overlaps can however be of two different types. In one of the types there can be different names to represent basically the same domain of knowledge. That is the case of *Mathematical Content Knowledge* in PTK, of *Content Knowledge* in TPACK, and of *Mathematics Knowledge* in KTMT. In the other type of cases the domain is entirely different, although it is possible to see some relation. That is the case of *Technology Knowledge* in KTMT and TPACK, and Technology Instrumental Genesis in PTK.

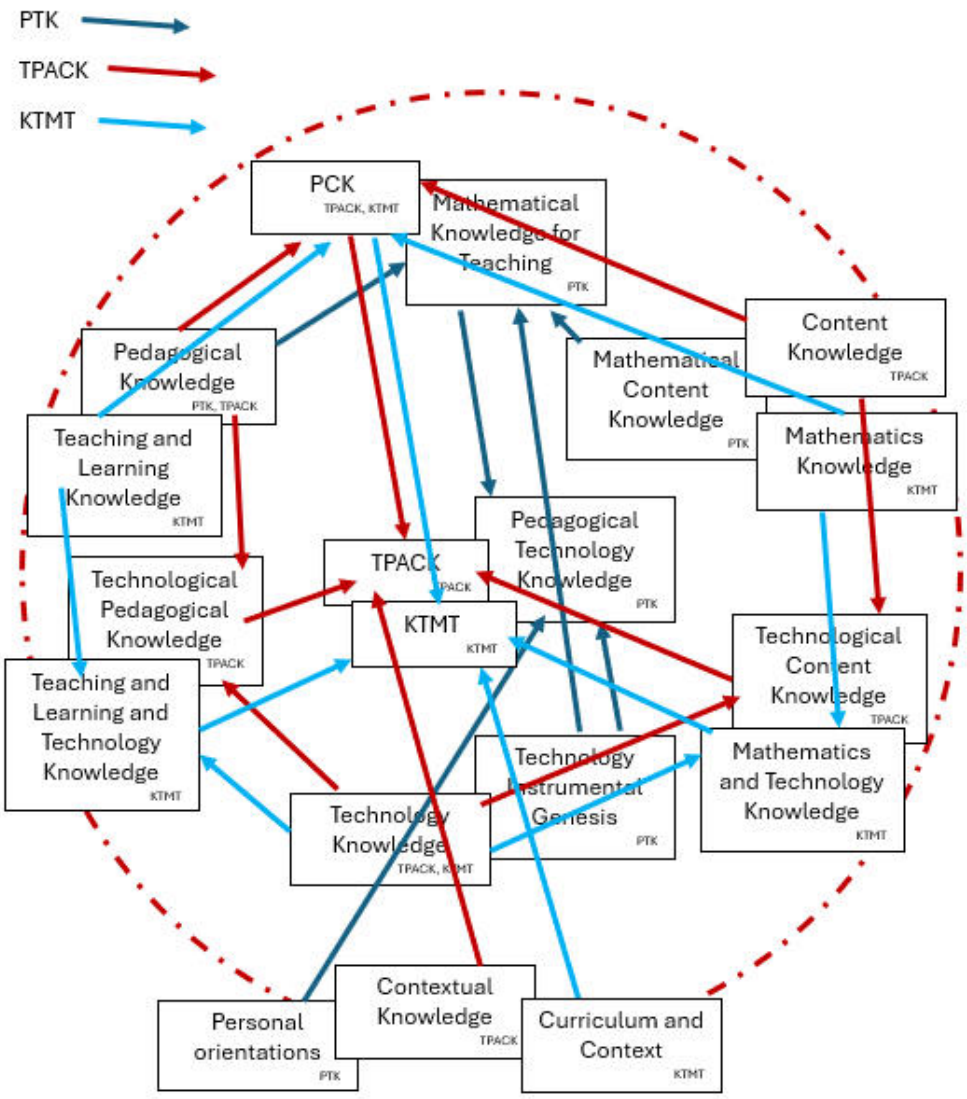

Figure 1: The Global Model (Rocha, 2024 – in press)

Conclusion

The conclusions achieved point to some similarities, essentially related to the mathematics knowledge and to the pedagogical knowledge, with a strong value given to PCK. But the conclusions also point to some differences. Some related to the name given to the domains (e.g., mathematical knowledge, mathematical content knowledge), but mostly related to the understanding of the domains (e.g., PCK vs MKT) or even to the domains considered (e.g., curriculum, context), and also to the way how the knowledge's development is conceived.

It is now important to refine the model, based on its use in the analysis of teachers' professional knowledge, and analyze whether this *Global Model* can make contributions to understanding teachers' knowledge and to promoting their professional development.

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Resonance of instrumental functions in geometric constructions: Unpacking teacher and student artifact use

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This ongoing research investigates the didactic potential of using multiple artifacts in the teaching and learning of geometric constructions. It aims to explore the interactions of instrumental functions by accounting for the diverse perspectives of the user(s) and the object(s) of their activities. We hypothesize that the same artifact can give rise to multiple instruments when integrated into the activities of different subjects, owing to their unique processes of instrumental genesis. In addition to examining the synergy of instrumental functions, we introduce the concept of 'resonance' which emerges when instruments mediate the activities of different subjects engaging with 'paired objects of activity'. It also applies to the set of semiotic resources activated within such context. To illustrate these emerging concepts, we present a case study highlighting their application and relevance.

Keywords: Instrumental genesis, instrumental functions, interaction of artifacts, semiotic bundle.

Introduction

Integrating multiple tools in mathematics education reflects a didactic need that underpins this diversity (Drijvers et al., 2016). Digital artifacts offer unique affordances compared to their non-digital counterparts, emphasizing the importance of understanding how technologies are used in technological environments and how both technological and non-technological artifacts interact to enhance the forms of mathematical teaching and learning in the classroom (Clark-Wilson et al., 2020). The teacher's decision to integrate an artifact into their teaching is shaped by its perceived didactic affordances and the anticipated obstacles students might encounter (Abboud-Blanchard, 2014). In this paper, interactions between artifacts are examined in terms of their functionalities within the different activities observed in the classroom. Beyond the synergy of instrumental functions (Rabardel, 1995), we introduce the concept of *resonance*, which can further be extended to the set of semiotic resources activated by different subjects within the same teaching and learning context. We explore and characterize this conceptual tool throughout a case study of a teacher conducting a sequence on the construction of parallelograms.

Theoretical frames and research questions

The study we present in this paper is structured around the following research questions: Why do teachers integrate multiple artifacts within their teaching? How do the functions of selected artifacts relate? How does the interaction of different semiotic resources leverage students' learning experience?

The overarching theoretical framework is *the instrumental approach*, which provides a solid foundation for examining how teachers and students adopt and adapt artifacts through the intertwined processes of instrumentation and instrumentalization, transforming them into instruments (Rabardel, 1995). This approach enables the analysis of an artifact's appropriation, both in terms of mastering the artifact itself - which involves technical proficiency independently of its context of use - and

mastering the use of the artifact enabling the subject to mobilize its functionalities in alignment with their object of activity. The dynamics of instrumental functions depend on whether a single artefact or multiple artefacts are mobilized, and whether one or several subjects are involved. A single artefact may serve multiple functions within a subject's activity (Rabardel, 1995), while the use of multiple artefacts can create synergies, with each activity enhancing the potential of the others (Faggiano et al., 2018). Furthermore, different subjects may be engaged with coexisting similar *paired objects of activity*. These are not identical since they are approached from distinct perspectives such as those of the teacher and the students. We introduce the *resonance of instrumental functions* that emerges when identical or distinct artifacts are used simultaneously by multiple subjects engaging with *paired objects of activity* (Figure 1).

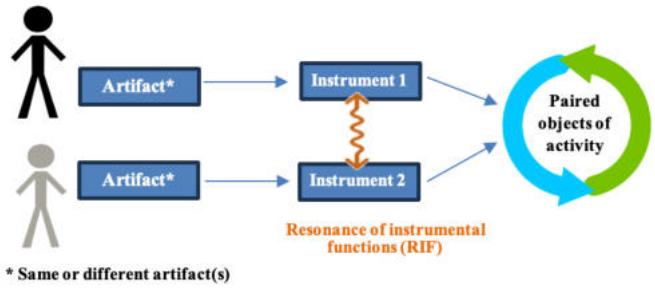


Figure 1: Resonance of instrumental functions

With this approach, we define *the resonance of semiotic resources* as the interaction within the set of semiotic resources mobilized by different subjects - including artifacts, utterances, gestures and inscriptions - when engaging with paired object of activity.

A second framework that offers analytical concepts to investigate teachers' practices in the context of digital tool use is the *Double Approach applied to Technology* (DAaT) (Abboud, 2024). It examines teachers' practices through three principal axes - cognitive, pragmatic, and temporal - while accounting for social, institutional, and personal factors as key determinants shaping them (Abboud-Blanchard, 2014). Our focus lies on the reasons underlying teachers' use of different artifacts within the classroom setting, the tensions they encounter in striving to achieve their teaching objectives and how they manage them (Abboud & Rogalski, 2021). As for aids provided to students, Abboud (2024) identifies two types: *manipulatory aids*, which involve using the artifact itself without explicit connections to mathematics, and *mathematical aids*, which establish direct links between the artifact and mathematical concepts.

We also refer to the *semiotic bundle* developed by Arzarello (2006) to explore the interaction of artifacts and different semiotic resources at play - utterances, gestures, and inscriptions – within teachers' and students' instrumented activities.

Methods

In our ongoing research, we study three cases of French teachers at lower secondary school designing and implementing parallelogram sequences using technological and non-technological artifacts. To illustrate our methods and analysis, we present the case of Nina, one of these three teachers.

Data were collected through naturalistic observations conducted in Grade 7 mathematics classes with students aged 12 to 13 years. Nina, has approximately 20 years of professional experience. She provided us with her detailed lesson plan that relates to the sequence of parallelograms. We conducted a pre-interview with her, comprising two phases. The first phase provided valuable insights into her overarching motivations for selecting artifacts in her teaching, both on a general level and within the context of geometric constructions. In the second phase, the focus shifted to a more detailed exploration of the mathematical content, investigating the interplay between her teaching objectives and the use of selected artifacts. Subsequently, we observed how artifacts' use was implemented within the classroom setting; seven teaching sessions were videotaped. Finally, we had a post-interview with Nina to reflect on whether the use of artifacts aligned with her stated intentions, to uncover challenges she faced and to point out any changes she would like to make.

To better organize our class observations, we initially categorize the teaching-learning classroom environment into three distinct types: (a) *Non-Technological Environment (NT)*, where no technology is involved; (b) *Techno-Latent Environment (TL)*, where technology is present but not directly engaged in mathematical tasks, instead serving to display textual documents or support the organization of the teacher's work (Clark-Wilson et al., 2020); and (c) *Techno-Operant Environment (TO)*, which fosters innovative ways of performing and representing mathematical activities.

The analysis was carried out under a three-tiered approach, encompassing macro, meso, and micro levels. The macro level analysis provides us with a synthetic view of artifact use by both the teacher and students. It highlights the overall synchronic and diachronic articulation of technological and non-technological artifacts, taking into account the mathematical task under study and the type of teaching and learning environment. At the meso-level, we focus on teaching episodes within TO environment. We closely examine the dynamics of instrumental functions. The fine-grained micro-analysis of key moments explores the resonance of semiotic resources activated by both the teacher and the students.

Teacher motivations for artifact selection and use

At a global level, Nina integrates multiple artifacts into her teaching practices for cognitive reasons directly tied to the mathematical domain itself - Excel for probabilities, Geogebra for geometry, Scratch for algorithms, etc. Her selection of artifacts aligns with institutional requirements including those for official examinations and accommodations for students with special needs. She also prioritizes accessibility by leveraging artifacts that are readily available to students, such as iPads. Pragmatic considerations play a predominant role in her choices, shaping the didactic strategies she adopts to best serve her teaching objectives. These choices help her overcome anticipated student difficulties, foster motivation, provide opportunities for consolidating knowledge beyond the classroom, manage the learning environment, and ensure access to instructional resources. The temporal dimension further influences her use of artifacts, as she adapts their selection to the different phases of a lesson - for instance, incorporating flash questions during the introduction. She also resorts to specific artifacts in response to time constraints, whether to compensate for delays in her overall annual progression or to manage limited time within a particular session.

At a local level, regarding geometric constructions, pragmatic reasons are prevailing. Nina emphasizes artifacts that help students overcome difficulties, particularly videos and animations that demonstrate the manipulation of geometric tools. She prioritizes online exercises with detailed feedback for construction tasks. Nina adapts Geogebra to her teaching goals, using projection mode for whole-class instruction and active mode on students' iPads for individual or collaborative work. She identifies Geogebra as a key artifact for generating conjectures about parallelogram properties, thereby making these properties more tangible for students.

Macro-level analysis

Classroom observations are synthesized in the macro-synoptic diagram illustrated below (Figure 2). The teaching and learning environment, along with the artifacts mobilized by both the teacher and students, underpins teacher practices along the pragmatic axis of the DAaT. The generic tasks related to the mathematical knowledge involved highlight the cognitive axis. The main contribution of this diagram lies in its integration of a temporal dimension, enabling a dual reading perspective – both diachronic and synchronic – regarding artifact use.

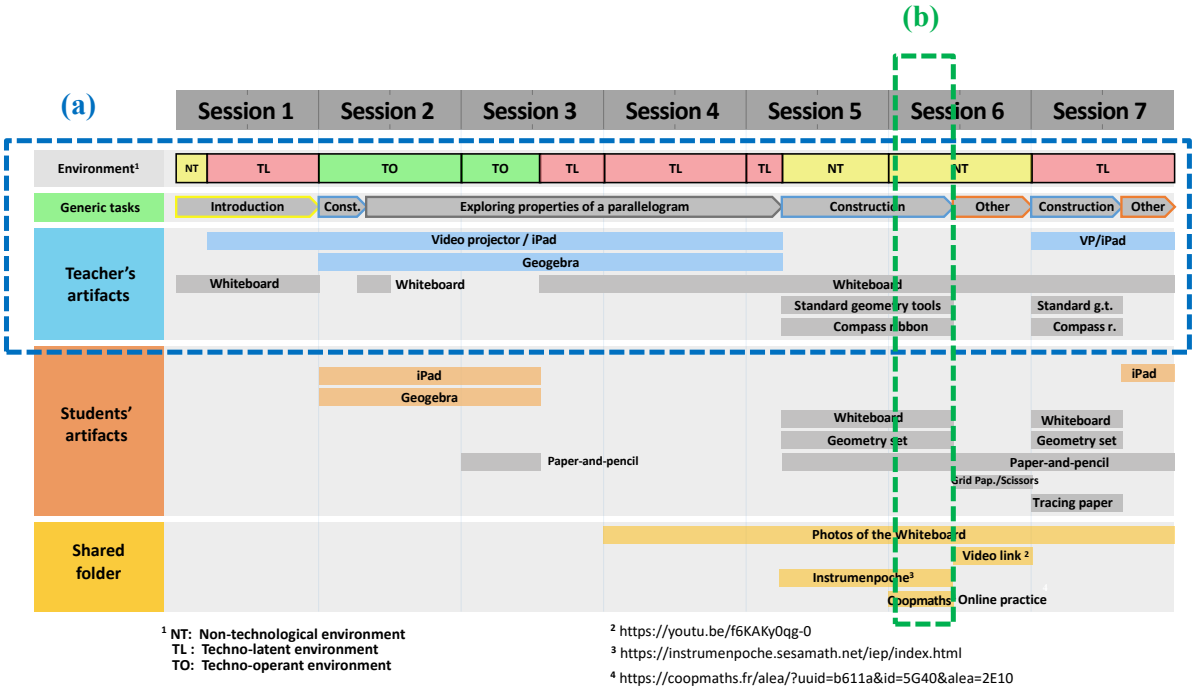


Figure 2: Macro-synoptic diagram. (a) Diachronic analysis, (b) synchronic analysis

The diachronic analysis at the teacher's level (Figure 2a) might initially suggest a mere juxtaposition of artifacts; however, this is not the case. The choice of artifact serves as a strategic means to support Nina's teaching objectives. For instance, Geogebra is used in parallelogram construction tasks based solely on the parallelism of its opposite sides, while it is also deployed to explore the properties of the parallelogram. When proving these properties, it is projected to display a pre-constructed figure without drawing upon any of its functionalities. As a result, three distinct instruments emerge from this artifact, each corresponding to a specific object of activity.

Conversely, a synchronic analysis offers a perspective on Nina's teaching approches, emphasizing three key dimensions: knowledge delivery, students' anticipated engagement with the assigned tasks, and the provision of resources to support learning and practice beyond the classroom. For instance, during the construction tasks in session 6 (Figure 2b), both the teacher and the students use standard geometry tools. Additionally, the teacher has developed her own personalized compass using a ribbon. Although no technology is integrated into the classroom setting, animations illustrating detailed construction procedures and online exercises with step-by-step corrections are made available on the shared folder for students' use outside the classroom. This approach aligns with the rationale behind artifact use, as previously discussed in the pre-interview with the teacher. However, in the post-interview, Nina notes that the animations were not particularly helpful to her students, despite accurately replicating the task assigned in the video - which did not correspond to the task she had given her students. In fact, the positioning of the missing vertex with respect to the given vertices of the parallelogram (Figure 3) was not the same - which ultimately required challenging adaptations from students. This highlights the importance of considering different subjects' perspectives when engaging with paired objects of their instrumented activity.

Figure 3: (a) Teacher-assigned task, (b) task as illustrated in the animation

Meso-level analysis

We investigate the dynamics of instrumental functions within the second teaching session (Figure 4). The teacher shares her iPad screen displaying Geogebra, while students work on the online version using their iPads. The first dynamic concerns the functional versatility of the dragging tool. Nina anticipated its use across different tasks (Figure 4a): checking the robustness of the construction in the construction task and developing conjectures while investigating the properties of parallelograms. For students, the functions of dragging varied within a single task (Figure 4b): adjusting constructions, validating conjectures, and revealing unexpected configurations. These observations highlight the instrumental functions shift according to the user's perspective and to their object of activity. The second dynamic is illustrated through the synergy between the *distance tool* and the *dragging tool*, each contributing a distinct dimension to the conjecture-making task (Figure 4c). While some students formulated the conjectures solely based on the displayed measurements, others used the dragging tool to explore covariation, demonstrating a more advanced level of reasoning. In this interplay, each artifact enhances the functional contribution of the other, fostering a more comprehensive learning experience.

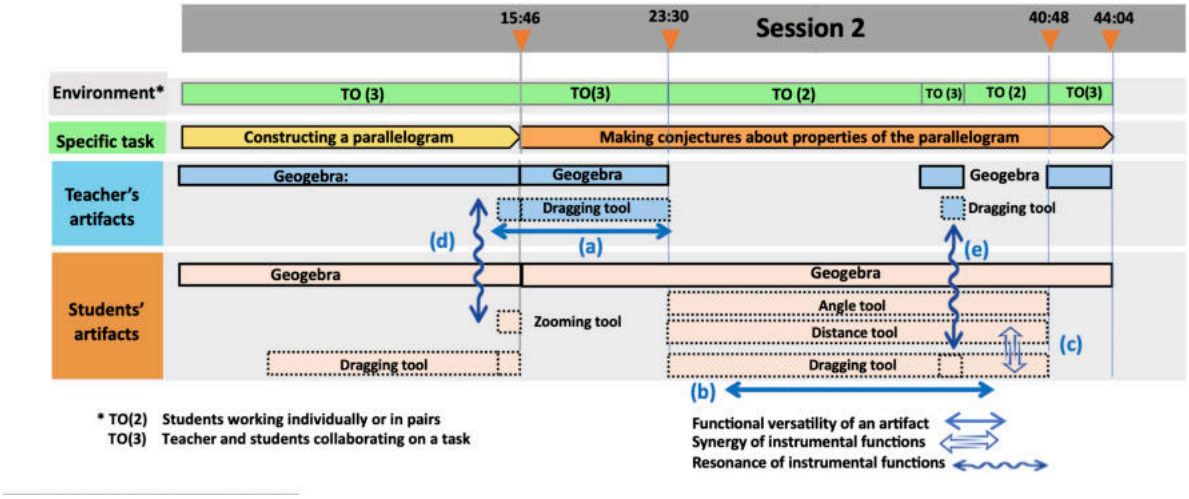


Figure 4: Meso-synoptic diagram

The third dynamic concerns the teacher and students collaborating on a construction task (Figure 4d). While the teacher was demonstrating how to use the dragging tool to check the robustness of a construction, a student simultaneously used the *Zoom tool*, believing he was engaging in the same process. This instance of resonance reveals a divergence in tool use and understanding that went unnoticed by the teacher and remained unresolved for the student. A second example involves the identification of an anomalous configuration (Figure 4d): the student shared his screen and dragged point A to check on the measures of opposite angles (Figure 5); however, the display of angle measures appeared abnormal. To illustrate the expected dynamic behavior of the construction, the teacher shared her own screen and performed the dragging test. The resonance of the dragging tool functions as a strategic means of making the student's difficulty explicit and supporting the teacher's intervention.

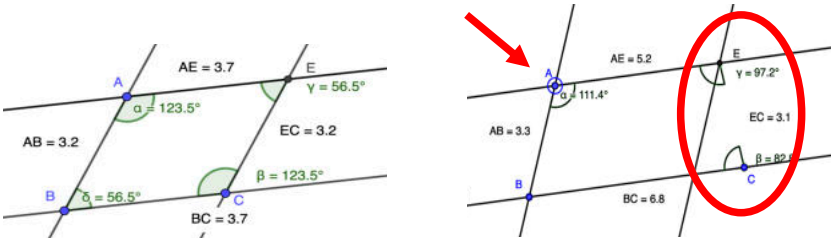


Figure 5: Meso-synoptic diagram

Micro-level analysis

We are interested in a student's interpretation of the teacher's gestures in two episodes within the second session and how he implemented them within his construction procedures (Figure 6). The first one relates to constructing a parallel line to (AB). Nina first pointed to (AB), then to the expected location for the parallel line. The student, independently exploring the parallel tool, touched (AB) on point A. Consequently, he obtained point C when pointing to the position of the parallel line as indicated by Nina. When the intended parallel line still did not appear, he undid his actions and, on his third attempt, successfully drew it - albeit accidentally - by touching (AB).

The second episode relates to drawing (AC) by joining points A and C, a situation not identical to drawing (AB) on an empty screen, even though the same artifact line is being used. The student linked the teacher's vertical hand movement to the perpendicular bisector icon, which depicts a vertical line and two connected points. The negative visual feedback prompted him to select the perpendicular icon instead, as it produced a visually similar configuration on his screen for the given position of point C. He then deemed his objective achieved.

The resonance of semiotic resources activated in these two episodes highlights the importance of considering the distinct perspectives of the subjects involved, as well as the crucial role the teacher plays in an open, dynamic technological environment (Abboud & Rogalski, 2021). Differentiated instrumental interventions by the teacher are necessary to foster meaningful learning experiences and to support students' instrumental genesis of the artifact in use.

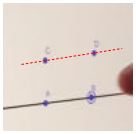
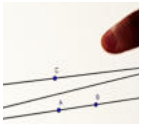
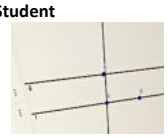
Timing	03:41	03:45	07:25	07:27	07:30	07:40
	Episode 1		Episode 2			
Subject	Teacher	Student	Teacher	Student	Student	Teacher/student
Utterances	Find out how you can draw the parallel line to this one (G1), which is located approx. here (G2).		Draw the line passing through points A and C (G3) then draw the //line to (AC) passing through B (G4). We'll get by then two pairs of // lines.			Teacher: When you make a mistake and you try to clear it, you know well this tool "going backwards"(G5). You do all master it.
Describing gesture and/or inscriptions						Student 
	G1: pointing to (AB) - her index is between points A and B G2: enacting the parallel line position	Picking the // tool. Touching (AB) on A, then touching the screen: Point C is displayed. Selecting point B. touching the screen so that (CD) appeared parallel to (AB)	G3: enacting the line passing through points A and C. G4: enacting a line // to (AC) passing through point B.	Picking the perpendicular bisector tool.	Picking point C, then point A. Then undoing his work.	G5: Teacher pointing to the "undo" S1: Picking the perpendicular tool, picking point A, then touching (g) *nearby* point C.
Teacher's tensions and students' difficulties		By touching line (AB), the student accidently picked point A.	The teacher limited students' strategies by imposing a construction procedure.	Misinterpreting the perpendicular bisector icon for it shows two connected points.	Direct manipulation feedback prompts S1 to undo his work.	Particular position of point C was behind S1 choice to draw a perpendicular line. The icon shows a vertical line as well.
Resonance of semiotic resources	**Resonance 1:** (Teacher; gesture) – (S1; parallel tool)- **Resonance 2:** (Teacher; board) – (S1; iPad screen)- **Resonance 3:** (Teacher; utterances) – (S1; screen touch gesture) **Paired objects of activity:** Drawing a line // to (AB)		**Resonance 1:** (Teacher; gesture) – (student; perpendicular bisector icon) **Resonance 2:** (Teacher; gesture) – (student; perpendicular tool) **Paired objects of activity:** Drawing a line passing through two existing points			

Figure 6: micro-synoptic diagram for selected episodes

Conclusion

The integration of an artifact into teaching practices is intrinsically linked to the teacher's objectives and contingent upon the specific function the teacher intends to leverage through its deployment. Three key points emerge when investigating the dynamics of instrumental functions. First, the necessity of accounting for the uniqueness of instrumental genesis, which relates not only to the artifact *per se* but also to the subject engaging with it, to ensure that actions effectively align with objectives. Second, when identical artifacts are activated by the teacher and students within a task, they are interpreted and implemented from distinct perspectives, leading to divergent procedural

approaches. Third, when teachers integrate an artifact without sufficiently recognizing its didactic potential and semiotic affordances, their ability to address emerging students difficulties is compromised, thereby hindering students' instrumental genesis and impeding their learning.

Within our ongoing research, two other case studies are in progress. Using the same methods and layers of analysis, we look forward to a broader exploration of the articulation of artifacts in mathematics classrooms. In particular, by introducing and testing the concept of resonance in various contexts, we aim to highlight its potential. Our goal is to enhance insights into teaching and learning experiences within these contexts and, hopefully, demonstrate its potential contribution to supporting teachers' professional development.

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Mathematical problem-posing according to defined educational needs utilizing generative AI as a lever to enhance pre-service mathematics teachers' TPACK

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This study was conducted during a course entitled Teaching Advanced Secondary School Mathematics. Fifteen pre-service teachers solved various mathematical problems and then were asked to pose additional problems and improve them to meet precise educational goals by furnishing "prompts" to a generative artificial intelligence environment and refining them as necessary according to predefined steps. They then answered a questionnaire (ten Likert-type statements; six open-ended questions). The study shows that this process contributed to the pre-service teachers' body of technological, pedagogical, and content knowledge (TPACK), exposed them to the pros and cons of the environment itself, and showed them how to generate effective prompts to obtain relevant responses. Overall, the teachers expressed satisfaction with using this environment and indicated they would use it in the future.

Keywords: Problem-posing, generative AI, ChatGPT, TPACK, pre-service high school mathematics teachers.

Introduction

The emergence of artificial intelligence (AI) technology environments have yielded new opportunities in all spheres of life. Its integration into the education system is still in its early stages, but it has reached a point where it can be effectively utilized to address specific challenges faced by teachers. One area is that of posing mathematical problems.

In the current study, 15 pre-service high school mathematics teachers were asked to pose mathematical problems that meet their pre-defined educational needs. To this end, they conducted dialogues with an AI "assistant" (in this case, ChatGPT). Through the use of mixed-method questionnaire, we aimed to identify how such a process can contribute to teachers' technological, pedagogical, and content knowledge (TPACK).

Literature review

Mathematical problem posing

In mathematics education, the twin processes of problem solving and problem posing (PP) stand as cornerstone objectives. These intertwined activities, as highlighted by scholars such as Polya (1954) and Silver (1994), are not just educational goals but also pivotal methodologies. Mallart et al. (2018)

suggest that PP may serve as a complementary approach to problem-solving, enriching the overall learning experience.

PP encompasses a wide range of opportunities, such as creating a new mathematical problem or re-formulating given ones (Silver & Cai, 1996). This may be done by implementing inquiry strategies such as the "what if not" approach proposed by Brown and Walter (1983), which has been implemented and explored in various studies (Lavy, 2015; Segal et al., 2018; Silver & Cai, 1996). This technique encourages learners to modify given problems, fostering a deeper engagement with mathematical concepts.

Posing mathematical problems is a challenging task for teachers (Cai & Jiang, 2017; Chan, 2023; Lim et al., 2023), who often too closely follow the mathematics textbook. This shapes the tasks and examples they choose to present in class (Johansson, 2006; Lepik et al., 2015). As a result, they often fail to recognize the opportunities textbook tasks can provide for exploratory mathematical investigation. The results of a study by Mallart et al. (2018) emphasized the challenges teachers encounter when posing problems related to educational needs, such as a specific educational level or problems that enable students to realize the connection between mathematical content and their everyday lives. Often, teachers require support that provides opportunities to broaden their mathematical and pedagogical approaches to posing mathematical problems.

Cai and Rott (2024), based on a comprehensive review, proposed a four-phase model for PP processes, namely, orientation (understanding the PP task); connection (linking the mathematical concept and developing ideas); generation (creating the actual problems); and reflection (evaluating and refining the posed problem).

Utilizing generative AI (ChatGPT)

The advent of digital technologies, especially the emergence of artificial intelligence (AI) technology environments have yielded new opportunities to support teachers in their endeavors (Baidoo-Anu & Ansah, 2023). Although its integration into the education system is still in its early stages, it has now reached a point where AI can be effectively utilized as a lever to address specific challenges faced by both pre-service and in-service teachers, particularly in areas where teachers may struggle, such as in posing mathematical problems.

The recent emergence of generative AI (GenAI) has aroused special interest among the education community. GenAI is a type of AI technology that is capable of performing actions that usually require human intelligence, such as creating new content, producing images, and more (Baidoo-Anu & Ansah, 2023; Chan & Hu, 2023). ChatGPT is a form of GenAI that involves discourse between the computer and the human. It can serve as a "partner" and offer personalized and interactive learning. It can understand the meaning of basic prompts and collect available online information, analyze data (either provided or gathered), and generate texts with meaning (Deng & Lin, 2022; Kasneci et al., 2023). GenAI can provide personalized feedback and specific support for learning mathematical subjects or understanding complex content (Jia, 2015). It can facilitate solving various mathematical problems (Rane, 2023; Wardat et al., 2023).

Researchers estimate that ChatGPT, among other things, may affect traditional teaching methods (Rudolph et al., 2023). However, along with the advantages that come with integrating AI, teachers have also noted disadvantages in its use. For example, using AI may create an emotionless learning environment, may threaten the security and privacy of its users, and may make learners passive, thus preventing the development of inquisitive thinking (Lee et al., 2024).

Teacher's technological pedagogical content knowledge (TPACK)

The TPACK framework serves as a cornerstone for understanding the complexities of integrating technology into educational practices. As highlighted by Caniglia and Meadows (2018) and Nantschev et al. (2020), TPACK outlines the knowledge that teachers need to effectively incorporate technology into their teaching.

Building upon Shulman's (1986) work, Koehler and Mishra (2009) conceptualized TPACK as a synthesis of three primary knowledge domains: Technological Knowledge (TK), Pedagogical Knowledge (PK), and Content Knowledge (CK). The TPACK model implies that all three bodies of knowledge intersect at various levels of complexity. For example, Technological Content Knowledge (TCK) is the knowledge teachers need to use technology for presenting mathematical objects and complex mathematical processes. It requires employing varied representations of mathematical concepts to facilitate the development of their students' content knowledge. Technological Pedagogical Knowledge (TPK) is knowledge of how to use technology during teaching and learning, such as to foster inquiry-based learning (Jahangiri et al., 2021; Segal et al., 2021).

Research goal

This research focused on how integrating GenAI – specifically ChatGPT – for mathematical PP contributed to pre-service teachers' knowledge. The TPACK model was used as a lens by which to examine and describe the knowledge acquired by them. The research question we aimed to answer was the following: how does utilizing ChatGPT to pose mathematical problems with defined educational needs enhance pre-service high school mathematics teachers' TPACK?

Method

Participants

Fifteen pre-service high school mathematics teachers participated in this study. They were studying in a preparatory course entitled Teaching Advanced Secondary School Mathematics in a teachers' college. After several lessons in which they solved various mathematical problems, they underwent two sessions that focused on posing problems utilizing ChatGPT. During these lessons, the participants were asked to follow specific steps to compose and refine prompts for ChatGPT so that it would assist them in posing problems that would meet their educational needs.

Research tools

The study used a mixed method (combined quantitative and qualitative data collection and analysis) protocol (Creswell, 2014). During the two sessions, the researchers documented all stages of the problems and prompts that the teachers gave ChatGPT by integrating the links to the AI environment into a Google Form file that guided the teachers, step by step, through the process.

Following the two sessions, participants were asked to complete a two-part questionnaire that had been developed and validated by three mathematics education experts. Its purpose was to determine the pre-service teachers' experiences with GenAI and to explore its impact on their TPACK. The first section of the questionnaire included 10 Likert-scale statements, where participants rated their agreement from 1 (strongly disagree) to 5 (strongly agree) regarding their feelings about working with GenAI. The second section contained six open-ended questions designed to further elaborate on their responses from the first part and to provide deeper insights, particularly regarding the contribution of utilizing GenAI to their TPACK through PP.

Stages toward the problem-posing process using ChatGPT

During the two sessions, the teachers were required to employ ChatGPT for the process of writing a PP task. The process included the four main stages (based on Cai & Rott, 2024).

Stage 1: Orientation. The teacher poses a mathematics problem that is applicable for an advanced-level subject. The teacher lists the knowledge required to tackle the problem and the potential difficulties a student might confront while solving it.

Stage 2: Connection. The teacher defines their educational needs and formulates prompts for ChatGPT in order to receive recommendations for improving the problem.

Stage 3: Generation. The teacher considers the ideas received from ChatGPT and decides if and how they relate to their defined educational needs. The teacher decides which suggestions are suitable and reformulates the problem accordingly.

Stage 4: Reflection. The teacher identifies the benefits and challenges of using ChatGPT for the process of mathematics PP and decides whether to continue working with ChatGPT for another task.

Data Analysis

Quantitative and qualitative data were collected via the questionnaires, and additional qualitative data were obtained from the documentation. Quantitative data analysis (frequencies, means, and standard deviations) of the Likert-type statements allowed analyzing the contribution that ChatGPT made to the mathematics teachers' technological-pedagogical and mathematical knowledge, as perceived by them. To qualitatively analyze the data obtained from the teachers' responses to the open-ended questions, their responses were divided into "units of meaning" (from one word to a sentence or paragraph that focused on one idea), themes were identified and categorized using "constant comparative analysis" (Charmaz, 2014; Glaser & Strauss, 2017), and then the themes were divided into primary and secondary content areas. In the framework of this presentation, we offer part of the results based on the Questionnaire.

Results

First, we present an example of the four stages documented for one participant's process.

- Stage 1: Orientation. Problem posed: "Find a horizontal asymptote for the following function…"
- Stage 2: Connection. Educational need: "To improve the problem so that students better understand what a horizontal asymptote is."

- Stage 3: Generation. Participant's decision in accordance with ChatGPT's suggestion. “I am considering changing the formulation of the task to: Why does the function approach a certain value as *x* approaches positive or negative infinity?"
- Stage 4: Reflection. "ChatGPT helped me understand how changing the wording of the task can promote a deeper understanding of a mathematical concept."

Regarding the quantitative results, the mean and standard deviation of the participants' responses to the 10 statements in the questionnaire is presented in Figure 1 (Likert scale: 1, strongly disagree, to 5, strongly agree). The mean scores for all statements range from 3.20 to 3.93, that is moderate to high agreement, indicating that the teachers were satisfied using ChatGPT.

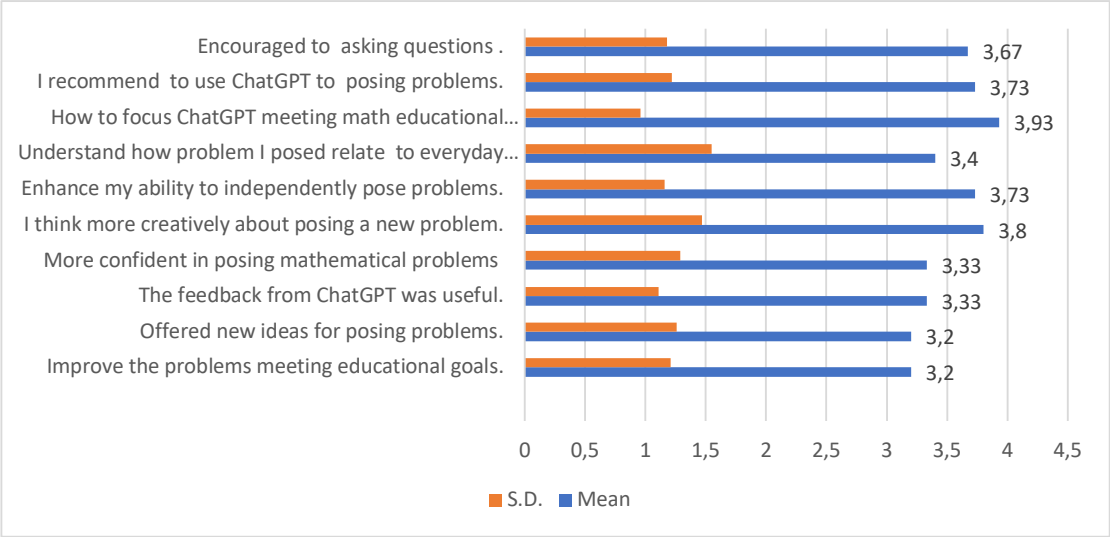

Figure 1. Chart showing mean (blue) and standard deviation (orange) of pre-service teachers' answers to the 10 closed statements on the questionnaire

The qualitative analysis of the open-ended questions revealed the five themes. The three with the highest frequency are presented below along with some sample statements. (The other two were challenges and limitations (n=6), and accessibility and efficiency (n=5))

(1) Exposure to new pedagogical ideas (n=10). “*ChatGPT gives new ideas such as how to connect mathematics to real life situations*”; “*ChatGPT helped me notice students' difficulties and formulate questions to help them, such as having them be more aware of the behavior of the function graph near its vertical asymptotes*” (TPK, TCK); “*ChatGPT offers ideas that promote deep understanding rather than procedures.*” (TCK)

(2) Develop critical thinking (n=8). “*ChatGPT suggests rich and varied opportunities to increase the challenge of a given task if appropriate for my students' abilities*” (TCK, TPK); “*ChatGPT gave me an incorrect answer related to function transformation, so I rejected that suggestion.*”

(3) Continued (future) implementation of ChatGPT (n=7). “*I will use ChatGPT to expand the repertoire of my pedagogical ideas*” (TPK); “*I intend to use ChatGPT to tailor existing textbook questions for students' daily situations because I believe it may increase their motivation.*” (TPK)

These qualitative findings, where teachers provided concrete examples of the benefits of ChatGPT, align with the quantitative data, as both indicate high satisfaction with ChatGPT. The teachers reported that the process gave them valuable pedagogical ideas and required critical thinking. They were encouraged to think about how to pose problems that were better related to their students and would be more interesting for them, whether by challenging them or by wording the questions to make the solution to the problems more accessible. All this provides strong evidence that ChatGPT enhanced pre-service teachers' TPACK.

Discussion

The results of this study indicate that the utilization of ChatGPT assisted the participants in posing mathematical problems according to their educational needs and provided a personalized and supportive environment where they could formulate prompts as much as needed. The pre-service teachers recognized the value of working with ChatGPT. These findings are consistent with the findings of Rane (2023) and Wardat et al. (2023), which showed that the personalized feedback the participants received from GenAI promoted their proficiency and capabilities in posing mathematical problems.

In addition, the results emphasized how the process of posing mathematical problems utilizing ChatGPT can serve as a lever to develop the pre-service teachers' TPACK as follows: they learned about an innovative technological environment that could aid in posing mathematics problems (TK); they were required to obtain in-depth familiarity with the problem they wished to pose for the chosen curriculum topic and to predict the difficulties that may arise in the process (PCK); their dialogue with ChatGPT helped them improve the clarity of the problem's wording (PK), deepen their mathematical understanding of the concepts by carefully examining the ChatGPT proposals and deciding if they were suitable for to their students and increase the relevance of mathematics by linking the problem to daily life (CK); they needed to formulate prompts that would generate problems that met their pedagogical needs (TPK); ChatGPT's answers led them to investigate whether all the mathematical data in a given task were required or if any mathematical data were missing (TCK); the process allowed them to expand their repertoire of mathematical tasks related to the curriculum and their mathematical needs (TCK).

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Teaching computational technology in mathematics education: a comparative analysis of three frameworks

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In this paper, we explore differences in how KOM (Niss & Højgaard, 2019), computational thinking (Weintrop et al., 2016), and computational literacy (diSessa, 2018) frame teaching mathematics with computational technology. We find that a main difference lies in their perspectives on the disciplinary stability of mathematics when using computational technology, whether considered a tool, added disciplinary knowledge, or a re-mediating infrastructure. We conclude the paper with a discussion on the implications of these differences for teaching mathematics with technology in terms of the promotion of teaching activities pertinent to each framework.

Keywords: Computational literacy, computational thinking, mathematical competencies.

Introduction

Computational technology[1] has become increasingly ubiquitous in mathematics classrooms in Europe (Bocconi et al., 2022). While there is consensus of the benefits of using such technology in mathematics teaching, there are different perspectives on how and whether they bring changes to mathematics as we know it and what didactical work, they confront teachers of mathematics with. One view is that computational technology should be considered merely as a relevant tool to be mastered in subordination to mathematics it (e.g., Niss and Højgaard, 2019). Another view is that computational technology represents new knowledge from computer science that we should strive to incorporate as part of mathematics teaching in response to the increasingly digitalized world (e.g., Kallia et al., 2021; Weintrop et al., 2016). The work presented in this paper stems from ongoing research in a newly established National Centre for Digital Technology Comprehension in Denmark. Part of our research within this centre explores the interplay between computational technology and mathematics education from different theoretical positions. In this paper, we present a study on how the Danish KOM-framework[2] (Niss and Højgaard, 2019), computational thinking (CT) (Weintrop et al., 2016), and computational literacy (CL) (diSessa, 2018) frame the relation between computational

[1] In this paper, we rely on Weintrop et al.'s (2016) description of computational technology as referring to graphical programming environments, computational modelling environments and electronic prototyping kits.

[2] KOM is short for "kompetencer og matematiklæring", which translates into competencies and mathematics learning.

technology and mathematics teaching. While each of the frameworks has had a significant impact in discussions in mathematics education, most of the debate has concerned what is relevant for students to *learn* rather than new requirements of how to *teach* mathematics. Our paper contributes by providing such an overview, which we seek to develop by addressing the following research question: *What are the differences in how KOM, CT, and CL frame teaching mathematics with computational technology?*

In what follows, we provide a brief overview of research related to our work in answering this question.

Related work

The potential of computational technology in mathematics teaching can be traced back to Papert's (1980) influential book *Mindstorms*. Papert (1980) proposed that intuitive programming environments can make otherwise abstract mathematical ideas more accessible and tangible for students. As an alternative to interact with geometry through Euclidian notation, Papert and colleagues developed a "Turtle Geometry", where students could navigate a robot via a programming language similar to natural language to explore mathematical concepts. Although widely spread from 1980 and well into the 1990's, Papert's (1980) ideas never led to sustaining curriculum policy revisions and slowly received less and less attention. In the beginning of the millennium, Wing (2006) revitalized the term CT, and sparked an agenda that since has resulted in a momentum in research and curriculum revisions. In Europe, 25 countries have embedded CT in the compulsory school curriculum (Bocconi et al., 2022). Researchers in mathematics education have been highly active in embracing this term and explored potentials of programming environments (e.g., Scratch) to scaffold students' learning (Benton et al., 2018). Moreover, CT has led to fundamental discussions on the relation between computational technology and the discipline of mathematics. The increased interest in CT has led to myriad definitions in the context of mathematics teaching, each of which comes with different viewpoints of how to consider the relation between computational technology and mathematics education. It is these differences among CT, CL, and KOM and their implications for mathematics teaching, we study in this paper. Below, we briefly outline these three perspectives.

Three takes on the relation between technology and mathematics: Competencies, practices, and literacy

Mathematical competencies

The KOM framework was introduced in 2002 and developed "the concepts of mathematical competence and mathematical competencies with particular regard to their possible roles in the teaching and learning of mathematics" (Niss and Højgaard, 2019, p. 10). KOM primarily consists of the description of eight mathematical competencies. Niss and Højgaard (2019) describe the mathematical competencies as an alternative to a traditional focus on curriculum content and as a framework that can be applied both descriptively and normatively. While technology plays a role in e.g., the tools and aids competency, no explicit attention is devoted to computational technology. Moreover, as indicated by the name, the aids and tools competency primarily describes technology as *tools* that are part of a mathematical activity that students either carry out or in other ways relate

to. KOM however includes an outline of mathematical competencies for teaching (Niss and Højgaard, 2002), which consists of a characteristic, didactical and pedagogical comments and examples. With respect to teachers' competencies for the tools and aids competency, Niss and Højgaard (2002) stress that they must master a broad spectrum of available and relevant tools and aids to those that fit for a particular group of students. Moreover, they describe how teachers should be able to identify didactical potentials of new aids and use them to initiate or furthers students' learning. Since KOM is a widely used framework and has been used in studies on computational technology (e.g., Bråting et al. 2022), we regard it as a relevant perspective to include in our study.

Computational thinking

CT was introduced by Papert (1980) and revitalized by Wing (2006) in an influential paper in which she argued that "thinking like a computer scientist" is a strength in all knowledge domains (Wing, 2006, p. 34). In the past decade, research on CT in mathematics education contexts has grown rapidly and led to several definitions (e.g., Kallia et al., 2021; Weintrop et. al., 2016). For this paper, we rely on Weintrop et al.'s (2016) definition. This definition features four main practices—data practices, modelling and simulation practices, computational problem-handling practices, and systems thinking practices. Each practice includes five to seven taxonomic levels built from existing literature, educational activities, curriculum documents from the United States, and interviews with mathematicians and scientists. They argue that the taxonomy provides a set of actionable guidelines for teachers to follow when introducing CT to mathematics education as well as a shared language for teachers to better understand and systematically develop their teaching practices. As described, the taxonomy consists of *practices* (what students should *do*), which mirror the practices employed by researchers and scientists. The work by Weintrop et al. (2016) mainly characterize these practices as being distinct from, but relevant for, other practices or content taught within mathematics education. We chose this framework due to its wide adoption in mathematics education research.

Computational literacy

diSessa (2000) defines a literacy as "a social widespread patterned deployment of skills and capabilities in a context of material support (...) to achieve valued intellectual ends" (diSessa, 2000, p. 19). diSessa stresses that computational literacy "do[es] not mean a casual familiarity with a machine that computes" (diSessa, 2000, p. 5). Rather, computational technologies are thought to lead to a new literacy as we adopt "a particular infrastructural representational form for supporting intellectual activities" (diSessa, 2018, p. 4). Computational literacy involves four "R's":1) re-mediation, 2) reformulation, 3) reorganizing the intellectual terrain, and 4) revitalizing the ecology of learning activities (diSessa, 2018). *Re-mediation* concerns that the implementation of computational technologies in classrooms involves new representation forms. *Reformulation* is considered an important human intellectual resource due to "the possibility of substantial changes in what, when, and how we teach subject matter" (diSessa, 2018, p. 26). *Reorganization* concerns that teaching mathematics will change due to cultural changes in our society, requiring rethinking teaching and learning mathematics – not simply using/adding computers. *Revitalization* regards awareness of the activities within the classrooms to ensure that teaching and learning in mathematics remains relevant and authentic considering the newly implemented computational technology.

Theoretical approach and method

Our analysis of how the above frameworks above frame mathematics teaching follows ideas proposed by Popkewitz (1984). For Popkewitz (1984), theories in educational research are conceptual lenses that enable researchers to think about empirical phenomena that otherwise seem difficult to grasp. The choice of theory to study such phenomena implies that the line of inquiry is based on epistemological assumptions and values, leading to specific analytical outcomes (Popkewitz, 1984). *Epistemological assumptions* concern questions regarding the type of social phenomena that the conceptual lens orients the researcher towards, and what explanations of the phenomena it offers. *Values* regard questions of what rationale for change the lens proposes. *Nature of outcome of analysis* concerns how the conceptual lens suggests resolving emerging questions, and what the answers to these questions imply for professionals to do (Popkewitz, 1984). A central point for Popkewitz (1984) is that to grasp differences among conceptual lenses, we must scrutinize their differences along the three aspects mentioned above. To operationalize Popkewitz's (1984) ideas, we structure our analysis of KOM, CT and CL by the following three questions:

1. *Epistemological assumptions:* What ways of understanding and explaining mathematical learning and teaching do the lenses orient the researcher towards?
2. *Value of computational tools:* Which rationales do the conceptual lenses present for why mathematics teaching should include digital technology in relation to new computational tools, and what direction are given to accommodate this?
3. *Analytical outcome*: Which approaches do the conceptual lenses propose to resolve the questions brought by the emergence of computational technologies?

According to Popkewitz's (1984), co-existing traditions are vividly engaged in discussions about defining the adequate approaches to study a given phenomenon. Until this point, it is however our not our impression that sufficient attention has been devoted to discussing how to teach mathematics with computational technology. It is this discussion we wish to open with our paper. Our choice to focus on CL, KOM, and CT reflects that these lenses are internationally influential and yet offer highly different starting points for research on teaching mathematics with computational technology.

Results

Epistemological assumptions

KOM explicitly considers mathematical learning as a cognitive process (Niss and Højgaard, 2019), bound by a conception of what it means to master the eight mathematical competencies. KOM describes the eight competencies to be of a general and stable nature that, as such, not are affected by emerging tools. The eight competencies thus regard how we access and interact with mathematics, e.g., *through* tools, representations, reasoning, etc. KOM's *aids and tools competency* is particularly devoted to the role of the materials through which we engage with mathematics, here among digital technology. This competency concerns the ability to put aids and tools to "constructive use in mathematical work, as well as to critically relate to one's own and others' use of such aids and tools" (Niss and Højgaard, 2019, p. 18). Aids and tools are thus considered instruments for interacting with an otherwise stable discipline, which fundamentally remains unaffected by whatever tools we deploy.

Moreover, the competency of mastering this technology is largely described from a standpoint where mathematics is regarded as stable. The role of computational technology is considered like any other tool, e.g., a ruler or a calculator. Hence, computational technology is considered a tool to interact with mathematics. The few indications given on teachers' roles in students' development of competencies primarily stress the pedagogical potentials of tools and aids, and that teachers should master a broad spectrum of tools and aids to select those appropriate to further or initiate student learning of particular competencies or subject matter (Niss and Højgaard, 2002). This reflects the view described above, where the value of tools and aids mainly are ascribed to their potentials of supporting students' acquisition of stable mathematical competencies and subject matter areas.

Weintrop et al. (2016) discusses mathematical learning with a primary emphasis on what new practices students should learn. They argue that the role of computational technology in scientific contexts has led to an urgency to define CT "(...) and providing a theoretical grounding for what form it should take in school science and mathematics classrooms" (Weintrop et al., 2016, p. 127). Thus, their approach to CT regards what *could* be taught. They describe their taxonomy as an inspiration for possible content and not as a normative framework as such. Moreover, Weintrop et al. (2016) describe learning of computation and mathematics as reciprocal in that they each provide tools and contexts for learning the other. Here, their study thus foregrounds a content-perspective on *what* knowledge students should have rather than the cognitive process of acquiring mathematics. Moreover, Weintrop et al. (2016) subscribe to an ideal of congruence where what is taught should align with what is practiced in industry and research. Implicitly, the disciplinary nature of mathematics and the content it consists of are therefore considered constantly evolving, particularly as computational technology is widely used in the contexts surrounding schools.

Whereas KOM and CT ascribe importance to cognition and content respectively, CL is characterized by focusing on representation. diSessa (2018) argues that representations to be considered highly significant—and ultimately transformative—infrastructures, and that new infrastructures imply a change in the disciplinary core of mathematics. In diSessa's view (2018), disciplinary knowledge and (representational) infrastructure are thus inseparable, and the emergence of new tools implies both attention to mastering these tools and the transformative nature of the discipline in which they are used. Hence, literacies (including mathematical literacy) are to some extend in flux. As opposed to KOM (Niss and Højgaard, 2019) and CT (Weintrop et al., 2016), diSessa's (2018) perspective implies that mathematical learning in the context of computational technology should not be reduced to neither content nor tool, but should reflect their reciprocal and transformative relation.

Value of computational technology

In KOM, technologies are mainly seen as tools for mathematics' sake, and secondly as tools which usage can be orchestrated by teachers to support student learning. Students need to build competencies with the purpose of assessing limitations and affordances of specific tools and to reflectively make use of such tools within mathematical activities. Computational technologies are not assigned particular significance in KOM but are included in the aids and tools competency in juxtaposition to other tools. Mathematics represents a domain of knowledge that allows a particular approach to model the surrounding world. Computational technologies are thus considered new

contexts in which mathematical competencies may be exercised and not tools that influence the nature of mathematical mastery. However, new emerging tools may confront users with new knowledge of such tools and artefacts in their assessment of when, how, and for what purposes to use computational technology. In the parts of KOM (Niss and Højgaard, 2002) on the role on teachers in building students' competency, they argue that teachers need proficiency in the tools and aids they use in their teaching. Here, technology is predominantly described as pedagogical tools that can initiate and further students' learning – not as an occasion to reconsider the disciplinary core of mathematics.

In CT, computational technologies are thought of as a key to achieve economic and intellectual growth, why we need to address them in teaching. Weintrop et al. (2016) specify this as two interconnected values. First, computational tools represent "powerful tools for understanding phenomena across all spectra of human inquiry" (Weintrop et al., 2016, p. 131), particularly in STEM subjects. Thus, integrating computational technologies into mathematics teaching will give "learners a more realistic view of what these fields [science and mathematics] are, [which] better prepares students for pursuing careers in these disciplines" (p. 128). Second, Weintrop et al. (2016) argue that "the thoughtful use of computational technologies and skillsets can deepen learning of mathematics and science content" (p. 128), especially by providing authentic contexts. Thereby, the value of computational technologies in mathematics teaching adds to *what* students learn, and the value provides authentic *contexts* where mathematics has relevant applications. Primarily, these values are determined from considerations of how mathematics education can maintain, and perhaps increase, its relevant role in preparing students to contribute to an effective and competitive workforce.

In CL, computational technologies are thought of as interconnected in our interaction with science and mathematics. Computational technologies are not described as external to mathematics, but as infrastructure that redefines what mathematics and mathematical practices are. Computational technologies thus imply a demand to constantly pay attention to how they change our understanding of mathematics and what is appropriate to teach students. In diSessa's (2018) view, education and literacy are committed to developing mastery in relation to the most powerful intellectual infrastructures that humankind develops. When these infrastructures change, so does (or, at least, should) teaching. In this way, the CL perspective implies that computational technology is considered a bearer of values to potentially constructively develop societies. However, CL rather provides a framework for *relating to* and *analysing* the infrastructural transformations of technology, including the values added, rather than generically specifying the exact nature of these values.

Analytical outcome

KOM analyses typically focus on students' cognitive mastery of mathematics, either in terms of the extent to which they have activated certain competencies, or how to design teaching that will enable students' acquisition of a competency. Relating to computational technology from a KOM point of view will likely lead to an outcome of an analysis of what it means to master or how to enable the acquisition of one of the mathematical competencies. As the competencies are considered stable, the outcome of analyses will regard computational technology as *means* to, e.g., represent, communicate, or engage with mathematics without altering the core of the competencies. From a teaching perspective, the KOM-framework can be used for various purposes, including to diagnose students

learning of mathematics or to plan, orchestrate and evaluate teaching (Niss and Højgaard, 2019). The premise is however still that the mathematics to be learned remains unaffected by new tools.

With regards to analytical outcome, CT offers framework to make informed choices about which new computational thinking practices to add to mathematics teaching to align with needs from the industry. Contrary to KOM, CT practices originate from outside a purely mathematical discipline. This implies that an analytical outcome of such a perspective will concern interdisciplinary configurations between an existing mathematical discipline and an external computational discipline.

Where KOM and CT offer perspectives that aids specifying cognitive mastery of mathematical competencies and CT practices, diSessa's (2018) perspective implies a more radical view of the two disciplines as being in flux. Consequently, an analysis following diSessa's (2018) four R's will lead to a new understanding of how a specific computational technology re-mediates intellectual activity, reformulates subject matter, reorganizes intellectual terrain, and revitalizes the ecology of learning activities. The outcome of such analyses will thus be technology-dependent and apply to a specific infrastructure. When new technologies emerge, new analyses are needed to adequately understand the implications for our understanding of what mathematics what it means to master mathematics.

Discussion and conclusion

As evident from the analysis section above, there are notable differences among KOM, CT, and CL in their epistemological assumptions, the value ascribed to computational technologies, and the nature of the analytical outcome they afford. In KOM, the task of mathematics teachers lies in which computational technologies to consider relevant within their current understanding of mathematics as a discipline. While this becomes increasingly different as the complexity of computational technologies increases, teachers can rely on their mathematical proficiency and knowledge of the curricular content. CT brings a different task for mathematics teachers, as CT practices are described as extra-mathematical disciplinary knowledge, for which room should be made in mathematics education. Teachers are thus confronted with a task of selecting practices from the CT-taxonomy to incorporate and to consider how such new practices should relate to existing components of the mathematics curriculum. This is a challenge since most mathematics teachers do not have proficiency in computer science practices. By far, CL however represents the most radical viewpoint and its associated implications for mathematics teaching. This perspective implies that teachers not only are faced by having to relate to new technology but to analyse the reciprocal, infrastructurally transformative relation between discipline and tool. In KOM, the mathematical discipline is stable, and so are the mathematical competencies. What is in flux is the aids and tools used. In CT, mathematics is described as needing to expand its disciplinary demarcation lines, and Weintrop et al. (2016) contribute by specifying what is relevant to add. However, the perspective described in CL leaves open which tools to assign significant potential to transform mathematics and ultimately society, and to provide a framework to engage with these questions. Albeit it might be timely to welcome computational technology in mathematics education, decisions to do so require careful consideration of the implications for mathematics education and mathematics teaching. It is our hope that this paper has brought an initial foundation, which future studies can continue to elaborate.

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Investigation of the use of AI as a scaffolding tool during mathematical discussion in geometry problem-solving

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This study aims to investigate using ChatGPT as a scaffolding tool for pre-service mathematics teachers while solving a GeoGebra-based geometry problem. In this case study design, two volunteer pre-service mathematics teachers worked on a problem whose solution was not provided directly by ChatGPT. The data consisted of GeoGebra solution files, ChatGPT screen recordings, transcriptions of task-based and post-implementation interviews, and researchers' observation notes. As a result of the data analysis, the participants benefited from ChatGPT with the aim of guidance, questioning, feedback, and cognitive challenge in the context of scaffolding components. Although ChatGPT scaffolded the participants during problem-solving, the fact that they were too dependent on ChatGPT and wanted ready-made solutions prevented them from reflecting on their reasoning for the solution.

Keywords: ChatGPT, GeoGebra, geometry, scaffolding.

Introduction

Studies emphasize that digital tools are critical in deepening students' conceptual understanding and developing their problem-solving skills, increasing their risk-taking tendencies, and enabling them to try new strategies (Mohamed et al., 2022; van de Pol et al., 2018). Among these tools, artificial intelligence (AI) stands out for its ability to personalize learning experiences, provide instant feedback, and encourage students to explore new learning strategies. In recent years, the increasing role of AI in education has also brought along discussions that AI can completely replace teachers. While AI-based learning tools provide significant advantages to students by providing personalized guidance, feedback, and support for students to learn individually (Mohamed et al., 2022; Zawacki-Richter et al., 2019), the reduction of teacher-student interaction can create some challenges in terms of social and emotional learning. Within the scope of these discussions, the rationale for this study lies in exploring how ChatGPT, an AI tool, can influence students' learning processes while addressing how to establish scaffolding to balance between technological support and AI-student connection. This study aims to investigate using ChatGPT as a scaffolding tool while solving a GeoGebra-based geometry problem by pre-service middle school mathematics teachers.

Theoretical background

Vygotsky (1978) explains the Zone of Proximal Development (ZPD) as tasks that an individual cannot do independently but can accomplish with the help of a guide or a more competent individual. Wells (1999) defines scaffolding as "Vygotsky's concept of working in the ZPD as the basis for making learning effective" (p. 127) and emphasizes the importance of the scaffolding function of the tools used in this process. Providing these supports, especially in discussion processes during problem-solving in small groups, not only helps individuals to progress in ZPD but also contributes to a more effective and more profound learning experience in the small groups. The support provided

by teachers during small group work also comes to the fore, and various studies (e.g., van de Pol et al., 2018) show that teachers sometimes have difficulties supporting small groups during mathematical discussions. Calor et al. (2022) developed a tool for teachers to support small group mathematics discussions and thus focused on explaining mathematics teachers' scaffolding processes with five key dimensions: Guidance, Questioning, Feedback, Cognitive Challenge, and Fading Support.

While AI tools have shown promise in education, studies also highlight gaps, particularly in their application for problem-solving in mathematics (Zawacki-Richter et al., 2019). This underscores the need to explore how AI can effectively support mathematical discussions during problem-solving in small groups. This study aims to investigate the support provided by ChatGPT to students during small group mathematical discussions with the teacher in the background. In this direction, the aim is to examine the use of ChatGPT as a scaffolding tool for pre-service middle school mathematics teachers (PMTs) to solve a GeoGebra-based geometry problem. A literature review reveals that AI tools take center stage in most studies, primarily analyzing AI-generated responses (e.g., Gurl et al., 2024; McGalliard & Otten, 2024). However, no study has been conducted in which an AI tool is used as a supportive resource that participants can consult when needed, emphasising the nature of the support provided by the tool. Therefore, studies examining teacher support during mathematical discussions conducted in group settings were reviewed to conceptualize AI support. Among these studies, the dimensions of scaffolding in the framework of Calor et al. (2022) were adapted to examine the support provided by ChatGPT: Guidance refers to ChatGPT helping PMTs focus on the correct information and strategies. Questioning involves providing PMTs with strategies that encourage them to deepen their thoughts and ideas. Feedback refers to ChatGPT offering feedback and hints to help PMTs improve their work. Cognitive Challenge entails ChatGPT challenging PMTs in a way that fosters the development of more complex thinking skills. Fading Support is the gradual reduction of the support PMTs receive from ChatGPT over time. In this context, the following research question guided the study: How do PMTs use ChatGPT as a scaffolding tool to guide them in solving a GeoGebra-based geometry problem?

Methods

In this study, we adapted the case study design (Creswell, 2007). The participants were two PMTs (PMT1 and PMT2) studying in a state university's third year of the Department of Elementary Mathematics Teacher Education. The participants were selected based on criteria such as having a good level of proficiency in using GeoGebra, actively using ChatGPT, having a Cumulative Grade Point Average (CGPA) of at least 3.00, and volunteering in the study. The identities and information of the participants were kept confidential. To determine the PMTs' use of ChatGPT as a scaffolding tool, we conducted a task-based interview during the solution process of the Ship Problem (Yazıcı, 2022). The Ship Problem is as follows: "Suppose you are on the shore of an island. You see a ship anchored some distance from the shore. You want to swim towards the ship because you need food. You know from experience that you can swim at most 600 meters at sea. You want to swim towards the ship but can't estimate how far away you are. There is only one question running through your mind. Is the distance to the ship more or less than 600 meters?"

The reasons why we chose the Ship Problem were that ChatGPT did not give the problem solution directly and that the problem could be solved using GeoGebra. During the task-based interview, we asked questions such as “Can you explain what you did here?” and “Why did you think this way?” to encourage students to articulate their explanations and think aloud. After the task-based interview, we conducted a post-implementation evaluation interview to evaluate the PMTs’ experiences of using ChatGPT by asking questions such as “How did ChatGPT influence your thinking process?” and “Based on your experience in this implementation, what are your thoughts on the use of ChatGPT during teaching process?”. The video recordings of the task-based interview were transcribed and supported by the researcher’s observation notes. The transcripts and observation notes constituted the primary data source of the study. The transcript of the evaluation interview was used as a secondary data source to support the primary data. The collected data were analyzed by integrating ChatGPT screen recordings and GeoGebra files. The researchers analyzed all data to examine the PMTs’ use of ChatGPT as a scaffolding tool using the framework of Calor et al. (2022). Approximately one month after the data analysis, two researchers reviewed the coding and compared the codes. The data analysis process was completed by examining different situations in depth. In this way, the reliability of the data analysis was attempted to be ensured.

Findings

After reading the problem, the PMTs copied the problem text directly into the ChatGPT. The PMT1 explained the reason for this action: “I mean, for example, in the beginning, not every student knows what to start with and how to proceed...”. Since ChatGPT presented a solution method, including physics, the PMTs could not use it. “As I already said, when you type the problem directly, it transfers it to other paths. It takes the friction force... without these, the problem cannot be solved.” (PMT1, 280). Then, the PMTs reduced the problem text to simplify the solution path and avoid unnecessary complexity due to the other discipline. These actions in the Guidance dimension enabled the PMTs to eliminate unnecessary information in the problem. After that, the PMTs asked ChatGPT to model the situation and discussed the steps of the presented mathematical model, including inequalities. “We can use a simple geometric approach to model the ship’s distance. Let’s represent the distance to the ship as x. If the distance to the ship is less than 600 meters, $x<600$, and if it is more than 600 meters, $x>600$”. In fact, ChatGPT replied, “We can use triangle similarities to model this situation.” but they only focused on the numerical values in the answer. They constructed sliders and the line segments in Figure 1. However, since they could not connect these sliders to anything, they needed ChatGPT support again and constructed a right triangle in Figure 1 in line with the suggestions by typing the prompt “How can we show it on GeoGebra?”.

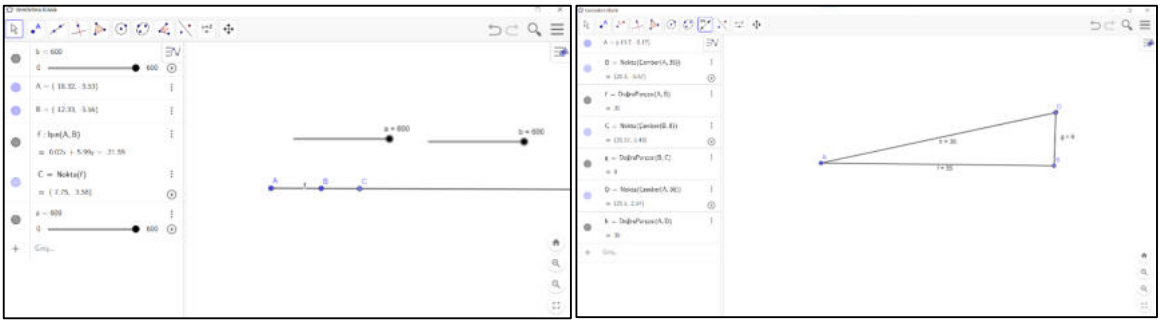

Figure 1: Initial attempts on GeoGebra

Although they followed the instructions, they did not know how to evaluate the triangle they drew, so they decided to proceed by asking ChatGPT, "If the distance from point A to point B is 600 maximum meters, is the distance from point B more or less than 600 meters?" and "How can we show it by using the slider". At this stage, the PMTs deepened their thinking within the Questioning dimension. They tried to interpret distances with the slider they constructed to assess whether the swimmer could reach the ship. Since they did not think that the swimmer had to turn back and they had to take the man's height into account as per ChatGPT instructions, they considered the man as a point (see Point A in Figure 1.b). So, their approach was unsolvable, and they fell into a Cognitive Challenge.

The PMTs realized they could not solve the problem by following ChatGPT instructions. As a result, they revisited the responses from the chat records and attempted to revise the problem text ("If 600 maximum meters can be traveled from point A to point B, is the distance to point B more or less than 600 meters?") using their reasoning in the context of the Questioning dimension. During their drawings, the researcher summarized what was done and provided scaffolding to the PMTs with the question, "So why did you take the man as a dot?" and the PMT2 explained their approach as "We did it from the direct viewpoint, and there should have been a height underneath. There is indeed a height there.". Therefore, they realized that the mathematical model they had constructed was invalid and decided to move to a different solution strategy, as follows:

PMT1: We will draw a line. We'll just draw a line to this point L and extend it to the line. I get it now.
PMT2: We'll set up a certain ratio based on the man's height.

Both ChatGPT chat records and the researcher's scaffolding enabled the PMTs to provide Feedback and an opportunity to evaluate the outcomes of their decisions and learn from these results.

The PMTs decided that they needed to construct a certain ratio based on the man's height and asked ChatGPT, "If we take into account the height of the man in the third answer you gave us, how can we construct a similarity relationship from there?" referring to the ChatGPT statement previously recommending the use of sliders. ChatGPT proposed a solution that included the use of the average height of a man: If we subtract the man's height (1.80 meters) from the distance from point A to point B (600 - 1.80 = 598.20 meters), we get the length of the hypotenuse of the second triangle. That is, if the distance to 600 meters is longer than 598.20 meters, it will have to exceed the swimming distance to reach the ship. To model this situation in GeoGebra, we can construct a triangle containing the distance from point A to point B and the man's height. Then, we can create a triangle containing the distance from point A to point B and 600 meters. We can find that these two triangles are similar and find the similarity ratio of the triangles. If the similarity ratio is less than 1, he must overcome the swimming distance to reach the ship.

They modelled the similarity relation by redrawing the mathematical model they constructed in GeoGebra (see Figure 2) following the steps provided by ChatGPT. They tested the model by moving the slider to change the man's height.

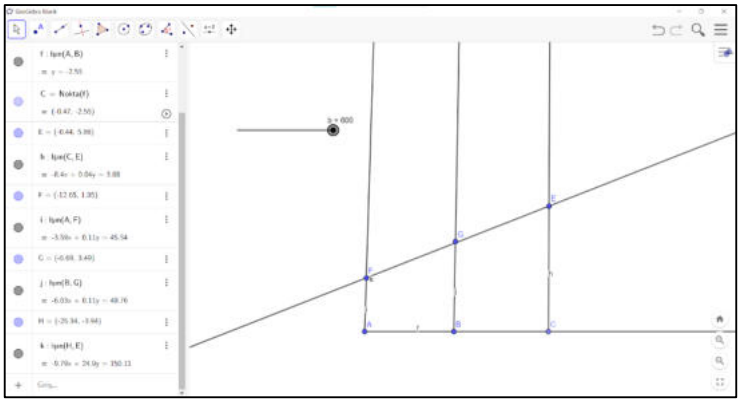

Figure 2: Final model regarding problem solution on GeoGebra

Having difficulty interpreting the meaning of the model, the PMTs again asked ChatGPT to find the similarity ratio to give a direct interpretation of the model they had constructed in GeoGebra. At this stage, it was observed that the PMTs wanted to reach a direct numerical value that was ready for problem-solving. In line with the response they received from ChatGPT, the PMTs said, "In case the similarity ratio is less than 1, it needs to reach the swimming distance to reach the ship. So, by looking at the |IF|/|HD| ratio, we get that if it is less than 1, it will reach, and if it is greater than 1, it will not reach.". Although the PMTs did not solve the problem, they thought they had concluded the solution process, and they presented their solution to the researchers for verification and tried to explain their reasoning. When asked why they could not solve the problem, the PMT2 said:

PMT2: I shut myself off from all my perspectives. Maybe if I had thought about it, I would have been able to find the height of the man and the ratio. I mean, although it [ChatGPT] is really very guiding in that regard, it always directed us toward a single perspective. So, it prevents me from seeing different things. When I looked at it, I ruled out my thoughts.

During the problem-solving process, the focus was on the dimensions of Guidance, Questioning, Cognitive Challenge, and Feedback in the context of the scaffolding purpose of the ChatGPT. Since the PMTs were so dependent on ChatGPT, they did not exhibit any approach in the context of the Fading Support dimension. The PMTs generally gave too much responsibility to ChatGPT and put their reasoning on the back. At the end of the problem-solving process, they were asked to interpret the influence of ChatGPT on their thinking process. The PMT1 explained their thoughts: "There were advantages and disadvantages. It was good for us to be available anytime and get short feedback. But not all answers were correct. There were stages where it led us to physics. For example, it takes the currents, wind, and wave conditions. I don't know what else happens in daily life... Since we try to solve the problem as if these realistic conditions do not exist, the steps it gives may confuse me. Someone may interpret the ChatGPT with the logic that the problem cannot be solved without them. But the prompts we enter in ChatGPT are also very important here. For example, if we had formulated appropriate prompts after interpreting the given instructions mathematically, we would have obtained more accurate answers." The PMT1 evaluated ChatGPT as a tool that provided accessible and quick feedback. However, she highlighted that the instructions given by ChatGPT were not always correct

and that the user needed to filter these answers. When the PMTs were asked about their views on the use of ChatGPT during instruction, the following responses were obtained:

PMT1: The teacher's responses in class are guiding and definitively correct. ChatGPT is also guiding, especially when we are limited by time, encounter a problem, or need immediate support. However, we must filter our thoughts through a certain reasoning process when writing prompts.

PMT2: I didn't think of a different point of view... Because I completely trusted what it said and did not think of such a possibility… While ChatGPT helped us progress to a certain extent in solving the problem, it was through your guidance that we became aware of the ideas we had neglected. This highlighted, for me, the significance of teacher support.

An analysis of these views reveals that the PMTs found ChatGPT helpful in terms of support but indicated that it would not be sufficient on its own, emphasizing the reliability of teacher guidance.

Discussion and conclusion

Calor et al.'s (2022) theoretical framework emphasizes the critical importance of scaffolding strategies in pre-service teachers' mathematical discussion. However, some studies advocating using AI tools and technological support in education offer an alternative perspective to these traditional guidance approaches. For example, Luckin et al. (2016) argue that AI-supported educational tools offer personalized support based on students' individual learning needs and thus can be more effective than teacher guidance. However, our study demonstrated that ChatGPT cannot surpass or substitute teacher support. We can conclude that ChatGPT has the potential to enhance education and student learning; however, it is essential to use it as a complementary tool in support of human teachers (Wardat et al., 2023).

This study has contributed to the current literature by examining the scaffolding dimensions in the problem-solving process with an AI tool. Within the framework of the Guidance dimension, the PMTs consistently aimed to reduce and simplify the problem text during the process. Although they obtained progressively more reasonable responses from ChatGPT, they could not effectively utilize them, as they did not engage in critical reflection or interpret them through their reasoning processes. In other words, the PMTs' resorting to copy-paste strategies without adequate interpretation caused them to fail in problem-solving. The Questioning dimension played a crucial role by encouraging the PMTs to think critically, reflect on their prior knowledge, and explore new avenues for solutions, as outlined in Calor et al. (2022). But they still concentrated on ready-made answers or solutions. The Cognitive Challenge dimension became evident when the PMTs recognized that the solutions they applied based on ChatGPT's instructions were invalid. This was a result of their direct reliance on ChatGPT. Similarly, Opesemowo and Ndlovu (2024) highlight that overreliance on AI technologies prevents students' problem-solving skills. Consequently, the PMTs required researcher support. The Feedback dimension (Calor et al., 2022) was focused on the hints provided by ChatGPT. Although the PMTs received appropriate feedback from ChatGPT when they engaged in their thinking to create correct prompts, they could not adequately assess this feedback due to insufficient interpretation in other dimensions. The Fading Support dimension was not applied as suggested by van de Pol et al. (2018). This situation reveals the importance of gradually reducing guidance. AI cannot provide the dynamic support the teacher molds to the student's learning needs; therefore, it cannot be an efficient

learning guide. Therefore, although AI tools complement guidance, human guidance is critical, especially in mathematical discussions (Neogrády-Kiss et al., 2023).

In conclusion, technological tools and teacher guidance should be used strategically to help students develop independent thinking skills. Future studies can be conducted with the recognition that ChatGPT alone will not be sufficient to support the instructional process. In these studies, it is recommended that a limitation be imposed beforehand to examine ChatGPT's support in the problem-solving process and that these limitations continue to be applied once they are input as prompts. This approach would allow for in-depth analyses, enabling the evaluation of participants' utilization of ChatGPT from different perspectives.

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A pre-service mathematics teacher's digital resource design capacity

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This study examines a pre-service mathematics teacher's (PMT) design capacity based on designing and implementing a technology-based task focusing on the objective of subtraction in fractions with unlike denominators through a ratio word problem. The recently developed Digital Resource Design Capacity framework provided the conceptual foundation of the study. This case study was conducted in the context of a practicum course in a mathematics teacher education programme in Türkiye. The data included the PMT's planning of a technology-based task and audio recordings of pre- and post-lesson interviews and classroom teaching. The results showed that the PMT planned detailed and sequential questions/prompts to teach the topic and utilised the affordances of GeoGebra, particularly the role of different sliders, which enabled her to support students' inputs and explorations in the classroom.

Keywords: Teacher design capacity, digital resource, pre-service mathematics teacher.

Introduction

In the evolving landscape of mathematics education, integrating technology into teacher education programs has become increasingly crucial. Pre-service mathematics teachers (PMTs) must develop both a deep understanding of mathematical content and the ability to effectively incorporate technological tools into their teaching practice (McCulloch et al., 2021). Effective integration involves not only familiarising PMTs with a range of technological tools but also fostering their ability to critically evaluate and select appropriate tools based on pedagogical goals (Clark-Wilson et al., 2023). Practicum courses, serving as a critical bridge between theoretical knowledge and practical application, offer opportunities for PMTs to evaluate the affordances of digital resources based on curriculum needs and to make decisions about how to use them efficiently (McCulloch et al., 2021). As teacher educators and mathematics education researchers, our long-term aim has been to develop PMTs' professional knowledge and experience regarding the successful integration of digital technologies (Bozkurt & Yiğit Koyunkaya, 2022). Our professional development (PD) ideas focused on orienting and conceptualising the PMTs' lesson planning and classroom practices allowing them to critically interact with digital resources throughout the practicum course and enabling them to make connections between planning and implementation (Bozkurt & Yiğit Koyunkaya, 2022). Effective use of technology requires more than just familiarity with tools; it involves the ability to design and adapt digital resources and activities that support curricular goals (Clark-Wilson et al., 2023; Trgalová & Tabach, 2024). PD and collaborative planning are essential for teachers to build and sustain their design capacity, allowing them to continuously innovate and refine their use of technology in the classroom (Pepin et al., 2017). Hence, we focus on PMTs' design capacity, particularly with the use of digital technology,

in the context of a practicum course. This study potentially contributes to mathematics teacher education by shedding light on how one PMT navigates design decisions when planning and implementing a technology-based task during a practicum course.

Theoretical framework

According to Pepin, Gueudet, and Trouche (2017), the process by which mathematics teachers engage with digital resources significantly influences their ability to design and implement effective instructional materials. The authors highlighted that teachers' interactions with these resources—encompassing their selection, adaptation, and integration—play a critical role in enhancing their design capacity. Recently, Trgalová and Tabach (2024) developed the Digital Resource Design Capacity (DRDC) framework by considering mainly Pepin et al.'s (2017) teacher design capacity conceptualisation and combined it with the notion of instrumental orchestration (Drijvers et al., 2010) that includes three main concepts: *didactical configuration* referring to the arrangement of the teaching with the use of technology; *exploitation mode* referring to teachers' planning decisions to reach their objectives; and *didactical performance* is about teachers' on-the-spot decisions based on the selected teaching setup and strategies. In this sense, the DRDC framework focuses on teachers' designing and teaching processes with the use of technology, and thus would serve a better theoretical tool for our study.

The DRDC framework includes three main components: designing for teaching (DefT), design-in-use (DeiU), and re-design (ReDe). The first component, DefT, is essentially related to teachers' skills in the planning stage of the digital resources. The second one emphasises teachers' actions on the spot by focusing on their didactic performance of the instrumental orchestration and their reflections and interventions during the actions. The third component, ReDe, is concerned with teachers' ability to assess the task design, considering the affordances and difficulties in the teaching and revising the design. Table 1 provides the descriptions of each component in the DRDC framework.

Table 1: The DRDC framework (Trgalová & Tabach, 2024, p. 662)

Design for teaching (DefT) (before enactment)	**(DefT0) To set up a mathematical learning goal** (a) Choose a mathematical content that the task will be aiming at. (b) Analyse how the mathematical goal is aligned with the curriculum.
	(DefT1) To choose relevant digital technology (DT) (a) Identify affordances of DT allowing to achieve the learning goal. (b) Identify affordances of DT allowing to address students' difficulties or misconceptions.
	(DefT2) To design tasks mobilising the chosen DT and justify the design choices (a) Analyse in what way the tasks contribute to the achievement of the learning goal. (b) Identify previously acquired (math. and inst.) knowledge necessary to engage in the tasks. (c) Highlight affordances of the DT that allow for a different approach to achieving the targeted learning (added value of DT).
	(DefT3) To plan classroom enactment of the tasks: (a) Organise phases of the task enactment (introduction, individual /group work, whole class discussion...).

	(b) Plan didactic configuration and exploitation mode. (c) Plan teacher's interventions in response to anticipated students' strategies and difficulties (planned didactic performance). (d) Plan the institutionalisation articulating math and instrumental knowledge. (e) Provide material to support the teacher's interventions (presentations, student worksheets..).
Design-in-use (DeiU) (during enactment)	**(DeiU) On-the-spot teacher's actions and decisions (improvised didactic performance) about:** what question to pose now, how to do justice to (or to set aside) any particular student input, how to deal with an unexpected aspect of the mathematical task or the technological tool, or other emerging goals
ReDesign (ReDe) (after enactment)	**To assess the task design based on an a posteriori analysis:** (a) Identify what went well and what did not. (b) Identify what can be improved and suggest modifications of the tasks.

We chose to employ this framework since our PD context included first PMT's task design, then teaching the task in the classroom, and finally re-designing the task based on the implementation.

Methods

This case study illuminated a PMT's DRDC in the context of a practicum course. By following Stake's (2005) suggestion "a case study is both a process of inquiry about the case and the product of inquiry" (p. 444), we explored and illustrated one PMT's DRDC by examining her thoughts and experiences related to designing and implementing a technology-based task.

Context, participants and procedure

The study took place during a compulsory final-year practicum course in a four-year mathematics education programme at a Turkish university. Twelve PMTs enrolled in the course were purposively selected based on their interest and ability in using dynamic technology for mathematics teaching and task design, as well as their willingness to participate.

The practicum requires PMTs to observe/teach six hours per week under the supervision of a mentor teacher in a school placement and attend a two-hour university seminar. Although technology use was not mandatory, the course content was designed to enhance PMTs' skills in using digital resources, focusing on task design (DefT), didactic performance during teaching (DeiU), and task redesign based on experience (ReDe). In the first weeks, we offered seminars on designing and implementing technology-based tasks. During the first 3–4 weeks, PMTs observed their mentor teachers and then planned a technology-based lesson aligned with the mentor's schedule. After reviewing their plans, we conducted individual pre-lesson interviews to explore their design decisions (DefT). PMTs then taught their lessons (DeiU) and participated in post-lesson interviews to reflect on their teaching, including challenges and adaptations (ReDe).

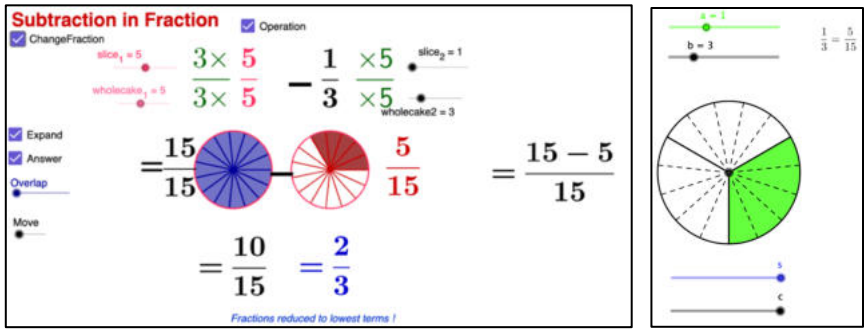
This study reports on one PMT's (pseudonym Nil) planning and implementation of a technology-based task involving a ratio word problem focused on subtracting fractions with unlike denominators. We chose to present her task because her planning reflected thoughtful consideration of multiple factors, including the different roles of sliders in the digital environment to support

students' thinking in reaching a generalisation, as well as her attention to students' prior knowledge and potential misconceptions.

Data collection and analysis

Nil's planning of a technology-based task and audio recordings of pre- and post-lesson interviews and her classroom teaching constituted the data of the study. We prepared interview questions by considering and revising Pepin et al.'s (2017) ten teacher design questions by focusing on using digital resources. Example pre-lesson interview questions (completed in 20 mins) included: "How will technology help you to achieve your goal?", and "What resources or tools did you use to plan the task?". Then, she implemented her plan and we audio-recorded her teaching (completed in 35 mins). Example post-lesson interview questions (completed in 25 mins) included: "Did you have any difficulties during the teaching?" and "What do you think about your response to the students' questions in the classroom?".

In her task, she initially planned to propose the following problem "*At the end of a busy working day, a pastry chef wants to calculate the amount of the cake sold during the day. 1/3 of the cake served for 5 people was sold during the day. What is the ratio of cake slices remaining for sale the following day compared to the whole cake? Why?*" and aimed to solve it with the use of technology. In her planning, she used two separate GeoGebra (GGB) files, and the sliders were the main technological tool in both files for reaching her aim. She selected ready-made files and re-designed them (i.e., changed the colours and language). The first GGB file was the main file (Figure 1.(1), https://www.geogebra.org/classic/apve8uwj) in her task, and she aimed to use the second file to remind of the equivalent fractions (Figure 1.(2), https://www.geogebra.org/classic/hazfkdw4), which are prerequisite knowledge for learning subtraction in fractions with unlike denominators.



(1) (2)

Figure 1: Nil's GGB files in her task

In her main GGB file (Figure 1.(1)), there were six sliders representing two numerators and two denominators in different fractions, overlapping the fraction models, and moving the final model of the subtraction. She used different colours to represent different fractions and four checkboxes for changing the operation (addition or subtraction), showing the fractions, expanding the fraction, and presenting the result. She planned to use the sliders to direct students in finding the subtraction and to use the checkbox for guiding the discussion or presenting the numerical values of fractions. Additionally, she planned to use the dynamic texts to support her teaching and verifying responses.

We analysed the data, considering all the components of the DRDC framework. While analysing Nil's planning and pre-lesson interview accounts, we focused on her design choices for teaching (DefT), including her emphasis on the objective in the curriculum, the rationale behind selecting relevant digital technology, her consideration of students' existing mathematical and instrumental knowledge, and her analysis regarding the affordances of the task and chosen technology. We also examined how she organised the phases of the task and planned her didactical configuration and exploitation mode. In analysing Nil's teaching (DeiU), we examined how she implemented her lesson plan, considering the components of the instrumental orchestration framework. Specifically, we identified key moments related to her ability in guiding students and her questions/prompts to facilitate the whole-class discussion. This analysis helped reveal how she implemented her plan and to identify the critical points related to her DRDC. In the analysis of the post-lesson interview, we focused on how Nil criticised her own teaching and how she planned to revise her task (ReDe). Throughout the analysis, triangulating her planning and the audio recordings of her interviews and teaching enabled us to validate and verify the data.

Results

Here, we report Nil's DRDC by presenting her planning and implementation of a technology-based task, as well as her reflections on planning and teaching the task in the classroom. While setting a mathematical learning goal (DefT0), Nil decided on the topic with the mentor teacher. In the pre-lesson interview, she explained that although she had other options, she intentionally chose to teach fractions due to her personal interest in the concept. She also emphasised that GGB is a valuable tool for enhancing both the teaching and learning of fractions. In her own words, she stated:

> Nil: It was not only that fractions were the next objective to teach in the school placement, but also, I am personally interested in teaching fractions with technology. I believe that using GGB is very relevant while teaching this topic as we can easily show equivalent parts in a fraction model. To be honest, I realised why we really equalise the denominators while working on the concept in GGB.

While preparing her plan, she examined the curriculum, digital resources, and existing literature, which helped her identify the existing knowledge and misconceptions of 6th grade students. Nil preferred to use GGB for designing the task (DefT1) indicating that the tools and features of GGB support drawing, dividing, or changing the fractions easily and enable students to visualise and concretise any model of any fraction. She also noted that the models in GGB could help overcome students' misconceptions, such as failing to divide fractions into equal parts and incorrectly expanding the denominator without adjusting the numerator when expanding fractions.

In terms of mobilising the tools in GGB and justifying her design choices (DefT2), she provided evidence by effectively planning her technological actions, particularly using the sliders for different purposes. For instance, with the help of sliders, she planned to initially present the cakes (Figure 2.(1)) and ask students to find the remaining part of the cake. Then, by using another slider,

she aimed to overlap the models (Figure 2.(2)) and guide students to consider whether it is possible to find the remaining part of the cake. At this point, she planned to open another GGB file (Figure 1.(2)) to help them think about dividing the cakes into the same number of slices by dragging the sliders. According to her, one of the biggest affordances of GGB was drawing, dividing, and expanding the fraction to find equal denominators and moving the models on top of each other to see the remaining part, which would have been impossible to do on a traditional board.

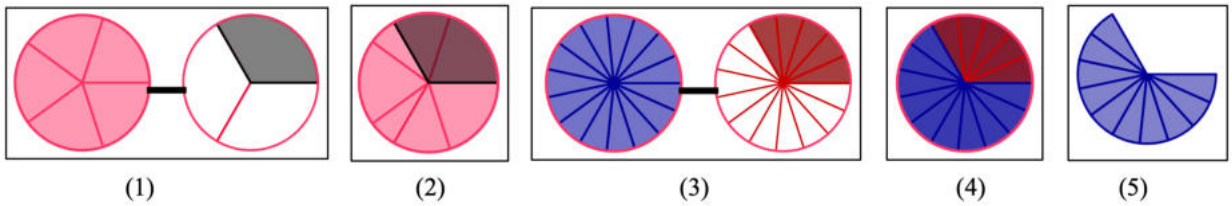


Figure 2: The fraction models obtained by dragging the sliders in the task

For the planning classroom enactment of the task (DefT3), Nil prepared a detailed, step-by-step plan for launching the task and facilitating classroom discussions. In her plan, she prepared detailed and sequential questions considering students' responses and misconceptions. Initially, she planned to present the problem and ask students to draw ⅓ of a cake, which was difficult for them. Afterwards, she planned to present and discuss how the cakes could be divided (Figure 2.(1)). Then, she aimed to overlap the models (Figure 2.(2)) and propose the question, "From the figure, could you infer how many slices of the cake was sold?". Nil anticipated that students might struggle to find the answer, and if they were unable to equally divide the models, she planned to guide them in equalising the denominators. In this process, she planned to help students remember the role of the least common multiple in equalising the denominators. She then planned to divide both cakes into 15 slices (Figure 2.(3)) and help students observe any changes in the slices that were sold. If students noted a change, Nil intended to open a second GGB file to remind the students of the equivalent fractions. In this part, Nil aimed to drag the sliders to help students focus on what changes and remains the same in the model, aiming to show how ⅓ equals 5/15. She then planned to return to the main file and discuss the reasoning of equalising the denominators. Afterwards, she intended to overlap the models and guide students to calculate the slices sold by focusing on the red and blue parts (Figure 2.(4)). Finally, she planned to use another slider to move the remaining part of the cake to a different place on the screen to present the answer (Figure 2.(5)). Throughout her teaching, she also planned to use the dynamic texts step by step to verify students' inputs. From the perspective of DefT component, it became apparent that Nil's task design was informed by both her pedagogical goals and the particular affordances of GGB. Her design process exemplifies how a deliberate integration of technology could enhance the teaching and learning of fractions.

Nil implemented her plan with a class of 6th grade students in a school placement. In terms of criticising her actions and decisions on-the-spot (DeiU), Nil stated that her didactic performance was good, and she was able to reflect on each step of her planning. Initially, Nil wanted students to draw ⅓ slices of the cake on paper with a pencil, but the students had difficulties dividing it into equal slices. Then, Nil introduced the fraction models on the screen by dragging the sliders and continued to discuss how to find the remaining part of the cake as follows verbatim:

Nil (N): The left model represents the whole cake, and the right model represents the cake that was sold, right? Could you observe what changes occurred on the screen when I drag the slider [she dragged the slider for overlapping the models]?
Student (S) 1: The pieces, slices. S2: But this does not work because the cakes are not equal.
N: They are not equal. Can I now find how much of the cake has been sold?
S3: No, the given slices are not equal. S4: About one and a half slices were sold.
S5: The slices are not equal; thus, they do not match up.
N: Well, I can't say exactly how much was sold. Maybe it is one and a half; it could be more than one and a half or less. What did your friend say before? He said the slices of the cakes did not match. What should I do to make them match?
S6: We need to expand. S7: We must equalise the denominators.
N: How do I equalise them?

Nil guided students step by step, adjusting her questions based on their responses to facilitate their progression through the task. She emphasised the reasoning behind equalising denominators when operating on fractions and used the other GGB file to demonstrate how ⅓ equals 5/15 by dragging the sliders. Then, she went back to the main GGB file and discussed the rationale for equalising the denominator with the help of the sliders. She also discussed the meaning of the blue and red parts to identify the remaining part of the cake. Finally, Nil used the sliders to overlap the models and moved the remaining part to a separate section to verify the solution. During her teaching, she supported her explanations and responses using dynamic texts on the screen. From the perspective of DeiU component, these actions highlight Nil's capacity to use affordances of GGB in ways that encourage students to engage actively with the learning process. In the interview, Nil stated that she was able to gain students' attention through the tools and features of GGB. Utilising the sliders with different aims was a key technological action for her to develop students' reasoning on the rule for subtraction in fractions. In the evaluation of her teaching (ReDe), Nil reported that she successfully implemented her plan without encountering unexpected issues or emerging goals and thus saw no need for revisions. However, this could point to a lack of experience in recognising challenges, identifying emerging goals, and critically reflecting on her teaching practice.

Conclusion

This study examines one PMT's design capacity by analysing her design choices and knowledge related to the selection and use of digital resources in both planning and teaching a technology-based task. The results indicate that the PMT's design decisions concerning the DefT component appear to be the key to an effective use of technology in the classroom. Nil carefully prepared and adjusted detailed sequential questions and coordinated them with the tools and features of GGB. According to Nil, her careful planning during the DefT stage contributed to a successful DeiU stage, where she effectively utilised different sliders for different aims, enabling students to reach generalisations on the topic. Nil emphasised that designing a comprehensive lesson plan—carefully considering all details—helped her anticipate potential challenges during instruction, allowing the lesson to proceed smoothly without unexpected situations. As a result, she did not perceive a need to revise her lesson plan following its implementation (ReDe stage). While thorough planning can indeed facilitate effective teaching, this result should be interpreted with caution. As Clivaz and Miyakawa (2020) highlight, cultural and institutional contexts play a critical role in shaping how

mathematics is taught and how teachers reflect on and enact designed lessons. Moreover, existing research (e.g. Cavanagh & Prescott, 2010) suggests that beginning teachers often struggle to engage in reflective practice, as their attention is typically directed toward technical skills and practical concerns, such as classroom management, rather than focusing on students' perspectives and learning processes. Overall, this study highlights the potential of practice-based practicum courses as important places for teacher educators to assess and foster PMTs' design capacities, particularly in terms of technology integration. The findings also suggest that the DRDC framework is a valuable tool for teacher educators—not only for course design but also for evaluating PMTs' planning and instructional decisions within such courses (McCulloch et al., 2021).

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Maths AIde: Can GenAI support mathematics teachers to plan engaging, student centred lessons?

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Keywords: Generative artificial intelligence (GenAI), second level mathematics education, professional development, in-service teacher education.

Introduction

Mathematics is a subject which students find difficult, it is perceived as hard, tends to be taught as abstract, unconnected topics (Boaler, 2022). A historical over-reliance on rote-learning and procedural fluency has led to a lack of conceptual understanding and increased disengagement throughout adolescence (Berry et al., 2021). Reforms in maths education have attempted to address these issues by focusing on student-centred, problem-solving activities (NCCA, 2017). However, change is hard and years after curricular reform teachers still require further professional development (PD) (Berry et al., 2021). For PD to be successful it needs to be experiential, situated in the teacher's own classroom, beneficial for their students and sustained (Byrne & Prendergast, 2020). There are many concerns around the emergence of GenAI in terms of education, such as its usefulness when cheating, it's propensity to lie and inbuilt biases (Luo, 2024). Notwithstanding GenAI is an emerging technology which could potentially support teachers to lesson plan, create resources and support critical thinking (Hu et al., 2024).

The intervention

Maths AIde is a PD offering which supports a collaborative dialogue between a teacher, their colleagues and the GenAI. The teachers initially survey their students to determine their interests outside of school. They engage in a collaborative workshop with both the GenAI and their colleagues where they complete a series of tasks using the GenAI. The outputs are reflected upon with their colleagues, refined and implemented in their classes.

Methods

The aims of the research are twofold to improve maths planning and teaching while also supporting teachers to gain confidence engaging with GenAI. This research is following a design-based/ action research methodology and will consist of at least four cycles of planning – acting – developing – reflecting (Mertler, 2017). As there are less than 30 participants involved in the research it will utilise a case study approach. Both quantitative and qualitative data will be collected to strengthen the understanding of the case (Creswell & Garrett, 2008). Quantitative data in the form of surveys and qualitative data such as interviews, GenAI transcripts and artefact analysis will be used.

Data collection and analysis

A baseline survey was administered (N=27) before the introduction of the intervention. Following one cycle of the intervention interviews were conducted using voluntary sampling

(N=6). Teachers from Ireland, Austria & Sweden took part in interviews conducted over Teams. Interviews were 8-10 minutes in length. Transcripts were downloaded to Word, and adapted to ensure conversations were accurately recorded. Transcripts were then uploaded to the coding software NVivo for initial coding. Following multiple refinements three areas emerged: GenAI uses, professional development and technology adoption.

Preliminary results

80% of respondents had never used or only used GenAI once to plan or create resources prior to the intervention. During the interviews teachers spoke about how they were using GenAI to structure lessons, create resources, tailor activities to groups of students and adapt previous assignments. Most of the teachers felt they needed to upskill in GenAI to ensure they were not outpaced by students. The teachers liked the supportive structure of the workshops. It was noted that older more experienced teachers in that school were most resistant to upskilling in GenAI while those newer to the profession were embracing it's potential.

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Chatbots performance in problem posing: An exploratory study

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Keywords: Artificial intelligence, problem posing, chatbots, mathematics education.

Introduction

Conversational agents, or chatbots, are gaining prominence in mathematics education for their potential to support teachers in various tasks by acting as "teacher-agents." These AI-driven tools are designed to generate coherent, human-like text (Kuhail et al., 2023). Their outputs, shaped by large language models (LLMs) and user inputs, often exhibit significant variability. While chatbots are well-documented in problem-solving contexts (Schorcht et al., 2024), their application to other mathematical practices, such as problem posing, remains largely unexplored. This study aims to address this gap by examining the quality of outputs generated by four widely used chatbots in the context of a problem-posing task including fractions.

Theoretical background

Problem posing involves creating new problems or reformulating existing ones according to specific instructions (Silver, 1994). Central to this practice is the quality of the problems posed. Desirable features for mathematics problems include motivating contexts, clarity, and diverse solution strategies (Leavy and Hourigan, 2022). A key characteristic is plausibility, meaning the problem must be free from mathematical contradictions and include all necessary information for complete understanding and solvability (Grundmeier, 2015).

This research focuses on the following question: To what extent do chatbots generate plausible math problems with all necessary information for solvability?

Methods

This qualitative exploratory study evaluated outputs from four chatbots: GPT-4, Claude 3.5 Sonnet, Copilot (precise mode), and Gemini. These chatbots were chosen for their relevance in educational AI applications and their diverse underlying LLM architectures. The study employed zero-shot prompting, a technique that elicits responses without additional contextual information, (Schorcht et al., 2024) to assess the chatbots' baseline performance in problem posing. Each chatbot was given the following prompt five times: "Formulate three problems of varying difficulty that include the numbers 1/4 and 3/8, which can be used as either data or solutions. You may incorporate any type of information (numerical, contextual, etc.)." Outputs were analysed using Grundmeier's (2015) coding scheme: problems with mathematical contradictions (non-plausible), plausible problems with omitted yet understandable information (P1), plausible problems requiring a single task to solve (P2), and plausible problems requiring multiple tasks (P3).

Results and discussion

Copilot frequently failed to include the required fractions, resulting in unacceptable problems. Claude produced the most plausible problems overall (13 out of 15, with 7 categorized as P3). ChatGPT

followed with 10 plausible problems, while Copilot and Gemini produced 7 plausible problems each. A detailed breakdown of the results is shown in Table 1.

Table 1: Chatbot performance and plausibility of problems generated

	Unacceptable	Non-plausible	P1	P2	P3
ChatGPT	0	1	4	8	2
Claude	1	1	0	6	7
Copilot	5	0	3	5	2
Gemini	0	3	5	7	0

These findings suggest that some chatbots can act as teacher-agents by generating plausible and complete problems, while others demonstrate significant deficiencies. The variability in AI-generated content underscores the need for robust validation mechanisms and improved training of chatbots for educational contexts (Schorcht et al., 2024). Further research should explore larger datasets, diverse prompts, and teachers' experiences using chatbots in professional contexts. Such studies will deepen our understanding of how emerging technologies can support mathematics educators in designing effective and engaging instructional materials.

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Computational literacy in mathematical problem solving

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Keywords: Computational literacy, mathematical problem-solving, computational thinking.

Introduction

As the integration of digital technology into everyday life accelerates, the urgent need to leverage its potential in education has been recognized (Kaup, 2022). This transformation underscores the necessity to equip future generations with essential technological skills, highlighting the importance of Computational Literacy (CL) in fostering a deeper understanding and innovative application of technology, particularly in mathematics education (diSessa, 2018).

The introduction of Computational Thinking (CT) in educational systems aim to meet this need. Wing (2006) defines CT as a focus on problem-solving (PS) and system design through computational principles, offering a framework for a deeper understanding of technology. Despite CT's potential contributions to mathematics education (Tamborg et al., 2022), its practical implementation has been difficult. CT is often reduced to an extension of traditional mathematical thinking, including algebraic thinking (Bråting & Kilhamn, 2021), rather than being applied to PS as Wing (2006) originally envisioned. diSessa (2018) criticises how current dissemination of CT often mirrors Pólya's (1945) older mathematical PS techniques. This limits its potential in education.

The PhD project from which this poster reports hypothesizes that introducing CT through PS in mathematics, rather than an extension of 'mathematical thinking' can better harness its potential to develop students' abilities to engage with complex problems in a digital context. CL offers a more expansive framework, focusing not just on learning programming or algorithms, but on empowering students to think critically, creatively and holistically about the application of digital technologies. By embedding CT within a CL framework, the project posits that students will develop not only technical skills but also the critical and adaptive mindsets necessary for effective PS in a world increasingly shaped by digital technology. The research is guided by the following research questions:

RQ1: In what ways can the integration of Computational Thinking in mathematics education support students' ability to solve (complex) mathematical problems?

RQ2: How can mathematical competence in problem-solving be rethought through a Computational Literacy perspective?

A response to RQ1 will serve as input for addressing and discussing RQ2.

Methods

This project uses Design-Based Research (DBR), which integrates design and research to improve educational practices and generate theoretical knowledge. This makes DBR suitable when examining the integration of CT in mathematics education and how this evolves into a broader CL framework. The project is structured around three iterative cycles, with the following phases:

1. **Activity Design**: Develop teaching activities that integrate CT and mathematics concepts. Adjustments will be made based on teacher input, with a focus on how CT supports PS (RQ1).
2. **Reflective Adjustment:** Students' work will be analyzed to assess the effectiveness of activities, particularly CT's impact on PS abilities. Discrepancies between planned and realized outcomes will be evaluated to refine activities in subsequent cycles.
3. **Theory Development**: Insights from the iterative cycles will contribute to theoretical advancements, specifically addressing how CL can further enhance PS competence (RQ2).

This project aims to deepen the understanding of how CT through CL can transform PS into mathematics. Initial challenges in integrating CT into mathematics education will be addressed, providing insights into how CT supports PS. These insights will then serve as the foundation for a broader exploration of CL, ultimately proposing a redefined framework for PS that moves beyond traditional mathematical thinking approaches.

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Engaging and explorative digital tasks designed by pre-service teachers: A teacher training course design

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Keywords: Professional development, digital materials, task design, cognitive activation.

Introduction and theoretical frameworks

As teachers and teacher educators, we aim to enhance learners' conceptual mathematical understanding to foster engagement, motivation, and sustainable learning. This can be achieved via effective integration of digital tools and resources into mathematics education (Hoyles et al., 2013). A central aspect of this purpose is the design of tasks, lesson planning, and student activities (Drijvers, 2018). However, a major challenge is the abundance of digital resources with varying quality. Successful implementation of digital media strongly correlates with the role of the mathematics teacher (Clark-Wilson et al., 2020), highlighting the need for pre-service teachers (PSTs) to develop the skills to competently evaluate the potential of digital materials (dMs) and integrate them to support conceptual understanding. In this study, dMs refer to pre-designed GeoGebra-based configurations embedded in online worksheets with accompanying tasks for learners. Assessing and designing such dMs is a complex process that requires focusing on various aspects such as content-specific pedagogical considerations, visualization, presentation, or task wording, which represent quality issues for technology integration (e.g., Trgalova & Jahn, 2013). Drijvers (2018) outlines three didactic functions of digital technology: doing mathematics, practicing skills, and *developing concepts*. The latter is particularly challenging to implement technologically but entails the potential to discover and explore mathematical objects by utilizing dMs (Drijvers, 2018). The emphasis on enhancing learners' conceptual understanding leads to a specific focus on dMs, which encourage the exploration of mathematical content and provide technology-added value compared to paper-and-pencil tasks.

The presented project investigates how to design aspects of a teacher training (TT) course that focuses on dM and task design to enhance PSTs' skills in creating (i) dMs for exploring mathematical content and (ii) accompanying tasks that engage students in high levels of cognitive thinking. The specific skills are informed assessment, design, and implementation of dMs. As theoretical foundation, we adopt, among others, the TPACK framework that encompasses teachers' knowledge and skills required for teaching with digital media (Koehler & Mishra, 2009).

Methods and context of the study

For approaching the research question, we analyse data from a pilot study and an exploratory qualitative study of a TT course at an Austrian university for secondary mathematics teacher education. This course focused on the design of two dMs with accompanying tasks. The PSTs worked in pairs, selected a lower secondary education topic, conducted an intensive review of relevant mathematics education literature, and engaged in a design and redesign process with integrating feedback from peers, teacher experts, and the mathematics teacher educator, finally presenting their

work. Qualitative data was collected in the form of PSTs' learning diaries, working notes, digital designs, and interviews, and analysed using thematic analysis procedures (Braun & Clarke, 2006).

Exemplary results and outlook

Thematic analysis enriched by theoretical literature review leads to the following successfully supporting design principles: (i) provide foundational mathematical knowledge for teaching and dM design, (ii) request intensive literature review of mathematical knowledge for teaching on the topic addressed by the dM, (iii) enable and foster peer exchange, (iv) implement reviews with experienced teachers and/or teacher educators. Further data analysis reveals two main challenges for the designing skill: First, PSTs repeatedly struggle to distinguish between materials for exploring versus practicing mathematical concepts, making it difficult for them to design tasks that encourage learners to explore mathematical content. Second, PSTs face difficulties in formulating engaging, cognitively activating tasks for the presented mathematical content. A theoretical model could support PSTs in this regard.

In a next course, we will thus use an adapted version of the DGTA framework (Trocki & Hollebrands, 2018; Lindenbauer et al., 2023) for non-geometric topics, refining it in the future with newly collected data. In addition, integrating this adapted task design model into the already complex TT course requires careful considerations. It will be essential to examine what makes tasks cognitively activating for learners, effectively supporting them in exploring and understanding mathematical concepts.

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Developing lower secondary mathematics teachers' ICT competencies: The case of China

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Keywords: Mathematics teachers, information and communication technology (ICT) competencies, secondary school, China, geometry.

Background and rationale

Integrating ICT into mathematics teaching has been widely promoted globally (Foster, 2023), including in China (Ministry of Education, China, 2022). However, Chinese mathematics teachers' ICT use remains superficial, and their ICT competencies are still limited (Sun et al., 2019). Inspired by Niss and Højgaard (2019), this study defines ICT competencies as teachers' ability to use ICT, act appropriately, and adapt their teaching in response to specific challenges in various mathematics topics for students with diverse learning needs to enhance their understanding.

Teachers' professional knowledge is shaped by teaching practice and developed through adaptation in response to experience (Ruthven, 2009). This highlights the importance of relevant and well-designed teaching tasks and training for developing teachers' ICT competencies. Such sign should be informed by an understanding of mathematics teachers' daily practices and experiences.

This poster presents preliminary findings from an exploratory study on Chinese lower secondary (12-15 years old) mathematics teachers' use of ICT, as well as their views of the opportunities and challenges ICT presents to their teaching practice and their students' learning of mathematics.

Methods

Data was collected via questionnaires and individual interviews with lower-secondary mathematics teachers in China. GeoGebra (GGB) was first chosen as the ICT and task design platform due to its popularity and potential in mathematics teaching. 11 teachers engaged with a GeoGebra-based activity, *Euclid: The Game* [https://www.geogebra.org/m/gzfn4pe3], followed by a questionnaire on their experiences and perceptions of the activity. This activity, featuring progressively challenging geometry tasks, aimed to assess teachers' familiarity and proficiency with GGB, particularly in constructing geometric shapes and solving problems using a limited set of tools.

The questionnaires revealed these teachers' low usage, unfamiliarity, and negative perspectives on GGB, which suggests the need for more in-depth interviews to explore their commonly used ICT and their experiences, challenges, and needs. Therefore, the focus was shifted to collect information about Chinese teachers' widely-used ICT, through individual interviews with another group of 15 teachers from 4 schools in 3 cities. The interviews also served to verify and further test the preliminary findings from the questionnaire responses, and the inclusion of this second group aimed to gather a broader range of the use and perspectives on ICT integration in classroom practices.

Findings and analysis

From the questionnaires, teachers expressed mixed views on GGB, with more disadvantages than advantages noted. Compared to traditional methods, the advantages include accurate and easy

drawing, integrating new tools that align with students' cognitive development, an intuitive interface, and supporting cross-platform compatibility. Disadvantages include insufficient instructions, high demands for plane geometry knowledge, no "undo" button resulting in messy drawings, and a lack of immediate feedback. More importantly, teachers' low use and negative attitudes to GGB make it impractical to continue to design ICT tasks exclusively on GGB, as it may increase teachers' workload and cause resistance. However, teachers' initial exposure to GGB still holds value in promoting the use of this powerful mathematics-specific ICT in China.

Following the shift in focus, the interviews showed that ICT hardware, such as laptops, projectors, and interactive whiteboards, is extensively used. Regarding software, PowerPoint is universally adopted, and Seewo [https://www.seewo.com/en/] and Geometer's Sketchpad (GSP) [https://sketchpad.keycurriculum.com] are the most frequently mentioned ICT. Hence, future task design could prioritise more familiar tools, such as Seewo and GSP, to better align with teachers' existing practices and reduce potential barriers to ICT integration. Teachers also reported challenges in using ICT, including a lack of dynamic presentation tools, insufficient interactivity, outdated activities, and inadequate training. Additionally, they faced difficulties in visualising geometric transformations and integrating traditional with digital methods. Expectations for future tools include better shape generation, dynamic visualisation, and generalisable activity templates.

Future plans

The next steps involve developing a quantitative tool to evaluate secondary mathematics teachers' ICT competencies. Using a design-based research approach, ICT-integrated geometry tasks will be created with different teacher groups in China, prioritising widely used ICT such as Seewo or GSP. The task design will address the challenges and needs identified in the teacher interviews and aim to enhance their ICT competencies through targeted, iterative implementation. Moreover, mathematics teachers' communication and collaboration around ICT use within professional learning communities will be observed and analysed to support continuing professional development.

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Group Instrumental Genesis

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Keywords: Instrumental genesis, Group cognition, Group Work

Introduction

Working in groups in mathematics today will oftentimes involve sharing tools (e.g., GeoGebra or similar) on a computer. To work *collaboratively* with these tools, students need to create a shared interpretation of what the tools are and what they should be used for. This poster has a theoretical contribution through presenting a bricolage between theories to describe how groups can engage in such an endeavor.

Since the mainstream adoption of digital tools in mathematics teaching around the millennium, a detailed conceptual language of distributed cognition has helped mathematics education researchers and teachers understand instrumented learning processes (Guin et al., 2005). This is most prominently articulated in the instrumental approach, which focuses on how artefacts are shaped into personal instruments by individuals over time. Collaboration and cognition in groups have garnered similar interests, with instrumental genesis being highlighted as a relevant framework to understand the relationship between group work and tools (Overdijk et al., 2012). However, the inherent focus on individual instrument processes in instrumental genesis limits its ability to conceptualize groups' tool use and view of tasks. Therefore, we pose the following research question: How can we conceptualize the influence of artefacts on group work in continuity with Instrumental Genesis?

Role of Others in Instrumental Genesis

The Instrumental Approach concerns the relationship between tools and users, often with a cognitive focus on how the tools influence cognitive schemes (Guin et al., 2005). Two central concepts describe the processes of continuation and dialectics between design and use: *instrumentation*, where the use of a tool shapes a user's goal-directed activity, and *instrumentalization,* where the pupil's goal-directed activity reshapes the tool (Rabardel & Bourmaud, 2003, p. 673).

'Others' are embedded in instrumental genesis, as the *artefact* is understood as a social construct mediating human activity. The concept of *instrumental orchestration* furthermore highlights an embedded social dimension, as it denotes the class-level introduction to artefacts leading to a more coherent instrumentalization process (Guin et al., 2005). A group-level introduction to the artefact can thereby change the foundation of the processes of *instrumentation* and *instrumentalization.*

The Instrumental Approach describes a social dimension, through the artefact and (potential) instrumental orchestration, as an institutional element which conditions how students use tools. While it adds to our understanding of the basis for the group work, it does not address the social dimensions of the dialectic processes between individuals, tools, and goals. In other words, we still

lack a conceptual framework for (1) how the artefact shapes the group constellation and (2) the role of the group in the instrumentation and the instrumentalization processes.

Group Cognition and Instrumental Genesis

Group cognition can be a relevant theoretical framework for understanding the social dimension of instrumental genesis (Stahl, 2011). Informed by activity theory, group cognition conceptualizes the process by which object-oriented group work establishes group knowledge. It describes group cognition through three dimensions; (1) a temporal dimension of ordered events, (2) a problem space of shared knowledge artefacts, and (3) an interaction space of positioned actors. Here, the second and third dimensions are particularly relevant, as they touch upon the role of artefacts for groups. Stahl (2011) describes that a (math) object is constructed through visual, narrative, and symbolic reasoning. The object exists as a shared knowledge artefact, with meaning embedded by and visible in the group interactions. Beyond conceptualizing the negotiation of the social meaning of the artefact, it allows for the creation of a *group instrument*, where the instrumentalization and instrumentation of the artefact occur at a group level. The temporal dimension and the interaction space between actors provide a conceptual framework to discuss the potential process of *group instrumental genesis* over time. Our pre-liminary definition of group instrumental genesis is the following: *the temporal order of events in which interaction between problem space, discourse, individual subjects' roles and position in a group of subjects lead to the emergence of a group-level instrument with a cognitive function distributed across the participating members of the group* .

The group instrumental genesis allows us to discern between groups by whether they engage in a process of group instrumental genesis. This allows us to understand both the artefacts' shaping of group constellations and the role of the group in the instrumentation and instrumentalization process. For future work, a pressing issue is how this process unfolds, particularly how group cognition includes body communication when applied outside the virtual space from which the group cognition theory originates.

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TWG 16: Learning Mathematics with Technology and Other Resources

Learning mathematics with technology and other resources: A short report from TWG16

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Abstract: This brief report summarizes the results of TWG16 based on the five themes identified from the contributions received. Overall, we observe that the contributions were distributed fairly equally across these themes. This distribution indicates that our field is interested in emerging digital tools while still valuing traditional digital tools among researchers. Based on the diverse contributions and our discussions during TWG16, we will outline potential areas for future research.

Keywords: Programming, 3D design, feedback, problem-solving, mathematical reasoning, AI, Eye tracking.

Overview of themes and contributions

The scope of the thematic working group TWG16 is to address the potential and limitations of incorporating digital technology and other resources to enhance students' learning of mathematics. As a targeted outcome, we aimed to establish an understanding of the current state of the art of technology in mathematics education. Therefore, we are looking to establish an overview, focusing on themes such as programming and 3D design, feedback and interaction, artificial intelligence (AI) and eye tracking, technology and mathematical problem-solving, and tools to promote mathematical reasoning. In addition, we aimed to suggest future directions for technology-rich mathematics education, including a research agenda.

Table 1: TWG16 themes and contributions

Theme	# accepted papers	# accepted posters
1. Programming and 3D design	4	2
2. Feedback and interaction.	4	1
3. AI and Eye Tracking	4	1
4. Technology and mathematical problem-solving.	4	1
5. Tools to promote mathematical reasoning	4	2

The contributions presented to TWG16 highlighted five primary themes to be addressed. Table 1 offers a summary of these themes along with the number of contributions for each. In total, we accepted 20 papers and 7 posters. Approximately 35 participants from various countries, both within Europe and beyond, participated in the sessions, which were characterized by a positive and productive atmosphere.

Main results per theme

Programming, 3D design, and 3D printing: Four papers and two posters addressed this TWG16 theme. The central focus across these contributions was the relationship between computational thinking and programming on the one hand, and mathematics learning on the other. The presented studies, conducted at various educational levels, from primary school to university, highlight the rich potential for cross-fertilization between computer science and mathematics through computational thinking. This intersection creates valuable opportunities for mutual enrichment. However, to fully realize this potential, a crucial element is the inclusion of a reflection phase following programming tasks. Such reflection helps students make explicit connections between computational and mathematical thinking, thereby drawing out the targeted mathematical knowledge. Several contributions emphasized the educational value of using robots in mathematics classrooms. Interacting with robots can create a safe and engaging learning environment that reduces the effects of the didactic contract. Nevertheless, achieving meaningful mathematical learning requires careful orchestration by the teacher, with clear and explicit learning goals to avoid distraction or off-task engagement. Programming robots, along with 3D design and 3D printing, offers students hands-on, concrete experiences that support mathematics learning in multiple ways. These activities serve as springboards to more abstract mathematical thinking; they encourage argumentation and explanation; they foster creativity, freedom, and flexibility; and they open up possibilities for interdisciplinary STEAM approaches that connect subject matter, real-world objects, and aesthetic dimensions.

Feedback and interaction: Five papers and one poster addressed this theme. The central focus across these contributions was the role of feedback and interactive formats in enhancing mathematical learning within digital environments. The presented studies highlight the complex interplay between feedback type, cognitive demands, learner autonomy, and emotional engagement. This intersection creates opportunities to design feedback mechanisms that support metacognition and meaningful task engagement. However, to fully realize this potential, feedback must be carefully aligned with students' prior knowledge, cognitive readiness, and motivational needs. Such alignment may help reduce overload, sustain attention, and foster independent reasoning. Across the contributions, different feedback types, such as correctness, attention, and explanation, were explored, showing that while correctness feedback can increase engagement, explanation feedback is often skipped, and attention cues may offer a balanced alternative. Research on interactive video formats revealed that embedded feedback is frequently ignored and that solutions are often reached through trial-and-error, suggesting that interactivity alone does not ensure effective learning. Mixed Reality environments offered less formalized feedback but supported spatial reasoning through embodied interaction. Collectively, these studies emphasize that effective feedback and interaction require intentional design and orchestration to support both cognitive and emotional aspects of learning. They also

suggest the value of expanding feedback beyond textual forms to include multimodal and embodied dimensions within classroom contexts.

AI and Eye Tracking: Four papers and one poster addressed this theme. The central focus across these contributions was the potential of AI and eye-tracking technologies to support mathematical learning and inquiry. The presented studies highlight how Gen-AI tools can promote productive engagement with mathematical tasks by shifting student behavior away from seeking quick answers and toward deeper reasoning and problem-solving. This intersection creates new possibilities for developing core mathematical skills and encouraging the use of multiple representations. However, to fully realize this potential, educational guidance is essential, as students often default to using Gen-AI as a basic search tool. Such guidance may help students use AI critically and reflectively, rather than passively consuming responses. In parallel, the contributions addressed how AI-based analysis of eye-tracking data can provide insight into students' attention patterns and conceptual difficulties. While promising, this approach requires triangulation with qualitative or physiological data, as current AI interpretations of gaze data are still limited in scope. Results derived from such analyses should inform instructional practice, for example, by helping redirect students' attention to key mathematical structures. Collectively, these studies underscore the need for further research on how to integrate AI and eye-tracking meaningfully, and how to design supportive environments that cultivate critical thinking and mathematical understanding through these emerging technologies.

Technology and mathematical problem-solving: Four papers and one poster addressed this theme. The central focus across these contributions was the design and impact of digital tools on students' mathematical thinking and problem-solving. The presented studies highlight how thoughtfully structured digital environments can support metacognition, engagement, and flexible problem-solving strategies. This intersection creates opportunities to enhance learning through intuitive interfaces, contextualized tasks, and interactive representations, while reinforcing the ongoing importance of teacher support. However, to fully realize this potential, digital tools must be designed to balance student autonomy with instructional guidance. Such balance may help students engage meaningfully with mathematical challenges while developing strategic approaches to learning. Across the contributions, a variety of tools and environments were examined. Augmented reality applications supported embodied exploration in geometry, enabling students to detect and correct conceptual errors. Eye-tracking studies revealed that non-routine problems encouraged deeper, exploratory strategies, particularly among higher-achieving students, while routine tasks led to more superficial engagement. Structured hint systems, such as heuristic trees, were found to promote ownership and desirable help-seeking behaviors. Short interventions using tools like spreadsheets, GeoGebra, and Scratch improved students' problem-solving abilities, though their capacity to articulate justifications remained limited. Collectively, these findings suggest that digital tools, when carefully designed and integrated, can play a powerful role in fostering problem-solving and reasoning, particularly when combined with explicit attention to students' explanatory practices.

Tools to promote mathematical reasoning: Four papers and two posters addressed this theme. The central focus across these contributions was the use of digital and non-digital tools to promote various forms of mathematical reasoning. The presented studies highlight how interactive resources, ranging from digital textbooks and dynamic geometry environments to physical models and collaborative

platforms, can support students in constructing arguments, identifying patterns, and developing conceptual understanding. The intersection of the contributions in this theme creates opportunities for integrating multiple representations into learning environments, allowing learners to transition between visual, symbolic, and verbal modes of reasoning. However, to fully realize this potential, tool design must attend to both the mathematical goals and the orchestration required to guide students through increasingly abstract forms of reasoning. Such orchestration may help learners connect concrete experiences with formal mathematical structures. Contributions involving interactive textbooks demonstrated how multimodal elements (e.g., handwriting, audio, and applets) can support generalization and proof construction. Other studies explored dynamic geometry tools that help students navigate between Euclidean and alternative geometries through covariational reasoning. Collaborative settings were shown to foster participation and reflection, with teacher guidance playing a critical role. Collectively, these studies underscore the importance of designing tool-rich environments that promote reasoning, while highlighting the complementary roles of technology, collaboration, and teacher mediation in shaping mathematical understanding.

Conclusion and outlook

The discussion in the TWG 16 aimed, first, to establish an overview of the current state of the art in technology-rich mathematics education. Several cross-cutting insights emerged from the thematic contributions. First, technology-mediated learning environments are increasingly being seen not just as transformative platforms but as catalysts for developing essential mathematical competencies such as abstraction, argumentation, and problem-solving. Interdisciplinary approaches that integrate programming, robotics, or 3D modeling with mathematical inquiry demonstrate the potential for meaningful learning when tasks are accompanied by reflection and pedagogical guidance. Additionally, the nuanced roles of feedback and interaction, especially in terms of aligning with learners' cognitive and emotional needs, underscore the importance of careful instructional design. While AI technologies hold promise for fostering critical thinking and engaging students through multiple representations, their integration must be structured and supported with effective scaffolding to avoid superficial usage.

The second aim of the discussion was to outline future directions and research agendas for technology-rich mathematics education. Looking ahead, several key areas emerge. These include designing learning environments that explicitly connect computational and mathematical thinking; developing adaptive feedback systems that respond to learners' individual profiles; integrating AI technologies in pedagogically meaningful ways; and supporting tasks that promote conceptual understanding through both digital and physical tools. A critical area for further research is examining how digital and non-digital tools work together in real classroom settings, where learning occurs through dynamic interactions between students, teachers, and resources. Future TWG 16 may focus on promising themes such as hybrid learning environments, students learning in multi-tool environments, and the role of embodied experiences in supporting abstraction. These directions encourage further theoretical development and empirical investigation into how technologies, along with their integration into broader pedagogical practices, can effectively support mathematics learning.

The icosahedron in the room: Revisiting pedagogical dilemmas of individual learning in collaborative activities

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Four adult study participants successfully built together a body-scale polyhedron with minimal instructions. Preliminary observation of their rapid construction process would suggest that they were generally "on the same page" and, therefore, perhaps all entertained the same geometrical ideas. However, deeper analysis of their multimodal behaviours and utterances, both during the activity and in post-intervention stimulated-recall individual interviews, suggests acute divergence in perceptual Gestalts, even when two or more participants referred in speech and gesture ostensibly to the precise same material element, voicing operational consensus. As such, our findings query cognitive–anthropological theory of organised social activity, e.g., Goodwin's Co-Operative Action, by producing empirical evidence of productive joint action on indexically referenced co-attended features of the environment yet in the absence of conceptual cohesion.

Keywords: Collaboration, co-operative action, geometry, groupworthy tasks, joint action.

> Let us imagine a language ...The language is meant to serve for communication between a builder A and an assistant B. A is building with building-stones; there are blocks, pillars, slabs and beams. B has to pass the stones, and that in the order in which A needs them. For this purpose they use a language consisting of the words 'block', 'pillar', 'slab', 'beam'. A calls them out; — B brings the stone which he has learnt to bring at such-and-such a call. — Conceive this as a complete primitive language. (Wittgenstein, 1953, §2)

Design problem: learning despite collaboration

It is not simple to facilitate equitable collaborative learning. When a group of students works on a challenging task, they may not enjoy equal opportunities to engage with the task's core cognitive challenges, as evidenced in process analyses and individual assessments. Instead, group members may spontaneously distribute the labour so as to maximise the group's productivity and ultimate success. Consequently, members best pre-equipped to tackle conceptual problems lead others down the production line, who design, manufacture, and assemble artifacts, thus engendering what Abrahamson and Wilensky (2005a) diagnose as a *stratified learning zone*. As such, group activities may not always hoist the lower-achieving students as hoped. Worse, teachers are liable to both congratulate students on their successful achievement as a group and critique struggling group

members for their unequal achievement on individual assessments. Something is not working in group work.

Well aware of this pedagogical tension between the quality of collaboration versus individual learning, Cohen (1986) developed the *complex design* framework, which included: (1) a set of heuristics for creating *groupworthy* tasks—tasks that authentically require a number of contributing participants, each with a distinct role, who must consistently coordinate their actions; along with (2) facilitation guidelines, such as rotation schedules, to ensure that each participant fills each of the roles. Approaching the same design tension from a constructionist perspective, Abrahamson and Wilensky (2005b) explored digitally enabled groupworthy tasks that create equitable participation, without prescribing rotation regimens, by ensuring that all students were operating similarly on the same types of objects (see also White & Pea, 2011).

Still, when students engaged in a shared task are handling the same objects together, are they necessarily perceiving and understanding these objects in one and the same way, such that, by and large, they are learning the same notions?

Sharing perception through the social enactment of cultural practice

According to the Theory of Co-Operative Action (Goodwin, 2013), human activity is intensely co-operative, emerging from interpersonally elaborative actions. New forms of co-operative action by a social group of common interest rely on their developing a *substrate*—a cumulative amalgam of iteratively laminated heterogeneous semiotic contributions respecting a shared domain of scrutiny. The substrate is the group's ever-growing resource of endorsed notions, routines, and perspectives for fluently managing their practice. Drawing on the substrate, co-operative action is coordinated through: (1) participants physically orientating their bodies towards each other and their shared domain of scrutiny and manipulation, creating a focus of perceptual attention and a sense of collective endeavor; (2) a spatial arena that includes phenomena of shared interest; (3) production and regulation of consensual linguistic tokens organising mutual interactions; and (4) gestures—deictic, iconic, and emblematic—for indexing features of the shared environment and imbuing them with contextual meanings.

Moreover, theories of joint action often underscore the vital function of joint intentionality, attention, and frames of reference, as well as semiotic interaction, in facilitating participants' coordination of effective simultaneous actions (Marschner et al., 2024). These theories of co-operative and joint action could be read as suggesting that when several students engaged in a groupworthy task appear to be doing literally the same manual actions, looking at the same objects, and even discussing these same objects, we could quite safely assume that they are developing similar conceptual understandings. But is that necessarily true? Our motto quotation from Wittgenstein's Philosophical Investigations suggests the possibility of joint action absent of shared meanings. We now look at resonant voices from systemic paradigms.

Critiques of joint-action and joint-attention theory

Alternative theorisations of coordinated action are rooted in scientific paradigms such as cybernetics, as well as phenomenological philosophy that promote a different fundamental sensibility of order, one that can embrace the complexity of dynamical systems in flux (Kelso, 2021). Embodiment, a rising exemplar of such post-Cartesian perspectives, has been evaluated as bearing dramatic implications for the learning sciences (Hutto et al., 2015; Macrine & Fugate, 2022; Nathan, 2021; Shapiro & Stolz, 2019). We now overview some recent cognitive science work in this vein that queries traditional models of joint action.

Rejecting reductive theories of collective intentionality, philosopher Seemann (2024) develops an *enacted theory of joint attention*, where, "The openness of joint agents' intentions and meshing subplans is explained by appeal to their practical knowledge of how to maintain the process by way of which they pursue the collective intention" (p. 259). The would-be golden link between joint attention and joint action is further interrogated by questioning the inter-cultural universality of previous assertions that were based on empirical work within WEIRD societies (i.e., Western, educated, industrialised, rich, and democratic; see Henrich et al., 2010). Investigating parent–child naturalistic interactions in non-industrialized societies, Taverna et al. (2024) demonstrated joint action in the absence of joint attention, thus posing questions for ontogenetic theories that foreground joint attention as a critical developmental marker (Tomasello, 2019). In a resonant view from the philosophy of sports, Birse (2024) rejects representationalist assumptions that what is shared in joint attention is mental content. Instead, he sides with relationalist interpretations that look to include individuals' aspectual perceptual perspectives as well as the actions that perception inherently enables (see also Story, 2021, on the emergence or joint action without a theory of mind). Finally, Loehr (2022) problematizes what she views as facile models of individuals' sense of agency as mediating joint action.

What might all this mean for pedagogical frameworks and, more generally, anthropological theories of sociogenesis and semiosis? We wish to leverage these alternative theorisations of joint action in the study of individual learning in groupworthy tasks. As such, we ask:

1. Can individuals engaged in the successful collective enactment of a group task entertain different perceptions of the objects they are jointly handling?
2. How would productive collective enactment be possible when individuals perceive differently the key objects they are jointly handling?
3. What causes such hetero-perception? What are its educational implications?

Methods

Our empirical context is a groupworthy task, in which 4 adults, several of whom had worked as math teachers, are tasked to construct and analyse a body-scale polyhedron. Whereas we had used this activity extensively in researching geometry education (Benally et al., 2022; Palatnik & Abrahamson, 2022, 2025; Rosenbaum et al., 2024), we had yet to closely examine individual

experiences. An auspicious opportunity emerged during our workshop on learning analytics (EDaMMLA 2024), where a team built a large icosahedron (see illustration in Figure 1, left) from long wood dowels and silicon-pipe joiners, with no instructions for construction.

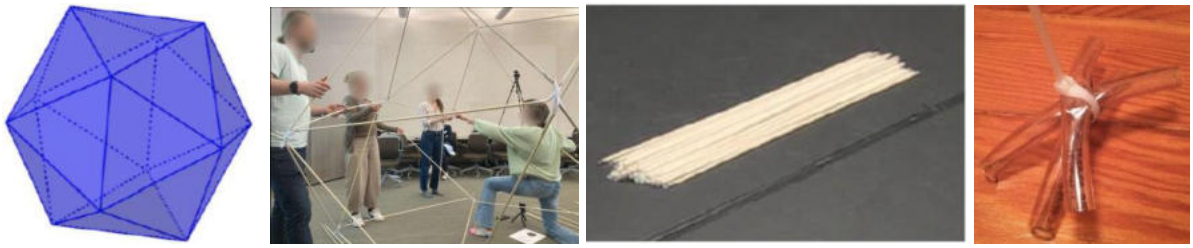


Figure 1: Building a body-scale icosahedron from wood dowels and joiner links

The session was audio–video recorded. Our data further comprise post-intervention individual stimulated-recall interviews, where participants (hence, e.g., P3) discussed their experience of collaboratively constructing the form. We analysed mutual physicality, language, gestures, and the surrounding phenomena. The video was coded by marking key moments in the construction process to capture elements of co-operative action. Utterances were transcribed and coded, marking aspects of perception and action. We later juxtaposed key video moments in the construction process with relevant testimonials from stimulated-recall interviews (hence, SRI).

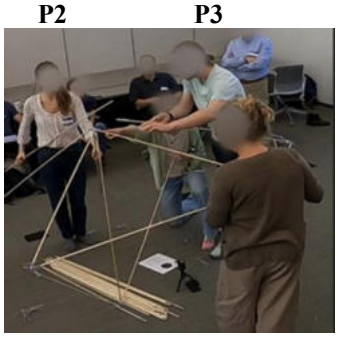
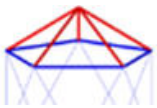
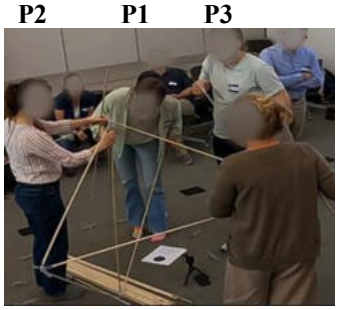
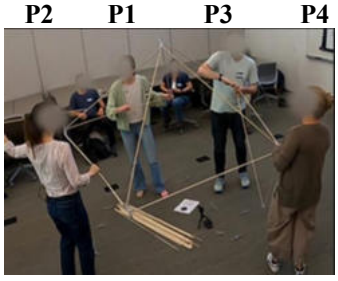
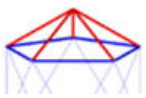
Results

To address our research questions, we examine 3 vignettes (see Table 1) that triangulate the activity's transcribed videography with the post-intervention individual SRIs.

- Vignette 1 shows four participants engaged in a groupworthy task, performing the same manual actions, looking at and discussing the same objects, while the SRIs of P1 and P2 indicate that they are not developing similar conceptual understandings, implying the possibility of *joint action in the absence of shared meanings*.
- Vignette 2 suggests that what is shared in joint attention is not mental content, as P1 and P2 evidence hetero-perceptual perspectives and actions that nevertheless support *joint action without a shared theory of mind*.
- Vignette 3 illustrates that individuals can engage in successful collective enactment of a group task while entertaining different perceptions of the objects they are jointly handling, such that they manifest and *learn different notions*.

In building the icosahedron, P1 and P2 followed different conceptual perceptions: P1 counted the five edges in each vertex, while P2 tried to build "the roof." Meanwhile, P3 deliberated over these two strategies—he would later admit that the activity's hardest aspect was "knowing who[m] to listen to in such moments of confusion." P4 followed on, attentive to the various framings.

Table 1. Collective enactment through hetero-perceptions in joint action

The vignettes (60 seconds)	The transcript (selected utterances)	P1: SRI	P2: SRI
Vignette 1 (98 s to 111 s) **P2** **P3**  **P1** **P4**	**P3 [leaning in, centre right]:** Ok, I'm making… [attempts to connect a sixth edge to a vertex] **P1 [kneeling, centre]:** But there is only…, is it…, should be… **P2:** But there's, yeah… **P1:** I think we're… is there five [points to a vertex]? I think there should be five vertices or five edges meeting. **P2 [on left]:** But then this one is extra, yeah. **P1:** Yeah…. **P2:** …we need to take this off [holds the edge going down in the middle of P1's vertex]	I don't know… I think it's just like "I was just doing the 5s patrol." I was just like "I have to say something now because this is a thing that…". Maybe I should have vocalized it earlier: "Oh, there's five," but I think in my head I was like, "Oh, there's five!" 	Now I can see that this is not how I imagined "the roof"—the roof is not working well. (...) Everyone is working on a different side, so I can see that it is not symmetrical. 
Vignette 2 (112 s to 121 s) **P2** **P1** **P3** 	**P2 [on left]:** [to herself] This extra one in the middle… [Instructor intervenes] (...) **P2:** Yeah… **P1 [leaning in, centre]:** [to herself] Oh, there should be... [looks carefully at what P2, on left, is doing]	(...) That was the first time that I thought, "We don't need to add any more of these, because there are five." (....) It seems like that was the first time that I actually felt like I had to say what I was thinking for the past two minutes.	(...) It came from "the roof" idea, because there was something in the middle, [an edge], that's why I thought that, "Okay, it's not the roof yet."
Vignette 3 (122 s to 160 s) **P2** **P1** **P3** **P4** 	**P2 [on left]:** (…) [removes an edge from one side of the "top" vertex," joins it into opposite side, rendering the top vertex visible] Yeah, this is the now "top" [raising the vertex]. **P1**: Oh, cool. **P2:** Yeah, we can think like this and then… **P1:** Yeah. **P3 [right]**: Yeah. **P2**: …we go from here. **P3**: Alright **P1**: Ok **P4**: [silent, looks at "top"]	I don't think I noticed this ["the roof"] in the activity itself, but watching it now, I see this shape of the five triangles at the very top. It is interesting that we didn't even finish that shape first—we started going into the next... 	Now, the idea of "the roof" is becoming more and more urgent for me, so now I'm…yeah…ye ah…" 

Our four "Rashomon" participants experienced different learning opportunities, pursuing self-assigned purviews ("5s patrol" versus "the roof") or following the group's actions. P1, who "just kept an eye on the five," maintained her role as "5 patrol"; she did not perceive the whole polyhedron "until the very end, when it all came together." P2, per the SRI, drew on prior knowledge that the shape comprises congruent triangles to form her "roof." P3 grasped the form as a whole only towards the end of the activity, "when the shape was already correct and there was very little missing." Even though P4 did not always understand the task at hand and was comfortable following the group, per her report, the "5s rule" prompted her sense-making. She stated, "OK, if I have to repeat it, I know how to do it." In sum, the vignettes described task participants who, while engaged in task-effective co-operative action, nevertheless: bear different *perceptual orientations* on the situation; highlight *different properties* of the environment as action affordances; entertain different *construction objectives*; and consider different *task constraints* which they announce, monitor, and enforce. We thus witness a set of individuals who, though inhabiting different environments (Shvarts & Abrahamson, 2023), nevertheless successfully perform an apparently well-coordinated collective task.

Discussion

Regarding RQ1, we have told the tale of four persons who, though successful in building together a complex geometric structure, each perceived it differently (Borovska et al., 2024) and, later, retained different conceptual notions about its geometry (Zamecnik et al., 2024). This tale demonstrates that participants in collaborative activities may not enjoy equal opportunities to engage with the task's core cognitive challenge, highlighting the pedagogical tension between the quality of collaboration and individual learning (Abrahamson & Wilensky, 2005a).

Regarding RQ2—the issue of how productive collective enactment is possible given a group's hetero-perceptions—our findings pose new questions to cognitive–anthropological theory of organised social activity, such as Goodwin's Co-Operative Action. Namely, we have produced empirical evidence of productive joint action on indexically referenced, co-attended features of the environment yet despite the absence of conceptual cohesion: When task members make heterogeneous semiotic contributions that are apparently endorsed by others, what may actually be endorsed are practical operations on designated *local* features, while colleagues' idiosyncratic *global* perceptual constructions of these features are filtered out in communication. Co-operation does not imply, and may not require, co-perception. When teachers conduct group activities in their classrooms, they should remain vigilant that effective coordination may nevertheless conceal hetero-perception and divergent understandings (Abdu & Schwarz, 2020). These divergent perspectives could be gainfully leveraged through facilitated post-activity reflection.

Regarding RQ3, we thus rekindle long-standing research concerns over the pedagogical optimisation of groupworthy assignments. Efficacious joint action may result not from explicitly negotiated engineering, distribution, and monitoring of individual roles, which would occasion pedagogical opportunities for engaging in disciplinary discourse around a lesson's target content. Instead, it may result from the emergence and stabilisation of multiple, loosely recurring, ad hoc

local transactions among spatially proximate participants, each with their self-assigned purviews, who temporarily and unwittingly become functionally co-dependent, thus requiring only brief, cursory coordination.

Developing interpersonally heterogeneous perceptions of shared objects may be an adaptive evolutionary social heuristic for collaborative problem solving in novel contexts under conditions of uncertainty, namely, disaggregating exploration so as to maximise discovery opportunities (Richardson et al., 2008). Educational designers, however, ultimately are charged with converging students towards more-or-less homogeneous understanding of the subject content. We believe that educators should keep deliberating how to utilize humans' atavistic genetic inclinations for hetero-perception—leveraging it, rather than inhibiting it—nevertheless while also ensuring equitable instruction. As such, we propose, if indeed we are to foster the skill of collaboration, we should appreciate, understand, embrace, and support humans' soft-assembly routines, including tinkering, divergence, emergence, and loose coordination, rather than impose regimens for the arbitrary distribution of labour that are liable to suppress students' powerful evolutionary proclivities. At the same time, we should surface the phenomenon of hetero-perception for the students themselves, so that they can learn through coordinating their divergent visualisations. Further investigations of equitable learning in collaborative activities are called for, because our speculations are inferred from a single group of adult participants working in a unique social context (an international workshop) on a rather irregular task, using unusual media.

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Students' use of feedback when working on fractions with a digital learning platform and its effects on mathematical performance

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The present study analyses the use of feedback – a factor that does not necessarily go along with higher performance – when working with fractions via a digital learning platform, and its impact on students' performance. In this context, in 2021/2022 a laboratory study was conducted with 137 seventh graders from German schools. The students were randomly assigned to one out of two experimental groups. Within the groups, all students worked on fraction tasks by using a digital learning platform. The groups differed in the complexity of the feedback (right/wrong feedback vs. elaborated feedback) presented by the learning platform. Results highlight a low utilisation of right/wrong feedback and a very heterogeneous use of elaborated feedback, with increased use being associated with lower fractions performance. This can be partly explained by the students' prior knowledge. Educational stakeholders should therefore consider how feedback should be designed and implemented in the classroom so that it is (1) used by learners and (2) supports their learning.

Keywords: Feedback, Intelligent Tutoring Systems, Fractions

Introduction

With the introduction of fractions, mathematics becomes a major challenge for many learners (Obersteiner & Tumpek, 2015). Feedback plays a crucial role in helping learners meet this challenge, as feedback has been highlighted being one of the most important factors in successful learning (Wisniewski et al., 2020). Studies have shown that elaborated feedback is superior to right/wrong feedback, even though a lack of elaborated feedback in teaching-learning processes has been identified (e.g. Harks et al., 2013). To address this issue, digital learning platforms offer potential for individual learner support through automated feedback. However, the results of studies conducted on the effectiveness of learning platforms have been inconsistent (Hillmayr et al., 2020). Given this heterogeneity in research findings, it is important to identify the factors that influence the effectiveness of digital learning platforms in general and feedback on digital learning platforms in particular. Current research results, for example, raise the question of whether and when increased use of feedback actually leads to higher performance or whether the effect of feedback use depends, for example, on the prior knowledge of the learners (e.g. Reinhold et al., 2021). Due to the research gap on when feedback is actually conducive to learning, this study will address the questions of whether and to what extent learners use two different types of feedback and to what extent the use of feedback has an influence on mathematical performance.

Theoretical Background

The topic of fractions is considered being a widely researched field (Siegler et al., 2013). In practical teaching, however, the topic is identified as a challenge for both teachers and learners (Obersteiner

& Tumpek, 2015). At the same time, fractions appear to be of particular relevance for other mathematical topics as well as for future live (Lortie-Forgues et al., 2015). One effective approach to addressing challenges in complex mathematical domains, such as fractions, is to provide students with feedback. The primary objective of feedback is to modify the recipient's thinking and/or behaviour, typically with the aim of enhancing performance by closing the gap between the current state of learning and the learning goal (Narciss, 2004; Shute, 2008). Feedback is considered being one of the most important factors influencing learning success. Various meta-analyses have demonstrated the superiority of feedback over no feedback. However, the study results vary widely (e.g. Hattie & Timperley, 2007; Mertens et al., 2022). Consequently, empirical research does not seek to determine whether feedback *can* be effective, but rather *how* feedback should be designed to support students in their learning. One factor that contributes to the variability in the effectiveness of feedback is the variety of ways in which feedback can be designed (Mertens et al., 2022). In essence, two types of feedback can be distinguished: simple feedback, such as information about whether an answer is correct or incorrect (= right/wrong feedback), and elaborated feedback. Elaborated feedback typically consists of an explanation indicating whether the response is correct or incorrect, accompanied by supplementary information. The information provided can be diverse (van der Kleij et al., 2015), with the degree of complexity of elaborated feedback varying considerably across studies. For instance, elaborated feedback may comprise illustrative examples of solutions or concrete help. Elaborate feedback seems to be superior to right/wrong feedback, although too much information can be overwhelming for learners (keyword: cognitive load) (Sweller et al., 2011; Shute, 2008).

Given the multifaceted nature of teaching roles, it can be postulated that the provision of elaborated written feedback and tailored support for students' learning is not a common practice. Digital learning platforms offer considerable potential in addressing the dearth of written feedback that is currently observed. As Petko (2020) notes, digital learning platforms are capable of generating reactions to learner input. Consequently, digital learning platforms are designed to store, process and record information in a manner that enables learners to interact with them both within and beyond the confines of the classroom. Digital learning platforms provide users with feedback on their input (Daniela & Rudolfa, 2019), thereby facilitating interactivity between the learner and the learning platform (Petko, 2020). The feedback from digital learning platforms can be characterised by a considerable degree of complexity. This can be attributed to the fact that, despite the prevailing view that learning platforms are conducive to learning, empirical studies have yielded disparate results with regard to their effectiveness (e.g. Mertens et al., 2022). However, the complexity of the feedback is not the only decisive factor for its effectiveness. Another factor is how learners deal with the feedback. This is because the desired target state can only be achieved when learners actively use feedback (van der Kleij & Lipnevich, 2020). Empirical evidence suggests a correlation between the use of feedback and (mathematical) performance. However, increased feedback use does not necessarily lead to higher performance (e.g. Reinhold et al., 2021). Operationalising the use of feedback is not easy. A variety of approaches have been identified in the literature for capturing the use of feedback. For instance, van der Kleij et al. (2011) refer to the duration of a window containing feedback in order to differentiate the use of feedback by learners. In contrast, Beal et al. (2010)

operationalise the use of feedback via the number of aids that learners look at. In a similar vein, Reinhold et al. (2021) employ process data to record the number of hints displayed and the feedback time. To gain further insight into (1) learners' use of feedback and (2) the effect of feedback use on performance, the study addresses the following research questions: (1) (To what extent) Do learners use two different feedback types (right/wrong vs. elaborated feedback) from digital learning platforms? (2) (To what extent) Does the use of two different feedback types (right/wrong vs. elaborated feedback) impact upon learners' fraction performance?

Methods

In 2021/2022, an experimental laboratory study was conducted with $N = 137$ seventh graders from different German middle-class schools. Before starting the intervention, at a first measurement point general background variables and students' prior knowledge of fractions (using a standardised test by Lenz et al. (2019); 37 items; dichotomously coded) had been assessed. The students were then randomly assigned to one out of two experimental groups. During the intervention, both experimental groups worked on fractions for around 30 minutes using a digital learning platform. While experimental group 1 (EG 1; n = 67) received feedback on correct and incorrect answers at the end of all tasks, experimental group 2 (EG 2; n = 70) received elaborated feedback continuously while working on the tasks. The elaborated feedback included feedback on the correctness of the answer as well as concrete help: show tip (e.g. "Expand to the main denominator."), look up ("usually consisting of a similar type of task including a sample solution"), show solution, definition of mathematical terms) for processing the task (see Figure 1). Following the intervention, at a second measurement point the students were required to complete the same test that was used to assess prior knowledge at the first measurement point in order to assess their performance in relation to fractions after the intervention.

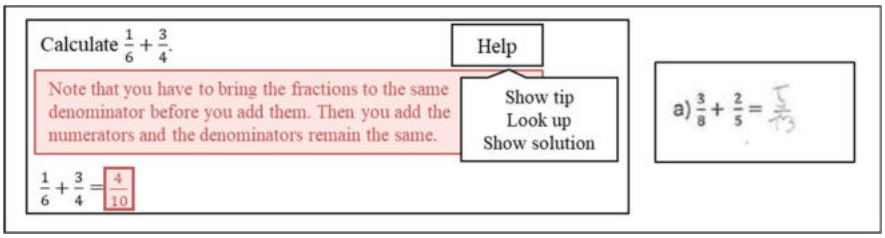

Figure 1: Exemplary fraction calculation task from the intervention including elaborated feedback (left) (own illustration based on bettermarks (2023)) comprising an exemplary task from the fraction calculation test including student solution (right)

The operationalisation of feedback use for EG 1 was based on the decision to categorise them as either having worked on the incorrectly solved tasks a second time after receiving feedback on correctly and incorrectly solved tasks (coded as 1), or as having not done so (coded as 0). The use of feedback for EG 2 was documented throughout the intervention using screen recordings. Subsequently, the use of feedback was operationalised based on the duration (time on feedback) and the number of times help has been viewed (frequency of feedback usage). The feedback usage behaviour is initially described descriptively. Unpaired t-tests examine whether different feedback types have an influence on fraction performance. Subsequently, regression models with moderation

analyses were carried out to investigate the relationship between the use of feedback, prior knowledge of fractions and fraction performance at the end of the intervention. The assumptions for carrying out the tests have been checked and are considered to be met. The analyses were carried out using IBM SPSS Statistics Version 28.0 (IBM, 2022).

Results

Research question 1: Of the 70 learners in EG 1, only 19 (27.1%) attempted the incorrectly completed tasks a second time. 50 learners (71.4%) did not use a second attempt and therefore did not use the feedback provided. In the case of one student, it was not possible to determine whether a second attempt was made. For EG 2 the high standard deviations indicate a considerable degree of variability in the use of feedback. Considering the variance observed, it is notable that the empirical minimum for all usage indicators is zero (see Table 1). A more detailed examination of the data indicates that six students did not utilize the feedback provided. One student used the feedback 60 times.

Table 1: Descriptive results of feedback usage of EG 2

	Min	Max	M	SD
Time on feedback (in sec)	0	344	73.90	90.29
Frequency of feedback usage	0	60	12.91	13.81

Research question 2: A comparison of the two experimental groups reveals that EG 2 exhibits superior performance. However, as Table 2 illustrates, this effect is not statistically significant.

Table 2: Descriptive results of fraction performance and mean differences by experimental group

	n	Min	Max	MW	SD	
Fraction performance	137	0	36	13.42	8.96	
EG 1	70	1	33	13.34	9.11	t(135) = -0.10, p = .46, d = - 0.02
EG 2	67	0	36	13.49	8.87	
Maximum score = 37; d was calculated with: Hemmerich (2015) (two-sided test)						

Table 3 shows models for the influence of feedback use on fraction knowledge (M1 & M1.1) and this influence when controlled by prior knowledge of fractions (M2 & M2.1). Models 3 and 3.1 examine whether prior knowledge has a moderating effect. The results regarding the influence of feedback use on fraction performance in the context of EG 2 show that the decisive factor is the frequency of feedback, rather than the time on feedback. The results demonstrate that 4% of the variance in fraction performance can be attributed to the frequency of feedback usage. It is notable that the significant

regression coefficients are negative, indicating that learners tend to perform less well on the fractions test as the frequency of feedback usage increases. However, the significant effects of feedback on fraction performance are no longer apparent when prior knowledge of fractions is taken into account.

Table 3: Results of the regression models with interaction effects for the prediction of the fraction knowledge

	Fraction knowledge					
	M1	M1.1	M2	M2.2	M3	M3.1
Time on feedback	-.90		-.31		.47	
Frequency of feedback usage		-2.22*		-1.50		.01
Prior knowledge			2.43	1.99	3.33**	3.48*
Prior knowledge*Time on feedback					2.76	
Prior knowledge* Frequency of feedback usage						3.43**
Significance	.43	< .05	.07	< .05	< .05	< .01
R^2 corr	-.01	.04	.05	.07	.09	.12
Shown are the unstandardized (B) regression coefficients; R^2 corr = corrected explained variance; * p < .05, ** < .01, the analyses were carried out with z-standardized values						

The Johnson-Neymann diagram demonstrates that learners with limited prior knowledge of fractions demonstrate a decline in fraction performance with increasing feedback usage (see Figure 2). However, as prior knowledge increases, the significant effect disappears and prior knowledge of fractions has no moderating effect. This indicates that, in instances of elevated prior knowledge, the use of feedback is less likely to exert an influence on subsequent fraction performance following the implementation of an intervention.

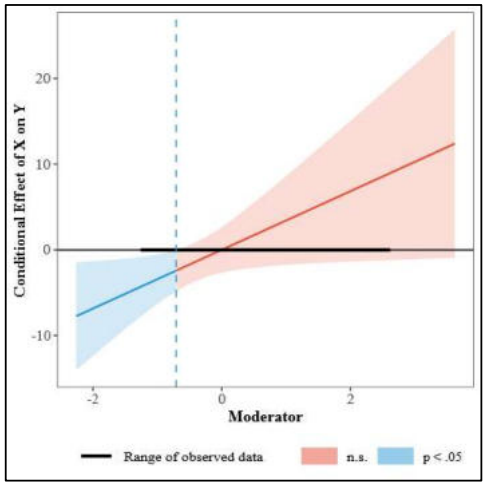

Figure 2: Moderating effect of prior knowledge (= moderator) on the relationship between frequency of feedback usage (= X) and fraction performance (= Y), Note: If the moderator is outside the interval [-0.71, 8.19], the conditional effect of X on Y is significant, p < .05, Hemmerich (2020)

Discussion

The results of the present study highlight that right/wrong feedback (EG 1) provided by a digital learning platform is rarely used by learners as an opportunity to revise incorrectly solved tasks. The low use of feedback may be explained by the delayed presentation of feedback (Winstone et al., 2017). However, a lack of information, e.g. on how to proceed, may also be the reason for the low use of feedback (Fong & Schallert, 2023). The use of elaborated feedback seems to be quite heterogeneous (EG 2). It can be assumed that too much information is offered, which not all learners (want to and/or can) use, as the large amount can be overwhelming (van der Kleij & Lipnevich, 2020). The analyses to answer the question of whether the use of feedback affects fraction performance showed no effect of right/wrong feedback. For EG 2, the more feedback is used, the lower the fraction performance. In relation to this result, it was assumed that the negative influence of the use of feedback on fraction performance is mainly explained by the prior knowledge. Graphical analyses confirm that learners with little prior knowledge of fractions have poorer fraction performance with increasing duration of use. The fact that learners with less prior knowledge do worse as feedback use increases seems to contradict previous research discourse at first (Fyfe & Rittle-Johnson, 2016). However, 'cognitive load' is a recurring theme in the context of feedback (e.g. Sweller et al., 2011). Learners with less prior knowledge may feel overwhelmed by elaborate feedback, and more frequent use may require more cognitive resources to process feedback, making it more difficult to focus on the task at hand. Therefore, teachers should be aware that learners with less prior knowledge in particular need additional support that goes beyond the functions of a digital learning platform. The task of feedback designers is to consider the different levels of prior knowledge of the users. Although the topic of 'feedback' has been extensively researched, the authors of this article agree with the call for further research into when feedback is conducive to learning. The focus should be on students and their individual needs. For example, qualitative (interview) studies could contribute to a better understanding of the use of feedback in general and of problems in implementing feedback information in particular.

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Supporting cognitive engagement in working with interactive video: Potentials and limitations of video interaction and feedback

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This design research study investigates students' processes in an interactive and conceptually oriented video on understanding variables. Data has been collected in design experiments with eight pairs of adolescents. In the drag-and-drop question in focus of this paper, they receive feedback and can answer multiple times until correct. The qualitative analysis investigates their different attempts regarding cognitive engagement and use of feedback. It reveals that video interactions can support cognitive engagement but also bears a risk of eliciting superficial processes. While feedback on the result seems to trigger several attempts, sometimes by trial and error, elaborated feedback was hardly read in our sample. This shows the importance of a research-based design of interactive videos in general and automated feedback in particular.

Keywords: Interactive video, cognitive engagement, student processes.

Introduction

Providing learning opportunities with instructional videos opens up potentials for learning at one's own pace and for illustrating dynamic processes, which has proven effective for the acquisition of different types of knowledge, including conceptual knowledge (Kulgemeyer, 2020). A key condition of success, however, is the depth of students' cognitive engagement (Chi & Wylie, 2014). In general, interactive elements (such as drag-and-drop questions with automated feedback) have been shown to support students' cognitive engagement (Moreno & Mayer, 2007). In our overarching project MuM-Video, we explore to what extent we can transfer these interactive supports to videos offering learning opportunities for deep conceptual understanding. In the current paper, we report from a qualitative analysis of process data from Cycle 2 of a design research study with a focus on potentials and limitations of drag-and-drop tasks and automated feedback for the particularly challenging case of adolescents who receive a second chance for learning algebra, with the following research questions:

(RQ1) How do students cognitively engage in their work on a drag-and-drop task?
(RQ2) How is the engagement supported by their use of the feedback provided by the video?

Theoretical background: Cognitive engagement and feedback

Cognitive demands that engage students in deep thinking processes are considered to be a key condition for successful mathematics learning. That is why cognitively demanding tasks and teachers' moves have intensively been researched in mathematics education (Henningsen & Stein, 1997). For our target group – adolescents in need of a second chance of learning an especially challenging content (understanding variables) – effective strategies to foster engagement are of particular relevance for two reasons: First, the association between cognitive activation and achievement is

stronger in classes with low average socio-economic status (Li et al., 2021) and second, transition points in school come along with the risk of a decline of engagement (Skilling et al., 2021). Yet, one cannot draw conclusions from the cognitive demand of a task to each student's cognitive engagement, so new instructional designs (such as for videos) must be investigated with respect to the cognitive engagement they actually initiate and support (Chi & Wylie, 2014).

The depth of students' cognitive engagement can be operationalized within the ICAP framework using students' overt behavior as an indicator for cognitive processes (Chi & Wylie, 2014). ICAP distinguishes four modes of engagement, from *interactive* (generating information and connecting ideas through dialogue) over *constructive* (generating information and connecting ideas) and *active* (manipulating information or materials) to *passive* (receiving information), or even less than passive, *disengaged* (if students are either off-topic or if they try to somehow get to the solution without actually solving the problem). Empirical studies have shown that higher modes of visible engagement correlate with deeper cognitive processes and higher learning outcomes (Chi et al., 2018), this allows the ICAP framework to connect elaborated knowledge construction with the constructive mode of engagement (Chi & Wylie, 2014). Hence, support for cognitive engagement is to be carefully designed in learning environments.

In fact, learning from video often goes hand in hand with the passive mode of engagement, not least within mathematics education: Students just receive information from the video and are not necessarily encouraged to give reasons or to come up with own considerations (Bersch et al., 2020). Integrating interactive elements, such as drag-and-drop or multiple-choice questions, into videos can initiate deeper engagement by providing time to elaborate, demanding physical action, or delivering automated feedback (Delen et al., 2014). Also, the activity setting of working in pairs can contribute to eliciting constructive and interactive student processes (Chi et al., 2018).

One important advantage of interactive video is its potential to give automated feedback. Feedback is defined "as information provided by an agent … regarding aspects of one's performance or understanding" (Hattie & Timperley, 2007, p. 81) and has been shown to enhance students' learning gains. Kulhavy and Stock (1989) classify feedback with respect to two main aspects: *verification* (students are only informed whether their result is correct) and *elaboration* (students receive information beyond that, either on the task or on metacognition). By pointing to specific aspects of the content in view, elaborated feedback has the potential to support focused cognitive engagement. In a meta-analysis on feedback across subjects and media, elaborated feedback was found to be associated with high effect sizes (Wisniewski et al., 2019). While the meta-analysis revealed differences between learning contents, no differences occurred between target groups. However, we assume that our target group might have more difficulties in exploiting the offered support.

Methods

Research context: Design of the learning environment and interactive video

The conceptually focused video investigated in the overarching design research study (Korntreff & Prediger, 2024; Korntreff et al., 2023) explains the meaning of variables as generalizers in a realistic context of costs for e-scooter rides that vary with the riding time. In the learning environment, students first work on a worksheet to complete a table, make sense of its columns and re-invent

variables for describing the costs for any riding time (Figure 1). The video itself serves to systematize the learning content after students' free inquiries and has a duration of 5:52 min. In order to foster cognitive engagement, we integrated interactive elements (implemented with the open-source software H5P), such as reflective questions, summaries, or cognitive prompts. For the analysis in this paper, we chose one drag-and-drop interaction on the interpretation of arithmetic expressions in the given table. When working on the worksheet, students tended to skip this task, because it is not obviously mathematical. In fact, it is an important intermediate step towards the symbolic generalization of the arithmetic expressions using variables.

Students receive feedback on their answers to the drag-and-drop question (cf. Figure 1) and are invited to answer multiple times until correct. The feedback refers to both: *verification* of correctness using colors and symbols (red and cross marks for incorrect matches, green and check marks for correct matches). Furthermore, *elaborated feedback* is provided through sentences or questions for incorrect answers (when no description is correctly dropped into a column of the table), partially correct answers (when one or two descriptions match correctly), and correct answers (all three match correctly). In the frequent case of partially correct answers, the elaborated feedback reveals two prompts: a *cognitive prompt (*"Did you make sure to not only describe what is new?") and a *metacognitive prompt* ("Explain to each other what the calculations in the columns mean"). Elaborated feedback was added as superficial active behaviors were found in earlier cycles (documented in Korntreff & Prediger, 2024). The cognitive prompt addresses a typical student error focusing recursive building rules instead of the whole expression. Due to the capabilities of the software, the feedback had some technical limitations, such as just one feedback for all patterns of partially correct answers or the spatial distance between the actual answer and the feedback. These limitations have been solved in the subsequent design experiment cycle (cf. Korntreff et al., 2023).

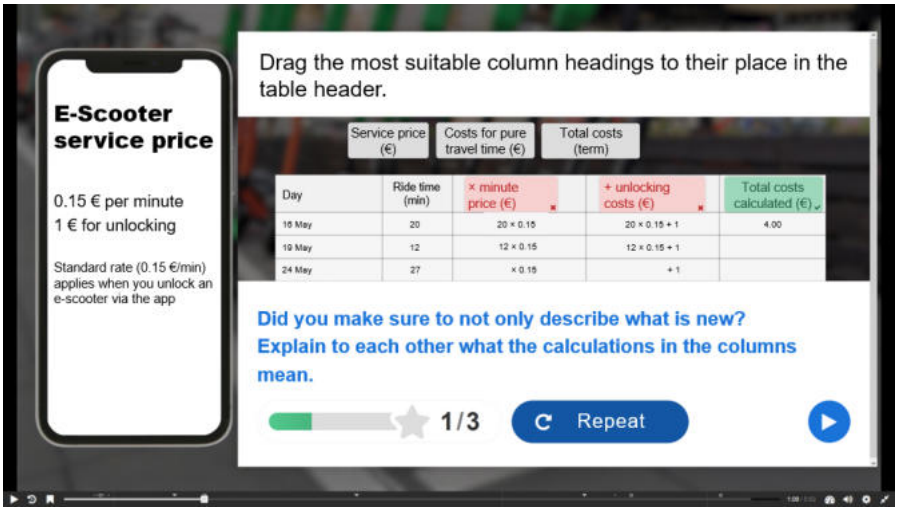

Figure 1: Drag-and-drop question with feedback, translated from German (Korntreff et al., 2023)

Data collection

For the second cycle from which the empirical part of this paper reports, data was collected in design experiments at vocational schools, with adolescents in need of a second chance to understand algebra.

Eight pairs of learners (5 females and 11 males between 15 and 33 years old, $M = 19.6$ years) collaboratively worked with the interactive video in a laboratory setting. After an individual inquiry and activation of prior knowledge with a paper-and-pencil worksheet, the learners worked with the video in pairs sharing one device (mostly smart board, some with tablet). The work was screen- and video-recorded and transcribed. The eight drag-and-drop sequences analysed here lasted between 1:00 min and 4:25 min ($M = 2:24$ min) from the first presentation of the task until the mouse click on continue.

Data analysis

The video data and transcripts of the drag- and drop sequences of eight pairs were qualitatively coded in a deductive coding procedure (Mayring, 2015). In *Step 1*, the sequences were segmented into *coding units* that comprise one solution step – typically one attempt or, if no further attempt follows, one engagement with a feedback offer. For each attempt, the correctness of the students' answer was tracked (correct vs. partially correct or incorrect). Three coding units were later split to capture engagement changes. In *Step 2*, students' cognitive engagement was deductively coded with codes derived from the ICAP framework (cf. Figure 2): *No engagement* corresponds to disengagement, *moderate engagement* to the active mode and *strong engagement* to the constructive or interactive mode (which are not distinguished because interactivity usually scratches over several coding units). The passive mode of engagement cannot occur as every coding unit ends with a click on either "check" or "continue" which is "overt motoric action or physical manipulation" (Chi & Wylie, 2014, p. 221) and coded in active mode. In *Step 3*, we deductively coded the use of elaborated feedback with parallel codes (cf. Figure 2), with the important difference that the passive mode of engagement *can* occur for feedback. Additionally, we analysed how the provided feedback aligned with students' thinking in the particular moment. This relates to the question how helpful the feedback could have been in the specific problem-solving situation.

Codes for cognitive engagement		Description
C3	Strong engagement	Constructive or interactive engagement with the task: Explaining, justifiying, making connections.
C2	Moderate engagement	Active engagement with the task: Reading out loud, repeating, asking questions, describing using similar words.
C1	No engagement	Disengagement with the task in terms of content: Superficial strategies (e.g., trial and error) or no focus on the task at all.
Codes for use of feedback		**Description**
F3	Strong use of feedback	Elaborated feedback is read and applied to the problem.
F2	Limited use of feedback	Elaborated feedback is passively read but not applied to the problem.
F1	No use of feedback	Elaborated feedback is not read.

Figure 2: Deductive codes for cognitive engagement and use of elaborated feedback

Results on students' engagement and use of feedback

Number and correctness of attempts

In Figure 3, we provide an overview of the analyses and findings which are presented in this section. With respect to the number and correctness of attempts, the boxes in the visual summary reveal that no pair succeeded to correctly drag all matching descriptions in their first attempt, one in the second,

two in the third, and one in the fifth attempt. Four pairs continued the video without an own correct solution (as visualised by the missing check marks in the visual summary).

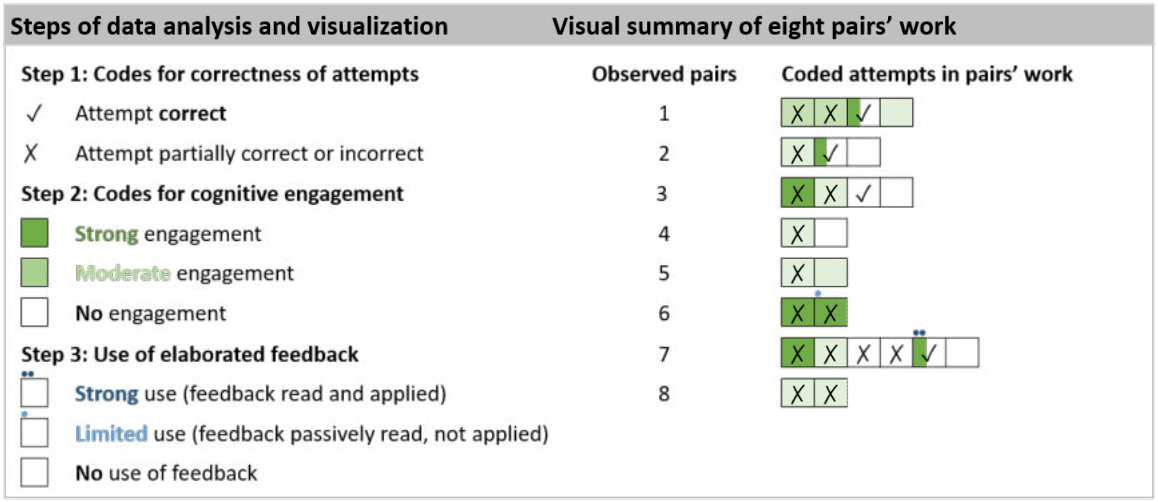

Figure 3: Visual summary of the analysis of eight pairs' processes

Students' cognitive engagement

RQ1 asks how the students cognitively engage in their work on the drag-and-drop task. As intended, the video interaction seems to have supported students' cognitive engagement: All eight pairs interrupted the passive mode of watching the video and engaged at least *moderately* (as visualised by the light green boxes in Figure 3) with the drag- and-drop question. Five pairs *strongly engaged* by performing deeper processes such as justifying or making connections. One example is given by Student 2B from Pair 2 in their second attempt:

Student 2B: The unlocking costs are actually coming next, because there is always the one euro added with the unlocking costs [*drags "+ unlocking costs (€)" into the 4th column*].

Nevertheless, although all pairs started with at least moderate engagement and thought about the task in terms of content, several switched to superficial strategies such as trial-and-error or answering by exclusion (coded as no engagement and visualised by white boxes in Figure 3) towards the end of the solution process. Especially, no pair finally found the correct answer by strong engagement, not even Pair 7. Also, none of the four pairs who answered the question until correct (check marks in Figure 3) engaged with the correct answer on a deeper level, as the example of Pair 7 illustrates:

Student 7B: We have had this one [*points to "service price (€)"*], that one [*points to "× minute price (€)"*] and that one [*points to "+ unlocking costs (€)"*]. So, it has to be this one [*drags "Total costs (term) (€)" in the 4th column, clicks on "check"*].
Video: [*Shows feedback for correct answer*]
Student 7B: Yes, yes, it's this one [*smiles*], we were so bad. [*Clicks on "continue"*]

Pair 1 was the only pair that at least copied the correct answer to their worksheet (moderate engagement in their last step after the correct attempt), the other three were satisfied with the green check marks on the screen and just continued, even though the correct answer had not been understood and only been found by exclusion or trial-and-error. In total, we see moderate and strong cognitive engagement, but also superficial ways of coming to and engaging with the correct answer.

Students' use of feedback

As each attempt receives an automated response by the interactive video, in Figure 3, each closed box with a check or cross mark stands for a feedback at the beginning of the subsequent step (Pair 8, for example, interrupts without asking for feedback again). In total, 17 feedback offers were analysed, four for correct and 13 for partially correct or incorrect answers. The mathematical adaptivity of the elaborated feedback to students' thinking in the moment could have been mostly high: Out of the 13 feedback offers for partially correct or incorrect answers, seven had a high fit, indicating that the feedback prompt precisely focused the difficulty in understanding, another three had a medium fit (the prompt matched one of the answers) and only three a low fit. This stands in noticeable contrast to students' actual use of the feedback prompts: Elaborated feedback was only *read* in two cases out of the 17 feedback offers analysed and only one pair went beyond reading and tried to apply the prompt to the actual problem. In the other 15 situations, the elaborated feedback was not even read, like for Pair 1 in Attempt 3:

Video:	[*Verification feedback: "× minute price (€)", incorrect | "+ unlocking costs (€)", incorrect | "total costs calculated (€)", correct.* *Elaborated feedback (unread): "Did you make sure to not only describe what is new? Explain to each other what the calculations in the columns mean."*]
Student 1B:	Huh? But it has to be this one. Because ride time times minute price really is "ride time times minute price". [*Drags "× minute price (€)" into Column 3*]
Student 1A:	But this is just "times minute price".
Student 1B:	Yes, but maybe … [*continues the discussion without getting back to 1A's point*]

Regarding verification feedback, after a partially correct or incorrect first attempt, six pairs took at least a second attempt and the feedback that the first solution was not fully correct seems to represent an important trigger here:

Video:	[*Verification feedback: "service price (€)", incorrect | "total costs, term (€)", correct | "total costs calculated (€)", correct.* *Elaborated feedback (unread): "Did you make sure to not only describe what is new? Explain to each other what the calculations in the columns mean."]*
Student 3B:	Crap! Can we find the correct answer anywhere or should we repeat the question?
Moderator A:	This is up to you.
Student 3B:	Then, we'll do it again …

Four pairs worked on the task until finding a correct solution while two continued the video on purpose, before they got to the correct answer. Another two pairs continued the video accidentally by clicking on "continue" instead of "check". On average, the students took 2.4 attempts on the question. We concede that our target group of students in need of second chances made only limited use of the provided elaborated feedback.

Summary and discussion

The small excerpts from eight students' working processes shows that integrating interactive elements into instructional video has the potential to foster cognitive engagement: All eight pairs at least moderately engaged with the task in one solution step. Thus, making instructional video interactive seems to be a potential support for cognitive engagement (Delen et al., 2014), also in our case of algebraic understanding for students with limited school success in need of second chances. As in earlier studies (summarized by Chi et al., 2018), the collaborative setting also appears to have

contributed. However, in particular for the target group in view, there is a risk of initiating only superficial processes just to get to the correct solution due to an actual or perceived lack of content-specific support. This pattern is mirrored in the engagement with the correct answer: There was no case of strong engagement with the actual solution to the problem. Quite the contrary, most of the time it was either simply accepted or even ignored even though it had not been understood.

In the meta-analysis summarizing studies with other target groups, providing specific feedback proved effective for students learning and was considered as a promising way to tackle these problems also in our design. However, while the simple verification feedback turned out to be an important stimulus for students to take more than just one attempt, the elaborated feedback was hardly used by the target group of adolescents in need of second chances after their regular high school pathway. Only one pair used the elaborated feedback by applying it to the actual problem (strong use), while it was mostly not even read. This stands in a strong contrast to the high mathematical adaptivity the feedback prompts could have provided, as most prompts precisely focused the students' problems in the particular moment. Yet, these target group's self-regulation strategies were not strong enough to make use of this opportunity. In the next design experiment cycle, we learned from these experiences and modified where and how the feedback was given. Embedding the interactive video in a learning environment with succeeding whole class discussions and follow-up tasks was a second consequence taken (Korntreff et al., 2023).

In more general terms, the observed need for focused cognitive and metacognitive support in order to prevent superficial processing, on the one hand, and the danger that the provision of elaborated feedback is not automatically used due to existing self-regulation obstacles, on the other hand, presents a recurrent challenge to teachers and instructional designers (Chi & Wylie, 2014). The interactive task itself as well as the feedback it provides have to be designed in ways that students can both, perceive it and apply it to the actual situation. When interactive videos are used in classrooms, teachers can make sure that superficial processing is deepened in a classroom conversation (Henningsen & Stein, 1997). In a more longitudinal perspective, students' self-regulation competences should be fostered as well.

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Investigating covariational reasoning: a multimodal analysis including eye-tracking data

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Covariational reasoning is a complex form of mathematical reasoning. Recently, eye-tracking data have been used to identify strategies in the field of research on covariational reasoning. However, in adopting such an approach, the richness and interlacing of multimodal resources involved in cognitive processes might be lost. Hence, in this methodological contribution we analyze an episode of a student reasoning covariationally by including all the multimodal resources involved in his learning process: speech, gestures, gazes, and representations. Our findings suggest an added value of such a multimodal approach for the identification of strategies for covariational reasoning. Benefits and limitations of this approach are finally discussed.

Keywords: Covariational reasoning, multimodality, eye tracking.

Introduction

Covariational reasoning is recognized as both an important and challenging type of mathematical reasoning (Thompson & Carlson, 2017). Supporting it in school practice is a relevant educational goal, but such a supportive approach also requires a better understanding of how students reason when engaging with varying quantities or relations between them. The extensive research done in this area identified *cognitive levels* (e.g., Thompson & Carlson, 2017) and *characterizations* of covariational reasoning (e.g., Bagossi, 2024). In recent studies, researchers have been looking into covariational reasoning from a socio-cultural perspective considering a multimodal approach to learning. Some studies have looked at students' gestures and language (e.g., Bagossi et al., 2022), while others have focused on students' interactions with representations while reasoning covariationally (e.g., Johnson et al., 2017).

The use of eye tracking (ET) has also been recently explored for the elaboration of strategies that can be detected from gaze patterns when students interpret contextual graphs (Thomaneck et al., 2022a). The value of ET in unraveling strategies has been widely acknowledged in different mathematical areas (e.g., Sthromaier et al., 2020; Van Dooren & Ufer, 2023), stressing the importance of triangulation of data - such as with stimulated recall interviews (e.g., Thomaneck et al., 2025; Van Dooren & Ufer, 2023) and the need for domain-specific interpretation (e.g., Schindler, & Lilienthal, 2019). Indeed, considering just eye movements in the elaboration of such strategies might result in an oversimplification in which the richness and interlacing of multimodal resources involved in cognitive processes are lost. Hence, in this contribution, we argue that the identification of strategies for covariational reasoning requires a triangulation of all the multimodal resources involved in the learning process: speech, gestures, gazes, and representations. Such a complex approach requires first a *methodological* reflection. Hence, this contribution argues for the necessity of such a multimodal

approach and does so by introducing an episode analyzed in this way. Benefits and limitations of this approach are finally discussed.

Covariational reasoning and its investigation

Interpreting functional relations as dynamic concepts, rather than static objects, requires disentangling the quantities involved and conceptualising how such quantities vary simultaneously. We call this form of functional thinking *first-order covariation*, and it turns out to be essential both when moving from a real-world context to its mathematical model and when interpreting a graph representing a real phenomenon (Thomaneck et al., 2025; Thompson & Carlson, 2017). In addition, the conceptualization of a mathematical relation within a variety of connected situations or scenarios requires a form of covariational reasoning, called *second-order covariation* (Bagossi, 2024). Reasoning at this second-order concerns the conceptualisation how a relation between co-varying variable-quantities varies simultaneously with a parameter-quantity. Even though a covariational approach has been shown to be important for developing a deep and meaningful understanding of functions, students struggle to develop and apply covariational reasoning (Thompson & Carlson, 2017). Hence, it is essential to investigate which forms of functional thinking students enact in contextual situations and which task design and digital tools may support them. Throughout the longstanding research in mathematics education aimed at investigating students' ways of covariational reasoning, several theoretical perspectives and methodological approaches have been adopted.

The foundational work by Thompson and Carlson (2017), which builds on decades of research contributions, stems from a *constructivist perspective* according to which knowledge is actively built by the subject by developing schemes of reasoning. Such work has led to elaborating a taxonomy of six cognitive levels describing how one can reason covariationally: no covariation, pre-coordination of values, gross coordination of values, coordination of values, chunky continuous covariation, and smooth continuous covariation. These levels have been investigated taking the teaching experiment as methodology (Steffe & Thompson, 2000) in which a teacher-researcher interacts directly with the investigated subject to better understand the students' hypothesis and ideas. Other works adopted a *socio-cultural perspective* in which knowledge is built with and through interaction with others (both peers and the teacher) as well as with digital and non-digital artefacts. In this approach, human activity is understood as being multimodal so all modalities should be analysed to understand cognitive processes (Gallese & Lakoff, 2005). Thus, to investigate short classroom excerpts adopting such a multimodal perspective on learning (Arzarello & Robutti, 2010), for example, Bagossi et al. (2022) considered all the relevant interactions between the subjects and artefacts together with all the semiotic resources involved (students' perceptuo-motor experiences, languages, signs, representations, and tools) by means of the Timeline as analytical tool. In addition to all the semiotic resources previously mentioned, also relevant gazes are considered meaningful to disentangle students' reasoning. This approach has contributed to a preliminary characterization of second-order covariation (Bagossi, 2024). Even though the analysis of gazes may sound peculiar within the multimodal perspective, ET has become a more and more appreciated methodological tool to investigate processes that we as researchers, cannot directly access by interpreting eye movements. The eye-mind hypothesis (Just & Carpenter, 1976) broadly states that the eyes fixate on what is

currently being processed as long as it is processed. With respect to the investigation of covariational reasoning, Thomaneck et al. (2022a, 2022b, 2025) have interpreted students' gazes when reading contextual graphs. They were able to detect gaze patterns and revealed emerging forms of covariational reasoning.

In light of all these previous considerations, in this study, we aim to investigate how a multimodal analysis, when integrating also ET data, could shed new insights into the investigation of covariational reasoning. Hence, through a feasibility case study, we will reflect on the following research question: *How is it possible to identify covariational reasoning from data integrating multimodal resources, including ET data?*

Method

Feasibility study and task design

This feasibility study was conducted with an Israeli 10th-grade public school student (16 years old), which we call here Yuval. The student indicated that he was male and that he studied according to the high level of the mathematics curriculum. The study presented here is part of a larger study in which participants worked with two animated task units containing both a dynamic applet and a function graph, and three purely inner mathematical task units showing a graph of a second or third-degree function. In this feasibility study, we focused only on the first animated task dealing with a ball rolling down an inclined plane focusing on the dependence of the distance traversed on time (Galileo unit). The animated tasks consisted of three parts, which we are going to refer to as (a) stimulus, (b) descriptive task, and (c) drawing task. At first, the participants could watch the video of the applet three times, i.e., a ball rolling down a plane[1]. This video acted as a *stimulus* to introduce the contextual situation. Then, in the *descriptive task*, the participants were asked to describe the dependence of the quantities involved, i.e. time and distance (Figure 1, left), with reference to the graph. In the *drawing task*, they were asked to sketch at least two more distance-time graphs when the length of the plane is kept fixed whereas its slope is changed (Figure 1, right) and to explain the shapes of the graphs.

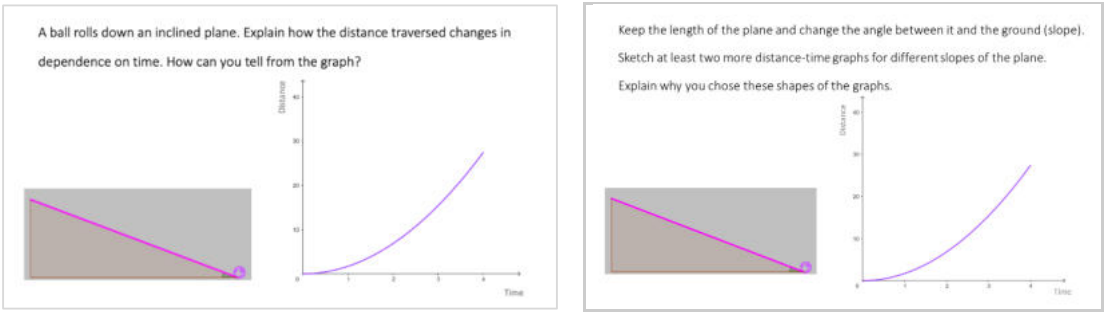

Figure 1: Galileo-unit (translated from Hebrew into English): descriptive task (left) and drawing task (right)

[1] The dynamic applet for the Galileo unit: https://www.youtube.com/watch?v=U2XfNJCJdkw

Setting and data collection

The study took place in a quiet room with the participant and the fourth author administering the tasks. There was no time restriction for working on the tasks. The participants sat on a firm chair in front of a table where a monitor was located. The tasks were presented on a 23" screen. The viewing distance was approx. 60 cm. Data was collected with a screen-based eye tracker: Tobii Pro Fusion (250 Hz, binocular). First, a 9-point calibration procedure was performed. Under optimal conditions, gaze accuracy is 0.3° (Tobii Pro Fusion, 2024). Inaccuracy was taken into account in the task design by placing the relevant objects far enough apart and data interpretation. In our study, the accuracy was 0.88° on average. In addition to the eye movements, we also recorded videos of the student: one facial video recorded by the webcam integrated into the monitor, and one video generated by a camera positioned behind the students' back recording both the student and the screen. After solving all five units, a *stimulated recall interview* (SRI) (Nguyen et al., 2013) was conducted, in which the student was shown the gaze-overlaid videos generated by the software Tobii Pro Lab (Tobii AB, 2022) as stimulus. Such a technique, also used in previous studies on covariational reasoning (Thomaneck et al., 2022a), allows participants to view a recording of the screen together with their gazes, animated as wandering red dots, jointly with the dialogues taking place while working on the tasks. The SRI took place immediately after Yuval was working on the tasks, so that his memory of what he was doing was still fresh. Both the interviewer and Yuval could stop the video: the interviewer to ask the participant to express his thoughts, and Yuval to explain what he was thinking.

Data analysis

The Galileo unit lasted in total 02:13 minutes and it was split into 20 segments. The analysis of each segment was conducted by means of an analytic table (see Table 1) in which columns 1-5 were devoted to collecting information on a specific semiotic resource each: 1. transcription of verbal *utterances* made while working on the tasks, 2. description of the *gazes*, as displayed in gaze-overlaid videos, transcribed by the researchers, 3. *gaze plot* created by Tobii Pro Lab for the corresponding time duration of the segment considered in that specific row, 4. *gestures* of the student, 5. the student's *representations* created while solving the drawing task. In a first step, the description of the gazes was related to the other semiotic resources so that hypotheses were developed on why Yuval fixated on the different locations of the stimulus. Where appropriate, these hypotheses were then elaborated into a *preliminary domain-specific interpretation* of the detected forms of covariational reasoning (column 6). In a second step, this interpretation was related to the respective utterances from the *SRI* (column 7) in order to elaborate on a *final interpretation* of covariational reasoning for the respective sequence (column 8). We finally considered all those segments where the final interpretation revealed some forms of covariation for this feasibility study. The analysis of the Galileo unit was conducted by the third and fifth author and then discussed with and validated by the first and second author. Rows that were presenting a discordant final interpretation were discussed and revised until reaching a shared interpretation.

Results

In this section, we will comment on one episode, corresponding to segment 6 in the overall analytic table, in order to elaborate on the methodological approach adopted in this study and elaborate directly on the final interpretation by addressing all the semiotic resources involved: utterances, gaze plots, gestures and representation. Table 1 (presented in a more compact way than used for data analysis) contains all the elements considered for the final interpretation.

In this episode, Yuval engages in answering the descriptive task (b) of the Galileo unit asking to elaborate on an explanation of how distance traversed by the ball changes in time elaborating on the provided graphical representation. The specific request leads Yuval to connect the quantities involved (distance and time) with the specific representation in the Cartesian system and so at first he directs his gazes to the two axes and specifically the points of coordinates (0,40) and (2,0) (Table 1, lines 2 and 5 in gaze description corresponding to gazes 12 and 13 in the gaze plot). In this short segment, Yuval claims that "As time passes, the distance increases". Yuval acknowledges covariation between time and distance and the direction of change of the two quantities expressed qualitatively ("passes", "increases") here. We interpreted this sentence as a form of gross coordination, i.e. first-order covariation (Table 1, column 6). Progressing with his explanation, Yuval moves his gaze to the graph itself starting from the point (1,2) and fixes it for a while (Table 1, gaze 14 in gaze plot). While following the graph with his gazes (Table 1, line 10 in gaze description) he adds that "because the ball is accelerating then we can see that the graph increases in this way". He simultaneously accompanies these words with an iconic gesture performed with his right hand tracing the course of the graph. Finally, his gazes jump back to the center of the graph (2.7, 12) (Table 1, gaze 20 in the gaze plot). This further multimodal explanation suggests a deeper understanding with respect to the first half of the sentence (Table 1, column 6), expressed by means of a combination of verbal and non-verbal resources. The hand gesture enables Yuval to refer to the increasing trend of the graph considering it as a dynamic mathematical object. The way in which Yuval addresses acceleration in his words is not clear from the data interpreted so far. Yet, the triangulation of these data with the SRI suggests a clearer interpretation. Yuval disentangles the variations of velocity and acceleration in this context (Table 1, column 7): "velocity is increasing" while "acceleration is constant" and this justifies the shape of the graph ("like this"). Hence, we can conclude that in this segment emerges a form of smooth continuous covariation (Table 1, column 8) in which the variation of distance in time (velocity) is increasing in a constant way and if the ball had continued travelling, in a possible interpretation of Yuval's words (SRI), "the slope would have been further and further increasing".

Concluding discussion

This contribution brings to the discussion the need for a deeper understanding of cognitive phenomena by means of a methodological approach that goes beyond the triangulation of ET data and SRIs: such an approach takes into account all the semiotic resources involved in the learning process and considers them globally. The *multimodal analysis* of the short segment of an experiment involving Yuval in the interpretation of a contextual task shows how the learner's conceptualization reveals a full understanding of the phenomenon which may not be grasped without such a multifaceted analysis. What is not fully expressed by words (Yuval cannot come up with a

mathematical term describing the trend of the graph, e.g., increasing) is supported by a gestural conceptualization reflecting the shape of the graph - performed while following the whole graph with his gazes. The lack of proper mathematical terminology is offset by a detailed embodied explanation using both arm and eyes. Yuval also elaborates on the variations of velocity and acceleration which correctly justifies the graphical relation between distance and time. Therefore, he reasons covariationally in a smooth and continuous way and properly argues for the motivations of such covariational relation. The gaze plot suggests that Yuval focused his gazes mainly on the y-axis and the beginning of the graph. He almost ignored the inclined plane despite mentioning the elements that refer to the contextual situation (ball, velocity, time, distance, etc.). The added value of such an approach does not end with a deeper understanding of the phenomenon investigated, in our case covariational reasoning. It aims to prepare the ground, theoretically and methodologically, for an unraveling of students' possible strategies in various contexts. Such an approach does not reduce to an oversimplification of the identified strategies but considers the interlacing of multimodal resources involved in cognitive processes by valuing an embodied perspective on cognition. However, we acknowledge the time-consuming nature of such an approach, which may be worthy of future discussion to distinguish those contexts in which such a rich analysis is needed or to better ponder how to use eye-tracking data. Even though the identification of covariation strategies goes beyond the purpose of this contribution - it is the final goal of the research project - the exemplificative analysis proposed here still provides an argument for the simplification of the context-related interpretation we would have run into without considering all the semiotic elements involved.

Ethics statement

The Human Subjects Research Committee of Ben-Gurion University has approved this study.

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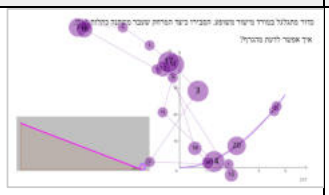
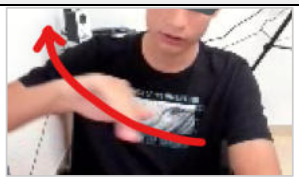
Table 1: Analysis of segment 6 of the experiment by means of the multimodal analytic table

<table>
<tr><th>1. Utterances</th><th>2. Description of gazes</th><th>3. Gaze plot</th><th>6. COV preliminary interpretation</th><th>7. SRI</th></tr>
<tr><td rowspan="6">00:01:15 - 00:01:25

“As time passes, the distance increases and because the ball is accelerating then we can see that the graph increases in this way.”</td><td rowspan="6">1. Gaze of non-meaningful point in the coordinate system (traversing gaze)
2. Gaze on y-axis label (or point on y-axis (0/40))
3. Gaze jumps back to the end of the 1st line of text (“time”)
4. Gaze jumps to non-meaningful point next to the y-axis (traversing gaze)
5. Gaze jumps to x-axis (2/0)
6. Gaze jumps to graph (1,2/3)
7. Gaze follows along the y-axis to the y-axis label
8. Gaze jumps to non-meaningful point (traversing gaze)
9. Gaze jumps to graph (1/2)
10. Gaze follows along the graph
11. Gaze jumps back to the center of the graph (2.7/12)</td><td>  </td><td rowspan="6">First-order covariation:
- Gross coordination: “As time passes the distance increases”
- (Gross or) smooth: “because the ball is accelerating then we can see that the graph increases in shape”. He can’t express it properly verbally but speaks of acceleration and thus shows more deeper understanding than in the first half of the sentence.</td><td rowspan="2">“And really it makes sense because that is, if the velocity was constant of the ball let’s say then the graph was supposed to look straight, but because it’s gaining momentum, meaning the acceleration is constant, but because the velocity is increasing as a result then the graph is like this, and if we continued then the slope would have been further and further increasing.”</td></tr>
<tr><th>4. Gestures</th></tr>
<tr><td> 
[8-11] He traces the course of the graph with the right hand.</td><th>8. COV final interpretation</th></tr>
<tr><th>5. Representations</th><td rowspan="2">Smooth continuous covariation between time and distance.</td></tr>
<tr><td>None (it is a descriptive task)</td></tr>
</table>

Identifying design criteria of digital learning environments for geometry learning: A student-centred approach

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In response to the Austrian Federal Ministry of Education's initiative to distribute digital devices to all students of lower secondary level, the demand for high-quality digital learning resources has increased. Despite this growing need, there is limited research on the optimal design of these materials. This study therefore investigates the design of digital learning environments (DLEs) for geometry learning at lower secondary level from students' perspectives. Employing a qualitative methodology through the Think-Aloud technique, this study engaged 17 fifth-grade students with digital geometry learning materials from the FLINK in Maths project. The research identified three critical design aspects: a) navigation, b) contextual framing of tasks, and c) fostering self-effective experiences. These findings emphasise the need for interactive exercises and the crucial role of teacher guidance in effective DLE design.

Keywords: Digital learning environment, design aspects, discovery learning, geometry learning.

Introduction

As digital technologies increasingly influence modern life, their integration into educational settings, particularly mathematics classrooms, has gained significant attention (Drijvers, 2022). Austria has actively embraced this trend with a thorough digitalization strategy that seeks to equip students from the fifth grade onwards with digital devices to foster active learning (BMBWF, 2024).

However, the increase of digital devices in classrooms necessitates the creation of high-quality digital learning materials customised to suit students' requirements. Therefore, more empirical research on the design aspects critical for digital mathematics learning experiences needs to be done, particularly for geometry learning. This study addresses this research gap by exploring key design aspects of digital learning environments (DLEs) for geometry learning in lower secondary education, focusing on students' perspectives. The research question guiding this exploration is:

What design aspects of digital learning environments for geometry learning hold relevance for students at the beginning of lower secondary level?

This research utilised a qualitative research approach employing Design-Based Research. The Think-Aloud method was used to observe fifth-grade students (ages 11 to 12) interacting with digital geometry materials from the FLINK in Maths[1] project. FLINK, an educational platform developed by the Center for Open Digital Education at Johannes Kepler University, provides a DLE that focuses on enhancing mathematical concept development and skills acquisition. This research aims to enhance the design of DLEs in digital geometry learning, bridging theoretical insights with practical applications.

[1] https://www.jku.at/flink-in-mathe/

Theoretical framework

This research adopts the frameworks of Constructivist Learning Theory and Artifact-Centric Activity Theory to investigate the design of DLEs for geometry learning at lower secondary level. These theoretical frameworks will help to provide a foundation for understanding how students interact with digital tools and how these interactions facilitate meaningful learning experiences.

Constructivist Learning Theory as described by Boyle (1997) is a framework that postulates that learners actively construct their knowledge through experiences and interactions with the world around them. This theory emphasizes the importance of students' active engagement in the learning process through exploration, experimentation, and the connection of new information to their existing knowledge (Boyle, 1997). In the context of geometry learning, digital tools such as graphing calculators and Dynamic Geometry Software (DGS) play a key role in fostering this active engagement. These digital tools allow students to manipulate and visualize geometric figures, thereby promoting independent exploration (Barzel et al., 2005). The underlying research draws on this theory to analyse the design of DLEs that encourage students to take an active role in their learning.

Artifact-Centric Activity Theory (ACAT) provides a complementary perspective by integrating principles from both Activity Theory (Vygotsky & Cole, 1978) and Instrumental Genesis (Verillon & Rabardel, 1995). ACAT focuses on the interactions among subjects (students), objects (mathematical tasks), and artifacts (digital tools) and how they mutually influence each other and the learning process (Ladel & Kortenkamp, 2016). By incorporating the social aspects of Activity Theory and the meaningful use of artifacts as described by Instrumental Genesis, ACAT allows for an examination of how digital manipulatives can facilitate both the internalization and externalization of knowledge. This theoretical framework provides a comprehensive approach to evaluating DLEs by ensuring that they serve as tools for learning and contribute to a meaningful learning experience that aligns with students' requirements (Ladel & Kortenkamp, 2016).

Literature review

This concise review of literature highlights key aspects of DLEs identified by scholars, relevant to our research question on design aspects for geometry learning at the lower secondary level.

The integration of digital technologies in classrooms has revolutionised the teaching and learning landscape, especially in the domain of mathematics (Drijvers, 2022). Geometry learning has particularly benefited from technological advancements (Henn & Filler, 2015), with DLEs becoming central to modern pedagogical strategies that emphasise discovery learning and student-centred approaches (Barzel et al., 2005). Digital tools enable dynamic visualisations and manipulations of geometric objects (Henn & Filler, 2015), allowing students to explore and experiment with mathematical relationships in ways that pencil-and-paper based methods cannot achieve (Haug, 2012). Additionally, automated feedback from digital tools enhances personalised learning by providing instant responses, similar to one-on-one tutoring (Barana & Marchisio, 2019).

According to Reiss and Hammer (2021), digital technologies are also fitting to promote discovery learning. Discovery learning, rooted in Constructivist Learning Theory, allows students to independently explore mathematical concepts to foster a deeper understanding of the materials (Reiss

& Hammer, 2021). However, balancing independent exploration with necessary guidance is crucial to prevent students from becoming overwhelmed (Kollosche, 2017).

Research on task design for digital learning in mathematics highlights the need for a strong conceptual foundation in developing digital tools. Bennison et al. (2020), as well as Yerushalmy and Olsher (2020), emphasize that tool design should be guided by clear educational goals to support students' learning outcomes. Additionally, Boyle (1997) outlines requirements for learning environments, that not only apply to DLEs itself but also to its implementation within a constructivist-oriented teaching framework. These requirements for DLEs include enabling individual experiences, embedding knowledge in real-world contexts, reinforcing learning through social interactions, incorporating diverse forms of knowledge representation, and fostering self-efficacy in independent knowledge construction (Boyle, 1997). Our study builds upon these insights by exploring how DLEs can be designed to enhance geometry learning, focusing on the integration of digital tools that promote self-efficacy, and independent exploration. We aim to connect theoretical insights with practical findings to better understand how DLEs can be optimized for lower secondary education in geometry.

Methods

This study focuses on the FLINK in Maths project, which aims to create high-quality digital learning resources for mathematics. FLINK resources are structured as interactive, thematic books that align with the Austrian mathematics curriculum and, in line with the principles of ACAT, serve as digital artifacts that mediate learning experiences. Each book includes sections for both discovery and practice and uses automated feedback and dynamic visualizations (Johannes Kepler University, n.d.). By analysing the discovery learning processes, this study draws on Constructivist Learning Theory to examine how FLINK resources aid students in building their understanding of geometry. The development of the FLINK resources follows an iterative Design-Based Research approach, including design, evaluation, and refinement (Bakker & van Eerde, 2015). This study specifically engages with the evaluation phase by analysing the resources in classroom settings using the Think-Aloud method. Through this approach, the research identifies areas for improvement and thereby directly contributes to the iterative refinement of the DLE.

Table 1 summarizes the observations conducted across five schools – both middle schools (MS) and lower secondary schools (AHS) located in urban and rural areas – where researchers had established contacts with teachers. Following the approval from the Education Directorate of Upper Austria, schools were chosen based on their consistent use of digital devices in mathematics and familiarity with FLINK resources. The study involved fifth-grade students (ages 11 to 12), and out of 63 observations conducted by three researchers between May and June 2023, 17 were selected for detailed analysis based on their relevance to the study's objectives in investigating geometric learning experiences. This selection aimed to capture a diverse representation across gender, school type, and geometrical topics such as cuboids and cube nets, geometric solids, and the introduction to circles.

Table 1: Overview of the conducted observations

School, Region	Mathematical content	Quantity male	Quantity female	Total number

MS, countryside	Cuboid and cube nets[2] Geometric solids[3]	1	2	3
AHS, city	Cuboid and cube nets	2	2	4
AHS, city	Circle introduction[4]	2	1	3
MS, countryside	Cuboid and cube nets Geometric solids	5	2	7

Data collection involved using the Think-Aloud method (Ericsson & Simon, 1984), which was chosen for its ability to capture students' immediate reactions as they engaged with FLINK materials. In line with Constructivist Learning Theory, this approach helps to comprehend how students actively build understanding by allowing them to verbalize their thoughts in real time to provide insights into their strategies and challenges when interacting with DLEs (Cotton & Gresty, 2006). Given the students' unfamiliarity with the Think-Aloud process, the observers sometimes posed targeted questions to direct the discourse toward the research question. Additionally, valuable insights emerged from observing students as they worked with the materials. By observing and analysing students' interactions with the DLE, this study aligns with ACAT's emphasis on considering also social and contextual dimensions of learning. The interactions were recorded in audio-visual format, transcribed, translated to English, and analysed by three coders using Qualitative Content Analysis. The analysis, guided by Mayring's (2022) summarising approach and Kuckartz's (2019) content-structuring approach, identified three key categories and eight subcategories of students' learning preferences. In developing the codes and categories, the analysis relied on Constructivist Learning Theory and ACAT to structure insights into students' learning experiences with DLEs. Constructivist Learning Theory informed codes related to active knowledge-building and captured how students used the digital tools to construct understanding independently. ACAT provided a framework for categorizing interactions and dynamics between students, tasks, and digital artifacts. Figure 1 outlines the resulting code and category system.

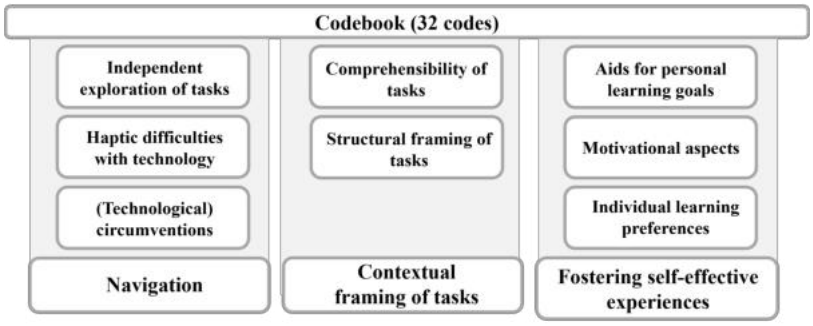

Figure 1: Code and category system

[2] https://www.geogebra.org/m/hqeaawte

[3] https://www.geogebra.org/m/daekxjzz

[4] https://www.geogebra.org/m/a4pppe7a

To ensure objectivity in assigning codes to subcategories and main categories, intercoder reliability was assessed. The resulting Cohen's kappa coefficient of .78 indicates substantial reliability of the research findings according to Landis and Koch (1977).

Results

The research question of this paper aims to pinpoint design aspects of DLEs for geometry learning, focusing on lower secondary level students' perspectives. The analysis identified three main findings: a) navigation, b) contextual framing of tasks, and c) fostering self-effective experiences.

Navigation

The analysis underscores the importance of navigation within DLEs for geometry, identifying three key aspects: independent task exploration, haptic difficulties, and technological circumventions. Students enjoy interacting independently with DLEs and find their interactive nature more engaging than traditional pencil-and-paper methods. They appreciate the opportunity to manipulate shapes, adjust parameters, and rotate objects to deepen their geometrical understanding.

> Student (AHS|m): This is an interactive exercise; it has more momentum. Without the interactive elements, learning is simply boring.

However, haptic challenges can detract from this experience, especially on touch devices without an external mouse, making certain interactions like drag-and-drop and scrolling frustrating at times.

> Student (MS|f): It's a bit of a pain when you have to drag [the slider] like that all the time.

Furthermore, the ease of accessing correct answers can prompt some students to take shortcuts instead of fully engaging with the material, which potentially hinders the learning experience within the DLE.

> Student (MS|f): I write something briefly, then I click on "check answer" and then I can simply copy the text.

Contextual framing of tasks

In addition to navigation, the study reveals that contextual framing of tasks is crucial for DLEs in geometry, as perceived by students. Two primary aspects emerged: comprehensibility and structural framing of tasks. According to the observations, clear and precise instructions, along with logical colour coding, significantly enhance task comprehensibility within DLEs.

> Student (AHS|m): I think it's good that […] they also instruct you step-by-step on how to do it.

For structural framing of tasks, students find that visual aids and 3D models within DLEs make abstract geometric concepts, like nets of cuboids and cubes, more accessible and engaging.

> Student (AHS|f): This 3D view, in general, helps me to visualise the concept.

Fostering self-effective experiences

Finally, the data shows that fostering self-efficacy is essential for DLEs for geometry learning, with key factors including providing aids for personal learning goals, motivational aspects, and catering to individual learning preferences. Observations reveal that students appreciate DLEs that promote autonomy, for instance through automated feedback.

> Student (AHS|f): I don't always have to ask the teacher whether it's right or not. That's why I find it very practical and good.

Additionally, the data indicates that using digital devices increases motivation, as many students prefer digital resources over traditional methods. Students find that digital interactions help them grasp mathematical concepts better and enjoy features like hints, which contribute to their independent learning experiences within DLEs.

> Student (MS|m): I actually find the materials very exciting because […] I understand it [the mathematical content] much better straight away.

The observations indicate that students generally navigate DLEs successfully and appreciate their autonomy when working with digital devices. However, it is essential to address any technological challenges that may arise during this process. Furthermore, by improving the contextual framing of tasks and fostering experiences that enhance self-efficacy, DLEs can enrich geometry learning experiences from students' perspectives.

Discussion, conclusion, and further research

Drawing from the insights gathered in our investigation into DLEs for geometry learning, our findings highlight several critical aspects of DLEs that impact students' learning experiences. Firstly, navigation proves essential, as students value the independence to explore tasks on their own and manipulate objects, reflecting principles from Constructivist Learning Theory (Boyle, 1997) that emphasize active self-directed exploration. Students' enthusiasm for tasks that allow them to manipulate and explore shapes aligns with previous studies on dynamic geometry environments (Henn & Filler, 2015), which highlight how these tools encourage experimentation and enhance engagement through interactive visualizations. However, this independence also reveals practical limitations, as students occasionally resort to technological circumventions, such as copying answers, which can reduce engagement. This finding underlines the need for integrating gradual task scaffolding to support discovery while reducing tendencies to circumvent the learning process.

Secondly, the study's findings emphasize the importance of contextual framing in task design within DLEs, which was highly valued by students for promoting comprehensibility. Clear instructions, visual cues, and 3D models help students to grasp abstract geometric concepts. In line with ACAT, the structured use of digital artifacts supports the internalization and externalization processes that are critical for meaningful learning experiences. These insights align with prior research (Henn & Filler, 2015; Haug, 2012) that emphasizes the value of visual aids and consistent task framing for facilitating comprehension. However, observed haptic difficulties, particularly with touchscreen interfaces, highlight the need to consider device compatibility to prevent frustration and maintain engagement. Addressing such challenges by designing interfaces that are more touch-friendly could significantly improve the overall learning experience.

Finally, fostering self-effective experiences emerged as essential for supporting students' motivation and confidence, as seen in students' appreciation of features that allow for immediate feedback and self-correction. These features align with the research findings on automated feedback (Barana & Marchisio, 2019), which enables students to assess and correct their understanding independently. Creating self-efficacy experiences within DLEs also resonates with Boyle's (1997) emphasis on increasing students' confidence in line with constructivist learning principles. To strengthen self-efficacy, DLEs could integrate reflective prompts that encourage students to assess their learning

strategies and progress. This feature would empower students to take ownership of their learning process and enhance their sense of control which boosts their self-efficacy as they work through geometric tasks within DLEs.

Thus, our study highlights the vital role of well-designed DLEs in enhancing students' learning experiences in geometry. By addressing key aspects such as enhanced navigation, effective contextual framing of tasks, and fostering self-efficacy, educators can create digital tools that support students' autonomous learning. As digital technologies become more integrated into education, these tools must be thoughtfully designed to meet the diverse needs of learners. While our study provides valuable insights, several limitations demand consideration. The sample size and specific demographic may limit generalisability. Moreover, the study primarily focused on student perceptions rather than objective learning outcomes, suggesting a need for further research integrating qualitative and quantitative approaches.

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Programming and mathematics – Experiences using programmable robots in a 6th grade mathematics class

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This paper reports on the first intervention of an action research study involving a series of four tasks designed to provide experiences for learning mathematics through programmable robots. We examined opportunities observed in a Norwegian 6th-grade class working on these tasks and propose recommendations for classroom implementation. We found that the programmable robots provided opportunities for open tasks where students can discern which mathematical content areas or algorithms are beneficial in a practical hands-on context and where the environment gives feedback on the mathematical solutions. We also found indications that perceptible experiences from programmed sounds and robot movements can help students focus on the mathematics. Nevertheless, technological distractions and task openness require a teacher's guidance towards the mathematical aspects.

Keywords: Programming, mathematics education, 6th grade students, programmable robots

Background

A school requested assistance from our university college to make use of their programmable robots, LEGO Education SPIKE, in their mathematics teaching. This was the start of an action research study (Cohen et al., 2018), and the current paper is based on the first intervention of this study. In Norway, programming has been a part of the curriculum since 2020, and is included in the mathematics education (The Norwegian Directorate for Education and Training, 2019). The role of programming in mathematics education is multifaceted, and programming can e.g. be seen, as a pedagogical tool, to increase engagement, as a new way of thinking (computational thinking), or as a way of learning mathematics (Kilhamn et al., 2021). Boaler (2001) pointed out that to learn how to use mathematics in different situations, students need to not only learn mathematical procedures, but also to work on open-ended tasks. In this study, we aim to investigate the mathematics learning opportunities provided by programmable robots and open-ended tasks that invite various solving strategies.

In a literature review, Zhong and Xia (2020) studied the use of robotics in mathematics education and pointed out the opportunity for hands-on learning experiences. As within mathematical modelling, there is not a given correct answer, but the aim is to find a sufficient way of solving a problem. This includes validating the answer and refining your solution. To validate their answers has previously been seen as a part of the teacher's role, and students might stop working on a task once they have an answer (Blum, 2015). When developing problems including robots, we focused on designing a task where the robots could give the students feedback on their solution and exploit the potential for hands-on learning experiences. Zhong and Xia (2020) also pointed out that the introduction of robotics in mathematics education can elevate students' cognitive load and divert their attention from mathematical problem-solving due to the novelty of robotics. In a research project where one of their aims was to find out if the students would perform better in mathematics after working with programmable LEGO teaching materials during a school year, Lindh and Holgersson (2007) found

that the teacher's role is important, and that the use of robots does not necessarily have a positive effect on cognitive development. It is important to further investigate how to teach programming in primary school mathematics, especially performing intervention studies and empirical classroom studies in collaboration with teachers (Holo et al., 2022).

In this paper, we present an activity in four parts that was developed for an intervention in a 6th grade at the school which invited us. We examine scenarios from their classroom, and our research question is formulated: *What opportunities for learning mathematics can be identified when using programmable robots in a 6th-grade classroom, and how can the execution of the tasks be improved based on these classroom experiences?*

Development of tasks

In collaboration with a teacher and three fellow researchers in mathematics education, we developed a four-step activity for learning mathematics. The class teacher, citing his limited experience with programming, requested that we propose activities and actively participated in discussions about them. The students had limited experience with programming and the LEGO Education SPIKE environment, which utilises a robot, a tablet device and block programming, as shown in Figures 1 and 2. Therefore, the task was designed to provide students with programming experience while using programming and robots as tools for gaining mathematical insights. Following Boaler (2001)'s recommendation, we aimed to present students with open-ended tasks to encourage the use of diverse solving strategies. We decided to create a task centred around measurement, with the objective of providing students opportunities to refine strategies and engage in mathematical generalisation.

Each pair of students received a robot equipped with motorised wheels and a light sensor, which they connected to their tablet device. We decided not to let the students build the robots, because we did not want them to focus on aesthetic purposes or building techniques, but rather on solving strategies. On the classroom floor, we marked a starting point and placed a finish line made of a red paper strip. The tasks were given orally, one by one. The first was formulated:

a) Ensure the robot halts as close as possible to the finish line, which is situated 100 cm away from the starting point.

We wanted the students to explore different strategies, e.g., to find the length of one rotation by testing, or by measuring the radius of the robot's tyre and adding numbers of rotations to run. They could also find out how many seconds the robot used on the distance and add numbers of seconds to run. In task a) the students could explore by trial and error and had the opportunity to change strategy several times. In the next task, the opportunities for trial and error were narrowed down, and we wanted them to generalise a strategy and program the robot to run any given length:

b) Program the robot to halt as close as possible to an unknown/secret distance from the starting point. The distance will be revealed 1 minute before the race. Leave your robots while figuring out a strategy.

The students are asked to leave their robots while figuring out a strategy following the teacher action "disable the run-command" presented by Berge et al. (2024). This aims to foster discussions and

reflection to generalise the findings from task a). We anticipated that the students would generalise by making use of the number of rotations/seconds found in task a). They could then either make a table with possible lengths and corresponding numbers of seconds/rotations or they could find the function expressing the number of seconds/rotations per distance. They can implement their general function expression by programming the input to be the number of meters to run. An alternative, simpler method for solving the task is to calculate (or read their table to find) numbers of rotations/seconds for the given length. The coding will then be to enter the number of seconds/rotations into a single command block, as shown in Figure 1.

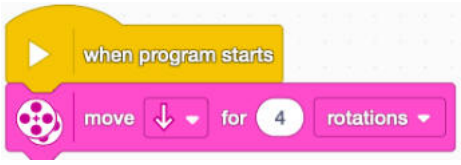

Figure 1: Example of solution task a). The number 4 can be changed before the race in task b)

In the next two tasks, we aimed to provide students with hands-on experience with the use of a loop, condition, and a variable as included in the curriculum (The Norwegian Directorate for Education and Training, 2019) and opportunities to refine their solutions. Through task c) the students will be challenged to predict, run and investigate, and in task d) modify, which is four of five components of PRIMM suggested as a way of learning programming (Sentance et al., 2019).

c) What does this programme do?

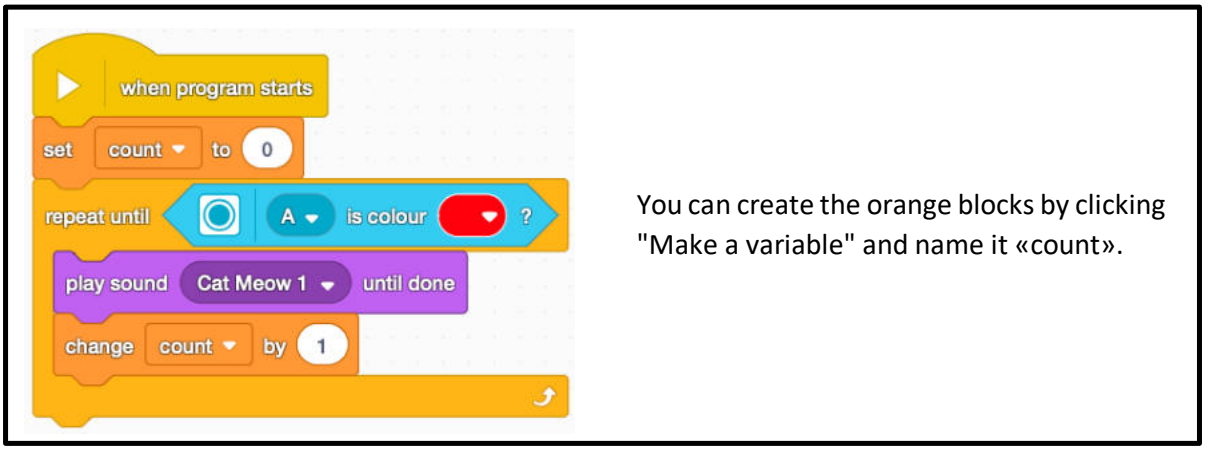

Figure 2: Handout task c), printed on paper

The programme makes the robot repeat a sound and counts how many times the sound is repeated until a red colour is identified through the light sensor. The students can make the programme on their tablets. This task provides a tangible experience with a variable, as the students can see the counter changing its value in correspondence with hearing the sound.

d) Utilise your experiences from tasks a)–c) to modify the programme in order to measure any distance from the starting point to the red finish line.

One solution to task d) is to remove the purple block for making the sound and insert a pink move-block as shown in Figure 1. We expected that most of the students would make the robot move one rotation per count and calculate the distance by multiplying the circumference of the wheel with the number of counts. If they used seconds in task a) and b), they could do the same in d). The students could either use the output to calculate the distance or, for a more generalised solution, make a programme that gives the length of any measured distance as output. The execution of the programme decides the value of the count variable, which later in the programme can be used in a length function dependent on the count. This type of generalisation might be within reach for sixth graders if the task specifies that the output should be given in metres. The task could potentially be extended to a relay race measuring different lengths for each round, where time matters, and they thus find it useful that the programme calculates the length for them. It has been found that programming languages might constrain students' development of algebraic thinking due to confusion of concepts, for example, the count variable changing its value during the execution of the programme, which is different from the variable concept in algebra (Bråting & Kilhamn, 2021). It is, however, found that for the block-based programming language used in this task, the concept of functions behaves as its counterpart in mathematics (Bråting & Kilhamn, 2021). This task might thus be suitable for working on the intersection between programming and algebra.

We made the finish line 10.5 cm thick, less than the circumferences of the wheels used. After running the loop, the programme instructs the light sensor to check if the condition for repeating the loop is true. If the students opted for a complete rotation, they might observe that the robot does not halt at the line if it is crossed during a rotation. Consequently, the robot would not come to a stop but continue onward. If they first, based on what they found in task a), instruct the robot to run for 1 m in each loop, they will discover that they need to reduce the length per loop to improve the accuracy. Thus, to generalise by expanding from natural numbers to rational numbers can be encouraged by asking the students to develop more accurate and refined solution methods.

Experiences gained and possible further directions

The classroom activities were carried out in a 6th grade where the teachers and students had limited experience with programming and LEGO Education SPIKE. The activities were conducted by their teacher. Five pairs of students were filmed by researchers, and their tablet screens were recorded. The teacher tried to solve the tasks himself before giving them to the students but had only time to work on the first two tasks. Concerning what is pointed out by Lindh and Holgersson (2007), that "[t]he role of the teacher, as a mediator of knowledge and skills, [is] crucial for coping with problems related to this kind of technology" (p.1097), this was not optimal. The teacher also expressed uncertainty about his role and found the tasks more open than he often presented to his students. They usually worked with one mathematical topic at a time. A strength of the data is authenticity. Many teachers have no experience regarding programming themselves, and do not know how to implement it in their mathematics teaching.

We have selected three scenarios in which we identify opportunities for learning mathematics. Now, we will discuss them and connect our findings to the literature used in the design process. These

scenarios are merely examples of potential opportunities and should not be considered exhaustive. The participating teacher has reviewed this paper and discussed the interpretations presented.

Opportunities for choosing appropriate strategies in the application of mathematics

In task a), one of the student groups ran one rotation and measured the distance from the starting point to where the robot stopped. They found it was approximately 17.5 cm, and Student 1 wrote 17.5 on their board (the groups had one board each) and said: “We just need to figure out how many times we need 17.5 to get to a hundred.” Student 2 replied: “Oh no, not multiplication.” Student 1 tried to mentally multiply and add up before he looked at the board of Student 3 and Student 4. They had a different size of wheels, with a circumference of 26 cm, and had written $26 + 26 + 26 + 26 = 104$ and were satisfied with their 4 cm too long solution. Student 1 said: “It's probably better to rather use seconds.” To add up was easier for the other group which added up 26 cm. Rather than using division, a more accurate way of finding out how many 17.5’s they needed to get 100, they switched strategy. According to the teacher, the students had primarily dealt with divisions involving whole numbers. The length 17.5 could have deterred them, especially since the teacher did not expressly permit the use of calculators. Or it may be due to their difficulty in associating the operation of division with the practical context. In task a), students applied arithmetic in a practical situation without predefined operations. The teacher noted that they typically focused on one mathematical concept at a time, but this task allowed for more open solution strategies. Boaler (2001) emphasises the importance of open tasks and diverse situations to help students use their mathematical knowledge beyond procedural repetition. It was positive that groups explored various strategies, though it was surprising that most used addition rather than more precise calculations in task b). One group used division to calculate "five point something" rotations per meter, while another group added rotations to reach a whole number. Task a) provided an opportunity for the teacher to discuss and compare these strategies with the class. Encouraging multiple strategies helps students evaluate their choices and understand the accuracy needed for the task. Moreover, this task provided them with feedback on the level of accuracy required when executing the race.

Waiting or refining the strategy?

In task b), the students needed to use their result from task a) and generalise their strategy to any distance. The students got 14 minutes to work on a strategy and were only given one minute after the distance was announced to prepare their programme and robot to run. One pair of students found that their robot did not reach the finish line in a). For task b), they therefore adjusted their estimate from 4.27 to 5 seconds per meter. Student 1 reasoned out loud: “Now we have how many seconds it took for one metre. So, if it's 2 metres, it takes about 10 seconds, because it takes 5 seconds on one metre,” while student 2 wrote “10 seconds on 2 metres” on the board. After careful consideration, Student 1 suggested that “if [the teacher] says like 10 cm, we need to start dividing right after he announces it. Because we have the basis of 1 metre.” From this, it seems like their general strategy is to multiply or divide 5 seconds depending on whether the distance is longer or shorter than 1 meter. The group now sat quietly and said nothing for around 9 minutes until the length was announced. It seems like the students were satisfied when they came up with one strategy, and did not discuss others. There is, however, potential for further generalisation and accuracy of their strategy.

Other groups also used approximated values expressed as whole numbers and struggled to find a general strategy. One of the groups used doubling as a strategy to find the number of rotations for two more distances. They added up from 104 cm (four rotations) and got 208 cm (eight rotations) that they rounded off to 200 cm. They seemed frustrated and in need of a better way to work with the problem. When they got the distance of 4.5 metres, they continued by doubling their previous calculation on the board to 416 cm (sixteen rotations) before adding 26 cm (one more rotation). They did not openly reflect on the fact that the robot would stop at 442 cm (8 cm before the finish line, according to their calculations). If the students are used to solving procedural tasks, they might have the understanding that "[c]hecking the correctness and suitability of a solution is exclusively the teacher' responsibility" (Blum, 2015, p. 79). As pointed out by Blum (2015), students need to be challenged to justify and improve their answers.

Both groups could have benefited from reflective questions posed by the teacher. He could have encouraged them to create a table or articulate their strategy before the secret distance was revealed. While they were formulating a strategy, the teacher could have presented hypothetical scenarios such as: What if the distance is 50 metres? Or 100? Or 33.7? Such queries could have aided them in recognising that they could multiply the secret distance d by 5 for any given d metres or that addition might not be the optimal strategy for all distances. Given their knowledge that the robot took between 4.27 and 5 seconds to cover 1 metre, they could have investigated methods to determine a more precise value. The teacher could have asked the students how the margin of error would change for longer or shorter distances and discussed if it is sufficient with an inaccurate value.

Students' hands-on experiences and troubleshooting

Just a few seconds after task d) was announced, and while the teacher was still speaking, a student modified the programme provided in task c) by replacing the "cat meow" block with the "move 1 rotation" block. We interpreted this as an intuitive idea from the student. The sound of the meow and the sound of the robot's wheel running for one rotation were easy to count individually. Therefore, even if they were not conscious of the variable and the counting within the programme, they still utilised their experiences.

However, this group, along with others that tested their programme, quickly realised that the robot did not always halt at the red finish line. The students grappled with understanding the concept of stepwise instruction in relation to a loop and made assumptions that we had not foreseen. Instead of evaluating the meaning of the programme to find that the light sensor registered colour only after a whole rotation, they suggested that the robot "drives too quickly over it" (Student 3). And "maybe it [the finish line] is not sufficiently red" (Student 5). They were both questioning the capabilities of the light sensor (too slow/wrong colour) and thus worked on troubleshooting technical issues. In this situation, the teacher alternates between the groups with the aim of guiding towards counting numbers of rotations/seconds and might not be aware that several groups are grappling with this issue. This could be a crucial moment to halt the exploration, reassure the groups about the functionality of the light sensor, and emphasise that the key to comprehending the problem lies in going back and understanding the programme. This could involve a discussion about the programming concepts of condition and stepwise instructions in relation to a loop. However, we should utilise the potential that

lies in hands-on experiences with the robot (Zhong & Xia, 2020), like the sound effect in task c). A more tangible experience with stepwise instructions within a loop could involve prompting the students to determine what the robot does when the counter alters its value (it halts to do a colour measurement). Another possibility is to conduct a plenary session to test where the robot halts when the finish line is made thicker and what happens if the starting position of the robot is shifted by 10 cm. The latter activity will also have the groups reflect on the concept of conditions. It was already tried out by one of the groups, but not in a systematic manner. It is also possible to make the concept of conditions even more tangible by adding sound blocks to the programme in task c, that read the result of the colour measurement. These realisations are pivotal to initiate work on refining solution strategies.

The students' interactions with the robot stimulated suggestions to surmount obstacles. Some of these suggestions were deviations from the given problems. The students could be directed to systematically investigate the situation to discern which suggestions are relevant and which are not.

Summarising

Through the given task in four steps, the students experienced mathematics to be useful for solving problems, and the hands-on experience with the robot aided their mathematical work. Working with the task provided opportunities for the students to engage in several mathematical content areas, in contrast to how they normally worked, one topic at a time. The nature of the tasks allowed for the precision of their solutions to be revealed, and the students could be given the opportunity to refine their strategies. The repeating sound in task c) gave an intuitive experience of what a loop is, and the stops by the robot between loops helped the students count the repetitions to predict and check their programmed measuring results.

When considering the improvement of task execution, we found that the teacher's role is crucial due to the open-ended nature of the task and the technical environment of programmable robots. While the teacher could have divided and restricted the problems, doing so would have compromised their openness. To enhance the learning experience in task c), the teacher could provide step-by-step instructions within a loop, prompting students to observe the robot's actions when the counter changes its value. Additionally, a plenary session could be conducted in task d) to test the robot's behaviour when the finish line is made thicker, or the starting position is shifted. There are opportunities for generalising the solutions in task b) by changing the input variable by including a general formula in their coding, but this was not observed during the intervention. We did not identify generalised solutions in task d) either, possibly due to time constraints. They could for example have programmed the output to display the race distance in centimetres instead of the number of rotations. In further interventions, we will therefore suggest placing greater emphasis on this part of the solution process to investigate the intersection of algebra and programming as part of generalisation in mathematics.

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Investigating the potential of GenAI as a scaffolding tool in Mathematics Education

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In the beginning of the 2020s, a number of Generative Artificial Intelligence (GenAI) systems emerged and started to be used by the students for their ability to process natural language input and give immediate feedback. Although schools are mainly struggling to reduce the possibilities of cheating, researchers in education are interested to understand if and how generative AI technologies might open up new possibilities and approaches to improving student learning. In this regard, we seek to explore the potential of GenAI to support students mathematical learning. This work presents an a priori analysis in which researchers simulated the use of a GenAI system, by undergraduate students, to explore if and to what extent GenAI can be used as a scaffolding tool for learning the concept of function continuity. Results reveal the high potential of the tool but underline the need to further investigate the nature of the changes that GenAI brings to mathematics teaching and learning.

Keywords: GenAI, scaffolding, function continuity.

Introduction and rationale

In recent years, the integration of artificial intelligence (AI) into education has expanded considerably, giving new impetus to Artificial Intelligence in Education (AIEd). AI can enhance teaching, learning, and decision-making processes by automating tasks such as student assessment, providing feedback, and identifying learners at risk. Moreover, AI can personalize learning experiences through tutoring, adaptive content, and support for self-regulated learning (Hwang et al., 2020). In mathematics education, AI can play a crucial role in addressing challenges such as generating problems, tracking student progress, and evaluating performance. As recently highlighted, these innovations particularly boost student engagement and empower educators within STEM education. In this context, three major paradigm shifts can be identified to define AI's evolving role in STEM education (Ouyang et al., 2021): a) *AI-Directed Education* – Based on behaviorism, this model sees AI as the main instructor, leading students through predefined learning paths. AI is seen as a "black box" teaching machine that can present the knowledge subject and guide the learning process through enquiry and by providing the learner with knowledge of correctness (Burton et al., 2004). b) *AI-Supported Education* – Grounded in cognitive and social constructivism, this model points on the collaboration between AI systems and the learner to focus on his/her learning process. AI can provide personalized feedback and real-time assessments to guide student learning, through the information collected from learner's input. The mutual interaction between AI and the learner moves toward a more learner-centered learning (du Boulay, 2019). A main issue is the complexity of the interaction due to the complexity of both the learner and the system. c) *AI-Empowered Education* – Based on the notion of learner agency (Bandura, 2006), this model sees AI as a tool to augment

human intelligence and foster collaboration. AI can assist in decision-making, offering personalized support and feedback, while learners take a leading role in their education. Central in this model is the promotion of a significant interaction between the learner and the AI system that can give as a result adaptive personalized learning (Hwang et al., 2020). Assuming these three paradigm shifts can be considered for mathematics education as well, we claim that the role of GenAI as a supporting learning tool still needs to be investigated.

In our work, with the aim to explore if and how the second paradigm—AI-Supported Education—can be realized, we consider the notion of scaffolding. As in the field of educational psychology, we refer to scaffolding as the process of providing structured support to learners in order to help them achieve tasks just beyond their current capabilities (Wood et al., 1976). As students gain proficiency, this support is gradually reduced, allowing them to become independent learners. In our view forms of scaffolding can be applied by using AI systems that not only provide personalized feedback and real-time assessments but also adjust the level of guidance based on the learner's evolving needs. We are interested in exploring if, by dynamically responding to student progress, AI can offer more targeted interventions, enabling students to engage with complex tasks while maintaining an appropriate level of challenge.

To understand whether and how GenAI could be used as a scaffolding tool to support learners in developing mathematical concepts, we conducted an a priori analysis of the tool's potential. It is important to note that we did not work directly with students: the first two authors rather tested a GenAI system through a simulated interaction. This decision was driven by the need to first evaluate the tool's effectiveness in addressing learning challenges before moving on to testing it in real context with students.

Results are analysed within the framework of cognitive apprenticeship (Collins et al., 1987) in mathematics education and reveal the high potential of GenAI in mathematics education. In addition, our findings highlight the need to think about how these tools can change the way teachers and students interact with the mathematics, and how to conceptualise and analyse the nature of the changes that GenAI brings to mathematics teaching and learning.

Conceptual Framework

To analyse the potential of GenAI as a scaffolding tool, this paper adopts the concept of cognitive apprenticeship grounded on Vygotsky's well-known concept of the Zone of Proximal Development (ZPD) (1972). Vygotsky's theory was extended to the K-12 educational system, by Wood et al. (1976), introducing the notion of scaffolding, according to which teachers provide support, which they call scaffolding, for students to master their learning. Then, when students have mastered the material, teachers remove the scaffolds, much like a structure, and as students learn new material using new scaffolds (tutorials, videos, visual aids, etc.), the ZPD expands outward. Collins et al. (1987) further developed the notion of scaffolding, introducing the model of *cognitive apprenticeship*, designed to make thinking processes visible and explicit in the learning environment. It emphasizes social interaction and the development of cognitive and metacognitive skills to 'use, manage and discover knowledge', through six main processes. The basic processes –modelling, coaching and scaffolding– support the development of both cognitive and metacognitive skills. The next two,

articulation and reflection, are designed to introduce novices to expert-like problem-solving and execution techniques. The final stage, known as exploration, aims to guide the novice towards self-sufficiency and the ability to autonomously solve and recognise problems within the field.

In our study, we analyse a simulated student-GenAI interaction exploring how and to what extent GenAI can perform each of the six steps that characterise cognitive apprenticeship.

Methods

With the aim to investigate the potential of GenAI as a scaffolding tool for students learning, in this paper we focus on the following research question: How can the model of cognitive apprenticeship allow us to investigate the potential of GenAI as a scaffolding tool for mathematical learning?

To answer our question, this paper offers an interpretive a priori analysis: two researchers (the first two authors) investigated the advantages and disadvantages of using GenAI to scaffold undergraduate students learning in a calculus course. The topic chosen for this study –function continuity– comes from a calculus course for biology students taught by one of the authors. Based on the instructor's experience, function continuity is one of the most challenging topics for students in this course, often requiring extensive justification and additional explanation to ensure understanding.

A lecture on the function continuity given in a calculus course for first-year students of the bachelor's degree in biology was audio-recorded and www.turboscribe.ai was used to automatically generate subtitles for the recording. The whole text was uploaded to a GenAI system (Capone & Faggiano, to appear) and, based on the transcription a conversation was developed with GenAI. The questions posed to GenAI during the simulation were based on existing literature and on previous experimentations (which aimed at investigating the common misconceptions and difficulties on function continuity encountered by undergraduate students not enrolled in mathematics courses). In what follows we briefly mention the relevant existing literature.

The learning of the concept of function continuity: related literature

It is well known that students are often unclear about the formal definition of a continuous function (e.g. Jayakody & Zazkis, 2015). A significant part of this difficulty stems from the relationship with the concept of limits which has been proven difficult to learn (e.g. Juter, 2005). Very common is the use of an intuitive (concept) image in which a continuous function is described throughout its domain as a function whose graph can be drawn continuously, i.e. without 'breaks', 'tears' or 'jumps' throughout the fixed domain (Tall & Vinner, 1981). Of course, such a description of continuity should not omit an explicit reference to the continuity of the domain. Students do not always understand how to adequately separate the concept of continuity from its intuitive graphical visualisation. It can happen, for example, that the evaluation of the function continuity whose Cartesian graph cannot be drawn becomes a difficult obstacle to overcome, precisely because it is impossible to visualise continuity or discontinuity through the characteristics of the graph. Moreover, the extensive use of continuous functions in examples and exercises in textbooks can lead to the misconception that continuity is the rule and discontinuity the exception. In other words, the student seems to implicitly associate the concept of 'function' with the property of 'continuity', without realising that a continuous function should instead be considered as a (very) special case of a function. The

interaction with GenAI has been developed taking into account this related literature concerning the learning of the concept of function continuity.

Investigating the potential of GenAI

In this section we report some extracts of the conversation between the researchers and the GenAI system. We then reflect and discuss on what emerged in the light of our framework, seeking to interpret what happened in terms of the different steps that characterise cognitive apprenticeship.

In the following transcript, the sentences attributed to the "User" were written by the researchers, acting like a student who wanted support in her learning. The answers processed by the GenAI system appear as given by "Assistant".

1	User:	Hi, I am enclosing a transcript of a lecture on the concept of continuity in mathematics. [The researcher pasted the transcript of the lesson.]
2	Assistant:	Thanks for sharing the transcript of the lecture. It seems very rich in examples and detailed explanations of the concept of continuity and limits in mathematics. Here is a list of key points from the lesson on the concept of continuity of functions. […] Would you like to make changes, adjust some parts or elaborate further on some specific concept?

We start asking GenAI to present a series of exercises based on the transcribed lecture.

3	User:	Can you give me some exercises on the lesson I gave you?
4	Assistant:	Of course! From the analysis of the lesson transcript, here is an exercise for you. Check the continuity of the function f(x)=|x| at the point x=0.
5	User:	I think the function is continuous at x=0? Is my solution correct?
6	Assistant:	Yes, your solution is correct
7	User:	Can you show me your solution to compare with mine?
8	Assistant:	Of course! […]

The system gives a correct and clear solution and then the researchers tested the system with some question on concepts that were not covered in the transcribed lecture.

9	User:	Yes, thank you, I did not understand one thing: is the function $1/x$ continuous?
10	Assistant:	The function $1/x$ is continuous in its natural domain, which is $\mathbb{R} \setminus \{0\}$, which is the set of real numbers except $x = 0$. However, at $x = 0$, the function is not defined (because you would have division by zero) and is therefore not continuous at that point.
11	User:	But if it is continuous, why can't I draw it without taking the pen off the paper as was said in the lecture?
12	Assistant:	Good point! […] When we say that a function is continuous 'without taking the pen off the paper', we are referring to a function that is continuous over an entire interval. In the case of $1/x$ the problem is that there is a point of discontinuity (i.e., a gap) at $x = 0$. Therefore, the graph has a gap and, to draw it, you would have to take your pen off the paper when you go through […]
13	User:	You told me that at $x = 0$, there is no continuity because the function is not defined. If there is no continuity, how can there be discontinuity?
14	Assistant:	You are right to raise this question, and it is a crucial point in understanding the relationship between continuity and discontinuity […] In summary, in the case of $1/x$, there is no discontinuity in $x = 0$ because the function does not exist at that point.
15	User:	So, if a function is defined at a point, will it also be continuous at that point?

16 Assistant: No, a function defined at a point is not necessarily continuous at that point. For a function to be continuous at a point, three conditions must be met: The function must be defined at that point. The limit of the function must exist when x tends to that point. The value of the limit must be equal to the value of the function at the point. If one of these conditions is not satisfied, the function is not continuous at that point, even though it is defined there.

The researchers then wanted to test the system's ability to generate new examples.

17 User: Can you give me an example of that?

18 Assistant: An example of a non-continuous but defined function is the step function:

$$f(x) = \begin{cases} 1, & if\ x < 0 \\ 2, & if\ x \geq 0 \end{cases}$$

In this case, the function is defined at $x = 0$ (its value is 2), but is not continuous at $x = 0$ because: The left and right limits do not coincide and the limit at $x = 0$ is not equal to the value of the function at $x = 0$, which is 2. Therefore, although the function is defined in $x = 0$, it is not continuous at that point.

As we were unable to provide a correct interpretation of the graph obtained, GenAI has been tasked with constructing a graph in a Cartesian plane with x and y axes.

19 User: Can you show me the graph?

20 Assistant: Here is the graph of the step function you asked for. [Fig. 1 was displayed] As you can see, the function takes the value 1 for x<0 and the value 2 for $x \geq 0$, with an open circle at $y = 1$ in $x = 0$ and a full circle at $y = 2$ in $x = 0$

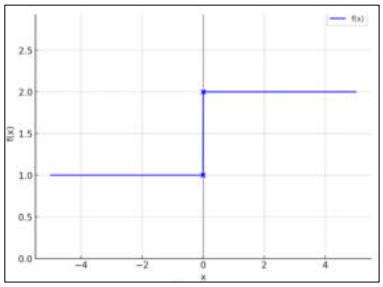

Figure 1. Graph of the step function

The researchers continued to test the capabilities of the GenAI by generating and solving new exercises, introducing variations in the numbers and increasing the level of difficulty.

21 User: Can you assign me some more difficult exercises?

GenAI therefore provides more complex exercises based on the topics discussed, enabling students to challenge themselves and deepen their understanding. A sample solution was then requested, as it could serve as a useful reference for comparison or offer new insights, especially given the increased complexity of the exercises.

GenAI was then asked to generate a summary of the lesson and based on this summary, to pose some questions to the user.

22 User: Finally, can you ask me some questions to verify what I understood of the lesson?

23 Assistant: Of course! Here are some questions to check your understanding:

Definition of continuity: What are the three conditions that must be fulfilled for a function to be continuous at a point c?
Limits and continuity: How can you check whether a function is continuous at a point where the denominator cancels? What do you need to check?
Functions and isolated points: What does it mean if a function is defined at an isolated point? Is the function always continuous at these points?
Graph and continuity: If you cannot draw the graph of a function without taking your pen off the paper, does this necessarily mean that the function is not continuous?
Discontinuities: What are the main types of discontinuities that a function can have? Can you describe one of them with an example?
Try answering these questions and let me know if you have any doubts!

Discussion

In this study, the researchers simulated the actions of a hypothetical student to explore how GenAI could serve as a support tool for their learning. For this reason, in this section, we put the word "student" in quotation marks, as we are referring to them in a hypothetical context. In order to thoroughly analyse the discussion with GenAI in relation to the six steps that characterise apprenticeship learning, we developed and identified a set of categorisation criteria (see Table 1) designed to systematically map the different aspects of the interaction.

Table 1: Identification Criteria for the conversation's analysis

Process	Description	Identification Criteria
Modeling	The expert demonstrates the desired behavior or concept.	Clear definitions, explanations of concepts and the key aspects.
Coaching	The expert provides personalized support and feedback based on the learner's progress.	Adaptive feedback on exercise solutions.
Scaffolding	Temporary support that helps the learner complete difficult tasks, gradually reduced as competence grows.	Breaking complex problems into simpler steps, context-specific suggestions based on particular requirements, progressive guidance that diminishes as the learner gains autonomy.
Articulation	The learner is encouraged to explain and articulate his/her thinking.	Questions that prompt the learner to explain his/her reasoning and justify his/her answers.
Reflection	The learner compares his/her reasoning with that of the expert and reflects on mistakes.	Comparisons between the learner's answers and correct solutions, reflection on choices made during the process.
Exploration	The learner explores independently, solving problems and discovering new applications of the concept learned.	Complex or more difficult problems, decreasing support to stimulate independence and self-directed exploration.

Analysing the conversation between GenAI and the researchers we can see that, when equipped with lecture's transcript as a knowledge base, GenAI can act as a useful *modelling* tool providing a summary of the key points (excerpt 2). Our results also show that GenAI can support individual *coaching* providing some exercises and feedback on their solutions (exc. from 3 to 6). When the "student" asks for support on things she cannot understand on her own (exc. 7, 9, 11, 13), GenAI provide customized *scaffolding* trough contextualized hints and detailed explanations based on their specific needs (exc. 8, 10, 12, 14). In response to requests for explanatory examples and representations (exc. 15, 17), GenAI provides accurate and precise responses, facilitating a deeper understanding of the key aspects (exc. 16, 18). However, it is unclear whether GenAI can always adapt its level of assistance in a manner that is responsive to the evolving competence of the student.

The question thus arises as to whether the assistance provided by GenAI can be gradually reduced as the student becomes more independent and capable of addressing problems autonomously, selecting their own approaches and solutions. When "student" asks GenAI to pose questions about the key conceptual points from the lesson uploaded by the lecturer (exc. 20 and 21), she is encouraged to *articulate* her own reasoning. This process helps her grasp the importance of asking the right questions and thinking aloud, fostering deeper understanding and active engagement. Through this kind of interaction, "student" can develop critical thinking skills and learn to navigate complex concepts more effectively. The request to generate more complex tasks based on the lesson uploaded (exc. 19), and to provide solutions, enables "student" to engage with more advanced problems. This approach allows her to self-assess the skills she has acquired and *reflect* on her own solutions, promoting deeper understanding and independent learning.

Conclusion

This study shows the role of AI within the paradigm of AI-supported education. With the aim to explore the potential of GenAI as a scaffolding tool for students learning, in this paper we attempted to investigate if there is such a potential and how this can be interpreted through the model of cognitive apprenticeship. The findings of this study reveal that when students share the text of a lecture with GenAI, it can provide them with advanced, targeted and personalised support. In fact, we were able to experience the ability of GenAI to summarize lessons (modelling), provide examples (coaching), to develop, solve problems and contribute additional information (scaffolding), simulate interviews (articulation) and propose more challenging activities (reflection). Referring to the six steps of cognitive apprenticeship, the analysis of our results seems to miss the last step, autonomous exploration (exploring). In fact, we were unable to assess this aspect, which is linked to the students' attainment of independence, as our experiment did not involve direct interaction with them.

Although our simulation explored the use of GenAI as a tool to give students the opportunity to explore ideas, ask questions, and receive real-time explanations, we acknowledge that this only represent one step in the broader process of learning a new mathematical concept. Despite the high potential of the tool revealed by this study, further investigation concerning the nature of the changes that GenAI brings to mathematics teaching and learning is needed. For example, we seek to understand how and if different types of prompts—ranging from open-ended questions to more structured guidance—can be used by the learners to support their mathematical learning. We also are interested in exploring the possibility of creating an inclusive learning environment where each student can progress at their own optimal pace with tailored support. To gather more valuable insights on the effectiveness of GenAI as a scaffolding tool, we have already planned additional experiments that directly involve students. This will help us understand how well the AI-supported framework works in a practical classroom environment.

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Examination of eye movements of primary school 4th-grade students during mathematical problem-solving*

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The study aims to examine differences between primary school students' eye movements during routine and non-routine problem-solving. Heat map data grouped by achievement levels were compared for routine and non-routine problems. Eye movement recordings and think-aloud data were coded as 'question-to-answer' and 'answer-to-question'. Problem solving time and number of fixations were also compared. Our findings show a difference between the two types of problems in terms of eye movements.

Keywords: Eye movements, primary school students, problem solving, routine problem, non-routine problem.

Introduction

The eyes are seen as a window to the mind. We perceive the world with our eyes, and almost everything we do during our waking hours involves eye movements (Karatekin, 2008). Eye movements are a vital part of our interaction with the world. They play a crucial role in perception, cognition, and education (Wade & Tatler, 2005). An eye tracker, as stated by Duchowski (2017), is the most used device for measuring eye movements. Eye tracking can detect where users look at a point, how long they look at something, and the path of their eyes (Schall & Bergstrom, 2014). Research by Schindler et al. (2016) showed that eye-tracking provides in-depth information about the processes that students follow when solving tasks. Studies in mathematics education show that eye tracking can provide information about the attentional processes involved in solving mathematical problems (Norqvist et al., 2019).

When classifying mathematical problems, a common distinction is made between routine and non-routine problems. It is challenging to draw a clear line between routine and non-routine problems, as they can vary from person to person. However, the key difference lies in whether the problem requires higher-order thinking skills. Problems involving logical reasoning, inference, or prediction often demand such skills and are likely to be non-routine, which can generally be identified using specific criteria. Routine problems can be solved by applying methods that students have previously learned step by step (Polya, 1957; Woodward et al., 2012). For example, consider the routine problems presented in Figures 1 and 2. These types of problems are structured and familiar, enabling individuals to solve them using their existing knowledge and skills. Since they can be resolved straightforwardly by applying specific formulas or algorithms, they are classified as routine problems. In contrast, non-routine problems, as illustrated in Figures, require individuals to think creatively, tackle unfamiliar

* This research was produced from the first author's master's thesis entitled "Examination of eye movements of primary school 4th-grade students during mathematical problem-solving".

situations, and develop solutions beyond conventional approaches. Solving such non-routine problems demands higher-level skills beyond basic operational abilities, such as organizing and classifying data or identifying relationships (Souviney, 1989). While routine problems can be solved using basic arithmetic skills, non-routine problems often allow for multiple solution pathways (Woodward et al., 2012). Although not classified as “non-routine problems” in the literature, there are some eye-tracking studies that examine such problems (Norqvist et al., 2019; Rożek et al., 2014; Sajka & Rosiek, 2015). There is a limited number of studies that combine eye-tracking techniques and non-routine problem-solving methods. This study aims to address this gap by examining the eye movements of fourth-grade primary school students during mathematical problem-solving. Accordingly, the central research question guiding this study is: "Is there a significant difference in the eye movements of fourth-grade primary school students when solving non-routine versus routine problems?"

Method

The method of this research is a holistic multiple-case study, which is a qualitative research design. The study group consisted of 12 fourth-grade students from a primary school in Istanbul/Türkiye. These students were categorized into three levels of mathematics achievement: high (six students), medium (three students), and low (three students), based on their math grades. All high-achieving students were male, while the medium-achieving group included two females and one male. The low-achieving group consisted of three female students.

Study procedure

The first author conducted the study in a laboratory setting, and participants were informed about the process. During the application, the participants were shown consecutive slides on the computer screen and asked to read the problems aloud and express their thoughts aloud during the problem-solving process. The researcher provided reminders as needed to encourage participants to keep thinking aloud, and the process was recorded with a voice recorder. The research was conducted in two sessions; in the first session, non-routine problems were asked, and in the second session, routine problems were asked. No time limit was imposed on the students during problem solving and the use of tools such as paper and pencil or calculators was not allowed. Due to the non-routine problems with individual differences described in the introduction, one-on-one interviews were conducted with the students after the problem-solving process. The students mentioned that they had not encountered the non-routine problems (from session one) before. In contrast, the routine problems (from session two) reminded them of questions from their textbook or those previously asked by their teacher.

Data collection and analysis

In this study, an SMI eye-tracking device was used. Participants were presented with seven routine and eight non-routine math problems in two separate sessions, and their eye movements were recorded throughout the process. The higher count of non-routine problems is due to one of them consisting of two stages. As part of the initial analysis, heat maps were used both to compare students' approaches to the two problem types and to analyze their performance according to their level of achievement. Heat maps were first compared with participants' success levels to understand the

distribution of their attention and focus areas during the problem-solving process. Since the space limit, only two routine and two non-routine problem heatmaps were provided in Figures 1 and 2. Secondly, scan path videos, gaze replay videos, and think-aloud data were categorized under two predefined codes—'question-to-answer' and 'answer-to-question'—to uncover participants' problem-solving strategies. By coding and analyzing these data, we examined the flow of information in students' problem-solving processes. This analysis provides important clues to understanding students' problem-solving strategies and how they access information. Finally, the comparison of the average durations and number of fixations between the two problem types was conducted to determine which problem type students found more challenging or time-consuming. SPSS version 25 was used to analyze the differences in problem-solving times and the number of fixations between routine and non-routine problems. After confirming that the solution time data were normally distributed, a dependent samples t-test was applied. However, since the fixation data were not normally distributed, the Wilcoxon signed-rank test was used for comparison. These statistical comparisons were part of the quantitative analysis, while the qualitative analysis involved coding and evaluating the eye movement videos. This analysis is necessary to understand how the students' cognitive load and problem-solving speed change according to problem types. In this way, it was possible to obtain information about how the problem types used affect students' thinking processes.

Validity and reliability

The problems used in the study were administered to 85 fourth-grade students who were not part of the application group, and the Cronbach's Alpha coefficient obtained from the reliability analysis of these problems was found to be 0.703.

Results

This section examines how students interact with routine and non-routine problems, highlighting differences in their responses and eye movement patterns across various achievement levels. First, the findings are supported by heat maps (Figures 1 and 2), which illustrate the variations in students' eye movements as they engage with each type of problem. To enhance the clarity of the figures, English translations of the problems are provided before the figures themselves.

English translation of the routine problem in Figure 1:

"314 male parents and 492 female parents attended the PTA meeting. In which of the options is the number of parents attending the meeting rounded to the nearest hundred?"

English translation of the non-routine problem in Figure 1:

"While a tourist was visiting a cave in Antalya, they saw the following code on the wall. He realized that there was a relationship between the top number and the bottom number. However, some parts of the code were left blank. What numbers should go in the blank spaces of the code?"

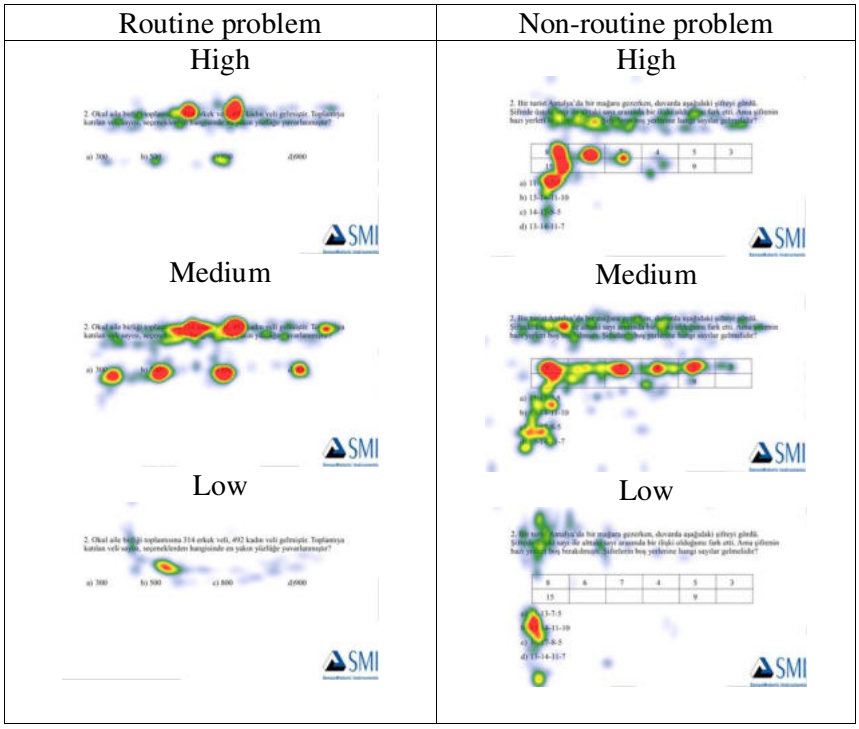

Figure 1: Heat maps of routine and non-routine problems

When the heat maps in Figure 1 are analysed, in the routine problem, the participants at high achievement and medium achievement levels made more fixations on the correct answer to the problem and the relevant statements of the problem. The low achievement level participant, on the other hand, made scattered fixations and did not focus on the relevant parts of the text for the solution of the problem. When the visual attention of the participant at the low achievement level was examined in the non-routine problem, it was observed that although he focused on the expressions "the number above" and "the number below" on the text, his attention on the visual figure was very little and he did not make a correlation between the numbers above and below on the figure. Especially when compared with the other achievement levels, it is seen that the participants at the high achievement level and at the intermediate achievement level looked for a relationship between the boxes at the top and bottom of the visual figure and focused on those parts. When the two problem types are compared, it is seen that participants made more fixations in the non-routine problems. This supports that non-routine problems lead to more cognitive effort.

English translation of the routine problem in Figure 2:

"Ayça and her friends are going to have pizza for lunch. There are 5 pizzas with 12 slices in the pizzeria. If Ayça and her 12 friends each eat 4 slices of pizza, how many slices of pizza will be left?"

English translation of the non-routine problem in Figure 2:

"Using two straight lines, divide the clock face below into three parts so that the sum of the numbers in each part is the same. What is the sum of each of these three parts?"

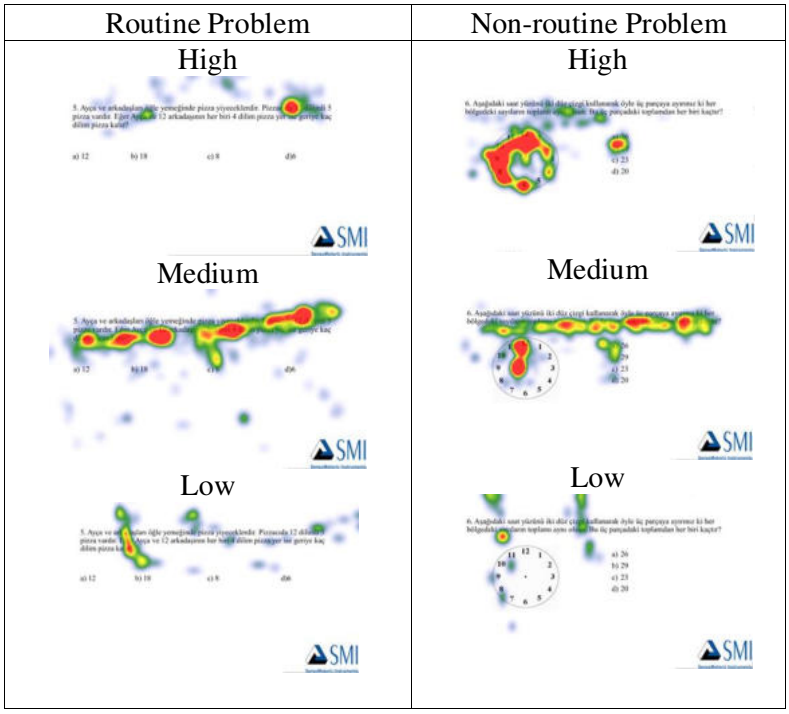

Figure 2: Heat maps of routine and non-routine problems

When the heat maps were analysed, in the routine problem, the participants with high achievement levels focused on the important parts ("12 slices", "Ayça and") of problem-solving. The participants at the medium achievement level focused on many parts of the problem. The low achievement level participant focused on the irrelevant parts of the problem. When the non-routine problem was analysed, the participant at the high achievement level applied the "guess and check" strategy and tried to find the sum of different parts and found the result. The participants at the middle achievement level focused on the problem text and solved the problem with a different strategy. The participants at the low achievement level focused on the irrelevant parts of the problem. When the heat maps were analysed for the routine problem, participants with high achievement levels focused on the critical aspects of problem-solving: the 'three parts' word, the 'same' word, and tried different numbers on the clock as part of the guess-and-check strategy.

This study was conducted to understand the problem-solving strategies of fourth-grade students in routine and non-routine problems. Accordingly, the responses to routine and non-routine problems were coded in terms of "question-to-answer" and "answer-to-question" approaches. The related coding is presented in Table 1. We needed to conduct such an analysis to examine the effects of multiple-choice questions on eye movements and thinking processes and to understand how students adopt different approaches to different problem types.

Table 1: Table on the coding of the answers to routine and non-routine problems from 'from question to answer' or 'from answer to question'

Problem	Routine Problem		Non-routine Problem	
	From question to answer	From answer to question	From question to answer	From answer to question
P1	9	3	11	1
P2	10	2	10	2
P3	12	0	8	4
P4	7	5	12	0
P5	9	3	2	10
P6	8	4	12	0
P7	8	4	12	0
P8			11	1
Total	63	21	78	18

Table 1 shows the classification of responses to routine and non-routine problems under two methods: 'from question to answer' and 'from answer to question.' For the seven routine problems, the participants solved them 'from question to answer' 63 times and 'from answer to question' 21 times. For the eight non-routine problems, they used 'from question to answer' 78 times and 'from answer to question' 18 times.

The paired sample t-test conducted to compare the average time (in ms) required to solve non-routine and routine problems revealed a statistically significant difference. On average, participants took significantly more time to solve non-routine problems (M = 144627,8 ms, SD = 50007,4 ms) compared to routine problems (M = 72251,4 ms, SD = 27041,6 ms). The difference in means was significant, with $t = 5{,}614$ and $p < .05$.

The Wilcoxon Signed Rank Test conducted to compare the number of fixations between routine and non-routine problems revealed a statistically significant difference ($p = 0{,}002$, $p < 0.05$). Negative ranks (routine < non-routine) showed higher fixations for non-routine problems, with a mean rank of 6,50 and a sum of ranks of 78,00. No positive ranks or ties were observed ($Z = -3{,}059$), indicating that non-routine problems demand more attention. Accordingly, more fixations in non-routine problems indicate that there is more cognitive load in solving these problems.

Discussion and conclusions

The heatmaps obtained from the study revealed that participants directed their attention towards specific elements of the problem statements when faced with routine problems, as they were able to predict the solution. This suggests that their focus was concentrated on predictable and specific points. However, participants adopted a more exploratory approach when solving non-routine problems. They carefully examined various elements of the problem statements to explore possible solutions. This indicates that their attention was spread across a wider range, and they developed more flexible strategies during the problem-solving process. These findings are consistent with the findings of Zhu et al. (2022), where it was observed that experts' eye movements covered a larger

area, and they transitioned between different thinking modes more flexibly compared to novices. The increased focus also suggests deeper cognitive processing (Jian, 2017). The flexibility and in-depth attention demonstrated by the participants when faced with non-routine problems support the idea that cognitive processes were more intense.

When the heat maps are considered, and although gender was not a focal point in this study, the observation that high achievement levels were only present among male participants may suggest a potential effect of gender on mathematical problem-solving performance. However, this finding cannot be generalized due to the limited sample size and distribution, emphasizing the need for further research. Another important finding of the study is that as the participants' achievement levels decreased when faced with routine problems, their ability to focus on critical parts of the problem text also diminished. It was observed that participants with low achievement levels struggled to adopt effective problem-solving strategies and could not focus enough on important details in the problem text. This can be explained by the distraction of their attention and loss of motivation when they experienced failure, which negatively impacted their problem-solving processes.

Based on the previously described coding process, routine problems were solved more often by the "from answer to question" method than non-routine problems. These findings indicate that the rate of transitioning from question to answer was higher in non-routine problems, suggesting that participants engaged more in gathering information and exploration during non-routine problem-solving. The results reveal that non-routine problems generally require more complex thinking and advanced problem-solving strategies. Such problems force students to focus on analyzing the problem rather than directly starting from the answer to find a solution.

The study also focused on the solution times and the number of fixations of routine and non-routine mathematical problems. The problem-solving time and number of fixations indicate that non-routine problems require more time and a higher number of fixations than routine problems, as they demand more mental processing. This finding supports the notion that non-routine problems impose a greater cognitive load during problem-solving.

In this study, the data collection sequence was applied in the same order for all participants, which might have increased the influence of factors such as fatigue and loss of attention. For future research, it is recommended to shuffle the order of routine and non-routine problems among participants. Future research could utilize eye tracking as a methodological tool to investigate changes in primary school students' gaze patterns when engaging with routine and non-routine problems, particularly before and after interventions aimed at enhancing mathematical thinking.

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Heuristic trees and their fruits

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Heuristic trees are hint cards for problem-solving, structured in a tree shape, following specific design principles. Cards near the trunk of the tree show help in a compressed form, and the next cards along the branches decompress the content. Students' use of heuristic trees, implemented in a digital environment, has been studied for several years, and this paper reviews the results of these efforts. In particular, we address how the use of heuristic trees impacts students' engagement, ownership of solutions, problem-solving ability, and metacognitive skills, such as process-structuring and help-seeking.

Keywords: Problem-solving, Heuristic trees, Help-seeking.

"Every good tree bringeth forth good fruit" (Matthew 7:15–20)

Introduction

In this introduction, we argue that technological progress underscores the importance of teaching problem-solving. Heuristic trees to support students' problem-solving constitute the paper's main topic. Increasingly, tasks in daily and professional life can be automated. We believe this has profound implications for education: what was previously taught as a task that required brain and hands can now increasingly be outsourced to technology. Teaching students only to perform tasks and solve problems that can be outsourced to technology would leave those students empty-handed for their future professional lives. This seems to be an old problem: the boundary between outsourceable and non-outsourceable tasks has been shifting with the development of each new technology, from the steam engine to the combustion engine, to the mechanical calculator, to the digital computer. The latter two perform cognitive work, and, with the development of artificial intelligence (AI), the nature of outsourceable cognitive work is increasingly complex. Whereas at first computers could perform strictly delineated calculations based on formal input, with the arrival of generative AI, computers can process a wider range of prompts, including images and natural language. Hence, an important domain of hitherto uniquely human mathematical work, namely (horizontally) mathematizing a problem situation or formalizing a problem, can now to some extent be outsourced. Undoubtedly this outsourcing boundary will continue to shift across the cognitive domain in the future, and education needs to adapt accordingly.

The main response of mathematics educators to the increasing opportunities to outsource complex mathematical tasks is to give more importance to teaching problem-solving (Liljedahl & Cai, 2021). The goal is for students to build skills and experience working in situations where invoking a mathematical routine does not suffice. What is considered routine is subjective: non-routine mathematical tasks become routine as students develop mathematically. What is considered a routine mathematics task in a given society also progresses, as mathematical results and approaches are structured and polished into theories, involving new techniques which after some time are considered routine. Finding an antiderivative was once a non-routine, creative art, but is now a routine if

sometimes complex task. Moreover, this task can nowadays be outsourced to computer algebra systems. This brings us to an important point: there is a close connection between routine tasks and technology: considering a task as routine often coincides with understanding how to outsource the task to technology. When a routine cognitive task requires data processing, this can often be performed by software. Of course, routine, outsourceable tasks are taught in every mathematics curriculum, as they are the stepping stones and building blocks for approaching non-routine tasks. In this light, it is not sufficient to teach outsourceable procedures without some accompanying insight into the concepts and ideas underlying these procedures. This insight is strongly intertwined with the ability to recognize contexts where a procedure is applicable and the flexibility to adapt and apply a procedure to a context (National Research Council, 2002).

We have discussed above how a shifting outsourcing boundary underscores the importance of teaching problem-solving, including the ability to recognize which techniques a problem requires, and the ability to adapt and combine these techniques accordingly, possibly outsourcing routine techniques. The main concern of this paper is how to support students during problem-solving. Two common exclamations of students while working on non-routine problems are: "I don't know what to do!" and "I don't know how to do it!". The first refers to being unable to recognize what general concepts, techniques, or theories apply in a specific problem situation; and the second refers to the challenge of recollecting details and adapting them to the specific problem situation. We hypothesize that these inabilities display a lack of *compression* or an inability to *decompress*. Compression is the cognitive process of reorganizing mathematical knowledge, considering comparable cases as instances of an overarching category, and shifting focus from details of those cases to properties of that category (Sfard, 2008; Tall, 2013). In mathematics, compression applies to concepts, procedures, and statements (Bos & van den Bogaart, 2022a). *Decompression* is the opposite cognitive process: applying a general concept, procedure, or statement to a concrete problem situation: retrieving relevant details or steps, and adapting them to a specific case (Bos & van den Bogaart, 2022a). In a world where the outsourcing boundary shifts at a rapid pace, students should learn to compress mathematical concepts, procedures, and statements, to experience how these compressed constructs can form part of an approach to non-routine problems. This is the more abstract level at which they will work in their future professional life, in many cases outsourcing decompression to digital technology.

This raises the issue of how teachers should foster compression-decompression dynamics during problem-solving. An approach introduced by Bos (2017) suggests promoting this through structured hints in help-seeking episodes. When students are stuck, hints are first offered in compressed form, i.e. applicable concepts, procedures, and statements are shared in compressed form. Further hints, if necessary, are offered in increasingly decompressed form. This idea has been elaborated into the concept of a *heuristic tree*. A heuristic tree offers help on cards placed on the nodes of a *directed tree* (in the mathematical sense: an acyclic, connected, directed graph). The three main branches are labeled "orientation", "making and executing a plan" and "completion", reminiscent of Pólya's phases of problem-solving (Pólya, 1945). Each successive card, moving away from the initial node, decompresses the help offered on the previous card. In the next section, we discuss an example of a heuristic tree in detail.

In this paper, we aim to review four studies on heuristic trees—Bos & van den Bogaart (2022a and 2022b), Stokkermans (2024) and Zwennes (2024)—discussing issues of students' learning, problem-solving skills, reasoning, and ownership, and teachers' design. The motivation for this review is to share the possibilities and challenges of heuristic trees with the international audience at CERME.

Heuristic tree design

Figure 1 shows an example of two main branches of a heuristic tree for the problem: compute

$$\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1+\frac{1}{4}\right)\ldots\left(1+\frac{1}{1000}\right).$$

The online version can be accessed here: https://edspace.nl/htree/heuristiekboom.php?boom_id=403.
Cards can only be opened from left to right.

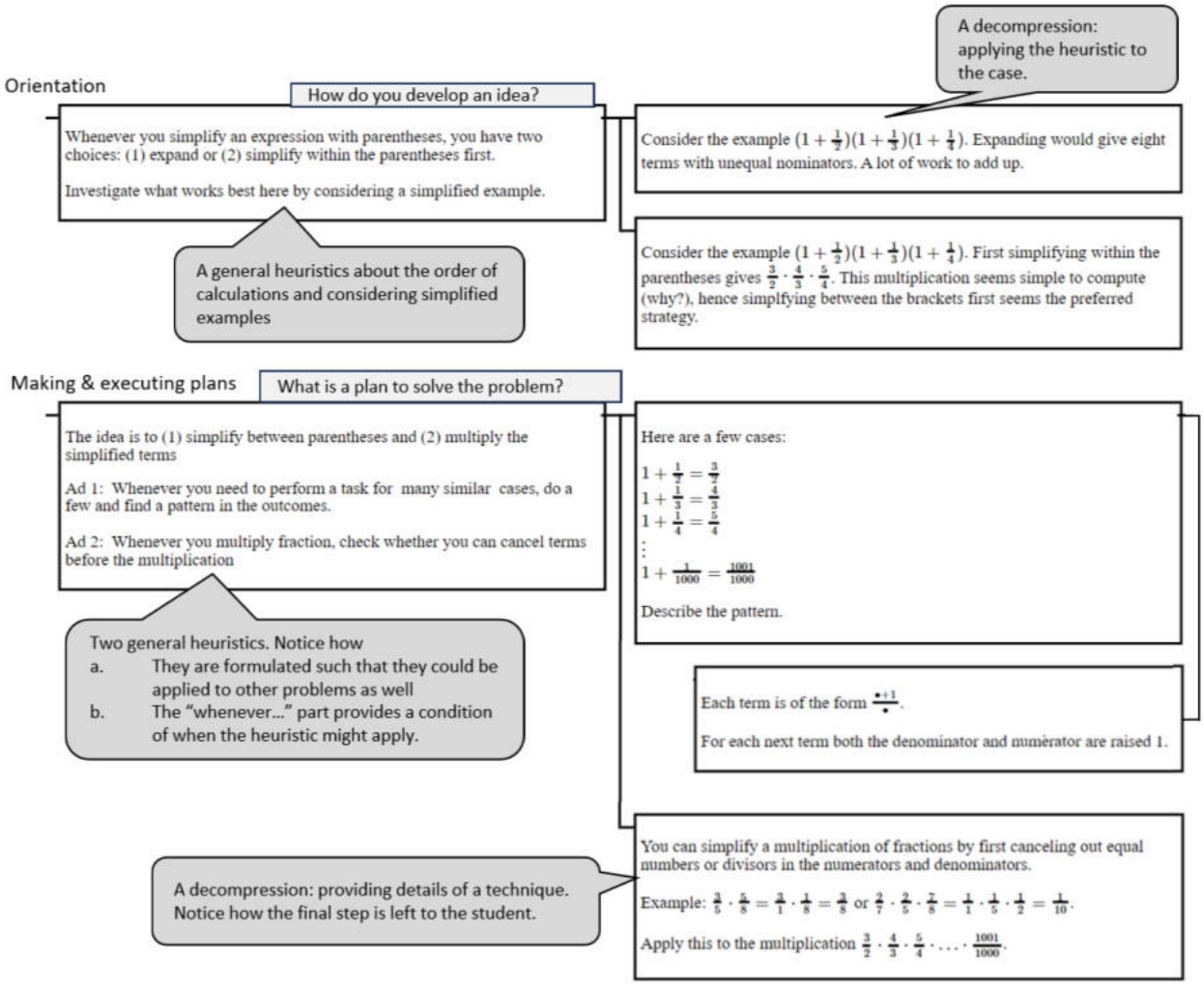

Figure 1: An example of a heuristic tree with all cards opened; with explanatory annotations

A unique heuristic tree is designed for each new problem, though leftmost cards can and should be recycled. The design of heuristic trees is based on six design principles (Bos, 2017):

P1. *Compression-decompression order*: The order along a branch should be from general to more concrete hints, thereby decompressing initial concepts, techniques, and statements.
P2. *Logical order*: The structure of the tree (both along and across branches) should represent a logical order of reasoning within a solution model. It should highlight the main structure of the argument and separate main and side issues.
P3. *Problem-solving phases order*: The branches should be ordered following the phases of problem-solving: orientation, making and executing plans, and completion.
P4. *Independence*: The help offered in any one branch should not require information from any other branch.
P5. *Rationing*: Each card should not give more help than needed.
P6. *Revelation*: The questions that are shown on unopened cards should indicate what help can be obtained along that branch, but not give away any content. In the same way, a hint on a card should indicate what can be expected in the next hint, but not give away the content.

Design principles P2, P3, P4, and P6 aim to facilitate students' navigation of the tree. Heuristic trees are intended to be used by students independently of the teacher. Students need to autonomously decide which cards to open, facilitated by cues on the back of the leftmost cards, and subsequently by cues on the front which indicate what help can be found on the next card. Design principles P5 and P1 aim to promote students' maximum ownership of their solutions. Whenever possible, solution steps and insights are left to the student.

Heuristic trees can be designed and shared with students in our online, interactive environment at https://edspace.nl/htree. Many trees designed by teachers are accessible here, mostly written in Dutch. Two English examples: https://edspace.nl/htree/heuristiekboom.php?boom_id=146 and https://edspace.nl/htree/heuristiekboom.php?boom_id=421. Though heuristic trees could be implemented with paper cards too, the digital version allows easy sharing among teachers and students, improving, and using wherever internet access is available.

Review study 1: students' help-seeking behavior

In a study conducted in 2019-2020, we studied students' help-seeking behavior during work on 55 problems in a number theory course over several weeks, while supported by heuristic trees (Bos & van den Bogaart, 2022a). Data was gathered through a survey, in-class camera observations of two small groups, an interview with those two groups at the end of the course, and an informal teacher's log. The latter showed that students in comparison with previous years posed approximately 90% fewer questions to the teacher while working on the problems, allowing the teacher to spend more time with students most in need of support. The videos showed how students can struggle to self-diagnose whether they need help, and what sort of help they need. However, many successful help-seeking episodes were registered. Sometimes students find that the hint cards confirm their chosen approach and this gives them confidence to continue their attempts. Both in the survey and the interviews, students indicate that they are less inclined to consult answer models and therefore stay engaged with the problem for longer. Also, they confirm that they experience more ownership of their solutions since the heuristic trees do not reveal the solution as a whole, as a model answer does. Finally, the data also support the hypothesis that students develop a more structured approach to problem-solving by being provided with help in a structured way. Their improved engagement, ownership, and problem-solving process are considered central results of the study.

The reviewed paper also addresses the issue of inflexibility in heuristic trees. The solution approach that is presented, the level of that presentation, the notation, and the assumed pre-knowledge are all fixed in the hints, whereas a good teacher can adapt to the student's needs. However, one of the reasons this was of limited impact was that students often sought help before they found a fruitful approach, and otherwise students often did choose the approach presented in the heuristic tree. Moreover, the designer of a heuristic tree (often the teacher) tries to adapt it to the expected approach and pre-knowledge of the students.

Review study 2: teachers' design of heuristic trees

A study in 2022 focused on the challenges of designing heuristic trees (Bos & van den Bogaart, 2022b). A group of teachers followed a workshop on designing heuristic trees given by the two authors of the paper, consisting of two meetings of one hour. In the workshop, the participants were acquainted with the design principles of heuristic trees, the underlying theoretical idea of compression, and a design protocol to shape the design process. The authors analyzed seven heuristic trees (totaling 153 cards in 48 branches) that the participants developed after the workshop. Below we discuss some of the main findings.

Figure 2 shows an example branch from an analyzed tree. The problem involved a geometric situation where a length had to be calculated. The example illustrates some frequently encountered issues. Firstly, the first card is not a heuristic and should have been phrased more generally, for example, "For a geometry problem try introducing a variable for a central unknown length, express the other relevant lengths in this variable, and express a relation as an equation in this variable". The questions "What do you want to find out precisely?" and "What is the consequence?" do not clearly indicate the content of the card to be opened.

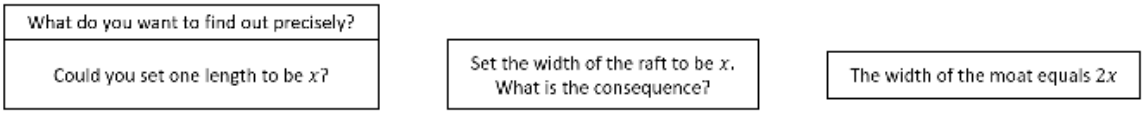

Figure 2: One branch of a heuristic tree designed by a teacher participating in the workshop

Implementing P1 is not straightforward. While 38 opportunities for decompression were taken, 24 opportunities were missed, alongside 12 cases where the first card could have been phrased more generally. A few additional principles should help implement P1 (Bos & van den Bogaart, 2022a):

P7. The first card of a branch should always also apply to other problems; no reference to the concrete problem situation is allowed.

P8. If possible, phrase the first hint on the branch in the form "If [condition], then …". The condition should indicate the class of situations where a certain heuristic or technique applies. Only in the next card is the suggestion applied to the concrete case (a decompression).

A related issue is that teachers tend to use each subsequent card on a branch for the next step of the solution (instead of decompressing anything). Instead, if a technique comprises three steps, then three cards should branch from one, such that students can go immediately to the specific step on which they require help. This inspires another new principle:

P9. Whenever possible, branch out vertically, not horizontally.

Issues with implementing other design principles were observed as well. For example, sometimes a hint gave away part of the solution without announcing this on the previous card, contradicting principle P5. Finally, formulating the hints on the cards proved a major challenge for teachers, as observed in the data (21 occurrences). Teachers tend to formulate in a conversational, somewhat imprecise way, without taking into account that students use the cards without them present to clarify themselves. All in all, the study reveals that designing a good heuristic tree is an art, with many challenges and pitfalls.

Review study 3: heuristic trees support proving

Next, we review the recent master's thesis by Stokkermans (2024), who studied whether and how heuristic trees can support the process of proving in secondary education. On the one hand, finding and writing proofs is just another kind of problem, so why would heuristic trees not help? On the other hand, in proof situations, students find it even harder to diagnose whether their "solution" is correct and whether they need help.

Stokkermans evaluated the students' proof skills based on five indicators:

1. Proof direction: does the reasoning lead to the right conclusions?
2. Assumptions: are assumptions made explicitly visible?
3. Conjunctions: is each step logically connected with conjunctions to the next?
4. Structure of the proof: is a cleaned-up step-by-step elaboration presented?
5. Justification: does each step contain a clear explanation of why it is allowed?

In each of the three iterations, approximately 20 students tried to prove either that the square of even numbers is even or that the sum of the angles in a triangle equals 180 degrees. In the first iteration, students failed on most indicators. A major issue was the lack of generality of their reasoning: many students thought providing a few examples was enough to prove a statement. The main finding of the study was that attention has to be paid to setting proof values and norms (Dawkins & Weber, 2017), both in the introduction to the lesson and again in the heuristic tree. In particular, students benefited from the addition of a fourth main branch representing a fourth phase, namely formulation and finalization. The hints in this branch guide students in formulating their proof, directing attention to the five indicators mentioned above. In this way, students improved their proof-writing skills, in particular in proof direction, conjunctions, and structure. Clarifying assumptions and justification remained a challenge. Stokkermans' work is exploratory and it raised many exciting challenges and questions about the opportunities of supporting proving using heuristic trees.

Review study 4: heuristic trees versus worked-out solutions

Does support from heuristic trees improve students' problem-solving ability? In a recent study, Zwennes and Bos studied this question through an experimental research design. They tested the use of heuristic trees against the use of worked-out solutions, the latter being the most common support during problem-solving in Dutch classrooms. The current paper describes the experimental design and presents the major result. The thesis (Zwennes, 2024) and a full paper (forthcoming) will present the full results, including findings on ownership and engagement.

Zwennes selected two 10th-grade groups that she teaches, with similar group sizes and mathematical achievements. Once a week for six weeks, the groups worked on a non-routine problem from their regular course book in a 15-minute session without the help of the teacher. The problem posed above Figure 1 is one of these problems; a selection of others is shown in Table 1. The experimental group ($n = 27$) was given a heuristic tree with each problem, and the control group ($n = 28$) a worked-out solution. In the seventh week, the students took a post-test, consisting of three problems similar to the ones they had practiced. For the post-test tasks, no help was available.

Table 1: A selection of the problems from Zwennes' study (2024)

	Problem	Link tree
Week 3	René buys fruit for exactly 10 euros. He buys some apples for 40 cents apiece, three times as many pears for 50 cents apiece, and with the rest of the money he buys bananas for 80 cents apiece. How many pieces of fruit did René buy?	#313 Link
Weeks 6	Harry writes down a sequence of numbers. He starts with 1 and the second number is 2. He computes every subsequent number as follows: he divides the second last number by the last number. Write down the 10th number in the sequence as a power of 2.	#322 Link
Post-test problem 2	Mr. van Dijk is 61 years old. He has four sons. Jan is twice the age of Piet. Piet is twice the age of Theo. Wim is five years older than Theo. Mr. van Dijk's age is the same as the sum of all his sons' ages. How old is Wim?	No tree

Zwennes graded the post-test results and Bos acted as the second grader, grading 20% of both groups. A Spearman's correlation was conducted to evaluate the relationship between the results of the first and second graders. There was a significant positive relationship between the results of the graders: $\mathrm{rs}(13) = .94,\ p \leq .001$, for the experimental group, and $\mathrm{rs}(13) = .91,\ p \leq .001$, for the control group. A pre-test was deliberately not included, but instead, the two groups were compared in a test only containing routine tasks on the same subjects as the non-routine problems they had worked on, a week after the post-test.

For the test containing routine tasks the groups had similar results: the experimental group $\mathrm{M} = 5.2$, $\mathrm{SD} = 1.7$, and the control group $\mathrm{M} = 5.2,\ \mathrm{SD} = 1.5$. Indeed, a Mann-Witney test showed there was no significant difference between the mean scores ($\mathrm{M} - \mathrm{W} = 349.5, p = .490$). In contrast, the mean scores on the post-test were as follows: the experimental group (heuristic trees) scored $\mathrm{M} = 6.3$, $\mathrm{SD} = 1.2$, and the control group (worked-out solutions) scored $\mathrm{M} = 4.6, \mathrm{SD} = 2.2$. From a Mann-Whitney test, we conclude that the experimental group scored significantly higher than the control group, $\mathrm{M} - \mathrm{W} = 91.5, p = .008$. This is an exciting result and more can be said to explain it, in particular from our results on the students' engagement and their learning of techniques (Zwennes, 2024).

Conclusion and discussion

From the presented reviews, we conclude that significant progress has been achieved in studying the impact of providing heuristics trees on students' help-seeking behavior and problem-solving skills. Evidence accumulates that promotes heuristic trees to support problem-solving when the teacher's availability is limited. At the same time, new avenues of research present themselves. Experiences during workshops suggest that teachers' scaffolding skills may benefit from designing and discussing heuristic trees with peers: a hypothesis worth investigating. Moreover, though Zwennes' work provides the first quantitative experimental evidence of the eligibility of heuristic trees over worked-out solutions, additional support through a larger experimental setup would strengthen this claim. It would be interesting to see whether heuristic trees could be designed to support more open-ended, explorative problem situations. Finally, while heuristic trees are gaining popularity within the Netherlands, it would be interesting to see whether similar results can be achieved across its borders.

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Changes in abstraction levels among undergraduate students during game programming using Scratch

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Computational thinking (CT) is a fundamental thinking competence strongly related to mathematical thinking (MT). Research shows that CT and MT share underlying skills, including abstraction. The current study aims to examine changes in abstraction levels among undergraduate students by introducing a sequence of programmable games in the Scratch environment. The scripts of these games share repetitive structures that reflect an inverse relationship between motion and rate. The degree to which students' scripts reproduce this inverse relationship throughout the game sequence can be indicative of a change in their abstraction level. The findings show that while most students implemented this inverse relation, their process of abstraction was not immediate. Rather, changes in abstraction were identified throughout the game programming sequence. Hence, this study introduces a novel way to enhance MT in a programming-based CT activity.

Keywords: Computational thinking, mathematical thinking, abstraction, Scratch.

Theoretical background

Computational thinking (CT) comprises "thought processes involved in formulating problems and their solutions, so that the solutions are represented in a form that can be effectively carried out by an information-processing agent" (Wing, 2011, p. 20). CT encompasses the development of four fundamental skills: (a) analytical thinking—the ability to analyze problems and decompose them into elements or parts; (b) algorithmic thinking—the ability to plan a sequence of steps to solve a problem; (c) debugging—the ability to monitor and correct errors in the execution of a plan; and (d) abstraction— the ability to decide which details should be highlighted and which can be ignored (Wing, 2011). Abstraction can also be defined as the ability to identify the most significant aspects of a problem and to create generalizable algorithms that make it possible to apply what has been learned in other problems (Montuori et al., 2024).

CT is closely connected to mathematical thinking (MT) "in the general ways in which we might approach solving a problem" (Wing, 2011, p. 3717). Studies stress the connection between CT and MT based on their shared skills, one of which is abstraction. Programming activities provide users with an opportunity to implement both CT and MT and to enhance their abstraction skills by identifying mathematical structures and patterns in codes, as described, for example, by Büscher (2025) and Miller (2019). Both studies focused on geometric tasks in elementary school carried out in different programming environments. Büscher (2025) showed that while programming a square, some students identified a pattern, used it to generalize about angles in the polygon, and subsequently applied this pattern to a hexagon. That is, they made an abstraction from the code of the square to the code of the hexagon. In Miller (2019), students were able to analyze repeating patterns when programming the act of drawing a square by inferring the connection between their codes and the

generalized structure of square's perimeter. Thus, through programming students were able to identify mathematical structures and patterns in codes and make abstractions. In addition to the focus on elementary students, a recent study examined in-service teachers' various methods for reusing Scratch code patterns (Ye et al., 2024). Integrating geometric knowledge regarding the parameters of regular polygons (such as size and shape), the teachers expanded their creation from a basic geometric shape made of a single polygon to composite shapes made of several polygons by reusing code patterns. The teachers discovered new ways to create drawings exhibiting rotational symmetry.

As these articles demonstrate, abstraction can be viewed as "the ability to see patterns, recognize the unit of repeat, deduce the pattern and abstract the general structure" (Miller, 2019, p. 924). This view of abstraction is also at the core of current research focusing on changes in abstraction levels among undergraduate students while programming a sequence of games in the Scratch environment. Throughout the sequence, several repetitive structures appear in the scripts, specifically mathematical structures that reflect an inverse relationship between motion and rate. The research examined the following questions: Will undergraduate students be able to express this inverse relationship in their game scripts? And what changes will emerge in their level of abstraction as they implement this inverse relationship throughout the sequence of games?

Methods

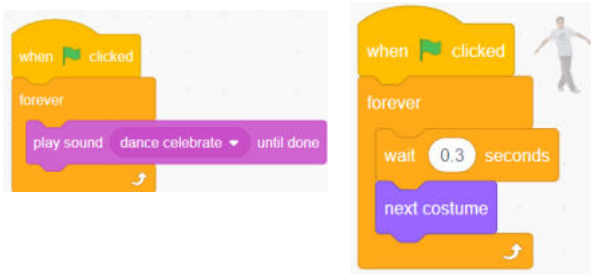
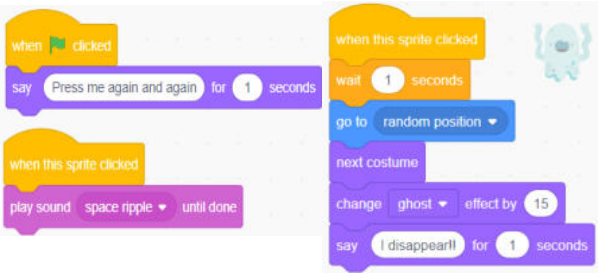
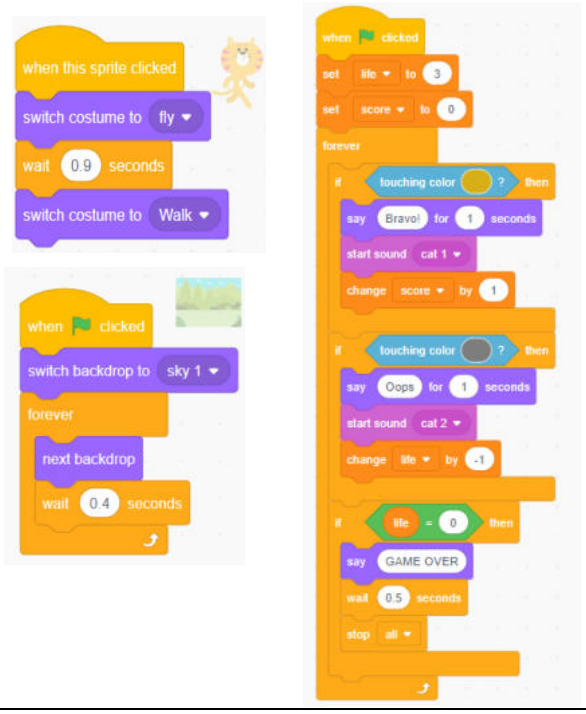
Participants and Procedure

The current study is part of a larger study that examines mathematical relations (such as direct relationship between motion and rate) in a sequence of programmable games. This qualitative study, which focuses on the inverse relation between motion and rate, explores changes in abstraction level among 47 undergraduate students majoring in education who were enrolled in a programming course on digital games in teaching and learning. The course included eleven 90-minute meetings. The Scratch environment allows users to program applications, animations and games by means of "blocks" that represent programming commands. Connecting the "blocks" generates a script, according to which the computer performs a sequence of actions. None of the students participating in this course had any prior programming experience in Scratch or in any other programming environment. The data collected at the end of each session included each student's script.

Task Design

The course consisted of a sequence of six games to be programmed in the Scratch environment. The sequence provided an opportunity to identify structures and mathematical connections that could be repeated in the sequence (i.e., abstraction). For example, three of the games (2-Dancing; 3-Ghost; 5-Cody Catches Coins) provide an opportunity to generalize about an inverse relation between motion and rate. The sequence also allows for the gradual use of computational concepts required for script programming (Brennan & Resnick, 2012), such as the use of loops (a command to repeat instructions in the script) and later to add conditions within the loops. Table 1 shows how the scripts of these three games can express the inverse relation between motion and rate.

Table 1: Possible game scripts and how they can express the motion/rate inverse relation

Game name and description	Game scripts	Expressing motion/ rate inverse relation
Dancing The dancer dances to the music without stopping.	Game scripts: Student 17 	Setting the rate of the costume change to a value *lower* than the default time of 1 second will generate a *faster* dance motion.
Ghost Repeatedly clicking on a figure will make it more and more transparent until it eventually disappears.	Game scripts: Student 4 	Setting the rate of disappearance to a value *lower* than the default value of 25% transparency will slow down the disappearance of the figure and make the game *longer* and more enjoyable.
Cody Catches Coins Clicking on the figure (Cody the cat) will cause the cat to jump and collect coins or leap over obstacles.	Game scripts: Student 42 	Setting the rate of the costume and backdrop change to a value *lower* than the default time of 1 second will generate a *faster* and more synchronized jumping and walking motion.

Data analysis

In accordance with the conclusions of Miller's qualitative study (Miller, 2019), we defined three levels of abstraction as reflected in the game scripts. In the current study, these levels described the extent to which the script reflected the inverse relation between motion and rate. Each student's script for each game was analyzed according to these levels.

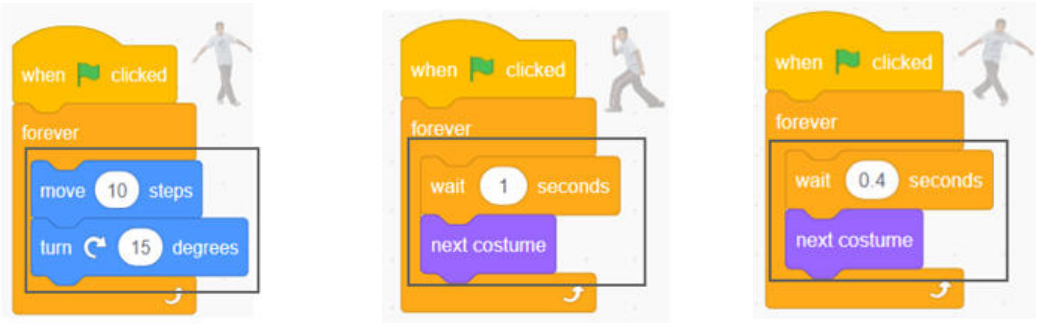
Level 0: No evidence of abstraction - the script does not include the desired animated motion.

Level 1: Partial evidence of abstraction - the script includes animated motion but does not include any reference to rate.

Level 2: Full evidence of abstraction - the script includes animated motion with reference to rate and describes their inverse relationship.

The data were analyzed in three steps. In the first step, the level of abstraction in each student's game scripts was determined according to the predefined level (0,1,2) for all three games in the sequence (Dancing, Ghost, and Cody Catches Coins).

The levels determined for the Dancing game were as follows: Figure 1a illustrates a script in which the abstraction level was rated 0. This script moved the character 10 steps and 15 degrees to the right without using a Scratch "switch costume" block. Hence, no dancing motion occurred. Figure 1b illustrates a script in which the abstraction level was rated 1. This script incorporated a "switch costume" block as well as a "wait _ seconds" block with a default of 1 second, resulting in a dance movement but at a rate that is too slow. Figure 1c illustrates a script in which the abstraction level was rated 2. This script uses a "switch costume" block as well as a "wait _ seconds" block set at a number lower than the default of 1 second. This script reflects an understanding of the inverse relation between motion and rate, because lowering the wait time between costumes changes the rate of costume changes, resulting in a faster dancing motion.



1a – Level 0 1b – Level 1 1c – Level 2

Figure 1: Three levels of abstraction, reflected in the dancing game scripts

At this stage, we also used Cohen's kappa coefficient (Cohen, 1960), to measure intercoder reliability, with the following interpretations: 0.60–0.79, Moderate agreement; 0.80–0.90, Substantial agreement; 0.91–1.00, Almost perfect agreement. Cohen's κ was calculated to determine the level of agreement between two researchers' judgments on three levels of abstraction as reflected in the 34 game scripts. The results indicated strong agreement between the two researchers' judgments ($\kappa = 0.864$, $p < 0.001$).

In the second step of the analysis, each student's sequence of game scripts was examined for changes in level of abstraction. We observed nine general patterns of change in level of abstraction: (a) remained at level 0; (b) remained at level 1; (c) remained at level 2; (d) advanced from level 0 to level 1; (e) advanced from level 1 to level 2; (f) advanced from level 0 to level 2; (g) dropped from level 2 to level 1; (h) dropped from level 2 to level 0; (i) dropped from level 1 to level 0.

In the third step, we examined the patterns of change for each student, first in the transition from the first and second game, and then in the transition from the second and third game. For instance, a level rated 0 in the first game and 1 in the second game establishes a pattern of "progress from level 0 to 1". Similarly, a level rated 2 in the third game creates a pattern of "progress from level 1 to 2". This indicates an increase in the level of abstraction. Based on these patterns, we drew conclusions about the overall change in students' level of abstraction.

Findings

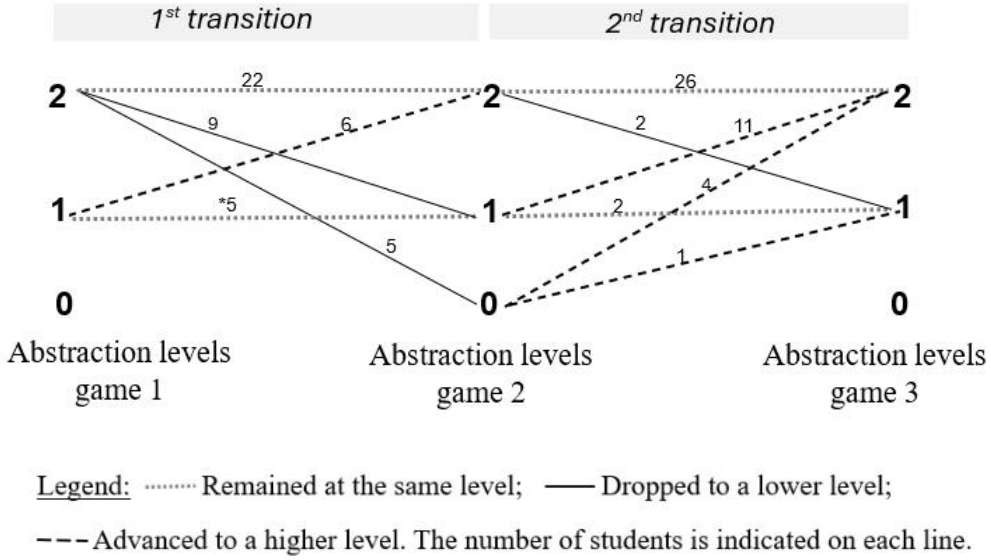
To address the research questions, we analyzed the level of abstraction in each student's game scripts according to the predefined levels of abstraction (0,1,2) for all three games in the sequence. The changes in each student's level throughout the programming sequence are shown in Figure 2 below. The patterns were illustrated through different line representations during two transitions: from the first game script to the second, and from the second game script to the third. Additionally, we provided the number of students who exhibited these patterns during each transition. For example, 28 students reached level 2 in the first transition, i.e., from the first to the second game script (22 of the students remained at level 2, while six others advanced from level 1). Subsequently, 26 of them also remained at level 2 in the second transition, while two students dropped to level 1 in the third game script.

Five of nine patterns of change emerged *in the first transition* (from the first and to the second game script). Patterns d, f and i (advanced from level 0 to 1; advanced from level 0 to 2; dropped from level 1 to 0) were not expressed in this transition, indicating that no student began the sequence at abstraction level 0. *In the first transition* 22 students remained at level 2 and five students remained at level 1. Six students advanced from level 1 to level 2; nine students dropped from level 2 to level 1; and five students dropped from level 2 to level 0. In total, out of 47 students, 28 students advanced to/remained at level 2, 14 students dropped to/remained at level 1, and five students dropped to level 0.

In the second transition (from the second to the third game script), most of the students who remained at level 2 in the first transition remained at level 2 in the second transition (21 students in total), while one student dropped to level 1. Most of the students who advanced from level 1 to level 2 in the first

transition remained at level 2 in the second transition (five students in total), while one student dropped to level 1. Most of the students who dropped from level 2 to level 1 in the first transition advanced to level 2 in the second transition (seven students in total), while two students remained at level 1. Everyone who remained at level 1 in the first transition advanced to level 2 (four students in total). Most of the students who dropped from level 2 to level 0 in the first transition advanced to level 2 in the second transition (four students in total), while one student advanced to level 1.

In summary, 41 out of 46 students reached level 2 in the second transition: 21 who started and finished the sequence at level 2 and 20 who gradually advanced to level 2 during the script sequence. Five students finished the sequence at level 1, whereas none dropped to level 0.



*One of the five students who exhibited abstraction level 1 in the first transition did not submit a script for the third game; thus, the subsequent second transition refers to one student less.

Figure 2: Patterns of change and corresponding number of students in transitions between scripts

Discussion

The aim of this study was to portray changes in abstraction levels among undergraduate students while engaging in a programming activity. The findings indicate that the ability to identify mathematical structures and patterns in codes and make abstractions about the inverse relation between motion and rate was not immediately implemented by the students. Although this inverse relation is within the realm of their mathematical knowledge, not all of them immediately exhibited the ability to identify when and how to use it (i.e., abstraction). Rather, they made gradual changes in abstraction throughout the programming process. Our findings are consistent with studies showing that abstraction may be reflected in geometry programming tasks (Büscher, 2025; Miller, 2019). Moreover, our findings suggest that a sequence of games can be designed to encourage students to implement mathematical thinking (MT) in a programming-based computational thinking (CT) activity, even when they are not explicitly instructed to do so.

This kind of task sequence provides an opportunity to make abstractions, thus addressing the education system's recent focus on CT in the curriculum and the aspiration to encourage students to become problem-solving citizens who can implement different skills (Montuori et al., 2024). While there are still gaps in our knowledge about how CT and MT are interrelated and how they can be intertwined in the school curriculum (Ye et al., 2023), recent study provided in-service mathematics teachers with essential tools to integrate programming into mathematics education, to enhance their teaching practices and support student learning. Similarly, we can infer from our findings that the sequence of games introduced in the study allows students to implement mathematical thinking in programming-based CT activities. Thus, it is highly beneficial to begin programming using Scratch to promote mathematical abstraction as early as undergraduate school of education.

In the future, this kind of task may also provide an opportunity for students to make abstractions about other mathematical relations, such as direct relationship between motion and rate, which involves defining angles to establish their direction of movement. This might provide more information about the way in which CT and MT can be used interconnectedly.

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Students' mathematical thinking through programming, animation and 3D printing: Early insights in the case of curvature

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We discuss 10th and 11th grade students' ideas about curve and its properties, and their competence in designing 3D curved objects in the programming environment of MaLT2, which supports intrinsically defined curvature animated in space. We consider curvature as a weakly addressed conceptual field in terms of its potential to create environments rich in opportunities to enhance students' mathematical thinking through a restructuration approach. We focus on how students use mathematics for both construction and reasoning when interacting with the digital models through a constructionist approach, and we explore the potential added value of the emerging 3D printing technology in students' mathematical thinking when combined with design through programming.

Keywords: Curvature, intrinsic geometry, programming, 3D design, 3D printing.

Introduction

The rapid evolution of technology in our society is fueling concerns in the mathematics education community about both what mathematics and students' skills are relevant to today's society and how new emerging technologies are being put to use to explore the question. One such case concerns the field of 3D geometry and space (Gravemeijer et al., 2017), along with the emerging technology of 3D printing, which is rapidly entering the field of education, and inviting us to rethink pedagogical practices in relation to this three-dimensional revolution (Ng et al., 2018). The construction and printing of digital 3D models is considered to support the educational process by engaging students with authentic problems, providing opportunities to cultivate their imagination and creativity, and opening a window to develop different skills through prototyping, such as testing and redesign, and design and spatial thinking (Dickson et al., 2021; Coşkun & Deniz, 2022). However, the studies exploring the potential of 3D printing exclusively in mathematics education are limited and mainly focus on the concepts of volume and area, properties of prisms and various 3D solids, while most of them either focus on other areas of STEM/STEAM or use ready-made 3D models, depriving students of authentic engagement with mathematical concepts in both Euclidean and non-Euclidean geometry (Ng et al., 2022), such as curves. Although curvature is a fundamental concept in mathematics and various applied fields, it hasn't found its way (e.g. into the Greek curriculum) for students to explore curves as geometric shapes created by covariational relations. Hence, in this paper we explore the potential of combining 3D design with the emerging 3D printing technology using the programmable tool MaLT2 to support students' mathematical thinking when tinkering with curved objects.

Theoretical Framing

The authors adopt a constructionist approach in their research view and task design, aiming to engage students with the geometric nature of the curve to generate meanings as they construct and tinker with digital models and discuss their mathematical ideas in relation to these artefacts and their behavior

(Kynigos, 2015; Papert, 1980). We discuss knowledge as an active construction by the learner through the exploration and symbolic expression provided by the MaLT2 programming tool, and consider a restructuring of the concept of curve (Wilensky & Papert, 2010) that makes use of the concept of function-as-covariation but does not ignore the actual concept of curvature, as is usually the case in school. With respect to the Greek curriculum, students may learn about the circle, arcs and various conic sections, but they don't have the opportunity to intuitively discover and explore curves as geometric shapes. The mathematical approach to 3D shapes in MaLT2 is through differential geometry, where curvature is described as an intrinsic property of curves and surfaces. In our case, the activity is based on Turtle Geometry (Papert, 1980), a kind of differential geometry linked to movements in which the main ideas are based on the intrinsic properties of geometric models, locality and the expression of mathematical concepts in terms of constructive procedures rather than equations (Abelson & diSessa, 1981). In this way, concepts such as curvature, which are reserved for more senior courses, can be made more accessible through intuitive ideas rather than axiomatic formalisms. Through exploring and tinkering with the 3D digital artefacts using a programming tool with dynamic manipulation of variable values, our intention was for the students to gain a local and intrinsic approach to curvature in space. At the same time, in studying student activity, we wanted to keep a wide lens in terms of the use of their mathematical ideas in the design and construction of 3D curved objects and to consider the concept of curvature in space as a conceptual field (Vergnaud, 1983) rather than as an isolated concept. For example, trying to design a triangle on the surface of a cylinder may involve different concepts such as torsion, rate of change, and the concept of curvature itself as a contrast to non-curved objects or as a property of an object (Stevenson, 1995), which are closely involved with the conceptual field of curvature in space and the intrinsically defined covariation function. Previous research has also explored the concept of curvature using programming through the construction of arcs in 2D space (Kynigos & Psycharis, 2003) and cylindrical helixes in 3D space (Kynigos & Zantzos, 2017), which have shown promising results in creating students' meanings around the concept of curvature, but also students' difficulties over the local nature of a curve (Montoya Delgadillo et al., 2018). Mathematical concepts can be understood in two different ways: structurally (as objects) and operationally (as processes). Learning mathematics results from an interaction between these two elements, while a mathematical concept is formed through a long sequence of transitions from operational to structural concepts, a process known as reification (Sfard, 1991). According to this approach, the use of mathematical concepts leads to a better understanding of them, which in turn leads to finding a wider range of uses for the concepts. In our case, the coding procedure aligns with an operational view of the concept of a curve, as it presents the curved model as a computational process. Conversely, the graphical figure promotes a structural approach, presenting the mathematical concept as a unified whole, with visualization making abstract ideas more tangible and easier to manipulate. It is true that students' learning typically begins with physical objects and later moves toward abstraction. In our case, the use of 3D printing works in the opposite direction, starting with the design of 3D models and ending with physical objects. We suggest integrating both processes in the learning journey, using them as tools to give meaning and structure to mathematical knowledge. In other words, we argue that an axiomatic definition is insufficient to convince a learner of the existence of a mathematical object, or when concepts are either too abstract or not clearly defined in students' experience, they can understand them through other concepts (e.g.

spatial relations or physical objects) through the repetitive cycle of imagining (coding), seeing (figural representation) and touching (3D printed model). In this research, the authors aim to explore the potential of integrating 3D printing with 3D design using constructionist media in the service of mathematics education regarding the concept of curvature in a process from students' meaning making to their mathematical competence in using this concept in different contexts. In this paper, however, the authors' first level of focus is on the study of (RQ1) *What perceptions of the concepts of curve and curvature emerged during students' 3D designs in MaLT2?,* and (RQ2) *How students' mathematical thinking might be enhanced as they use constructionist media for 3D design and printing of curved objects?*

Design of the activity

MaLT2 is an online programmable tool for symbolic representation in mathematical activities through textual programming for the creation and exploration of 3D graphical models with the affordance of dynamic manipulation of variable procedure values (Kynigos & Grizioti, 2018). It provides the following interrelated representations: i. symbolic (code), ii. figural representation (scene), and iii. dynamic manipulation of generalised models in the form of executed parametric procedures (a variation tool activates a slider for each variable of the procedure responsible for the selected linear trace) (Figure 1a). It has recently been extended to include the functionality of 3D printing to enable students to connect the digital models with the physical world and to promote the development of different skills and meaning-making processes for complex concepts (Grizioti et al., 2024). A task was given to the students involving the construction of a cylinder through a curved quadrilateral without reference to the mathematical properties of radius, center or any other external frame of reference. Using only the *forward* move command, and *right* and *down* turn commands, the *repeat* command, and two variables (Figure 1a), the rationale behind the design was to emphasize the local nature of the curve and its construction as a result of its rotation in space, contrasting it with a straight line. Students were challenged to explore the 3D curve created by dynamically changing the values of the variables and to further create their own artefacts on the surface of the cylinder in an intrinsic way. During the intervention, the researcher encouraged the groups to save the different artefact prototypes they were working on, as students could print out any prototype they wanted to see in physical form at the end of each session. The purpose was for the students to use the mid printout as feedback before their final artefact, and for the researchers to gain more insight into the use of 3D printing and its potential added value.

Methods

This study involved five 10th-grade students and three 11th-grade students, divided into three groups. They voluntarily participated in the intervention as part of an after-school Math Club for 4 hours split in 2 sessions. The students had some experience with the MaLT2 tool. In terms of their performance, the groups were mixed to provide extra motivation and support for all. Each group was given a laptop and a notebook. The teacher and the participating researcher acted as facilitators, encouraging the students to share their thoughts openly with the group. In order to have a comprehensive view of students' actions and interpretations throughout the activity, we collected several types of data, including computer screen and audio recordings, students' notes, MaLT2 codes and photos, and we

analysed them using a thematic analysis approach, starting from an initial scheme and remaining open to the emergence of new codes. After we triangulated the transcribed dialogues with the rest of the data, we analyzed the data based on critical episodes (Tripp, 1993) of students' meanings around the concepts of curve and curvature that emerged during their interaction with the 3D models that were relevant to the research questions. To answer RQ1, we looked for incidents in which students expressed their perceptions around concepts closely related to curvature that emerged during their design in MaLT2 and in interaction with their group or the researcher. To answer RQ2, we sought to identify ways in which the 3D design and printing of curved models enhanced students' mathematical thinking using the theory of reification. After coding of the dialogues, we created the main themes of the results. As the intervention was a pilot study for a larger design-based research project (Bakker, 2018), the findings will inform the next cycle of the study and the redesign of the activity.

Results

The thematic analysis revealed four main themes: 1. Approach to curve, 2. Approach to curvature, 3. Characteristics of curve and 4. Challenges. Regarding the RQ1, Table 1 summarizes the codes for each of the above themes, followed by some of the students' quotes during the critical episodes, while in the coding phase we always kept in mind the added value of the technology to the meanings the students generated or the challenges they faced in relation to each code in order to answer the RQ2. For the sake of brevity and clarity, we will describe briefly the 'torsion' code that responds to RQ1 and the 'locality' code that responds in depth to RQ2 using an indicative critical episode.

Table 1: Codes emerged from the data analysis followed by some students' quotes

1a. As a contrast to straight line	St. F: Something that turns, something that is not straight. St. C: Let me think… It's a curved line.
1b. As a property of an object	St. D: Curve is the circle.
2a. Size of curvature	St. C: It's opening up, the curve is widening… Before we wrote down 160/360. Now, we put 50/360. So, the smaller the fraction, the smaller the curvature we have.
2b. As a rate of change	St. D: The [fd 1 down :n/360] tells the bird by how much to turn.
2c. Continuity	St. A: Let's change the step of the slider to 0.1.
3a. Torsion	St. A: The bird has to go up-down, change its direction and also... turn its body.
3b. Locality	Critical episode is quoted below
4a. Orientation in 3D space	St. B: When we say down, I don't understand why we say degrees.
4b. From 2D to 3D	St. B: Can we use the Pythagorean Theorem in curved geometry, in curved space?

Regarding the first research question, the meanings that emerged during the students' engagement with curve models concerned the concept of curve itself which students intuitively encountered as a

contrast to non-curved objects and as a property of an object (Stevenson, 1995), approaches to describe and use the concept of curvature in their designs related to concepts unknown to them (i.e., rate of change, continuity) and characteristics of a curve in 3D space such as torsion, which emerged through the use of the concept of the angle and body syntonicity (Papert, 1980). For example, student A mentally put himself in the bird's position and used not only his hands but his whole body to explain to his classmate his proposed *roll_left* command (Figure 1b), which denoted rotation around the straight line of the bird's movement. Whereas curvature shows how far the curve moves away from being straight, torsion shows how much the curve twists out of the osculating plane in 3D space (e.g. moves away from being flat). Both concepts are associated with a change of direction in space.

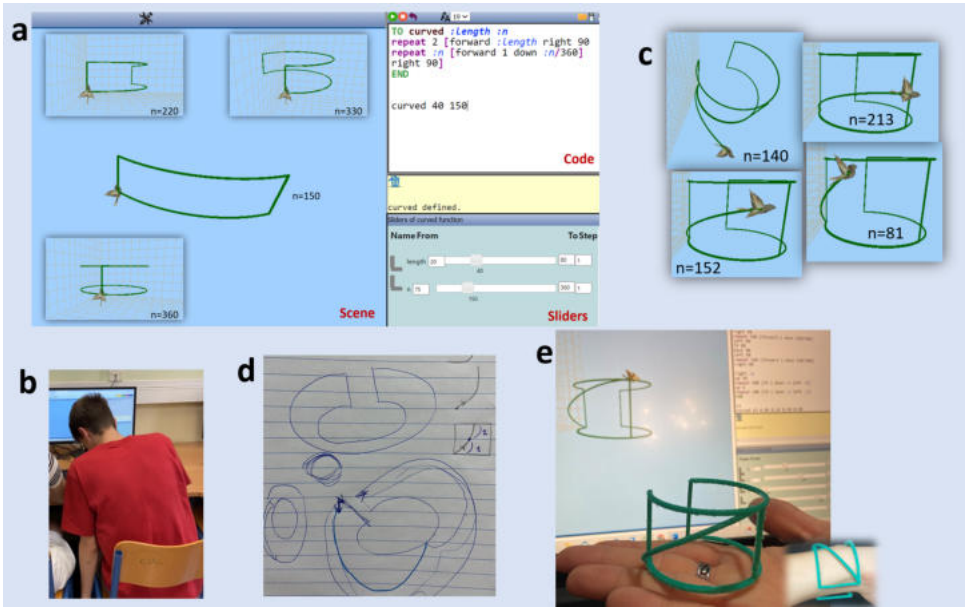

Figure 1: a. The three representations of the microworld in MaLT2, b. Student's use of body connected to his idea of torsion, c. Student's testing using variables, d. Students notes of the 'curtain' idea, e. Final digital and physical model of group B

To address the second research question, we report and analyse one selected case that stood out in combination with the impact of the physical touch of the object. The following excerpt concerns group's B attempt to construct the curved line on their cylindrical bracelet by connecting the bottom point of the vertical line to the top point of the middle one. The challenge that they faced was that the curve was closing towards the inside of the cylinder. After many attempts in changing the code and the values of the variables through the sliders, they decided to print their model. Although up to this point the students had been working on the code using the *forward* and *down* commands in a *repeat* command, they had been concentrating on the twist they wanted to give their curve (torsion) and had not realized, even from the graphical representation of the code in the tool (Figure 1c), that this curve would never reach the end point they wanted. Student C commented when looking at the printout: *"It looks like it wants to go in a circle, whereas we want it to stretch upwards more smoothly"*. This showed that the students had approached the concept of curvature through an operational understanding that was based on the aforementioned commands and their perception of it as a curved line. They had discussed trying a smaller or higher curvature in their model, but they had difficulty

realizing that their curved line would always create a circle with different, but constant curvature, which seemed a challenge for them in approaching curvature structurally. They seemed to had used so far, the concept of curvature as a measure to their curve and as a tool to describe it in their digital model. The next incident in the progress of this group's work was triggered by a student looking around the physical model in her hands.

Student E: The curve we want has the slope of the cylinder and also an upward slope (shows several times the path the curved line should follow with her finger).
Student C: That's right. How can we make the whole line move upward to reach our point?
Student E: What if… (still touches the 3D printed model) What does this move remind you of?
Student D: I don't know. A wave? An uphill?
Student E: A curtain for me. Doesn't the curtain curve upwards like this?
Student C: Yeah, it could. But... The slope at the beginning is steep, and then it tends to be marginally straight.
Student E: That's right! What if we break the movement into 2 curves? What if from the beginning we take it up to about here (i.e. the middle of the trajectory) with a curve that is steeper and from here to the end point with another smoother curve?

Student C's first observation about the circle, combined with the dialogue that followed, triggered the group's discussion around two axes. The first axis concerned the issue of the local nature of the curve, as the students pointed with their hands to specific points along the path of the curve in both the physical and digital model, and proceeded to deconstruct more consciously the code they had created earlier, discussing around the use of the *repeat* command of the code that gives this 'movement' of the curve. The second axis involved the concept of curvature as a property of the curve. They had now focused on curvature as a fixed and perhaps more abstract concept of their model, firstly recognizing the curvature of the circle as a constant and secondly trying to relate curvature to their model as an attribute to describe it. This incident highlights the effect of the 'digital-physical' approach, as this combination led the students to a deeper engagement with the mathematical idea of curvature that might not have happened without the possibility of 3D printing. The process of the construction of a curve with different curvatures or, as they put it, two curves (breaking the bird's movement into two parts) with different curvatures seemed to open the path towards a structural understanding of the concept for the students. The physical model, with its strong sense of touch, prompted students to think of a physical object that resembled the desired curve (e.g. curtain), and to 'break' their curve in one point (Figure 1d) in order to build their final model of a bracelet (Figure 1e). Therefore, it is argued that the alternation between the 'predict-see' approach in the digital model and the 'touch' approach in the physical model could provide opportunities for students to engage with the local nature of curvature and serve as a channel between the procedural understanding and use of concepts around curvature to a more conscious and general one, leading to finding a wider range of uses for them in the models.

Discussion

Students brought to the surface and used, either in the construction of their model or in the justification of their thinking, different concepts involved in the conceptual field of curvature in space, such as the concept of torsion, the locality and the concept of curvature itself. This was mainly triggered by the use of programming commands, the students' body syntonicity and the dynamic manipulation of the models through the use of variables in commands the students wanted to explore

further. The activity of this first cycle was quite open and aimed at discovering students' intuitive ideas and meanings in the field of curvature, since some of the concepts that emerged were either taught in a fragmentary way at school (e.g., curves), or not taught at all (e.g., rate of change), or related only to 2D space (e.g., orientation) in order to inform the design of the next cycle, which will be targeted at specific characteristics of curves and have more cycles of printed prototypes. As for the use of 3D printing, the results showed that it played an interesting role in enhancing students' thinking as the printed model designed by them created a link between abstract ideas and tangible models, giving them the opportunity to engage with the local aspect of curvature. This 'digital physical' alternation acted as a bridge, where the use of curvature facilitated a deeper understanding of it, providing students with opportunities to use concepts related to curvature in both an operational and structural approach. Despite the limitations of this research, it seems worth exploring the possibilities of integrating 3D printing with 3D design using constructionist media in the service of mathematics through repetitive cycles of imagining (coding), seeing (digital figure) and touching (3D printout) to enhance students' mathematical thinking in 3D space.

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Opportunities for low-achieving students to learn mathematics through design and 3D printing of usable objects

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Low-achieving students rarely encounter opportunities to engage deeply with mathematics in ways that connect abstract concepts to tangible, real-world applications. This case study of a 10th-grade classroom examines how 3D design and printing (3DP) activities can offer meaningful opportunities to learn mathematics for these learners. The saved work of eleven students, who used 3D design software to build a biased die, was analysed to reveal the dynamic interplay between concrete experience and abstraction in the activity. We illustrate this process through Nathan's example, who employed an iterative design process, progressing from concrete to sophisticated design by effectively engaging with mathematical concepts such as symmetry, geometry attributes, and spatial reasoning while fostering perseverance and motivation. These findings stress the potential of 3D design and printing to provide low-achieving students meaningful opportunities to learn mathematics.

Keywords: Mathematical modelling, 3D design and printing, opportunities to learn, low achievers.

Introduction

Mathematical modelling entails converting real-world situations into mathematical terms to analyse and solve problems effectively. For students, mathematical modelling is cognitively demanding, requiring continuous transitions between reality and mathematics (Borromeo Ferri, 2018). Although mathematical modelling tasks are typically done on pen and paper, in this study we explore the use of 3D design and printing (3DP) activities as modelling tasks settings.

As a result of recent technological advancements, 3D printers have become popular and can now be found in many schools and households. Although 3DP activities have demonstrated the potential to foster applied mathematical skills and engage students in mathematical modeling and problem-solving (Levin & Verner, 2021), the specific ways in which students learn mathematics through these activities require further investigation. Moreover, low-achieving students' learning through these activities has rarely been studied. This study addresses this gap by exploring the mathematical learning opportunities arising from 3DP tasks among low-achieving students attending a "last-chance" secondary school.

Hence, the goal of this study is to explore opportunities to learn mathematics among at-risk youth through 3DP activities. As a proof of concept, we describe a case study in which a class of low-achieving students at a high school for at-risk youth engage in a 3DP activity. To further exemplify the design and learning process, we describe the example of Nathan, a 10th grade student participating in the activity. Using the students' 3D designs as recorded in the Onshape CAD (Computer-Aided Design) software, we explore the opportunities to learn mathematics that emerged from the activity by examining the following research question: *What opportunities arise for low-achieving students to learn mathematics as they work on a 3D design and printing activity of a biased die?*

Literature review

Mathematical modelling requires identifying and applying relevant mathematical concepts and methods to address real-life contexts (Borromeo Ferri, 2018). Researchers who used authentic tasks that combined mathematical modelling and 3DP activities as an integrative context for real-world applications found that such activities can enhance mathematical learning and help students develop practical application skills (Guler et al., 2019). Creating a 3D printable model requires planning an object that does not yet exist and digitally designing it using CAD software. The students must be able to understand ill-defined problems, collect and quantify relevant data, construct mathematical models and inquire about them using mathematical methods, and derive a solution to the real-world problem from the results of the mathematical analysis. This process demands selecting and defining the geometric shapes that form the design, making decisions that consider technological and task constraints, and evaluating the planned product against the requirements (Levin & Verner, 2021).

Further, when engaging in a 3DP task, students must move back and forth between the physical model they are planning and the mathematics that enables them to carry out their design. That is, students must switch back and forth between abstraction and concrete experiences, which constitutes the relation that "lies in the core of mathematics education studies" (Shvarts et al., 2022, p.3024). This dynamic movement yields meaningful schematizing and algorithmizing, which ultimately turn into mathematical models and formalized rules. Through 3DP, students can apply mathematical concepts to physical objects that they can touch, explore, and use, thereby bridging the gap between theoretical mathematics and its practical applications.

Yet mathematical modelling using 3DP can be challenging for students due to the time-consuming nature of the tasks, the need for group work, uncertainty in the solution process, and cognitive barriers such as identifying unknown quantities and making numerical assumptions (Böswald & Schukajlow, 2023). Educational research indicates that when working on time-consuming tasks low-achieving students tend to exhibit more work avoidance and boredom than high achievers, consequently affecting their engagement and motivation in mathematical modelling activities (Hirt et al., 2021). Nevertheless, we strongly believe that such learning experiences should be accessible to all students. Hence, here we focus on implementing/designing such experiences for low-achieving students enrolled at a so-called "last-chance" secondary school. Building on this perspective, we focus on opportunities to learn mathematics through 3DP activities among those students.

Theoretical perspective

As noted, this research assumes that the emerging opportunities to learn during 3DP mathematical modelling tasks should be accessible to as many students as possible, regardless of teachers' conceptions of students' achievement level.

We define *opportunities to learn* mathematics as "the extent to which genuine opportunities can be created, facilitated, and promoted for different groups of [...] children and young people" to learn mathematics (Ng et al., 2018, p.12). These learning opportunities are realized at the instructional level through the curriculum as delivered by the teacher and at the learning level through student activities. We adopt an enactivist view when considering how students can realize these learning opportunities. Knowing does not arise in the mind from representations of the world but rather through action and interaction with the world (Varela, et al., 1991). From this perspective, if a student designs a model

that "works" in terms of fulfilling its ascribed functionality, this can be viewed as evidence of learning. A 3D designed and printed model constitutes the physical evidence that students know how to use the mathematics underlying the design. In fact, it is "doing that instantiates this knowing" (Abrahamson, 2024, p.158). Consequently, in this study, we examine the learning opportunities emerging from the dynamic movement between abstraction and concrete experiences within a group of low-achieving students who worked on the biased-die task (described below) using 3D CAD software (Onshape). We also provide a detailed analysis of the work of a specific student—Nathan.

Research context – the biased die activity

The biased die activity is part of the DoLearn3D project, an initiative to enhance mathematics learning through 3DP activities for 10th-grade students at a last-chance secondary school. The project includes nine learning units aligned with the low-level mathematics curriculum, covering topics such as measurement, polygons, probability, area, volume, equations, ratios, and spatial orientation. Using Onshape, students design objects, which are 3D printed and analysed for potential redesign. Each design starts as a 2D sketch that is used to create 3D shapes through extrusion. The 3DP activities were described in real-life contexts without explicitly stating the underlying mathematical concepts, thereby encouraging students to apply previously learned mathematics, discover and explore new mathematical connections, and develop original mathematical strategies. The teacher provided targeted support with software use and mathematical concepts while encouraging student autonomy.

This study focuses on the biased-die activity. Students were asked to design and print both fair and biased dice. They then exchanged dice with their classmates, played with them, and attempted to determine which die is biased and which is fair, distinguishing between the dice using the rolling outcomes distribution. The teacher also discussed important concepts connected to dice, such as shape (a cube), symmetry, fairness, chance, and center of gravity. The activity included a worksheet for students that was meant to promote individual work. Figure 1 illustrates Nathan's final design (A), and a cross-section of the biased die that exposes its interior (B).

Several learning opportunities related to the aforementioned mathematical concepts emerged through this activity. One is using the origin of the Cartesian coordinate system as a point of reference for the design. Another is using symmetry to extrude a die symmetrically around its center, which becomes much easier when the die is placed at the origin. A third involves placing the dots on the face of the die. This could be done by measuring an equal length from all sides of the face, which requires using concrete thinking. Another option is taking advantage of the geometric properties of the intersection point of the diagonals in a square and symmetry, which indicate an abstract way of thinking.

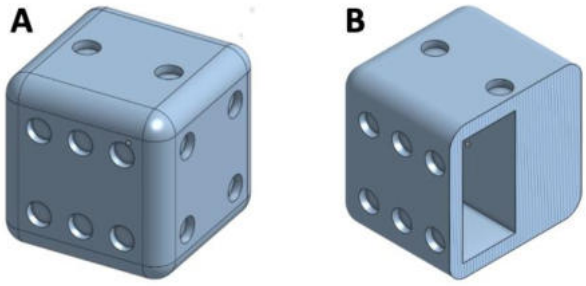

Figure 1: The biased die design and its cross-section view

Methods

To achieve the study goal, we selected the case study methodology (Yin, 2018). The case examined in this study is a group of eleven 10th graders (six boys and five girls, aged 15) with low academic achievement enrolled at a high school in central Israel serving at-risk youth who engaged in the biased die activity described in the previous section. The students used Onshape software to design the objects. Then they 3D-printed, analysed, redesigned, and printed them again if necessary. We selected Nathan's example for detailed analysis as his work illustrated the typical progression we observed across students - moving from basic visual approximations to sophisticated mathematical concepts. His extensive engagement with the task (over 5 hours across 6 days) provided rich data for analyzing the transition between concrete and abstract thinking. Additionally, Nathan's initial struggles and subsequent progress make his example particularly relevant for understanding how low-achieving students can develop mathematical understanding through 3DP activities

Data collection and analysis

The Onshape version used by the students automatically records all data related to the design process, including student interactions and design iterations, and allows teachers to monitor students' progress. This study used the collected data, including the final designs and the students' design process steps, recorded in an audit log format. The platform also provides the option to view historical snapshots of the digital designs for a thorough examination of the design process.

The data analysis was conducted using inductive content analysis methods (Mayring, 2015). To analyse the data, we began by examining the designs and referring to the logs to gain a deeper understanding of the sequence of the design process. We began by collaboratively coding three designs to ensure consistency in our approach and to align our interpretations of the data. Subsequently, each researcher individually coded four additional designs. We then discussed our coding until reaching agreement and presented our interpretations to the class teacher, who provided additional context and information beyond what was visible in the data gathered from the OEE platform. Table 1 presents the final coding schema and the results showing the code's frequency.

Findings

The analysis of students' final designs revealed opportunities to learn mathematics through spatial reasoning, geometric strategies, and functional design. As shown in Table 1, nine students aligned their dice with the origin of the axes (1.4 Center-Origin), demonstrating spatial reasoning, while two placed them randomly (1.1 Random-Placement), highlighting varying levels of understanding of reference points. Strategies for dot placement ranged from mixed geometric methods (3.4 Mixed-Methods-Placement, four students) to diagonal-based precision (3.3 Diagonal-Guides, three students) and visual approximation (3.1 Visual-Approximation, two students). Symmetry was intentionally integrated by nine students (5.2 Symmetry-Integrated), reflecting their grasp of geometric consistency. Six students engaged in biasing their dice (6.2 Biasing-Process), linking their designs to functional outcomes. However, five did not attempt biasing (6.1 No-Biasing-Process), reflecting varied task engagement. These findings suggest that 3DP activities scaffold mathematical concepts, bridging abstract principles like symmetry and spatial reasoning with tangible applications. Nathan's case illustrates how these strategies and learning opportunities emerged in practice, providing deeper insights into the interplay between abstraction and design.

Table 1: Categorization and frequencies of mathematical and CAD processes in students' die designs

Category Name/Description	Code Number, Name and Description	Frequency
Mathematical reasoning and spatial thinking involved in designing the die.		
1. **Placement of die in space**: How die is positioned within the 3D space relative to coordinate axes.	1.1 **Random-Placement**: Die is placed randomly, without specific alignment with the coordinate axes.	2
	1.2 **Corner-Origin**: Corner of die is placed at the origin of the coordinate axes.	1
	1.3 **Bottom-Origin**: Center of bottom face is aligned with origin of the coordinate axes.	1
	1.4 **Center-Origin**: Center of die is aligned with origin of the coordinate axes.	7
2. **Order of working on faces**: Sequence in which student works on each face of the die, arranging dots from face to face.	2.1 **Random-Face-Order**: Faces are worked on in random order, without following any specific logic.	1
	2.2 **Sequential-Face-Order**: Movement from face to face follows numerical order (e.g., 1, 2, 3, etc.).	3
	2.3 **Opposite-Sum-Order**: Arrangement follows die's regular logic, ensuring that the sum of opposite faces equals 7.	3
	2.4 **Idiosyncratic-Face-Order**: Faces are worked on systematically according to another logical order, not related to the sum of opposite faces, but resulting in opposite faces adding up to 7.	4
3. **Placing dots on a face**: Method used to distribute dots on each face of the die.	3.1 **Visual-Approximation**: Dots are placed based on visual approximation without precise methods.	2
	3.2 **Parallel-Guides and Measures-Combination**: Dots are placed using parallel guiding lines and precise measurements.	2
	3.3 **Diagonal-Guides**: Dots are placed based on properties of the square, using diagonals.	3
	3.4 **Mixed-Methods-Placement**: A mixed method, combining several approaches to dot placement.	4
4. **Size of dots**: How size of the dots is determined on each face and across faces of the die.	4.1 **Random-Dot-Size**: Size of dots is randomly determined (every dot is different size).	1
	4.2 **Visual-Dot-Size**: Size of dots is determined visually, without precise measurement.	1
	4.3 **Inconsistent-Dot-Size**: Size of dots is determined by measurements on each face but is not consistent across all faces.	1
	4.4 **Measured-Dot-Size**: Size of dots is determined according to specific measurements.	8
5. **Use of symmetry as part of design process**: How symmetry is applied or ignored during die design.	5.1 **Symmetry-Not-Integrated**: Symmetry is not observed or maintained during design process.	2
	5.2 **Symmetry-Integrated**: Symmetry is intentionally used in design process.	9
CAD Process Related: Aspects related to interaction with CAD software and overall design approach.		
6. **Process of biasing die**: Steps taken to intentionally bias the die, making it unfair.	6.1 **No-Biasing-Process**: Student does not engage in process to bias the die.	5
	6.2 **Biasing-Process**: Student explicitly engages in process to bias the die, making it unfair.	6
7. **CAD design process**: Overall workflow in designing the die.	7.1 **Software-Struggle**: Student appears to struggle with software control, making multiple errors or unnecessary sketches.	2
	7.2 **Iterative-Trial-Error-Approach**: Student makes multiple attempts, often undoing actions and experimenting with different approaches.	5
	7.3 **Systematic-Approach**: Student follows a systematic approach: constructing the die, creating holes, emptying, and filling to bias the die.	4

Nathan's Example. Nathan worked on the biased die activity for six days, for 5 hours and 17 minutes. We describe Nathan's learning opportunities in chronological order as he designed and

printed the die. According to the audit log, on day one (27 minutes of work), Nathan created a rectangular prism, placing its corner at the origin of the 3D coordinate system (1.2 Corner-Origin) (Figure 2A). He initially distributed dots using visual approximation (3.1 Visual-Approximation), starting with one dot and sequentially progressing to adjacent faces (2.2 Sequential-Face-Order). As he rotated the object, he ensured the dots were consistently sized (4.4 Measured-Dot-Size), demonstrating an early attempt at uniformity. Realizing his shape was not a cube, he adjusted its height, disrupting the dots' distribution. The use of symmetry was not manifested (5.1 Symmetry-Not-Integrated). At this stage, Nathan demonstrated a novice approach to 3D design based on visual approximation with little planning and precision.

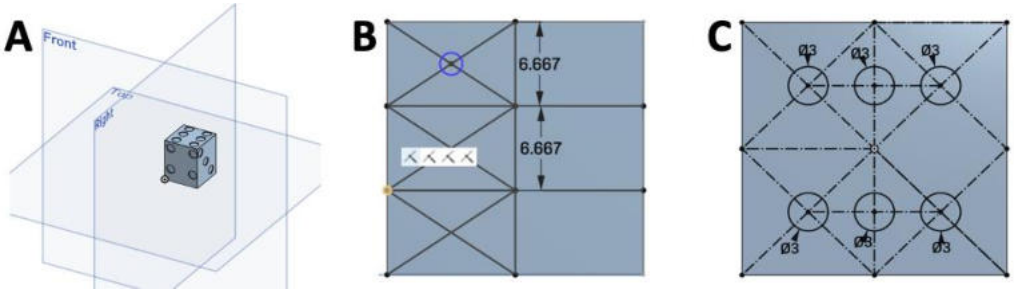

Figure 2: Nathan's designs

On day two (1 hour and 3 minutes), Nathan deleted all previous actions, demonstrating his motivation to improve and awareness of his previous sub-optimal performance. Starting over, he created a die with the corner at the origin of the axes (1.2 Corner-Origin). Initially, he placed dots arbitrarily (3.1 Visual-Approximation) but soon erased them. He sketched equally distanced parallel and perpendicular lines, dividing the face into six equal parts (3.2 Parallel-Guides and Measures-Combination). Then, he added diagonals to determine the center of each part (3.4 Mixed-Methods-Placement) (Figure 2B). This sequence shows Nathan's progression through the design process, incorporating mathematical concepts such as division, measurements, parallelism, and rectangle properties such as the intersection of diagonals. After consulting the teacher, Nathan started over again, sketching a square centered at the origin of the axes (1.4 Center-Origin) and incorporating another mathematical concept: symmetry (5.2 Symmetry-Integrated).

Two weeks later, on day three (37 minutes), Nathan used diagonals to place the dots (3.3 Diagonal-Guides), mirroring their placement for symmetry (5.2 Symmetry-Integrated) and maintaining consistent dot sizes (4.4 Measured-Dot-Size), demonstrating consolidation and application of a previously learned concept. His frequent use of undo-redo actions reflected an exploratory learning approach (7.2 Iterative-Trial-Error-Approach), indicating the ongoing development of CAD proficiency and geometry understanding.

A week later, during day four (2 hours and 3 minutes), Nathan created a dot at the centre of one face, using the origin as a reference after getting teacher assistance. Then, he systematically used diagonals for placing equally sized dots, ensuring opposite faces summed to 7 (7.3 Systematic-Approach, 3.3 Diagonal-Guides, 4.4 Measured-Dot-Size, 2.3 Opposite-Sum-Order) (Figure 2C). Then, Nathan added rounded corners to embellish the die, which was finally ready (see Figure 1A above). To create a biased die (6.2 Biasing-Process), Nathan modified the interior by creating an internal void and adding material to one side (Figure 1B above). This modification represents a mathematical

modelling translation from the digital design to the physical object to be created and used in a real-world scenario, thus showing Nathan's ability to create 3D designs for a specific functional outcome. On days five and six (7 and 9 minutes each plus idle time), Nathan tested different options for discretely marking his biased die, downloaded digital files, and 3D printed them.

Discussion and Conclusions

In this study, we explored low-achieving students' opportunities for learning mathematics when working on a 3DP biased die activity. The class findings revealed varied levels of engagement and success. All the students revealed the application of mathematical reasoning and spatial thinking during the design process. For example, nine of the eleven students demonstrated understanding of the coordinate system, aligning the die' corner with the coordinates origin and a similar number applied symmetry to their designs. On the other hand, only six students created a biased die. Nathan's example illustrated the students' progress during the 3DP activity, from concrete geometric manipulations using visual approximations to precise measurements and then to more abstract conceptual applications of square properties. These findings are in line with previous research showing that 3DP can be combined with mathematical modeling activities to promote mathematical learning (Guler et al., 2019).

This learning trajectory, progressing from arbitrary placement to a more systematic method, occurred organically due to the iterative trial-and-error approach and the teacher's scaffolding and mentoring. The students incorporated various mathematical concepts, such as symmetry, parallelism, and the intersection of diagonals in a rectangle/square. Nathan's design process exemplifies how students discover and apply mathematical patterns, supporting the enactivist framework, which emphasizes knowing as emerging through embodied actions and interactions with the environment (Varela et al., 1991). From this perspective, Nathan's progression shows that he perceived the feedback received from the 3D design software and then improved the model, enacting mathematical concepts through repeated iterations of actions and feedback. Moreover, Nathan's ability to transfer the mathematical patterns across different parts of the design task reflects his evolving mathematical understanding, suggesting, according to the enactivist view (Abrahamson, 2024), that repeated enactment indicates as well as develops a deeper understanding of mathematical concepts. Nathan aimed to create a model that "works", fulfilling its ascribed functionality, and in that sense, his doing reflected his knowing.

Further, Nathan evaluated and validated his 3D model and improved it iteratively. We propose that this assessment, which is an inherent yet often overlooked step in mathematical modelling, was fostered by the concrete and immediate feedback provided by the CAD software during the design process (Levin & Verner, 2021). Additionally, Nathan exhibited notable perseverance and motivation throughout the activity, continuing to improve his designs despite initial struggles. This suggests that 3DP tasks can help students engage with challenging mathematical concepts.

Furthermore, building on Nathan's example we suggest that 3DP tasks can be effective for low-achieving students by providing them with hands-on experiences that bridge the gap between abstract mathematical concepts and tangible applications. We propose that this study makes novel contributions to the field in two ways. First, the use of 3D design and printing among low-achieving students to learn was rarely studied. Second, it shows the unique trajectory by which mathematics was learnt even though it was not explicitly instructed. Nathan did not learn because he was taught

but rather because he experienced. Finally, building on the growing role of 3D printing in industry and STEM education, we suggest further exploration regarding how these activities can be scaled and adapted for other mathematical concepts and diverse student populations.

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Promoting Euclidean-Taxicab covariational reasoning to enhance understanding of trigonometric concepts in a digital environment

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Research in mathematics education shows that activities promoting students' covariational reasoning, as well as the transition from one geometry world to another, favour the meaningful learning of mathematical concepts and the construction of knowledge. In this work, we explore the intertwining of both these educational potentials by engaging undergraduate students in activities aiming to enact covariational reasoning processes within and between two different geometries, specifically Euclidean and Taxicab geometry, in a digital environment. We aim to investigate how students connect trigonometric elements (quantities, functional relations) to identify their covariational positioning within and between the two worlds supported by GeoGebra.

Keywords: covariational reasoning, theoretical covariation, covariational positioning, Taxicab trigonometry, undergraduate university students.

Introduction and conceptual background

Covariational reasoning and the notion of variation or change of a quantity within a scientific phenomenon have been identified by research over the last decades as a powerful way of thinking that helps students construct meaningful learning of a wide range of mathematics concepts (Bagossi & Swidan, 2023; Carlson et al., 2002; Jaber et al., 2024; Moore, 2014; Thompson & Carlson, 2017). The concept has been defined in a variety of ways in the literature, but they all agree that covariational reasoning is the simultaneous coordination and imagination of changes in two quantities. Carlson (1998) describes covariational reasoning as "the cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other" (p. 354). In Thompson's theory of quantitative reasoning, a person reasons covariationally when "she/he envisions two quantities' values varying and envisions them varying simultaneously" (Thompson & Carlson, 2017, p. 425). Thompson and Carlson (2017) argue that "ideas of continuous variation and continuous covariation are epistemologically necessary for students and teachers to develop useful and robust conceptions of functions" and that "variational and covariational reasoning are fundamental to students' mathematical development" (p. 423). According to Carlson et al. (2002), the development and interpretation of models of continuously changing events are significantly aided by students' covariational reasoning skills. Students' struggles stem from their inability to visualise and coordinate the simultaneous changes in variables or their lack of variational reasoning skills. The mental images they refer to are not limited to visual images but encompass all types of mental representations that represent past experiences, thoughts, emotions, and sensations related to the quantities. Creating mental images is cognitively demanding, requiring the conceptualisation of quantities (quantity in mind, not the real world), quantification (an act of conceiving an object and assigning a unit of measure to one of its attributes), and relationships among quantities. Thompson (1994) asserted that students' reasoning about covarying quantities needed to form a series of mental

images: an image of the change in one quantity, then an image of the coordinated change between two quantities, and finally an image of the two changing quantities covarying simultaneously. Understanding and knowledge should be strengthened when the processes manifest by changing the theoretical world in which to reason and reflect on how to coordinate concept images in different theoretical registers. Research shows that comparing different geometric worlds aids in the advancement of Euclidean knowledge (Kemp & Vidakovic, 2023; Miranda & Saliceto, 2024). In our ongoing experience, university students are involved in exploratory cooperative learning activities within Taxicab geometry (Krause, 1975) using digital tools to better understand concepts in Euclidean geometry. Taxicab geometry relies on an alternative definition of distance to that used in Euclidean geometry. Given two arbitrary points, $A = (a_1, a_2)$ $B = (b_1, b_2)$ the Taxicab distance is defined by $d_T(A, B) = |a_1 - b_1| + |a_2 - b_2|$ and the induced geometry is the totality of notions preserved by Taxicab-isometries. This geometry challenges Euclidean intuitions and encourages reflection on definitions, axioms, and properties. Furthermore, exploring the properties of figures and making conjectures through dynamic geometry software helps to visualise the simultaneous change in quantities and promote students' covariational reasoning. The educational implications of altering the theory within which a concept can be described are seldom ever reported. We argue that switching between geometries stimulates and supports covariational reasoning. We explored the effects of university students' engagement in covariational reasoning in different geometries with the aim of developing meaningful learning of trigonometric concepts and investigating how the use of digital technology supported the covariational processes involved. The aim of this study is to identify the different positionings in students' covariational reasoning emerging from their discourses. We face the issue of promoting undergraduate students' covariational reasoning by giving them theoretical and digital opportunities to explore a trigonometric concept within and between different geometries.

Approaching definitions of theoretical covariation and covariational positioning

From a theoretical perspective, we make a first attempt to define the following notions.

Theoretical covariation. Representations play a fundamental role in the learning of mathematics and are essential for an understanding of mathematical concepts, as they allow students to think mathematically. Moreover, changing theory, exploring the meaning of the concept within each mathematical theory, and transforming from one theory to another makes the semiotic representations set richer and strengthen the understanding of a concept (Duval, 2006). The simultaneous variation of two representations (quantities, relationships, etc.) within distinct theories is said to be *theoretical covariation*. It undoubtedly represents a covariation order different in nature from those already introduced in literature (Bagossi, 2024; Bagossi & Swidan, 2023; Jaber et al., 2024).

Covariational positioning. Positioning theory is the study of how people assign themselves and others a position in discourse (Harré & Van Langenhove, 1999). Wagner and Herbel-Eisenmann (2009) showed that the norm in mathematics education research was to identify students' positioning in relation to mathematics, which is a departure from positioning theory. In line with this research, the idea to investigate how students position themselves in relation to covariational reasoning takes shape. Thompson and Carlson (2017) characterised six different major levels of covariational

reasoning through some mental images and behaviours inducing them: *L5-Smooth continuous covariation*. The person envisions increases or decreases (hereafter, changes) in one quantity's or variable's value (hereafter, variable) as happening simultaneously with changes in another variable's value, and the person envisions both variables varying smoothly and continuously*; L4-Chunky continuous covariation*. The person envisions changes in one variable's value as happening simultaneously with changes in another variable's value, and they envision both variables varying with chunky continuous variation; *L3-Coordination of values.* The person coordinates the values of one variable (x) with the values of another variable (y) with the anticipation of creating a discrete collection of pairs (x, y); *L2-Gross coordination of values.* The person forms a gross image of quantities' values varying together, such as "this quantity increases while that quantity decreases." The person does not envision individual values of quantities going together. Instead, the person envisions a loose, nonmultiplicative link between the overall changes in two quantities' values; *L1-Precoordination of values.* The person envisions two variables' values varying but asynchronously—one variable changes, then the second variable changes, then the first, and so on. The person does not anticipate creating pairs of values as multiplicative objects; *L0-No coordination.* The person has no image of variables varying together. The person focuses on one or another variable's variation with no coordination of values. We call *covariational positioning* the way in which a student positions himself/herself in a mathematical discourse with respect to covariational reasoning; that is, a positioning process (a sequence of positions in the speech) within the mathematical discourse from which it can be clearly deduced that a certain level of covariational reasoning has been or is going to be reached, according to Thompson and Carlson's classification (2017). When someone exhibits a particular level of covariational reasoning, it indicates that she/he can reason consistently at lower levels but not at higher ones. A Position Act PA0, PA1, …, PA5 is one or more sentences in the mathematical discourse from which it is clear that a certain level of covariational reasoning L0, L1,…, L5 has been or is going to be attained, respectively. When the discourse exhibits a reasoning in terms of theoretical covariation, we refer to this act as *theoretical covariational positioning* (TCP).

Research questions

The research started from the hypothesis that shifting geometry enacts covariational reasoning at different orders and positionings. In this paper, we focus on the concept of sine and other concepts related to it (angle, cosine, sinusoid), within the Taxicab trigonometry. Our research questions are:

How do undergraduate students enact reason covariationally when working on knowledge construction involving the shifting of geometries in digital environments? Specifically, which kind of covariational positioning emerges in students' discourses about trigonometric quantities?

Methods

The experiment, which took place at a university in the south of Italy, involved 12 undergraduate mathematics students taking a short course organised in two phases, shifting the students' position from "being a student" to "feeling a teacher". According to the design, each group had access to a vertical non-permanent board as well as a digital environment that included a collaborative tablet, a digital software for design, a collective document in which to place the solution product and process,

and a personal document in which to report on their experiences. Undergraduate students were given a task to introduce and explore the definition of the Taxicab metric starting from a real-life situation:

Model a real-life problem (reasoning on Taxicab paths). Define the Taxicab metric; define Taxicab objects (circles, conics, angles, trigonometric functions), one or more. Choose them and investigate their Taxicab world (relationships, statements) by reasoning on definitions and statements analogous to those already known in Euclidean geometry.

Figure 1: A generic task addressed to undergraduate students

One of the working groups focused on trigonometry concepts. The task, once solved the real-life problem, was structured in instructional episodes described as: I. Illustrate the taxicab distance; II. Explain what a circle is (what is fixed and what varies) and draw an example; write the Taxicab equation of a circle; draw the unitary circle; III. Illustrate the taxicab radian; explain what the sine of an angle is; IV. Graph the sine function; graph the cosine function; V: Investigate the fundamental trigonometric identities; VI. Choose a trigonometry theorem and investigate its validity in the world of taxicab trigonometry. We focus our attention on excerpts from Episodes III, IV, V and VI.

Data collection and Data Analyses

Our collected data, specifically students' audio-recorded discourses and GeoGebra creations, were qualitatively analysed through a systematic identification of some levels of covariational reasoning, labelling the covariational positionings, deriving from Thompson and Carlson's classification (2017), that emerged in their speech acts. Specifically, we only give a taste of the findings selected from the discourses made within a four students group exploring the Taxicab trigonometric world. We labelled through a covariational positioning lens the discursive positioning acts (PA0, PA1, PA2, PA3, PA4, PA5) occurring within a collective discussion carried forward by the working group.

Preliminary findings and discussion

Let us look at the covariational processes stimulated by the transition between different geometric contexts shifting from traditional tools, such as non-permanent boards, to digital tools like GeoGebra. Excerpts were selected from students' recorded discussions related to Episodes III, IV, V, and VI, in which the processes of covariational reasoning, also according to the theoretical covariation and other orders of covariation, emerge through positioning. We discuss here the most significant moments in the discussion in the exploratory immersion in Taxicab trigonometry.

Episode III- Exploring the definition of the Taxicab-radian and envisioning the changes with respect to the Euclidean radian. Students begin with the angles as quantities and identify corresponding quantities in Taxicab and Euclidean geometry through acts of theoretical covariational positioning. The teacher (T) stimulates a correspondence between quantities, the measure of Euclidean radians and the corresponding measure of the same portion of a plane in Taxicab radians (Figure 2), to enact a reflection about the definition of t-radians:

T: When the measure of an arc on the Taxicab circle is one? […] Right! It happens when the Euclidean angle is half of a Euclidean right angle. Why? And, what about the value of the Taxicab angle π_T corresponding to π Euclidean radians?

Student St4 coordinates the value of one variable with changes in another one: (1) pairs of Euclidean-Taxicab angles, producing a discrete relationship, and (2) arc-radius discovering π_T by a ratio (PA4):

St4: […] As a circle of radius 1 has four sides, each of length 2, a Taxi-straight angle π_T is 4. Also, from $2\pi r = circle\ length$ you have $\pi_T = \frac{circle\ length}{2r} = \frac{8r}{2r} = 4$

St2 forms a general image of the relationships between arc, angle, and distances, but without an explicit connection between specific values. He attempts to link geometric concepts such as arc length, angle, and sine qualitatively. This goes beyond the mere separate consideration of variables:

St2: […] A Taxicab radian (t-radian) is an angle whose vertex is the centre of the unit Taxicab circle and intercepts an arc of Taxicab length 1. Look at the figure!! [*pointing by fingers*] So, the length of an arc is the Taxicab distance of that points!! Is the sine of an angle the ratio of the side opposite to the hypotenuse? (PA2)

The discussion goes toward the corresponding definition of Taxicab-angle in analogy with the Euclidean world. The definition subtends the mental coordination of the amount of change of a Taxicab angle with changes in the corresponding Euclidean angle, a discrete set of pairs (PA3):

St1: The measure of a Taxicab-angle θ is the number of t-radians subtended by the angle on the unit circle with vertex in the centre. It's analogous to the Euclidean!!! The definition, but the values are different. Let us take a line through the origin: $\frac{\pi}{4}$ corresponds to 1 t-radian, $\frac{\pi}{2}$ corresponds to 2 t-radian, and so on (PA3).

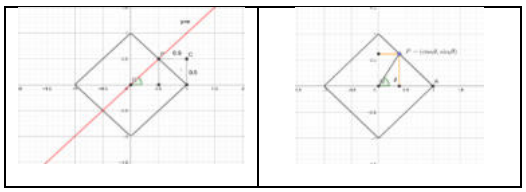

Figure 2: The Taxicab radian, sine, and cosine

The exploration proceeds with GeoGebra (Figure 2), with which students gain awareness. St2 coordinates the variables more deeply than St1, moving from discrete exploration to a more continuous approach, even though the covariation envisioned is not completely smooth (PA4):

St2: The length of the arc must be 1, but the arc is a segment…π/2 becomes 2 t-radians, so π becomes 4 and the circle is 8 radians. See also the values between them! (PA4).

Episode IV - Discovering the Taxicab-trigonometric functions and the related covariations. The exploration concerns the coordination of the t-radian values along the circumference of the unit circle and along the *x*-axis and the coordination of this measure with the sine value. Student St2 expresses clear coordination of the two variables, saying that the sine changes with changes in the radian *x*, and this seems a covariational positioning corresponding to PA4:

St2: When the t-radian value equals the value 0 the sine is 0, the value 1 the sine is 1/2, as the sum of the catheti is 1, and so 2 [*pointing by fingers on the sheet*] (PA4)

St3's speech shows awareness of the sine while considering changes in the Taxicab angle. St3 describes the relationship between variables in terms of simultaneous variations and coordinates geometric and algebraic concepts in a continuous but not yet smooth manner (PA4):

St3: The *sine* of a Taxicab-angle is the Taxicab-distance from the *x*-axis. And when the t-angle changes then the distance from the *y*-axis (and the *x*-axis) changes (PA4).

St1 speech act verbalises that while considering uniform increments of the angle, the rate of change of the sine varies uniformly, increasing or decreasing the function, and this corresponds to coordinating the average rate of change (PA4) anticipating periodicity (towards PA5). The reflection on maximum, minimum, and periodicity suggests advanced reasoning, not yet completely smooth:

St1: If so, let us graph it with GeoGebra!! Oh. [..] It is edgy!! It is not the same, but in the same way, here it increases, then it decreases (PA4). Right at the maximum and minimum points. Better to say angular! Mh… (Figure 3) In this world, the sine function is an open, broken line [*reflecting*]. Is it periodic? (PA5)

Students reflect to define the Taxicab sine and sinusoid (PA3, PA4, PA5). Going back helps students to conjecture the image of the sine graph (PA5) and develop awareness and critical thinking. St2 merely suggests a reflective process without directly considering the variation of values:

St2: So […]. Why don't we reflect on a good definition of sine? (Towards PA5)

St1 coordinates the different sine values in various geometries through a definition.

St1: The simplest is "the distance from the *x*-axis in the unitary circle" […] In Taxicab geometry the sine doesn't take the same values of Euclidean geometry (PA3).

St3 coordinates variations between variables in a continuous but not completely smooth manner, showing an advanced but not global understanding:

St3: If we change the angle length and consider the distance, the maximum is 1, […] is it really getting closer to be a sinusoid? (PA4)

St4 integrates local and global observations, an advanced understanding of the variable relationships:

St4: As the circle is a polygon, the Taxi sine and cosine functions have angularities in the maxima and minima, unlike what happens in Euclidean trigonometry (PA5).

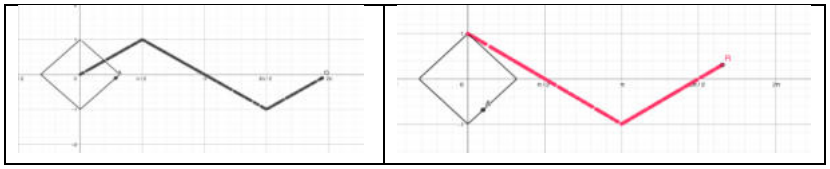

Figure 3: The Taxicab sine and cosine function

Students show awareness of the instantaneous changes in the domain of the functions (Figure 3):

St3: To simplify, I would see sine and cosine while the point moves on the Taxicab unitary circle […] and study its variation [*moving her fingers*]. Let me change the angle *α!* Rather than running through a finite number of points…(PA5)

St4: GeoGebra gives us the possibility to choose a point of the circle, and to make it move on it, visualizing the variation of its distance from the axis […] (PA5)

Episode V - Taxicab sinusoid and cosinusoid. The advanced way of reasoning induced by the shifting helps students engage in covarying sine and cosine (Figure 4), what can be called third-order covariational reasoning (Bagossi, 2024). Coming back to the Euclidean setting, St1 says:

St1: In this world what could be said about fundamental identities? Pythagorean Theorem is no longer valid, so what are the relationships between the sides of the right triangle? [*after a long discussion*] The relation is $|\sin\theta| + |\cos\theta| = 1$. This makes it clear that when sine varies, cosine also varies pointly. Let's graph it!! (PA5)

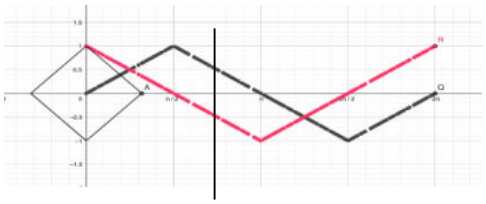

Figure 4: The Taxicab sine and cosine covariation: a third-order covariation

Episode VI - Theoretical covariational positioning in the discourses about Taxicab sinusoids and Euclidean sinusoids. In this episode, students begin verbalising a higher covariational positioning, a double *theoretical covariation* since two measures (T-E-angles) and two relationships (T-E sinusoids) simultaneously covary but in different geometries (Figure 5). St1 and St3 envision a relation between both corresponding angles and function values, both locally and globally:

St1: In this world the same behaviour, when a t-radian varies also the corresponding Euclidean-radian varies. When I move the point on the circles and I imagine representing it on my sheet, I describe the graphs of a function. The sine! (TCP)

St3: We can do the same in the Taxi-world. Let us try to see this in GeoGebra […] But?

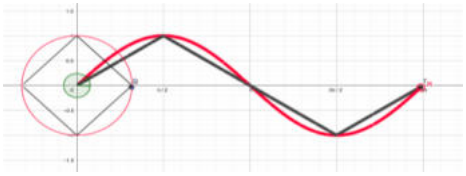

Figure 5: The Taxicab sine and the Euclidean sine covarying: a double theoretical covariation

Shifting to a different geometry opens the construction of new knowledge, reinforcing the old one.

Conclusion

In this experience, the geometrical transition challenges students to reinterpret mathematical concepts, relationships, and representations within a new theory. Such transition stimulates their covariational reasoning by requiring the coordination of analogous quantities across geometries, fostering the ability to observe and analyse how these quantities vary simultaneously. Furthermore, it supports covariational reasoning by encouraging students to create mental images and compare variations, enhancing their conceptual understanding and promoting critical reflection on the properties and definitions that govern each geometry. The reflections stimulated by the transition between Euclidean and Taxicab geometries provide valuable insights into the types of covariational positioning verbalised by students. Across Episodes III to VI, students showed varying levels of covariational positioning, ranging from basic coordination of values (PA3) to higher covariational positioning up to theoretical covariational positioning (TCP). The role of the digital environment in validating covariational reasoning reveals fundamental: students use GeoGebra software to dynamically explore simultaneous variation. They use the slider function to control the independent variables, the animation function to visualise continuous changes, and the trace tool to record the paths of the dependent variables. These tools allow students to observe the continuous changes between quantities supporting covariational understanding through interactive visual representations.

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Chat GPT as a didactic tool to improve students' abilities in managing different semiotic representations: a first study.

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The large multimodal model GPT-4 with the chatbot ChatGPT can operate with text, images etc. With the further aim of understanding how to employ GPT-4 to design a classroom activity, we want to investigate if ChatGPT's multimodal functionalities allow it to effectively handle different (semiotic) representations while solving an assigned mathematical task with operations such as treatment and conversion of data, and provide potentially useful new representations for students. We conducted a qualitative test in which two tasks containing graphs were submitted to GPT-4, and we analysed its responses with a focus on used and produced representations.

Keywords: Artificial intelligence, multimodality, semiotic system, mathematics education.

Introduction

The Large Multimodal Model of artificial intelligence GPT-4 (OpenAI, 2023) with the interface ChatGPT is capable of operating with - and generating - human natural language, programming languages, images, audio etc. interlaced together. Although it does not have didactical purposes and has not been specifically implemented with mathematical knowledge, has become popular in mathematics classrooms, used by teachers and students. It is commonly known that these kind of models are not reliable: the developers reported that "it "hallucinates" facts and makes reasoning errors" (OpenAI, 2023, p. 10), and it is affected by bias and disinformation, which can lead to ethical issues. In computer science field GPT-4 is considered a black box, as it is impossible to see all the processes that conducted to a precise answer; therefore it is a didactical black box, "a system whose inner-workings are (partly) unknown to the user" (Greubel & Siller, 2022, p.1), this representing a limit in education given the importance of argumentation and proper use of sources. Before its release, GPT-4 was tested in question answering with few mathematical tasks with diagrams to be interpreted; developers reported that its capabilities in questions with images are as effective as for text-only inputs (OpenAI, 2023), but we point out that the model's logical and mathematical performances are not solid when images are involved (Nieddu et al., 2024). It is possible that ChatGPT is subject to mathematical biases, which results in greater ability on certain topics than others (due to greater representation in the data used for training) or preference of some solving strategies over others; for example Getenet (2024) in his research involving pre-service teachers finds that GPT primarily relies on algebraic equations and formulas, giving less importance to visual solutions. Variety of representations is a crucial focus in mathematics education, as the emergence of mathematical knowledge is linked to the production of different semiotic representations, for Duval (2006, p.103) "something that stands for something else". We address then the following research question: Can ChatGPT handle different (semiotic) representations

while solving an assigned mathematical task containing graphs and provide potentially useful new representations for students?

Our long-term goal is to design classroom activities in which students work collaboratively with different registers of semiotic representation and interacting with multimodal ChatGPT; This paper constitutes an initial phase of our work, necessary to outline our idea based on the results that we will show here. After a description of the framework(s), we will analyse two tasks performed by ChatGPT with the aid of students' pre-existing data.

Theoretical Frameworks

As we focus on the design of collaborative activities with the use of ChatGPT and mobilization of representations, we employed the theory of registers of semiotic representation (Duval, 2006) and humans-with-media (Borba & Villareal, 2005); the former will help us to recognize the registers used and the relationships between them, the latter will be needed to understand the possibilities that ChatGPT offers in terms of sharing knowledge in a community with technologies.

Theory of registers of semiotic representation

According to Duval (2006), each representation can be generated automatically by a device or an organic system, as a memory image, a reflection or a photograph (non-semiotic) or produced with an organized system of signs as algebraic language, natural language etc. (semiotic). Systems which allow us to describe a representation with operations as treatment (modifying a given representation using the rules of the system in order) and conversion (translating the representation to one into another system) are called registers of semiotic representation. Although ChatGPT's representations are "automatically" generated, some of them could actually be expressed with precise registers of semiotic representation. This relates to our research question.

Humans-with-media

The assumption that knowledge is produced by humans together with the media, whether it is writing, orality or a multimodal technology is the basis of humans-with-media (Borba & Villarreal, 2005): computer technology rearranges human activity in a unique way as in activity theory (Tikhomirov, 1981), allows knowledge to be shared and a collective intelligence to be created (Lévy, 1993). This approach also focus on the importance of representations for mathematics learning: Borba (2012, p. 803) states that "knowing can be understood as coordination of multiple representations" and

> "Based on a combination of images, other kinds of visual feedback, writing and sound, and possibilities for sharing ideas in multimedia environments, multimodal discourse becomes a new means of expression when digital media are readily available in the production of knowledge" (Borba, 2012, p. 803).

Methodology

We selected two tasks from the Italian national INVALSI tests for 10th grade students containing text, tables, pie charts and bar charts as we wanted to include various registers, operations to be carried out and types of questions (and answers to be given). Mathematical contents are mean and

frequency (absolute, cumulative, percentage). We submitted each task to ChatGPT as an image (see Figure 1) with the prompt (in Italian): "You are a student of the second class of a second grade, secondary school. Solve the following task", as it performs better provided with a context (Lee et al., 2023).

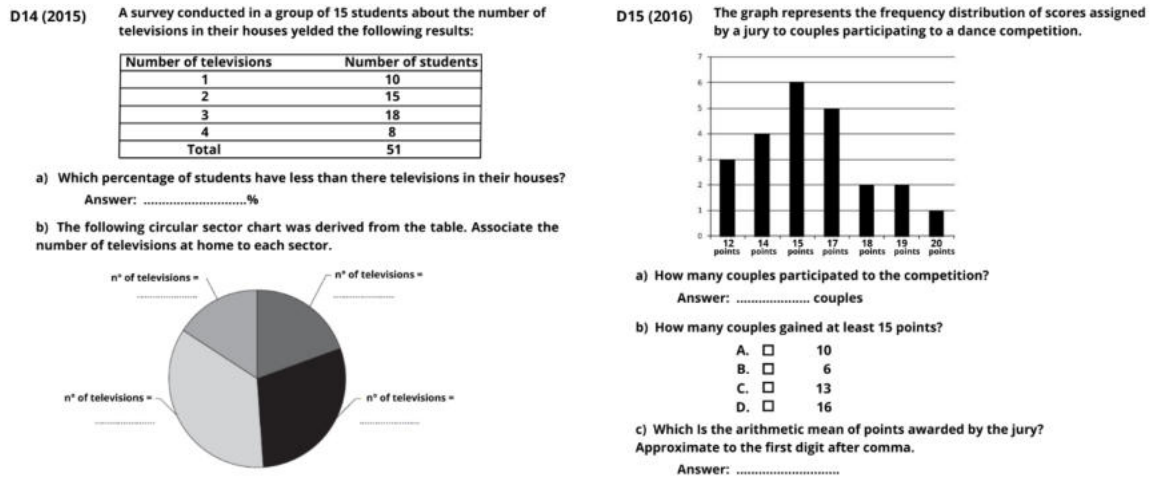

Figure 1: tasks D14-2015 and D15-2016 as examples (original versions are in Italian).

We employed the available pre-existing data (https://gestinv.it/Index.aspx) to understand whether the answers and produced representations could respond to students' potential difficulties. In Table 1 some data about students and ChatGPT (first answers) are collected. A description of GPT's answers and some transcripts follow.

Table 1: General results. Rows 2-3 refer to students' answers.

task	D14-2015 (a)	D14-2015 (b)	D15-2016 (a)	D15-2016 (b)	D15-2016 (c)
Correct	48,3%	61,5%	65,3%	44,8%	24%
Wrong	44,8%	35,1%	29%	51,8%	52,6%
Students' number	27207	27207	33992	33992	33992
ChatGPT	correct	unclear	wrong	wrong	wrong

Task D14-2015

ChatGPT correctly addresses part (a): "In order to calculate the percentage of students with less than three televisions at home, we sum the number of students with one or two televisions"; it calculates 10 +15 = 25 and then the percentage diving by 51 and multiplying for 100. The script for part b) follows:

ChatGPT: [..] we will need to determine the proportion of each category based on the data in the table. Remembering that the sum of the sectors in the graph must equal 100% and that each sector represents a proportion of the students with a certain number of televisions, we proceed to calculate the percentages for each category: [..] we need to associate each sector with the calculated percentages: The largest sector (about one-third of the graph) corresponds to 35.29%, representing students with 3 televisions. The next largest sector corresponds to 29.41%, representing students with 2 televisions. The smallest sector corresponds to 15.69%, representing students with 4 televisions. Finally, the remaining sector represents 19.61%, which corresponds to students with 1 television.

It was sufficient to pick the higher frequency and so forth, without calculating percentages; we also wondered which one is for ChatGPT the largest area it refers to, and asked for more explanations:

Researcher: Are you sure that for part b you need to calculate the percentages corresponding to the various frequencies?

ChatGPT: You are right; [..] since the circular sector graph you provided should already represent the percentage distributions based on the data in the table. Therefore, my goal was more to confirm that the proportions displayed in the graph match the original data. [..] we can rely on the visual distribution to determine which sector corresponds to which category, based on the relative sizes of the sectors. [..] the largest sector should correspond to the category with the most students (in this case, those with 3 televisions), and so on.

Researcher: Can you complete the given pie chart? produce an image of the pie chart where the missing data are inserted.

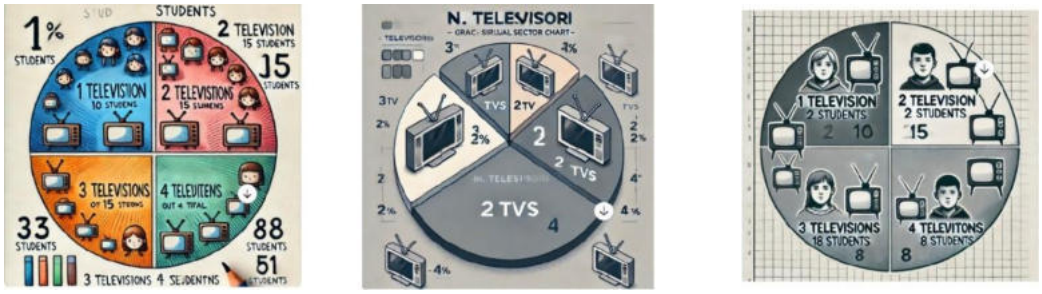

Figure 2: three attempts of ChatGPT to complete the pie chart, task D14-2015, part (b).

We suggested to associate the frequencies referring to sectors' greyscale as in the picture (we asked for the lightest sector, the second, the third and the darkest), but the answer still was meaningless.

Task D15, 2016

ChatGPT misunderstood the frequencies (it uses 1,4,6,6,5,3,1 instead of 3,4,6,5,2,2,1) in the chart.

ChatGPT: In order to determine the total number of participant couples, we will sum the frequencies of points in all columns in the graph: 12 points: 1 couple; 14 points: 4 couples; 15 points: 6 couples; 17 points: 6 couples; 18 points: 5 couples; 19 points: 3 couples; 20 points: 1 couple. Summing these values we obtain: 1+4+6+6+5+3+1=26. So, 26 couples participated to the competition.

For part b), it says: "We consider the couples which obtained 15 points or more", then proceeds to list the data (wrong as before) corresponding to 15, 17, 18, 19, 20 points and sums them. It also states that the result does not correspond to any option. In part (c), it explains "we have to multiply each score for its frequency and divide the total sum of points for the total number of couples". The calculation is addressed correctly (but with wrong data as before), and it also rounds the result as

requested. We asked "are you sure you used the correct frequencies" again obtaining wrong data (but different from the ones before).

Discussion

Using text and image input, one can effectively interact with ChatGPT and obtain new approaches when a mistake occurs. In task 14 GPT shows its understanding of the problem and its ability to find solving strategies, in particular it partially solved part (b) in another way when requested. It does not explicitly use the word 'frequency', this being an adaptation to the language used in the given task or a limitation in the natural language register, which remains colloquial and not specifically mathematical; It could extract the necessary data from the table, showing some handling of the tabular register, a good conversion to natural language and symbolic registers, used explicitly to perform the necessary calculations. On the other hand, 51.7% of the students could not calculate the frequencies for less than 3 televisions, showing some difficulties with the cumulative frequencies. Part (b) was the one solved correctly by the largest number of students, we can assume that it was quite natural for them to proceed assigning the sectors in increasing or decreasing order; ChatGPT initially shows some understanding of the meaning of a pie chart, stating that the sum of the sectors must be equal to 100%; then, it maintains the link between a pie chart and percentages, associating them with the sectors. When we asked about this, it specifies that it is not necessary to calculate percentages. Finally, when asked to complete the graph, it changed the given one, 'forgetting' that the graph must visually represent the percentages (modalities) that it wrote down. It seems that priority is given to the numbers, that is the symbolic register, over the pie chart register.

Dealing with task 15, we can affirm that GPT had a hallucination, resulting in a bad conversion from the bar chart into the natural language and algebraic register. We assume that GPT either misread the chart or randomly generated the data not knowing how to derive them. The natural language and symbolic registers are used in a consistent manner dealing with the concepts of frequency and mean; a weighted mean was not explicitly requested, but GPT was able to recognise the need for it. Furthermore, approximation is carried out correctly using the symbolic register. Students in general obtained better results in part (a); in part (b), it was found that most of the errors were for option B. (39.8% of total answers), corresponding to 6 couples. Being 6 the number of couples that gained exactly 15 points, we can suppose that the cumulative frequencies of 'at least 15 points' might have been mistaken for the frequency of the '15 points' modality; this shows a probable students' difficulty in decoding natural language and in transitioning from natural language register (at least 15 points) to the algebraic register (P>=15). Finally, only 24% of students performed correctly part (c); We can assume that it can largely be attributed to the presence of cumulative frequencies.

Conclusion

Despite GPT-4 not being explicitly designed for didactical purposes, its performance in handling operations such as treatment and conversion between different registers has proven promising in certain contexts, such as processing natural language, tabular and symbolic data. However, limitations remain, as evidenced by its struggles in accurately converting pie charts and bar charts into meaningful representations, leading to mistakes, hallucinations and misinterpretation of data.

Difficulties with pie chart register were both in replication and in analysis, because specific references to sectors in the image are never made, not even when suggested to indicate them using grayscale. It showed no difficulties in recognizing the questions' layout, not even options in question 15 (b). Dealing with the second part of our research question (production of useful representations) ChatGPT provided answers that may or may not be correct, but still well expressed, argued and explained in detail, including the use of bulleted lists, formulas and mathematical symbols used appropriately. This could provide students with useful examples of how a solution is argued and expressed using natural language and symbolic register. It is possible to edit a previously sent message, obtaining two different paths within the same conversation; this ensures the possibility of recalling previous elements or providing new representations to GPT. The model's ability to change its response once it receives a suggestion often turns out to be fictitious, as the second response will be a reworking of the first. When observing ChatGPT outputs, we can see how the model generates - types - its answers word by word, as each one is produced based on the previous one; this constitutes a huge difference between the knowledge produced, for example, by a book, or even a web page, as these media are produced before students' use, and not during it. We argue that, as expressed by Borba and Villareal (2005) in their framework, the media in which a task takes place is relevant, and solving problems with ChatGPT (or other chatbots) is different to do it with every previously known media. It allows students to access information produced, expressed in a certain way, with certain modalities, exclusively for them; as we discussed before, the lack of control on generated contents can be dangerous, and for this reason we believe that teachers should guide students to a fruitful use, and educational research should investigate on best practices.Although the model cannot (and should not) replace human guidance in the classroom, it may enhance learning experiences by offering diverse representations and fostering critical thinking. As argued by Yoon et al. (2024) ChatGPT can be considered as a resource in students' proving process: students were effectively able to take authority in their reasoning through activities such writing a prompt, revising it based on the outcome, evaluating GPT's responses and deciding what to take from them, and finally write a final proof. Barana et al. (2023) explained how ChatGPT supports and activates strategies of critical thinking and problem solving when students use it: as well as being helpful when they could not find a solution approach, students were able to find personal strategies such as dividing the problem into smaller tasks and analysing them with GPT or using it as a confirmation in case one or more solutions had already been found. Dilling et al. (2024) in their research showed how ChatGPT can stimulate interaction among students in various ways such as identifying errors, discussing responses, comparing different solutions; these processes, as the authors affirm, recall and reflect the competences of mathematical communication and argumentation. This is consistent with what we have been saying, but we believe that in order to introduce AI into education, it is necessary to prepare students; as expressed by Arzarello and Robutti (2010), to use effectively the new technologies available in education it is necessary to introduce new methodologies and not simply repeat traditional activities with the new devices. For example, it is necessary to address to students ethical issues, mistakes, hallucinations and other limitations, but also to provide some basic information about how chatbots work, such as the need of asking questions (prompt engineering) in a purposeful way and specifying the context; indeed, as found by Yoon et al. (2024), teachers can influence students' conceptions of genAI and stimulate

constructive use by providing examples of errors and inaccuracies of artificial intelligence and emphasizing its use as a support and not as a solver. We think that various established best practices are suitable for activities with chatbots; for example, it is possible (as we showed) to promote multiple representations, problem solving and collaborative learning: students can be encouraged to conjecture solutions, evaluate and check AI solutions and discuss with peers, for example with the organization of a final discussion to summarize the work done and stimulate the metacognition, focusing on discerning and comparing personal contributions from GPT's ones. Multimodal ChatGPT can be integrated into activities that exhibit a Concrete Pictorial Abstract approach; it could guide the various phases by providing customized cues to reinforce the transition between levels of abstraction; in particular, it can prove a support in the pictorial phase for images and virtual drawings, but also in the abstract phase, since, as we have seen, GPT can consistently handle symbolic register. The practice of scaffolding can also be applied with ChatGPT: the teacher can stimulate students to ask questions (that is writing prompts) to ChatGPT to address complex problems; As the student progresses, ChatGPT support can be gradually reduced, encouraging autonomous responses and using it only at a final stage. Future research should investigate its use in real classroom settings, applying more refined strategies for multimodal learning.

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Students' written and oral verbalizations of a numerical pattern using the tool *BookCreator*

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Numerical patterns are crucial for mathematical learning. However, it is often difficult for students to verbalize their discoveries of a pattern in a generalizing way. In this study, we give insights into a learning environment, in which primary school students are encouraged to verbalize their discoveries on a numerical pattern in both analogue and digital formats using the tool BookCreator. Results from a preliminary design experiment, in which students used a paper-pencil note, a typed text, and an audio recording, reveal differences among these methods, with the digital ones being more generalizing. Every verbalization includes specific information on the pattern. The findings suggest that the BookCreator may offer potential for verbalizing numerical patterns in a generalizing way.

Keywords: BookCreator, numerical pattern, verbalization, generalization, primary school.

Introduction

Mathematical patterns are an important basis for the mathematical thinking of primary school children (e. g., Mulligan & Mitchelmore, 2009; Warren, 2005). Activities such as describing, generalizing, and justifying patterns can make mathematical structures accessible to students, but are often not clearly separated from each other. Therefore, students' verbalizations regarding patterns can include both descriptive and justifying elements (Akinwunmi & Steinweg, 2022). This paper does not aim to differentiate which parts of a verbalization are descriptive and which are justifying. Therefore, we use the term "verbalization" to address all products (written or in a sound record) that the students create when they are tasked to describe a pattern.

Even without any knowledge of variables, it is possible to describe a pattern in a generalizing way, but for young students, it can be difficult due to challenges in the use of (written) language (Akinwunmi, 2012; Coles & Ahn, 2022; Warren, 2005). Research (e. g., Warren 2005) implies, that oral verbalizations may sometimes be more precise than written ones, so working with oral utterances concerning a pattern may be helpful for some students. The use of digital media is one way to fix those oral expressions as a base for further work (Cooper & Warren, 2008; Warren, 2005).

In this paper, we will investigate how students describe a numerical pattern written and orally using analogue and digital media. Therefore, according to the Design Research Method (Nührenbörger et al., 2019), a learning environment with a focus on two numerical patterns was developed and tested in a design experiment. The learning environment uses analogue media and the digital tool *BookCreator*. During the design experiment, two fourth graders produced three different verbalizations that will be analysed by using qualitative content analysis (Kuckartz, 2019). The underlying research question is: *How do students' verbalizations of a pattern differ in paper-pencil notes, typed text, and audio recordings*, and *what impact does the transition between analogue and digital media have on the content and the way of generalization in these verbalizations?*

Theoretical and empirical background

Patterns function as a mediator to understand mathematical structures and are therefore essential for mathematical learning (Akinwunmi & Steinweg, 2022; Mulligan & Mitchelmore, 2009; Warren, 2005). According to Mulligan and Mitchelmore, "A mathematical pattern may be described as any predictable regularity, usually involving numerical, spatial or logical relationships" (2009, p. 34). Discoveries related to patterns can be verbalized orally or in written form, often involving description and justification (Akinwunmi & Steinweg, 2022). Research suggests that verbalizing these discoveries may be challenging for students, particularly when expressing them in written form (Warren, 2005). Oral expression seems easier for students because, for example, it can include gestures and material interactions, elements that are missing in written tasks (Warren, 2005).

As patterns are not limited to specific cases, they can be verbalized in a generalizing way. Such a generalization is important because, for example, it contributes to the development of algebraic thinking (Cooper & Warren, 2008). By using and discussing different representations, students can be supported in abstracting their discoveries and verbalizing them in a general manner (Schifter & Russell, 2022; Warren, 2005). Even young students without any knowledge of symbolic variables can express the general character of a pattern by using a representative example with labelling it such as that, or using their own words as variables, and other (Akinwunmi, 2012; Coles & Ahn, 2022).

Summarizing, students' difficulties with the verbalization of patterns could be rooted in language use, especially when tasked with writing down their ideas. Activities with material seem to support students when verbalizing their discoveries orally (Cooper & Warren, 2008; Warren, 2005); thus the use of representations and the opportunity to work with oral utterances appear to be helpful.

In this study, we use the digital tool *BookCreator* in addition to analogue material to provide these opportunities. The tool enables the creation of multimedia books[1] with features such as text input ("text function"), drawing or handwriting ("pen function"), audio recordings ("sound function"), and photo and video integration ("camera function") (Schäfer, 2023). Integrating videos in *BookCreator* allows the presentation of dynamic processes and activities with concrete material, which can be accompanied by oral utterances, gestures, or notes. As previously discussed, these factors may contribute to oral verbalizations being more precise than written ones (Warren, 2005). While the text and the pen functions allow to integrate handwritten or typed expressions, the sound function enables fixing oral utterances with or without reference to other representations in the eBook. Therefore, when creating an eBook about a pattern, students have several options to verbalize their discoveries. In summary, *BookCreator* as part of a learning environment offers an opportunity to provide students with different representations, even dynamic ones such as videos, or to create these representations themselves. Using multiple external representations in digital media has the potential to enhance mathematical learning (Walter, 2018). Similar to working with analogue media, the students need to actively establish connections between those representations since they are not automatically linked by the tool.

[1] In the following referred to as eBooks.

Method

According to the Design Research method (Nührenbörger et al., 2019), a learning environment focused on numerical patterns was developed and tested in a design experiment. The design experiment involved two pairs of students (grades 3 and 4) participating in three clinical interviews each. The interviews were videotaped and transcribed, and the students' products were collected.

In the following, we provide a brief overview of the learning environment using sample products and recordings from the interviews with one pair of students (Jona and Marco, grade 4). Afterward, three verbalizations produced by the two students in three different formats (paper-pencil note, typed text, audio recording) will be analysed regarding the content and whether there is a general character of the expression recognizable or not. To address the research question, we developed categories using a mix of concept-driven and data-driven development according to qualitative content analysis (Kuckartz, 2019). By classifying smaller parts of the verbalizations, we will focus on the content and the level of generalization within the different products, and examine their differences in terms of both content and general character. Table 1 displays the main categories and subcategories with respect to content. Within each main category, we differentiate between the five subcategories.

Table 1: Categories regarding the content

Name of category: Information about …	**Definition:** The (part of the) verbalization …
Main categories	
I the current pattern	… focusses on the pattern that is currently dealt with, without any information on previous interview sessions or representations that might be used.
II previous sessions	… focusses on previous sessions (e. g. patterns that were discovered), without any references to the current session.
III the current pattern with a reference to previous sessions	… draws a connection between the pattern that is currently dealt with and previous interview sessions (e. g. connection between the patterns that were discovered).
IV the current pattern with a reference to representations	… draws a connection between the pattern that is currently dealt with and a representation that is currently used (e. g. how to recognize regularities of the pattern in a representation)
V other aspects	… focusses other aspects that cannot be assigned to main categories I to IV
Subcategories	
(1) Characteristics of one number	… focusses on characteristics of one number without mentioning characteristics of the other number or the relation to the other number (e. g. the quantity of tens in one number).

(2) Characteristics of two numbers separately	… focusses on characteristic of two numbers without relating them to each other (e. g. the quantity of tens in two numbers).
(3) relation between two numbers	… relates two numbers to each other (e. g. comparison of the quantity of tens in the two numbers).
(4) criteria for matching number pairs	… establishes criteria that must be fulfilled by numbers to match the pattern (e. g. a minimum number of tens that is necessary).
(5) other	… focusses other aspects that cannot be assigned to subcategories (1) to (4).

Concerning generalization, we use category (A) where the general character is not recognizable, or (B) where it is recognizable. Further analyses could explore how exactly the students generalize in the verbalization. However, our focus in this paper is solely on whether the general character in the verbalization becomes clear or not. To categorize a phrase as making a general character clear, we refer to Akinwunmi's (2012) ways of generalization: the use of an example and labelling it as such, the use of several examples, the formulation of conditional phrases, the use of quasi-variables, or the use of variables (Akinwunmi, 2012).

Learning environment

The learning environment consists of three sessions. Figures 1 to 3 show impressions from the sessions, using the example of the interviews with Jona and Marco. The students are tasked with discovering, describing, and justifying numeric patterns using different representations. In sessions 1 and 2, the students work with number pairs such as 25 and 34. In this pattern, the tens and ones differ by 1 and the numbers as a whole differ by 9. In the third session, the students discuss number pairs like 34 and 52. Here, the tens and ones differ by 2 and the difference between the whole numbers is 18. While the students work completely analogue in session 1, they use the tool *BookCreator* to discuss and improve an existing eBook with a focus on the pattern in session 2. This way, the second session gives the students the opportunity to familiarise themselves with the digital tool and its functions on the one hand, and to further develop and question their discoveries on the pattern on the other hand. In session 3, the students build on their previous work by exploring the new number pairs with analogue material and creating an eBook about the pattern.

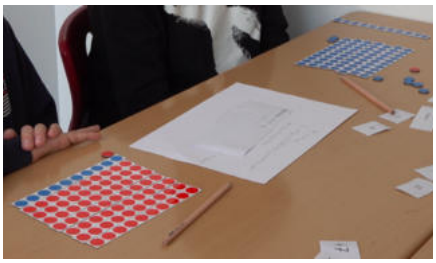

Figure 1: Impression from session 1 - students use counters and stripes of tens

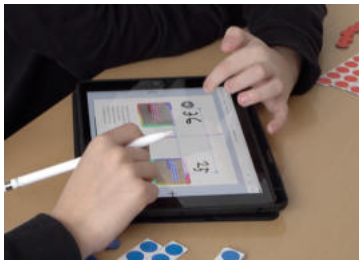

Figure 2: Impression from session 2 - students add a page to the given eBook

Figure 3: Impression from session 3 - the eBook's title page (translated)

Jona's and Marco's verbalizations of the numerical pattern

The verbalizations we will analyse in the following were produced by Jona and Marco in their third interview session. The students discovered the pattern within number pairs such as 34 and 52 and verified it using self-chosen number pairs at the beginning of the session. Then, the students verbalized their discoveries in a handwritten paper-pencil note (analogue), as well as in a typed text and an audio recording on their eBook's title page (digital). These verbalizations were chosen for analysis because they were all created at the point of transition from the analogue to the digital media. Between typing the text and making the audio recording, the students drew an iconic representation (an empty number line with the calculation of plus 20 and minus 2) which is included on the eBook's title page (see Figure 3). Table 2 shows the three verbalizations translated by the authors. The coloured numbers were added to categorize smaller parts of the products.

Table 2: Jona's and Marco's three verbalizations, translated by the authors

Handwritten verbalization	[1] You swap 2 tens and 2 ones. [2] The smaller number has 18 difference to the large one. [3] You swap the 34 becomes the 52 and the 52 becomes the 34. [4] The smaller number has two ones more. But the larger number has 20 more.
Typed verbalization	[1] That with different numbers like 34/52 you can also swap [2] that we had named before as swapping cards [3] but that now it is that you swap 2 tens and 2 ones.
Audio recorded verbalization	[1] We had already made an eBook before. But with this eBook it is like that, before it was with, but the swapping cards were with nine. [2] This time it was simply calculated again plus nine, so two times nine, so that it's eighteen and um, [3] here you can see, that it's twenty minus two and the eighteen and [4] it actually only works with eighteen. [5] And what you also cannot do is, um, now calculate ten plus eighteen, [6] because there has to be a two, there has to be at least a two behind it and or a higher number, because it otherwise, um, because otherwise you can't deduct it so well.

Results

Figure 4 illustrates the categorization of verbalizations into two levels: (A) where the general character is not recognizable and (B) where it is. Both levels assign phrases from the verbalizations into main categories I to V and subcategories 1 to 5.

In the handwritten verbalization (see Figure 4, green parts), most expressions can be understood in a general way. They are not limited on a specific number pair, as the students use their natural language as variables with expressions like "the smaller number". Only phrase [3] is placed at level (A) as it refers to a concrete number pair. Content-wise, this verbalization falls under category I, as it focuses solely on the pattern from the third session without linking to other sessions or representations. On the level of subcategories, the verbalization concerns the relation between two numbers of a pair (subcategory 3), highlighting a swapping process of tens and ones separately (part [1]) and of the

whole numbers (part [3]). With this concept of "swapping", the students have come up in the first session naming their discovery *the swapping theorem* (original "Der Tauschsatz"). As they used the expression "to swap" before, a reference to the first pattern can be recognized but is not explicitly drawn by the students. Part [2] discusses the difference of 18 within a number pair, whereas part [4] compares tens and ones, with tens expressed as "20" rather than "2 tens."

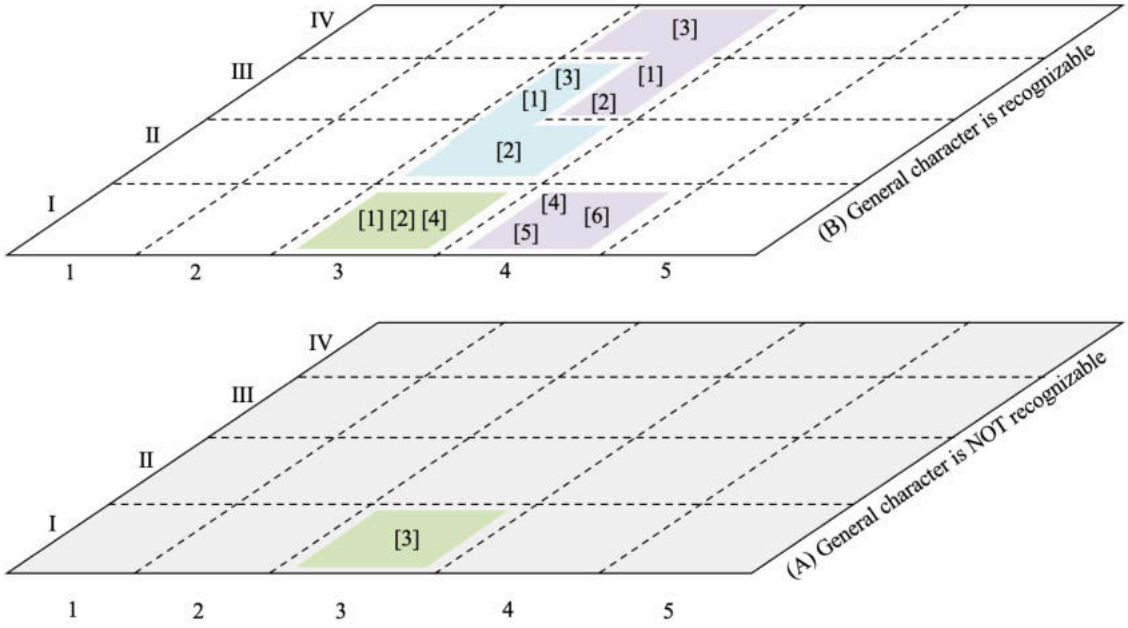

Figure 4: Categorization of the verbalizations (green: handwritten, blue: typed, purple: audio)

The typed verbalization is entirely general (see Figure 4, blue parts). Again, 34 and 52 are used as an example, now labeling it such as that by using the expression "with different numbers like" (part [1]). All parts mention the relation between numbers in a pair (subcategory 3), with swapping processes of tens and ones (part [3]) and whole numbers (part [1]). While part [2] refers on previous sessions alone, parts [1] and [3] draw a connection between the current pattern and previous interview sessions.

The audio recording is again located on level (B) (see Figure 4, purple parts) as all phrases reflect a general character with no specific number references. Phrases [4], [5], and [6] establish criteria for number pairs that are matching to the pattern in the current session (main category I, subcategory 4). Stating that "it actually only works with eighteen" (part [4]), the students underscore the importance of 18 without explication. We suspect that, in part [9], the students try to make clear that the numbers 10 and 10 plus 18 might not fit this pattern, referring to the need for at least two ones, presumably in the smaller number (part [10]). Phrases [1] and [2] draw a connection to earlier sessions (main category III), relating to the swapping process and differences 9 and 18 (subcategory 3). By expressing 18 as *9 plus 9* and *2 times 9* (part [2]), the differences 9 and 18 are set in relation to each other. In part [3], the students express the difference 18 between two numbers of a pair (subcategory 3) as *20 minus 2* with a reference to their drawn empty number line (main category IV).

Comparing the three verbalizations reveals that the handwritten one is less generalizing than the ones in *BookCreator*. The handwritten product focuses on the current pattern itself, discussing the relation

within a number pair (difference of the numbers and of tens and ones, swaps of the whole numbers and of tens and ones) without drawing a connection to prior sessions or representations. The typed verbalization connects ongoing activities to earlier interview sessions and addresses relations between two numbers of a pair. Here, they focus on swaps, excluding the difference of 18. The audio recording again makes links between the sessions, novelly connecting the differences 9 and 18, expressing 18 as *20 minus 2* with reference to the iconic representation, and establishing criteria for number pairs. However, they omit previously mentioned relationships between tens and ones.

In summary, students' handwritten verbalization remained specific and focused, while the digital formats encouraged broader connections and generalizations, with varied emphasis and omissions across media types.

Conclusion and Outlook

In our paper, we presented initial results from a design experiment involving a learning environment focused on numerical patterns using qualitative content analysis (Kuckartz, 2019). We observed differences in content and generalization across three different types of verbalizations: handwritten, digitally typed, and audio recorded.

The verbalizations mostly are expressed in a general way, suggesting that the students do not perceive the pattern as tied to specific numbers. While the handwritten product includes a part that is specifically referring to one number pair, the typed text and the audio recording can completely be understood in a general manner. The handwritten verbalization included key pattern information, while the typed and audio versions referenced previous patterns but varied in detail. Interestingly, in the audio recording, the students established criteria for matching number pairs, which they did not before, and connected to an earlier created iconic representation. The expression of these further ideas is consistent with Warren's (2005) findings that, for some students, oral expression is less challenging and therefore more comprehensive than written expression.

While we cannot definitively link the differences to digital tools, the inclusion of new aspects in each product suggests that creating the eBook inspired new ideas or led to fixation of existing ones. The process of creating fresh verbalizations for the eBook without including their previous products may have enhanced the students' ideas in the digital versions. The creation of a drawing before making the audio recording possibly sparked further ideas captured in the audio. Notably, the audio recording uniquely referenced the empty number line, perhaps due to its production sequence. The written verbalizations omitted such references despite prior use of different representations.

The findings suggest that using *BookCreator* in such learning environments may enhance the ability to verbalize numerical pattern discoveries in a generalizing manner, as it seems to encourage students to articulate ideas that might otherwise remain unexpressed. However, these results are specific to this student pair and context. Our next steps involve further analysis of the verbalizations, refining the learning environment to better encourage generalization and diverse verbalization of pattern aspects, and retesting with more students. In future research it could also be interesting to try elements of the learning environment in a different setting, for example by starting with the digital tool and concluding with analogue media to scrutinize the role of *BookCreator* in our results more closely.

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Can you help us? – Task-specific LLM-based support in a digital mathematical escape game

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This research explores the integration of large language models (LLMs) into digital mathematical escape games to provide individualised student support during problem-based tasks. The study employs a design-based research approach to develop a task-specific LLM-based assistant for a digital escape game focusing on symmetry. The task-specific LLM-based assistant is shaped by principles of minimal help, multiple representations, enriched language use for conceptual understanding, and cognitive activation. Through analysing student interactions with a task-specific LLM-based assistant, the research aims to uncover effective design principles for task-specific LLM-based assistants, assess the potential and limitations of task-specific LLM-based assistants in supporting individualised learning, and refine the approach for broader educational use.

Keywords: LLMs, digital escape games, support-element, problem-solving.

Introduction

Digital educational escape games represent a form of game-based learning environments that offer various potentials, including a high level of motivation for mathematics education. The games typically include different types of (mathematical) puzzles that students must solve to achieve a goal that fits within the narrative context (Nicholson, 2015). Solving these puzzles can enhance the students' content-related knowledge as well as process-related competencies, such as problem-solving and communication. However, alongside these benefits, there are challenges in using digital escape games in mathematics education. For example, students may struggle to solve a difficult task independently, necessitating additional support. Designing these support elements can be both difficult and time-consuming. An innovative approach to address these challenges is to integrate a large language model (LLM) into a digital escape game, as LLMs like ChatGPT can provide specific responses to students' attempted solutions or queries in natural language. However, current research on the use of LLMs to assist learners in problem-based mathematical tasks is limited, particularly in the context of escape games.

This paper presents a design-based research project focusing on the integration of LLM-based assistants to support learners in problem-based tasks within a digital mathematical escape game. After a brief overview of the theoretical aspects, the paper introduces design principles grounded in theory for developing an LLM-based assistant. Then it presents the results of an initial pilot study.

Theoretical Background

Digital Escape Games

> An educational escape room / game can be defined as an instructional method requiring learners to participate in collaborative playful activities explicitly designed for domain knowledge acquisition or skill development so that they can accomplish a specific goal [...] by solving puzzles

linked to unambiguous learning objectives in a limited amount of time. (Fotaris & Mastoras, 2019, p. 2).

This game-based learning approach has the potential to enhance mathematical learning processes, for example, by fostering students' motivation and collaboration when engaging with problem-based mathematical tasks (Jiménez et al., 2020; Sidekerskiene & Damaševicius, 2023). In this project, we focus on the use of digital escape games for mathematics education, as they offer additional and specific learning advantages. One such advantage is the integration of interactive and dynamic tasks within the digital escape game (see Figure 1) that can enhance cognitive activation and encourage a deeper focus on conceptual understanding. Moreover, digital escape games enable the incorporation of immediate feedback mechanisms (see Figure 1), such as responding to students' solution attempts (Sidekerskiene & Damaševicius, 2023). This feedback can be delivered indirectly, for instance, by preventing progression to the next task until the current one is solved correctly.

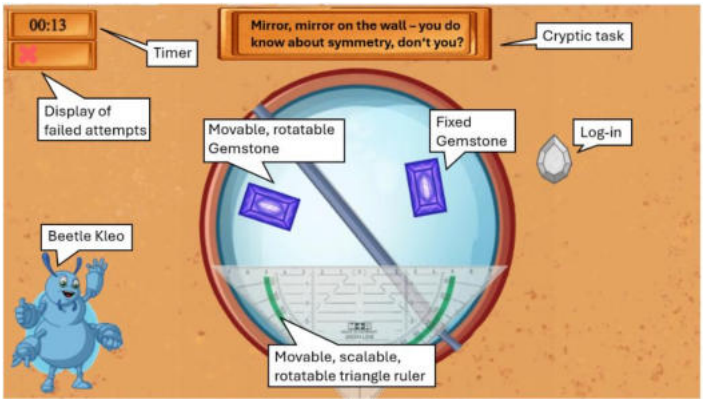

Figure 1: Interactive task with immediate feedback

When such problem-based tasks are integrated into a digital escape game, students may require support when facing the task. Research indicates that adaptive designed educational games can lead to higher learning outcomes and that well-designed learning support in educational games can enhance both learning and student engagement (Shute et al., 2020). However, creating individualised support elements for a digital mathematical escape game is a major challenge. Support elements can be realised, for example, by displaying text. These hints often appear after a predefined time or a certain number of failed attempts. Another option is to employ a pedagogical agent to provide guidance during the task. Wouters and van Oostendorp (2013) show, however, that none of the educational games they analysed integrated a pedagogical assistant to provide support. It can be assumed that the design of such an agent is very complex with previous possibilities (without artificial intelligence (AI)). An innovative approach to enhance the adaptiveness of educational games, including digital mathematical escape games, is the integration of LLMs. The following chapter will explore this in more detail, highlighting the potential of LLMs to offer adaptive support in a digital escape game.

LLMs as support in problem-based tasks

LLMs are a type of generative AI designed and trained to process and produce human language. Recent advancements in LLM technology (e.g., ChatGPT 4.0) have extended their capabilities to

include image and speech processing. The input and output formats can be freely chosen, allowing users to select modes that align with their preferences. Custom LLMs, or "customized GPTs," can be created via prompts, leading to a distinction between generic models (e.g., ChatGPT), education-specific models (e.g., customized GPTs guiding learners through reflective processes), and subject-specific tools (e.g., Khanmigo). Khanmigo, a subject-specific tutor, assists students in solving mathematical problems by avoiding direct answers and instead encourages independent problem-solving through targeted questions and guidance, thereby promoting active learner engagement. Within the category of subject-specific tools, we can further differentiate between three dimensions: cross-topic, topic-specific, and task-specific tools. Generic tools have the advantage of broad applicability across diverse problems, whereas more specialized tools offer potentially more precise support, for example, by taking into account learners' prior knowledge and common misconceptions. In the following sections, these tools will be referred to as LLM-based assistants.

Research on the effectiveness of LLMs and LLM-based assistants in supporting problem-based tasks in mathematics education is limited. While several theoretical studies (e.g., Matzakos et al., 2023) highlight the potential advantages and disadvantages of LLMs, empirical research remains scarce, likely due to the emerging nature of the field. A previously published study conducted with Taiwanese seventh-grade students explored the use of ChatGPT within a digital learning game for physics education (Chen & Chang, 2024). This study examined three groups: The first group ('Game only') used the digital learning game without any LLM-based assistance and served as a control group. The second group ('GameGPT') played a version of the game that incorporated ChatGPT, with adjustments to fit the game context. The third group ('GameGPT_examples') also used a ChatGPT-integrated version but with the addition of context-specific examples and questions provided by the LLM-based assistant. The study found that both experimental groups reported significantly higher perceived competence, and in some cases, experienced a reduced cognitive load. Post-test results revealed that the third group ('GameGPT_examples') achieved significantly better outcomes compared to the control group. It can be assumed that integrating LLM-based assistants could similarly enhance the performance of students working on problem-based mathematics tasks within a digital learning game, such as a digital escape game.

Research Goal and Research Questions

In this research project, we aim to develop and investigate customised task-specific LLM-based assistants that students can interact with during a digital mathematical escape game when they require assistance with challenging task. Employing a design-based research approach (Prediger et al., 2015), we will address the following research questions.

1. What are relevant design principles for a task-specific LLM-based assistant to support the students' working on a challenging task in a digital mathematical escape game?
2. How do students use the task-specific LLM-based assistant working on a challenging task while playing a digital mathematical escape game and which potentials and limitations can be identified?

To address these questions, we will develop task-specific LLM-based assistants for a digital mathematical escape game that focuses on symmetry and reflection.

Development of an LLM-based assistant

Design-principles

For the development of the task-specific LLM-based assistant to support students in solving challenging tasks, four theory-based design principles were derived. The first design principle draws from research in game design, while the remaining four are grounded in research in mathematics education.

1. Non-player characters (NPCs) are in-game characters that players cannot control, often enhancing the immersion of the game world (Ermi & Mäyrä, 2005; Rogers et al., 2018). In the digital escape game, the task-specific LLM-based assistant functions as a NPC, integrated into the narrative context. We assume that this will enhance immersion within the digital escape game, thereby promoting the learning benefits associated with the game flow (Shute et al., 2020).
2. The LLM-based assistant follows the principle of minimal assistance (Zech, 2002). Tips and hints progress gradually, starting with motivational and strategic guidance, and eventually offering more content-specific support. Minimal help is a well-established strategy in mathematics education (Heitzer & Weigand, 2020).
3. Visual aids are employed by the LLM-based assistant to support the problem-solving process. Visualizations can help students to establish connections between mathematical concepts and are particularly crucial for understanding geometry (Hattermann et al., 2023). Presenting multiple representations related to the mathematical task can further enhance conceptual understanding (Prediger et al., 2022).
4. The assistant employs precise technical language while also responding to students' everyday language. The goal is to bridge three language registers—everyday language, educational language, and technical language (Prediger, 2020)—thereby supporting subject-specific language development. Additionally, the LLM-based assistant serves as a linguistic role model by consistently using appropriate technical terminology (Leiss et al., 2023).

Exemplary realisation in a prompt

In the digital escape game, the students find themselves trapped in an ancient temple during a school trip. They are guided by a friendly beetle named Kleo, who lives in the temple and directs them to an alternative exit. However, the doors are locked and can only be unlocked by solving a series of puzzles. Kleo accompanies the students throughout their journey in the temple. One of the puzzles they encounter involves a task (see Figure 1), where students must position a movable, rotatable gemstone in reflection-symmetric with the fixed gemstone. Students have three attempts to solve the task correctly. Kleo provides guidance and encouragement, but solving the puzzles requires students to apply their mathematical knowledge and spatial reasoning skills.

According to Götz and Gasteiger (2019) the difficulty of reflection tasks can vary based on specific features. The task depicted (see Figure 1) includes several difficulty-generating elements: students must reflect a non-adjacent object with no sides parallel to the axis on an oblique axis. Additionally, there is no structured underground (e. g., a square grid), which is assumed to further increase the task's difficulty. These features suggest that the task may pose a challenge for students.

To assist students who may struggle with the task, a task-specific LLM-based assistant was developed using a customised GPT, guided by the outlined design principles. In the game, this task-specific LLM-based assistant takes the role of the friendly beetle Kleo (see Figure 1, bottom left). At the beginning of the prompt, the task and the task goal are described. Based on insights from a preliminary pilot study conversation starters, specifically prepared prompts to initiate dialogue (e.g., "We don't understand the task. Can you help us?"), were integrated into the task-specific LLM-based assistant. Then the five design principles are implemented:

1. In alignment with *design principle 1*, the role is described (e. g., "You take on the role of a friendly, helpful beetle and act as a tutor.") and the name ("Kleo") is given.
2. *Design principle 2* is implemented throughout the prompt. For instance, the customised GPT is instructed to first offer motivational help ("You motivate the students to solve the task on their own.") before progressing to strategic or content-related guidance. The prompt also requires the customised GPT to refer to the triangle ruler as a mathematical tool to help content-strategical.
3. To incorporate *design principle 3*, the prompt includes instructions for the assistant to offer a content-specific visualization as part of its minimal support. This visualisation intends to encourage students to reflect on the orientation of the reflected object. The LLM-based assistant should direct the learners' focus to the two ladybugs. (see Figure 2).
4. *Design principle 4* is integrated by instructing the assistant to use technical language ("Use the following technical terms for the tips and queries: [...] orientation, rotation, size"). Furthermore, the assistant is guided to respond in short, concise sentences without complex subordinate clauses, ensuring the language aligns with the linguistic capabilities of the target audience.

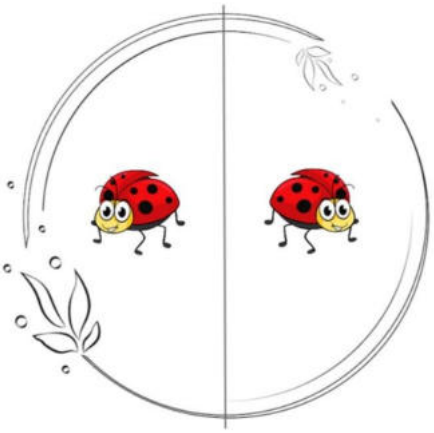

Figure 2: Exemplary visualisation of the LLM-based assistant

Data collection and analysis

The data were collected in July 2024 at a secondary school in Germany, with fourteen 7th and 8th-grade students participating in the study, all participants were video recorded. They attend a school for students with a lower performance in general. This target group was chosen because low-performing learners often require significant support in completing tasks, and the teacher's resources

are insufficient to meet these needs. Therefore, the aim is to specifically address the needs of these learners. The students worked on the task depicted in Figure 1 in a clinical setting at their school, using the task-specific LLM-based assistant previously described. At the time of the study, the LLM-based assistant had not yet been fully integrated into the digital escape game. As a result, students completed the task on a computer screen and used an iPad to interact with the LLM-based assistant. All chat transcripts were recorded, serving as an additional data source alongside the video recordings.

None of the seven groups successfully solved the task. Six of the seven groups utilised the task-specific LLM-based assistant while working on the depicted task. The students were free to choose the input and output format. Communication between students was generally minimal, both among themselves and with the LLM-based assistant. However, a few noteworthy interactions emerged.

Till and Anton, two 7th-grade students, initiated the conversation with the LLM-based assistant using a conversation starter ("We don't understand the task. Can you help us?"). Their prompts were primarily single-word responses, and they spoke little to each other. When the LLM-based assistant presented a visual representation, Till remarked, "*If the beetle sits like this and the other one like this, then the other has to go like this*", then he moved the mouse over the movable gemstone and rotated it to change its orientation to the axis. This marked the first instance where Till took an active role in solving the task.

Jeanette, another 7th-grade student, reflected after working on the task that the task-specific LLM-based assistant helped her to understand the task (‘*Because otherwise, I wouldn't have come up with the task, i.e. that you have to reflect it’*). She and her partner Samuel also communicated little with each other during the task.

Conclusion and Outlook

Working on problem-based tasks (in the context of digital escape games) is challenging for students, so integrating support is an important aspect to assist in completing the task (successfully). Task-specific LLMs, which incorporate didactic dimensions, present innovative possibilities for designing such support in digital escape games. This project aims to identify design principles for task-specific LLM-based assistants to offer support for problem-based tasks and to focus on their use by students with low performance in mathematics and the associated opportunities and hurdles.

Regarding the first research question, it can be assumed that the described design principles can be implemented through prompt design. The task-specific LLM-based assistant largely adhered to the intended structure, utilized the provided technical language, and consistently maintained a friendly, helpful, and motivating tone. This suggests that further design principles could be successfully implemented through prompts, enabling the development of additional subject-specific tools.

In response to the second research question, both opportunities and challenges were identified. As an opportunity, one can see that the interaction with the LLM-based assistant appeared to aid in students' task comprehension and facilitated discourse around the mathematical content. Furthermore, the ability of students to verbalize their difficulties proved important, a task that appears to present a significant challenge for the selected target group. This promotes the described active role in their

learning process, in which subject-specific LLM-based assistants should strengthen the students. Despite the opportunities, several challenges were also noted. Students often found it difficult to initiate conversations with the LLM-based assistant, despite the inclusion of conversation starters. Even when the interaction had already started, communication was frequently limited. It can be assumed that, due to the novelty of LLMs, many students do not yet have the necessary competencies. It seems necessary to learn these competencies explicitly.

The next steps envisage the further development of the existing task-specific LLM-based assistant as well as prompting at least three new ask-specific LLM-based assistants. Based on the results of the pilot study, a training phase is being developed to familiarize the learners with the use of the LLM prior to the data collection phase. During this phase, potential questions and inputs are anticipated together with the researcher. The goal of this training phase is to encourage more effective use of the LLM-based assistant in the digital escape game. One avenue will involve testing a different mathematical task to gather more generalizable data and assess whether the design principles are effective across various tasks. Another focus will be exploring the role of the task-specific LLM-based assistant, specifically investigating how positioning the assistant in a peer role may influence student interaction.

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Three forms of feedback in digital learning material in mathematics

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This study reports on findings from an empirical investigation into the use of three different forms of feedback in digital material for learning mathematics: correctness feedback, attention feedback, and explanation feedback. All feedback types are received immediately by the students. In total 267 upper secondary school students used the material during a lesson and data from the system, a questionnaire and observations were collected and analysed. The results show that correctness feedback drives the learners' engagement, while explanation feedback helps learners to build understanding but is often skipped by learners as it is time-consuming. This study suggests that attention feedback, in which the essentials or critical aspects are highlighted without giving all details, is often sufficient in this setting and retains learner autonomy.

Keywords: Mathematics learning, feedback, immediate feedback, digital learning material,

Introduction

At the Akelius Math Learning Lab[1], digital learning materials are being developed to teach mathematics, specifically targeting 7 to 18-year-olds in refugee camps and under-resourced schools in the Global South. The material will be made accessible offline on simple tablets in multiple languages. The digital environment offers educational opportunities to learners with limited access to teaching, ensuring the availability of course materials and feedback. The learners move at their own pace and revisit lectures if they want.

Feedback is considered an important and effective aspect of learning (Black & Wiliam, 1998; Hattie & Timperley, 2007; Shute, 2008; Wiliam, 2016). When creating a digital system for giving feedback there is a need to understand how learners use and benefit from different types of feedback. This study seeks to explore the impact of different feedback forms on learners using digital learning material for learning mathematics. Specifically, we examine learners' interactions with and reactions to the feedback. The research aims to answer the following research question: How do learners react to three feedback forms: correctness feedback, attention feedback, and explanation feedback, in a digital learning environment for learning mathematics?

Theoretical framework

To frame our analysis, we begin by examining the concept of feedback. According to Hattie and Timperley (2007), feedback can be defined as "information provided by an agent (e.g., teacher, peer, book, parent, self, experience) regarding aspects of one's performance or understanding" (p. 81).

[1] https://www.akelius-math.com/en/about

Meta-analyses show that feedback significantly influences learning (Hattie & Clarke, 2018), but its effect can vary, it can both help and slow down progress, depending on the situation, form, and content of the feedback. It is therefore important to conduct a fine-grained analysis to develop productive feedback adapted to the specific learning situation. In particular, feedback in a digital setting must be tuned in advance, for use in the absence of a teacher.

Effective feedback, as outlined in Hattie´s model, should address three critical questions: "Where am I going?" (the goal), "How am I going?" (current progress), and "Where to next?" (steps to improve). For feedback to be instructional, it needs to offer specific information about the task or learning process, helping to bridge the gap between the learner's current understanding and the desired level of mastery (Sadler, 1989).

The literature differentiates between different forms of feedback (Barana et al., 2021), particularly between elaborate feedback, which provides detailed guidance, and corrective feedback, which merely indicates correctness. Studies suggest that elaborate feedback, such as explanations, reading links, and worked examples, is more effective in enhancing learning.

Barana et al. (2021) conducted a study on student performance in a digital learning environment for mathematics. They analysed results for three types of tasks, with associated forms of feedback: simple tasks with *corrective feedback*, i.e. information on whether the answer was right or wrong, more complex tasks with *interactive feedback*, which provides step-by-step guidance to students who have reached a wrong answer (or have answered right but chosen to verify their answer), and *guided activities*, which lead the student in an interactive exploration process, without building on a previously asked question. Their finding, derived from analysing platform data, show that the tasks with interactive feedback result in frequent improvement when students reattempt tasks they initially failed, whereas corrective feedback only, or guided activities, yielded minimal improvement. The framework developed by Barana et al. (2021) provides a starting point for how we classify different forms of feedback. Similarly to the authors we use corrective and interactive feedback.

Baker et al. (2010) studied, among other things, the impact of boredom, frustration, confusion, engaged concentration, delight, and surprise on students' results in different learning environments. Interestingly, emotions such as frustration seem to have little to no effect on learning outcomes. The authors even suggest that "frustration may not need remediation" (p. 227). This is echoed in the work by Mentis (2007) who suggests that frustration only requires intervention when it stems from factors unrelated to cognitive learning efforts.

Shute (2008) discusses the impact of formative feedback on learning, including how timing affects students' emotions. While the article covers various subjects, mathematics examples are prominently featured, showing how immediate feedback can mitigate negative emotions and enhance learning. Hannula's (2012) study delves into how emotions like frustration and enjoyment influence students' attitudes and behaviour in mathematics learning, highlighting the significant impact of these emotions on student engagement. Fyfe & Rittle-Johnson (2016) specifically examine how the timing of feedback in mathematics tasks affects learning and retention, with a focus on how immediate feedback can help manage frustration and improve problem-solving skills.

The implementation of feedback in the digital learning material

Active learning is one of the pedagogical foundations for the digital learning material used in this study. This approach emphasises that learners should engage in activities beyond listening and reading. They should actively express their thoughts and reasoning (Freeman et al., 2014). Active learning is widely recognised and empirically supported within educational research as a powerful approach to improving learning outcomes (Freeman et al., 2014). This has been implemented in this learning material by abstaining from instruction videos or texts and instead prompting learners to answer questions continuously throughout their learning process. The questions come in various formats such as multiple-choice, sorting objects, connecting objects or asking for a numerical value. Notably, there are no free text questions or questions where the full calculations are asked for. The design aims to guide learners incrementally to facilitate understanding of mathematics through engagement and immediate and varied feedback mechanisms.

The most frequent form of feedback is *correctness feedback*. This is done by showing the learner their answer alongside the correct one. Correctness feedback is given after each question. It does not include intermediate steps or guidance on how to arrive at the correct answer.

At the other end of the feedback spectrum, we have *explanation feedback*. If the learner submits an incorrect answer, they can receive an elaborate and full explanation. This is done in alignment with the principles of active learning. The learners are guided through small intermediate steps by answering questions which guide them through all the steps required for answering the original question. Once the learner completes these steps, they are given the opportunity to attempt the original question again before moving forward. For questions where explanation feedback is available, it is optional for learners who has answered correctly but automatic for those gave a wrong answer.

The objective of explanation feedback is to ensure that learners actively engage with all steps in the solution, find where their thinking was faulty, reflect on their errors, reinforce correct steps and support the consolidation of the whole process. Explanation feedback is similar to what Barana et al. (2021) call interactive feedback. However, since interaction is the core of the digital material we choose to use another name for this form of feedback.

Finally, we have the feedback form *attention feedback*. This is a form of feedback that intends to give the learner some idea on how to find the right answer without giving a full detailed explanation. Instead, attention feedback intends to shift the learner's attention to some critical aspects. In this material, this is done by asking thought-provoking or leading questions or by directing the learner's attention to key features in an image by circling or using arrows. Sometimes only audio is used to highlight some key aspects. The attention feedback does not require the learner to do anything. The learner can disregard it and move on directly. Attention feedback includes correctness feedback, as seen in Figure 1.

For questions where attention feedback is available, it is given to all learners, also to those who answered correctly. The idea is that this form of feedback can ensure those who answered correctly that their thinking, not only the answer, was correct. The learner is free to move on at any time.

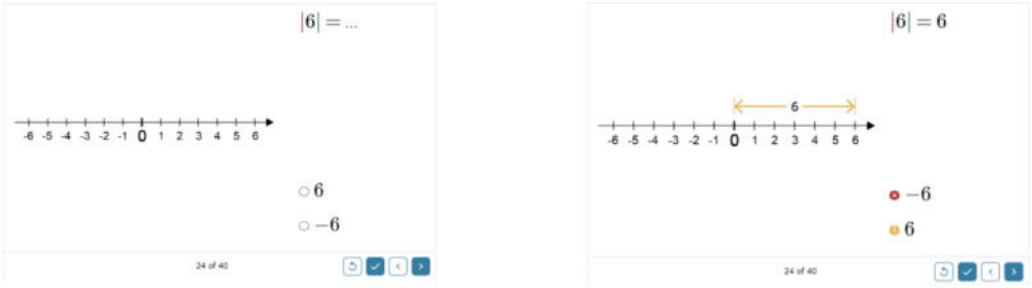

Figure 1: An example of attention feedback. To the left the question is given and to the right the feedback given. A yellow arrow draws the learner's attention to the distance to the origin.

Method

The study's empirical approach involved using the learning material in a classroom setting, allowing for an analysis of how students interacted with the material and responded to the various feedback types. Data was collected in ten classes of 17- to 18-year-old students in upper secondary school in Sweden, during the spring of 2024, with 267 students in total. This was a convenience sample where the respondents were of the right age and appropriate pre-existing knowledge, but from a more homogenous background compared with the intended target group.

Some of the students study mathematics-intensive programs. They worked with lessons about the absolute value, a concept they would normally learn about a few weeks later. Other students studied in programs with less exposure to mathematics. They worked with lessons on systems of two linear equations with two unknowns, something they had not met quite yet. Hence, neither of the groups had been taught the chosen topic in class beforehand. Both groups had in total around 40 minutes to complete as many tasks as possible from the 60 tasks available.

Three types of data were collected. System data consisting of learners' results and time spent on tasks in the learning material, a questionnaire about their experiences with the material, and finally observations in class. The total amount of respondents in the learning material and questionnaire was 246 and 267 respectively.

The questionnaire consisted of 18 questions. In the first section, the respondent consented to being part of this study and entered anonymous code making it possible for the researchers to connect the responses in the digital learning material to the answers to the questionnaire. The second section was on the overall experiences of the material. In the last part of the questionnaire, the focus turned to experiences of separate tasks. Responses to free text questions related to two of the multiple-choice questions are analysed in this report. These two questions are the following (the relevant multiple-choice options will be given in the result section):

Question 17: Sometimes, when you gave an incorrect answer, you entered a sequence with step-by-step explanations on how one can think. What do you think about that?

Question 18: Sometimes you got the option to get an explanation when you had given the correct answer. What do you think about that?

In each class, the observations were made in two different settings: in full classroom settings, where the learners worked individually, and in pair settings, where the learners worked in pairs. In the classroom setting, there were approximately 25 students who worked individually with the learning material on laptops and earphones connected for sound. The observers, consisting of both the authors of this paper and other staff at Akelius Math Learning Lab, walked around and made notes on a protocol focusing on three main themes: how the system was experienced, how the mathematics was perceived and what emotions and reactions were evoked. The protocol was developed by the observers. While preparing for data collection the observers discussed what themes to focus on to ensure reliable and uniform data collection. The classroom setting was used to get many individual reactions from learners using the system in a way that the system most often will be used.

In the pair setting, two students worked in front of a laptop. They were encouraged to collaborate and talk out load about their thinking. This setting was used to get better insight into the respondents' experience by listening to the learners. These groups of students only left the trace of one user on the learning system, while answering the questionnaire individually. This explains why there are more respondents in the learning material than in the questionnaire.

To avoid issues of privacy and integrity, each student was randomly given an anonymous code only indicating which class the student attended. No names or other personal data about the students were collected. The anonymous code was only used to connect the students' results in the learning material with their responses to the questionnaire. Students that worked in pairs both used the same anonymous code when answering the questionnaire.

We used qualitative content analysis to analyse the data from the observations and the written comments given in the questionnaire (Krippendorff, 2018). Two authors participated in coding the data. The procedure for coding started with identifying all questionnaire comments and observations that were related to feedback. These were then categorized into one of the three forms of feedback.

The researchers then searched for recurring themes within each feedback category. This was done inductively. At the end of each phase of analysis, the entire research group validated the categories that emerged from the content analysis. The validation process involved comparisons of the researchers' categorisations, and in cases of disagreement, the categorisation was discussed until a consensus was reached.

Some results

The proportion of correct answers overall was 84 %. Hence, the level of difficulty was such that most students did okay without direct instructions. Nonetheless, some students expressed a desire for more direct instruction and less reliance on feedback.

> Learner Vin44: Some tasks were like "here you get a task, try to solve it" as I said, I think it's better if someone explains how to do it before I make a mistake and not after.

Many learners reacted emotionally strongly when they received *correctness feedback*. They were keen on getting everything right and generally reacted with discomfort when they were incorrect.

> Observer NF: "Facepalm" when he gave an incorrect answer. I got the feeling that he thought it was a careless mistake. He was concerned about getting it right.

The correctness feedback in this learning material do not only show if the answer is correct or not, but also gives the correct answer. Therefore, the material typically does not allow learners to redo the questions they failed, which quite a few respondents wanted.

> Learner Box77: That you could redo the questions you did wrong (What can be improved?)

On the other hand, the observations showed that learners did not always bother to check what the right answer was.

> Observer KT: Clicks past without checking what he did wrong on several pages.

Explanation feedback was only given sparsely in the material. Learners requested more opportunities to receive this type of feedback.

> Learner Tid11: Ask after each answer if you [the learner] want an explanation of what you did wrong.

For those who received the explanation feedback, it often helped them understand the material better. Out of 267 respondents to question 18 in the questionnaire 110 selected the option “The explanation feedback helped me understand”. This was also seen in the observations.

> Observer NF: “Now I understand!" Understood after the explanation feedback.

However, the explanation feedback is time-consuming. For example, at one of the questions that offered explanation feedback, 39 out of 149 learners answered incorrectly. The explanation feedback given consists of five extra questions guiding their thinking before they could answer the original question again. These learners spent in mean spent 80 seconds on this explanation feedback, including redoing the original question.

Some of these would have liked to have continued without explanation feedback.

> Observer AH: They make mistakes but immediately understand why, are still "forced" into the explanation feedback.

Among the remaining 110 learners who answered correctly to the same question, 11 chose to get the explanation feedback. This is mirrored in the questionnaire where 142 out of 267 respondents to question 18 chose the option “I always choose to skip the explanation feedback”. This was the most common respons to this question.

The third form of feedback, *attention feedback*, gives more information than correctness feedback but is less instructive and time-consuming than explanation feedback. Students normally spend just a few seconds looking at the attention feedback and are free to continue at any time.

The data indicated that this was sufficient for many learners.

Observer NF: [The learner …] Continued with only correct answers after looking at the attention feedback on page 8.

For other students, attention feedback was not enough.

Observer SB: They had difficulty understanding the attention feedback given through audio concerning the absolute value of -5.

That the learners could skip the attention feedback had consequences. There were many observations of learners who failed the question but still did not spend much time on the attention feedback. However, data shows that learners that answered incorrectly spent more time, in most cases double the time or more, on the attention feedback.

Discussion

Common types of feedback in a conventional classroom setting are notifying the student if the answer is right or wrong and giving the learner access to the correct answer. In this study we examined the former feedback type, labelled correctness feedback. The results from this study indicate that students view the use of correctness feedback as beneficial and/or helpful.

Explanation feedback is given to students who have responded incorrectly on certain questions. Students' general opinion on this feedback type was quite divided. Some expressed frustration that they were forced into a long loop while others wanted this type of feedback more often. However, the most frequent response was that this feedback type helped students' understanding. Although some students were frustrated that they were forced by the system to stay on this feedback type, it might have been to their benefit. In contrast, students were able to skip the attention feedback type, and this appeared to be a common occurrence, even to the detriment of their own understanding. One student mentioned that they skipped a lot of these feedback types and then realized that they lacked vital information to solve upcoming questions.

Attention feedback was given to all students, regardless of whether their answer was correct or not. Students were provided key insights regarding the studied topic. We considered this feedback type to be an appropriate compromise between correctness and explanation feedback. Some students considered this feedback type to be insufficient, asking for explanation feedback instead. Others thought that they would have liked to have this information before they answered the question – making the information more like a hint than actual feedback. This can perhaps be attributed to how they have been taught mathematics previously. However, an interesting follow-up on this topic would be to compare how students fared when getting the same information before and after responding to a question.

A future iteration should, in the case of an incorrect response, include the possibility of a new chance on the same question after receiving attention feedback. This might encourage students to absorb the information in the attention feedback, knowing they can make use of it.

Student reactions, as observed by the researchers and in the questionnaire responses, suggest that adding a possibility to return to a question that they answered wrong, and correct their answer. could help dissipate the frustration caused by negative feedback. It would be interesting to test a set-up

where students get the possibility to answer a new, similar question to correct themselves in case they have answered wrong. A risk with such a set-up is that some students may not make the effort to try to answer right on their first try, knowing that they will get correctness feedback and a new chance.

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Enhancing students' techno-mathematical problem-solving skills: Results from a quasi-experimental study with 5th graders

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This paper discusses a teaching experiment aimed at enhancing 5th-grade students' performance in solving mathematical problems using digital tools like spreadsheets, GeoGebra, and Scratch. Employing a quasi-experimental design with a control group, the study assessed the impact of four 100-minute, inquiry-based lessons. Each lesson focused on a problem involving different concepts and employed specific digital tools. The effectiveness of the intervention was evaluated based on students' abilities to grasp the problem situation, implement a techno-mathematical approach, and explain the solution. While the intervention significantly improved students' comprehension of problems and application of techno-mathematical strategies, it was less effective in enhancing their skills in justifying solutions, highlighting strengths and areas for improvement, as well as portraying how the design principles of effective teaching experiments can be grounded in theoretical models.

Keywords: Mathematical problem-solving, Digital tools, Digital problem-solving, Teaching experiment, Quasi-experimental study.

The use of digital tools for mathematical problem solving

There are several ways in which one can address the issue of integrating technological tools in the teaching and learning of mathematics. One of the key distinctions refers to technology as an instrument to engage in mathematical activity, namely mathematical reasoning and problem-solving, versus technology as a device used to outsource procedural routine calculations or perform symbolic operations (Hoyles, 2018). Another division may be found between the utilization made by the teacher, namely for the purpose of presentation, demonstration, or explanation, and the utilization made by the student, consistent with exploring, investigating, and obtaining the solution to a mathematical problem (Drijvers & Sinclair, 2024). In this article we present and discuss a methodological proposal to achieve digital technology integration in the mathematics class, explicitly considering the technological tool being used by the students (5th graders) and aiming to develop their ability to solve different non-routine mathematical problems. Theoretical advances and research results showing the opportunities offered by digital tools to improve students' learning results in mathematics, particularly in solving non-routine problems (e.g. Bray & Tangney, 2017), have driven the study. A background motivation is the growing need to develop students' techno-mathematical fluency (Jacinto & Carreira, 2017a), which means empowering them to effectively interlace mathematical and technological competencies to solve mathematical problems (Liljedahl & Cai, 2021).

Our goal is to assess the impact of a teaching experiment carried out in a Portuguese public school and focused on increasing 5th-grade students' performance in solving non-routine mathematical problems using various digital tools. To measure the impact of the teaching experiment, we targeted

students' ability in: a) grasping the problem situation, that is, finding out how well they understood and framed the given problem; b) implementing a techno-mathematical approach, that is, their ability to apply appropriate methods and technological tools to solve the problem, and c) finding and explaining the solution, that is, knowing how effectively they articulated the reasoning behind their solutions.

The theoretical framework underpinning a teaching experiment on digital problem solving

In today's world, success in solving mathematical problems entails the ability to use appropriate mathematical resources and, at the same time, the ability to use digital technologies as equally indispensable tools (Carreira et al., 2016). The theoretical model MPST - Mathematical Problem Solving with Technology (Jacinto & Carreira, 2017b) aims to describe how students integrate both mathematical and technological knowledge to effectively solve and communicate solutions to mathematical problems. The MPST model comprises ten descriptive processes: *Grasp* (getting initial ideas about the problem); *Notice* (recognize fitting mathematics and useful tools); *Interpret* (considering affordances of digital tools and mathematical approaches); *Integrate* (combining mathematics and technology); *Explore* (using technological and mathematical resources for exploration); *Plan* (outlining an approach to achieve a solution); *Create* (carrying out the plan and recombining tools); *Verify* (explaining and justifying with the resources available); *Disseminate* (presenting the solution and outputs); *Communicate* (interacting with others in solving the problem).

Yet, the model was developed from empirical studies taking place within a beyond-school context, raising the issue of whether and how it could be applied in regular mathematics classrooms. More recently, in a later research phase also fuelled by the operationalization of the MPST model, Jacinto (2023) investigated how to enhance middle-grade students' ability to solve mathematical problems with digital tools by providing moderate mathematical challenges and fostering collaborative work in the Tecn@Mat Club, an online after-school program. The Club fostered the sharing and discussion of strategies and solutions to non-routine problems, with an emphasis on collaboration and closely aligned with an inquiry-based learning approach. Another important feature was the disclosing of several digital tools' affordances in matching the conceptual structure of some kinds of mathematical problems (Leung & Bolite-Frant, 2015; Monaghan & Trouche, 2016). It enabled students to engage in problem-solving, develop proficiency in using technology to find solutions, and enhance their ability to explain and justify them. This movement from researching MPST in the beyond school context (Jacinto & Carreira, 2017a, 2017b) to the after-school program (Jacinto, 2023), made clear that the theoretical model could guide the design of an instructional sequence focused on enhancing students' techno-mathematical problem-solving skills.

When moving towards the implementation in a classroom setting, it became necessary to develop an instrument to measure skill development. Muir et al. (2008) developed a rating scale specifically designed to assess several aspects of problem-solving in educational settings. The scale focuses on evaluating the student's understanding of the problem, the selection and implementation of a strategy, and the accuracy of the final answer. Based on the ten processes established by the MPST model and the assessment rating scale, we devised a rubric for classifying individual responses under three criteria:

i) *Appropriateness* focuses on how well the student grasps the problem, identifies relevant data, and selects the appropriate digital tool for the task. It concerns the initial engagement with the problem and the preliminary strategic decisions made by the student.

ii) *Implementation* involves the adoption of solving strategies, evaluates the degree of tool integration, and the exploration of the results of this integration. The focus is on the practical application of the selected strategies and digital tools, assessing how effectively the student integrates these resources to find a solution to the problem.

iii) *Explanation* examines the student's ability to communicate the solution process and justify the answer, evaluating how well the technology was leveraged to improve understanding or make the solution process more efficient.

The teaching experiment

The teaching experiment was designed to enhance students' problem-solving skills using several digital tools and was conducted by the first author (the teacher of the classes). Embracing the new mathematics syllabus in place for elementary education in Portugal (Canavarro et al., 2021), a set of four lessons for 5th-grade students, each lesson lasting 100 minutes, was planned (Simões, 2024). Figure 1 presents a summary of the teaching experiment conducted in 2023. Table 1 lists the mathematical content and capabilities that were the focus of each task of the teaching experiment.

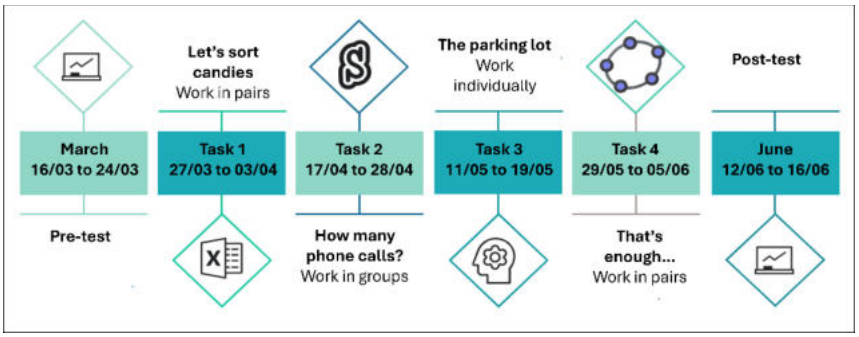

Figure 1: Structure of the teaching experiment

Table 1: Mathematical content and capabilities in focus in each task of the teaching experiment

			Mathematical capabilities					
#	**Task**	**Theme.** Content.	**PS**	**R**	**CT**	**Com**	**Rep**	**Con**
1	Let's sort candies	**Numbers/Natural numbers.** Multiples and divisors. Prime numbers.	X		X		X	
2	How many phone calls?	**Algebra/Patterns and sequences.** Formation rule.	X		X			
3	The parking lot	**Patterns and sequences.** Growth sequences. Formation rule of a sequence.	X		X	X		X
4	That's enough…	**Geometry and measurement.** Construction and classification of triangles.	X	X		X	X	

PS: problem solving; R: reasoning; CT: computational thinking; Com: communication; Rep: representation; Con: connections.

Each lesson adopted an inquiry-based approach: students were introduced to a new problem and worked autonomously for about 40 minutes; then a whole class discussion took place and, finally, the teacher summarized key points regarding efficient problem-solving strategies and highlighted

affordances in the tool used. The four problems involved concepts related to Algebra, Numbers, and Geometry. In three of the problems, students were asked to use a specific tool - Spreadsheet, GeoGebra, and Scratch - and in one of them, they could choose the tool they felt more appropriate. As an illustration, the problem "How many phone calls", related to patterns and sequences, stated: *Five students won a contest. When they got the news, they called each other. Find out how many calls the five friends had to make to congratulate each other. Discover a rule for any number of friends.*

Research method

This study employed a quasi-experimental design with a control group, using a pre-test and an equivalent post-test to assess the impact of the intervention on students' abilities to solve mathematical problems with technology.

The participants in this study were 54 students from three 5th-grade classes attending a school in a rural area of the country. Two classes, comprising 34 students in total, were designated as the Experimental Group (EG), while the third class, with 20 students, served as the Control Group (CG). The majority were 10 years old, yielding a mean age of about 10.2 years. In the interim evaluation, the percentage of students in the EG requiring additional support was 25%, while 35% in the CG were in that situation. Regarding the students' results on mathematical reasoning and problem-solving, more than half of the students in both groups achieved satisfactory ratings. While the groups were not fully homogeneous, efforts were made to ensure the sample size was sufficient to produce reliable results.

The development and validation of the research instruments followed a rigorous process to ensure reliability and alignment with the aims of the study. A pre-test and a post-test were designed to assess students' techno-mathematical problem-solving performance. Both tests included non-routine problems on geometry, numbers, and algebra, and were designed to ensure a good matching (Monaghan & Trouche, 2016) with specific tools such as spreadsheets, dynamic geometry environments, and visual programming platforms. The tests were administered via Google Forms, a familiar platform to the students, facilitating usability and reducing potential technical difficulties.

The classification rubric, designed to evaluate students' techno-mathematical problem-solving skills, comprised three main criteria split into a few indicators: *Appropriateness* – A) interpretation of the problem statement, and B) selection of relevant data and tools; *Implementation* – C) adoption of strategies, D) integration of technological resources, and E) exploration of the results of integration; *Explanation* – F) obtaining the solution, and G) justifying the obtained solution. Each indicator was measured on a scale from 0 to 3, reflecting the level of proficiency in handling each aspect of the problem-solving process.

To validate the instruments, a comprehensive multi-step process was undertaken, beginning with reviews by two experts in mathematics and technology education to ensure that the content and structure were conceptually clear and aligned with the research goals. Feedback was then provided by a mathematics teacher and two 5th-grade students who were not participating in the study, leading to revisions that simplified language and adjusted task difficulty. These collaborative efforts focused on enhancing usability, optimizing timing, and ensuring technical accessibility.

Data collection and analysis

The data collection process began with an adaptation phase, held in December, where students from both the EG and CG engaged in diverse tasks, termed "Adaptation Tasks" using tools such as Excel, Scratch, and GeoGebra. These tasks aimed at helping students become accustomed to the technology and mitigating any potential frustration. Following this phase, a pre-test comprising three non-routine mathematical problems was administered in March, allowing students to use their preferred tools to solve the problems and discuss their solutions. In June, after a 3-month period in which the teaching experiment was conducted, a post-test having a similar structure was applied. The data analysis involved the teacher-researcher classifying each test item using the rubric. The students' scores from both tests were analysed using descriptive and inferential statistics utilizing SPSS.

Results

Table 2 shows the results of the statistical analyses comparing the EG and CG pre-test and post-test scores along with the measures of *Appropriation*, *Implementation*, and *Explanation*. On the pre-test, the mean scores were 18.06 for EG and 14.85 for CG, with standard deviations of 6.84 and 8.1, respectively. The differences were not statistically significant ($t = 1.55$, $p = 0.126$), suggesting similar levels of baseline capabilities to solve problems using digital tools between the groups. However, on the post-test a significant improvement was observed in the EG with a mean of 32.38 compared to the CG's mean of 22.25. The t-test was statistically significant ($t = 3.56$, $p < .001$), indicating a substantial improvement in EG relative to CG, with a mean difference of 10.13.

Table 2: Comparative analysis of pre-test and post-test results between EG and CG

	EG (N=34)		CG (N=20)		Shapiro-Wilk test		Levene test		T-test			
	Mean	**SD**	**Mean**	**SD**	**df**	**Sig**	**Z**	**Sig.**	**t**	**df**	**Sig**	**MD**
Pre-test (T1)[a]	18.06	6.84	14.85	8.1	54	.855	0.304	0.584	1.55	52	0.126	3.21
Post- test (T2) [a]	32.38	11.67	22.25	6.54	54	.461	4.684	0.035	3.56	52	<.001	10.13
Appropriation (APT1) [b]	1.33	0.55	1.21	0.62	54	.037*	0.384	0.538	0.71	52	0.477	0.12
Appropriation (APT2) [b]	2.24	0.64	1.72	0.60	54	.027*	0.060	0.807	2.93	52	0.005	0.52
Implementation (IMPT1) [b]	0.66	0.35	0.54	0.42	54	.014*	0.835	0.365	1.15	52	0.255	0.12
Implementation (IMPT2) [b]	1.42	0.65	0.86	0.28	54	.014*	9.689	0.003	3.68	52	<.001	0.56
Explanation (EXPT1) [b]	0.68	0.24	0.44	0.24	54	.065	0.101	0.752	3.95	52	<.001	0.23
Explanation (EXPT2) [b]	1.02	0.48	0.70	0.26	54	.054	6.458	0.014	2.79	52	0.007	0.32

SD: Standard Deviation; MD: Mean Difference; * p < .05 ** p < .01 *** p < .001
a: classification between 0 and 63; b: level between 0 and 3

The results also indicate that the EG revealed statistically significant improvements over the CG regarding *Appropriation* ($t = 2.93$, $p = 0.005$), *Implementation* ($t = 3.68$, $p < .001$), and *Explanation* ($t = 2.79$, $p = 0.007$), reflecting enhanced proficiency in solving the problems using digital tools. Hence, the data indicate that while the groups started with similar capabilities as shown in the pre-test, the EG showed significant improvements in post-test scores which suggests the effectiveness of the intervention in enhancing their techno-mathematical problem-solving skills.

Table 3 provides a detailed statistical analysis of the results from pre-test and post-test assessments across the multiple indicators within the criteria *Appropriation*, *Implementation*, and *Explanation*. The Shapiro-Wilk normality test is used to assess the distribution of data and the Mann-Whitney Test to compare differences between groups.

Table 3: Comparison between the pre- and post-test results on Indicators between EG and CG

		Shapiro- Wilk Normality test		Mann-Whitney test		
Criteria	**Indicators**	**df**	**Sig**	**U**	**Z**	**Sig**
Appropriation	Indicator A Pre-test	54	.034*	316.000	- .437	.662
	Indicator A Post-test	54	.008*	186.000	- 2.795	.005**
	Indicator B Pre-test	54	.008*	306.000	- .620	.535
	Indicator B Post-test	54	.013*	174.500	- 2.999	.003**
Implementation	Indicator C Pre-test	54	.001*	252.500	- 1.616	.106
	Indicator C Post-test	54	.002*	372.000	- 3.245	.001**
	Indicator D Pre-test	54	<.001***	278.000	- 1.157	.247
	Indicator D Post-test	54	.001**	138.500	-3.672	<.001***
	Indicator E Pre-test	54	<.001***	301.500	- .728	.466
	Indicator E Post-test	54	<.001***	148.000	- 3.514	<.001***
Explanation	Indicator F Pre-test	54	<.001***	165.000	- 3.288	.001**
	Indicator F Post-test	54	.024*	183.000	- 2.870	.004**
	Indicator G Pre-test	54	<.001***	253.000	- 1.651	.003**
	Indicator G Post-test	54	<.001***	243.500	- 1.785	.074

* p < .05 ** p < .01 *** p < .001

Significant improvements were observed in the post-tests for *Appropriation* indicators A and B, with p-values of .005 and .003 respectively, which suggests that the intervention was effective in enhancing skills in interpreting the problem statement and in data selection. The pre-test results showed non-normal distributions, highlighting the initial variability in these skills among participants. The intervention also improved *Implementation* skills, as demonstrated by significant enhancements in the post-test scores for indicators C, D, and E, both yielding p-values of .001, <.001 and <.001, respectively. While the pre-test data reveals existing disparities prior to the intervention, the post-test results indicate a strong impact of the teaching experiment on the adoption of strategies, integration of technological resources, and exploration of the results of the integration. Additionally, there was a significant improvement in *Explanation* skills for indicator F in the post-test, evidenced by a p-value of .004, suggesting effective skill development of the EG in obtaining the solution. However, the changes for indicator G from pre-test to post-test did not reach statistical significance, with a p-value of .074, indicating that this aspect of *Explanation* – G) *justifying the obtained solution* – may require additional focus or alternative instructional strategies.

These results, altogether, suggest that the teaching experiment was particularly effective in enhancing *Appropriation*, *Implementation*, and certain aspects of the *Explanation* skills. These improvements are statistically significant, illustrating that the intervention apparently had a positive impact on students' abilities to apply the learned skills about using technology to solve mathematical problems.

Discussion and conclusion

The results of this study provide solid evidence about the effectiveness of the designed teaching experiment in enhancing 5th-grade students' performance in solving mathematical problems with digital tools. The experimental group showed significant improvements across all three key criteria assessed: *Appropriation*, *Implementation*, and *Explanation*. The improvements were particularly notable in the students' ability to interpret problem statements, select relevant data, adopt appropriate strategies, integrate technological and mathematical resources, and explore results.

The teaching experiment encouraged the use of technological tools combined with problem solving, which led students to interpret statements, select necessary data and appropriate the tools to be used, that is, to find mathematical properties of objects and ways to solve a problem using a certain technological tool (Santos-Trigo et al., 2021). Furthermore, the teaching experiment intentionally led students to recognize the affordances of each tool for each given problem (leading to integration) and to explore the different functionalities of each tool in developing successful strategies for solving problems (promoting exploration). Overall, the results reinforce the potential of the intervention design in enhancing students' ability to leverage digital tools for mathematical problem-solving. While the intervention proved highly effective in most indicators, the results also pinpoint areas needing further attention, particularly the ability to justify the obtained solutions (indicator G). This suggests the need to improve explanation skills through additional or different instructional strategies. For example, more emphasis may be placed on tasks that require students to develop explanations or justifications for their solutions. Additionally, providing opportunities for students to present their solutions and reasoning to the class could further reinforce these skills.

A final important highlight is the observation that the design principles of an effective pedagogical intervention can be rooted in theoretical principles and models, as was the case with the MPST model. The latter was used as a guide for the design of productive problem-solving activities with digital technologies, both in line with curricular topics and with the mathematical skills to be developed in the mathematics classroom.

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Analysis of mixed-types in pre-service teachers' argumentation using a digital tool

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This paper examines pre-service teachers' transition from empirical to deductive argumentation when they work with a digital tool. For this purpose, an interactive book page about the interior angle sum of a triangle was created to assist students in the development of different proof types. Using a structural approach that combines a proof type model and the Toulmin scheme, the pre-service teachers' argumentations were categorized as empirical, preformal and symbolic-generic. The results show that working with the interactive book page encourages students to develop proofs in which argumentation elements of different proof types are mixed. These mixed-types express first deductive approaches and mark an important step in the development from empirical to deductive proving.

Keywords: Geometric proofs, digital tools, deductive thinking, proof types, Toulmin scheme.

Motivation

Numerous studies have highlighted the significant challenges students face when attempting to conduct deductive proofs (e.g. Healy & Hoyles, 1998). A possible approach to support students in the development of deductive proofs can be the use of digital tools (see e.g. Christou et al., 2004). However, current approaches often focus on conjecturing that utilize the empirical nature of digital tools to help students determine whether a theorem is correct or not (Campbell & Zelkowski, 2020). Even so, the research focus on the proving processes of students while working with digital tools can also provide insights into the transition from empirical to deductive reasoning. From a mathematics education perspective, the concept of a proof process can be grounded in the idea that proving can be learned when empirical understanding is a starting point of a proof process and deductive reasoning is built upon it (Brunner, 2014). Thus, a genetic approach is assumed in which empirical argumentation and deductive proving form a continuum, consisting of different proof types (e.g. empirical proofs or preformal proofs). To clarify this, it is more appropriate to refer not just to a proof process, but rather to a proof-type-process. Based on these fundamental assumptions about the developmental process associated with proving and the impact of digital tools on proving, the following research question can be derived:

Which proof types emerge when pre-service teachers work with an interactive book and how do the proof types reflect pre-service teachers' transition from empirical to deductive proofs?

Understanding of the term *deductive* in relation to the Toulmin Scheme

The Toulmin scheme (Toulmin, 2003), which is a tool to structure and analyse argumentation, will be presented in this section, because, for the analysis of different proof types, a structural approach is pursued. Focusing on the Toulmin Scheme also assists in clarifying how deductive reasoning is understood in this study. Toulmin (2003) assumes that argumentation can be structured in a certain way and he distinguishes different argumentation elements. In the Toulmin scheme one infers from

a datum (which is the starting point of the argumentation) to a claim. This connection can be legitimized by a warrant, and the warrant explains the connection between the datum and the claim.[1] The Toulmin scheme could be transferred to mathematical theorems and proving. To illustrate this transfer, Figure 1 gives an example of a mathematical proof structured by a Toulmin scheme.

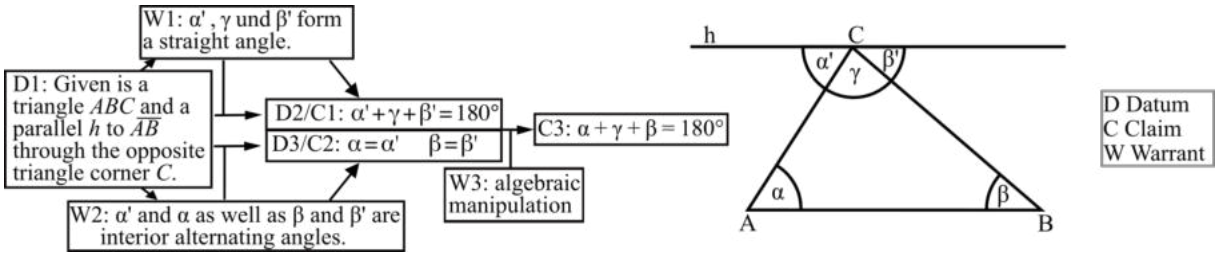

Figure 1: Toulmin Scheme about the proof of the interior angle sum of a triangle

Figure 1 shows the interior angle sum of a triangle, because this geometric proof is focused in this study. The warrants (W1, W2 and W3) of the Toulmin scheme reflect what is meant by deductive thinking in this study, that is the recourse to mathematical theorems, axioms and definitions (see Duval, 1991, for a similar approach). In Figure 1, this is realized by the warrants, which refer to alternating angles (W1) and angles on a straight line (W2). These warrants could also be legitimized by other warrants whereby a deductive network of axioms, definitions and theorems expresses the self-referentiality of mathematics. This structural perspective emphasizes the challenges associated with learning deductive proving, because students not only have to associate the suitable theorem but also have to deploy the mathematical theorem at the correct place in their argumentation. In a rigorous view on mathematical proofs, only the deductive approach counts as acceptable in mathematics, but especially in a didactic context other proofs may exist in which the warrant is not a mathematical theorem, definition or axiom. For example, the warrant of an argument can also be based on experience and perception, which also means that experiences with a digital tool can be used as a warrant (e.g. Hollebrands et al., 2010). This perspective implies that there is not one acceptable proof but different proof types, which are described in the following.

Proof-type-processes applied to an interactive book page

In order to develop a proof type model, which can be applied to geometric proofs for this study, different sources of literature are involved and compared to filter proof types that are suitable for geometric proving with digital tools. The combination of Reid and Knipping's (2010) and Biehler and Kempen's (2016) classifications of different proof types, along with additional literature cited in these sources, lead to three main proof types (adapted to geometric proofs). These three proof types will be explained in the following.

Empirical proof: The empirical proof concentrates on one example which only stands for itself. In geometric proofs, actions like cutting the corners of a triangle and placing them together are often performed. Empirical proofs could be separated into empirical proofs with or without propaedeutic

[1] In the Toulmin scheme there are also the elements rebuttal and qualifier, which are not in the focus in this paper (Toulmin, 2003).

elements. An empirical proof with propaedeutic elements includes elements preparing for the following proof types, which will be exemplified later in the paper.

Preformal proof: In the preformal proof also one or more examples are used and actions are performed. The difference from empirical proof lies in the fact that every step and every argument of the preformal proof can be transferred into a symbolic-generic proof (which is especially demanded in Semadeni, 1984). The preformal proof does not require deductive conclusions but can be translated into them, because it is a concretization of the symbolic-generic proof.

Symbolic-generic proof: The symbolic-generic proof consists of deductive conclusions, but also refers to a figure which is specific for geometric proofs (Reid & Knipping, 2010). An example of a symbolic-generic proof is shown in Figure 1, where for instance symbols are also used to denote equal angles.

In order to illustrate the different proof types, reference can be made to the digital tool used in this study. For the purpose of analyzing proof-type-processes of pre-service teachers working with digital tools, a digital tool was designed that should encourage students to develop proof types in a specific order (from empirical to preformal to symbolic-generic). Therefore, in this study, an interactive book created with the software H5P (https://h5p.org) was deployed. An interactive book consists of different digital pages, and on every page different digital book elements (like videos, drag-and-drop tasks or dynamic geometry software) can be integrated. This study focuses on an interactive book page about the interior angle sum of a triangle. In order to reach the goal of evoking the development of different proof types, the book page follows a certain structure. Apart from a short introductory video, the first task (on the book page task 7) is to find a first argumentation for the theorem that the interior angle sum of a triangle is 180°. This task is open because a GeoGebra applet is integrated in which only a triangle is given. In relation to the proof type model, it is evident that every proof type could be developed for such an open task. This is different for the following tasks (task 8 and task 9), as the development of specific proof types is intended.

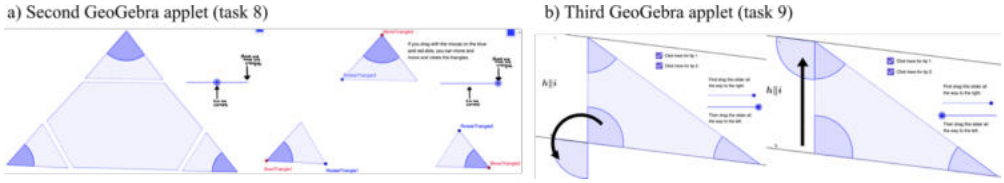

Figure 2: GeoGebra tasks on an interactive book page

In the second GeoGebra applet (task 8, see Figure 2a) a triangle and two sliders are presented. Through the usage of the sliders the corners of the triangle are cut off. Additionally, the students are invited to move and rotate the cut corners. This GeoGebra applet should help students discover that the angles can be placed together to form a straight angle. In relation to the three proof types this GeoGebra applet intends the development of an empirical proof with propaedeutic elements. The propaedeutic element of the proof is that a straight angle could be discovered which is a warrant of the symbolic-generic proof (Figure 1). This points out that this proof can prepare for the symbolic-generic proof. The third GeoGebra applet (task 9, see Figure 2b) should evoke the development of a preformal proof. Here, a parallel is already given and students can move and rotate one angle so it

fits next to the other corner of the triangle (Figure 2b). With the help of this GeoGebra applet students can develop an argumentation, which has the same structure as the symbolic-generic proof, but they do not necessarily have to refer to other mathematical theorems. For example, it is possible to warrant the equality of the alternating angles by referring to the copying and moving of the angles and not to the alternating angle theorem (Figure 2b). At this point, students do not need to argue deductively, but the symbolic-generic proof is prepared, as the warrant of a preformal proof just needs to be replaced by a deductive warrant. Because of the parallel line in the GeoGebra applet, students could also develop a symbolic-generic proof, so both proof types are intended at the third GeoGebra applet. Apart from these specific GeoGebra applets at the end of the book page a solution video is integrated in which the symbolic-generic proof is explained. After this video students are invited to revise their argumentation.

The 'proof-type-structure' of the book page should invite students to develop proofs in a genetic way from empirical to preformal to symbolic-generic proofs. In the results section of this paper pre-service teachers' proof types and their way to deductive argumentation are described.

Methods

In order to answer the research question, task-based interviews (Maher & Sigley, 2020) were conducted with seven pre-service elementary teachers at the university level. During the interviews the pre-service teachers worked (among others) on the interactive book page about the interior angle sum of a triangle, and meanwhile they were invited to articulate their thoughts as a part of a think-aloud process (Charters, 2003). Since the interviews were conducted and videotaped via Zoom, all actions performed by the pre-service teachers in the interactive book (such as a mouse click) could be observed. The interviews were transcripted and afterwards they were structured with the help of the Toulmin scheme (Toulmin, 2003). For this purpose, the different roles in the Toulmin scheme were assigned to the argumentation elements. Next, the entire Toulmin schemes and also the individual argumentation elements were categorized as (part of) a proof type through the utilization of the proof type model described above. To address this, every element of a student's argumentation was assigned to a proof type. An example of a categorisation is shown in Figure 3. Additionally, all proof types of the students were considered in their developmental sequence and they were compared to the structure of the interactive book page. In general, the usage of the digital book elements was consistently examined, taking into account the specific location on the page where students develop their argumentation.

Results

Through the analysis of the Toulmin scheme different proof types could be identified according to the proof type model for geometric proofs. In general, empirical, preformal and symbolic-generic proof types could be reconstructed in students' argumentation. Apart from these expected proof types, mixed-types, in which proof elements of different proof types are integrated in one argumentation, could be identified. The following sections will present these mixed-types, along with a discussion of the complete proof-type-processes of the students, considering the book elements of the interactive book page.

The following figure gives an insight into the mixed-type of a student called Carla. Carla develops the mixed-type at the end of the book page. Initially, with less support from the book page at the open task, she creates an empirical proof without propaedeutic elements in which she argues by halving a square. Afterwards, she develops the intended empirical proof with propaedeutic elements by arguing that the corners of a triangle can be cut off and moved to one point.

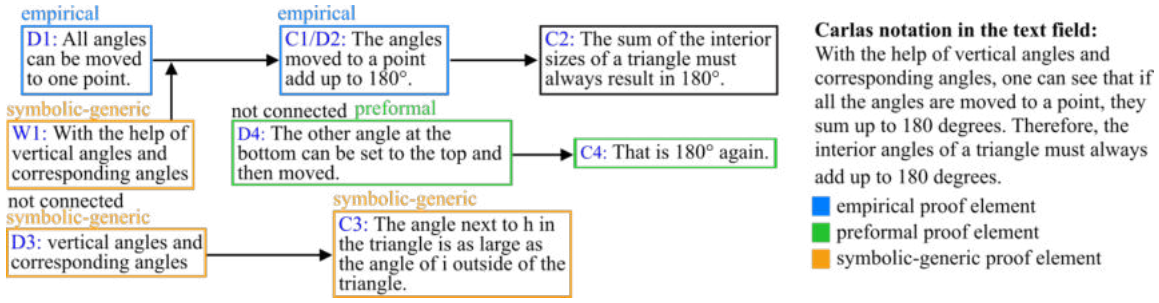

Figure 3: Mixed-type of the pre-service teacher Carla

In Figure 3, Carla combines empirical (D1 and C1/D2), preformal (D4 and C4) and symbolic-generic proof elements (W1, D3, C3) in one argumentation. Carla develops this argumentation while working on the third GeoGebra applet (see Figure 2b) in which the development of a preformal or symbolic-generic proof is intended. Nevertheless, Carla's incorporation of empirical elements into her argumentation (D1 and C1/D2 in Figure 3) becomes apparent, because these argumentation elements reflect the work with the first GeoGebra applet in which all angles could be placed to a straight angle. From this the conclusion can be drawn that the structure of the interactive book page invites students to integrate their previous argumentation from other tasks in the mixed-type. Apart from the empirical elements, there are also first symbolic-generic elements (W1, D3 and C3). Especially the argumentation element W1 expresses that Carla tries to warrant her argumentation with a mathematical theorem. This suggests that the interactive book supports Carla in developing and discovering symbolic-generic proof elements, which is an important step on the way to deductive thinking.

However, the mixture of empirical and symbolic-generic proof elements indicates that students have problems detaching from the empirical part of the proof and that they have difficulties integrating symbolic-generic proof elements and conclusions in their argumentation. This becomes particularly evident when comparing Carla's Toulmin scheme with the text she formulates for her argumentation. Although Carla infers from vertical and corresponding angles on the angles' congruence (D3→C3 in Figure 3), she does not integrate this conclusion in her written argumentation, as she only writes *with the help of vertical angles and corresponding angles*. This can also be demonstrated by the structure of Carla's mixed-type, as two of the symbolic-generic elements could not be integrated into the rest of the argumentation.

The difficulties with incorporating symbolic-generic argumentation elements are also evident in the mixed-types of other students, because many students interpret their discoveries in the GeoGebra applets mathematically, but likely do not understand what their discovery means for the argumentation. This can be seen in the following statements of a student called Amalia.

Amalia: Well, that's when, once—so alternating angles are like a vertical angle shifted. Therefore, this here would now be an alternating angle (points to α') to this one here (points to α). Um, yeah (...) I don't know. Maybe I'm just totally confused right now, but somehow (..) alternating angles, vertical angles (9). Hmm, maybe if you somehow make all three angles (..) into alternating angles, by shifting and rotating them (..), then you get 180 degrees altogether, but that's not really an explanation, so I have no idea.

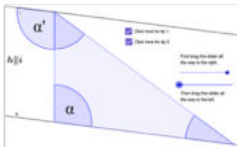

Figure 4: Transcript of the pre-service teacher Amalia

This transcript shows that Amalia associates alternating angles on the basis of the movements revealed by the sliders, but she has problems using the alternating angles and their congruence for her argumentation. This could be regarded as the main challenge in the development of symbolic-generic proofs.

In the following figure, abstracted structures of all mixed-types are considered. This structured analysis highlights that there is not just one mixed-type, but that the mixed-types are constructed in an individual and different way. These individual mixed-types will suggest a fine-graded transition process from empirical to deductive proof types, consisting of different stages.

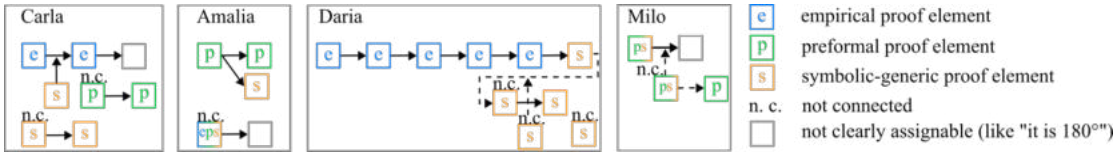

Figure 5: Mixed-type structures of all students

As seen in Figure 5, the mixed-types are mixed differently. While Carla and Amalia mix proof elements from all proof types, Daria connects empirical with symbolic-generic proof elements and Milo connects preformal and symbolic-generic proof elements. Furthermore, the proof elements themselves also can be a mixture. The 'eps-element' in Amalia's argumentation (see Figure 5) can also be seen in Amalia's transcript in Figure 4 in the phrasing 'maybe if you somehow make all three angles (..) into alternating angles, by shifting and rotating them (..), then you get 180 degrees'. The shifting and rotating of (all three) angles reflect the movements in the first and the second GeoGebra applet and can be regarded as empirical or preformal. This emphasizes the deep connection between the different proof types. Focusing on the structure of the Toulmin schemes, further differences can be identified, for example, in the comparison of Carla's and Daria's mixed-type (see Figure 5). The structure of Carla's argumentation reveals a first sign of the integration of symbolic-generic argumentation elements, as Carla attempts to integrate the vertical and corresponding angle theorem in her warrant. In contrast, Daria's approach shows that symbolic-generic proof elements are attached to empirical ones or not completely connected. This does not necessarily indicate a deductive approach in which the connection from data to claim is legitimised by a mathematical theorem. This shows that proving with digital tools can not only be seen as a development of different proof types, but that there are different, finely structured stages of development even within the mixed-types. With regard to the comparison of Carla and Daria, their mixed-type structures suggest that Carla is a little more advanced on the path of developing deductive proofs than Daria, because integration approaches are evident in Carla's proof. At this point, however, further research is needed to categorize such fine-graded stages of development and to derive direct design implications for the interactive book page.

Figure 6 shows not only the mixed-types, but also students' overall development across the book page. This overall development and the inclusion of the book elements in Figure 6 emphasize that students have problems with the development of symbolic-generic proofs, as the mixed-types are developed at the end of the book page and thus at the end of students' proof-type-processes.

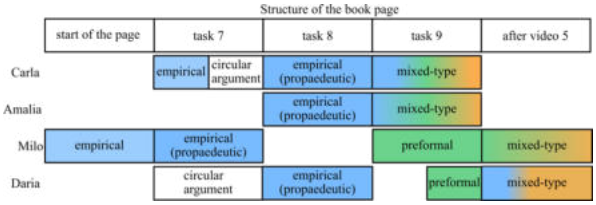

Figure 6: Overview of developed proof types through the work of the interactive book page

A possible explanation for students' development of mixed-types at the end of the book page is that the various findings from the other tasks are combined with each other at this point. With regard to the book elements, the students Milo and Daria develop mixed-types at the very end after video 5. This indicates that despite the fully provided solution in the video, the students develop a mixed-type, even though they could have simply copied the video solution. This again emphasizes that students have difficulties with deductive proofs, even when they are only required to rephrase them in their own words. At this point further research is needed to determine how instructional videos can be designed so that students do not transfer the solution into a mixed-type, but into their own deductive proof. Finally, another similarity of students' proof-type-processes is that their development ends with the mixed-type and that no symbolic-generic proof is developed after that.

Conclusion and Outlook

The analyses reveal different mixed-types that extend the existing proof type model. These mixed-types reflect students' transition from empirical to deductive proofs, as they show students' starting points in deductive thinking. The results of this study indicate that an interactive book with its specific 'proof-type-structure' can evoke the development of mixed-types, which demonstrate the strength of the use of digital tools in proving situations. However, it is not only the structure of the book page that has to do with the development of the mixed-types, because the detailed analyses of Carla's mixed-type show that the features of the GeoGebra applet (angles can be rotated and moved) can help to discover vertical and corresponding angles. In addition, Carla's mixed-type also emphasizes that students try to integrate symbolic-generic proof elements in empirical or preformal proofs. However, Carla's and Amalia's results indicate that symbolic-generic elements were either not incorporated in the rest of the argumentation or in a superficial way. In addition, Figure 5 shows the diversity of mixed-types and their different structures, which suggest a finely graded developmental process with various stages. The individuality of structures also implies that students need individual support and personalized feedback. From this, one can conclude that the use of digital tools at university must be actively supported by a teacher in order to provide individualized feedback. Furthermore, the overall development of students' proof types in Figure 6 stresses students' problems in developing further to a symbolic-generic proof type. This presents new challenges in the use of digital tools, as it is important to not only assist students in discovering the mathematical theorems necessary for a proof, but also to provide individual support in such a way that students perceive these theorems as helpful

for supporting their arguments and apply them as warrants in the appropriate part of their argumentation. This raises the question of how digital tools should be designed to support students in continuing their development beyond the mixed-type. Here, the results suggest that it could be helpful to integrate tips on the book page, motivating students to reflect on the meaning of their mathematical discoveries. Indeed, more research is needed to further develop interactive books.

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Learning success while working with (interactive) videos on multiplication derivation strategies

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Keywords: Interactive video, primary school children, multiplication, derivation strategies.

Videos are an integral part of our lives and are used as a learning medium. This paper attempts to find an answer to the question, which effect educational videos have on children's mathematical academic achievements compared to interactive educational videos, using the example of multiplication derivation strategies. Do educational videos generally have a positive influence on learning? In order to investigate these questions, the change in performance before and after two different forms of video intervention is examined in a pre-post design.

Introduction

Media education is a part of the primary school curriculum in Germany. Teachers don´t have to understand only how to use media, they must also be able to select suitable media for the individual situation. In my work I focus on the medium of video in the context of learning. The use of video in the classroom is very controversial, which is the starting point for my research desideratum. On the one hand, it is said that videos offer more opportunities for interaction than traditional media (Berk, 2009). On the other hand, it is described that there is a lack of interaction when watching a video and that the viewers are passive recipients (Zhang et al., 2006). However, one study has shown that dynamic and interactive dynamic visualisations have a significantly better effect on learning than static visualisations. As there were no differences between the interactive dynamic and dynamic visualisations, the influence of interactivity needs to investigated (Rolfes et al., 2020).

Theoretical Background

Educational Videos

Children have a lot of contact with (educational) videos outside of school, so schools need to provide guidance on how to use them wisely. In the school context, videos are mainly used to prepare for tests, to help with homework or to revise content that they have missed due to illness, for example. But teachers fear a loss of control and a loss of role (Arnold & Zech, 2019). That's why teachers need to recognise the potential of videos in school. When used sensibly in schools, educational videos should complement rather than replace teaching, as they can provide new learning opportunities depending on the situation (Arnold & Zech, 2019).“Videos materials are supplements to the class session, not a substitute to the class session presentation” (Bento, 2000, p. 111).

Video allows for a change of perspective, internal differentiation, working at your own pace and can present different topics in a very dynamic way (Arnold & Zech, 2019; Modlinger, 2020). Emotions are addressed and actions and three-dimensional facts can be visualised well. On the one hand, videos increase the viewer's motivation, which increases attention. On the other hand, it is said that motivation is high when working with videos because the pupils are convinced that they have to make

less effort (Arnold & Zech, 2019; Clark & Feldon, 2014). Video is often consumed passively as it is perceived as an entertainment medium. (Zhang et al., 2006). This requires a high degree of self-discipline is required, partly because there is no incentive to engage in intensive dialogue, as ready-made solutions are often presented.

Types of educational Video

There are different types of educational videos. In Germany, the names of the most common videos are explanatory videos and discovery videos, which are often have a short duration of usually three minutes. These videos are summarised here under the term *educational video*. These videos often lack real interactivity because the viewers are passive recipients (Römer & Nührenbörger, 2018; Zhang et al., 2006). The supposed interactivity refers to controlling the speed of the video (*control of pacing)*, the Control of the order of performance (*sequencing)* and the (*control of content)* (Merkt & Schwan 2016). These forms of interactivity are merely basic functions of any video. Interactive educational videos are a sub-section of educational videos and are understood here as a sub-category, as the educational videos have been enriched with interactive elements. The interactive learning videos that I have created have been evaluated by an expert rating. They enriched by tasks (multiple choice and drag and drop) and selection options (back, explanation, next). In the latter case, viewers can choose whether they want to see an introductory example again, listen to an explanation of the solution to the task or move on to the next task. Nevertheless, interactive learning videos are intended to activate the viewer, but do not guarantee (Lehner, 2011).

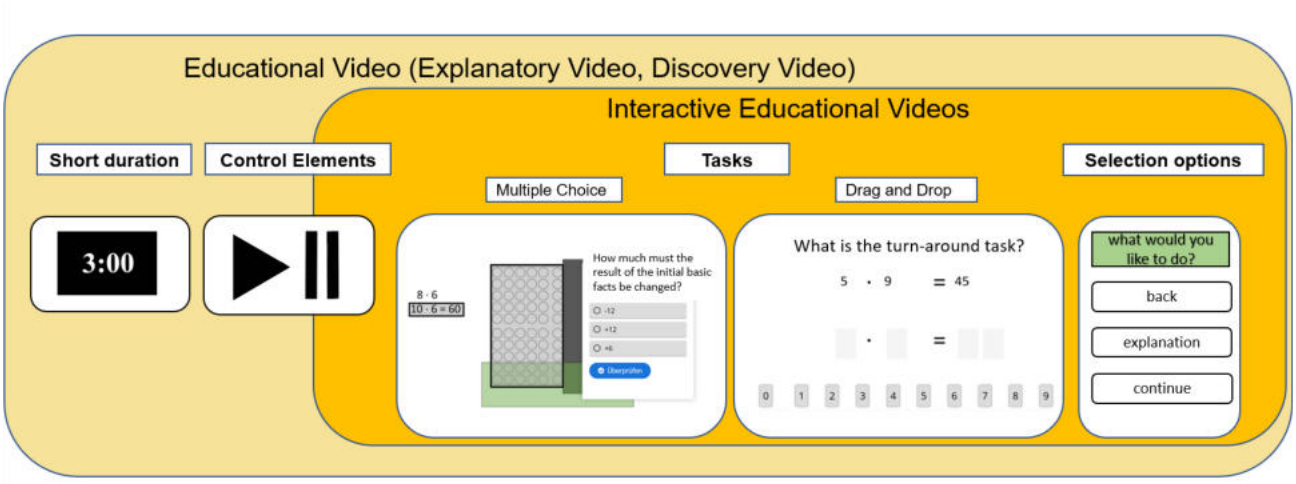

Figure 1: Differentiation of educational video types

Educational videos in primary school mathematics

It has been found that young children's attention is linked to formal features such as movement, voices, fast-paced action or funny scenes and not to themes (Krapp, 2010). Moving images attract more attention than text due to their depiction of movement, but they do not actually carry of the essential information. Teachers can be relieved by the use of video, as video can be used well for differentiation, but only if the passive consumption of video is counteracted. Videos are particularly suitable for dynamic topics and activities. They can be used in geometry lessons to develop spatial

awareness and to visualise spatial relationships. In arithmetic lessons, for example, arithmetic laws can be illustrated well or, in relation to multiplication, consecutive actions can be demonstrated. A static book can only illustrate this dynamic basic idea with several pictures and it is difficult to visualise the derivation of an arithmetic law. Therefore, derivation strategies for multiplication were considered as a suitable example for a content that can/ should be visualised dynamically, what cannot be presented in an appropriate way in textbooks. Furthermore, this is a central topic in the second school year in Germany (Gaidoschik, 2014).

Multiplication derivation strategies

It is not a new discovery that children use many different strategies when working on multiplication problems (Anghileri, 1989). Nevertheless, an exploratory study was able to show that repeated addition was the predominant strategy up to the middle of third grade, especially for lower-achieving children (Köhler & Gasteiger, 2016). As the multiplier becomes larger, children cannot maintain the strategy of repeatedly adding the addends. If different calculation strategies are developed in the class, and they can overcome the unsustainable approach of repetitive addition (Köhler & Gasteiger, 2016). Through the automation of initial *basic facts* (1x, 2x, 5x, 10x and the Square numbers) and the development of operational relationships, children can derive unknown multiplication problems from known ones. It is not sustainable if multiplication is only based on procedural counting (Anghileri, 1989). The systematic development and consideration of interrelationships helps to remember the basic fact tasks (Köhler & Gasteiger, 2016). Even if the long-term goal is to automate all multiplication tasks, it has been shown that a holistic approach produces better results in terms of the transferability of the tasks to larger numbers and in terms of estimation skills (Gaidoschik, 2014). The focus should never be exclusively on finding a result, as this would only distract from discovering multiplicative situations and possible changes in representation (Gaidoschik, 2014). In order to give multiplication a real meaning, the primary aim of multiplication (and division) is to find and discover multiplication situations in the environment.

The derivation of tasks can be illustrated well using a 100-dot field, with two sheets of paper. Dot fields are not only used to represent numbers in different number ranges in relation to the number range extension, they are also suitable for visualizing operations and operational changes and regularities (Kuhnke, 2012; Wittmann & Müller, 2016). Starting from the initial basic fact $5 \cdot 8$ in Figure 2, the task $6 \cdot 8$ can be derivate by moving the transparent green piece of paper one line down. On the left we can see a coloured 100-dot field, for understanding the task. The multiplier indicates shows how many rows there are and the multiplicand shows how many dots are in a row. By moving one or the other of the slips of paper by one or two lines, you can easily visualise the references. Figure 2 on the left first shows how the factors are represented on the 100-dot field. There are 5 x 8 5 rows in the task and there are always eight dots in each row. After understanding the meaning of the factors, the illustration should be shown again in order to direct the focus to a process of change.

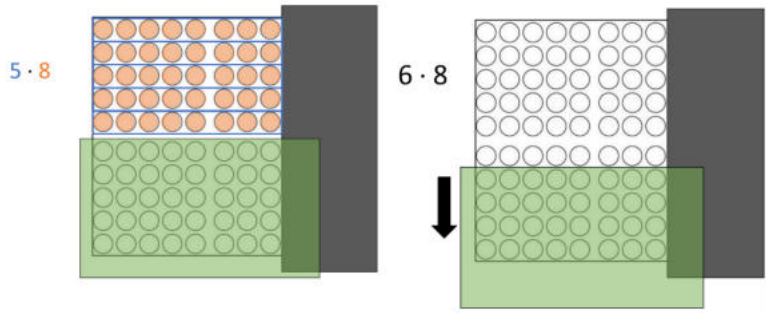

Figure 2: Use the task 5x8 to derive the task 6x8 on the 100-dot field

Aim of research

The theoretical background raises the question of the actual (learning) effect of educational videos. The paper compares the effect of linear educational videos is compared with interactive educational videos and a control group using the derivation strategies of multiplication as the content. The research question is: What is the effect of linear educational videos compared to interactive educational videos on children's performance in multiplication derivation strategies? The focus is not on solving problems, but on understanding derivation strategies.

H1: The group membership (interactive, linear, control) has a significant effect on the test performance of TMU2 (*Time Measurement Unit)*.

Method

The survey period was March 2024 to April 2024. The study analysed 129 second grade children from three primary schools in North Rhine-Westphalia. During the survey, demographic data (mother tongue, gender) were collected, basic mathematical skills were assessed using the *Heidelberger Rechentest* (HRT), and skills in multiplication derivation strategies were assessed using a test specifically designed for the study. Both, the video and the test were validated by an expert rating, consisting of people with doctorates and post-doctoral habilitations. I worked with a pre-post-follow-up design. Therefore, the test for the derivation strategies was used for all three TMUs. Each school was visited for five consecutive days, with the exception of TMU3, which is not discussed in detail in this paper. For logistical reasons, a quasi-experimental design was used.

Figure 3: Procedure for the video groups

On the first day, all children were tested with the HRT and the multiplication derivation strategies test, with the derivation strategies test taking place on both the first and the last day. The intervention took place over a period of 4 days, starting on the second day. The videos are based on the children's prior knowledge, because the children already know the dot-field from school. I use a dynamic visualisation by moving one of the two pieces of paper, as you can see in Figure 2. This shows how the results of the tasks change.

There were three experimental groups that received exactly the same input but processed differently. A group with the *interactive* educational videos (n=41), a group with *linear* educational videos (n=47) and a *control group* (n=41) with lessons, in which the content was presented by the researcher using a document camera. The children worked independently with the eight videos on different facets of multiplication derivation strategies. In addition to the quantitative data from the test results, qualitative data was also collected by examining how the children interacted with interactive educational videos. Screencasts were used to observe patterns in the children's use of the videos and whether they were working with the videos as intended.

I want to elaborate the influence of group membership. The advantage of this *linear mixed model* is that it does not only analyses the differences between the groups, at the same time it also can analyse the individual, random differences between the children. The individual differences of the children, the influence of the mother tongue or the influence of HRT can be integrated into the model. Even if the qualitative data and the covariates of mother tongue and the results of the HRT cannot be discussed in this paper.

Results

In this study, the dependent variable is the children's score in TMU2 and the independent variable is group membership and timing. The interaction diagram can be used to gain an initial impression of how the score of the children in the respective group have changed.

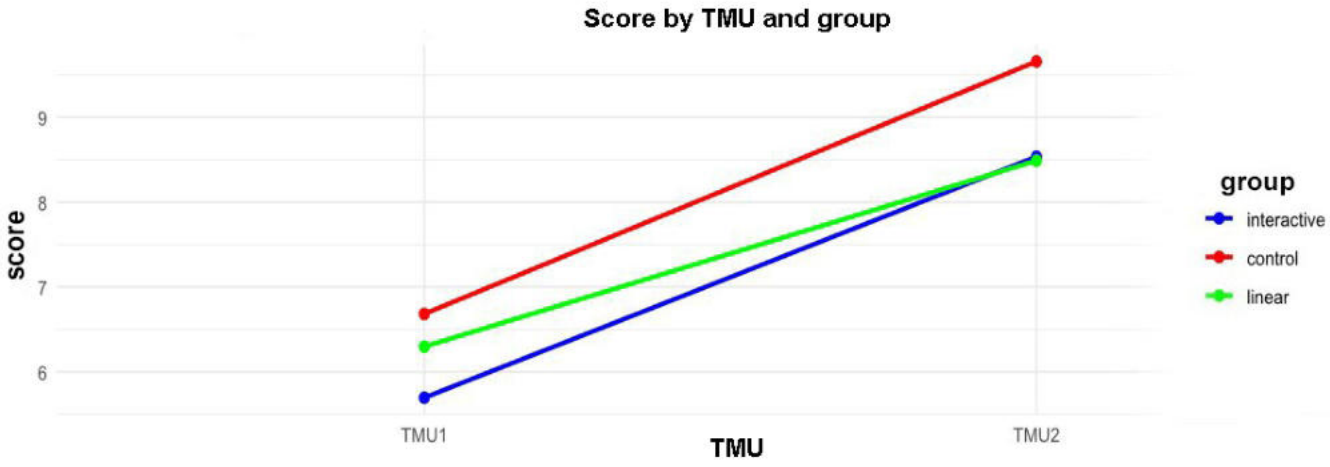

Figure 4: Interaction diagram of the three groups between TMU1 and TMU2

The interaction diagram shows that the starting position of the groups for TMU1 was not quite the same. The interactive group (blue) was weakest at TMU1 and the control group (red) was strongest. The lines in the interaction diagram are not completely parallel. If the lines were parallel, this would mean that there are no significant differences in the interaction effects. However, it can be seen that the line of the interactive group and that of the control group run approximately parallel, which means that the children in the two groups have a comparable performance development. In order to be able to make precise statements about the development of the groups, the dependent variable (test performance) is analysed with the independent variables (group membership and time). The control group is the reference group.

Table 1: Linear Mixed Modell

	Regression coefficient (β)	Standard error (SE)	Confidence interval (CI)	p-value
Intercept	6.6829	0.4331	[5.8394, 7.5265]	0.00***
interactive group	-0.9878	0.6125	[-2.1808, 0.2051]	0.108
linear group	-0.3815	0.5927	[-1.5393, 0.7692]	0.517
TMU2	2.9756	0.3533	[2.2860, 3.6652]	7.08e-14***
interactive group TMU2	-0.1341	0.4997	[-1.1093, 0.8410]	0.789
linear group TMU2	-0.7841	0.4835	[-1.7277, 0.1594]	0.106

*p< 0.05, p**<0.01, *** p<0.001

The intercept (6.6829) shows the points from the reference group for TMU1. It can already be seen in Figure 4, in the section on the y-axis, that the starting position of the values of TMU1 is not the same. The interactive group has a weaker score than the reference group (6.6829 + (-0.9878) = 5.6951). The linear group has an estimated initial value at TMU1 (6.6829 + (-0.3815) = 6.3014).

TMU2 has a significant influence (7.08e-14***) on the reference group. The estimated coefficient of TMU2 (2.9756) describes the average change in the value from the baseline TMU1. This means that the reference group improves by around 3 points. The interaction between the reference group and the interactive group shows that the interactive group does not improve quite as much as the control group (2.9756 + (-0.1341) = 2.8415). Compared to the linear group, the reference group has an estimated beta value of -0.7841. The group also improves, but less (2.9756 + (-0.7841) = 2.1915).

The following standard error (SE) and confidence interval (CI) indicate how accurate the estimated values of the regression coefficients are. The SEs are approximately the same in all areas, which means that the uncertainty in the estimation of the beta coefficients is relatively constant across all predictors. This means that the estimate for these values is relatively accurate. This is also confirmed by the CI values. Looking at the CIs it is noticeable that when comparing the reference group with the interactive group or the linear group, the lower limit ($\beta - 1.96 \cdot SE$) and the upper limit ($\beta + 1.96 \cdot SE$) both contain zero. This means that it is unclear whether the true effect is positive or negative or

whether the effect is actually zero. This means that it is possible that the intervention had no effect. The p-values provide information on how significant the estimated values of the regression coefficients are. When comparing the p-values of the reference group with the interactive or linear group are all non-significant, according to convention ($p<0.05$). Since the p-values are not significant, which could be due to the small number of participants, no statements can be made about the effectiveness of the interventions. In relation to the hypothesis, we cannot give a conclusive answer based on the large p-value. However, a first impression is that group membership seems to lead to different test performances in the TMU2.

Discussion

The first impression from Figure 4, that the control group and the interactive group improved equally, could not be directly confirmed by the data in Table 1 using the linear mixed model, don't indicate significance, which is not yet apparent in the graph. The high p-values are probably due to the small sample size. The power of the study is also limited by the non-randomised groups. Furthermore, the covariates (mother tongue and basic mathematical knowledge form the HRT) were not analysed at this point. The same applies to the long-term effect, the results of the TMU3. Qualitative data were also collected, which cannot be mentioned here in this paper. This qualitative supplement to the quantitative data offers a further explanatory approach for the interpretation of the results. The aim of the qualitative data is to check whether the children in the interactive group used the interactive elements as intended. In addition, this study worked exclusively with multiplication derivation strategies. However, no generalisations can be made about other mathematical topics. Despite the limitations, especially the high p-values, and the restriction to the derivation strategies of multiplication, there is a tendency that linear videos cannot be used as profitably in primary school math's lessons as interactive videos. It could be shown that the mere use of videos in the classroom, at primary school level, does not guarantee learning success. This first approach to the research desideratum should be the starting point for further investigations. Subsequently, the characteristics of a successful use of educational videos in the classroom should be investigated, as there have been no effectiveness studies to date. The main questions to be answered are how the videos are prepared and how well they are embedded in the classroom.

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Digital Game-Based Learning in Moroccan Mathematics Education: Insights and Challenges

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Keywords: Digital Game-Based Learning, mathematics education, Moroccan schools.

Introduction

Employing video games for educational purposes, known as Digital Game-Based Learning (DGBL), offers a promising solution to challenges in Moroccan mathematics education, such as student disengagement, mathematics anxiety, and persistent misconceptions. While educational technology adoption has grown in Morocco (Taam et al., 2024), its integration into mathematics remains limited (Nejjari & Bakkali, 2017). This significantly influences the integration of DGBL in Moroccan schools. This contribution draws on studies of DGBL in the Moroccan context to highlight its impact on mathematics learning outcomes and identify practical pathways for effective implementation.

Key Findings from the Moroccan Context

A systematic review of 38 studies on learning mathematics through DGBL in Morocco reveals its transformative potential:

- Engagement and Motivation: 'JeuTICE', an educational arcade game, provides ludic training in arithmetic, geometry, and measuring - core domains in Moroccan primary mathematics education - increasing pupils' engagement and motivation, though performance gains vary (Tazouti et al., 2019).
- Addressing Misconceptions: 'Didactic Balance', a simulation game, uses the analogy of a balance to approach algebraic equality, allowing pupils to test conjectures, solve for the unknown in first-degree equations, and effectively address algebraic misconceptions (Lhachimi et al., 2022).
- Reducing Anxiety: 'Arithmetic Bomb' generates progressively challenging arithmetic tasks within a time constraint, engaging pupils through competitive play and reframing mathematic anxiety triggers into fun, ludic errors in a sanction-free environment, resulting in enhanced motivation and reduced Mathematics Anxiety. Fostering a positive Mathematics learning (Bouzid et al., 2021).

However, challenges such as curriculum misalignment, limited teacher training, and resource constraints hinder widespread adoption.

Practical Recommendations and Conclusion

- Design and Implementation: Use effective game design models and mobile learning approaches to enhance accessibility and engagement (Bennis & Benhlima, 2017), while addressing cost and complexity barriers.
- Learner Modeling: Tailoring games to individual needs (Hamdaoui et al., 2021) and AI-based adaptive learning (Ezzaim et al., 2023) show promise for personalized education.

- Evaluation and Improvement: Use evaluation tools to assess and improve the effectiveness of DGBL. Intelligent tools can reduce bias and improve assessments (Omari et al., 2021). Consistent tools are needed for the Moroccan context.
- Teacher Training: Train educators to effectively use DGBL to support student learning.
- Curriculum Alignment: Ensure games align with national curriculum objectives
- Resource Allocation: Prepare schools with technological and logistical infrastructure.

DGBL holds transformative potential for Moroccan mathematics education. Success depends on addressing challenges in teacher training, curriculum alignment, and resources. By focusing on these areas, schools can harness gamification to improve student outcomes.

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Enhancing mathematics learning through Mixed Reality: The Function Virtuality prototype

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Keywords: Mixed Reality, Virtual Reality, Augmented Reality, Artificial Intelligence.

Research Context

The integration of emerging technologies, such as Mixed Reality (MR), into educational practices is a growing area of interest, particularly in mathematics, where abstract concepts often pose significant challenges to students (Bulut & Borromeo Ferri, 2023; Lei et al., 2018). Recent studies suggest that MR, which includes both Virtual Reality and Augmented Reality, could enhance learning by providing immersive and interactive environments that support cognitive engagement and conceptual understanding. Function Virtuality is a MR prototype developed by Mosè Colangelo within the Widera TransEET1 project (Transforming Education with Emergent Technology). This study is grounded in embodied cognition theory, which emphasizes the role of bodily interactions in cognitive processes within MR environments (Johnson-Glenberg, 2018). This theoretical foundation supports the use of MR to facilitate embodied interactions with mathematical objects, potentially improving conceptual understanding.

The research question is how Function Virtuality, leveraging an embodied approach, enhances the exploration and understanding of mathematical concepts within a MR environment.

Function Virtuality and its affordances

Function Virtuality offers various affordances designed to make the exploration of mathematical concepts accessible and engaging. Within the virtual environment offered by Function Virtuality users can interact with 3D mathematical surfaces in real-time using hand gestures. This embodied interaction enhances the learning experience by allowing users to engage with mathematical functions in a 3D space through gestures and hand movements (Johnson-Glenberg, 2018).

The prototype integrates Artificial Intelligence to support natural language commands, enabling users to easily create and modify mathematical surfaces without needing to rely on complex formulas or syntax. This affordance is particularly helpful for those users who may not be familiar with advanced mathematical notation, making the tool more accessible to a wider audience.

Method

To analyze the impact of Function Virtuality on student engagement and interaction with mathematical concepts, a didactic experiment was conducted with two groups of four undergraduate mathematics students at the University of Turin. Two different approaches were tested: a "sequential activity," where students completed tasks in a fixed order using three different technologies (MR headset, GeoGebra 3D, and pen and paper), each linked to a specific task; a "parallel activity," where students could freely choose the order of the three different technologies. Both activities focused on exploring the same mathematical concepts, such as paraboloids and their intersection with planes.

Feedback on Function Virtuality was also gathered from researchers during the CIEAEM 75 workshop to evaluate the tool's affordances and potentiality. The study utilized qualitative observations and performance assessments to analyze how Function Virtuality, as an emerging technology that lets possible embodied approaches, supports the exploration and understanding of mathematical concepts. Observations focused on student interactions within the MR environment, their engagement levels, and their ability to solve related mathematical problems.

Preliminary results

While the structured approach of the sequential activity helped build a systematic understanding, it limited flexibility and spontaneous collaboration. In contrast, the parallel activity allowed students to switch freely between tools. This flexibility enabled students to engage with the same mathematical concepts from multiple perspectives, using the MR headset for exploration and GeoGebra 3D or pen-and-paper for deeper analysis and verification. In both activities, Function Virtuality facilitated embodied interactions with mathematical objects, enhancing spatial reasoning for a more intuitive understanding of mathematical surfaces and served as an effective exploratory tool, increasing students' engagement.

At the CIEAEM 75 workshop, researchers provided feedback on Function Virtuality's affordances and out of 19 responses, key aspects noted were its interactivity, visual capabilities, multisensory engagement, and embodied experience.

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LLMs as a tool for problem solving – A case study on the role of generative AI in the solution of Fermi problems

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Keywords: Artificial intelligence, Fermi problem, Large language model, Problem solving

Introduction

Since the public release of ChatGPT, generative AI has become a major part of the daily life of young people, especially students. For mathematics education, different studies were able to show the potentials large language models and generative AI offer (Dilling & Herrmann, 2024; Schorcht et al., 2024; Yoon et al., 2024). Among the different technologies within the field of AI, a major focus of current research in mathematics education is on large language models and their implementations in chatbots as ChatGPT. This study looks at Fermi problems, as they require problem solving and modelling skills, while making assumptions about the real world. The capabilities of LLMs for the prior were demonstrated by Schorcht et al. (2024) and giving world knowledge is among the tasks they seem to perform reliably. For these reasons, in this study it is of interest, how students use LLMs while solving Fermi problems (RQ1). Beyond the way they use it, it is also of interest, why they decide to do so. Trying to understand their reasoning, the students' beliefs are analysed (RQ2).

Theoretical framework

According to Schoenfeld (1985) characterizing the mathematical problem-solving performance of students, the four factors *resources, heuristics, control* and *belief systems* are components of knowledge and behaviour suitable to describe the process. Belief systems are "one's 'mathematical world view', the set of (not necessarily conscious) determinants of an individual's behavior" (Schoenfeld, 1985, p. 15) and influence students mathematical processes. This has been shown for beliefs concerning the learning of mathematics (e.g. Witzke & Spies, 2016), which were originally meant by Schoenfeld, as well as technology-related beliefs and their influence on the learning with technology (e.g. Pepin et al., 2024). While there are many different constructs, terms and definitions in the area of beliefs or affective concepts (Philipp 2007), for this study the notions and differences of these will not be discussed. The broad definition of Schoenfeld (1985) shall suffice hereafter.

Fermi problems are a special kind of problem-solving tasks, that require the students to estimate values by making assumptions about problems and the real world. Albarracin & Ärlebäck (2019) describe four activities of students making these assumptions: *Guesstimations*, *experimentation*, *looking for data*, and *statistical data collection*. Besides describing these activities, the authors also link the problem solving of Fermi problems to mathematical modelling, pointing out similarities and differences. Their views on the matter are shared by the author of this submission.

Methods & materials

To answer the research questions an exploratory case study was conducted in December of 2023 in a secondary school in North Rhine Westphalia in Germany. The participants were multiple groups of

students of an 8th class, all of whom participated on their own volition. The students sat around a table in groups of 1-3, on which a calculator, pen and paper and a laptop with ChatGPT 4.0 were placed. They were given basic information about ChatGPT, as the way to interact with it in a chat and that it may produce misleading results. Among other Fermi problems, the students were asked to estimate the amount of water they use per year or how often they blink in one day. The problem-solving process and interviews were filmed, transcribed and analysed with an inductive qualitative content analysis (Mayring, 2000).

Results

Summarizing the results for the first research question, Adam (group 1) used the chatbot by copy-pasting all the problems into the chat and copying down its answers without questioning them any further. The students Berta and Claire (group 2) decided to use the AI as a resource of knowledge for some numerical estimates they used in their calculations. For some problems they decided not to use AI at all and for some they also asked ChatGPT for the full answer after giving up on calculating it on their own. Considering the second research question, all participants stated a high trust in AI and the correctness of its answers, possibility connected to their misconceptions about its operating principle. The self-concept of the students in group 2 seemed to have been the deciding factor for their approach to try and solve the problems on their own as much as possible, only relying on AI as a source of knowledge or once convinced, they could not solve a problem on their own.

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Programming to inscribe geometric entities in circle and in sphere: an embodied approach of textual programming towards assembling a new Conceptual Field

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Keywords: debugging, dynamic manipulation, Design-based Research, 3D-printing.

Introduction

Digital environments, according to Papert (1980), are fields where, with appropriately designed activities, conditions fertile for rich experiences of meaning-making are ensured. Notable emphasis is put on embodiment, through what he called 'body-syntonicity' and its significance for students' mathematical engagement when working in a programming environment requiring the coordination of one's body posture, physically or imaginarily, with the turtle-vehicle to navigate it and solve geometric problems, drawing upon personal embodied motional experiences. Following this line of thinking, I have started conceiving digital projects and exploratory activities that integrate dynamic manipulation and modelling using a textual programming tool to challenge children to collaborate to debug existing digital models and to produce their own personally meaningful artefacts (diSessa, 2001), posing the question whether -and under what conditions- a geometric entity can be inscribed in another. For this purpose, the Conceptual Field "Inscribability" was assembled, intending to construct a network of related ideas (mainly drawn from Geometry and Trigonometry), representations and problem situations emphasising the interconnectedness of mathematical concepts, arguing against the isolationist approach in education (Vergnaud, 2009). The central objective is to examine how engaging with transformative technologies, redefines learning conditions, affects the production, expression and communication of ideas and, subsequently, how it enriches mental constructions and connections between abstract mathematical ideas and concrete programming implementations. Utilising Design-Based Research (DBR), I mean to iteratively refine the intervention design in real educational settings to enhance learning outcomes and pedagogical practices.

The learning environment

The learning environment MaLt2 (available here: http://etl.ppp.uoa.gr/malt2/#Gr) integrates textual programming with the affordances of dynamic manipulation of parametric procedures, 3D graphics and camera navigation. This online Logo-like tool includes the following interrelated representations: i. Symbolic (code), ii. Figural and iii. Dynamic manipulation of generalised models controlled by procedures with variables (Kynigos & Grizioti, 2018).

Research questions

- What meanings do children construct within the Conceptual Field of Inscribability, while tinkering digital models and linking real-space lived experiences with the embodied turtle metaphor and the symbolic expression?
- How, dynamic manipulation of models, and the parameters that define them, affect the ways in which children **a)** perceive and express Inscribability as a relationship and **b)** explore how it is maintained - or not - as certain parameters are altered within the MaLT2 digital environment?

Study design

This study was designed as a series of open exploratory activities on the issue of "Inscribability" of 2D and 3D objects in a circle and in a sphere, addressed to 9th-grade students (age 15-16) as a task of "Designing the interior space of a crazy ball"; this first pilot conducted with 26 students with prior experience with MaLT2, over two sessions. Initially, children explored given procedures, in which incorrect mathematical relationships were embedded; they worked on 'discovering' these errors and correcting them (debugging) to successfully inscribe the models in a given circle. In the second phase, each group chose a 2D object (e.g. letter, shape) to inscribe in the circle and then in the sphere-crazy ball. The final phase extended to selecting and inscribing a 3D object within the sphere-crazy ball. Their work was documented via screen-recording software and video recordings including audio; moreover, their digital artefacts (codes) were collected and analysed.

Preliminary results and next steps

Students used gestures as a spatial form of action in order to navigate and act in the virtual space and harmonically combined verbal and symbolic expression to interact with the 2D and 3D figural representations, aiming to inscribe one in another. The results gained from this pilot implementation are highly promising suggesting that students' engagement with this Conceptual Field in MaLT2 could create rich opportunities for interconnecting ideas, relating mathematical concepts, properties and knowledge from different contexts. Among these first findings lay the meaningful use of mathematical formalism while modelling and programming as well as the variety of students' creative solutions to the problem. One group, after debugging the given procedure of a triangle, used it as a unit to base on and construct their own star-shaped model by rotation of the triangle. Another group generalized the debugged triangle procedure and created a regular hexagon and a regular decagon, inscribed in the same sphere. For the following cycles of development, I am planning to gradually deepen students' engagement within this mathematical-computational experience by asking them to 3D print their models posing the challenge of creating classes of printable models that could be easily and dynamically modified.

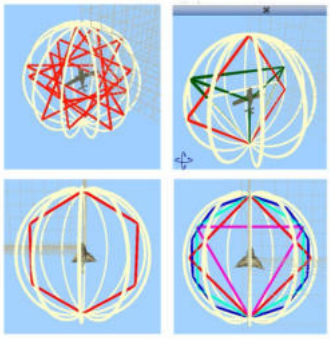

Figure 1: Students' artefacts

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The impact of technology-based collaborative learning on enhancing students' fraction comparison skills

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Keywords: Learning mathematics with technology, collaborative learning, fraction comparison

Technology-based collaborative learning in mathematics education

Recent technological advancements have significantly impacted mathematical education, prompting schools to integrate technology into teaching to enhance learning and student achievements (Ran et al., 2021). 'Collaborative learning' describes an educational approach wherein students benefit from complete participation in the learning process. Collaborative learning occurs in small groups characterized by mutual engagement and interaction among participants. This enables students to assist one another, exchange ideas, articulate diverse viewpoints, and foster interpersonal social ties (Lahann & Lambdin, 2020). Technology-enhanced collaborative learning environments facilitate student learning in computer and internet-supported contexts via cooperative interactions (Ozdamli et al., 2013). Numerous research has shown that collaborative learning in mathematics improves students' comprehension (Ozdamli et al., 2013). Various studies demonstrated that collaborative learning in mathematics enhances students understanding (Retnowati et al., 2017). These studies described the interactions that occur when students solve math problems in small groups, which include asking questions to get help, confirming ideas, and clarifying progress toward a solution. Previous studies also addressed the positive group synergy when students solve mathematical problems together with technology (Clark et al., 2014; Roschelle et al., 2010). The purpose of the present study was to examine the impact of an innovative technology-based collaborative learning program on enhancing students' fraction comparison skills and to determine whether there is any correlation between gender and knowledge with respect to comparing fractions. Until today, few studies specifically examined the impact of technology-based collaborative learning on students' understanding of fractions. This study can contribute to mathematics teachers by introducing a unique intervention program that combines technology and collaborative learning. The study can make a significant contribution to the field of mathematics education by showcasing how technology-based collaborative learning can improve students' conceptual understanding of fractions comparison.

Method

The study used a quantitative method and involved 125 fifth-grade students (59 boys and 66 girls) studying in a primary school in Israel. Two classes participated in a technology-integrated intervention program, and two classes were taught in the traditional way. The intervention program included three stages: (1) All the students received a preliminary fractions knowledge test; (2) Students in the intervention group were taught about comparing fractions while integrating collaborative technology, while the students in the control group were taught in frontal lessons. The students in the intervention group worked together on various technology-based activities such as solving computerized tasks, playing in computer-based games including real-time competitive quizzes, building a Padlet board with solutions for different tasks, and solving word problems in small

groups and discussing their reasoning in real time through group chats. The intervention duration was about eight weeks. The students solved the tasks jointly in small groups and offered their conclusions regarding rules for comparing fractions using computerized representations. (3) A second test was administered to all the students. Fractions knowledge tests were specially designed for this study and were based on the national mathematics curriculum and a nationwide standardized test. The first test, administered before the intervention began, covered basic fractions knowledge (i.e., understanding of fraction magnitude, identifying numerator and denominator, fraction nomenclature, representing the fraction as an illustration). The second test, administered after the intervention period, focused on skills in fractions comparing.

Results

Repeated measures ANOVA indicated that on students' scores on the fractions knowledge tests both Time and Time × Intervention interaction were significant, [$F(1,121)=33.316$, $p=.000$; $\eta^2=.21$; and $F(1,121)=89.245$, $p=.000$; $\eta^2=.42$, respectively; respectively]. The intervention group significantly improved their mean-score between the pre-and post-intervention [*pre*: $M=67.32$; $SD=21.11$ *post:* $M=76.62$; $SD=19.75$]. In contrast, the control group's mean score remained stable [*pre*: $M=63.22$; $SD=23.36$; *post:* $M=61.07$; $SD=20.45$]. No significant interaction was found regarding Time × Intervention × Gender ($F(1,121)=0.026$, $p=.873$; $\eta^2= .00$).

Conclusions

This study's findings highlight the benefits of technology-based collaborative learning in supporting students' ability to compare fractions. The results demonstrate technology-based collaborative learning has the potential to meet the varied needs of primary students that struggle in studying the subject of fractions comparison.

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Moving forward with AR: Effects of augmented reality on learners' solution behaviour in analytic geometry

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Despite the fact that new technologies are used in teaching there is a need to investigate the impact of Augmented Reality (AR) on learning outcomes in specific areas (Chang et al., 2022). Using AR in mathematics education can be particularly enriching when three-dimensional representations are taken into focus. In analytical geometry, a technical and less descriptive approach is predominant (Borneleit et al., 2001). Starting here, the MalAR (Mathematics learning with AR) app allows users to place mathematical objects of analytical geometry (e.g. points, planes and straight lines in space) together with a coordinate system in one's real environment through the smartphone's camera view. In detail, learners can use MalAR to bodily explore mathematical situations of analytic geometry to either discover given mathematical situations (e.g. textbook tasks) or verify their calculated results (Reit, 2022). Studies using extended reality in STEM education show that learners who moved their bodies to grasp and experience concepts, made learning gains, were more motivated and had more positive attitudes (Lindgren et al., 2016).

Building on these results and the very technical view of the content in the field of analytical geometry described above, the MalAR app was used to investigate the influence of AR support when working on mathematics tasks in analytic geometry. The study described here focuses on two main research questions:

RQ1: What types of body movement related to interaction with virtual objects occur when solving a textbook task in the field of analytic geometry?

RQ2: How does MalAR influence the learners' approach to their written responses?

Method

A video study documenting and analysing the learners' body movements, together with screen recording of the virtual objects used in the MalAR app, was conducted with 25 pre-service mathematics teachers at the University of Education Karlsruhe who agreed to take part in the study in June 2024. The study involved two tasks, adapted from a standard German textbook, on analytical geometry. To ensure that technical difficulties did not have too great an influence on the completion of the tasks, in task one the participants were asked to follow a sample solution to a task step by step and decide whether the solution presented was correct. In task two, the participants were first asked to explore a mathematical situation consisting of 2 points, a straight line and a plane, using the MalAR app. This exploration phase was followed by written solutions to several sub-questions of the task asking to calculate the equations of straight lines using the given points, to determine the point of intersection of the calculated straight line with the given plane and then to analyse the mutual position of the straight line and the plane. Finally, these written solutions were to be verified by exploring the mathematical objects with the MalAR app.

A qualitative content analysis of the videos and screen recordings was conducted to construct a system of movement type categories, drawing upon and extending the findings of Reit (2022). Subsequently, the identified movement types were associated with the virtual mathematical objects that the learner engaged with in MalAR app, addressing research question 1. The quality of the written solutions was analysed to investigate how varying degrees of correctness relate to distinct movement patterns, as outlined in research question 2.

Results and Discussion

The qualitative analysis of the video and screen data revealed that all learners performed various types of body movement while using MalAR: circling objects, verifying intersection points, and manipulating the visibility of virtual elements. These findings answer RQ1 and complement the existing category system of movement patterns (Reit, 2022; Reit & Wachter, in print). An analysis of the learners' written solutions revealed that approximately one-third of the students were initially wrong. After being able to validate their initial incorrect solution with MalAR, they recognised their mistake, recalculated their solution and finally obtained the correct one. This allows to conclude that the support provided by MalAR enables learners to independently assess the correctness of their solution and to make corrections if necessary (RQ2).

Further analyses of the data will show whether specific movement patterns are initiated by the use of MalAR and which of these support learners when solving problems in analytic geometry.

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Students' acceptance of the multimodality of the social robot Pepper as an educational technology for learning mathematics

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Keywords: Social robot's acceptance, multimodality, educational robotics, mathematics learning.

Introduction

Social robotics is one of the most promising approaches to changing the way that teachers and students interact with mathematics content. Allowing students to have interactive and hands-on learning experiences, robotics can inspire a new generation of mathematics learning (Zhong & Xia, 2020). On the one hand, the use of robots can improve students' understanding of mathematical concepts. In this regard, Baccaglini-Frank et al. (2020) show how the use of the robot GeomBot makes mathematics lessons dynamic and interactive and facilitates students' understanding and visualization of geometric properties. On the other hand, robots can help students see real-life applications and develop lasting practical experience in a social context. Moreover, robots offer new opportunities to support the development of social and cognitive skills and to personalize the educational experience (Belpaeme et al., 2018). Understanding how social robots could enhance the students' learning experience is essential for their implementation in educational contexts. However, there are crucial aspects to be taken into account when using new technologies, such as social robotics. According to Venkatesh et al. (2003), the user perception of a new technology as easy to use is essential for its adoption. Therefore, for the integration of social robotics in the school environment, it is necessary to investigate their acceptance by students in order to have clear guidelines for their effective implementation.

The research study

In our research we are using the robot Pepper. Pepper is a semi-humanoid robot equipped with a tablet display. It is able to speak, move, and relate to the user by reacting in a personalized manner according to what the user does or says. Our research aims to evaluate the interaction between students and Pepper by focusing on multimodality, usability and acceptance in order to investigate students' comfort while interacting with it. Specifically, we are implementing inclusive learning activities in mathematics through storytelling, using Pepper as an inclusive, multimodal educational technology. Pepper is used to improve student engagement and to foster learning. This work is part of a larger project. The focus of this work is to investigate the level of acceptability of the students' interaction with Pepper. Specifically, our research question is: how does the multimodal interaction with Pepper influence its acceptance by students in a mathematics educational context? For an initial investigation, we involved eight students (two from middle school and six from high school) in a learning activity regarding narrative, in particular a story problem on the Pythagorean Theorem. Pepper welcomes the

student and introduces itself. Then it begins to tell the story problem. The student is asked to help the protagonist of the story solve a task. Pepper begins to ask the student questions related to understanding the story problem and provides feedback to her answers. The student begins to solve the problem. If she has difficulty, Pepper provides suggestions. In particular, at the student's request, Pepper provides contextualized examples on the use of the Pythagorean Theorem and properties of right triangles, helping with gestures and pictures that the student can see on the tablet robot. The activity ends with Pepper thanking the student for helping the protagonist of the story and saying goodbye. At the end of the activity, we asked students to answer the Robot Acceptance Questionnaire (RAQ) (Esposito et al., 2019) to investigate the students' interest in talking to Pepper and their emotional involvement during the interaction. In addition to the RAQ, students were asked to answer other questions concerning Pepper's multimodal abilities with the aim to investigate the extent to which Pepper's multimodality affects its acceptance in mathematical educational contexts. For all students, Pepper's comprehension of verbal expressions, its gestures, and the relationship between comprehensibility and gestures were outstanding. All students stated that Pepper had a high ability to understand their spoken expressions. They enjoyed its ability to give appropriate feedback according to their specific requests and to interact in a friendly way, especially in the initial getting-to-know-you process, making them feel comfortable. Moreover, all students stated that they appreciated Pepper's clarity in telling the story problem, its ability to offer personalized support in case of difficulties, and the effectiveness in making mathematical concepts accessible. This study seems to show a high student acceptance of Pepper as an inclusive educational tool in mathematics.

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TWG 17: Theoretical Perspectives and Approaches in Mathematics Education

Introduction to thematic working group 17: Centring the function and purpose of theory in mathematics education research

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Thematic Working Group 17 (TWG17) addressed theoretical perspectives and approaches in mathematics education research. To come together amidst a diversity of theories, the TWG17 centred theory as the object of discussion, guided by three main questions around the function and purpose of theory for mathematics education researchers (MERs). The questions were: What motivates MERs' theoretical working and related empirical study? How do MERs' methodological and methodical approaches cohere with their theoretical perspectives? How do MERs' units of analysis relate to their theoretical perspectives (i.e., what counts as material to be studied)? Comparing theoretical approaches and respective research methodologies and methods served as vantage points to initiate dialogue among theories. Implications include the work to broaden audiences for different theoretical perspectives as well as the development of new theories in mathematics education research.

Keywords: Theories, Methods, Research, Mathematics education.

Introduction

At the 14th Congress of the European Society for Research in Mathematics Education (CERME14), the Thematic Working Group 17 (TWG17) responded to the urgent need not only to develop and use theories in mathematics education research, but also to make explicit the reasons for and purposes of choosing specific theoretical approaches. Mathematics Education Researchers (MERs) draw on a diversity of theories to make sense of their respective objects of study. As has been the case for nearly two decades, such a diversity was thus also a defining characteristic of the CERME14 TWG17. To come together amidst this diversity, the TWG17 proposed a change in perspective: Centre theories as the objects of discussion, making *them* (rather than the theorised objects of study) the focal point. This not only made room to discuss specific theories from new angles but also offered space for dialogue between MERs espousing different theoretical traditions.

We wish to relate this change in perspective to MERs' conversations on mathematical objects. MERs have long distinguished between *mathematical meanings* and *objects*. The *concept* of circle, defined, for instance, as the set of all points on a plane equidistant from a given point, can be distinguished from the geometrical *objects*—circles—that are meant and determined by this very concept. The Pythagorean *theorem* can be distinguished from the geometrical *state of affairs* that is meant by virtue of its propositional meaning. Consequently, it is one thing to refer to mathematical objects through their meanings, and quite another to make these meanings the focus of study and attention themselves. To achieve the latter, a meta-mathematical perspective must be adopted, involving the introduction

of new terms. In this regard, Pimm (1993, pp. 261–262) stated: "The term *definition* is one of a handful of meta-mathematical marker terms (others include *axiom, theorem, proof, lemma, proposition, corollary)*, terms which serve to indicate the purported status and function of various elements of written mathematics." And just as meta-mathematical terms such as 'definition', 'theorem', or 'proof' refer to a meta-aspect of mathematical research, our initiative to talk about 'theories as such' points to meta-aspects of mathematics education research. In both cases, the transition to considering meta-aspects gives rise to new questions, which, if addressed, have the potential to affect how the intended and theorised objects are approached in research. This is evidenced, for example, by the case of a preservice teacher reported by Johnson et al. (2014), whose approach to mathematical objects reflected her view on meta-mathematical aspects, namely the function of definitions in mathematical activity. Similarly, MERs' views on meta-aspects of mathematics education research, such as the function of theory-use, has potential to impact their work with theories (see Sinclair & Herbst, 2024).

When meta-aspects of research become objects of discussion among MERs, questions emerge that cut across a range of theories. One question that MERs have asked for some time and have not found consensus is how the concept of theory may be defined in mathematics education research. To look across different definitions, Bikner-Ahsbahs and Prediger (2014) distinguished between static and dynamic views of theory. From a static view, a theory becomes a theory only after it has a developed an organized hierarchical structure and a well-defined field of application that it explains. In contrast, from a dynamic view, a theory is something that evolves through humans' investigation of problems. Bikner-Ahsbahs and Prediger (2014, p. 168) espoused a dynamic view of theory, stating: "Adopting this dynamic view on theory, we are not basically interested in defining what a theory *should be* but in describing how theories *are used* by different researchers and research groups." This interest in describing not just what theories are, but also addressing researchers' interactions with theories foreshadowed questions emerging a decade later. One such question related to how MERs promote dialogue between theories. In a systematic literature review of mathematics education research, Scheiner and Bosch (2025, p. 712) examined *theoretical dialogue*, which they defined as "the intentional and systematic engagement between theoretical approaches that actively informs, challenges, or transforms the delineation and understanding of phenomena in mathematics education." Hence, theoretical dialogue not only connects theoretical approaches, but also impacts knowledge in mathematics education.

One way to initiate such a dialogue would be to ask questions about the function of theory in mathematics education research. It is in the spirit of what constitutes the function of theory in mathematics education research, that we found ourselves in the CERME14 TWG17. To ground our discussion, we posed three broad questions: What motivates MERs' theoretical working and related empirical study? How do MERs' methodological and methodical approaches cohere with their theoretical perspectives? How do MERs' units of analysis relate to their theoretical perspectives (i.e., what counts as material to be studied)? By posing questions that interlinked theory, methodology, and method, we made efforts to examine not only what theory is, but also what theory does.

Extending from Previous Thematic Working Groups

The CERME14 TWG17 rests on the shoulders of the TWG17s that have preceded this group. Our focus on theory, rather than specific theories, grows from the distinction between horizontal and vertical theorising put forward by Bikner-Ahsbahs et al. (2022) at CERME12. A key distinction between these forms of theorising is the object under study. Bikner-Ahsbahs et al. (2022, p. 2916) characterised horizontal theorising in this way: "With horizontal theorizing, researchers focus on the *how* of theory use, for the purpose of illuminating new aspects of complex phenomena." Put another way, with horizontal theorising, MERs use theory to solve problems related to the teaching and learning of mathematics, such as technology use, task design, or classroom practices. In contrast, with vertical theorising, theories themselves are the object of study: "With vertical theorizing, researchers focus on meta-issues of theory use for the purpose of *understanding theories as entities in and of themselves*." (Bikner-Ahsbahs et al., 2022, p. 2916). In our view, Bosch and Scheiner's (2025) definition of theoretical dialogue demonstrates the compatibility between horizontal and vertical theorising. The end goal is not simply to connect different theories for the sake of connecting them, but also to advance knowledge of the teaching and learning of mathematics.

Discussing meta-issues of theory is not trivial, as people need to make efforts to understand and be understood through different theoretical perspectives. Using Lotman's term *semiosphere*, Radford (2008, p. 318) described this shared space as an "uneven multi-cultural space of meaning-making processes and understandings generated by individuals as they come to know and interact with each other." At CERME13, the TWG17 took steps toward discussing meta-issues of theory, and they found that negotiating shared meanings for terms used in English (and in turn terms that do not translate well into English) was essential to making progress (Johnson et al., 2023). If theory, not just specific theories, was to become an object of discussion for the CERME14 TWG17, then there needed to be tangible ways for MERs to communicate across different theoretical perspectives.

Drawing on the work of previous TWG17s, at CERME14 we leveraged theoretical approaches and research methodologies and methods as points of dialogue, to connect discussions of theory to results of research that advance knowledge. To explain what we mean by methodology, we appeal to Tashakkori and Teddlie (2010, p. 276), who state: "We believe that a research methodology is a general approach to scientific inquiry involving preferences for broad components of the research process, whereas research methods are specific strategies for conducting research." From this view, a methodology is something more than a particular strategy; it is an overall approach to research. Not only are MERs' methodologies intertwined with their theories, in a broad sense, methodology functions like theory as a meta-aspect of research.

At CERME13, Johnson et al. (2023) identified theoretical approaches to be fruitful for organizing discussion across theories. One theoretical approach was the networking of theories, useful for drawing connections between different theories for the purpose of solving problems by bringing together different theoretical perspectives (Bikner-Ahsbahs & Prediger, 2010). To broaden discussion, the networking of theories was positioned among a range of different approaches concerning work with individual theories or across theories (Johnson et al., 2023). At CERME11, relationships between theory and methodology were examined (Bikner-Ahsbahs et al., 2019). Chan

and Clarke (2019) offered Gibson's (1986) term of affordance to characterise a mutually beneficial relationship between theory and methodology. Rather than prescribing a methodology given a particular theory, a relationship of affordance means that theory and methodology need to cohere, such that the methodology makes sense in relationship to theoretical principles. By leveraging theoretical approaches and methodologies/methods in our discussions on theory at CERME14, we meant to focus not only on what theories are, but also on what theories do. With this action-focused move, we adopted the caution from Bikner-Ahsbahs et al. (2022), to stay grounded in our discussions of theory, so that we draw on theory to solve problems in mathematics education research.

Call for CERME14

In the call for the CERME14 TWG17, we invited contributions addressing four areas (Johnson et al., 2024, p. 54):

> 1. What approaches are being developed and/or used to navigate a diversity of theoretical perspectives? What practical and/or conceptual problems are solved by the approaches?
> 2. How can theoretical and methodological working be leveraged to disrupt the status quo in mathematics education research?
> 3. How are theories and methodologies being taught in mathematics education? And how are new scholars being supported in learning new theories and methodologies?
> 4. What new theories and methodologies are emerging in mathematics education research? How are they being developed and used?

Underlying each question was the notion of what MERs do with theory (and why they do what they do). The first focused on the approaches and the problems those approaches address. The second focused on the ways in which theory and methodology may impact what counts in mathematics education research. The third focused on new scholars' enculturation into theory and methodology in mathematics education research. The fourth focused on the creation of new theories and methodologies.

What was learned from CERME14

Sixteen papers and one poster were presented. Not only were there a diversity of theories, the topics under study also diverged. Participants theorised around four broad topics, namely mathematical content, the teaching and learning of mathematics, mathematics teacher education, and the practice of mathematics education research. The mathematical content spanned grade levels, including whole number multiplication in elementary school to analysis and topology in postsecondary school. In what follows, we use the three broad questions to organize the contributions from the CERME14 TWG17. While contributions may be organized under more than one question, we chose to present each contribution within a single question, to illustrate how we operationalized the questions in practice.

What motivates theoretical working and related empirical study?

Two contributions illustrated how existing perspectives could be used for new purposes. Lien et al. demonstrated how an existing framework, Skott's Patterns of Participation, was viable for the study of preservice teachers' professional identity. Asenova and Capone demonstrated the utility of category theory for examining interdisciplinarity from a perspective of mathematics (i.e., how

mathematics is employed in other subjects, such as physics). Two contributions showed how a critical theoretical perspective could offer new avenues to examine (or re-examine) mathematics teaching and learning, to disrupt the status quo. Yolcu explained how Foucault's postfoundational theory could be used to reframe what is meant by opportunities to learn mathematics. Xenofontos drew on Foucault's concept of heterotopia to explain how mainstream mathematics classrooms could learn from classrooms outside the mainstream (i.e., special education, gifted education, and prison education). One contribution demonstrated the need to network theories to explain phenomena. Günther et al. reported on a case study in which they networked the Theory of Transformative Education (TTE) and the Anthropological Theory of the Didactic (ATD). By combining theories, they explained a student's transition from secondary to postsecondary mathematics from individual (TTE) and sociocultural (ATD) perspectives. Making explicit the motivations for theoretical working provided new insights into MERs' purposes for their theoretical working.

How do methodological and methodical approaches cohere with theoretical perspectives?

Three contributions demonstrated how theory and methodology/method could work in conjunction. Herbst et al. presented further advancements to their "middle-range theory" of practical rationality, which explains mathematics teaching as a rational, sensible practice. They demonstrated how theory development can go "hand in hand" with the development of instruments to measure theoretical constructs. Drawing on an interactionist perspective, Kollhoff and Gerlach developed a framework that aims to describe and explain the interactive dynamics in which meanings of mathematical representations are negotiated in classroom settings. Bini et al. blended theoretical perspectives from mathematics education and digital sociology to put forward a hybrid experimental design to study students' creation of and work with mathematical memes. Two contributions articulated links between theoretical perspectives and decisions on the use of research methods. Lombard presented methods for ongoing work to archive fundamental resources on the Theory of Didactical situations (TDS) and the ATD. The encyclopaedia will be organized like a thesaurus, linking key constructs to literature explicating those constructs. Drawing on Bachelard's construct of epistemological obstacle, Sebök and Lensing examined the potential of Husserl's eidetic method to examine such obstacles in students' learning of mathematics. Two contributions examined how theory use could connect to the study of the practice of mathematics education research. Shinno and Miyakawa addressed the problem of transposing theoretical and methodological knowledge, from one culture to another. They employed the ATD to explain the transposition from Western culture to Far East Asian culture. Johnson et al. demonstrated the viability of the use of a theory sourced from one tradition to examine a research method sourced from a different tradition. They employed positioning theory to examine the teaching and learning of the research method of task-based interview. The connection between theory and methodology/method grounded our discussions in the work to use theory to solve problems (and what the nature of those problems might be).

How do units of analysis relate to theoretical perspectives?

Units of analysis represent what counts as material to be studied from different theoretical perspectives; these included not only individuals and classroom but also institutionalised knowledge. As evidenced in three contributions, the ATD provided strong grounding for the study of

institutionalised knowledge. Hochmuth et al. examined choices made in a textbook's exposition of an equivalence statement on function continuity. Espín et al. employed diachronic analysis to examine modalities through which research mathematicians studied the real numbers. Nicolás et al. examined the potential for modalities of teaching to promote creativity, illustrating with the use of the derivative to examine the monotony of functions. One contribution demonstrated the utility of quantitative methods for the study of institutionalised knowledge. Drawing on Brousseau's construct of didactic variable and Da Ronch's construct of research variable, Mili et al. examined problems in the official Swiss textbook, to statistically measure the potential of problems to support students' mathematical activity. Another contribution addressed the difficulties MERs often experience when trying to adapt conceptual tools developed in other areas of research for the questions and purposes specific to mathematics education research. In their contribution, Kollosche and Wang dealt with the question of how to conceptualise the relevance that students attribute to their engagement in institutionalised mathematics education. They argued that psychological concepts often employed by MERs to study questions of relevance—such as the concepts of belief, motive, and utility value—cannot account for key dimensions of the subjective relevance of mathematics education. Therefore, they called for a triangulation of psychological and sociological approaches to overcome these limitations. Making explicit our units of analysis offered a different angle to discuss what theories can (and cannot) do in mathematics education research.

Discussion and conclusion

At the CERME14 TWG17, we centred the function and purpose of theories in mathematics education research. Accordingly, this meant that we focused on what MERs do with theory (and why they do what they do). The three guiding questions provided us with tangible ways to dialogue across different theoretical perspectives and across contributions addressing different aspects of the call for proposals. Something new that we made explicit in the call for proposals was the enculturation of new scholars into theories and methodologies in mathematics education research. In our view, the three guiding questions can serve as productive starting points to organize seminars for new scholars. In our discussions, we found the questions to provide footholds for MERs with different degrees of experience with theory to come together, because responses to the questions illuminated our reasons and purposes for our theoretical working.

The three guiding questions led us to consider different roles that MERs may take on with respect to theory, namely *producers*, *consumers*, and *distributors* of theory. By *producers*, we mean those who develop theory, or, to put it another way, those who engage in 'vertical theorising'. By *consumers*, we mean those who put theory into practice to conduct research in mathematics education, or 'horizontal theorising'. By *distributors*, we mean those who expand theories beyond the contexts and cultural traditions in which they were originally developed, providing a broader audience for the theory in question. When we talk about distribution, we do not mean that theories are something that MERs can pick up and "plug in" to a new setting. Theories are rooted in cultural traditions, and it is necessary to honour those traditions in our work with theory (see Jablonka et al., 2013). In addition, we see these three roles as interconnected. If MERs are to use theories specific to mathematics education research, then they need to develop such theories. The contribution from Herbst et al. makes explicit the ongoing work of theory development in mathematics education research. If theories are

to expand beyond the cultural tradition in which they were developed, then MERs need to work to transpose those theories from one tradition to another. The contribution from Shinno et al. provides insights into the work of such transposition. In our discussions, we also critically examined the affordances and constraints of adopting terminology that stems from a different domain (e.g., economics) when seeking to conceptualise possible relationships between MERs and theory.

We provide three implications from the work of the CERME14 TWG17, to advance mathematics education research. The first is to develop points of dialogue among MERs espousing different theoretical perspective. We found theoretical approaches and research methodologies/methods to be productive starting points. For example, our discussions around units of analysis offered a way to examine what counts as material to be studied across different theories. The second is for MERs to make explicit connections between our theorising and our motivations for our studies. The third is to broaden audiences for existing theoretical perspectives, to expand the reach of different theories.

A consequence of our focus on theory as a meta-object of research is a need to be vigilant to ground our conversations so that we do not only discuss theory for theory's sake. We found that moving back and forth between specific theories and broader notions of theory allowed us to ask questions that cut across different perspectives. To illustrate, a feature of the ATD is the use of formal symbolism to articulate theoretical constructs. This feature of a specific theory led us to a broader question about the role of symbolism in theorising, which we left as an open question. While we made strong progress, there is still much work to be done. We close with some comments around future possibilities. One area for future work is to examine what theoretical development looks like in mathematics education research. During a discussion, we considered what progress in theory development might look like if we did not conceptualise it as making finer and finer distinctions. These considerations let us to conclude that there is a need for more guidance around different forms that theoretical growth progress may take. In the call for the CERME14 TWG17, we invited contributions that demonstrated how theory and methodology could disrupt the status quo in mathematics education research. To illustrate, we asked how MERs addressed ontological, epistemological, axiological, and/or ethical dimensions of theorising. We also asked how the power and status of different theoretical traditions among MERs may impact the scope of mathematics education research. We concur with Sinclair and Herbst (2024), that there is a need for more explicit treatment of axiological and ethical dimensions of theorising. We welcome such treatment at future iterations of the CERME TWG17.

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Interdisciplinarity in STEM education: A category-theoretical model from the viewpoint of Mathematics

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Interdisciplinarity, especially within STEM and STEAM education, has been emphasized to foster the technical and creative skills required in an innovative, technological world. Despite various models addressing interdisciplinarity, a theoretical framework that explicitly articulates the structure of interactions between disciplines from the perspective of Mathematics remains lacking. This paper proposes such a framework by leveraging fundamental concepts from category theory to provide a structure interpretable across specific contexts. The model employs category-theoretical concepts metaphorically and symbolically rather than formally, intertwining them with the notions of semiosphere and border. The paper introduces and interprets the diagram representing the model and exemplifies its functionality by analysing interdisciplinary interactions in a classroom experiment, highlighting how Mathematics retains its identity through these interactions.

Keywords: Interdisciplinarity, Mathematics, category theory, semiosphere, border, STE(A)M.

Rationale

To answer the formidable challenges of globalisation and the complexity of life on both national and worldwide levels, it is necessary to educate on complexity and overcome separations between disciplines (Lensing & Pilous, 2022). Indeed, the importance of interdisciplinarity has been increasing since 1990, when the acronym STEM has been coined. The Ministry of Education recently issued guidelines on teaching STE(A)M disciplines in Italy. According to these guidelines, the interdisciplinary approach, together with the comingling of theory and practice, is at the heart of the teaching of STEM disciplines, which are particularly suited to fostering the development of technical and creative skills necessary in an increasingly technological and innovative world. Thus, giving theoretical grounding to the interdisciplinary approach is of primary importance. Sanders (2009) introduced the concept of STEM as a 'meta-discipline,' a holistic approach where STEM subjects are integrated and applied in real-world contexts. Kelley and Knowles (2016) proposed a framework that emphasizes the integration of STEM subjects, highlighting the importance of interdisciplinary teaching and real-world problem-solving. In Physics Education, different approaches and models have been developed to address interdisciplinarity (e.g., Satanassi et al., 2023). For instance, in the European research context recently, two projects carried out about interdisciplinarity between Mathematics and Physics, called IDENTITIES Project and FEDORA (Levrini et al., 2021) focus on interdisciplinarity in teacher training, highlighting the identities, the borders, and the forms of integration of the different disciplines reflecting on their epistemological and linguistic structure. Despite various approaches to the theoretical foundations of interdisciplinarity, what is still lacking is a model that can make explicit the abstract relations between the disciplines from the point of view of mathematics. This paper proposes such a model. The theoretical framework is based on the metaphorical interpretation of some basic category-theoretical concepts (Asenova, 2021) that help to

represent symbolically the interdisciplinary interactions in the model's diagram. Indeed, interdisciplinarity is based on recognizing analogies between different areas of knowledge, and category theory provides a language that can grasp analogies between different domains. We focus on such analogies, taking a cue from the category-theoretical terminology. This technical part of the model is intertwined with the concepts of semiosphere (Lotman, 2005) and border (De Luca Picione et al., 2022), which interpret the model's diagram.

The research question we focus on is: To what extent can the model be based on the metaphorical use of category-theoretical tools, semiosphere, and border frame interdisciplinary interactions while keeping the focus on Mathematics?

After presenting the theoretical framework, we describe the adopted method, present and interpret the model's diagram, and exemplify how it works by analysing the interactions between the disciplines in a classroom experiment, keeping the focus on Mathematics and its identity characterisation.

Theoretical framework

The theoretical framework is composed of a first part referring to concepts taken from cultural semiotic (e.g., semiosphere, border, etc.) and needed to interpret the model's diagram, and a second part that focuses on the metaphorical introduction of the technical terms coming from category theory (e.g., category and representable functor) and needed for structuring the diagram.

The notion of *semiosphere*, created by Lotman (2005) in cultural semiotic research, helps us explore interdisciplinary education by focusing on how different subjects interact. A semiosphere denotes the space where different cultural semiotic systems (e.g., Mathematics, Physics, etc.) can subsist and generate new information (Lotman, 2005). Lotman believed that interdisciplinarity, or combining different fields of study, is essential to understanding the complexity of reality. So, he introduced the concept of the border, not just as a dividing line but as a space where things gain meaning. These borders help organise how we understand time, space, and relationships. Lotman also described the semiosphere as a space within these borders, where different cultural codes like language, art, and science coexist and exchange information, defining the disciplines and establishing identities while encouraging translation and understanding of others. What defines a semiotic system is that it allows us to interpret and construct meaning. Lotman (2009) suggests that what turns a structure into a thinking system is its dialogical interaction with the outside world, facilitated by the organisation of borders. The movement between inside and outside involves processes of translation and interpretation. To understand how what exists at the border interacts with its surroundings—and how semiotic systems interact—we can look at the concept of enantiomorphism. According to Lotman, all meaning-making processes begin in a state of symmetry or balance, which becomes more complex over time by creating a mirrored symmetry. For example, one semiotic system, called culture C1, starts in a state of symmetrical balance. When it encounters texts from a different semiotic system, culture C2, it interprets them, breaking them down and reorganising them. Through this process, culture C1 generates an image of culture C2. This image, which Lotman refers to as an

enantiomorphic model, acts like a mirror, allowing culture C1 to see both similarities and differences between itself and culture C2 (Figure 1).

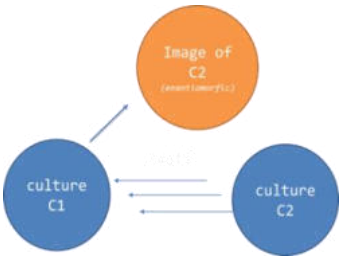

Figure 1: Scheme of the enantiomorphism between disciplines

Imagine Physics as one culture (C1) and Mathematics as another (C2). Physics might take a mathematical formula from mathematics and analyze it to form an understanding of what mathematics is like. This "mirror image" helps Physics compare itself to Mathematics, seeing similarities and differences in their methods and ideas. By doing this, Physics can better understand how to use Mathematics to solve real-world problems. Conversely, Mathematics might take examples from Physics to see how its formulas work in practice, creating its own 'mirror image' of Physics. This exchange at the border between the two disciplines not only helps solve problems from different perspectives but also deepens each discipline's self-understanding through comparison with the other. Working at the intersection of disciplines offers more opportunities than focusing on just one field. This study draws on De Luca Picione and Valsiner's concept of the *border as a metaphor*. They approach the idea of the border not as a fixed entity but, from a psychological standpoint, as a semiotic tool that drives dynamic growth and mental development processes. From this perspective, borders help establish our position within a symbolic field. Aligning with this view, we conceptualise disciplinary borders as psychological tools that support ongoing development and meaning-making processes. Human experience is characterised by a continuous construction of meaning through actions like classification, categorisation, and interpretation, which are essential in shaping behavior, thinking, and relationships. In this context, borders create a symbolic field where these semiotic processes occur, starting with differentiating this field from the outside world. Although borders often represent separation, definition, and closure between distinct disciplines, they also play a crucial role in forming dynamic relationships and structures that encourage comparison, exchange, and dialogue. In this work, we metaphorically connect the idea of the border to the concept of interdisciplinarity (Capone, 2022).

Now, let us come to the technical part of the model. It allows us to schematise the relations between the concepts of semiotic system, translation between semiotic systems, enantiomorphism, border, and semiosphere, taking the cue from the category-theoretical tools category, functor, embedding functor, and representable functor. The diagram resulting from the representation of these technical tools allows us to represent and interpret in detail the relations between the concepts of the involved disciplines and between the disciplines themselves. Our proposal considers the category-theoretical notions involved as universal tools of thought, not as formal mathematical tools (Asenova, 2021; Asenova et al., 2023; Zalamea, 2012). In the following, we explain the metaphorical meaning of each of these technical tools.

In the model, a discipline intended as a cultural semiotic system (e.g., the cultures C1 and C2 in Figure 1) is named 'category'. In this context, a *category* is therefore not a mathematical structure but a relational context where concepts represent the objects and the relations between them represent the arrows of the category, e.g., the concepts involved in a mathematical theorem and the relations between them, explained in its proof, represent such a category. The process representing the translation of aspects of meaning in Culture C1 into aspects of meaning in Culture C2 is named here 'functor'. In this context, a *functor* is not a formal mathematical object but a tool of thought that establishes an analogy from one cultural semiotic system to another, i.e., from one discipline to another, mapping concepts in concepts and relations in relations. This translation between cultural semiotic systems could be complete in the sense that there is a one-to-one mapping between the meanings in the first and the second semiotic system or only partial (e.g., only some elements of mathematical proof can be made to correspond to elements in an experimental proof in Physics). The concept of enantiomorphism introduced above allows us to associate objects belonging to a culture C1 with objects belonging to another culture C2. It is considered a special kind of functor, named 'embedding functor,' that emerges by embedding one cultural semiotic system into another. An *embedding functor is thu*s a tool of thought based on an analogy that allows interpreting a relational context within another relational context and gives rise to a metaphorical use of aspects coming from the first relational context in the second one. In this sense, it is a way to describe the metaphorical use of concepts from a discipline in another discipline (e.g., the physical term 'center of gravity' when dealing with the intersection point of the medians of a triangle in Mathematics). In an interdisciplinary context, the identity of a chosen discipline is characterised by the images sent back from its mirroring in the other involved disciplines through the interactions with them that take place on its border. From a category-theoretical viewpoint, this idea is the core of the 'representable functor' concept. A representable functor is an embedding functor that expresses the relational context it starts from, assuming a particular viewpoint. In our model, a *representable functor* is a tool of thought that expresses the interactions between the concepts belonging to the different disciplines from the viewpoint of one of them, in our case, Mathematics (e.g., when we examine the interaction between a mathematical proof and a physical experiment associated to it, and we assume the point of view of the mathematician who is asking for the meaning of a physical concept). A representable functor focuses on relations rather than on objects. Indeed, it "replaces" the objects, in our case the involved disciplines, by their relations to one chosen object, in our case Mathematics. The concept of semiosphere, where the interactions of the disciplines take place, is modeled by a special kind of category, whose objects are functors and whose arrows are mappings between them. In our model, this special category is the metaphorical space where the interdisciplinary activity takes place and where the representable functor is embedded. The interplay of the technical tools described above allows us to express our point of view on Mathematics in an interdisciplinary context that could be composed of some or all of the STEAM disciplines. More precisely, all the relations that express how a discipline is related to Mathematics within an interdisciplinary approach represent components of a border as a metaphor of interdisciplinarity (Capone, 2022) that can be conceived of as a representable functor.

Methods

In the following, we present the diagram of the model that allows us to frame interdisciplinary settings and provide its interpretation. Then, we describe an interdisciplinary classroom activity from a secondary math classroom and analyse it according to the model.

The model's diagram and its interpretation

The model's diagram (Figure 2) has three main elements: one irregular form named C_{INT} on the left; one rectangular form named C_{INT}^{MATH} on the right, and a curved arrow between them named r_{MATH}.

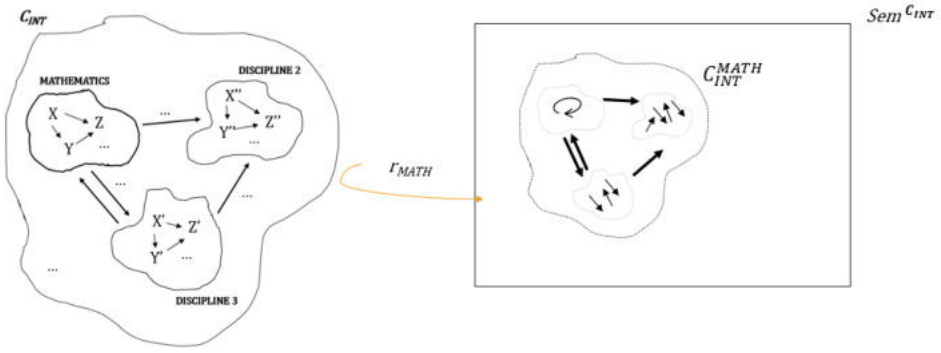

Figure 2: The model's diagram. The interdisciplinary context (C_{INT}) is embedded by the representable functor (r_{MATH}) in the semiosphere ($Sem^{C_{INT}}$) highlighting the borders from the point of view of Mathematics (C_{INT}^{MATH})

C_{INT} is the category representing the interdisciplinary relational context composed of the cultural semiotic systems involved in the activity (Mathematics, Discipline 1, Discipline 2, …) and the relations between them that are relevant to a designed interdisciplinary activity. In each cultural semiotic system, concepts and relations are involved in the activity (X, Y, Z, for Mathematics, X', Y' and Z' for Discipline 2, and so on). $Sem^{C_{INT}}$ stands for the special category representing the semiosphere in which the interdisciplinary activity is operationalised during the classroom activity. Finally, r_{MATH} is the representable functor that embeds a copy of CINT, named C_{INT}^{MATH} in $Sem^{C_{INT}}$. C_{INT}^{MATH} represents the disciplines in C_{INT} and their relations from the viewpoint of Mathematics. So, for instance, if Discipline 1 is Physics, so in C_{INT}^{MATH} it is not defined in itself but as the discipline that uses mathematical results for modeling natural phenomena or other similar relations. This aspect is expressed for each discipline by the straight, thin arrows within the dotted forms in C_{INT}^{MATH}; only Mathematics is expressed by an identity arrow (a loop) that relates it to itself. This underlines that the viewpoint of Mathematics is privileged, and the model aims to characterize its identity by interacting with the other disciplines on its border. The bold arrows between the dotted forms stand for the borders crossed between the disciplines that emerge during the activity in the semiosphere ($Sem^{C_{INT}}$).

An example of interdisciplinary activity analysed according to the model

The interdisciplinary activity presented here is taken from Adesso et al. (2019) and Capone and Faggiano (2024). It focuses on Ceva's theorem. The teaching activity was designed and experimented with 10th-grade students in different schools across Italy. The design of the activity was based, among others, on the following principles: students work in a fixed disciplinary contest but foster the connection between Mathematics and other disciplines; linguistic practices that can be meaningful to

connect the different activities are promoted. The activity was carried out in seven phases. *Phase 1*: Students are provided with the original Latin text of the theorem and divided into small groups; they are requested to translate the text into Italian language. *Phase 2*: Guided by the teacher, students are invited to discuss whether the problem they are faced with is a Physics problem, a Mathematics problem, or both, by answering the following questions: *Which elements of the text can be related to a Mathematics problem? Which elements of the text can be related to a Physics problem?* According to their arguments, they are then asked to identify the strategies and the tools that could be used to give a proof of the theorem. *Phase 3*: Students are asked to prove the theorem with the classical approach and tools of axiomatic Euclidean geometry. The teacher then orchestrates a discussion to bring students' attention to the analogies between the original text of the theorem, the experimental activity, and the geometrical proof of the theorem. *Phase 4*: Students are asked to construct and verify Ceva's theorem using dynamic geometry software. *Phase 5*: Students are invited to choose one of the three approaches to present and prove the theorem. Then, according to their preference, they are divided into three groups and asked to prepare themselves for a debate concerning the reasons for their choice. *Phase 6*: Students in each of the three groups are invited to argue why they consider the chosen approach the best to prove the theorem. The debate is followed by a final collective discussion orchestrated by the teacher. In the following section, we interpret the interdisciplinary activity according to the model presented in Figure 2.

Data analysis and discussion

In the activity presented above, the category representing the interdisciplinary context C_{INT} is composed of Mathematics, Physics, and Latin. The Mathematics context is characterized by four concepts that represent the objects of the related category: the mathematical statement of Ceva's theorem (X), the Euclidean Geometry context (Y), the proof provided in the traditional Euclidean context (Z), the proof in the GeoGebra context (V). Examples of relations between these concepts, as they were conceived by the researcher who designed the activity, are the following: relation from X to Y (X is a theorem in Y) and from Y to X (Y is the theory where X 'lives in'); relation from Z to X (Z is a means to proof X) and from X to Z (X is a part of Z). Further relations between the involved concepts could be singled out. The Physics context (Discipline 2) is characterized by three concepts: the physical statement of Ceva's theorem (X'), the experimental setting context (Y'), and the weight experiment that checks the equilibrium (Z'). Examples of relations between these concepts are: (1) from X' to Y' (X' provides the instructions for the experiment in Y'); from Y' to X' (Y' is the verification of X'). The Latin context (Discipline 3) is characterized by two concepts: the Latin text of Ceva's theorem (X'') and the Latin Grammar (Y''). The relations between Mathematics and Physics are characterized by the assignments formulated by the researchers for the students. Latin to Mathematics and Latin to Physics: read the text and translate the statement from Latin to Italian; Physics to Mathematics: construct a cardboard triangle and apply weights to it that represent the relations of the Cevians. The representable functor r_{MATH} acts as an embedding of the interdisciplinary context C_{INT} within the semiosphere $Sem^{C_{INT}}$, where the interdisciplinary activity is taking place. It expresses the viewpoint on this activity from the perspective of Mathematics. Let us look now to C_{INT}^{MATH} which represents the copy of C_{INT} in the semiosphere $Sem^{C_{INT}}$ assuming the viewpoint of Mathematics. As we stated above, in C_{INT}^{MATH} each discipline is replaced by its relations

to Mathematics, and Mathematics itself is characterised by the images mirrored from the other disciplines. In this sense, examples of images sent back by Physics to Mathematics are the following: (1) Physics provides a concrete and manipulative context for Mathematics that allows one to experiment and thus sends back an image of Mathematics as a discipline that allows to generalise and validate experimental results; (2) the Physics experiment is a way to render more convincing the mathematical relations by experiencing them metaphorically in a concrete context; another image send back by Physics to Mathematics is thus that Mathematics benefits from physical models for being better understood. Images possibly sent back from Latin to Mathematics are the following: (1) Latin provides the opportunity to reflect deeply on mathematical relations by translating them into another language, and thus, Mathematics is a discipline with an abstract language whose understanding can benefit from the interpretation in different languages, including the historically original one. These images are just generic examples of borders crossed during the interdisciplinary activity, emerging through the interplay of the semiotic resources of the involved cultural semiotic systems (the disciplines) in the semiosphere, represented by $Sem^{C_{INT}}$. They are specific to the designed activity and can be nearer defined by analysing the students' points of view. However, the structure expressed in the model's diagram (Figure 2) does not depend on the activity and would always be the same. This characterises the 'syntax' of the interactions of the involved disciplines, keeping a privileged point of view of Mathematics. By syntax, we mean the elements involved in the interdisciplinary interactions and their relations, represented by the symbols associated with the technical tools in the model diagram, while the interpretation that lends the technical tools and symbols related to them with meaning represents the model's semantics. Furthermore, the model allows us to take into account the activity design (the entities and the relations between them in C_{INT}^{MATH}), as well as the analysis of the activity in the classroom, considering the interdisciplinary borders that emerge within the semiosphere ($Sem^{C_{INT}}$).

Concluding remarks

The model presented and exemplified in this paper seems to be able to frame interdisciplinary interactions while keeping the focus on Mathematics, as it makes evident how its disciplinary identity is shaped by its interaction with the other involved disciplines. It allows for clarifying the relations in an interdisciplinary context by expressing them in a symbolic, context-independent manner. We conclude by pointing out that the prioritisation of Mathematics in the model concerning the other disciplines is not necessary. Indeed, we could choose another discipline as a privileged one; in that case, Mathematics would contribute to defining the identity of the privileged discipline. We believe this is not a disadvantage but emphasizes the model's versatility and the generality of the relationships highlighted in it. This aspect represents a possible future research direction pointing to evolving the model towards an interplay between the identities of the involved disciplines, expressed as borders emerging from multiple privileged perspectives within the semiosphere they are interacting in.

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On the diachronic analysis of a modality of study

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The aim of this work is to emphasize the relevance of the diachronic analysis in order to understand a current modality of study of a certain field of knowledge in an institution. This kind of analysis requires the construction of three didactic paradigms to model three modalities of study: the current one, an alternative possible future one and a possible past one. To exemplify a diachronic analysis, we use the case of the study of real numbers in the institution of researchers in Mathematics. Moreover, we suggest that the diachronic analysis is a suitable procedure to tackle a wide range of problems in didactics.

Keywords: Didactic analysis, didactic paradigm, modality of study, real numbers.

Introduction

From the perspective of the Anthropological Theory of the Didactic (ATD), we understand the 'study' in a broad sense: it includes processes of knowledge creation and reorganization, not only those aimed at enabling the dissemination of knowledge but also those designed to facilitate its use in specific situations and within different institutions (Chevallard, Bosch, & Gascón, 1997). We assume that didactics is the science of study and, by extension, of the modalities of study, leading us to consider the following problem: What types of models allow us to characterize the structure and dynamics of a current modality of study in a didactic institution related to a specific field of knowledge? How can such modelling be used to propose possible changes to the modality of study?

In the second section, we introduce some concepts from the theoretical framework that we will use in the work: we explore the concept of the didactic paradigm, as introduced by Gascón and Nicolás (2021a, 2021b), analysing its structure and dynamics. Didactic paradigms serve as theoretical models for certain systems, specifically, those composed by the study of a particular field of knowledge within a given didactic institution (i.e., didactic paradigms model what we refer as *modalities of study*). Our main objective is to propose an expanded approach to the didactic analysis of a current modality of study by examining three didactic paradigms in parallel: a possible past, the current one, and a potential future. This idea, which forms the core of our methodology, is outlined in the third section via the *diachronic analysis*. In the fourth section, we implement the diachronic analysis in the case of the real numbers in the institution of researchers in Mathematics. We conclude the paper summarizing key insights and identifying open didactic challenges.

Theoretical framework: the structure and the dynamics of didactic paradigms

Let us begin by clarifying some key concepts. When we speak of field of knowledge, we do not mean just disciplines (mathematics, history, etc.), but also some specific parts of them (real numbers, French Revolution, etc.). Moreover, the field of knowledge can be quite wide, including everything what is to be studied at a certain didactic institution. When we speak, of didactic institution, we refer to a social institution with the declared aim of studying certain fields of knowledge. A social

institution consists in a set of constitutive[1] rules which fixes a series of institutional positions, allowed, compulsory and forbidden actions for each position, and rewards and punishments for certain actions (Searle, 2010). Examples of didactic institutions are Primary Education, Secondary Education, Research in Mathematics, etc.

Social sciences want to understand the working of social institutions. In didactics, we want to understand the working of didactic institutions; that is, given a didactic institution, I, and a field of knowledge, F, we want to understand how the system formed by the study of F in I works. In science, the study of systems is made by using theoretical models. Here, we propose as a theoretical model the idea of didactic paradigm. Specifically, we say that a didactic paradigm for the study of F in I is a triple, $DP_I(F) = [EM_I(F), DE_I(F), DM_I(F)]$, where: the epistemological model, $EM_I(F)$, is a certain description of F (Gascon, 2014), the didactic ends, $DE_I(F)$, are the answers to "What are the reasons for studying F in I?", and the didactic means, $DM_I(F)$, are the actions (which may involve the usage of certain objects, like textbooks) carried out in order to reach the didactic ends.

We thus have a structural description of didactic paradigms. Our use of the term "paradigm" goes back to Kuhn (1962). In our work, a paradigm does not refer to a canonical example to imitate, but rather to a series of shared ideas that strongly conform the way things are being done. Thus, a didactic paradigm DPI(F) aims for being a description of the shared didactic ends, didactic means and epistemological model concerning the study of F in I. However, this is too ambitious and, in practice, instead of aspiring to describe all the shared elements, a didactic paradigm describes just some features, giving rise to a partial didactic paradigm. Hence, there might be several didactic paradigms modelling the study of F in I.

When we use a didactic paradigm to describe the current modality of study, we speak of current didactic paradigm, and we write $CDP_I(F)$; similarly, if we are describing a reference modality of study (i.e., a conceivable modality of study alternative to the current one), we speak of reference didactic paradigm, and we write $RDP_I(F)$.

We will see that to describe the dynamics of a $DP_I(F)$ it is necessary to use the notion of didactic phenomena, $D\varphi_I(F)$. We cannot say what a didactic phenomenon is[2]. We will simply say that a didactic phenomenon (or study phenomenon) is expressed through a series of features of a modality of study of F in I, modelled by a certain paradigm DPI(F). It manifests through a set of events or processes related to study that are typifiable, regularly recurring under certain circumstances and across different institutions, and are surprising (from a certain perspective), thus requiring an explanation. The events related to the study involve relationships between the different aspects of the modality of study, which translate into relationships between the components of the $DP_I(F)$.

Didactic phenomena should not be confused with praxeological features (features of the epistemological model) that suggest or indicate their existence. For example, interpreting

[1] By 'constitutive' rules we mean they are not merely rules observed within the institution; rather, the institution is fundamentally made up of these rules (i.e., the rules define the institution).

[2] It is not possible to give a definition of a didactic phenomenon, just as it is not possible to define a physical phenomenon.

proportionality as a purely arithmetic relationship between numbers, separating it from other functional relationships (instead of considering it as functional relationship between quantities of magnitude that requires the systematic use of algebraic tools), is a praxeological feature that is surprising from the perspective of the ATD and suggests the existence of didactic phenomena. Among these phenomena, we can mention the following ones: confusion in interpreting language related to ratios, proportions, means, extremes, the rule of three, etc.; systematic technical mistakes in the use of these concepts; the identification of "direct proportionality" with "increasing function"; and, due to the absence of algebraic tools, the difficulties experienced by the school system (and therefore by students and teachers) in interpreting the relationships between "direct proportionality", "inverse proportionality" and "compound proportionality".

As explained in Gascón & Nicolás, (2021a, 2021b), frequently in research we use reference didactic paradigms to describe current didactic paradigm as follows. First, we wonder how to avoid a certain didactic phenomenon3, $CD\varphi_I(F)$, we vaguely detect in the current modality of study. In order to describe this current didactic phenomenon, $CD\varphi_I(F)$, with more precision, we need to specify the current didactic paradigm, $CDP_I(F)$. This, in turn, is done by having in mind an alternative modality of study, described by a reference didactic paradigm, $RDP_I(F)$, which has among its didactic ends the avoidance of $CD\varphi_I(F)$. The constructions of $CDP_I(F)$ and $RDP_I(F)$ are done almost simultaneously, emphasizing their differences. A reference didactic paradigm, $RDP_I(F)$, can be regarded to be an hypothesis as follows: a modality of study ruled by the $RDM_I(F)$ and the $REM_I(F)$ would achieve the didactic ends RDEI(F) and, in particular, would avoid the didactic phenomenon $CD\varphi_I(F)$ which takes place in the current modality of study described by $CDP_I(F)$.

Of course, one can think of the possibility of providing an interpretation of the current didactic paradigm without having explicitly an alternative, a reference didactic paradigm. However, typically, our interpretation of something (a modality of study, in this case) do not rely on obvious and necessary things, but rather are done by pointing out those features that could be differently. For example, when describing a function, we do not point out that it has an input and an output, but rather specify properties such as continuity, differentiability, periodicity or other things that could be differently. In other words, our descriptions seem to be done by keeping in mind, as a reference, a possible alternative reality. This is the role played by the RDP with respect to the CDP.

Since institutions can change a lot along time, in the specification of a didactic institution one has to fix a certain time interval. Indeed, for instance, the didactic institution of University Education in the 12th century in Bologna was very different from the present one. Therefore, it might happen that the current didactic paradigm $CDP_I(F)$ of an institution I was previously a reference didactic paradigm for this very institution I in former times, alternative to the current one at that time. But then, the

[3] The didactic phenomena we study are not always necessarily 'undesirable'. However, our decision to study a phenomenon is motivated, first of all, by the need to explain it, which already indicates that a phenomenon always contrasts with what would be expected from the perspective of our postulates. Secondly, we typically choose to study a didactic phenomenon because we aim to act on the system in which it appears, intending to modify it in some way. Therefore, we tend to choose phenomena that, from our perspective, generate undesirable consequences.

current didactic paradigm $CDP_I(F)$ aimed at that time to avoid certain didactic phenomena that could be described with a certain didactic paradigm current at that time. In this situation, and speaking from present times, we say that the didactic paradigm current at former times is the antecedent didactic paradigm, $ADP_I(F)$, with respect to the current one, $CDP_I(F)$. Hence, we have three didactic paradigms: $ADP_I(F)$, $CDP_I(F)$ and $RDP_I(F)$: $RDP_I(F)$ tries to avoid a certain didactic phenomenon $CD\varphi_I(F)$ which takes place in $CDP_I(F)$, and $CDP_I(F)$ tried in the past to avoid certain didactic phenomena $AD\varphi_I(F)$ taking place in $ADP_I(F)$.

Methodology: didactic analysis as diachronic analysis

With the idea of these three didactic paradigms, the antecedent ADP, the current CDP, and the reference RDP, in mind, we see that the *interpretative analysis* of the current didactic paradigm CDP, can be diachronically complemented by considering $ADP_I(F)$, which explains the origin of $CDP_I(F)$, and $RDP_I(F)$, which points out to a possible change. The links between these three paradigms are the didactic phenomena to be avoided: $AD\varphi_I(F)$ is what $CDP_I(F)$ tries to avoid, and $CD\varphi_I(F)$ is what $RDP_I(F)$ tries to avoid.

Let us show some questions connected to the three different types of analysis:

Interpretative analysis: It consists in providing an interpretation of the $CDP_I(F)$. As said before, this is typically done by comparing it with a reference didactic paradigm, $RDP_I(F)$. In our description of $CDP_I(F)$, we aim to represent the $CEM_I(F)$, the $CDE_I(F)$ and the $CDM_I(F)$, and this would entail to answer the following questions: 'how is the $CEM_I(F)$?', 'what possible elements of F are included in the study of F in I?', 'how is the $CEM_I(F)$ different from the $REM_I(F)$, not only concerning the knowledge involved itself, but also concerning the link of this very knowledge with other fields of mathematics?', 'which are the $CDE_I(F)$?', 'how are the $CDE_I(F)$ different from $RDE_I(F)$?', 'which are the $CDM_I(F)$?', 'are these $CDM_I(F)$ really suitable to reach the $CDE_I(F)$?', 'if these $CDM_I(F)$ are not useful in order to reach the $CDE_I(F)$, how should them be changed?', 'what are the didactic phenomena, $CD\varphi_I(F)$, revealed by $RDP_I(F)$?'

Forwards analysis: It essentially consists in constructing an alternative didactic paradigm, $RDP_I(F)$, and study the conditions that would make possible to go from $CDP_I(F)$ to $RDP_I(F)$. Among other things, the interpretation of $RDP_I(F)$ should show that a modality of study ruled by $RDP_I(F)$ would avoid the $CD\varphi_I(F)$. As said when explaining the interpretative analysis, the construction of $RDP_I(F)$ is done in parallel with the interpretation of $CDP_I(F)$. Thus, some parts of the interpretative and the forwards analysis are done simultaneously. But there is a part of the forwards analysis that can be done before the interpretative analysis. Namely, the study of the features of the institution that would hinder the implementation of $RDP_I(F)$, and how would affect the institution. Those features can appear at different levels of the scale of didactic codetermination (Chevallard, 2002).

Backwards analysis: It essentially consists in explaining the origin of $CDP_I(F)$, and thus, it is connected to the following questions: 'Can $CDP_I(F)$ be regarded as a reaction to a previous paradigm, $ADP_I(F)$?', 'How was this $ADP_I(F)$?', 'How are the didactic phenomena, $AD\varphi_I(F)$ that $CDP_I(F)$ tries to avoid?', 'How to describe $AD\varphi_I(F)$ in terms of $CDP_I(F)$?', 'Are $AD\varphi_I(F)$ really avoided by $CDP_I(F)$?', 'Why are $AD\varphi_I(F)$ non-desirable by the didactic institution I in current times?'

In summary, the didactic analysis of a current modality of study of a certain field of knowledge at a certain institution entails, along with a interpretative analysis, the analysis of a hypothetical past and a possible future, which help to *interpret* the origin and a possible development of $CDP_I(F)$.

Sketch of a diachronic analysis of the current modality of study of the real numbers in the research of mathematics

We will briefly consider the case $CDP_{ResMath}(\mathbb{R})$, where the field of knowledge is that of real numbers, $F=\mathbb{R}$, and the didactic institution is the one formed by Research in Mathematics (included the Degree in Mathematics, regarded as a tool to educate possibly future mathematicians), written I=ResMath. Of course, space restrictions will only allow a sketch of what would be a complete didactic analysis of $CDP_{ResMath}(\mathbb{R})$. In this sketch the emphasis is made on the epistemological model, which is typically the entrance door for the didactic analysis of a modality of study.

Let us first show some features of the antecedent didactic paradigm, $ADP_{ResMath}(\mathbb{R})$, mainly concerning the antecedent epistemological model $AEM_{ResMath}(\mathbb{R})$. We could say that irrational numbers were used, until well into the 19th century, as a *paramathematical* object, in the sense of Chevallard (1985). Indeed, the institution ResMath was happy with a merely practical use of the idea of irrational number, without going deep into the meaning (Edwards, 1979). The use of $\mathbb{R}$ was based on geometric intuitions concerning the magnitudes length and area. However, a worrying didactic phenomenon, $AD\varphi_{ResMath}(\mathbb{R})$, happened. Indeed, the appearance of mistakes and paradoxes, the difficulty in proving some elementary results of mathematical analysis, and the loss of the status of truth of geometry (due to the discovery of non-euclidean geometries), led ResMath to look for a more solid foundation of $\mathbb{R}$. This is how the so-called *arithmetization of analysis* started (Kline, 1990).

Let us illustrate the kind of problems connected to $AD\varphi_{ResMath}(\mathbb{R})$ by showing two mistakes that took place during the 19th century. For much of the 19th century, it was wrongly accepted in ResMath that every continuous function was differentiable at every point, except for, perhaps, at some isolated points. Some books even pretended to prove this fact. In this context, the example presented by Weierstrass in 1861 of a function that is continuous on the whole real line but not differentiable at any point was a great surprise. Apparently, already in 1834 Bolzano showed an example of such a function, but it was overlooked (Edwards, 1979). On the other hand, in a paper published in 1869, Charles Méray pointed out a circular reasoning accepted in ResMath since the time of Cauchy: the limit of a sequence was defined as a certain real number, but, at the same time, a real number was defined to be the limit of a certain sequence of rational numbers (Boyer & Merzbach, 2011).

To avoid the phenomenon $AD\varphi_{ResMath}(\mathbb{R})$ that was taking place in $ADP_{ResMath}(\mathbb{R})$, ResMath constructed the current didactic paradigm, $CDP_{ResMath}(\mathbb{R})$, in which the process of arithmetization of analysis was carried out. The culmination of this was in the year 1872, with the simultaneous publication of four constructions of real numbers in purely arithmetic terms by Dedekind, Cantor, Méray and Heine (Edwards, 1979). These constructions are part of the $CEM_{ResMath}(\mathbb{R})$.

Among the constructions of $\mathbb{R}$, the most extended ones are the Dedekind cuts (due to Dedekind) and the equivalence classes of Cauchy sequences of rational numbers (due to Cantor). As can be checked by consulting the standard bibliography of elementary courses in mathematical analysis in the Degree

in Mathematics (e.g. Apostol, 1991; Rudin, 1976; Spivak, 2008), the introduction of real numbers is as follows: (1) one constructs the field $\mathbb{Q}$ from the set of natural numbers; (2) one proves that $\mathbb{Q}$ is archimedian but not complete (as there are non-empty and upper bounded subsets of $\mathbb{Q}$ with no supremum); (3) one constructs (e.g., by using Dedekind cuts) an extension of $\mathbb{Q}$, called $\mathbb{R}$, which is an archimedian and complete field; (4) one proves that this extension is unique up to isomorphisms. It is also important to mention that the operations in $\mathbb{R}$ are based on those of $\mathbb{Q}$, and, in turn, the operations in $\mathbb{Q}$ are not shown to be as they are due to arguments using magnitudes, but rather they are defined axiomatically. Axiomatic definitions and avoidance of non-deductive arguments are distinctive features of the current epistemological model, $CEM_{ResMath}(Math)$, in ResMath not only concerning $\mathbb{R}$, but the whole discipline of mathematics, Math. This shows that sometimes (we would even say "frequently"), in order to understand a current didactic paradigm for the study of a certain field of knowledge, one has to pay attention to didactic paradigms at higher levels of the scale of didactic codetermination, see (Chevallard, 2002). Hence, in this case, some features we encounter in our analysis of $CDP_{ResMath}(\mathbb{R})$, which is at a sub-disciplinary level, are consequences of some features living at $CDP_{ResMath}(Math)$, which is at a disciplinary level.

The current didactic paradigm, $CDP_{ResMath}(\mathbb{R})$, has among its didactic ends the avoidance of $AD\varphi_{ResMath}(\mathbb{R})$, and this is done. However, a new didactic phenomenon, $CD\varphi_{ResMath}(\mathbb{R})$, appears: lack in the current epistemological model, $CEM_{ResMath}(\mathbb{R})$, of a *raison d'être* for the apparently artificial constructions of $(\mathbb{R}, +, \cdot, \leq)$, with the unanswered question of why decimal expressions are not enough; disappearance of the working with magnitudes in the current epistemological model, $CEM_{ResMath}(\mathbb{R})$; and authoritarian didactic means $CDM_{ResMath}(\mathbb{R})$.

We wonder, as researchers in didactics, how to avoid this didactic phenomenon, $CD\varphi_{ResMath}(\mathbb{R})$, how to construct a reference didactic paradigm, $RDP_{ResMath}(\mathbb{R})$, having among its didactic ends that of avoiding $CD\varphi_{ResMath}(\mathbb{R})$. The idea is that it should be coherent with the idea of mathematics regarded as a modelling activity (Gascón & Nicolás, 2021a), which would allow to connect $\mathbb{R}$ with magnitudes. The $RDP_{ResMath}(\mathbb{R})$ is still under construction, but we already have constructed the reference epistemological model $REM_{ResMath}(\mathbb{R})$. The idea is to understand real numbers as general decimal numbers (finite or infinite, periodic or non-periodic). Positive real numbers can represent measurements of certain objects with respect to specific magnitudes (such as length, area, etc.). These objects may be physical (like a rope) or ideal (like a geometric entity, such as the diagonal of a square). The methods for operating with positive rational decimal numbers, as well as their properties, can be easily explained in terms of measuring magnitudes. On the other hand, one can prove that real numbers (considered as decimal numbers), together with lexicographical order, satisfy the least upper bound property. Using this property, addition (and similarly, multiplication) of two positive real numbers is defined as the supremum of the sums (or products) of the positive rational numbers obtained by truncating the original real numbers. These operations turn out to satisfy the desired properties. When these operations are extended to the entire $\mathbb{R}$ while preserving these properties, the sign rules can be deduced, resulting in a totally ordered complete field (Espín et al., 2024).

This $REM_{ResMath}(\mathbb{R})$ seems to avoid $CD\varphi_{ResMath}(\mathbb{R})$ since: (1) it bases the operations of $\mathbb{R}$ in those of $\mathbb{Q}$ which, in turned, are based in magnitudes; (2) it avoids the artificial construction of $\mathbb{R}$ (for instance, in term of Dedekind cuts on the rational numbers), as it identifies real numbers with certain decimal

expressions. Thus, this $REM_{ResMath}(\mathbb{R})$ seems suitable for a $RDP_{ResMath}(\mathbb{R})$ having among the didactic ends $RDE_{ResMath}(\mathbb{R})$ that of avoiding $CD\varphi_{ResMath}(\mathbb{R})$. Notice that, a priori, many different types of didactic means, $RDM_{ResMath}(\mathbb{R})$, are compatible with this epistemological model $REM_{ResMath}(\mathbb{R})$. Which is the most suitable would depend on the whole set of didactic ends $RDE_{ResMath}(\mathbb{R})$.

It is also worth mentioning that the modality of study ruled by this new paradigm could give rise to new unexpected didactic phenomena, hard to predict now.

Conclusion and open problems

The diachronic analysis of the current modality of the study of a certain field of knowledge F at a didactic institution I not only describe this modality of study, and proposes a change of paradigm, but also explains why the current modality of study is as it is. We have illustrated this with the case of the study of real numbers in the institution of Research in Mathematics. Thus, we have explained why certain apparently artificial constructions of $\mathbb{R}$ are still taught, and why magnitudes are missing, and we have sketched a paradigm to support an alternative modality of study of $\mathbb{R}$.

If we interpret the diachronic analysis as a procedure to investigate modalities of study, a huge range of research problems opens up before us. Let us propose just three examples connected with the investigation of antecedent didactic paradigms that could explain certain didactic phenomena:

- Concerning the current modality of study of *proportionality* in Secondary Education, why is proportionality interpreted as a purely arithmetical relationship between numbers, segregating it from the rest of the functional relationships, instead of considering it as a functional relationship between quantities of magnitude whose study requires the systematic use of algebraic instruments?
- Concerning the modality of study of analytic geometry in Secondary Education, why does it concentrate on the resolution of quasi-algorithmic problems whose data are given in a transparent and unmodifiable reference system, instead of connecting it with the problem of synthetic geometry and aiming at the study of geometric systems expressed with the use of parameters?
- Concerning the modality of study of *differential calculus* in the step from Secondary to Tertiary Education, why does it prioritize technical work aimed at studying the characteristics of a few types of functions, instead of proposing differential calculus as a tool for building functional models of systems of all kinds, working on them, and interpreting the results obtained in terms of the modelled system?

The answer to these questions, and many others of the same type, would require constructing in each case a hypothetical antecedent didactic paradigm to be compare with the current didactic paradigm and to get an explanation for those phenomena.

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Crisis-driven transformation in tertiary mathematics – A case study connecting TTE and ATD

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This paper examines the potential for connecting the Theory of Transformative Education (TTE) and the Anthropological Theory of the Didactic (ATD), with a case study serving as the basis for this investigation. On the one hand, using TTE an individual perspective of a mathematics student in her inaugural semester is taken, whereby specific moments of a crisis and a reaction can be worked out on an individual and socio-cultural level. On the other hand, the mathematical content dimension of the crisis based in the courses Analysis I and Linear Algebra I is analysed using ATD. Based on the findings, assumptions can be made as to why the student fails the crisis and does not undergo a mathematical enculturation.

Keywords: Enculturation, Theory of transformative education, Anthropological theory of the didactic, Networking of theories, Biographical-narrative interviews.

Introduction

As part of a research project at University of Hannover, the socio-cultural transition from school to university mathematics was analysed from the perspective of the Theory of Transformative Education (TTE; Günther & Hochmuth, 2023; Günther, i.p.; Koller, 2023). Here, the transition is understood as a transformation of a culturally shaped relation of an individual to the mathematical world, to others, and to oneself. Individual crisis in the confrontation with the unfamiliar university mathematics culture can override the routines of action familiar to first-year students from school mathematics and can lead to a resignification of experienced explicit, implicit and social school or university mathematics knowledge. In TTE, such crisis are described by Lyotard's discourse theory (1989), with which the experience of crisis can be summarised as a conflict of incommensurable discourses. Using TTE, the mathematical world can be grasped in significant subject-related moments. However, the addressed mathematical actions and knowledge elements remain abstract. This is not a deficit of this specific approach but a general characteristic of educational theories. To further investigate these actions and knowledge, we draw on the Anthropological Theory of the Didactic (ATD) and its notions when concretising relevant aspects of the mathematical world from the perspective of the subjects. In our opinion, this is fruitful because, among other things, ATD and TTE adopt a dialectical view on the relationship between the individual and the institutional-social. We will demonstrate the fruitfulness of a methodological connection between the two approaches using a case study based on an interview excerpt.

In the following, we will briefly introduce our vantage point, TTE and its perspective on university mathematics. A case study that has raised an interesting phenomenon in the context of this theory is then presented and analysed within TTE. This is combined with an ATD analysis of the corresponding excerpt from the teaching materials from the Analysis I lecture. Finally, we will conclude with a discussion on the connections between the two perspectives in terms of an initial local integration (Prediger, 2008).

University mathematics in the perspective of TTE

Based on a range of theoretical perspectives, TTE is an approach widely used in the German educational sciences to describe education (as in the German notion of *Bildung*) as a transformation of a culturally shaped relation of an individual to the world, to others, and to oneself in that world (Koller, 2023). From that perspective and in alignment with a socio-cultural understanding of mathematics (e.g. Prediger, 2004) mathematics is to be understood as a socio-cultural space that mediates dispositions of perception, thought and action required for interactions (Bourdieu, 1985). The subject (e.g. a student) is constituted in this social context of mathematics by replicating the dispositions and hierarchy of social power within interactions (Butler, 2001), thereby acquiring a position in the social space of mathematics relative to others (e.g. peers, lecturers). This positioning determines the subject's possibilities and limits of perception, thought and action (Oevermann, 1991). The constitution of the subject is thereby always contingent upon interactions with others who are part of the mathematical world. The subject's relation to themselves and its dynamic and structuring is on the one hand dependent on the relationship to others, on the other hand marked by a fundamental misrecognition and deception of this dependency (Lacan, 1975). It can be argued that dependence on others must be denied just as much as the difference between oneself and others. With regard to mathematics, this is evident in the constant feedback loops between the subject and others that take place in the various formal and informal interaction spaces and in which each member of the social space must constantly reposition themselves in relation to others and the world (Günther, i.p.). The resulting "condition of possibility for the questioning and shifting of established meanings as well as for the emergence of new meanings" (Koller, 2023, p. 54) opens the possibility of transformation.

Accordingly, a certain mathematical world – it being school, university or academic mathematics – conveys specific mathematical dispositions in the form of explicit, implicit and social mathematical knowledge, and thus can be understood as representing a specific socio-cultural space. The transition between two such mathematical worlds, e.g. from secondary to tertiary mathematics, can be described as an intercultural, processual transformation. But also transitions within such a world, e.g. between different courses or areas of content, can require the adaptation to different cultural knowledge (Hochmuth et al., 2021). The subject must either discard or alter their established dispositions or evolve new ones to develop a system of dispositions that is appropriate to the specific mathematical context and, thus, to become integrated into a new socio-cultural space of mathematics. If this process is successful, we may describe the subject as having been enculturated (Bishop, 1988) into a specific culture of mathematics. This transformation process is initiated by crises (Clark & Lovric, 2008). They emerge through confrontation with problematic situations in which the established mathematical dispositions are overridden and new/other dispositions are required for further actions. Since the subject's engagement with mathematics cannot be fully determined, the emergence of crises is individual and emergent but also unpredictable and uncontrollable from a pedagogical perspective.

TTE allows for various ways of describing crises as triggers of transformations (Koller, 2023). Günther and Hochmuth (2023) used the notion of experiences of foreignness (Waldenfels, 1997), whereby possible reactions of students were described as rejecting, assimilating or (re)productive. Expanding on this, a discourse-analytical perspective will be adopted here. According to Lyotard (1989), "discourses" (p. 9 & 59) are systems of rules governing the combination of sentences,

each serving a specific purpose and involving referent, sense, receiver, and sender. When discourses with distinct sentences and rules interact, they cannot be combined, compared, or reconciled. They are incommensurable, leading to inevitable conflicts, so-called "differend[s]" (p. 9). Lyotard distinguishes those from "litigation" (p. 9), conflicts that can be resolved within the same discourse. He emphasises the importance of keeping differends open and searching for new idioms of expression to avoid silencing the affected party (pp. 32–36). Within TTE crises can thereby be understood as differends of incommensurable discourses which require a productive reaction in the form of a search for new idioms.

Based on this understanding, establishing new mathematical knowledge can be considered a process with three phases: (i) The experience of a crisis initiates a protracted phase of engagement with the differend and a gradual search for new ways of acting and thinking. (ii) This results in the abrupt manifestation of mathematical dispositions in the form of potential actions within the social domain of mathematics that have not yet been actualised. (iii) This is followed by a longer phase of further development and adaptation towards new dispositions of the self-other-world relation. Thus, new mathematical knowledge arises from the divergences between the discourses and can be comprehended as resignification of cultural possibilities for action in mathematics. The advent of new self-other-world relations can be understood as a culmination of social constituted signifying practices that reiterate and adapt established relations in unfamiliar contexts.

Following, we present a case study in which the TTE perspective is used to analyse a problem situation of a student in the first semester of tertiary mathematics.

The case study Johanna

In a series of interviews conducted with first-semester mathematics students at the University of Hannover, the student Johanna (pseudonym) reports on a crisis that was analysed from the perspective of TTE. The following section is part of a longer discussion of the two courses, Analysis I and Linear Algebra I. In brief summary of the interviews, Johanna largely identifies with the discourse of logical and deductive development of theory in Analysis I, whereas she describes the discourse of Linear Algebra I as chaotic and less accessible. Furthermore, Johanna states that both lecturers in Linear Algebra I and Analysis I made references to each other's subjects, which resulted in confusion and a sense of crisis for her (Günther, i.p.).

> Joh.: So maths is somehow still a lot like that, this yes, and then they also do it in a similar way in Analysis, and they do it in a similar way in linear Algebra, and it somehow blurs like that and then somehow not really. And I also noticed that the other day in calculus our professor said, yes, you work with matrices in linear algebra. Why don't you tell me what- what- uh linear function translates to a uh matrix. I look at him like this. What are you talking about, what translates to where? So that has - oh, that has something to do with it? and that's just this – I've also realised a bit... In maths at the moment I'm not looking at how much I enjoy this subject, but how good - how good is the professor (translated by the authors)

In this segment, Johanna refers to what she sees as the two incommensurable discourses in the Analysis I and Linear Algebra I courses, as she has already done in previous sections of the interview series. The professor's attempts to establish links between the subject matters of Analysis I and Linear Algebra I, especially regarding the use of matrices, are met with a lack of understanding by Johanna,

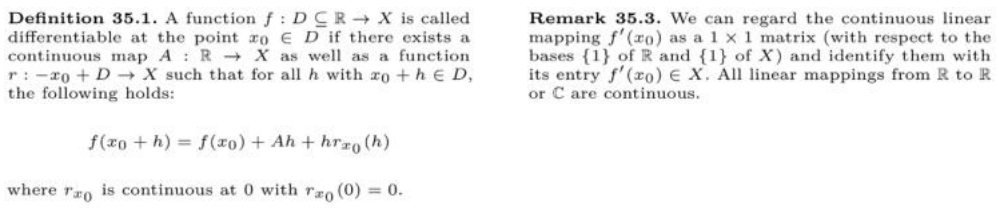
as she makes clear by retrospectively asking the lecturer questions in direct speech. From the perspective of TTE and taking other interview sections into account, the lecturer's attempt to establish a link to Linear Algebra I can be interpreted as an idiom for expressing and advancing the argumentation of Analysis I with the help of Linear Algebra theorems, but also – from a cross-curricular and educational perspective – to illustrate larger connections between the different discourses of mathematics. However, for Johanna, the idiom employed by the professor is unable to establish a connection between her receptions of the discourses of Analysis I and Linear Algebra I, leaving them incommensurable which is made clear by the crisis developing here. At the same time, the link presented is provided with a claim of truth through the representation of the mathematical world by the lecturer. Nevertheless, Johanna's reaction can be interpreted as a transformation of the resulting differend into a litigation in favour of her own perception of the two discourses. Given Johanna's disposition of self-assured demeanour as evidenced in various segments of her interviews, this inability to connect the discourses is attributed by her to the quality of the teaching, rather than to a lack of interest, understanding or enjoyment on her part. The lecturer in particular is to blame for not addressing Johanna's interest or understanding during the discussion of content (cf. Günther, i.p.).

The concise presentation of the TTE analysis illustrates Johanna's perspective in relation to the depicted situation, whereby crises and transformations at specific points in her engagement with mathematics can be elucidated. In the following section, we will examine the content-related aspects of the Analysis I and Linear Algebra I courses that Johanna refers to in the aforementioned segment and led to her crisis, based on the materials presented in the lectures and the perspective of ATD.

The case in the perspective of ATD

ATD is a research programme (Chevallard, 2019) that allows mathematical practices to be analysed from two intertwined perspectives (Winsløw et al., 2014): *The institutional dependence of mathematical practice* states that institutional conditions influence which mathematical actions and which associated mathematical reasoning discourses are appropriate. This allows an external perspective that focuses on those conditions, that can vary across institutions. The internal perspective focuses on concrete mathematical subject matter, which is modelled as *praxeologies* that depend on the respective institutions. A praxeology consists of two interrelated components: The *praxis component* and the *logos component*. Praxis is made of *types of tasks T* and of *techniques* τ, that is, ways of performing the tasks. Logos contains two levels of justification discourse. At the first level *technology* θ describes, justifies, explains, etc. the technique. At the second level, *theory* Φ organises, supports and explains technology. In ATD, institutional conditions are differentiated in relation to certain social levels and modelled with the levels of codetermination (Artigue & Winsløw, 2010). The social mediation of mathematical actions is thus already inherent in ATD but is not further differentiated with regard to the specific organisation of this mediation. The two intertwined perspectives on mathematical practice make a connection between ATD and TTE seem fruitful to us. In the following, we illustrate connections between ATD and TTE using the case of Johanna mentioned above. To illustrate the ATD perspective and for examples of praxeological analyses, we refer to the work of Winsløw et al. (2014). They describe a general praxeological model of the transition from school mathematics to university mathematics and then further within university mathematics. When students transition from school mathematics to university mathematics, the

praxeologies of the student acquired at school must change to become consistent with university mathematics praxeologies. An important part of the Winsløw et al. (2014) model for our considerations here is the new meaning of the logos component of praxeologies for mathematical practices at university. In university mathematics the logos component of mathematical praxeologies must be (further) developed by the student. It can't be compared with school, where it is essentially the teacher who connects mathematical notions through logos aspects. The second transition takes part, when in more advanced courses, former logos elements become new objects of students' praxis.



Definition 35.1. A function $f : D \subseteq \mathbb{R} \to X$ is called differentiable at the point $x_0 \in D$ if there exists a continuous map $A : \mathbb{R} \to X$ as well as a function $r : -x_0 + D \to X$ such that for all h with $x_0 + h \in D$, the following holds:

$$f(x_0 + h) = f(x_0) + Ah + hr_{x_0}(h)$$

where r_{x_0} is continuous at 0 with $r_{x_0}(0) = 0$.

Remark 35.3. We can regard the continuous linear mapping $f'(x_0)$ as a 1×1 matrix (with respect to the bases $\{1\}$ of $\mathbb{R}$ and $\{1\}$ of X) and identify them with its entry $f'(x_0) \in X$. All linear mappings from $\mathbb{R}$ to $\mathbb{R}$ or $\mathbb{C}$ are continuous.

Figure 1: Definition and remark from the Analysis I lecture notes (translated by the authors)

The situation addressed by Johanna in the interview relates to the definition of differentiability formulated in the Analysis I course attended (Figure 1). The version favoured for school mathematics, i. e. the differential quotient, is not used here. In the definition (see Definition 35.1. in Figure 1), the limit value is hidden in the continuity condition for the function r. The expression for the differential quotient can be obtained by dividing the equation by h. In relation to Linear Algebra, the following special features can be identified: (1) The definition speaks of a continuous linear mapping A, which is denoted in the equation by Ah and not by $A(h)$. The expression recalls that a linear mapping from $\mathbb{R}$ to $\mathbb{R}$ can be represented by a real number (in the complex case, i.e. $X = \mathbb{C}$, by a complex number), whereby in $\mathbb{R}$ the canonical basis with basis vector 1 is taken as a basis (technique). (2) Within the standard discourse at school, this would be seen as an inadequate technique. On the one hand, it would be argued that a clear distinction should be made between mappings and numbers, which can also be seen as a permissible criticism of the chosen formulation in university discourse. (3) On the other hand, this was clarified in the lecture (as mentioned by Johanna) by referring to the connection between matrices and linear mappings in Linear Algebra. A corresponding reference can be found on the same page as the definition of differentiability in the Analysis I script (see Remark 35.3 in Figure 1). We will now reflect on the connection assumed in the Analysis I script and presumably explicated in the lecture with the knowledge covered in Linear Algebra I: (1) There, as usual, the relation between matrices and linear mappings is treated in detail. However, no special reference is made to the 1-dimensional case, which is the case here. It may therefore come as a surprise that a number is written as a 1×1 matrix, albeit without matrix brackets, and is therefore interpreted as a linear mapping. (2) Fundamentally, the question of the underlying basis, regarding which the number represents a linear mapping, should also be discussed.

To summarise, it can first be stated that the reference to Linear Algebra I addressed here is not immediately clear and requires independent reflection on content and notation, which can be located in the logos component of mathematical praxeologies. Within the school discourse of mathematics, it is usually the responsibility of the teacher to foster connections within logos components of praxeologies. If such connections are missing, one would speak of a teaching deficit in the school

discourse in view of the hurdle and the effort required to overcome it. In the university mathematics discourse, it represents an (intended) learning opportunity. Fostering such connections becomes a responsibility of the student at university, therefore former logos aspects transition to the praxis component within the university mathematics discourse. With her criticism, Johanna positions herself within the school discourse.

A complementary perspective on Johanna

The helplessness and disorientation expressed by Johanna indicates that she does not see dealing with the discursive difference as a responsibility of herself. From an ATD perspective it is not possible to determine whether she is unable to recognise the gap as a responsibility and therefore does not mobilise any techniques to deal with it, or whether she does not have the techniques at her disposal. Looking back on the TTE analysis of the interviews, Johanna would – in principle – have had the opportunity to elucidate her problem in informal discussions with her fellow students, her tutors or the lecturer. The fact that she does not engage in this approach but instead adheres to the school discourse is a sign of failed or yet not experienced enculturation into university mathematics. This prompts the question of why Johanna possibly failed to capitalise on the opportunity, which, beyond what has already been concluded, is no longer the primary focus of the ATD. With TTE, however, it becomes clear that Johanna's actions throughout the semester are characterised by a distinct self-imposed requirement to work independently. She defines her success at school and subsequently in university mathematics by establishing a clear distinction between herself and her fellow students. She attempts to elevate her positioning within the social space of university mathematics through her achievements. The TTE perspective also indicates that the actions undertaken by Johanna are based on a relatively stable and robust conceptualisation of the nature of mathematics and the characteristics of mathematical activity. This conceptualisation cannot be reconciled with the cultural circumstances of the university mathematics she encounters. The nature of these ideas indicates that they possibly serve as a stabilising force for Johanna. This might be interpreted within TTE in Lacanian terms, as within the hysterical discourse a subject "seeks to be determined by the other, turns to him, desires his desire in order to present himself as his object" (Widmer, 1990, p. 138). As a result, the subject is more concerned with the symbolic value of the object and will never be able to understand its very essence. This becomes apparent as the questioning of Johanna's own dispositions, and the necessary transformation and generation of new ones would be associated with a destabilisation of the established structure. For Johanna this would result in a more profound crisis based on her relation to the mathematical world and herself. Thus, a resignification of existing knowledge (in the sense of praxeological equipment, as shown in the following) cannot take place in this way. For this, in the sense of the first part of the transformation process, the problem would have to be worked through slowly, i.e. in a time-consuming manner, which is not encouraged at school. However, in school it is considered successful if it is done quickly and teaching is considered good if this is prepared accordingly. E.g. the technology is ready with the students and a problem does not have to be prepared technologically, which not only requires having technology, but also having to endure the fact that it is not clear a priori how and what (Winsløw et al. 2014, p. 100). In addition, the discussion and action on the technology level is a new disposition or was demonstrated at school at best by the teacher, less formally and, for example, based on basic ideas (Hochmuth et al., 2024). This also happens at

university. However, the didactic contract applicable to the course stipulates that students understand and critically reflect on this. Ideally, exercises should support this process. However, this would require freedom of studying and time. Recordings from the exercises at least suggest that they are well organised and that so-called face-to-face exercises, which are less technologically demanding, are worked on with the understandable aim of practicing techniques. This at least indicates that the transition to university mathematics not only creates something new but also necessarily resignifies previous knowledge, here in particular aspects of the school mathematics logos component.

Conclusion

We have taken a case study as the starting point for two different perspectives on a problem situation of Johanna, a first-semester mathematics student. By using TTE, it was possible to work out the specific point of a personal crisis in the interaction with mathematics in the extensive material of the interview series and to present Johanna's perspective as well as the socio-cultural context of this problem. The ATD analysis then focused the mathematical content in Johanna's statements in relation to the institutional context. This allowed us to elaborate Lyotard's abstract discourse-related characteristics. The typology of the discourses was then used to emphasise their emerging constellation, which could then be traced back to the mathematical content modelled using ATD. This tracing back and the interrelation of the elements thus revealed in the discourse goes beyond the scope of possible statements within the framework of ATD. On the other hand, the content and the specific institutional context of the discourse could not be reconstructed with TTE alone. Rather, the specific structuring of the discourse represents an institutional-societal possibility. The generalisability of our interpretation is also located at this level: it formulates conditions and restrictions that enable and determine the specific constitution of the mathematical content. TTE and ATD formulate statements at an institutional-social level, which, we can't work out in detail, for reasons of space alone, and appear compatible in this respect. We think that our results demonstrate the fruitfulness of connecting the two.

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Didactic transposition of structuralist praxeologies at the interface between analysis and topology: a case study

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Mathematical reasoning based on structures is important for the transition from beginner to more advanced mathematics: it reflects the epistemology of the body of mathematical knowledge in more advanced mathematics courses that progresses from local reasoning about particular instances of classes of objects to more global reasoning about underlying mathematical structures through historically endorsed abstraction. We examine this type of argumentation in the sense of "structuralist praxeologies" at the interface between analysis and topology, taking as an example the equivalence of the continuity of a real-valued function on the real line with the path-connectedness of its graph. In particular, we identify phenomena of external didactic transposition in the context of options for reasonings and their implementation.

Keywords*: Structuralist praxeologies, didactic transposition, analysis, topology, continuity.*

Introduction

A significant aspect of the transition between calculus and analysis or elementary and abstract algebra is a change in attention from the concrete (e.g., specific functions or arithmetic properties) to the abstract (e.g., theorems about function spaces or abstract rings) (Hausberger, 2018; Winsløw, 2008). A similar issue arises at the interface between analysis and topology. Here, undergraduate mathematics education may foster reasoning based on mathematical structures (e.g., topological properties of metric spaces) rather than merely concretising and thus redeveloping such reasoning locally each time (e.g., arguing for a specific metric space). Emphasising this transition from the concrete to the abstract may thus ultimately raise students' awareness for the relevance of structural reasoning, and this reasoning reflects the historical evolution of mathematical domains such as abstract algebra, real analysis, or topology.

In this paper, we continue our previous work in this direction (Hanke, 2018; Hausberger & Hochmuth, 2023). More concretely, we focus on the role of mathematical structures in *external didactic transposition* (EDT; Bosch et al., 2021): external didactic transposition refers to the process of selecting, adapting, organizing and declaring the knowledge to be taught—beginning from scholarly knowledge and ending with all the teaching materials that can be proposed in a given course. In this process, a series of textbooks, drawing on previous textbooks, can be used to analyse the EDT on specific topics. While Bosch et al. (2021) analysed EDT regarding the organisation of entire study programs in a cross-national comparison, we examine here more local phenomena based on an example at the interface between analysis and topology. Specifically, we analyse the presentation of the following equivalence in a textbook by Ross (2013, p. 182): *for an interval* $I \subseteq \mathbb{R}$*, a function* $f: I \to \mathbb{R}$ *is continuous if and only if its graph* $\Gamma_f = \{(x, f(x)) : x \in I\}$ *is path-connected.*

Occasionally, this equivalence is presented as an exercise in topology courses (for the case of $I = \mathbb{R}$ in Gathmann, 2023, pp. 25–26), and Hanke (2018) examined it in this context (thus considering the next step of *internal didactic transposition*, including the everyday teaching experience, which leads to the actually taught knowledge). However, this task could also be dealt with in analysis courses. It is thus a case for a topic at the interface between analysis and topology, which can arise at different moments in a study programme. As a result, several options for the presentation of the equivalence open up and textbook authors must make choices regarding the selection, adaptation, and organisation of elements of reasoning. To shed light on the role of mathematical structures in the choices made by Ross (2013), we will draw on the model of structuralist praxeologies examined by Hausberger and Hochmuth (2023) and analyse them against the background of further texts for analysis and topology as well as research literature. In doing so, we exemplarily answer the following research question: "Which phenomena in terms of structuralist praxeologies arise in the context of external didactic transposition considering thematical and organisational issues in study programmes (in our case, the interface between analysis and topology)?"

Theoretical framework and method

The *Anthropological Theory of the Didactic (ATD)* analyses relationships to an object of knowledge by praxeologies where the object intervenes at the level of at least one of the following components: "The anthropological theory of didactics considers that, *in the end*, all human activity consists of *accomplishing a task* t of a certain *type* T by means of a certain *technique* τ, *justified* by a *technology* θ which at the same time makes it possible *to think about* it, or even *to produce* it, and which is in turn *justifiable* by a *theory* Θ" (Chevallard, 2001, p. 3, own transl.).

In the context of abstract algebra, Hausberger (2018) introduced *structuralist praxeologies*: these are praxeologies $P_s = [T^g, \tau, \theta, \Theta_{\text{structure}}]$, where T^g is a generalisation of T in a not-yet-structural praxeology $P = [T, ?, ?, \Theta]$ (at least when compared with the structure for P_s), where it is unclear which technique to apply as we signify with ?. Here, structuralist refers to the use of structures in the sense of Bourbaki (e.g., mobilising results about general metric spaces rather than a specific metric space). In the type 1 transition from P to P_s, generally, the rationale of the praxeology P_s is still related to solving concrete tasks from P. We further distinguish between level 1 structuralist praxeologies, where structures mainly appear as vocabulary and through definitions, from *level 2* structuralist praxeologies, where a technique mobilises general results about the structures. A dominant level 1 type of task is "Show that X is Y (or has property p)", which can be achieved with the technique "Show that the definition of Y (or p) is fulfilled". A level 2 task may be to show a theorem about the technology used in a level 1 praxeology. In the teaching and learning of structures, a new stage is reached when praxeologies are based on purely abstract types of tasks that are anchored on reasoning with concepts originating in the logos of previously developed structuralist praxeologies P_s. This is called the type 2 transition, typically situated between the Bachelor's and Master's programmes.

The general view of structures thus allows to demonstrate properties of objects by making them appear as consequences of more general facts (theorems about structures). Conversely, generalisations are put to the test of objects, hence a *dialectical relationship between objects and structures*" (Hausberger & Hochmuth, 2023, p. 3). Through the interplay of didactic variables of tasks

(e.g., functions, domains and their properties) the dialectic of objects and structures is a dialectic of contextualisation and decontextualisation.

In the following, we use this praxeological model to analyse the choices made by Ross (2013) against the background of further literature from analysis and topology. In particular, we discuss options with respect to didactic variables (e.g., connectedness vs. path-connectedness, $\mathbb{R}^n$ vs. metric space, domains or codomains of functions) and reflect on reasons for choices regarding structuralist praxeologies in relation to aspects of the external didactic transposition.

Praxeological analysis of the example

To our knowledge, the equivalence is not widely presented in real analysis or topology textbooks. We will follow Ross (2013), van Rooij and Schikhof (1982), and a post from an online mathematics forum (JOJO, 2020). We also remark that characterising functions in terms of properties of their graph is a topic of still active research in topology and analysis (e.g., Wójcik, 2008), which considers many more properties than continuity and path-connectedness only.

Ross (2013) bases his proof on a characterisation of continuous functions by Burgess (1990) in a journal addressed to university teachers. In the latter, Burgess gives three properties characterising continuous real functions over $\mathbb{R}$, including the *connectedness* of the graph (*not* the path-connectedness) under the assumption that the graph is closed (see Figure 1). In the spirit of the dialectic of objects and structures, he shows that the general statement that connectedness of a graph implies continuity is not valid (1990, p. 339) by taking as a counterexample the function f defined on $\mathbb{R}$ by $f(x) = \sin x^{-1}$, $x \neq 0$, and $f(0) = 0$, which is discontinuous at 0 and has a connected graph. This motivates the additional hypothesis of *closedness* of the graph. Moreover, it is easy to prove that the graph of a continuous real function over $\mathbb{R}$ is closed. Interestingly, closedness is not necessary for the implication of the function's continuity under the hypothesis of path-connectedness. In fact, connectedness is a more general notion than path-connectedness (any path-connected set is connected but not the other way around) with better topological properties (e.g., the closure of a connected subset is still connected, unlike in the case of path-connectedness). This may explain why Burgess (1990), referring to a research paper by Moore (1920), does not consider path-connectedness, so that Ross (2013) had to adapt the proofs.

THEOREM. *If f: $\mathbb{R} \to \mathbb{R}$ has a closed graph in $\mathbb{R}^2$, then the following four conditions are equivalent*:

(a) *f is continuous.*
(b) *f is locally bounded.*
(c) *f has the intermediate value property.*
(d) *The graph of f in $\mathbb{R}^2$ is connected.*

Figure 1: The theorem from Burgess (1990, p. 338)

Ross uses vocabulary from topology (open, closed, compact, …), but the definition of a topological space and thus methods independent from the use of metrics are not developed. This corresponds to a certain learning stage in the analysis path, before topology is taught: multivariate analysis is based on the elaboration of $\mathbb{R}^n$ as a metric space that generalises the real line. Overall, Ross follows praxeologies from the context of metric space topology. This means, that open sets and continuity are characterised in terms of metrics, in particular, continuity is characterised by the well-known ε-δ- or

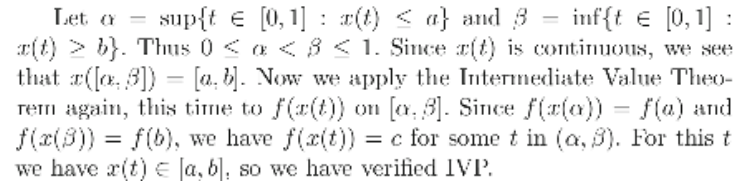
sequence-definitions. Connectedness is introduced in the last section of the chapter on continuity as an advanced topic (indicated by a star symbol) but it does not serve in any subsequent chapter.

The direct implication of the above equivalence points to an institutional type of task from a topology course T: *prove that a subset Y of a topological space X is path-connected*. Available techniques include τ_1: *show that the definition of path-connectedness* (the technology θ_1) *is fulfilled*, hence a level 1 structuralist praxeology from topology. Ross' (2013, p. 182) proof for this implication follows this line of argumentation: "If f is continuous, then for each $\big(a, f(a)\big)$ and $\big(b, f(b)\big)$ in $[\Gamma_f]$, $\gamma(t) = \big(t, f(t)\big)$ is a path connecting these points. Note that γ is continuous by Proposition 21.2". The latter proposition states that continuity of vector-valued functions is equivalent to the continuity of its component functions. It is based on properties of distances and the ε-δ-definition of continuity. On a higher level 2, structuralist theorems may be used, for example θ_2: *path-connectedness is preserved by continuous mappings*, in relation to the technique τ_2: *show that Y is the image of a path-connected set Z under a continuous mapping F*. With the above notations, we have $Y = \Gamma_f$. To apply the level 2 structuralist praxeology, supposedly the more powerful (economy of argumentation) and enlightening (mathematical understanding), the following values $F = \mathrm{id} \times f$ and $Z = I$ must be assigned. Another key element here is the technology θ_3: *intervals in* $\mathbb{R}$ *are the only path-connected subsets of the real line and vice versa*. A familiarity with product mappings and topologies is needed (a product of continuous mappings is continuous for the product topology), hence supplementary elements of logos from topology. It results from these two proofs that the task t is not a good example to underline the strength of the structuralist methodology: the first proof at level 1 is as economical in terms of length as the other, so that the gain of concepts and abstraction is not clear.

Burgess (1990) divides his proof of the theorem in Figure 1 into three lemmas, in particular (c) ⇒ (a) (lemma 2), and (d) ⇒ (c) (lemma 3), where the closedness of the graph of f is required in lemma 2 only (lemma 1 proves the implication (b) ⇒ (a)). The implication (a) ⇒ (d) (the graph of a continuous real function over $\mathbb{R}$ is connected) is not proved "since there are well-known easy proofs" (p. 338). Here, the author likely refers to the level 2 structuralist praxeology described above where path-connectedness is replaced by connectedness, which is equally preserved under continuous mappings. In the sequel, we continue examining lemmas 2 and 3 since our goal is to study structuralist arguments for (d) ⇒ (a) (a real function over $\mathbb{R}$ whose graph is connected and closed is continuous).

Lemma 3 is proved by showing first that $\Gamma_{f|[a,b]}$ (the graph of the restriction of the function to a compact interval) is connected if Γ_f is connected. However, Burgess (1990) does not give the details for the needed conclusion that f has the intermediate value property (IVP) afterwards. Probably, the previously mentioned level 2 structuralist praxeology is again left to the reader: by projecting to the second coordinate, we get that $\{(x, f(x)) : x \in [a, b]\}$ is also connected (the projection is continuous), hence an interval, so the IVP is fulfilled.

Ross partly follows this sequencing and begins his proof by showing that the path-connectedness of Γ_f implies that $\Gamma_{f|[a,b]} \subseteq \gamma([0,1])$, where $\gamma(t) = \big(x(t), y(t)\big)$ defines a path connecting $\big(a, f(a)\big)$ and $\big(b, f(b)\big)$. He then uses properties of a certain supremum and infimum and the continuity of x to show that $x([\alpha, \beta]) = [a, b]$ for some $\alpha < \beta$ in $[0,1]$ such that $x(\alpha) = a$ and $y(\beta) = b$ (see Figure 2). The application of the intermediate value theorem (IVT) to $y(t) = f\big(x(t)\big)$ yields the IVP.



Let $\alpha = \sup\{t \in [0,1] : x(t) \leq a\}$ and $\beta = \inf\{t \in [0,1] : x(t) \geq b\}$. Thus $0 \leq \alpha < \beta \leq 1$. Since $x(t)$ is continuous, we see that $x([\alpha,\beta]) = [a,b]$. Now we apply the Intermediate Value Theorem again, this time to $f(x(t))$ on $[\alpha,\beta]$. Since $f(x(\alpha)) = f(a)$ and $f(x(\beta)) = f(b)$, we have $f(x(t)) = c$ for some t in (α,β). For this t we have $x(t) \in [a,b]$, so we have verified IVP.

Figure 2: Ross' (2013, pp. 182–183) proof of the path-connectedness of a restriction of the graph[1]

In fact, an affine reparametrisation of $[\alpha,\beta]$ to match $[0,1]$ may be used to produce a path inside $\Gamma_{f|[a,b]}$, which shows that $\Gamma_{f|[a,b]}$ is path-connected. The structuralist input that path-connectedness of graphs of real functions is preserved by restriction of the function to a compact interval is interesting per itself (and equivalent to proving the IVP), but it is not emphasised since the primary goal is the IVP. Moreover, Ross could have used Burgess' proof since Γ_f is path-connected hence connected. It seems that Ross decided to totally avoid connectedness (for reasons see above) although the latter notion brings a simpler proof for the property that relates to restrictions of mappings.

In fact, Burgess explains that he introduced the IVP since he did not find such a characterisation of continuity in textbooks and "it is my main purpose in this note to present an elementary proof" (1990, p. 337). Complementarily, he stresses that, on a more abstract basis, the equivalence of continuity and connectedness and closedness of the graph for real functions "follows from the characterization of a topological line as a locally compact metric space that is separated into two connected sets by each of its points" (p. 337). We see these choices and comments as signs of the didactic transposition operated from the scholarly structuralist practice that serves as a reference, while elementarisation of proofs is a key driver for the didactic transposition. More precisely, Burgess points to a paper by Moore from 1920. Contemporary mathematical research (e.g., Wójcik, 2008) about the characterisation of continuity of functions between topological spaces in terms of properties of their graphs helps to clarify further the situation from a structuralist point of view: topologists define a *connectivity function* by the property that the connectedness of the graph of the function is preserved by restriction of the latter to any connected subdomain (Wójcik, 2008, p. 5). Burgess has then proved that a real function with a connected graph is a connectivity function. Moreover, it is known that a connectivity function $f: \mathbb{R} \to Y$ to any topological space Y has a locally connected graph at a point $\big(x, f(x)\big)$ if and only if it is continuous at x (Wójcik, 2008, Thm. 4). In other words, local connectedness of the graph rather than its path-connectedness enlightens the property of continuity whenever the domain is $\mathbb{R}$ but the codomain is generalised to any topological space.

Burgess' proof of lemma 2 is based on the negation of continuity at a point b to produce a sequence of points of the graph that converges to a point outside the graph (using the IVP) (see Figure 3). The same argument could be applied to the equivalence stated at the beginning of our paper, if we knew that Γ_f was closed. We can find such a proof in van Rooij and Schikhof's (1982) textbook: they provide a similar characterisation of the continuity of a real function by its graph, however, in the case of a compact domain $[a,b]$ (see Figure 4). As a matter of fact, in this case the path-connectedness

[1] In the argument the definition of β must be changed to $\beta = \inf\{t \in (\alpha,1] : x(t) \geq b\}$.

of the graph implies that it is compact. The key structuralist ingredient is that compactness is preserved under a continuous mapping. As a consequence, the graph is closed and we may conclude that f is continuous on any compact interval, hence globally continuous as well.

Proof. Suppose that f is not continuous at $b \in \mathbb{R}$. Then for some $\varepsilon > 0$, there is a sequence $\{b_n\}$ converging to b such that, for each n, $|f(b_n) - f(b)| > \varepsilon$. For convenience, we assume that each $f(b_n) > f(b) + \varepsilon$. The intermediate value property implies that, for each n, there is a number c_n between b_n and b such that $f(c_n) = f(b) + \varepsilon$. Thus the sequence of points $\{(c_n, f(c_n))\}$ converges to the point $(b, f(b) + \varepsilon)$, contradicting the hypothesis that the graph of f is closed.

Figure 3: Proof of Lemma 2 in Burgess (1990, p. 338)

Van Rooij and Schikhof (1982) prove the three implications in Figure 4 by verifying the properties path-connectedness, compactness, and continuity each time going back to the definition (level 1 structuralist praxeologies). Moreover, their proof uses classical real analysis techniques on sequences and compactness.

THEOREM 8.2. *Let* $f : [a, b] \to \mathbb{R}$ *and let* $\Gamma_f := \{(x, f(x)) : x \in [a, b]\}$ *be*
(α) f *is continuous.*
(β) Γ_f *is compact.*
(γ) Γ_f *is arcwise connected.*
Proof. (α)$\Rightarrow$(γ). Let $(p, f(p)), (q, f(q)) \in \Gamma_f$ and let $a \leqslant p < q \leqslant b$. Let $\sigma : [0, 1] \to [p, q]$ be an increasing continuous function for which $\sigma(0) = p$, $\sigma(1) = q$, and let $\psi : [p, q] \to \Gamma_f$ be the map $x \mapsto (x, f(x))$. Then $\phi := \psi \circ \sigma$ is a continuous map $[0, 1] \to \Gamma_f$ with $\phi(0) = (p, f(p))$ and $\phi(1) = (q, f(q))$.

(γ)$\Rightarrow$(β). There is a continuous $\phi : [0, 1] \to \Gamma_f$ such that $\phi(0) = (a, f(a))$ and $\phi(1) = (b, f(b))$. For $x \in [0, 1]$, let $\phi_1(x)$ and $\phi_2(x)$ be defined by $\phi(x) = (\phi_1(x), \phi_2(x))$. Clearly, $a \leqslant \phi_1(x) \leqslant b$ for all x, and by the (Darboux) continuity of ϕ_1 we see that ϕ_1 maps $[0, 1]$ onto $[a, b]$. But then $\phi : [0, 1] \to \Gamma_f$ is surjective. It follows that Γ_f is compact.

(β)$\Rightarrow$(α). Let $x_1, x_2, \ldots \in [a, b]$ with $\lim_{n\to\infty} x_n = x$. We prove that $\lim_{n\to\infty} f(x_n) = f(x)$. The sequence $f(x_1), f(x_2), \ldots$ is bounded (Γ_f being bounded). Let $\alpha := \limsup_{n\to\infty} f(x_n) = \lim_{i\to\infty} f(x_{n_i})$ for some subsequence $x_{n_1}, x_{n_2}, \ldots$ of $x_1, x_2, \ldots$ Then $(x, \alpha) = \lim_{i\to\infty} (x_{n_i}, f(x_{n_i})) \in \Gamma_f$, so that $\alpha = f(x)$. Similarly we can prove that $\beta = f(x)$, where $\beta := \liminf_{n\to\infty} f(x_n)$. Thus the sequence $f(x_1), f(x_2), \ldots$ converges to $f(x)$.

Figure 4: Proof of the equivalence of the continuity of a function on $[a, b]$ and compactness of its graph (van Rooij & Schikhof, 1982, pp. 52–53)[2]

We highlight that Ross also uses a compactness argument to adapt Burgess' proof of lemma 2: he introduces a sequence $(t_n)_{n\in\mathbb{N}}$ of points in $[0,1]$ and uses the Bolzano-Weierstraß theorem from real analysis (that unfolds to the topological notion of compactness in the topology course). Moreover, Ross (2013, §21) has already proved, that the continuity of the function is equivalent to the compactness of the graph. Thus, he could have adopted the short argument for $(\gamma) \Rightarrow (\beta)$ by van Rooij and Schikhof. We will not trace here the complete argumentation but instead point to an alternative proof (JOJO, 2020) that mimics and combines lemma 2 and lemma 3 from Burgess (1990) shorter than Ross' proof:

> Assuming that f is not continuous at some x_0, the definition implies that there is a sequence $(x_n)_{n\in\mathbb{N}}$ converging to x_0 in some compact interval $[a, b]$ such that $|f(x_n) - f(x_0)| > \varepsilon$ for a given $\varepsilon > 0$ and for all $n \in \mathbb{N}$. Then, there is a path $\gamma: [0,1] \to \Gamma_f$, $\gamma = (u, v)$, with $\gamma(0) = \big(a, f(a)\big), \gamma(1) = \big(b, f(b)\big)$ and $v = f \circ u$. Since u is continuous, the intermediate value theorem yields $t_n \in [0,1]$ with $u(t_n) = b_n$ for all $n \in \mathbb{N}$. By compactness of $[0,1]$, $(t_n)_{n\in\mathbb{N}}$ can be chosen such that it converges to some $t_0 \in [0,1]$. Then, the continuity of v implies that $f(x_n) = f\big(x(t_n)\big) = v(t_n) \to y(t_0) = f\big(x(t_0)\big) = f(x_0)$, which contradicts the assumption $|f(x_n) - f(x_0)| > \varepsilon$ for all $n \in \mathbb{N}$.

[2] We note that van Rooij and Schikhof (1982) use the word "arcwise connected" instead of "path-connected".

For Ross, it may not have been worthwhile to present a shorter more purely real analysis based proof but to maintain the structuralist organisation in Burgess' paper. This and his choice of didactic variables may be interpreted as a sign of the type 1 transition.

Finally, the use of compactness arguments can be understood on a more advanced level with a structuralist result (Wójcik, 2008, p. 13) that replaces both the domain and the codomain of the function with arbitrary topological spaces, the codomain being compact, and reemphasises again the hypothesis of closedness of the graph.

Conclusion

Related to the equivalence of the continuity of a real-valued functions on intervals and properties of their graphs, the didactic transposition, in particular represented in choices regarding didactic variables, achieved by university teachers (Ross, 2013; van Rooij & Schikhof, 1982) seems to prioritise simpler and less abstract arguments, rather than more conceptual structuralist arguments that may be organised in the form of lemmas as in Burgess (1990) when addressing university teachers. Indeed, Burgess (1990) and Wójcik (2008) point to or contain several possible structuralist generalisations of the theorem concerning real functions over a real domain in scholarly works. Each of them enlightens different aspects of more elementary analysis based proofs, and they can be established using structuralist praxeologies from topology. But applying them to deduce the result for real functions nevertheless require praxeologies from real analysis to prove that the hypotheses are fulfilled (such as the connectedness or closedness of the graph of a restriction of the function). The cost to achieve this seems as high as a direct proof of the theorem itself, so that the gain of generality and abstraction is not clear.

In terms of structuralist praxeologies and their evolvement, a struggle between contextualisation (the context of the task and more generally the background of real analysis) and decontextualisation (the development of level 2 structuralist praxeologies based on topological knowledge) can be observed, which arises at least in part from the use of textbooks and the organisation of study programmes. All the structuralist praxeologies applied by Ross (2013) or van Rooij and Schikhof (1982) are level 1. On the one hand, the most elementary proof for the direction "path-connected implies continuous" is not chosen by any of them, but on the other hand, none of the proofs bases the arguments on structuralist principles (this would have required considering abstract domains and/or codomains and connectedness and/or other properties instead of path-connectedness). These observations support the view that real analysis cannot be reduced to its structuralist dimension, but that the fruitfulness lies in the dialectic of objects and structures and in the interplay of real analysis and topology to increase mathematical understanding.

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The professional obligations of mathematics teaching: How measurement and theoretical progress work hand in hand

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We discuss the evolution of the professional obligations construct by looking at how the development of operational definitions and measurement instruments has allowed the making of some claims about groups of instructors. We suggest how further refining of the instruments can optimize the way those operational definitions of obligations inform the role that professional obligations play not only as constraints but also as resources for the work of mathematics teaching.

Keywords: practical rationality, measurement, obligations, theorizing, resources, constraints

Middle-range theorizing (Merton, 1967) involves bidirectional traffic between proposed theoretical constructs and empirical research: The former offers lenses for how to read experience and the latter provides justification to propose theoretical distinctions and corrections (Herbst & Chazan, 2022, 2023). We exemplify this using the development of the theoretical notion of professional obligations.

Practical rationality

The theory of practical rationality (Herbst & Chazan, 2003, 2023) aims to describe the grounds upon which mathematics teaching can be understood as a rational or sensible practice. Following Bourdieu (1990), this practice theory avoids reducing an account of teaching either to merely an objectivist or structural perspective, which would account for the work of the teacher as determined by social structures, or to merely a subjectivist or individualistic perspective, which would explain this work in terms of individual goals, knowledge, and beliefs. Instead, practical rationality reconciles both perspectives to account for teaching in ways that require considering both structure and agency: A teacher's position conditions them to act in particular ways, but those positions also allocate individual agency to them that they experience as they make choices and provide reasons for their actions. Their position creates conditions for them to feel disposed to see some actions as desirable; reciprocally, dispositions to act in line with personal inclinations may become defensible if the characteristics of their position afford it. The theory assumes that the practice of teaching is neither determined by structural characteristics, nor a free expression of individual will, but a defensible choice among available possibilities. The theory posits that a habitus of mathematics teaching, a "system of durable, transposable dispositions, structured structures predisposed to function as structuring structures" (Bourdieu, 1990, p. 53) regulates what's sensible for teachers to do.

Practical rationality has developed as a middle-range theory—neither a local theory (i. e., a synthesis of empirically confirmed relationships between specific constructs), nor a grand theory (i.e., a speculative language of description and interpretation that precedes empirical research practice). Merton's (1967) notion of a middle-range theory applies to practical rationality because this theory starts from the identification of a realm of social activity (i.e., the activity of mathematics teaching) and a perspective on social theory (i.e., social activity as practice, action as embodiment of a habitus)

and adds notions that support theoretical development in interaction with empirical research: Empirical research is theoretically motivated and serves as a source of exploration material (to generate theoretical ideas) and as mechanism for theoretical feedback (i.e., provisional confirmations and refutations). A central strategy for theorizing is the development of operationalizations of theoretical concepts that can be used to develop protocols for data collection, including measures.

This paper further advances a component of the theory–the notion that the position of teacher obligates individuals to stakeholders of their work and that such obligations are perceived as dispositions to act (Herbst & Chazan, 2012; Chazan et al., 2016). Four stakeholders–knowledge, the client, society, and organization–are posited as invested in the work teachers do with students and content in classrooms, holding teachers accountable to the stakeholders' interests. This structural claim can be fed by aggregate expectations expressed by classes of people associated with the interests of those stakeholders. *Knowledge* refers to the system of production and use of mathematics and is identified empirically in the expectation that mathematicians and STEM practitioners have that mathematics teachers should represent mathematical knowledge, practice, and applications faithfully. *The client* refers to students and is identified in the expectations that parents and children have that teachers treat students as unique individuals. *Society* refers to the needs and values of society and its communities and is identified in the expectation that prospective employers, politicians, and taxpayers have that teachers foster prosocial behaviors and values in classrooms (e.g., fairness, honesty). *Organization* refers to the institutions that organize mathematics classroom instruction, including departments, schools, districts, and school systems and is identified in the expectation that organization leaders (e.g., government officials, superintendents, principals, department chairs) have that mathematics teachers will be effective and efficient, and operate within the law.

In positing the four obligations, we make a structural claim that these four stakeholders obligate mathematics teachers and that the norms of instruction (i.e., the didactical contracts that are viable in practice; e.g., the expectation that problems assigned to students have all the information needed to be solved and require the use of knowledge recently learned; Brousseau, 1997) result from historical adaptations to the combination of those obligations. The obligations ascribed to the position of school mathematics teacher thus complements prior structural characterizations of the role of the teacher in mathematics instruction, according to which the communication of knowledge to students is regulated by instructional norms (e.g., the claim that a didactical contract exists). In what follows we discuss how these professional obligations have been used in research, how this use has contributed to theory development, and argue for further development of its associated measurement instruments to support further theoretical growth. We develop further the distinction between two empirical manifestations of these obligations so that these can help capture the complementarity between structure and agency of the notion of obligations and support further empirical research on obligations.

Empirical research on obligations–reviewing the first cycle

The proposition that mathematics teachers are obligated to four stakeholders entered the theory as a hypothesis in the mathematical sense. Its origin was as an abductive inference from listening to teachers justify the actions of a teacher who the former saw on video acting in ways that departed from what they expected (Nachlieli et al., 2009). The majority of the studies that have relied on the

obligations construct, have used it to organize the sources of justification that individual practitioners call forth, for example in interviews or focus groups where they consider aspects of someone's practice (Bieda et al., 2015; Webel & Platt, 2015; Lande & Mesa, 2016; Shultz et al. 2021). Practitioners recognize the influence of an obligation in the way they speak about why what they see being done is or is not justifiable. Individual distinctions among practitioners can be made regarding what obligations individuals invoke and how they argue for the justifiability of actions. And the obligations have been used to investigate the various circumstances in which individual mathematics teachers recognize those obligations. In all this, the obligations seem to be resources available to individuals who wield them if they are so disposed, to justify conceivable actions.

But structural empirical questions could also be posed to understand more about the obligation construct and its relationship with other constructs. Our own work inquired on whether further considerations of theoretical structure were needed within each of the obligations. For instance, in the context of the individual obligation, would it make sense to distinguish an obligation to students as cognitive beings from one to students as emotional beings? But alternative structural differentiation could be conjectured too. For instance, could the obligation to individual students distinguish two ways in which a teacher could defensibly recognize this obligation, say as an obligation to cater to the needs of individuals or as an obligation to challenge individuals. Herbst and Chazan (2003) noted how practical rationality could accommodate discussions about the commitments involved in different kinds of teaching precisely by assuming that opposite dispositions (e.g., to coddle or to challenge) could attest to the existence of a common ground (viz., the obligation to the student).

To be clear, though these and other distinctions might make sense and could be used in interpretive work with individuals, incorporating these distinctions into the theoretical account of the obligations required more than consideration of their sensibility. The notion that practical rationality is a middle-range theory suggested instead that the sensibility of those claims, possibly backed up by case studies in which that sensibility was grounded, would need to be tested further against more stringent empirical benchmarks: It became important to further develop the hypothesis that the obligations exist by creating measures that could be used to describe how they are recognized and which could be used to discern how obligations made a difference in making claims at the level of structure.

Even in taking for granted that mathematics teachers would be obligated to the four stakeholders, some structural questions could be formulated empirically and addressed if we had measures. Among those, we could ask whether the obligations hold equally across groups of mathematics teachers from different school levels (e.g., university vs. high school), teachers from different countries, or teachers from different types of schools (e.g., public vs. private). Also, do teachers' recognition of obligations moderate the role that norm recognition plays in teacher decision-making? (Herbst et al., 2016).

The PROB and PROSE measures

To address those questions, we developed two instruments (PROB and PROSE[1]) to be completed by individuals and that posed different questions relating individuals to the sources of justification whose

[1] PROB stands for Practical Rationality Obligations, PROSE for Practical Rationality Obligations Scenario Evaluation.

hold on individuals we wanted to gauge. We operationalized the relationship between individual teachers and each obligation by hypothesizing the construct "recognition of an obligation" to name the extent to which an individual would recognize the existence of specific external expectations on their teaching. PROB (Herbst & Ko, 2017), asked teachers to indicate their agreement with statements that explicitly identified an expectation, sampling from among the various aspects of each obligation statements with which they had to indicate agreement (Figure 1). Respondents had to rate the degree to which mathematics teachers are expected, as professional educators, to act in the manner that the statement describes. The logic of the item responses was that the more likely the respondent recognized themselves as accountable to the obligation in question, the higher they would agree that teachers are expected to act in the manner described. Herbst and Ko (2017) reported on an exploratory factor analysis on item responses that suggested internal factor structure within each of three obligations (disciplinary, individual, interpersonal). For example, items written to gauge recognition of the disciplinary obligation might indicate 3 factors: (1) Obligation to the discipline as member of a community contributing to increase and extend appreciation of knowledge outside of the classroom (9 items) (2) Obligation to the discipline as responsible for its correct representation in study resources (3 items), and (3) Obligation to the discipline insofar as responsible representation in classroom interaction (5 items). Analysis showed poor reliability for the institutional scale, and a variety of significant but low correlations between scores on the obligations and on the various factors of Stipek et al.'s (2001) beliefs instrument, suggesting that obligations and beliefs are different constructs.

Obligation	Statement
Disciplinary	Mathematics teachers describe new procedures to students in ways that a mathematician would endorse.
Individual	Mathematics teachers are responsive to individual student's emotions.
Interpersonal	Mathematics teachers ensure that they address questions from all students.
Institutional	Mathematics teachers assign grades that represent how much of the curriculum students have learned

Figure 1. The PROB surveys

The PROSE instrument attempted to get at whether individuals would recognize opportunities to do something that deviated from a norm as justifiable on account of an obligation they had been primed for. Each item included a priming statement anticipating that the respondent would see a teacher make a decision that considered a given obligation[2] and a scenario such as shown in Figure 2, rendered with nondescript cartoon characters, in which the teacher avowedly acted in response to the primed obligation. After perusing the scenario, the respondent was asked to consider a statement such as "The instructor should stick to the mathematics at hand, rather than take class time to make

[2] The priming statement for the disciplinary obligation at the college level was "In the following storyboard we invite you to consider a scenario in which a college instructor deviates from the lesson in order to attend to issues of mathematical importance." Other obligations were primed using equivalent phrasing.

connections to other mathematical ideas," which indicted the teacher for doing so and to which respondents had to indicate their agreement using a 6-point Likert scale ranging from strongly disagree to strongly agree. As a rule, items had statements indicting the teacher for what they had done and offering instead what would have likely been the norm in a U.S. classroom. PROSE scenarios sampled domains of practice–e.g., in Figure 2, the disciplinary obligation would justify making connections to higher-level mathematics. PROSE items did not directly ask for agreement to be expected to do so, but the question amounted to ask whether the decision was justifiable.

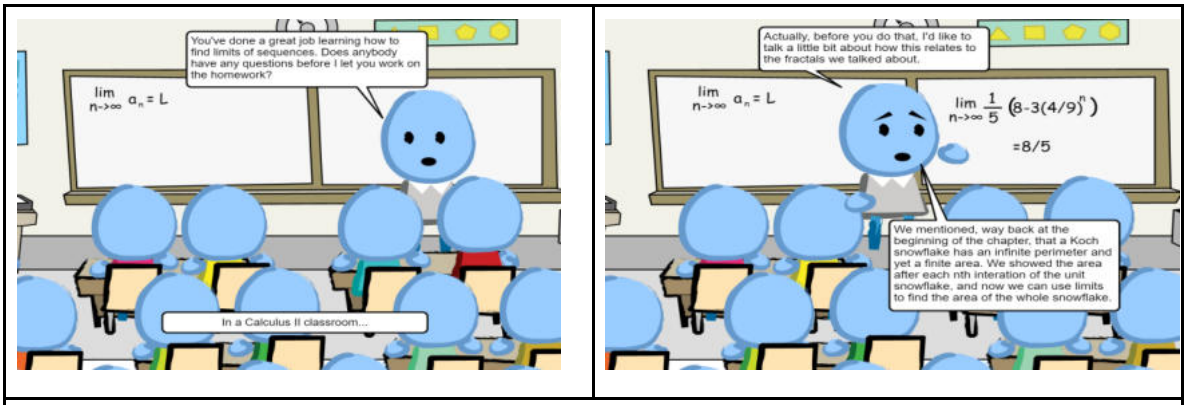

Figure 2. Two slides of the scenario for item A1115, primed for the disciplinary obligation

Sets of parallel items were developed for high school and lower division undergraduate mathematics levels. Matched items across levels would sample the same domain of practice, but the content being handled corresponded to courses in the level of schooling under consideration. Additionally, some linking items were developed that did not show explicit mathematical content and could be included with the other items at all school levels. Ko et al. (2024) used these versions of the PROSE instrument to inspect differences in recognition of the obligations across the undergraduate and high school levels. These examples add to the use of PROB in showing the kind of structural questions that can be asked. In both uses, recognition scores were used to understand empirically what sort of structural claims the obligation construct could help make–that is, the measures were used for basic research and not to make claims about individuals or to inspect any theory of individual differences. At the same time, because recognition of obligations varies and it is arguably a valued asset individual teachers might have that might support the transformation of their practice, one could raise the question of whether an individual's level of recognition is variable or stable, and, if the former, how it varies (e.g., developmentally, situationally). This, in turn, requires us to think carefully about the experience of obligation recognition and the possible variability in individual recognition.

Using obligations measures to make claims about groups and individuals

Despite the productive role that the PROB and PROSE measures have played in confirming that the obligations construct adds to our understanding of mathematics teaching, they could be further improved by refining how their scores are to be interpreted and used. The construction of PROB and PROSE items tap into different operational versions of the obligation construct, to the point that the kinds of claims that each instrument might make possible could contribute differently to improving

the theory of practical rationality. Whereas PROB items elicit recognition of obligations as constraints on individual action, the PROSE items gauge whether recognition of obligations can be resources for justification of actions that might otherwise be seen as problematic. Though both instruments are completed by individuals, the variability in the elicited responses can serve different theoretical ends.

One theoretical end is to better understand what the claim that the stakeholders obligate mathematics teachers means in practice: Does the recognition of obligations by groups of teachers depend on the institutions in which they work, the students they serve, the societies to which they send their graduates, or their relationships with the discipline of mathematics? Among the many different aspects of practice in which such accountability may show up, are there empirically identifiable aspects that may explain variability across groups of teachers? The use of the obligations construct in qualitative research is likely to illustrate affirmative answers to those questions. For example, teachers working in districts where administrators are involved as instructional leaders may experience the institutional obligation differently than those where administrators exercise their supervision through classical means (e.g., test scores, passing rates). Similarly, individual teachers may bring up different aspects of the obligations they recognize as reasons to do or not to do what they do in classrooms. But structurally, some of that individual variability may be explainable in relation to other factors while some other may not. The redevelopment of an instrument like PROB could be oriented to detecting those differences. This instrument could also be used with stakeholders It seems that a next step in the study of whether mathematics teachers are, as a group, obligated to those stakeholders should include a PROB-like instrument being administered to experienced mathematics teachers as well as stakeholders (e.g., mathematicians, students, employers outside of STEM, and administrators who don't know mathematics) to gauge correlations in recognition. A PROB-like instrument could be designed to attend to hypothesized factors within each obligation (either sub-obligations or competing dispositions, or both).

A second theoretical end is to better understand how the obligations become resources for teachers to justify their actions. The PROSE instrument's basic design seems to allow for such detection. Scores on these items not only could vary on the basis of individuals' increasing socialization into teaching (e.g., teacher education, professional development) but also on the basis of individual recognition of events in the class as associated with a particular stakeholder. Indeed, investigation of pilot data with PROSE items (Milewski et al., 2021) documented that individuals varied in their recognition that the circumstances in which a given obligation was applied actually warranted deviation from the norm–oftentimes they thought justification required extra information. Rather than thinking of the source of this variation as item-specific noise, we would like to think of these contexts as avenues through which recognition of the obligation might develop, somewhat in analogy as how weather patterns constitute climate or how mood patterns constitute personality. In other words, a PROSE-like instrument could be sensitive to contextual circumstances, in which recognition of an obligation might not merely be applied if it exists but also be elicited as an agentic disposition.

Conclusion

The pairing of personality and mood is proposed as an analogy with recognition and justification disposition for two reasons. Clearly, both personality and mood patterns develop over time and are

latent traits whose measurement rely on items that indicate them. But personality is stable over longer stretches of time while mood patterns can be more variable not only according to local context but also as those contexts become more pervasive (e.g., with age, changes in financial status). The analogy with obligations suggests that we should use the different constructions of PROB and PROSE as a stimulus to consider two ways in which the obligations are manifest in individual teachers.

On the one hand, teachers may recognize an obligation as a professional constraint or requirement. This recognition may vary across institutions, communities, and cultures and may admit to factor structures that allow for detecting teachers of different groups (e.g., teachers that recognize some dispositions or some sub obligations, etc.). An explicit survey instrument such as PROB could allow us to associate with each practicing teacher a vector of recognition scores that describes their way of being obligated to the stakeholders. This instrument should be able to detect differences between large changes in a teacher's professional life such as before teacher education, toward the end of teacher education, five years into professional practice, and close to retirement. This type of instrument should be usable to examine structural characteristics of the obligation construct and variations in obligations across groups of teachers (e.g., teachers of different countries, levels of schooling, types of schools, teachers with and without strong mathematics preparation, etc.).

On the other hand, teachers may recognize an obligation as a professional resource or support, particularly to justify doing something that feels appropriate at a moment even though it may not be the expected thing to do. A scenario-based survey such as PROSE could be redesigned to include items which differ in regard to both how justifiable a teacher's decision is and in regard to how recognizable the opportunity is to invoke an obligation. This PROSE-like instrument would not measure an individual's recognition of an obligation but rather an individual's present disposition to use an obligation to justify a decision that might need justification. One would expect that this disposition would vary within a timescale smaller than recognition of obligations varies, as individuals accumulate more experiences through practice or professional learning. Thus, the PROSE instrument could be used to detect individual changes over time and possibly also to make claims about the effects of professional learning opportunities, for example those that promote instruction that departs from customary norms.

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Using theory as a tool to cultivate and analyse research methods: Theorising positioning to examine task-based interviews

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Researchers use theory to solve problems of practice, which can include problems related to the practice of conducting research. We put forward a use of theory, namely, as a tool to cultivate and analyse research methods in mathematics education. To illustrate, we use the theorising of positioning as a tool to analyse power dynamics involved in task-based interviews. We explain how theorising can contribute to the implementation and further development of research methods, such as the task-based interview. Implications include the training of new researchers in the use of research methods and the interrogation of the use of established research methods.

Keywords: Research problems, Mathematics education, Research methods, Theory use.

Introduction and purpose

In mathematics education research, theories can serve a variety of purposes. One prominent purpose is to provide "*a safeguard against unscientific approaches* to a problem, issue, or theme" (Niss, 2007, p. 1309). This purpose encompasses the use of theories to guide empirical research studies (e.g., drawing on theory to frame a study) as well as to inform the practice of research (e.g., selecting and implementing research methods). Here, our focus is on the latter; we put forward a use of theory to cultivate and analyse research methods in mathematics education.

Two questions frame our approach. First, how can theory use inform the implementation and evolution of research methods in mathematics education? Second, how can theory use inform the training of new mathematics education researchers in research methods? To respond, we start broadly, discussing how theory can be used as a tool to cultivate and analyse research methods. Next, we turn to a specific instance of theory use, namely how the theorising of positioning (Davies, 2023; Davies & Harré, 1990; Davies & Hunt, 1994; Herbel-Eisenmann et al., 2024) can be used as a tool to examine the well-known research method of task-based interviews (Goldin, 1997; 2000). To ground the argument, we provide a vignette to illustrate how theorising can inform the training of new researchers in the practice of conducting research. We conclude with discussion and implications.

Using theory as a tool to cultivate and analyse research methods

Theories are vital for connecting problems of practice and the practice of conducting research in mathematics education (Silver & Herbst, 2007). Problems of practice often include questions or ponderings that arise in the teaching and learning of mathematics (e.g., How may students' reasoning impact their performance on an assessment?). Here, we extend problems of practice to include questions or ponderings that may arise in the practice of conducting research, with a specific focus on research methods, such as task-based interviews.

Theory, whether in the background or foreground (Mason & Waywood, 1996), informs the kinds of problems people may consider as well as potential solutions to those problems. To illustrate, the

research method of task-based interviews involves interaction between an interviewer and an interviewee, in relation to tasks that the interviewer poses to the interviewee (Goldin, 2000). A theory (or theories) guiding researchers' actions informs the interpretations they make when analysing interviews. For example, the theorising of positioning may lead to questions about how young children may use nonverbal actions to communicate their unwillingness (or discomfort) to participate in an interview (see Parks & Schmeichel, 2014). Elaborating on a relationship between theory and problems, Johnson (2024, p. 3121) wrote:

> Theories can bound the kinds of things that people question, and in turn, the kinds of things that can count as answers to those questions (Mason & Waywood, 1996). Put another way, theories shape where people look (or listen) and what they see (or hear), as well as how they make and act on their interpretations.

To explain what we mean by theory, we draw on Radford's (2008) characterisation of theory as a dynamic whole comprising three interrelated aspects. From this perspective, a theory involves a system of basic principles, a methodology that includes techniques/methods that cohere with the principles, and a set of paradigmatic research questions. By using "system" to describe the principles, Radford communicates that the collection is structured, and the structure provides meaning for the principles. While there is not a lockstep relationship between a theory and a methodology, the methodology needs to cohere with the principles and have operability to answer research questions. From this view, it is possible for research methods, such as task-based interviews, to be associated with different theoretical perspectives and related methodologies. Consequently, those methods undergo evolution to cohere with principles underlying the different perspectives.

Like theories, research methods are not static; they evolve as researchers continue to employ them. This evolution includes researchers' use of methods (enacted method) as well as discussion of methods in the literature (intended method). By cultivation, we refer to the nurturing of this evolution. By analysis, we mean the examination of assumptions (tacit or explicit) underlying a "business as usual" approach to the method. Cultivation and analysis can focus on both the intended and enacted method. Analysis of the intended method involves examination of original sources. Analysis of the enacted method involves examination of instances of implementation of the method, which may be retrospective or ongoing. Cultivation of the intended method involves advancement of the method, which may occur through recommendations or further elaboration on the method. Cultivation of the enacted method involves advancements in the approach to the method. Theorising guides cultivation and analysis.

Theories emerge in response to problems, with questions to be answered (Radford, 2008). As theories evolve, the questions also evolve, while marks of the early question remain. Here, we employ this "problem-based" approach, using the metaphor of a tool (Simon, 2009) to describe our use of theory to do a particular kind of work. Our approach responds to a twofold problem—a need to engage in the purposeful practice of research and train new researchers in such practice. We focus on developing expertise in the implementation of research methods more broadly, with a particular interest in task-based interviews. Our author team is composed of researchers (two White men, one White woman) who grew up in different geographical locations in the U.S., in different generations, and in different

social classes. Two have a longer history of designing and conducting task-based interviews and drawing on those interviews as a primary source of data for empirical studies. One is a new researcher, developing expertise in the use of task-based interviews as a research method. As researchers using task-based interviews, our goal is to position ourselves as listeners and to (re)distribute power to participants, communicating our belief that they are the expert in their thinking and reasoning, and our role is to learn from them. Our intention is to foster participants' sense of agency and autonomy, which we have found to be effective to encourage sustained engagement and feelings of success.

Theorising positioning to examine task-based interviews

> The positioning of self in relation to another is not simply or only an act that reveals who that individual has invested him- or herself in being. While it may reveal the investment they have made in one set of values or another, or one discourse or another, it is also, always, an act in relation to the power/knowledge assemblages through which credence in one's own eyes and the eyes of others is accomplished. Positioning is relational, not just the immediate relation with one's interlocutor, but relations with/in the assemblages of power/ knowledge that make up academic/intellectual and everyday life. (Davies, 2023, p. 473)

We intentionally use the phrase "theorising positioning" rather than "positioning theory." Like Herbel-Eisenmann et al. (2024), we do this to communicate that the theory is dynamic, rather than static, having multiple potentialities. In our view, the phrase "theorising positioning" begs the question "theorising positioning in what way(s)?" For our purposes, we adopt the theorising of positioning that Herbel-Eisenmann et al. (2024) put forward; that is, theorising positioning as praxis. With this approach, a relationship between theory and practice is one of reciprocity; theory informs practice and practice informs theory. A reciprocal relationship between theory and practice dovetails well with our use of theory to inform the practice of research. In our view, "theorising positioning as praxis" suggests an active use of theory, something researchers do for certain purposes; it is a verb rather than a noun. We embrace this action-oriented approach to theory.

According to Herbel-Eisenmann et al. (2024), when researchers theorise positioning as praxis, they engage in five commitments, acknowledging that their list is neither exhaustive nor entirely mutually exclusive. The commitments are to "historicize," "reflect on positionality," "foreground/honour participant perspectives," "address tensions," and "acknowledge hopes and fears" (p. 291). Next, we elaborate on these commitments, appealing to ideas presented by Herbel-Eisenmann et al. (2024). To "historicize" involves learning histories of people involved, which can happen in different scales and scope. This work entails understanding that histories, like positions, are something negotiated and contestable. As researchers, to reflect on our positionality means examining how "our histories, languages, experiences, and cultures relate with our participants and give an account of how tensions shape our interactions and interpretations" (p. 294). By foregrounding/honouring participant perspectives, researchers attend to how participants may experience or interpret positionings, which may continue to evolve and can happen at different scales. To address tensions means not to oppose or squelch them; in contrast, it entails welcoming and navigating tensions as part of the process, and these tensions can include acknowledging participants' hope and fears. We view these commitments to embody early work around positioning. In their study, Davies and Hunt (1994, p. 392) describe

their efforts to "reveal the positions from which and from which we have struggled to gain some understanding of the positioning of marginal students."

Storylines are a key construct to the theorising of positioning, and people's hopes and fears frame their stories (Herbel-Eisenmann et al., 2024). Storylines refer to "the ongoing repertoires that are already shared culturally or they can be invented as participants interact" (Herbel-Eisenmann et al., 2015, p. 188). Like positioning, storylines are dynamic, rather than static, and they constantly evolve. Storylines also can operate at different scales (Herbel-Eisenmann et al., 2015); they may happen moment-to-moment, and they can be products of accumulations of experiences.

Wagner and Herbel-Eisenmann (2009, p. 2) have called for researchers' attention to positioning in their practice: "The way a researcher theorises positioning (the way she sees and thinks about it) affects the way she interprets actions, including her own." When researchers describe and implement research methods (intended and enacted methods), they create conditions that result in sources of data to be interpreted. When research methods, such as interviews, involve communication acts between researchers and participants, both contribute to the enactment of the method. In this vein, Solomon (2012, p. 177) contends that interviews are something more than "a giving up of information by Joanne and Roz [participants], but are joint constructions in which I played a part". We concur with Solomon (2012)'s characterisation of interviews as "joint constructions" between at least a researcher (interviewer) and participant (interviewee), and view this to be applicable to the mathematics education specific research method of task-based interviews.

Task-based interviews have a long history as a method in mathematics education research (Goldin, 1997; 2000; Maher & Sigley, 2020), being used primarily to understand students' thinking and reasoning while solving problems. Thus, it is not surprising that many research studies employing task-based interviews are framed by cognitive theories (Goldin 1997; Maher & Sigley, 2020). Yet, social aspects are inherent within the method, as evidenced by Goldin's identification of questions for researchers to provide explicit consideration in advance of conducting task-based interviews:

> What constraints or limitations are imposed by the social, cultural, and psychological contexts of the interviews? How may the student's expectations, presumptions, apprehensions, and intentions interact with mathematical cognitions and affect (and with task variables) to influence the interview outcomes? (Goldin, 1997, p. 42)

Parks and Schmeichel's (2014) study illustrated how researchers may theorise positioning to respond to Goldin's questions, as they examined how a researcher (interviewer) and student (interviewee) worked to "position themselves and each other" during task-based interviews used for assessment purposes. A purpose of the study was to draw on students' embodied actions to provide a way to analyse power dynamics that can unfold when researchers from dominant populations interview participants from nondominant populations (in their case, young children). As a result, Parks and Schmeichel (2014) advocated for researchers to develop awareness of their positions of power, to allow more nuanced reporting that may provide insight into how children felt during the interview.

Theorising positioning can uncover power dynamics, as both the researcher and the participant(s) co-construct task-based interviews, and their interactions can manifest a range of storylines. Johnson et

al. (2024) have identified three researcher-participant storylines that emerged during an interview: expert-novice, questioner-answerer, and listener-expert. If the "expert-novice" storyline dominates a task-based interview, it has potential to be problematic, because the storyline preserves the *status quo* power dynamic of researcher as expert and participant as novice.

We distinguish the theorising of positioning from the theorising of decentring, a theoretical construct widely used to examine researcher actions during task-based interviews. Decentring, the "ability to see a situation as perceived by another human being" (Confrey, 1990, p. 108), is crucial for researchers conducting task-based interviews. It includes researchers' active efforts to make sense of situations from a student's perspective, in the moment or retrospectively (Hackenberg, 2010). Through the theorising of positioning, researchers can examine power dynamics at play during task-based interviews, even when researchers are actively engaging in decentring.

Using theory to inform the training of new researchers: a vignette

The questions that frame our approach ask how theory use can inform the practice of research (i.e., the implementation of research methods, specifically task-based interviews) and in turn, the training of novice researchers in such practice. We contend that the theorising of positioning demonstrates promise to inform the training of researchers to learn to implement task-based interviews. To illustrate the potentiality for such theorising, we share a vignette of Roger, a new researcher developing expertise in leading task-based interviews, during multiple cycles of preparation, implementation, and reflection. Across the cycles, in the implementation phase, there is a back and forth between Roger's roles as observer and interviewer, guided by reflection on the implementation.

To prepare to implement task-based interviews for his dissertation study, Roger apprenticed with his dissertation advisor. This included acting in the role of observer for task-based interviews his advisor conducted, as well as watching videos of existing interviews with tasks similar to those he intended to implement. Roger and his advisor discussed key principles (intended method), including ways in which he may decentre, to make sense of students' perspectives on tasks, which may be different from his own. Along with this, Roger planned questions (and follow-up questions) to guide task-based interviews he would conduct. To practice, he interviewed a few friends and family members. While on the surface, Roger's enacted method followed the principles of task-based interviews, after the interview, one person said that they were trying to remember what they thought they were "supposed" to do. Others said they felt the interview was very formal, like an evaluation, and they were self-conscious about getting the right answers. This created space for Roger to reflect, and talk with his advisor, thinking that if people with whom he was close felt this way, student participants he did not know as well likely would feel uncomfortable, perhaps even more so. Across conversations with a colleague, Roger's advisor came to wonder if Roger's positioning during the interview might be perpetuating an expert-novice storyline, in which participants may be trying to figure out what Roger (the expert) thought would be appropriate for them (the novices) to say/do during the interview. In response, Roger and his advisor hatched a plan, in which Roger would observe his advisor conduct new interviews, for the purpose of supporting him to (re)distribute power to students. In these observations, focus shifted between interviewee and interviewer. First, Roger observed an interview with a focus on what the student said and did, considering how long the student's responses were and

how comfortable the student seemed to feel during the interview. Second, Roger observed another interview, this time with a focus on follow-up questions that his advisor asked in response to student's statements. Then, Roger applied the different foci to each of the interviews. These targeted observations were intended to allow for an exploration of conditions under which students might feel safe and welcomed to engage in the vulnerable act of sharing their thinking during an interview.

In response to the observations, Roger described his emerging awareness of power dynamics during interviews. First, he began to consider how interviewer-interviewee interactions could communicate who holds power during an interview. He wrote that he could feel how an interview became more comfortable and "low-stakes" as it continued. Second, his debriefs with his advisor after the interviews allowed him to share things that he noticed, and made room to discuss the intentionality behind those moves to (re)distribute power to students (e.g., If a student says that they don't know, one follow-up question could be to ask them if there are things they are wondering about.). Third, Roger watched and reflected on his own interviews, catching times when he might sound judgmental, or jump in too quickly, and making notes for future interviews. In Roger's words:

> The more interviews I conduct, and the more comfortable I get with asking students questions, the more I realize how complex and rich their thinking is. When students feel comfortable and in control during an interview, they really can "open up" and provide deep insight into their thinking.

Discussion and implications

At the outset, we asked how theory use can inform the implementation and evolution of research methods in mathematics education. In turn, we asked how theory use may inform the training of new mathematics education researchers in research methods. We argued for a use of theory as a tool to cultivate and analyse research methods, both intended and enacted. To illustrate, we theorised positioning to analyse task-based interviews, demonstrating how the theorising could make explicit power dynamics having potential to impact the course of an interview. Through Roger's vignette, we illuminated the utility of such theorising not only for the advancement of research methods, but also for the training of new researchers to viably implement research methods.

Analysing research methods from different theoretical perspectives can create opportunities to problematise the intended and/or enacted method. Drawing on Radford's (2008) notion of theory, a theory's principles inform the logic underlying a research method. When a research method is developed in one theoretical tradition, then imported into a different theoretical tradition, the logic of the method carries residue of the theory in which it was generated. From this view, research methods (e.g., task-based interviews) may only be viable with respect to certain theoretical principles. Different theories can illuminate aspects of research methods that otherwise may be tacit.

Focusing on the research method of task-based interview, we sourced a theory outside of the tradition associated with the development of the method. We view this use of theory to be consistent with Scheiner's (2020) mode of juxtaposing potentially competing explanations, to reveal conflicts and tensions. While decentring explains researchers' active efforts to understand participants' perspectives, theorising positioning explains power dynamics that can be at play during interactions between researchers and participants. Despite Roger's efforts to decentre, power dynamics present in

interviews (enacted method) remained tacit for him. Theorising positioning, together with cycles of observation/implementation and reflection, made room for those dynamics to become explicit. The navigating, rather than squelching, of this theoretical tension also aligns with commitments to the theorising of positioning, as laid out by Herbel-Eisenmann et al. (2024). In our view, embracing the tension is a key aspect of the use of theory to cultivate and analyse research methods.

In conclusion, we return to a purpose of theory put forward by Niss (2007); that is, to guard against unscientific approaches to problems. A crucial aspect of scientificity is the ethical application of research methods. Theorising positioning to analyse power dynamics in task-based interviews has potential to enhance the ethical ways in which researchers act (see also Parks & Schmeichel, 2014), and to centre the humanity of researchers and participants in task-based interviews.

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An interactionist perspective on the negotiation of meaning of mathematical representations

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This article presents the theoretical and methodological framework of an empirical research programme that aims to develop an interactionist perspective on the question of how representations acquire meaning in the context of interaction in mathematics classrooms. In order to address this question, we propose a conceptualisation of the interpretation of representations as situationally established rules that are negotiated in classroom interactions. By way of an illustrative example, we demonstrate that the processes of interaction in the classroom manifest on two levels: the interactional negotiation of the meaning of a mathematical representation, which is orientated towards norms that are in turn shaped and established in the process of interaction.

Keywords: Representation, Norms, Symbolic Interactionism, Interaction Analysis

Representations as tools for doing mathematics

Mathematics is an abstract science. In contrast to other sciences, we cannot experience the objects of mathematics empirically or directly with our senses (Duval, 2000). It is therefore particularly important for the learning of mathematics to visualise the abstract mathematical objects with representations, e.g., illustrations, diagrams or symbols, that mediate between the abstract objects of mathematics and the learner's world of experience. In this way, learners can utilise representations to discern mathematical objects, experiment with them, develop hypotheses about them, verify and justify them. Through the utilisation and manipulation of representations, learners may be enabled to experience abstract mathematical objects with their senses (visually and, if appropriate, also haptically) to conceptualise them and mentally anchor them as ideas regarding flexible application (Kollhoff, 2022; Finesilver, 2022; Tiedemann, 2020). Representations thereby become a means of doing mathematics, of understanding and of being understood (cf. Bauersfeld, 2002).

However, representations are generally not self-explanatory. There is no direct, isomorphic correspondence between representations and their associated mathematical objects. Representations do not convey mathematical objects; rather they represent mathematical objects through an assigned similarity between the object and its representation. In many instances, mathematical discourse employs the use of symbolic artefacts that have been specifically designed for the purpose of facilitating mathematical communication, in the sense of didactical materials. Such representations refer to entities that do not exist in the physical world (Sfard, 2008). Duval (2000) characterises this discrepancy between the abstract nature of mathematical objects and the concrete nature of representations as the "paradoxical character of mathematical knowledge" (Duval, 2000, p. 61). The use of mathematical representations to represent abstract mathematical objects results in the creation of essentially artificial artefacts. Consequently, learners must develop an increasing awareness that mathematics education operates with representations and not with the represented objects themselves (Duval, 1999).

The interpretation of a representation involves a process of abstraction from the surface features of the representation itself (cf. Söbbeke, 2024; Kollhoff, 2022). This interpretation of meaning is an individual and constructive process, whereby learners interpret a representation on the basis of their individual previous experiences. Furthermore, the mathematical objects and relations depicted in representations are often initially unfamiliar or unknown to learners, and their individual interpretations frequently diverge from the normatively intended ones. In order to utilise representations for the purpose of mathematics learning, it is essential that learners develop the ability to interpret representations in a way that is aligned to the intended interpretation as closely as possible. Söbbeke (2024) characterises this necessity for the learners as an epistemological conundrum: On the one hand, learners require the concrete quality of a representation to facilitate the conceptualisation of mathematical objects. Conversely, learners must also abstract from this concrete quality to interpret the underlying mathematical object represented.

From an interactionist perspective, we put forth the argument that such learning necessitates intersubjective negotiations of meanings between the learners and the teachers. Consequently, representations become "social products" (Blumer, 1969, 2013), which are negotiated situationally in and through the interpretations and defining activities of the interacting individuals.

Negotiation of meaning from an interactionist perspective

From an interactionist perspective, as applied in mathematics education research, particularly in interpretative research (e.g., Bauersfeld et al., 1988; Krummheuer, 2011; Schütte et al., 2019; Kollhoff & Gerlach, 2023), mathematics learning is understood to transpire within and through social interaction. The fundamental assumptions of this approach are derived from the sociological theory of symbolic interactionism (Blumer, 2013), as well as from approaches to collective learning (Miller, 1986) and ethnomethodology (Garfinkel, 1967).

Blumer posits that human coexistence is constituted by symbolic interactions, whereby the actions of the individual are always based on the previously interpreted actions of the other person. In addition to the concept of interaction, the concept of interpretation and the meaning resulting from these interpretations form the basis of symbolic interactionism. In consequence, the meanings of mathematical objects, operations, and representations in particular, can be characterised as 'social products' that are generated in reciprocal collective interpretation processes (Blumer, 2013, p. 67; cf. Fetzer & Tiedemann, 2018). By engaging in such collective processes of meaning-making, individuals can transcend their limited individual abilities to construct meaning, thereby establishing a foundation for individual mathematical learning (Schütte et al., 2021). From an interactionist perspective, however, it is a fallacy to assume that learners enter collective negotiation processes devoid of prior experience and knowledge, and that they construct only situation-specific concepts in these processes. Consequently, at the outset of an interaction, all learners establish preliminary situation-specific interpretations of the situation in which they find themselves, termed *situation definitions*, based on their individual experiences and knowledge (Krummheuer, 2011).

In order to facilitate effective communication and collaboration, those engaged in the negotiation processes invariably prepare the definitions of the situation in anticipation of potential interpretations by their counterparts. These are then modified or adapted during the process of mutual negotiation of

mathematical meaning (Krummheuer, 2011). In this process of mutual alignment of individual definitions of a situation, a moment of 'interpretive agreement', that is an instance of shared interpretation between the learners, can be achieved. This is referred to as an 'interpretive interim' by Schütte et al. (2021). The term is also discussed by Voigt (1995) in the context of 'taken-as-shared' understanding.

In line with Miller's (1986) perspective, the interpretive interim has the potential to be a source of innovation for the individual, enabling them to transcend their existing interpretive capabilities in a systematic manner. From the perspective of an interactionist understanding of learning, the interpretive interim can be seen as a source of 'stimulation potential' for fundamental, situation-transcending cognitive construction and restructuring processes at the individual level. Consequently, a one-time collective construction of an interpretive interim does not necessarily lead to individual cognitive construction or restructuring processes. Rather, it may be more accurate to view this as a process that requires repeated negotiation of the same interpretive interim, potentially in different groups of learners.

The underlying situation definitions of the individual that led to the production of the interpretive interim can be considered to 'converge' into more *standardised* and *routinised* retrievable situation definitions (Schütte et al., 2021). These enduring cognitive edifices, or their reorganisation, are referred to as the construction or modification of *frames* in accordance to Goffman (Goffman, 1957; Krummheuer, 2011). On the basis of such frames, learners reinterpret the situation (here: the representation) in subsequent mathematical negotiation processes, thereby acquiring altered opportunities for participation. In the context of classroom discourse, it is to be expected that this convergence of individual interpretations – cf. 'conception-frames' as introduced by Salle and Schütte (2023) – of representations towards a conventional normative meaning will be driven by the negotiation of sociomathematical norms.

Norms and Processes of standardisation

Sfard (2008) categorises rules and norms in mathematics education into two different types. On the one hand, she identifies 'object-level rules', which orientate the content level of a conversation, and on the other hand, 'metadiscursive rules', which orientate what to do, when and how (Sfard, 2008). Sfard uses the following example of an object-level-rule: "To divide a by b, you have to find a number c such that $b \cdot c = a$" (Sfard, 2008, p. 201). The metadiscursive rules are no longer concerned with the content of a verbal utterance, but with its situationally appropriate integration into the course of a classroom discussion, such as 'Always explain how you arrive at the result of a calculation'. Sfard (2000, p. 166) identifies sociomathematical and social norms as instances of metadiscursive rules. These provide a promising foundation for the examination of interpretation-related norms as pursued here.

Voigt (1994) and Yackel and Cobb (1996) employed the same data set to reconstruct sociomathematical norms. Voigt (1994) posits that sociomathematical norms serve as value criteria for mathematical activities within the classroom setting. These norms, he asserts, do not function as definitive rules within the social dynamics of the classroom; rather, they typically operate as implicit orienting principles (ibid., p. 107). A sociomathematical norm, for instance, may delineate the

parameters of an acceptable interpretation, a desirable linguistic description, or a successful operational modification of a representation. As Voigt asserts, the criteria in question are negotiated within the classroom setting and cannot be reduced to the expectations of the teacher or the individual objectives of the students. As Voigt (ibid., p. 106) notes, learners typically regard the teacher as a knowledgeable representative of mathematical culture and, as a result, tend to take the teacher's presumed expectations into account when formulating their own statements (cf. ibid., p. 106). However, he emphasises that mathematical norms are not constraints imposed upon the students by the teacher (cf. ibid., p. 106f.). While teachers have the opportunity to indicate their demands through their assessments, it is ultimately up to the students to decide whether they orientate their actions and interpretations accordingly. For Voigt (ibid., p. 105), only those rules that are recognised as valid by both sides are considered norms.

Yackel and Cobb (1996) emphasise that the norms are always contingent upon the specific actors involved in the given situation: "[…] what becomes mathematically normative in a classroom is constrained by the current goals, beliefs, suppositions, and assumptions of the classroom participants" (p. 460). The negotiation of norms through interaction enables the development of a shared interpretative frame of reference, or an interpretative interim, which facilitates the joint interpretation of representations.

Two challenges for classroom interaction

An interactionist perspective on the interpretation of representations poses two fundamental challenges that must be addressed to facilitate effective mathematical learning through representation in classroom interaction:

1. The negotiation of meaning: Representations do not convey mathematical content. Therefore, it is necessary to interpret abstract mathematical objects into representations or to ascribe meaning to representations.
2. The establishment of value criteria for the interpretation of representations: The purpose of introducing representations is typically to represent a specific abstract mathematical object. In the context of classroom interaction, the negotiation of value criteria is significant for determining if an interpretation of a representation is appropriate in a given situation.

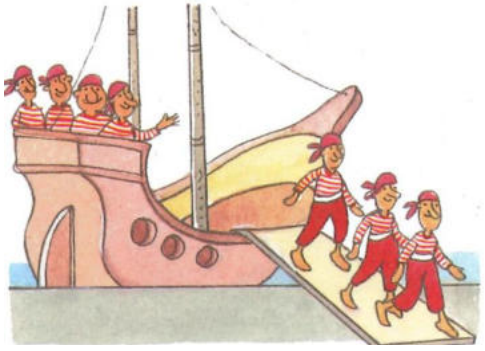

Figure 1 taken from the textbook „Welt der Zahl 1" [World of Number] (Rinkens et. al, p. 40)

We would like to exemplify these two problems using an illustrative representation. This illustration, depicted in Figure 1, appears in a widely used German mathematics textbook in the section on the introduction of subtraction. This indicates that the editors and authors of the textbook consider this representation to be an appropriate means of conceptualising the concept of subtraction of natural numbers and of identifying and developing corresponding mathematical structures within it. From an objective standpoint, it is evident that the depicted action can be interpreted in a multitude of ways.

For instance, it can be expressed as 7-3, 4+3, 4-3, and so forth. Conversely, the objective of the authors or instructor is for the learners to discern that, of the seven pirates on the ship, three pirates disembark, leaving four pirates on the ship. The discrepancies between the feasible and the intended interpretations of the representation can be contemplated in the context of the tension between the representation and the abstract mathematical concept. In consideration of the position in the textbook, it becomes evident that there is meant to be a single appropriate interpretation, accompanied by a multitude of situationally inappropriate ones. This necessitates the negotiation of value criteria that determine which interpretation is preferable in the context of the classroom interaction in accordance with the requirements of the present situation. The following example of a classroom interaction serves to illustrate this assumption.

Illustrative Example: Reformulation

The following scene is taken from a classroom observation of a second-grade lesson on the introduction of multiplication.

Teacher:	My aim in the weeks before the March holidays is to make sure that you understand what a multiplication task is. So, who's going to go ... six times and get two cubes each time? (02:46-06:02) Okay, what do you see in the centre now? ... Hannah.
Hannah:	That's twice as many as six.
Teacher:	Mmh. That's quite interesting information. Liam.
Liam:	A bunch of cubes.
Teacher:	Mmh. What do you see in the centre? ... What is there? ... Emma.
Emma:	That's ... um ... That's six times two. So there are always two cubes, but if you put them together, there are always six.
Teacher:	Well, what would the plus task be called, a suitable one? (...) Jakob.
Jakob:	Two plus two plus two plus two plus two plus two plus two.
Teacher:	[writes '12' on the whiteboard] But now it's like this and you've already said 'Phew, that's quite long!', that's too tedious for me and it could be written differently, namely as a multiplication task. ... Stevan.
Stevan:	You write down six. Then one point. And two.
Teacher:	Exactly. Great. [writes '$6 \cdot 2 = 12$' on the whiteboard] So, these six twos become six times two. (...)

Figure 2 Six times two cubes

The scene opens with the staging of a representation, for which the instructor requests that a student retrieve "six times [...] two cubes" and place them on a table within the view of the entire class. The teacher provides precise instructions regarding the repeated addition of two cubes, enabling learners to build on their existing knowledge and interpret the enacted representation within the context of addition as "adding". After preparing the representation, the teacher requests an interpretation of the representation (Figure 2): "Okay, what do you see in the centre now?". In consequence, she does not respond to the multiplicative interpretations put forth by Hannah and Emma, but rather concretises her expectation: "Well, what would the plus task be called?" She makes an intermodal alteration of the representation from the concrete cubes to the symbolic term by noting the repeated addition of 2 on the whiteboard. She requests the result, "12", and repeatedly emphasizes that this calculation is

"too lengthy" and "too tedious", thus sociomathematically motivating a shorter and more concise method of writing the equation.

Subsequently, the teacher initiates an intramodal alteration of symbolic representation by indicating the corresponding multiplication task associated with the repeated addition and explicitly articulating the transformation of representation: "$2 + 2 + 2 + 2 + 2 + 2 = 12$" becomes "$6 \cdot 2 = 12$". Although the teacher provides the additional clarification that "these six twos become six times two", she does not offer a reinterpretation of the additive structure as a multiplicative structure in the cube representation, thereby omitting any substantial justification for the reformulation of the equation.

Discussion

This scene on the interactive standardisation of representations is to be understood as an illustration of a key insight. The process of developing shared meanings from individual interpretations is inherently complex. It is shaped by the interplay between learners' prior experiences, teachers' normative guidance, and the negotiation of sociomathematical norms within collective argumentations.

The teacher's reactions to the learners' contributions and her own argumentative behaviour in this scene indicate her expectations. In particular, the enacted representation should be interpreted as a case of repeated addition. Although she appears to accept the learners' divergent interpretations and acknowledges them by marking them as "interesting information", she largely disregards the learners' interpretations of the representation and explicates the intended meaning as the foundation for further thematic development. In doing so, she makes a decisive change to the object of study, abstracting from the cube representation to the symbolic notation of the calculation, which from then on is at the centre of the discussion. There is no explicit connection made between representation and term, which would be desirable in terms of understanding the concept of multiplication. Instead, the term is reformulated on a symbolic level. On this note, the teacher introduces the objective of maintaining terms in the shortest possible format.

From a methodological standpoint, the theoretical framing of norms for the interpretation of representations as situationally negotiated binding rules allows us to reconstruct thematic developments on both levels: the negotiation of meaning and the negotiation of value criteria to characterise a situationally appropriate interpretation. The observation-based assumption that the reconstructed processes exhibit a certain regularity in terms of interaction patterns motivates the conceptualisation of the interactional negotiation of representation meaning as a discursive practice from a functional-pragmatic perspective. A discursive practice is defined as an always situated, socially regulated linguistic-interactional procedure for solving a repeatedly occurring interactional problem (cf. Tiedemann, 2020). In this case, the problem arises from the necessity of interactively establishing representations in mathematics lessons in a manner that ensures their interpretation is sufficiently similar by teachers and learners. The rationale behind the development and utilisation of such practices can be attributed to the existence of discrepancies between the learners' individual interpretations and the normatively intended interpretations of representations (cf. "conflicting interpretations" in Schulz and Tiedemann (2020); "deviating subjective interpretations" in Kollhoff (2022)).

From this perspective, processes of sociomathematical standardisation in classroom interactions are not merely situational solutions; they also represent an opportunity to establish rules for interpreting representations in the field of tension between mathematical conventions and individual constructions. This allows for the interactive stabilisation of scientific consensus in the classroom, thus facilitating the resolution of tensions between mathematical conventions and individual learners' construction processes.

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Conceptualising subjective relevance of mathematics education

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In this essay, we understand subjective relevance of mathematics education as the relevance that students attribute to their engagement in institutionalised mathematics education. After presenting reasons for studying this concept, we ask how to conceptualise it. We propose to test possible theories, all of which have already been used in publications in the field, against a list of features of subjective relevance. Here, we discuss the suitability of the concepts of beliefs, motives, and utility value. We conclude that these concepts may be promising to capture some cognitive aspects of subjective relevance but fail to explain the developmental nature of subjective relevance. Also, these concepts struggle with explaining the social entanglements of subjective relevance and do not facilitate reflection on the role of subjective relevance for mathematics education as a social institution.

Keywords: Subjective relevance of mathematics education, beliefs, motivation, expectancy-value.

Introduction

The last decades have seen a growing interest in the relevance that students see in mathematics education. For example, Kislenko et al. (2007) compared two Likert-scale questionnaire studies with Norwegian students from grades 6 and 9, including questions on their "own evaluation of importance of mathematics" (p. 352), and found that students mostly agreed to the statements that "mathematics helps those who make important decisions" and that "good mathematical knowledge makes it easier to learn other subjects", but not that "mathematics helps me to understand life in general". Dobie et al. (2021) collected questionnaire and interview data from 84 US-American seventh-grade students, analysed it with the concept of "perceived utility", and found that utility was predominantly associated with the use of mathematics in daily life. They highlight the case of Liliana, who finds mathematics useful for money-related activities and sees relevance in learning mathematics in order to be able to help friends and family with money-related challenges. However, Liliana also expressed "some scepticism, as she worked to make sense of whether math was truly useful" (p. 419). Chronaki and Kollosche (2019) studied subjective relevance as an aspect of identity work of students of mathematics in order to make sense of the case of Anja, a ninth-grade student from Germany who denied any relevance of mathematics education for her life.

In this paper, we understand "subjective relevance of mathematics education" as the relevance that students attribute to their engagement in institutionalised mathematics education. One could just as well study "subjective relevance of mathematics" as related to the academic discipline in broader terms. We decided for the narrower focus on mathematics education, because our main research interest lies in understanding mathematics education as a social institution. Especially, we refuse to narrow "engagement" down to "learning", as students can find mathematics education relevant for very different reasons (see, e.g., Kollosche, 2017), including the self-discipline that mathematics requires, the objectivity of mathematical knowledge, and the utility of mathematics in daily life, in targeted vocations, or in society in general. The categorisation of such reasons is subject of a different essay (Wang & Kollosche, 2025). Nonetheless, while this distinction serves conceptual clarity, it

becomes blurred when put into practice: On the one hand, experience and our (yet unpublished) data show that most students identify the discipline of mathematics with what is happening in their mathematics classrooms. On the other hand, even if someone is able to distinguish mathematics as an academic discipline from mathematics education in school, it is fair to assume that subjective relevance of mathematics and subjective relevance of mathematics education would interact strongly. This blurriness is present when reporting on empirical studies in the field which use different concepts to frame their results, for example, Kislenko et al.'s (2007) concept of "importance", which could mean subjective relevance in our sense just as well as societal instead of individual importance.

The subjective relevance of mathematics education is worth studying for various reasons:

- We can ask in how far a student's subjective relevance fits to what is offered in mathematics education. Different studies from Western countries (e.g., Sachdeva & Eggen, 2023) have repeatedly shown that students have very limited and fragile accounts of the relevance they see in mathematics education. Studies show that a lack of subjective relevance constitutes an obstacle to identifying with or becoming motivated for mathematics education (e.g., Chronaki & Kollosche, 2019). If more aspects of the relevance of mathematics education were available for students, they might face better conditions to productively engage with mathematics.
- We can also ask in how far what is offered in mathematics education fits what students find to be relevant for them. Such a perspective might reveal that mathematics education includes offers for some interests but not for others – for example, offers for students who want to follow rules and experience flow in batteries of exercises, but no offers yet for students who want to work on complex problems collaboratively (e.g., Andersson et al., 2015). Ultimately, an awareness of possible aspects of the relevance of mathematics education would be a condition for emancipating students to express what they expect from mathematics education.
- Eventually, subjective relevance can be seen as a functional part of mathematics education as a social institution, helping as a device to train specific subject positions towards mathematics. For example, if students disengage from mathematics because they have learned to find it irrelevant to them, then they become more easily subjected to socio-political decisions which are allegedly supported by mathematics (Fischer, 1984). More generally, such an institutional perspective on subjective relevance can help us reflect on the roles that mathematics and its education play in society, for example, concerning the question if it is a function of mathematics education to train and measure the ability to concentrate. A deeper understanding of such functions might also prove important to legitimise school mathematics.

We hope that this short introduction to the nature and importance of the subjective relevance of mathematics education convinces the reader that this concept is worth studying. However, although it has been touched upon in various studies under different terms, there is no common conceptual basis for what we understand as the subjective relevance of mathematics education. Kislenko et al. (2007) framed their research along the concepts of *belief* and *attitude* in mathematics education. Dobie et al. (2021) used the psychological concept of *utility value* to inform their approach on "perceived utility", and other concepts from motivational psychology, such as the concept of *motives,* appear promising to illuminate the motivational aspects of subjective relevance. Vollstedt (2011) approached subjective relevance through the concept of meaning (German: *Sinn*). Chronaki and

Kollosche (2019) understood subjective relevance as a part of students' mathematical *identities*, which are constantly developed within a social discursive field. Given this variety of theoretical framings, we ask *what these theoretical frameworks offer to conceptualise the subjective relevance of mathematics education and where its limitations lie*.

The search for a helpful theory is not straightforward. In fact, it could be that there is no fitting theory at all, which can easily be adapted to explain what subjective relevance is. More likely, we will find that some theories serve well to explain some aspects of subjective relevance but not others. But how to determine that? We propose to lay down some key features of the concept of subjective relevance. A theory fits if it does not contradict these features, and it helps all the more if it is able to explain them well. This process does not have to be chronologically linear. In fact, a feature of subjective relevance might not be striking until facing a theory that contradicts it, whereupon this feature can be described and serve as a basis to judge that theory. In our written presentation of the results of such processes, in order to have a basis on which to judge the suitability of such theories, we first present key aspects of the subjective relevance of mathematics education, which a suitable conceptualisation will have to capture. Due to space limitations, we will only discuss the concepts of beliefs, motives, and utility value here. This approach not only provides insights into which aspects of these concepts can help to conceptualise subjective relevance, it also helps us to further develop the features of subjective relevance and allows us to judge the fruitfulness of this approach in general.

Features of subjective relevance of mathematics education

Past research has shown that there are features of the concept "subjective relevance of mathematics education" that a theoretical conceptualising should be able to address. These include:

1. Subjective relevance of mathematics education has both a *cognitive* dimension in that it describes what individuals think and how they behave on that basis as well as a *social* dimension in that it is constructed in a field of social discourses.
2. A conceptualisation of subjective relevance should address its importance for motivation (in providing reasons to engage with mathematics education), emancipation (by allowing to formulate expectations in mathematics education), and our understanding of mathematics education as a social institution.
3. Between individual accounts of the subjective relevance of mathematics education, there are differences, but also recurring patterns, which can be described as *dimensions* of subjective relevance. Such dimensions include the idea that mathematics education is relevant because mathematical qualification will help oneself in private life. Students hold different sets of such dimensions, and the dimensions can be formulated as ideal types to study them further.
4. Different dimensions of subjective relevance can be based on one and the same feature of mathematics education. For example, some students find mathematics education relevant because mathematics "knows only right and wrong", while other students find mathematics education irrelevant for the same reason (Kollosche, 2017). A conceptualisation of subjective relevance will have to consider that different people judge similar experiences differently.
5. Subjective relevance of mathematics education develops (e.g., within an interview) and can be fragile (e.g., notwithstanding spontaneous questioning) (Chronaki & Kollosche, 2019).

Mathematics-related beliefs

Subjective relevance of mathematics education has been studied in research on *beliefs* in mathematics education research. In their seminal "quest for conceptual clarity and a comprehensive categorization", Op't Eynde et al. (2002) arrive at the definition that "students' mathematics-related beliefs are the implicitly or explicitly held subjective conceptions students hold to be true about mathematics education, about themselves as mathematicians, and about the mathematics class context" (p. 27). They shortly refer to cognitive theories of learning, which identify "conative and affective factors […] as constituting elements of the learning process" (p. 14), and offer a categorisation where beliefs stand side-by-side with knowledge, heuristics, metaknowledge, and self-regulatory skills as factors influencing learning. Further elaborations of how beliefs integrate into cognitive theories of learning are scarce, but there are studies illuminating how beliefs influence problem-solving activities in mathematics education (Schoenfeld, 1989).

Although Op't Eynde et al. (2002) stated that "a person's beliefs about what counts as a mathematical context and what (s)he finds interesting or important will, as such, have a strong influence on the situations (s)he will be sensitive to, and whether or not (s)he will engage in them" (p. 15), and although prompts such as "I find mathematics important" had been studied in beliefs research (p. 24), their quest for a categorisation shows that subjective relevance was no explicit focus of beliefs research, nor was it in any way conceptualised. Nevertheless, a few studies use the concept of beliefs to address such issues (e.g., Kloosterman, 2002, see below). Other studies did not prove relevant here and usually only refer to the concept of beliefs to loosely locate their research within a research field without using the features of the concept (e.g., Gijsbers et al., 2020).

We will use the above features of the subjective relevance of mathematics education to judge how fruitful it is to understand subjective relevance as a part of students' mathematics-related beliefs. Beliefs research allows to understand recurring dimensions of subjective relevance of specific beliefs and acknowledges that students hold different beliefs. However, while beliefs are often considered to be enduring and difficult to change, the same cannot be said about subjective relevance, which has been found to be able to change quickly and drastically. The description by Op't Eynde et al. (2002) proposes that beliefs might be suitable to capture the cognitive aspects of subjective relevance, but the cognitive features of the concept of beliefs are not well developed, for example, by expanding an existing psychological theory. A definition of beliefs as "subjective conceptions students hold to be *true* about mathematics education" (Op't Eynde et al., 2002, p. 27 emphasis added) would be a serious problem, as we might hold something to be true (e.g., that cigarettes are bad for us) and still not act accordingly. Likewise, it might be possible to hold it to be true that mathematics education is relevant for oneself for a specific reason, but this idea might have no influence on how one engages with mathematics education. A possible social dimension is not discussed, and their categorisation of factors influencing learning does not refer to the social environment of students. Further, beliefs research does not provide useful methodological input, as most studies in beliefs research use closed instruments such as Likert-style questionnaires and would thus not be open to the identification of new dimensions of subjective relevance, nor do the minimalistic answers to such questionnaires allow for a study of the social circumstances and implications of specific dimensions (Kloosterman, 2002). Eventually, even qualitative studies on beliefs focus on the manipulation of students in order to

improve learning outcomes (e.g., Kloosterman, 2002) and neither problematise this goal nor take any emancipative or institutional perspectives. We conclude that understanding subjective relevance of mathematics education as a part of students' mathematics-related beliefs would dramatically curtail our understanding of what we want to conceptualise.

Motives and expectancy-value theory

Motivation is the willingness of an individual to perform an action, such as learning mathematics or refusing to learn mathematics. Motivation theory assumes that motivation is negotiated not only by the situation where individuals are urged to act, but also by their motives, which are features of the individual. Reiss (2016) understands motives as "universal goals that motivate significant behavior" (p. 576), and Wigfield and Rosenzweig (2016) point out that "motivation and cognition influence one another" (p. 571). On the one hand, motivation influences how far a person engages in cognitive activities, which cognitive resources this person activates, how persistent this person is, and how much this person can learn. On the other hand, how people think about certain actions, goals, and about themselves will influence their motivation.

Motivation theory is a field with dozens of partly conflicting and yet prominent theoretical offers. It is clear that we cannot cover these offers here, but we want to highlight two very active strands in the field: First, there are a number of theories which Wigfield and Rosenzweig (2016) argue "have had strong social cognitive foci in that individuals' beliefs, values, and goals are the major components of these theories" (p. 570). They list Weiner's attribution theory, expectancy-value theory, Bundura's social-cognitive theory, and self-determination theory as examples. Second, there is a research strand focussing on universal needs of humans – universal in the sense that these needs appear frequently among humans. Reiss (2016) speaks of 328 different goals discussed in the literature, including, for example, such needs as acceptance ("the desire for positive self-regard"), curiosity ("the desire for understanding"), dominance ("the need to control and lead others"), infavoidance ("the need to avoid failure/shame"), or nurturance ("the need to protect the helpless") (p. 577).

Different from the basic needs, whose significance for mathematics education has, as far as we know, not yet been studied, theories from the strand that Wigfield and Rosenzweig (2016) refer to have already been applied to mathematics education research, for example, by Kloosterman (2002). He assumed "that motivation is a cognitive activity and that students make specific choices about how and when they apply effort to learn" and that "these choices are based on students' beliefs" (p. 248).

For an exemplary look into one special theory, we will discuss the expectancy-value theory. In their study on how students perceive the "utility" of mathematics, Dobie et al. (2021) referred to the concept of utility value. Alongside attainment value and intrinsic value, utility value is one category of values in expectancy-value theory (Eccles, 1983). In its simplest form, this theory assumes that achievement motivation is a function of both the probability of success and the values associated with the goals of action in question. Thereby, the attainment value is understood as the importance for someone to do something well, the intrinsic value is understood as the pleasure that an individual feels while performing an activity, and the utility value is understood as the importance of an activity in order to achieve future goals. Since its introduction by Atkinson in the 1950s, expectancy-value theory has seen further conceptualisations of expectancy, such as self-concept of ability, perception

of task difficulty, perception of other's expectations, and causal attributions of success, as well as further conceptualisations of value (ibid.). Dobie et al. (2021) use the theory to theoretically attest the importance of utility values. This perspective would be too narrow for a conceptualisation of the subjective relevance of mathematics education, as not all of its dimensions are directed at future achievements. However, expectancy-value theory, in general, would be open to include further values in order to cover dimensions of subjective relevance beyond utility.

Again, we will use the above features of the subjective relevance of mathematics education to judge in how far it is fruitful to understand subjective relevance through motivation and expectancy-value theory. The dimensions of subjective relevance can be understood as motives for engagement in mathematics education. It has become clear that motivation theory offers different approaches to understand the cognitive dimension of subjective relevance, for example, through expectancy-value theory. Also, there are motivational theories that acknowledge that motives can be socially influenced (Kumar, 2016). Nevertheless, in descriptions of motivation theory (e.g., Eccles, 1983; Kumar, 2016; Reiss, 2016; Wigfield & Rosenzweig, 2016), the question of how the social environment influences motives is not central, if posed at all. Also not central to motivation theory is the question of how motives develop and in how far they can collapse. Instead, motives are often regarded as more or less stable properties of the individual (Reiss, 2016). Eventually, though not surprising, it has to be noted that motivation theory does not easily allow to focus on emancipation through mathematics education or on the understanding of mathematics education as an institution. Although it seems fair to assume that motivation theory has this potential – that, for example, a better understanding of students' motives would allow us to change mathematics education for the better or at least to understand its institutional confinements –, critical psychology points out that the predominant goal of motivation theory is to manipulate workers to be better workers, students to be better students, etc. (e.g., Osterkamp, 1975).

Conclusion

First of all, it proved fruitful to depart from features of subjective relevance of mathematics education as a basis to judge the suitability of possible theories for the conceptualisation of this concept. During the process, some features were found to be irrelevant, while others had to be added or elaborated. This allowed us to develop judgements that we hope to be transparent.

The list of features helps us to understand what we mean by subjective relevance. Where easily possible, we included references to studies where the relevance of certain features has been established. More references could be added for all listed features if space limitations allowed.

It appears to be possible to understand dimensions of subjective relevance as "beliefs" in the sense of subjective conceptions students hold to be true about why mathematics education is relevant for them. The concept of beliefs might allow to capture the cognitive aspects of subjective relevance, but the benefits of such a conceptualisation are not easily visible as the concept of beliefs appears not to have a strong theoretical foundation. Likewise, it appears to be possible to understand dimensions of subjective relevance as "motives" for one's engagement with mathematics education. This approach is more promising, for there are many theoretical offers on motivation theory, which have been developed over decades. Expectancy-value theory would be one such offer. Its strength lies in not

only focusing on motives as values but also on expectancy, which is largely influenced by the experiences in the mathematics classroom. Concerning values, the distinction between attainment, intrinsic, and utility value does not appear to be sufficiently accurate, but different approaches in motivation theory might allow the development of a more detailed categorisation of possible motives (Reiss, 2016).

Both beliefs research and motivation theory seem unfit to explain the fragile and developing nature of subjective relevance, as their concepts are considered to be more stable and difficult to change. Further, both fail to describe the social dimension of subjective relevance and its importance for students' emancipation and the understanding of mathematics education as an institution. This might not come as a surprise, as both theories are psychological in origin and thus focus on an understanding of the individual. From such a perspective, social questions are only relevant as long as they allow for a better understanding of the individual, but not for their own sake.

An alternative might be to use sociological theories. For example, Chronaki and Kollosche (2019) used discourse theory to understand subjective relevance of mathematics education as a part of the discourse that an individual develops about the relationship between the self and mathematics. Locating subjective relevance in such a discursive field allows to better understand the fragile and developing nature of subjective relevance (as discourses are easily changeable), its entanglements with the social environment (e.g., with stereotypes about mathematics), and its functions for mathematics education as an institution (e.g., concerning the question which sentiments towards mathematics are developed in school). Yet, such a sociological theory might leave open questions concerning the conceptualisation of the cognitive dimensions of subjective relevance, which might, in the end, call for a triangulation of psychological and sociological approaches.

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Pre-service teachers' developing professional identity as mathematics teachers – a theoretical perspective

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In this paper we will launch the theoretical perspective Patterns of Participation as a lens to study pre-service teachers' developing professional identity as secondary mathematics teachers, through the notions of being, becoming, and belonging. It is documented in the research literature that pre-service teachers experience a gap between the university and the school as learning contexts. In our study we address tensions that manifest themselves in the development of professional identity, as the pre-service teachers participate in these two learning contexts. Furthermore, we introduce the idea of a resource bank, theorised as a boundary object between these contexts. Drawing on an example from another study, we argue that by reflecting on the content of a resource bank, one pre-service teacher testifies her being, becoming and belonging.

Keywords: Communities of practice, patterns of participation, pre-service teachers, professional identity, secondary school

Introduction

In this paper we will launch the theoretical perspective Patterns of Participation (Skott, 2013, 2019) as a lens to study pre-service teachers' (PSTs) developing professional identity as secondary mathematics teachers. The importance of studying professional identity formation among pre-service teachers has been advocated by Arvaja and Sarja (2021). Skott (2019) argued that teachers draw on previous and current practices as they participate in classroom interactions. Unlike in-service teachers, PSTs engage in practices both at the university and in schools through school placements. Their professional identity develops across these two learning contexts, distinguishing them from in-service teachers whose identity is primarily shaped within the school context. Moreover, it is well documented in research that the PSTs experience a gap between the university and the school as learning contexts (Goos, 2009; Jenset, 2020). In our study we address tensions that manifest themselves in the development of professional identity as mathematics teachers, as the PSTs participate in these two learning contexts. Furthermore, we build on the idea of a resource bank, hypothesised as a boundary object (Star & Griesemer, 1989) between these contexts.

These tensions stem from the particularities in the teacher education that the PSTs in question participate in. In Norway, those PSTs who are becoming mathematics teachers in secondary school, follow a teacher education in which the mathematics subjects are separated from educational and didactical aspects of mathematics teaching and learning. These PSTs also study mathematics subjects together with undergraduate students in mathematics. Furthermore, the school placement part of their teacher education is more spread compared to primary teacher education. We argue that these PSTs' professional identity formation as mathematics teachers deserve attention in research. Thus, we launch a framework for studying identity formation in this context adopting Patterns of Participation (PoP) (Skott, 2013, 2019) as a lens, accompanied by the construct of boundary object.

Below we will give a theoretical account of Communities of Practice (CoP) (Lave & Wenger, 1991;

Wenger, 1998), as well as PoP, since PoP take CoP as its theoretical point of departure. However, we will first give an account of (mathematics) teacher education in Norway, its regularities and expectations. We also introduce the idea of a resource bank for PSTs to bridge their experience from the two learning contexts university and school.

Teacher education for grades 8–13 in Norway

According to the National Council for Teacher Education (NRLU), the teacher education for grades 8–13 is "integrated, professionally oriented, research based, and experience based and offer the student professional deepening within two school subjects" (2017, p. 4, our translation). For PSTs for grades 8–13 in mathematics, the first or second year of their studies consists of mathematics as a discipline subject, which means they study mathematics alongside students from other study programs. In the third and fourth year, PSTs encounter mathematics education courses and general education courses. During their studies, PSTs must spend at least 100 days in school placement, where they will teach mathematics and another chosen subject (Subject 2). It is the institution's responsibility to organise the school placement so that it contributes to a coherence in the education. That requires collaboration between PSTs, mathematics teacher educators both in the institution and in school, to plan, implement and evaluate school placement. The content of the school placement should be connected to pedagogy and didactics, and build on the PSTs' knowledge in mathematics and Subject 2, to develop their professional knowledge through both theory and practice. Through school placement, PSTs should "gradually be introduced to various aspects of the teaching profession and strengthen their development of a professional identity" (NRLU, 2017, p. 15).

Following the national guidelines presented above, Norwegian teacher education for grades 8–13 should be both research-based and practice-oriented. Moreover, the PSTs are expected to learn and develop professional identity from consolidating experiences in school placement and in university coursework. However, research indicates that PSTs tend to view the two sites as separate and unconnected, appraising practical experience as more valuable than theoretical aspects (Bjerke et al., 2013). In Norway, the disconnection between theoretical and practical aspects of teacher education is a persistent criticism and there is a call to better prepare PSTs to take on the practical work in classrooms (Jenset, 2020). This experienced disconnection also causes deficits in the PSTs' developing professional identities as mathematics teachers.

During the teacher education programme, PSTs participate in the practices of the university site and the school site. When participating in these practices, PSTs are both students themselves and (novice) teachers. The two roles sometimes hold different expectations for and images of teaching mathematics (Goos, 2009). Norwegian teacher education programmes tend to operate on the assumption that laying a foundation of theoretical principles that PSTs are to apply in practical situations will sufficiently prepare them for the practice of teaching (Solomon et al., 2017). However, Gainsburg (2012) observed that PSTs struggle to connect theoretical and practical principles of teaching if not supported in this process.

Resource bank as a potential boundary object

A recent study introduces a resource bank for teaching mathematics, exploring how the resource bank can support PSTs in connecting experiences from the practices at university and the practices at school placement (Sødal, 2023). The approach builds on Gueudet et al. (2019), who theorise teacher learning

and professional development through interaction with resources. Sødal (2023) inquired into how a resource bank may support PSTs in connecting experiences from coursework and school placement, thereby bridging the gap between the university site and the school site. Each PST developed a resource bank which contained for example mathematical tasks, activities, lesson plans and research related to mathematics teaching. Moreover, experience from interactions with students, mentor teachers, peers and teacher educators were also considered and potentially included in the resource bank through entry-slips (short reflections on why a resource was included and how it was to be enacted). The resource banks were developed both in connection to coursework and during school placements, and in this way, they were developed on a continuum across the boundary between university and school. Sødal (2023) studied how the practices of university and of the schools shaped the PSTs' process of developing the resource bank. The resource bank mediated PSTs' learning across practices, for example as the PSTs updated their entry-slips following the enactment of a resource.

We thus propose the idea of a resource bank as a means to explore PSTs' development of professional identity as they participate in different university and school practices. Star and Griesemer (1989) launched the analytical concept of boundary object to address scientific objects that encompass intersections between different social worlds. The authors argue that boundary objects "have different meanings in different social worlds" (p. 393). We hypothesise that the PSTs' resource bank may become a boundary object between different practices. At the university site, the resource bank (including resources from courses and school placements) serves the purpose of operationalising theoretical ideas, while at the school site the resource bank serves the purpose of implementing these operationalisations into the classroom practice.

In their review of boundary crossing and boundary object, Akkerman and Bakker (2011) identified four learning mechanisms in crossing boundaries: identification, coordination, reflection, and transformation. Identification addresses central demarcations between the intersecting practices of university and school. Coordination allows means and procedures to be used across the intersecting practices. Reflection is "coming to realize and explicate differences between practices and thus to learn something new about their own and others' practices" (pp. 144–145). Transformation involves confronting the tensions that emerge between the intersecting practices, resulting in changes in the practices. All four mechanisms play a role in developing professional identity as mathematics teachers. As we will argue, professional identity development may be viewed through the lens of Patterns of Participation (PoP), as being, becoming and belonging (Skott, 2019).

Professional identity as a social construct

Wenger (1998), in his seminal book 'Communities of practice. Learning, meaning, and identity', gives an account of what he calls a community of practice. As we, in this study, address the development of professional identity amongst pre-service mathematics teachers for grades 8–13, the communities we are talking about are the varied social constellations that pre-service mathematics teachers participate in during their education and the practices are what these communities do.

According to Wenger (1998), a community of practice is characterised by mutual engagement, a joint enterprise, and a shared repertoire. It is beyond the scope of this paper to do full justice to making explicit these features. Nevertheless, we would like to mention that mutual engagement is what defines the community, «what makes a community of practice a community is mutual engagement»

(Wenger, 1998, p. 75) – in our case the PSTs' participation in the community of mathematics, mathematics education, teaching and learning of mathematics in the classroom, etc. In order for a community to be sustainable, a joint enterprise is needed. Despite plausible disagreements, participants still invest their energy into discussions aiming for some agreement. In a community, participants make efforts to achieve agreement on the essence of the enterprise, since «defining a joint enterprise is a process» (Wenger, 1998, p. 82). In this process, a separate discourse develops comprising terminology, expressions and metaphors, use of tools and actions. This discourse is 'local' in the sense that each community will be characterised by 'local' negotiated expressions and metaphors, some of which overlapping with more general ones. Wenger calls this a shared repertoire. "The repertoire of a community of practice includes routines, words, tools, ways of doing things, stories, gestures, symbols, genres, actions, or concepts that the community has produced or adopted in the course of its existence, and which have become part of its practice" (Wenger, 1998, s. 83).

In the process of developing mutual engagement of a joint enterprise, characterised by a shared repertoire, Lave and Wenger (1991) argue that participation takes place at various levels. However, among several constituents of a community, Wenger (1998) emphasises that the development of personal identity plays an important role in a person's participation in a community. Due to space limitations, we cannot possibly give a full account of identity as a construct. However, we adopt Wenger's point of departure, that identity is profoundly connected to a practice entailing "the negotiation of ways of being a person in that context" (p. 149), in our case the practice of mathematics teaching. Furthermore, "our identity includes our ability and our inability to shape the meanings that define our communities and our forms for belonging" (p. 145). Thus, identity is characterised as negotiated experience, membership in a community, a learning process, and membership of multiple communities in which belonging is globally and locally negotiated.

Skott (2013, 2019), in his framework Patterns of Participation, draws in part on social practice theory (Wenger, 1998; Holland et al, 1998) to understand the teacher's role in classroom interactions. Rather than seeing the teacher in the classroom as a strategic individual acting from a firm set of beliefs, knowledge and skills, the teacher is seen as a reflexive responder to pupils' needs and the evolving classroom situations, drawing on previous experiences of being, becoming and belonging, again providing new experiences that allows for continues transformations of the teacher's role. Thus, for Skott (2013, 2019) identity is not static but in flux.

A teacher is and has been participating in a multitude of practices that may form his reflexive acts in classroom interactions (Skott, 2013), including local communities of practice such as mathematics classes in school, courses in teacher education, professional development programs, colleagues at former and current schools, mathematics clubs, etc. While participating in these different practices, the teacher may also have encountered cultural and more generalised practices. For example, reformed mathematics teaching or ideas from theories of learning, such as zone of proximal development, the importance of student talk, and conceptual learning. To incorporate such practices Skott used the concept of figured worlds drawing on Holland et al. (1998). They argued that identity is the views one has of oneself in worlds of action and involved two contexts of activity: (1) the immediate here and now interaction; and (2) figured worlds, i.e. socially and culturally constructed realms of interpretation inhabited with actors, significations of acts and different values assigned to different outcomes.

Thus, considering the many practices and figured worlds that may form a teacher's role in classroom interactions, Skott (2013) argued that social practice is not sufficient as the unit of analysis, which is often used within social practice theory. Therefore, he argued, a recentring of the individual is necessary to understand how a teacher's professional identity take form as PSTs or teachers move in and out of different social constellations. To do so, Skott incorporated the idea of self. The self is simultaneously the process and product of action and reflection on action and has two phases: the *I* that act; and the *me* that takes the attitude of others and interpret the acts instantaneously. Participating in interaction consists of two processes, one of acting including indicating others' action, and one of interpreting others' actions symbolically and ones' own actions in terms of models of others' perspective. Action and meaning are inherently linked and are informed by previously established norms and histories. The unit of analysis is the individual in discourse, including current and previous practices and figured worlds that are actualised as generalised others.

Skott (2019) argues that professional identity develops through "shifting experiences of being, becoming and belonging as they relate to the profession" (p. 469). In terms of PSTs, this may include experiences of being competent or challenged in using resources for teaching mathematics, experiences of becoming autonomous as a mathematics teacher in the classroom, and the experiences of (not) belonging to a community. In the development of a professional (mathematics teacher) identity, several tensions can emerge.

According to Pillen et al. (2013), novice teachers face three main categories of tensions: "(1) the change in role from student to teacher; (2) conflicts between desired and actual support given to students; and (3) conflicting conceptions of learning to teach" (p. 660). In their study, they found that these tensions usually were accompanied by significant emotions experienced by the novice teachers. Thus, as Skott (2019) argues, to study the development of (professional mathematics teacher) identity, it is vital to study (mathematics) teachers' experience as they draw "on a multiplicity of prior and present practices and figure worlds when facing the challenges" (p. 472) or tensions emerging in their profession.

Approaching the development of professional identity as mathematics teachers

As mentioned in the introduction, we aim to address the tensions that manifest themselves in the PSTs' development of professional identity as mathematics teachers, as they participate in practices at university and school. In a Norwegian context, this may involve studying PSTs from the beginning of their teacher education when they study undergraduate mathematics and make initial experience with school placement as well as educational and didactical issues with respect to mathematics teaching and learning.

To exemplify how to identify tensions in PSTs' development of a professional identity as mathematics teachers, we will provide an example from data collected for a different purpose than ours. Nevertheless, the analysis of excerpts from a questionnaire and an interview of Ingrid, a primary school PST, testifies that Patterns of Participation shows promise for addressing PSTs' developing identities as mathematics teachers, despite not providing insights into all aspects of being, becoming and belonging (Skott, 2019).

Example

Ingrid worked with her personal resource bank in her fourth year of teacher education. She included

resources from different practices, some were adapted from her previous school placements while others were developed based on inspiration from fellow PSTs or lecturers at the university. During school placement (five weeks), Ingrid struggled to put the resource bank into use, partly because she experienced her mentor teacher to be fixed on a particular resource for teaching mathematics (a textbook called Dragonbox school). This led to a tension between what Ingrid wanted to do and what she was allowed to do in her teaching (experience of not belonging to the same community as her mentor teacher). Ingrid was concerned about her students' learning of mathematics, and in her opinion the chosen textbook material did not encourage learning ("drag and drop" which instantly gave feedback on whether the answer was correct or not, thus allowing the students to guess).

Prior to school placement, Ingrid was particularly concerned with including manipulatives in her teaching (exemplifying her being). Most of the resources in her resource bank included the use of manipulatives. After her school placement, the use of manipulatives was still important for Ingrid, but now she also stressed the importance of not being dependent on a textbook, but rather being able to develop her own teaching content (testifies her becoming):

> What's important for me when I start working as a mathematics teacher is to have manipulatives available and not be dependent on the textbook. I want to be able to do things my own way, following the curriculum and competence aims without having to follow the textbook to the letter (our translation).

It may seem that Ingrid's lack of freedom and pressure to use Dragonbox school in her teaching, has given her clarity on how she does not want to be as a teacher, in addition to possibly reinforcing her positive perspective on the use of manipulatives in teaching. She also reflects on her own school experience and how her teachers mostly had students work with problems from the textbook. This also reinforces Ingrid's desire to put the textbook aside, and she notes that the resource bank will be valuable for facilitating other teaching activities. She points out that working with the resource bank has helped her become aware of the type of mathematics teacher she is (testifies her being):

> Because I actually thought I was very much … just following the book. But I see that I have become better at not following the book to the letter and at using other things, and that I use a lot of manipulatives. I haven't really noticed this until I started writing down these tasks. I really enjoy using manipulatives, and it helps me in my teaching. I find it easier to explain [the mathematics].

Concluding remarks

In this paper we have made efforts to address tensions in PSTs' developing professional identity as mathematics teacher through the lens of Patterns of Participation, through the notions of being, becoming and belonging. Furthermore, we have attempted to theorise the gap in the PSTs' participation in various practices at the university site and various practices in school placements, materialised through the boundary object of a resource bank. As a participant (being and belonging) in various practices (seminar sessions with the subject teacher and fellow PSTs, and in school placement with mentor teachers), we argue that Ingrid was inspired to develop her resource bank, which in turn made her more aware of her own development as a mathematics teacher (becoming).

We have argued that professional identity development may be viewed through the lens of PoP, as being, becoming and belonging (Skott, 2019). Akkerman and Bakker's (2011) four learning mechanisms in crossing boundaries (identification, coordination, reflection, transformation) may be

linked to Skott's definition on professional identity. Identification addresses how teachers acknowledge demarcations between different practices and figured worlds, while transformation involves confronting the tensions that emerge between these practices. Ingrid was dragged between the practice of using Dragonbox school for teaching, and what she wanted to do in her teaching. This led to a tension, as Ingrid experienced that she did not belong to the same practice as her mentor teacher. Coordination includes Ingrid's experience of belonging to the various practices. Based on the analysis, there was a lack of coordination, in that she could not use many of the resources she had collected at the university. Reflection is about realization (being) and about broaden one's perspective (becoming). Ingrid realized that she was particularly concerned with including manipulatives in her teaching, and during school placement, she stressed the importance of being able to develop her own teaching content.

The framework discussed in this paper, we argue to be appropriate for studying PSTs' professional identity as secondary school mathematics teachers as they participate in various practices. A particular research scope may be to examine the tensions that arise as the PSTs engage in these various practices and reveal insights into their development of professional identity. By providing such insights, perceived disconnections within mathematics teacher education programmes might be mitigated.

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Rationale and methods for an archive and encyclopaedia of "fundamental didactics"

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"Fundamental didactics" designates a line of development of didactics of mathematics which appeared in France in the late 1960s, before being developed around the world. Despite its historical importance, complete bibliographic and documentary resources do not exist yet. This is the main raison d'être of the recent project of elaborating its archives and encyclopaedia. Carried along by a group of researchers under the supervision of a scientific committee, it aims at assembling a body of works together with its commentary, to give an account of the genesis and development of fundamental didactics up to the present day, for expert as well as novice readers. Above and beyond, this project raises the question of how a research community can make progress, by showing how its results can cumulate into a coherent body of conceptual and methodological resources.

Keywords: Fundamental didactics, Theory of Didactical Situations, Anthropological Theory of the Didactic, Reference materials.

Introduction

"Fundamental didactics" designates a line of development of didactics of mathematics, otherwise referred to as the "epistemological programme" (Gascón, 2003), which appeared in France in the late 1960s before being developed in Spain and around the world. It was initiated by Guy Brousseau as the Theory of Didactical Situations and was later characterised by him as the "science of the conditions for the production and dissemination of mathematical pieces of knowledge useful to societies and human affairs" (Brousseau, 1995, p. 2, our translation). In line with this project, Brousseau and his successors have stressed the importance of developing this fundamental facet of research in didactics of mathematics. Assuming certain theoretical and experimental detours, it aims at developing theoretical and methodological means to study didactic phenomena per se, as independently as possible from the direct and sometimes pressing demands of society.

In this spirit, Brousseau inaugurated the Centre for Observation and Research on Mathematics Teaching (COREM in French) in 1972, a kind of "didactic accelerator or didactron" (Barquero & Bosch, 2015, p. 258). Such an observational device quickly enabled previously undetected phenomena to be brought to light, while an intense theoretical work was conducted to account for them. Two connected theoretical frameworks thus emerged in the French context, the Theory of Didactical Situations (TDS) and the Anthropological Theory of the Didactic (ATD, initiated by Yves Chevallard in the 1980s). The result today is a coherent set of concepts and methods for analysing the phenomena of dissemination of knowledge, with an important bibliographic collection. Its function as a fundamental science of didactics makes it very distinctive in educational research.

However, despite its historical importance, complete and well documented bibliographic and documentary resources on fundamental didactics do not exist yet. Consequently, the exact historical

(conceptual and sociological) roots of this significant way of doing didactics of mathematics are not always well known. Half a century after its birth, at a time when the main players in its development are gradually withdrawing from the scene, the preservation of this scientific heritage is at stake. Even if this line of scientific research is still vivid nowadays, worldwide, some of its constitutive texts and concepts remain poorly spread, because they are not available in digital form and are mostly written in French. Furthermore, the notions involved are numerous, representing a vast and interconnected conceptual network, of which today's researchers can for the most part only catch a glimpse.

This is the main raison d'être of the recent project of elaborating the archives and encyclopaedia of fundamental didactics. Carried along by a group of researchers, under the supervision of a scientific committee, it aims at assembling a body of works together with their commentaries, to give an account of the genesis and development of fundamental didactics up to the present day, for expert as well as novice readers. The project raises, however, several questions. What is the exact extent of fundamental didactics? Where to find its original sources and how to make them accessible? Given the corpus we are dealing with, how to design and then write an encyclopaedia reflecting it? More generally, and beyond the strict scope of fundamental didactics, we want to address the question of how a research community can make progress, by taking advantage of the different approaches developed throughout the years and letting the ensuing results cumulate into a coherent body of conceptual and methodological resources.

What is "fundamental didactics"?

As we mentioned already, "fundamental didactics" refers to the ambition, embodied and then formulated by Guy Brousseau to establish an autonomous science of didactics of mathematics, in a context where psychology and linguistics were the predominant disciplines in the field of mathematics education. Brousseau started out as a primary school teacher, with a deep interest in mathematics and its social relevance. In the late 1960s, he gathered a team of colleagues at the IREM[1] of Bordeaux (France) to enquire about issues posed by the teaching of mathematics from an epistemological point of view, particularly at primary school. Through the support of colleagues at both French and European levels, he managed to create the Centre for Observation and Research on Mathematics Teaching (COREM, 1972-1999), associated with the primary school Jules Michelet (Talence, France). There, through a vibrant theoretical-experimental work, together with his team of teacher trainers and school teachers, he elaborated a novel experimental method for mathematics education based on the didactic engineering of situations, the theory of which was carefully worked out (see for instance Brousseau, 1997, p. xv and p. 275; and also Artigue et al., 2019, section 2.1.2; Barquero & Bosch, 2015, section 8.2.2). This experimental method, together with its theoretical grounds, enabled a new scientific practice which raison d'être was to answer fundamental questions on what school mathematics is and how it disseminates, hence the term "fundamental didactics".

Didactics of mathematics in France thus "emerged with the aim of building a genuine field of scientific research and not just a field of application for other scientific fields such as mathematics or

[1] The *Instituts pour la Recherche sur l'Enseignement des mathématiques*, French for Institutes for Research on the Teaching of Mathematics, were founded in 1970 to bring teachers of all levels together with mathematics researchers to discuss issues related to the teaching of mathematics.

psychology." (Artigue et al., 2019, p. 14). Of course, France is not the only country where such an enterprise took place (for instance, see Blum et al., 2019, for the Czechoslovak and German cases). However, due to its size and scope, we have decided to restrict ourselves for the time being to the corpus of the branch of fundamental didactics which originated in France (and has been widely developed internationally since its creation). In the French context, besides the seminal work of Brousseau's group in Bordeaux, another channel of development of fundamental didactics was made possible by Yves Chevallard's breakthrough on didactic transposition phenomena and their subsequent formalisation in anthropological terms. Indeed, the ATD was born from the close collaboration of Chevallard (born in 1946) with his elder Brousseau (1933-2024) around 1980. Since then, the two frameworks evolved in parallel, while always sharing the same theoretical foundation, as Brousseau (2005) himself pointed out during his opening plenary at the first International Conference on the ATD:

> I have the feeling, a priori, that there is no contradiction between their statements, which means that they can be associated. They could even prove to be equivalent: the axiom of choice has a large number of expressions whose equivalence is not obvious. What differentiates them, then, is the choice of the objects they mainly deal with and those they seem to neglect. If none of them contains the other, they should at least be able to "complement" each other (Brousseau, 2007, p. 50, our translation).

Chevallard also explicated in what respect the ATD is grounded in the TDS: "anthropological sensitivity is, as opposed to what one might think, constantly present in the work of G. Brousseau, and has been from the start — even if it is often situated slightly in the background compared to explicitly formulated analyses." (Chevallard, 1992, p. 137). Today, the TDS and the ATD form the bulk of the "epistemological programme" in didactics of mathematics (Bergsten, 2020; Gascón, 2003), which, following Brousseau, we call in this text "fundamental didactics".

Methods and outlook

The goals of our project include a better understanding of the historical interplay between the TDS and the ATD as well as the subtle conceptual differences in their epistemological constructs. As such, it should contribute to identifying a research paradigm (in the sense of Kuhn, 1962). However, many questions arise, as this ambition touches upon the making of an epistemological and social history of a field of knowledge. To go beyond Lakatos' programmes and Kuhn's paradigms, we may rely on the sociology of knowledge constructed by authors such as Norbert Elias (2009) or Pierre Bourdieu (2004). Of course, Gaston Bachelard (2002) himself is also a reference when it comes to studying the epistemological history of a scientific community. Given the size and scope of the corpus at hand, such analyses could be performed by different people. So, the goals of our project also include collecting, sorting and presenting what we consider the necessary sources for such historical and epistemological studies of fundamental didactics, though they may be carried out by others. We consider such surveys to eventually contribute to the development of fundamental didactics itself and its dissemination. However, our undertaking goes beyond the sole case of didactics of mathematics. This project actually raises the question of how our research community in mathematics education can make progress, which is an especially sensitive issue in contemporary social sciences (Lahire, 2023, chapter 2). By assembling this archive and encyclopaedia, we wish to tackle the problem of integrating past results in mathematics education and going forward from a scientific

point of view. Indeed, our project should lead to an in-depth structural work on a huge corpus, which could very well foster new insights on our research methods and theories. Over the years, it could question our research practices at a wider scale. Not necessarily by adherence, but because of the questions it will raise for other approaches as well: What are their roots and rationales? What are their conceptual histories? How do they permit to ground subsequent research practices?

In the following, and to substantiate the previous claims, we will develop the facets of the work as it was launched last year, how it goes on, and what are its prospects for the future.

The stages of the archival work

The historical evolution of the publication of the major texts in the TDS and the ATD followed a similar pattern. In their first years, many (sometimes hand-written) documents were printed on the occasion of a research seminar, the National Seminar of didactics of mathematics (starting in 1978) or a Summer School of didactics of mathematics (starting in 1980). Some of them where then edited by each university's own IREM, and only rarely would they also be published in academic journals. This way, Guy Brousseau and his team in Bordeaux published most of their works until the 1980s at the IREM of Bordeaux, followed by Yves Chevallard and his colleagues at the IREM of Marseille. At that time, the French journal *Recherches en Didactique des Mathématiques* was founded and published by *La Pensée sauvage* in Grenoble (France), where several books were published as well (Brousseau, 1998; Chevallard, 1985).

Fortunately, each one of the two lead authors built and maintained a website, which, though not complete, give an idea of the scope of the corpus at stake. Brousseau's personal webpage (guy-brousseau.com) comprises more than a hundred papers for a total of circa 1,500 pages, while Chevallard's website (yves.chevallard.free.fr) features around 300 articles for some 2,500 pages. From the 1990s, the text written by the numerous contributors to each of the frameworks have been published in research journals, which they have since made available in digital form. Another aspect greatly facilitating our archival work is the growth of open archives such as the French online repository HAL[2], where many of the previously mentioned contributions may be found.

However, given their historical conditions of production, some of these texts are neither on the above-mentioned websites nor otherwise online. Three possible locations then exist. First, some references may still be found in the libraries of French universities, especially in the libraries of the IREMs, despite regular storage space savings. The Centre for Resources in Didactics of Mathematics (CRDM-Guy Brousseau), hosted by University Jaume-I in Castelló (Spain), is a second location where the teaching and learning material coming from the Centre for Observation and Research on Mathematics Teaching (COREM) was archived, by Pilar Orús and her team. Many master's and PhD theses from the 1970s may also be found there. Though its purpose makes it focused on a particular period (1972-1999) and place (the COREM), the CRDM is a significant place for the history of fundamental didactics. Thirdly, a crucial archive is constituted by the personal libraries of the researchers themselves. Indeed, some documents concerning scientific events from the 1970s and 1980s may only be found there. Therefore, what may be important contributions to the early history of fundamental didactics sit in attics or basements somewhere in France, waiting to be sought for. This is even more important that, at that time, foundational issues were discussed, which resonate

[2]*Hyper Archive en Ligne*, which means "hyper online archive" in French (https://hal.science/ HAL).

well with current debates on methods in our community. Finally, pieces may have been lost. At best we can then identify them by their reference in the bibliography of other documents.

After having recovered existing source texts, the archival work will then consist in preparing a digital version of them to be uploaded on an open archive (the French online repository HAL, in our case). Secondly, a mirror archive will be constructed thanks to a reference management software (Zotero in our case) for the users to better navigate it. Given the extent of the archival work, the practical value of the previously mentioned twice-restrictive hypothesis might be better understood: first, to assimilate fundamental didactics in the French context to the coupling of the Theory of Didactical Situations and the Anthropological Theory of the Didactic; secondly, to focus primarily on the works of their respective lead authors, Guy Brousseau and Yves Chevallard.

So, the stages of the making of the archives are the following: 1) collect all the texts written by either Brousseau or Chevallard, 2) reference them in the Zotero library, 3) create their digital version (when it does not already exist), 4) upload them on HAL. Other kinds of documents (audio recordings, videos, photographs) would require a special treatment, which we will not detail here. Then, the same kind of work is to be done on the collective events which punctuated the scientific life of fundamental didactics in the XX century. In particular, the still-ongoing National Seminar (1978 to the present) and Summer Schools (1980 to the present) could be quite momentous at the time, triggering the production of significant texts (courses or seminar articles). The last and open-ended stage of the making of the archive is the adjunction of the complete works of other contributors than Brousseau and Chevallard, starting in chronological order. It is a practical necessity, to organise this vast archival work. Based on the already known literature, together with the crucial testimonies of the colleagues who participated in it, a first historiographical model of the early development of fundamental didactics should be established.

This is why a scientific committee was asked to oversee this archival work. Such a collective, through its combined expertise and perspective, should indeed compensate for the limitations of decision-making by the small group of individuals initially implementing the project.

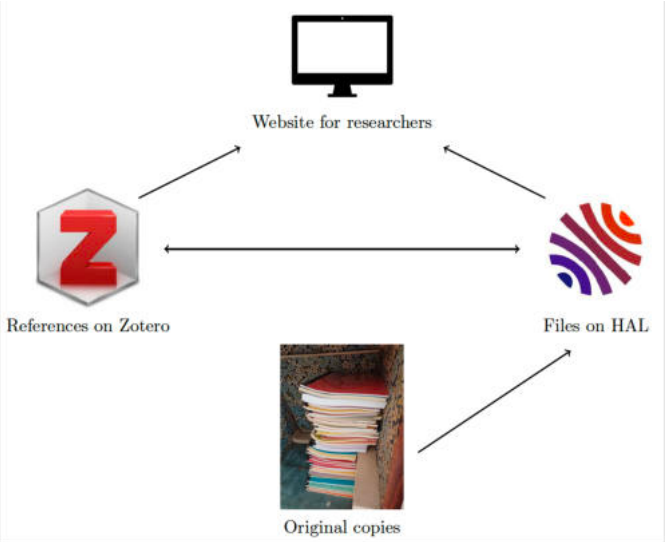
Writing an encyclopaedia: why and how?

The overall aim of this project is to produce digital resources accessible via a website and providing both an overview and the contents of half a century of contributions in fundamental didactics. But the mere library of articles, plethoric, would not be of much help for colleagues investigating, were it merely displayed in chronological order. When writing a paper, one often needs the literature just on a given topic or concept. So, in the case of fundamental didactics, one would rather need a thesaurus, together with references to the most important articles elucidating each term. These are the grounds for the development of the encyclopaedia.

Both the TDS and the ATD put at play a large number of concepts organised as dense conceptual networks, which exhibit a high level of internal coherence. These are crucial epistemological features, when it comes to designing their thesaurus. Of course, each of these terms have been shaped by its epistemological and social environment, in the course of time. Consequently, the stages of the encyclopaedic work are the following: 1) select the entries of the thesaurus; 2) for each entry, compile a list of extracts from the archived corpus reflecting the development of this notion and its possible facets; 3) redact a presentation of these extracts, adapted to both specialists in fundamental

didactics and less experienced colleagues. Concretely, each article (e.g. “institutionalisation”) will have three parts: an introduction aimed at non-specialists (young researchers, teacher educators, teachers interested in research in didactics of mathematics); a presentation of the historical development of the term; a selection of excerpts from the main sources relating to it, lightly discussed. Some terms, which are widely used by other research communities (e.g. “model”, “knowledge”) should have their meaning specified in relation to their most prominent uses in mathematics education. The articles of the encyclopaedia will naturally be linked when they are conceptually related, but also when they need to be clearly distinguished. Finally, we could think of articles dedicated to individuals (bibliographical notes) or historical episodes (a given Summer School, a famous controversy, etc.).

Figure 1: Organisation of the encyclopedia in relation to the archives



Two legs of the same project

Clearly, only by meticulously collecting the sources of fundamental didactics will we be able to properly reconstruct the historical development of its notions and present them in their full extent. So, the archival work is a necessary stage on the path to writing the encyclopaedia. On the other hand, the texts of the encyclopaedia will discuss excerpts of the articles of the archives. In particular, they should feature hyperlinks to consult them directly on the open archive online (HAL), or to visualise the collection of papers relating to a given notion on the reference management software (Zotero). So, the encyclopaedia is also a portal towards the archives. This tight link between the website of the encyclopaedia, HAL and Zotero (figure 1) is made primarily to facilitate the investigation work of colleagues.

Conclusion

This project of establishing an archive and encyclopaedia for fundamental didactics has a main objective: to produce a privileged source for anyone whose personal investigation leads them to come

across concepts or works in the field of fundamental didactics. In addition to its function as an overview and search engine for researchers in didactics, it should also be aimed at a wider public and provide an image of this particular approach to mathematics education. However, as we have seen, it also raises numerous issues, at both technical and epistemological levels, which go beyond the project and affect the future evolution of the field of mathematics education.

First, the very definition of fundamental didactics, even restricted to the French context, is problematic. For the time being, equating it with the duo formed by the TDS and the ATD seems sufficient, because accurate enough and practical. However, further historical studies would be in order. This raises the important issue of how the different branches, areas or domains of a field like mathematics education are defined and by whom. Then, most of its original sources, if they still exist, are accessible in France and Spain. Technical solutions exist to make them accessible, via an articulated use of an open archive (HAL) and a reference management software (Zotero). Even if this question can appear as only practical at first sight, the language used in the development of research frameworks and its dissemination among the international community in English as lingua franca requires a specific attention, especially to prevent negative effects coming from the dominance of English-speaking communities.

Regarding the encyclopaedia, a significant research work is still necessary, in order to list its entries and to select the most relevant excerpts from the literature. But this necessarily collective work, supervised by our scientific committee, might also lead to very fruitful debates, historical or etymological discoveries, and further writings. In this sense, this project is also a way to further develop the theory and practice of fundamental didactics.

Ultimately, this project deals with the problem of integrating past results in mathematics education, and going forward from a scientific point of view. By presenting in one place a comprehensive network of the concepts of fundamental didactics, coupled with a collection of articles both defining them and exemplifying their use to probe didactic phenomena, this project will help researchers build their investigations on this amassed, common ground. Because starting from scratch would become absurd, this project should foster truly cumulative contributions, which is an essential sign of doing science. Furthermore, this project raises the question of how a research community can make progress. This article indeed indicates another way to bring out “solid findings” in mathematics education. Instead of being declared directly, we suggest they can also emerge from the body of knowledge produced by a given community of researchers, that bespeak the accumulation of its results into a coherent body of conceptual and methodological resources.

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Combining a theoretical framework and a statistical measure to assess the reliability of didactic criteria in the analysis of large corpora

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The aim of this article is to present the articulation of a theoretical framework and a statistical measure using Fleiss' Kappa to assess the reliability of didactic criteria. These didactic criteria have been developed within the framework of the analysis of a corpus of consistent resources and aim to evaluate the potential of problems enabling students to develop a mathematical activity of research. The results show that the criteria constructed are reliable, with significant inter-coder agreement, and provide interesting perspectives for the analysis of resources in the context of teacher training.

Keywords: Mathematical activity, problem, data analysis, reliability, Fleiss' kappa.

Background and research question

Internationally, recommendations from research and the field suggest that the teaching and learning of mathematics should take place through problem solving, preferably linked to the activity of the mathematician (see in particular OECD, 2014; Ministère de l'Education Nationale, 2019; National Research Council, 2001). But what is the reality? Do the resources available to teachers encourage such activity? And how can we identify that a teaching resource has the potential to encourage mathematical research activity? Our contribution aims to propose a theoretical framework and a methodological tool for analysing the potential of resources to encourage mathematical activity.

This contribution follows on from an article published at CERME 13 (Da Ronch et al., 2023), in which initial criteria for assessing the potential of school level problems were defined. Since that prospective study, these criteria have been revisited, broadened and refined, and the corpus studied extended. The research question now focuses on methodological aspects: can these criteria be considered "reliable" (in the sense of concordance between ratings made by several coders, see Sim & Wright, 2005)?[1]

To do this, we begin by defining mathematical research activity and the notion of problem. In the second part, we detail the development of criteria for identifying, within a teaching resource, whether a mathematical problem is likely to encourage mathematical research activity. In the third part, we put these criteria to the test by applying them to the analysis of a corpus from French-speaking Switzerland, in order to test their reliability. Finally, in the fourth part, we open up perspectives on

[1] In our view, this is an essential step before testing the reliability of the criterion in terms of predicting student activity.

measuring the validity and reproducibility of these criteria, as well as their reinvestment in teacher training.

Theoretical Background

Da Ronch's (2022) epistemological study of the notion of mathematical activity as a human activity shows that for most mathematicians and mathematics didacticians, this activity is strongly linked to problem-solving (see Brousseau, 1997; Chevallard, 1998; Da Ronch, 2022; Gardes 2013; Halmos, 1980; Perrin, 2007; Thurston, 1994). During this activity, there is no question of limiting oneself to the use of definitions and theorems for the sole purpose of applying them to specific situations (Brousseau, 1997). Neither is the production of a mathematical result the ultimate goal of this activity. It is above all the processes involved in solving the problem that are indicative of genuine mathematical activity. The finished product, *i.e.*, the result of the problem, is in itself only a secondary objective of this activity (Chevallard, 1998). Consequently, mathematical activity involves different processes leading to the elaboration of a finished product, as the result on the problem, which are linked to the dialectics, which Brousseau qualifies, of action, formulation and validation. This means proposing problems that allow us to articulate, during these phases, "the reasoning of plausibility, purely inductive, and that of the necessities attached to deductive reasoning" (Da Ronch, 2022, p. 11) in the search for mathematical truth and its explanation in order to answer the problem. Doing mathematics therefore requires solving problems and accepting a certain form of scientific responsibility in the face of the challenge of truth and the need for proof (Da Ronch, 2022; Da Ronch et al., 2023). These different dialectics emerge if, and only if, specific knowledge is mobilized in the resolution of a problem (e.g., Da Ronch, 2022; Da Ronch et al., 2023). For example,

> […] the mobilization of specific knowledge linked to the practice of the mathematical activity. For example, entering into the resolution of a problem by studying particular cases is a knowledge linked to the experimentation process. The formulation of conjectures resulting from the study of these cases, the change of the register of representation of the mathematical objects at stake, sometimes leading to modelling premises, are examples of knowledge linked to the formulation process. Finally, validation is based on hypotheses (e.g., law of excluded middle, law of noncontradiction) and rules of logic (e.g., *modus ponens, modus tollens*) which allow, thanks to different mathematical reasoning (e.g., direct implication, contradiction, induction, counterexample...) to enter into the argumentation and proof process and thus to rule on the truth or falsity of a mathematical statement. This knowledge is therefore knowledge related, this time, to the validation process […] (Da Ronch et al., 2023, p. 97).

Furthermore, the processes of action, formulation and validation and their associated knowledge are indicative of a real articulation between inductive and deductive reasoning, testifying to a genuine experimental approach in mathematical activity (see, Da Ronch, 2022; Gardes & Durand Guerrier, 2016; Giroud, 2011; Perrin, 2007).

These processes are visible only through the encounter with a problem and the proven need to solve it for those who intend to investigate it (Da Ronch, 2022). The etymological and epistemological study carried out in Da Ronch's work (2022, chap. 2) has enabled us to characterize a problem according to a double aspect, syntactic and semantic. The syntactic aspect concerns the structure of

the problem presented in the form of a general question and a set of instances, in reference to Garey and Johnson (1979).

> For our purposes, a problem will be a general question to be answered, usually possessing several parameters, or free variables, whose values are left unspecified. A problem is described by giving: a general description of all its parameters, and a statement of what properties the answer, or solution, is required to satisfy. An instance of a problem is obtained by specifying particular values for all the problem parameters. (Garey et Johnson, 1979, p. 4)

The general question refers to the existence of a search variable (Grenier & Godot, 2004; Da Ronch, 2022), *i.e.,* a problem parameter whose value is not fixed and which is therefore the responsibility of the problem solver.

The semantic aspect of a problem refers to its epistemological quantity, determined by a detailed mathematical analysis of the problem (Da Ronch, 2022). This quantity can be identified by proximity to other related problems and by changes in the scope of the question and instances of the problem studied. The proofs used are also indicators of proximity to other problems that mobilize similar invariants in their resolution process (Da Ronch, 2022; Giroud, 2012; Da Ronch et al., 2023). All this highlights the non-isolated nature of the problem and its richness. To carry out such an analysis in order to determine the epistemological quantity of a problem, we pointed out a deficiency in the method of conduct to be adopted. Indeed, during the preliminary analysis (Artigue, 2015), no method of conducting a mathematical analysis with a view to building solid didactic engineering, with real consistent problems, leading to real mathematical activity was made explicit. This has led to the development of a method for carrying out such analyses (see Da Ronch, 2022; Da Ronch & Gravier, 2024). However, this method is relevant for the construction of didactic engineering, but remains too complex and not very operationalizable to evaluate the mathematical potential of problems on large corpora of data. Indeed, it is not possible to carry out such fine analyses for a wide variety of problems. This calls for the development of a new pragmatic evaluation tool that can effectively analyse a large corpus of problems for their potential in the implementation of mathematical activity.

Development of didactic criteria

In order to determine the potential of a problem to foster mathematical activity as we have defined it, we use the concepts of research variable (Grenier & Godot, 2004; Da Ronch, 2022) and didactic variable (Brousseau, 1997).

As mentioned in the previous section, a research variable is a problem parameter that is not initially fixed in the statement. It is up to the problem-solver to set different values for this variable. For example, if we wish to determine whether a rectangle of size p×q can be tiled with 1×2 dominoes, the search variable here is the size of the grid, which is not fixed in the statement. It is therefore up to the subject to appropriate it, *i.e.,* to set its own values in the process of solving the problem. The existence of a search variable thus guarantees that the problem is syntactically correct and semantically rich. When solving the problem, the problem-solver's mathematical activity is focused on the study of particular cases (as sub-problems) and then on generalization.

A didactic variable is a parameter of a learning situation that the teacher can vary to influence the way students interact with the proposed problem. The choice of a didactic variable value by the teacher can thus enable him or her to encourage mathematical activity in the pupils, in particular to take them gradually from the study of particular cases to the study of the generalization of a problem (see Figure 2).

The analysis of a problem's potential to encourage mathematical activity is based on a four-level criteria, relating to the nature of the variables present in the problem statement. We have formulated and coded them as follows:

- *Code 3: existence of a search variable in the problem statement*

For example, in the exercise Ribambelle[1] (see Figure 1a), the research variable is the number of "men" (where "man" is the basic figure created with the matches and then reproduced iteratively). It's up to the student to make this number of men vary, in order to study a few special cases that will enable him to determine a general formula.

- *Code 2: presence of a didactic variable with several values set by the problem statement*

For example, in the exercice Les étapes[2] (see Figure 1b), the number of steps is a didactic variable that the statement makes vary (20, 50, 100) to encourage students to study particular cases in order to determine the general formula.

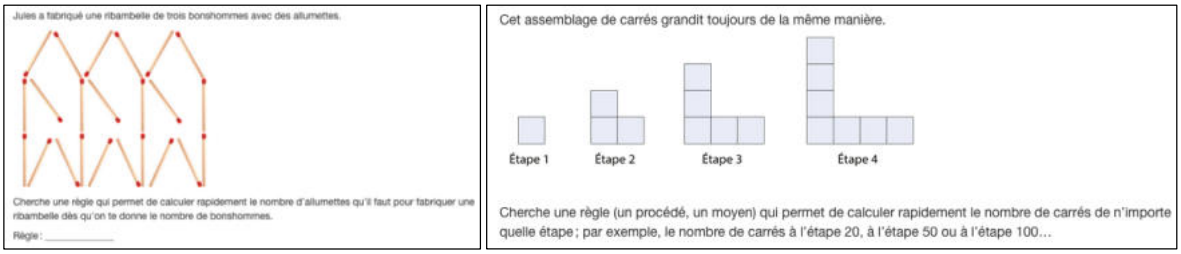


Figure 1: Statement of "Ribambelle"[2] (1a) and "Les étapes"[3] (1b) – fifth grade textbook in french-speaking Switzerland

- *Code 1: presence of a didactic variable with a value fixed by the problem statement, but which can be modified by the teacher.*

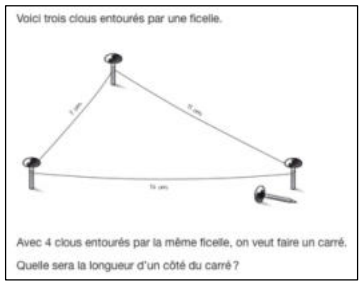
For example, in the exercice The string (see Figure 2a), a didactic variable is the length of the string. The teacher could vary this to study different cases and guide the students' activity towards a more general question, e.g. what are the triplets of three numbers such that the perimeter of the triangle is equal to the perimeter of the square?

[2] Jules made a string of three men out of matches. Find a rule to quickly calculate the number of matches needed to make a string once you've been given the number of men.

[3] This set of squares always grows in the same way. Look for a rule (a process, a way) that allows you to quickly calculate the number of squares in any step; for example, the number of squares in step 20, step 50 or step 100...

- *Code 0: no didactic variable*

For example, the exercise Achats (see Figure 2b) is a verbally stated problem that can be solved with simple arithmetic operations. There is no didactic variable that would enable us to develop a mathematical research activity in our meaning of the term.



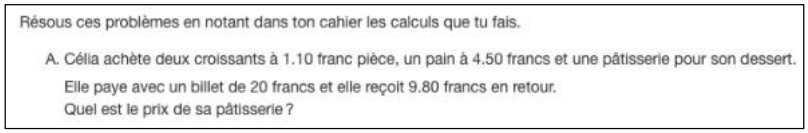


Figure 2: Statement of "La ficelle"[4] (2a) – fourth grade – and "Achats"[5] (2b) – sixth grade

Testing criteria on a corpus, choice and method

In addition to the fact that the authors work in Swiss institutions, the French-speaking Swiss corpus was chosen for the following three reasons. Firstly, because of the country's administrative structure, only one "official" teaching resource (textbook and book) is available to teachers: the Moyens d'Enseignement Romands (hereinafter MER). Although teachers have pedagogical freedom to supplement these MERs according to their needs, they are compulsory at ministery level, as they have been designed, developed and written to cover the subject content of the official curricula, grouped together in the Plan d'Études Romand (hereafter PER). This uniqueness of resources makes it possible to quickly get an overview, without having to analyse several different manuals. Secondly, some of the PER's injunctions are similar to our definition of mathematical activity. Indeed, in the "General comments" of the "Mathematics" domain presentation text, we find that

> "[Mathematics] promotes [...] an attitude of research through trial-and-error, generalization, conjecture and validation. In this way, the practice of mathematics develops the ability to imagine strategies, and to organize and structure knowledge" (free translation, General comments on the MSN field, CIIP, 2024).

This is in line with the components of mathematical activity presented above. Other recommendations are along the same vein, such as the organization of scientific debates in mathematics, the formulation of questions, the development of reasoning, the use of trial and error to reconstruct a thought process and understand missteps, and the analysis of the relevance and limitations of a chosen model, among others (see "Contribution au développement des Capacités Transversales", CIIP, 2024)

[4] Here are three nails surrounded by a string. With 4 nails surrounded by the same string, we want to make a square. How long will one side of the square be?

[5] Solve these problems by writing down the calculations you make in your notebook. A. Célia buys two croissants at 1.10 francs each, a loaf of bread at 4.50 francs and a pastry for dessert. She pays with a 20-franc bill and receives 9.80 francs in return. How much does her pastry cost?

Thirdly, a section of the MER entitled “Aide à la Résolution de Problèmes” (Problems solving support, hereinafter ARP), distinct and circumscribed and constituting around 1/5 of all the statements, is explicitly intended to “work on problem solving”. The editors justify this section by pointing out that “it is not enough to let students find problems and then correct them on the basis of those who have succeeded, for everyone to learn to solve problems”, and that it is “necessary to put in place real teaching in problem solving” (free translation - ARP comments, CIIP, 2024).

Using our criteria, we analysed the entire Cycle II ARP corpus (students aged 8 to 12). The analysis was carried out in parallel by the three authors on a corpus of 312 items, with an initial sampling of 10 items per level (over 4 levels) to calibrate the coding. Inter-coder similarity, an indicator of criterion reliability, was measured using the appropriate Fleiss' Kappa coefficient (see Fleiss, 1971). It measures the agreement between several coders when they classify items into discrete categories. The Fleiss' Kappa coefficient ranges from 0 to 1, where 0 indicates a match equivalent to that expected by chance, and 1 corresponds to a perfect match between coders. A medium match is considered to exist when $\kappa > 0{,}41$, and a large match when $\kappa > 0.61$. The fact that four categories are available increases confidence in the interpretation of the coefficient obtained. Moreover, testing it on a wide range of problems (312) reinforces this confidence (Fleiss, 1971; Landis & Koch, 1977; Sim & Wright, 2005).

Results and discussion of the reliability of the criteria used and research perspectives

The results of our study show the reliability of the didactic criteria obtained by calculating Fleiss' Kappa coefficient. The latter indicates a significant concordance value ($\kappa \approx 0.7$) according to Landis and Koch's (1977) table. Indeed, of the 312 problems coded in triple blind according to our four criteria, 44 obtained a different coding with each time exactly 2 coders in perfect agreement out of the 3 with only a deviation of 1 in the coding. Thus, 268 problems obtained perfect agreement in coding. Consequently, the results show that the criteria constructed are reliable, with significant inter-coder agreement. This paper has thus presented a reliable method for processing and analysing resources in order to identify the potential of problems enabling students to develop a mathematical research activity. This method of conducting research is based on epistemological and didactic criteria for categorizing problems in the study of a corpus involving a significant quantity of resources. Finally, our criteria-based processing and analysis method seems effective for analysing the potential of problems to generate mathematical activity in students, since it enables us to process a large corpus of data, without having to carry out an exhaustive *a priori* analysis of the problems to measure their epistemological quantity.

A first perspective provided by these results is to pursue the articulation of our theoretical framework with the use of statistical measures to question the validity of the didactic criteria developed, as well as the reproducibility of their use. Validity could be assessed through an analysis of student activity during problem solving, showing that the nature of the variables involved does indeed have an impact on students' mathematical research activity. With regard to reproducibility, initial results suggest that these criteria are sufficiently precise and operational for other coders to take them up and use them on different data corpora. However, this remains to be proven. A second perspective is to deepen the

analysis of the corpus studied. We can already state that this corpus includes few problems that allow students to develop a mathematical activity of research. Indeed, there is only one problem with a search variable (Ribambelle, see Figure 1a) and five problems with a statement proposing several values for a didactic variable. We can thus see that the MERs do not really enable teachers to implement the PER injunctions concerning problem-solving. Indeed, with the exception of these six problems, the statements do not propose any work on the formulation and proof of conjectures, generalization is little worked on, the issue of truth is absent, as are discussions on the existence of solutions, their multiple or universal nature. A third perspective is to consider the use of these didactic criteria in teacher training, for analysing resources or designing situations. Indeed, since the institutional resource only partially addresses the requirements of curricula regarding mathematical research activity, teachers must be able to identify problems that allow students to engage in mathematical activity. The developed didactic criteria could thus be a relevant tool for refining their didactic analyses.

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Didactic paradigms to promote creativity in the study of mathematics

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This paper asks what it means to be creative in mathematical activity, why one might want to promote creativity in mathematics education, and how this might be achieved. For this we use the notion of a didactic paradigm, consisting of didactic ends, didactic means and an epistemological model. We show how including the promotion of creativity among the ends affects the means and the epistemological model.

Keywords: Creativity, didactic paradigm, didactic ends, epistemological model, didactic means

Introduction

The aim of this work is twofold. On the one hand, it aims to explore how the study of mathematics could be planned to promote mathematical creativity. On the other hand, it aims to carry out an analysis of the very idea of creativity in mathematics. This analysis should contribute to a better understanding of how creativity can be promoted. Thus, beyond its theoretical importance (e.g. as an aspiration to the understanding of certain notions), the practical importance of the second aim lies in the practical importance of the first one. As for the practical importance of the first goal, it lies in the fact that the aims of education should include not only the acquisition of certain knowledge, but also of certain skills and attitudes. Indeed, it seems that formal education is aimed at maintaining (or even increasing) the welfare state of our societies. In turn, it seems clear that this state of well-being is due not only to the accumulated knowledge itself, but also to the fact that some members of our society have attained this knowledge at some point. The attainment of new knowledge for humanity, when it is not a rare case of serendipity, is obtained through an act of creativity.

In order to achieve these two goals, we will use the theoretical framework provided by the Anthropological Theory of the Didactic (ATD). Of course, it would be interesting to compare the way in which we address questions about creativity in ATD with the way in which they are addressed in other approaches in mathematics education. Unfortunately, the limited space available prevents us from doing so. Instead, we address the two objectives in sufficient detail to show the challenge they pose to the mathematics education research community.

Didactic paradigms rule didactic institutions

Didactic institutions are those social institutions stablished with the aim of studying something. In order to analyse how didactic institutions work, we will use the notion of didactic paradigm as introduced by Gascón and Nicolás (2019, 2022). Given a didactic institution, I, and a certain field of study, F, in I, we say that a didactic paradigm for F in I is made of: didactic ends, which explain the purpose of studying F in I, an epistemological model of F in I, written $EM_I(F)$, which provides a description of F, and didactic means, which state what is done in order to study F in I.

Of course, there should be strong links between these three components. For instance, the didactic ends should determine to a large extent the epistemological model. The didactic ends should also determine the didactic means, because different ways of teaching boost or complicate the acquisition of certain types of knowledge or attitudes or skills which might be among the didactic ends. This is also the reason why there is a strong relationship between the didactic means and the epistemological model, or, in other words, between how to study and what is studied. In what follows, we will show how to include the promotion of creativity among the didactic ends affect the epistemological model (the way knowledge is organized) and didactic means (how this knowledge is studied).

Epistemological models

As said before, in a didactic paradigm the epistemological model describes what is specifically studied about F in I. For this we use the notion of praxeology. This is a fusion of the terms praxis (related to the idea of procedural knowledge) and logos (related to the idea of propositional knowledge), and it has been used in different places in social sciences. Following Chevallard (1999), we say that a praxeology is a 4-tuple Π = (T, τ, θ, Θ), were: T is a family of types of tasks, τ is a family of techniques (typically described in terms of actions), devoted to carry out the types of tasks, θ is the technology (that is, the logos of the techniques), devoted to explain why the techniques of τ work, and to assess the scope, the economy and the reliability of those techniques, and Θ is the theory, devoted to specify everything else concerning the knowledge of F in I.

In order to specify certain ingredients in Θ we will use the general description proposed by Gascón and Nicolás (2024), according to which a theory would have, at least:

- An ontological component, denoted by $\mathcal{O}$, which provides the language L used to speak of the field of study F, and a list of axioms, which are statements we assume to hold without any supporting argument beyond the evidence of 'experience'.
- A nomological component, denoted by $\mathcal{N}$, made of theorems. Each theorem is a statement expressed in terms of L, and, contrary to what happens with axioms, theorems are not a starting point in the study of F, but the conclusions of valid arguments with true premises (either because they are, in turn, theorems, or because they are axioms).
- An epistemological component, denoted by $\mathscr{E}$, which states what kinds of arguments are valid to support theorems. Notice that, depending on the institution, certain kinds of argument might not be allowed. For instance, a drawing is not regarded as an argument in the institution formed by professional mathematicians, but it can be taken as a proof in Primary Education (for instance, we make a draw to show that in the case of an acute triangle, the three heights lie inside the triangle). Moreover, a single institution might have, at different historical periods, different criteria of validity for arguments. Indeed, the institution of professional mathematicians has considerably narrowed the family of allowed arguments after Hilbert's proposal (Zach, 2023), as a way of fighting the crisis in the foundations of mathematics (Snapper, 1979).

Therefore, in order to specify the epistemological model $EM_I(F)$ for the study of F in I, to describe what is specifically studied of F in I, we need to describe a certain set of praxeologies, Π = (T, τ, θ, Θ), with Θ = ($\mathcal{O}$, $\mathcal{N}$, $\mathscr{E}$) and with $\mathcal{O}$ = (L, Axioms).

Didactic means

In a didactic paradigm, the component devoted to the description of how the study is carried out is called didactic means. In our view, a good way of conceptualizing it is by using the idea of game, very much in the spirit of mathematical Game Theory (von Neumann & Morgenstern, 1944; Luce & Raiffa, 1957). The idea of game as a tool to understand the working of certain social situations goes back to Wittgenstein (1953). Games are useful in order to describe the dynamics of situations governed by prefixed rules, where several agents (players) take part, where there might be something that (some of) the agents want to get (so that they can lose or win), and where the actions of the agents affect each other. Thus, in order to describe a game, one has to specify which are the players, which are the allowed moves for each player, which are the rewards and punishments (if any) for each move, and what does it means for a player to win the game. The use of games in didactics started with Brousseau (Brousseau, 1986, § III) and Hintikka (1982), who used the notion to describe the wide variety of interactions that can take place in a classroom between teacher, Y, and students, X, around the study of a certain field F. Here, however, we prefer to distinguish between different, more specific, types of games: imitation games, argument games and inquiry games.

An imitation game consists in the teaching of a certain technique (in the sense of ingredient of a praxeology) by putting it into practice with examples (ideally covering all cases). An example of imitation game could be the teaching of the technique for solving a quadratic equation by means of solving particular cases. The students are then encouraged to solve other cases themselves in a similar way to the teacher's approach.

An argument game consists in Y showing X valid arguments to justify a certain proposition. In argument games, although Y seems to be the only participant, X is supposed to complete the argumentation with Y, actively collaborating in the development of the game, so that Y can observe if X follows the steps of the argumentation. We could even come up with an alternative definition of argumentative games: X is the one who has to win the game by understanding the premises used by Y and why those premises can have the initial assertion as a conclusion.

Finally, in an inquiry game students X are challenged to answer an initial question by probing and being guided, to a greater or lesser extent, by the teacher Y. Inquiry games have a high degree of uncertainty, both for the students (who are less guided than at other times) and for the teacher (who does not know in advance how the students will act). In any case, imitation games and argumentation games can also be extremely difficult for the teacher. Choosing the examples to imitate and the steps to take in the arguments requires a deep understanding of mathematics.

What is creativity in mathematics

We have used praxeologies to describe epistemological models, and the ingredients of praxeologies point out in how many different ways one can be creative in mathematics:

- Creativity in proposing open problems, for instance, conjectures
- Creativity in developing techniques, which cannot be overestimated in maths.
- Creativity in proving theorems, which cannot be overestimated either!

- Creativity in setting axioms. We can think of Euclid's Elements, the foundation of Euclidean Geometry by Hilbert (2015/1899), or the axiomatization of Set Theory by Zermelo and Fraenkel (Zermelo, 1908, 1930; Fraenkel, Bar-Hillel, and Lévy, 1973/1958).
- Creativity in the description of the ontological component of the theory. Of course, there are really extraordinary occasions where some people are creative enough to propose a completely new ontology. This was the case with the study of sets and their relationships, which was created at once by Cantor (1874), and was used to set the foundations of the whole contemporary mathematics. This is also the case of Grothendieck's creation of the theory of schemes and topos theory in his series of works called Éléments de géométrie algébrique, which entailed a great unification of Number Theory, Geometry and Topology, previously cut off from each other.
- Creativity in setting the epistemological component of the theory. Among the most sharp mathematicians there is, actually, a constant reflection on the validity of methods of proving the validity of statements. For example, Euclid's distinction between axioms and theorems, and his carefully way of deriving theorems out of the axioms, started a new way of thinking for the whole humankind, beyond mathematics. Concerning the strict realm of mathematics, there have been some proposals originated at the beginning of the 20th century as a response to a crisis in the foundations of mathematics. The most prominent are logicism, intuitionism and formalism (Snapper, 1979).

Didactic paradigms that allow creativity

Promotion of creativity as one of the didactic ends

It is well known that, from the very beginning, formal education has to do with the transmission of knowledge to future generations, but also with the transmission of beliefs, attitudes and skills from which society in general can benefit. Among the skills that deserve to be encouraged we can think of creativity. But we have to keep in mind that genuine mathematical creativity requires to carry out a systematic, rigorous, patient and long-term work. However, current didactic paradigms, in order to entertain and not eliminate pupils from the school system, disregard many of these aspects of mathematical work, needed for creativity to appear. Therefore, in practice, real creativity is replaced with an apparent one, staged through a continuous change of activity. As a consequence, there is a mismatch between didactic ends (which include promoting real creativity) and didactic means (which only promote fictitious creativity), giving rise to the creativity paradox pointed out by Chevallard, Bosch and Gascón (1997). Thus, the question is: How to avoid the phenomenon of creativity paradox and how to encourage real creativity while teaching knowledge well established in advance?

Epistemological models that allow creativity

There do not seem to be magic formulas for promoting creativity, but there are epistemological models more or less likely to leave room for creativity. Teaching of mathematics is based on an epistemological model previously designed (or at least assumed) by the teacher. An epistemological model is a way of organizing the knowledge to be taught, and that organization may be more or less conducive to the creativity of those who learn. Thus, in order to favor creativity in the learning of mathematics, the epistemological model should include the possibility for the students of conceiving

new types of tasks, developing their own techniques to solve types of tasks, reflecting on the ontology they are dealing with, questioning and making explicit the basic properties assumed, proving new results (linked to the nomological component), and reflecting on the way they prove results (linked to the epistemological component). But, for this, there must be a previous design of the epistemological model, conceived in a diachronic way. Let us explain it further.

One can use a single (possibly big) praxeology to provide a synchronic description of $EM_I(F)$, that is to say, a description in one go of all that students must learn along the study process. In a synchronic description there is, in general, no space for types of tasks that cannot be accomplished, or defective techniques, or bad candidates for being theorems, or mistaken proofs, since they are not supposed to be learned. But it is surprising not to include all these features in our study processes, since they are innate to real mathematical activity, and they really foster creativity.

Therefore, if one wants all these features to take place in the study of mathematics, one has to use a diachronic description of $EM_I(F)$, taking into account the order in which the steps could be achieved along the study process. For this, one might use a directed graph, where the nodes are praxeologies, and the arrows indicate that the praxeology in the tip is a single step evolution of the praxeology in the tail. Such a diachronic description of $EM_I(F)$ is said to be a diachronic epistemological model of the study of F in I. In a diachronic epistemological model, we have what we call initial praxeologies, which are those praxeologies which are not in the tip of any arrow, and final praxeologies, which are those praxeologies which are not in the tail of any arrow. Among final praxeologies, we can distinguish between failed praxeologies, which are discarded attempts to advance in the study of F in I, and successful praxeologies, which provide the answers we were looking for at the beginning of the study process. Those praxeologies which are neither initial nor final are called transition praxeologies. Progressing through a diachronic EM involves finding praxeologies that offer increasingly better solutions to the issues raised by the initial praxeologies. Failed praxeologies represent dead ends in the study process, while successful praxeologies offer conclusive solutions. Of course, not every collection of praxeologies placed in the nodes of a directed graph is capable of describing the praxeological development of a possible study process. On the contrary, there must be some kind of epistemological plausibility in the passage from one praxeology to another. However, once one has organized knowledge through a diachronic epistemological model, it is easy to see where are the opportunities to foster creativity in a study process based on that model. The proposal of basing the study of mathematics on diachronic epistemological model is very much in the spirit of Lakatos' proposal (1976).

Didactic means that allow creativity

In an imitation game there is space for a certain degree of creativity. This is mostly the kind of didactic games and the kind of creativity currently observed in lessons in Primary and Secondary Education in Spain. However, it seems clear that inquiry games allow a huge degree of creativity. These games may aim to have students reconstruct pre-established knowledge through an inquiry process or to develop enquiry skills like creativity. It is natural to wonder to what extent it is feasible to get students to carry out a successful inquiry without the teacher's inevitable inference distorting it. Indeed, inquiry is a process in which a complex interplay between deductions and questions takes place, and

in an inquiry game not all the questions can be asked at any moment. Fortunately, Genot and Gulz (2015) explain how the teacher can strategically manipulate what they call students' range of attention, so that they can pose the right questions in the course of the research, and bring it to fruition.

Let us sketch now two different ways of teaching the use of derivatives to study the monotony of functions: One way could be to explain first how to calculate the derivative of a function (via an imitation game), and then, how to use this derivative to find the extremum of the function (again, via an imitation game). Note that in the course of these imitation games, pupils will have learned new types of tasks (finding the derivative of a function; using it to find the extremum), the corresponding techniques (including the derivatives of elementary functions; derivative of the composite function, etc.), will have enlarge their nomological component (incorporating, for example, a theorem saying that the derivative vanishes in the extremum of the function), and will have enlarge also their ontological component (incorporating to their language the notions of extremum of a function and the notion of derivative). In addition, in the course of these games, students can employ a certain degree of creativity in imitating the teacher. For example, if the teacher says that the derivative of $f(x)=x4$ is $f'(x)=4x3$, students can creatively imitate it and deduce that the derivative of $g(x)=x5$ is $g'(x)=5x4$. However, the monotony of functions can be studied in a way that promotes much more creativity. Let us very briefly outline a diachronic epistemological model for this. One could start by seeing that to check whether a linear function $f(x) = mx + n$ grows in a certain interval [a, b], it suffices to look at the sign of $f(b) - f(a)$. This incorporates a new notion to our ontological component, the rate of change, $RC(f, a, b) = f(b) - f(a)$, which tells us whether $f(x)$ grows in [a, b] and how much it grows, and we can compare growths of different functions. However, to compare the growth of $f(x)$ relative to the length of the interval [a, b], we must consider the average rate of change, $ARC(f, a, b) = (f(b) - f(a))/(b - a)$. This, which is a new ingredient in our ontological component, works perfectly well for linear functions, and for monotonic sections of any function. But it is not useful to detect monotony in intervals with monotony changes. So, for example, if $f(x) = x2$ we have that $ARC(f, -1, 2) = 4 - 1 = 3 > 0$, but $f(x)$ is not increasing in [-1, 2]. Once it is observed that the reason for the failure is that the interval [-1, 2] is too large, because a monotony change occurs within it, one can consider the option of dividing [a, b] into smaller intervals, and then calculate the ARC of $f(x)$ in those smaller intervals, in the hope of reducing the probability of encountering a monotony change within those smaller intervals. However, the teacher can direct the students' attention to functions and intervals for which such a division of the interval does not guarantee that ARC solves the problem. Indeed, such an interval splitting before knowing the monotonicity of the function cannot guarantee a solution, because it is not known how to choose suitable subintervals that avoid a change of monotony. The solution could come from the idea of regarding x and $f(x)$ as representing amounts of magnitudes. Let $f(x)$ be the polynomial $f(x) = ax + bx2$ describing the height $f(x)$ of an object as a function of time, x, along a certain trajectory. When working with magnitudes there is, in practice, a degree of precision. Well, to study the monotony of $f(x)$ near a point $x=c$ one could choose a number h such that bh is a negligible amount of length, i.e. less than the degree of accuracy. We can then use ARC to see the monotony of $f(x)$ in the interval [c, c+h]. Indeed, $ARC(f, c, c+h) = a + 2bc + bh$ m/s. This means that $a + 2bc + bh$ meters is travelled in 1 second. As bh is a negligible amount of meters, we can say that $ARC(f, c, c+h) = a + 2bc$ m/s. Thus, for those c where $a + 2bc > 0$ the function will

be increasing, and for those c where a + 2bc < 0 the function will be decreasing. We could say that ARC, with h small enough, is an instantaneous rate of change. We can use this method to say that f(x) = 6x - x2 is increasing for x < 3, and decreasing for x > 3. We obtain the same result if we study the behavior of f(x) only a little to the left of x = c, that is, in the interval [c-h, h]. This is how our inquiry game on the study of the monotony of a function can naturally lead us to enlarge our ontological component with the idea of rate of change, average rate of change and instantaneous rate of change (derivative), to consider new types of tasks (is ARC useful for non-linear functions?, does it make sense to arbitrarily divide the interval?, can the degree of accuracy help us in our study of monotony?, what is the derivative of a polynomial of degree n in general?, etc.), and new techniques (together with its technology) that provide answers to these types of tasks. The same enquiry can be extended beyond rational functions, by considering questions as the following one: is there anything like the techniques used for, say, f(x) = sen(x)? It seems clear that this game of enquiry, full of possible dead ends, but with students asking questions and checking their conjectures against examples (sought by them or suggested by the teacher), is more conducive to creativity than a game of imitating certain unfounded procedures showed by the teacher and leading in a straight line to the resolution of types of tasks that appear out of nowhere.

Conclusion

The use of the notion of praxeology to analyse mathematical knowledge helps us to understand the different ways in which one can be creative in mathematics. By using the notion of didactic paradigm for the analysis of the functioning of didactic institutions, we have seen to what extent what is taught (the epistemological model) and how it is taught (the didactic means) should be conditioned by the raison d'être of what is taught (the didactic ends). Thus, if promoting creativity in the study of mathematics is assumed as one of the didactic ends, the choice of the epistemological model and the didactic means must be made accordingly, being a diachronic epistemological model most appropriate, and being the use of inquiry games advisable.

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On the concept of epistemic obstacles and its significance to mathematics education (research)

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Obstacles to learning mathematics are of natural interest to mathematics education research. In this paper, we examine a particular strand of research which has its origins in Gaston Bachelard's concept of epistemological obstacle. The paper is divided into two parts: In the first part, we start by giving a brief historical sketch of how the concept found its way into mathematics education research, and then focus on a specific subset of these obstacles which we will term 'epistemic obstacles'. The second part of the paper presents examples of results that can be obtained from analysing such epistemic obstacles and argues for their significance for mathematics education (research). We conclude by discussing the potential of Edmund Husserl's eidetic method for a systematic analysis of epistemic obstacles.

Keywords: Obstacle, epistemological obstacle, epistemic obstacle, epistemology, Husserl.

Introduction

Mathematics education research seeks to identify the conditions under which the teaching and learning of mathematics can be successful. In this context, however, it is repeatedly confronted with the fact that many students do not learn the mathematics that is taught. Hence, it seems natural to look at the processes of mathematical learning and teaching from the perspective of obstacles that may stand in the way of students acquiring mathematical knowledge.

In this paper, we would like to take a closer look at a strand of research that originated in Gaston Bachelard's (2002) work on epistemological obstacles. To do this, we first provide a historical sketch of how the concept has found its way into mathematics education research, and then focus on a particular subset of these obstacles, which we propose to call 'epistemic obstacles'. In the second part of the paper, we illustrate the kinds of results that can be gained by investigating epistemic obstacles by providing some examples, aiming to show the relevance of such results for mathematics education (research). Having done this, the methodological question of an appropriate methodical approach for conducting investigations of epistemic obstacles remains, and so we conclude the paper by briefly outlining how Edmund Husserl's eidetic method could provide the basis for such an approach.

Epistemological obstacles and mathematics education research

When analysing obstacles to learning and knowing, the adjective 'epistemological' is often used to distinguish one's research interest from the investigation of obstacles that also hinder knowledge, but which do not originate in characteristics of the knowledge being sought after. Examples of such non-epistemological obstacles to mathematical knowing would be mental obstacles such as learned helplessness (Yates, 2009) or social obstacles such as socioeconomic status (Valero et al., 2015).

Bachelard (2002) introduced the term 'epistemological obstacle' in 1938, while "looking for the psychological conditions in which scientific progress is made" (p. 24). He identified obstacles like *opinion*, *conservative instinct* and *explanatory preferences* such as animism or substantialism as causes of inertia and stagnation in the genesis of scientific knowledge.[1] While Bachelard (2002) looked to the history of science to identify the mental tendencies that could obstruct acts of knowing, as an epistemologist, he insisted on taking a normative stance, rather than choosing the descriptive methods of a psychologist or historian: "[Epistemologists] have to evaluate [historical] documents from the standpoint of reason and indeed from the standpoint of developed reason for it is only now that we can really judge the errors of the mind's past" (p. 27).

In the 1970s and 80s, Brousseau (2002) transferred the concept to mathematics education research. He observed that many errors that students make in the process of learning mathematics „are not only the effect of ignorance, of uncertainty, of chance, […] but the effect of a previous piece of knowledge which was interesting and successful, but which now is revealed as false or simply unadapted" (p. 82). Such obstacles are considered pieces of knowledge (as opposed to a lack of knowledge) which produce responses that are appropriate within some typical contexts but inappropriate outside of these. Pieces of knowledge that are obstacles in this sense appear to be resistant to the occasional contradictions they cause and continue to reappear persistently even after their inadequacy for a more universal understanding has been recognised (Duroux, 1982, as cited in Brousseau, 2002).

While obstacles, in general, "are made apparent by errors" (Brousseau, 2002, p. 84), epistemological obstacles understood as this kind of illegitimate transfer of knowledge to new domains can be distinguished by the fact that the later established system of knowledge will bear signs of the obstacle that had to be overcome in order to reach it. In these kinds of obstacles, Brousseau and his followers thus see a constitutive part of the mathematical knowledge which students should learn. As a consequence, such obstacles should not and cannot be avoided and the role of the teacher is to provide didactic situations in which students apply their previous beliefs to a more general problem, realise its limitations and modify their beliefs by incorporating the obstacle into an adequate generalisation. One approach to this kind of 'didactical engineering' can be to selectively restore some of those conditions in the historical development of a concept that have "disappeared in curriculum processes such as decontextualization and sequentialization of knowledge" (Bergsten, 2020, p. 502).

Typical methods of uncovering epistemological obstacles include reconstructions of the history of mathematics and content-focused analyses of common teaching challenges, since aside from the pervasiveness of an obstacle, the role it plays in the knowledge system of mathematics is crucial. In the next section, we shall argue that one can add to this strand of research by introducing further distinctions within the sphere of epistemological obstacles and that for a subset of these obstacles, a different method of analysis is required.

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[1] Even though Bachelard explicitly excluded mathematics from his investigations (see Bachelard, 2002, p. 32), we believe that the 'generic extension principle' (Tall, 1991), according to which properties of objects in a restricted context are assumed to be more generally valid, could be considered an obstacle of a similar kind in the mathematical domain.

The nature of epistemic obstacles and how to identify them

In the broadest and most literal sense, any obstacle that prevents a subject from coming to know could be called an epistemological obstacle. However, besides didactic obstacles and ontogenetic obstacles, for example, which are contingent upon specific characteristics of the teaching situation and the developmental level of the students, one can identify certain obstacles of "really epistemological origin" (Brousseau, 2002, p. 87).

These can be found among the obstacles that mathematicians had to overcome in history in order to open up certain areas of mathematical knowledge or the obstacles that students of mathematics face again and again as they acquire certain pieces of mathematical knowledge. Epistemological obstacles are obstacles that are not a result of suboptimal conditions (e. g., if mathematicians in a certain region of the world and at a certain point in history were not familiar with a particular set of mathematical methods, or if a topic is presented to students at an age in which the necessary cognitive skills aren't sufficiently developed) but specifically those obstacles which have been empirically shown to occur even in processes of knowing that are ultimately successful.

However, when obstacle seeking reconstructions of the genesis of mathematical knowledge remain grounded in mathematical knowledge as it was *actually* discovered or is *actually* acquired, this leaves room for certain contingencies. For it does not follow from the fact that certain obstacles have arisen in the history of mathematics, or repeatedly arise in educational contexts, that their arising could not have been avoided. For example, a student might expand their understanding of multiplication as repeated addition to include an understanding of multiplication as scaling in order to make sense of an expression like '$0.75 \cdot 0.4$'. In this instance, the previous understanding may well be an obstacle, which, once engaged with and overcome, can be contextualised and incorporated into a correct system of knowledge (see Brousseau, 2002, pp. 109–110). Yet it is by no means clear whether this – the path which leads through this understanding which can become an obstacle – is the path which must be taken *en route* to the multiplication of rational numbers.

In light of this observation, one might wonder whether there are any obstacles which not only have factually prevented certain groups of human beings from knowing, but *that would necessarily prevent any rational subject from attaining the piece of knowledge in question*. Accordingly, we set out to identify those obstacles to knowing which make acquiring mathematical knowledge of a particular kind impossible. To distinguish this subset from epistemological obstacles in Brousseau's sense, we will use the term 'epistemic obstacles'.

Note that empirical observation cannot yield answers to such questions of strict necessity, which means one has to find other methodically controlled ways of investigating them. We will return to this methodological issue toward the end of our paper. First, we will provide some examples of the kinds of results that a *rational epistemic obstacle analysis* can arrive at, and thus illustrate its relevance to mathematics education and its research.

Epistemic obstacles and their significance for mathematics education

In order to get a better impression of which obstacles to (mathematical) knowledge can be called 'epistemic obstacles' in our sense, we will now, in the second part of the paper, discuss some

examples. A large group of such epistemic obstacles can be summarised under the formula of 'the failure to distinguish the different'. This formula refers to the numerous cases where distinctions that are necessary for progress in the act of knowing are overlooked. We shall now show how a *failure to distinguish between the meaning and the object of a mathematical expression* can become an epistemic obstacle in the depicted sense.

We start directly with a mathematical example and develop the necessary distinctions along the way: In 'What is mathematics?', Courant and Robbins (1996) write about division by zero: "Expressions such as 1/0, 3/0, 0/0, etc. will be for us meaningless symbols" (p. 56). Although they acknowledge that it may be useful in some cases to replace "such expressions by the symbol ∞ (read, 'infinity')" (ibid.), they immediately go on to add that one should not then try to deal with "the symbol ∞ as though it were subject to the ordinary rules of calculation with numbers" (ibid.). If one were to do this, after all, one could infer, for example:

$$3 = 0 \cdot \infty = (0 + 0) \cdot \infty = 0 \cdot \infty + \ 0 \cdot \infty = 3 + 3 = 6$$

The assumption that 3/0 = ∞ thus – by applying the distributive law – results in a contradiction. From a *mathematical* point of view, this is an ordinary permanence argument: One demands the permanence of certain laws (here: the 'ordinary rules of calculation with numbers') and then shows, by deriving a contradiction, that the operation of division cannot be extended to the divisor zero under the demanded condition. From an *epistemological* point of view, however, it is illuminating to take a closer look at the explication that Courant and Robbins provide for their own mathematical activity. For this passage shows quite clearly the difficulties in mathematical knowing that can result from not distinguishing between the *meaning* and the *object* of a mathematical expression.

We may first observe that every mathematical expression has a certain meaning and by virtue of this meaning, it relates to ('means') an object. "In meaning", writes Husserl (2001), "a relation to an object is constituted" (p. 201). Note that "it makes no difference whether the object exists or is fictitious or even impossible" (ibid.). That the mathematical expression relates to a mathematical object through its meaning therefore says nothing about the existence of the expressed and meant object. Rather, it only indicates that the expression relates to its respective object *by* the particular meaning. However, this then leads to the important consequence that the question of the *existence of the object* corresponding to the mathematical expression must not be confused with the question of the *existence of its meaning*. To put it another way, the *meaninglessness* of an expression is something quite different from its *objectlessness* (see Husserl, 2001, p. 202).

"But if one" – as Courant and Robbins (1996) seem to do in the passage above – "gives a very rigorous interpretation to the proposition that an expression, in so far as it has meaning, relates to an object, i.e. in a sense which involves the existence of the object, then an expression has *meaning* when an object corresponding to it exists, and it is *meaningless* when no such object exists" (Husserl, 2001, pp. 201–202).

We will now show that this view is not tenable. While Courant and Robbins (1996) deny expressions such as '3/0' or '1/0' any meaning, they paradoxically also use the meanings of these expressions to justify their statement that they are meaningless. After all, to deduce a contradiction, one has to

interpret division as the inverse operation to multiplication and further demand that this purported operation satisfy the 'ordinary rules of calculation with numbers'. In doing so, a meaning *is* assigned to the expressions involved, namely a meaning which could be formulated as follows in the case of '3/0':

'3/0' means as much as 'the number which, when multiplied by 0, gives 3'

And only by utilising *this* meaning is it then possible to show that the number that the expression '3/0' means is determined in a contradictory way and that this purported number therefore cannot exist (see the proof above).

We now want to show that such considerations are not merely epistemological quibbles, but that we are dealing with an epistemic obstacle that can have tangible consequences for the teaching and learning of mathematics. For this, we build on an example stemming from Hans Freudenthal (Freudenthal, 1973, p. 232): Imagine a mathematics class which is dealing with the extension of the natural number domain to the integers. Sooner or later, the class will face the question of why the multiplication of two negative numbers yields a positive result. On this occasion, the teacher starts from the paradigmatic example $(-3) \cdot (-4) = 12$ and presents the following argument for it:

'We know that -a is the number which, when added to a, gives 0, so we know that the following two equations hold:

$$(-3) + 3 = 0 \text{ and } (-4) + 4 = 0$$

If we now multiply the first equation by 4 and the second by -3, we get

$$0 = \big((-3) + 3\big) \cdot 4 = (-3) \cdot 4 + 3 \cdot 4$$

and

$$0 = (-3) \cdot ((-4) + 4) = (-3) \cdot (-4) + (-3) \cdot 4.$$

This allows us to deduce that

$$(-3) \cdot 4 + (-3) \cdot (-4) = (-3) \cdot 4 + 3 \cdot 4.$$

And therefore:

$$(-3) \cdot (-4) = 3 \cdot 4 = 12.'$$

The teacher finishes off this presentation with the conclusion: "Of course, this was no proof, it was a mere plausibility argument, because in our so-called proof we have been so reckless as to use the distributive law for negative numbers which we never proved" (Freudenthal, 1973, p. 232).

Freudenthal (1973) analyses the situation as follows: "The teacher denies that what he has done was a mathematical activity though the pupil must have received the impression that it was rather profound mathematics. The pupil is right. In denying it, the teacher causes harmful confusion" (p. 232). But where exactly does this 'harmful confusion' come from? And on what basis may the teacher have arrived at their mistaken conclusion? *One* way of making sense of the teacher's actions is to build on the previous considerations: The teacher could be led to their conclusion because they failed to distinguish between meaning and object of the mathematical expressions involved. How so? Well,

having this issue in mind, the teacher's comment might be indicative of a correct intuition that *something* remains unproven, namely the existence of the realm of negative numbers and the operations upon them – but this does not mean that nothing is proven at all.

The argument shows that *if* there exist negative numbers and an operation of multiplication upon them that satisfies the ordinary laws of arithmetic, *then* this operation necessarily must yield 12 in the case of -3 and -4. This constitutes a *proof*, and not only a plausibility argument. But it constitutes a proof that is played out entirely on the *level of meaning.* Whether the expressions '-3', '-4', and '·' relate, through their meaning, to *existing* objects is not at issue in the proof. It is not a proof of existence, but only a proof of uniqueness, so that the proof for the existence of mathematical objects with properties satisfying all the demanded conditions remains to be given. But the proof does unfold the implications of fixing the meanings of the expressions involved in this specific way.

The fact that proofs based on the principle of permanence do not allow for positive conclusions about the *existence* of the expressed and meant mathematical objects in question hence does not mean that no 'progress' is made in terms of knowing. On the contrary, in these kinds of extension problems, one must *first* analyse the desired meaning of the new expression and find out in which way the object in question is determined by this meaning. This can lead, as the analyses above have shown, for instance, to the result that said object is determined in a unique way. In these cases, then, a proof of existence remains to be given and is, as Freudenthal (1973) notes, fundamentally important; but it comes second, "both as regards the temporal succession, and as regards the importance" (p. 234).

In the example, the teacher's unawareness of this necessity of dealing with the task of elucidating the *meaning* of an expression before advancing to the question of the existence of its *object*, leads to an ill-structured presentation followed by a misleading conclusion. As such, a *didactic* obstacle in the instructional situation might occur that could stand in the way of students' own epistemic processes.

Concluding and looking ahead

Conceptualising certain constitutive elements of the mathematical knowledge being sought as potential 'obstacles' to knowing has a longstanding history (see Brousseau, 2002; Prediger, 2008) within mathematics education research. In our paper, we aimed to present a case for the potential of considering a more narrowly defined category of *epistemic obstacles*. These are not dependent on any specific past or present circumstances of acts of knowing or characteristics of real or imagined knowing subjects. As such, epistemic obstacles are relevant to all students of mathematics. As we have mentioned, a rational inquiry into such conditions for knowing mathematics and the resulting obstacles to knowing mathematics poses a certain methodological question (see Kollosche (2023) on the need for a methodology for rational didactic content analyses): How can one analyse mathematical knowing in a purposeful and systematic manner to arrive at conclusions about epistemic obstacles?

In the analyses we have conducted so far, we found that the "eidetic method" (Husserl, 1960, p. 71) might provide an answer. Its core idea is to 1) *hold a* (perceived, remembered or imagined) *instance of an act of knowing in one's mind,* while 2) *systematically varying* features of the act, carefully paying attention to when the altered act stops being an act of knowing. Features which can be altered without the act ceasing to be a knowing act cannot lead to epistemic obstacles in our sense. However,

some features seem to be necessary for us to still consider the modified act an 'act of knowing'. Whenever those features are absent, an epistemic obstacle occurs.

To illustrate the working of the eidetic method, let us now retrospectively provide a firm foundation for our analyses from the last section. In order to get our analysis off the ground, we began by introducing some distinctions that, we believe, are relevant to all acts of mathematical knowing. We first distinguished between a *mathematical expression* and that which it expresses. Then, on the side of what is expressed, we distinguished between the *meaning* of the expression and the *object* to which it relates through that meaning. Using the eidetic method, it can be shown that these conceptual *distinctions* are actually grounded in objectual *differences* in the sphere of mathematical acts.

The starting point of the eidetic analysis is a mathematical act in which these three aspects – expression, meaning and object – are given. Imagine that I, Kata Sebök, am confronted with the expression '12' and interpret it as a numeral within the decimal system. '12', then, is construed by me as a mathematical expression that has the same meaning as '$1 \cdot 10^1 + 2 \cdot 10^0$' and that thus, through that very meaning, relates to the number twelve. To 'hold this act in one's mind' now involves considering this act not as an event that occurred at a certain point in time and space in my specific mind, but rather conceive of it "merely as exemplifying a pure possibility" (Husserl, 1960, p. 71). And the eidetic insight that expression, meaning and object are actually distinct features of this possible act can be obtained by systematically varying the act. First, it becomes evident that the expression does not impose its meaning on me, but rather that the meaning can be varied while adhering to the expression. I could, for example, also interpret the expression '12' in another place value system, for example base 3. '12' would then have the same meaning as '$1 \cdot 3^1 + 2 \cdot 3^0$' and would thus relate to the number five. However, it is also evident that the expressed object is not necessarily altered by different interpretations of a mathematical expression: For example, the expression '3', when interpreted in place value systems of bases 4 and 10, relates to the same object, the number three, but does so via different meanings, namely the respective meanings of '$3 \cdot 4^0$' and '$3 \cdot 10^0$'. In other words, it is possible to retain the expression and the object, but vary the meaning. Yet, it is also possible to retain the meaning and the object while varying the expression. I could, for example, use the English word 'five' or the German word 'fünf' and thereby use different expressions to relate to the same number via the same meaning.

This series of variations, which could of course be taken further, shows that expression, meaning and object are really three different aspects of mathematical acts. In the last section, we developed some of the problems which can arise when these crucial differences in mathematical acts are not taken into account – or, as we have put it succinctly: if a subject *fails to distinguish the different*.

It is important to point out that, while varying the components of mathematical acts in such a way takes us away from any particularities of a situation, the results of this kind of analysis have the benefit of applying to any human activity with the aim of performing such acts. Rather than being interested in a particular group of students with specific characteristics, the level of abstraction serves us in achieving results that apply to a most diverse range of students of mathematics.

Mathematics education researchers have already recognised that an awareness for the results of epistemological analyses of the mathematics being taught and learned can be of benefit both to

researchers as well as teachers in mathematics education (Prediger & Hefendehl-Hebeker, 2016). Rational epistemic obstacle analysis can reveal the *common* obstacles which students of all backgrounds and in all educational contexts can always face. As such, it complements empirical research that seeks to identify effective ways in which educators can help individual (groups of) students overcome these obstacles given specific sets of resources.

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Transposition of research knowledge: From the West to Far East Asia

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Theories and methodologies in mathematics education are disseminated into different international contexts. In this paper, we aim to explore conditions and constraints underlying the transposition of theories and methodologies from the West to East Asia (Japan in particular). Reflecting on two case studies in terms of the anthropological theory of the didactic, we discuss some cultural specificities of researchers' work in Japan that contrast with those in the West: regarding the relationship between researchers and teachers, the nature of theories in relation to the practices, and so forth.

Keywords: Diversity of theories, transposition, culture, language, research knowledge

Introduction

Theories and methodologies in mathematics education are seen as 'research knowledge (scholarly knowledge)' produced by a community of researchers. We assume that any theory in mathematics education cannot be seen as an absolute or finished product but can evolve and change over time through its use and adaptation to the different local contexts in which researchers work. However, it is often implicit what conditions or constraints may underlie local educational contexts or scholarly traditions, which are more or less invisible to those outside them. This is also the case for methodologies in mathematics education. The variations in the methodological frameworks may bring diversity to researchers' theorising work (García, 2019). Although there have been many discussions of the diversity of theories at international conferences such as CERME and ICME, most are about the theories that emerged in the West, in particular, within the European tradition of didactics (Blum et al., 2019). Indeed, different scholarly and cultural traditions exist within the West (Gascón & Nicolás, 2017; Jaworski et al., 2018). However, there are few contributions from East Asian colleagues to such discussions. Theories and methodologies originating from the West are often imported and disseminated into mathematics education research in East Asia. Traditionally, East Asian researchers have translated seminal books and papers for dissemination in their local communities. However, research knowledge sometimes undergoes changes when adapted to a country with a very different language and culture (e.g., Far East Asia such as Japan), due to issues of linguistic and cultural origins. In this paper, we conceptualise such theoretical or methodological dissemination as the *transposition of research knowledge* and illustrate conditions and constraints that may affect the transposition. For this purpose, we use two case studies that the authors have recently worked on. One case illustrates the transposition of a theoretical framework (Documentational Approach to Didactics; hereafter DAD; Trouche et al., 2020) developed in Europe. The other demonstrates the transposition of a methodological framework (design research) that is also well developed in the West (e.g., Gravemeijer & Prediger, 2019). To explore the cultural specificities of research knowledge under the influence of transposition, in the first case, we reflect on the researchers' translation works on the terminologies of the DAD, and in the second case, we consider local theories produced from design research. This theoretical paper makes a new

contribution to the extant literature on the diversity of theories from cultural and institutional perspectives. To this end, we draw some concepts based on the Anthropological Theory of the Didactic (hereafter ATD). One of the reasons for adopting ATD is that it is a research programme with a basic assumption that admits the 'institutional relativity of knowledge' (Chevallard, 2019), which could be suitable to achieve the purpose of this paper.

Theoretical perspectives

Understanding different researchers' works on theories is challenging but promising (e.g., Sierpinska & Kilpatrick, 1998). For instance, Bishop (1992) introduced the notion of research traditions and distinguished three traditions: the pedagogue, the empirical scientist, and the scholastic philosopher traditions. Thanks to the initiative to develop mathematics education as a scientific discipline, the scientific progress of our field has shown a multi-faceted nature of scholarship over the last five decades (Niss, 2019). As the field has developed, theories per se have become more intensively discussed; for example, the roles of theories (Silver & Herbst, 2007), the networking of theories (Bikner-Ahsbahs & Prediger, 2014), and the dialogue between scholars (Gascón & Nicolás, 2017). However, there are far fewer contributions from East Asian colleagues to those discussions. This is not only because they use non-dominant language, but also because they work in a very different research tradition. This is what we would like to explore in this paper.

Some perspectives from ATD

In this section, we offer three perspectives based on ATD (*didactic transposition*, *praxeology*, and an *ecological perspective*) to explore and illustrate the cultural specificity underlying transposition.

Transposition. Didactic transposition conceptualises diffusion of certain mathematical knowledge across institutions, such as in a community of mathematicians (scholarly knowledge), an educational system (knowledge to be taught), a mathematics classroom (taught knowledge), or a community of study (available knowledge) (Chevallard, 1985/1991). Although most existing studies have focused on the transposition between 'knowledge to be taught' and 'taught knowledge', the relativity of 'scholarly knowledge' is rarely discussed. However, scholarly knowledge also varies according to institutions: it is obvious that there is different scholarly knowledge in the history of mathematics. The concept of transposition can be extended to the transposition of research knowledge (Wang et al., 2023). Research knowledge refers to the scholarly knowledge in researchers' communities of mathematics education. In this paper, we assume that research knowledge exists in different research institutions and consider the transposition between contemporary institutions from an international perspective. In particular, we focus on the transposition from the West to Japan. It should also be noted that the transposition does not necessarily lie in one direction, but both directions are possible.

Praxeology. To better characterise research knowledge, we draw the praxeology, which is a model of human activity (know-how) and knowledge (know-why). Artigue et al. (2011) applied this notion to researchers' practice and knowledge. This praxeology is called *research praxeology* (hereafter RP), and it consists of a *praxis* block describing research problems and research methods, and of a *logos* block providing a rationale and discourse to explain and justify the praxis blocks. Certain didactic theories (e.g., DAD) and methodologies (e.g., design research) are considered parts of the logos block

of RP. Developing mathematics education research in international contexts, those RP can then be transposed into different research institutions. The focus of this paper is on the transposition from Western to Japanese research institutions. In fact, the transposition of the logos of RP is an important work that constitutes RP in East Asian countries, such as China and Japan (Wang et al., 2023).

Ecology. In an institution, praxeology is shaped by *conditions* that support its organisation and *constraints* that hinder it. Those conditions and constraints constitute the ecology of lived praxeology (Gascón & Nicolás, 2019). In the case of RP in the West, the logos block is developed within the research institution, which is subject to its ecology (e.g., European traditions of didactics). In Japanese research institution, RP exists with some specificity shaped by ecology in its local contexts. Exploring such conditions and constraints underlying the transposition allows us to understand the cultural specificities of research knowledge in each institution.

Reflecting on two case studies

Case 1: Transposition of DAD

DAD is a theoretical framework in mathematics education that aims to understand teachers' interactions with the resources used and designed in/for their teaching practice (Trouche et al., 2020). The internalisation and the diversity of researchers working on this theory have led to the DAD Multilingual Project (Trouche et al., 2023). One of the aims of this project was to deepen and diversify didactic theory by adapting it to different local contexts. We were involved in this project and engaged in the Japanese translation. One of the outcomes of this project was published in the special issue of ZDM, in which we contributed with Chinese colleagues (Wang et al., 2023). In the following, we summarise some principal aspects of our work in terms of the transposition of DAD. Translation involves different issues, including not only the linguistic aspect but also considering the possible understanding of concepts in the target institution. We therefore assume that "the translation work of a didactic theory is considered a process of transposing a logos block of research praxeology from one research institution to another" (Wang et al., 2023, p. 707). We have experienced various difficulties during translation at different levels. At the linguistic level, there are four main issues: 1) the development of new terminology; 2) multiple meanings and candidates; 3) phonetic expression; and 4) the rules of derived words. These issues stem from the distance between the Western and Japanese languages. Therefore, the translation is not always equivalent to the original term because the choice of the term is strongly affected by the researchers' perspective.

Regarding issues related to cultural origin, the translation of technical terms of DAD is affected by the local educational contexts in which teachers and researchers work when choosing terms. For instance, the word resource is a key term in DAD, and our Japanese translation is リソース(risōsu) which refers to a phonetic expression of the original term (but cannot retain the English connotation '*re-source*' suggested by Adler (2000)). In professional communities in Japan, most teachers are not very familiar with this term. One of the reasons for this is that other terminologies are more commonly used in the communities. An example is the term 教材 (kyōzai), meaning teaching material, which is sometimes used similarly to the term resource in DAD. Kyōzai is the one that is usually provided to students, not teachers. For instance, mathematical tasks and textbooks are seen as kyōzai as well as resources for students' learning. However, curricular documents such as teacher's guides on national

curricula are not considered kyōzai. Additionally, the use of the term kyōzai is practice-oriented without theoretical underpinning, in contrast to the term resource in DAD. Therefore, the use of the term リソース(risōsu) in Japan is not necessarily the same as in the West. There are some Japanese terms that refer to teachers' work with resources. For example, the term 教材研究(kyōzai kenkyū), which denotes studying kyōzai (teaching materials), could be viewed as teachers' documentation work with different resources. The term kyōzai kenkyū is often used to describe teachers' lesson preparation work, including task design and lesson plan drafting. During kyōzai kenkyū, teachers work on various curricular resources; for instance, textbooks and their supplementary materials are very influential in teachers' lesson planning in Japan. Most Japanese educators and educational researchers acknowledge that kyōzai kenkyū is a promising activity for developing teachers' professional and mathematical knowledge for teaching. Although kyōzai kenkyū is a well-known terminology used to describe teachers' practice, the use of the term has no theoretical underpinnings. From a research perspective, kyōzai kenkyū is an object of study that can be analysed or characterised through a particular theoretical lens such as DAD. Consequently, the translation work went beyond the linguistic issues, leading to an understanding of the extent to which the target texts would be evident to them in an institution (see further examples in Wang et al., 2023).

Case 2: Transposition of the design research

Design research is a methodological framework developed across disciplines. In the field of mathematics education, different variations of design research have been developed in different scholarly traditions: such as *instructional design research*; *design experiment*; *topic-specific design research*; *didactic engineering*. One of the reasons for this diversity or dissemination is that design research is a methodology that combines the practical aim of improving designs with the theoretical aim of generating theories (Prediger, 2019). It seems that the implementation of design research is therefore strongly influenced by the local and cultural contexts in which the researchers and teachers work. This is also true for task design research in mathematics education. In the following, we report and reflect on some results of an ongoing project in Japan (Komatsu, 2024; Tsujiyama & Kato, 2022).

This project aims to develop task design principles (TDPs) that allow researchers to design mathematical tasks that make proving activities explorative and collaborative (Komatsu, 2024). Komatsu (2024) defined TDP as follows: "theoretically and empirically grounded prescriptions that guide the design of tasks for specific learning contents or activities" (p. 2). This project adapted a design research model from the West (e.g., Gravemeijer & Prediger, 2019) and offered a research framework comprising the following elements: 1) specifying the learning goal, 2) constructing TDPs, 3) designing tasks, 4) implementing the tasks, 5) analysing the implementation. This framework also includes two fields: "research" and "context (of the implementation)". "Research" is a scientific field in which researchers refer to relevant theoretical references to provide the underpinnings of designs and contribute to the literature. "Context" is a practical field in which the designed tasks are implemented. The project also offers a task design format that allows researchers to work on elements 1, 2, and 3. The following presents examples of learning goals and TDPs taken from Tstujiyama and Kato (2022) during the project.

- Learning goals: Realising explorative proving activities that allow interactions between 'statement (conjecture)', 'proof', and 'looking back'
- TDP1: Use tasks whose conditions are purposefully implicit, in order to allow the production of a conjecture whose hypothesis is implicit. TDP2: Provide a problem that considers if the conjectured statement is true, in order to allow the production of proofs with 'irrational' diagrams. TDP3: Increase students' recognition of the contradiction between the 'irrational' diagram and the valid proof, in order to allow the revision of the statement and the proof

Specifying learning goal(s) and designing tasks are researchers' or teachers' activities that contribute to the practical aim of the design research. There are various methods to design tasks such as 教材研究(kyōzai kenkyū). A set of TDPs supports their task designs to realise learning goals. The development of TDPs is considered a research-based activity because some theoretical references are used to justify TDPs. In terms of the theoretical aim of design research, a set of TDPs produced during the cycle could be the main outcome. TDPs are considered prescriptive theory elements (Prediger, 2019) that guide task design to achieve learning goals. To generate such elements, the theorising steps (e.g., connecting, transforming, refining) are crucial (Prediger, 2019). For example, Tsujiyama and Kato (2022) stated that "when teaching proof problems with diagrams through Lakatosian style, it is difficult for some students to distinguish if one can construct a diagram and if one can prove" and justify this statement with the reference. In terms of the categorisation of the theory elements (Prediger, 2019). This statement is considered an explanatory theory element generated by connecting it with some descriptive elements that they identified in the preliminary classroom practice. This element is then transformed into a conjecture for a prescriptive element "to provoke students' awareness of a contradiction 'while the diagram is irrational, the proof is valid'" (Tsujiyama & Kato, 2022, p. 154). This element was then refined and incorporated into one of the TDPs (see TDP3 above). Those successive steps of theorising (connecting, transforming, and refining) contribute to the generation of TDPs, which are prescriptive theory elements.

Discussion and conclusions

Conditions and constraints underlying the transposition of research knowledge

Reflecting on our transpositive works on research knowledge in the two case studies illustrated in this paper, common conditions seem to support transposition, and common constraints hinder it. A close relationship between researchers and teachers may constitute a condition underlying the transposition. In the first case, some DAD terminologies were translated into Japanese with reference to Japanese professional terminologies. Teachers' documentational work corresponds well to the practice related to classroom teaching in Japan, and Japanese terms already exist to denote such practices. In the second case, a well-established system in Japan allows researchers and teachers to work collaboratively. Thanks to the tradition of lesson study, which is similar (not identical) to the practical aim of design research, the implementation of the cycle in design research seems to work naturally in the school context in Japan. In this case, the framework of the project described earlier is not seen as a product of the 'one-directional' transposition from the West but rather as a product of the 'double' transposition (cf. Cabassut, 2009). As will be mentioned later, mathematics education research in Japan has a strong orientation towards development and practice. This orientation may

constitute a constraint that a theoretical framework is received as a prescriptive or normative one that can be used for developing, designing, and improving educational practices, rather than for understanding phenomena. In the first case, although DAD is a didactic theory to understand teachers' work with resources, it might be conceived as a strategic approach to improving teachers' practice such as kyōzai kenkyū. In the second case, the generation of TDPs (a set of prescriptive theory elements) is the main theoretical outcome, however, this is also an explicit contribution to improving designs. This is because TDPs as prescriptions seem to be acceptable outcomes of research in Japan. In terms of the dialectic between the theoretical and practical aims of design research, the TDPs in the project have two interwoven aspects: TDPs were first produced to support task design and then discussed how these principles could contribute to the theoretical aim of design research. However, this is not the case in other countries' research projects on task design or didactic engineering (García, 2019).

Research praxeology in different research institutions

What is the nature of the logos block of RP in the West? There are many earlier works for developing mathematics education as a scientific discipline (e.g., Sierpinska & Kilpatrick, 1998). The evolution of this discipline requires the creation of new scientific knowledge and terminologies in the theoretical world other than those spontaneously developed in the practical world (see also Margolinas, 1998). To get a deeper understanding of didactic phenomena at stake, the use and role of such theoretical terminologies constitute the logos of RP in the West (e.g., the European tradition of didactics). In other words, there is an *epistemological rupture* (Bachelard, 1938/2002) between theoretical/scientific terminologies, comprising RP in the West, and practical/professional ones, comprising teaching (didactic/paradidactic) praxeology. In this regard, Gascón and Nicolás (2019) offer a clear view on didactics as a science that "is legitimised to present, as results of research, neither normative prescriptions nor value judgments of any kind" (p. 42). In contrast, in the East Asian tradition of mathematics education research, teachers and researchers often share and develop terminology for collaborative work. Hirabayashi (1987), who obtained the first PhD in mathematics education in Japan, stated: "Mathematics education research is, in a sense, a practical science. Its problems are always identified within the educational practice, and its results should always be reflected in the educational practice" (p. 28; our translation). Consequently, RP in Japan has been strongly oriented towards the improvement of teaching (see more examples; Shinno & Mizoguchi, 2023), in which teaching and research ends share the same sphere of educational research. Our contribution could open a new dialogue between scholars working in different research institutions, which can be exposed when research knowledge is transposed from one institution to another.

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Mathematical heterotopias: Lessons for the mainstream classroom

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This paper explores the application of Michel Foucault's concept of heterotopia to mathematics education, examining alternative learning environments such as special needs classrooms, gifted education programmes, and prison education. These "mathematical heterotopias" challenge traditional educational structures, fostering inclusive and flexible approaches to learning. By highlighting the potential of these environments to inspire critical thinking and personalised instruction, this paper emphasises the importance of reimagining mainstream mathematics education. The insights gained from these alternative spaces offer valuable lessons for creating more empowering and inclusive learning environments in mainstream classrooms.

Keywords: Mathematical heterotopias, special needs education, gifted education, prison education.

Introduction

This paper draws on the concept of heterotopia, rooted in Michel Foucault's framework of spaces that exist outside conventional social norms yet remain closely tied to them. Through an exploration of special needs classrooms, gifted education programmes, and prison education as "mathematical heterotopias", I stress the potential of these spaces to challenge dominant narratives and foster inclusive, critical approaches to learning. The theoretical importance of this paper lies in its application of Foucault's heterotopia to mathematics education, revealing how these alternative environments not only disrupt traditional pedagogies but also offer insights that could transform mainstream education. Through this lens, I argue that these spaces present a model for more inclusive, flexible, and empowering learning environments, ultimately suggesting a reconceptualisation of how we approach mathematics education within conventional classroom settings.

Foucault's concept of *heterotopia*

Michel Foucault's concept of *heterotopia* refers to real, physical spaces that exist outside conventional social and cultural norms, yet remain closely connected to them (Foucault, 1986). Heterotopias are spaces that mirror and invert societal structures, representing a kind of "otherness". Unlike utopias, which are idealised or imagined places, heterotopias are real locations that serve specific roles within society. For Foucault, heterotopias are defined by several principles. Firstly, every society creates them, although their forms and functions vary across cultures and time. Secondly, they combine multiple spaces or meanings within a single location. For instance, a garden might represent nature, while also being a constructed, controlled space. Thirdly, heterotopias often play a role in managing time, such as museums or cemeteries, which preserve objects or people for the future. Fourthly, they are spaces that are both accessible and restricted, with entry governed by specific conditions. Finally, heterotopias often reflect or challenge dominant ideologies, offering spaces of deviation or critique, like psychiatric hospitals, prisons, or even carnivals. These spaces serve as mirrors, reflecting a society's norms while also providing an alternative view. Through

heterotopias, Foucault invites us to consider how spatial organisation can reveal the power dynamics and tensions that shape our everyday lives.

Classrooms as heterotopias

Classrooms have often been described as heterotopic spaces where dominant narratives and traditional structures are challenged, offering opportunities for critical engagement and transformation. In this context, Zembylas and Ferreira (2009) describe classrooms in conflict-ridden societies as spaces where students engage with their identities beyond entrenched conflict narratives, allowing for reimagined forms of belonging. These heterotopic spaces resist normative identities grounded in polarised trauma, fostering new possibilities for understanding identity and coexistence. Similarly, Liddle (2021) reveals how a British classroom became a space to question war and the normalisation of military values, enabling students to critically examine dominant discourses around conflict. This classroom, functioning as a heterotopia, facilitated deeper discussions that contrasted with mainstream school environments. Focusing on physical education, Larsson et al. (2014) explore how classrooms become spaces where gender norms can be disrupted. By challenging the traditional masculine and feminine roles typically reinforced in such settings, these spaces provide opportunities for students to explore non-normative identities. In the same vein, Gourlay (2015) discusses educational spaces as heterotopias of compensation, where gaps in mainstream educational practices can be addressed. Classrooms are not merely spaces for conventional learning but dynamic environments where normative structures can be questioned, fostering intellectual and personal growth.

Three examples of mathematical heterotopias

In the context of mathematics education, the mainstream classroom is often viewed as a normative space, structured by traditional pedagogies, standardised assessments, and prescribed curricula. While some may argue that it functions as a heterotopia due to its controlled and regulated environment, I consider it the norm from which alternative learning spaces deviate. In this article, I discuss three distinct heterotopias where mathematics learning occurs outside the mainstream classroom, focusing on the positive and empowering lessons they offer. These spaces challenge traditional structures, foster inclusivity, and encourage critical thinking in ways that can enrich the mainstream classroom. Although I acknowledge the potential negatives and shortcomings of these heterotopias, my focus will remain on the opportunities they present to improve mathematics education within the mainstream classroom setting.

The mathematical heterotopia of the special needs classroom

Teaching mathematics in special education settings presents unique challenges and opportunities. Students in these classrooms often have a diverse range of needs, from learning difficulties to cognitive disabilities like dyscalculia, which can make even basic numeracy skills difficult to grasp. There is sometimes a tendency to set lower expectations for these students, resulting in a diluted curriculum. Instead of being encouraged to engage with challenging content, they are frequently assigned tasks that are overly simplistic (Kurth et al., 2024). In some cases, classrooms spend an

excessive amount of time on basic concepts such as counting or letter recognition, neglecting more complex, age-appropriate material (Cascales-Martínez et al., 2017).

Recent technological advancements are helping to bridge these gaps in mathematics education. Tools like augmented reality (AR) have been shown to enhance engagement and learning outcomes significantly. By making mathematical concepts more visual and interactive, AR allows students to understand better practical ideas like financial management (Cascales-Martínez et al., 2017). For example, AR can transform abstract concepts like percentages and fractions into visual, manipulable objects, making them easier to comprehend. However, when students with special educational needs are integrated into mainstream classrooms, they often face additional hurdles. The pace in general education is usually too fast, and these students may require more time to process and internalise new material. McCarthy and Shevlin (2017) observe that mainstream classrooms frequently lack the necessary repetition and explanation needed for these students to fully grasp mathematical concepts. This emphasises the importance of differentiated support, ensuring that all students, regardless of ability, have the opportunity to succeed.

Mainstream classrooms can benefit greatly from practices commonly used in special education, particularly in personalising instruction and fostering inclusive environments. One notable practice is co-teaching, where a general education teacher and a special education teacher collaborate to deliver lessons. This approach ensures that students with a range of abilities receive the support they need within the same classroom. Rexroat-Frazier and Chamberlin (2019) argue that this method allows teachers to combine their expertise to better address diverse learning needs, and its implementation in mainstream classrooms could be especially beneficial. Another important lesson is the value of maintaining high expectations for all students. In mainstream education, there is often a tendency to lower expectations for students perceived as struggling, but this can have detrimental effects. Kurth et al. (2024) emphasise that when students with disabilities are provided with grade-level content and the necessary support, they often rise to the challenge. This suggests that rather than simplifying material, mainstream mathematics teachers should focus on offering appropriate challenges while providing the necessary scaffolding for students to succeed.

Technology, which has proven transformative in special education, also holds promise for mainstream classrooms. The use of AR, for instance, has been effective in helping students grasp complex mathematical concepts. Expanding the use of such tools could benefit not only students with special needs but also those who struggle with more traditional teaching methods (Cascales-Martínez et al., 2017; McCarthy & Shevlin, 2017). Finally, the principle of inclusion, central to special education, should be more widely embraced in mainstream classrooms (Roos, 2023). In special education, individualised lesson plans such as Individualised Education Plans (IEPs) are tailored to meet each student's unique needs. While this level of formal individualisation may not be feasible in general education, teachers can still adopt more flexible and differentiated strategies to support all learners. Techniques like differentiated instruction, continuous assessment, and small curricular adjustments can ensure that every student is given the opportunity to succeed (McCarthy & Shevlin, 2017).

The mathematical heterotopia of gifted education programmes

Mathematics education in gifted programmes often focuses on both acceleration and enrichment to accommodate the advanced cognitive abilities and unique learning styles of gifted students. These programmes provide opportunities for students to engage with abstract and complex problems that require higher-order thinking, creativity, and independence. Students in these settings benefit from instructional approaches that introduce deeper mathematical content earlier than their peers, fostering creative problem-solving and a more advanced understanding of mathematical concepts (Dimitriadis, 2016). Koshy et al. (2019) also highlight the importance of recognising individual learning speeds in mathematics, encouraging students to move beyond routine tasks and engage in conceptual and creative thinking.

A key element in these programmes is allowing students to create their own problems rather than simply solving existing ones. As Matsko and Thomas (2014) concluded, when gifted students generate their own mathematical problems, their engagement, motivation, and metacognitive skills are enhanced, helping them see mathematics as a creative and dynamic discipline. This approach not only deepens their understanding but also allows them to explore mathematical concepts in a more personal and meaningful way. Mainstream classrooms can learn valuable lessons from these practices, particularly in fostering creativity and autonomy in mathematics. Encouraging all students to create their own problems, as is often done in gifted programmes, can lead to a deeper understanding of mathematical concepts and greater engagement in the learning process.

Acceleration, that is, enabling students to progress through the curriculum at a faster pace, is another strategy proven effective for mathematically gifted students, in allowing them to cover content more efficiently and keeping them engaged by offering appropriate levels of challenge (Maggio & Sayler, 2013). However, the success of acceleration depends heavily on carefully assessing students' readiness and providing the necessary support throughout the process. Differentiation, a central aspect of gifted education, is also highly applicable to mainstream settings. Grouping students by ability or offering tiered assignments based on their readiness allows for more personalised instruction. Koshy et al. (2019) point out that teachers need to be aware of the varying levels of mathematical ability in their classrooms and adjust their teaching methods according to these diverse needs. Acceleration can also be implemented in mainstream education, allowing students who are ready to progress through mathematical content at a pace that aligns with their abilities (Maggio & Sayler, 2013).

Inquiry-based learning and enrichment activities, commonly used in gifted education, can also enhance mainstream mathematics instruction. Leikin (2021) discusses how mathematically gifted students benefit from enrichment tasks that promote deep thinking and challenge them beyond the standard curriculum. Incorporating such activities into mainstream classrooms can encourage all students to engage more creatively with mathematics, fostering engagement and the development of higher-order thinking skills.

Finally, maintaining high expectations for all students is an important principle that mainstream classrooms can adopt from gifted education. Research shows that when students are presented with challenging tasks and given the necessary support, they often rise to the occasion, showing improved performance and confidence (Smedsrud, 2018). Instead of simplifying content for students who

struggle, mainstream teachers should aim to provide suitable challenges along with the support needed to help all learners succeed.

The mathematical heterotopia of prison education

The constraints of incarceration and the specific educational needs of inmates often shape a unique set of challenges and opportunities for mathematics education in prisons. Prison classrooms often serve individuals with diverse educational backgrounds, many of whom have disengaged from formal education early on. Incarcerated students frequently enter prison with significant gaps in basic skills, particularly in mathematics, a subject often associated with frustration and failure in their past (Chrysikou et al., 2023). As a result, prison educators must address both the academic and emotional barriers students bring to the classroom, making it essential to adopt alternative teaching methods tailored to this population (Vasquez, 2017).

Creating a supportive and empathetic learning environment in prison classrooms is vital, often referred to as a "pedagogy of care". When incarcerated students are treated with respect and humanity, they are more likely to engage meaningfully with the subject matter (Wolf, 2020). This approach is especially important in mathematics education, where students often feel inadequate or fear failure. A caring, student-centred approach can help build confidence and change their relationship with mathematics, encouraging them to view it as a tool for personal growth rather than another hurdle to overcome. Moreover, peer learning plays a crucial role in prison mathematics education. Stathopoulou et al. (2024) describe a project in Greece where young prisoners worked collaboratively with prospective teachers to co-create mathematical knowledge. This collaborative approach not only improved the mathematical understanding of all involved but also fostered a sense of community and mutual respect. Such initiatives show the potential for mathematics education to be empowering, helping incarcerated students reclaim their identities as learners and thinkers.

When it comes to teaching methods, hands-on, discovery-based learning has shown great promise in prison settings. Battaglia and Ferrarello (2021) discuss the use of the Mathematics Laboratory approach in Italian prisons, where students engage with mathematical concepts through physical manipulation of objects and collaborative problem-solving. This method, known as "horizontal teaching", places teachers and students on equal footing, encouraging a partnership in the learning process. In this model, students are encouraged to explore mathematical ideas at their own pace, which has proven particularly effective in restoring a sense of agency and self-worth in incarcerated learners (Ahl & Helenius, 2021).

The experiences of teaching mathematics in prison offer valuable insights that could benefit mainstream education. One of the key lessons is the power of a supportive, student-centred approach to teaching mathematics. In prison classrooms, where many students have been let down by the traditional education system, creating a caring and empathetic learning environment is essential for re-engaging them in the learning process (Wolf, 2020). Mainstream classrooms could benefit from adopting similar teaching principles, recognising that students thrive when they feel supported, respected, and treated as capable learners, regardless of their past academic performance. Another important lesson is the value of hands-on, discovery-based learning. The success of the Mathematics Laboratory approach in prisons, where students engage with mathematical concepts through practical,

real-world applications, highlights the need for more experiential learning opportunities in mainstream education (Battaglia & Ferrarello, 2021).

Mainstream classrooms can also learn from the role of peer learning in prison education. In prison settings, students often rely on each other for support, working together to solve problems and build mathematical understanding (Chrysikou et al., 2023). This collaborative learning model can be just as effective in mainstream classrooms, where students with different strengths and abilities can help one another succeed. Encouraging peer learning fosters a sense of community within the classroom and promotes active engagement with the material, leading to better outcomes for all students. Finally, prison education highlights the value of personalised, flexible learning pathways. In prison, students benefit from individualised instruction that meets them in their mathematical development, allowing them to progress at their own pace (Ahl & Helenius, 2021). Mainstream education could adopt similar strategies by offering differentiated instruction and flexible pacing to accommodate diverse learning needs. By moving away from a one-size-fits-all approach, mainstream classrooms can create a more inclusive environment where every student has the opportunity to succeed.

Conclusions - Moving forward

Considering the concept of mathematical heterotopias offers an insightful perspective on how alternative learning environments can contribute valuable lessons to mainstream education. The heterotopias explored, ranging from special needs classrooms to gifted education programmes and prison education, each presents a distinct context. Yet, they converge in their capacity to challenge conventional educational structures and introduce innovative, inclusive pedagogical practices. The common thread across these environments is the capacity for flexible, responsive teaching that prioritises the individual needs and abilities of learners. These settings exemplify the limitations inherent in uniform instructional models within traditional classroom settings, by embracing individualised and personalised approaches

A particularly noteworthy aspect is the role of creativity in fostering autonomy and engagement, as observed in gifted education. This invites us to reconsider how student potential is conceptualised within mainstream education. Providing opportunities for all students to explore stimulating, problem-solving-oriented mathematical tasks has the potential to spark curiosity and critical thinking, irrespective of prior experience or ability levels. Equally significant is the relational focus in prison education, where care, collaboration, and mutual respect are foregrounded. These elements have transformative potential, particularly for re-engaging marginalised learners through peer learning and experiential methods, an approach that could significantly enrich traditional classroom settings.

The insights gained from these unique spaces draw attention to the need to rethink how we approach mathematics education, placing inclusion, diversity, and emotional well-being on an equal footing with academic achievement. These lessons have the potential to transform the mainstream mathematics classroom into a space that meets students' diverse needs, but also empowers them in ways traditional approaches often overlook. The future of mathematics education lies, in my view, in moving beyond rigid, standardised methods towards a teaching that values every student's potential to engage with challenging and meaningful content. Future research could benefit from deeper, more focused investigations into specific heterotopic environments. In-depth ethnographic studies within

a single heterotopia could explore the lived experiences of learners and teachers, examining how mathematics is taught, understood, and valued in these settings. This could provide richer insights into the specific conditions that enable effective mathematical learning outside mainstream structures. Another promising avenue is longitudinal research tracking learners who experience mathematics in heterotopic environments. For example, a study could investigate whether students who engage with mathematics in gifted education, special education, or informal learning spaces carry these alternative approaches into later academic or professional contexts. Furthermore, future research should consider broadening the definition of mathematical heterotopias to include less commonly studied spaces, such as mathematics within Indigenous knowledge systems or apprenticeship programmes. A wider scope could refine our understanding of how mathematical learning operates across different cultural and professional contexts, potentially revealing overlooked yet powerful pedagogical models.

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Reconceptualizing opportunities to learn mathematics through postfoundational theories

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This paper is a theoretical reworking on the concept of opportunities to learn mathematics. Drawing on postfoundational approaches, particularly Foucault's work, the relationalities between systems of thought and pedagogical practices for school mathematics are investigated through teachers' statements. This theoretical approach decentres individuals but foregrounds the materiality of knowledge as normalisations of mathematics education and society. Our reconceptualization reveals the relationalities between organisation of school mathematics and the inscription of kinds of children, and makes visible normalised practices in mathematics teachers' everyday lives in schools.

Keywords: Learning opportunities, Foucault, systems of thought, pedagogical practices, in/equity.

Introduction

The concept of opportunities to learn has widely been used to understand teaching and learning situations in mathematics education. It is taken up by transnational studies that attend equity issues, as students' mathematics achievement disparities are explained through unfair distribution of opportunities to learn (OECD, 2016). Beside large-scale comparative research, opportunities to learn mathematics are also taken as an analytical tool for classroom studies in which student learning, teaching practices and mathematical tasks are considered as the main pillars (Cai et al., 2020). Despite the utility of the concept opportunities to learn in relation to immediate research and practical needs of mathematics education, the field acknowledges that this concept is vaguely defined and yet to develop a comprehensive theorization driven from the practice.

This paper aims at reconceptualizing opportunities to learn mathematics through postfoundational theories, particularly Foucault's analytic approaches. Part of a larger ethnographic project with mathematics teachers in Turkey, our purpose is to produce a new understanding of opportunities to learn mathematics, accounting for relationalities between systems of thought and pedagogical practices embedded in teaching and learning situations. As we explain later, the focus is not on the individuals and their intentions, but on the prevalent discourses that govern ways of thinking and acting in the social sphere of school mathematics. In doing so, our goal is not to develop a fixed and stabilised framework that is applicable in everywhere. Rather, we aim to foreground the relationalities and to make visible the taken for granted discourses and practices. Unpacking the normalising effects in mathematics education offers crucial implications for issues of equity and access.

Opportunities to Learn Mathematics in the Literature

The concept of learning opportunities has occupied the educational literature for more than a half century. Influenced by the psychometrics, the concept is first formulated as the actual time spent in the instructional activities to learn a topic (Carroll, 1963) and the content taught in the classrooms (Husen, 1967). Several studies discussed opportunities to learn by increasing the number of school

days or lesson hours (Wiley & Harnischfeger, 1974), as well as extending the scope of mathematics content and curriculum (Törnroos, 2005). Later, in mathematics education, Tate (2005) pointed to the quality in addition to the time, as predictive of opportunities to learn mathematics in relation to access and equity. For Tate, the quality of instructional activities includes pedagogical methods, techniques and strategies positively affecting students' learning. In relation to this, the bulk of research has worked for the quality of instruction such as cognitive demand of mathematical tasks (e.g., Hadar, 2017) or using daily life examples to improve access to mathematics (Andrews, 2003).

Mathematics education research considers mathematical learning opportunities not only through the quality of mathematical tasks but also in a classroom interactional level, including participation structures, roles and positions of students and teachers during teaching (Gresalfi, 2009). Relatedly, studies direct attention towards how to educate teachers with an equitable vision, including high expectations for historically marginalised groups and asset-based views for students' mathematical thinking (Flores, 2007). Research focuses on changing authority relations by positioning students as producers of knowledge and as capable of doing mathematics (Christiansen et al., 2023).

The concept of opportunities to learn mathematics has been subjected to constant change with the needs of educational research and practice and it is modified in relation to the access and equity concerns. Different conceptualizations offer immediate solutions to the existing inequalities, the research has yet to provide long term alternatives. Despite the equitable intentions and inclusive efforts of teachers providing equitable learning opportunities, classroom participation structures of students are still differentiated based on gender, race or immigration status (Reinholz & Shah, 2018). Particularly, the research exclusively focuses on quality of mathematical tasks, students' participation structures or teachers' positionalities, moving away from holistic approaches that account for teachers, students and mathematical tasks altogether. This separation yields to the individualistic explanations for the current inequity problems in mathematics education, where the unintentional result is blaming individuals or individual groups of students or teachers (Chen & Horn, 2022). Additionally, the recent studies reported that a culture of exclusion emerges in relation to normalised notions of mathematical ability and competences even in the classrooms of equity-committed mathematics teachers (Louie, 2017). These limitations urge us to rework on the concept of opportunities to learn mathematics, a term historically familiar to the mathematics education community. This paper reconceptualizes opportunities to learn mathematics through Foucault's (1995) analytics and aims at opening up new understandings and possibilities of actions.

Theoretical Approach and Analytical Strategies

This paper reconceptualizes opportunities to learn mathematics through postfoundational theories. Postfoundational theories challenge taken-for-granted understandings, systems of thought and normalised practices in social and material settings. We consider postfoundational theories are not anti-foundational, rather similar to how Butler (1995) directs attentions to the eventuality of current understandings and practices that are built on the contingent foundations, which simultaneously excludes or forecloses. In this sense, postfoundational theories rework on the shortcomings and limitations of traditional modern concepts that explain realities and seek to develop new associations and relationalities among thoughts and things that are usually taken for granted.

Drawing on Foucault's (1995) work, this paper aims to make visible the normalised understanding and practices that constitute a regime of truth embedded in practising mathematics teachers' statements regarding learning opportunities. Particularly, our postfoundational approach is concerned with the systems of thought that make, differentiate, classify and order particular kinds of people on the hierarchy (Hacking, 2007). System of thought is not about people's intentions and consciousness but it is considered as the normalised ways of thinking about schooling, students, teaching and learning mathematics. Postfoundational theories along with Foucauldian tradition interrogate the limits of rationality, place the current regimes of truth in doubt and open up possibilities to create alternative understandings with contingent foundations. Our efforts to reconceptualize opportunities to learn mathematics are concerned with questions and critique of normal and taken-for-granted in mathematics education settings. Nonetheless, postfoundational inquiries initiate a shift to creation than merely deconstruction and generates the unthought, with its contingency embedded in social, cultural and institutional contexts (Jackson & Mazzei, 2024).

In mathematics education, research from a postfoundational perspective is few in number, but advancing (Stinson & Walshaw, 2017). Particularly, studies drawing on Foucault's work focus on the cultural and historical processes of subjectivation through mathematics education as well as they make visible how differences are fabricated (Diaz, 2017; Yolcu & Popkewitz, 2019). These studies collectively argue that mathematics does not stay as it is in the spaces of schooling, rather the subject is transmuted with pedagogical tools and techniques, instructional aids and materials. Pedagogical tools and techniques are not merely enabling a better access and opportunity to learn mathematics but also embodying a comparative thought that inscribes differentiations, normalizations and hierarchies among students. In this sense, opportunities to learn mathematics is reconceptualized through the relationality of systems of thought and pedagogical practices that potentially can address mathematics teaching and learning in a broader context instead of focusing on individuals' own efforts.

This paper presents preliminary findings of a larger postfoundational ethnography (Jackson & Mazzei, 2024). This on-going project consists of two phases (first is interviews with 12 mathematics teachers, second is a semester-long observation of 3 teachers from three different schools) and aims to develop a cultural and situated understanding of opportunities to learn mathematics in relation to fabrication of differences in Turkey. We are in the first phase of the project. During the interviews, we ask teachers about purposes of teaching and learning mathematics, and also typical pedagogical strategies for providing access for all. Particularly, our questions focused on the possible classification and ordering of children and the pedagogical interventions for those groups of students.

Rather than capturing individual's beliefs, intentions and capacities of providing learning opportunities, the analytical focus is on the how these opportunities are organised by the practices and tools deemed necessary to provide access and equity in mathematics education. We deliberately seek to understand what pedagogical tools, strategies are reported as effective to provide access for all students for learning mathematics and under what circumstances. Decentring social actors and their intentions (Popkewitz & Brennan, 1997), the analytical focus is on the interactions between ways of thinking and pedagogical tools and practices. The relationality between these two is sought through the data to make visible the normalised statements for children and also the assumptions about teaching and learning mathematics.

Consistent with our theoretical approach, the analysis has started with open-ended questions: What kind of a child and society is envisioned by teaching mathematics in schools? What are the systems of thought that govern the type of mathematics incorporated in teaching and learning situations? How pedagogical practices are assembled with the making of children and their differences? What kind of pedagogical practices are designated for effective mathematics teaching and for whom? These questions enable our research team to conduct an inductive coding that captures participants' statements regarding mathematics, child, society and teaching and learning. In the analytical process, we continuously problematize the self-evident claims made by teachers and consider their multiplicity, partialness and contingency of the statements that govern the possibilities for learning opportunities. For example, when teachers talk about using real-life context to mathematics accessible, we consider what kind of mathematics is becoming accessible for whom and in what purpose the contextualization is serving. This kind of inquiry help us to make interactions between systems of thought and pedagogical practices visible. We consider the data and the relationalities emerged from the data in relation to self and society. That is, when we notice statements, such as "learning mathematics is for happiness or hardworking" as productive of kind of citizens and their differences. We are still in the process of refining and producing these themes that focus on the relationalities. In this paper, we present the preliminary findings emerging from one of the participant teachers. We chose Yıldız, because her data seems intensive of providing relationalities that assist to reconceptualize opportunities to learn mathematics from a postfoundational approach.

Relationalities between Systems of Thought and Pedagogical Practices

Organizing School Mathematics and Cultural Inscription of the Child

The analysis of Yıldız's statements continually reveals differential categories of mathematics that are organised in relation to contexts such as daily life and occupation. One of the prevalent purposes of school mathematics is to maintain the daily life, from shopping to measuring the objects at home:

> In daily life, for example, if they are going to buy a curtain or a tablecloth, they should know the length measurements or… They should know the litres at home and their conversions, for example, know how many litres of fuel a vehicle will take, they should know how to pay for shopping in instalments, we need to teach them enough to know the basics, at least they should learn these.

How mathematics permeates daily life of individuals is normalised not only through shopping situations but also calculating the amount of gas for driving a car or a sense of measurement for random household items. Knowing mathematics is considered as indispensable to pursue an adult life. Here, basic mathematics is deemed as a prerequisite and a baseline for all individuals in society.

Further, mathematics is divided based on the type of professions that require and do not require higher education. In Turkey, students are selected in higher education after taking centralised exams in which mathematics is part of it. Mathematics for passing high stakes exams or the advanced mathematics for a STEM related career are elevated in relation to the mathematics needed in skill-based professions such cook, mechanic or for agricultural work:

> We have children who will study advanced mathematics, and become engineers. It is beneficial for them to know [mathematics] in order to be good at their jobs in the future. [...] Also, according

> to today's needs, […] we tell them, 'If we want to move forward, if we want to get a certain education, if we want to use these in our daily lives, even if we don't study, you will all do jobs in certain places, for example, you will be a cook and you will use mathematics or you will be a mechanic and you will use it. Knowing something never hurts, if you know it and you add it to your daily life, you will do your job better' When we talk like this, it becomes more convincing. [...] So, I think [mathematics] is necessary in many areas.

Above quotes from Yıldız demonstrates how the occupations and so the mathematics needed for those occupations are ordered on a hierarchy. While calculating practices for cooking work is positioned at the lowest, the mathematics needed for engineering careers are referred to as "advance level." Here, the type of mathematics differentiates the kind of students in relation to their future vocational possibilities in the efforts of being responsive to students' interest or needs.

Mathematics is not only organised for occupational or daily life needs, but also different types of mathematics are ordered in relation to the career trajectories of students. Future possibilities are arranged along a hierarchy of the type of mathematics ranging from daily life needs towards STEM based career requirements. Nonetheless, the organisation of mathematics is not merely about career trajectories but also highly associated with cultural rules and norms of the society where for example Yıldız says "There is never any harm in knowing something, if you know it and incorporate it into your daily life, you will do your job better […] be honest people, be happy people. So, do your job well." Specifically, she mentions children "who are more hardworking, more planned and more disciplined" learning mathematics "more clearly." Here, mathematics goes beyond differential career requirements but also it is accompanied with cultural norms and values regarding what kind of people are desired in this culture. Learning mathematics is associated with personal qualities such as hardworking, planned and disciplined. As we explained next, this has crucial implications on how opportunities to learn are enacted with pedagogical practices.

Organising School Mathematics and Pedagogical Inscription of Child

Organisation of opportunities to learn mathematics is found to be associated with how teachers organise the types of mathematics, how they perceive the kinds of children and their future possibilities in relation to that type of mathematics. As discussed above, different contexts of mathematics are associated with the cultural inscription of the child. Second relationality we want to discuss is about the pedagogical inscription of a child. The way teachers consider school mathematics is not merely concerned with teaching and learning mathematics better, but also embedded with how categories of children are perceived and corresponded with particular pedagogical practices. For example, below quote from Yıldız is exemplar of how learning opportunities are arranged with exam-mathematics focusing on speed of test taking for students who are conscious about high stakes centralised exams are gatekeeping for future professions:

> Yes, as I said, from those student groups, the conscious students are like 'let me solve the problem directly, and let me move on to another problem', in other words, 'the more questions I solve, the better for me', you know, generally those who know a lot are not interested in the story part, they are like 'let me solve the problem and be done with it'. There is no such thing like asking a question, responding, explaining the solution or defending opinion. You know, they are like 'I

solved it one by one, if I learn this, I will do it in the exam in the future, this is important for me, the rest is not important'. You know, there are those who are in a hurry…

Rather than contextualising the problems, spending time with discussion of questions or explaining the solutions, the instructional emphasis is put on the ability to gain competency to test taking those "conscious" students. On the other hand, Yıldız mentions students in the "lowest" group who do not want to participate in those activities. Opportunities to learn for those students are organised through notebook keeping and control, using daily life examples for basic mathematics rather than solving questions that might appear in the centralised exams:

> The students in the lowest group, as I said, do not want to participate at all. At most, they are on the notebook keeping part of it. You know, they don't participate in the lesson. I check the notebook with a little bit of force [when they need to keep] notebooks… or in very simple parts, they can give examples from daily life, depending on the content. But other than that, they are not involved in the parts where they [are expected to] solve questions, find solutions, generate ideas. So, they only are involved in the parts where they listen and keep the notebook.

Here, notebook keeping, a practice that needs time, is allocated with the students in the lowest group compared to 'conscious' others in the classroom. These students are subject to 'at least' keeping their notebooks properly, using daily life examples for basic mathematics instead of preparation for centralised exams that are perceived as gatekeeping for their future lives. Our position here is not to overemphasise the preparation for the exams than the notebook keeping, rather, through the analysis employed in this paper, we want to make visible how these two pedagogical practices are productive of constructing children in two distinct and unequal categories of being. That is, preparation for the exams or notebook keeping have less to do with teaching and learning mathematics effectively or increasing mathematical participation of students in the classrooms. Rather, these practices produce two distinct and hierarchical spaces of preparing "conscious" students for the next level in future life and restraining "lowest level" students with simple activities. Here, opportunities to learn mathematics are produced through the kind of pedagogical practice and the constructed categories of mathematics, which are also generative of differences and inequalities.

Conclusion

This paper aims at reconceptualizing opportunities to learn mathematics through postfoundational theories. We deliberately seek for relationalities between systems of thought and pedagogical practices of teaching and learning mathematics as organising learning opportunities. Grounded on interview data yet decentring teachers and their intentions, the analytics unpacks the normalised practices that govern the possibilities for mathematics, children and also teaching and learning. This has been helpful to disentangle the for granted knowledge and practices that deemed effective.

In our cultural and institutional context, particularly, we have observed an emphasis on mathematics that are necessary for the competitive centralised examinations. Yet these priorities are not merely considered for children's future career options but also, they are coming together with cultural norms and values such as being good at work, planned or hardworking. Here, particular norms and values are desired than the others. Systems of thought regarding the type of mathematics assemble with the

cultural inscriptions of child and in effect, these processes create hierarchies and inequities between students. The pedagogical practices are embedded in the broader cultural and institutional context and the systems of thought for the subjects: mathematics and child. Particularly, preparation for the centralised exams and notebook keeping appeared as pedagogical techniques that inscribe students in two distinct categories of being and acting, which simultaneously differentiate them on a hierarchy. Rather than arguing one is better than the other, our aim is understanding how these are generative of inequalities by subjecting students to limited opportunities despite the inclusive aspirations.

We, further, aim to find more relationalities through our data collected from mathematics teachers and more refined associations between the types of mathematics, groups for children and families and the pedagogical practices and strategies. According to the preliminary findings, opportunities to learn mathematics consist of multiplicity of pedagogical practices organised with the systems of thought about mathematics, children and society. Our continual questioning to detect the relationalities and to make visible the presumed categories of mathematics, students and pedagogical practices. Through pointing to the contingency of foundations, this reconceptualization extends the possibilities of action both in practical and research settings in mathematics education.

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Research mixology: Blending mathematics education & digital sociology for a hybrid approach to experimental design

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Keywords: Mathematical memes, class cohesion, cultural capital, hybrid research design

Introduction and rationale: When memes meet maths

This work presents the early stages of the MeMa project, led by the first author, connecting research in Mathematics Education (ME, the first and second author) and Digital Sociology (DS, the third and fourth author) to explore the role of mathematical memes (i.e., Internet memes with a mathematical content) to foster learning and cohesion in the classroom. Mathematical memes are a complex phenomenon, operating at the intersection of various communities, including teachers and students as in Bini (2022), and disciplines like ME and DS. To address this complexity, rather than networking theories within a specific research perspective (such as ME, as in Prediger et al., 2008), we adopt a novel hybrid approach to preserve each research perspective's distinctiveness while nurturing their interaction for complementary understandings. We illustrate this through the *mixology* metaphor, drawing on the art of *blending* elements enhancing the peculiarities of each component. Here, *blending* refers to aligning insights from ME and DS to create a cohesive yet multidimensional analysis of how mathematical memes function as educational cultural artifacts: ME examines the potential of memes to enhance learning opportunities, while DS offers a lens on their social impact.

Research mixology: Blending theoretical frameworks and methods

Theoretically, we blend Bourdieu's capital (1986) with Carron and colleagues' multidimensional model of cohesion (1985). Bourdieu defines capital as the set of skills, knowledge, cultural and social experiences providing individual advantages. Archer and colleagues (2015) extend this concept defining *science* capital as the resources and dispositions related to science that shape an individual's learning opportunities. Building on this, students' *mathematics* capital (MC) can influence how they perceive their mathematical abilities, shape their aspirations, and affect their likelihood of succeeding in mathematics. Carron and colleagues' model distinguishes between task cohesion (collaboration on academic goals) and social cohesion (interpersonal bonds), both of which are key to examining how memes affect classroom dynamics and perceived class cohesion (PCC). Blending these perspectives, the study explores the role of mathematical memes in education guided by the research hypothesis that memes can enhance students' PCC and engage students with low MC. Specifically, we ask: How does the creation, sharing, and discussion of mathematical memes interact with students' MC and PCC? Methodologically, the hybrid experimental design blends methods to produce meaningful data for both ME and DS researchers (Table 1). Given the one-year project duration, design choices reflect time constraints for experiments and data analyses. Experiments will occur from Dec 2024 to Feb 2025 in an Italian secondary school, exploring two dimensions: vertically, across grades 9, 11, and 13, to observe if students' maturity affects results; horizontally with three grade 11 classes, one participating in the full experiment, one in problem-solving activities only, and a control class.

Table 1: The hybrid experimental design

Schedule	Experimental design	Hybrid elements	Underlying Rationale
Nov 24	MCC questionnaire	Individual MCC and PCC questionnaires based on literature (Archer et al., 2015; Leo et al., 2002). Completed at home for free responses (DS) and non-anonymous for mutual (DS + ME) and classroom interventions correlation (ME)	MCC (ME) measured quantitatively before the activities, with qualitative observations during activities to assess engagement at different MCC levels. PCC (DS), expected to vary, is measured quantitatively pre- and post-activity. Cross-analysis (DS + ME) will examine correlations between MCC and PCC
Week 1 at home	PCC questionnaire pre-activity		
Week 1 at school	1h group meme creation activity; topic and groups chosen by teachers 2-hour class discussion	Student groups create mathematical memes on a previously covered topic (ME) Memes are shared on Padlet with non-anonymous comments (DS + ME) Memetic & mathematical discussion (DS + ME)	
Week 2 at school	1-hour group problem-solving activity; same topic, spontaneous groups 1-hour class discussion	DS focuses on spontaneous group dynamics, ME examines if students reference memes or earlier discussion during problem-solving, and observes the following mathematical discussion (ME)	
Week 2 at home	PCC questionnaire post-activity	PCC questionnaire post-activity is the same as pre-activity (DS)	

Following our first objective to establish a hybrid research approach to investigate memes, we look ahead to sharing insights that deepen our understanding of digital culture's role in education.

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TWG 18: Mathematics Teacher Education and Professional Development

Subjects and objects of research and theories of learning in the education and professional development of mathematics teachers

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This group report focuses on two key aspects of research on teacher learning in TWG18: (1) what is being studied—the main topics of interest, that is, the subjects and objects of research; and (2) how these topics are investigated—particularly in relation to various theories of learning. Our goal is to map the contributions to CERME14 through these two lenses, offering both a descriptive overview of the current landscape and a critical perspective on our research community. Such a perspective is essential for identifying blind spots, questioning taken-for-granted assumptions, and pushing our collective work forward.

Keywords: Mathematics teacher education, professional development, learning theories.

The scope of TWG18

Research on teacher education (TE) and professional development (PD) centres on what and how mathematics teachers learn throughout their careers. Research from TWG18 explores the preparation of pre-service teachers (PSTs) and ongoing development of in-service teachers (ISTs), emphasising frameworks, methods, and tools that shape their professional learning across a variety of mathematical topics and classroom settings. As a group, we welcome research that explores and conceptualises mathematics teachers' growth and the development of their professional competences, knowledge, attitudes, and identities. We encourage studies firmly grounded in theories of teacher learning that enhance teachers' professional growth and development at all stages of the profession. The research presented in TWG18 took place in diverse contexts across the world. These included university courses, field practice, online platforms, school-based PD and spanned TE institutions, museums, and professional learning communities in both formal and informal settings. Participants comprised prospective and practising teachers across various international contexts. Group sizes ranged widely, from single case studies to cohorts exceeding 170 participants.

The subjects and objects of research into teacher learning (the *what* of TWG18)

The question of *what* PSTs should learn in university programmes to master their professional tasks and how they can be supported in their professional learning is of major interest to teacher educators and related research. A similar question arises for ISTs: What knowledge, attitudes, professional competences and practices should be addressed in PD programmes to promote students' learning of

mathematics? And how can ISTs be supported in analysing and reflecting their daily practice? A recent systematic review on PST education (Matsumoto-Royo & Ramírez-Montoya, 2021) revealed that many TE programmes are based on practice-based approaches, however with very different inventories of what is considered as most essential, different content-specific substantiations, and heterogeneous conceptualizations of practices in different grain sizes (from atomistic moves to holistic complex combinations of actions and theoretically grounded decision-making). A current review on research related to PD programmes for ISTs (Shure et al., 2025) showed a growing emphasis in PD research on tasks of teaching and corresponding teaching practices focusing on promoting conceptual understanding, using and connecting different representations and revealing and initiating student thinking. There has also been a call to further investigate the connection between the role of teacher collaboration and changes in teaching practices. Accordingly, we were interested in *what* was researched in our TWG in relation to the professional learning of prospective and practising mathematics teachers. Analysing the 35 papers and 10 posters of our group, we identified seven key topics that will be explained in the following.

Professional learning related to teachers' identity, knowledge and instructional practices

Veldhuis et al. investigated PSTs' awareness of their developmental trajectories in TE. Focusing on practicing teachers, *Izzi-Prato and Cordero* aimed at providing empirical evidence on how the ISTs develop their disciplinary identity as they engage in a PD experience. Examining the development of ISTs' professional identity during a PD programme, *Eraky* integrated eye-tracking technology with reflective practices. *Søyland and Jakobsen* examined whether Norwegian PSTs' foundational mathematical knowledge at the beginning of their teacher training could predict their academic performance in university courses. *Grünig and Vogel* explored the extent to which a German preparatory service fosters the development of technological pedagogical content knowledge (TPACK) among PSTs, identifying key factors that significantly contributed to variations in TPACK acquisition. *Avgeri et al.* reported on the initial phase of a topic-specific intervention aimed at enhancing PSTs' knowledge of fraction multiplication and division. Their study focused on developing learning arrangements to help PSTs construct meaningful and transferable understanding. *Hein et al.* conducted a case study on a PD module designed to enhance teachers' use of representations combining open-ended activities with structured multiple-choice tasks that introduced new concepts and provided immediate feedback. They examined how these complementary strategies influenced teaching practices. *Parasha* investigated the areas of growth and dimensions of change in the trajectories of secondary ISTs. The PD aimed at fostering argumentative dialogue in the classroom and the related study identified the programme components contributing most to these developments. *Lit et al.* reported on a design study exploring how PSTs can develop effective teaching strategies for problem-solving instruction. *Digel and Roth* explored how technology can support student learning when benefits are used to increase teaching quality. They developed large a PD programme on the concept of derivative and investigated mathematical understanding on student level.

Growth in teachers' knowledge and identity in connection to other disciplines

Čech and Samková conceptualised ITSs' ability to notice mathematical concepts in chemistry tasks as an integral part of mathematics teachers' conceptual knowledge. They investigated how secondary

teachers educated in both subjects solved ratio application tasks in chemistry, what the teachers detected about corresponding strengths and weaknesses and how this was connected to their teaching practices. *Maugesten et al.* were interested how the integration of programming into the mathematics curriculum may influence PSTs' identities. The study compared PSTs' designated and actual identities as programmers in mathematics to reveal how these reflect knowledge and attitudes related to programming and computational thinking. *Fojcik et al.* investigated the development of competencies of PSTs enrolled in a programming-based mathematics course. They explored the initial development of a PST's schemes when reflecting about her own learning experiences.

Professional learning of teaching practices: planning lessons, analysing textbooks

Lesson planning was examined as an instrument in TE, but also as a process. *Yilmaz Aslan and Ozmantar* researched how elementary school PSTs structured their lesson plans and described the challenges PSTs encountered throughout the process, while *Knaudt et al.* investigated how the design of a planning phase facilitated ISTs' planning actions. *Jones*, in relation to lesson planning, analysed learning objectives from all mathematical pedagogy courses in Sweden. Textbook analysis papers focused on critical thinking: *Østby* studied how ISTs would reflect on textbooks after using a corresponding theoretical framework, and *Ortiz-Laso et al.* investigated whether and how PSTs were able to analyse tasks from geometry textbooks to recognise their limitations, emphasising three different analytical perspectives: attributes, geometric processes, and modes of representation.

Professional learning through approximations and representations of practice

We found a large collection of various design formats: educational vignettes that were text-based *(Geldermann)*, cartoon-based *(Kuntze et al.)* or video-based *(Morales-Reyes & Zakaryan)*, silent video tasks *(Galiç et al.)*, video-clubs *(Didis Kabar & Walkoe)*, simulations *(Shaughnessy & Boerst)*, role playing *(Ødegaard; Sidenvall)*, and scriptwriting *(Rotem & Eraky)*. These representations and approximations of practice were often intended to support teachers in developing noticing (e.g., *Didis Kabar & Walkoe*; *Galiç et al.*; *Morales-Reyes & Zakaryan*; *Wirth et al., Schaap et al.*) or diagnosing student misconceptions (*Rieu et al.*) but also to help teachers learn to provoke meaningful classroom discussion (*Ødegaard*), elicit and respond to student thinking (*Shaughnessy & Boerst*), or lead students through productive struggle (*Rotem & Eraky*).

Mathematics teachers' professional learning within collaborative settings

Skott investigated the reflection phase of lesson study in two schools, examining how teacher groups adapted lesson study goals and tools to fit their local contexts. Similarly, *Redder Honoré and Falkenberg*, also working within a Danish context, reported on a continuing PD project aimed at implementing inquiry-based mathematics teaching (IBMT). They explored how a PD programme structured around a three-phase IBMT model supported implementation and identified the challenges and successes teachers experienced in each phase. *Casi and Sabena* explored a PD programme that fostered collaboration between teachers and museums. By amplifying Italian teachers' voices and reflective practices, they examined informal mathematics education in non-scientific museum contexts, with a focus on teachers' perceptions of emergent learning. *Bjørnebye et al.* investigated how the large-format ratio triangle could enhance PSTs' collaborative reasoning in missing-value

problems. By integrating full-body movement, speech, and symbolic representation, they explored how various semiotic resources supported collective proportional reasoning.

Professional learning through interaction between teachers and their environment

Ebbelind and Helliwell, through a meta-analysis of prior TWG18 papers, drew attention to the ways in which teachers' individuality is portrayed in relation to their institutional and social environments, suggesting a need to better account for this embeddedness when studying professional growth. Another group of contributions focused on the beliefs and values that teachers bring to their practice. These studies (e.g., *Bjuland et al.; Serrazina et al.*) explored how such beliefs—regarding the nature of mathematics, teaching, and learning—interact with the opportunities offered in both PST and IST settings, and how they shape teachers' professional trajectories. How teachers engage with instructional practices in various learning environments was explored in another set of studies: The papers by *Pozhegu-Ermeni et al., Wirth et al., and Maksym'yak* examined how teachers approach activities such as problem posing, interpreting student thinking, and negotiating the role of social and epistemic considerations in mathematics education. These practices were situated within structured learning experiences and were analysed as opportunities for reflection and development.

Exploring design aspects of PD programmes

Carlsson examined various PD designs and their impact on teachers' instructional practices. In contrast, *Schwarts and Friesen* investigated the shift from in-person to online PD environments. Their study explored how key design principles for effective PD—such as centering mathematical content, incorporating representations of practice, fostering collaboration, and adapting to participants' needs—can be translated into asynchronous online mathematics PD programs. They also identified the challenges and opportunities associated with designing such online formats.

Conclusion I: The subjects and objects of research into teacher learning in TWG18

Investigating professional growth related to teachers' knowledge, identity and practices is at the centre of TWG18. Reflecting current trends in mathematics TE research, about half of the studies addressed practice-based approaches. The connection to other STEM subjects, the use of technology and the use of various approximations of practice in research or to foster teacher and student learning were other key topics. There appears to be a growing interest in collaborative PD settings, the interaction between teachers and their environments and research questions related to design aspects of effective TE and PD programmes, e.g., related to scaffolding, online learning and opportunities for reflection. Although many key topics are represented in our TWG, we see that impacts of PD programmes on student learning are rarely addressed and that the mathematical content delivered in courses and programmes is often neglected when learning goals, contents and contexts are reported. Since this seems to be an ongoing and therefore alerting trend (cf. Friesen et al., 2023), we take our findings as opportunity to encourage studies relating teacher learning to student learning and centring the mathematics in mathematics education research.

Theories of teacher learning, growth and development (the *how* of TWG18)

In this section, we examine *how* research on mathematics teacher learning is framed through theoretical perspectives. Our premise is that articulating the underlying theoretical commitments—

even briefly—is essential for situating contributions within broader scholarly conversations and for making visible the assumptions that inform design and analysis. In what follows, we identify how theories are taken up across the papers in our group (whether explicitly named or implicitly guiding) and consider the implications of these patterns for the field. The theoretical perspective is the chosen viewpoint used to comprehend and enhance the research process, suggesting that it directs the reasoning in research as a fundamental criterion for the research endeavour. Theoretical perspectives include predefined structures, such as concepts and assumptions that guide ideas and the research design. Eisenhart (1991) refers to these structures as frameworks. We select from various frameworks, including theoretical, practical, and conceptual frameworks. According to Eisenhart (1991), a theoretical framework employs a formal theory to establish explanations about a phenomenon. A practical framework is informed by identifying what works in practice. Furthermore, one can view a conceptual framework as a scaffold that justifies the study in relation to its aim – a set of assumptions about reality that underpins the research. Eisenhart (1991) emphasises that the research interest itself does not determine the theoretical direction of the study. The researcher needs to choose an explicit conceptual framework. The selection of a framework is crucial for how the research process is conducted and what findings the researcher ultimately presents. Once a theoretical perspective is chosen, researchers must consider how to transform empirical material into meaningful data aligned with that perspective.

At its core, TWG18 focuses on the learning, growth, and development of mathematics teachers at all stages of the profession. Theoretical perspectives, in this context, help us to make sense of developmental processes. We maintain that articulating a clear theory of teacher learning is essential for designing effective TE and PD environments. For this reason, we are particularly interested in how researchers in TWG18 draw on and apply such theories in their work. Our review of the TWG18 contributions revealed that most papers were explicitly or implicitly informed by sociocultural theories of learning, or by theoretical hybrids that included sociocultural components alongside other frameworks of sorts. Sociocultural theories of learning, grounded in the work of Lev Vygotsky and others, have become foundational in mathematics education, particularly for understanding how learning is mediated through social interaction, language, tools, and cultural practices. These theories are especially influential in TE and PD, where learning is seen as participation in social and cultural practices. Rather than occurring in isolation, knowledge develops through participation in shared practices and dialogue with others, particularly those with greater expertise. Key constructs such as Vygotsky's Zone of Proximal Development (ZPD) and scaffolding underscore how learners grow through guided support that enables them to move beyond their current capabilities. Several key frameworks in education and in TE—such as situated learning (e.g., Brown et al., 1989), communities of practice (Wenger, 1998), and pedagogies of practice (Grossman et al., 2009) —are grounded in sociocultural theory. These frameworks build on the idea that knowledge is constructed through participation in social practices and that learning is embedded in the cultural and relational contexts in which it occurs. Other frameworks are less explicit in naming their sociocultural foundations, yet they are deeply informed by this perspective through their emphasis on participation, mediation and contextualized activity. A similar trend can be observed in papers submitted to TWG18: while many appear to be grounded in sociocultural theory or its derivatives, this theoretical lens often remains

implicit. In the following, we describe different ways in which sociocultural perspectives were used in our group.

Social practice theory, identity development

Eraky, *Maugesten et al*., and *Skott* investigate mathematics teacher identity and professional development through different theoretical lenses, all of which are connected to the concept of identity. *Eraky* combines cognitive science with a situated view of identity, using Looney et al.'s (2017) five-component model of teacher identity. *Maugesten et al*. draw on Sfard and Prusak's (2005) narrative identity theory and Beijaard et al.'s (2004) categorisation of teacher roles, while *C*. *Skott* uses Holland et al.'s (1998) concepts of figured worlds to analyse how teachers adapt lesson study practices, combined with J. Skott's (2013) Patterns of Participation framework. All three studies view identity as a dynamic and developmental concept. *Eraky* emphasises cognitive-affective and reflective dimensions, *Maugesten et al*. focus on narrative and role-based constructions, and Skott foregrounds socially situated practices. Through the use of theory, *Eraky* shows how technology can externalise cognitive processes and support reflective identity development. *Maugesten et al*. reveal how PSTs navigate the gap between current competence and future aspirations. *Skott* highlights the importance of cultural and institutional contexts in shaping PD practices.

Values

In 1988, Alan Bishop introduced the mathematics education community to the idea of values, arguing that mathematics is a deeply value-laden cultural activity, shaped by and embedded within the norms and practices of the societies in which it is developed and taught. Since then, the role of values has become increasingly recognised as central to a sociocultural perspective on mathematics education, particularly in the fields of TE and PD, as scholars have recognised that teachers' values influence their pedagogical decisions, classroom interactions, and the kinds of mathematical knowledge they prioritise, ultimately shaping students' learning experiences and mathematical identities. *Serrazina et al*. draw on Bishop's work, presenting an exploratory study to identify and discuss what first-year undergraduate students in Education value in the teaching and learning of mathematics. These authors argue that the process of valuing is inherently sociocultural since it is shaped by experiences, backgrounds, and social interactions over time as members of particular cultural communities. *Izzi-Prato and Cordero* theorise a PD proposal for mathematics teachers that is grounded in the development of their disciplinary identity. They draw on Cantoral's (2016) Socio-epistemological Theory of Mathematics Education (STME), which aligns with sociocultural perspectives by examining how values shape the social construction and institutionalisation of mathematical knowledge. STME is a deeply value-laden theory that challenges dominant, decontextualised views of mathematical knowledge and emphasises the inclusion of diverse cultural and social practices, reflecting a commitment to epistemic justice.

Anthropological Theory of the Didactic, and Instrumental Genesis

Several papers draw on theoretical frameworks that, while distinct in origin, share key assumptions with sociocultural theories —particularly the emphasis on context, mediation, and practice. For example, Anthropological Theory of the Didactic (see *Maksym'yak's* and *Jones'* papers) focuses on

the institutional organization of knowledge and didactic transposition. Instrumental genesis as a perspective examines how learners shape and are shaped by engaging with artefacts (in *Fojcik's* paper, with programming). These frameworks, though not sociocultural in foundation, offer complementary perspectives on learning as shaped by cultural and material contexts.

Noticing

While early research on noticing was grounded in a cognitivist perspective, recent work increasingly draws on sociocultural perspectives, framing noticing as a socially situated, culturally mediated process. This lens emphasizes that what teachers notice—and how they interpret it—is shaped by broader cultural, institutional, and relational contexts. Five papers (*Galiç et al.*; *Kabar & Walkoe*; *Kuntze et al.*; *Morales-Reyes & Zakaryan*; *Ortiz-Laso et al.)* and one poster (*Sumeyye*) explored teacher learning through the lens of noticing. While all draw on the core components of attending, interpreting, and responding (Jacobs et al., 2010), they vary in focus: *Sumeyye* considers noticing before teaching; *Galiç et al*. and *Kabar & Walkoe* examine student thinking; *Morales-Reyes & Zakaryan* analyze classroom videos using Sherin et al.'s (2011) framework; *Kuntze et al.* introduce multicriterion noticing, including how students use representations, and students motivation; and *Ortiz-Laso et al*. study curricular noticing (Dietiker et al., 2018) in relation to textbook tasks. Taken together, these contributions suggest that noticing frameworks are increasingly being applied beyond the analysis of student thinking, expanding the range of phenomena studied through this perspective.

Conclusion II: Theories of teacher learning, growth and development in TWG18

While sociocultural perspectives are widely represented in this year's TWG18 contributions, we observe limited engagement with other important theoretical lenses for understanding teacher learning—such as commognitive theory, socio-political and socio-ecological perspectives, or embodied cognition. These approaches could offer richer accounts of how, for example, language, power, and political contexts shape teaching practice and professional development. Their relative absence suggests an opportunity for the group to further diversify its theoretical repertoire and attend to dimensions of learning that remain underexplored. We encourage continued expansion of theoretical approaches to support research on emerging and understudied phenomena.

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Teaching fraction multiplication and fraction division to prospective teachers: A design research study

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We present findings from the first cycle of an ongoing topic-specific design research study focused on fraction multiplication and division. We designed a course unit addressed to prospective teachers to support them in building productive meanings for multiplication and division that are transferable to rational number contexts. Thirty-four prospective teachers participated in the first enactment of the course unit. They were pre- and post-tested via a paper-and-pencil test comprising 11 tasks, examining various aspects of knowledge in this area that are necessary for teaching. The results showed that their performance improved significantly after the intervention, but substantial conceptual difficulties were identified at both test moments.

Keywords*: Fraction multiplication, fraction division, conceptual change, teacher education.*

Introduction

Multiplication and division of fractions present teachers with many challenges (Copur-Gencturk, 2021; Izsák, 2008; Ma, 1999). Typically, teachers are able to perform computations, but face difficulties in problem solving, problem posing as well as in representing and interpreting the two operations and their algorithms (Ma, 1999; Olanoff et al., 2014). In terms of the theoretical framework of Ball and colleagues (e.g., Ball et al., 2008), such evidence indicates that teachers lack in content knowledge necessary for teaching mathematics, either Common Content Knowledge (i.e., mathematical knowledge necessary for problem solving in various contexts, including teaching; hereafter CCK); or Specialized Content Knowledge (i.e., knowledge that is used specifically in the context of teaching and allows, for example, for explaining, exemplifying, and representing mathematical content for students; hereafter SCK). Such limitations are bound to have adverse effects on the quality of teaching and, consequently, on students' learning.

In this paper we present findings from the first cycle of a topic-specific design research study (Gravemeijer & Prediger, 2019) aimed at developing materials and teaching-learning arrangements to enhance pre- and in-service teachers CCK and SCK on fraction multiplication and division.

Theoretical Framework

A major source of conceptual difficulties with fraction multiplication and division is the changes in meaning required in the transition from natural to rational numbers. The dominant meanings of these operations within natural numbers—namely "repeated addition" and "fair sharing"—do not generalize for fractions (Fischbein et al., 1985; Tirosh, 2000). In addition, the relationship between multiplication and division (as inverse operations) is reframed via the introduction of reciprocals, allowing for division to be replaced by multiplication by the reciprocal of the divisor. This reframing is crucial as it underpins the fraction division algorithm commonly known as "invert and multiply".

It is widely known that these changes are a source of difficulty for students as well as for pre-and in service teachers (Ma, 1999; Olanoff & Tobias, 2014; Tirosh, 2000). This phenomenon can be

theoretically approached by conceptual change perspectives on learning that is focused on problems faced by learners when their prior knowledge in an area is incompatible with the new content to be learned; we thus draw systematically on principles for instruction stemming from conceptual change approaches to mathematics learning, in particular rational number learning (Vamvakoussi, 2017) to inform our design. More specifically, in the phase of prospective analysis (Gravemeijer & Prediger, 2019) of our study, we looked for productive meanings for multiplication and division that are transferrable from natural to rational numbers (see also Simon et al., 2018). To this end, we invested in a relational approach to multiplication, interpreting an expression of the form a x b=c as depicting a relationship between three physical and/or numerical quantities (Polotskaia & Savard, 2021). For example, the expression 3x5 =15 is typically interpreted as "3 times 5 is 15"; from a relational perspective, it could be interpreted as "the triple of 5 is 15" and, simultaneously, "5 is one-third of 15". This is the same multiplicative relationship viewed from two perspectives, that allows for assigning a similar role to the natural and fractional number as multiplicative operators. The multiplicative relation between similar quantities can be established through measurement, a notion crucial also for the operation of division, since this meaning is transferable to rational numbers. A meaning for multiplication that is also transferable to rational numbers draws on the "rectangular area" model (Iszák, 2008), where the factors are represented as the lengths of the sides of a rectangle and the product as its area. The inverse problem situation, where the length of one side is the unknown quantity, affords a viable meaning for fraction division as well (Yim, 2010).

To foster the intended meanings for multiplication and division, we designed a course unit targeting prospective elementary teachers, consisting of short lectures, in-class activities, and self-study tasks. Consistent with our theoretical lenses (Vamvakoussi, 2017), the key ideas underlying the design of all components of the course unit were the following: a) explicitly problematizing the meaning of multiplication and division already in the context of natural numbers; b) systematic cross-domain mapping (i.e., an analogy between dissimilar domains) between the domain of physical quantities, specifically length and area, and the domain of number, established through measurement and implemented in terms of the representations as well as the problem situations employed; c) highlighting (deep) similarities and differences between natural and rational number contexts by systematically varying the types of numbers in the same problem situations (see also Sun, 2017). All instructional tasks for students were framed as classroom scenarios (Biza et al., 2007), with a view to making the tasks relevant to their professional interests. Students were asked to evaluate problems and problem solutions provided by hypothetical students, compare different representations, provide feedback, solve as well pose problems related to fraction multiplication and division; and to evaluate and modify instructional tasks. The course unit was enacted with prospective teachers. In this paper, we only present findings regarding students' performance in fraction multiplication and division tasks before and after the intervention. Data collected during the intervention are currently under analysis and will inform the retrospective analysis of the first implementation.

Method

The participants were 34 prospective teachers (hereafter, PTs) from a Department of Primary Education in Western Greece (26 female). According to their Study Program (8 semesters in total), the PTs completed two compulsory courses related to elementary mathematics and mathematics

education in the primary years (Grades 1 to 6). The participants attended a 7th semester elective course on mathematics education, within which the unit was embedded. They gave their informed consent for their research data to be used anonymously for research purposes.

We designed a test consisting of 11 tasks (T1 - T11) focusing on fraction multiplication and division. Except for 3 of the tasks (T1, T10, T11), all were framed as classroom scenarios (Biza et al., 2007). The PTs were presented with hypothetical students' responses to mathematical tasks in the form of solutions, representations, or statements, and were asked to evaluate them in terms of mathematical accuracy; to explain students' thinking; and to provide feedback, if necessary. In this paper, we will refer only to PTs' evaluations which reflect their CCK (T1, T2, T5, T6, T7) or SCK (T3, T4, T8, T9, T10, T11; Ball et al., 2008). Specifically, the first group of tasks examine whether the PTs can themselves perform fraction multiplication and division; recognize the appropriate operation for solving a problem; and can exchange dividing by n with multiplying by 1/n (CCK). The latter group of tasks examine whether PTs can correctly evaluate alternative solution processes, verbalize a rule accurately, and exemplify fraction multiplication and division via word problems or non-symbolic representations, which are actions that are particular to teaching (SCK).

In tasks T1 and T2, the PTs were asked to calculate the results of the operations 2/5×7/3 and 2/5: 7/3, respectively, writing down the calculation process in detail. In T3, the PTs were asked to evaluate the following calculation process: 3/4×5/2 = 3/4×10/4 = 30/16. If a PT acknowledged that the calculation process is correct despite the fact that converting to equivalent fractions is unnecessary, the response was deemed correct. In T4, the PTs were asked to evaluate the statement, "When I want to divide two fractions, I invert one and multiply". If a PT acknowledged that it matters which of the two fractions is inverted, the response was deemed correct. In T5, the PTs were asked whether the problem "5 kilograms of cherries cost 10 euros. How much does half a kilogram cost?" reflects the operation 5:1/2. Similarly, in T6, they were asked to judge whether the problem "A carpenter measures the length of a plank with a stick and finds it to be 18 sticks long. What would he find if he measured the length of the plank with two-thirds of this stick?" can be solved via the operation 2/3x18.

In T7, a circular diagram (Figure 1) was presented as a task given to students to fill in the missing number, and the PTs had to evaluate a statement claiming that this diagram is incorrect. This task examines whether PTs could substitute division with multiplication by the reciprocal of the divisor. Responses stating the diagram is correct and/or mentioning the missing number were deemed correct. In T8, the PTs were asked whether the representation in Figure 2 is a valid representation of the operation 2/3×3/4. This representation is, according to Iszak (2008), a powerful representation for fraction multiplication. In T9, the PTs were asked to create their representation for this operation.

Finally, in tasks T10 and T11, the PTs had to pose a problem matching the operations 3/4×2/5 and 4: 2/3, respectively. In these tasks, we examined which types of problems were created and whether they were appropriate (i.e., they matched the operation in question and used the given numbers).

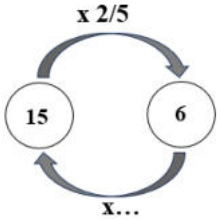

Figure 1: Circular diagram (T7)

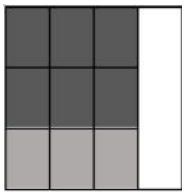

Figure 2: Fraction multiplication representation (T8)

The PTs completed the test before the intervention. The course unit was enacted in 6 sessions, approximately 2 hours each. The PTs worked on the in-class activities in groups of 3-5 members. Three research team members (first, third, and last author) were present to support student work and document their difficulties. The last author provided feedback in the form of short interactive lectures. After the intervention, the pretest sheets were returned to the participants who had the opportunity to review their initial responses and revise them if they judged it necessary. An identical empty sheet was provided to fill in the revised responses. Posttest responses included the unrevised and the revised responses.

Results

Responses were coded as correct/incorrect and scored by 0 and 1, respectively. The total score was calculated as the sum of correct responses across tasks. The Shapiro-Wilk test (sample size<50) showed that the data tested were normally distributed (p=>.05). The paired sample t-test showed a statistically significant difference between the pre- and post-test scores, $t(33) = -6.849$, $p = .000$. Specifically, there was a significant increase in the mean scores from pretest (M=4.76, SD =1.759) to posttest (M=7.35, SD=1.454). Table 1 presents the percentages of correct responses per task in the pre- and post-test. In all tasks, an increase in the percentage of correct answers is observed at post-test, with tasks T7, T8, T9, and T10 showing the most substantial improvement.

Table 1: Percentage of correct answers per task, in the pre- and post-test

	T1	T2	T3	T4	T5	T6	T7	T8	T9	T10	T11
Pre-test	85,3%	82,4%	35,3%	64,7%	82,4%	41,2%	44,1%	17,6%	0	11,8%	11,8%
Post-test	97,1%	94,1%	44,1%	73,5%	88,2%	61,8%	85,3%	61,8%	52,9%	35,3%	35,3%

Table 1 shows that the PTs reached their highest scores in T1 and T2 (i.e., in the calculation tasks), at both test moments. However, evaluating a calculation process for fraction multiplication (T3) as well as an inaccurate expression of the "invert and multiply" rule (T4) proved more challenging. Further, it appears that at pretest the majority of PTs recognized that the given operation did not match the given problem in the case of division (T5), but not in the case of multiplication (T6). At posttest, there were still about 40% incorrect responses in T6.

T7 elicited a substantial number of incorrect responses at pretest. The PTs stated that the second operation in the diagram should be division, or that the second operand should be the same as the first; or they explicitly claimed that it is impossible to find a number that, when multiplied by 6,

results in 25. Despite the large increase in correct responses at posttest, there were still about 40% incorrect ones in this task. In Task 8, a small number of correct evaluations were made at pretest that increased substantially at posttest. T9 elicited only incorrect responses at pretest; about half of the responses were correct at posttest. In most of the responses deemed incorrect (23 and 12 at pre- and post-test, respectively), the PTs calculated and represented the product (1/2), indicating that they misinterpreted the task. Two PTs represented the given fractions separately and put the multiplication sign in between. Appropriate representations were either grounded on the "part of a part" meaning of multiplication, or the rectangular area model, similarly to T8.

T10 and T11 proved challenging at both test moments. Fifteen PTs at pre-, and 6 PTs at pos-test did not pose any problem for T10 (and were scored by 0). The number of inappropriate problems at pretest was 15 and 3 of them were meaningless or unsolvable. The remaining 8 problems had a common feature: The given fractions were identified as parts of a quantity but were not combined multiplicatively, resulting either in addition problems (e.g., *Paul bought 3/4 of a kilogram of coffee, and Maria gave him another 2/5 of a kilogram. How many kilograms of coffee does Paul have in total?*); or fraction comparison problems (e.g., *John ate 3/4 of a pizza, and Helen ate 2/5. Who ate more*?). At posttest, there were 9 such problems in a total of 15 inappropriate ones. On the other hand, the number of appropriate problems increased from 4 to 13 after the intervention. Two problems at pretest and 5 at posttest were grounded on the rectangular area model of fraction multiplication (e.g., *The length of a rectangle is 3/4 m, and its width is 2/5 m. How much is its area?)*. The remaining were grounded on the "part of a part" meaning (e.g., *Konstantina bought 3/4 of a kilogram of apples and gave 2/5 of that to her friend Maria. How many kilograms did Maria get?*).

In T11, 16 PTs at pretest, and 3 PTs at posttest did not pose any problem. The number of inappropriate problems increased from 14 to 19 after the intervention. In many of these problems (6 at pretest, 12 at posttest), the given fraction (2/3) functioned as a multiplier of a quantity. Some of these problems (4 at pretest, 10 at posttest), were multiplication problems (e.g., *A kilogram of cherries costs 4€. How much do 2/3 of a kilogram of cherries cost?*). In the remaining such problems, arbitrary data were introduced to pose problems that did not correspond either to multiplication or to division with the given numbers, or they were unsolvable; however, 2/3 still functioned as a multiplier of a quantity (e.g., *4 kilograms of potatoes cost €5. How much do 2/3 of a kilogram cost*?). In one problem (at pretest), the roles of 4 and 2/3 were reversed, with 2/3 functioning as the dividend (*4 crackers cover 2/3 of a small baking tray's surface. How many centimeters does 1 cracker cover?*). The remaining inappropriate problems were mostly meaningless or unsolvable; corresponded to other operations, particularly subtraction; or a combination of the above (e.g., *I have 4 liters of milk, and I want to distribute them into 2/3 of a cup. How many liters of milk will I have left?*).

On the other hand, the number of appropriate problems increased from 4 to 12 after the intervention. Most of them were grounded on the measurement meaning of division (e.g., *A baker distributes 4 kilograms of dough into containers with a capacity of one kilogram, but he uses only 2/3 of the capacity. How many containers will he need?*). Only one problem at posttest was grounded on the rectangular area model (*I have a rectangle with an area of 4 square meters. If its length is 2/3 meters, what is its width?*). Additionally, 2 participants posed problems of the type "known part, unknown whole" at pretest, and retained them at posttest (e.g., *If 2/3kg of lemons costs €1, find the cost of 4kg*).

We cross-tabulated the individual total scores (max 11) at pretest with the corresponding ones at posttest (Table 2). Each cell of Table 2 represents the number of participants reaching the particular scores at pretest and posttest. The individual scores range from 1 to 9 at pretest; and from 4 to 11 at posttest, pointing to inter-individual differences with respect to performance at both test moments.

Table 2: Cross-tabulating the individual total scores at pre- and post-test

	Individual total score -Posttest												
Individual total score-Pretest	0	1	2	3	4	5	6	7	8	9	10	11	Total
0							1						1
1							2		1				3
2													
3						1		1	4				6
4							1	2					3
5					1	2	1	3		2	1		10
6							1	1		2		1	5
7							1	1	2				4
8									1				1
9											1		1
10													
11													
Total					1	3	7	8	8	4	2	1	34

Most participants (27/34) improved their score at posttest. The score of 14 of them increased by 1 to 3 points; the score of the other 13 increased by 4 to 6 points. Five participants retained the same score; two did not revise any of their initial responses; the others revised incorrect as well as correct responses. Finally, 2 participants received a lower score at posttest. Further, there were two participants who had a relatively high score already at pretest (8 and 9); one of them was among the ones who retained the same score, and the other increased the score by 1 at posttest. Finally, only one participant reached the maximum score (11), with an initial score of 6.

Conclusions - Discussion

We presented findings from the first cycle of a topic-specific design research study (Gravemeijer & Prediger, 2019) focused on fraction multiplication and division. We designed a course unit addressed to pre-service teachers, aiming to enhance their CCK and SCK (Ball et al., 2008). We targeted the challenges that relate to the transition from natural to rational numbers, thus the design of the course unit as well as the intervention was based on principles from instruction stemming from conceptual change perspectives on learning (Vamvakoussi, 2017). The key idea was to foster productive meaning for multiplication and division already in the context of natural numbers which can be transferred to rational number contexts (see also Simon et al., 2018). In this paper we focused on results regarding the participants' performance before and after the intervention in related tasks.

Consistently with prior research, before the intervention, most participants were able to execute the algorithms of fraction multiplication and division but had very poor conceptual understanding of the two operations and their relationship (Ma, 1999; Olanoff et al., 2004; Tirosh, 2000). This is particularly evident in problem-solving and problem-posing tasks in which, similarly to students, the participants appeared to confuse fraction multiplication with fraction division, fraction multiplication with addition, and fraction division with subtraction. Evaluating and constructing representations pertaining to fraction multiplication and division also proved very challenging (Iszák, 2008). Further, the participants were mostly unaware of the possibility to replace multiplication with division by the reciprocal of the divisor (Ma, 1999), with some of them stating explicitly that it is not possible to multiply a number and get a smaller result (Fischbein et al., 1983, Tirosh, 2000). Finally, a correct, even if atypical, calculation process for multiplication as well as an inaccurate formulation of the "invert and multiply" rule was evaluated incorrectly to a large extent. These results depict limitations in various aspects of Common Content and Specialized Content Knowledge (Ball et al., 2008), and underly the need for substantial support to prospective teachers. After the intervention, the participants' performance increased significantly. However, this should not be taken to imply that they have mastered the intended content. Indeed, many of the difficulties identified by the pretest, persisted after the intervention. This is not uncommon in conceptual change research because revising prior knowledge and building new knowledge in a complex area such as fractions is a challenging and time-consuming endeavor.

Our findings also indicate that there were inter-individual differences with respect to the participants' performance before and after the intervention, as well as to the extent they benefited by the intervention. Analysis of the data collected during the intervention will shed more light on the conditions that favor or constraint learning and will be examined in the phase of retrospective analysis of the study (Gravemeijer & Prediger, 2019) to inform our design for future implementations.

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Preservice teachers' collective proportional reasoning in large-format ratio triangles

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A goal in teacher education is to help preservice teachers (PSTs) develop skills in using multiple representations to foster mathematical reasoning with their future students. However, there is limited understanding of how different semiotic resources can support students' collective reasoning about proportions. To explore this, the paper presents two excerpts from a design study involving PSTs working collaboratively on proportional reasoning within a large-format ratio triangle (LFRT). The semiotic analysis demonstrates that LFRTs can facilitate the integration of movement, gestures, speech, and symbols, enabling them to build on each other's ideas while reasoning through missing-value problems. The study also reveals that the level of participation in the collective reasoning process among PSTs within the LFRT can vary, ranging from active integration of semiotic resources to a more non-verbal role, where they translate verbal reasoning of peers into physical reasoning.

Keywords: Design research, teacher education, proportional reasoning, semiotic bundle

Introduction and research question

The ability to reason about proportional situations is fundamental to students' mathematical competence; however, it is a topic that many students find difficult to grasp and teachers find challenging to teach (Lobato et al., 2010). These challenges stem from superficial teaching methods focusing on procedures, and a lack of relevant context and visual aids, which hinder students' understanding of proportion concepts. Research shows that collaborative discussions help students understand and apply mathematical reasoning more effectively by encouraging them to articulate and justify their thinking (Hansen, 2022). In teacher education in Norway, a key objective is to equip the preservice teachers (PSTs) with the skills necessary to facilitate creative and mathematically grounded reasoning among their future students. A part of this goal is helping PSTs develop the ability to use multiple representations (concrete, embodied, visual, symbolic) to foster collective proportional reasoning. This approach not only enhances their pedagogical techniques but also ensures that they can teach these critical concepts, enabling their students to understand and apply proportional reasoning in diverse and meaningful ways. Some research has shown how multiple representations in proportional tasks, such as double number lines, ratio tables, and strip diagrams, can influence students' reasoning (Arican, 2018; Beckmann & Kulow, 2018; Orrill & Brown, 2012). However, Prahmana and Nusantara (2024) systematic review on ratio and proportion learning design revealed a limited exploration of modeling tools beyond ratio tables, resulting in a lack of variety in approaches to modeling ratios and proportions. Particularly, a gap remains in understanding how tasks involving full-body movement can foster collective proportional reasoning.

A ratio triangle features a V-shaped design, where each axis lies within a common measurement space, and proportional relationships are represented by corresponding sides maintaining consistent ratios, as seen in similar triangles. This visualization of proportional relationships is less clear when using a double number line or a table of proportional values. Based on a larger design study in teacher education, this paper addresses the following question: How does the use of various semiotic

resources in a large-format ratio triangle (LFRT) support PSTs collective proportional reasoning? The research seeks to gain deeper insights into how the LFRT, as an artifact, can enhance PSTs' collaborative reasoning in missing-value problems by integrating full-body movement, speech, and symbols.

Theoretical background

Mathematical reasoning can be defined in various ways, encompassing both individual and collective processes and products (e.g., Lithner, 2008; Niss & Højgaard, 2019). Collective mathematical reasoning is defined as lines of connected arguments emerging from collaborative efforts (Sumpter, 2016), which can be mediated through various semiotic signs (e.g., speech, movement, symbols, drawings, use of tools). Within a semiotic lens, Arzarello et al. (2009) introduce the concept of a semiotic bundle, which refers to a dynamic system of signs generated by one or more interacting individuals over time. This system typically includes the signs produced by a student or a group of students while solving a problem or discussing a mathematical question. The signs included in a bundle may evolve, and the relationships among these signs can also vary as the students or the interacting students produce them. As noted by Arzarello et al. (2009), collaboration enables students to share semiotic bundles, facilitating the transformation of informal and formal mathematics. Based on the view of learning as a semiotically mediated process (Presmeg et al., 2016), we use semiotic bundle to explore PSTs collective mathematical proportional reasoning in a LFRT.

Proportional reasoning involves a complex set of skills that intersects with various concepts in rational numbers and algebra, such as ratios, rates, fractions, and functions (Lamon, 2007). For PSTs, it is essential to understand what constitutes robust proportional reasoning, identify strategies used in proportional reasoning, and learn how to effectively support their future students in developing proportional skills (Weiland et al., 2021). A proportion is defined as a statement asserting the equality of two ratios, symbolically expressed as a/b=c/d. According to Lamon (2007), proportional reasoning involves providing justification for claims about the structural relationships among the four quantities of a proportion within a context that involves both the covariance of quantities and the invariance of ratios or products. This entails recognizing a multiplicative relationship between two quantities and extending that relationship to other pairs of quantities. Proportional reasoning is usually assessed using comparison or missing-value problems. In comparison problems, all four values of the proportion are provided, and the task is to find equivalent ratios. In missing-value problems, three of the four values are given, and the challenge is to identify the missing value to complete the proportion. For example, consider a situation where 4 coffee mugs cost $20, and you want to determine the cost of 6 coffee mugs. In this context, there are two measurement spaces: one involves the quantities of coffee mugs, and the other involves the corresponding monetary values. The within-measures strategy compares quantities within the same measurement space, forming ratios like 4/6 for the mugs and 20/x for the cost. In contrast, the between-measures strategy compares values across the different spaces, forming ratios such as 4/20 for mugs to money and 6/x for the missing value. Solutions to proportional problems can be achieved using both formal and informal strategies. Formal methods include cross-multiplication, while semi-formal methods encompass the rule of three or using a fixed factor for multiplication between ratios. Informal, flexible scalar strategies involve reasoning up and down within ratios. For instance, we can reason down by first determining the cost of 1 mug, which

would be $5 (20/4). Then, reasoning up to 6 mugs, we multiply the cost per mug by 6, arriving at $30 ($5 \cdot 6$).

According to Weiland et al. (2021), a robust understanding of proportional reasoning for PSTs involves using various representations to support reasoning, including ratio tables or triangles, double number lines, and strip diagrams. Ratio tables represent elements within proportional situations in a symbolic form and support thinking strategies for solutions by encouraging the use of multiplicative strategies such as halving, doubling, and multiplying by 10 (Dole, 2008). The double number line (DNL) is a visual tool that coordinates quantities in a proportion by using two parallel lines. For instance, if 2 kilograms of fruit cost 1.5 Euros, the DNL shows that doubling the amount of fruit to 4 kilograms would correspondingly double the cost to 3 Euros. The ratio triangle is designed to help visualize and solve problems involving proportional relationships through a triangular arrangement of numbers. Specifically, the ratio triangle can be used to solve missing-value problems by either identifying the scalar ratio as a within-measures strategy or identifying the functional rate as a between-measures strategy (Kazunga & Bansilal, 2016). Unlike the double number line, which uses parallel lines, the ratio triangle features a V-shaped design where each axis is within the same measurement space. In this research, we will explore how the large-format ratio triangle (LFRT; Figure 1) can support the integration of semiotic resources for students positioned along each measurement space, enhancing their collaborative proportional reasoning.

Arican et al. (2023) examined the strategy repertoire of 91 PSTs in solving missing-value word problems, finding that cross-multiplication was most used. Additionally, they discovered that PSTs who employed less common strategies in proportional problem-solving had a significantly larger strategy repertoire compared to those who favoured more common strategies. In another study, Arican (2018) showed that providing practical contexts for missing-value problems helped PSTs recognize proportional relationships in their solutions, thereby reducing the reliance on error-prone strategies like cross-multiplication. Copur-Gencturk et al. (2023) explored 238 teachers proportional reasoning on the non-routine task “which rectangle most closely resembles a square”: (a) 1 foot by 4 feet, (b) 17 feet by 20 feet, (c) 7 feet by 10 feet, or (d) 27 feet by 30 feet. They identified four categories of reasoning: incorrect (28%), additive (11%), relative (17%), and proportional (45%). Based on their findings, Copur-Gencturk et al. (2023) suggest that similar tasks can help teacher educators guide teachers in identifying covarying quantities and transitioning from additive to proportional reasoning. Orrill and Brown (2012) investigated teachers participating in a professional development course focused on proportional reasoning using DNL, a representation that was new to them. The participants demonstrated a stronger ability to reason about relationships within a single measurement space (e.g., comparing the number of kilograms in different scenarios) than to coordinate values across two different measurement spaces (e.g., comparing kilograms to Euros). The findings indicate that teachers require extensive support and guidance to effectively apply their proportional knowledge to DNL-related tasks. Berk et al. (2009) investigated 148 preservice elementary teachers' flexibility in proportional reasoning before formal instruction and the impact of two intervention versions. The study found that participants initially exhibited limited flexibility in proportional reasoning, and that the intervention significantly improved strategy flexibility, with gains retained six months later.

Method

This paper is part of a larger design project conducted in a mathematics didactics course attended by fourth-year students in five-year teacher education programs qualifying them to teach mathematics for grades 1-7 and 5-10 in Norway. The project involved four researchers (the authors) and 25 PSTs participating voluntarily. The design principles guiding this research included the integration of movement with (proportional reasoning as) mathematical content (Skulmowski & Rey, 2018), flexible strategy usage through the creation of original solutions (Lithner, 2017), collective reasoning (Sumpter, 2018), and supporting abstraction through the use of symbols. Prior to the group activities, an introductory lecture on the use of the LFRT was held. In this paper, we examine how a group of five PSTs utilized various semiotic resources within the LFRT to enhance collective proportional reasoning and explore different solutions in two tasks: a) Calculating 27% of 480 NOK, and b) Designing and solving their own proportional reasoning problem. The excerpts have been chosen to emphasize contrasting elements of collective reasoning using the LFRT, rather than the complexity of the mathematical tasks.

One PST presented the tasks, and one PST was positioned on each axis of the LFRT (i.e., the task's measurement spaces; Figure 1). Any group member was welcome to contribute when working on the tasks. The presentation and the solution of the tasks were videotaped by one of the group members. The videos were transcribed verbatim, and non-verbal actions were enclosed in brackets. The transcripts were translated from Norwegian to English. To capture how the PSTs' semiotic-mediated arguments build on each other in the excerpts examined, we employ synchronic and diachronic analysis of the semiotic bundle (Radford et al., 2017). The synchronic analysis captures how various semiotic signs are bundled together in brief moments of a particular PST's action, while the diachronic analysis examines how these semiotic bundles interconnect to form a coherent line of collective arguments.

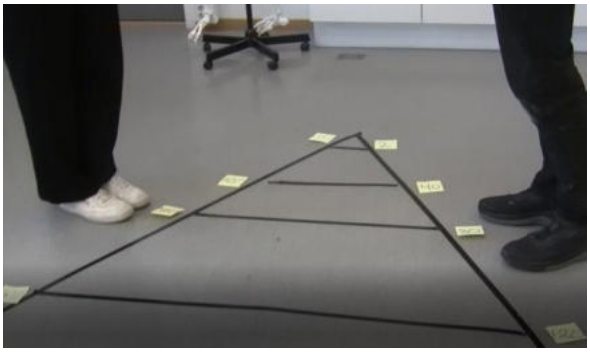

Figure 1: The LFRT is taped on the floor

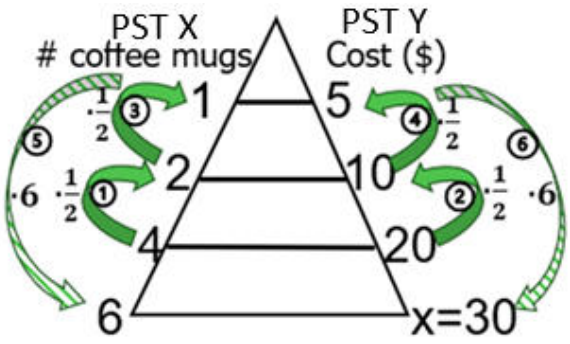

Figure 2: Example of coding

The transcripts were supplemented with coded illustrations, reflecting the PSTs use of semiotic resources in the reasoning process (Figure 2). These illustrations show numbers along the axes that correspond to values written on Post-it notes (or verbally stated if not symbolized), with the multiplicative structure of the movement-based verbal reasoning represented along curved arrows (Figure 2). Striped curved arrows indicate multiplicative reasoning without physical movement. To support the diachronic analysis, the reasoning steps within the ratio triangle are sequentially numbered in circles. Referring to the earlier example where 4 coffee mugs cost $20, and you want to

determine how much 6 coffee mugs cost, Figure 2 illustrates the positions of PSTs X and Y on the left and right axes of the LFRT, respectively, with the scaling process shown through six ordered movements. For instance, PST X's initial movement from the note labelled "4" to "2" (accompanied by the statement, "half of four coffee mugs is two coffee mugs") is represented by an arrow connecting these numbers, demonstrating multiplicative reasoning (i.e., ·1/2).

Results and analysis

We present and analyse two excerpts from the collaborative problem-solving activities carried out by the PSTs, focusing on their proportional reasoning within the semiotic framework. The excerpts were selected to highlight contrasting aspects of collective reasoning when using a LFRT. The selected group of PSTs had set up the LFRT using Post-it notes, placing percentages on the left side and corresponding numbers on the right to outline their steps in proportional reasoning (cf. Figure 1). Both excerpts present the groups third solution of the two missing-value problems examined.

Excerpt 1 begins with PST A presenting the task while PSTs B and C stand at positions representing 27% and x, respectively, and PSTs D and E observe.

A: We need to find 27% of 480 [B and C move to 100% and 480, respectively].
A: We first choose to find 50% [① and ②], which is?
C: 240.
A: Then we go down to find 25% [③ and ④].
C: Which is 120.
A: Yes, we took the half of 50% since 25% is half of 50%.
A: And then we needed 2% more to reach 27%, so we first find 1 percent which is 4.8 [⑤ and ⑥].
A: Then we multiply it by 2, which is 9.6 [⑦ and ⑧].
A: Adding it to 25% gives us 27% [B and C move to 27% and x, respectively], which is 129.6.

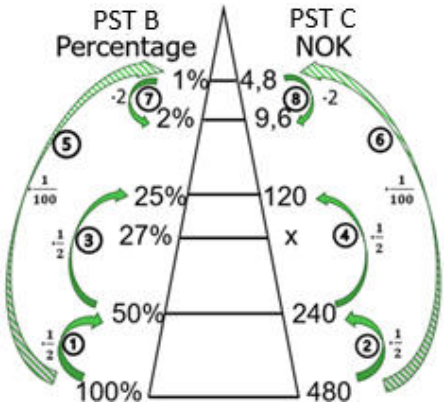

Figure 3: Coding of excerpt 1

Excerpt 2 addresses the task of finding 80% when 60% equals 120 NOK. The scene begins with PSTs A and D positioned to represent 60% and 120 NOK, respectively, while PST C introduces the solution, and PST E observes.

C: Now to solution 3 [A and D at 60% and 120, respectively].
A: We have 60%. We divide by 3 to get 20 [①].
D: And 120 divided by 3 is 40 [②].
A: Then we double 20% to get 40% [③].
D: And doubling 40 is 80 [④].
A: Then we multiply by 2 to find, or double to find 80% [⑤].
D: And 80 doubled is 160 [⑥].

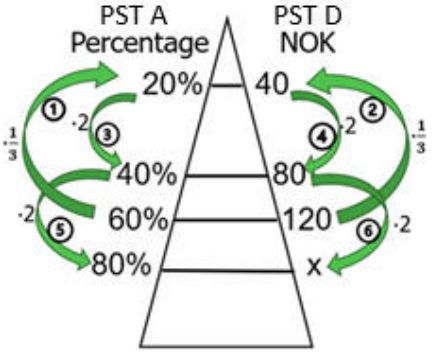

Figure 4: Coding of excerpt 2

In both excerpts, the PSTs collaboratively solve missing-value problems using the LFRT as an artefact while integrating various semiotic signs such as movement, symbols, speech. The reasoning

primarily follows a within-measures strategy, scaling up and down within the same measurement space, such as calculating 50% and then halving it to determine 25% (240/480 and 120/480, respectively), as seen in Excerpt 1. The PSTs' reasoning is both synchronic, connecting different semiotic resources in real-time, and diachronic, building on previous steps to form a coherent line of argument. For instance, in Excerpt 2, PST A's division of 60% by 3 to get 20% is followed by PST B's division of 120 NOK by 3 to get 40 NOK. The sequence of the six movement-based scaling steps ①-⑥ in Excerpt 2 illustrates how the PSTs' semiotic-mediated arguments accumulate to form a coherent line of reasoning (Figure 4).

In Excerpt 1, PSTs B and C positioned within the LFRT adopt a more passive verbal role, primarily translating PST A's verbal reasoning into physical reasoning. Excerpt 1 also demonstrates that not all reasoning steps need to be aligned with movement (the steps ⑤ and ⑥ in Figure 3). In contrast, Excerpt 2 presents a more balanced interaction, with both PSTs A and D, positioned in the task's measurement spaces, actively contributing to the reasoning process by building on each other's arguments and closely integrating movement into the scaling up and down the percentages and NOK, respectively. Consequently, the two excerpts differ in the level of PST participation and the semiotic resources employed in their collective proportional reasoning.

Discussion and concluding remarks

We analysed video recordings of PSTs group work to explore how various semiotic resources, particularly movement within the LFRT, support collective proportional reasoning. The research reveals two major insights. First, the LFRT facilitates the exchange of arguments through multiple semiotic resources, fostering the development of collective proportional reasoning. Second, the roles taken by PSTs during collective reasoning in the LFRT, as well as the extent to which movement is integrated into the reasoning process, can vary. Consequently, this research contributes to the understanding of collective reasoning (Sumpter, 2016), particularly in how it develops within proportional tasks using the within-measures strategy (Kazunga & Bansilal, 2016). The use of the LFRT in the PSTs reasoning processes mirrors the application of tools like the double number line in teaching proportional reasoning, as discussed by Orrill and Brown (2012). Both tools help visualize proportional relationships, making abstract concepts more concrete. However, our analysis shows that the PSTs are not merely using the LFRT to perform rote calculations; they are collectively engaging with it to construct and validate their reasoning processes. This reflects Orrill and Brown's (2012) finding that teachers need guidance in effectively using such tools to bridge their proportional knowledge with the task at hand. The collaborative nature of the problem-solving process, as seen in the two excerpts, aligns with the findings of Copur-Gencturk et al. (2023), who emphasize the importance of guiding teachers in recognizing covarying quantities to transition from additive to proportional reasoning. In the excerpts, PSTs work together to build on each other's ideas, moving from simple to more complex reasoning. This process of collective reasoning in practical contexts is crucial for reinforcing proportional thinking and moving beyond cross-multiplication and additive strategies, which are often error-prone (Arican, 2018; Arican et al., 2023; Copur-Gencturk et al., 2023). In line with Berk et al. (2009), who found that PSTs often struggle with proportional reasoning but can benefit from targeted interventions to improve strategy flexibility, our research highlights the value of the LFRT. This tool supports PSTs in understanding and working with ratios and proportional relationships both visually and physically, while fostering collective reasoning and the

use of various semiotic resources. This work contributes to the existing knowledge on task design aimed at improving collective proportional reasoning in teacher education, highlighting the value of incorporating semiotic resources like movement into the learning process.

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Tensions emerging when discussing tasks for problem-solving lessons

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As provoking tensions, talking about tensions, and managing tensions are found to be effective ways to promote teachers' professional development (PD), identifying tensions in PD initiatives is important. This study explores tensions that emerge in the 'task-centric' PD and research initiative Partners in Practice. Data from one small-group discussion, involving three teachers (Grade 8) and one teacher educator is explored. A main tension in the work of leading small-group discussions was the contrasting force between a safe practice where students solve routine tasks (tradition) versus a practice where they solve problem-solving tasks (innovation). A second set of tensions were found at the teacher-researcher level, within the research-practice partnership. A third set was at the teacher-teacher level or related to the teachers' work of teaching. This set might be illustrated by contrasting forces like using non-curriculum tasks versus curriculum tasks.

Keywords: Professional development, tasks, tensions, problem solving, mathematics teaching.

Introduction

Twelve lower secondary mathematics teachers and three teacher educators are involved in *Partners in Practice*, a professional development (PD) initiative with a problem-solving approach to mathematics teaching. In the beginning of the fifth PD session (six months into the project), one of the teacher educators shares a concern that we (as PD providers) have not succeeded in selecting tasks that are relevant for the teachers' practice. One teacher confirms that the teachers and the teacher educators have various inputs to the sessions, expressing that the tasks discussed are not relevant to the mathematical topic they focus on "just now". Another teacher expresses: "I feel we are in two different worlds, our daily work is from minute to minute, it is hectic, and the exam [Grade 10] is approaching, and suddenly we get a task from you at a session that we should try out [with our students]." After this discussion, the teachers and the teacher educators agree to discuss and try out tasks suggested by the teachers in the coming sessions.

Tasks have a fundamental role in mathematics teaching (e.g., Hiebert et al., 2003), in attempts to understand teaching and student thinking and learning in research on mathematics classroom practices (e.g., Tekkumru-Kisa et al., 2020), as well as in teachers' PD (e.g., Boston & Smith, 2011; Johnson et al., 2016). *Partners in Practice* is a 'task-centric' initiative where we seek to establish a collaborative space for continuous development of mathematics teaching practice. The vignette above illustrates how tensions might emerge, and understanding tensions is important for PD providers (e.g., Johnson et al., 2016). The present study aims at exploring these tensions.

Background

Tensions are often described as dilemmas where no ultimate solution can be found (Katz & Raths, 1992), and dilemmas have been extensively discussed in the research literature. As an example,

Lampert (1985) defines teacher dilemmas as problems to be managed, and she does not consider it a weakness that some dilemmas of teaching are not resolvable. Berry (2007) builds on Lampert's (1985) work and considers tensions as dichotomous forces acting on a teacher (educator). In a self-study on improving her own teacher education practice, Berry used the notion of tensions to capture the feelings of being pulled in different directions by competing pedagogical demands. The self-study identified twelve tensions, or dichotomous forces. One of these tensions, *Telling and growth*, is embedded in teacher educators' learning "how to balance their desire to tell prospective teachers about teaching and providing opportunities for prospective teachers to learn about teaching for themselves" (Berry, 2007, p. 120). Building on Berry's (2007) understanding of tensions as dichotomous forces acting on a teacher, others have identified comparable pairs of forces within the lived experiences of mathematics teachers. For instance, Andrà et al. (2019, p. 3) describe tensions as "pairs of contrasting forces that pull a teacher in two different directions." An example of a pair of contrasting forces might be a safe practice where students solve routine tasks (tradition) versus a practice where they solve problem-solving tasks (innovation) (Liljedahl et al., 2015). Unlike prior research on tensions, Liljedahl et al. (2015) found that "teachers do not simply manage these opposing forces, but also work at, and seek help in, resolving them" (p. 193). They argue that their extension of how tensions are understood (i.e., by including affective aspects) has implications for the work of providing PD for mathematics teachers and suggest that an emphasis on tensions is important "since provoking tensions, talking about tensions and managing tensions are effective ways to promote teachers' professional development" (Andrà et al., 2019, p. 5). In a more recent study aiming to better understand the way tensions might contribute to teachers' beliefs, decisions, and actions, how teachers live with tensions and how they resolve them, Liljedahl et al. (2023) understand the source of forces at play in tensions as personal/internal, meaning that tensions are strongly connected to past experiences and values arising from inside a teacher. The source of forces at play in tensions are also contextual/external, meaning that they arise from "norms, education and stakeholders" (Liljedahl et al., 2023, p. 1666). The present study defines tensions as contrasting forces (Andrà et al., 2019; Berry, 2007), and we aim at answering the following research question: What tensions emerge when teachers discuss tasks used in a problem-solving lesson?

Methods

This study is part of an ongoing project that aims at developing a problem-based approach to mathematics teaching, and it involves twelve Norwegian lower secondary teachers and three teacher educators (the first three authors of this paper). The PD sessions involve discussing key concepts related to problem solving, getting firsthand experience with solving mathematical problem-solving tasks and inviting teachers to try out problem-solving tasks with their own students. The discussions in the PD sessions range from how problem solving is defined in the Norwegian national curriculum from 2020, to tasks as presented in the textbooks used. When trying out problem-solving tasks, the focus this far has been on presenting tasks and facilitating group and whole-class discussions when students solve problem-solving tasks. In between sessions, the teachers are challenged to do some work like observing their own students' mathematical thinking when solving tasks. Tasks have not been designed in the PD sessions. However, characteristics of tasks to be used in the teachers' own

classes have been discussed, both tasks brought to the sessions by the teacher educators and by the teachers themselves. All sessions are recorded by audio and video, and transcribed verbatim.

The teachers participated in twelve two hours PD sessions over a period of 18 months. In this paper, data from a small-group discussion at the sixth session—involving three teachers (Grade 8) and one of the teacher educators—is explored. Based on the discussion from the previous PD session exemplified in the introduction's vignette, the teachers had selected a task related to the ongoing mathematical topic and tried it out in their classes. In this small-group discussion, the teachers reflect on their experiences from using the task with their students. The mathematical focus of the task is algebraic calculations with parentheses, and the students are going to find the area of a rectangle with a given length and width or a square with a given side. The task consists of a string of exercises where students are supposed to find the area of rectangles where the side lengths become increasingly more complex, from simple areas like 8×10 to more complex situations where the area can be expressed as $(5x + 4)(4x + 3)$. For instance, one of the teachers suggests: "Say that it says *10* on one side and *5x* plus *3* on the other side. What I thought they were going to do, that they just do like that, and then you get *50x* plus *3*. That it will be that order of calculations. That they forget to take it in parentheses."

The analysis of the discussion (15 minutes) started by identifying the parts of the transcription where tensions, or pairs of contrasting forces that might pull the teachers in different directions (Andrà et al., 2019), emerged when the task was discussed. This was followed by inductive coding of all utterances in the teachers' exchanges. In addition, memos were written. After one researcher inductively coded the whole transcription, two other researchers double-checked the coding, deleted and added codes and wrote memos where they disagreed. All three then discussed and agreed about the final coding before the coding was analyzed with the purpose of identifying patterns of tensions in the teachers' talk. As a supplement to this analysis, the transcript of the small-group discussion was divided into 14 episodes by one of the authors.

Findings

In addition to the main tension where students solve routine tasks (tradition) versus a practice where they solve problem-solving tasks (innovation), we found three levels of tensions in the teachers' talk: researcher-teacher, teacher-teacher and teacher-student. When dividing the transcript into episodes, the analysis revealed that sub-categories of contrasting forces within these overall patterns were interwoven, but we focus on the first two levels here due to space restrictions. Tensions related to the research-practice partnership were particularly prominent in two episodes while sub-categories of tensions related to the teachers' work of teaching were interwoven in all episodes.

The first episode is presented below to illustrate the complex pattern of tensions. Prior to the small-group discussion, one of the teacher educators has introduced a list of talk moves (Kazemi & Hintz, 2014) that might facilitate teacher-student dialogues in classrooms. As a starting point for reflecting on "what you've done" (1), the teacher educator (Raymond) challenges one of the teachers (Nina) to reflect on how this task could be used or revised to elicit student participation. Raymond seems to have noticed a possible tension emerging in Nina, between a wish for student dialogue and the student dialogue observed:

1	Raymond:	You talked about how you thought there was little dialogue, I heard you say. With this task, how could you have invited them into more of a student dialogue? Now that you have done something, based on the talk move, are there other steps you could have taken? Is there anything about the task that you could do something about? So, it's a good starting point for reflecting on what you've done. I noticed what you said about missing that there was a bit more student-dialogue.
2	Nina:	I would have had it later, but I thought we had to do it before today. So, I would've had it later in the chapter, where they had…
3	Kai:	So, you want to learn the rule first, and then the proof afterwards?
4	Nina:	Not necessarily, but…
5	Kai:	Because I want to.
6	Nina:	I want to have worked at least with parentheses. And then afterwards to have had that aha experience, oh, that's how it all goes together.
7	Kai:	Yes, but then you must have done it many times first. Okay, now you know the rule, but don't quite know why it gives an answer, and then you see, oh yeah, that's why you do it like that.

Nina's response (2) illustrates how different levels of tensions are interwoven. She seems to experience a tension between when to use tasks (especially tasks provided by the PD providers) and her teaching plan. Participating in the problem-solving session "today" forced her to use the task with her students before it was suitable concerning her teaching plan, indicating that there was a wish from the teacher educators to bring her classroom experience to the session. At one level, we identify two contrasting forces, teacher practice versus participating in PD sessions, within a research-practice partnership (Johnson et al., 2016). Related to the same level, there also seems to be a tension which relates to Berry's (2007) *Telling versus growth*: between the PD providers informing and creating opportunities to reflect and self-direct or between acknowledging teachers' needs and concerns and challenging them to grow. We also identify signs of Berry's tension *Safety versus challenge*: between providing teachers with a constructive learning experience and an uncomfortable learning experience.

We also identify a tension related to her work of teaching. For Nina this specific task seems to fit later, when the students have worked on more mathematical content from a particular textbook chapter, also indicating her view of teaching, she wants students to know the content they need to solve the task. We can illustrate this tension with the two contrasting forces: know content before working on a task versus learn content while working on a task.

In utterances 3–6, Kai and Nina seem to agree that their students must know some content before working on tasks like the one discussed. However, while Kai would like his students to "know the rules" before proving (3), Nina highlights some content the students would need for her students to have an "aha experience" while solving the task (6). Kai (7) seems to think that practicing routine tasks is important. After practicing "many times", you learn the rule, but not necessarily the reason why the rule works. The utterances indicate tensions between these teachers' views of teaching.

At the end of the group session (Episode 14), Nina's talk also made the tension of research–practice partnership (Johnson et al., 2016) visible when she expresses:

143	Nina:	But I would like to think that the tasks that are also presented here (in the sessions), and likely the ones that we have used, are perhaps given more for a grade four and up. So, it makes it difficult for the weak (students) to join. How will you manage to keep up if you barely know that…

144 Kai: You can't explore until you…
145 Nina: Yes, you must have some prior knowledge to do it. So, I think you aim too high with these tasks.

Here Nina challenges the PD providers for presenting tasks in the PD that could not be used for all students (143). Nina's last sentence "I think you aim too high with these tasks" might also be seen as a tension between the PD providers ideas about these tasks and the reality of the classroom. Kai challenges the idea of exploration in mathematics teaching (144) and Nina concretizes by highlighting the need for students to "have some prior knowledge" before exploring (145). These exchanges indicate the tensions, tradition (prior knowledge before exploring problems) versus innovation, and routine tasks (for all students) versus problem-solving tasks (for some students).

Tensions at a teacher-teacher level were identified in all episodes. In episode 6, Nina expresses that she would have liked to have "an hour or two" focusing on parentheses before working on the task where use of parentheses was of importance. She also thinks the students need repetition of "how it is when we multiply x by x". After this "hour or two" she would start introducing the task by only showing "the figure with four squares" and saying/asking "$5x$ and 3 and $2x$ and 5, and simply said, find the area! What should I do? How many ways can you find the area?" So, if Nina had the opportunity to let the students work on parentheses beforehand, she would make the introduction to the task short. Nina summarizes by stating that "the task is not open enough. It's an overtly teacher-directed task." Per agrees, but he expresses that "you have to go through some of the non-open tasks as well." Here the tension seems to include not only routine tasks versus problem-solving tasks, but which type of task comes first. It seems like they agree that "non-open tasks" (routine tasks) might be important to do before more open tasks (problem-solving tasks). In episode 7, these tensions are strengthened when the teachers agree that there are mathematical themes the students must master before working on more open tasks. Later in the group discussion (episode 8), Kai also says that he feels that the number of mathematical themes in the curriculum limits the time for exploration, indicating a tension related to mathematical themes versus lack of time for explorations. The issue of time is also emphasized (episode 9) when Nina finds it easier to invite students into dialogues by using non-curriculum tasks (or tasks not relevant to the mathematical topic they focus on "just now", see vignette in the introduction), but she finds it challenging to have time to work on such tasks: "That's when we have a dialogue, not when we start with the syllabus and work on it. Then it won't be that way, because you feel a bit… Yes, the time that isn't enough either."

To summarize, we have identified the contrasting forces, to solve routine tasks (tradition) versus a practice where they solve problem-solving tasks (innovation) (Liljedahl et al., 2015) as superior to other tensions. Patterns of tensions related to research-practice partnership can be divided into three sub-categories: a) teacher practice versus participating in PD sessions, b) *Telling versus growth* (Berry, 2007), and c) *Safety versus challenge* (Berry, 2007). Patterns of tension related to the overall category teachers' work of teaching are related to the sub-categories: a) student dialogue versus missing dialogue, b) know content before working on a task versus learn content while working on a task, c) know rules for managing a task versus learn rules while working on a task, d) non-curriculum (open) tasks versus curriculum (open) tasks, e) mathematical themes versus lack of time for explorations, and f) non curriculum tasks (and time to work on such tasks) versus curriculum tasks.

Concluding discussion

Selecting, discussing and trying out tasks for problem-solving lessons is vital to a PD initiative like *Partners in Practice*. Building on Andrà et al.'s (2019) finding that provoking, talking about and managing tensions are effective in promoting teachers' PD, we explored tensions in this PD initiative by identifying pairs of contrasting forces that might pull teachers in different directions (Andrà et al., 2019; Berry, 2007; Liljedahl et al., 2015, 2023). We identified complex collections of contrasting forces related to both the selection and enactment of tasks, and we found some overall patterns of tensions in the teachers' talk. First and most prominent was the tension between a practice where students solve routine tasks (tradition) versus a practice where they solve problem-solving tasks (innovation). A tension between tradition and innovation is previously discussed by Liljedahl et al. (2015), but whereas these researchers identified a tension between the traditional wishes of a teacher's practicum and her desire to be more innovative and the teacher's wish to enact her own teaching program and following the school's workbooks more generally, our findings relate more specifically to tasks (e.g., routine tasks vs. problem-solving tasks). Since tasks are fundamental in mathematics teaching (e.g., Hiebert et al., 2003), and in understanding teaching and student thinking and learning (e.g., Tekkumru-Kisa et al., 2020), this tension is important for PD providers to be aware of and discuss with teachers in PD sessions (e.g., Boston & Smith, 2011; Johnson et al., 2016). PD providers must be aware that (problem-solving) tasks might produce tensions.

In relation to the main tension between tradition and innovation, we found two other overall patterns of tensions. We identified tensions between 1) what the PD providers brought to the sessions and teachers' ideas, and between 2) the teachers, both having interwoven sub-categories of contrasting forces. The first one relates to Johnson et al.'s (2016) tensions within research-practice partnership and can be situated at the teacher-researcher level. Like Berry (2007, p. 133), we acknowledge that tensions "are not necessary *per se*, but they offer a useful way of conceptualizing and communicating practice." Berry (2007) also suggests that even though it is not expected that a series of tensions could provide guidance in all situations, tensions offer a way of understanding and improving practice. At a teacher-researcher level, we emphasize the importance of being aware that different participants bring different experiences and purposes when participating in PD sessions. More specifically, our analysis has illustrated how teachers and PD providers have different approaches when selecting tasks for use in PD. While the teachers want their students to work on tasks related to the syllabus or curriculum that are relevant to the mathematical topic they focus on "just now" (see 'Teachers' in Table 1), we as PD providers have suggested problem-solving tasks which are not necessarily related to the mathematical curriculum the teachers teach the exact time we meet for a PD session (see 'PD providers' in Table 1).

Table 1: Matrix for categorizing tasks for use in PD

Tasks	Non-Curriculum	Curriculum
Routine tasks - Tradition		Teachers ↓
Problem-solving tasks (open) - Innovation	PD providers →	Goal

The shaded cell in the 2 by 2 matrix in Table 1 suggests that PD providers might bring problem-solving tasks into the sessions which can also be related to the mathematical curriculum the teachers

teach at the time for the PD (curriculum tasks). We could consider this as a compromise between teachers and PD providers, bringing different perspectives to the PD sessions. The goal for future PDs might be to discuss problem-solving tasks more directly connected to curricula. On the other hand, Liljedahl's (2021) research based on empirical data from many classrooms revealed that the use of non-curriculum (open) problem-solving tasks in classrooms invited more students to think, and they were more engaged in the problem-solving activity compared to when students were introduced to curriculum tasks. We suggest that when PD providers bring problem-solving tasks (or non-curricular/open tasks) to the PD sessions, categorizing the task within the matrix in Table 1 might be an idea for further discussions in PD to highlight the challenge of selecting appropriate tasks for use in classrooms.

The other overall pattern of tensions was at the teacher-teacher level and was related to the teachers' work of teaching in which we identified six subcategories (see finding section). These tensions (e.g., know rules for managing a task versus learn rules while working on a task) illustrate the importance of identifying and unpacking the special mathematical work of teaching (Ball, 2017) in PD and consider the work of teaching to be "the core tasks that teachers must execute to help pupils learn" (Ball & Forzani, 2009, p. 497).

We acknowledge that our study is limited by the data we have analyzed from just one small-group discussion, and the small sample size is a limitation. However, we believe that our unpacking of this specific group discussion, highlighting tensions arising from teacher talk in a PD context might be useful for further studies and for understanding and improving PD practice. One promising direction for further research could be to delve into tensions that arise at a teacher-student level, bringing ideas from PD sessions into the teachers' practice in classrooms in which data also involves student voices.

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On making mathematics flourish for students to grasp it: Teachers designing informal mathematics education workshops in museums

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This paper presents findings from the InformalMath program, which aims to integrate Informal Mathematics Education Workshops (IMEWs) into teacher education through collaboration with non-scientific museums. Through a three-phase structure, teachers engaged in collaborative IMEW design, addressing challenges, and enhancing their understanding of integrating mathematics into informal contexts. Interviews with participating teachers revealed significant learning experiences, including group collaboration skills and the importance of hands-on activities. Moreover, teachers reported integrating elements of InformalMath, such as teaching through artifacts and emphasizing experiential learning, into their classroom practices. Overall, the program demonstrated the potential to transform mathematics education by fostering creativity, exploration, and meaningful engagement with mathematics beyond traditional classroom boundaries.

Keywords: Informal mathematics education, emergent learning, professional development, museums.

Introduction and theoretical background

Out-of-school settings may host educational activities intentionally designed to support mathematical learnings, within the perspective of the Informal Mathematics Education (IME). IME activities differ from most mathematics classroom activities because they offer learners greater freedom to pursue their interests, blur the boundaries between disciplines, and lack of traditional forms of academic assessment (Nemirovsky et al., 2017). The type of learning generated by IME experiences has been described by Nemirovsky (2018) as "emergent learning". Emergent learning, in contrast to teleological learning, focuses on the spontaneous and often unpredictable nature of gaining knowledge. Teleological learning is goal-oriented, structured around achieving predefined objectives, such as passing specific tests or accomplishing set targets. One limitation ascribed to the teleological learning approach is that it often limits learners by confining them to specific outcomes and disregarding the natural, organic process of learning. On the contrary, emergent learning approach allows for a richer educational experience by acknowledging and integrating the memories, sensations, and emotions of learners. By doing so, it fosters a deeper connection to the material, making learning a personal and impactful experience that can adapt and evolve in response to the learners' motivations and contextual interactions. This makes learning more engaging and ensures that it is more aligned with the individual's cognitive and emotional development.

We believe that teachers can serve as IMEducators, and we agree with Nemirovsky and colleagues (2017) that the education of IMEducators requires different approaches from those prevailing methods in mathematics teacher education (TE). This suggests the need for designing and implementing TE programs in IME, as well as investigating them from a research perspective.

To address this research gap, we explored IME in the context of non-scientific museums and designed a TE program named InformalMath. Seeking an approach that diverges from the prevailing paradigms

in TE and aligns with the perspective of emergent learning characteristic of IME, we identified Korthagen's (2017) framework as a valuable reference. Korthagen offers a critical analysis of traditional models, specifically those they term Professional Development (PD) 1.0 and 2.0. PD 1.0 primarily emphasizes theoretical knowledge dissemination, often leaving a disconnect between conceptual understanding and practical application. PD 2.0 shifts focus to practice-oriented training but still struggles to bridge the gap between the technical and personal dimensions of teaching.

In contrast, Korthagen's PD 3.0 marks a transformative shift by integrating professional skill development with personal growth. This model acknowledges that effective teacher learning involves not only cognitive processes but also emotional and motivational dimensions. Furthermore, PD 3.0 incorporates principles of socio-constructivism, recognizing the critical role of social interactions and collaborative practices in the learning process. Teachers are not seen as isolated learners but as participants in a dynamic, interactive community where shared experiences and reflections deepen understanding. This perspective aligns with Vygotsky's (1978) emphasis on the social context of learning, where collaboration with peers and mentors fosters the co-construction of knowledge.

Within the InformalMath program, the socio-constructivist elements of PD 3.0 are particularly relevant. The collaborative and inquiry-based nature of IME environments benefits from this framework, as it encourages teachers to engage in reflective dialogues that integrate theory with practice and to co-develop adaptive teaching strategies. By delving into the personal aspects of teaching, PD 3.0 supports teachers in navigating the complexities of informal learning environments, such as those encountered in non-scientific museums. This approach also resonates with Schön's (1987) concept of the reflective practitioner, wherein ongoing reflection bridges theory and practice, cultivating a richer, more responsive pedagogy. Ultimately, this synthesis of professional and personal growth equips teachers to create meaningful, learner-centered experiences that are adaptable to diverse educational settings.

In this paper, we give voice to the teachers and highlight their reflective practices at the conclusion of the InformalMath program, focusing particularly on their perceived emergent learning.

InformalMath teacher education program

The InformalMath program, which ran from December 2021 to June 2024, was structured into three distinct phases with the goal of providing in-service training for primary and middle school mathematics teachers on the unique context of non-scientific museums. The core of the program is to collaboratively design IME Workshops (IMEW, Casi, 2024) within a team of teachers, mathematics education experts, and museum education specialists.

The training program is structured into three distinct phases through which teachers acquire progressive autonomy, according to a cognitive apprenticeship model (Collins et al., 1989). In the first phase (Discovery Phase) teachers explore various IMEWs to gain understanding and inspiration. In the second phase (Guided Co-design Phase) teachers are guided by teacher educators in designing new IMEWs within museums proposed by the educators. Again, in the third phase (Scaffolded Co-design Phase) teachers designed an IMEW in a museum, but in this case they also chose the museum itself. This design is then shared with the museum so that it can be offered to schools interested in

providing this unique educational experience to their students. In Phases 2 and 3 teachers collaborate in small groups. In our presentation at CERME13, we shared the initial findings from research carried out on the Guided Co-design Phase (Casi & Sabena, 2023). In the current paper, we focus on the Scaffolded Co-design Phase, which corresponds to Phase 3 of InformalMath.

Among the twenty-two teachers participating in the second phase, seven opted to continue into the third one. We explored the reasons provided by the fifteen teachers who decided to exit the program to determine if there were specific issues with the course that influenced their decision. The feedback from these teachers revealed that their departure was influenced by changes in their personal lives (such as pregnancy or relocating outside the region) or professional circumstances (like a career shift, increased responsibilities in school administration, or changing schools).

Phase 3 started in March 2023. The teachers were divided into two groups: Group P composed by four primary school teachers, and Group M composed by three middle school teachers. Group P chose to explore and design at the Royal Castle of Moncalieri (castellodimoncalieri.it/), one of the UNESCO World Heritage castles located close to Turin. Group M's choice was the Museum of the Soundscape (museopaesaggiosonoro.org/), a small museum in Riva presso Chieri, about 20 km from Turin.

The co-design method was the same as that already used Phase 2. It involved, in different roles and at different times, teachers (T), teacher educators (TEd) and museum educators (MEd). Unlike the previous phase, however, agency in the choices was devolved more to the teachers, drawing on the principles of cognitive apprenticeship (Collins et al., 1989). Each group decided the modalities of co-working, for instance time and space for their meeting. At certain moments each group discussed the design with T, TEd, and MEd. Table 1 details the actions of this phase.

Table 1: Actions composing the Scaffolded Co-design Phase

Actions	Participants
1. Group formation and Museum choice	T; TEd
2. Museum visit and sharing of first ideas	T; TEd; MEd
3. Design: first draft	T
4. Scaffolding activity	T; TEd
5. Design: second draft	T
6. "*Dress Rehearsal*": T presented the IMEW to TE and ME	T; TEd; MEd
7. Design: finalising	T
8. Whole group presentation	T; TEd

As a result of the work of the two groups, two IMEWs were designed. Due to the limited space available in this paper, we simply give a brief suggestion of the topics covered in the two IMEWs, deferring their description to the project website www.informalmath.unito.it.

Group P designed 'Eye... A Game at the Castle' ('Occhio... Un gioco al Castello', in Italian) for the Royal Castle of Moncalieri. The IMEW is structured as a treasure hunt in the rooms of the castle, where the object of the search are geometric shapes and geometric properties.

Group M designed 'A Matter of Ratio' (‘Una questione di rapporti’, in Italian) to guide students to investigate the relationships between sound and mathematics while visiting the impressive Museum of the Soundscape. This small and original museum collects and make memories of sounds, noises and melodies that have characterized the local landscape, from rural culture to contemporary times.

Research questions and methods

We acknowledge that designing IMEWs is a demanding endeavor for teachers. We are also convinced that through engaging in this process, teachers can learn a lot. Furthermore, while the IMEWs, along with the entire InformalMath project, are conducted outside the school context, we believe that teachers' participation in such endeavors may also yield benefits for their everyday practice. Based on these considerations, we formulate three research questions:

RQ-1. What critical issues and difficulties did the teachers perceive, related to the third phase of the program?
RQ-2. What learning did the teacher report as a result of participating in the program?
RQ-3. What aspects related to IME did the teachers perceive as being filtered into their classroom practice?

RQ-1 specifically focuses on the design processes during the third phase of the program (for results regarding Phase 2, see Casi & Sabena 2023). Since Phase 3 is the final phase of the program, RQ-2 explores the teachers’ emergent learning throughout the entirety of the program. Lastly, RQ-3 connects the learning experiences within IME to the classroom context.

To answer the research questions, we collected the voices of participating teachers through individual semi-structured interviews. The interviews took place between Actions 7 and 8 (see Table 1) in March 2023. Interview questions were shared with teachers five days before the interview date. The interviews were conducted via online calls by the first author of the paper. Below is the English translation of the six questions that shaped the interview:

Q-1. What difficulties did you face in the third phase of the program? How did you overcome them?
Q-2. What do you think you learned in this phase?
Q-3. Considering your experience in InformalMath, what do you think about IMEWs in museums?
Q-4. Are there aspects of InformalMath that you think you have brought into the classroom in your daily teaching? If yes, which ones? If no, why?
Q-5. What do you feel you have learned from active participation in InformalMath program?
Q-6. As teacher educators, should we have done anything differently? Can you give us any suggestions for repeating the experience with other teachers?

Q-1 and Q-2 address challenges and insights from the third phase of the program, repeating questions posed in focus groups at the end of the second phase (Casi & Sabena, 2023). Q-3 solicits teachers' views on IMEWs after being involved in InformalMath for more than two years. Q-4 and Q-5 prompt teachers to reflect on the entire TE program. These questions aim to capture teachers' perspectives at a more advanced stage of their autonomy within the project. Lastly, Q-6 seeks to identify critical issues that we as teacher educators can address in future iterations of InformalMath.

The interviews were recorded and fully transcribed. However, one interview deviated from the planned setting: a teacher chose to include a colleague who was not involved in the InformalMath project. Due to this methodological bias, this interview was therefore excluded from the analysis. The transcriptions underwent qualitative content analysis, with entire sentences as units of analysis. Categories were developed inductively, and each sentence was assigned a label summarizing the intended expression of the teacher. These labels were then grouped according to their content, resulting in the final list of categories.

Results

Tables 2, 3 and 4 report the categories identified as described in the previous section and indicate in which interview were detected. All teachers' names are pseudonyms.

Table 2 reports the categories related to difficulties, responding to RQ-1.

Table 2: Categories of difficulties perceived by teachers

Difficulties perceived reported by teachers	Ada	Anna	Bianca	Dario	Lea	Lina
D1. Working in groups	✓	✓		✓		✓
D2. Accepting that the math in the IMEW is too basic	✓		✓	✓		✓
D3. Designing required extensive study	✓		✓			
D4. Choosing a meaningful museum for doing mathematics					✓	✓

D1 refers to the experience of working in groups; this was also found in second phase of the project (Casi & Sabena, 2023). D3 concerns the study needed to design an IMEW. The teachers described this need related to the peculiar themes of the chosen museum. D2 and D4 refer to the mathematics present in the IMEW, highlighting a tension between the simplicity of the topics and the significance of the relationship between mathematical topics and museum content.

The response to RQ-2 is reported in Table 3, which shows the categories related to the learning described by teachers.

Table 3: Categories of learning identified by teachers

Learnings reported by teachers	Ada	Anna	Bianca	Dario	Lea	Lina
L1. Group working when there is not full agreement		✓	✓		✓	✓
L2. How to design an IMEW, from the beginning to the end		✓	✓	✓		✓
L3. How to make mathematics flourish in a context where it does not seem to be present			✓	✓		
L4. In mathematics it is important the doing, possibly by hands	✓					✓

L1 can be directly traced back to D1, indicating how a perceived difficulty was exploited for learning. L2 and L4 highlight different nuances of learning in mathematics education, such as the importance of hands-on activities in the mathematics laboratory and how to design an IMEW. L3, which we named after one teacher (Bianca)'s words, underscores the significance of recognizing and leveraging the mathematics already present in informal contexts, specifically in the two chosen museums, rather than merely adding mathematical content to these contexts.

Table 4 allows us to answer RQ-3, showing the elements of InformalMath that teachers claim to have let filter in their teaching practice in the classroom.

Table 4: Categories of elements filtered in the classroom teaching practice reported by teachers

Elements filtered in the classroom teaching practice	Ada	Anna	Bianca	Dario	Lea	Lina
E1. Teaching mathematics through artifacts	✓		✓	✓		✓
E2. Emphasis on informal activities and outdoor education		✓		✓	✓	
E3. Practice first, then theory. Experiencing precedes formalizing				✓	✓	✓
E4. Creation of interdisciplinary teaching paths from science to mathematics through formalization				✓		✓
E5. Activities designed were taught to students in the classroom		✓	✓			

E1 is closely linked to L4 and represents a key element of InformalMath and IMEWs nature, drawing from the established research in Italian mathematics education, particularly regarding the mathematics laboratory and the semiotic mediation of artifacts (see Casi, 2024). E2 reflects teachers' effort to move some lessons outside the classroom (e.g., garden, gymnasium, schoolyard), integrating typical IME spaces into formal classroom practice. E3 and E4 both refer to mathematical formalization, which must take place after the experience, both with respect to laboratory activities and interdisciplinarity with science. Here, the emphasis is on the modalities of the IME rather than its spaces. E5 concerns the direct use of activities constituting IMEWs at school, facilitated by teachers for implementation in the classroom context.

As highlighted in the comments of the tables, the categories emerging as answers to the RQs are intertwined. This is evident also in the teachers' words form the interviews, which were carried out by the first author. We report here some excerpts from Bianca and Dario. Both teachers referred in explicit way to what they had expressed during focus groups or personal essays in the previous phases.

In answering Q-5, Bianca says

Bianca: I've learned how to make mathematics flourish, so to speak, in every place. That means looking with a different eye, in this case the mathematical eye, at the things around us. Then ensuring that the students, when they visit the castle, can see them too. So that the students understand when they look at the carpet, and they see the birds put in symmetrical positions, they understand that there has been weaving, but it's not just weaving: plus, there are mathematical properties that the person who wove the carpet put in! Our job as teachers is to make sure that students see and grasp these things, I really understood that in this course.

Interviewer: Tell me more about this.

Bianca: It had happened before: you see an object, that time it was the *Metrocubo di Infinito* [pièce by M. Pistoletto] in the museum in Rivoli [in Phase 2], you understand it mathematically, and you want your students to get that experience. It's a sudden and beautiful thing; we had talked about it before.

Bianca initially refers to her learning, linked directly to her IME experience, describing how she learned to observe the reality around her with mathematical eyes. Later, she shifts the focus of the talk to the role of the teacher, and how it is important to her to make sure that students can have the same learning. When asked to elaborate on this concept, she refers to an episode that happened during Phase 2 of the program (discussed in Casi & Sabena, 2023, p. 3357). In that episode, Bianca referred

to the process of discovering mathematics in the museum from the students' perspective as the most difficult aspect encountered in designing an IMEW. Now, one year later, after having co-designed the IMEW in the Castle of Moncalieri, she appears to have overcome such a difficulty, as she says that she learnt how to make mathematics flourish in every place. Moreover, she recognizes this feature as central in her teacher role (Our job as a teacher is…)

The following is an excerpt from the interview with Dario, in response to Q-3:

Dario:	[This program] has stimulated me a lot to conceive for my students more and more interdisciplinary paths, because once you unlock this mechanism [seeing mathematics in the museum] it is easy to see mathematics in other contexts as well, such as the sciences. As I said earlier, I am not trained as a mathematician, and it has always been difficult to think about how to teach it [math] well.
Interviewer:	And how do you manage this?
Dario:	I always did this thing of trying to connect math to science. But now I've noticed that I always start in lessons from the sciences, and then get to the math. I'll give you an example: to work on ratios, I start from speed experiments, without explaining the formal part first. I start from that, then do the formalization. So, this inversion perhaps was given to me by this very program: no longer explaining and then making experiments, but the other way around: making people discover and then going to formalize later. This has definitely increased a lot in my work.

Dario expresses his views about IMEWs in museums by relating his experience back to his work in the classroom context, and how the opportunity to see mathematics in museums allowed him to learn a way of working (unlocking a mechanism) that led him to seek connections with mathematics in other contexts. As a result, he says he began to design interdisciplinary classes for his students, which enabled him to overcome difficulties related to his lack of initial training in mathematics. When Dario mentions his previous discussion about his non-mathematics background, he is referring to a conversation that took place during the focus groups at the end of Phase 2 (Casi & Sabena, 2023, p. 3358). In that discussion Dario expressed difficulty in finding a teaching approach that do not overlook fundamental aspects of mathematics. However, in the interview above, it appears that he has found a method that suits him well, leveraging his education (Dario has a MS degree in Geology), to support students in meaningful mathematics experiences. This change in Dario's perspective appears to have been strongly influences by his experience in Phase 3 and by subsequent reflection on those experiences.

Conclusions

The structure of the InformalMath program proved to be effective in guiding teachers through the process of designing IMEWs collaboratively. By allowing teachers greater autonomy and agency in the design, particularly in the Scaffolded Co-design Phase, they were able to overcome obstacles and develop a deeper understanding of how to integrate mathematics into informal contexts. The results of our research revealed that participating teachers reported significant learning experiences throughout the program. From gaining practical skills in group collaboration and IMEW design to recognizing the importance of hands-on activities and interdisciplinary connections, teachers expressed a newfound appreciation for teaching mathematics in diverse contexts.

Furthermore, the interviews conducted with participating teachers provided valuable insights into the impact of the InformalMath program on their teaching practices. Teachers described how they had

integrated elements of InformalMath, such as teaching mathematics through artifacts, emphasizing informal activities and outdoor education, and prioritizing experiential learning over formalization, into their classroom teaching. These findings highlight the transferability of skills and concepts learned through the program to the broader educational context.

However, it is important to acknowledge that the limited number of teachers who participated in the third phase of the research constrains the statistical robustness of the findings. Nonetheless, it was never our intent to conduct a nomothetic study; rather, we aimed to explore the potential of IME elements for teacher education. In this sense, we believe that this work provides a promising foundation for future investigations in the field, offering preliminary insights and actionable frameworks for developing innovative approaches to mathematics education.

Overall, the InformalMath program has demonstrated its potential to transform mathematics education by equipping teachers with the knowledge, skills, and confidence to engage students in meaningful mathematical experiences both inside and outside the classroom. As we move forward, further research and exploration into the implementation of IMEWs and their impact on student learning will be essential for continuing to enhance mathematics education practices.

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Investigating how secondary school teachers perceive mathematical concepts in application contexts: The case of ratio and chemistry tasks

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This paper presents a qualitative empirical study investigating how secondary school mathematics teachers perceive the concept of ratio in application contexts given by lower secondary school chemistry tasks on stoichiometry. Considering the ability to notice mathematical concepts in application contexts as an integral part of conceptual knowledge of mathematics teachers, especially those who specialise both in mathematics and the school subject covering the application context, the aim of the study is to analyse various ways of how such specialised teachers solve the application tasks, detect potential strengths and weaknesses in their ability to notice mathematical concepts in the tasks and connect the ways of solving to their current teaching practices. As a consequence of the study, we discuss its implications for initial teacher education and professional development courses that educate all mathematics teachers together regardless of their other specialisation.

Keywords: Mathematics teachers, conceptual knowledge, mathematics teacher's specialised knowledge, application tasks, ratio.

Introduction

With this contribution, we follow up on our previous investigations (Čech & Samková, 2024) that concerned lower secondary school students solving tasks with both mathematics and chemistry content, namely the topics of ratio and stoichiometry. Since stoichiometry is defined as "the determination of the proportions in which elements or compounds react with one another" (Encyclopaedia Britannica, 2024), the concept of ratio is in the background of all stoichiometry tasks, including lower secondary school tasks on molar mass and the law of mass conservation. The respondents of the previous study were ninth graders who had already completed the topic of ratio (in the seventh grade) as well as the topics of molar mass and the law of mass conservation (in the eighth grade). The study revealed that the students were mainly unable to notice the concept of ratio in stoichiometry tasks and that their solution procedures were usually based only on purely chemical algorithms. To better understand the circumstances, we redirected our focus to teachers, namely, to teacher knowledge and practices that come into play when dealing with similar tasks. Therefore, the qualitative study presented in this paper involves five teachers whose students were respondents of the previous study and, to get a broader picture, one more teacher with similar background (specialisation, teacher training). As the neglected topic was the mathematical concept of ratio, we address the issue from the perspective of didactics of mathematics and construe stoichiometry tasks as applications of this concept. The research question is: *How do secondary school teachers specialising in both mathematics and chemistry perceive the concept of ratio in application contexts given by lower secondary school chemistry tasks on stoichiometry*? This question consists of two sub-questions. *Which procedures can the teachers use to solve the problems,* and *how do the teachers relate the procedures to their own teaching practice?* Our goal is to obtain feedback relevant not only

to participating teachers, but also to teacher educators. In this way, our contribution addresses two of the TWG18 key themes identified at CERME13 (Friesen et al., 2023): crossing the boundaries between different school subjects during initial teacher education and mapping the relationship between what takes place during initial teacher education and what happens in classrooms. The relationship goes both ways, and we take them both into consideration by discussing implications for classroom practice as well as initial teacher education.

The study takes place in the Czech Republic, where the initial teacher education of secondary school teachers takes the form of a five-year master's degree programme. Before entering the programme, each applicant chooses two school subjects in which to specialise. During the studies, content courses and didactic courses of the two school subjects are run separately. For mathematics, the offer of the other subject is usually open so that mathematics can be combined with any other secondary school subject (chemistry, biology, various languages, arts, etc.). Courses on mathematics content and didactics then educate all prospective mathematics teachers regardless of their other specialisation. The first author of this paper is a young researcher who is a secondary school teacher specialising in mathematics and chemistry. The second author is a researcher and mathematics teacher educator.

Ratio in lower secondary school stoichiometry tasks

In this paper, we discuss lower secondary school stoichiometry tasks that deal with molar mass and the law of mass conservation. These tasks typically include the process of balancing chemical equations by adding stoichiometric coefficients to them, calculating the molar mass of compounds, or calculating missing quantities of reactants and/or products in problems where quantities are given in terms of mole or mass. According to the categorisation of proportional stoichiometry tasks made by Ramful and Narod (2014), such tasks belong to the two lowest categories. Each of these tasks can be solved in different ways, using not only procedures learnt during chemistry lessons, but also mathematical procedures based on the concept of ratio. For instance, in the task presented at the top of Table 1, the solver can base the solution on a purely chemistry algorithm for oxidation-reduction (see the left side of Table 1) or proceed from the fact that chemical equations inform us in a special way about ratios of individual elements present in the compounds. In the latter case, no oxidation numbers are needed to solve the problem, only equivalent ratios (see the right side of Table 1).

Table 1: A sample of a stoichiometry task (top), a chemistry procedure using oxidation-reduction (left) and a mathematical procedure using ratios (right).

<table>
<tr><td colspan="2">Add the stoichiometric coefficients to the chemical equation: $Al + SiO_2 \longrightarrow Al_2O_3 + Si$</td></tr>
<tr><td>$Al^0 \rightarrow Al^{III}$ 3 4
$Si^{IV} \rightarrow Si^0$ 4 3
$4\ Al + 3\ SiO_2 \longrightarrow 2\ Al_2O_3 + 3\ Si$</td><td>$Si : O = 1 : 2 = 3 : 6$
$Al : O = 2 : 3 = 4 : 6$
$Si : Al = 3 : 4 \Rightarrow 3\ Si, 4\ Al$
$\Rightarrow 3\ SiO_2$
$\Rightarrow 2\ Al_2O_3$</td></tr>
</table>

Empirical research focusing on the mathematical side of lower secondary school stoichiometry tasks is rather scarce. In addition to Ramful and Narod (2014) mentioned above, this issue of mathematics and stoichiometry was also addressed in detail by Gabel et al. (1984); however, both papers follow the perspective of didactics of chemistry and at higher levels of schooling. Ramful and Narod (2014) studied solutions provided by prospective chemistry teachers and found that it was not easy for them to transfer their knowledge of ratios from mathematics to chemistry. Gabel et al. (1984) showed that high school chemistry students preferred chemistry algorithms to mathematically based procedures and, at the same time, displayed weak knowledge of included chemistry concepts.

Teacher knowledge

Teacher knowledge is a complex matter that can be construed from different perspectives, all of them reflecting in its own way how and why content knowledge is or is not passed out to students. As most useful for the presented study, we found the perspectives of conceptual knowledge in mathematics (Hiebert & Lefevre, 1986) and mathematics teacher's specialised knowledge (Carrillo-Yañez et al., 2018). Conceptual knowledge is defined as "knowledge that is rich in relationships" (Hiebert & Lefevre, 1986, p. 3) and as such can generalise to other situations. Also, when procedures are learnt conceptually, they can be applied in new contexts. In our case, when mathematical knowledge of the topic of ratio is conceptual, the concept of ratio can be noticed in stoichiometry tasks and so procedures known from mathematics can be applied in such tasks. Mathematics teachers' specialised knowledge (Carrillo-Yañez et al., 2018) is divided into (i) mathematical knowledge consisting of knowledge of topics, knowledge of the structure of mathematics, and knowledge of practices in mathematics; and (ii) pedagogical content knowledge consisting of knowledge of mathematics teaching, knowledge of features of learning mathematics, and knowledge of mathematics learning standards. *Knowledge of topics* includes not only definitions, properties, models, representations, and procedures, but also different associated contexts, awareness of applications, and connections to content from other disciplines. *Knowledge of mathematics teaching* includes knowledge of different ways of representing content and also knowledge of resources, teaching materials, textbooks, tools, etc., their potential and limits; *knowledge of mathematics learning standards* includes sequencing of the topics, that is, ways of connecting prior, current, and future knowledge. In our case, *knowledge of the topic* is relevant for knowing the context of stoichiometry tasks as associated with the topic of ratio, *knowledge of mathematics teaching* for knowing different ways of representing ratio within the context of stoichiometry tasks, and *knowledge of mathematics learning standards* for connecting the seventh-grade topic of ratio with its eight-grade application in stoichiometry.

Materials and methods

We conducted a qualitative study (Holton, 2007) with six in-service secondary school teachers of mathematics and chemistry as participants; we labelled them R1 to R6. Four of them have a shorter teaching experience (up to 6 years; R1, R2, R3, R5), and the other two have a long one (more than 20 years; R4, R6). All except R4 studied the teacher education programme at our university. All except R5 were mathematics or chemistry teachers of students from the previous study (Čech & Samková, 2024), but none of them taught both subjects to them. Having a different teacher for mathematics and chemistry is a common practice in secondary schools in our country.

To collect data, we assigned the participants a paper and pencil worksheet containing a mixture of application tasks related to ratios, all suitable for the lower secondary school level. Some of the tasks were located in usual everyday practical contexts used in our country during mathematics lessons (e.g. mixing muesli ingredients in a given ratio), and others in scholarly contexts from chemistry lessons. The tasks were ordered by complexity. The five less complex tasks at the beginning of the worksheet served as a warm-up, and the participants could solve them in any way they were comfortable with. The two most complex tasks at the end of the worksheet (Table 2) formed the main data collection instrument. To obtain detailed information on conceptual and specialised knowledge, we employed comparisons of multiple solution strategies (Durkin et al., 2017) and asked the participants to solve each of the tasks in three different ways and then relate the procedures to their own teaching practice.

Table 2: The assignment of the two most complex task

T6	Add the stoichiometric coefficients to the chemical equation. Find the ratio of the mass of the elements in the CaO compound. How many grammes of calcium and how many grammes of oxygen are produced from 490 grammes of CaO? element | g/mol O | 16 Ca | 40 $CaO \longrightarrow Ca + O_2$
T7	Using the chemical equation, find how many g of Al and SiO_2 were used to prepare 408 g of Al_2O_3. First, add the stoichiometric coefficients to the chemical equation. element | g/mol O | 16 Al | 27 Si | 28 $Al + SiO_2 \longrightarrow Al_2O_3 + Si$

During data analysis, we first open-coded all data materials related to T6 and T7. When coding procedures, we divided them into individual steps and monitored instances where an explicit mathematical or chemistry approach was used. We also highlighted all the places that seemed interesting from the perspective of conceptual knowledge. Then, we compared the codes across all procedures provided by individual participants and across all participants. Afterwards, we went through the texts relating the procedures to teaching practice and monitored instances relevant to teacher knowledge. Finally, we compared the procedure codes with the teaching practice codes and sought relations and patterns to create code categories. At the end of the process, we applied theoretical coding with respect to the two knowledge frameworks (Carrillo-Yañez et al., 2018; Hiebert & Lefevre, 1986). In the Results section, we focus on data related to T6, as this task had higher data density.

The two authors cooperated during the coding process: first, we worked separately, then we discussed and compared our codes and categories and reflected on the comparisons. This process also included several semi-structured interviews conducted by the second author, in which the first author (who has the same specialisation and graduated from the same university as most of the participants) explained and clarified some aspects and circumstances related to the course of initial teacher education at our

university, to textbooks, and to common teaching practice. This additional perspective allowed for a more accurate interpretation of collected data. We took written records of the interviews.

Results

As shown in Table 3, the six participants presented solution procedures of various quantities and backgrounds. Particular samples for task T6 are presented below.

Table 3: Types of solution procedures provided by participants for tasks T6 and T7; shaded are procedures declared as "I would use it in my own practice"

	T6			T7		
	procedure 1	**procedure 2**	**procedure 3**	**procedure 1**	**procedure 2**	**procedure 3**
R1	ratios	molar mass & rule of three		molar mass & rule of three		
R2	molar mass	ratios	rule of three	molar mass	molar mass	
R3	molar mass	rule of three	ratios	molar mass	rule of three	ratios
R4	molar mass	rule of three		rule of three		
R5	ratios	ratios	molar mass	ratios	rule of three	molar mass
R6	rule of three	ratios		rule of three	ratios	

For T6, the participants altogether provided four different types of solution procedures: based on a chemistry approach with molar mass (for a sample, see Figure 1), on the rule of three (a mathematical algorithm commonly used in our country to solve tasks on proportionality; Figure 2), on ratios (Figure 3 left), and on a combination of chemistry and mathematical approaches (Figure 3 right).

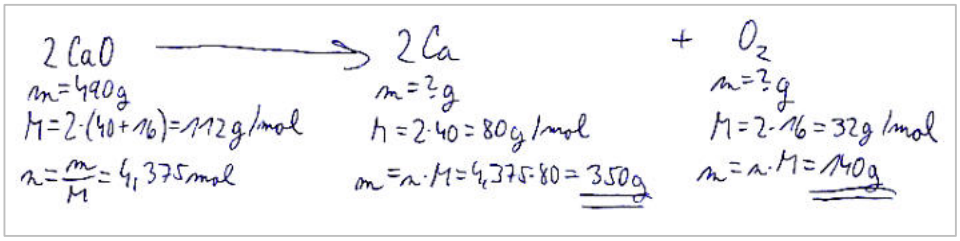

Figure 1: A chemistry-based procedure with molar mass provided by R3 for the task T6

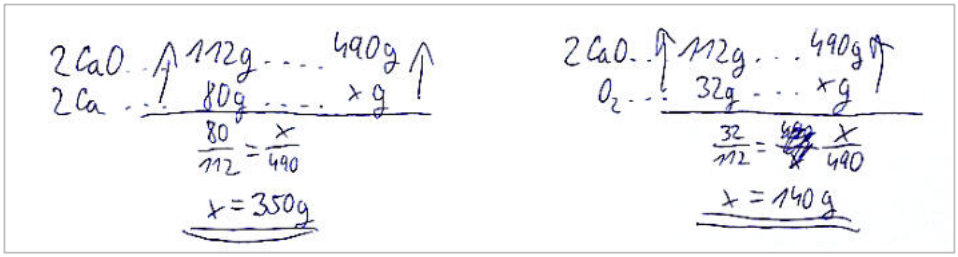

Figure 2: A mathematically-based procedure with the rule of three provided by R3 for the task T6

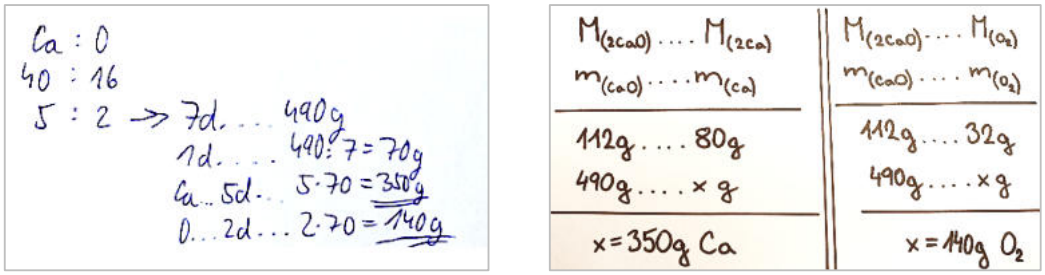

Figure 3: A mathematically-based procedure with ratios provided by R3 for the task T6, "d" is a Czech abbreviation for the word that has the same meaning as "part" or "parts" in English (left); a mixed-type procedure with molar mass and the rule of three provided by R1 for the task T6 (right)

Responses relating the procedures to participants' own teaching practice varied a lot. We found three patterns in them, represented by participants R3, R6, and R2. Participants R3 and R2 provided similar solution procedures but with opposing opinions on those that were based on ratios:

R3/molar mass: I use this procedure in my teaching because students know the formula

R3/rule of three: I use this procedure in my teaching because some students prefer the rule of three to the formula.

R3/ratios: I do not use this procedure in my teaching because students see only mathematics in it. Chemistry is removed from this procedure.

R2/all: I use all these procedures and decide between them individually according to the needs of the student.

R6 presented just mathematically-based procedures and wrote:

R6/rule of three: I use it because it is easy. Students know the rule of three.

R6/ratios: I use it because it uses ratios that students work with in mathematics.

According to the findings mentioned above, the interview between the second and the first author focused on aspects that could impact the awareness of the procedure based on the ratios and its active use in the classroom. A detailed discussion on the content of the courses on didactics of mathematics at our university revealed a possible key moment. During the courses, individual mathematical topics were discussed, prospective teachers gave presentations about them, and, among others, sought and showed various textbook tasks related to the topics. These textbook tasks usually served as the outline of the presentation. Since the textbooks include only applications to other disciplines already familiar to students (e.g. a map scale at the time when ratio is taught), the seventh-grade mathematics textbook with the topic of ratio does not include the application to the eighth-grade chemistry topic of stoichiometry. In addition, eighth or ninth grade mathematics textbooks do not return to the ratio topic. Therefore, from the perspective of mathematics, such future applications remain unaddressed in the textbooks and remained unaddressed during the courses on didactics of mathematics.

Discussion

As an answer to our research question *How do secondary school teachers specialising in both mathematics and chemistry perceive the concept of ratio in application contexts given by lower secondary school chemistry tasks on stoichiometry?* we may state that the teachers participating in our study are mostly aware of the possibility to apply the concept of ratio in the context of chemistry

stoichiometry tasks (so, the relationship between ratio and stoichiometry is fully perceived) and so that they learnt the topic of ratio conceptually (Hiebert & Lefevre, 1986). This is in contradiction with the findings of Gabel et al. (1984) as the teachers in our study are able to transfer their knowledge of ratios from mathematics to chemistry. However, some of the teachers do not consider such applications suitable for their own teaching practice (e.g. R3/ratios) or implement them in a way that would result in students noticing the concept of ratio in stoichiometry tasks (Čech & Samková, 2024).

From the perspective of *mathematics teacher's specialised knowledge* (Carrillo-Yañez et al., 2018), the teachers in our study know different contexts associated with the concept of ratio and are aware of its applications. That is, their *knowledge of topics* is sufficient. For themselves, they can connect mathematics to other disciplines, namely chemistry. However, our findings indicate that, at least for some of the teachers, there are weaknesses in *knowledge of mathematics teaching*: the teachers know different ways of representing content and various textbooks, but they may not be aware of their limits. In this particular case, applications to other disciplines appearing in available textbooks address current and prior knowledge only, but not future knowledge. In that sense, the lack of awareness of such limits also relates to weak *knowledge of mathematics learning standards*. During the interview in our study, it turned out that the courses on didactics of mathematics at our university could also have such weaknesses. These revelations bring the following implication for initial teacher education courses: To fix the weaknesses found, the theme of possible future applications of individual mathematics topics should be explicitly added to the content of the courses on didactics of mathematics. Ideally, it can be done together with the instruction for prospective teachers to look for materials and resources outside the textbooks, cover lower secondary school textbooks for higher grades and other school subjects, and create new materials that would reflect various possible future applications of the topic. Another implication concerns the timing perspective, both for the initial teacher education and for classroom practice. It remains to be explicitly acknowledged that future applications of the sixth to eighth grade mathematics topics can be addressed during the ninth grade, where a final repetition of all lower secondary school topics takes place. As a part of the repetition, a new perspective on various lower secondary school topics can be introduced to students, for instance, in a format using comparisons of multiple solution strategies (Durkin et al., 2017). In particular, the ratio-based procedures to eighth-grade stoichiometry tasks can be discussed and contrasted with other applications of the concept of ratio already known to students from seventh-grade mathematics.

Having the tasks in our data collection materials related to chemistry, we also have to consider the perspective of didactics of chemistry. We do not suggest forgetting the chemistry procedures that are important for future and more complicated upper secondary chemistry tasks (e.g. for the higher-level tasks in Ramful & Narod, 2014). However, we suggest enriching them by including also the mathematical procedures, that is, by including the perspective of didactics of mathematics. In this way, conceptual knowledge in both mathematics and chemistry can be enhanced by acknowledging connections between concepts in different disciplines (Hiebert & Lefevre, 1986).

As educators of mathematics teachers, we cannot influence what happens in chemistry classes and courses on didactics of chemistry, but we can still include chemistry applications in mathematics teaching and courses on didactics of mathematics. If these applications are lower secondary school applications, they have been completed by all prospective secondary school mathematics teachers

(regardless of their other specialisation), because they all had attended lower secondary school. Thus, chemistry applications can be included in initial education courses for mathematics teachers, even if most of the attendees of these courses do not have chemistry as their other specialisation.

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Pre-service mathematics teachers' conceptualizations and noticing of algebraic thinking

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This study investigated how an intervention based on video clubs designed using a novel perspective on algebraic thinking supported middle school pre-service mathematics teachers' (PSTs) noticing of students' algebraic thinking. This research was conducted as part of a mathematics methods course that integrated the intervention in the course context. At the end of the intervention, post-interviews were conducted with five PSTs. The findings indicated that the intervention had supported PSTs in interpreting students' algebraic thinking from a broad perspective, including both students' early nascent algebraic reasoning and their more advanced conceptual algebraic ideas.

Keywords: *Algebraic thinking, seeds of algebraic thinking, noticing, pre-service mathematics teachers, video clubs.*

Introduction

The National Council of Teachers of Mathematics [NCTM] (2000) underscores the importance of algebraic thinking, stating that all students should learn algebra, as algebraic competence crucial in post-secondary education and in the workforce. Therefore, algebra plays a critical role in school mathematics. NCTM (2000) emphasizes that algebra includes more than symbol manipulation by displaying the four fundamental ideas of algebra: "Students' understanding of patterns, relations, and functions; students' representing and analyzing mathematical situations and structures using algebraic symbols; students' using mathematical models to represent and understand quantitative relationships and students' analyzing change in various context" (p. 37). Furthermore, Kieran (2004) points out that early algebraic thinking focuses on activities such as proportional thinking, problem solving, noticing structure, and analyzing relationships among quantities. Different characterizations of algebra and algebraic thinking offered in the literature (e.g., Kaput, 2008; Kieran, 2004; NCTM, 2000) provide a broad perspective on the scope of algebraic thinking, and algebra is approached as a way of thinking in a dynamic structure.

Students can display various algebraic reasoning while engaging in mathematical tasks. Driscoll and Moyer (2001) stress that when teachers have insight into students' algebraic thinking, they can design their courses to enhance and further their learning. So, it is essential that teachers can notice and reason about students' emergent algebraic thinking during classroom instruction. A teacher's capacity to recognize and interpret student's ideas thinking and make decisions based on student thinking has been referred to as "teacher noticing" (Jacobs et al., 2010). Teacher noticing has been identified as an essential teaching skill. Jacobs et al. (2010) define professional noticing of children's mathematical thinking as involving three interrelated skills: "attending to children's strategies, interpreting children's understandings, and deciding how to respond on the basis of children's understandings" (p. 172). Teacher noticing has been a topic of interest in mathematics

teacher education research; however, more research is needed, focusing on specific content domains. Walkoe (2015) examined the development of pre-service teachers' noticing of algebraic thinking rather than on particular algebraic concepts during a methods course that engaged PSTs in a video club. Seven pre-service teachers watched and discussed videos from algebra classes using an online video tagging tool. The findings displayed those PSTs tended to be more evaluative and descriptive while discussing students' algebraic thinking in early sessions; however, their discussions about student thinking were more substantive in the later sessions.

Pre-service teachers' noticing skills, particularly in a specific mathematical domain, should be supported through different interventions or professional development activities designed to support developing their noticing skills. As Walkoe (2015) stated, when teachers' noticing in algebra is developed, teachers could be better supported to engage with students' algebraic thinking in classroom. Therefore, investigating teachers' noticing of student thinking in algebra is crucial. This study aims to investigate how a video club intervention supports middle school pre-service mathematics teachers' (PSTs) noticing of students' algebraic thinking. The intervention focused on pre-service teachers' attention to children's formal and pre-formal algebraic thinking by incorporating a "Seeds of Algebraic Thinking" framework that allowed teachers to reason about students' nascent algebraic reasoning (see Walkoe and Levin, 2020). PSTs' noticing was enhanced and tracked using a video annotation tool (www.anotemos.org). This study extends prior research on teacher noticing of student algebraic thinking in a video club context. The following research questions guided this study: *To what extent does an video-based intervention designed using Seeds of AT perspective support PSTs' noticing of students' algebraic thinking?; What is the level of PSTs' noticing of students' algebraic thinking as they participated in an video-based intervention that leverages the Seeds of AT perspective?; What is the level of PSTs' noticing of students' algebraic thinking who did not participate in the video-based intervention?*

The Seeds of Algebraic Thinking

The Seeds of AT perspective, a theoretical perspective introduced by Walkoe and Levin (2020), highlight sub-conceptual cognitive resources that stem from children's early, pre-instructional experiences. This perspective suggests that larger concepts in algebra are constructed from sub-conceptual pieces, with the roots of these pieces originating from a child's lived experiences. These seeds, as pre-instructional resources, are not inherently related to algebraic concepts. Walkoe and Levin (2020) highlighted the crucial role of children's pre-instructional experiences in their later mathematics learning. "Balance", "in-betweeness", and "closing-in" are three examples of seeds of algebraic thinking the authors discuss (Levin & Walkoe, 2022; Walkoe & Levin, 2020). For instance, balance is a resource children gain early in life as they learn to walk or balance on a teeter totter. Later, in formal algebra, balance serves as a productive resource for relational understanding, a concept that can be confusing early on for algebra students (Walkoe & Levin, 2020).

Methods

This study utilized a case study research design. Two different cases—the noticing levels of PSTs who participated in the intervention, those who did not—were investigated.

Research Context and Participants

This research was conducted as part of a mathematics methods course offered to middle school mathematics teachers in their third year in the 2023–2024 academic year at a large Mid-Atlantic, public university in the US. The course had various general and specific goals. One of the particular goals of this course was to explore the impact of a Seeds of AT framework (Walkoe & Levin, 2020) on the video club (VC) intervention. This study involved five pre-service middle school mathematics teachers. Three volunteer PSTs who attended the intervention were chosen for interviews (PST1, PST2, and PST3). Additionally, to explore the impact of the intervention, two volunteer PSTs, who did not enroll in this methods course, were also selected for the interviews (PST4 and PST5). PST4 and PST5 were in their fourth years in the same program. These PSTs had enrolled in the methods course in previous year; however, they did not participate in the intervention discussed in this study, while taking the methods course. As this interview was conducted, PST3, PST4 and PST5 had taken the mathematics course "Algebra for Pre-service Middle School Teachers".

Intervention

The PSTs participated in a series of six VCs (see Figure 1). Before each VC session, as part of the VC professional development, PSTs worked on a mathematical task (e.g., Rat-a-Tat Cat, Flag Area,) that they would see students engaged in in the video. During the VC sessions, PSTs individually watched short video clips (4–10 minutes), which included students/kids' discussions or classroom excerpts while they were engaging in these mathematical tasks. These videos were selected by the second author based on her video library. PSTs watched the videos on an online platform where they could observe students' algebraic thinking. Then, they tagged and commented on moments when they noticed students thinking algebraically. The tags and annotations were logged in the system and the facilitator had access to PSTs' tags and comments.

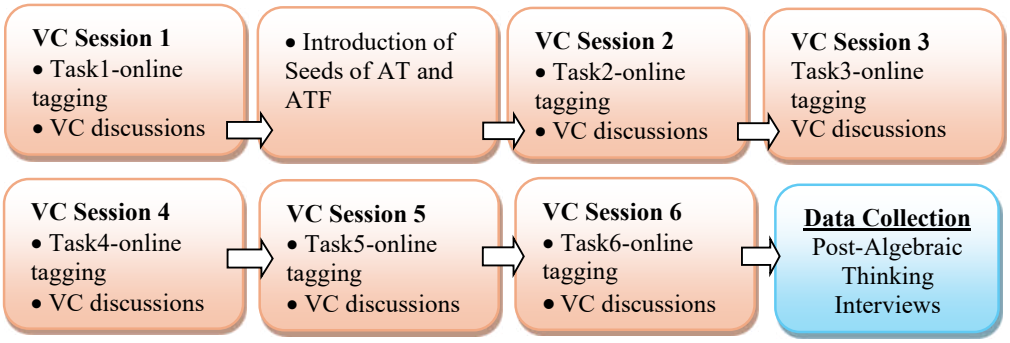

Figure 1: The design of VC professional development and data collection

Furthermore, as seen in Figure 1, PSTs were introduced to the Seeds of AT perspective and an Algebraic Thinking Framework (ATF) (Walkoe, 2015) before attending the second VC session and a class meeting was held to discuss these frameworks. The two instructors of the course facilitated the VC sessions. Finally, when PSTs finished annotating the videos, the PSTs and the facilitators discussed the nature of the students' algebraic thinking and the seeds they observed in the video clips. During the VC discussions, the facilitator pointed out PSTs' tags and invited them to share their thoughts. The facilitators pushed PSTs to articulate what they noticed and why they found that

particular moment interesting using prompts such as, "What about this video is possibly algebraic? What about the balance is algebraic for you?".

Data Collection Process

After the VC sessions concluded, to investigate PSTs' noticing of students' algebraic thinking, the post-interviews were conducted with five volunteer PSTs. In the post-interviews, PSTs were asked to examine five mathematics questions (e.g., see Figure 2 and 3) selected from previous research studies or adapted from the online resources along with possible student solutions of these questions. PSTs were asked the following questions to focus more on the PSTs' noticing of students' algebraic thinking. These questions were: Describe in detail, "Where do you notice students' thinking algebraically? How can you interpret this student's response? What did you learn about this student's understanding?". The data for this study were PSTs' post-interviews.

Data Analysis

We used content analysis to analyze the data obtained through post-interviews. To determine each PST's attending and interpreting levels, we coded for the depth of PSTs' explanations, where they described and interpreted students' algebraic thinking on the student solutions to five mathematics questions. We used the codes described in Table 1, which were informed by teacher noticing research, particularly Jacobs et al. (2010), Walkoe (2015) and Sherin and van Es (2009). We traced how PSTs' explanations connected with students' algebraic thinking.

Table 1: Noticing students' algebraic thinking coding scheme

<table>
<tr><th>Attending</th><th>Interpreting</th></tr>
<tr><td>Level 0: No evidence. Participants do not provide details of students' algebraic thinking.</td><td>Level 0: No evidence. Participants interpret students' strategies/thinking ways in a general sense without attending to students' AT; not connected with student's algebraic thinking. They talk about general teaching and learning issues.</td></tr>
<tr><td>Level 1: Participants provide some evidence of attention to students' algebraic strategies/ways of thinking by either describing or evaluating.</td><td>Level 1: Limited evidence. Participants superficially interpret students' strategies/thinking ways; connected with students' AT (formal algebra, procedural), providing less evidence and narrow level (one idea).</td></tr>
<tr><td rowspan="2">Level 2: Participants provide much evidence of attention to students' algebraic strategies/ways of thinking.</td><td>Level 2: Limited evidence. Participants interpret students' strategies/thinking ways using some evidence; connected with students' AT (procedural or conceptual), not so wide (at most two ideas).</td></tr>
<tr><td>Level 3: Robust evidence. Participants interpret many details of student strategies/thinking using much evidence, connected with students' AT in terms of formal algebra and early algebraic ideas (e.g., seeds); Mainly conceptual, broad level (more than two ideas).</td></tr>
</table>

Results

The analysis of the interview data showed that PSTs, who participated in VC intervention, could provide evidence of attention to students' algebraic thinking strategies. However, even though one PST (PST1) was able to provide evidence of attention to students' algebraic strategies, she interpreted students' algebraic ways of thinking superficially and provided less algebraic evidence. Therefore, PST1's noticing student algebraic thinking level was at Level 2 (L2) in terms of attending and mostly at Level 1 in terms of interpreting. She generally interpreted students' algebraic ideas at narrow and procedural levels by focusing only on algebraic ideas. On the other hand, two PSTs (PST2 and PST3) noticed the level of student algebraic thinking in terms of attending and interpreting were at the top levels (see Table 2).

Table 2: PSTs' noticing level of student algebraic thinking

	Attending					Interpreting				
Questions	PST1	PST2	PST3	PST4	PST5	PST1	PST2	PST3	PST4	PST5
Q1	L2	L2	L2	L2	L2	L1	L3	L3	L2	L2
Q2/S1*	L2	L2	L2	L1	L2	L2	L3	L3	L2	L1
Q2/S2	L2	L2	L2	L1	L2	L2	L3	L3	L1	L1
Q3	L1	L2	L2	L2	L2	L1	L3	L3	L2	L1
Q4/S1	L2	L2	L2	L2	L2	L1	L3	L3	L2	L2
Q4/S2	L2	L2	L2	L1	L2	L1	L3	L3	L1	L2
Q5	L1	L2	L2	L2	L2	L2	L3	L2	L2	L1

* Q2 and Q4 include two students' solutions (S1 and S2). Q1, Q3 and Q5 include only one student solution.

On the other hand, the analysis of the interview data showed that like other PSTs, the PST4 and PST5 who did not participate in VC intervention could provide ample evidence of attention to students' algebraic thinking strategies. Particularly, PST5's attending level was consistently at Level 2. However, unlike the PST2 and PST3, who participated in VC intervention, these PSTs interpreted students' strategies/thinking at a limited level. As seen in Table 2, their interpreting levels were either at Level 1 or Level 2. That means these PSTs mostly did not interpret the details of students' algebraic strategies using evidence and they did not attend to conceptual aspects of students' strategies.

Q1: What number goes in the __? $37 + 54 = \_\_ + 55$ (Stephens, 2008).
Student's solution: "36 goes in the box. 55 is one more than 54, so the number in the box has to be one less than 37, so it's 36".

Figure 2: Solution (S) for Question 1 (Q1)

When PST2 examined the student's solution in Figure 2, she noticed how the student's understanding of algebraic thinking was evident in their solution. The student's thought process, "36 goes in the box; because 55 was more than 54, the number in the box must be less than 37" was a demonstration of relational understanding. According to PST2, the student's focus on the relationship between numbers demonstrates this understanding. Furthermore, PST2 provided the conceptual underpinnings of algebraic thinking by mentioning the balancing idea behind the student's solution. The excerpt of PST2 for this solution was as follows.

> PST2: …What the student did is they focused really on the balancing portion, and they really focused on just seeing the numbers as individual things they needed to compare them to, umm, I feel like this works with this equation…What I can tell from the students' solution is that they might not be used to missing variable equations, so because balancing is something that is an algebraic seeds that kids learn from a young age through many different forms…

Like PST2, PST3 noticed the algebraic idea behind the student's solution and interpreted the student's idea conceptually. PST3 pointed out the student's use of relational understanding of the equal sign. PST3 interpreted this as an early form of algebraic thinking, referencing the student's use of "the balance idea" in the solution. This interpretation by PST3 not only acknowledged the student's progress but also focused on the early algebraic ideas at a broad level. On the other hand, PST4 could recognize student's relational understanding of equal sign and interpret student solution by pointing out student's understanding, too; however, PST4 did not focus on the balance idea and interpreted student's thinking at a less broad level than PST2 and PST3, as shown in the following excerpt.

> PST4: I can learn that they have a good understanding of what the equal sign means here. And an understanding of what we must do to keep those numbers equivalent, meaning keeping 37 plus 54, equivalent to blank plus 55. And they understood that, if we added one somewhere, we take it away to keep it at the one.

When PST1 examined the student's solution, as shown in Figure 3, she just described it by comparing her thinking. While interpreting, PST1 connected with the student's algebraic thinking by pointing out why the student might have constructed algebraic expressions rather than equations. However, PST1's interpretation was narrow, ambiguous and did not include much more understanding of the student's algebraic thinking details. PST1's attending and interpreting level for this question was Level 1.

Q3: Classify all numbers that leave a remainder of 3 when divided by 5 and a remainder of 1 when divided by 3.
Student's solution: Students writes $5x+3$ and $3x+1$ on her paper and stops there.

Figure 3: Student solution (S) for Question 3 (Q3)

Similarly, PST5's interpreting level was at Level 1. As presented below, PST5 pointed out the student's formulation of an equation by using a variable and focused on the student's good understanding of algebra. Even though she connected with algebra, her interpretation was general and procedural without providing more detail about the student's strategy.

> PST5: I think they show a pretty good understanding of algebra and how to use algebra, to kind of set up things to then be able to solve. So, I think a lot of the times we work problems like this, the hardest part for students...I think that students did the hard part, which is making an equation out of these words, and putting a variable in there.

> Although they didn't solve for x, or anything, I think it shows pretty good algebraic understanding.

On the other hand, PST3 viewed the students' solution from a broader perspective, recognizing the potential of their abstract thinking: "I think they're definitely thinking algebraically in terms of thinking a little bit more abstractly…" and "use of working backwards idea", "... I think maybe they're trying to use algebra and the idea of working backwards, and that's where the $5x$ plus 3 and the $3x$ plus 1 comes from".

Discussion and Conclusions

The findings showed that although the attending levels of all PSTs who participated and non-participated in the intervention were nearly the same, their interpreting levels differed. All PSTs could notice and provide some evidence of attention to students' algebraic strategies in their solutions; however, not all could provide robust evidence by connecting with students' conceptual algebraic thinking while interpreting students' solutions. We found that the intervention supported two PSTs (PST2 and PST3) in interpreting students' algebraic thinking from a broad perspective, and their interpretation of students' solutions algebraically included both students' early algebraic perspective and conceptual algebraic ideas used in their solutions. For instance, in the solution of the open box (Q1) question, they could notice and interpret students' use of balance, which is a productive resource that comes into play when thinking about the relational understanding of the equal sign (Walkoe & Levin, 2020, p.30). In the solution to the Q3, they could notice students' working backwards idea to set up an equation from a word problem. Teachers' noticing of students'early algebraic ideas not only helps them understand where these ideas may originate based on students' experiences but also supports students' formal algebraic thinking in later stages (Levin & Walkoe, 2022). One of the reasons why the intervention might have supported two PSTs' interpreting is related to their rich class discussions on PSTs' noticing of students' use of small-grained algebraic ideas. Through online tagging tools, PSTs attended to students' algebraic thinking. In the VC discussions, PSTs deeply interpreted students' algebraic thinking in terms of not only fully formed conceptions, but also nascent ideas. In addition, they discussed various seeds, such as balance, replacement and greater or less in algebraic or non-algebraic contexts, which they observed and tagged in the videos. So, all these factors may impact on two PSTs' high-level interpretation.

On the other hand, we did not observe any prominent impact of the intervention on PST1's interpreting level. The data displayed that the attendance and interpretation of PST1s were nearly the same as that of PST4 and PST5. PST1 could interpret students' solutions by connecting with the algebra; however, she mainly provided limited evidence of students' algebraic thinking from a narrower perspective than PST2 and PST3. There may be various reasons why the intervention did not have a distinctive impact on PST1's interpreting level. PST1 might have displayed less effort in attending to students' algebraic thinking while watching the videos or discussing their noticing, or the intervention time may not be enough to make sense of the Seeds of AT perspective. This study suggests that the video-based intervention designed using the Seeds of AT perspective has potential to support PSTs' noticing students' early and nascent/developing algebraic thinking. However, the findings of this study were limited only to post-interview data, and the analysis of individual video

tags and class discussions will provide more robust evidence to understand the support of intervention.

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Teaching with technology to foster conceptual understanding: Student level evaluation of a professional development programme

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The digital transformation of schools is imminent, yet numerous obstacles persist that hinder the effective integration of technology-enhanced teaching (TET). As pointed out in the PISA framework, technology can support student learning when benefits are exploited for teaching quality. Teachers themselves report a lack of skills for technology integration to foster understanding and learning, and call for professional development programmes on technology-enhanced teaching. This paper outlines the concept and evaluation of the professional development programme MaTeGnu for upper secondary schools fostering understanding through TET with a focus on basic mental models. In the evaluation of student learning as the most remote level of PD success, students from classes of MaTeGnu teachers demonstrated higher understanding of the concept of derivative than their peers. Consequently, a focus on basic mental models appears to be promising for PD in this context.

Keywords: Technology-enhanced teaching, professional development programme, evaluation, conceptual understanding, digital tools.

Technology-enhanced teaching

The International Computer and Information Literacy Studies 2018 and 2023 show that schools in Germany have adopted technology-enhanced teaching (TET) only at a low level (Fraillon et al., 2020). Even when using TET, teachers often do not fully exploit the potential of technology but use it as substitute in traditional teaching processes (e.g., for presentation or as textbook; Backfisch et al., 2021). At the same time, teachers report a lack of agreed TET principles for beneficial learning and corresponding quality teaching material, as well as a high demand of PD programmes (Mußmann et al., 2021). Since research in mathematics education has carved out and validated several findings on how technology enhances student learning (e.g., Digel et al., 2023; for an overview see Hillmayr et al., 2020), a major leverage to foster TET is to implement effective teacher professional development programmes for teaching with technology, enriched with adequate teaching material.

MaTeGnu – a professional development programme on TET focusing on basic mental models

A relevant shortfall of upper secondary school mathematics teaching is the focus on formalisms, which is often not accessible to students. This is often justified as a preparation to higher levels of abstraction at universities, but Rach and Ufer (2020) have shown that procedural knowledge without a conceptual basis is not sufficient for academic success while at the same time formal representations are dispensable. A conceptual foundation needs basic mental models that guide the teaching and learning process (Greefrath et al., 2023). The professional development programme MaTeGnu (Teaching mathematics with technology effectively focusing on basic mental models) aims at fostering understanding in upper secondary school through TET with a focus on basic mental models. The PDP is outlined as a three-year course, accompanying teachers with their math course through a

complete cycle from the beginning of upper secondary school to the A-levels. Each module is dedicated to the understanding of one concept (derivative, integral, vectors and matrices, hypothetic-prognostic probability, exponential growth) using TET to learn the respective basic mental models.

The concept of derivative

This study evaluates the module on derivatives. Greefrath et al. (2023) describe four basic mental models (BMMs) characterizing the concept of derivative: (RC) local rate of change, (TS) tangent slope, (LA) linear approximation, and (AF) amplification factor. RC uses the understanding of the mean rate of change, but unlike that it is not a quotient, but the limit of quotients. TS uses the idea of a tangent as a clinging straight line to the graph. LA can be described with a "function microscope": when zooming in on the graph it appears as a straight line and the relationship represented by the graph can be described locally by a linear relationship. AF views the derivative as a proportionality factor, indicating how (small) changes in the independent variable affect the dependent variable.

Teachers' professional knowledge on technology-enhance teaching (TET)

As previously indicated, the integration of technology does not automatically enhance the quality of teaching. The teaching quality itself can be operationalized with the four aspects cognitive activation, individual learning support, classroom management and cognitive support (Kleickmann et al., 2020). To exploit the benefits of technology for teaching quality, teachers need to build professional knowledge on TET as described in the TPACK model (Mishra & Koehler, 2006). First, teachers need technological knowledge (TK) to handle digital educational tools, like for example the dynamic mathematics software GeoGebra which is widely used in mathematics classes. Teachers also need technological content knowledge (TCK) to use these tools, for example to calculate derivatives or integrals. They also need technological pedagogical knowledge (TPK) to organise students' interaction and documentation of their work with a digital tool. Finally, the facet TPCK describes teachers' crucial technological pedagogical content knowledge, necessary to use features of a digital tool deliberately to foster the learning process of their students (e.g., the use of multiple representations and systematic variations in GeoGebra to foster students' representational flexibility and their basic mental models of functions (Digel et al., 2023). TPCK thus enables teachers to exploit the potential of technology to enhance teaching quality.

Changing teaching practice in a professional development programme

Participating in a TET professional development programme can significantly enhance teaching practice, mediated by collaboration (Konstantinidou & Scherer, 2022). In their research synthesis on effective professional development programmes related to TET, Lipowsky and Rzejak (2021) developed an offer-and-use model describing influential factors of teacher PD (see Figure 1).

Changes in teaching practice do not happen instantaneous but undergo a transfer process (Fütterer et al., 2024). This process forms the centre of the offer-and-use model. At first teachers review their lesson design regarding how technology serves the learning objectives (unfreezing). From there teachers iteratively try to reconceptualise their planning, revise the practical implementation in the context and, if convinced of the utility, implement it in the classroom (changing). As soon as they are convinced of the practical results, they consolidate their concept (freezing). This transfer process can

be fostered by a PDP, if a collaborative network is established, to accompany and support the transfer. Each module in the MaTeGnu PD consists of one-to-two-day-workshops on how to foster basis mental models with TET for one mathematical concept (e.g., derivative). The workshop combines professional knowledge on TET and basic mental models (BMMs) with evaluated teaching material and initiates the unfreezing phase. It is followed by meetings in a professional learning community (PLC), formed by 16-20 participating teachers and coached by two teacher educators. These meetings accompany the changing phase through guidance and exchange on reconceptualization and planning as well as reflection on implementation and collective revision, leading to individual freezing phases.

Evaluation of MaTeGnu

Lipowsky and Rzejak (2021) incorporate the four levels to evaluate training programs by Kirkpatrick and Kirkpatrick (2016) in their offer-and-use-model (Figure 1).

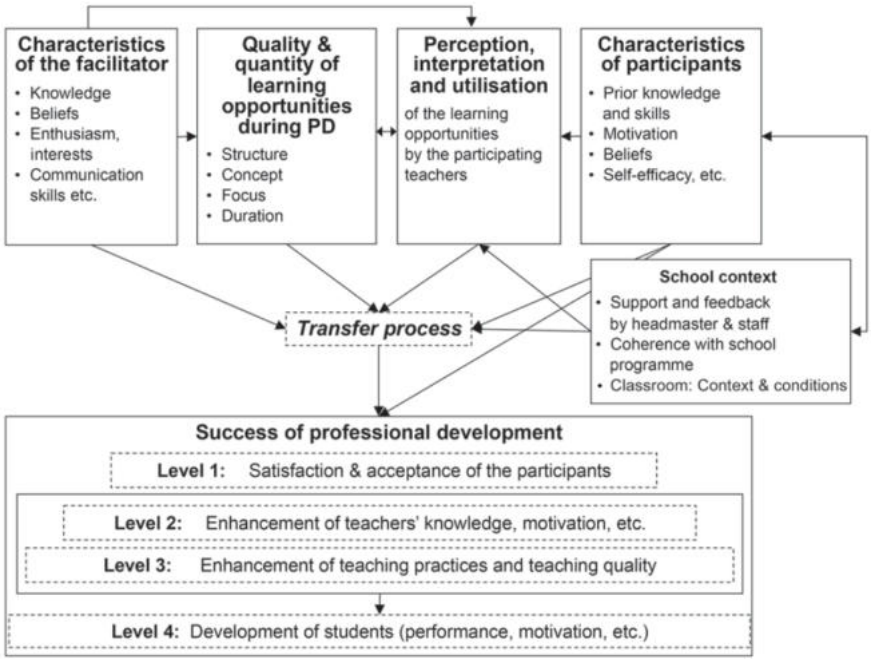

Figure 1: Offer-and-use-model on influential factors of teacher PDPs (Lipowsky & Rzejak, 2021)

The evaluation component of MaTeGnu presented in this study is located on level four: results, i.e., the development of students. Since MaTeGnu has the goal to foster understanding through TET with a focus on basic mental models, two main research questions guide this evaluation study:

RQ1: Do students in MaTeGnu math classes show different preferences for explanations of derivative with basic mental models or formal expression than students in regular math courses?

RQ2: Does the understanding of the concept of derivative of students in MaTeGnu math classes differ from that of students in regular math classes?

Level four is the most remote in the effect chain of the PD and considering the complex transfer process and the manyfold influences on the process as outlined in the offer-use-model (Figure 1), hence rather low effects are to be assumed. Nonetheless, it is a central objective of MaTeGnu with

the three-year perspective as well as intensive peer interaction and coaching network to initiate, accompany and foster the transfer process.

Method

Greefrath et al. (2023) provide a test on basic mental models of derivatives with eight questions and explanations from the perspective of each basic mental model (GV-A). We added another formal-symbolic explanation (FS) to each question and asked the students to rate the suitability of each explanation for their own explanation to a classmate on a unipolar four-point Likert-scale (see Figure 2 for sample item). Additionally, a test on conceptual understanding of derivatives (FALKE, Klinger, 2021) was used. The combined test (GV-A, FALKE) is outlined in a multi-matrix design, with a total of 58 items, distributed in six different booklets with 14 anchor items. The test time was 45 minutes.

Figure 2: Sample item: Test on basic mental models of derivatives (Greefrath et al., 2023)

The derivative f' of a function f at point x can be explained differently.					
Please rate the suitability of the following explanations for use when explaining it to a classmate.		very suitable	suitable	less suitable	not suitable
RC	The derivative indicates the local rate of change at the point x.				
TS	The derivative indicates the slope of the tangent at a point on the graph of f.				
LA	The graph can be approximated well by a straight line in the vicinity of x. The derivative indicates the gradient of this straight line.				
AF	If you move a small distance Δx from x to the right or left, the value of the function changes by $\Delta y \approx f'(x) \cdot \Delta x$.				
FS	The derivative at point x is the limit of the difference quotient $\frac{f(x+h)-f(x)}{h}$ for $h \to 0$.				

Sample of the study

To answer our research questions an experimental-vs.-control-group design has been used. The teaching material, workshop content and PLC concept have been piloted with a group of 18 expert teachers during their qualification as MaTeGnu teacher educators. The qualification has been outlined in the same way as the MaTeGnu PDP and the experts have also implemented the teaching material of the modules in their math classes. The students of these classes participated in the combined test mentioned above (GV-A, FALKE) after completing the MaTeGnu lesson series on derivative, forming the experimental group (EG). This sample consists of *n=151* students *(76* female*, 75* male*, 0* diverse) with an average age of *M=18.9* and tests were performed from June to December 2024. Before the MaTeGnu qualification of teacher educators started, each school of the teacher educators participated in the combined test with mathematics classes that had just completed the subject

derivative. This sample formed the control group (CG), since the teachers have not participated (yet) in the MaTeGnu PDP. The sample of the control group consists of *n=571* students (*256* female, *260* male, *0* diverse) with an average age of *M=17.1*. Tests were performed from June to November 2023.

Results

The reliability of both tests (GV-A, FALKE) was analysed using IRT. The GV-A Likert-scale was aggregated (suitable/not suitable) for dichotomous coding to answer RQ2. The two-dimensional Rasch model with the aggregated GV-A on the first dimension and the concept test FALKE on the second showed acceptable reliabilities (Wilson, 2005) ($rel_{WLE} = 0.64 / 0.72$ (CG); $rel_{WLE} = 0.68 / 0.74$ (EG)). The one-dimensional generalised partial credit model (GPCM) of the four-point Likert-scale GV-A (to answer RQ1) showed good reliabilities (ibid.) ($rel_{WLE} = 0.88$ (CG) and $rel_{WLE} = 0.78$ (EG)).

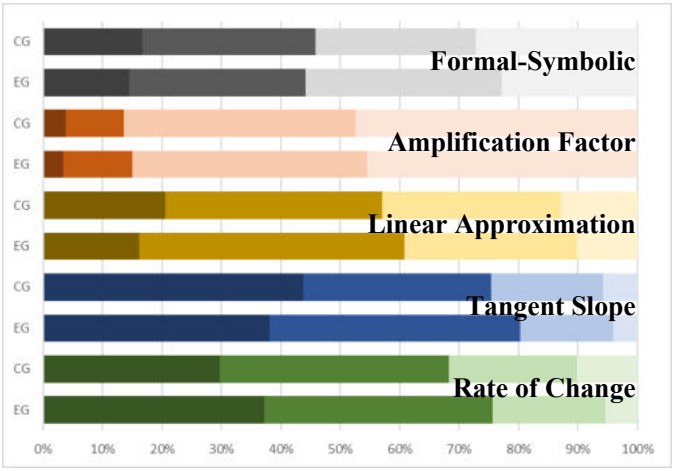

Figure 3: Likert-scale suitability ratings of the presented sample item in CG (top) and EG (bottom)

Figure 3 compares the ratings for the suitability of each explanation presented to explain the derivative to a classmate. in the control group (top) and experimental group (bottom) of the sample item. Darker colours represent higher suitability (very suitable, suitable, less suitable, not suitable).

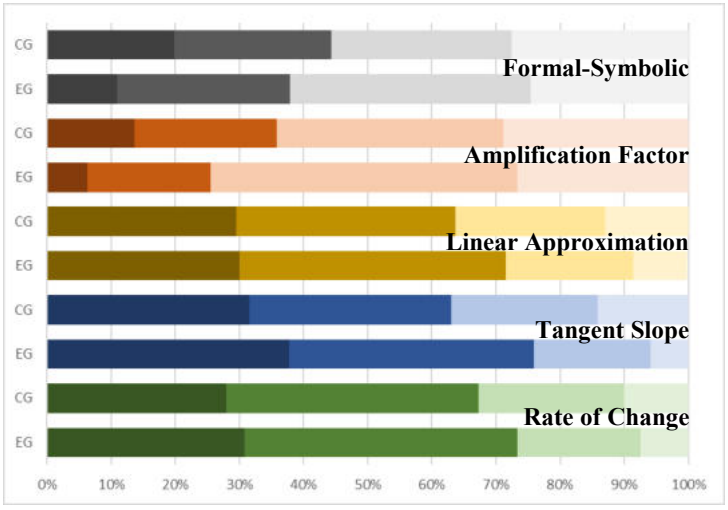

Figure 4: Aggregated ratings of all items in control group (top) and experimental group (bottom)

Figure 4 shows the aggregated ratings of all items for the suitability of each explanation presented to explain the derivative to a classmate. The ranking is similar in both groups with the highest suitability given to tangent slope (TS), followed by local rate of change (RC), linear approximation (LA) and formal-symbolic (FS), while amplification factor (AF) is rated a much lower suitability in the sample item. Comparing the ratings between control and experimental group, we identify higher ratings for tangent slope and rate of change in the experimental group. This distance is even bigger in contextualised items, where lower ratings for the formal-symbolic explanations with a considerable distance between CG and EG can be observed as well.

The overall solution rates for the FALKE test on conceptual understanding of the derivative are lower in CG (*M=0.54*) than in EG (*M=0.58*). The solution rates of individual items vary considerably but show a similar pattern in both groups (Table 1). Looking at the content of items with low solution rates (marked in Table 1) there are two clusters: 1) Graph of f given, interpret f' (Z8PC, X4TP, S3AB): e.g. graph of a function f is given, decide which of the following three properties ($f'(0)>0$; $f'(1)<0$; $f'(x)<0$ for all x) does f' possess; 2) Determine the slope in context (JSE, Y2VKc): e.g. the graph of f has to be drawn for a cone-shaped vessel that fills up with water. Items with high solution rates either 3) present a graph for f' (U3TP, W7CKa, V3RKa) or 4) use a linear relationship in context (Y2VKab, N1FR, D6LGa).

Table 1: Solution rates for FALKE test items in control and experimental group

	Z8PCa	Z8PCc	Z8PCd	Z8PCe	X4TP	S3AB	JSE	Y2VKc	U3PT	W7CKa	V3RKa	Y2VKa	Y2VKb	N1FR	D6LGa	Z8PCb	V3RKb	V3RKc
	Cluster 1)						2)		3)				4)			others		
CG	.27	.37	.25	.36	.36	.33	.46	.34	.75	.76	.78	.71	.68	.83	.84	.50	.51	.51
EG	.41	.62	.34	.44	.43	.40	.53	.35	.71	.87	.82	.73	.76	.78	.82	.49	.47	.48

Discussion

Explanations employing basic mental models (tangent slop, linear approximation and local rate of change) elicit noticeably higher suitability ratings than formal symbolic explanations in both the experimental and control group. While the presence of tangent slope in numerous schoolbooks and of local linear approximation in several German A-level tasks might explain these findings, local rate of change is not a common approach for teaching derivatives. One explanation is that the item stem have captured the accessibility of the explanation rather than reflecting the familiarity with the explanations from their math classes. In this context, the findings of this study demonstrate the significant explanatory value of basic mental models for students, thereby substantiating the comprehensive strategy of MaTeGnu to combine TET with a focus on basic mental models. There is an increase in tangent slope, rate of change and linear approximation in the experimental group, despite the high level already evident in the control group. Furthermore, the ratings for formal-symbolic explanations in the experimental group are even lower than the mediocre ratings in the control group. Regarding our fist research question, we can infer that MaTeGnu students exhibit a higher preference for explanations with basic mental models. It appears that MaTeGnu facilitates a

transition in focus from formal-symbolic to basic mental models. Furthermore, the emphasis on the rate of change in MaTeGnu renders all three basic mental models more accessible to students.

The concept test results expose two levels of difficulty across experimental and control group: while easy items present the graph of *f'*, difficult items ask for properties of *f'* without a graph. This finding suggests a strong correlation between conceptual understanding of derivatives and graphical representation. The interpretation of a given graph of the derivative is therefore considered part of their conceptual competence. To determine the derivative only from the function or situation (as in difficult items) is considerably less developed. The experimental group demonstrates higher solution rates particularly in these items. Regarding research question two we can state that MaTeGnu students demonstrate higher conceptual understanding, particularly in more intricate aspects.

Conclusion

The findings of this study suggest that MaTeGnu supports teachers in transitioning the emphasis from formal-symbolic to basic mental models in the learning process of the concept of derivatives. Emphasising the basic mental model *rate of change*, as outlined in MaTeGnu, appears to facilitate the integration of all three basic mental models in the teaching and learning. Furthermore, our results demonstrate that fostering basic mental models broadens and deepens students' conceptual understanding.

The high explanatory value of basic mental models for students supports the approach of MaTeGnu to use technology-enhanced teaching to set a focus on understanding with basic mental models. This focus could serve as an effective strategy to exploit technology benefits for learning and overcome (beliefs of) mindless working when teaching with technology (Thurm & Barzel, 2022). The approach has proved itself even in the more formal setting of upper secondary school mathematics. Furthermore, the experimental group's lead in difficult items of the concept test indicates that the MaTeGnu approach appears to be most beneficial for complex conceptual competencies.

Our distinct findings at the most remote level of PDP success (i.e., student learning) demonstrate the impact of the MaTeGnu PDP concept. The combination of a topic-focused workshop, concatenated with evaluated teaching material, a professional learning community and the just-in-time procedure accompanying teachers in one of their math classes over a long period, seems to foster the complex transfer process in teaching practice considerably.

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Exploring the notion of individuality in research on mathematics teacher education and professional development

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In this paper, we present findings from our initial analysis of TWG18 papers published in the CERME13 post-conference proceedings. In response to the call by Jeppe Skott to re-centre the individual in research on and with mathematics teachers in his plenary address during CERME12, we look into the notion of individuality and the individual-context relationship in research on mathematics teacher education and professional development. We present our results having analysed the abstracts of 40 papers to explore how the interaction between teachers and their environment are portrayed.

Keywords: Teacher education, professional development, individuality, situatedness, context.

Introduction

During CERME12, Jeppe Skott delivered a plenary lecture entitled: *Conceptualizing individual-context relationships in teaching: Developments in research on teachers' knowledge, beliefs and identity* (Skott, 2022). In that plenary, Skott traced the evolution of research concerning teachers' knowledge, teachers' beliefs, and teachers' professional identities over recent decades, asking "how and to what extent aspects of 'the social' have been taken into account" (p. 35) in each of these fields of research, with particular concern for how relationships between individuals (i.e., teachers) and their contexts are conceptualised. Skott claims that while studies concerning teachers' knowledge and beliefs have predominantly drawn on constructivist underpinnings and are primarily cognitive in their approach, contextuality has "gradually been taken into account in both fields" (p. 47). However, the context is mostly framed regarding external constraints (p. 52). On the other hand, studies concerning teachers' professional identities have been rooted in more profound social perspectives on human functioning, emphasising the situated and structural aspects which can influence or even constitute, professional identities (p. 47). Having acknowledged the positive outcomes from each field, Skott observes the lack of personal variability in research on teachers. He argues for interactionist approaches as a complement to what he calls 'variable oriented approaches' (that prioritise agency or structure and ignore the significance of local practices) and 'community/social oriented approaches' (that prioritise situation or structure and overlook the individual), contending that interactionist approaches "re-centre the individual, while maintaining [a] participatory stance" (p. 51). According to Skott, interactionist approaches focus on how meaning emerges in local contexts (e.g., classrooms, schools, professional development programmes) through interactions between groups of individuals and how teachers adjust their actions accordingly in relation to these situations.

Lutovac and Kaasila (2018) have made similar accounts elsewhere. Having prioritised the social aspects of identity, these authors suggest more studies are needed that re-focus on the individual element that the notion of identity was set out to illuminate in the first place. In other words, the focus

on the social constitution of identity has led to a need to re-emphasise how individuals react to and engage with the multiple challenges and commitments they face on a daily basis in their localised situations. One aspect of teachers' multiple challenges and commitments is what we call everyday work-life. Skott (2022) stresses the need to understand the complexity of mathematics teachers' everyday encounters, especially since most individuals cannot access sustained subject-specific professional development (PD). For mathematics teacher education (TE) and PD to be effective and impactful on teachers' practices, researchers need to consider the tensions, challenges and successes that teachers face within their individual contexts to usefully inform the design and implementation of future TE programmes and PD initiatives in new and potentially unpredictable ways. This might be especially valuable in the light of sustainable careers.

We (Andreas and Tracy) were group leaders and co-leaders of Thematic Working Group 18 (*TWG18: Mathematics Teacher Education and Professional Development*) at the last two CERME conferences (CERME12 and CERME13). We have read, reviewed and edited many papers and posters in these roles and organised and facilitated several group discussions. These experiences have made us curious about how studies presented in TWG18 are situated concerning the call made by Skott (2022). We therefore ask *in what ways do researchers conceptualise the individual-context relationship in studies presented in TWG18*? By asking this, we are also asking *whether (and in what ways) individual teachers feature as part of the unit of analysis in the research on mathematics TE and PD presented at CERME, and what other relationships exist*? In this paper, we report specifically on our analysis of TWG18 papers from the CERME13 post-conference proceedings in light of these overarching questions.

Learning: acquisition or participation

Research on how mathematics teachers learn can be grounded in two main metaphors: learning-as-acquisition and learning-as-participation (Sfard, 1998; 2009). Learning-as-acquisition, often linked to certain constructivist theories (e.g., Piaget's model of cognitive development), emphasises the individual process of *acquiring* and *possessing* knowledge, beliefs, or skills such as noticing. The focus here is on mental processes, positioning learning as occurring within the head of the learner. According to Skott (2022), learning-as-acquisition is the basis of a range of frameworks relating to mathematics' teacher knowledge, including some aspects of the 'Knowledge Quartet' (Rowland, 2013) and 'Mathematical Knowledge for Teaching' (Ball et al., 2008). These frameworks emphasise content acquisition, mental schemas, and knowledge transmission (Stetsenko, 2008).

In contrast, the metaphor of learning-as-participation centres around a social community or becoming a practice member. Here, learning is seen as a process of engagement in social practices, where the individual and the community shape one another. Learning-as-participation, aligned with situated learning theories, involves key concepts such as emergence, apprenticeship, cultural embeddedness, and cooperation (Stetsenko, 2008). It highlights the social nature of learning, situating it within a broader socio-cultural and historical context (Wenger, 1998). As Skott (2022) noticed, learning-as-participation is beginning to receive more attention in research on mathematics teachers. While learning-as-acquisition focuses on internal cognitive processes, learning-as-participation emphasises the social, relational, and dynamic nature of learning. However, the dichotomy between these

perspectives is often criticised for overlooking their similarities (Sfard, 1998). For example, some aspects of the Knowledge Quartet can be interpreted through a situated lens, particularly contingency, which involves adaptive teaching decisions in real-time situations (Ebbelind, 2020).

The rise of identity as a focus in mathematics education reflects a shift towards participation, described by Lerman (2000) as 'the social turn' in mathematics education. Identity is increasingly seen as a dynamic, participatory process, consistent with a sociological paradigm (Darragh, 2016) rather than a static attribute corresponding to a psychological paradigm (Darragh, 2016). The psychological perspective views identity as an internal construct (i.e., learning-as-acquisition). At the same time, the sociological frame sees identity as shaped by social interactions and the individual's actions within specific contexts (i.e., learning-as-participation). This sociological approach is often linked to symbolic interactionism, in which identity is constructed and reconstructed through participation in social practices (Skott, 2022). While Skott (2022) advocates for re-centring the individual by ensuring their part within the unit of analysis in research on and with mathematics teachers, discursive research and research that draws on social practice theories, for example, have tended to turn their gaze away from the individual toward dominant discourses and, for example, 'Communities of Practice' (Wenger, 1998) as opposed to the individuals who participate within them (Ebbelind, 2020).

In our post-conference report on TWG18 at CERME13 (Friesen et al., 2023), we noted that even though teacher learning is now widely acknowledged as situated, i.e., "a product of the activity, context, and culture in which it is developed and used" (Brown et al., 1989, p. 32), the way this situatedness is conceptualised can differ:

> From some perspectives, the learner is viewed as being situated *within* a particular environment or context, for others, the learner is considered *part of* the environment or context. How this relationship is conceptualised informs researchers' methodological decision making. During TWG18 discussions, participants grappled with the consequences of their own methodological decisions (Friesen et al., 2023, p. 3173, emphasis original)

The motivation for writing this paper thus lies partly in the call made by Skott (2022) and in our desire to explore the ways in which individuals and contexts are related in research on and with mathematics teachers. In doing so, we hope to identify whether dominant discourses seem to exist within research presented at TWG18, and to share these findings with TWG18 participants at CERME14 as a way of prompting further reflection and discussion.

Methodology

The study of TE and PD has been a major focus of research in mathematics education over recent decades. Various research activities on this topic have been presented and discussed at previous CERME conferences. Of the thematic working groups that focus on mathematics teachers, TWG18 has concentrated on research into the preparation of prospective mathematics teachers and the PD of in-service mathematics teachers from any phase of education. Discussions within TWG18 cover models and programmes of TE and, including their contents, methods, tools, and impacts. TWG18 provides a communicative, collegial, and critical forum for discussing these and related issues,

embracing diverse perspectives and theoretical approaches, and contributing to our development as researchers, educators, and practitioners.

To develop insights relating to the questions set out in the introduction (summarised here as: *in what ways do researchers conceptualise individual-context relationships?* and *in what ways do individual teachers feature as part of the unit of analysis?*), we retrieved all TWG18 papers from the CERME13 post-conference proceedings which consisted of 40 papers in total and began with an analysis of the abstracts from each paper. In our joint reading of the set of abstracts, we noticed that each one referred to the basis of research. By 'basis of research', we mean the motivation or starting point that the study is based upon and/or that justifies the need for the study. We identified four broad categories for the basis of research (each category is explained in the results section): 1) an assumption; 2) a lack of something; 3) exploring a phenomenon; 4) developing something.

Having established the broad research basis of each study, we then looked in the abstracts to identify the unit of analysis when describing the results. Here, we identified three broad categories (again, the categories are explained in the results section): 1) group(s) of teachers, 2) individual teachers, and 3) objects. Since our focus is on the notion of individuality, we then looked more closely at those studies where we categorised the unit of analysis as 'individual teachers' by going into the full papers. With these studies, we looked at how the individual-context relationship is conceptualised. We appreciate the limitations of performing an initial analysis based on the content of a set of abstracts and use this approach primarily to identify papers considering individual variability in their findings. Having said that, we believe the results from this initial stage of analysis are illuminating in terms of providing a broad categorisation of TWG18 research that can usefully provoke reflection and discussion amongst TWG18 participants.

Results

In this section, we present the findings from our analysis. After presenting an overview, we explain the seven broad categories introduced in the previous section, exemplifying them with examples from selected TWG18 abstracts before presenting more nuanced findings in relation to those papers identified as recognising the individual variability of participating mathematics teachers. Table 1 presents our broad categorisation of TWG18 research from the CERME13. For example, there were six studies based on 'an assumption' where the unit of analysis was 'group(s) of teachers'. There were three studies that used the basis of 'developing something', where the unit of analysis was an 'object'. Altogether, we found that based on our broad categorisation, there were 10 different types of study (denoted by the shaded cells within Table 1).

In terms of the basis of research, there were seven papers where the abstract begins with *an assumption*. For example, Frey and Sproesser (2023) state, "Functional thinking plays a key role in mathematics education and beyond" (p. 3195). An assumption like this, often justified with references to earlier work done by other researchers, is used to justify doing the research. Most papers that start with an assumption report on pilot studies or are framed as initial stages of more extensive studies or doctoral projects.

Table 1: Summary of TWG18 papers (CERME13) by research basis and unit of analysis

		Unit of analysis			
		Group(s) of teachers	**Individual teachers**	**Object**	
Basis of research	**An assumption**	6	1	-	**7**
	A lack of something	7	4	-	**11**
	Exploring a phenomenon	7	2	2	**11**
	Developing something	7	1	3	**11**
		27	**8**	**5**	**40**

Eleven papers start by articulating a need for something that is lacking (i.e., a deficit), such as knowledge, variability in research, noticing skills or particular beliefs. Examples include Ebbelind et al. (2023), who state, "there remains an undocumented variability in how teachers change that identity studies try to document" (p. 3187), and Crisan (2023), who states, "an explicit model of subject-specific mentoring is needed" (p. 3371). Many of the papers that present *a lack of something* specifically concern a lack of teacher knowledge in relation to desired knowledge, putting the teacher in a deficit position. We have written elsewhere about the deficit view of mathematics teachers and teaching in research (Ebbelind & Helliwell, 2024; Helliwell & Ebbelind, 2024). 11 papers begin by stating their intention as to explore something such as teachers' noticing skills, teachers' knowledge, teachers' beliefs, teachers' identity, discourses, content or tools. This *exploring a phenomenon* group includes Mizoguchi et al. (2023), who explore "the conditions and constraints [...] from the perspective of the didactic tools proposed to teacher-students for the design of teaching and learning sequences" (p. 3251). Papers within TWG18 that set out to explore a phenomenon tend to go in diverse directions. Finally, 10 papers emphasise the development of a tool, intervention or design. For example, Casi and Sabena (2023) "design informal mathematics workshops, with the support of teacher educators and museum experts" (p. 3175). The category of *developing something* includes various things: developing learning opportunities or potential, developing learning spaces, developing design principles to use when planning PD programmes, and developing tools.

Concerning the unit of analysis reported in the results of the TWG18 studies, 27 papers reported their results in relation to a *group(s) of teachers*. These studies can be considered as having a 'variable oriented' approach (Skott, 2022) since they are not particularly concerned with the way in which different individuals respond in relation to the local context. In reporting on findings in relation to a group of teachers, results tend to be presented in more generalised terms. In research on a group of teachers, the issue of context can be dealt with by increasing the sample size (i.e., with the effect of averaging out the effect of the local contexts) or by paying attention to the details of a particular context (Wiliam & Lester, 2006). Five papers report results in relation to *objects* such as tools, resources or instructions for supporting TE and PD including role-play as a tool for developing

teaching. Finally, and most significantly in terms of our questions, eight papers report on *individual teachers*, describing individuals' commonalities and differences. However, from this initial analysis, it is not yet clear how the individual-context relation is conceptualised (i.e., how individuals attend to and adjust their actions in relation to local situations and social practices). For these eight studies, we looked closer into the full papers, asking how the individual-context relationship is dealt with and/or conceptualised.

In doing so, we focused on the relationship between the individual subjects and the context of the situation to understand how papers position the individual and whether a participatory stance is taken, classifying each paper in relation to the 'learning-as' metaphors used by Sfard (2009) and Stetsenko (2008). We found four of the eight papers rooted in the learning-as-acquisition metaphor. These studies seem to be concerned with the knowledge needed to foster the mathematical knowledge considered as being essential for success, teachers' explicit knowledge for teaching, and teachers' noticing skills. Although these four studies focus on individual subjects, we consider them to be framed by a 'variable-oriented approach' (Skott, 2022) since individuality is not considered in relation to the local social contexts. The remaining four papers were found to be rooted in the learning-as-participation metaphor. They are concerned with how participation in a development project influences the identity of teachers; how the stories influence individuals told of experiences gained in social practices; how the perception of being a part of mathematics courses influences descriptions of the subject of mathematics, and how moments of dissonance arising from participation in teaching and TE guides teachers' narratives.

Concluding thoughts

A close relation to TWG18 is Thematic Working Group 20 (*TWG20 Mathematics Teacher Knowledge, Beliefs and Identity*). Given the nature of the research in TWG20, we speculated whether this may be the forum where individuality is most prevalent in research on and with teachers. This remains a question for us, yet we read in the CERME13 report on TWG20 that:

> Despite the increasing number of research studies in the educational literature that investigate preservice and in-service teacher identity development and the factors affecting that process, there were few presentations at CERME13 that focused on this topic, which tends to be approached from a narrative perspective. Instead, many of the studies presented focused on beliefs [...] (Martignone et al., 2023, p. 3677)

From this we conclude that the participatory perspective is underrepresented in both thematic working groups. Through the process of analysis we have new questions for the TWG18 community to consider. For instance, how might we support one another in questioning the underlying assumptions behind our research on mathematics TE and PD and how might be develop different framings and approaches that open up new possibilities. We close by acknowledging that in setting out to analyse part of a field of research, we have tried to avoid reporting on another 'lack of' in research on mathematics TE and PD, and yet are aware of the challenges in doing this.

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Preservice teachers' professional instrumental genesis for integrating programming in the math classroom: the beginning

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We investigate the development of professional competencies of preservice teachers enrolled in a programming-based math investigation course designed for future teachers. We discuss a case study of a student's experience in this course. Grounded in the instrumental approach, the analysis of the student's scheme elements suggests her initial professional instrumental genesis, including for example her early development of a scheme of preparing a programming-based math task aiming at students to complete it mindfully.

Keywords: Programming, preservice teachers, professional instrumental genesis, schemes.

Introduction and context

In recent years several countries and jurisdictions have integrated programming as a part of mandatory math curricula (Broley et al., 2023), which in turn creates the need to develop pedagogical programming competencies for math teachers. Some recent studies of inservice (Grimeland et al., 2022) and preservice teachers (Maugesten et al., 2023) argue that even after some initial introduction to programming-based math activities, teachers still report having limited teaching competencies in that area. In fact, inservice teachers' "lack of knowledge about programming is clearly an obstacle for developing good teaching plans that would enhance pupil knowledge" (Grimeland et al., 2022, p. 3129). Similarly, Maugesten et al. (2023) have concluded that preservice math teachers highlighted programming as a tool for motivation "rather than developing students' math competence" (p. 7). This points to the need to understand more about pre- and inservice mathematics teachers' development of professional knowledge and beliefs in the context of exploring math through programming.

This study contributes to this need by focusing on preservice teachers' initial professional development in a math content course integrating computer programming, that is targeted for future teachers (MICA III*, Mathematics Integrated with Computers and Applications III*). MICA is a sequence of three math project-based courses (I-II-III/III*) in which math majors and all preservice secondary math teachers conduct pure and applied math investigations supported by programming (Buteau et al., 2015). For example, MICA I students investigate, numerically and graphically, the behavior of the discrete dynamical system based on a two-parameter cubic; and MICA III* students solve, through creating simulations in Python, a probability problem with seemingly multiple answers ('Bertrand paradox'), of a random chord being longer than a side of the inscribed triangle (for MICA III* project guidelines, see Brock University, n.d.).

In 2021, the MICA III* course was revised in response to the recent integration of "coding" in school mathematics curricula in Ontario (Canada). In this course, future teachers complete the math investigation projects, regular reflection tasks, and a final didactical project in collaboration with local schoolteachers through which pre- and inservice teachers co-learn and develop some professional competences in this area. This study explores the development of preservice math teachers'

professional competencies by focusing on a case of a MICA III* student, Nadia (pseudonym). By framing this research in the double instrumental genesis (Haspekian, 2017), we examine the initial development of elements of Nadia's schemes of integrating programming in the math classroom.

Theoretical framework

Our research is grounded in the instrumental approach as articulated in mathematics education (e.g., Trouche, 2004), especially the double instrumental genesis that brings together personal and professional geneses (Haspekian, 2017). The instrumental approach allows us to study the user's appropriation of an artefact as an instrument to reach a goal through a specific activity. The user's appropriation of an artefact happens through instrumental genesis, a dialectical combination of two processes: instrumentation (from the artefact to the learner) and instrumentalization (from the learner to the artefact) (Trouche, 2004) where the learner is transformed by the artefact and the artefact is transformed by the learner. Thus, an instrument is composed of an artefact and schemes. A scheme is defined by Vergnaud (1998) as a stable organization of the subject's activity for a given goal, with four components: goal of the activity (goal, purpose); rules of action (RoA, information gathering, strategies); operational invariants (OI, concepts, propositions considered true, beliefs, justifications); and possible inferences (inferences, new structures and patterns).

To look more closely into the user's appropriation of an artefact (in our case programming) for professional use, we frame our work on the double instrumental genesis (Haspekian, 2017). This framework considers personal and professional instrumental genesis and links them together (Figure 1). In the context of math teachers, the personal instrumental genesis (IGpe) involves the development of schemes by the teacher when they interact with the artefact for their own learning (personal goal, personal instrument). The professional instrumental genesis (IGpro) entails the development of schemes by the teacher when they interact with the artefact for their teaching (professional goal, professional instrument; e.g., to guide students to develop their own personal genesis). As part of IGpro, Haspekian (2017) considers the teacher's instrumental orchestration, which Drijvers et al. (2010) describe as the "intentional and systematic organization and use of the various artefacts available … in order to guide students' instrumental genesis" (pp. 214–215). This instrumental orchestration consists of three components (Drijvers et al., 2010):

- Didactical configuration – e.g., selecting artefact, classroom format
- Exploitation mode – teachers' decision on how to use artefacts chosen, e.g., preparing activities
- Didactical performance – ad hoc decisions when teaching

The components are not considered linear but rather intertwined and connected, where didactical configurations influence and are a part of exploitation mode, and both of those components are again intertwined in didactical performance.

Previously, Haspekian (2017) used this framework by looking at experienced math teachers who have started to introduce a new artefact (e.g., programming) in their math teaching. In this study, we use this framework, specifically IGpro, in a case study of a preservice teacher who has experience in programming for math but has limited pedagogical knowledge and teaching experience. Therefore, we are not looking for stable schemes for IGpro, but rather at the initial development of elements of such potential schemes that a MICA III* student starts developing. By "potential" schemes, we mean

schemes in construction with tentative stability. Finally, we view the initial development of professional competences through our framework as the initial development of potential schemes for IGpro.

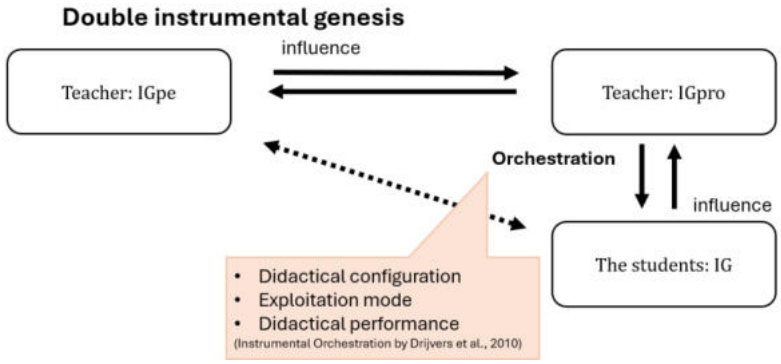

Figure 1: Double instrumental genesis, based on Haspekian (2017, p. 25)

In our work, we explore the framework of double instrumental genesis, in the case of programming-based math investigations (Buteau et al., 2024; St-Cyr et al., 2025). In this study, we focus on preservice teachers' development of professional competencies, exploring and deepening our understanding of future teachers' professional genesis, namely: *What elements of schemes of integrating programming in the school math classroom are developed by preservice teachers as they reflect about their own learning experiences in programming-based math investigations course?*

Method

This study focuses on exploring the case of Nadia, a preservice teacher enrolled in a MICA III* course. In this course, students write reflection tasks that involve guiding questions, integrating selected didactical literature, to think about the way they worked on the investigation projects, with the purpose of building their awareness about learning processes for them and for others. The final project in this course involves a collaboration with local schoolteachers during which the students in pairs prepare and teach a programming-based math lesson in the teachers' Grade 3–9 class. For a majority of students, this is their first experience of teaching math in school.

Nadia was selected, among other volunteer participants, due to her deep reflections on her professional experiences. The data was collected as a part of a larger naturalistic, mix-methodology and iterative design research study focusing on university teaching and learning of programming as a meaningful tool for mathematics (e.g., as done in MICA courses). Data for this study consist of: Nadia's responses to pre- and post-course questionnaires that included some questions related to students' future teaching career; semi-structured post-course interviews that sought elaboration on selected questionnaire responses and inquired about her overall MICA experience; and five written reflections tasks that she wrote during the course. An example of a reflection task is: "Describe one key moment where you received or gave help: What was the issue, what was the strategy used by the 'helper,' and in what ways did it help the person who needed help (in terms of cognitive, interpersonal, and/or intrapersonal domains)?" (Brock University, n.d., Part B – The Reflection section, Item ii).

Data is coded by identifying elements of Nadia's IGpro schemes when she "explicitly makes a connection between their or others' learning experience in relation to the teaching or the learning environment, or when they explicitly comment about their future teaching role" (Buteau et al., 2024, p. 4); that is, their future instrumental orchestration. The first two authors identified individually

excerpts that highlight Nadia's reflections or views that mentioned or were connected to teaching and learning. The authors then compared the individually marked excerpts they had selected and kept only the ones they both identified and coded these excerpts. The coded excerpts were thereafter discussed with the rest of the team, to come to a joint understanding of elements of Nadia's potential schemes for IGpro. This stage required many iterations. In the last stage, when all authors agreed on the elements of potential schemes, we analyzed further the identified schemes and their elements and categorized them by their relation to the instrumental orchestration components.

Findings

In this section, we present Nadia's early development of IGpro where, through completing the MICA III* course, she starts to see herself as a math teacher capable of integrating programming in a math classroom. We first investigate Nadia's initial development of programming as a professional instrument, and we then delve into details of this instrument by highlighting some of Nadia's potential professional schemes. We do so by selecting three illustrative examples associated with each of the three components of instrumental orchestration.

Nadia's initial development of programming as a professional instrument during the course

In the pre-course questionnaire, Nadia was asked to indicate her level of confidence "to teach a math activity with programming in a Grade 6–12 class", for which she answered "not confident", stating:

> I do not feel as of right now I am equipped to teach any programming for Grades 6–12 as I am not aware of the many softwares out there and would not know how to make a lesson out of it/teach it to young students. (S1)

She reported her lack of knowledge and pointed to the competencies (i.e., schemes) she thought she would need for such a professional activity, namely (i) to choose an appropriate programming language, and (ii) to make/teach a programming-based math lesson. However, by the end of the MICA III* course, Nadia answered "very confident" to the same question in the post-course questionnaire. During the interview, Nadia grounded her response with her experiences in the final project whereby she had the opportunity to teach six classes, and she now felt more equipped for her future teacher role. We interpret that Nadia acknowledges having developed her professional instrument to a certain extent for integrating programming in school math classrooms. Next, we investigate Nadia's professional instrument by presenting potential schemes related to her IGpro, and associated to the three components of instrumental orchestration. These schemes actually address Nadia's lack of competencies she mentioned at the start of the course.

Nadia's potential scheme, associated with her didactical configuration

An example related to selecting resources, as part of didactical configuration, was mentioned by Nadia in the interview when she reflected on the programming languages she could use in the future. After having used VB.net in MICA I-II, Nadia was introduced in MICA III* to Scratch (one assignment), and Python (three assignments). In her final project, she chose to make a programming-based math activity in Scratch, explaining that it fits well with geometrical shapes and angles that the students can easily modify, which were the concepts for her planned math lesson. When Nadia was asked in the interview if she would use Python to teach math in high school, she stated:

> Yeah, absolutely. ... I don't think that it's something that's too advanced. It depends on, kind of, [students'] experience with programming from before. (S5)

We interpret from this excerpt that Nadia is starting to develop her scheme of "Choosing a programming language (to teach selected math concepts)" (Goal), with a RoA "I assess the experience of the students I work with", and grounded on an OI of "What programming software is useful depends on students' previous knowledge" (see Table 1). This illustrates how Nadia considers students' previous experiences and thereby willing to adapt her choice of the artefact based on the goal of the activity. Furthermore, Nadia's explanation about her choice of artefact in the final project suggests another RoA "I select a language that provides a math representation (in code, in output) aligned with my didactical aim", grounded on an implicit OI "Different programming languages afford different representations of math concepts (in code, in output) at 'different effort cost'".

Table 1: Nadia's elements of a potential scheme, associated with her didactical configuration

Goal	Choosing a programming language (to teach selected math concepts)
RoA	(i) I assess the experience of the students I work with; (ii) I select a language that provides a math representation (in code, in output) aligned with my didactical aim
OI	(i) What programming software is useful depends on students' previous knowledge; (ii) (Implicit) Different programming languages afford different representations of math concepts (in code, in output) at "different effort cost"

Nadia's potential scheme, associated with her exploitation mode

Once the programming language and a math topic are selected, the teacher then needs to prepare (or select) a task. Reflecting on experiences of collaborating with peers explaining math to each other, Nadia commented during the interview on why when teaching a programming-based math activity it is important for her to guide the learner. She explained how she prepared for this in the final project:

> So, now [some students] have the answer. But I just asked them why? But I didn't necessarily even need to do that because in our activity there was questions afterwards that ask what does the turn angle do? What does this do? What does this do? And so, they actually kind of had to demonstrate, um, their, their learning. (S40)

We interpret Nadia's excerpt as elements of her potential schemes for IGpro, of "Preparing a math-programming task that aims at encouraging the students to do the activity mindfully" (Goal) with a RoA "I prepare a worksheet with questions for students to complete to demonstrate their learning", and grounded on the OI "Teacher should ask 'why is it working?' (a) to encourage students to understand what they do, (b) to have the opportunity to see if the student understands the connection between the code and the math" (see Table 2). We interpreted further two implicit OIs: "Programming in math is about exploring math concepts, learning them, not just calculating or programming the answer" and "One can code the right answer by tinkering (without understanding)." This illustrates how Nadia is building her IGpro on the reflections about her own learning of math, where she plans to prompt students to investigate the math in the programming-based task by guiding students both during her preparation of the task (task guidelines) and during implementation when she plans to walk around and ask students when looking at their work.

Table 2: Nadia's elements of a potential scheme, associated with her exploitation mode

Goal	Preparing a math-programming task that aims at encouraging the students to do the activity mindfully
RoA	I prepare a worksheet with questions for students to complete to demonstrate their learning
OI	(i) Teacher should ask "why is it working?": (a) to encourage students to understand what they do, (b) to have the opportunity to see if the student understands the connection between the code and the math; (ii) (implicit) Programming in math is about exploring math concepts, learning them, not just calculating or programming the answer; (iii) (implicit) One can code the right answer by tinkering (without understanding)

Nadia's potential scheme, associated with her didactical performance

Nadia did not have classroom teaching experiences prior to the participation in the MICA III* course. Yet, during the interview, she reported the following conversation when preparing with her peer partner and her two associate schoolteachers for the classroom implementation (final project):

> So, like, we kind of explained to [the schoolteachers]: "please, when going around the classroom, don't try to tell students to change their block. Ask [students] what parts they think are wrong. Ask them." And so, that's kind of the way where we were almost teaching [the schoolteachers], how to teach coding within the [math] classroom. (S17)

We interpret this as Nadia's potential professional scheme of "Intervening with an individual student to address an error with their math program" (Goal), with two RoAs: "I ask the students what part they think is wrong" and "I do not tell students what to change", grounded on the OI "To teach math-programming, teachers do not have to tell the answer to the students, but should ask the students what part is wrong" (see Table 3) and an implicit OI "It is more helpful for the students if I help them identify the issue and guide them to find a solution, rather than directly tell them what to fix."

Table 3: Nadia's elements of a potential scheme, associated with her didactical performance

Goal	Intervening with an individual student to address an error with their math program
RoA	(i) I ask the students what part they think is wrong; (ii) I do not tell students what to change
OI	(i) To teach math-programming, teachers do not have to tell the answer to the students, but should ask the students what part is wrong; (ii) (implicit) It is more helpful for the students if I help them identify the issue and guide them to find a solution, rather than directly tell them what to fix

Discussion and concluding remarks

In this paper, we presented the case of a preservice teacher's (Nadia's) initial professional development in a math content course (MICA III*) focused on programming-based math investigations, that is targeted for future teachers. The research question guiding this study is: *What elements of schemes of integrating programming in the school math classroom are developed by preservice teachers as they reflect about their own learning experiences in programming-based math investigations course?*

Our case study shows that this kind of course can contribute to the development of schemes of integrating programming in the math classroom; that is, to the initial development of professional competencies in programming-based math investigations course for future teachers. Our analysis showed that, at the start of the course and despite her previous two MICA courses, Nadia was aware of her lack of competence for integrating programming in a school math classroom. She specifically identified two missing competences: knowing a programming language and the kind of math activity appropriate for the school classroom. Additionally, our analysis showed that, throughout the course, Nadia has started developing those competences to the extent that she reports, at the end of the course, feeling "very confident" to integrate programming in a school math classroom. More specifically, we identified elements of professional schemes related to the above two competences, that Nadia developed in the course, and that are part of the three instrumental orchestration components.

By highlighting a preservice teacher initial development of IGpro for integrating programming in a math classroom, our study contributes to the research about preservice teachers' competencies development through a programming-based math investigations course approach. It also contributes to the literature on the double instrumental approach (Haspekian, 2017) by focusing on preservice teachers' IGpro where programming is the artefact and by detailing their schemes in connection with their associated components of instrumental orchestration. In fact, we argue that the instrumental orchestration organizes the teacher's overall activity of integrating programming in a math classroom.

A limitation of this study is that the data was taken from a larger research project and not gathered with this study in mind, so questions were not specifically targeted to seek insights into participants' IGpro; in particular, we do not directly look at participants' activity, but rather at their reflections on the activity. This study utilized a case study research design and focused on a math content course integrating computer programming that is targeted for future teachers. The implications of the study should not be read in terms of generalizability, but of transferability to other cases (Creswell, 2008).

Our results align with other studies (e.g., Maugesten et al., 2023) that argue there is still more to learn about the development of preservice teachers' competencies for implementing programming-based math activities in their future classrooms. Our study offers one way of understanding the development of such preservice teachers' competencies through the double instrumental genesis. Our ongoing work (e.g. St-Cyr et al., 2025) seeks to deepen our understanding of ways by which preservice teachers' development of programming-based mathematical competencies (i.e., personal instrumental genesis) can support, and is enhanced by, their development of professional competencies in this area (i.e., professional instrumental genesis).

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Exploring the impact of using silent video task on teacher noticing professional development in mathematics education

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Teacher noticing refers to the ability of teachers to attend to, interpret, and make instructional decisions based on students' mathematical understanding. This paper examines the role of silent video tasks in supporting teacher noticing in mathematics education. A silent video task is a short, dynamic mathematical task presented without audio, text, dialogue, or explanations, designed to highlight a specific mathematical concept. The study is a preliminary part of an ongoing project focused on designing teacher noticing professional development (TNPD). In this study, five in-service mathematics teachers participated in TNPD using silent video tasks focused on the concept of slope. The results show improvement in teachers' ability to attend to students' strategies and interpret their understanding, although making instructional decisions based on this noticing remains a challenge.

Keywords: Teacher noticing, silent video task, professional development, mathematics education.

Introduction

Teacher noticing (TN) has recently been highlighted as a crucial element in the understanding of professional competence (König et al., 2022). Jacobs and her colleagues (2010) defined teacher noticing as an ongoing ability to attend to the students' strategies, interpret their understanding, and decide how to respond based on their cognitive processes. TN improves teaching practice by enabling decision-making on where and how to focus attention to address classroom complexities (Blömeke et al., 2022), yet despite its critical role, much work remains in its enhancement and assessment (Sherin et al., 2011). This emphasizes the necessity for innovative approaches in teacher education and professional development that prioritize the improvement of TN skills (Amador et al., 2021). Therefore, we propose using of silent video task (SVT), comprising brief, dynamic mathematical animations devoid of audio, text, or dialogue, as a means of facilitating teacher noticing and addressing the need for an innovative method. SVT enables teachers to concentrate on a specific mathematical concept, facilitating focused engagement with students' cognitive processes and reasoning. It also encourages students to articulate, explain, and justify their ideas using mathematical language (Kristinsdóttir, 2023), thereby enabling teachers to comprehend students' mathematical reasoning. Professional development for teacher noticing provides teachers with opportunities to notice key aspects of student thinking and classroom dynamics (Jacobs et al., 2010). This study is part of a long-term ongoing project aiming to design a professional development program aimed at enhancing TN in mathematics education through the use of SVT. This paper aims to explore the impact of using silent video task on teacher noticing professional development (TNPD) in mathematics education to identify benefits, challenges and needs for improvement for designing TNPD.

State-of-the-art and theoretical framework

Teacher noticing is the process through which teachers become aware of the complexities in mathematics classrooms, including students' thinking and responses, and interpret these events within a pedagogical framework (Mason, 2021; van Es & Sherin, 2002). Rather than addressing all classroom interactions, Jacobs et al. (2010) examined teacher noticing specifically in terms of students' mathematical thinking from a cognitive-psychological perspective. The authors focused on students' mathematical thinking and expanded its conceptualization beyond attending and interpreting by incorporating instructional decision-making as a key component. Accordingly, teacher noticing consists of three interrelated dimensions: attending to students' strategies, interpreting their understanding, and deciding how to respond (Jacobs et al., 2010). Attending involves focusing on details in students' work, which provides insights into their strategies. Interpreting requires teachers to analyse students' understanding based on these observations. Finally, deciding entails making pedagogical decisions. It is important to highlight that there is no single best way to make these decisions (Jacobs et al., 2010). Jacobs et al.'s (2010) framework serves as a basis for our study.

Silent video tasks are innovative tools for capturing and analysing students' mathematical reasoning. These short, text-free animations illustrate mathematical concepts, prompting students to collaboratively generate voice-over explanations (Kristinsdóttir, 2023). SVT promotes critical thinking and deeper engagement but also presents challenges such as technological literacy requirements and classroom management. Despite these challenges, SVT provides valuable insights into students' reasoning, making it a promising tool for enhancing teacher noticing (Galiç et al., 2024). By focusing on how students think and justify mathematical ideas, teachers can better understand reasoning processes and identify misconceptions (Kristinsdóttir, 2023). The combination of video-based interventions and written responses supports teacher noticing of students' thinking.

Although research on SVT in teacher noticing is still emerging, studies suggest its potential for assessing and improving teacher noticing (Galiç et al., 2024). In this study, SVT was used into a TNPD program to enhance teacher noticing in mathematics education. Our aim is to enhance teacher noticing through the use of silent video task for in-service mathematics teachers in a professional development program. In this respect, we aimed to answer the following research questions: To what extent do in-service mathematics teachers improve teacher noticing of students' mathematical thinking in the context of slope through the use of silent video task in teacher noticing professional development program?

Method

This study employs a Design-Based Research (DBR) methodology, following Gravemeijer and Cobb (2006). DBR is an iterative process consisting of three phases: (1) preparing for the experiment, (2) experimenting in the classroom, and (3) conducting retrospective analyses. This study represents the first experimentation of the TNPD based on the literature, with subsequent retrospective analysis guiding its revision. The design of the TNPD is based on the theoretical framework of teacher noticing as proposed by Jacobs et al. (2010). Teacher noticing is a domain-specific skill that requires both pedagogical and content knowledge to enable teachers to notice students' mathematical thinking and

interpret the students' understanding behind their strategies within a specific learning domain area (Ivars et al., 2020). This study focuses on slope due to its role in mathematics curricula and its connections to advanced topics like derivatives, ratios, rates of change, and real-world applications (Stump, 1999). Given the challenges in teaching slope, including its multifaceted conceptualizations and potential for misconceptions, professional development programs are essential for enhancing teachers' ability to notice (Styers et al., 2020).

We designed silent video tasks utilizing GeoGebra, a dynamic geometry software, to enhance the understanding of mathematical concepts considered a basis for understanding slope conceptualization by Nagle and her colleagues (2019). The procedure for designing silent video tasks is outlined in the study by Galiç et al. (2024). We shared the silent video tasks with lower secondary students (8th grade, age 14 students), requiring them to provide a voice-over for the videos in order to define the mathematical aspects of the video. We named students' voice over of silent video tasks as students' videos in this study.

Before and after the TNPD process, mathematics teachers were asked to implement researcher-designed silent video tasks in their classrooms and collect video recordings of their students. Each teacher collected two students' videos from their classroom. The teachers then analysed these videos by using a reflection paper including the following questions:

(i) Attending – Please describe in detail what you think each child did in their voice-over of the given silent video task.
(ii) Interpreting – Please explain what you learned about students' understanding of the slope concept in their voice-over of the given silent video task.
(iii) Deciding – Pretend that you are the teacher of these students. Explain what problem or problems you might pose next on the basis of students' understanding in their voice-over of the given silent video task.

During TNPD, mathematics teachers were asked to individually make a voice-over for the researcher-designed silent video task. Next, they were provided with two students' videos featuring unfamiliar students and were asked to analyse these videos using a reflection paper. After completing their analysis, the teachers were required to re-make voice-over the same videos to refine their analysis. This process was repeated weekly for four weeks, ensuring a structured and iterative approach to enhancing teacher noticing. TNPD is guided by four conceptualizations of slope: geometric ratio, algebraic ratio, functional properties and calculus. Each conceptualization was determined each week.

Silent video tasks were developed using GeoGebra to represent the four slope conceptualizations (see Galiç et al., 2024) based on design principles of silent video task by Kristinsdóttir (2021). The videos were created after reviewing curricula, designing prototypes, and refining them based on expert feedback. The final videos were transcribed, analyzed, and filtered for mathematical reasoning and technical quality, essential for teacher noticing. Videos of unfamiliar students were selected to offer an authentic, unbiased opportunity for teachers to analyze student thinking. The selection process emphasized diverse student approaches and reasoning. The participants in this study were five in-service mathematics teachers with diverse teaching experiences, selected based on their varying levels of experience and backgrounds in mathematics education to ensure a rich sample.

In this paper, we analyzed their reflection on students' videos before and after TNPD in order to explore the impact of using silent video task on enhancing teacher noticing. We have found it helpful to adopt the same framework for both the design of the TNPD and analyzing data, namely Professional Noticing of Children's Mathematical Thinking by Jacobs et al. (2010). This framework has two categorizations of attending to students' strategies in silent video task. However, in our case, the third emerged as limited evidence and was added. Therefore, we analyzed the data under three classifications for each dimension of teacher noticing, namely lack of evidence, limited evidence and robust evidence. For attending to students' strategies, we coded as lack of evidence when teachers did not describe students' strategies. On the other hand, teachers' explanations that included descriptions of students' strategies without details were classified as limited evidence. When teachers presented all the mathematical concepts and details of students' strategies and correctly identified the strategy, we coded it as robust evidence. In terms of interpreting students' thinking, teachers who showed robust evidence of interpretation were able to explain the key concepts of slope in detail, using appropriate mathematical language. In contrast, teachers' interpretations that showed a lack of evidence included no or incorrect interpretations using general statements without using the conceptualization of the slope concept. When teachers correctly interpreted students' understanding of the context slope in less detail, we classified it as limited evidence. With similar categorizations of attending and interpreting, we classified deciding from general to specific pedagogical decisions. When teachers decide to respond to students with general statements without using students' thinking, we code it as lack of evidence. Attempts to ask general questions, explain students' videos to students, and ask the same question using different numbers were classified as limited evidence. If the teacher's decisions included further elaboration with rationale, asking students some questions to make them think mathematically or extending students' thinking, we coded it as robust evidence.

Results

We present the comparison of the three dimensions of teacher noticing of five in-service mathematics teachers before and after TNPD. We provide an overall summary of the results in Figure 1.

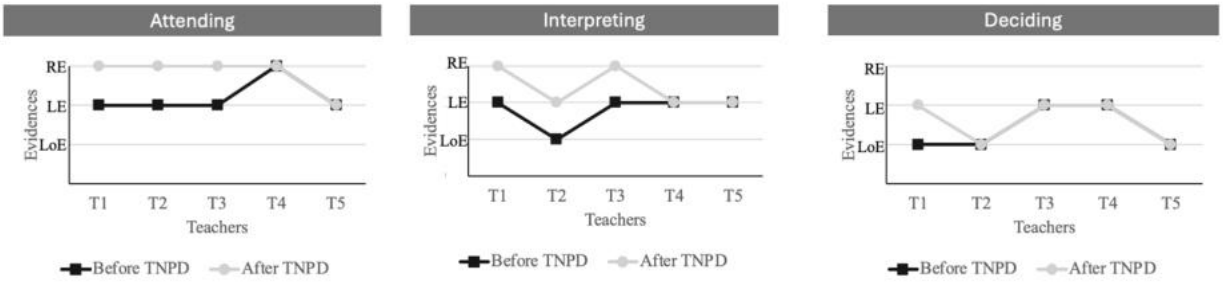

Figure 1: Teacher noticing of five in-service mathematics teachers before and after TNPD

As Figure 1 illustrates, teachers mostly demonstrated a lack of evidence or limited evidence before TNPD. However, it is encouraging to note that mathematics teachers improved their level of evidence in noticing skills at least one level. Three mathematics teachers have been demonstrating a growing ability to attend to students' strategies in the given silent video task after TNPD. In terms of interpreting students' understanding, we can see two distinct outcomes. We observed an improvement

for three mathematics teachers. On the other hand, two teachers did not show any improvement in this skill. Surprisingly, we did not observe any changing in decisions to develop students' thinking on the basis of their understanding at a robust evidence level.

We present T1's statements as an example for each dimension of teacher noticing. Table 1 illustrates T1's improvement for attending to students' strategies from limited evidence to robust evidence.

Table 1: Improvement of T1 in attending to students' strategies

Before TNPD	The students successfully related the falling speeds of the balls to the differences in slope. However, they were unable to precisely articulate the point at which the triangles overlap.
After TNPD	The students stated that the task involved a scenario related to the concept of slope. They pointed out that climbing a steeper mountain is more challenging. Although their use of mathematical terminology was not entirely accurate, their interpretations and approaches were appropriate. However, they did not thoroughly examine the details. In particular, they did not analyze the changes in slope during the ascent and descent on the ramp. Specifically, the part of the scenario in which the slope increases to a certain point, then decreases and approaches zero, was not addressed.

As can be seen in Table 1, T1 first described the students' strategies by providing limited and general evidence related to the students' explanation, such as word errors, and wrong expressions. However, she ignored the use of slope conceptualization. After TNPD, she described the students' strategies with specific evidence from the students' videos. Table 2 illustrates T1's improvement for interpreting students' understanding from limited evidence to robust evidence.

Table 2: Improvement of T1 in interpreting students' understanding

Before TNPD	The students did not relate its velocity to the steepness of the slope.
After TNPD	Although the students correctly stated that slope is the ratio of vertical change to horizontal change, they did not mention when and why the slope is positive or negative. They did not address the relationship between the right triangle and the concept of slope. This lack of understanding may stem from the real-life examples of slope commonly used in our national curriculum.

Before TNPD, T1 could not interpret the students' understanding. She could not make any claims about students' understanding or students' thinking. On the other hand, T1 interpreted students' understanding of slope with details after TNPD. For instance, she mentioned that students' understanding of the geometric ratio of slope and the characteristic of slope as positive or negative. Table 3 illustrates T1's improvement for interpreting students' understanding from lack of evidence to limited evidence.

Table 3: Improvement of T1 in deciding how to respond

Before TNPD	I realized that more concrete examples are needed for a meaningful understanding of this topic.
After TNPD	The student did not provide a detailed explanation and offered only a partial understanding of the concept. I am capable of teaching the notions of positive and negative slope.

As evidenced in Table 3, T1 was unable to determine an appropriate response to enhance students' thinking about slope, based on their existing understanding, before the implementation of the TNPD. She merely reminded the students of the missing point in the video, which did not facilitate further student thinking. When we examined T1's decision after TNPD, it is evident that she attempted to make a decision based on the students' understanding. In the earlier dimension, interpreting, she identified the students' lack of understanding of negative and positive slope. Consequently, she decided to teach it to the students. Despite her decision being based on the students' explanation, it did not it did not provide adequate support for students' thinking.

Discussion and Conclusions

The study yielded significant insights into teachers' attending, interpreting, and deciding skills. While some participants demonstrated growth in attending skills (e.g., T1, T2, T3), others, such as T4, maintained consistently high initial levels. Interpreting skills improved for certain teachers (e.g., T1 advancing from limited evidence to robust evidence, and T2 progressing from lack of evidence to limited evidence). However, deciding skills exhibited minimal change overall, except for T1. Teachers frequently struggled to distinguish among the dimensions of teacher noticing. Consistent with prior studies (e.g., Santagata & Yeh, 2016), this research highlighted the challenge of separating attending and interpreting, as these dimensions often overlap. In some cases, teachers interpreted students' understanding within the deciding dimension of noticing. To address these challenges, professional development programs should explicitly define the dimensions of noticing, offering clear guidance on what to attend to, how to interpret mathematical thinking, and how to make informed instructional decisions. Researchers might also consider examining teacher noticing through a holistic lens to allow for comparative insights and a broader understanding of the phenomenon.

Teachers need to articulate the mathematically significant aspects of students' solutions in order to assess their understanding (Jacobs et al., 2010). Although all teachers were able to focus on students' strategies, some teachers in this study found it difficult to interpret students' understanding. The findings indicated that attending to students' strategies is not enough to interpret their understanding (Jacobs et al., 2010). The ability to interpret mathematical ideas accurately may require an understanding of the subject matter (Fernandez et al., 2013). We also observed that teachers had difficulty interpreting students' understanding, particularly concerning calculus conceptualization. As a next step, we plan to interview mathematics teachers to further explore these difficulties. Additionally, we will delve into their awareness of their own noticing improvement during these interviews.

The findings suggest that while silent video tasks are effective for assessing noticing (Galiç et al., 2024), they alone are insufficient for enhancing teacher noticing without additional support. Teachers

showed some improvement in their decision-making skills during the TNPD, but this was not strongly supported by robust evidence. The "deciding" dimension seemed to be less developed compared to "attending" and "interpreting". This finding aligns with the findings of previous studies (König et al., 2022). This may be because making informed instructional decisions is more complex and requires more support and practice. In the next phase, interviews will be conducted with teachers to gather their feedback and perspectives, and updates will be made to improve the decision-making component of the TNPD. Overall, working with student videos had a greater impact on teacher noticing development than the production of voice-overs. The videos provided authentic opportunities for teachers to analyse and interpret student thinking, fostering a deeper understanding of student reasoning and refining their noticing skills.

As Amador et al. (2021) pointed out, methodological concerns may have arisen when researching teacher noticing. The framework used for analysis faced challenges in classifying responses due to a lack of evidence or clarity in teacher explanations, especially in interpreting students' mathematical understanding. This issue highlights the need for a clear definition of mathematical thinking to guide teachers in the processes of noticing for future studies, as pointed out by Ivars et al. (2020). Nevertheless, we suggest in focusing on what to notice instead of the level of evidence for teacher noticing in the future studies. In the next cycle of DBR (Gravemeijer & Cobb, 2006), we will first determine students' mathematical thinking and go deeper into what mathematics teachers notice to improve TNPD.

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Development of self-reported technological pedagogical content knowledge (TPACK) during the preparatory service of prospective mathematics teachers in Germany

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In this study the self-assessed technological pedagogical content knowledge (TPACK) growth of 91 prospective mathematics teachers during the 2017-2018 preparatory service in Baden-Württemberg, Germany, was surveyed. In addition, data about hypothetically beneficial factors of the preparatory service were collected and their influence on knowledge acquisition was analysed using multiple regression analysis. Over the study period, a small to medium effect was found for the increase in knowledge. Multiple regression analysis revealed that the number of seminar sessions about technology integration in the mathematics classroom during the preparatory service and the extent of digital media use in teaching as significant factors influencing the TPACK acquisition of the prospective teachers.

Keywords: Teacher competencies, technological pedagogical content knowledge, teacher induction, workplace learning.

Theoretical background and research questions

The integration of digital technologies into mathematics education presents a significant challenge for prospective mathematics teachers. Technological pedagogical content knowledge (TPACK; Mishra & Koehler, 2006) is considered essential for effective classroom technology implementation, encompassing knowledge about effectively teaching with technology in ways that are appropriate for specific subject matter and pedagogical approaches.

In Germany, the "Vorbereitungsdienst" (preparatory service) bridges theoretical knowledge learned at university and teaching practice (Kultusministerium Baden-Württemberg, 2014b). This 18–24-month induction program follows university education. During the preparatory service, prospective teachers are supported in two main ways: through mentors at training schools and by trainers in state seminars for teacher education. At schools, they progress from observation to supervised and independent teaching. Mentors provide feedback and guidance. Simultaneously, at the state seminars they attend seminar sessions on pedagogy and didactics.

Research on the development of (not necessarily technology-related) professional knowledge during this phase has shown mixed results. Some studies suggest the preparatory service is effective in enhancing professional competence (e.g., van Driel et al., 2002), while others report little to no improvement (e.g., Kirschner et al., 2017). Alongside evaluation studies, large-scale surveys underscore the importance of the preparatory service in developing technology-related teaching skills.

The Bund-Länder-Indikator 2016, a series of studies assessing digital education across Germany's federal states, reported that over 80% of in-service secondary school teachers felt that the preparatory service should include better training on using digital media in teaching (Eickelmann et al., 2016). This highlights a perceived gap in technology-related preparation during this phase of teacher education. Similarly, the International Computer and Information Literacy Study (ICILS) 2018 study emphasized preparatory service's role in fostering prospective teachers' digital competencies. Younger teachers (aged up to 39) were more likely to acknowledge that the preparatory service had encouraged them to think about how digital media influences their teaching methods (Drossel et al., 2019). This suggests that recent changes in the implementation of the preparatory service may positively impact TPACK development.

However, questions remain about which specific aspects of the preparatory service support TPACK acquisition. To understand how technology integration in mathematics education is addressed, various curricular documents must be examined. In the German context relevant resources include: Reference frameworks for media education, such as the European Framework for the Digital Competence of Educators (Redecker, 2017) and the strategy paper of the Conference of the Ministers of Education and Cultural Affairs for education in a digital world (Kultusministerkonferenz, 2016); training standards for teachers, either federal or state-specific (e.g., Kultusministerium Baden-Württemberg, 2014b, 2016; Kultusministerkonferenz, 2012); and media education plans or seminar curricula (e.g., Kultusministerium Baden-Württemberg, 2014a, 2018).

An analysis conducted by the authors on preparatory service curricula uncovered a substantial gap in fostering technology-related teaching competencies for mathematics education. No explicit curricular components addressing subject-specific technology integration. The media education plans and seminar curricula examined contain only vague literature references about media use in mathematics classrooms, offering minimal practical guidance. Due to space limitations, the analysis cannot be presented in detail here.

On a broader scope, research on TPACK has identified several factors that are beneficial for professional development. Based on prior studies, the following learning opportunities (Lohse-Bossenz & Vogel, 2021) are considered relevant for fostering TPACK during teacher education

- Discussion of mathematics-specific digital tools and media in formal seminar sessions (Ning et al., 2022; Wang et al., 2018)
- Practical use of digital tools and gaining firsthand experience by incorporating them into one's own teaching (Li et al., 2022; Mueller et al., 2008; Tondeur et al., 2017)
- Mentor-guided discussions about technology use in mathematics instruction, allowing mentees to share experiences and receive targeted feedback (Ning et al., 2022; Tondeur et al., 2017; Zhang & Tang, 2021)
- Collaborative reflection on teaching with digital media, enabling teachers to critically assess and improve their practices (Harris, 2016; Zhang & Tang, 2021)
- Basic technological knowledge (TK) at the start of training, which serves as a foundation for developing TPACK, though the extent of its influence remains debated when other TPACK components are considered (Li & Li, 2024; Li et al., 2022; Schmidt et al., 2009)

Given the current state of research and the lack of explicit curricular elements targeting subject-specific technology integration within the German teacher induction system, this study makes a unique contribution by conducting a detailed, longitudinal assessment of TPACK growth during the preparatory service and by systematically examining the relative impact of various learning opportunities on technology-related knowledge development during the critical preparatory service phase. The research questions guiding this investigation are:

RQ1: Does the preparatory service of prospective mathematics teachers lead to knowledge acquisition in the technological pedagogical content knowledge (TPACK)?

RQ2: Which factors within the preparatory service significantly contributes to explaining the variance in the knowledge acquisition?

Methods

The study utilized a longitudinal design with two measurement points over the course of the 2017-2018 preparatory service in the state of Baden-Württemberg, Germany. The sample included 91 prospective mathematics teachers preparing for the (non-gymnasium) secondary school track. During their preparatory service, participants completed a self-assessment questionnaire at both measurement points to evaluate their TPACK. The questionnaire contained subscales for technological knowledge (TK) based on Schmidt et al. (2009), and technological pedagogical content knowledge (TPACK) based on Handal et al. (2012). Both instruments were adapted for the German language by the authors.

The TK subscale consists of seven items that assess general technological skills knowledge rather than specific technologies. They cover aspects such as the ability to solve technical problems, the ease of learning new technologies or past opportunities to work with different technologies. Responses are rated on a 5-point Likert scale, ranging from “Strongly Disagree” to “Strongly Agree”.

The TPACK subscale for mathematics teachers includes seven items adapted from Handal et al. (2012). They are also rated on a 5-point Likert scale, ranging from “Strongly Disagree” to “Strongly Agree”. Topics focus on the integration of technology, pedagogy, and mathematics content knowledge and cover aspects such as using technology to assist students in developing math problem-solving skills, representing math problems linking symbolic, numerical, and graphical data or incorporating authentic tasks into math lessons.

At the second time point, participants additionally reported on factors hypothesized to influence TPACK development. These factors aimed to explore the potential mechanisms and opportunities contributing to the acquisition of TPACK throughout the preparatory service:

- *Seminar sessions on technology integration:* Participants reported the number of seminar sessions that addressed the use of various digital tools in mathematics teaching, including computer algebra systems, dynamic geometry software, graphing calculators, and general computer/tablet use. Response options ranged from “None/Never” to “5 or more” sessions.
- *Collaborative case analysis:* Participants indicated whether they engaged in analysis of their own concrete teaching examples during seminar sessions.

- *Mentor discussions:* Participants reported the frequency of discussions with their mentors about using digital media in mathematics lessons. This was assessed for different types of technology, with response options ranging from “None/Never” to “5 or more” discussions.
- *Teaching experience with digital media*: Participants estimated the typical duration of technology use in their own teaching, reported in minutes for a standard 180-minute teaching unit. This was collected for various types of digital tools and software. This item was taken from the MiU project (Ostermann et al., 2019) and adapted for this study.
- *Initial technological knowledge:* The TK subscale from the first measurement point was used to assess participants' general technological knowledge at the start of the induction year.

To evaluate the influence of hypothesized factors on TPACK growth, a score was aggregated for each factor. The frequency of both seminar sessions and mentor discussions related to digital tools was coded on a scale from 0 to 5. A sum score was then calculated for different types of digital tools discussed. The occurrence of collaborative case analysis was coded dichotomously (0 = no occurrence, 1 = occurrence) based on whether participants engaged in collaborative analysis of cases involving technology use. To acquire a score for the teaching experience with digital media per participant, the specific technology was identified which was used the most. The duration of media use was quantified as the proportion of its use within a 180-minute unit. For the TK score as well as the TPACK scores at both measurement points a mean score was calculated across all items.

Multiple linear regression analyses were conducted to determine the impact of these factors on TPACK growth while controlling for baseline TPACK scores. The initial regression model included the TPACK score at the first measurement point as a baseline predictor along with all five hypothesised factors. A backward selection approach was used to identify relevant factors. Predictors were removed step-by-step based on statistical significance. At each step, model comparisons ensured that removing a predictor did not significantly reduce the model’s explanatory power. To address missing data, analyses were conducted across 100 imputed datasets. The pooled results from these datasets were used for both model selection and reporting.

The following analysis reveals the outcomes of these methodological approaches, providing insights into TPACK development among prospective mathematics teachers during their preparatory service.

Results

This study involved $n = 91$ prospective teachers from six state seminars, with between 1 and 31 participants responding to the questionnaire at each seminar. Addressing RQ1, the analysis of the self-assessed TPACK scores showed a statistically significant increase from $M = 2.88$ ($SD = 0.80$, $\alpha = .90$) at the first measurement point to $M = 3.20$ ($SD = 0.77$, $\alpha = .90$) one year later, $t(77.096) = 3.765$, $p < .001$. This represents a small to medium effect size (Cohen's $d = 0.45$).

Participants reported attending a median of 2 seminar sessions ($M = 2.38$, $SD = 2.75$) on technology integration in the mathematics classroom. They also reported having a median of 2 discussions ($M = 3.10$, $SD = 3.99$) with their mentors on the same topic. The most frequently mentioned technology in seminar sessions was dynamic geometry software (DGS), reported by 52 participants. In contrast, graphical calculators and scientific calculators were the least mentioned, with only 6 and

22 participants, respectively, noting sessions involving these technologies. However, scientific calculators were the most frequently discussed technology in mentor discussions, with 50 participants reporting at least one discussion on the topic. In their own classrooms, the prospective teachers reported using digital media for an average of 36.20 minutes per session (SD = 35.99 minutes).

Addressing RQ2, the baseline model, which included only the TPACK score at the first measurement point, explained $R^2 = 24.93$ % of the variance. At the beginning of the backwards selection process a model with all predictors was computed. It explained an additional $\Delta R^2 = 18.80$ % of the variance beyond baseline prediction. During the selection process, predictors that did not significantly contribute to the model were removed one by one. First, the frequency of mentor discussions was excluded ($\beta = -0.008$, $p = 0.730$). Next, the collaborative case analysis variable was removed ($\beta = 0.149$, $p = 0.431$), followed by the initial TK score ($\beta = 0.106$, $p = 0.304$). At each step, comparisons confirmed that removing these variables did not significantly reduce the model's ability to explain variance ($p > 0.282$ for all comparisons).

The final model incorporated the initial TPACK score ($\alpha = 0.426$, $p < 0.001$) and two significant predictors: seminar session frequency ($\beta = 0.082$, $p = 0.001$) and digital media teaching experience ($\beta = 1.032$, $p = 0.004$). The model explained 39.70% of variance in TPACK, with additional predictors contributing 14.77 percentage points beyond the baseline score.

Discussion

The findings of this study provide insights into the development of TPACK among prospective mathematics teachers during their preparatory service. The small to medium effect observed in self-reported TPACK growth suggests that the preparatory service does contribute to the development of technology-related competencies, albeit to a modest degree. A detailed survey measured various aspects of the induction experience, such as formal learning sessions, informal discussions, and hands-on teaching practice, to understand their influence on TPACK development.

Using a backward selection approach, the study identified two key factors that significantly affected TPACK growth: the frequency of technology-focused sessions in seminars and the extent of digital media use in the prospective teachers' own teaching These findings emphasize the importance of combining theoretical training with practical application to foster TPACK acquisition.

The impact of technology-focused seminar sessions highlights the value of structured learning opportunities in teacher training. This suggests that the preparatory service should allocate time and resources specifically to teaching how to integrate technology into the mathematics classroom. Additionally, the positive impact of prospective teachers using digital tools in their own teaching demonstrates the importance of learning through reflective practice. Offering supervised opportunities for technology use in real classrooms is essential for developing TPACK.

Interestingly, factors such as collaborative case analysis, mentor discussions, and initial technological knowledge did not significantly predict TPACK development when controlling for other factors. This unexpected result suggests a need to reevaluate how these elements are implemented in the preparatory service or how they were measured in this study. For instance, the assessment of case analysis did not focus explicitly on technology-related reflections, which may have limited its

relevance to TPACK development. General case work might not be specific enough to explain TPACK outcomes. Although TK scores are often identified as relevant factors in other studies, they did not show significant effects here. Further analysis revealed no extreme low or high scores (floor or ceiling effects) and no significant influence on TPACK growth when TK was the sole predictor. Recent studies also questioned the relevance of TK in TPACK development (Li & Li, 2024; Li et al., 2022). Mentor discussions, while theoretically valuable, may not have stood out due to variability in mentoring quality and appears to be masked by other factors. In many schools, mentor selection is often delegated to school leadership, who may prioritize factors such as general teaching experience or workload considerations over specific qualifications for (technology-related) mentoring.

The small sample of 91 participants from a single German state limits generalizability. Additionally, since the participants were drawn from different state seminars, variations between these groups could have influenced the findings. Variations between state seminar groups couldn't be properly assessed due to sample size. This implies a need for caution when applying these findings to broader contexts. The study relied on self-reported indicators for TPACK, which, while well-established as predictors of teaching practice, do not correlate perfectly with objective measures of knowledge. This limitation suggests that the study might not have captured the full spectrum of TPACK development, particularly with respect to objective and demonstrable competencies.

The study highlights the need to balance theoretical training with practical opportunities in teacher preparation programs. However, the limited variance explained by the model suggests that other factors, not examined here, may also influence TPACK growth. Future research should explore additional variables that might influence TPACK growth. Longitudinal studies extending beyond the induction year could give further inside which factors support TPACK acquisition in the long run.

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From "anyway using many representations" to targeted practices: A teacher's professional learning in an asynchronous online PD

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Asynchronous online teacher professional development (PD) courses have a long tradition. However, only a few studies have investigated the content-related learning pathways that such courses can initiate. We present a case study from the larger PD design research project Mastering Math, which follows a teacher's professional learning pathway initiated within a PD module. The content of the PD module connects representations to deepen students' understanding of subtraction; with the case study, we can show that professional learning of representations is also possible based on focused asynchronous PD activities.

Keywords: *Online professional development, multiple representations, professional learning*

Introduction

Although multiple representations are widely used in many mathematics classrooms, teachers' practices in dealing with them have been shown not to be always sufficiently targeted to profoundly enhance students' conceptual understanding of concepts and operations (Dreher & Kuntze, 2015; Prediger & Wischgoll, 2023). Therefore, there is a need for scaled professional development.

Asynchronous online teacher professional development (oTPD) courses have been developed and promoted for two decades in order to scale up PD offers and to make them more flexible, for example, in rural areas (Bragg et al., 2021; Dede et al., 2009). However, there is still a lack of research that captures the content-specific professional learning pathways that have been initiated, particularly for oTPD in mathematics education (Friesen et al., 2023). In this paper, we present a PD design research study of an oTPD module that addresses multiple representations, which aims to answer the following design research question: *What professional learning pathway can be initiated by an oTPD module designed to promote teachers' practice with multiple representations?*

Theoretical background

Research gap on teachers' content-specific professional learning in oTPD

Within the last 20 years, research on oTPD designs has primarily focused on reducing dropout and enhancing teachers' cognitive and communicative engagement in content-independent research perspectives. For example, studies have demonstrated the potential of combining inputs with PD activities using interactive features in closed or open formats (reviewed by Bragg et al., 2021). While many studies have shown the overall effectiveness of oTPD, the authors of the review call for disentangling the conditions and effects of using particular design elements differently and tracing the professional learning pathways in more detail (Bragg et al., 2021; Seago & Knotts, 2021). Dede et al. (2009) and Friesen et al. (2023) also call for more content-specific research into the different PD contents and the content-specific professional learning pathways of individual teachers.

Teachers' content-specific learning pathways can be investigated with respect to changes in teachers' simulated practices in carefully designed PD activities, with vignettes designed to simulate particular content-specific instructional demands (Dreher & Kuntze, 2015). Simulated practices can be captured by teachers' suggestions about what they perceive and how they would react to the situational demands simulated in the vignette. As the extensive research on teacher noticing has shown, what teachers perceive and how they react is shaped by their overall orientations (e.g., the more graphic representations are used, the better) and by the categories that they implicitly or explicitly apply to their noticing and decision making (Sherin, 2014). For the practices in view of our PD project, the unpacking of learning goals into sub-goals serves as an important diagnostic category. Teachers who clearly articulate learning sub-goals tend to address enhancement practices that are more specifically targeted at carefully chosen sub-goals (Prediger et al., 2023).

Teachers' practices for dealing with multiple representations

For enhancing students' conceptual understanding of mathematical concepts and operations (such as subtraction), the use of multiple representations (manipulatives, graphical, verbal, and symbolic) is widely promoted (Lesh, 1979). However, the use of multiple representations does not always lead to deeper conceptual understanding, only when the representations are explicitly connected, with an explanation of where to find the relevant elements and structures that represent the operation (ibid.). In our German context, almost all teachers appreciate the work with multiple representations but have heterogeneous practices in its implementation: Dreher and Kuntze (2015) investigated teachers' underlying views and noticed in dealing with multiple representations and found evidence that many German teachers are unaware of typical challenges in dealing with multiple representations. For the case of multiplication, Prediger & Wischgoll (2023) showed that many teachers do not know the underlying structures and rarely use the diagnostic category of surface translations (that not only the two numbers need to be found in both representations) for their noticing, more concise learning goals were not addressed as topic-specific diagnostic categories. In order to enact enhancement practices that leverage students' ability to explicate the underlying structures and to connect representations, teachers' categories for unpacked learning sub-goals should also include the typical *models for the operation*, such as counting in units for multiplication (Prediger & Wischgoll, 2023). For subtraction, the two crucial models are *taking away* and *determining the distance* (van den Heuvel-Panhuizen & Treffers, 2009). In summary, the relevant categories for enhancing students' understanding by dealing with multiple representations include whether the student translates only the involved numbers or also the underlying structure modelled by the operation and the particular models for the content in view of this paper, for subtraction *taking away* and *determining the distance*.

Design of the overall PD program and the oTPD module in view of this paper

Research context of the Mastering Math program

Since 2010, formative assessments, teaching materials and PD materials have been developed for the program Mastering Math so that teachers can foster their at-risk students' conceptual understanding of basic arithmetic concepts such as place value understanding, the meaning of addition/subtraction, multiplication/division, etc. in conceptually focused foundation lessons (Prediger et al., 2023). In a two-year PD program, teachers experiment with the materials and work in twelve PD modules,

consistently targeting practices for three situational demands, *specifying learning contents/goals*, *monitoring/noticing students' learning progress* and *enhancing students' understanding* (of basic concepts). In the third PD, in view of this paper, the three demands are addressed in terms of dealing with multiple representations for subtraction, which has already been addressed in the previous modules, but for the basic concepts of understanding place value understanding and the number line. As the Mastering Math program (with face-to-face PD modules conducted by facilitators) has shown overall effectiveness for students' mathematical learning and teachers' professional learning (Prediger et al., 2023), we can now investigate more focused aspects for the scaling-up process with oTPD.

Design of the oTPD module on work with multiple representations for subtraction

As there are too few facilitators available to conduct face-to-face PD modules, we have started to develop asynchronous oTPD modules for scaling up (Tester et al., 2024; so far, four modules). Participating volunteer teachers can work through the oTPD modules and experiment with the teaching materials in between. Each three-hour PD module consists of a number of thematic blocks, each of which addresses one or two of the situational. Figure 1 documents the design of Blocks 1 and 4 (out of 12) of the oTPD that the teacher in our case study worked on, with the content and formats. Inputs (with video sequences or slides) and PD activities are combined with different formats. Module 3 deals with typical challenges and specifying learning goals for representations of subtraction. PD activities use closed formats (such as multiple choice or drag-and-drop) and open formats (such as reflection questions, e.g., the teachers have to identify typical challenges or suggest enhancement practices), partly based on written vignettes or classroom videos (Seago & Knotts, 2021). Figure 2 shows further activities and inputs of later blocks.

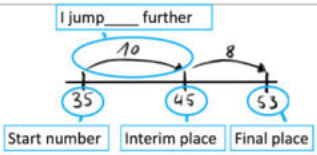
Relevant steps in the trajectory of the oTPD Module 3 on dealing with representations for subtraction (key slides or questions)	**Formats of oTPD elements in these blocks (each block combines activities and inputs)**
Monitor students' learning progress / enhance students' understanding	***(Sequence of inductive-deductive blocks)***
Block 6/7: Discussing conditions for the match of representations **b)** Which representations do not match to the story? (Subtraction) ... **e)** Analyse how the teacher works with the students. **f)** Plan your next prompt to make the students aware that the two numbers cannot be represented independently. 	**a)** Introduction to the task and challenge in the video case ("Why does this representation fit? How can the minus be represented within the representations?") **b)** Drag-and-drop activity to monitor incorrect representations **c)** Text input about representations that match **d)** Text input about learning needs in conditions for matching of representations (exemplified for representing 305-30 with base-10 blocks) **e)** Video of an enhancement teacher and student dialogue **f)** Activity: Preparing oral prompts to support students
Block 8: Supporting the connection of representations a) What do students need to articulate for explaining why these representations match? How do you plan to support the students' explanation? 	**a)** Activity: Preparation of oral prompts to elicit students' explanations of connections between representations **b)** Text input on possible prompts with for explanations and underlying structures
Block 9: Language support for articulating meanings Extracts from a poster with language scaffolds 	**a)** Input by expert video about phrases needed to articulate meanings and connections **b)** Cloze text activities: "Which phrases can support the articulation of meanings?" **c)** Text input about language such as "What number is searched?"

Figure 1: Design of the oTPD module: Example inputs and PD activities about subtraction

Methods of the case study

The overarching design research project: Given the dual aims of developing an oTPD and investigating professional learning processes, the overarching project adopted a design research approach to teacher PD, in which designs and the qualitative investigation of their outcomes are iteratively refined, so far in three cycles of design experiments (Bragg et al., 2021; Prediger, 2019). In the empirical section of this paper, we present a qualitative case study of one teacher's professional learning pathways, as outlined in the design presented above. Similar intended learning pathways have been described for teachers in face-to-face teacher professional development programs (Prediger et al., 2023), but rarely for asynchronous teacher professional development programs (Tester et al., 2024). The case is from Cycle 2, in which 20 teachers participated.

Methods of data collection for the case study in this paper: As recommended by Dede et al. (2009), our methods of data collection "make use of the data streams collected in technology-mediated interactions… [when and] to what ... they respond, in what order they solve problems, and their trajectory of learning" (Dede et al., 2009, p. 15). The data were collected in the Moodle learning management system, where the oTPD modules of the Mastering Math program (Prediger et al., 2023) are implemented. In addition to teachers' responses and self-corrections, Moodle tracks the chronology of work with timestamps. The teachers' professional learning gains are assessed by capturing their simulated practices in parallel open-ended diagnostic vignettes in situated ways (Friesen et al., 2023), here with Block 1 a) from Figure 1 and a parallel vignette in Block 12. For this paper, we chose the case of the pseudonym Helen, a secondary school teacher in North Rhine-Westphalia, who attended a synchronous kick-off meeting with her colleague at her school to receive a general presentation of the Mastering Math program. Helen's example illustrates how the oTPD can enhance professional learning.

Methods for qualitative data analysis: *Teachers' responses* within the oTPD activities were qualitatively analysed regarding teachers' simulated practices and the underlying content-specific categories for perceiving and thinking (methodologically based on the framework for content-related teacher expertise; Prediger, 2019). These categories include, for example, goal categories such as meaning-related language in "Which number is the whole? Which number we seek?" and appropriate representations. For productive enhancement practices, categories for reaching deep connections might include, for example, seeing the mathematical structures in the materials. To analyse the process over time, the time stamps were analysed in relation to the content of the blocks and accomplished activities. By comparing the categories and processes over time, we intend to investigate the teacher’s professional growth within the oTPD.

Empirical insights into Helen′s professional learning pathway

Time structure and chronology of Helen's work: After the kick-off meeting (October, 20) of the PD program, Helen worked within Module 2 on number lines (November, 1, 2.01-2.42 pm; November, 7, 9.28-9.29 pm) and looked at the beginning of Module 3 (meanings of addition and subtraction: December, 14, 9.55 pm). Helen worked within Module 3 while skipping the activities marked as complimentary (January, 4, at 9.24-10.26 pm). Helen completed the open-ended questions when they were relevant to the certificate (vignettes) and the activities saved for the school practice

(here, stimuli and learning goals). Helen looked again at Block 3 on models for addition/subtraction (January, 18, 10.26-10.27 pm), where she had skipped the activity, and then at Block 1 with the pre-vignette. In summary, Helen first previewed the introduction, later worked through the module linearly without the complimentary activities and later looked back at the overviews of the models and the pre-vignette.

The starting point of Helen′s professional learning pathway in Block 1: At the beginning of Block 1, Helen was asked to analyse Finn′s translation of the symbolic expression 35– 19 into a graphic representation with two unconnected amounts of money, 35 € and 19 € (see Figure 1). Finn′s drawing exemplifies a typical student challenge, as it represents both numbers (subtrahend and minuend) separately, without explication of a subtractive structure, neither *taking away* nor *determining the distance*. Teachers are invited to identify this challenge when analysing Finn′s work and to suggest a targeted approach to overcoming it, for example by starting a conversation about what it means to represent an operation graphically. Helen's answer to the pre-vignette was as follows:

> Helen (Block 1a): "I would ask him to represent the subtraction also with (drawn) base-10 blocks, with ten rods and unit cubes. So, we could talk again about the total quantity. Subsequently, I would make him draw the total quantity on the number line so that he can conduct the calculation on the number line."

Helen did not make explicit the main challenges that she saw in Finn′s drawing, so we have to infer her diagnostic categories from the suggested enhancement practice. The main strategy of her enhancement practice was to change the representations: She proposed two more different representations, base-10 blocks and the number line. On the number line, she intended to solve arithmetic problems, which might refer to a perceived challenge of decomposing a ten into 35 before removing 19 from 35. Helen may have planned the base ten blocks to address this decomposition problem but without articulating it explicitly. Instead, she wrote about her intention to talk about the "total quantity" ("Grundmenge" in the original German), a technical term for interpreting the minuend in sets. While she did not address the main challenge of representing the subtrahend with its own money drawing, she may have alluded to the implicit idea that Finn needed to learn to represent a subtraction by explicating the taking away structures: the minuend should represent the whole (the "total quantity") and the minuend a subset that is taken away, but she did not articulate these plans explicitly. These observations are consistent with the teacher practices identified for multiplication (Prediger & Wischgoll, 2023): Many teachers had some key ideas of the model in mind but rarely explained that students should explicitly articulate how the mathematical structure that is modelled by the symbolic expression is represented graphically. Rather than planning for these deeper learning opportunities, teachers suggested changing the representation (again, without an explicit plan for targeted enhancement practices with this new representation).

Advanced point of Helen′s professional learning pathway in Block 12: Post-vignette: The post-vignette in Block 12 was designed to run in parallel with the pre-vignette, with a more explicit invitation to formulate prompts that may be useful later in the class (which are automatically integrated into the teacher′s personalised planning document). Helen′s answers to the post-vignette:

> Helen (Block 12: "I would ask him what we have to do in general (taking away vs. add). Then I would ask him from which number we have to take away something, and, I would say, that it is not possible to find everything that he drew in the second

> wallet in the first. In this way, he would have to 'decompose' the notes by himself. Subsequently, I would ask him, how he can draw that we can take something away. Maybe we can cross out/circle better?"

Helen′s suggested prompt: "Mark in the left wallet, what you have to take away!" The analysis shows that after working through the oTPD module, Helen explicated the need to talk about the taking away structure that is modelled by a subtraction. Instead of talking about total quantities in two further representations, she addressed the targeted ideas by leveraging Finn′s resources. While she still felt the need to address the calculation obstacle of decomposing tens for the subtraction with carry, her main attention went to the key idea: When representing an operation, the two numbers should not be addressed separately, but the action of taking away (modelling subtraction) should be represented. For this much more targeted learning goal, she provided graphical scaffolds (by suggesting a way of to draw the action) and conceptual scaffolds in her suggested prompt. In this way, her simulated practices became much more targeted to the student′s detailed learning needs. The targeted practice drew upon new categories explicated in her writing: the model taking away and the explicit requirements for drawings to represent not only the numbers separately, but also the structure.

Helen′s professional learning pathway between pre-vignette and post-vignette: Helen′s simulated practices changed considerably from pre- to the post-vignette in terms of making implicit ideas and targeted learning sub-goals explicit. What happened in between? Already the activity that followed the pre-vignette in Block 1b (printed in Figure 1) provided a learning opportunity as multiple choices of interpretations provided categories for articulating interpretations more concisely. Helen chose the correct one. In the drag-and-drop activity in Block 4, Helen wanted to express the complementary relationship to addition which is important, yet not in the simulated situation. Figure 2 depicts some of the further professional learning opportunities in Blocks 6 to 9 that focus on monitoring and enhancing students′ ability to explain how representations are connected.

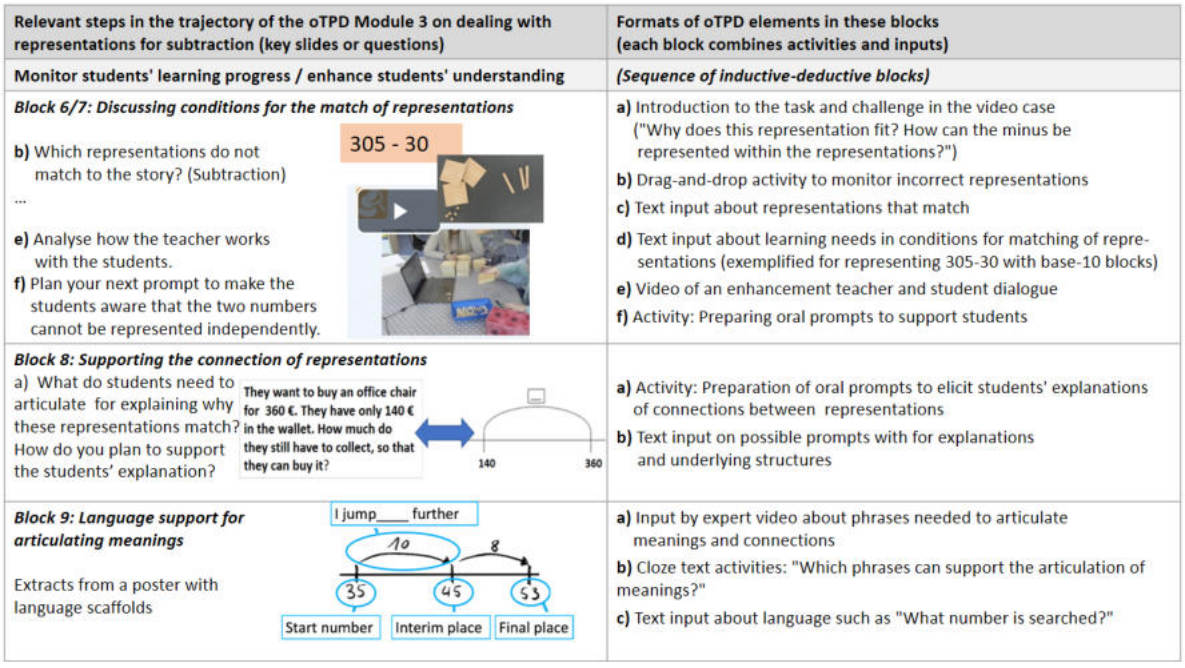


Figure 2: Further professional learning opportunities in between connecting representations

In Block 6, Helen correctly identified the problematic representation of both numbers, indicating that the learning in Blocks 1b and 4 was having an effect. When asked to prepare oral prompts for the students in the video case (Block 7), Helen focused on decomposing the hundreds rather than the key ideas about connecting representations. In Block 8, she correctly identified the model of determining the distance and unfolded it for the students as in the videos and texts given earlier. In Block 9, she chose appropriate language scaffolds to support students′ articulation of meanings. In this way, she exploited the professional learning opportunities, even though there is still room for improvement their design for clarity.

Summary of Helen′s professional learning pathway and implications

The analysis of Helen's case shows that, although her work was fragmented over several sessions and took unexpected turns, she was following an interesting professional learning pathway. The practices she articulated in the pre-vignette were guided by superficial categories and not yet articulated with explicitly targeted sub-goals of conceptual understanding. In particular, general requirements for connecting representations or the concrete model of *taking away* were not addressed (similar to the findings of Prediger & Wischgoll, 2023). In her intermediate activities, she was distracted by additional ideas that were not core to the tasks (e.g., decomposing bundles of tens). First, she used "total quantity" in her formulations and also expressed it to the students as "amount that is still missing". In this way, she unconsciously jumped between models. In the post-vignette, she addressed the need to graphically represent also the structure of *taking away*, not just individual numbers. In a later oTPD activity, she also elicited explanations of connections from the students that were still missing in the post-vignette. Overall, we can see a professional learning pathway characterised by successive unpacking of learning goals and successive targeted prompts.

In this single case study, we see the first modest indications that the rich open-ended activities that ask teachers to articulate their initial ideas can be successfully combined with more focused multiple-choice formats that introduce new ideas or categories for learning goals in a cognitively activating way with immediate feedback (Seago & Knotts, 2021). Although Helen did not make immediate use of all the inputs and activities, she did pick up on goal categories offered (e.g., appropriate connecting representations). Helen took advantage of the opportunity to work time independently in the evening and returned to relevant input on the most important learning goals. However, there is also the obstacle of going to school in between to use the material for diagnostic judgements or to enhance students. While the case study of Helen′s pathway suggests that it is possible to trace professional learning pathways in a qualitative way and that it is fruitful to combine data on the content and the time to understand more about how teachers can learn with oTPD, a single case should not be over-interpreted in terms of generalizability. Further research is needed to investigate more cases and examine the design elements individually, as suggested by Bragg et al. (2021).

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Development of the disciplinary identity of the mathematics teacher: A professional development proposal

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Socio-epistemological theory defines the notion of mathematical school discourse as a system of reasoning that governs the practices, actions, and decisions of the actors within the educational system. Teachers being immersed in this system of reasoning, accept certain established practices as predominant paths that are difficult to question. From the Soltsa socio-epistemological program, it has been pointed out that this is one of the reasons why a distance is established between school mathematics and the realities of both students and teachers. In this context, the program seeks to propose the development of the disciplinary identity of mathematics teachers by strengthening three aspects of their practice: resistance, legitimation, and project. The objective of this paper is to discuss a professional development proposal for mathematics teachers, which is grounded in the development of their disciplinary identity.

Keywords: Teacher professional development, socio-epistemology approach, functional Mathematics.

Research problems and some theoretical aspects

The report presented here describes an ongoing doctoral research project in the Department of Mathematics Education at Cinvestav, Mexico. The theoretical framework considered is the Socio-epistemological Theory of Mathematics Education (STME) (Cantoral, 2016; 2019). This scientific theory within the field of Mathematics Education falls under the sociocultural paradigm of the discipline and studies the processes of social construction of mathematical knowledge and its institutional diffusion. It considers that mathematical knowledge is constructed in diverse social contexts, whether in academic knowledge (scholarly), technical (professional), or popular (everyday life), as a socially produced entity with context and heritage. The theory suggests that incorporating the sociocultural dimension of mathematical knowledge into didactic interventions could be beneficial for mathematics education in schools.

Specifically, this research is immersed in the socio-epistemological program known as *forgotten subject and the transversality of knowledge* (Soltsa for acronym in Spanish) (Cordero et al., 2022; Cordero, 2023). This program aims to recognize the functional uses of mathematical knowledge within various communities by documenting their relationship with the peculiarities of their everyday lives. The goal is to acknowledge the plurality of mathematical knowledge, that is, to give epistemological validity to different ways of doing mathematics. In doing so, the objective is to influence institutionalized school mathematics and its teaching and learning practices.

Within the framework of the Socio-epistemological Theory of Mathematics Education (STME), the theoretical notion of *mathematical school discourse* (dME for acronym in Spanish) has been developed, characterizing it as a system of reasoning (Bourdieu, 1997 in Soto & Cantoral, 2014) that

governs the practices, actions, and decisions of individuals, as well as the regulations that shape the school system around the teaching and learning of mathematics. A system of reasoning shapes a social structure that sets boundaries on what is considered acceptable and what is not within that system. Bourdieu suggests that our actions and decisions are influenced by social and cultural dispositions we have acquired throughout our lives. These dispositions include ways of thinking, perceiving the world, and acting, which are socially constructed and often operate unconsciously. In the educational sphere, the system of reasoning can determine which knowledge is valued and how students' skills are assessed, thereby influencing their success or failure within that system.

In this sense, the dME hegemonically imposes certain meanings, procedures, and arguments associated with what is conceived as doing mathematics. One of the characteristics of dME is that it prioritizes the abstract, formal, and procedural treatment of pre-established concepts of the curricula.

The dME produces certain phenomena, one of which is *adhesion* (Cordero et al., 2015). This notion of *adhesion* refers to the process by which the normative nature of the dME leads to the internalization and reproduction of its traits as norms that are rarely questioned. This dynamic establishes a framework for the teaching and learning of mathematics, where teachers, operating within institutional and cultural constraints, adopt and replicate these practices as part of their professional role. In this way, the dME perpetuates itself, not through individual choice, but as a result of the broader structure in which educators and students are embedded. This results in certain knowledge being accepted as truth, certain procedures and arguments being favored over others, and the lack of recognition of mathematical knowledge constructed in other contexts than school (Cordero et al., 2015). Figure 1 presents a schematic representation of the structure of the dME, reconstructed based on Soto and Cantoral (2014).

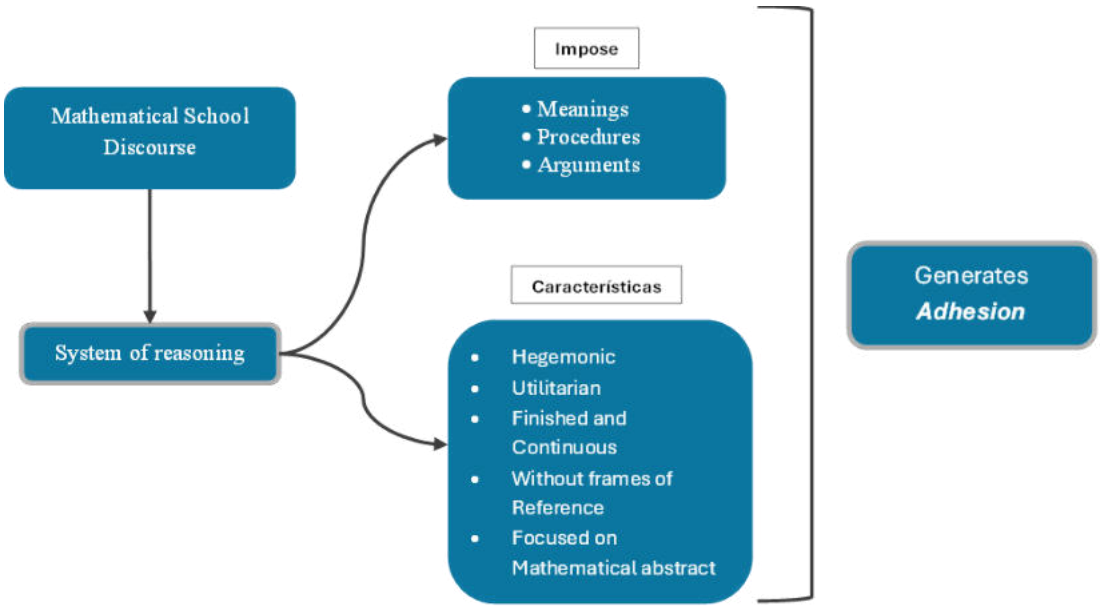

Figure 1: Mathematical School Discourse

This issue of *adhesion* is sought to be addressed through the theoretical construct of *disciplinary identity* of mathematics teacher's (Opazo-Arellano & Cordero, 2021; Opazo-Arellano et al., 2022). Cordero (2023) explains that the notion of *disciplinary identity* aims to describe the actions teachers take to "value the mathematical knowledge of people in order to influence the practice and teaching of mathematics" (p. 169). The model of *disciplinary identity* consists of three systematically interrelated components: *resistance, legitimation* and *project.*

Resistance involves a critical stance towards dME. This component is expressed by valuing epistemologies of mathematical knowledge that challenge the established norms of the dME, shifting the focus away from the mathematical abstract concepts without rejecting them. *Resistance* manifests in new perspectives and practices in mathematics teaching, questioning adherence to a dominant epistemology and promoting epistemological plurality.

Legitimation refers to the process through which the teacher recognizes and validates different types of knowledge, not just traditional school knowledge. This recognition leads to a reconsideration of mathematics teaching, valuing the plural use of mathematical knowledge, which can be reflected in the teacher's practices and the design of the school activities they implement.

The *Project* component represents the construction of new, autonomous proposals and arguments in the teaching of mathematics, which emerge from the teacher's creation of school activities. This component highlights the development of innovative uses of mathematical knowledge, reflecting the teacher's autonomy and creativity.

Within this framework, the objective of this research is to provide empirical evidence on how practicing mathematics teachers develop their *disciplinary identity* as they engage in a professional development experience. And, the objective of this paper is to discuss this professional development proposal.

The Experiment

We characterize this research project as an implementation (Artigue, 2021) of some results obtained from studies framed within the socio-epistemological Theory of Mathematics Education (STME) (Cordero et al., 2022). By implementation (Artigue, 2021), we refer to the type of application that involves translating theoretical results with an empirical basis into strategies, methodologies, or practical interventions that can be used for the benefit of society, or at least as applications to better understand possible pathways for innovative educational actions in practice. Artigue (2021) reflects on the inherent challenges of conducting this type of research. For instance, she mentions that applied studies are expected to serve as a bridge between basic research and its implementation in education in a rigorously scientific way. In this sense, applied research, while it can draw on findings from scientific studies conducted under controlled conditions (coming, a priori, from any study within the discipline), must seek, as part of its disciplinary nature, to face the challenge of amplifying the relevance of those results. In figure 2, we present a global outline of the experiment as we are currently conceiving it.

The experiment we are designing consists of three stages, as shown in figure 2. The first stage will involve an initial immersion with a community of in-service mathematics teachers. While we have

not yet defined the exact population, we do have certain candidate groups of mathematics teachers from different states in Mexico or Uruguay. The objective is to have an approach on their daily professional practice and register aspects of the dME.

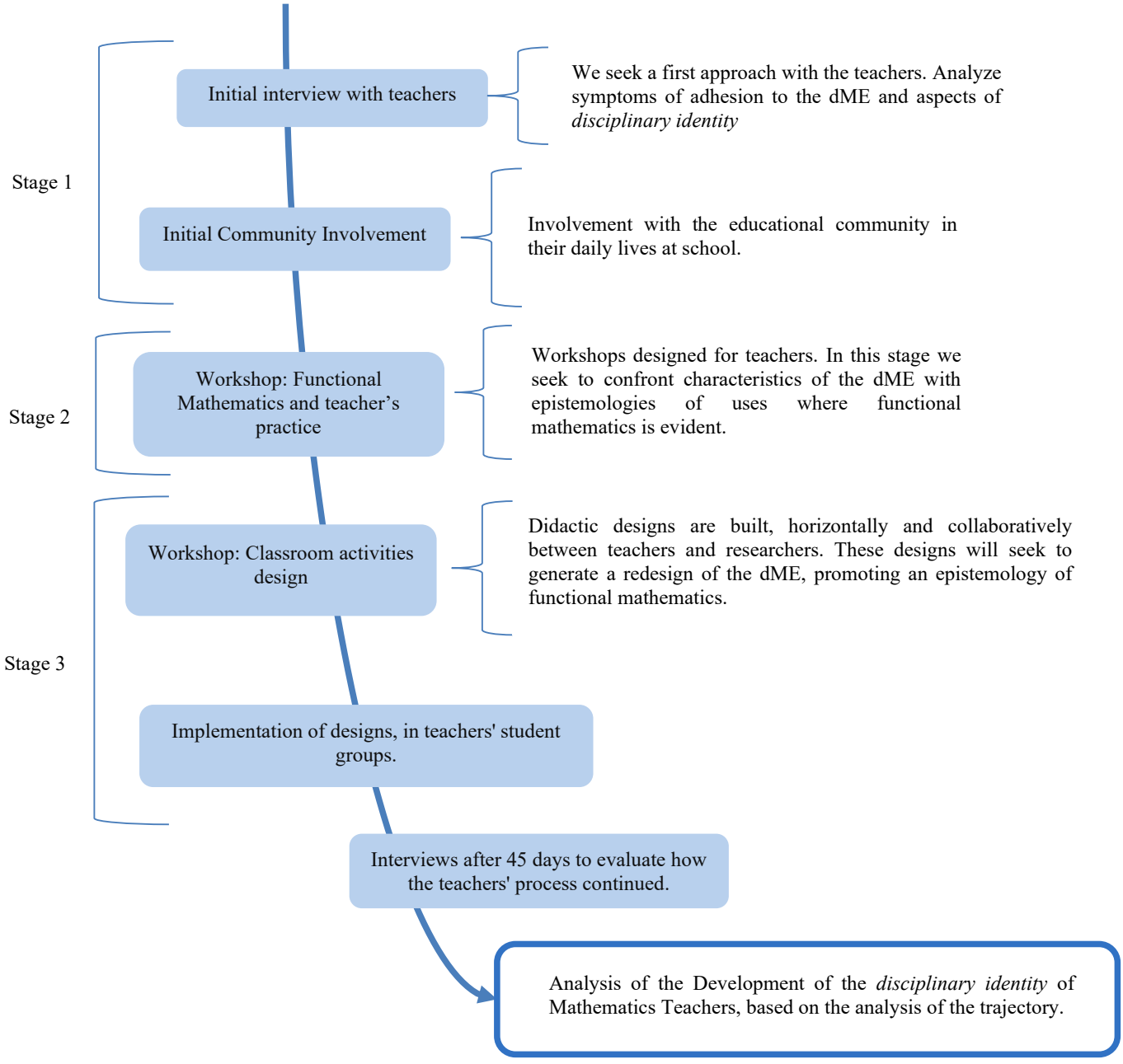

Figure 2: Overall structure of the experiment design

In the second stage, workshops will be conducted for the teachers. The goal of these workshops is to confront characteristics of dME and to facilitate this discussion, examples from the teachers' own

everyday professional practices, observed during the first stage of the research, will be brought into focus to problematize these theoretical aspects based on what is familiar to them. We consider that the success of this stage will be crucial to achieving the objectives of the final stage of the experiment.

In the last stage, based on the workshops in which the teachers participated, they will design didactic activities for their students with the aim of questioning certain established norms of dME.

Some theoretical foundations and methodological considerations for each of the stages

From the perspective of the Soltsa program, individuals generate knowledge by being immersed in and influenced by the communities they belong to. To better understand the teacher groups we will be working with, it will be necessary to become involved with the educational community in their everyday school environment. The methods considered for achieving this include interviews, classroom observations, and audio/video recordings (Rockwell, 2009; Guber, 2013). These considerations will guide the entire duration of the experiment, with a particular focus during the first stage.

In the first stage of the experiment, we aim to identify and record the dME in concrete situations that we can revisit to problematize during the workshops in the second stage. These characteristics can be observed, for example, in the opinions expressed by teachers, in how they organize their classes, in the tasks they assign, in how they validate or invalidate knowledge, and in how they value certain student contributions, among other aspects. During the last months of 2023, we had the opportunity to conduct an initial exploratory experience with practicing teachers in Uruguay. With them, we discussed and reflected on notions related to dME, and what we observed was promising. Teachers with certain orientations generally become aware of this phenomenon, because they recognize it from their practice, and achieving this awareness is a good starting point.

For the second stage, in the workshop design, we have considered two crucial theoretical methodological aspects for this project. The first is that the epistemological foundation of the workshop is based on the *category of modelling* proposed by the Soltsa program (Cordero et al., 2022; Cordero, 2023)

This *category of modeling* is proposed as a variety of the concept of mathematical modeling, as a theoretical way to characterize the uses of mathematical knowledge in specific situations. These situations occur within mathematical knowledge communities and are structured by sequential elements that build mathematical thought: meanings, procedures, and instruments, which lead to a situation-based argument (Cordero et al., 2022; Cordero, 2023). The core of the category does not lie in the mathematical model itself but rather in how the mathematical features of a specific situation can be meaningful and useful to a particular community. This perspective implies that the value of mathematics is not solely in its abstraction or formalization but in its ability to adapt to and address the needs, problems, or contexts relevant to the community that engages with it. This situation can be framed within two axes: institutionalization and the transversality of knowledge, where situations, domains, and alternations between settings occur, such as school-academia, work-profession, and city-everyday life (Cordero et al., 2022; Cordero et al., 2024). For example, Cordero et al. (2024)

shows how the category of *reproduction of behaviour* and its transversality can be interpreted across various domains and specific situations.

In second Stage, to apply this epistemological approach in the workshops for teachers, we consider the concept of *design of socialization school situation* (DSES, for acronym in Spanish), as proposed by the Soltsa program (Cordero et al., 2022). According to Cordero et al. (2022), the goal of these processes is to affect the dME by creating opportunities to emphasize the uses of mathematical knowledge within specific situations. This approach counters the phenomena induced by the dME while also enabling the analysis of teachers processes of developing of their *disciplinary identity* (Cordero, 2023).

The DSES concept is supported by two main aspects: an epistemological base rooted in the *category of modelling*, and a perspective that includes various factors contributing to the teaching and learning of mathematics depending on the intention, in this case it's the perspective of *disciplinary identity*. Rodríguez-Amador and Cordero (2023) provide an example of how this framework can be applied in a middle school context, focusing on the real-life situation of rainwater harvesting. This example highlights how mathematical knowledge can be used functionally, rather than simply acquiring abstract concepts. This design approach prioritizes the functionality of mathematics in context over the formalization of mathematical definitions or concepts. The idea of DSES is essential in the workshops because it shifts the focus from traditional, concept-driven teaching to a more contextual, use-oriented approach. This allows teachers to explore how mathematics can be relevant and practical in the everyday lives of students, promoting a more meaningful engagement with mathematical knowledge.

Gaete-Peralta et al. (2024) provide an epistemological foundation for developing the DSES intended for mathematics teachers in this research, which sets the second Stage. This work describes recent research from the Soltsa program, which focuses on the uses of quantification, through the analysis of how mathematical knowledge is used in specifics situations. It offers an alternative to the traditional dME in this topic, which is predominantly centered around formulas and primarily associated with the measurement of geometric figures. Additionally, the master's thesis by Izzi-Prato (2021), developed from a socio-epistemological perspective, recorded uses of measurement and quantification with epistemologies that also confront the traditional school concepts of measurement.

In the third stage of the experiment, the participating teachers will be invited to create their own designs. The aim is to observe the development of elements related to their *disciplinary identity*. We believe that in this stage, it will be possible to assess whether they have developed the ability to redesign aspects of dME and appreciate other mathematical epistemologies, as well as apply them in activities with their students. The methodological details of this stage are still in the design phase.

Conclusion

In this research, we explore a path to developing a *disciplinary identity* for mathematics teachers. We argue that it is crucial for teachers to develop a *resistance* to the norms of dME and to be knowledgeable about alternative approaches. However, approaches that have developed these mathematical epistemologies, as proposed by Soltsa, within the educational system, have not yet been

widely implemented. Throughout the paper, we highlight that this research aims to make an impact by implementing results derived from empirical studies, focusing on a professional development trajectory of teachers that is intended to be discussed in this paper.

In this regard, we believe that documenting the development of the *disciplinary identity* of mathematics teachers, as we propose here, could be beneficial to their professional practice. Teachers are consistently required to connect with the reality of their students or to work in interdisciplinary approaches, among other demands. However, the elements of the formal mathematics curriculum often make this task seem artificial and challenging. We believe that this approach, through collaborative work with practicing mathematics teachers, could help bridge the gap between reality and school mathematics, and counteract the hegemony of dME and some of the unfavorable phenomena it produces, such as *adhesion.*

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Planning considerations stated in intended learning outcomes for primary teacher education in Sweden

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Intended learning outcomes (ILOs) in mathematics pedagogy courses describe the competencies preservice teachers must demonstrate to pass the course. This study uses the notion of praxeology from the Anthropological Theory of Didactic (ATD) to analyse ILOs concerning "planning for mathematics teaching" in Swedish teacher education syllabi. The results showed that the tasks (T) of planning varied in what preservice teachers had to consider in their planning, with the national curriculum being most prevalent, followed by utilising theoretical perspectives, and considering the learners. Few ILOs clarified techniques (τ) to use for planning. Eleven out of 19 universities required preservice teachers to provide justification for their plans. Some ILOs put additional restrictions on the preservice teachers, such as using a particular teaching approach. The results suggest that mathematics teacher education courses operate with somewhat different praxeological equipment.

Keywords: Mathematics teacher education, planning, praxeologies, intended learning outcomes.

Introduction

Planning mathematics teaching is essential for high quality instruction (Corey et al., 2021; Stein et al., 2008) and can be described as the making of a detailed description of what to be taught, how, and for whom, perhaps by using a template guiding teachers on what to include. But planning can also be explained as a cognitive process in which the teacher makes choices and decisions related to the context in which they find themself (Grundén, 2020), which is how the concept will be defined in this paper. For teachers in Sweden, certain aspects are already decided in the national curriculum, such as what mathematical content is to be taught and what mathematical competencies students ought to acquire. However, there is nothing written about the extent to which the specific content is to be taught, and nothing regarding pedagogical methods or theoretical perspectives that should be adopted (Swedish National Agency of Education, 2022). So, in a world with a wide range of different mathematical teaching approaches and innovative tools of varying quality, it will thus be the task of teacher education to provide preservice teachers (PSTs) with praxeological equipment (knowledge, skills, and attitudes) to make decisions rooted in subject-specific and general pedagogical theories and research results that support their choices (Chevallard, 2022).

PSTs' practice of planning mathematics teaching has long since been considered an important element of teacher education by teacher educators at universities, by mentor teachers in field practicums, by researchers, and by the Swedish government (Kastberg et al., 2017; Rusznyak & Walton, 2011; Swedish Ministry of Education, 1993). This is in line with Swedish values for teacher education. This study found that planning competence is addressed in the syllabi of mathematics pedagogy courses at all Swedish primary teacher education programs for grades 4–6. Yet despite planning being a learning objective in PSTs' mathematical pedagogy courses and possibly even a practice to enhance PSTs' pedagogical content knowledge and teaching competence (Rusznyak & Walton, 2011), a majority of Swedish PSTs experience that they have not gained sufficient knowledge

about planning mathematics teaching – particularly in areas such as selecting appropriate teaching materials or adapting and modifying mathematical tasks (Christiansen & Erixon, 2021).

An on-going mapping review study (Jones et al., manuscript) suggests that there is limited research on how planning as a multi-faceted activity is addressed in mathematics teacher education, with more research addressing subdomains of planning, for example, designing and adjusting mathematical problems or predicting student responses. This observation led to an interest in examining how Swedish mathematics teacher education addresses planning practices in its mathematical pedagogy courses. It seemed reasonable to begin the investigation by analysing intended learning outcomes (ILOs) related to planning as stated in the syllabi because they are closely linked to the course content and what is assessed in PSTs' process of planning. Furthermore, from an anthropological perspective, ILOs also reflect what is foregrounded in teacher education practices in the institutions, and a comparison across institutions in a country can facilitate the understanding of conditions and constraints imposed nationally on the institutions (Chevallard & Bosch, 2020).

Swedish mathematics pedagogy courses and intended learning outcomes (ILO)

In Sweden, mathematics pedagogy courses are mandatory within primary teacher education programs for grades 4–6. The organisation and number of these courses taught within the four-year programs vary between universities, but all programs include a minimum of 30 mathematics credits, which equates to 800 study hours (see also Christiansen et al., 2021). The reason for the variations between universities has to do with whether practical components are included in these courses or kept as separate courses. In most cases, the mathematics pedagogy course content includes both mathematics and mathematics pedagogy. The syllabi for these courses are official documents and are available on the universities' websites. One of the purposes of the syllabi is to outline the intended learning outcomes, which includes specifying the knowledge and skills that PSTs need to demonstrate to pass the course – therefore also informing course assessment (Christiansen et al., 2021).

Notions within the anthropological theory of didactic

The anthropological theory of didactic (ATD) states that any human activity can be described in terms of praxeologies, where the activity in question comprises a *praxis* block and a *logos* block (see Figure 1). The praxis block relates to the practical aspects of the activity – the know how – and is further divided into two components: The type of *task* (*T*) which addresses the problem or problems the praxeology aims to solve, and *technique* (τ) which describes strategies used to carry out a specified task (*T*). The logos block relates to the knowledge about the praxis – the know why – and is also separated into two components, *technology* (θ) and *theory* (Θ), where the notion of technology (θ) describes and gives justification for the praxis, and theory (Θ) which is perceived as a higher level of justification, often invisible, taken for granted and part of a cultural context (Bosch et al., 2020). Analyses within the component of theory (Θ) are for this reason beyond the scope of this study and only the concept of technology (θ) will be used. Within ATD, any learning situation, no matter what

the level or kind, is explained as a *didactic system*[1] (*S(X, Y*, ♥). This system (*S)* includes learners (*X)*, study assistants (*Y*), and a didactic stake (♥) — "the thing [praxeology] that is to be learned" — (Boch & Gascón, 2014 p. 71).

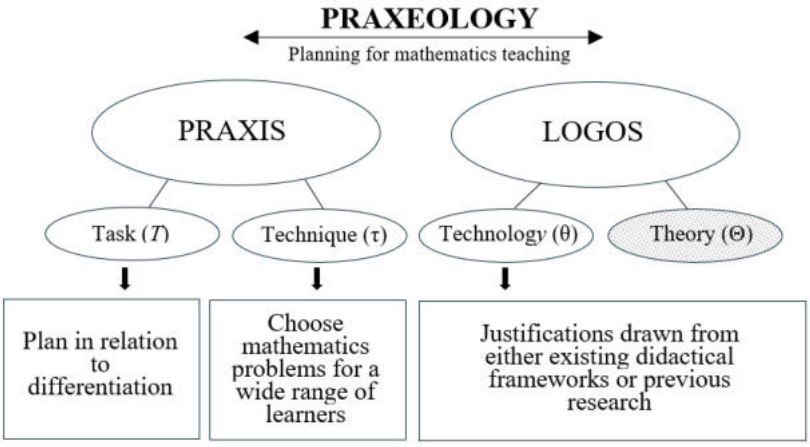

Figure 1: Didactic praxeology model

When teacher education is viewed as a didactic system (*S*), the learners (*X*) are pre-service teachers, the study assistants (*Y*) are teacher educators (including cooperating teachers), and the didactic stake (♥) consists of the teacher knowledge of praxeologies that PSTs are to acquire. **Figure 1** illustrates an example of a didactic praxeology model where the didactic stake (♥) involves *planning for mathematics teaching with a particular focus on inclusion*. A technique (τ) to plan for inclusion might involve deciding on the sequencing of the content, choosing activities for students, etc., or using known strategies for adjusting the difficulty level of an existing task. The technology (θ) used to justify the adjustments might relate to either existing didactical frameworks or previous research. PSTs can practice such a planning task in mathematics pedagogy courses through exercises that might say: *Plan a series of three 40-minute lessons where students work with proportionality expressed in decimals, fractions, and percentage representations. Vary between inquiry-based and traditional teaching and explain and justify how the activities and problems you choose are aligned with the competencies expressed in the national curriculum, as well as how an inclusive perspective is integrated* (fictive example).

A didactic analysis within ATD and the use of a praxeology model seeks to describe didactical systems (*S(X, Y*, ♥) present in any social situations at a given time and understand how these systems or their components evolve over time within broader contexts of institutions, societies, and cultures (Chevallard & Bosch, 2020). The notion of praxeology is a useful tool in didactic analysis, because it encompasses discourse on both knowledge and practice, "it provides a unitary vision of these different activities, without considering some of them as more 'intellectual', 'abstract', 'difficult' or

[1] In ATD, letters and symbols are used to clarify that a concept's meaning is specific to ATD's definition, which may differ from its general interpretation. For example, in ATD the word task (*T*) has a broader meaning and refers to a verb followed by an object, e.g., drinking water or writing a letter.

theoretically based than others, and thus without assuming the scale of values usually given to them" (Bosch & Gascón, 2014, p. 69).

This paper describes the praxeology—planning for mathematics teaching as reflected in ILOs in Swedish teacher education—for grades 4–6 through the lens of the praxeology model, which enables a comparative analysis of potential variations between universities. In so doing, I hope to further the discussion of using ATD's notion of praxeology to provide a more nuanced discourse on PSTs' practical and theoretical knowledge related to the planning process. The research questions guiding this study are: *What praxeological components, related to the didactic stake (♥) planning for mathematics teaching, can be identified in the intended learning outcomes (ILOs) stated in the syllabi for mathematics education within primary teacher education programs for grades 4–6, and are there variations between universities?*

Method

To identify praxeological components (T, τ, θ) related to the didactic stake (♥) *planning for mathematics teaching* as specified in the ILOs outlined in the syllabi for mathematical pedagogy courses in primary teacher education programs for grades 4–6, and to compare variations between universities, the study was conducted as follows.

From the websites of the 19 Swedish universities offering teacher education programs for grades 4–6, 53 obligatory mathematic pedagogy course syllabi were collected during the spring of 2024, including courses with a mathematics focus within field practicum. In the first phase of the analysis, all the syllabi were reviewed to identify ILOs that specified what PSTs should be able to do in relation to planning, designing, or structuring mathematics teaching and activities. While not all the collected syllabi addressed this praxeology in their ILOs, every university had at least one mathematical pedagogy course that did. ILOs from 37 syllabi were extracted for further analysis.

The second phase was to analyse ILOs in greater detail and categorise their content in relation to the components of the praxeology model. For example, one ILO stated, "After passing the course, the student should be able to plan, implement, and analyse teaching based on the national curriculum, as well as didactic and subject-specific theoretical aspects" (University A). In this example, the subtasks that are visible in relation to planning for mathematics teaching are (T_{pc}): Plan considering the "national curriculum" ($T_{pt)}$, and plan considering the "didactic subject-specific theoretical aspects". "Implement and analyse *teaching*" could also be interpreted as Tasks (T_i and T_a); however, these tasks (T) do not fall within the praxeology of planning for mathematics teaching in this study because they can be interpreted to fall within a praxeology of teaching. Techniques (τ) explaining strategies for how to incorporate the national curriculum are not visible in this IOL, and neither are the components of the logos block, because we cannot identify the specific theories that PSTs are expected to use that justify the techniques (τ). The tasks (T) identified in the ILOs were characterised according to the aspects that PSTs were expected to consider in their planning, which were grouped into three major categories to allow for easier comparison between universities.

An example of an ILO that included technique (τ)- and the technology (θ)- component stated: "After passing the course, the student should be able to, based on a didactic analysis, plan a teaching situation

and justify the choice of teaching materials, working methods and working approaches" (University K), "didactic analysis" is understood as a technique (τ) and "justify the choice of…." as technology (θ). Institutional constraints are also evident in this ILO because "plan a teaching situation" implies a specific, limited context that influences how the task (*T*) is carried out. In the first ILO example, the word "plan" was not linked to a limitation of either scope or teaching method.

Result

The results are presented in two tables. Table 1 summarises what was identified in the ILOs associated with the components of the praxeological model, but also institutional constraints on PSTs planning practices. The tasks (*T*) related to the didactic stake (♥) *planning for mathematics teaching* required PSTs to consider: 1) the national mathematics curriculum, 2) theoretical aspects, and 3) learners. Table 1 also displays subcategories observed within the tasks ($T_{pc,\ pt,\ pl}$). Only one technique (τ) was identified – the encouragement to use didactic analysis as a strategy for planning. In terms of the logos block and the technology (θ) component, some of the ILOs stated the need for PSTs to critically justify their planning. Institutional constraints are not part of the praxeological model, but they still warrant attention because such constraints will affect the PSTs' decisions in the planning process. Table 2 shows the distribution of the main categories (from Table 1) among the Swedish universities.

Table 1: Aspects of praxeologies identified in IOLs

Component of praxeology	**Types of** (*T*) (τ) (θ)	**Types, thematised into tasks** (*T*) **Technique** (τ) **Technology** (θ)	
Task (*T*): Planning considering …	National curriculum (T_{pc})	• Lgr22 • Mathematical abilities and grading criteria stated in Lgr22 • Supportive material from The Swedish National Agency for Education	
	Theoretical aspects (T_{pt})	• Didactic perspectives • Subject didactics • Current research • Literature	• Didactic and subject-theoretical aspects • Documented experience
	Learners (T_{pl})	• Inclusive perspective • Equity perspective • Students' diversity • Students' prerequisites	• Students' different interests and backgrounds • Accommodations • Additional support needs
		• Assessment of students • Progression	• Formative assessment • Reasoning competence
Technique (τ)	Didactic analysis		
Technology (θ)	Critical justifications	• Justify choices • Evaluate • Concretise	• Argue • Reflect • Explain
Institutional constraints on PSTs' planning practice	Teaching method	• Problem-based approach • Laboratory approach • Aesthetic learning process	• Formative student-oriented teaching • Various ways of working • Teaching materials
	Scope	• Mathematics activity • Full unit • Delimited section of unit	• Lesson series • Teaching situation

Table 2: Praxeology components of planning identified in different universities

University		A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	Sum
(T)	National curriculum	x	x	x	x		x	x	x	x	x		x	x	x	x		x	x	x	**16**
	Theoretical aspects	x	x	x		x	x	x	x	x				x	x	x	x		x	x	**14**
	Learners		x	x		x	x	x	x	x	x		x		x	x	x			x	**13**
(τ)	Planning strategies											x									**1**
(θ)	Critical justification		x	x			x	x	x			x	x			x	x		x	x	**11**
Institutional constraints	Teaching method		x	x	x	x	x		x					x	x		x			x	**10**
	Scope		x				x	x	x	x		x	x	x		x	x	x	x		**12**

Table 1 and Table 2 enable us to answer the questions: *What praxeological components, related to the didactic stake (♥) planning for mathematics teaching, can be identified in the intended learning outcomes (ILOs) stated in the syllabi for mathematics education within primary teacher education programs for grades 4–6, and are there variations between universities?*

The results show that the tasks (T) of planning mathematics teaching vary in what has to be considered in a Swedish teacher education context. National curriculum is the most prevalent (16 universities), followed by utilising theoretical perspectives (14), and considering the learners (13). Nine universities expected PSTs to take into consideration all three of these contextual aspects, one none of them, and two universities only one aspect. Few ILOs clarified techniques (τ) for planning that PSTs were expected to master, only didactic analysis was referred to, and one university explicitly mentioned this. Although there appears to be a general agreement among universities to consider the national curriculum in the planning process, less uniformity is seen in the ILOs concerning the inclusion of critical justification for the decisions PSTs make, where only 11 out of 19 universities make that explicit. All but two universities also put additional restrictions on PSTs, such as using a particular teaching approach or plan for a specific scope.

Discussion

The choice of ILOs as research data clearly did not provide a sufficiently robust foundation for making all the components of the praxeology model visible and thus may not have offered a fair reflection of either the intended learning or what may be covered within the pedagogy courses. To gain deeper insight into this, data such as examination tasks and interviews with teacher educators are needed. For instance, we identified only one ILO indicating that PSTs were encouraged to use a

specific technique (τ) to plan their teaching. Regarding the *logos* and the component of technology (θ), it was evident that most universities required PSTs to justify the choices made in the planning process, but the grounds for such justifications (e.g., frameworks or research evidence) were not specified in the ILOs. A possible reason for the general and ambiguous articulation of ILOs could be the pedagogical freedom that characterises the Swedish educational context at all levels which allows for a variety of techniques (τ) to be employed. This also facilitates future developments of the pedagogy course without requiring changes to official documents.

The anthropological perspective in research aims to describe rather than rate and evaluate what becomes visible in the data (Bosch & Gascón, 2014). However, descriptive studies can serve as a basis for discussions on consequences of different approaches. For instance, the result of this study indicates that in the Swedish teacher education context, three domains seem to receive particular emphasis in the planning process: the national curriculum, theoretical aspects, and learners. The prominent focus on the National Curriculum could be attributed to pedagogical freedom, as well as the fact that teaching resources are no longer reviewed by a governmental agency. Similarly, the emphasis on learners could be related to the goal within the Swedish education system to ensure all students' right to an inclusive and equal education, which highlights that "teaching shall be adapted to the circumstances and needs of each pupil" (Swedish National Agency for Education, 2022, p. 6). However, if certain areas receive greater attention within teacher education, it may imply that others are given less, and we need to consider the potential consequences of this distribution of focus. For example, none of the ILOs referred to considering lesson activities, and at this point, we do not know if there is a correlation between this and PSTs' experiences of lacking knowledge regarding selecting appropriate teaching materials or adapting and modifying mathematical tasks (Christiansen & Erixon, 2021). Table 2 suggests that Swedish teacher education programs emphasise different aspects for PSTs to consider when planning for mathematics teaching. This diversity may reflect institutional constraints but could also stem from insights developed within institutions regarding effective ways of supporting PSTs' learning. Cross-institutional discussions could positively influence the development of mathematics pedagogy courses (see Kastberg et al., 2017). However, this is the first stage in a more comprehensive analysis. Future stages will include detailed case studies and nuanced discussion.

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Intertwining theory and practice: Design of planning phases in PD training to support effective teacher planning processes

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Professional development (PD) represents an opportunity to enhance teaching quality and transfer innovations throughout the school system. For this, PD must meet certain quality criteria, such as intertwining theory and practice. In this study, a practical phase lies between two PD training sessions, and a 30-minute planning phase is embedded within the PD training itself. This study aims to investigate how the design of the planning phase can facilitate teachers' planning actions. In an explorative study, 11 groups (N=28) of teachers were observed and videotaped. The interviews were transcribed and categorized by applying qualitative content analysis. First results indicate that substantial planning includes interlinked quality criteria of good (mathematics) teaching and an explanation of actions. Also, not every group of teachers seems to achieve complete and finalised planning due to several framework conditions. Thus, the design of the phase requires adaptation.

Keywords: Professional development, facilitators, planning phase, mathematics, primary school.

Introduction and Theoretical Background

Professional development (PD) in tertiary education is an important part of enhancing the quality of teaching. PD allows addressing the constant change and evolvement of schooling conditions (Fussangel et al., 2016). These changes are reflected in, for example, new curricula, digital media, and new didactic findings. To encounter these changes and design and plan effective lessons, both general didactic and specialized didactic quality criteria for good (mathematics) teaching were developed. General didactic criteria include mainly organizational and methodological aspects, which may be applied to every subject (Meyer, 2018). These include, among others, the proportion of learning time, clarity of content, competence orientation, variety of methods, and promotion of meaningful communication by proper tasks. In addition, subject-specific didactic content must also be included when planning good mathematics lessons. Prediger et al. (2022) have identified five quality criteria for this and point out that lessons should be designed with a *conceptual focus* and a *cognitive demand,* be planned considering *longitudinal coherence*, take into account students' heterogeneity, and, thus, allow *adaptivity* to the learners' level of knowledge, and enhance content-specific *communication* between the students.

Teachers should be constantly trained to meet these standards for high-quality mathematics teaching. PD is one way to do so. To ensure the effectiveness of PD training, several quality criteria have been developed based on various studies (Barzel & Selter, 2015; Desimone & Garet, 2015; Lipowsky, 2020). All researchers agree on the following four criteria of effective PD training (Knaudt et al., 2023), although naming them differently: (1) learning in professional learning communities and stimulation of cooperation, (2) intertwining of theory and practice, (3) experiencing the effect of one's actions, (4) case reference and participant orientation. Also, PD training should be designed for the long term and enable coaching and feedback, guide understanding of the profession, focus on students' competencies, and enable input (Darling-Hammond et al., 2017; Desimone & Garet, 2015).

It is important to note that when applying these criteria, they should be combined to provide effectiveness and not be understood as standalone measures (Sims & Fletcher-Wood, 2021). One way to address these different criteria and integrate them into PD training is to incorporate a practical phase between PD training sessions and embed a 30-minute planning phase within the PD training itself (Knaudt et al., 2023). Yet, in current research, it remains open how a planning phase should be structured and how concrete tasks should be designed for the effective planning of lessons.

A previous study analyzed the content of planning phases and its qualitative impact on the design of the phase. Knaudt et al. (2023) found that a planning phase was mainly used to review the material and then to focus on ‘what to teach’ rather than ‘how to teach’. It was also observed that during the planning phase, the teachers pursued different aims depending on the compositions of the teacher teams. For instance, the practical phase can be implemented directly in one's lessons if the teachers have their class or it must first be transferred to colleagues, if not. The authors recommended changes in the design of the planning phase. However, it remained open whether or not these changes led to effective or purposeful planning of mathematics lessons.

Research and Design Questions

This design-based study builds on these previous findings by addressing the new design of the planning phase and by investigating whether or not a certain structure or task leads to effective planning. It aims to investigate whether the design of the planning phase is conducive to the objective or whether other elements should be changed. This leads to the following questions:

- Research question: To what extent do teachers plan a practical phase purposefully with the tasks and material provided to them in the planning phase of the PD training?
- Design question: How should the design of the planning phase be adapted to best support teachers in planning and implementing the practical phase in the classroom?

Design of the Study

This study was conducted within the broader project ‘School makes strong’ ['Schule macht stark']. One aim of this project is to foster the basic mathematical competencies of primary school students by developing and carrying out various PD training modules. One module covers the basic numeracy skills of third- and fourth-graders. Each module consists of six training sessions, and in between are inserted a practical phase, transfer to the teaching staff, and digital self-study modules (Figure 1).

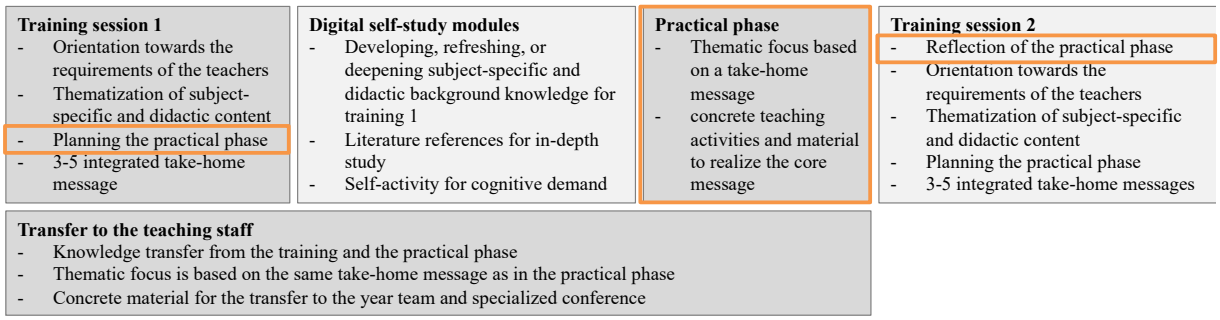

Figure 1: Professional development series and its elements

During the **planning phase**, the practical phase should be planned for implementation in the class. The practical phase includes an activity that can be integrated into any mathematics lesson. This ensures that the activity can still be implemented independently of the series of lessons, and the teachers are supported in their knowledge acquisition. Therefore, this phase fulfils certain features:

- Take-Home Message (THM, core content in one sentence): The practical phase relevant for this study deals with analogies of place values promoting the THM 'I promote a material-based understanding of analogies of place values through action-prompting activities'.
- Mathematical content: Analogies of place values mean that the simple task 3+4 also helps with supposedly difficult tasks such as 30+40 or 3000+4000.
- Mathematical activity: The activity includes a ritualized initial game in which students are required to solve a challenging task with the assistance of a relatively straightforward one. They may draw upon the analogy of place values in doing so. (Figure 2).
- Material package: All material relevant to the THM is bundled in a material package, which includes a mathematical activity, language support, further tasks, digital and haptic materials.

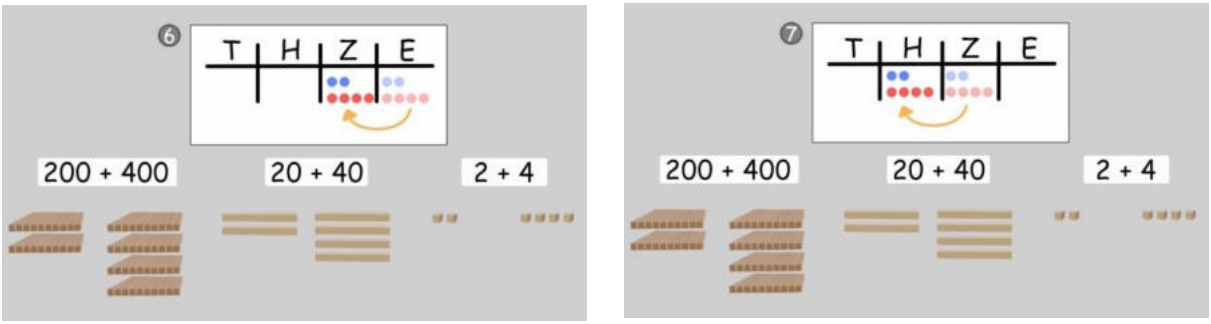

Figure 2: Approach with a place-value chart and the Dienes-material

The planning phase follows a certain **design structure**. It is 30 minutes long and embedded at the end of each training session. This phase aims to implement the practical phase in a lesson the next day without the need to start the planning process anew. The teachers may choose their objective: (1) plan the implementation of the activity from the practical phase for their class, or (2) plan the transfer of material to the teaching staff. It is necessary to address these different objectives in the planning phase due to the considerable heterogeneity of the participating teachers. Independent of their chosen objective, they can (1) gain an overview of the material package, (2) use a focused planning instruction to plan its implementation (e.g., 'If not all of the prerequisites (e.g. tenfold) have been met, how should this be addressed? What specific forms of support are provided during difficulties (stimuli, materials, etc.)? How is linguistic support provided to the children when using analogies?', and (3) clarify questions.

Data Collection

The data of this study was collected and analyzed in the second of two cycles. 11 planning phases with two to four teachers in a group (total of 28 teachers) were observed and videotaped (Figure 3). Each group consisted of teachers from one teaching staff; hence, they were not mixed. Each planning phase lasted between 20 and 30 minutes (a total of 280 minutes) and was fully transcribed.

Additionally, the facilitators (N=4) who led PD training sessions were interviewed guideline-based about the practical and the planning phases. Those interviews were videotaped and fully transcribed.

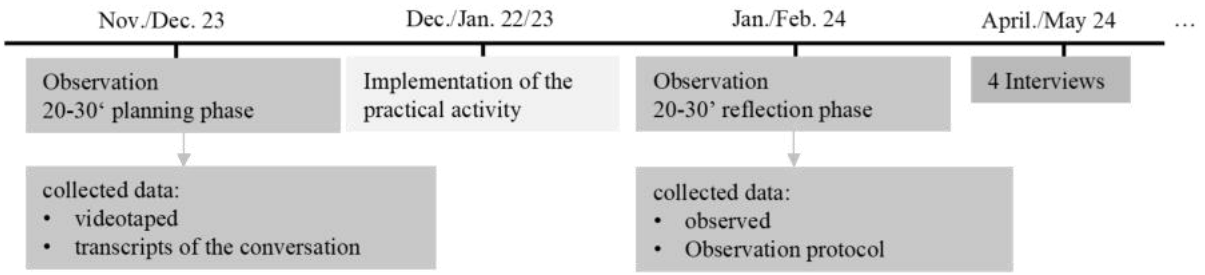

Figure 3: Timeline of the data collection in the second cycle

Data Analysis

All transcripts were analyzed using a coding system (Table 1) (Knaudt et al., 2023). This system is based on deductive codes of effective (mathematics) teaching (see C1 and C2) and inductive codes generated through the observation of the planning phases (see C0 and C3) (Kuckartz & Rädiker, 2020). Applying these categories and their subcategories to the provided data allows for categorizing and analyzing all teachers' statements. After that, all coding of each planning phase was illustrated within progress plans (Figure 4), which show in which order and at which point in the conversation the statements were coded. This way, those diagrams provide an overview of the data and allow investigations into the interconnectedness of the coded categories and their relative frequency to each other.

Table 1: Overview of categories

Category	Description	Example
C0: No planning	Statements that did not result in planning	‘I had forgotten again that someone was listening to us.’ (Teacher C, transcript 27)
C1: Planning good teaching	This includes all planning statements that were made on a general didactic level.	‘Or you make groups of four, that they do such a group work and then …' (Teacher B, transcript 40)
C2: Planning good mathematics teaching	Statements that refer specifically to the planning of mathematics lessons.	‘… Some children may see the balloons, some may see, well, everyone has a different focus.’ (Teacher A, transcript 25)
C3: Framing planning actions	Statements that framing the planning actions, like overview the material package.	‘Oh no, open the planning grid first.’ (Teacher B, transcript 4)

Results

A review of the data revealed three research findings that have a significant impact on the design.

Research Results

Figure 4 exemplifies four of the 11 progress plans and shows typical distributions of the categories. Some groups (Team I) planned in terms of general didactics (C1), subject didactics (C2), and framing (C3) during the planning phase. Others (Team III) concentrated on planning good (mathematics) teaching (C1 or C2), while others (Team II and IV) focused on framing planning actions (C3).

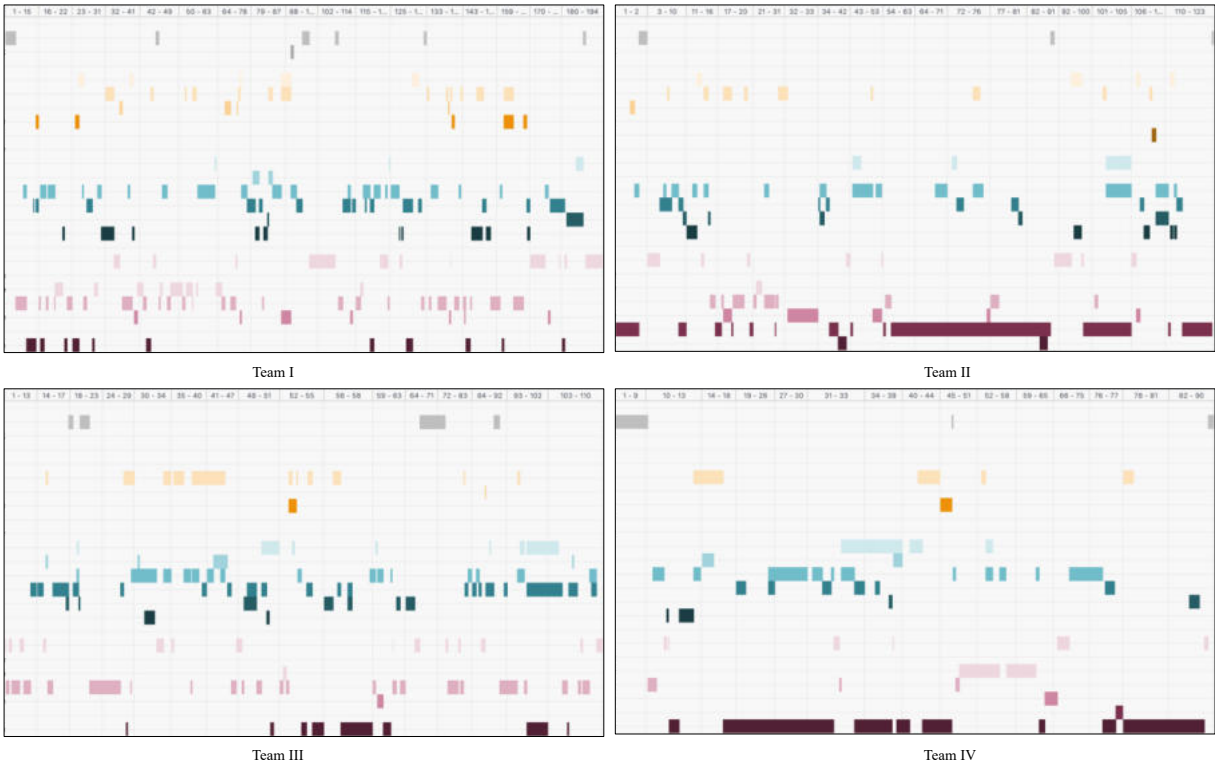

Figure 4: Progress plan of planning actions of four in-service mathematics teacher groups

Addressing the transfer to the teaching staff within the planning phase is used meaningfully. Following the results of the previous study, the transfer was included in the planning phase. It has now been confirmed that this design change is a beneficial addition. Analyzing the progression diagrams, frequent coding of framing planning activities (C3) is revealed. For example, Team II focused on transferring the material to the teaching staff. Due to unknown reasons, they cannot carry out the activity by themselves. Instead, they must pass on the content and consider how this is best realized (C3, light berry). Within the planning phase, discussions are then held on the content level (C2). For example, the teachers talk about specific actions and tasks for the children (line 72). Also, they included the organizational level (C1), talking about the presentation to their teaching staff (line 57) by distributing the take-home message and the corresponding material package.

57	Person 1:	Present the slides. Alternatively, print out individual ones [reads out]. But we can throw that on the wall. That would now be the first slide you throw on the wall, say: "The take-home message", So what's it about?
72	Person 1:	Exactly, then first, actually, with this visualization. The one becomes a ten, and the colleagues then also guess, 'Let the children lay it out, the small task and then the big task'

Planning good teaching (C1) and planning good mathematics teaching (C2) are mutually interlinked and result in a substantial planning phase. Comparing the four teams, the categories for effective teaching (C1) and those specific to mathematics teaching (C2) occur throughout the planning phase, indicating interlinked planning actions of the teachers. However, their frequency varies. Both categories alternate or can be double-coded. If a teacher's statement can be double-coded, the focus is on 'how to teach'. The double-coding provides a more detailed explanation of the implementation process. For example, in line 111 (Team II), the teacher's objective is to adopt a long-term perspective on understanding (C2) and, at the same time, to promote this through a structured activity (C1). In line 112, the conceptual focus ("building understanding") and longitudinal coherence ("regular practice") in good mathematical teaching are emphasized. Thus, the double coding continues. The categories for good teaching are interlinked with those for good mathematics teaching.

111 Person 2: Yes, and as I said, with the children, it is also more likely that you repeat this two to three times every day.

112 Person 1: A ritualised introduction, actually, nh? Building understanding is not about, I'm going to do this once anyway, but about regular practice. And then initially without bundling and unbundling, right? Because thirteen plus eight or something (Person 2: or eight plus five) is difficult.

In contrast, the following statement (line 30) is coded in only one category. The teacher considered the levels of knowledge (C2) based on the heterogeneity of the students. However, it remains unclear how the activity could be adapted to address this. The planning would be richer if the teacher interlinked the thought of adaptivity (C2).

30 Person 2: So, my fear now is that if I only do it [the activity] with tiles, the children won't understand it.

Statements that allow double-coding show more substantial planning actions than the ones that can only be coded with one category.

Prevailing circumstances determine the extent to which the planning phase can be completed in a meaningful and aim-oriented way. Due to several reasons, some teams do not consider the practical phase and the *corresponding* planning phase as meaningful. The progression diagram of Team IV reveals a high frequency of framing planning activities (C3, dark berry). Yet, their emphasis is different from Team II. They have already covered the content in class, do not see a need in the activity given, and instead use the planning phase to report on previous experiences. Thus, they focus on sharing experiences (Team IV, line 31) and not on planning the activity.

31 Person 2: I had the same thing today in my fourth year at school. There were tasks where the aim was to do clever maths, i.e. to use strategies. And then there were always tasks like plus 599, and the children then really had to calculate plus 600 and minus one. And then I took out the Dienes-material again to lay it out.

Furthermore, in interviews with the facilitators, they reported several factors that prevent teachers from engaging in those planning activities. For instance, only single teachers are present at the PD training session instead of the entire team. They also report on materials that are missing or not available for their class, e.g., mathematics textbooks. Further, they mention that the timing of school content does not always match the timing of our PD training sessions. If the content is covered too early or too late in the PD training, the activity cannot be planned properly. The current objective of

the planning phase is to plan an activity for one's lessons in 30 minutes to be implemented on the next day. The facilitators' interviews indicate that a planning phase can only be effective with a more flexible objective. Possibly, the effectiveness of the planning phase is higher when planning is initiated within a PD training but finished within a school team.

Design Result

The three research results prompted the redesign of the planning phase. First, based on the positive results of the transfer, the transfer to the teaching staff should be planned by all teams and not offered as an alternative objective. Hence, the planning phase is divided into two 15-minute slots to plan the transfer to the teaching staff and to plan the practical phase. Second, the focus of the planning phase should be on planning good (mathematics) teaching. Therefore, general didactic criteria for effective teaching and subject-specific didactic criteria for effective mathematics teaching should be directly addressed within the planning process. To achieve this, the task within the phase is formulated more precisely, and the focus is directed towards one of the five quality criteria of good mathematics teaching. Third, the high frequency of framing planning activities (C3) enriched with the facilitators' report resulted in a modification of the timeframe from 30 minutes to 15 minutes. The mathematical activity is integrated into an activation phase within the PD training session, which allows addressing initial questions promptly. This way, teachers may start content planning immediately. The planning phase itself is limited to 15 minutes, and the objectives of the planning phase are more flexible concerning the timing of when to carry out the practical phase or the transfer to the teaching staff. This way, the planning is not finalized within the PD training, but the implementation of the activity is initiated. In sum, with these design changes in the planning phase, teachers should be enabled to use the planning phase meaningfully and aim-oriented for themselves without perceiving it as unnecessary.

Discussion and Conclusion

Considering the generalizability of the findings in this study, two main limitations should be taken into account. The results are specific to the mathematical content and structure of the PD training in the broader project. Also, only a small number of teachers participated in this study. Nonetheless, the results of the current study hold implications for future research and teacher education.

The aim of this study was (1) to gain a profound understanding of planning actions undertaken by teachers during a planning phase integrated into a PD training program and (2) to ascertain how the structure of this phase can facilitate the planning process. Studies have already shown that the connection between theory and practice is important (Barzel & Selter, 2015; Desimone & Garet, 2015; Lipowsky, 2020), which can be confirmed by our results. The results of this study indicate that a high connection between planning good teaching (C1) and planning good mathematics teaching (C2) leads to rich planning actions. Yet, a simple existence of the planning phase to provide an opportunity to connect theory and practice (Darling-Hammond et al., 2017; Lipowsky, 2020) is not sufficient. Instead, the planning phase needs to be integrated into a regular activity within PD training. Also, the participants' needs concerning their school curriculum and other circumstances should be considered. A flexible but obligatory way of planning has to be found. Based on the research findings of this study, the design of the task, structure, and timeframe of the planning phase were modified.

Future research should ascertain whether these modifications lead to better planning actions. Additionally, it would be of interest to observe the implementation of the practice phase after planning to identify factors to support teachers during the planning phases. In this study, integrated planning phases of practical phases exemplify the implementation and purposefulness of the quality criterion 'intertwining theory and practice' of effective PD programmes by demonstrating qualitative insights. It remains open whether this leads to a transfer of PD content into the teacher's lessons and whether the results can be transferred to other PD concepts and content.

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Noticing of students' thinking – Is it sufficient for teachers' analysis of interaction in the mathematics classroom? – A vignette-based study

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The key role of teacher noticing regarding students' thinking has been emphasised in recent mathematics education research. Even if corresponding methodological approaches differ, it can be assumed that noticing aspects of the learner's thinking and reasoning affords access to teachers' learner-adapted reactions and help, among other. However, the articulation of a student's thinking is embedded in classroom situations and content-related interaction processes, thus there is a need to explore whether the noticing of students' thinking is sufficient for an adequate analysis of classroom situations and which analysis foci should complement this focus. Through a vignette-based explorative study, we address this research need: The findings indicate cases in which an additional focus on content-related interaction aspects, in particular on teacher reactions, is needed.

Keywords: Noticing, teacher noticing of student thinking, vignettes-based study, classroom cartoons, teachers' analysis of classroom interaction.

Introduction

Teachers' ability to notice relevant situation aspects of the mathematics classroom has received increased attention in mathematics education research (e.g., Fernández et al., 2018), above all for its relevance for responding adaptively to demands of the situation and providing help to learners (Kuntze et al., 2021; Sherin et al., 2011). In this context, under the label of *teachers' noticing of students' thinking*, various studies have pointed at the significance of the teachers' focus on aspects of the individual mathematical understanding of the learners. Focusing on students' thinking is associated with high learner orientation and assumed to enable teachers to react adaptively to the students' learning needs (e.g., Hardy et al., 2019). As noticing in the sense of knowledge-based reasoning is a knowledge-based process (Sherin et al., 2011), its foci can be based on various criteria (e.g., student's use of representations, student's motivation, …), which are connected to professional knowledge elements and can be assumed to be in a competing relationship (Kuntze et al., 2021). Under this perspective, there is a need for research into the interplay of noticing foci, such as exploring their potentially conflicting role when making sense of processes in a classroom situation. Consequently, finding out about teachers' noticing of aspects of students' thinking together with other noticing foci which can play an additional role for interpreting classroom situations is of high interest.

This study thus responds to this research need, by using a vignette-based design: Purposefully designed vignettes afford examining whether teachers attend to specific situation aspects which can be implemented in the corresponding vignette in relatively controlled ways (e.g., Skilling & Stylianides, 2020). Profiting from these methodological possibilities, we asked teachers to analyse a vignette in which specific analysis requirements were implemented. In the vignette, a vignette teacher reacted to a student in a way which was rather disconnected to the student's thinking. The research interest focused on whether an analysis of the student's thinking also led to a correspondingly critical

evaluation of the vignette teacher's reaction. The study further explored which other noticing foci may have inhibited the teachers from a corresponding analysis of the teacher's reaction, in case of successful analysis of the student's thinking.

Theoretical background

Approaches to mathematics teachers' *noticing* and related constructs such as *professional vision* are numerous (e.g., Amador et al., 2021; Berliner, 1991; Choy, 2014; Fernandez & Choy, 2020; Sherin et al., 2011; Vondrová & Žalská, 2013) – a common feature of the approaches is that observations in classroom situation contexts are made by teachers and connected with elements of professional knowledge. In particular, the notion of *knowledge-based reasoning* (Sherin et al., 2011) emphasises this core of teacher's noticing. In our research, we build on these advances in research into teacher expertise and often use the notion of teachers' *analysis* of classroom situations, emphasising that in noticing, teachers connect observations with specific professional knowledge elements (e.g., Kuntze & Friesen, 2018; Kuntze et al., 2021). The use of criterion knowledge relevant for such observations and hence for the teachers' analysis is facilitated by criterion-related *awareness* (Kuntze & Friesen, 2018), which facilitates teachers' access to the corresponding criterion-related knowledge. In case of several relevant criteria, so-called *multi-criterion noticing* denotes the corresponding requirement for teachers' analysis of classroom situations (Kuntze et al., 2021) – namely to include several different criteria in noticing processes. In prior research we have explained the supporting role of criterion-related awareness for noticing in more detail and explored the associated prerequisites for teachers' noticing empirically (e.g., Kuntze & Friesen, 2018; Kuntze et al., 2021). In the present paper, we connect this scope with approaches to noticing students' thinking (e.g. Fernandez et al., 2018; Jilk & Crespo, 2015; Scheiner, 2022). In the lens of our theoretical framework, noticing students' thinking requires awareness of how the students think mathematically, as well as corresponding content knowledge and pedagogical content knowledge (Shulman, 1986) which has to be connected with situation observations. However, further awareness criteria might be beneficial for a more comprehensive analysis of the respective classroom situations, including the quality of teacher reactions – in line with the research need introduced above.

Vignette-based methodologies offer a broad spectrum of possibilities in the context of exploring this research need. By the notion of *vignettes*, we understand *representations of practice* (Buchbinder & Kuntze, 2018), which by definition stand for classroom situation contexts and carry demands relevant for the profession of mathematics teachers. Vignettes can be taken from authentic contexts such as video recordings of classrooms or they can be designed so as to exemplify elements of mathematics education theory, enabling a normative approach for defining and coding analyses as "adequate" in case an analysis fulfils the corresponding built-in analysis requirement. Vignettes can combine analysis needs in complex settings (Kuntze et al., 2021) and be an opportunity for analysis and profession-related learning (Kuntze et al., 2022; Skilling & Stylianides, 2020). Design possibilities and characteristics for vignettes and their use are described in Kuntze et al. (2022) or Skilling and Stylianides (2020) in more detail.

Research interest and research questions

For exploring the role of aspects of noticing the students' thinking in relationship to noticing related to other potentially relevant analysis criteria, vignettes in which student thinking can be analysed and plays a role for the subsequent interaction are particularly suitable. The combination with a teacher reaction, which does not optimally connect to the student's thinking, is a vignette design feature which adds a further profession-relevant analysis requirement to the analysis of the student's thinking, which is however connected to this analysis step, as the teacher's reaction has to be analysed against the background of how the student thinks. Being able to evaluate the helpfulness of a teacher reaction is a profession-related requirement (e.g. Hardy et al., 2019; Rotem et al., 2019) which necessarily builds on noticing related to the student's thinking. Consequently, this mainly explorative study based on specifically designed vignettes focuses on the following research questions:

(1) Does an adequate analysis of students' thinking warrant also an adequate analysis whether teacher reactions to the learners' needs are appropriate?

(2) Which other analysis foci beyond aspects of students' thinking can be identified, which may strengthen teachers' analysis of classroom situations in this context?

The study aims in the first place at collecting proofs of existence for phenomena, namely (according to the first research question) cases, in which noticing of the students' thinking coincides with inappropriate analyses of the rather unhelpful vignette teacher reaction, and (as far as the second research question is concerned) insight into the potential role of other analysis foci. As noticing and analysis competences can be assumed to be still under development, a focus on pre-service teachers is particularly interesting for this study, as implications for vignette-based learning opportunities in mathematics teacher education might be drawn from potential findings.

Design and methods

The vignette set this study is based on has been designed by the second author and been further developed in collaboration with the first author of this paper, checked for validity under the scope of school culture aspects from the Israeli and German perspectives. The design features mentioned above have been implemented. This paper focuses on the vignette shown in Figure 1. In this vignette, the students reason in an example-based way and appear to have difficulties in translating their approach into a general, variable-based algebraic formula. The teacher's answer mainly takes up the student's "n+1" utterance like a formula conjecture and provides two falsification variants. The student probably interprets the teacher's answer as not having understood their approach and continues with a new attempt of his explanation. The vignette does not offer evidence that the teacher's conjecture-falsification approach is useful for the learners, in relationship to their mathematical thinking.

Together with the vignette presented in Figure 1, the pre-service teachers were asked (a) to explain what the student(s) meant mathematically by their statement and to justify by referring to their observations from the classroom dialogue. Further, they were asked (b) about their interpretation of the teacher's intention in her response, to judge (c) on the helpfulness of the teachers' reaction for

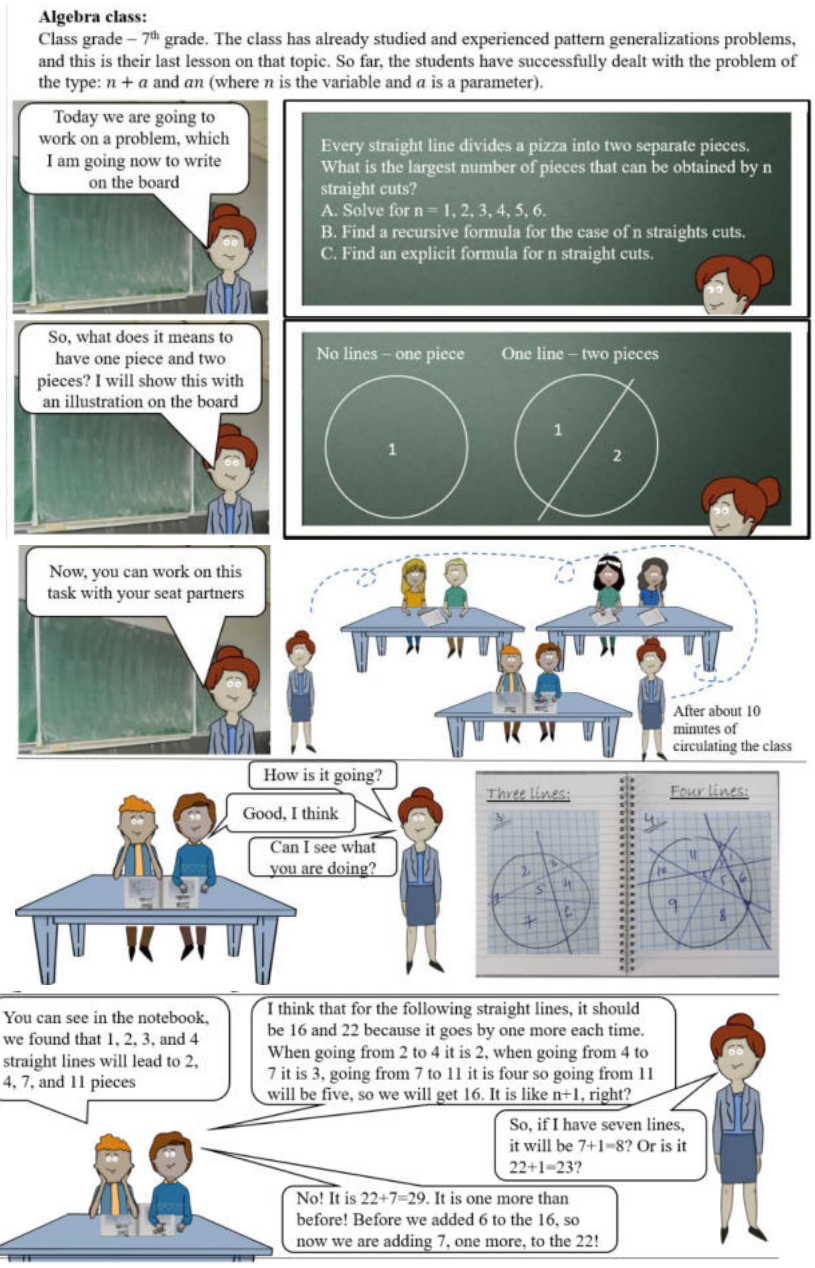

Figure 1: Vignette (inspired by Palatnik & Koichu, 2015) in English translation

the students' learning, and to describe (d) how they would have reacted in the classroom situation. Again in these sub-tasks, the pre-service teachers were explicitly encouraged to justify their answers with their observations.

The vignette was handed out to pre-service teachers in Israel and Germany. As a first step in the collaborative research process, this paper focuses on the data from a sample of 30 pre-service teachers from a South-German university of education, who were asked to answer the vignette questions in written form and hand them in through a *Moodle* environment associated to a seminar, in which no specific intervention related to the vignettes took place. According to a criteria-based interpretive

analysis (Mayring, 2015), a first coding procedure focused on whether the student's thinking had been analysed in a way that shows that the student's mathematical reasoning has been captured. On the basis of this selection of cases according to the adequate analysis of student's thinking, the cases were further examined with respect to the analysis of the teacher's reaction related to the student's thinking, using a bottom-up interpretive approach in accordance with the explorative research interest. As laid out in the previous section, this analysis led to the identification of cases that play the role of proofs of existence of phenomena according to this study's research aims.

Results

In line with research question (1) and the analysis presented in the previous section, cases of adequate analyses of the student's thinking were identified, in which, however, the teacher's reaction was evaluated positively. In order to illustrate the results of the analysis, we present the (complete) answer data of one of these cases. The following were the answers to the analysis tasks (a)-(d) introduced above:

> "(a) The student is able to explain well how he proceeds and thinks that it continues. At the end he says a wrong sentence, which the teacher takes as an opportunity to "feign ignorance", encouraging the student to explain it again. Mathematically, he means that each step between the number increases by one. One could represent it like this:
>
> $a_1 \overset{+1}{\rightarrow} a_2 \overset{+2}{\rightarrow} a_3 \overset{+3}{\rightarrow} a_4 \overset{+4}{\rightarrow} a_5$ ... with a_n as the elements of the sequence, $n \in N$. As the students explains quite precisely how he calculates, this formula can be deduced from his statements.
>
> (b) I think the teacher has the intention that the student explains his calculation and his chain of thought even more precisely. If he manages to express his thoughts more precisely, he can deduce the formula directly from that. As it is, it still appears to be a little confused, but I am almost sure the student knows what he does, just without being able to express it in a better way. The teacher gives him the chance by her comment to find the formula himself and to improve his language abilities.
>
> (c) I think the question was helpful in the sense that it will like this be easier for the student to set up the recursive formula, as they are already on the right way by reformulating it. I see this as very helpful, as formulae make life easier in general and in task b) on the blackboard, precisely this is asked.
>
> Moreover, the student learns to express himself in a more elaborated way and more precisely and to explain, which can bring him forward on the language level.
>
> (d) I like the approach and would not have done it differently. I would build on the better formulation and have it expressed in written form, respectively ask for the corresponding formula. Maybe something like: 'Can you write this also mathematically?' or 'Is there a formula with which you can write this down?'." [Sample answer from a pre-service teacher]

Given the length limits of this paper, we will in the following have to limit the presentation of results of the interpretive analysis to key aspects.

In particular the answer part in section (a) indicates that key aspects of the student's mathematical thinking have been analysed adequately: In the first sentence, an overall interpretation of the student's reasoning is given ("The student is able to explain well how he proceeds and thinks that it continues."). The pre-service teacher judges on the vignette student's question "It is like n+1, right?" in a somewhat questionable way ("At the end he says a wrong sentence"), as the vignette student might not see the formula requested in the task in the expression "n+1" included in his question. However, the third sentence shows a consistent mathematical interpretation of the student's thinking: "Mathematically, he means that each step between the number increases by one." This is formalised in the follow-up sentence: "One could represent it like this: …", this section of the answer even showing awareness of a difference between the vignette student's representation and the arrow-based representation chosen by the pre-service teacher. The pre-service teacher links the analysis back to the observations made in the vignette ("As the students explains quite precisely how he calculates, this formula can be deduced from his statements").

This analysis of the student's mathematical thinking is however not sufficient for a critical analysis of the helpfulness of the vignette teacher's reaction. The pre-service teacher completely agrees with the vignette teacher's reaction ("I like the approach and would not have done it differently") and justifies the helpfulness (e.g., "the student learns to express himself in a more elaborated way"). At the same time, the pre-service teacher's answers to (c) and (d) hardly connect to observations from the vignette.

A possible interpretation is that the pre-service teacher neglects an analysis of how mathematically distant the vignette student's thinking is from the solution required by the task and from the vignette teacher's response, as dealing with formal algebraic expressions and variables might still be a severe obstacle for the vignette student. An additional analysis focus on relating different "kinds" of mathematical reasoning of the participants in the classroom interaction and on the ways of representing mathematical thinking involved in the classroom interaction could have been helpful here (cf. research question (2)). In the case of this vignette, such a noticing focus could have led this pre-service teacher to perceive the potential gap between reasoning with specific numbers vs. reasoning with algebraic expressions including variables.

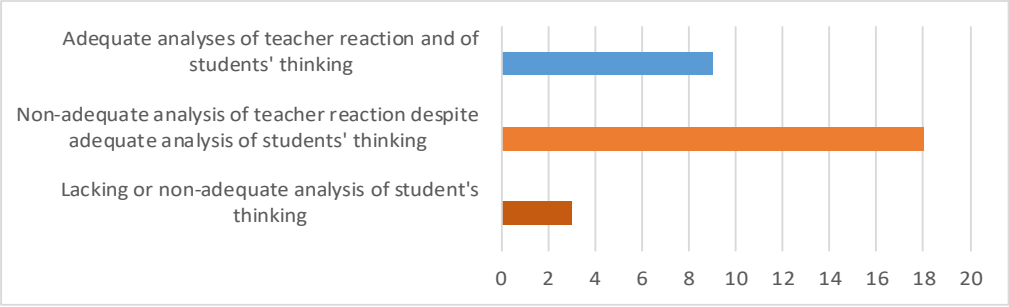

Figure 2: Frequencies of code combinations

Further examples of pre-service teachers' noticing in this sample support that an additional focus on aspects of the classroom interaction can play a key facilitating role for analysing the helpfulness of

the vignette teacher's reaction, and there are also examples of adequate analyses with evidence that such noticing had taken place. Figure 2 shows frequencies of code combinations, revealing that more than 50% of the pre-service teachers did not notice the non-helpfulness of the teachers' reaction despite an adequate noticing of the student's thinking.

Discussion and conclusions

Research question (1) focused on whether an adequate analysis of students' thinking warrants also an adequate analysis of the appropriateness of teacher reactions given the learners' needs. In the sense of a proof of existence, this research question can be answered by the case presented above, in which a pre-service teacher provided evidence of relatively deep noticing of the student's mathematical thinking in a written summary of her/his analysis of the vignette. However, the approval of the vignette teacher's reaction appears to be in contradiction to this analysis of student thinking. Analysing the helpfulness of the vignette teacher's reaction thus requires using further noticing criteria, such as a focus on comparing and relating different mathematical approaches of the participants in the classroom interaction or a focus on the quality of classroom interaction processes (cf. research question (2)). Such discourse-oriented foci of noticing and analysis should receive increased attention, both in teacher education/professional development and in empirical research into teacher noticing. When including several foci of noticing and classroom situation analysis, multi-criterion noticing (Kuntze et al., 2021) should be considered, both in the design of profession-related learning opportunities for teachers and in research into aspects of teacher expertise.

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Discourses of diversity in mathematics teacher education research: An analytical framework for meta-didactic praxeologies

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Mathematics teacher education research engaging with discourses of diversity reflects visions of mathematics education with complex relations between pre-service teachers, students, and mathematics. This requires practices that cultivate these relations. Utilising the Anthropological Theory of the Didactic, this methodological paper proposes an analytical framework for meta-didactic praxeologies. This is done by identifying the relations informed by discourses of diversity envisioned within the field. Preliminary results indicate that the proposed framework makes it possible to identify meta-didactic praxeologies by pinpointing the envisioned relations aimed at generating them. This invites a dialogue within the field about praxeological variations produced through the discourses of diversity in mathematics teacher education.

Keywords: Anthropological theory of the didactic, discourses, diversity, mathematics teacher education, praxeologies.

Mathematics teacher education pedagogies and education models

The analysis presented in this paper is an attempt to develop an analytical framework for mathematics teacher education practices based on an analysis of recent research publications engaging discourses of diversity in mathematics education. Discourses of diversity echo across political agendas and in educational and research spheres. Through the conceptualisation of equity, inclusion, and social justice, these discourses inform institutional practices, teachers' dispositions, and students' positions in mathematics and mathematics education across diverse contexts (e.g., OECD, 2018). These discourses further reflect relations and practices within mathematics education, including mathematics teacher education (e.g., Skog & Valero, 2021). In relation to mathematics and mathematics education, discourses of diversity focus on expanding multiple cultural, linguistic, and contextual perspectives in mathematics teaching and learning. They aim to represent mathematical concepts as contextually relevant to diverse identity markers, address social justice issues, or position mathematics at the core of social justice concerns.

The mathematics teacher education research field, engaging in discourses of diversity, addresses various concerns and visions of current mathematics education in schools. Although the aim of improving teacher preparation is shared, what constitutes knowledge and effective teacher education pedagogies differ. By discussing teacher education models and positioning them as co-determined by both theory and practice, Brooks et al. (2023) provide a valuable addition to widely used teacher education models, focusing on pedagogical approaches used in teacher education and their possible theoretical justifications. Even though the analysed models are positioned within a specific teacher education context, they do not address the mathematical teacher education pedagogies driven by specific discourses, such as those of diversity. This paper seeks to address this area of concern by examining how discourses of diversity inform teacher education practices. Mathematics teacher

education practices informed by these discourses do not fall under a specific model, making it difficult to comprehensively understand their pedagogical structures using Brooks et al.'s (2023) approach. However, recognising how these discourses inform teacher education practices and reflect envisioned relations is crucial to addressing the equity-driven and inclusion-oriented challenges in mathematics teacher education.

The multifaceted discourses of diversity continuously inform mathematics teacher education pedagogies, diffusing equity-driven and inclusion-oriented knowledge, visions, practices, and relations. As these discourses gain prominence in educational research as a way to address equity and inclusion, mathematics teacher education institutionalises them, encountering various pedagogical challenges (e.g., Guðjónsdóttir & Óskarsdóttir, 2020). Engaging with research publications raises critical questions about what is produced by the research field and how the discourses of diversity envision knowledge and inform mathematics teacher education pedagogies. As equity-driven and inclusion-oriented discourses are central to mathematics teacher education research, understanding their role in shaping practices is essential. Hence, there is a need to develop a framework that allows systematic analysis of envisioned mathematics teacher education pedagogies.

Theoretical Framework

The Anthropological Theory of the Didactic (ATD) considers "doing, teaching, diffusing, creating, and transposing mathematics, as well as any other kind of knowledge, […] as human activities taking place in institutional settings" (Bosch & Gascón, 2014, p. 68), where each activity is defined as praxeology. The theoretical notion of praxeology can be considered within any institution, regardless of the grain size. A *praxeology* consists of the praxis and logos blocks, and the praxis block can be further understood by the type of tasks (T) it entails and the ways of doing them (techniques – τ). The logos block further consists of technology (θ) – a discourse for the techniques – and theory (Θ) that justifies the technological discourse grounded in theories, principles and/or assumptions (Bosch & Gascón, 2014). However, these justifications do not need to be explicit; they can also include conjectures and taken-for-granted ideas residing in belief and value systems articulated by discourses that are reproduced by praxis.

In the context of mathematics teacher education, *meta-didactic praxeologies* refer to the teaching practices enacted in teacher education, and the notion is used to distinguish the praxeologies teacher education engages in, considered "the practices and the theoretical reflections developed in teacher education activities" (Aldon et al., 2013, p. 101). As this paper is positioned in the field of mathematics teacher education, meta-didactic praxeologies (MDP) will be the theoretical notion of praxeologies referred to. Thus, the type of tasks and techniques described and discussed in the paper refers to the praxis involved in teaching pre-service teachers (PSTs) how to teach the desired knowledge as envisioned in mathematics teacher education. Similarly, the logos is discussed as a discursive justification of the proposed praxis.

As praxeologies are considered a human activity of knowledge dissemination, the theoretical notion of *didactic transposition* refers to transforming knowledge from its scholarly state into content suitable for teaching in educational settings (Figure 1).

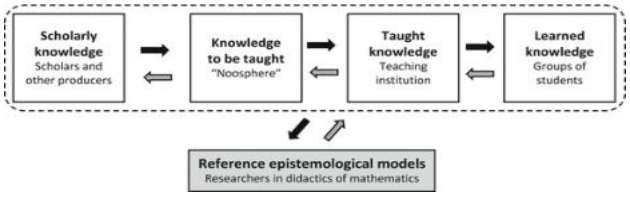

Figure 1: Didactic transposition (Bosch & Gascón, 2014, p. 71)

Produced scholarly knowledge is further addressed in the *noosphere* of the didactic transposition. The noosphere can be considered an institution or a field that deals with issues and concerns regarding all teaching processes concerning knowledge to be taught. A reference epistemological model, or knowledge reference model, acts as a heuristic and analytical tool to explore knowledge and its transposition within and across spheres and institutions (Gascón & Nicolas, 2022). Within the scope of this paper, the proposed analytical framework is positioned in the equity-driven and inclusion-oriented teacher education noosphere, and the aim is to describe the MDP as produced within the field. This is done by examining the relations connecting and linking PSTs' institutional position toward an object of primary inquiry, as well as the proposed praxis in empirical research.

It is essential to incorporate the theoretical notion of *relation* into the analytical framework as it models personal and institutional connections and links with various knowledge objects (Miyakawa, 2024). Thus, the notion allows us to identify and discuss the discourses that are the justifications on which the proposed praxis is grounded. The notation of relation is represented as $R(x, o)$, where x is a person, and o is an object – anything that exists for x. This can further be understood as "a set of links of any kind between x and o" (Bosch et al., 2020, p. xix). Similarly, from the institutional perspective, such a relation can be denoted as $R_I(p, o)$, where p is an institutional position (e.g., teachers) in the institution I (e.g., school). To exemplify, $R_I(p, o)$ models connections teachers (p) make to students' mathematical thinking (o), or critical awareness teachers (p) have towards mathematics teaching for equity (o).

Engaging with the envisioned relations that link PSTs' position to the desired objects allows for the analysis of diversity discourses within the given field. This process addresses the logos block of the praxeology driving the privileged praxis, as it, through discursive nuances, articulates the principles, assumptions and taken-for-granted ideas. In the context of this paper, the notion of relation can be elaborated as $R_{I'}(p', R_I(p, o))$ where $R_{I'}$ models links and connections that any position (p'), within the proposed noosphere knows, feels, or privileges towards the envisioned relation R_I that models what PSTs (p) should know or feel about some desired object (o). Thus, $R_{I'}(p', R_I(p, o))$, shaped by discourses of diversity, informs the praxis of MDP, which is then justified by $R_{I'}(p', R_I(p, o))$.

The aim and the research question

This methodological paper examines scholarly works in the mathematics teacher education research field that engage with discourses of diversity. The aim is to develop an analytical framework to describe meta-didactic praxeologies by elaborating on the relations and mathematics teacher education practices envisioned in the field. The guiding research question is as follows: *What meta-didactic praxeologies are produced in mathematics teacher education research engaging with discourses of diversity?*

Methodology

The analysis presented in this paper utilises a *researching research* approach, which enables the investigation of the conceptualisation of objects of inquiry and goes beyond simply summarising and mapping empirical findings insofar as it critically examines what is produced and envisioned by empirical work (e.g., Skog & Valero, 2021). The analysis identifies the nuances in diversity discourses through envisioned relations and privileged teacher education practices, although it does not constitute a comprehensive review of the literature in the field. From Skog and Valero's (2021) database (2006–2020), which focuses on equity, inclusion, and social justice in mathematics teacher education contexts, eight empirical articles (2008–2019) were selected for thorough reading and analysis, four of which are mentioned in this paper. Articles that did not have mathematics in explicit focus or did not explicitly address the relation between PSTs and a mathematics education object of inquiry were excluded. In other words, the articles focused on the knowledge of teaching mathematics for equity and inclusion envisioned for PSTs and how this knowledge is disseminated in mathematics teacher education programmes. No distinction was made between primary or secondary mathematics teacher education in the proposed analysis.

Analysis

The data processing and analysis operationalised theoretical notions of *relation* and *meta-didactic praxeologies*. The discourses captured through the notion of *relation* identified links PSTs were positioned to manifest and towards what *objects* of knowledge, as envisioned within the research field – $R_{I'}(p', R_I(p, o))$. Through the inclusion criteria for the empirical studies, institutions (I') and (I) and positions (p') and (p) were already pre-determined: (I') – mathematics teacher education noosphere; (I) – primary/secondary educational institutions; (p') – any position within the proposed noosphere; and (p) – mathematics PSTs (primary/secondary). Thus, the analysis aimed to identify the connections and links $R_I(p, o)$ emphasised in the empirical work $R_{I'}$ – implied or focused – between PSTs (p) and an object (o). To exemplify, a relation R_I (p, o) models the "connections" PSTs (p) make towards "students' multiple mathematical knowledge bases" (o). Thus, the relation between PSTs' position and "students' multiple mathematical knowledge bases" ("MMKB") is linked by "connections". Similarly, a relation R_I (p, o) models the "critical awareness" PSTs (p) develop towards "mathematics teaching for equity" (o). Consequently, the relation between PSTs and "mathematics teaching for equity" is linked by "critical awareness". Thus, through the operationalisation of the notion *relation*, the analysis of diversity discourses allows for the description of the *logos block* of MDP where the discursive justifications can be identified; it is through the precision and nuances of the produced discourses and a systematic approach to analysis that the logos block can be identified within these discourses. The examples below will further elaborate on the operationalisation of the notion in analysis.

To operationalise the theoretical concept of *meta-didactic praxeologies*, the analysis sought to identify the *praxis block* of the praxeologies as manifested in the empirical studies. This included the praxis directly addressed and those implied through broader context. Firstly, through content analysis, this entailed the *type of tasks* mathematics teacher educators engaged with to facilitate the establishment of the envisioned relation, where the desired object of knowledge was identified. In

other words, mathematics teacher educators' type of tasks was identified by the created conditions for PSTs to engage with the praxis at hand. Next, the *techniques* that were carried out to accomplish the mathematics teacher educators' tasks were determined. To exemplify, the mathematics teacher educators' type of task (T) is *to create conditions* for PSTs to develop and plan problem solving-based mathematics lessons that "connect" to students' "MMKB". Consequently, the means (τ) of accomplishing this type of task, as presented in the empirical data, is to facilitate PSTs' engagement in cultural communities and the analysis of mathematical lesson plans. To connect this example to the logos block, the justification presented by the relation modelling "connection" is grounded in social processes, establishing effective teaching of mathematics as "culturally relevant" through instructional implementations and practice-based pedagogies, for instance, "[…] teacher learning as a sociocultural activity wherein PSTs develop understandings and practices as they interact over time in multiple communities, such as mathematics methods courses […] and children's homes and communities." (Aguirre et al., 2012, p.180). In another example of MDP, the type of task (T) is *to create conditions* for PSTs to develop "critical awareness" of issues and concerns of "equitable and socially just mathematics teaching and learning", where the techniques (τ) utilised are facilitation of self-study narratives, discursive observations, emotional engagement and critical dialogue, as well as intervention and creative actions. As for the logos block, the justification lies in the relation modelling "critical awareness", where professional knowledge of mathematics teaching and learning is grounded in reflexive work, for instance, "[pedagogy of discomfort] in generating critical dialogue […] feelings of discomfort are crucial aspects of transformative learning about self and society." (de Freitas, 2008, p. 46). Together, the praxis captures the equipment utilised in mathematics teacher education to generate the envisioned relations. Positioning relations as the justification discourses of the praxis block provides an overview of the whole praxeology, praxis and logos of MDP. At this stage of the analysis, the logos block is discussed as a whole, omitting the elaboration of *technology* and *theory*.

Following content analysis and coding, two groups were established, guided by relational correlations (desired object of knowledge and relations modelling links) and the generated practices and justifications shaping these relations. This resulted in the identification of two primary praxeological approaches - *praxis-driven* and *logos-driven meta-didactic praxeologies*. Meta-didactic praxeologies are categorised as praxis-driven when they are grounded in mathematical activities and emphasise praxeologies aimed at addressing and enacting these activities within the given context. These are justified by context-specific discourses rooted in diversity and theoretical, conceptual, and instructional frameworks. In contrast, logos-driven meta-didactic praxeologies are categorised as such when they are grounded in theoretical and pedagogical principles, as reflected in the discourses of diversity and critical frameworks. They emphasise praxeologies of mathematics teaching and learning, focusing on context-specific key issues and concerns of the field. The two categories of meta-didactic praxeologies are further discussed below with supporting excerpts from the sample articles.

Results and discussion

In investigating MDP within mathematics teacher education research informed by diversity discourses, variations are observed across the empirical data examined. As a result of the analysis,

the analytical framework discussed here provides possibilities to identify and discuss the envisioned relations and desired object of knowledge that shape the discourses and justifications of the privileged practices of mathematics teacher education. Consequently, the analytical framework serves as a lens into the field engaged with discourses of diversity in mathematics education, providing an overview of meta-didactic praxeologies that are produced.

The praxis-driven MDP generate relations R_I that model didactic practices, choices, and decision-making towards (o) students' "mathematical thinking", "mathematical communication", and/or "multiple mathematical knowledge bases". Such relations model, for instance, "connections" towards children's "MMKB", as in: "[…] PSTs to explore the practice of making connections to children's MMKB" (Aguirre et al., 2012, p.188). Praxis-driven MDP moreover engage with the type of meta-didactic tasks (T) that are to create conditions for PSTs to effectively teach, develop understandings, practices, capabilities, and/or manage practices. The techniques (t) of carrying out the type of tasks vary accordingly and can be seen through, for example, facilitating situated learning and simulations "[t]o simulate teaching in diverse settings, we developed a middle school classroom in Second Life [software]" (Brown et al., 2011, p. 273). Consequently, the praxis-driven MDP are justified by practice-based pedagogies and co-construction of learning, such as community engagement that is seen in Aguirre et al. (2012):

> […] to engage PSTs with the local community and to increase their knowledge of students' out-of-school activities and practices, particularly those that might relate to mathematics instruction. (Aguirre et al., 2012, p. 181)

On the contrary, logos-driven MDP produce relations that model "critical awareness", conceptions", "becoming", "identity", and "relationships", for instance, "[…] to consider and reflect on classroom relationships and practices that can promote social justice in mathematics classrooms" (Boylan, 2009, p. 1). Logos-driven MDP also consider the type of tasks (T) that create conditions for PSTs to engage in issues and concerns of equity and social justice, for example, creating conditions that encourage PSTs to consider and reflect on social and personal ideologies, challenge beliefs and identity, develop critical perspectives, and engage in critical reflection of the self as well as social power structures. This includes providing opportunities to develop and deepen one's understanding of mathematics education, critically informed through various critical perspectives, for instance, "[…] to trouble or contest the dominant belief expressed by my PSTs […]" (de Freitas, 2008, p. 44). Corresponding to the tasks, the techniques (t) are troubling discourses and dispositions and eliciting perceptions to critically examine current beliefs and practices by interventions and critical dialogue engaging educational experiences and enactments. Such logos-driven MDP are therefore justified by transformative learning and pedagogies of discomfort.

Informed by discourses of diversity, the *relation* in the empirical studies examined is often explicitly elaborated and given a purpose and contribution within diverse contexts concerning all and/or a diverse body of students, mathematics and mathematics education, and communities/society. Driven by the discourses of diversity, the proposed MDP analytical framework could potentially be discussed using Brooks et al. (2023) categorisations, thus contributing to a more comprehensive understanding of pedagogical models. Logos-driven MDP often centre around relations that emphasise PSTs'

positions. These praxeologies address issues and concerns of diversity by fostering critical awareness and identity work and by challenging PSTs' traditional beliefs, practices, and dispositions. Thus, the examined studies deemed it necessary to situate these connections and links as part of the internal endeavour. In contrast, praxis-driven MDP tend to centre on PSTs' positions, which also emphasise improvement and change. However, the focus often lies on teaching methods, new didactical instructional strategies, and choices. Although incorporated, reflective work frequently does not engage with interpersonal aspects. Consequently, these studies situate connections and links envisioned for PSTs to establish as external knowledge to be acquired.

Engaging with the ATD and operationalising the notions of *relation* and *praxeologies* provides research strategies for understanding how equity-driven and inclusion-oriented mathematics education is informed and disseminated in mathematics teacher education, thus contributing to the production of knowledge in noosphere and the further development of a knowledge reference model of meta-didactic praxeologies driven by discourses of diversity. This analytical framework helps to grasp the nuances of praxeologies produced to engage with envisioned relations toward desired objects of knowledge. While mathematics teacher education addresses issues of equity and inclusion, these practices have yet to be systematically investigated. By engaging with the discourses of diversity, the analysis shows how these discourses inform mathematics teacher education praxeologies, resulting in logos- and praxis-driven meta-didactic praxeologies. In line with Brooks et al. (2023), this work further contributes to understanding how various models inform and shape mathematics teacher education pedagogies. Through the complex reciprocity of relations and praxeologies, the proposed analytical model opens discussions on the complexity of MDP, advancing equity-driven and inclusion-oriented mathematics teacher education. Furthermore, it can be utilised in mathematics education institutions to examine context-specific institutionalised practices and the relations that inform them.

The presented methodological paper attempts to develop an analytical framework of MDP by analysing a small sample of research. This factor might reduce reliability and validity. However, the aim is to discuss a perspective that addresses questions of diversity discourses that ATD has not yet addressed and offers pragmatic nuances that address the struggles of diversity discourses in mathematics teacher education, thus enabling the multifaceted practices of mathematics teacher education to be addressed systematically.

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Preservice teachers' identity as programmers in mathematics

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In this study we investigate preservice teachers' actual and designated identities. Analyses of preservice teachers' reflection notes written to complete a compulsory task about programming reveal both similarities and differences between preservice teachers' actual and designated identities. Both identities reflect a positive attitude towards programming and the importance of computational thinking. However, actual identity highlights preservice teachers' knowledge of problem-solving and their perception of programming as challenging but rewarding. In contrast, designated identity emphasizes the importance of programming knowledge to enhance students' learning, and knowledge of different programming tools to develop the students' knowledge.

Keywords: *Preservice teachers (PT), actual and designated identity, programming, mathematics.*

Introduction

Programming and Computational thinking (CT) have received increasing focus and important role in schools. CT represents one of the "21st century" key skills (Bocconi et al., 2018). In Europe, most countries locate CT within the general subject information and communications technology, while some countries have integrated CT in other subjects and/or as cross curricular approach (Bocconi et al., 2018). In 2020, the Norwegian mathematics curriculum for grades 1–10 was revised, and programming was integrated in mathematics. Exploration and problem solving are related to CT in the curriculum (Ministry of Education and Research, 2020). Programming is related to computational thinking, and CT is described as the thought process involved in formulating problems and their solutions so that the solutions are represented in a form that can be effectively carried out by an information-processing agent (Wing, 2006).

The introduction of programming in Norwegian mathematics curriculum poses an interesting implementation challenge. This introduction delegates the responsibility of including CT and programming in mathematics to mathematics teachers. There is a need to know more about how mathematics teachers and preservice teachers (PT) perceive programming as part of mathematics education in relation to their developing professional identities (Kaufmann et al., 2023; Nordby et al., 2024).

In their research studies, Nordby et al. (2024) found that primary teachers' lack of knowledge of CT affects their adoption of CT in mathematics teaching. Therefore, mathematics teachers need to learn more about how and when to include CT tools in mathematics. While Nordby et al. (2024) focus on primary teachers, other studies have focused on PT (Kaufmann et al., 2023; Maugesten et al., 2023). The findings of these studies reveal that the PT were motivated to teach programming, and they identify themselves as digital natives. Also, the findings revealed that not until their fifth year of teacher education, they realized that they need knowledge about both programming and experience. Kaufmann et al. (2023) found that PT had a positive attitude toward programming. Even though they

expressed a limited understanding of how programming can be integrated into their mathematics teaching, they consider themselves as digitally competent.

In this paper, we analyse PTs' reflection notes written to complete a compulsory task about programming and computational thinking. Reflective practice is essential to the Norwegian teacher education. PT are encouraged to continually reflect on their experiences, the teacher role and student perspective. We follow Lutovac and Kaasila (2018b) in understanding that narrative identity is a reflective and dynamic process involving individuals and their social context, i.e. teacher education. Therefore, social interaction and communication is essential to one's narrative identity. This reflective approach helps PT to critically evaluate and reshape their professional identities over time. The Norwegian context interacts to create a supportive and dynamic environment for PTs' identity formation. Our aim was to explore how PT in their fifth year of teacher education identify themselves as mathematics teachers using programming to solve mathematical problems. The research question addressed is: What characterizes PTs' actual and designated identity based on their written reflections in a compulsory task when solving a mathematical problem through programming?

Teachers' identity

Considering the new requirement to incorporate programming into mathematics education, we explore how this curriculum change may influence future teachers' identities as mathematics teachers teaching programming. Over twenty years ago, da Ponte et al. (2002) highlighted that "ICT may actually play an important role in teacher education, thus contributing to the professional and personal development of prospective teachers" (p. 97). However, the identities of PT are likely still evolving as they transition from being PT to becoming teachers (Beijaard et al., 2004). These identities will be influenced when they encounter new challenges, such as integrating programming, a subject they did not experience in their own schooling (Arslan & Haser, 2024).

In recent decades, educational research has increasingly focused on teachers' professional identity, recognizing its crucial impact on their performance (Philipp, 2007). Lutovac and Kaasila (2018a) define mathematics teacher identity as teachers' understanding of themselves regarding mathematics and its teaching. Teacher identity is the essence of the teaching profession, as it helps teachers formulate their own notions about "how to be," "how to act," and "how to understand" their roles and responsibilities in society (Sachs, 2005). Beijaard et al. (2004) assert that teacher identity greatly affects the decisions teachers make about their instructional strategies, subject matter, and interactions with students. Understanding teacher identity in mathematics education and teacher education programmes has an important role in developing teacher identity in line with contemporary understandings of mathematics teaching and learning (Arslan & Haser, 2024).

Following Darragh and Radovic (2020) and Sfard and Prusak (2005), we consider teachers' professional identity to be the narratives that individuals use to justify, explain, and understand themselves in relation to their work as teachers. These narratives are constructed from social discourses and provide insights into how those individuals currently view their identities (Lutovac & Kaasila, 2018a; Sfard & Prusak, 2005).

In this paper, our understanding of identity is guided by the work of Sfard and Prusak's (2005) narrative identity framework. According to Sfard and Prusak (2005), actual identity consists of narratives about the actual state of affairs and these narratives are usually told in present form. A statement such as "I am quite good at programming in mathematics," is an example of actual identity. Designated identities are narratives that have the potential to become a part of one's future actual identity. Such identity can be recognized by the use of future form or words that express wish, commitment or obligation. A statement such as "I must learn programming because one day I will teach my students about programming in mathematics" is an example of designated identity. According to Arslan and Haser (2024), closing the gap between actual and designated identity is described as a learning process, and failing to close the gap between these two identities negatively affects identity development. Therefore, exploring PTs' identity is important to understand how they identify as a mathematics teacher teaching programming.

Method

The data material is a compulsory written task for PT in their fifth year of teacher education. The topic was to develop and solve a problem-based task in mathematics related to computational thinking and programming suitable for future students in primary and lower secondary school. The PT chose their own mathematical problem to solve based on the criteria given by the teacher. PT had some experience with programming in previous workshops, but this is their first written task.

The 31 respondents (15 from 2022 and 16 from 2023) were PT studying to teach grades 1–7 (8 respondents) and grades 5–10 (23 respondents). The group of PT included both males and females aged between 23 and 30 years old. The PT gave their consent for data collection and were informed about the possibility of withdrawing from the study whenever they wanted. The project was also reported to SIKT (Norwegian Agency for Shared Services in Education and Research).

PTs' identities were explored by using Sfard and Prusak's (2005) concepts of actual and designated identity. These identities were further analysed by using the categorisation of Beijaard et al. (2004). They developed three relevant categories to describe teachers' identities: The teacher as subject matter expert (SME), as pedagogical expert (PE), and as didactical expert (DE). SME teachers have profound knowledge on the subject matter. PE concerns teachers' engagement with students and supporting the growth of students. DE teachers possesses the necessary expertise and deliberate intent to design and deliver lessons related to curriculum that facilitate students' deep comprehension of the subject matter. We have not had the opportunity to conduct a longitudinal study on changes in identity, such as the one by Arslan and Haser (2024).

The first author divided and placed narratives relating to actual identity in one category and narratives relating to designated identity in another category. The narratives belonging to actual identity often started with the words "I can" or "I am," while the narratives in designated identity started with "I will do ... in the future". The first and second authors (who do not teach the PT) discussed narratives in each category and placed them in the subcategories SME, DE and PE as described both by Beijard et al. (2004) and in Arslan and Haser (2024). Narratives categorized as SME are related to PTs' written reflection focusing on their own and future students problem-solving and programming skills. Narratives within DE are written reflections about teaching and learning mathematics and

programming. Narratives related PE are written reflections about students' growth and their own role as a teacher. The difference between DE and PE is that DE is related to mathematics and programming, while PE is more general without being connected to specific subject. However, it was not always easy to distinguish between PE and DE in the analysis process. In cases with uncertainty between PE and DE, these were discussed by the first and second author until agreement was reached. Each narrative was assigned with the year, the program 1–7 (B) or 5–10 (U) and a number (e.g., 23B1 was a PT from the batch of 2023, grades 1–7 and number one).

Findings

The table below presents a brief overview of the findings.

Table 1: A summary of the findings

Category/Subcategory	Actual identity	Designated identity	Total
Subject matter expert SME	11	7	18
Didactics expert DE	0	7	7
Pedagogical expert PE	8	13	21
Total	19	27	

From the table, we see that, except for Actual identity, DE, all subcategories are represented, and the total number of narratives identified varies from 0 to 13. According to the main categories, Actual identity and Designated identity, more narratives are identified as designated identity (27) than actual identity (19). This finding may indicate that the PT have chosen tasks with content that can be linked to their future profession because they are so aware that they are going to become teachers.

The identification of the subcategories Subject Matter Expert (SME) and Pedagogical Expert (PE) were 18 and 21, respectively. This can be explained by the fact that most PT knew little about programming in mathematics beforehand and therefore described the academic challenges they encountered in their problem-solving process (SME). Teacher education in Norway is not a subject-specific education in mathematics, which may be why the narratives are more general and apply to multiple subjects (PE). In the following sections, we present some illustrative narratives from the data and their categorization into the subcategories presented in Table 1.

Subject Matter Expert, SME

Regarding PTs' SME, many of their narratives focus on their own problem-solving skills (actual identity) and how they should as a becoming teacher focus on the relationship between programming and problem-solving (designated identity). The excerpt below is a narrative that illustrates PTs' SME and actual identity:

> 23U6: This is a task that has made me a better problem-solver, through patience and debugging, and which will help me further when it comes to programming.

The PT had little experience with programming during their study, and therefore it seems important for them to develop knowledge about computational thinking when they are programming.

The PTs' designated identity reveals a focus on their future students:

22U6: For students to develop computational thinking, it is required that they identify a problem and find algorithms to solve this problem using digital tools.

Another important part of their designated identity is the importance of an in-depth knowledge of various computational thinking tools/programs. This PT emphasises Scratch as a suitable program for grades 1–7.

23B6: Given that I am specializing in grades 1 to 7, I think Scratch will be the best-suited program and I will gain more knowledge and competence in it.

The PTs' narratives are seldom about mathematical content knowledge. They want to develop skills in programming and see that problem solving can be useful for that.

Didactical Expert, DE

The narratives of DE were all designated identities. In the narratives in this subcategory, the PT emphasize diverse teaching methods and how CT could be related to the curriculum.

22B2: This also fits both after grade 4 and grade 5 to achieve the competence goals of 'creating (and programming) algorithms and expressing them using variables, conditions, and loops'.

This PT emphasizes that the task she has chosen is meaningful. It is relevant for future planning and content in lessons. Students need different learning styles, and variation is important.

22U8: Another advantage is working with different methods. It introduces a type of variation in teaching, where students can engage with mathematics in a different way.

According to the narratives categorised as DE, it seems that didactic competence is important as becoming teachers in their students' learning processes in programming.

Pedagogical Expert, PE

Supporting the growth of future students and themselves is a hallmark of the narratives in the subcategory PE. Narratives categorized as actual identity are for instance connected to perseverance.

23B5: I have also gained insight into my own ability to handle challenges and to have perseverance in working to find solutions and to use programming tools to solve a mathematical problem by practicing with different algorithms until I found the solution I wanted.

Another PT linked the challenges to course literature.

23U1: I found it quite challenging to work with programming as I repeatedly discovered errors, had to fix the problems, and then continue the process until I got it right. This can be related to Herheim and Johnsen-Høines' (2021) research on productive struggling, even though I had no one to spar with. I thought a lot to myself and used the internet to find good solutions on how I could solve the problem along the way.

With respect to their future profession, PT refer to both students and their role as teachers. With a focus on students, the PT refer to motivation, creativity, interest, and ownership that support personal growth as this narrative shows:

23U5: I believe one advantage of programming is that there may be students who do not excel in mathematics usually but have an interest in this. I also believe this type of

work can give students the feeling that they can be creative and create something of their own. They gain ownership of the task/problem.

The role of the teacher and future teaching becomes apparent in this PTs' reflection:

23B5: I have chosen this language because it has great relevance for future teaching. The algorithmic way of thinking involves a systematic approach to problem-solving and decision-making. The key concepts and methodologies that characterize this approach play a central role in tackling challenges in a structured and efficient way. The algorithmic way of thinking is a valuable approach to developing problem-solving skills and promoting efficiency in various areas of life.

A summary of the findings shows that through their work with the self-chosen task, PT reflect on their identity as learners both in terms of academic, pedagogical, and didactical aspects in their narratives. We identified that most instances which were categorised as actual identity included narratives of PTs' knowledge about problem-solving and their development as problem solvers by using programming as a tool. The PTs' designated identity gets revealed in the narratives with both knowledge and reflection around students, the teacher role, and the need for knowledge. We did not identify any differences between the two programs, may be because they are coordinated.

Discussion and conclusion

The current curriculum in Norway (Ministry of Education and Research, 2020) requires the integration of programming into mathematics education. To prevent programming from becoming a tool that teachers struggle to incorporate or reject (e.g., Kaufmann & Stenseth, 2021), it is important to understand how the identities of prospective teachers influence their perceived capabilities as mathematics instructors using programming. Our research question was: What characterizes PTs' actual and designated identity based on their written reflections in a compulsory task when solving a mathematical problem through programming? Having a clear and decisive designated identity is critical in shaping actual identity, because designated identity serves as a roadmap in the identity development process (Arslan & Haser, 2024). Our research offers additional and valuable insights into the limited studies on PTs perspectives regarding the integration of programming into mathematics education, particularly by examining how this introduction may influence their actual and designated identities. The results might have been different if the reflections were oral.

Our findings indicate that PTs' identity as SME, the actual and designated identities is sometimes consistent, and other times different. The consistency is due to their narratives about SME presented in a positive way both in the actual and the designated identity. Knowledge in computational thinking is important in both SMEs, their own CT in actual identity and the students' CT in designated identity. We identified that most instances which were categorised as actual identity included narratives of PTs' knowledge about problem-solving and their development as problem solvers. However, in the instances categorised as designated identity focused on knowing programs to develop the students' knowledge. The instances of PTs' actual identity are characterized by a general impression that programming is difficult. It requires hard work to learn programming, which can sometimes be frustrating. Despite finding it challenging, they remain positive about working with programming. Our results are in accordance with Kaufmann et al. (2023) and Nordby et al. (2024). In Kaufmann et al. (2023) PT maintained a positive attitude towards programming. None of them opposed its

inclusion in mathematics education, likely because they felt the requirement of integrating it into their teaching due to curricular mandates. According to Nordby et al. (2024) despite teachers encountering some resistance, and their struggle to embed programming in their mathematics teaching, they were positive to include programming in mathematics education.

Our findings indicate that PT who used DE narratives focused on their role to support the learning of their students. This is in accordance with Arslan and Haser (2024), who found that PT whose metaphors were classified as DE emphasized the significance of employing diverse methods and being mindful of students' needs based on their varying knowledge levels. They highlighted their aspiration to be mathematics teachers capable of effectively utilizing different approaches to support their students' mathematical learning in a meaningful manner.

Our findings indicate a difference between PTs' actual and designated identities with respect to how they identify as PE. Their actual identity is characterized by the emphasis on perseverance and highlighting productive struggle as an important trait when working with programming. When it comes to designated identity, they prioritize motivating students in programming, sparking their interest and creativity, and recognizing that programming can support diverse teaching methods. This is consistent with Kaufmann et al. (2023) and Nordby et al. (2024) who also found that the PT and teachers emphasize the teacher's motivational role.

Our results contrast with Arslan and Haser's (2024) findings. According to their research, the PTs did not feel ready as mathematics teachers. They stressed their lack of experience. Their negative emotions as a teacher emphasised the need for professional development, and they had feelings of fear and anxiety due to this lack of professional development. We do not find any of this in our data. Although the PTs emphasise that they have little knowledge and experience with programming, none of them underscore frustration. Nevertheless, for teacher education we suggest more knowledge about programming in mathematics before the written task. Considering that the inclusion of programming in the Norwegian mathematics curriculum is still in its infancy and that our participants are just beginning their journey, it seems based on their narratives that they are accepting the inclusion of programming in mathematics education. Our analysis identifies an inspiration and a willingness to embrace new ideas and activities with programming. Focusing on and nurturing these efforts could lead to PT becoming facilitators who welcome programming into mathematics education.

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Exploring mathematics teachers' informal use of an online self-learning platform while planning a digital learning environment

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This case study explores primary school mathematics teachers' informal self-directed online learning experiences in the context of lesson planning for digital combinatorics learning environments. The study provides first insights into how teachers interact with online meta-resources supporting their informal professional development (PD), addressing the gap in understanding teachers' actual usage of online resources and the lack of foundational knowledge for user- and needs-oriented development of online platforms. Three teacher teams were recorded during their lesson planning sessions while using an online self-learning platform. Their topics of communication and digital actions were qualitatively analysed, revealing patterns in platform usage and topics of communication. These findings offer first insights into teachers' informal PD with online resources that might be useful for the design of user scenarios for future online self-learning platforms for teacher PD.

Keywords: Informal professionalization, online professional development, self-directed learning.

Introduction

The design of learning environments using mathematical action technologies, aimed at fostering a deep understanding of mathematical content, presents a significant challenge for teachers as it requires Technological Pedagogical Content Knowledge (TPACK) (Mishra & Koehler, 2006). In terms of teacher professionalization in this area, the approach 'Learning technology by design' (Koehler & Mishra, 2005) has been proven to be successful, providing teachers with materials driven by their immediate requirements while solving pedagogical challenges. However, the demand for support typically arises when teachers plan lessons for their own classrooms, rather than during formal professional development (PD) programs or workshops. In today's digital age, the abundance of online resources allows teachers to direct their professional growth according to their specific needs (Beach, 2017). Meta-resources (MR) specifically designed to assist mathematics teachers in planning teaching and learning activities, can fulfil a dual purpose: supporting lesson planning while simultaneously promoting teacher PD (Gueudet et al., 2023). Teachers' actual usage and related challenges while using online MR are not well understood yet. This results in a lack of foundational knowledge for user- and needs-oriented development of online platforms, despite the growing importance of this form of PD. Therefore, there is a desire for research that observes and analyses teachers' actual use of digital platforms in their work (Gueudet et al., 2023). These insights inspired the author and co-developer of the *Kombi* app[1] to create an online learning platform providing MR for all facets of TPACK required for teaching combinatorics in primary schools using the *Kombi* app, which teachers can access for informal PD while planning their lessons. The teachers' use of this platform will be in the focus of this study. After providing a brief overview on the theoretical

[1] The German version of the free iOS app for iPads is available at https://apps.apple.com/de/app/kombi/id1626071081.

background of informal PD, the use of MR during lesson planning and online self-directed learning, as well as introducing the self-learning platform for teaching combinatorics with the *Kombi* app, this paper addresses the research question: What usage patterns can be observed among teachers using an online self-learning platform with meta-resources while planning a digital learning environment for teaching combinatorics?

This question will be addressed through a qualitative analysis of the teacher teams' actions and discussions throughout their video-recorded lesson planning process. The goal is to identify usage patterns that can inform the design of user scenarios for future online self-learning platforms for teacher PD and guide the evaluation of design decisions for the *Kombi* platform used in this study.

Theoretical background

Informal PD is conceptualized as self-directed and self-initiated learning activities, primarily driven by intrinsic motivation with the goal of professional growth or improvement, occurring outside of organized learning events. Therefore, these informal learning activities are not planned or curriculum-based, but intentional, open, practice- and action-oriented, with a focus on reflective experiential processing (Röhl et al., 2023). Evers et al.'s (2016) five categories of teachers' informal learning activities can be applied to classify these activities: (1) *keeping up-to-date* (e.g., reading relevant literature and online resources), (2) *experimenting* (e.g. trying out new teaching methods, materials or technologies), (3) *reflecting and asking for feedback* (by themselves, with colleagues or students), (4) *collaborating with colleagues with the aim of improving lessons* (e.g., peer-planning, discussing of lessons), and (5) *collaborating with colleagues with the aim of improving school development* (e.g., discussing ideas about improvement and innovation on a school level). The role of digital technologies is emphasized in each category.

One opportunity for teacher informal PD and the ongoing improvement of teaching practices is during lesson planning (Gueudet et al., 2023), which is an important part of the everyday teacher job. The collaborative design of lessons by teacher teams can be particularly effective for professional learning, especially when they anticipate students' solution strategies and responses and engage in discussions about the lesson plans (Wake et al., 2016). Even though lesson planning is a routine aspect of teachers' work, it can be demanding due to the cognitive effort required before they engage in teaching and interact with students. Following the Documentational Approach to Didactics (Trouche et al., 2020) (digital) MR can support teachers in selecting appropriate materials and open up new possibilities for both individual and collaborative lesson planning (Gueudet et al., 2023). PD is particularly needed in combinatorics: While determining the number of elements in a set is crucial for both probability and calculations (Höveler, 2018), teachers often struggle with this topic, facing challenges in both content knowledge and didactics (Matitaputty et al., 2024).

While planning MR to support teachers' lesson planning, it needs to be considered that teachers do not follow prescriptive frameworks or normative guidelines when planning lessons and the typical linear steps—such as defining objectives, selecting activities, organizing, and evaluating lessons—do not reflect how teachers actually think (König et al., 2021). This suggests the presentation of MR in a nonlinear form. However, when digital MR are not presented in a sequenced order, only available on different websites or not always available on trustworthy websites regarding their quality, it

requires teachers' self-directed online learning (SDOL, Beach, 2017) to independently seek out and acquire knowledge according to their own needs, taking responsibility for their own learning. Since teachers do not typically follow linear steps, it is crucial to understand how they approach SDOL. Beach (2017) developed a model for teachers' SDOL, describing the central phenomenon 'Navigating a PD Website' with categories like, amongst others, *Focusing on Student Needs and Instructional Goals*, *Evaluating Information*, *Encoding Information for Future Retrieval*, *Reflecting on Learning* or *Intentions for Practice* that will inform this paper's analysis. This understanding can help tailor self-learning platforms to teachers' needs and thereby optimize teacher informal PD.

Context, content and design of the online self-learning platform

Teachers' use of MR in lesson planning is explored via an online self-learning platform that integrates MR to enhance combinatorics lessons with the *Kombi* app. Developed within a topic-specific design research project (see e.g., Mense et al., 2023), *Kombi* supports interactive problem-solving, strategy recognition, and structured solutions. It also includes a tool for creating digital task series.[2] Since its December 2022 release, *Kombi* has surpassed 350,000 downloads, increasing demand for teaching resources. In response, the author developed a platform integrating online MR to support *Kombi*-based lesson planning as part of her PhD project (see Figure 1, right), aligning with claims that mathematics teachers need MR to support their individual and collective design work with digital resources. The resources designed using the 'Learning technology by design' approach (Koehler & Mishra, 2005), cover essential TPACK facets with text, videos, and publications.

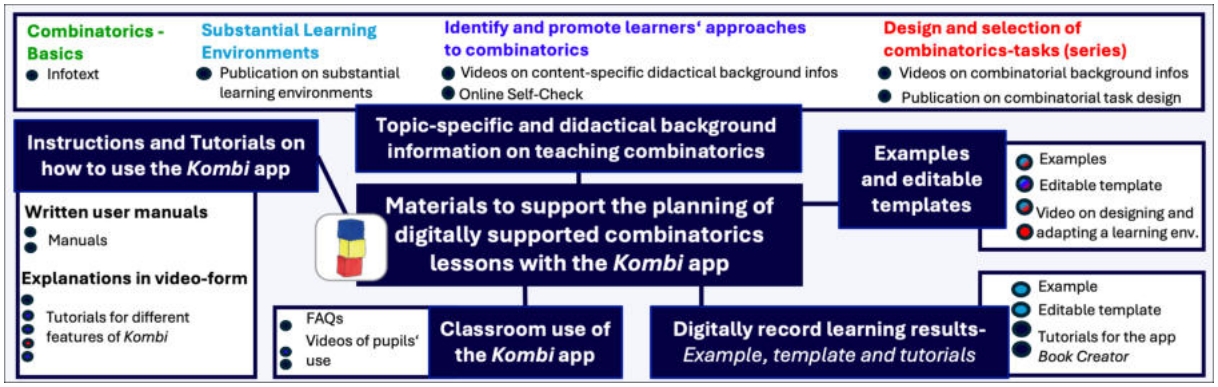

Figure 1: Brief overview of the content and design of the self-learning platform (translated)[2]

On a CK and PCK level the platform provides teachers with information on learners' possible individual approaches to combinatorics and how to promote systematic approaches to combinatorial listing and counting strategies, as well as information on important factors for sets of problems and the consequences of task design on the set of outcomes (cf. English, 1991; Höveler, 2018). On a TK level it provides manuals and tutorials for the app *Kombi*, and on a TPK level insights into how children use *Kombi* and its digital potentials, as well as which operational difficulties should be anticipated. On a TPACK level it provides an exemplary learning environment with the app *Kombi* and didactic explanations on how to digitally adapt it. All MR except the publications were designed

[2] Detailed insights into the self-learning platform and the app can be found under https://tinyurl.com/FiguresCERME14

by the author specifically for this platform and reviewed by colleagues and her supervisor. They are saved within a learning management system (LMS) course, and a document provides an organized overview with links to each resource. This approach was deliberately chosen to provide teachers with both flexibility and guidance, allowing them to navigate the LMS course without adhering to a linear sequence. Greater flexibility can better align individual needs with the content of online programs, thereby accommodating the diverse backgrounds and preferences of participants within the same course (Dille & Røkenes, 2021). However, not much is known about how teachers use this flexibility and their needs for support, leading to this study's research focus. Currently, access is limited to study participants, with plans for public release after finalized data collection for the PhD-project.

Methods

Setting and participants

The study examines collaborative lesson planning and informal PD of teacher pairs using an online self-learning platform. From a PD perspective, this approach aims to support professional learning through collaborative work. Methodologically, it has the added benefit of allowing thinking and planning processes to be verbalised and analysed within an authentic setting. In March 2024, teacher pairs designed combinatorics learning environments using the *Kombi* app, with full access to online MR. They set learning objectives, determined *Kombi*'s integration, and decided when to use MR, ensuring an authentic usage context. The sample of the three teacher pairs examined in this case study from the seven teams surveyed so far was selected to illustrate varying usage patterns. It includes three pairs of primary school mathematics teachers (aged 28 to 49, with 5 to 19 years of teaching experience) from four schools in western Germany. These teachers volunteered to participate after contacting the author for additional resources, making them representative of the intended audience for the online self-learning platform and informal PD. Team A consisted of Year 4 parallel class teachers, Team B of Year 2 parallel class teachers, and Team C of teacher friends from different schools who teach Year 2 and Year 3. All participants have previously explored the *Kombi* app, but none have yet used it in their classrooms.

Data collection and analysis

The teacher pairs met in person after school to plan their learning environments, while the author joined their meetings online via Zoom. The teachers used two devices: one computer for accessing the self-learning platform and making planning notes, and an iPad to use the *Kombi* app and other apps if needed. Both screens were shared via Zoom. After a brief introduction, the researcher muted her microphone and turned off her camera to quietly observe the planning process and record both the conversations and the two shared screens (cf. Beach & Willows, 2014).The recordings were then layered into one video per team. There was no time limit for the planning process, leading to planning times between 1,5 and 4 hours. Data collection also included a pre-survey with demographic and self-assessment questions as well as a subsequent interview after the lesson planning but those will not be in the focus of this analysis. To capture variations in discussions during lesson planning that could arise from using the resources (or the other way round), both the digital actions on the screens and the conversations were analysed in 30 second intervals using a qualitative content analysis (Kuckartz, 2019). With two screens and two teachers per team, up to two codes could be assigned per interval

and subtheme: one for each teacher's digital action and one for their communication topic, if working individually. The categories for analysing the usage patterns of teachers using an online self-learning platform with MR while planning a digital learning environment for teaching combinatorics were based on both Evers et al.'s (2016) categories for PD at work and Beach's (2017) themes for teachers' SDOL, explained in the theoretical background. Evers et al.'s (2016) category *collaborating with colleagues with the aim of improving lessons* naturally influenced all categories in this study due to the setting of data collection. The categories from literature were then inductively expanded to better grasp the teachers' digital activities and topics of communication as explained in the following section.

Results

The inductively expanded categories and (if applicable) their theoretical bases are presented in the upper half of Figure 2. These categories with their assigned colours respectively were then used to sequentially analyse the teachers' planning process which is visualised in the bottom half of Figure 2. The sequential visualisation of the color-coded categories reveals patterns in the three teacher pairs' usage of the resources and their topics of communication, serving as illustrative examples rather than representative findings. These patterns will be labelled and further described in the following.

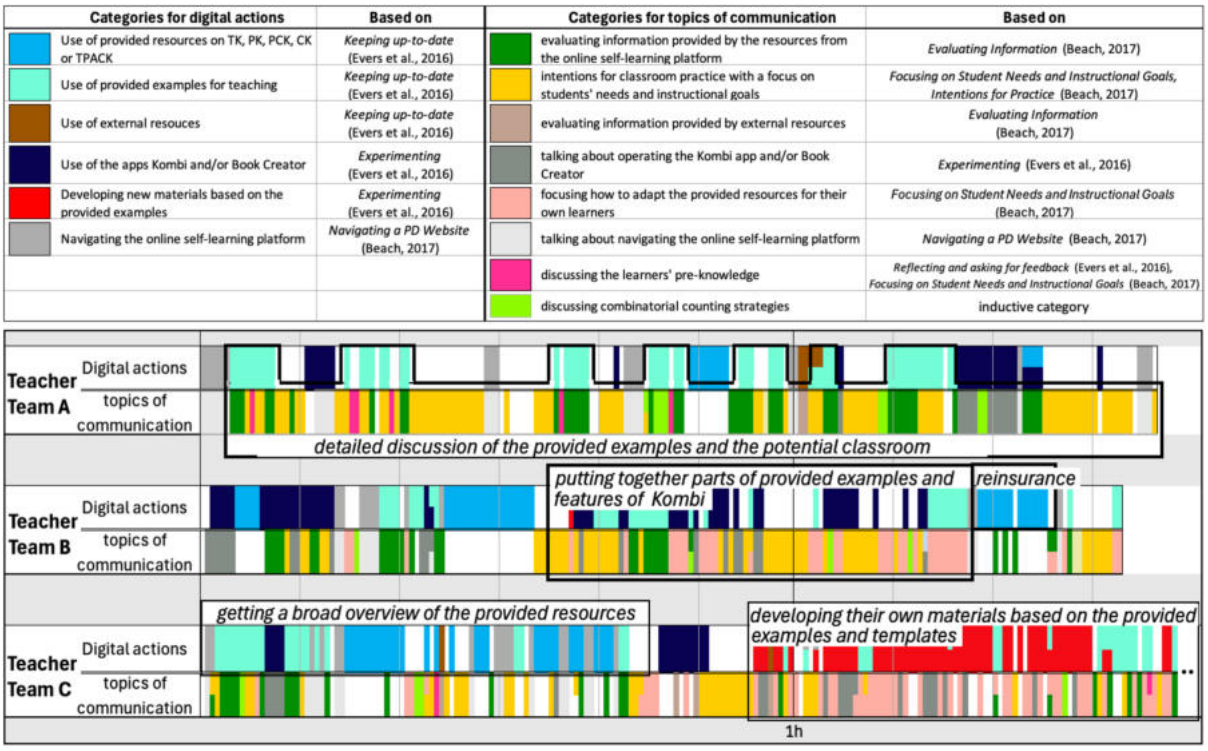

Figure 2: Inductive categories and their theoretical background and sequential visualisation of codes during the planning process with highlighted specific features of the usage patterns

The reflective example users: Team A extensively utilizes resources from the self-learning platform, particularly the provided examples (turquoise codes), to engage in deep discussions and reflections on the insights gained and their intentions for classroom practice (dark green and yellow codes).

Rather than adapting the examples, they plan to use them as they are, dedicating a significant portion of their planning time to anticipating student solutions to the combinatorial tasks and their use of the app and strategizing how to address them in their teaching like in the following illustrating statement:

> Ms. E.: The question might then arise: Do I always have to fill in the placeholders from scratch, or can I reuse the ones I already have? That's a structure I wouldn't specify for the children. It's something the more capable ones develop themselves, like: 'I don't want to drag everything in there anew every time; I just want to make adjustments because I know these three colours always go here, and three different ones above them.' For example—these three lower blocks. That's why I wouldn't address this top bar. The idea is that they recognize: I have this placeholder of two blocks, which I fill in with colours by dragging them in.

Those seeking reinsurance: Team B spends more time in the *Kombi* app (navy codes), anticipating learners' use of the app, compared to the other groups. They decide that they will have to adapt the examples for their year 2 learners, however, when discussing whether to design a new worksheet and digital research journal for their learners based on the templates, one of them states "I personally wouldn't dare to approach that just yet." Instead, they spend their remaining planning process making notes on how to put together different parts of provided examples and features of the *Kombi* app and implement them in their classroom practice (turquoise and navy codes, combined with coral and yellow codes). Since they often had difficulties consolidating their teaching ideas, they decided to validate their planning decisions with combinatorics-specific resources from the platform on how to promote systematic approaches to combinatorial listing and counting strategies, which they stated, "really helped to bring everything together, to summarize those scattered thoughts, and to help reformulate the goal in a different way."

The informed creatives: Team C focuses on acquiring an overarching understanding of the available resources during the initial 45 minutes (blue and turquoise codes). Following a comprehensive discussion centred around prospective pedagogical applications, they decide to design their own teaching materials based on the provided examples (turquoise codes) and templates (red codes). The remaining three hours of their planning process (partially omitted in Figure 1 for clarity in sequence comparison) they spend discussing their adaptions (coral codes), the operation of the apps (grey codes) and their intentions for classroom practice. This iterative process involves frequent transitions between revisiting the original examples and developing their own materials.

Discussion and Conclusion

The three cases discussed above, *The reflective example users*, *Those seeking reinsurance*, and *The informed creatives,* reveal distinct usage patterns of the MR on the online self-learning platform, leading to different documents according to Trouche et al.'s (2020) Documentational Approach to Didactics. First analyses of other teacher pairs of the PhD-study suggest that the described patterns can be found in other teacher teams' usage of the platform, however due to the small sample size, no statements can yet be made about the representativeness of the results. While Team A did not modify the provided examples, distinct adaptation processes emerged with Teams B and C, highlighting how they personalized the examples and required different resources at various planning stages. These patterns may be specific to planning combinatorics lessons, and further research is needed to explore whether these findings are topic-specific, focusing on the designed lessons and the teachers' use of

MR to support CK and PCK. In further analyses, the combinatorics-specific perspective will be brought to the forefront and interconnected with the technical perspective on lesson planning. Although these findings are preliminary, they emphasize the importance of allowing teachers flexibility in selecting resources, suggesting that a non-rigid sequence for accessing MR is well suited for informal use of online platforms aimed at supporting learner-centred lesson planning.

While Beach's (2017) model of SDOL and Evers et al.'s (2016) categories offered helpful insights for initial analysis, they had to be inductively expanded to be suitable for informal PD during lesson planning in a teacher team. The observed differences in usage patterns are likely influenced by factors such as teachers' experience with combinatorics, familiarity with digital technologies, expertise in digital learning design, TPACK, and students' knowledge levels. These differences, typical of teachers engaging informally with online platforms, reflect real-world teaching diversity and reinforce the validity of the findings. Future research could explore how these factors shape resource usage and how the platform can better support teachers with varying backgrounds, enhancing learner-centred lesson design. It is important to note that the sample is biased, as it consists of self-selected, highly motivated teachers; however, intrinsic motivation is central to informal PD.

The identified patterns might be useful user cases for the future development of online self-learning platforms to support lesson planning. Further analysis of the data will focus on teachers' design and redesign processes of digital MR and the challenges experienced by the teachers in the planning process. Additionally, important aspects yet remained unexamined in this study, such as the quality and effectiveness of the lessons designed by the teachers, as well as their professional learning outcomes. It is still unclear whether all observed usage patterns led to improved instructional practices. Future analyses will not only explore these questions but also aim to uncover key factors that impact the success of design and adaptation activities involving the provided MR. The overarching goal of the project is to gain a deeper understanding of the conditions that support informal teacher PD through these processes and to derive insights for the effective design of an online platform incorporating MR.

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Levels of noticing competence among preservice secondary mathematics teachers in a video analysis of functions

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This study examines the dimensions of attending and interpreting within noticing competence and their respective levels, as demonstrated by 35 preservice secondary mathematics teachers through a narrative analysis of a video on fundamental concepts related to functions, using the Learning to Notice Framework and content analysis. The results indicate that, in the attending dimension, participants predominantly exhibit mixed or focused levels, while in the interpreting dimension, they tend to display a mixed level of competence. A detailed example of how these levels were identified is provided in the paper. Further research could explore the possibility of assigning a single level to each dimension, as some cases revealed varying levels within the same narrative, particularly given the preservice teacher status of the participants. Additionally, future studies could refine the descriptors of noticing levels to better capture the nuances of mathematical content.

Keywords: Teacher education, prospective teacher, function, narrative, content analysis.

Introduction

Numerous studies in mathematics education have highlighted the importance of developing preservice teachers' competence in perceiving and interpreting students' mathematical thinking (Scheiner, 2023). This competence, known as noticing, is closely linked to the noteworthy events that mathematics teachers observe in the classroom and how they interpret these events to support student learning (Lopez & Zakaryan, 2023; Mason, 2002; van Es & Sherin, 2021).

According to Sherin and van Es (2005), noticing should be integrated into teacher education programs, providing preservice teachers with opportunities to engage in practices that develop this essential competence, which is considered a key professional element (Mason, 2002; Sherin et al., 2011). Given their limited classroom experience, preservice teachers often face challenges in focusing on students' thinking and other noteworthy didactic events, interpreting their observations, and making decisions based on these insights (Mason, 2011). Therefore, it is crucial to foster and examine the noticing competence of preservice teachers (Sherin et al., 2011; van Es, 2011).

This paper is part of a larger project aimed at establishing relationships between the development of noticing and the specialized knowledge of preservice secondary mathematics teachers in Chile. We focus on a specific aspect of the project: determining the noticing levels of preservice teachers, and illustrating the complexities involved in this process. Scheiner (2016) notes that while some studies propose a linear relationship between noticing skills, others suggest a cyclical and ongoing dynamic. For instance, El Mohauyar (2024) identifies three distinct profiles of preservice elementary teacher noticing levels: average to high, low to moderate, and moderate.

Determining the noticing levels is relevant in teacher education programs because it provides insights that can inform decisions about potential improvements to help preservice teachers attend to students'

mathematical thinking. Consequently, the objective of this study is to determine the noticing levels of preservice secondary mathematics teachers through video analysis, using the levels adapted by Lee and Choy (2017) from van Es (2011). Videos have proven to be an effective tool for developing noticing dimensions, as van Es and Sherin (2021) argue that providing preservice mathematics teachers with opportunities to observe and analyze teaching practices fosters the development of these competencies. Additionally, this focus is particularly important, as research on noticing has predominantly concentrated on elementary school teachers.

Theoretical considerations

From a cognitive-psychological perspective, teacher noticing can be defined as the process of attending to relevant classroom events, interpreting these events, and, in some conceptualizations, using decision-making processes to act or respond based on those interpretations (Bastian et al., 2024; Sherin et al., 2011).

This paper builds on the *Learning to Notice Framework* proposed by van Es (2011), who identify two key dimensions of noticing: (1) attending, which refers to recognizing noteworthy features of classroom interactions, and (2) interpreting, which involves drawing on one's knowledge and experiences to make sense of what is observed. This framework is adopted because it provides a lens through which to examine what a teacher notices and interprets in a video recording of a lesson, including students' thinking, the didactic and pedagogical strategies employed, and the attention given to specific events.

Furthermore, Lee and Choy (2017) propose indicators that facilitate the identification of varying levels of attention and interpretation of noticing, building upon the work of van Es (2011). For each level, there are indicators of actions that could be evidenced by teachers (see Table 1), allowing for the determination of their level in each dimension.

Table 1: Dimensions and levels of Noticing

Dimension	Level	Level indicators
Attending (What teachers notice)	Baseline	Attend to irrelevant details that do not have direct impact on student learning Attend to whole class environment, behavior, generic content and learning and to teacher pedagogy
	Mixed	Primarily attend to teacher pedagogy Begin to attend to particular aspects of mathematical concepts and the difficulties associated with them Begin to attend to particular students' mathematical thinking and behaviors
	Focus	Attend to particular aspects of mathematics and relate students' confusion to the teaching approaches Attend to particular students' mathematical thinking
	Extended	Attend to the relationship between particular students' mathematical thinking and between teaching strategies and student mathematical thinking

Dimension	Level	Level indicators
Interpreting (How teachers notice)	Baseline	Form general impressions of what occurred Provide descriptive and evaluative comments Provide little or no evidence to support analysis
	Mixed	Form general impressions and highlight noteworthy events or details Provide primarily evaluative with some interpretive comments Begin to refer to specific events and interactions as evidence
	Focus	Provide interpretive comments Refer to specific students' difficulties, events and interactions as evidence Elaborate on these specific students' difficulties, events and interactions
	Extended	Provide interpretive comments Refer to specific events and interactions as evidence Elaborate on these specific students' difficulties, events, and interactions Make connections between events and principles of teaching and learning On the basis of interpretations, propose alternative pedagogical solutions

Note. Adapted from Lee and Choy (2017)

It is important to note that there are key indicators that distinguish between the different levels of noticing. These indicators are as follows: (1) at the baseline level, there is no consideration of students' mathematical thinking; (2) at the mixed level, attention to students' mathematical thinking is general, whereas at the focused level, it is directed toward specific instances of students' thinking; and (3) the distinction between the focused and extended levels lies in the fact that the extended level not only addresses specific instances of students' mathematical thinking but also connects them to teaching strategies. In other words, the extended level involves an analysis that goes beyond mathematical thinking, relating it to elements of didactic action.

Similarly, in terms of interpretation, the baseline level consists of descriptions without concrete evidence, while the transition to the mixed level involves presenting specific situations, usually accompanied by evaluative comments. As these comments become more interpretative and the specific situations are described in greater detail, the level progresses to focused. The highest level is reached when these interpretive comments lead to solutions for the events observed.

Accordingly, these categorizations will be used to assess the level of noticing competence demonstrated by preservice mathematics teacher in a video on the topic of functions.

Method

This descriptive study focuses on 35 preservice secondary mathematics teachers enrolled in a Bachelor of Mathematics Instruction program at a university in Chile. The program spans 10 semesters, during which students complete concurrent courses in mathematics, general pedagogy, and specialized didactics. At the time of data collection, participants were in their seventh semester, having already completed courses in calculus (three courses), real analysis, and calculus didactics,

among others. They were enrolled in the Classroom Situations Workshop, a course designed to engage them in video clubs and related activities that foster reflective practice.

In this paper, we present the results of an analysis based on a video clip and a single questionnaire question. The objective was to identify the elements that preservice teachers considered noteworthy in the observed video. The question, presented in written form, asked: "Identify three moments in the class that caught your attention (specify the time in the video), provide a detailed interpretation of each moment, and explain your position regarding each moment (e.g., whether you agree, whether you would have done something differently, etc.)." The request for three distinct moments was intended to guide preservice teachers in their task and to assess whether their identification of noteworthy events varied across different stages of the reflections. The final part of the question aimed to evaluate whether the participants' interpretations aligned with the criteria for the extended level, particularly in proposing alternative pedagogical solutions. Participants were allowed to watch the video as many times as needed and had the option to pause or revisit specific moments.

The video clip, which has a duration of six minutes and 27 seconds, features a secondary school mathematics teacher delivering a lesson on the review of basic concepts of functions. The teacher uses a projector and a PowerPoint presentation to explain the verbal definition of a function (see Figure 1), illustrating the relationship between sets through two representations: a Venn diagram and a graphical representation. Additionally, the concepts of image, pre-image, output set, input set, domain, co-domain, and range are discussed.

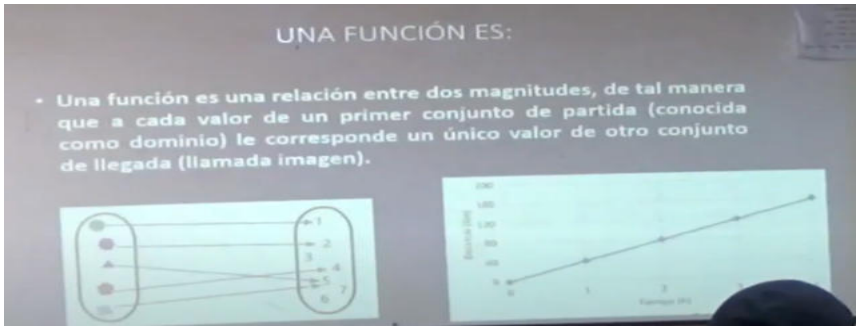

Figure 1: Capture of the initial moment of the video

Note. The following is a translation of the Spanish text from the figure: "A function is a relationship between two quantities, such that each value in the input set (the domain) corresponds to a single value in another set of outputs (the range)."

The narratives (Chapman, 2020) written by preservice teachers were selected for analysis to identify their noticing levels in the dimensions of attention and interpretation. Sentences and paragraphs that provided evidence of the level indicators outlined in Table 1 served as units of analysis. The operationalization of the noticing level indicators is based on content analysis (Bardin, 1996) of the notable events that preservice teachers observe and interpret. A higher frequency of any given indicator is sufficient to assign a corresponding level. For example, narratives that lack reflections on students' mathematical thinking are classified at the basic level, while those that include more detailed reflections are assigned a higher level of noticing. It is assumed that the level indicators follow a linear progression.

The authors of this paper identified evidence of the dimension and level indicators for the three narrated moments on an individual basis. These elements were then discussed, and consensus was reached to determine the final level assigned.

Results

Overall, the narratives of the 35 preservice teachers suggest that in the attending dimension, their responses are primarily categorized as either mixed or focused, while in the interpreting dimension, they tend to exhibit a mixed level of competence. It is also important to note that some participants demonstrated varying levels of noticing within the same narrative. Due to space constraints, this paper provides a detailed analysis of only one case to illustrate how the levels were assigned, as it was particularly relevant in displaying three distinct levels of interpretation.

The narrative produced by the selected preservice teacher, corresponding to the three moments in the video that captured his attention, are presented below.

Moment #1: 00:00-01:30

1 I would like to highlight a particular moment in the class when an interesting question came
2 up from a student: “Are the range and codomain the same thing?” Although some students
3 and the teacher answered that they are not the same thing, they pointed out that sometimes
4 they can coincide. In addition, the teacher emphasized the importance of having a clear
5 understanding of the concepts of domain, range, and codomain in order to move the topic
6 forward. In my opinion, although the student's question was addressed, it would have been
7 beneficial to provide a concrete example on the board to completely dispel the doubt at that
8 moment (later when an example is presented), but I want to emphasize the need to address
9 students' doubts in a timely manner (at the moment they ask the question) because I have
10 observed that sometimes it is difficult for them to express their concerns and, when they do,
11 they are often left with more questions if we do not provide a complete and concise answer.
12 This can discourage them from asking more questions or make it difficult for them to keep up
13 with the pace of the class because they start with unresolved doubts from the beginning.

In the initial four lines, there is evidence that the preservice teacher prioritizes the students' thinking processes, as evidenced by the highlighting of the question posed by the student and the subsequent emphasis placed on the necessity for the teacher to analyze the students' questions (lines 8 and 9). Furthermore, there are indications that the teacher is attending to aspects of mathematical concepts, as evidenced by the emphasis placed on the importance of understanding the basic concepts of functions for later topics (lines 4 and 5).

Additionally, the text points to elements of teacher pedagogy, as illustrated by the discussion of the benefits of using examples (line 6) and the incorporation of students' questions as part of the learning opportunities (lines 11, 12 and 13). Considering these indicators, it can be concluded that the teacher displays a mixed level of attention and interpretation.

Moment # 2: 02:30-03:45

14 First of all, what struck me about the teacher is that he points to the diagram on the left [Venn
15 diagram] as a “circle diagram”, although it is understood that for the students to understand
16 which [of the diagrams] he was referring to; it is necessary to promote technical language in
17 mathematics. Personally, I would have used that moment to teach the students that the diagram
18 has a name “mapping diagram” and that they can create such diagrams using functions.
19 Secondly, the teacher asks questions to make sure that the students correctly understand the

20 interpretation of the sagittal diagram, questions such as “What would be the domain in this
21 diagram, what would be the range, in this diagram do the range and codomain coincide?”, to
22 which the answers are quite enriching and correct. The students understand the diagram
23 perfectly and give a complete answer, although it was evident that they have difficulty in
24 differentiating “image and preimage”. Based on this, I emphasize the importance of
25 “evaluating” the course of the class according to the questions used by the teacher, since all
26 these data could be obtained thanks to the teacher's questions and the students' participation.

In this case, in lines 14 and 15, the preservice teacher expresses an intention to address aspects of mathematical concepts by emphasizing the use of appropriate mathematical language. Later, in lines 17 to 19, the teacher's pedagogical approach is described more concretely, with a particular focus on the use of questioning (line 20).

Furthermore, in line 12, he begins to recognize the difficulties the students are facing. At this point, the teacher’s level of attention is mixed, and his level of interpretation is at a baseline level, as evidenced by his evaluative rather than interpretative explanations. For example, he states that the students understand perfectly and provide complete answers, and that the data were obtained thanks to the teacher's questions.

Moment #3: 04:20-05:44

27 In this part of the video, I noticed that one student answered a question of the teacher
28 incorrectly, but most of the course recognized the error, to which several students answered
29 correctly, actions like this show that the course is really committed to the subject they are
30 teaching, as they can demonstrate the errors themselves and be able to justify them.
31 Afterwards, the teacher took the opportunity to give general feedback on the concept of
32 codomain and range, explaining with a concrete example from the blackboard where these
33 concepts do not coincide; personally, I think it was a good instance to emphasize this
34 difference of cases, as it gives feedback to students who may not have understood it so clearly,
35 and confirms the correct answers of the other students. Finally, a student asked, "What is the
36 difference between image and preimage?" Instead of answering directly, the teacher asked the
37 other students to give their answers. This practice is very important because it encourages
38 feedback among students, promotes collaboration, builds trust among peers, and contributes
39 to the assessment of student learning. Together, these practices create an enriching learning
40 environment that benefits the progress of all students.

In this concluding narrative, the preservice teacher identifies errors as a potential avenue for learning (lines 31 to 33) and demonstrates an awareness of student behavior, in this instance, by underscoring their engagement in the classroom (lines 30 to 32). Furthermore, he discusses the use of examples and counterexamples to contrast or clarify ideas (lines 32 and 34), thereby initiating attention to mathematical concepts. In lines 37 to 41, the trainee teacher emphasizes the importance of utilizing students' questions within the classroom setting and provides insights into the advantages to be gained from such scenarios. This illustrates a mixed level of attention and a focused level of interpretation. A summary of the findings is presented in Table 2.

Table 2: Levels of noticing evidenced in the narratives

Dimensions	Moment 1	Moment 2	Moment 3
Attending	Mixed	Mixed	Mixed
Interpreting	Mixed	Baseline	Focused

The results show consistency in the level of attention of this preservice teacher, but diversity in the levels of interpretation (see Table 2), despite the use of the same video for analysis.

Conclusions

This paper presents partial results of the research on the levels of noticing competence exhibited by preservice Chilean mathematics teachers. In the case presented, the level of attention demonstrated by the preservice teacher was mixed at the selected moments, while their level of interpretation differed at each of the three moments: moment 1 – mixed, moment 2 – basic, and moment 3 – focused. This final scenario renders the assignment of a noticing competence level within this dimension challenging, particularly when there is a diversity of levels within the same analyzed situation.

However, given that preservice teachers are still in training and developing their noticing competence (Kuntze et al., 2022), the evidence from the selected narrative, which shows a focused interpretive level in one of the moments described, suggests that this may eventually stabilize into a more consistent level of noticing. In general, these findings align with those of El Mouhayar (2024), who found that most preservice teachers exhibited noticing levels in the medium range, with a tendency toward moderate levels, which in this case corresponds to a mixed level. Another important point, which warrants further analysis, is the possibility that when observing other teachers, preservice teachers may concentrate on the actions of the teacher as a model for their future practice. This concentration could explain why some of them displayed baseline and mixed levels of attention.

It is also relevant to note that this study identifies certain challenges. Firstly, there is the complexity and yet unresolved issue of developing a strategy to establish a single level of noticing, rather than focusing on individual dimensions. This issue presents a valuable direction for future research, benefiting both scholars and those involved in teacher training programs. In these programs, decisions aimed at enhancing teaching competencies must be guided by tools that can accurately detect changes in teachers' noticing competence. Furthermore, this paper underscores the need for future research to refine the indicators for noticing dimensions, specifically to address particular mathematical objects.

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Supporting pre-service teachers in learning to promote students' proving: the case of general conjectures

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Leading mathematical discussions is an important teaching practice that allows students to develop their mathematical reasoning in collaboration with others. Aiming to provide PSTs with opportunities to learn this practice, four lessons for a mathematics teacher education course for elementary pre-service teachers (PSTs) were designed and implemented in an intervention in three iterations. Based on preliminary analyses, the design of the lessons was revised between each iteration. However, the design of the fourth lesson never successfully supported the PSTs in learning to lead discussions that reached a proof of a general conjecture. In this paper, the design of the fourth lesson and the challenges encountered in each implementation are discussed to provide directions for future interventions concerning classroom discussions and proving.

Keywords: Validity, class discussion, preservice teachers.

Introduction

An aim of teacher education is to prepare PSTs to teach with ambitious learning goals for the students, grounded in the expectation that all students are sense-makers and can be supported in reaching the goals (Lampert et al., 2013). For mathematics, ambitious teaching entails engaging students in problem-solving, sense-making, reasoning, and justifying (Forzani, 2014). Thus, providing students opportunities to develop a deep understanding of mathematics by focusing on reasoning and proving is central to ambitious teaching. However, learning to teach ambitiously is challenging due to the complexity of the mathematical content and the improvisational nature of teaching in a way responsive to students' thinking (Forzani, 2014). Moreover, learning to teach mathematics in a way that emphasises reasoning and proving is challenging (Stylianides & Stylianides, 2017). Efforts in teacher education should support PSTs in overcoming these challenges.

The practice-based approach to teacher education combines pedagogies of investigation and enactment of practice to support PSTs in learning to teach ambitiously (Ball & Cohen, 1999). By investigating representations of practice (e.g., videos, lesson plans, or students' work), PSTs are offered opportunities to observe and understand professional practice. Through approximations of practice (e.g., animations, role-plays, or school placement), PSTs are offered opportunities to enact professional practice in less complex conditions than authentic teaching (Grossman et al., 2009). Several studies have demonstrated that a practice-based approach to teacher education provides PSTs (and novice teachers) opportunities for learning to teach mathematics ambitiously (e.g., Lampert et al., 2013). However, varying degrees of complexity and authenticity in representations and approximations of practice provide PSTs with different learning opportunities (Grossman et al., 2009; Sapkota & Max, 2023). Thus, teacher educators need to consider the balance between more or less complex pedagogies for investigating and enacting practice.

As reasoning and proving is known to be challenging to teach (Stylianides & Stylianides, 2017), considerations of complexity and authenticity can be particularly important when implementing pedagogies for investigating and enacting practices related to reasoning and proving. In approximations relatively far from authentic practice, Arnesen and Valenta (2023) found opportunities for learning to promote proofs in mathematical discussions. However, Campbell and Elliot (2015) argue that approximations far from authentic practice risk oversimplifying teaching. At the same time, in a more authentic context, Buchbinder and McCrone (2020) found that PSTs struggled to enact a precise language for proving.

This study reports on a designed lesson in which PSTs investigated and, in a role-play, enacted a classroom discussion to prove a general conjecture. The lesson was implemented through an intervention in three iterations, and the design was revised based on initial analyses before the second and third iterations. However, the changes in design did not reach a balance in the complexity of the approximation to successfully support the PSTs in enacting a discussion that concluded with a proof of a general conjecture. In this paper, I discuss the lessons in terms of complexity and authenticity, answering the research question: *How do different aspects of complexity play out in three iterations of implementing role-plays of a discussion aiming at proving a general conjecture?* By discussing unsuccessful lesson designs, I make implications for the design of future interventions aiming to support PSTs in leading discussions that reach a proof of a general conjecture.

Theoretical framework

The complexity and authenticity of pedagogies in teacher education provide PSTs with different opportunities to learn to teach (Sapkota & Max, 2023). Authenticity and complexity have been defined in various ways. I follow Sapkota and Max's (2023) definition of authenticity as depending on the extent to which the pedagogies preserve the complexity of authentic teaching. As such, complexity and authenticity are codependent variables (Codreanu et al., 2020). The complexity of pedagogies for implementing investigation and enactment of practices in teacher education can be determined based on three dimensions: the nature of the practice, the context, and the scaffolding (Sapkota & Max, 2023). First, the nature of the practice varies from less complex when working with parts of practices, like practicing some teacher moves to elicit students' thinking, to more complex when enacting compound practices, like leading classroom discussions. Second, the context preserves less of the complexity of authentic teaching when the approximations are enacted in contexts outside of school. Such approximations include scriptwriting of imagined lessons, simulations, or role-plays. Third, the amount of scaffolding affects the complexity by providing, e.g., tools, frameworks, or teacher educator guidance, which is not available in authentic teaching.

A recognised pedagogy for supporting PSTs in learning to teach ambitiously is the learning cycle (McDonald et al., 2013), which combines representations and approximations of practice. The learning cycle is centred around a teaching practice and consists of four phases: investigating, planning, enacting, and reflecting (McDonald et al., 2013). The complexity and authenticity of the learning cycle are due to the adaptions and choices made by teacher educators. Learning cycles can evolve around more or less compound practices, be situated in teacher education or actual classrooms, and different kinds of support can be provided in the different phases of the cycle. Hence, teacher

educators need to balance the need for authentic learning experiences with the need to reduce complexity when implementing the learning cycle. PSTs need authentic learning opportunities to transfer the practices they learn into actual teaching (Codreanu et al., 2020). At the same time, lower complexity allows for isolating crucial aspects of the practice, removing the messiness of authentic practice (Grossman et al., 2009).

In this study, the design of a lesson implementing the learning cycle as a pedagogy for learning to lead classroom discussions that aim at proving a general conjecture is discussed. Leading mathematical discussions is a demanding teaching practice (Shaughnessy et al., 2019). Moreover, planning and enacting instruction about reasoning and proving is challenging (Stylianides & Stylianides, 2017), and proving general conjectures (statements about infinitely many cases) in a manner suitable to school mathematics is particularly challenging for PSTs (Rø & Arnesen, 2020). In this case, the complex nature of this practice was balanced with a less complex context: role-play, where the PSTs enact the roles of the teacher and students as if it was authentic teaching (Stürmer et al., 2024). Still, the scaffolding provided during the learning cycle plays an important part in balancing the complexity of the learning cycle, which is the focus of the following analyses.

Methods

This study is an intrinsic case study (Creswell & Poth, 2018), where the design of one lesson is a case of special interest because of the design challenges encountered. The lesson design analysed in this study was the last in a series of four lessons designed to provide PSTs opportunities to learn to elicit and promote students' reasoning and proving by implementing the learning cycle. The lessons were designed for a mathematics course in the first year of a five-year teacher education programme for primary school PSTs. Before the implementation of the intervention, the PSTs attended two lessons about reasoning and proving for primary school.

The lessons were implemented in an intervention through three iterations. Hence, each lesson was implemented three times. The four designed lessons were all implemented before moving to the next iteration. For example, after the fourth lesson was implemented the first time, lessons one, two, and three were implemented for the second time before the fourth lesson was implemented again. Since the overarching aim was to improve the design of the lessons, the iterations were implemented in a new group of PSTs, each group of approximately 30–40 PSTs. In each lesson, the PSTs were presented with a case from a classroom involving students' work presented as written work or in a video/transcript of a classroom discussion. Next, the PSTs planned how they would build on the students' work in classroom interactions before they enacted the plans in role-plays with one or two PSTs enacting the teachers, some PSTs enacting the students, and some PSTs observing the role-play. Last, the PSTs reflected on the enactments based on their experiences and observations. Each lesson lasted approximately two hours.

The four lessons involved different complexity. In the first two lessons, the nature of the enacted practice was that of a teacher eliciting one or two students' reasoning. In the last two lessons, the enacted practice was leading a mathematical discussion, which is a more complex practice. The context of the enactments was equal for all four lessons: role-plays situated in teacher education courses. The scaffolding provided varied in terms of the support provided by the teacher educator in

helping the PSTs unpack the mathematical content of each lesson. Moreover, the complexity of the mathematical ideas the classroom interactions were centred around varied. In the first lesson, the PSTs were asked to support students in developing conjectures based on similarities and differences. In the following two lessons, the PSTs were asked to support the students in explaining a strategy for an arithmetic problem. In the last lesson, the PSTs were asked to lead a whole class conjecture, reaching a proof of a general conjecture using a generic example. In generic examples, proofs of general conjectures are based on explaining a specific case and arguing for the properties of the case that are general (Rø & Arnesen, 2020). For the fourth lesson, which is the focus of this study, experiences from each iteration of implementation led to changes in the design regarding teacher educator support. These changes are described in the Results, along with the analyses of the lessons.

Changes were made based on initial analyses and reflections on the implementation of the lesson. Transcripts of the videotaped enactments were analysed in terms of problems encountered regarding general conjectures or mathematical discussions. Encountered problems were identified in instances where the PSTs explicitly or implicitly stated that they found it difficult, the discussion did not reach the goal of proving a general conjecture, or the enacted role-play did not meet the requirements of mathematical discussions. For instances identified as encountered problems, the enacted role-plays were analysed in terms of the design considerations made and discussed based on how the design played out in the enactment.

Results

For each iteration of implementation, I present the aim of the fourth lesson and the lesson design. For each of the three iterations of implementing the fourth lesson, the PSTs were presented with a transcription of a classroom discussion where students explained their strategy for an arithmetic problem. The PSTs were asked to build on the students' explanation when planning and enacting a discussion that generalised properties and proved a general conjecture. In the excerpts, T and S indicate that the PST enacted the teacher or student.

The first iteration

For the first iteration of implementing the fourth lesson, the aim was that the PSTs enacted a discussion that reached a proof of the general conjecture about distributivity in division: that you can split the dividend and divide each of the parts by the divisor before you add the quotients. In the first phase of the learning cycle, the PSTs were presented with a transcription of a fictive classroom discussion in which the students worked with the problem string 70:7, 21:7, and 91:7. The transcription of the classroom discussion ended with an explanation of the division strategy 91:7=70:7+21:7, using a measurement division context with bags of candy as a representation. The excerpt shows how the PSTs went from discussing validity to discussing the extent of validity.

Ane (T):	But why can we split it this way? I mean, does it apply to all division problems?
Simen (S):	Yes, if the number is very high.
Vilde (T):	Yes, because 91:7 is difficult, perhaps. So, we have made it easier. We have 70 and 21. But did we need to do it this way?
…	
Ane (T):	As long as it is 91 in total, is it true? Could you also split 50 candies? Is it possible? Does it work for all numbers?

Siri (S): Not if it isn't in the 7-times table.
Ane (T): Because then it doesn't work. So, what is important is that it is a number that is divisible by the number of candies in the bag.
Vilde (T): Yes, so we did talk about that no matter how we split 91, we still have the same amount. But then we can split it in 50?
Ane (T): Yes, we can actually split it in 50. And then realise that "oh, I miss some candies". We still have the 91, even if we take some from this pile of candy here … (hesitates)

Twice, Ane, enacting one of the teachers, asked if one can always split the dividend in a division problem. However, an explicit formulation of the general conjecture was never provided to support the PSTs. Consequently, the conjecture was never stated explicitly in the enacted discussion, and the PSTs' answers about the validity of the conjectures address the usefulness and the extent of validity (for the representation using candies). Furthermore, the enactment went from being a role-play to being a conversation about unpacking the mathematics. Hence, in this first iteration, the lack of scaffolding, in terms of not formulating a general hypothesis for the PSTs and using a representation for which the conjecture is not always true, led to the enactment losing track of the goal for the discussion and eventually being a discussion for the PSTs out of character.

The second iteration

Before the second iteration, the design of the lesson was changed to having the teacher educator present the given general conjecture. Additionally, the proof of the single-case conjecture that the PSTs were to generalise was based on a representation that would be true for all numbers. The aim of the lesson in the second iteration was that the PSTs enacted a discussion that reached a proof of the general conjecture "1 divided by any fraction is the inverted fraction". In the first phase of the learning cycle, the PSTs investigated a transcription of a classroom discussion about the problem 1:2/3, ending with an explanation of why the answer to 1:2/3 is the inverted fraction, 3/2. However, generalising and proving that 1 divided by any fraction is the inverted fraction is a complex mathematical idea, which led to problems in unpacking the mathematics, as illustrated below.

Nadia (T): OK, but what if we use another fraction, what happens then? Is it similar?
Anders (S): Yes.
Nadia (T): Why?
Lilja (S): Because it is the numerator bigger, sort of, so you need to divide by the numerator.
Nadia (T): So, you divide the answer from 1 divided by … Can you explain again?
Lilja (S): If you take another problem, for example, 1 divided by 3/5, then you first find that you get five 1/5s. Then you divide it by 3 because it is three times bigger than when the numerator is 1.
Dina (S): You divide 5 by 3, that's why it is 5/3?
Lilja (S): Yes.
Dina (S): Just checking that I follow.
Nadia (T): Okey, but will it always be this way? Or is there a relationship between the fractions and the answer? Anyone?

After Lilja justified that the conjecture was true, Nadia attempted to revoice her justification without being able to finish, so she asked Lilja to explain again. After Lilja explained again, the PSTs temporarily went out of their roles to unpack the mathematical content, e.g., seen in Dina's interest in understanding the justification herself, not as the student she enacted. At the end of the discussion, Nadia asked again if the conjecture was true, even though Lilja had already justified it. Moreover, she asked about the relationship between the numbers in the problem, which is a question that prompts

students to develop a conjecture, which, in this case, was already developed. Hence, the lack of scaffolding for unpacking the complex idea of inverse proportionality led to challenges in understanding the proof and what counts as proof.

The third iteration

The third iteration involved reasoning and proving processes in measuring. Based on experiences from the second lesson, the design of this lesson was changed to involve a much less complex mathematical idea and provide scaffolding by proving a similar conjecture in the given transcription of a mathematical discussion. The PSTs were asked to lead a discussion proving the conjecture that as long as the side lengths of the base rectangle (including lengths of decimal numbers) add up to the same circumference and the height is constant, the surface area of the walls of a prism remains unchanged. In the transcription investigated by the PSTs, the discussion reached a proof of the conjecture for side lengths of whole numbers. Providing the proof for whole numbers was intended as scaffolding for when the PSTs were asked to lead a discussion proving the extended conjecture, including rational numbers. In the excerpt below, the PSTs reached a proof of the general conjecture, but the PST who enacted the teacher did not position the students as sense-makers or ask them to explore or formulate the conjecture.

Amund (T): What do you think? Do you think it will always be 864 square meters when we have these measures? Yes?
…
Isak (S): Because we know that the height is 24 no matter what, and the circumference is 36. So, that is, you cannot change it, and if you multiply them, you get 864.
…
Amund (T): If I take five and a half and add five and a half, plus twelve and a half plus twelve and a half. What do you think the answer is then? Do not calculate it, what do you think? (Points at Gry).
Gry (S): 36
Amund (T): Why?
Gry (S): Because that is the circumference.
Amund (T): Yes, correct, it should always be 36. Then it is, then the height is 24, so we get 36 times 24, and that is 864.

In the excerpt above, Amund's questions directed the discussion towards the argument for the conjecture already given in the transcription they investigated. The PST's questions did not highlight the new conjecture: that the circumference can be made up of any (not necessarily integer) number as the side lengths. The new conjecture was mentioned by Amund as an example, and Amund also gave the proof. Suppose the PSTs considered the conjecture trivial (given the already proven similar conjecture). In that case, they might not have seen the need for formulating and validating it, and the teacher repeating could be a strategy to summarise the discussion. Hence, the scaffolding, making a conjecture about less complex mathematical content and providing a proof of a similar general conjecture, may have reduced the complexity of the approximation to such an extent that it lost authenticity and lacked opportunities for learning to teach.

Discussion

This study reveals the challenges of supporting PSTs in approximations of practices that involve complex mathematical processes. The scaffolding of proofs of general conjectures played out in three

different ways in the three iterations of implementation. First, not enough scaffolding was provided for the PSTs to unpack the mathematical content. Second, although providing more scaffolding to prove general conjectures, the mathematical ideas were so complex that more scaffolding was needed. Last, too much scaffolding regarding proofs of general conjectures was given, making the conjecture trivial to the PSTs. For the first two lessons, the lack of scaffolding consequently led to the approximation being too complex to provide opportunities for learning to lead mathematical discussions aiming at proofs of general conjectures. For the last lesson, too much scaffolding led to a less authentic approximation, hindering the PSTs' opportunities to learn a complex practice.

Although the changes in the design did not reach a balance in scaffolding suitable for supporting PSTs in learning to lead discussions reaching proofs of general conjectures in role-plays, implications for further interventions can be made. Based on the first iteration, I recommend providing a conjecture (in contrast to making the PSTs formulate it) and a representation on which to base the proof. Furthermore, the second and third iterations provide information about the amount of scaffolding used to unpack the mathematical content, demonstrating that the scaffolding must be on a continuum between preserving the mathematical complexity and making the mathematical ideas trivial. In line with Sapkota and Max's (2023) recommendations, further research should strive to find a balance in complexity that supports PSTs in learning to lead discussions aiming to prove general conjectures.

The results of this study contribute to the literature about practice-based approaches supporting PSTs in learning to teach reasoning and proving in several ways. First, by discussing authenticity in terms of complexity in detail, new insights about the design decisions teacher educators need to make is obtained. Although Grossman et al. (2009) suggest that less complex approximation allows for isolating and learning some aspects of the practice, the results in this study demonstrate that the complexity can be reduced too much. Second, using role-play is another approximation of practice than those reported earlier, as it is more complex than scriptwriting tasks (Arnesen & Valenta, 2023) and less complex than enactments in school (Buchbinder & McCrone, 2020). Last, the results contribute to already known challenges of learning to teach proof. In addition to struggling with enacting a precise language for proving (Buchbinder & McCrone, 2020) and proving general conjectures themselves (Rø & Arnesen, 2020), the PSTs met challenges regarding formulating conjectures, deciding the validity domain, and seeing the need for proving.

Several factors contribute to the complexity of pedagogies of enactments. In this study, scaffolding was the only factor experimented with, and future interventions should experiment with the nature of the practice and the context and possible relations between these dimensions when aiming to support PSTs in learning to promote students' reasoning and proving in classroom interactions.

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Pre-service primary school teachers' curricular noticing: interpreting a sequence of activities about 3D

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Teachers primarily use textbooks as resources for preparing their mathematics lessons. Consequently, initial teacher training programmes must develop skills to critically analyse a variety of mathematical aspects in textbooks. This study examined how 85 Spanish pre-service primary school teachers (PTs) analysed a sequence of activities from a textbook about three-dimensional (3D) geometry using a theoretical lens. Particularly, it is focused on how they analyse three aspects: attributes, geometric processes, and modes of representation. The study revealed that all pre-service teachers were able to find limitations in the sequence of activities at least in one of the three aspects and more than 60% in the three aspects. This result seems to indicate that the professional task and the theoretical lens provided could focus PTs' attention on these aspects.

Keywords: Curricular noticing, preservice primary school teachers, textbook, three-dimensional geometry.

Introduction

Teachers use curricular materials, primarily textbooks, as resources for preparing their mathematics lessons (Blanco et al., 2021; Glasnovic Gracin, 2018). Nevertheless, adopting an excessively dependent approach to textbooks when preparing geometry lessons may result in limited teaching. Barrantes-López et al. (2014) evidenced that textbooks present most pyramids and prisms prototypically. Díaz-Levicoy et al. (2019) suggested that textbooks rarely support the concrete mode, including manipulative materials.

Consequently, initial teacher training programmes must develop skills to critically analyse a variety of mathematical aspects in textbooks and other curricular materials (Amador et al., 2017; Ivars & Fernández, 2023; Zorrilla et al., 2024) in order to recognise affordances and limitations and to develop strategies on how to use them. Therefore, developing the skill of curricular noticing (Dietiker et al., 2018) focused on the interactions between teachers and the curricular materials is relevant in teacher training programmes.

Curricular noticing

Curricular noticing is defined as the process through which "teachers make sense of the complexity of content and pedagogical opportunities in written or digital curricular materials" (Amador et al., 2017, p. 427). Curricular noticing comprises three intertwined skills: curricular attending, curricular interpreting, and curricular responding (Dietiker et al., 2018). Curricular attending describes teachers' skills in viewing information contained in curricular materials to inform teaching. Curricular interpreting is a sense-making activity whereby teachers connect the curricular material information to their own mathematical and pedagogical content knowledge for teaching. Curricular responding corresponds to the skill of making curricular decisions considering the previous interpretation.

Previous studies on curricular noticing have investigated the establishment of similarities and differences between tasks (Amador et al., 2017). These authors concluded that pre-service secondary school teachers noted that tasks, even when sharing the same goal and context, resulted in different students' solution pathways and mathematical elements. Amador and Earnest (2019) showed that pre-service teachers noticed discrepancies in the presentation of fraction content in an inquiry-oriented textbook compared to how they had initially learned that content. McDuffie et al. (2018) reported findings on teachers' noticing of features in the teacher resources of mathematics curriculum programmes. These authors found that the interpretation and decisions varied based on teachers' orientations and curriculum type.

Our study is part of a larger research project that aims to characterise the development of pre-service primary school teachers' curricular noticing. This paper is focused on curricular interpreting, particularised, in this study, as the ability to use mathematical and pedagogical content knowledge to recognise limitations in the curricular materials. To support pre-service teachers in developing curricular interpreting, they were provided with theoretical lenses since they have been shown as potential tools to support their noticing (Fernández & Choy, 2020). In a previous study, Zorrilla et al. (2024) have examined the coherence between how pre-service teachers interpret a sequence of activities from a textbook about fractions and how they complete the sequence using a hypothetical learning trajectory as a theoretical lens. The results showed that more than half of the pre-service teachers were able to complete the sequence of activities considering the limitations interpreted.

Our objective is to examine how pre-service teachers analyse a sequence of activities from a textbook about three-dimensional geometry to recognise limitations. Particularly, the paper examines whether pre-service teachers recognise limitations regarding the attributes of geometric solids (e.g., convex/concave), geometric processes (e.g., classifying) and modes of representation (e.g., concrete).

Method

Participants and context

Participants were 85 pre-service primary school teachers (PTs) from the University of Alicante (Spain). When conducting the study, participants were in their third year of the degree to become primary school teachers. They were enrolled in the course 'Learning and Teaching of Mathematics in Primary Education'. The main objective of this course is to develop professional competencies for mathematics teaching, such as interpreting students' mathematical thinking, analysing curricular materials or planning lessons in different mathematical domains (e.g. geometry). Participants had already completed two mathematics courses related to number sense and geometric sense in previous years.

Instrument: Professional task

PTs were asked to analyse a sequence of five activities from a 5th-grade (aged 10-11) mathematics textbook (Figure 1, Ferrero de Pablo et al., 2018, pp. 178–179) related to 3D geometry. The activities address the concepts of polyhedron, prism and pyramid.

1. Answer the following questions:

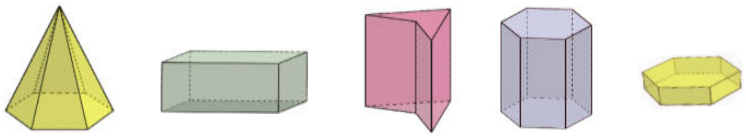
a) What kind of polyhedron is a shoebox?

b) What do the faces of a polyhedron look like?

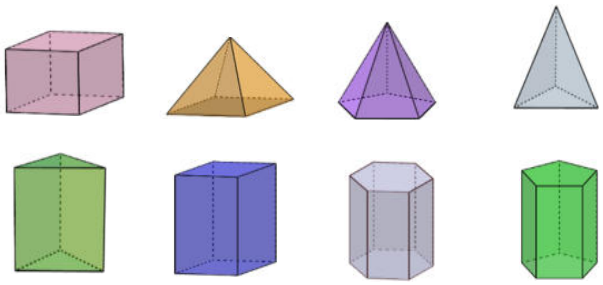
c) How many bases does a prism have? and a pyramid?

d) What do the side faces of a pyramid look like? What do the side faces of a prism look like?

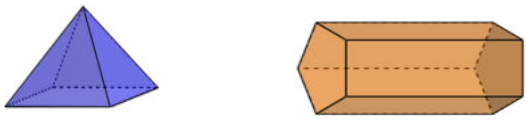
2. Classify the following polyhedra:



3. Indicate which of the following polyhedra are prisms and which are pyramids.



4. How many faces, vertices and edges do the following polyhedra have?



5. This is the flat development of a prism. Draw the flat development of the pyramid on the right.

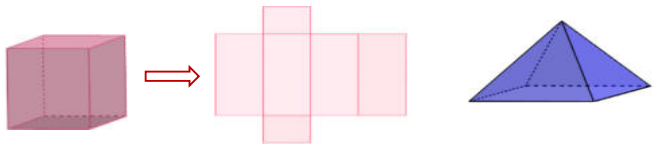


Figure 1: Activities from the mathematics textbook (translated into English from Ferrero de Pablo et al., 2018, pp. 178–179)

To analyse the activities, PTs had to answer some guide questions related to the curricular noticing. The guide questions related to curricular interpreting: (1) what other attributes of geometric solids could be incorporated to promote the understanding of the polyhedron, pyramid and prism concepts? (2) what other geometric processes could be introduced to promote the understanding of the

polyhedron, prism and pyramid concepts? And, (3) what other modes of representations and connections between them would help to promote the understanding of polyhedron, prism and pyramid concepts? PTs answered the guide questions related to the three skills individually in a 100-minute lesson of the course. The sequence of activities from the textbook was selected since PTs could identify some limitations, such as they could add oblique pyramids or prism as other attributes of geometric solids. Furthermore, they could add non-examples of polyhedra. They could add the process of defining or measuring and, regarding the modes of representation, the use of concrete materials.

PTs were provided with a theoretical document (theoretical lens) containing information related to the teaching and learning of geometrical contents in primary education. This document was designed considering previous research on mathematics education in this area (Battista, 2012; Bernabeu et al., 2021). For instance, the document encompasses information about the geometrical processes (recognising, analysing, classifying, defining, constructing, or proving), modes of representation (verbal, symbolic, concrete or graphic) and the importance of using a great variety of examples with different attributes of geometric solids (e.g., convex, concave, oblique, and right) and non-examples.

Data analysis

Data were the answers of the PTs to the three guide questions of curricular interpreting. Since the objective is to examine how pre-service teachers analyse a sequence of activities from a textbook about three-dimensional geometry, the four researchers performed an inductive analysis (Strauss & Corbin, 1994), identifying categories about the attributes of geometric solids, processes or modes of representations that PTs would incorporate in the sequence of activities to promote the understanding of the polyhedron, prism and pyramid concepts. First, a set of 30 PTs' answers were analysed individually, emerging initial categories. The four authors discussed agreements and disagreements until they reached a complete consensus. Second, the other set of answers was analysed, discussing and refining the categories, again achieving full agreement on the final categories. In table 1, two examples of the data analysis are presented.

Table 1: Examples of data analysis

Pre-service teacher's answer	Attributes categories	Processes categories	Modes of representation categories
"As for other attributes of pyramids, polyhedra and prisms, we could work on the following. In order to understand the concept of polyhedra, we could propose activities to work on convexity and concavity. These attributes can also be worked with prisms and pyramids. […] Regarding geometric processes, we could propose activities to define figures in different ways. […]	Concavity Convexity	Defining	Concrete materials

In order to support understanding, we could make use of concrete materials such as Meccano and Polydron". (PT1)			
"We can include oblique geometric bodies, bodies of revolution such as spheres, cylinders or cones to reinforce the concept of polyhedron and figures with different bases such as the truncated pyramid. To make other classifications, i.e. with other criteria; tasks of constructing and defining. To promote understanding it would help to use concrete materials that allow them to manipulate the figures". (PT3)	Oblique Non-examples	Classifying Constructing Defining	Concrete materials

Results

Our results show that 88% (75 PTs) would add attributes that can support understanding the concepts of polyhedron, pyramid and prism. They proposed including 3D shapes with the attributes of convex/concave (46 PTs, 54.2%), oblique/right (45 PTs, 52.9%), regular/irregular (39 PTs, 45.8%), or different 3D shapes according to the polygon of the base (18 PTs, 21.2%). These participants perceived that these attributes were not explicitly targeted in the sequence of activities of the textbook. However, it is noteworthy that the 3D shape representations in the sequence of activities considered some of these attributes (convex/concave; regular/irregular and 3D shapes according to the polygon of the base). However, oblique prisms or pyramids were absent, highlighting the concerns raised by 52.9% of the PTs who pointed out this attribute. Furthermore, other PTs reported the need to integrate non-prototypical figures (13 PTs, 15.3%) and non-examples (11 PTs, 12.9%). In this sense, PT30 alluded to both, expressing, "I would include the same figures in an oblique position, as well as solids of revolution such as sphere, cone and cylinder to consolidate the concept of polyhedron".

96.5% of the PTs (82 PTs) proposed some geometric processes to support students' understanding. These processes were constructing (49 PTs, 57.6%), measuring (32 PTs, 37.6%), analysing (26 PTs, 30.6%), classifying (26 PTs, 30.6%), defining (25 PTs, 29.4%), recognising (12 PTs, 14.1%), and proving (5 PTs, 5.8%). The majority of PTs recommended construction activities that should enable primary school students to not only obtain a flat development from its polyhedron but also reverse this process or combine a collection of polyhedra to obtain a new one. For instance, PT60 explained: "The sequence of activities works the geometric process of constructing but in the form of decomposition; therefore, some composition activities [with polyhedra] could be introduced". Furthermore, a relevant number of PTs noted the absence of measurement activities (e.g. obtaining the volume of a prism or measuring elements of polyhedra such as the height).

The number of PTs who identified modes of representation that can support students' understanding decreased compared to the other two aspects (attributes and geometric processes). 76.5% of PTs (65) proposed a different mode of representation or connections between them. The concrete mode of representation was the most frequently mentioned (62 PTs, 72.9%), followed by the verbal (27 PTs,

31.7%), graphic (8 PTs, 9.4%) and symbolic (3 PTs, 3.5%). Many PTs realised that the textbook does not support the use of physical objects (manipulative materials) and suggested the inclusion of structured (e.g., polydron, or Meccano) and unstructured (e.g., real-life objects and toothpick) materials. Additionally, some PTs suggested strengthening the verbal mode, highlighting the need for oral communication. For instance, PT4 would foster teacher-student interactions throughout the activity-solving process, whereas PT27 would encourage students to verbalise the solutions they obtained.

All PTs were able to identify at least limitations with regard to one of the aspects (attributes, geometric processes or modes of representation) in the sequence of activities to support the understanding of the concepts. As illustrated in Figure 2, 55 PTs (64.7%) identified some limitations in the three aspects, 27 (31.8%) in two aspects, and 3 (3.5%) in one aspect. Table 1 shows the answers of two PTs who identified limitations in the three aspects (PT1 and PT3).

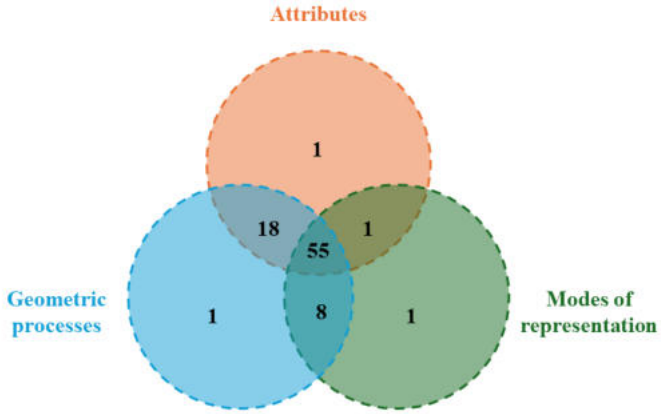


Figure 2: Aspects considered by pre-service teachers when interpreting

In the group of 27 PTs who identified some limitations with regard to two aspects, 18 focused on attributes and geometric processes, 8 on geometric processes and modes of representation, and 1 on attributes and modes of representation. For instance, PT10 successfully identified some attributes and geometric processes that could support the understanding of the concepts.

> We can introduce different types of pyramids considering the polygon of the base and regular polyhedra. With regard to geometric processes, activities of defining, constructing and classifying (PT10).

PT69 suggested some attributes that could support the understanding of the concepts and the use of non-examples. This PT did not propose any geometric process or mode of representation.

> In order to support understanding, bodies of revolution such as the cone, cylinder or sphere could be incorporated. Non-regular polyhedra or concave polyhedra could be introduced. Also, oblique polyhedral (PT69).

Discussion and conclusions

All PTs were able to find limitations in the sequence of activities at least in one of the aspects (attributes, geometric processes or modes of representations) and 64.7% in the three aspects.

Therefore, this result seems to indicate that the professional task and the theoretical lens provided (Fernández & Choy, 2020), could focus PTs' attention on these aspects (Zorrilla et al., 2024). The task and the theoretical lens used in this study can be considered in teacher education programmes as tools that can enhance PTs' curricular interpreting.

The number of PTs who identified modes of representation that can support students' understanding was lower compared to the other two aspects (attributes and geometric processes). It was expected that PTs identified the concrete materials (manipulatives) as a mode of representation that is rarely promoted by text (Díaz-Levicoy et al., 2019). This result suggests that PTs try to foster an approach similar to the one promoted by the curricular material and, as Blanco et al. (2021) indicate, the curricular material often outlines the teaching practice. In view of this result, teacher training programmes should emphasise professional tasks aimed at recognising the significance of incorporating manipulatives into primary school classrooms. With regard to the attributes, most pre-service teachers would add convex and concave polyhedra, and oblique prisms and pyramids. Therefore, these PTs were able to recognise the importance to work non-relevant attributes and a variety of non-prototypical examples. Considering the geometric processes, most of them proposed construction or measurement activities since the sequence of activities focused more on recognising, classifying and analysing.

This paper focuses on examining one of the three curricular noticing skills: curricular interpreting. The next objective is to investigate the relationships between the three skills to gain a deeper understanding of PTs' curricular noticing.

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Using textbook analysis in teachers' professional development

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This study investigates how ten elementary school in-service teachers reflect and conclude after using a theoretical framework to conduct a textbook analysis as a part of a professional development course. By using Duval's system of representations, the teachers did a textbook analysis of all tasks in a part of a textbook in use at their school. The teachers presented and discussed their analyses with each other. Audio recordings of presentations, discussions and a group interview were subject to a thematic analysis. The findings indicate that the textbook analysis stimulated the teachers' reflections in task selection, and they concluded that it is important to let go of the textbook and to be aware and critical in task selection. Further research is needed to explore this practice of professional development.

Keywords: Professional development, mathematics teaching, task analysis.

Introduction

It is widely accepted that professional development (PD) can improve teaching (Kennedy, 2016). A core part of a teacher's planning involves selecting, modifying, and formulating tasks as they prepare for lessons. This requires awareness and reflection around the choices made, and it is important to investigate practices that train teachers to meet this challenge. However, despite its significance, planning lessons is difficult to teachers. Cevikbas et al. (2024) presented the results of a systematic review of 20 empirical research studies on teacher competence in mathematics lesson planning and showed that teachers face difficulties in planning lessons. They call for courses in teacher education and PD that can provide teachers with a reflective experience to become proficient in lesson planning.

In Norway, educational policy grants teachers a high level of autonomy (Eriksen et al., 2022), which means that they are given pedagogical freedom to select methods, tasks and content they evaluate to be the most beneficial to their students' learning. Nevertheless, this autonomy is juxtaposed with a challenge: the current national curriculum offers limited directives about content and how it should be taught (Sundby & Karseth, 2021), so that the content and teaching methods depend on the individual teacher. This means that it is important to train teachers to fulfil this responsibility.

In situations where teachers do not find sufficient guidance from the curriculum, they may seek content information from the textbook (Bolstad, 2019). Fan et al. (2013) summarized 111 studies related to textbook research in mathematics the past six decades. The study revealed that researchers generally agree that textbooks have a significant influence in modern teaching of mathematics (Fan et al., 2013). Recent studies from a Norwegian context indicated that although teachers use different resources, they still use textbooks as their main source in their mathematics teaching (Bjørnset et al., 2020). These results are similar to those reported by Bolstad (2019) who reported that mathematics teachers seem to be aware that relying solely textbooks is not the best way to plan teaching, but the lack of specific guidance and objectives in the curriculum makes it difficult for them to disregard the textbooks. Together, these studies indicate that textbooks are widely used, so it is important to discover methods for teacher education and PD to train teachers in using textbooks wisely.

Mathematics textbooks consists of more than just tasks, but in this study I have investigated teachers' reflections after analysing textbook tasks. Riard and Kaur (2022) have investigated which and how teachers' resources influence their choices during mathematics task selection and modification. Their findings showed that the teachers' beliefs on the teaching and learning of mathematics, and the teachers' impressions of students' capacities and capabilities are dominant when choosing tasks. This aligns with Gustafsson et al. (2024) who investigated resource use in upper secondary school in Sweden and reported that when the structure of the textbook does not align with teachers' ideas of mathematics lessons, they use tasks from other resources. Siedel and Stylianides (2018) studied what kind of instructional resources teachers choose. Their findings showed that resource selection is complex, and teachers' reasons for selecting tasks are influenced by, among other things, resources at hand, collegiality, own beliefs of mathematics, students, and teaching mathematics. Taken together, these studies support the notion that beside curriculum resources, teachers' own beliefs of mathematics teaching are important when preparing tasks.

Remillard and Kim (2017) proposes that the knowledge activated by teachers when using curriculum resources is a central part of knowledge for teaching mathematics that needs comprehension and development. Reading, interpreting, and reflecting about mathematical concepts in curriculum resources represent an important aspect of teachers' work which directly affects lesson planning (Remillard & Kim, 2017). In this study, the aim is to understand how in-service teachers reflect after using a theoretical framework to do a textbook analysis. The present study seeks to answer the following question: *How do teachers reflect and conclude after conducting a textbook analysis with a theoretical framework as a pedagogical development practice?*

Context: systems of representations

In this study, the teachers were assigned to do an analysis of representations from a part of their textbook that contained fractions. The Norwegian curriculum states that pupils should use mathematical representations in various contexts through their own experiences and in mathematical conversations (Kunnskapsdepartementet, 2020). According to Duval (2006) "a representation is something that stands for something else" (Duval, 2006, p. 103). He points out that the difference between mathematics and other sciences is that mathematics does not have objects to observe directly. The only way to do mathematics, is by using symbols or other representations. Representations can be different, while representing the same mathematical object. Pupils' learning in mathematics is influenced by how they manage to understand and use representations, but also by how they learn to see connections between them (Duval, 2006).

Duval (2006, p. 111) has grouped representations based on the properties and proposes four systems: natural language, illustrations, symbolic language, and graphs/diagrams. Natural language contains both oral and written language and can, among other things, be a written text or an oral expression. Illustrations are referred to as visual figures and involve everything from simple sketches to geometric figures with very specific properties. Symbolic languages are all kinds of mathematical notation, both arithmetic and algebraic. Graphs and diagrams are usually presented in a coordinate system (Duval, 2006). This includes coordinate systems with one dimension, like a number line. Each system offers very specific and unique possibilities.

Method

A total of ten in-service elementary teachers participated in this study, teaching at ten different schools in the southeastern part of Norway. They had 2 to 27 years of teaching experience. At the time of the data collection, all participants were enrolled in a one-year part-time credit-bearing course (30 ECTS) for in-service teachers at Oslo Metropolitan University. Using in-service teachers in this study provided an opportunity to see how their teaching experience influenced their reflections and conclusions of their analysis.

Prior to this study, the in-service teachers had been introduced to Duval's systems of representations, and they were given the following assignment (translated from Norwegian and slightly shortened): "You are going to do a textbook analysis, based on fraction tasks in the 5th grade mathematics textbook used at your school. Look at all the tasks in that section of the book and use Duval's framework to decide whether students have to use language, illustrations, symbols, or diagrams to understand and solve the task. Create a table of the representations used in the textbook and present the result in an appropriate diagram." A quantitative approach to analyse the tasks provided the teachers with an opportunity to compare the use of representations in their textbooks. It also forced them to look at each task with new eyes, only looking for representations. The teachers' experience of doing a textbook analysis is the focus of this study, and a qualitative approach allowed me to explore their meaning and understanding.

The participants were divided into three groups of three or four teachers which had studied different textbooks. The group members compared their individual findings and together prepared a presentation, which ended with a discussion about their findings with all ten teachers. Following the discussions, a semi-structured group interview was conducted with the whole class. The intention of the interview was to explore the teachers' reflections, and frequent questions asked were "What have you learned from doing this?" and "What do you think about the results you found?" There were follow up questions, asking them to explain more carefully or elaborate. The intention was to let the teachers' ideas be the focus of the interview. The presentations, discussions and the interview were recorded with an audio recorder and transcribed in entirety in their original language (Norwegian), lasting totally for about 120 minutes. The teachers handed in documents from the textbook analysis, including results, written reflections, and presentations. Notes were also taken during data collection.

To answer the research question, the transcribed presentations, discussions and interview were subject to a thematic analysis, which offered the possibility for an inductively developed analysis, capturing the experiences of the teachers. Thematic analysis is a method for identifying, analysing and reporting stories about specific patterns (themes) of shared meanings within the data (Braun & Clarke, 2006, 2019). To acquire an overview of the data collected, all statements of potential interest to the research question were organized into a table and coded as clearly as possible. As one can see in Table 1, the words chosen when coding the statements were close to the original.

Table 1: Example of naming teachers' statements

00:41:02 Speaker 10 That is true. […] So definitely use multiple textbooks or try to find tasks and be much more aware of tasks.	Use multiple textbooks Be aware of tasks

The coding produced multiple codes, but these codes often captured small differences. For example, "Use multiple textbooks" and "find tasks from different places" was two statements with almost similar meaning. Clustering reduced codes to a more workable number, and the analysis went into a phase of generating, reviewing, defining, and naming themes. It initially ended up with three broad patterns of meanings: reflections upon task selection; reflections upon task implementation; and comparisons of textbooks. For this paper, I focused on developing the analysis of the data connecting to the reflections upon task selection. After reviewing relation to the codes, the coded data, and the full dataset, a structure with two themes appeared, with each theme describing an aspects of task selection. These are "let go of the textbook in task selection" and "to be aware and critical in task selection". For maintaining the teachers' anonymity, all names presented below are pseudonyms. Although every step of the analysis has been object of discussion with colleagues, it is important for me as a researcher to be aware of how my own subjectivity influences the analysis and interpretation of the data material. It will not be possible to capture all the reflections made by the teachers during this experience, and the analysis is based on teachers' own words, the presentations they made and written notes. I present my findings as possible tendencies, without making general conclusions.

Findings

Let go of the textbook in task selection

Examples of the codes included in this theme are "pick tasks from different places", "let go of the textbook" and "being tied to one book". When the teachers did their textbook analysis, it became clear to them that there were great differences between the textbooks, both in their use of representations, but also in content. The teachers' reactions can be exemplified in Figure 1, which shows digital messages sent between two teachers, Emma and Hedda, in an early stage of their analysis. The teachers shared these messages in their presentation. They asked question to their own textbook's use of representations and if it is true to the curriculum.

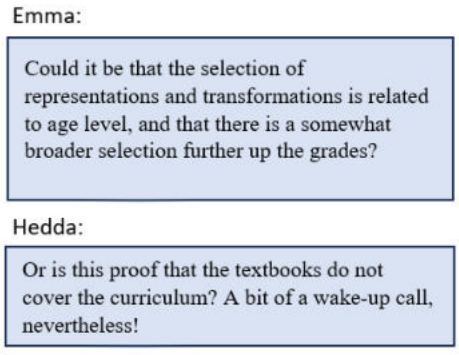

Figure 1: Messages

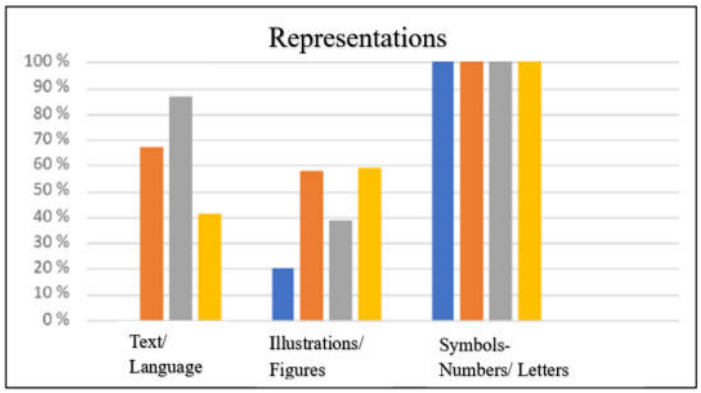

Figure 2: One slide from the teachers' presentations

Several quotes similar to “teachers really prefer following the textbook and the teacher’s guide” and “there are so many teachers that are bound to the book” made it look like the participants thought that following the textbook is a widespread practice among teachers. This align with recent studies of teachers’ use of resources (Bjørnset et al., 2020). It was also clear that they seemed not to approve of this way of teaching. When presenting their textbook analysis, the teachers emphasized the differences they found between the books, and they compared the textbooks with the curriculum, pointing out mismatches between the curriculum and the textbooks. After having presented their results, last slide shown in Figure 2, Lisa answered the question “what have you learned?” like this:

Lisa: So, we have learned a lot from working with this and gained awareness about what kind of task we use and also that the textbooks have a bit different approaches, and then of course when the students encounter different representations, they must also be encouraged by us to transform between them. So, it's our responsibility and not just the book's. Mostly our responsibility actually.
Tutor: What do you think these books require from a teacher?
Hedda: Ability to let go of the book.

These statements are examples of how the teachers both during the presentations and interview emphasized that the textbook analysis had illustrated for them the importance of avoiding a scenario where the textbook dictates the lesson planning. Instead, they advocated for using the textbook as a source as one of several sources to find the right task for each lesson. Eva commented on the textbook represented in blue in Figure 2, in which had illustrations in 20% of the representations, and the rest of the tasks were just represented with symbols. She said:

Eva: Focus on what you want the children to learn; if you want them to only be able to solve tasks and have quantity training, then you can use this book, but if you want them to have a deeper understanding, if you want them to explore mathematics, then you may have to use other tasks. Perhaps you must create your own tasks that are not in any textbook.

The teachers did not conclude with one book being the best or even better than the others. They stated strengths and weaknesses and compared the books. The importance of finding tasks from different sources, make their own tasks, and not rely solely on a textbook was clearly stated.

To be aware and critical in task selection

During the interview, the teachers emphasized the need to be critical and aware when choosing tasks for their lessons. Some of the codes included in this theme are “be aware of textbook being different”, “be aware of tasks” and “be critical while choosing tasks”. When asked about her findings, Eva said:

Eva: It appears that what we have actually found is the need to be aware in the selection of tasks, because all reflection is absent; no tasks facilitate collaborative learning or discussions, and none prompts reflection or exploring, like "What were you thinking?" Those aspects that, in a way, the new curriculum emphasizes, and the purpose of the subject are significantly missing in these tasks.

In this case, Eva stressed the importance of being critical in task selection. Although she did an analysis about representations of fractions, her argument shows that she has discovered other deficiencies in the textbooks as well. When discussing the need of being critical in task selection, the teachers stressed the importance of selecting tasks based on the goal of the lesson. Teachers must be aware of the qualities of the tasks and be critical when choosing which one to use. Hedda explained

this by asking “What does this task offer the students? Is it useful or not? How does this task differ from others? Which one should we choose?” These questions reflect a thinking process in selecting the right task to achieve a lesson’s goal. At the same time, the teachers refer to their experience of classes being different, asserting that a teacher needs to be critical in selecting tasks, tailored to their specific class. For instance, Ole explained what he had learned in these words:

> Ole: This means that you as a teacher must be critical to everything in the book, you must be critical to how to use it. And then I think that the book ... it is a book, and it is intended to be suitable for all students throughout Norway, but at the same time I know that each class and each group is different, so it is important to keep that in mind too. It is not certain that your book or those examples are exactly right. Perhaps you must present it in a different way for that particular group of students. […] So, I think no matter how you use the book, you must be critical of what you use and how.

Ole underscores the importance of being critical in choosing tasks that will both work for the class and the lesson’s purpose. He emphasizes the need to choose tasks critically, while recognizing that different tasks can be presented and exemplified in various ways. After having presented the differences between the books regarding their use of representations, Lisa highlighted how the experience from the textbook analysis shows how conversations about tasks should become integral to teachers’ discourse among colleagues:

> Lisa: But I think it really shows the importance of emphasizing that we are critical of the tasks we choose, and at my school there has always been discussions when purchasing a textbook; […] perhaps there needs to be more focus on the tasks we select ourselves than on which textbook we use; that is a discussion that might need to be brought to the table, then.

There is no clear distinction between the themes. When the teachers suggested letting go of the textbook, they didn't mean to stop using it altogether; instead, they advocated for its critical use. Being aware and critical in choosing tasks is necessary to let go of the textbook. At the same time, it is possible to be critical to tasks and still rely solely on a textbook. When asked about what they learned, Hedda said: “You become more aware yourself, I think. About what you choose. With the textbook, it's about letting go a bit and instead select tasks with care. In other words, thinking more about what you want.”

Discussion

The findings indicate that the textbook analysis stimulated the teachers’ reflections in task selection, concluding that it is important to let go of the textbook and to be aware and critical in task selection. This is important because we need to find ways to develop teachers’ professional knowledge in planning lessons (Cevikbas et al., 2024), and this might be a possible way to support teachers with a high level of autonomy (Eriksen et al., 2022) and a curriculum with limited directives (Sundby & Karseth, 2021). Mathematics teachers are influenced by their textbook (Bjørnset et al., 2020; Fan et al., 2013), and this study might offer a practice to train teachers to be more reflective in the process of selecting tasks for their lessons. If the teachers become more reflective in their selecting of tasks, it will make them more proficient in lesson planning. In the long term, this might lead to better instructions in mathematics classes and better opportunities to learn for students.

Getting to know a theoretical framework and use it for a textbook analysis seems to have transferability for these teachers. Using Duval's (2006) systems of representations as a framework for analysing textbook tasks appears to have started thought processes and reflections aside from the topics. The teachers did not only talk about representations of fractions, but selecting tasks and quality of lessons in a more general manner. Just by analysing a small part of their textbooks, the teachers experienced that the authors had made different decisions and that the textbooks differed from what the teachers perceive the curriculum to address. Although Duval's system of representations worked as lenses to see their textbook tasks in a different way in this research, other theoretical framework might also work to evoke similar reflections. More research need to be done on such designs to find out if similar practices of PD will support teachers' knowledge when using curriculum resources (Remillard & Kim, 2017).

This is an example of a practice that can train teachers' reflections upon task selection. Research indicate that teachers' belief of teaching has an impact when choosing tasks for a lesson (Gustafsson et al., 2024; Riard & Kaur, 2022; Siedel & Stylianides, 2018), and this knowledge activated by teachers when using curriculum resources needs development and comprehension (Remillard & Kim, 2017). The findings indicate that using a theoretical framework to do a textbook analysis can be an effective practice for PD to develop such knowledge, making the teachers reflect upon task selection and support their knowledge for lesson planning (Cevikbas et al., 2024).

In the interviews, the teachers were given open questions about what they have learned through this experience. This might have been important in catching the teachers' own experience doing the analysis. The teachers themselves kept coming back to task selection as an important eye opening when doing the analysis. It is a limitation that this study has only been with ten teachers who have attended a PD program voluntarily. Furthermore, it is not certain that doing a textbook analysis with pre-service teacher will show a similar result, since they do not have the same experience to reflect upon. More research needs to be done with other teachers and pre-service teachers as well. This study is an example of how PD can support and train teachers in choosing tasks when planning lessons.

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Inquiry-based learning through problem posing in pre-service teachers' teaching practice: a study of discursive manifestations of contradictions

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Inquiry-based learning and problem posing in mathematics education are often seen as two different learning approaches. In this study we combined these two approaches and examined them as integrated in pre-service teachers' field practice in order to explore how problem posing could simplify the inquiry-based learning implementation. We used discursive manifestations of contradictions from Engeström's Cultural Historical Activity Theory to discuss the themes of challenges and attitudes of pre-service teachers' practice. Data is interpreted using content analysis. We found that the understanding of contradictions increases the possibility to develop inquiry-based learning through problem posing in mathematics education, and they can be interpreted as opportunities for future professional development programs.

Keywords: Inquiry-based learning, problem posing, pre-service teachers, activity theory, manifestations of contradictions.

Introduction

Our research aimed to investigate the use of problem posing (PP) activities in inquiry-based learning (IBL) and explore how PP could simplify IBL implementation. By PP, we mean activities where the student is given opportunities to engage in reformulating, extending, or creating new problems. By IBL, we mean a student-centered teaching strategy in which the students' role is to engage in their own learning, explore situations, inquire, pose questions, and seek explanations in a collaborative setting (Artigue & Blomhøj, 2013). We do this by encouraging pre-service teachers (PSTs) to support IBL lesson plans with PP activities and see how these lesson plans change with the inclusion of PP. Research has shown that there is an interest in integrating PP into classroom practice (Li et al., 2020). The same applies to IBL, but studies indicate that there is a need for professional development programs for integrating IBL in teaching because teachers find it difficult to implement it in the classroom (Sikko et al., 2012). PRIMAS (2011) identified IBL as crucial for successful education, introducing five aspects and their features, including students posing questions. Therefore, we focused on PP to help PSTs engage students in IBL, simplifying IBL implementation and fostering a student-centered environment. While researchers have extensively studied IBL features (PRIMAS, 2011), the key to easy implementation lies within these features. Therefore, our research addressed PSTs' practical challenges and attitudes, to make IBL more accessible and effective by emphasising PP, and to explore how IBL can be developed through PP in mathematics education as an integrated approach. PP is the process of creating new problems for investigating a situation or reformulating existing problems (Silver, 1994). Research (e.g., Zhang & Cai, 2021) has highlighted a lack of studies analysing the content and features of teaching through PP. Studying how PP can be integrated into IBL to facilitate learning and understanding the challenges in this process can inform research to

support mathematics teachers in teaching through IBL. Thus, it is important to explore IBL and PP as integrated parts, not separately, and to clarify the challenges and attitudes of joining the two approaches.

IBL is a multifaceted activity that may guide students to pose meaningful questions and generate relevant answers. PP activities may result in an increased use of IBL. Whether PSTs tend to incorporate IBL and PP in their teaching depends on their attitude towards what they learned from their own experiences, their education and instruction experiences, and their formal knowledge experiences (Richardson, 1996). Attitude is commonly defined as "a predisposition or tendency to respond positively or negatively" to things, people, places, events, or ideas (Vargas-Sánchez et al., 2016). PSTs' attitudes are a key element in understanding classroom practices. Changes in teachers' attitudes are also closely related to changes in teachers' practices, which may result in improved student learning (PRIMAS, 2011). It is therefore essential that we gain a deeper understanding of the attitudes of PSTs toward IBL approaches through PP activities during their school practice periods.

Using PP as a tool, IBL may influence the development of preservice teachers' experience. Studying this connection contributes to the research field by investigating PSTs' challenges and attitudes while they use IBL through PP lesson plans in their teaching practice. This study used the framework of Cultural-Historical Activity Theory (CHAT), based on the discursive manifestations on contradictions (Engeström & Sannino, 2011). In accordance with Engeström and Sannino (2011), contradictions are manifested in people's actions and speeches, and they cannot be directly observed. The aim of this paper, is to identify the discursive manifestations of contradictions as PSTs pursued their practice period where they would implement IBL through PP. Thus, we look at challenges and attitudes as contradictions, and contradictions as opportunities for future professional development programs of teaching IBL through PP. Our research question is: "What contradictions were manifested by problem posing in pre-service teachers' school practice within inquiry-based learning?"

Theoretical framework

In CHAT, contradictions are seen as opportunities for learning and development (Engeström, 2001). Contradictions cannot be directly studied empirically but must be approached through their manifestations (Engeström & Sannino, 2011). The manifestations of the contradictions can be detected from the actions and speeches of people, and four types of manifestations of contradictions have been identified by Engeström and Sannino (2011):

Double binds "arise when requirements and conditions are incompatible, often expressed as helplessness or desperate rhetorical questions" (Harvey & Nilsson, 2022, p. 7). They are manifested in expressions that reveal that individuals don't know what to do, feel under pressure, and would prefer alternatives to what they are doing. Double binds cannot be handled individually but need to be addressed collectively. Most commonly, this is expressed in a collective action such as "we need", "we have to" and as well as rhetorical questions such as "what can we do?".

Critical conflicts "are situations in which a person encounters inner doubts on contradictory but mutually dependent motives and meanings that are unsolvable by the person herself" (Harvey &

Nilsson, 2022, p. 7). During critical conflicts, individuals feel violated, guilty, silenced, paralysed and communication occurs using "I", and strong metaphors are often used.

Conflicts occur when participants oppose each other's opinions and actions; and there is resistance, disagreement, argumentation, and criticism involved. Conflicts are accompanied by phrases such as "no" and "I disagree" and are typically resolved by compromise or by taking the majority's side.

In a *dilemma*, incompatible evaluations are expressed or exchanged, either between individuals or within a single discourse, and are usually expressed as hedges and hesitations, such as "on the one hand[...] on the other hand" and "yes, but".

Methods

Our data was gathered from ten groups of three to four Norwegian PSTs for grades 5-10, with one participant from each group being selected for interview. PSTs were in their second semester of their first year, engaging in their teaching practice period in actual classroom settings. This involves supervised teaching experiences, allowing them to apply theoretical knowledge, develop teaching skills, and understand classroom dynamics. During a workshop about IBL through PP on campus, the participants were divided into ten field practice groups, so the participants for interviewing were selected on a purposive sampling approach, while each participant represents one group. The interview guide for the semi-structured interviews was based on themes of PSTs' challenges and attitudes related to IBL through PP revealed from prior campus workshops as well as literature. The themes of the interview were jointly discussed by the three authors. The first author of this paper conducted the interviews, and all interviews were conducted in English. It should be noted that English was not the first language of any of the participants. The interviews were audio-recorded and fully transcribed. The data were analysed using qualitative content analysis. We used the framework proposed by Engeström and Sannino (2011) to identify the discursive manifestations of contradictions. First, we identified segments relevant to our research question and we reduced our data in which PSTs mentioned challenges or attitudes (see Table 1). Second, we categorised each significant segment according to the four manifestation types (Engeström & Sannino, 2011). Additionally, key words for each manifestation were emphasised, along with arguments for categorisation. A better understanding of contradictions was provided by analysing their type and theme together.

Table 1: Examples of themes

Attitude	Challenge
I think the students enjoyed it very much because it was something different because maybe they thought that math wasn't their like favourite subject… like…I think it was fun to do something else that didn't feel like a normal mathematics lesson and so it helped their motivation a bit.	That was hard! That's what I think is the hardest thing: Those questions you ask, the students or the groups.... To make them... Ehhmm... work more or realise themselves what they can do without you telling them what to do.

Results

This study identified 80 discursive manifestations of contradictions. Table 2 gives an overview of manifestations of contradictions and their types. Based on the nature of the contradictions (Engeström, 1987), it is obvious that we have a higher frequency of contradictions in the challenges than in the attitudes of the PSTs. Most noticeable is the high frequency of "dilemma" and "double bind" manifestations, especially in the challenge theme and there was a much lower frequency of "conflict" manifestation.

Table 2: Types and the themes of discursive manifestations of contradictions

	Double bind	Critical conflict	Conflict	Dilemma
Attitude	9	3	0	17
Challenge	12	17	2	20

Double binds

Double binds were less frequent in PSTs' attitudes than in challenges. PSTs were unwilling to use IBL through PP without prior instruction. PSTs declared that they need more help to implement these approaches in their teaching. They pointed out that it is important to practice it and not just being told about it. Thus, PSTs identified the developmental needs in their mathematics teaching with IBL through PP with the help of experts in the field:

> PST: I wouldn't wish that I knew more about PP and IBL, but I think by just reading or someone telling you about it, I don't think for myself, I don't learn more from that than actually doing it with the students because.... For me, seeing it in person… seeing it…come to life.

Time resulted in a double bind in the following way: during the practice, students had to abide by the schools' regular time schedule, which restricted their implementation of IBL through PP as they felt this required more time:

> PST: We have planned this, you know, so you only have like 45 minutes that it's just way too little time… That was stressful, and I also think the kids also got a little stressed because I think 2 minutes to write is way too.... It's not good at all, and they probably should have had way more time, I think, to write the problems. And also, I think you should have much time they like compared to like writing the equations and writing the text, you know...

Examples of double bind manifestations include rhetorical questions, the use of the pronoun "we," and confirms that contradictions need a collective action to get solved. Despite these seemingly insurmountable obstacles, PSTs managed teaching IBL through PP in their practice. Thus, manifestations of the double bind were characterised by an urgent need for action thinking that something must be done to improve the situation.

Critical conflicts

We found situations where PSTs felt guilty for asking students to work on tasks where they did not have sufficient prior knowledge, like using equations to solve word problems:

PST: I'm not sure, I think…and maybe not all of them really understood what an equation was or like they.... But at the same time all of them could be solved as equations, but they were kind of big task. I think maybe a little bit of we hadn't had like these types of problems that they had to solve beforehand because they only had like.... Just like solve this equation and not like text.... Yeah, so I think maybe we should have had like a lesson before where they solved these types of problems before they had to like to make them. But I don't, I don't know.... We got to know what could have made it clearer.

Another contradiction manifested as a critical conflict of PSTs was when PSTs were unsure how to guide the students without just telling them what to do. The following example illustrates PSTs' feeling of powerlessness:

PST: That was hard! That's what I think is the hardest thing: Those questions you ask, the students or the groups.... To make them... Ehhmm... work more or realise themselves what they can do without you telling them what to do.

Conflicts

Conflicts were the rarest type of manifestation of contradictions. The next example shows a challenge of a disagreement between a PST and a practice teacher:

PST: And I also think.... Time is also, can be a little challenging, time because…. Yeah, time was very challenging. We only have 45 which is way.... We were actually having 90 and then 45 minutes.... But then, like...Class like the week before she was like ...our practice teacher was like "yeah, we can't do that those minutes because we have to go do something else" or whatever they were.... And I was like, oh....

Another example is when there were also many different points of view about the contexts of the posed problems, resulting in a high incidence of conflicts between students:

PST: At least we had young students, and they also thought it was really cool to make a difficult task like "look, I made the most difficult task you can solve this", and then I asked, "can you solve it?", and then they're like "Oh no, I can't.". So yeah, that could be a problem as well. And also, how those students were because there was a class with many difficulties outside of the lesson itself because they yeah, as soon as they met a challenge, they were like jumping around and running around then yeah. So, they weren't dealing well with like meeting a task they didn't feel like they managed, yeah.

Dilemmas

Dilemmas were the most frequent type of manifestations of contradictions. Most of the dilemmas were about PSTs' expectations and responsibilities during their practice. One dilemma manifested when PSTs reflected on whether they could use IBL through PP in all lessons or whether this method should only be reserved for special lessons:

PST: I think it's… it could kind of explore things because I think you get a lot of a much deeper understanding too; you know solving the problems when you actually have to make them. It's often difficult to know how-to take-out information, but if you actually make them, you kind of know how they're built up, but I don't think we can like only have problem posing in IBL in like every lesson.... Because I think a lot of students need to have some structure and needs to be kind of guided in some ways as well.... hmmm…Important to have that habit in mind when you may....

Researcher: And listen to students that maybe sometimes it' s good to use problem posing into IBL....

PST: Yeah, yeah, and problem posing, because I think problem posing is kind of helps with IBL because IBL is very big and you don't know where to start, is like OK, you're going to be exploring things, but what do you do? Where do you... where do you begin ...?!

When IBL through PP teaching method is used in PSTs' practice, as new and more advanced than their previously used teaching methods, they become contradictory to PSTs' expectations of how the students understand the problem context and how this context has sense in real life. This contradicts the norms of creating mathematical problems that have sense in the real-life context. This contradiction is manifested as dilemma because PSTs were confused between the different ways in which students conceptualise the context of tasks:

PST: Choose harder [PST probably meant longer, not harder] length on the pool because...a pool that's 2 centimeters?! And yes, you can't swim in that, so you kind of get them to understand the logic that a swimming pool isn't just 2 centimeters, but it should be....

PST: Not everyone, I remember there was one girl that had this problem like oh, it didn't make sense, but it was so interesting to see how she came up with that problem. We also have one of these where like: Uh..., one half of a boyfriend and 1/4 of another boyfriend. How many boyfriends did you have like?

PST: Yeah, it didn't make any sense so but yeah, but some of them didn't understand it really good, then made this happened.

Researcher: Yeah, and why do you think this happened?

PST: It isn't a misunderstanding. Maybe they don't understand what the numbers mean or. Yeah, that is pieces of the whole enough that you can take one 3/5 of a person. And yeah, so these tasks are good to figure out and maybe things like this that the students don't know what it means or…. Yeah, yeah, common misunderstandings maybe!

Discussion and Conclusions

We show in the present study how activity theory with a contradiction's perspective contributes to understanding and shed light on the underlying structures for establishing PP in inquiry teaching. Our study implies that many PSTs are open to teach IBL through PP in their teaching. PSTs were ambiguous about IBL implementation, and this resonates with other findings from DiBiase and McDonald (2015). In our study PSTs were not confident regarding their own knowledge and skills. Blomhøj et al. (2022) emphasise the importance of supporting teachers in planning inquiry-based mathematic teaching, especially for students' independent inquiry. Our analysis of discursive manifestations of contradictions support Blomhøj et al. (2022). In our study, the PSTs felt that they needed more prior knowledge of IBL and PP before being able to implement it in schools in a way that enhances student learning. This points towards the need for professional development, at both pre- and in-service level, which is consistent with Sikko et al. (2012), who suggested that continuous professional development courses which promote improvements in mathematics and science teaching and classroom practice are needed.

We found that time constraints are an obstacle when trying to implement IBL through PP lesson in practice, and this turned up in different manifestations. Calleja et al. (2023) found that teaching through inquiry requires teachers to balance integrating inquiry with contextual challenges within sufficient time. It's typical for teachers to calculate their decisions in terms of time, but in this study, we also see time as an opportunity for PSTs to learn and develop their teaching. Firstly, in our study

the PSTs realised that lessons need to be organised so that students have sufficient time for posing problems and solving tasks. This means that lessons should be concentrated around a few carefully selected tasks/problems. PSTs saw this as important for their future work as teachers, and this was manifested as a double bind in our results. Secondly, a conflict about time occurred between a PST and a practice teacher. In line with Ruthven (2009), we see our results as an opportunity for PSTs to measure didactic time in terms of student learning progress and not just as a requirement to cover a curriculum. Ruthven (2009) calls this "time economy", which is a coordinating resource system consisting of a working environment, activity format, and curriculum script to support classroom activities. In our study, the PSTs felt guilty for not developing students' knowledge of textual problems with equations beforehand. Therefore, in this way they express their willingness to invest time in familiarising students with equations so that IBL teaching with PP activities is as practical as possible for students. This finding resonates with what Ruthven (2009) called "instrumental investment". The understanding of discursive manifestations of contradictions found in our study increases the possibility to develop IBL through PP in mathematics education as an integrated approach. We see potential for further study how IBL through PP approaches may improve teaching and learning. The results of the present study concerning PSTs challenges and attitudes, provide critical information about how teachers will implement teaching IBL through PP. Contradictions identified in this study could be interpreted as opportunities for professional development in the future. Also, this study could provide further insights into the relationship between contradictions and inquiry in mathematics education.

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Utilizing a three-phased didactical model in a continuous professional development program for the implementation of inquiry-based mathematics teaching

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Inquiry-based mathematics teaching (IBMT) has a potential to motivate and develop students' mathematical competencies but proves challenging for teachers to implement. This paper presents results from a continuing professional development (CPD) project focusing on implementing IBMT through lesson study cycles centered around a three-phased model for IBMT. A framework of essential IBMT-ingredients was used to identify essential IBMT-ingredients and challenges related to this. Our analysis shows teachers adopting IBMT-values and implementing the structure from the three-phased model. They succeed in initiating and engaging students in inquiry, thus creating opportunities for dialogue, justification and shared reflections. However, the latter rarely happens. This highlights the need to find ways to consistently support teachers in managing student responses through dialogue and facilitating whole-class discussions based on students' mathematical thinking.

Keywords: Inquiry based learning, professional development, mathematics teachers.

Introduction

Inquiry-based teaching has had a significant focus within the field of mathematics education during the last decades. Although there is no simple definition of inquiry-based mathematics, there is broad agreement that it is a student-centered approach that provides opportunities for problem-solving, reasoning, and communication, and has the potential to motivate students (Artigue & Blomhøj, 2013). The Danish curriculum is oriented towards developing students' mathematical competencies and it highlights the importance of students' engaging in mathematics through dialogue and in collaboration with others. It is therefore clear that the curriculum, in both goals and form, emphasize inquiry-based mathematics teaching (Ministry of Children and Education, 2019). Many mathematics teachers in Denmark find the inquiry-based and competency-oriented approach to teaching both inspiring and meaningful. However, they find it difficult to translate these ideas into their own teaching practice (Højgaard & Sølberg, 2023). In a large national survey, teachers highlighted several perceived challenges. They encountered difficulties generating inquiry-based problems to pose to students and finding time for planning. Furthermore, the teachers have an experience of "loss of control" when they cannot predict the direction of students' inquiry-based activities (Dreyøe et al., 2017).

In 2020 University College Copenhagen (KP) together with 10 primary and lower secondary schools in the Municipality of Copenhagen, developed a CPD-program aimed at enhancing motivation and student learning through inquiry-based mathematics teaching allowing students to engage in problem-solving, communication, and reasoning. For two years, mathematics teachers participated in a CPD collaboratively developing didactic strategies to facilitate inquiry-based mathematics teaching in their classrooms. They were supported by researchers and the school's mathematics supervisor.

In a similar project from Norway teachers highlighted a three-phased didactical model for IBMT as a crucial enabler for successful implementation of IBMT in school (Haavold & Blomhøj, 2019; Blomhøj et al., 2022). The three-phased model therefore became a central element in the CPD-program. In this paper, we will investigate if and how a CPD-program centered around the three-phased model can enhance the implementation of IBMT. The research questions are: *How can a continuing professional development program focusing on a three-phase model for Inquiry-based mathematics teaching strengthen its implementation? What challenges and successes do teachers encounter in each of the three phases?*

The framework of the CPD

Professional learning communities where teachers discuss instructional decisions based on shared goals and educational values, have shown promising in improving both teaching practices and student learning outcomes (Cobb et al., 2018; Akiba et al., 2019). Especially lesson studies have gained significant attention worldwide, inspired by how Japanese teachers collaboratively explore and investigate curriculum, teaching practices, and student learning through live classroom lessons (Lewis et al., 2006). The strength of lesson studies as a professional learning process lies in its strong focus on student learning and the enactment of teaching. This involves collaborative planning, observation and reflection, all centered around understanding how students learn and how teaching practices affect learning. It is well documented that this is crucial to promote mathematical knowledge for teaching and embed knowledge into daily teaching practice (Akiba et al., 2019). Lewis et al. (2006) illustrate two alternative ideas about how lesson study improves instruction: 1. by refining lesson plans and 2. by strengthening three pathways to instructional improvement – teacher knowledge, teacher communities and learning resources. While the first is easily recognized by teachers, it is the second idea our CPD program aims to promote.

In the project, a teacher learning community was established by teachers collaborating through lesson study-inspired learning loops. Mathematics teachers and supervisors participated in four loops. A learning loop consists of Shared Planning, Teaching/Observation and Joint Reflection (Figure 1).

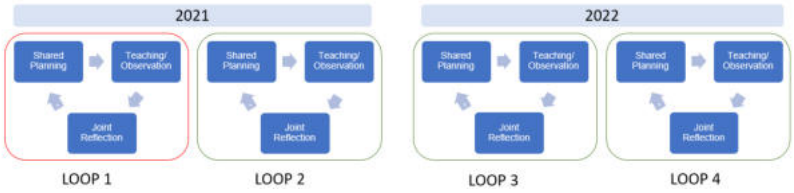

Figure 1: The framework of the continuous professional development program

The *shared planning* was led by both a researcher and a mathematics supervisor and was integrated into a course on current research in IBMT. It involved collaborative planning within smaller teacher teams. The *teaching and observation* followed a format of a research lesson. It was conducted in the classroom where one teacher teaches the lesson while the team, along with researcher and the mathematics supervisor, observe the lesson. The *joint reflection* was divided into two parts, Joint Reflection 1 being similar to a traditional lesson study post lesson reflection where observations are shared and reflected upon. This session is summarized by either the researcher or the mathematics supervisor. In Joint Reflection 2 all mathematics teachers and supervisor(s) at school gather to share

findings from their research lessons and to identify successful instructional approaches to IBMT. Research presented during their *shared planning* was revisited and connected to the experiences from the research lesson, ensuring that theoretical insights informed practical outcomes.

The content of the CPD

The content of the CPD focused primarily on operationalizing the ambitions of IBMT. The three-phased model for IBMT has proven to be an effective planning tool for teachers, and it also offers the potential to explore the instructional challenges inherent in each phase (Haavold & Blomhøj, 2019). The first phase involves setting the scene for students' activities. This includes creating engaging and clear inquiry questions, challenges or amazement to spark their curiosity. In the second phase, students engage in independent, inquiry-oriented activities. The third phase is a whole class discussion where results, and reflections from the activities are systematically organized and shared among students. Mathematical insights are extracted from these shared results and reflections, and these insights are linked to institutionalized knowledge, such as the curriculum. The model is illustrated in figure 2. The model served as a framework for sequencing the IBMT lessons and for discussing and reflecting on the teacher's role and challenges in IBMT.

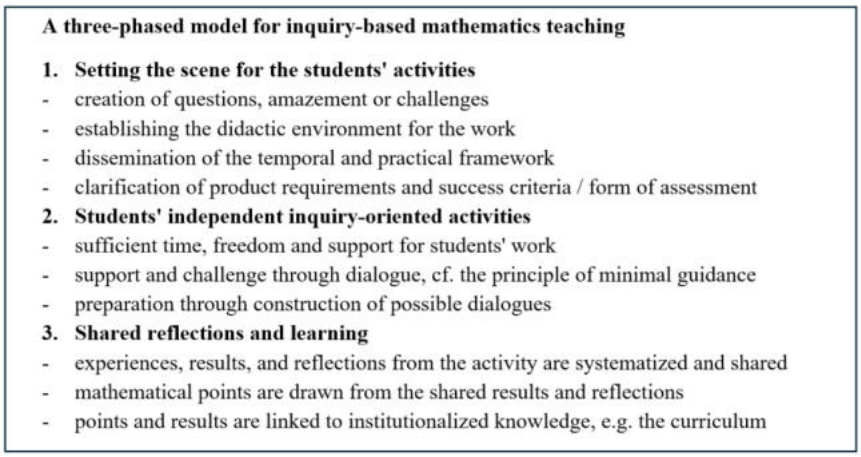

Figure 2: Three-phased didactical model for IBMT

To support teachers in noticing and enacting the classroom practices embedded in each of the three phases we found it useful to integrate the Five Practices framework into the three-phased model. Five Practices for Orchestrating Productive Mathematical Discussions is a framework designed to help teachers use student responses to advance the mathematical understanding of the class by providing teachers with some control over what might happen during the lesson. The five practices are *anticipating, monitoring, selecting, sequencing and connecting student responses* (Smith & Stein, 2018).

Each learning loop focused on one of the three phases and provided the teachers with research, examples, videos, and discussions addressing central elements within each phase. The teachers planned the entire lesson in each loop, but with particular emphasis on one of the phases. In loop 1 we focused on Phase 1 *setting the scene for the student activities*. Beside selecting tasks and setting goals, the teachers were engaged in how to launch a task in a way that motivates students to

investigative problems and provide all students with opportunities for participation. In loop 2 the central focus was Phase 2, *Students' independent inquiry-oriented activities***.** We addressed two central issues: to anticipate student response and support and challenge through dialogue. In loop 3 the central focus was Phase 3*, Shared reflections and learning*. Building on the teachers' anticipation of student responses, teachers engaged in how to organize a whole-class discussion where students ideas and results are shared, and the mathematical point stands clear. In loop 4 we addressed special issues the teachers identified as most difficult, e.g., the shared reflection (Phase 3) and how to identify the central mathematical idea or point in a task. We encouraged the teachers to base their lesson planning on IBMT activities that we introduced and worked on thoroughly during the course.

Methods

Our analysis focusses on the implementation of IBMT and the associated challenges encountered during the CPD-project. Two primary schools (grades 1–6) were selected as critical cases to obtain information that allows logical deductions of the type, "If this is (not) valid for this case, then it applies to all (no) cases." (Flyvbjerg, 2006, p. 230). These schools are considered critical cases because all mathematics teachers at both schools were actively involved in the project, each school had a highly experienced and dedicated mathematics supervisor and both schools received strong support from the head of school, which, as noted by Cobb et al. (2018), is a key factor in the success of a CDP. Our data consists of video observations from six IBMT lessons (1st grade (1), 2nd grade (2), 3rd grade (1) and 6th grade (2)) conducted during loop 2–4, as well as interviews with one teacher and three supervisors (who are also teachers). The video observations were partially transcribed, while the interviews were fully transcribed. The analysis and findings in this paper rely on data from both video observations and interviews. To identify the presence or absence of successful implementation of IBMT, we used the framework of five categories of essential ingredients in inquiry-based education shown in Figure 3: *valued outcomes, teacher guidance, classroom culture, type of questions* and *what student do* (Artigue & Blomhøj, 2013). The analysis followed a deductive approach, specifically looking for these essential ingredients. For instance, we coded the following statement from a teacher to his class "Just try. No one gets angry when you try. And if we make mistakes, we learn from them." (2nd grade) as *Valued outcomes, inquiring minds*. Like Figure 3 indicates, the five categories overlap, and this example could also be coded as *classroom culture, value mistakes*. Examples of coding of the remaining four categories. *Type of questions, open, multiple solution strategies*: Teacher: "It's about subtracting a one-digit number from a two-digit number. If I tell you the result [difference] must be three, which two-digit and one-digit numbers can I use?" (1st grade). *Teacher guidance, values and builds upon students' reasoning/scaffolding*: Teacher: "You've come to a standstill over here. Sander, tell us what you said before and how you arrived at that conclusion" (6th grade). *What students do, explain*: Student: "We started by giving 10. Then we found out that everyone could get 3 more, so everyone could get 13. Then there were 4 lefts." (3rd grade). *Classroom culture, dialogic*: Teacher: "Now we've been searching for rules. What have you come up with?" Student: "We've found a system." Teacher: "What's happening?" (1st grade)

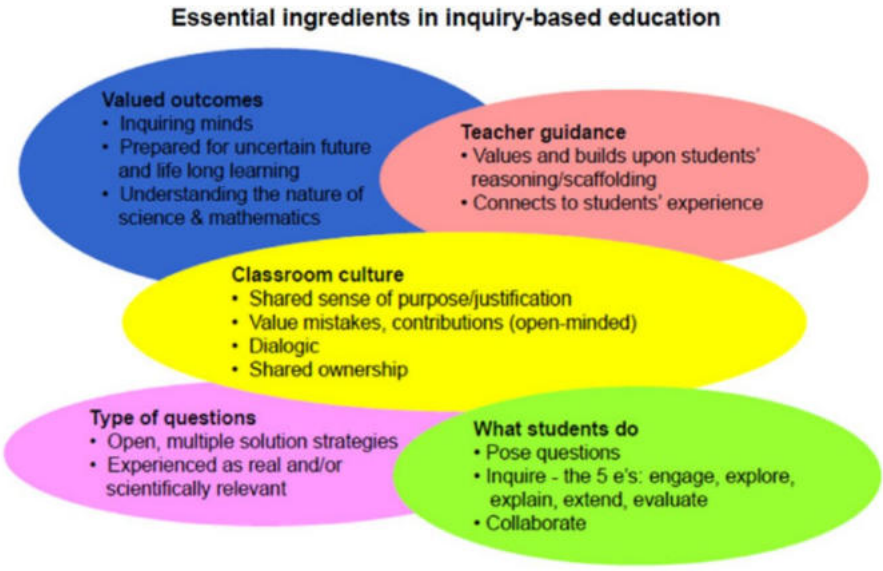

Figure 3: Essential ingredients in inquiry-based education

This coding was followed up by an examination of where these findings emerged during the phases of the lessons. Finally, we searched for elements of the three-phased model (Figure 2) to see what parts of the model was present (or absent) in the lessons and to identify possible challenges related to these. For example, "Is there a shared reflection and if so, does it contain sharing of results and systematization?"

Results

The analysis shows that all teachers adopt IBMT-values. In the interviews the teachers express the potential of IBMT in cultivating a *classroom culture* where students' contributions are seen as *valued outcomes.* One teacher emphasizes the affective potentials of IBMT: "…working together on something and feeling like one is contributing is significant for the students" (teacher 1). Another teacher highlights potentials regarding student participation "both high-achieving and low-achieving students can participate" (teacher 2). This teacher also emphasizes the potential of the *type of questions* in IBMT: "I like the idea that there are many ways to arrive at a solution" (teacher 2). The teachers' perceptions are also reflected in the IBMT lessons. They are being explicit with students about inquiry values by telling students mistakes are productive and welcome, that there are multiple paths to the answer, and encouraging them to experiment. Like in a 2nd grade class, where the teacher encourages students to take risks and again at the end of the lesson emphasizes this as a valued outcome: "Wow, you were so great. You must trust yourselves when you do something like this." (2nd grade)

The teachers use the terminology from the three-phased model when discussing teaching. For example, "We have opened our approach to setting the scene and shared reflections, and how we provide support along the way" (teacher 4). This indicates teachers having adopted the model and finding the concepts derived from it meaningful for describing teaching practice. The lessons are

clearly structured according to the three-phased model. However, in the following sections, it will become clear that some phases are easier for teachers to implement than others.

In phase one, we can identify a high amount of essential IBMT-ingredients, especially in terms of *type of questions* posed by teachers, *classroom culture* and *teacher guidance*. The teachers were exemplary in posing imaginative initial problems with multiple possible solution strategies. They received numerous positive reactions like laughter, smiles and students generally paying attention and listening to the teachers' initial question. This show that the problems are *experienced as real and relevant.* The teachers establish a *classroom culture* with a *shared sense of purpose and shared ownership* and are attempting to connect the mathematical problems to student's experiences for clear *teacher guidance*. Like in the following sequence from a 3rd grade, where the teacher introduces the class to a problem related to division with remainders through the context of Grandma Olga and her sausage-shaped dog. The teacher draws Grandma Olga on the board and tells the students she wants to distribute 95 gingerbread cookies equally among her 7 grandchildren. Olga has a dog named Sausage. Students are attentive as the teacher draws the dog. They laugh shortly and calm returns quickly. "Sausage will get the leftover cookies after a fair distribution… Sausage should keep his shape", the teacher says. (3rd grade) The teacher uses a familiar and fun context and uses drawings to capture students' attention, making the problem relevant and creating a shared sense of purpose. In the interviews the teachers also emphasize the first phase (setting the scene) as valuable. For instance, one teacher says, "…I think the idea of tying a story to an investigation really captures the students' interest" (teacher 4) thus telling us that this approach to starting an investigation is somewhat new to him. After a successful introduction of the problem to the students, the lesson progresses to the second phase - students' independent inquiry-oriented activities. Most of the children are actively participating in the task and *collaborate* with classmates. The teachers attempt to support and challenge students through dialogue, but maintaining a dialogic approach proves difficult. We observe teachers creating a *Classroom Culture* that is partly *dialogic.* For example, from video observation, 3rd grade:

Teacher:	Could you manage it [the method] with [distributing] 14 [gingerbread cookies to each]? how did you actually do it?
Student:	We just wrote 10 under each and then tried to write 4 under each, then we found…
Teacher:	…found out you were missing some gingerbread cookies.

Here we see an initially open question ("how did you actually do it?") which leads to the student explaining his method. However, the teacher interrupts the student and takes over his explanation taking away the possibility for the student to validate his solution. We see some elements of *teacher guidance* in trying to value and build upon student reasoning: "Okay, you find pairs of numbers. Aha. How do you know that 16 and 12 are a pair?" (2nd grade) But the questions are posed quickly and only partially answered by the students before the next question is introduced. Although the questions have the potential to enhance students' reasoning, insufficient time is provided for a genuine dialogue to develop. In phase three, all teachers gather students for shared reflections and learning at the end of the lesson. But it is mostly carried out as an arbitrary round of individual presentations of methods shortly validated by the teacher, e.g., "Great, good solution." (6th grade) without a need to justify.

Although focusing on the practices of anticipating and sequencing in the CPD we see few signs of teachers sequencing student solutions. On the contrary, one teacher opens the shared reflections by asking the student who found the task most easy to explain her most complex solution for the class: "Julia, you have solved all the tasks. Which part did you find the most difficult?" (2nd grade) This left the rest of the class losing focus and appearing disengaged in Julias explanation. Moreover, students' results are rarely connected to one another, and the lesson's mathematical points are seldom clear and linked to institutionalized knowledge.

However, in the 1st and 2nd grade we see students being able to identify and describe the mathematical system in the task (e.g., "It's a number pattern – you keep adding 1 each time." (1st grade student)), and a teacher who systematically guides students in this direction. Unlike the other lessons, this lesson was derived straight from a Japanese lesson study introduced at the CPD. Here the mathematical system and idea in the lesson was described in detail as well as the teachers' organization of the shared reflection. This could imply a need for more robust scaffolding for the teachers.

Discussion and conclusion

We have examined how a CPD program focusing on a three-phased model for IBMT can strengthen its implementation, and what challenges can be identified. Aligned with Højgaard and Sølberg (2023) we find teachers adopting IBMT-values. And as Blomhøj et al. (2022), we show that the elementary teachers adopt the three-phased model as a didactical tool for planning, conducting, and reflecting on their IBMT lessons. In this paper we show what kind of challenges the elementary teachers face translating IBMT ideas into their own teaching practice. The nature of these challenges appears to differ from the secondary teachers in the study from Blomhøj et al. (2022). Our analysis shows that the elementary teachers in phase one, clearly set the scene and hand over the problem to students in a way that allows all of them to engage in phase two with a shared sense of purpose. This result contrasts with the findings of Blomhøj et al. (2022), who report that secondary teachers face challenges in phase one, particularly in establishing a shared sense of purpose, where a mismatch between students' and teachers' perceptions of clarity is a key difficulty. We show how the elementary teachers struggle to sustain a dialogic approach to teaching that promotes students' reasoning, sequencing and connecting the class's solutions to the lesson's mathematical goals. The secondary teachers in the study of Blomhøj et al. (2022) have a clearer understanding of the mathematical purpose of the given task and focus on linking the task to the mathematical content. This suggests that the teachers in the two projects focus their efforts on different phases of the IBMT lessons.

Our study is based on two critical cases, defined by the fact that the teachers in our study have optimal conditions for implementing IBMT. These conditions include strong, supportive mathematics supervisors and a school leadership that prioritizes and facilitates teachers' time for professional development. Despite these favorable conditions and the strong three-phase framework for implementation, fundamental IBMT values such as creating opportunities for students' reasoning remains challenging. This suggests that even under favorable conditions, a CPD-program demands persistence to embed IBMT values into daily teaching practices.

However, our study also suggests that a CPD structured around well-developed IBMT lessons (e.g., from Japanese lesson studies) proves more successful for the implementation of IBMT than teachers

developing their own IBMT lessons. This is especially true when it comes to the shared reflection (Phase 3). While this finding is not surprising, it is to some extent still a rather provocative result in a Danish context, where we highly value teacher autonomy (Højgaard & Sølberg, 2023).

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Fostering the perception of ambiguity and the processing of information in complex diagnostic situations

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Human perception and information processing is not always rational. For instance, it has been shown that teachers' judgment processes are subject to cognitive biases: Teachers sometimes fail to perceive the ambiguity of diagnostic situations, or they process irrelevant information in order to make their final diagnosis. Both biases may result in inaccurate judgments. Since accurate judgments are a basis for adaptive teaching, it is a concern of teacher training to improve the diagnostic competence of teachers. The present study describes two structurally comparable one-shot debiasing interventions designed to improve either situational awareness or the processing of information during decision-making in a simulated pedagogical situation. To this end, we randomly assigned prospective teachers (N = 73) to two experimental or one sensitization group and had them diagnose misconceptions in the area of decimal fractions using vignettes of virtual learners. The results demonstrate a significant effect of a debiasing training in the area of information processing, but no effect of a debiasing training on the perception of the ambiguous diagnostic situation. These results are discussed with regard to the necessary cognitive processes and their possible consequences for teacher training.

Keywords: Teacher training, Diagnostic competence, Information processing

Diagnostic competence and adaptive learning

Brühwiler and Vogt (2020) show that teachers' adaptive teaching skills effect significantly the learning success of students, because it increases the quality of teaching. In addition to general didactic skills, i.e. the use of suitable classroom methods, diagnostic competence is an important component of this adaptive teaching competence. Diagnostic competence is defined as the ability to assess the requirements of tasks or learning settings in the classroom on the one hand and the learners' prerequisites on the other hand (Artelt & Gräsel 2009; Rieu et al., 2022; Schrader, 2013). The quality of this assessment can be measured via judgment accuracy and thus expresses the teacher's ability to make personal (teacher's knowledge of different learners), task-related (task difficulty) and person-specific (personal characteristics or behaviours of individual learners) judgments (Urhahne & Wijnia, 2021).

Judgment accuracy

Meta-analyses show that teachers' judgment accuracy varies in different diagnostic situations and report moderate to high correlations between teacher ratings on students' individual abilities and students' individual performance on a standardized test (Urhahne and Wijnia, 2021). Various moderators could be identified such as characteristics of the teachers, the tests or tasks or the learners (Funder, 2012). These results categorize and systematize the influencing factors on judgment accuracy, but they do not provide more specific theoretical assumptions about the genesis or the

underlying cognitive processes of teachers' judgment. In recent years, studies have therefore focused on the genesis of diagnostic judgments and have shown, for example, the influence of situational characteristics such as time pressure or stress (Rieu et al., 2022) and personal characteristics such as subject-specific didactic knowledge (PCK) on the judgment processes. PCK was identified as a necessary prerequisite for the complex judgment processes and thus influences the teachers' judgment accuracy (Rieu et al., 2022).

Judgment biases

Another factor influencing judgment accuracy can be cognitive biases, such as the confirmation bias. Due to the confirmation bias people tend to select and interpret information in such a way that it confirms their own expectations and opinions. Cognitive biases are usually unconscious and often occur in complex diagnostic situations, e.g. where a lot of information are to be processed, the elements are interconnected, etc. (Marewski et al., 2010). Such complex diagnostic situations also arise in the area of pedagogical judgments. For instance, systematic occurring errors can have its origine in different misconceptions of the learner. Thus, in order to determine the existing misconception, further information must be obtained to generate an accurate final diagnosis.

More concretely, an incorrect solution to a task often cannot be clearly attributed to a misconception but can be the result of various causes. Therefore, the teacher should first consider all possible causes (misconceptions) for the occurrence of the error as hypotheses. Then the teacher needs further information to diagnose the existing misconception. By selecting relevant tasks, the teacher can obtain the necessary information for an unambiguous final diagnosis on the basis of the solutions obtained or by observing the solution process. In the described situation, a cognitive bias can be assumed both 1) in the perception of the ambiguous (multi-causal) situation and 2) in the complex processing of information (Rieu et al., 2024). That is, the teacher may fail to perceive the ambiguity of the situation (possibly because he or she already has a hypothesis regarding the underlying misconception in mind) or the teacher may fail to select relevant tasks or misinterpret the information obtained from these tasks and their solution (process).

Consequences for teacher training

Based on the state of research, the question arises as to what extent teachers can be prepared during their university training for the professional challenges in the field of diagnostic judgments. Various studies (Morewedge et al., 2015; Sellier et al, 2019) indicate that debiasing interventions have a positive influence on judgment processes, e.g. in forensic psychiatry or economics. However, a transfer of this approach to teachers has not yet been documented.

In the present study, two brief debiasing interventions to foster teacher diagnostic judgments were conducted. One debiasing intervention focused on the perception of ambiguous diagnostic situations, the other on deductive information processing (see van Brussel et al., 2020, Sellier et al., 2020, etc.).

1. Intervention concerning the perception of the ambiguity (multi-causality) of the diagnostic situation

In general, it can be assumed that systematic errors or occurring problems of learners in different mathematical learning topics can have different causes. It is therefore necessary to recognize and

carefully analyse all possible causes before making a final diagnosis. Also in the school context, diagnostic situations are often characterized by ambiguity (multi-causality). An intervention in this area should therefore first impart knowledge about the multi-causality of learners' behaviour and then select formats in which this knowledge can be applied in concrete situations. Specifically, this means that a school-related topic (e.g. school anxiety) is presented and explained as a phenomenon with various causes. The participants are then confronted with case description in which they are asked to recognise the multi-causality and also receive feedback on their answers (multi-causality recognised vs. not recognised, correct cause(s) named vs. not named).

2. Intervention concerning the information processing (selection of relevant information)

An intervention in the area of unbiased and deductive information processing aims that persons select and process relevant information with rational conclusions. The *Wason Task* can serve as an application example, in order to train human information processing with conditional statements. Four cards are presented, and the participants are told that each of the cards contains a letter on one side and a number on the other. From a formal point of view, the following logically valid conclusions arise: *if p then q and if not-q then not-p*.

Studies that applied the *Wason Task* as an intervention have shown that the unbiased and deductive information processing is reached much more frequently when the content is presented realistically, i.e. less abstractly (Beller, 2012). In the school context, the *Wason Task* has never been applied in teacher training situation but can be conceptualised with school references.

Research questions

The perception of the situation and the information processing of teachers in diagnostic situations, which are often multi-causal, are subject to cognitive distortions such as confirmation bias. This means, for example, that teachers do not perceive the ambiguity of the diagnostic situation, or they process irrelevant information in order to make a final diagnosis. The present study investigates the influence of two structurally comparable one-shot debiasing interventions that are designed to improve either situation perception or information processing during decision-making in a simulated practice situation.

The following research question arises:

To what extent

(1) a multi-causality intervention and
(2) an intervention for undistorted and deductive information processing

improve the judgment process and the judgment accuracy.

It is assumed that (1) leads to an increase in the number of alternative hypotheses formulated and that (2) leads to an increase in the proportion of processed relevant diagnostic tasks and that both interventions (1) and (2) lead to a higher judgment accuracy compared to a sensitization group.

Method

To investigate the research questions, a design was chosen in which vignettes are presented to prospective teachers, each with an incorrectly solved task by a pupil in the area of decimal fraction comparison. The context is based on an everyday diagnostic situation in the classroom or in group work, when the teacher identifies a systematic error and uses other tasks from the textbook or a test sheet to diagnostic purposes. The diagnostic goal was to determine the underlying misconception. In all vignettes, the presented incorrect solution to the solved task can be the result of several misconceptions. The participants were asked to formulate initial hypotheses regarding the misconceptions of the learners. Afterwards, they could select further tasks in order to determine the underlying misconceptions of each pupil. The participants were informed that the diagnostic cases shown to them were attributable to virtual learners and that they always solve the tasks using the same misconception. In the case of real students, random solutions would also have to be considered, which was ruled out here for methodological reasons.

Sample

The survey was conducted with student teachers for primary schools (81% identified as female) at the University of Education Freiburg. All participants were studying mathematics, were on average 22.6 years (SD = 3.13) old and were completing a practical semester at the time of the survey. This group of students was deliberately selected as diagnostic judgments was the main topic of the course that accompanied the practical semester. An overview of misconceptions (i.e. the specific, necessary PCK) was always present to the participants at any moment of the survey to ensure that all participants had access to the same specific didactic knowledge base during the intervention.

Implementation of the study

The data collection was computer-based. First, socio-demographic data (age, gender, course of study, tutoring and practical experience) of the participants were collected. The second part was constituted by the intervention, in which the participants were randomly assigned to the sensitization group or one of the interventions (multi-causality or Wason task). The sensitization took place via a short, written statement concerning the question of whether logical human thinking always represents rational thinking and addresses the occurrence of cognitive illusions. The two interventions took place as described above.

Finally, standardized cases (N = 5) were presented and had to be diagnosed. In these cases, virtual learners had solved a decimal fraction comparison incorrectly. By clicking on further tasks, the participants obtained a solution to these further tasks by the virtual student. The required PCK (Rieu et al., 2022) about the typical misconceptions in decimal fractions was shown permanently to the participants by means of a text. In all diagnostic situations, the initial incorrect solution cannot be clearly traced back to a single misconception. It therefore is an ambiguous diagnostic situation.

Data collection

The prospective teachers first formulated an initial hypothesis on the existing misconception based on the request *Which of the misconceptions could be present in the following case?*, whereby a free

text field allowed one or more hypotheses to be named. Participants were then able to select further tasks and received the students' solutions to these tasks in coherence with their misconception.

The tasks offered for selection were presented in groups of four and differed in terms of the relevance of their information for the final diagnosis. In a block of four tasks to be selected, there were always two tasks that provided evidence for a possible misconception and contradicted another possible misconception (relevant tasks). Two other given tasks did not carry any additional relevant information: These tasks can either be solved correctly with every misconception in the present study setting or the tasks correspond in structure and thus in information content to the task already shown at the beginning of the vignette. The order and placement of the relevant and irrelevant tasks changed from block to block.

The participants were free to decide on the number of further tasks to be completed by selecting any number of tasks from a block of four and by deciding to obtain further tasks in further blocks of four after viewing the solutions of the learner to the selected tasks. At any time, they could proceed to submit their final diagnosis. If the individuals opted for a further block of four tasks, a new block was displayed, which again contained two tasks with diagnostic potential and two tasks without additional diagnostic information. No time restrictions were formulated; the participants were simply asked to make an accurate final diagnosis. In summary, to answer the research questions, the type of initial hypothesis (single or multiple hypothesis), the amount and the type of tasks (diagnostically relevant or irrelevant), and the final diagnosis was recorded.

Data analysis

The overarching research question refers to the effects of two structurally comparable one-shot debiasing interventions that focus either on situational awareness or on information processing during decision making in a simulated practice situation. For this purpose, case-related descriptive statistics are reported, and hypothesis test were calculated to compare the influence of the interventions on diagnostic activities and information processing based on the initial hypothesis of the participants as indicator for the recognition of the ambiguity of the situation, the information search processes with regard to the relevance of the additional information, and the accuracy of the final diagnosis. All participants who completed the survey tool were included in the analysis.

Results

The randomization into sensitization, multi-causality, and Wason groups was carried out using the online tool and generated comparable groups with regard to the socio-demographic data recorded (see Tab 1). Only in the case of tutoring experience, a few outlier data in one group (multi-causality) distort the mean value.

Table 1: Mean values (and standard deviation) of all participants in the Sensitization, Multi-causality and Wason intervention Groups

collected data	Sensitization Group	Multi-causality Group	Wason task Group

main subject mathematics	0.90 (0.30)	0.95 (0.21)	1.00 (0.00)
teaching experience	68.65 (59.31)	73.57 (41.42)	66.33 (58.05)
tutoring experience	10.52 (21.12)	77.90 (192.33)	23.33 (108.57)

In order to investigate the effect of the intervention on situation perception and information processing, the mean values of the persons participating in an intervention are calculated across all 5 cases and then compared with all other persons using t-tests. Due to this new point of observation, the number of analysed cases changes to 105 (5x21). The underlying hypothesis of this procedure is that one intervention has an exclusive effect on perception and the other exclusively on information processing and it can therefore be assumed that the multicausality intervention has no effect on information processing (and conversely the Wason intervention has no effect on perception).

The comparisons show a hypothesis-compliant result for the Wason intervention, after which the information processing (based on the proportion of relevant tasks processed (T(2,363) = -2.027, p = .022, d = . 0.33) and judgment accuracy (T(2,363) = -1.674, p = .047, d = .47) of the participants improved significantly (Table 2). However, the results of the multi-causality group are not as expected: an effect of the intervention on the perception of the situation (based on the formulation of multiple hypotheses) could not be proven (Table 3).

Table 2: Mean values of participants in the Wason intervention (Wason) in comparison to the other participants

Information processed in 5 cases	Control groups (n = 260)	Wason (n = 105)	Group comparison
Perception of situation	0.28 (0.45)	0.30 (0.46)	T(2,363) = -0.383, p = .351, d = -.45
Information processing	0.46 (0.20)	0.50 (0.18)	T(2,363) = -2.027, p = .022, d = -.33
Judgment accuracy	0.64 (0.48)	0.73 (0.44)	T(2,363) = -1.674, p = .047, d = -.47

Table 3: Mean values of participants of the multi-causality intervention (multi-causality) in comparison to the other participants

Information processed in 5 cases	Control groups (n = 260)	Multi-causality (n = 105)	Group comparison
Perception of situation	0.32 (0.47)	0.22 (0.42)	T(2,363) = -1.913, p = .028, d = -.34
Information processing	0.48 (0.20)	0.44 (0.16)	T(2,363) = -1.878, p = .031, d = -.21
Judgment accuracy	0.68 (0.47)	0.64 (0.48)	T(2,363) = -0.782, p = .217, d = -.47

Discussion

It was shown that an intervention for information processing based on the Wason task achieved an improvement in the proportion of relevant tasks processed by pre-service teachers, which also increased judgment accuracy. In contrast, a further, structurally comparable intervention to improve the perception of ambiguity in diagnostic situations did not achieve any effects. The question therefore arises as to whether the cognitive process of perceiving the ambiguity of a situation is more complex than information processing and therefore requires a special intervention. Müsseler and Rieger (2019) describe perception as a phase of early visual processing, which is characterized by a high proportion of stimulus-driven bottom-up processes (automated) without reference to more cognitive functions. However, when perceiving the ambiguity of a diagnostic situation, a process of identification and categorization, i.e. information processing of task features with regard to the depicted PCK, is expected. The perception of ambiguity required in diagnostic situations therefore requires top-down processing of the given information, which only a few participants achieved in our study.

A successful intervention in the area of situational awareness (in the sense of identification and categorization) must be cognitively more profound than that of information processing, since here both the complex situation (here: with the available information) and personal characteristics (here: the necessary PCK) are the necessary basis for recognizing the ambiguity (see Rauthmann, Sherman & Funder, 2015). One answer to this challenge can be found in the instructional design of the training curses: initial studies in the area of sensitisation to multicausal diagnostic situations show the expected effects (Özel et al., subm.) using the 4C/ID model (van Merrienboer, 2020) and thus point to a possible form of successful teacher training.

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Using scriptwriting to examine how pre-service mathematics teachers anticipate and respond to students' productive struggle

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This study is a part of a larger research project that seeks to characterise scripts written by pre-service mathematics teachers (PSTs) in terms of student productive struggle and to understand how this characterisation helps reveal their perceptions in managing this type of struggle. We analysed 17 scripts written by PSTs around one problematised classroom situation involving fraction representation. Using the productive struggle framework (Warshauer, 2015), the analysis reveals that PSTs often do not expect struggle in the problematised situation, and any struggle they do anticipate focuses mainly on students' uncertainty in explaining their work. The PSTs usually use two teaching strategies to respond to struggle: direct telling and affording opportunities to continue thinking about the task. These two strategies are somewhat contradictory in terms of the way they preserve the task's cognitive demand. Implications are discussed.

Keywords: pre-service mathematics teachers, teacher education, scriptwriting, productive struggle.

Introduction and Literature Background

Responsive teaching, a core practice in mathematics education, posits that for teachers to be able to develop students' mathematical understanding beyond current knowledge they must first be attuned to their students (e.g., Barnhart et al., 2024). Warshauer (2015) proposed a framework of productive struggle through which responsive teachers can attend to and respond productively to their students' struggles. This framework offers three lenses to examine how teachers respond to their students: 1) how the task's cognitive demand is preserved; 2) how the students' struggle is addressed; and 3) how the students' mathematical thinking is supported.

Accordingly, teacher preparation programmes include responsive teaching strategies to prepare pre-service mathematics teachers (PSTs) to use these strategies in their future professional lives (e.g., Barnhart et al., 2024; Grossman et al., 2009). One approach to imparting responsive teaching strategies involves the use of simulation-based tools that enable PSTs to engage with their students' struggles and to consider various ways to respond. This paper focuses on one such tool—scriptwriting tasks (Zazkis et al., 2009).

Using scriptwriting, teachers and PSTs write dialogues that outline the progression of a lesson centred on a mathematical task (Zazkis et al., 2009). As a simulation-based tool, scriptwriting tasks offer several affordances: they enable teachers to envision their role in the classroom, including their actions and reactions, without the need for extensive resources in teacher education programmes. Indeed, scriptwriting enables PSTs to create representations of how they anticipate they will enact their teaching practices (Grossman et al., 2009). Moreover, these scripts provide valuable data for teacher educators to learn how PSTs perceive their future teaching. Recent research has examined

mathematical knowledge (e.g., Marmur et al., 2020), particularly mathematical knowledge for teaching among PSTs and teachers (e.g., Shure & Liljedahl, 2023), across various domains, as reflected in the teachers' turns in these scripts.

Scriptwriting tasks were developed to overcome the frequent oversimplifications and overgeneralisations in the lesson plans PSTs write in teacher education programmes (Zazkis et al., 2009). These tasks reveal how PSTs plan to teach content in the classroom, and how the PSTs take into consideration what both teachers and students contribute to the lesson. Further, these tasks are designed to problematise teaching and highlight classroom complexity by beginning with a complex classroom situation, to which a teacher can respond in multiple ways to advance students' understanding. Problematising teaching does not necessarily refer to students' difficulties but rather can include any type of struggle (e.g., how to begin working on a task, address misconceptions, or explain work). In responding to such struggles, teachers can continue the struggle and use it to foster further mathematical learning, or they can diminish it with direct instructions that decrease the task's cognitive demand. Framing these problematised teaching situations in terms of productive struggle (Warshauer, 2015) helps characterise both student and teacher turns in the discussion script, potentially facilitating an exploration of the dialogic relationship between teacher and student turns.

Current research often analyses teacher and student turns in scripted discussions separately (e.g., Shure & Liljedahl, 2023), overlooking the dialogic communication between teachers and students. Yet responsive teaching necessitates the interplay between student struggles and teacher responses. Indeed, a student's struggle influences the teacher's response, which in turn affects whether the struggle remains productive for the student's learning. Scriptwriting gives PSTs the opportunity to experiment with this dynamic interplay. Through scriptwriting, PSTs can imagine how students will handle struggles, how they as teachers will respond, and how they can preserve the cognitive demand of the task while addressing the struggle in a supportive manner. Hence, we build on studies that used written scriptwriting tasks to learn how PSTs planned to enact their teaching practices (e.g., Shure & Liljedahl, 2023), with specific focus on how PSTs managed their students' productive struggle (Warshauer, 2015). Moreover, here we adopt a different approach: Not only do we focus on the teacher's moves but rather we simultaneously focus on the students' contributions and the teacher's responses. We begin with a problematised classroom situation (Markovits, 2003) that introduces a non-traditional representation of a fraction that may potentially trigger misconceptions among students and teachers (described in detail in the Methods section).

This study makes several contributions. Its analysis of the interplay between student and teacher turns in the scripts expands the research knowledge about how PSTs perceive student struggles and how they will use these struggles to advance students' learning. This knowledge also has practical implications for teacher preparation programmes in that it can help teacher educators plan teaching preparation programmes. Lastly, the methods described here can be used to examine scriptwriting tasks written for different classrooms situations and other mathematical topics. We examine these contributions by exploring the following research question: *What does analysis of PSTs' scriptwriting tasks in terms of productive struggle reveal about how PSTs plan to manage future student struggle involving student's suggestion for representing a fraction?*

Methods

Task starting point –problematised classroom situation

The task in this study consisted of 1) a short, problematised classroom situation that served as a starting point (Figure 1); 2) several prompts aimed at facilitating the PSTs' understanding of the problematised classroom situation; 3) instructing the PSTs to write a script; and 4) prompts aimed at encouraging the PSTs to reflect on their script. Due to space limitations, here we focus only on part 3 of the task in which the PSTs were asked to imagine that they were the teacher, respond to the student mentioned in the short situation (part 1), and continue the dialogue between the teacher and the student(s).

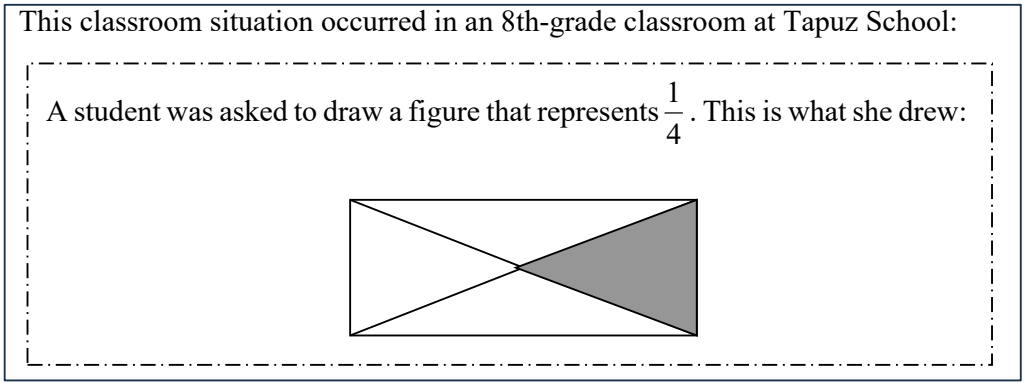

Figure 1: Starting point for scriptwriting task (Borrowed from Markovits, 2003)

The task was designed to problematise the classroom situation by showing a correct fraction representation that might potentially evoke misconceptions both about fractions and about geometry. In the case of fractions, a possible misconception is that students might claim that the triangles formed by the rectangle's diagonals are not equal because they might think about the congruence of the triangles rather than considering the areas of these triangles. A possible geometric misconception is that students might claim that the triangles do not appear to be equal in area. To address this misconception, a teacher or a student might suggest proving that the triangles are equal in area, showing that triangles that are equal in area are not necessarily congruent. In writing the scripts, the PSTs were instructed to imagine themselves in the 8th grade. We expected they would engage with the problem at a more advanced level than the simple part-whole relationship typically taught in 4th grade. Thus, we anticipated the PSTs would analyse it from an 8th-grade perspective, focusing on the geometric properties of the triangles formed by the rectangle's diagonals. By using this design, we aimed to reveal how PSTs navigate between different mathematical domains and grade-level expectations when confronted with a seemingly straightforward yet potentially complex situation.

Data collection and analysis

We collected 17 scripts written by 17 third-year undergraduate students (three male, 14 female) enrolled in a semester-long course about cognitive aspects in mathematics education. The course was part of a bachelor's degree programme (B.Ed), and the scripts were part of their final course assignment.

The scripts were coded according to the theoretical framework proposed by Warshauer (2015) (Tables 1 and 2) that incorporates the students' productive struggle and the teacher's responses to this struggle. As noted, we analysed only part 3 of the task (PSTs' scripts). Each turn in the script served as an analysis unit and was coded according to Tables 1 and 2. The scripts ranged from 14 to 98 turns. We coded a total of 467 talk-turns: 209 student turns and 258 teacher turns. The numbers in the square brackets appearing before the sub-categories in each category are used later in the examples in the Findings section.

Table 1: Coding scheme for students' struggle

Category name	Definition
Getting started	This type of struggle occurs when students have difficulty starting to work on a task. It includes [1-1] confusion about what the task is asking, [1-2] claims of forgetting how to approach this type of problem, [1-3] gestures of uncertainty and resignation, or [1-4] doing no work on paper.
Carry out a process	This type of struggle arises when students reach an impasse while trying to implement a mathematical process. It can involve [2-1] inability to implement a method from a formulated representation, [2-2] difficulty carrying out algebraic procedures, [2-3] forgetting facts or formulas, or [2-4] executing an algorithm.
Uncertainty in explaining and sense-making	This type of struggle is characterised by difficulty explaining work or solutions, such as [3-1] expressing uncertainty, [3-2] being unclear about reasons for strategy choice, or [3-3] being unable to make sense of their work.
Express misconceptions and errors	This struggle reflects fundamental misunderstandings about mathematical concepts, particularly misconceptions related to [4-1] probability, [4-2] fractions, [4-3] proportions or [4-4] geometry.

Table 2: Coding scheme for teacher responses

Teacher responses	
Telling	In this type of response, the teacher directly provides the needed information or instruction to the student by: [5-1] supplying information, [5-2] suggesting a strategy, [5-3] correcting an error, [5-4] evaluating student work, [5-5] relating the problem to a simpler one to decrease processing time.
Directed guidance	In this type of response, the teacher guides the student more explicitly towards a solution path by: [6-1] redirecting student thinking, [6-2] narrowing down possibilities for action, [6-3] directing an action, [6-4] breaking down the problem into smaller parts, or [6-5] converting the situation to an analogy.
Probing guidance	In this type of response, the teacher encourages students to elaborate on their thinking and reasoning by: [7-1] asking for reasons and justification, [7-2] offering ideas based on student thinking, [7-3] seeking explanations that could

	reveal an error or misconception, or [7-4] asking for written explanations of student thinking.
Affordance	In this type of response, the teacher provides opportunities for students to extend their thinking by: [8-1] asking for a detailed explanation, [8-2] building on student thinking, [8-3] pressing for justification and sense-making with a group or individually, or [8-4] affording time for students to work.

Because not all of the conversations revolved around struggles, we also used specific codes for other types of talk-turns. Here are some examples: student explains correct approach or reasoning (without any struggle); teacher asks an open question to promote classroom discussion (usually at the beginning of the lesson); teacher asks a student to share or explain answers.

To ensure the validity and reliability of the analysis, (Lincoln, 1995), both authors independently coded 15% of the data. After this individual coding, the authors compared their coding, reaching more than 80% agreement. We deliberated all discrepancies until reaching consensus and refined the codes accordingly. Then, the second author completed the rest of the coding process.

After all the analysis units were coded, we sought patterns within the data set. We calculated the frequencies for each student struggle and each teacher response. Then we sought patterns showing which type of student struggle is followed by which type of teacher response to reveal the interrelationships between them.

Findings

Our systematic data analysis of the PSTs' scripts detected some sort of struggle in 57% of the student turns, whereas the remaining 43% of the student turns did not exhibit any difficulty. Of the 57% that expressed some struggle, 39% were characterised as student uncertainty in explaining or sense-making, 9% expressed misconceptions, 7% were struggles in carrying out a process, and 2% were struggles in beginning to work on the task. Additionally, 10% of the teacher turns were coded as open invitations or requests by a specific student to share solutions or reasoning. The remaining teacher turns were aimed at responding to the students' struggles. Thirty-two percent of these teacher's responses were coded as affordances and 32% as telling. Eighteen percent of the teacher responses entailed probing guidance and 21% direct guidance. The diagram in Figure 2 describes these results. The arrow between the students' struggle and the teacher's response points to the dialogic relationship between the student turns and the teacher turns, as elaborated in the example below.

Two examples (Rana and Sarai) exemplify how this relationship is expressed in the script. Rana's script demonstrates the students' uncertainty in explaining their work, followed by teacher affordances. Sarai's script demonstrates the students' uncertainty in explaining their work, followed by teacher telling.

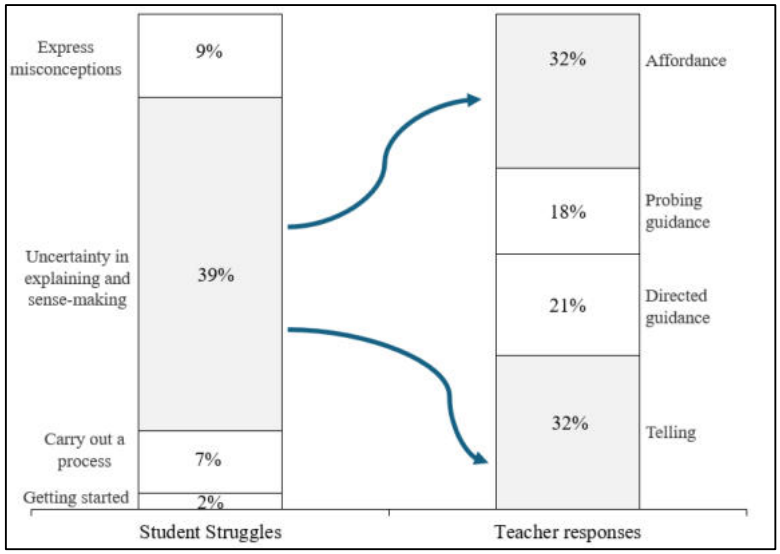

Figure 2: Students' struggles and teacher's responses in the PSTs' scripts

Rana's script. Rana's script included 32 talk-turns: 11 student turns and 21 teacher turns. In the script, Student 1 suggests a representation based on the starting-point situation. The numbers in the square brackets refer to the sub-categories delineated in Tables 1 and 2. The excerpt is followed by a full explanation.

Student 1: I divided the rectangle into four parts and coloured one part to represent a fourth.
Teacher: Dividing the rectangle into four parts is excellent work. However, it seems that the parts are not the same size. To accurately represent a fourth, the four parts need to be equal in area. Can you explain whether the areas of the triangles in your drawing are equal? [6-2] [8-1]
Student 1: Umm, I didn't really think about the areas of the triangles. I just divided the rectangle into four parts and coloured one part [3-1].
Teacher: That's a good start, Student 1. Let's explore the idea of equal areas together. Class, what do you think about the areas of the triangles in Student 1's drawing? [8-3]

The above student's turn, taken from the full script, suggests the student knows how to represent a fourth. The student's understanding appears to be somewhat intuitive as she does not know how to explain her reasoning. The teacher first responds by saying that this is the correct answer. Then, she *narrows down possibilities for action* [6-2] by saying the triangles need have equal area. Lastly the teacher *asks the student for a detailed explanation* [8-1], giving the student an opportunity to articulate her thinking. The student's struggle specifically emerges when she admits that she did not really think about how to reason this. Indeed, *uncertainty in explaining her work*, marked as [3-3] in the script, is her main struggle. The teacher then suggests that the whole class should explore why the areas of the triangles are equal. She *presses for justification and sense-making with a group* [8-3].

Figure 2 included two arrows. The first goes from *uncertainty in explaining their work* to *affordance,* as demonstrated using Rana's script. The second arrow goes from *uncertainty in explaining their work* to *telling*, which we now demonstrate using Sarai's script.

Sarai's script. Sarai's script included 48 talk-turns (17 student turns and 31 teacher turns). Here also, the student was uncertain when explaining the reasoning behind her response to the starting-point situation.

Teacher:	For the triangles to be equal, their areas must be the same. Let's think about the area of each triangle… calculate [its] area… [5-1] First, let's mark some points on the diagram. Before we calculate the area of the triangle, what do we know about a rectangle? [5-2]
Student 4:	It has 4 right angles.
Student 5:	Each pair of opposite sides is parallel and equal in length.
Teacher:	What else? Do you know anything about the diagonals? [5-2]
Student 1:	The diagonals bisect each other.
Teacher:	Very good. What about the point where the diagonals intersect? Is it a random point? [5-1] [5-2]
Student 1:	No, the diagonals meet at the midpoint of the rectangle.

As seen in the script above, the teacher provides a great deal of information (marked as *supply information* [5-1]) to help the students work towards the correct explanation. Then the teacher *supplies a specific strategy* [5-2] for marking some points on the diagram and thinking about what is known about the rectangle. After Students 4 and 5 stated what they now know (without exhibiting any struggle), the teacher directs the students' thinking even further by *supplying the strategy* [5-2] of focusing on the rectangle's diagonals. In the last teacher turn in the above excerpt (only a part of Sarai's script), the teacher *supplies information* in the form of *supplying strategy* – the teacher directs the students' focus towards discussing the diagonals' intersection point. Consequently, the teacher in Sarai's script focuses on *telling* the students what to do to reason about the starting point situation.

Discussion

The goal of this study was to characterise scripts written by PSTs in terms of productive struggle and to understand how this characterisation helps reveal PSTs' perceptions in managing this type of struggle. We systematically analysed 17 scripts written by PSTs that described classroom discussions around a problematised starting-point situation in which a student suggests an unusual representation for a fourth (Markovits, 2003). Analysis of the students' talk-turns in those scripts revealed that 57% of the turns reflected some sort of struggle, whereas in 43% of the turns the students did not express any kind of struggle.

Furthermore, in the PSTs' scripts identification of the students' struggle suggests that the struggle mainly revolves around expressing *uncertainty in explaining in sense-making*. This result surprised us. The problematised starting-point situation was designed to elicit various misconceptions, both in fractions and in geometry. Yet only 9% of the scripts perceived the students' struggle as connected to misconceptions. This finding has several possible explanations. First, talking about equal parts without emphasizing that the parts are also equal in area is a common misconception in fractions, and PSTs may also hold this misconception. Yet this explanation should be viewed with some skepticism in that the statements about equal parts in the PSTs' scripts may be somewhat inaccurate and they may actually mean equal in area. Second, with respect to the geometric misconception that triangles that are equal in area must be congruent, the PSTs may not perceive this in terms of student struggle. Nevertheless, one challenge for 8th-grade geometry students lies in recognizing the need to prove claims. The PSTs' lack of experience may have prevented them from seeing this as a possible struggle

for the students. Another possible explanation could be that the scriptwriting task format is challenging for the PSTs as they are not used to such a task, which in turn inhibits them from articulating their ideas; thus pinpointing some of the study's limitations.

Nevertheless, we are encouraged by these results exemplified by the scripts written by Rana and Sari. Rana and Sari's scripts were representative of the majority of the scripts while also pinpointing their variations, in which a similar starting-point situation and a similar struggle yielded two different teacher responses — *telling* and *affordance* — echoing what might happen in the classroom. Indeed, as both strategies are important and promote students' understanding (e.g., Warshauer, 2015), one of the contributions of this study is that it provides a nuanced understanding of how PSTs perceive and respond to student struggles. This in turn offers valuable insights for expanding teacher education programmes to include exploration of the dialogic relationship between student and teacher turns in the context of student struggle. Further, the small number of scripts addressing misconceptions highlights PSTs' "blind spots" in thinking about student struggle in this context, thus underscoring the affordances emerging from using scripts in teacher education. Further research can continue exploring the use of PSTs' scripts to examine teacher competencies, such as the flexibility inherent in moving back and forth between two the teaching strategies - telling and affording opportunities to think about the task. Another possible interesting line of future research would be to examine the PSTs' growth reflected in the scripts they write at different stages of their preparation program.

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Shifting from on-site to on-line: How can key design criteria for effective PD be transferred to online, asynchronous programs?

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We review recent developments and future opportunities in the realm of online, asynchronous professional development (PD) for mathematics in-service teachers, focusing on key design criteria. While teachers' asynchronous learning has gained traction in other educational fields, it has received limited and fragmented attention in mathematics education. We lay the foundation for a scoping review by examining how key design principles for on-site, practice-based PD programs can be used and adapted for the design of self-paced, online formats, emphasizing four key criteria: centering mathematical content, using representations of practice, facilitating collaboration, and being adaptive to participants' needs. In line with TWG18's focus, we discuss how on-site PD models, frameworks, and research methods can evolve in asynchronous spaces, expanding possibilities for teacher learning and related research.

Keywords: Asynchronous *online PD, mathematics teacher learning, teacher collaboration, adaptivity, representations of practice.*

Introduction

In recent years, accelerated by the COVID-19 pandemic, online platforms have become powerful learning environments — not only for students but also for teachers. In mathematics education, online professional development (PD) programs are increasingly popular (Koellner et al., 2023), whether conducted synchronously or asynchronously. Previous research on in-service teachers' professional learning has identified several key design principles for effective in-service PD, such as *centering content*, using *representations of practice* or facilitating *collaboration* amongst participants (e.g., Desimone, 2009). However, most of this research has been conducted in face-to-face settings. With the rise of asynchronous online PD, it is essential to explore how these elements can be effectively implemented in digital spaces and whether they require adaptation, especially since "It is still unclear whether online PD in general is as effective as face-to-face options" (Koellner et al., 2023, p. 554). Although several studies in the broader educational literature address this issue (e.g., Meyer et al., 2023), it has received scant attention in mathematics education research. Since mathematics teaching have unique characteristics, it is key to examine the particularities of shifting from on-site to on-line within mathematics PD. We focus on asynchronous PD — often referred to as "self-paced" or "on-demand", though each term carries nuanced distinctions — because these formats highlight the transition from on-site to on-line. In asynchronous settings, where participants engage with content independently, the impact of digital tools in prompting and directing becomes more evident, emphasizing the importance of their thoughtful design. Furthermore, it is important to distinguish fully asynchronous PD from synchronous online PD: While the latter shares key design features with on-site PD, such as duration, structure, and human infrastructure (Meyer et al., 2023), fully asynchronous PD may differ significantly in these aspects. Recognizing this distinction is crucial for

understanding the unique mechanisms at play in asynchronous environments. While the literature on massive open online courses (MOOCs) offers some valuable insights — such as principles of self-directed learning, peer support, and job-connected learning (Hollebrands & Lee, 2020) — MOOCs' focus on large, diverse groups and content delivery highlights the need for more research to understand the unique features of asynchronous online PD.

Following calls to improve the accessibility of PD and recognizing the potential for scaling-up PD programs, we anticipate a growth in the popularity of digitally mediated, asynchronous PD courses. However, we must not simply follow technological trends; while embracing innovations, we must examine whether key elements such as focus on content-specific practices, practice-based approaches, collaborative learning, and adaptivity to participants' needs (see Desimone, 2009) remain central to the design of asynchronous mathematics PD. For these reasons, we ask:

How can key design criteria for effective PD be transferred to asynchronous online mathematics PD programs? What challenges and opportunities are associated with designing asynchronous online PD in mathematics education, in particular with regard to centering mathematical content, using representations of practice, facilitating collaboration, and being adaptive to participants?

Modes of inquiry

Our aim in this paper is to take a first step in mapping the landscape of asynchronous online PD for mathematics teachers. To achieve this, we identified relevant papers published since 2020 in international journals or in proceedings of CERME and PME. Guided by the principles of scoping reviews, we sought to identify key themes, gaps, and trends in the literature, synthesizing the current state and highlighting future directions for research and design about asynchronous online PD. We focused on research papers describing projects that are fully asynchronous or have a particular focus on asynchronous design elements in their research and that are targeted to mathematics teachers. During this process, we faced challenges in selecting papers for review due to the broad (and ambiguous) definition of PD in asynchronous contexts. For example, we questioned whether a website offering resources for mathematics teachers qualifies as PD (see Copur-Gencturk & Orrill, 2023). Additionally, when a PD format resembles a lengthy survey, distinguishing between a PD project aimed at supporting teachers, and a research initiative primarily focused on collecting data, becomes complex. Ultimately, we decided to adopt an inclusive approach, to cover a wider range of PD formats and activities – realizing this is not noise, but signal: asynchronous spaces allow for innovative contexts that do not directly map to existing models and definitions.

Mapping the terrain: Key design criteria implemented in asynchronous PD

In the following, we review how four key design criteria identified in the literature for effective PD (e.g., Desimone, 2009) are addressed within asynchronous online PD. To do so, within each criterion, we first draw on the literature about on-site PD, and then compare the contexts.

Centering mathematical content

There is broad consensus on the importance of content-focused PD for mathematics teachers in enhancing their content knowledge (CK), pedagogical content knowledge (PCK), mathematical discourse practices, or teaching expertise. This emphasis is evident in PD designs targeting specific

content areas (e.g., statistics, Hollebrands & Lee, 2020), as well as in process-oriented programs not confined to a single topic, but aiming at fostering teaching practices or professional discourse (e.g., Schwarts et al., 2024). What is often not unpacked is *how* mathematical content is engaged within these PDs and how the PD structure shapes specific types of content engagement.

Engagement with mathematical content in synchronous versus asynchronous PD settings can differ significantly. In synchronous PD, real-time discussions allow for immediate feedback, clarification, and co-constructions of new meanings. Moreover, in terms of collaborative group work, the structure of synchronous PD often mirrors that of a mathematics classroom, enabling facilitators to plan activities where teachers engage in collective problem-solving. This live format fosters spontaneous mathematical reasoning that can go beyond the planned activities, and provides facilitators with opportunities to be responsive to participants' emerging needs. For instance, if facilitators observe that the mathematics at hand is too difficult, they can revisit key concepts.

Some of the elements described above can be addressed in an asynchronous format by drawing on asynchronous collaboration (see below). Matranga and Silverman (2024) provide evidence of such design in an asynchronous online course that was proven useful in fostering shifts in teachers' collective mathematical discourse, moving from focusing on visual features of function graphs to explaining function behavior. Another way to emulate the benefits of synchronous mathematical collaboration is through the careful design of prompts and feedback given by the system, for example by using avatars (AI-based or pre-programmed) to represent facilitators, peers, or students. For example, Copur-Gencturk and Orrill (2023) built on their research on common misunderstandings about proportional relationships to design an AI-based facilitator (based on an intelligent tutoring system) that offers targeted feedback to foster teachers' CK and PCK.

Asynchronous PD provides several distinct advantages over synchronous PD, including enhanced flexibility and interactivity in exploring mathematical content, as well as a convenient platform for experimenting with the integration of technological tools in lessons. In terms of flexibility, asynchronous PD allows teachers to engage with mathematical content at their own pace and revisit complex ideas based on their individual needs. This contrasts with synchronous PD, where teachers and facilitators may avoid challenging mathematics to protect their professional image. Moreover, a versatile design enables teachers to choose the level of complexity they prefer to engage with. In terms of interactivity, a major advantage is embedding mathematical applets as 'pop-up' activities directly within the course (Hollebrands & Lee, 2020), which enhances engagement by maintaining immersiveness and flow. Some asynchronous environments provide multilayered resources related to mathematics lessons (Hollebrands & Lee, 2020; Polly & Martin, 2020), enabling participants to swiftly switch between student and teacher roles. For example, teachers can use mathematical applets to solve problems (a student perspective) and then watch videotaped lessons around these problems, analyse student work, or access lesson plans (a teacher perspective). Finally, if the PD goal is to enhance technology use in the classroom, the online nature of asynchronous PD supports this aim by providing a space for experimentation. This is reflected in Polly and Martin's (2020) design of formative assessment digital tools, and the asynchronous PD framework built around them, providing a setting where teachers can examine these tools easily.

Using representations and approximations of practice

A practice-based approach to teacher learning emphasizes that teachers should gain insights from actual practice rather than relying solely on theoretical knowledge. Grossman et al. (2009) identify three different ways in which practice is taken up in teacher education: Representations of practice (e.g., videotaped lessons, storyboards of lessons), Approximations of practice (e.g., rehearsals, teaching simulations), and Decompositions of practice. These pedagogies demonstrate aspects of practice to allow teachers to enhance their repertoires, and to provide a public point of reference in teachers' discussions and imitations of practice. A wealth of research highlights the affordances of representations and approximations of practice for synchronous online teacher learning. However, only a few studies have explicitly examined their use in online, asynchronous PD. In particular, it is crucial to explore which approximations of practice can be enacted without the need for human infrastructure, as is required by synchronous approximations like rehearsals or micro-teachings.

Asynchronous PD includes some affordances compared to synchronous PD. For example, it enables smooth transition between different representations of practice, and easy scrutinization. In terms of the former aspect, Hollebrands and Lee's (2020) design uses several representations: videos of students working on a digital mathematics task (the video shows both the students and their shared screen), animations of students' reasoning, and videos of teacher launching a statistical investigation. Overall, similar to the literature about on-site PD, we found evidence for a wide use of video cases (Copur-Gencturk & Orrill, 2023; Polly & Martin, 2020; Seago et al., 2022) as well as student work and other classroom artifacts (Copur-Gencturk & Orrill, 2023; Schwarts et al., 2024). There are also uses of cartoons and storyboards of practice (Friesen & Knox, 2022; Schwarts et al., 2024). We also reviewed genres which are less common in the on-site PD literature, such as videos of experts explaining concepts (Hollebrands & Lee, 2020; Wischgoll & Prediger, 2024).

Returning to the second affordance mentioned above, the nature of asynchronous PD allows teachers to control the pace of going through the course, including engaging deeply with representations of practice. Teachers can review videos, artifacts, or storyboards at their own speed, pausing and replaying as needed to reflect on specific aspects of the practice, and decide with agency where they want to focus their attention within the representation. This flexibility allows for a more personalized experience, where participants choose the elements most relevant for them.

The lack of real-time discussion may limit opportunities for reflection on representations. To address this, asynchronous PD often integrates structured reflection activities, ensuring that participants can critically engage with the representations of practice in a meaningful way that aligns with the PD goals. Another option is the integration of AI-based facilitators or peers to scaffold engagement with the representation. For approximations of practice, such as digital simulations, interacting with non-human avatars (e.g., simulated students) replaces the need for costly and logistically challenging human infrastructure (e.g., as required for simulations with actors), providing a controlled environment to refine teaching (Schwarts et al., 2024).

Fostering collaboration

Collaborative settings provide teachers with the opportunity to form interactive learning communities and play therefore a crucial role for effective PD (e.g., Desimone, 2009). It can be assumed that collaborative activities, such as group discussions, enable teachers to make connections between practice and theory, provide opportunities to learn from peers and create alternative approaches to instructional practice (Meyer et al., 2023). For this reason, many PD programs support the collective participation of groups of teachers from the same school or grade and provide ample opportunities for participants to engage in collaborative tasks and reflection. Prominent settings highlighting collaboration in mathematics PD programs are, for example, lesson study approaches engaging teachers in co-planning and co-reflection, or video clubs engaging participants in peer discussions and joint reflections on critical incidents in teaching to develop teacher noticing (e.g., Amador et al., 2021). Recent studies have highlighted the key role of participants' collaboration also for the successful design and delivery of online PD formats: In a systematic review of the literature, Bragg and colleagues (2021) found that opportunities for collaboration positively affected learner satisfaction. Meyer and colleagues (2023) reported that collaboration in online PD did not only predict participants' satisfaction, but was the only quality characteristic besides cognitive activation that could also predict changes in teachers' professional practices. However, there was also evidence that in many of the PD online programs under investigation, collaboration was fostered to an only moderate degree and PD activities often lacked sufficient opportunities for teachers to collaborate (ibid). Since asynchronous online PD programs differ significantly from traditional face-to-face PD formats, the question of how collaboration among participants can be supported has to be reconsidered in a fundamental way.

Consequently, there is not yet a large enough body of research on how to design and implement effective collaborative settings in online PD (Koellner et al., 2023). Whereas synchronous online formats can still implement collaborative activities comparable to face-to-face PD programs (e.g., group discussions), asynchronous PD programs require the careful design of specific opportunities to allow for collaboration among the participants. A recent study in this context reported promising practices for online professional learning focusing on mathematics and coding, using a maker-pedagogies approach to teaching and learning (Morrison & Hughes, 2024). In the online PD, asynchronous collaboration was, for example, fostered by the use of a virtual platform hosting threaded discussions and an instant chat feature where teachers could share links, videos, and other resources. Teachers appreciated that they could post questions any time and received responses and feedback from their peers and the PD facilitator. Although online discussion forums as asynchronous communication tools are already widely used and researchers have tried to identify effective design elements, research-based applications in the context of online PD in mathematics are still rare (a notable exception is Matranga & Silverman, 2024) and will benefit from more investigation.

Being adaptive to participants' needs

Whereas adaptive teaching and differentiated instruction have become prominent research topics in mathematics education research on the classroom level (e.g., Gallagher et al. 2022), adaptivity in the PD level for in-service teachers has so far received much less attention. On one hand, there is a

consensus that what teachers learn in PD should be consistent with their knowledge and beliefs and connect to their prior experience to initiate cognitive activation and active learning (Desimone, 2009; Meyer et al., 2023). On the other hand, providing participants in face-to-face PD programs with learning opportunities that are tailored to individual needs is complex and challenging for teacher educators. Asynchronous online formats provide new possibilities in this context, since they can provide self-paced, on-going and work-embedded support for in-service teachers while also fitting better with teachers' busy schedules. Research shows that providing opportunities for teachers to engage in self-directed learning according to their own pace does not only affect their satisfaction with the related PD program, but can also lead to increased content knowledge as positive outcome (Bragg et al., 2021). However, adaptivity can not only be considered in the light of making choices about *when and how long* to learn, but also *what* and *in which format*: Friesen and colleagues (2023) designed an asynchronous online module on the teaching of algebra where the participants could follow individual learning trajectories. These were a combination of recommendations received by the online platform (based on responses to vignettes about typical demands when teaching variables, terms, and equations) and the self-paced engagement in corresponding instructional videos and interactive quizzes. Demmler and colleagues (2022) developed and investigated an asynchronous PD course in the context of problem-solving, where the participants could choose to study each of the presented topics (e.g., how to engage students in different problem-solving strategies) in different representation formats, namely as instructional video, collection of problem-solving tasks with students' solutions highlighting different strategies, or as classroom video, showing the teaching and use of different strategies in practice. PD which is not designed to be taken linearly also address the need for adaptivity (Copur-Gencturk & Orrill, 2023).

Concluding remarks and future directions

We join others in emphasizing that the quality of a PD program depends on its instructional design, not on whether it is online, in-person, or hybrid (Dede et al., 2016). We extend this argument by examining how shifts in context and structure — or, in PD space and time — are reflected in the PD instructional design. In this concluding section, we also touch upon the implications for research.

We demonstrated that all four key design criteria (mathematics, representations and approximations, collaboration, and adaptivity) are altered to varying degrees in the shift to online PD. Collaboration, in particular, undergoes a significant change, as asynchronous collaboration relies on different structures compared to synchronous collaboration. These shifts are mirrored in research, which gradually focuses more on system feedback than on skillful facilitation. Additionally, in online self-paced settings, teachers can choose the depth of their engagement with representations of practice and mathematical content, with additional resources (like mathematical applets) available at a click.

The shifts described above have implications for the frameworks and methodologies used to study teacher learning, calling for adopting methods from Learning Analytics, such as researching teachers' log data. Future research should also explore cultural differences in compliance with asynchronous format, as most current studies come from Europe and the US, and more is needed from countries with different norms around teacher learning and collaboration. Furthermore, we join other researchers who suggest that the study of asynchronous PD can advance the field by "disentangling

the contribution of specific design elements" (Wischgoll & Prediger, 2024, p. 734), by using controlled randomized trials and assigning teachers to different conditions that assess particular design components.

To conclude, we return to the point made earlier about the need to broaden and increase the flexibility of PD definitions in digital spaces. We see this as a crucial next step for research on mathematics teachers' professional learning, as it opens up new possibilities for designing adaptive and scalable PD experiences that better meet teachers' diverse needs.

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Values in mathematics education of potential future teachers on entry to a degree in basic education

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This article presents an exploratory study involving 1st year undergraduate students in Basic Education from public and private higher education institutions, which aims to identify and discuss the aspects they value in teaching and learning mathematics. The study adopted a qualitative methodology and used content analysis to categorize the responses of the 125 participants who could become kindergarten or elementary school mathematics teachers. In two open-ended questions, the respondents were asked to indicate three aspects they consider the most important in teaching and learning mathematics. With regard to learning, these students prioritize the values 'Process', 'Mathematics in Context' and 'Motivation and dedication'. With regard to teaching, these students prioritize the values 'Motivation and dedication', 'Exposition' and 'Process'.

Keywords: Values in mathematics education, future elementary school teachers, teaching and learning mathematics.

Introduction

Values in mathematics education are the principles that individuals consider fundamental to the learning and teaching mathematics, functioning as a "window" that reveals what is prioritized by students, teachers, parents, principals, and society in general about these processes (Barkatsas & Seah, 2015). Since attitudes, preferences, choices and actions related to the teaching and learning of mathematics reflect what students and teachers value, but also what is valued by society (Fan, 2021), identifying these values and knowing how some are prioritized more than others can be a key component to improving the teaching and learning of mathematics (Barkatsas & Seah, 2015; Seah & Andersson, 2015). Knowing the values that future teachers defend allows them to be discussed and questioned during their teaching education course, thus helping them to become better teachers in the future. So, this paper intends to respond to the question: What aspects do 1st year undergraduate students who will become kindergarten or teachers who will teach mathematics in elementary school up to the age of 12 value in the learning and teaching of mathematics?

Theoretical Framework

From a cultural perspective, the idea of "value" gained prominence in mathematics education at the end of the 1980s, introduced by Bishop (1988). In this century, Seah (2008) broadened its scope, proposing that values are influenced by the sociocultural environment in which the individual finds themselves, stating that values and the process of valuing them are not exclusively cognitive or affective since they do not represent only mental processing, nor do affections exclusively guide them. According to Seah (2018), the process and act of valuing invariably involves reasoning and thinking. It is sociocultural in nature, as it reflects years of learning and the influence of our experiences, histories and social interactions as members of the cultures to which we belong.

In the context of mathematics education research, Bishop et al. (1999) consider three categories of values, specified by Seah (2016) as: (i) values through mathematics education in which teachers defend and teach their students to develop values that structure their personality, such as justice and human rights, through the way they plan or integrate collective discussions in the classroom; (ii) values of mathematics education that represent sociopolitical and cultural values, are incorporated into mathematics through educational policies and social norms, and that may, for example, value the ability to calculate as opposed to visualization; and, (iii) values for mathematics education where what teachers and students value can be taken into account to optimize mathematical understanding and competence, for example, when teachers recognize that if students value 'play' they can take advantage of this appreciation to optimize student learning by using exploratory activities with manipulative materials. For Seah and Andersson (2015), attitudes, preferences, choices and actions related to the teaching and learning of mathematics reflect what students and teachers value and what is valued by society. Kinone and Seah (2015) mentioned there is a huge range of attributes in mathematics pedagogy, which are the mathematics educational values and adopts seven pairs of mathematics educational values that were used in the formulation of the WIFI questionnaire[1]. These seven pairs of mathematics educational values can be seen to exist in complementary ways. We would likely value one value more than the other.

A study involving teachers and future teachers in South Korea identified the following as values related to teaching and learning: fun lessons, problem-solving, representations, calculation, aptitude and explanation, being problem-solving the most valued by teachers and future teachers (Yim et al., 2020). More recently, in a study in Malaysia regarding the alignment between teachers' values and those of their students in relation to learning mathematics, Chia and Zhang (2023) concluded that teachers prioritize process over product, hardship over well-being, effort over aptitude, application of knowledge over calculation, ideas and practice over facts and theories, exploration over exposition, and memorization over creation. Studying what 7th and 10th grade mathematics teachers in Portugal value in teaching and learning mathematics, Silvestre et al. (2023) state that teachers prioritize motivation and dedication, well-being, and curriculum and school organization.

[1] A large-scale questionnaire study which maps out what students' value in their mathematics learning experiences, developed by the Third Wave Lab, an international consortium researching values and valuing in mathematics education. https://thirdwavelab.education.unimelb.edu.au.

Research specifically focusing on pre-service teachers' values seems scarce. The existing literature has been carried out in contexts very different from Portugal (and most European countries), which requires some caution in their interpretation. Among these studies, Haciömeroğlu (2020) found that the future teachers' experiences obtained while they were students influence their values of about teaching and learning mathematics. Osman et al. (2024) argue that future teachers need to be aware of their values regarding the teaching of mathematics before they begin teaching as these can influence the way their teaching practices are shaped. It is also recommended that educators of future teachers identify their students' values regarding mathematics and its teaching, to align their practices and provide opportunities for them to become aware of their own values, how they affect teaching, mathematical learning and the construction of their students' values. This exploratory study is a first contribution to this field that is still so little explored, especially with potential future teachers who will teach mathematics in the first years of schooling.

Methods

Students attending the 1st year of LEB[2], in January 2024, at different Higher Education Institutions (HEIs) in Portugal were invited to take part in the study "What do I consider important in the teaching and learning of mathematics?" through a link sent to their cell phones by the course leader at each institution. For this communication, we only analysed the data related to the responses of 1st year LEB students from the four participating HEIs in the Lisbon area. The students had started their course last September. As every HEIs defines its own curriculum, the majority had not yet taken any mathematics subject. The response rate was 54%.

Data was collected using an online questionnaire consisting of closed-response items (5-level Likert scale) and open-response items. The questionnaire is an adaptation to the Portuguese context of the questionnaire developed by The Third Wave Project consortium (Kinone & Seah, 2015), which was translated from English and adapted by the Portuguese team, namely by adding three more pairs of values (Silvestre et al., 2023). This paper focuses on the items that aimed to collect sociodemographic data about the participants, as well as the items where these future teachers were asked to indicate the three main aspects that influence, respectively, the students' learning and the teaching of mathematics. It focuses exclusively on the qualitative analysis of the participants' open answers of those items. It was assumed that the students, having to mention three aspects as the most important, would choose those they valued the most in learning and teaching mathematics.

In the invitation addressed to 1st year LEB students, participants were assured that their contribution would be anonymous and voluntary, and the data collected would be treated confidentially and would not interfere in any way with their academic performance. 129 students responded but four were not considered as they did not respond at all, or the answers given did not correspond to the questions. A

[2] The degree in Basic Education (LEB) refers to the 1st Bologna Cycle of Studies with a total of 180 ECTS - "the degree aims to train graduates in the various areas of knowledge, enabling them to intervene in the fields of educational action, namely kindergarten teachers, elementary teachers, ATL, Museums, Municipalities, Libraries, Pediatric Services, among others." To become a teacher they have to attend a master's degree after the LEB.

total of 125 LEB students' answers were analyzed. Most respondents were female (94%) and aged between 17 and 23. Around 57% (73) of the future teachers had no professional experience in the field of education. Of those who did, only eight (6.3%) had experience as a math tutor in the early years in tutoring centers.

As the answers to the two items were statements, a content analysis was carried out after the data cleaning process. The data was analysed using the codebook produced by the Portuguese team of The Third Wave Project consortium (Silvestre et al., 2023), with slight adaptations to analyse the two items aimed at potential future teachers. Some open codes included were "learning from mistakes", "critical thinking", "flexible thinking", "inclusion" and "mathematical communication". Figure 1 shows the pairs of values that make up the codebook, initially adapted from Kinone and Seah (2015), as well as a summary of their meaning in the teaching and learning of mathematics.

	Values	Content	Values (complementary way)	Content
1	Aptitude	Values talent, aptitude, ability, capacity in the teaching and learning of mathematics.	Motivation and dedication	Values effort, dedication and motivation in teaching and learning mathematics.
2	Well-being	Values situations of calm, well-being and enjoyment in the teaching and learning of mathematics.	Hardship	Values disciplined behavior, difficulty and a tense environment in the teaching and learning of mathematics.
3	Process	Values the carrying out of processes in the teaching and learning of mathematics.	Product	Values the achievement of products in the teaching and learning of mathematics.
4	Application of knowledge	Values the application of knowledge in solving exercises and problems in the teaching and learning of mathematics.	Calculation	Values the calculation and execution of algorithms in the teaching and learning of mathematics.
5	Mathematical facts and abstraction	Values mathematical facts and abstraction in the teaching and learning of mathematics.	Mathematics in context	Values the context of everyday life or the real world in the teaching and learning of mathematics.
6	Exposition	Values explanation by the teacher in the teaching and learning of mathematics.	Exploration	Values students' own exploration and discovery, including collaboration, in the teaching and learning of mathematics.
7	Memorization	Values memorization, the recall of mathematical knowledge as it has been transmitted, in the teaching and learning of mathematics.	Construction	Values the creation of ideas, processes and products by the student in the teaching and learning of mathematics.
8	Technological resources	Values the use of technology in the teaching and learning of mathematics.	Non-technological resources	Values the use of non-technological resources in the teaching and learning of mathematics
9	Curriculum and school organization	Values aspects related to the curriculum and school organization in the teaching and learning of mathematics.	Equipment and physical conditions of the school	Values aspects related to the equipment and physical space of the school in the teaching and learning of mathematics.
10	Teachers' professional education	Values teachers' initial and in-service education in the teaching of mathematics.	Teachers' knowledge and practice	Values aspects of teacher knowledge and practice in mathematics teaching.

Figure 1: Values framework for teaching and learning mathematics

Results

In this section, we present the results of the data analysis. The percentages always refer to the total number of valid responses and not the total number of participants in the study. It should be noted that any student gave three answers for each item (7 and 8). The graph in Figure 2 shows that these

students respond differently when referring to learning or teaching.

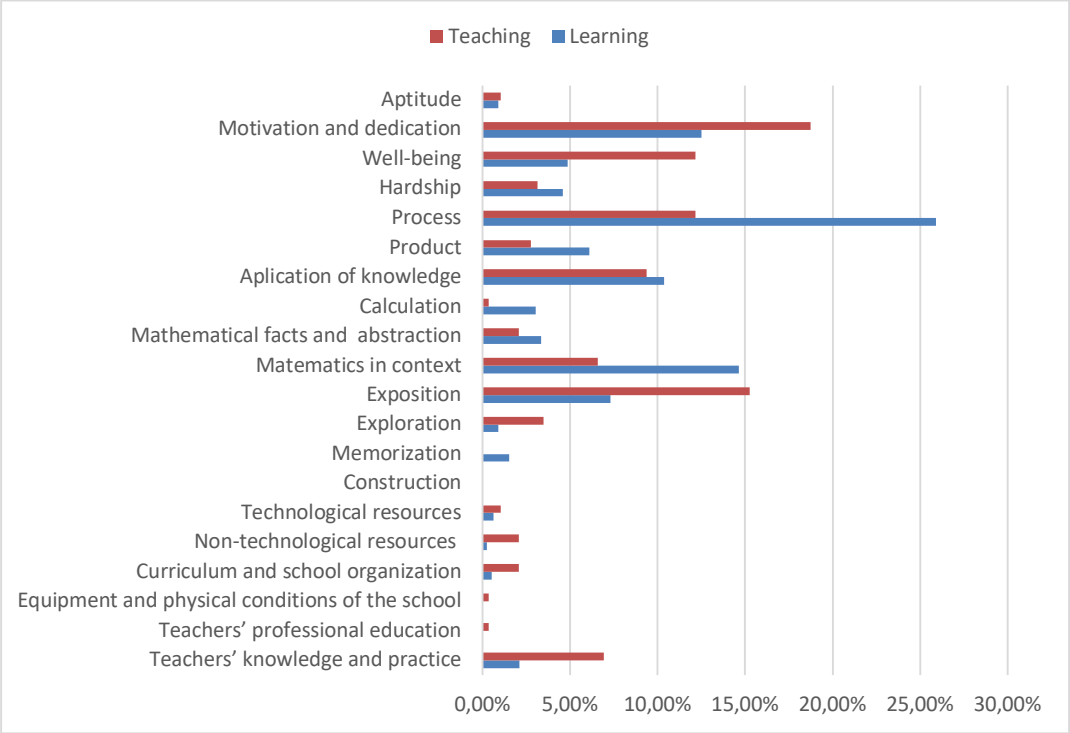

Figure 2: Percentages of the occurrence of values about learning and teaching mathematics expressed by the potential future teachers surveyed

From the graph, we identify the six aspects most valued by students in both teaching and learning. Four of them are common, but in a very different order. *Process* is the most mentioned value in learning, with 85 answers (25.91%), while in teaching it is in third place, with only 35 answers (12.15%). *Motivation and dedication* have the highest number of responses in teaching, 54 (18.75%), and are in third place in learning, with 41 (12.50%) responses. Still in teaching, *Exposition* has the second highest number of responses, 44 (15.28%), having 24 (7.32%) responses in learning (5th place). *Application of knowledge* has 34 responses in learning (4th place) and 27 responses in teaching (5th place). Also, among the six most valued are *Mathematics in context,* which is in second place in learning, with 48 answers (14.63%), *Well-being* in teaching (4th place), *Product* in learning (6th place) and *Teachers' knowledge and practice* in teaching (6th place).

To better understand the reason for the priorities set by LEB students, we analyzed the values corresponding to the top choices in learning and teaching. About *Process* (Table 1), the three most valued aspects coincide in learning and teaching, but not in the same order, with *Reasoning* being the most valued in learning and *Problem solving* in teaching. In learning, the three aspects identified are *Reasoning*, followed by *Understanding* and *Problem-solving*. It should be noted that the idea of *Reasoning* often mentioned is "logical reasoning". In teaching, the same three aspects are also prioritized, but in a different order and with a much smaller number of responses, *Reasoning* now appearing in third place, but not at a significant distance from *Problem solving* (1st place) and *Understanding* (2nd place).

Table 1: Aspects of valuing Process in learning (1st position) and in teaching (3rd position)

Learning		Teaching	
Reasoning	47 (56.63%)	Problem solving	12 (34.29%)
Understanding	27 (32.53%)	Understanding	11 (31.43%)
Problem solving	11 (13.05%)	Reasoning	10 (28.57%)

Related to *Motivation and dedication*, ranked first in teaching and third in learning (Table 2). A first analysis identifies a wide range of responses. *Dedication* has the highest number of responses in teaching, while in learning it is *Patience*, but with a number close to *Motivation* and *Dedication*. The data analyzed shows that, even in learning, students seem to be referring to the teacher with expressions such as "engaging students" or "students need to feel motivated to enjoy learning".

Table 2: Aspects of valuing Motivation and dedication in learning (3rd pos) and teaching (1st pos)

Learning		Teaching	
Patience	6 (14.63%)	Dedication	13 (24.07%)
Motivation	5 (12.20%)	Motivation	10 (18.52%)
Dedication	5 (12.20%)	Pactience	8 (14.81%)

These students seem to see learning as something they are still very much involved in, hence the value placed on *Reasoning*, *Understanding*, and *Problem-solving*. When they refer to teaching, they consider *Motivation and dedication* first, followed by *Exposition*. We, thus, conjecture that the values they attribute to teaching seem to be related to their experience as subjects who have been taught and not to their future role as a teacher, which is still a long way off.

Discussion and conclusion

This exploratory study sought to identify the aspects that 1st year students in LEB consider important in the teaching and learning of mathematics. The results show that these potential future teachers predominantly value the learning process, emphasising *Reasoning* and *Understanding*. This result aligns with the study by Chia and Zhang (2023), where *Process* is also highly valued. Still about the *Process* in learning, *Problem solving* appears relatively undervalued, contrary to what occurred in the study by Yim et al. (2020) also with future teachers, but, when refer to teaching is the most valued. These data suggest that it is important for these students to have opportunities to reflect on the meanings of the different terms involved. For instance, what *Reasoning* means to them when they associate it with logical reasoning, or what is for them Problem solving. This could be a way to expand the idea they have about the different terms and consequently the concepts involved.

Regarding teaching, they prioritize *Motivation and dedication* first, like the 7th and 10th grade teachers in the study by Silvestre at al. (2023). In contrast to this study, values such as *Curriculum*

and school organization are almost absent, which may be because participants are 1st-year LEB students and their values are marked by their experience as school students (Haciömeroğlu, 2020; Osman et al., 2024). In second place, they value *Exposition*, being *Quality of explanation* referred in a large number of responses, unlike the teachers studied by Chia and Zhang (2023), who value *Exploration*. It is worth noting that these students value *Teachers' knowledge and practice*, placing it in sixth place.

Given that teachers' values significantly influence their teaching (Bishop et al., 2006; Haciömeroğlu, 2020), and the values of future teachers may influence their future teaching (Osman et al., 2024), mathematics educators should be aware of the values prioritized by future teachers so that they can provide them with opportunities to reflect on those values. Thus, educators can align their practices by prioritizing the aspects they consider essential to be discussed. This alignment may challenge existing values and also facilitate the integration of these values with the principles of effective mathematics teaching as recommended in the literature and emphasized in curricular guidelines.

This exploratory study is a first approach to the topic with undergraduate students attending the 1st year of LEB. A first limitation of the study concerns the fact that we were only analysing responses from prospective 1st year LEB student teachers from four institutions in the Lisbon region. The fact that the data was collected through a questionnaire, answered by the students on their personal cell phones, constitutes another limitation of the study that should be completed following the students through their LEB course and/or collecting data using interviews. We consider it important to continue working on the topic and we propose to survey these students again at the end of the LEB.

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Teaching Simulations as Formative Assessments: Generating Validity Evidence Related to the Test Content

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In the context of a project to design, study, and use teaching simulations focused on eliciting and interpreting student thinking to formatively assess prospective teachers, we investigate questions focused on simulation validity. We investigated (a) the degree to which the student role protocol that guides a simulated student in the simulation covered the range of questions prospective teachers asked during the simulation, and (b) the degree to which simulated students were able to enact the role of the simulated student with fidelity. The results are promising and implications are addressed.

Keywords: Preservice teacher education, teacher educator, simulations, formative assessment.

Introduction

Formative assessment has been shown to impact learning significantly (Black & Wiliam, 1998) and it has been identified as a crucial component in teacher preparation (Darling-Hammond et al., 2005; Association of Mathematics Teacher Education [AMTE], 2017). Formative assessment provides prospective teachers with the type of feedback they need to improve their practice. As undeniably useful as it may be, providing formative feedback on most teaching requires seeing the teacher in action, and it can be challenging to facilitate such opportunities when field sites are limited in terms of in-person or video accessibility and/or opportunity to work on particular facets of teaching. In recent years, there has been an increased focus on using teaching simulations in teacher preparation (Mikeska et al., 2021). For over a decade, our research project has developed and studied the use of teaching simulations focused on the teaching practices of eliciting and interpreting student thinking as formative assessments to support prospective teachers' learning.

By teaching simulation, we mean an approximation of practice that places a prospective teacher in a situation that requires authentic engagement in the work of teaching while at the same time standardizing or controlling the context of that teaching in ways that support focus on a particular practice, element of content, or student interaction (Shaughnessy et al., 2025). As we have articulated elsewhere (Shaughnessy et al., 2019), simulations present several advantages for assessing prospective teachers in relation to the where, when, and what is assessed. They can focus on teaching without the need to be in a field context. Further, since simulations do not involve prospective teachers directly with students, simulations enable teacher educators to collect information about the knowledge, skills, commitments, and resources that prospective teachers bring to initial preparation and enable multiple collection points across a program. They can be constructed to embody particular teaching situations that might not be uniformly available. They position teacher educators close to the action (in terms of time and distance), which makes them amenable to providing feedback and gathering information useful for subsequent learning experiences.

While teaching simulations present important formative assessment opportunities, they also have limitations that must be carefully factored into studies of their use in teacher education. We focus on one such limitation. To enact teaching simulations requires resources that are not readily available and cannot simply be rechanneled from other assessment approaches. Special protocols and training are needed for those in the "student" role in a teaching simulation. These substantive limitations help define the nature of validation research needed to make teaching simulations a meaningful and usable mode of providing formative assessment.

The field would benefit from investigating the validity of teaching simulations, specifically when used to formatively assess teaching. As we noted earlier, an affordance of teaching simulations is that they can be designed to focus on a particular teaching situation. This stands in contrast to casually or informally observing teaching as a form of formative assessment where different aspects of teaching might be seen for different prospective teachers. To support the broader use of teaching simulations for formative assessment that present the same opportunities to demonstrate and appraise teaching, we need to design and then study the extent to which they make the situation repeatable across prospective teachers and doable by people beyond its developers. This is crucial because formative assessments are intended to support learning through appraisal of performance and actionable feedback (Cizek, 2010), and we need to ensure that all prospective teachers are encountering the same simulation so that it is possible to appraise their performance in that situation and provide feedback. Studies generating evidence for validity arguments (AERA et al., 2014; Kane, 2001; 2013) can support our understanding of how the simulation assessments are intended to work and what it takes to support effective use in varying contexts, including direction, preparation, support, and reassurance for those making an effort to use simulations as formative assessments. We report findings from a study that focuses on one source of validity evidence: test content.

Teaching Simulation

The teaching practice foci of our teaching simulation are eliciting and interpreting student thinking. By this, we refer to the fact that in teaching, teachers pose questions or tasks that encourage students to share their mathematical thinking. Teachers do this to assess student understanding and use what is learned for a variety of purposes, including instructional decision making. These teaching practices are central to everyday classroom teaching.

In the simulations we design, a prospective teacher interacts with a "standardized student" (a teacher educator taking on the role of a student using a well-defined set of decision rules for responding) around a specific piece of written work (see Shaughnessy & Boerst, 2018 for a more detailed description of how we design the simulation). Our live actor teaching simulations involve two teacher educators. One teacher educator takes on the role of the standardized student (referred to as the student or simulated student). The second teacher educator (referred to as the teacher educator) facilitates by appraising the performance and providing formative feedback.

Our simulations have four parts from the vantage point of a prospective teacher. First, the prospective teacher examines a piece of student work and plans questions to ask the student to learn about their process and understanding of the mathematics.

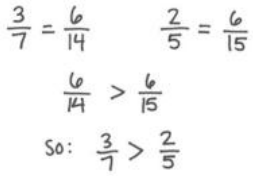

Figure 1: Sample student work

Second, the prospective teacher has five minutes to interact with the simulated student to elicit the student's thinking. In the data we feature in this paper, the student has used a procedure, sometimes known as "common numerators," to solve the fraction comparison problem shown in Figure 1. We train the "simulated student" to take on a particular student's way of reasoning about and understanding the mathematics task. To do this, we have developed a set of response guidelines that specify the student's process and the student's understanding of the ideas underlying the process (e.g., the student knows you have to multiply (or divide) the numerator and denominator by the same number to generate an equivalent fraction, but does not understand why that process works). These response guidelines (see Figure 2) include general guidance and specific responses to anticipated questions. As the prospective teacher interacts with the simulated student, the teacher educator uses an observational tool to capture key aspects of those interactions.

Third, the teacher educator interviews the prospective teacher to learn about their interpretations of the student's thinking. As the prospective teacher responds to questions about their interpretation of the student's process and understanding, the teacher educator uses an observational tool to capture key aspects of those interpretations.

Fourth, the teacher educator engages the prospective teachers in a feedback conversation about the performance in which they aim to support prospective teacher development through attention to considering when, how, why, and whether they might use particular eliciting moves.

Present Study

In this study, we explored two questions related to generating validating evidence for our teaching simulation as a formative assessment. We focused on the student role protocol (see Figure 2) that guides the simulated student's responses in the teaching simulations. First, we examined the match between the student role protocol design and the questions posed by prospective teachers. More specifically, we asked, "How well does the student role protocol cover the range of questions prospective teachers ask during the simulation?" This question is crucial to understand because simulated students will only be able to be standardized in their responses if the student role protocol supports responding to questions actually posed by prospective teachers. Second, we examined the match between the student role protocol design and the enacted student responses. We asked, "How standardized are "student" responses during the simulation?" In other words, when enacting the role of the "student," how closely were the "students" able to adhere to the student role protocol?

Characterization of the student's process: The student is using the common numerator method. To determine which numerator to use, the student finds the least common multiple of the numerators of the original fractions. The student generates equivalent fractions by multiplying the numerator and denominator of each fraction by the same number. Then the student compares the denominators to determine which fraction is larger.

Characterization of the student's understanding of the ideas involved in the problem/process: The student knows you have to multiply (or divide) the numerator and denominator by the same number to generate an equivalent fraction but does not understand why that process works. Once the student has common numerators, the student understands that when you have the same number of pieces, you can use the denominator to determine which fraction has larger pieces and therefore which fraction is larger.

Other information about the student's thinking, language, and orientation in this scenario: The student knows of other strategies for comparing fractions, but the student thinks that the common numerator methods works best with the given example.

The student's way of being: The standardized student gives the least amount of information that is still responsive to the prospective teacher's question.

Specific responses (a subset of them):

Prospective teacher prompt	Response
What did you do first?"	"I wanted to change the fractions so that there would be something in common."
Asks about how 6 was identified as the common numerator	"I wanted the numerator to be 6 because it was the least common multiple."
"Why can you compare fractions by comparing the denominators when the numerators are the same?"	"When you make the numerators the same it means you have the same number of parts so all you need to think about is how big the parts are."
"Why is 6/14 greater than 6/15?"	"Fourteenths are larger than fifteenths."

Figure 2: Excerpt of the student role protocol

Methods

Participants included both teacher educators and prospective teachers in the United States. Four teacher educators were recruited to take on the role of the "student" in the teaching simulation. These teacher educators, henceforth referred to as "simulated students," received the document containing the student role protocol (see excerpt in Figure 2) and engaged in a two-hour training in which they viewed and discussed video examples of simulations, discussed common questions, and practiced enacting the role of the "student" with coaching by project team member. Thirty-six prospective teachers enrolled in an undergraduate elementary teacher education program participated.

The assessment was administered at the mid-point of the teacher education program and prospective teachers consented to using this data for research purposes. Two teacher educators participated in two consecutive years and each served as the "student" in six teaching simulations each year. In addition, two teacher educators participated for one year, and each served as the "student" in six teaching

simulations. There was a total of 36 teaching simulations. The performances in teaching simulations were video recorded, and written artifacts were collected.

We used the software package Studiocode© to code the teaching simulation video. We began by parsing the video into talk turns. Then, we identified what we refer to as "instances," which contain a question posed by a prospective teacher and the "student" response to that question.

To address our first research question focused on the comprehensiveness of the student role protocol, for each instance, we characterized the question posed by the prospective teacher using one of three codes: (1) scripted response available, (2) no scripted response, but there is general guidance, and (3) guidance not available. *Scripted response available* means that a scripted response exists in the student role protocol and could be used to respond to the question posed. *No scripted response, but general guidance* means no scripted response exists in the student role protocol that could be directly used to respond to the question posed; however, the general guidance provides information that might be used in response. *Guidance not available* means there is no scripted response in the student role protocol that can be used to respond to the prospective teacher's question and no guidance for responding to the student was available in the general guidance section.

To address our second research question focused on the fidelity of the student, for each instance, we characterized the "student" response to the question posed by the prospective teacher using one of five codes. *Shares appropriate information* means that the "student" response was aligned with the student role protocol. In other words, the "student" used the scripted response or followed the general guidance in instances when a scripted response did not exist. *Omits important information* means that the "student" response left out important information. For example, the student role protocol includes a scripted response to "What did you do first?" The scripted response is, "I wanted to change the fractions so that there would be something in common." If the student said, "I wanted to change the fractions," and then stopped, we coded this response as "omits important information" because it did not include the fact that the student was changing the fraction because they wanted something to be in common. *Offers information that was not elicited* refers to instances where the response provided by the "student" shared information about the student's thinking that was beyond what was indicated in a scripted response. For example, if a prospective teacher asked, "What did you do next?" and the "student" responded with information about the next step and why the process works, it would be coded as "offers information that was not elicited." *Response was not aligned with the elicitation* refers to instances where the "student's" response did not address the prospective teacher's question. *Other* was used for instances that did not fall into the categories above, including instances where the question was unclear or there was no guidance in the student role protocol for responding.

Two coders independently coded each video. Discrepancies in scoring were examined by the full team, referencing the codebook as needed to reach consensus.

Results

First, we report the findings related to the comprehensiveness of the student role protocol. Specifically, we asked, "How well does the student role protocol cover the range of questions prospective teachers ask during the simulation?" Our analysis focused on the extent to which the

student role protocol provided guidance in responding to questions posed by prospective teachers. Across the 36 performances, there were 469 instances in which prospective teachers posed a question to the "student." In 69.5% of the instances (326 of the 469), a scripted response was available in the student role protocol. For example, one prospective teacher asked, "How did you pick 6 as the common numerator?" There is a scripted response for "asks about how 6 was identified as the common numerator." In another 25% of the instances (117 of the 469), there was no scripted response available, but the "student" could respond by drawing on other guidance provided in the student role protocol, such as information provided about the student's understanding of relevant mathematical concepts and general demeanor. For example, one prospective teacher posed an additional problem for the "student" to solve. In this problem, the "student" generated equivalent fractions with a common numerator of 12. The prospective teacher then asked the "student," "How did you use 12 as the numerator?" While the response to this exact question was not scripted, the "student" could reasonably use the general information available in the student role protocol and the scripted response for a common numerator of 6 to generate a response to the question. Another way of interpreting these findings is that for 94.5% of the questions posed, some guidance was available in the student role protocol for responding. These results suggest that the student role protocol is sufficient for providing guidance to the "student" for responding in standardized ways.

In the remaining 5.5% of the instances (26 of the 469), no guidance was available in the student role protocol for responding to the question. In these instances, the "student" had to improvise to construct a response to the question. We examined how these 26 instances were distributed across prospective teachers. Because the prospective teachers asked varying numbers of questions, we calculated the percentage of "no guidance" instances for each prospective teacher. We found that the percentage of instances where no guidance was available ranged from 0% to 29%, with a mean of 5% and a standard deviation of 10%. There were five prospective teachers for whom the percentage of instances with no guidance available was higher than 15%. This suggests that instances of "no guidance" were somewhat idiosyncratic and clustered around a small subset of prospective teachers.

Next, we report the findings related to the match between the student role protocol design and the enacted student responses. For this analysis, we looked across 24 teaching simulations. This was six teaching simulations for each teacher educator, and all were in their first year using the assessment. In 70% of the instances (229 of 327), the "student" shared an appropriate amount of information. In other words, the "student" responded in a way that matched the guidance in the student role protocol. This could have been responding with a scripted response or using the general guidance in the student role protocol. In another 12% of the instances (39 of 327), the "student" omitted important information in their response. In another 3% of the instances (10 of the 327), the student offered information that was not elicited by the prospective teacher's question. In 6% of the instances (20 of 327), the student response was not aligned with the question posed by the prospective teacher (e.g., the response answered a different question). The remaining 9% of instances were coded as other, and most of these cases were instances in which the question posed by the prospective teacher was not covered in the student role protocol, and thus, there was no guidance on how to respond.

Discussion

This study generated validity evidence related to the test content of a teaching simulation as a formative assessment. Our study results suggest that it is possible to design a "student role protocol" with scripted responses that address the vast majority of questions posed by prospective teachers and that it is important to include a general frame for how the student was thinking in addition to scripted responses to anticipated questions. Further, our results suggest that it is possible to train teacher educators to carry out the role of the student in a way that adheres to the student role protocol. In 82% of instances, the teacher educator carrying out the role of the student shared appropriate information or, in a small number of cases, omitted information, which created space for a prospective teacher to ask an additional question to learn about student thinking. We view these findings promising and we note that this is an exploratory study. The prospective teachers all came from one teacher preparation program and it is possible that a broader range of prospective teachers might pose different sorts of questions and that other teacher educators might need different supports to take on the role of the student. We and colleagues are investigating these and additional questions related to the validity of the teaching simulation as a formative assessment.

Taking a step back, as the field moves towards approaches for formatively assessing the teaching of prospective teachers beyond observations in the field, it is crucial to invest in conducting validity investigations into the approaches to ensure that the investment in such approaches is warranted. Formative assessment of prospective teachers demands the same level of attention to assessment quality given to high-stakes summative assessments while ensuring that we tailor validity investigations to the purposes that formative assessments serve. Specifically, formative assessments must provide information that educators can act on to support learning (Hopster-den Otter et al., 2019; Van Haneghan et al., 2009). In the context of our teaching simulation, it is crucial to ensure that all prospective teachers have opportunities to engage in a situation in which their eliciting of student thinking is being elicited by the test situation. We suggest that the field continue to explore both additional approaches to formative assessment and invest in such validity.

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Teacher reflections in lesson study: Transition from external to internal support

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Though teacher reflection with colleagues obviously has potentials, it is often a complicated endeavour especially in countries unaccustomed with the new forms of teacher collaboration like lesson study. In Denmark, where this is the case, some schools have moved beyond the initial adaptation of Japanese lesson study and now use it consistently as an approach to professional development of their mathematics teachers without including external support. In the paper, I investigate the reflection phase at two such schools from a social-practice perspective. The aim is to identify how one group of teachers at each school improvises purposes and artefacts of lesson study in their local setting. Although the schools are alike on comparative school measures, the conclusions are that the two groups improvise these parts of lesson study very differently and that the differences are strongly influenced by cultural and social aspects of their local school settings.

Keywords: Teacher reflection, lesson study, mathematics teaching, social practice theory, figured worlds.

Introduction

Teachers' joint reflections on aspects closely related to their classroom teaching are at the heart of lesson study, a relatively new form of teacher professional development in Denmark based on collaboration. However, it is often complicated for teachers to reflect productively together (Horn & Kane, 2015). Research on teacher reflection in lesson study tends to focus on the characteristics of teachers' discourse, especially the presence or lack of productive speech acts (Suzuki, 2012; Vrikki et al., 2017). However, 'discourse' is generally used in two senses, one that focuses merely on speech, and another that includes broader perspectives (e.g., Gee, 2005), such as social and cultural aspects of the specific educational settings. As such contextual aspects seem to be crucial for teachers' ways of participating in lesson study, there is a growing tendency to include these aspects in lesson-study research (Clivaz & Miyakawa, 2020; Ebaeguin & Stephens, 2014; C. K. Skott, 2024).

It is well-known that teachers tend to return to their old routines, when professional development initiatives end or extern support withdraws from them (Hennessy, 2014). In this light, it appears important to explore what happens when schools outside East Asia use lesson study consistently as an approach to professional development without external support. Though important, this focus is only rarely studied (C. K. Skott, 2024). The present paper aims to contribute an exploration hereof with a specific focus on the reflection phase and viewed from a holistic, contextual perspective.

The paper reports a multiple case study, in which a group of teachers and their lesson study at each of two schools – Topas and Hilltop – in the Copenhagen area constitutes a case. Both Topas and Hilltop have moved beyond the initial adaptation of lesson study and now use it consistently as professional development of their mathematics teachers. Topas and Hilltop included initially external support (among other the author) for longer periods but are now continuing without. At both schools,

lesson study has been designed as consisting of three phases: planning a research lesson collaboratively; conducting and observing the lesson; jointly reflecting on the lesson based on observations of student activity. The teachers found particularly reflecting challenging as they often found themselves in situations for which they did not have a set of immediate responses. Using social practice theory, inspired by Holland et al. (1998), the paper focuses on the two groups' joint reflections and their negotiation of purposes and artefacts of lesson study in their local school setting. The paper addresses the research question: How do two teacher groups improvise the reflection phase when they engage together in lesson study in their local school setting without external support?

Teacher reflection in lesson study

Studies argue for the importance of building teacher reflection into professional development (Hennessy, 2014), as it is generally believed to have significant potential for supporting teacher learning (Horn & Kane, 2015). According to Myers (2013), lesson study is "ideal for nurturing reflectivity" (p.1), as it is collaborative, cyclic, and focuses on messy, challenging and real-life teaching situations. However, reflecting with colleagues can be challenging, and often it does not lead to the expected improvements in classroom teaching or teacher learning (Horn & Kane, 2015).

Research on teacher reflection in lesson study is scarce (Myers, 2013). Among the existing empirical studies, I have identified three strands. The first strand focuses on generic levels of reflection, such as descriptive, analytical and critical levels (e.g., Myers, 2013). Teachers' speech acts are taken at face value without regard to how the content can be interpreted in relation to classroom events, teachers' sense-making or contextual aspects. Although the highest level, critical reflection, is found to be most productive in terms of teacher learning, Myers (2013) shows that this level is rare in both in-service and pre-service teachers' reflections.

The second and third strands focus on identifying the most productive aspects of reflections for teacher learning. The second strand has a quantitative focus, while the third has a qualitative focus and a stronger focus on the mathematics. The second strand combines an analytical focus on speech acts with teachers' dialogic moves and learning processes. One aim of its research group (i.e., Vrikki et al., 2017; Warwick et al., 2016) is to develop a reliable coding protocol for quantitative analyses of teacher discussions. A second aim is to unpack the black box of teacher learning and investigate "the specific dialogic mechanisms that enable teacher learning to take place" in a lesson study context (Warwick et al., 2016). The group adopts a sociocultural perspective, and claims learning happens when teachers *interthink* together – a dialogic analogue to interact (Warwick et al., 2016).

The protocol comprises three coding dimensions: 1) dialogic moves (i.e., teacher contributions that move the dialogue forward, such as requesting information, building on ideas, providing evidence), 2) scope of discussion (i.e., the extent to which teachers address groups or selected case-study pupils), 3) learning processes (i.e., descriptive or interpretative). For instance, the code, 'interpretive learning process' is assigned when "teachers use words of evaluation [of teaching] such as: good, nice, […] fun, […] difficult" (Vrikki et al., 2017, p. 223).

The third strand interprets teachers' speech acts in relation to classroom events and teaching stances. Suzuki (2012) identifies six discourse modes that led to the self-declared learning of a Japanese

teacher participating in several lesson studies. Suzuki categorises the six modes as belonging to a teaching stance either a technical expert or a reflective practitioner. For example, the mode "Did the children learn what the teacher intended them to?" (p. 220) belongs to the technical expert, whereas the mode "Did the teacher teach what the children actually learned?" (p. 224) belongs to the reflective practitioner. Suzuki concludes that the latter mode best supports teachers' learning.

To summarise, the three strands focus on discursive characteristics of teachers' speech. The first focuses on generic levels of reflection, whereas the second presents a protocol for quantitative analyses of teacher discussions. In contrast, the third offers categories that directly address the purposes of lesson study, which is to inquire deeply into teaching/learning problems. However, none of the strands applies a holistic, contextual approach, which is called for by recent research on lesson study adaptations. This research emphasises the importance of contextual concerns, such as teachers' use of artefacts, the collaborative setting of their school (e.g., social norms, work relations), and wider social, cultural and power-related questions (e.g., the national teaching culture outlining general obligations and values for the teaching profession) (Clivaz & Miyakawa, 2020; Ebaeguin & Stephens, 2014; C. K. Skott, 2024). Aiming for a holistic, contextual approach, I draw on social practice theory.

Theoretical approach

I use Holland et al.'s (1998) framework, especially their concepts of *figured worlds* and *improvisations*, to conceptualize the Danish teachers' adaptation of lesson study into their local school setting. The framework is rooted in social practice theory that highlights the mutual and formative relationship between persons and their social world. By using the framework, I can capture how teachers' behaviour form and are formed by their cultural and social contexts, and how they orient and respond to the contexts' prevailing interpretations. A figured world is defined as "a socially and culturally constructed realm of interpretation in which particular characters and actors are recognized, significance is assigned to certain acts, and particular outcomes are valued over others" (Holland et al. 1998, p. 52).

For my present purpose, it is important how teachers learn to act and engage in a new world of lesson study and to (re)produce and assign meaning to the artefacts in this world. According to Holland et al. (1998), people can form such cultural knowledge heuristically by continual rehearsal of how to act and engage in figured worlds, as those "move through us as spoken discourses and embodied practices" (p. 251). However, in educational contexts such as the Danish, where the underlying cultural features of lesson study are largely unknown, it is not straight forward how teachers may learn this cultural knowledge heuristically. To conceptualise this challenge, I will draw on the concept of improvisation. A person improvises when confronted with a situation for which he/she has no set responses (e.g., chairing a reflection session for the first time) by reacting impromptu and drawing upon cultural artefacts (verbal, gestural and material) that are available in the moment. In addition to these artefacts, a person also improvises in relation to the sediments of his/her past experiences and the subject positions afforded him/her in the situation (Holland et al., 1998).

An artefact relates to a figured world as they are constructed as "parts of collectively formed systems of meaning [and as] products of social history" (Holland et al., 1998, p.35). Thus, people construct artefacts by engaging in social activities of a figured world and by negotiating and assigning meaning

to the artefact in this process. However, artefacts can also be constructed as emerging products of an improvisation and possibly "be taken up and become conventionalized or made into culture" (p. 17). Holland and her colleagues emphasize two aspects of an artefact: a material aspect as some kind of an instrument or reification and a conceptual aspect, as a collective memory of the artefact's substance and meaning as it is embedded in a figured world. When people improvise, the two aspects are constantly changing.

In this paper, I focus on the groups' improvisations of both the purposes and the artefacts of lesson study including their material and conceptual aspects, and on how these improvisations are related to each groups' dominant figured worlds. For this, I draw on J. Skott's (2013) perspective of how a teacher relates dynamically to a multitude of shifting prominent practices and figured worlds, when participating in social interactions, such as in the reflection phase. The teacher's participation may be inspired or transformed by figured worlds and practices, which the teacher has engaged in previously or by broader social and cultural practices and worlds at the school and beyond. This perspective allows me to conceptualize the part of the teachers' improvisation that relate to their experiences from past, present and new figured worlds, and practices at their school and beyond.

Methodological approach

The study is a multiple case study (Cohen et al., 2011) with a teacher group at each of Topas and Hilltop as a case. The schools are in the same middle-class neighbourhood of Copenhagen and are alike on comparative school measures. Acting as participant observers, my colleagues and I facilitated lesson study at Topas for 3 years (2014–17) and at Hilltop for 2½ years (2016–18). The designs of the lesson study were continuously adapted to each school's setting and the last year of our collaboration, all the mathematics teachers participated. Hereafter, the schools continued with lesson study without external support. They devoted two hours to each teacher group to plan the research lesson and the lesson was only taught in one class. A year after our collaboration, we selected data from three lesson studies at each school and for this paper, I draw primarily on one lesson study in grade 5 at Topas with 8 teachers (December 2017), and one lesson study in grade 8 at Hilltop with 5 teachers (March 2019). The data consists of two reflection sessions (2 times 60 min., audio-recorded, transcribed), two lesson plans and two research lessons (2 times 45 min., video-recorded).

As "it is not apparent at the outset what practices and figured worlds are significant for the teacher" (J. Skott, 2013, p. 552), my analytical objective was to deduce empirically what figured worlds and practices the teachers tended to draw on in their reflections and to investigate the fluctuating relations between these figured worlds and practices. Conducting a double analysis, I first constructed a set of possible dominant figured worlds and practices by analysing the teachers' acts and utterances and assembling these into 'as-if' realms using our previous analyses as a backdrop (C. K. Skott, 2024). Second, I re-analysed the data to confirm, possibly delete, or add new worlds, and to investigate shifts in their prominence for the groups, when they improvise the purposes and artefacts of lesson study. For Topas, the dominant figured world was *School*, which encompasses disciplinary issues, students' general behaviour and interactions (e.g., group dynamics), student mood and other general pedagogical concerns. For Hilltop, the most prominent worlds were: a new world of *Lesson study,* which comprises values and acts related to planning for, investigating and reflecting on student

learning, and a well-established world of *School mathematics*, that includes values and acts related to the teachers' ordinary mathematics teaching which tended to focus on students' mastery of skills and procedures. I will briefly describe each research lesson before I present the results of the analysis.

Mathematical communication in grade 5 at Topas: Teacher Bo began the lesson by presenting its learning goals: learn to communicate about addition, subtraction, multiplication, and division, and use these Latin terms rather than the Danish ones. There were three activities: 1) categorise 20 word-problems in relation to the involved operation, 2) match an operation with a word problem, 3) explain each operation individually and talk about them in groups. Bo tended not to listen to the students' mutual talk and in classroom discussions, he did not unpack their contributions nor provide mathematical explanations or ideas and the discussions regenerated into show-and-tell. The students struggled with the last activity, writing for example "Subtraction is when you minus". Finally, the students should indicate with smileys if Bo had communicated clearly and if they had been motivated.

Mathematical inquiry in grade 8 at Hilltop: Focusing on tessellation, teacher Jan began by showing a picture of a street paved with dragon-squares tiles. Then, he asked for properties of the pattern and for reasons why dragon-squares can tessellate. The students did not specify any properties, but one student suggested that dragon-squares can tessellate, because they can be divided into two triangles and all triangles can tessellate. Jan responded that he was unsure if it was correct. Another student suggested that all quadrilaterals can tessellate, and Jan then asked the class to investigate this conjecture using paper and scissors. After working half an hour, Jan asked about their investigations, but none of the students were able to confirm or reject the conjecture. Both Jan and the students were clearly frustrated as they wanted a valid argument for why all quadrilaterals can or cannot tessellate.

The two teacher groups' improvisations

The protocol is an important artefact for structuring teachers' joint reflections in Japanese lesson study. Both groups appointed a chair to lead the reflections and the chair used the protocol to structure them, encouraging first Bo/Jan to recount their experiences of the lesson and then the other teachers. The Hilltop-group had a stronger focus on the other parts of the protocol than the Topas-group, such as selecting themes for discussion, improving the plan, and summarising the reflection. Both groups thus improvised the protocol's material aspect, and the Hilltop-group also improvised its conceptual aspect to some extent. For example, when the chair specifically asked for observations that could inform the plan's research focus (i.e., how to guide students' inquiry without supporting them too much) thus encouraging the participants to analyse deeply this focus based on their observations.

When reflecting together at Topas, the teachers distinguished strongly between their ordinary classroom teaching and teaching a research lesson and repeatedly negotiated the purposes of the latter as compared to the former. Bo stressed that lesson study was "a tour de force in how you can DO some things". He had wanted to show how to organise student work with the three activities and to show a new assessment method. Though feeling stressed, he was concerned by "signalling [to the class] that I was not [stressed] (…) it should be clear to them what the goal was (…) and my expectations". Therefore, the lesson "became more shallowly that normally", and he "did not walk to each group of students (…) to get an impression of what they were talking about (…) I would get it when we reflect together". Here, Bo improvises the purpose of the lesson study as demonstrating new

teaching approaches and assessment tools, and deliberately without paying attention to the students' mutual talk, nor to a great extent to their mathematical contributions. These improvisations were supported by the chair: "You will get these [insights into student thinking] now". None of the teachers questioned these improvised purposes, although only a few learning opportunities emerged for the students in the research lesson. One reason may be the group's orientation towards *School.* In this world, a teacher's autonomy and own decision-making regarding teaching methods are highly valued.

Another important lesson-study artefact is observations. The chair asked for observations of "what you have experienced" and not for observations that could shed light on the plan's learning goals or other central aspects. The teachers provided observations about students' use of the Latin terms as asked for in the plan, but none of their observations concerned the students' mathematical learning. Rather, the observations were detailed descriptions of group dynamics and organization of the activities, which all relate to *School*. One teacher used her observations to argue that the last activity was not sufficiently challenging but added immediately after "we need to adjust it", taking it upon her and the others to fix it, not suggesting that Bo should have reacted differently or otherwise criticising him. To respect colleagues and not criticise each other is also part of *School,* and the group can be seen as drawing heavily on this world in their ways of improvising the purpose and artefacts of lesson study. The group thus used observations, but they did not use them to analyse students' learning nor other emerging issues in the research lesson. As this is part of the artefact's conceptual aspect, the group only improvised its material aspect.

At Hilltop, the group improvised a double explorative purpose of lesson study. The first was to explore how to guide students' mathematical inquiry without providing them the solutions. The second purpose was purely mathematical and was to explore themselves whether all quadrilaterals can tessellate. The group did not solve the problem before the research lesson, and during the reflection they repeatedly expressed uncertainty about it, as when Jan said, "Are you convinced that they [quadrilaterals] can [tessellate]? I do not know if you are, but I believe they can". Jan's uncertainty strongly influenced his participation in the research lesson as is evident in the above description of the research lesson. He also emphasised it in the reflection, "there were too many uncertainties for me (…) both the mirror figure and the concave figure (…) we should have made some specific choices" and "we had agreed that I should use the picture to talk about what to be attentive to (…) I forgot it and I think it had an influence". Despite this, the teachers did not question the appropriateness of their second improvised purpose. Certain aspects of the group's *Lesson study* world were inspired by the municipality's involvement, and the group's improvisation of the second purpose and how they dealt with it seemed to be influenced hereby. Also, the first purpose was inspired by their orientation towards *Lesson study*, but more in line with a general picture of the figured world and its practices.

The Hilltop-group linked their observations to the research focus and the focal points of the lesson plan, skilfully guided by the chair. They used their observations, which were more reflective and less descriptive than the Topas-group's, to analyse the focus and selected themes. One teacher observed for example "They had drawn one figure, but they did not dare to draw more. Then you came [Jan] and you drew more figures (…) and they realised that they needed not to see through the problem before drawing it". As such use of observations relates to the cultural model of lesson study, the

Hilltop-group improvised both the material aspect and the conceptual aspect of the artefact. Occasionally, the group judged students' answers superficially, such as "if she said the same to an outsider, it would be nonsense" not analysing the meaning of the utterance. Such practices relate to *School mathematics*. Compared to the Topas-group, the Hilltop-group primarily oriented itself towards *Lesson study* and their well-established *School mathematics,* and much less towards *School*.

Discussion and concluding remarks

This study contributes an exploration of teacher reflection in lesson study without external support from a holistic, contextual perspective. The two Danish schools, Topas and Hilltop, have both moved beyond the initial adaptation of lesson study and now use it to develop their mathematics teachers professionally. My analysis showed that the two groups improvised the purposes and artefacts of lesson study significantly differently in the reflection phase. The Hilltop-group improvised two explorative purposes, one didactical and one mathematical, and tended to use both the material and conceptual aspects of the artefacts, while the Topas-group improvised lesson study as a way to demonstrate new teaching approaches and methods and tended to use only the material aspects of the artefacts. The analysis suggests that these differences are primarily due to social and cultural aspects of the teachers' local school setting as indicated by their different dominant figured worlds and practices. The significance of these contextual aspects is supported by the fact that the schools are similar on comparative school measures, such as parents' socio-economic status and national ranking, and these measures can thus not account for the differences between the two groups.

The study's contextual perspective differs from the above-described three strands identified in research on teacher reflection in lesson study (Myers, 2013; Suzuki, 2012; Vrikki et al., 2017). If I had used the first strand's generic categories (Myers, 2013), it would probably have led to the same result: fewest critical reflections. However, as indicated in my analysis, it is often difficult to grasp the meaning of teachers' utterances based on their face value as there can be many layers of meanings and understandings beneath. The second strand's interthink-concept seems productive to describe how teachers learn together in lesson study (Vrikki et al., 2017). However, the strand's quantitative approach and in particular the protocol's learning part appears rather superficial. Using it on my data, most of the teachers would learn interpretively as they tended to evaluate the teaching by using the codes fun, motivating, etc. However, as the analysis showed, the teachers' observations, especially at Topas, were rarely connected to the students' learning nor the content. The third strand's discourse modes focus on the relationships between mathematics, student learning and teaching (Suzuki, 2012), but there are important aspects of teacher reflections that the modes do not capture such as teachers' use of artefacts. Concludingly, compared to these strands, this paper's holistic, contextual approach encompasses to a larger extent the complexity of teacher reflection, and includes further teachers' (re)productions of existing or new artefacts which seem to influence their reflections significantly. Therefore, it appears important to further explore teacher reflection in lesson study by using contextual approaches. This study also sheds light on the sustainability of lesson study as a continued teacher-led professional development without external support. Both groups have profoundly developed their ways of working together, especially the Hilltop-group. However, there are aspects of both groups' improvised ways of doing lesson study that from a student-learning perspective seem inappropriate, such as the explorative purpose in relation to mathematics at Hilltop. These aspects

should preferably not develop into stable practices. Thus, both schools might still benefit from external support, for instance from a teacher with lesson study experiences from another school, who is able to question such parts of the groups' reflections.

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The impact of pre-service teachers' basic mathematical knowledge on their university mathematics grades: A case study from Norway

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In this paper, we investigate whether Norwegian pre-service teachers' (PSTs) basic mathematical knowledge and skills upon entering teacher education can predict their performance in mathematics and didactics courses at the university. To measure their basic mathematical knowledge, 127 PSTs were assessed using a test designed for grade 8 students. The PSTs' scores on this test were compared with their performance on two university-level mathematics examinations and a national test in algebraic thinking. The findings reveal a moderate but significant correlation between the PSTs' basic mathematical knowledge and their performance on both the university examinations and the national test. This paper contributes to the discussion on how well PSTs' basic mathematical knowledge at the start of teacher education predicts their future academic outcomes in university-level mathematics courses and how this insight can be used to improve the quality of teaching and learning for PSTs.

Keywords: Teacher education, pre-existing knowledge, mathematics achievement.

Introduction

In Norway, the debate over teacher quality and the entry requirements for teacher education has become a pressing issue. There is a shortage of qualified teachers, with 26 percent of primary school mathematics teachers not being qualified to teach the subject during the 2021–2022 academic year (Arnesen et al., 2023). Some attribute this shortage to the national entry requirements, which initially mandated that applicants to primary teacher education programs have a minimum of 35 school points from upper secondary school, along with a grade of 4 in mathematics and 3 in Norwegian (Forskrift om opptak til høgare utdanning, 2017, § 4-7). However, in 2022, the requirement was adjusted to 40 school points, while the minimum grades were reduced to 3 in both mathematics and Norwegian (Forskrift om opptak til høgare utdanning, 2017, § 4-7). Despite these changes, the number of applicants to teacher education remains low. In 2024, the requirements were further relaxed when the Norwegian Ministry of Education introduced legislation allowing universities to admit pre-service teachers (PSTs) with only a passing grade from upper secondary school (Forskrift om opptak til høgare utdanning, 2017, § 4-7A) after application to the ministry. Twelve universities applied, and 35 teacher education programs have accepted all applicants with minimum passing grades from upper secondary school to begin studying teacher education.

The opposition to lowering the requirements for entering teacher education argues that grades from upper secondary school are a good predictor of how well PSTs perform in teacher education. The argument is that quality teaching requires well-qualified teachers with a solid understanding of both content knowledge and pedagogical content knowledge. However, studies investigating how well students' grades from upper secondary school predict their performance in higher education show

mixed results. For instance, Cyrenne and Chan (2012) report that the average of students' high school[1] grades "is a strong predictor of their university GPA [grade point average]" (p. 541). Similarly, studies from the US have compared students' high school GPA with their ACT scores for college readiness, finding that GPA has a stronger relationship with students' college performance compared to the scores resulting from the ACT admission test (Allensworth & Clark, 2020).

In Norway, apart from a study investigating the relationship between engineering and calculus students' performance in mathematics at the university level and their basic mathematical knowledge at the start of their studies (Nortvedt et al., 2019), little is known about PSTs' readiness in terms of content knowledge. Except from a non-peer-reviewed report from Norwegian Council for Quality in Education (NOKUT) (Haakens & Bråten, 2017), we have found no research investigating the relationship between Norwegian PSTs' basic mathematical knowledge upon entering teacher education programs and their performance in mathematics and didactics courses at the university level. The report by Haakens and Bråten (2017) suggests a connection between prior formal knowledge in mathematics, such as grades and courses from upper secondary school, and PSTs' results on a national administrated mathematics examination. The report also suggests that the link between Norwegian PSTs' prior knowledge when entering university and their performance at university courses should be further explored.

This gap in knowledge motivated our study, which investigate the PSTs' mathematical knowledge using a standardized test designed for grade 8 learners. The study is limited to one university in Norway, where we aim to answer the following research question:

> What is the relationship between Norwegian PSTs pre-existing basic mathematical knowledge at the start of their teacher education and the grades they receive in university-level mathematics courses?

Previous research and context of the study

Common content knowledge is a component of Mathematical Knowledge for Teaching (MKT) (Ball et al., 2008). Good teaching skills and the ability to understand young children's mathematical thinking require that PSTs have a solid understanding of the content they are going to teach—referred to as basic mathematics. Several studies have shown that teachers with weak MKT provide lower-quality instruction, which also negatively affects their students' learning outcomes (Hill et al., 2008).

According to the Organization for Economic Cooperation and Development (OECD) and its 2022 Program for International Student Assessment (PISA) report, grades are "positively related to the expectation of completing a university degree in all participating countries and economies" (OECD, 2022, p. 11). However, the report does not provide information on students' performance or completion rates in specific study programs, nor does it explicitly address university mathematics courses.

[1]We use the term "upper secondary school" students except from when we refer to literature from the US that use the term "high school" students.

Research in the United States has examined both university entry examinations, such as the SAT, and grade point averages (GPA) from upper secondary school. These studies show that GPA is a better predictor of university mathematics performance than SAT scores (e.g., Allensworth & Clark, 2020). It's important to note that the SAT is a composite test, where basic mathematical knowledge from upper secondary schools in Norway contains only mathematics (Norwegian Centre for Mathematics Education, 2024). In Norway, it is only the grades from upper secondary school that are used as entry requirements to universities, and there are no national standardized tests regulating university admissions.

We therefore suggest that a test related to the content which the PSTs are going to teach is a minimum of what is expected as basic mathematical knowledge to succeed in the task of teaching mathematics. A test which is designed to unveil misconceptions related to mathematics for 8-graders will hence be suitable for PSTs that are training to teach 1–7-graders.

A project related to the mathematical pre-knowledge of economic students demonstrates how a diagnostic test at the beginning of their studies can be a useful tool to uncover frails in the students' mathematical knowledge (Landgärds, 2019; 2021). A pre-course related to the diagnostic test also seems to be an effective measure to help the students to succeed in university level mathematics courses (Landgärds, 2019; 2021).

The Norwegian context

In Norway, Initial Teacher Education (ITE) consists of four program types: (1) Primary ITE (1st–7th grade), (2) Lower-Secondary ITE (5th–10th grade), (3) Teacher Education Program (8th–13th grade), and (4) Post-Graduate Certificate in Education (Jakobsen & Munthe, 2020). This study focuses on PSTs enrolled in the first program—Primary ITE, grades 1–7—and in terms of the number of PSTs, it is one of the largest ITE programs in Norway. It is a five-year master's program that educates the vast majority of primary school teachers in Norway and is guided by national curriculum regulations (National Council for Teacher Education, 2016).

In this program, 30 out of 300 ECTS credits are in Mathematics, and it is compulsory. This study is situated at one Norwegian University where the 30 ECT mathematics course are divided into two: one course of 15 ECTS with an oral examination created locally by the ITE institution (LE-1), and one course of 10 ECTS with written examination, which is also locally made (LE-2). In addition, there is a written national examination worth 5 ECTS in algebraic thinking (NE), which is the same for all PSTs in all ITE programs in Norway. The content of all mathematics courses is a mix of mathematical and didactical content, preparing the PSTs to teach the subject.

Method

Sample

All PSTs enrolled in the ITE program for primary school teachers grade 1–7 from one Norwegian university were invited to participate in the study. 74 PSTs from the cohort that started August 2021 and 53 from the cohort that started August 2022, a total of 127 PSTs, opted to participate. The basic mathematics knowledge test (BMK-test) was given at the beginning of their first mathematics course (see Table 1 for an overview of the two first year of studies, including BMK test and examinations).

Table 1: Overview of the first two years of the ITE program, mathematics courses, BMK test, and examinations

Semester	Topic	Examination
1	Other ITE courses	
2	Mathematics (15 ECTS) + other ITE courses	BMK-test (at the beginning of the semester) + LE-1 (at the end of the semester)
3	Mathematics (10 ECTS) + national examination (5ECTS) + other ITE courses	LE-2 and NE (at the end of semester)
4	Other ITE courses	

Basic mathematical knowledge (BMK) test, locally and national made examinations

The BMK test was administered during the PSTs' first mathematics lesson. It consisted of tasks developed by Professor Alistair McIntosh (Norwegian Centre for Mathematics Education, 2024). The framework aims to assess Norwegian 8th graders' basic mathematical knowledge and identify common misconceptions, primarily related to number sense. Number sense is an essential part of the common content knowledge dimension of MKT, as it strengthens the teacher's ability to calculate numbers quickly and correctly. Therefore, it enhances their ability to "Evaluating the plausibility of students' claims (often quickly)" (Ball et al., 2008, p. 400). The test contained 35 questions in total, each graded as either "right" or "wrong," with a maximum score of 35. Eighteen of the questions were multiple-choice, each with only one correct answer, while the remaining seventeen were single-answer questions.

For both the locally administered and national examinations in algebraic thinking, the A–F grading scale was used. The grades were translated as follows for measurement purposes: A = 6, B = 5, C = 4, D = 3, E = 2, and F = 1.

The relationship between the PSTs' scores on each of the examinations and their results on the BMK test was analysed by calculation Pearson's correlation coefficient, calculated with SPSS Version 29.0.1. The correlation coefficient indicates the strength of the relationship between the pretest and examination results. The aim of the analysis was to either confirm or reject the hypothesis of a relationship between BMK test results and examination scores. Additionally, we calculated a weighted total score (WTS) for all three examinations (LE-1 + LE-2 + NE) and examined its relationship with the BMK score. It should be noted that we expect that WTS will be more influenced by, and probably more strongly correlated with the LE-1 test, while being less so with the LE-2 and NE. However, this aligns with the overall grades given to the students and will therefore be the most useful measure. We followed Cohen's (1988) guidelines for interpreting correlation strength: values

above 0.50 indicate a strong relationship, values between 0.30 and 0.50 indicate a medium relationship, and values between 0.10 and 0.30 indicate a weak relationship.

Results

The average scores and the standard deviation of BMK, LE-1, LE-2, NE and WTS are presented in Table 2, and shows that PSTs had around 77 % score in average at the BMK-test, and that that it was easier to get a good grade in the oral examination (LE-1), than the two written examinations (LE-2 and NE).

Table 2: The average scores and the standard deviation of BMK, LE-1, LE-2, NE and WTS

Test or examination	Point scale	Average score	Standard deviation
BMK	0–35	26.95	4.8
LE-1	1–6 (A–F)	3.75	1.52
LE-2	1–6 (A–F)	2.86	1.50
NE	1–6 (A–F)	3.00	1.48
WTS	1–6 (A–F)	3.28	1.31

The correlation between BMK and LE-1, LE-2, NE and WTS is respectively presented in table 3, and shows that there is strong relationship between BMK and both LE-2 and WTS, while there is a medium relationship between BMK and both LE-1 and NE. As predicted, there is a strong correlation between LE-1 and WTS, but also between LE-2 and WTS.

Table 3: Correlation matrix showing the relationship between the different examinations/test

		1	2	3	4	5
1	**BMK**	1				
2	**LE-1**	.44*	1			
3	**LE-2**	.56*	0.51*	1		
4	**NE**	.46*	0.40*	0.58*	1	
5	**WTS**	.57*	0.88*	0.82*	0.67*	1

* $p < .001$ (two tailed)

Discussion

The results indicate a medium to strong correlation between the PSTs' pretest scores and their examination grades (Cohen, 1988). While the correlation coefficient reveals a stronger relationship between the BMK test and the WTS score compared to the BMK test and any of LE-1, LE-2 or NE, it is also notable that the relationship between the BMK and LE-2 is stronger than that between the BMK and both LE-1 and NE. This may suggest that the BMK test is a better predictor of performance on the LE-2 examination than on the others. Further research is needed to get a better understanding of these relationships.

Although one should be cautious about establishing a direct connection between PSTs' basic mathematical knowledge and their performance in the mandatory mathematics course in teacher education, the scores suggest that PSTs with stronger prior knowledge tend to perform better in these courses. This is consistent with findings by Ball et al. (2008) and Hill et al. (2008), who argue that mathematics teachers need both a solid understanding of mathematics and specialized content knowledge to succeed.

The results of the BMK test indicate that several PSTs in this study struggled with basic primary school mathematics at the start of their teacher training. These PSTs were admitted based on their upper secondary school grades, which included a grade of 3 in both mathematics and Norwegian, and a total of 40 school points. However, as of fall 2024, twelve teacher education institutions in Norway have removed these admission requirements due to a shortage of qualified applicants (Forskrift om opptak til høgare utdanning, 2017, § 4-7A). After the previous intake requirements were introduced, the Norwegian Ministry of Education reported a higher completion rate among PSTs enrolled in teacher education program, with fewer PSTs dropping out (Ministry of Education, 2024). It is therefore likely that the reduction in admission requirements will have the opposite effect.

We also anticipate that PSTs enrolled at the teacher education institutions that have removed the intake requirements in 2024, may struggle with a lack of basic mathematical knowledge. In an environment where the formal requirements for admission to teacher education are lowered or even removed (Forskrift om opptak til høgare utdanning, 2017, § 4-7A), we foresee one of two outcomes: either PST success rates will decline (Allensworth & Clark, 2020), or the standards for passing teacher training will be lowered over time, as studies from other countries have documented (e.g., Nieswiadomy & Kim, 2024; Rojstaczer & Healy, 2012).

To address these challenges, we suggest that a pretest on PSTs basic mathematical knowledge—a test like the one presented in this paper—has the potential to serve as a tool for identifying mathematical difficulties among PSTs. Once these challenges are identified, targeted support should be provided to help PSTs begin their teacher training with a stronger foundation in mathematics, as Landgärds (2019; 2021) has had successfully implemented for the mathematics course for economics students.

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Teacher professionalisation through the use of a digital diagnostic tool in settings with different forms of additional support

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As teachers do not implement formative assessment practices frequently, there is a need for support. To understand how this may be realised, we investigated in which ways teachers show professionalisation in the work with a digital formative assessment tool in different support settings. In the analysis of six teacher interviews from three treatment groups, we found that all teachers show a form of professionalisation by applying and enlarging different facets of their professional knowledge leading to a reflection on or change in their teaching practices. Hereby, the attendance in a professional development session as well as an engagement with the test items and didactical explanations provided by the tool allow for similar forms of teachers' professionalisation.

Keywords: Digital formative assessment, professional development, algebraic thinking.

Introduction

Learning algebra, especially understanding the meaning of variables, is crucial for students to be successful in secondary mathematics education as algebraic competencies form the basis for many subsequent themes. However, students have great difficulties in this topic (e.g., Arcavi et al., 2017). Formative assessment (FA) is one possibility for teachers to address these problems as it might help to identify and respond to students' individual needs. Implementing FA can be complex and challenging for teachers, making the use of a digital FA tool a potentially valuable resource. This raises the question in which way such digital tools provide support for teachers and which forms of professionalisation they undergo when they are working with a digital FA tool.

Theoretical Background

FA is defined as an activity in which "evidence about student achievement is elicited, interpreted, and used by teachers, learners or their peers, to make decisions about the next steps in instruction" (Black & Wiliam, 2009, p. 9). Empirical studies show that, on the one hand, FA may have positive effects on students' learning, depending on the subject and the concrete form of implementation (McLaughlin & Yan, 2017). But on the other hand, it is rarely realised by teachers (Bennett, 2011). Research shows two main reasons why German teachers do not frequently practice FA: expenditure of time and a lack of expertise and experience (Bürgermeister & Saalbach, 2018). To overcome the lack of expertise, teachers could participate in a professional development (PD) program. But in Germany teachers rarely participate in such programs (Jude et al., 2024), so other options need to be considered. Cusi et al. (2024) point out that teachers might benefit from the use of digital FA tools. In their model of technology enhanced FA, they differentiate four areas of teachers' practices in FA which they combine with the three phases of preparing lessons as well as with three functionalities of technology in the process of FA.

The *digital tool SMART* (Specific Mathematics Assessments that Reveal Thinking), used in this study, was developed at the University of Melbourne and is currently being adapted for use in

German-speaking countries. It consists of about 130 multiple-choice-tests covering various mathematical topics. The tests elicit students' understanding of mathematical concepts by focusing on their conceptual knowledge. Based on response patterns, the tool carries out an advanced analysis shown in individual stages of understanding and possible misconceptions displayed in the automatic evaluation for the teacher. The tool provides teaching suggestions and additional materials for the teacher to support further instruction. Lastly, teachers decide whether and how to use the materials and teaching suggestions (Price et al., 2013).

In this process, the teacher as a decision maker plays an important role. Therefore, working with SMART requires a high level of teachers' professional competence. To define what we mean by *teacher professional competence*, we rely on two theoretical models. Blömeke et al. (2015) are modelling professional competence as a continuum, differentiating between dispositions, performance, and situation-specific skills. In the COACTIV model of teacher professional competence, Baumert and Kunter (2013) distinguish between four aspects of teacher professional competence: motivational orientations, beliefs/values/goals, self-regulation, and professional knowledge. In this paper, we focus on the professional knowledge which can be further divided into five domains: content knowledge, pedagogical content knowledge (PCK), pedagogical/psychological knowledge, organisational knowledge, and counselling knowledge. PCK is especially relevant when utilising a digital FA tool in a specific topic within a mathematics classroom. It can be realised in three facets: explanatory knowledge, knowledge of students' mathematical thinking, and knowledge of mathematical tasks. Combining these two models, we comprehend professional competence, and especially professionalisation, in this study as the application and enlargement of professional knowledge as a part of teachers' dispositions that leads to a reflection on or change in teachers' behaviour as performance.

While working with SMART demands teachers' professional competence, it can also contribute to teacher professionalisation. It offers didactical explanations on particular aspects of a topic (e. g. information on different roles of variables) supporting knowledge acquisition and reflection. Additionally, it offers matching teaching suggestions and material that may encourage teachers to reflect and adapt their teaching practices (Stacey et al., 2018).

Research Question and Methods

A first analysis of a few teacher interviews with the focus on tool elements used has shown that SMART contributes to teacher professionalisation (Eumann et al., 2024). But still little is known about the concrete forms of professionalisation that occur when using SMART. Therefore, this paper aims to explore the following research question: In which ways do teachers show professionalisation with regard to the application and enlargement of professional knowledge when they work with elements of SMART in different support settings?

To answer this question, we interviewed teachers after having used the tool in their lessons for the first time, implementing two tests on two different algebraic topics: substitution of values for letters in simple algebraic expressions and understanding the meaning of letters. The interviews are part of the project SMART[alpha] in which we investigate the algebraic thinking of students and the development of FA competencies of teachers while working with the tool. Teachers were randomly

assigned to three treatment groups, each receiving different forms of additional support: Group 1 (G1) received the fully automated evaluation of the SMART system, including the test items and their students' answers to each item, explanations about stages of understanding and misconceptions and teaching suggestions and material. After the implementation of the test, this group also took part in a PD program focusing on teaching algebra. In the first PD session, teachers were engaged in activities to learn about the different roles of variables and misconceptions that might occur during algebra lessons (PCK: knowledge of students' mathematical thinking). They also evaluated strategies and procedures for teaching algebra (PCK: explanatory knowledge) and tried different tasks related to algebra learning (PCK: knowledge of mathematical tasks). To investigate teachers' potential professionalisation using only the SMART tool, Group 2 (G2) received the fully automated evaluation of the SMART system, including the same elements as G1, but without an additional PD program. As a sort of control group, Group 3 (G3) received a reduced version of the SMART system with all the test items and their students' answers, but only feedback on the correctness of students' responses, without the fully automated evaluation, explanations, and material. Also, this group did not participate in any PD program. The interviews were conducted after the implementation of the two tests and for G1 after the attendance of the first PD session. In all interviews, we followed a guideline consisting of two main topics: teachers' interpretation of the test results and the consequences for their teaching drawn from it.

In the following, we present preliminary results, arising from an exemplary qualitative content-analysis of six teacher interviews, with two teachers from each treatment group[1]. These teachers worked in different regions in Germany at different secondary schools in grades 7 or 8. Before the test, all classes had worked on the meaning of variables and on how to formulate, manipulate, and substitute values for letters in an algebraic expression. Teachers worked with the SMART tool independently and autonomously in their classes and used the provided information at their discretion for lesson planning. Due to organisational issues, at the moment of the interview, some teachers had already taught some lessons after the SMART test, while others had not. Therefore, some teachers report on lessons they had already taught, while some describe only plans for their upcoming lessons.

Data analysis was conducted using qualitative content analysis. In a first step, we identified the central passages of the interviews, resulting from a previous analysis (Eumann et al., 2024). These passages were in a second step analysed in detail. Hereby, we used a deductive-inductive category system, where deductive categories arise from the material teachers had available (depending on their treatment group: items and students' answers, results, explanations on stages and misconceptions, teaching suggestions and material, insights from the PD session) as well as the facets of teachers' PCK (explanatory knowledge, knowledge of students' mathematical thinking, knowledge of mathematical tasks (Baumert & Kunter, 2013)). We inductively developed four categories indicating the different aspects of professionalisation, focusing on the form of reflection on or change in teachers' behaviour.

[1] The teachers are named by T, followed by the number of their treatment group and then numbered serially, e.g. T2.1 is the first teacher from G2.

Results

Results will be presented following the four aspects of professionalisation found during data analysis: explicit knowledge acquisition, reflection on tasks and learning activities, fostering diagnostic activities, and reflection on attitudes. Each aspect will be explained and illustrated with examples, hereby referring to the material mentioned by the teachers, the facets of PCK involved, and highlighting similarities and differences among the three treatment groups.

Explicit knowledge acquisition

The first aspect is realised only by teachers in G1 and G2 and predominantly, but not exclusively, in an unintended but conscious manner. All four teachers gained new PCK in the facet of *students' mathematical thinking*.

They state that they have gained new knowledge about possible misconceptions and students' ways of thinking referring to different material: While T1.2 and T2.1 draw this knowledge from the *explanations*, T2.2 relates to the *teaching suggestions* and T1.1 refers to the first *PD session*. As the teachers mentioned to be surprised by what they could learn from working with the tool, we concluded that they did not aim for gaining new knowledge, but noticed their own learning. Therefore, we name this an unintended but conscious knowledge acquisition.

Furthermore, the aspect of explicit knowledge acquisition also becomes apparent at the end of the interview with T2.2. He states that the *explanations* and the *teaching suggestions* were helpful for him and that he would use the tool again, but with constraints. For him, the decision to work with the tool depends on two factors: the subjective relevance and his own PCK of the concrete topic. This leads to the conclusion that he also actively and intentionally uses the tool to gain new knowledge.

Reflection on tasks and learning activities

The second aspect of professionalisation is shown in two forms: in a reflection on concrete situations and in general considerations on mathematics education practices. In both forms, teachers apply different facets of their PCK.

Both teachers from G2 apply *knowledge of mathematical tasks* when they report about the implementation of *teaching material* offered by SMART. T2.2 even adapted *explanations* of misconceptions and some questions of the *test items* for discussion in the classroom.

By reflecting on concrete classroom situations, teachers from all groups demonstrate a form of professionalisation in an expansion of their *explanatory knowledge*. T1.1 reports that at the beginning of the next lesson series on solving equations, he realised that particular students did not show the deep understanding of variables that is addressed by the SMART test. He takes this as an initial point to reflect on the conducted lesson, concluding that the introductory example might have been too complex. Relating to the first *PD session*, T1.1 reflects on the use of "fruit salad" algebra to explain simple algebraic transformations, concluding that it would improve his teaching to avoid this practice. T2.2 draws similar conclusions, reporting that he "felt caught" because he read about "fruit salad" algebra in the *teaching suggestions*. T1.2 reflects on his use of Cuisenaire rods for visualising algebraic expressions by areas and scopes of rectangles, concluding that this practice might encourage

misconceptions. Hereby, he draws on his own observations of the *test results*, as well as on the *explanations* and the discussion in the *PD session*: "and as the results were like this and as exactly this misconception was diagnosed for a lot of students, it came to my mind spontaneously in the last PD session that the Cuisenaire rods might not be meaningful." In addition, T2.2 describes that in future lessons he wants to emphasise more explicitly the different roles of variables. He sees a lack of students' understanding in this aspect as a probable reason for problems in the SMART test addressing the meaning of letters. Hereby, he relies primarily on the *teaching suggestions* provided by the tool.

Teachers in all groups also demonstrate an application of *knowledge of students' mathematical thinking*. T1.1 retraces a discussion about handling teaching material in the tool held in the *PD session*. He concludes that the main aspect of good mathematics teaching practice is not to provide students with opportunities for a lot of exercises and thus quickly develop procedural routines, but to initiate a deep understanding of mathematical concepts. Looking at the *results* of his student group, T3.2 comes to a similar consideration: "Now I recognise that in my own lesson, I introduced a procedure that maybe overloads them a bit. I have to take care that we focus on the meaning once again." T2.1 also relies on the general *results* of the group, using them as a source to prioritise ideas and concepts that he will focus on in future lessons. T2.1 also describes the reflections concerning his own language practice in classroom as an "aha-effect". He formulates the consequence that in future lessons he will pay more attention in the class to being a role model in the use of language and to pay more attention to the students' use of language as well.

Only T3.1 does not use his reflections on tasks and learning activities for his own professionalisation. He describes a similar situation as T1.2, using straws as Cuisenaire rods to visualise algebraic terms. But in contrast to T1.2, T3.1 does not reflect this practice as possibly promoting misconceptions. Instead, he states that this practice works well, showing that he is not aware of its underlying risk.

Fostering diagnostic activities

The third aspect of professionalisation is demonstrated in two different ways: comparing the results of the automatic diagnosis to expectations based on their own classroom observations and trying to understand the diagnostic potential of SMART.

When comparing the results of the automatic diagnosis to the expected outcome, teachers essentially focus on divergences in the *results* of the whole group or of individual students. In most cases, they try to find explanations for the observed discrepancies, providing short insights into their *knowledge of students' mathematical thinking*. Concerning the sort of explanations for discrepancies, especially for individual student results that are worse than expected, there are clear differences between individual teachers. On the one hand, T1.1, T3.1 and T3.2 only give superficial explanations, such as an unfavourable point in time for mathematics lessons, copying the neighbours' answers, or even students' characteristic traits: "That's also inattentiveness. Wanting to finish quickly, not reading adequately. Well, that's what he is like." Similar kinds of explanations are also given for individual results that are better than expected, such as luck or coincidence. On the other hand, T3.2 also tries to find reasons that show deep insights into students' thinking: "For that person, I would say that she did not understand exactly what p and b stands for." In a similar way, T2.1 and T2.2 give profound

possible explanations. They search for reasons in their own lesson practices as well as in the fact that students "do not rely on their own correct thinking and thus are sometimes misdirected by different response options." (T2.1)

This last statement already leads to the second way of fostering diagnostic activities, as all teachers (except T1.2) are trying to understand the diagnostic potential of the *items*, thus applying and enlarging their *knowledge of students' mathematical thinking*. While T1.1 only claims that for him it was quite obvious what the test items aim at, the other four teachers go far deeper in the analysis of particular items. T2.1 tries to retrace the automatic evaluation by trying to find connections between different response options in the *items* and the *explanations* about the stages of understanding and misconceptions, hereby taking into account students' thinking: "Well, I found that very fascinating. […] that you can somehow understand what could be the way of thinking that a student had." T2.2 also looks at some individual answers given by students to comprehend the automatic analysis. He looks deeply into the content of the items and the didactic information, which allows him to understand that it is not the number of correct answers that leads to a certain stage of understanding but the type of answers given. Of course, teachers from G3 could not analyse the results as deeply since they did neither receive the diagnosis of the fully automated evaluation nor the explanations about stages of understanding and misconceptions. Nevertheless, they also gained new knowledge by analysing the *items* and students' responses. T3.1 gains very deep insights into his students' thinking by investigating individual students' answers over several items in the test. In this process, he describes his observations very closely to the explanations in the stages of the tool. T3.2 also states that he seems to understand what the items aim at. By analysing the items and response options, he identifies that different typical errors and possible misconceptions are addressed. On the basis of some of his findings, he concludes that he has to change his own practice, especially concerning exact use of language when defining variables, and the necessity to clearly work out the meaning of variables. As a consequence, he sees the necessity to analyse students' responses more deeply to get reliable findings.

Reflection on attitudes

The fourth and last aspect shows a form of professionalisation that occurs aside from concrete facets of professional knowledge and is identified only in G1 and G2. Apart from the fact that all teachers in these two groups find it interesting to see the *results* and possible misconceptions, and are curious to get new results from other tests, they also reflect on their own habits. T1.2 realises that sometimes teachers have difficulties in anticipating particular misconceptions as they are very familiar with the mathematical topics. Moreover, T1.2 reflects on his attitude concerning the work with the *teaching suggestions* in the tool. He explains that even though it might be feasible for teachers to read all the explanations and choose adequate teaching material by themselves, he wishes that the tool would assign concrete tasks to individual students. He admits that this wish might be due to the fact that he did not engage enough with the material. T2.2 expresses a similar wish, explaining that an individual distribution of the proposed teaching material to students might be too time-consuming. He also shows a process of reflection of his work with the *teaching suggestions* and the *explanations,* reporting that, first, he was a little bit annoyed by the length of the texts, but then admits: "I also took a lot from it." T1.1 comes to the same conclusion, acknowledging that it is worth the time spent on

reading the teaching suggestions even if it seems quite a lot. He explicitly explains: “But in fact, I mean, it is basically your own attitude how you deal with it then. And if you want to gain something from it, then it is worth investing the time.”

Discussion and Outlook

In the analysis of six teacher interviews from three treatment groups working with the SMART tool in different support settings, we found four different aspects of professionalisation. All teachers apply different facets of their PCK (Baumert & Kunter, 2013) and use their insights to reflect or change their performance in mathematics teaching. Thus, teachers demonstrate professionalisation in different parts of the underlying continuum model (Blömeke et al., 2015).

Teachers in G1 and G2 show widely similar forms of professionalisation in all four aspects. In both groups, the aspect of *reflection on tasks and learning activities* is predominant. But we see that teachers recapitulating an activity or a discussion in a *PD session* (G1) might draw the same conclusions for their teaching practices as teachers only accessing *teaching suggestions* from the SMART tool (G2). Only teachers from G2 show an application and enlargement of the facet of *knowledge of mathematical tasks*. The facets of *explanatory knowledge* and *knowledge of students' mathematical thinking* are shown by teachers from both groups, even though they refer to different material.

Meanwhile, teachers from G3, that do not receive the information from the tool, show very different forms of professionalisation. In this group, only the aspects of *reflection on tasks and learning activities* and *fostering diagnostic activities* could be found, with a predominance of the latter one. An intensive analysis of the *test items* allows these teachers to gain deep insights into students' thinking. Furthermore, conclusions about good mathematics teaching and the necessity to foster students' understanding of concepts instead of setting priority to procedures are drawn from interactions in the first *PD session* (G1) as well as from a deep analysis of the *items* of the SMART tool even without the automatic evaluation and the supplementary explanations (G3).

Therefore, we may conclude that an intensive investigation of well-designed and research-based items of a diagnostic tool allows for an application and enhancement of PCK, especially in the facet of *knowledge of students' mathematical thinking*. But for professionalisation that includes more facets of PCK and allows for different forms of reflection on and change in behaviour, more support is needed. The preliminary results of this study indicate that the additional elements in the SMART tool might have similar potential for such forms of professionalisation as the attendance in the accompanying PD session. In addition, the engagement with the *explanations* in the SMART tool allows for reflective processes that go beyond concrete practices. Therefore, the use of this tool might offer a low-threshold option for teachers' professionalisation beside PD sessions (cf. Cusi et al., 2024).

Due to the small number of interviews analysed per group, these results are preliminary. To confirm these findings, an analysis of further teacher interviews is needed. Moreover, further analyses could also focus on other aspects, such as teachers' lesson plannings or realisation of FA practices and their interconnections with professionalisation processes. As the results of this study are snapshots that are based on self-reported practices, there is also a need for further research investigating teachers'

application of professional knowledge by directly observing their work with SMART and the implementation of the practices in their classrooms.

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Learning to assess and respond to children's mathematical thinking in early arithmetic education: a vignette-based study with pre-service teachers

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A central goal in early arithmetic education is to develop a solid conceptual understanding of numbers and operations. For this, teachers face the challenging tasks of assessing children's mathematical thinking and of responding to it in an adaptive way. Essential prerequisites are the abilities to form precise diagnostic judgments about children's existing number and operation concepts and to make adaptive decisions for further instruction. With a vignette-based research instrument and N=172 participants, this study investigated the quality of pre-service teachers' diagnostic judgments about children's mathematical thinking, and how these judgments were related to subsequent instructional decision-making. The results for example show that pre-service teachers' tend to formulate general and imprecise statements in their diagnostic judgments about children's mathematical thinking leading to difficulties in choosing the next adaptive task for a child.

Keywords: Adaptive teaching, diagnostic judgment, early arithmetic education, teacher education, vignettes.

Introduction

Children's heterogeneous learning requirements are a central characteristic of primary school education (Liebers, 2023). Among other things, selecting and designing adaptive learning settings based on children's individual learning requirements is an essential prerequisite for successful learning (Beck, 2008). This also applies to understanding-oriented early arithmetic learning settings, which aim to ensure that every child acquires solid basic concepts of numbers and operations. Therefore, teachers first must precisely assess children's mathematical thinking to be able to offer adaptive learning settings in subsequent instruction. This emphasises the need for teachers' abilities to make diagnostic judgments and instructional decisions (Brandt, 2022; Schulz, 2014; Streit et al., 2019). However, supporting children in developing their mathematical understanding from counting to solid concepts of numbers and operations is a challenging task (Anghileri, 2006). Due to uncertainties in assessing children's concepts and making subsequent instructional decisions, German primary school mathematic teachers do not yet seem to be prepared to cope with this task to a satisfactory extent (Hagena & Besser, 2023). National surveys have repeatedly shown that a significant proportion of children in Germany do not have an adequate understanding of numbers and operations at the end of primary school (e.g., Stanat et al., 2022). Consequently, the question arises on how necessary changes can be brought about in early arithmetic education and how teacher education and related educational research can contribute. Since studies have shown that adaptive

teaching in early arithmetic education is characterized by an interplay between assessing children's mathematical thinking and designing adaptive, understanding-oriented learning settings (Moser Opitz, 2022), current efforts focus on future teachers' abilities to form diagnostic judgments and to make instructional decisions (e.g., Brandt, 2022).

Theoretical background

There is a variety of different terms and discussions around the abilities that teachers need to form precise diagnostic judgments and to make instructional decisions for adaptive teaching. Three interrelated but distinct abilities have emerged, on which there is widespread agreement: based on one's professional knowledge, important information must be perceived in a targeted and selective manner, then interpreted, and finally a decision must be made (Enenkiel et al., 2022; Jacobs et al., 2010; Loibl et al., 2020; von Aufschnaiter et al., 2018). The first two abilities are needed to form a diagnostic judgment, and the last ability is needed to make an instructional decision. Applied to the task of designing adaptive early arithmetic learning settings, teachers must use their professional knowledge to

- *precisely identify children's thinking as shown in their approaches* when solving an arithmetic task and *interpret the identified approach regarding specific underlying concepts* of numbers and operations (diagnostic judgment formation)

- *make an adaptive instructional decision* to build on children's existing concepts (instructional decision-making).

Regarding the connection between the processes of diagnostic judgment formation and instructional decision-making, Moser Opitz (2022) assumes that they are closely related, and, from an analytical point of view, the diagnostic judgment formation precedes the instructional decision-making. In favour of considering the two processes separately, she also argues that teachers need subject-specific didactic knowledge about learners to form precise diagnostic judgments and need additional knowledge about understanding-oriented instructional methods to make adaptive instructional decisions. However, van Ophuysen (2010) emphasises a reciprocal interplay between the formation of diagnostic judgments and instructional decision-making. She argues that knowledge about instructional methods does not only become relevant during instructional decision-making but also directs teachers' perception and interpretation during judgment formation.

Despite the high relevance of teachers' abilities to form diagnostic judgments in connection with instructional decision-making for adaptive teaching in early arithmetic education, studies in this field are still scarce. Especially the interplay between diagnostic judgment formation and instructional decision-making in early arithmetic education is not yet fully understood (Streit et al., 2019). As far as we know, only Brandt (2022), Jacobs et al. (2010), Schulz (2014) and Streit et al. (2019) have explicitly investigated this complex interplay. Regarding the abilities of pre-service teachers, the findings of these studies indicate that pre-service teachers often fail to precisely identify children's approaches, to interpret underlying, specific concepts and to make appropriate instructional decisions for adaptive learning settings. Moreover, comparisons of the three abilities mostly show a decrease in capability from identifying to interpreting and again to decision-making. Although all three abilities have been described as interrelated, the connections appear to be only weakly developed. In

particular, there seems to be a lack of coherence between pre-service teachers' diagnostic judgments and instructional decisions.

Due to the few and predominantly qualitative studies with a small sample or only one vignette as a research instrument, the results are still limited. In order to substantiate the findings, further studies with larger sample sizes are needed, which also enable quantitative evaluation methods such as correlation analyses. Therefore, in the present study we used a vignette-based research instrument with four vignettes and a sample of N=172 pre-service teachers to further investigate the quality of and interplay between pre-service teachers' diagnostic judgment formation and instructional decision-making. The main research questions guiding this work are:

1) How well do pre-service teachers succeed in forming diagnostic judgments about children's mathematical thinking in early arithmetic education? 2) How appropriate are their instructional decisions to build on children's existing concepts? 3) How are pre-service teachers' diagnostic judgments related to their subsequent instructional decisions?

Sample, design, and methods

For answering the research questions N=172 pre-service teachers at the beginning of their bachelor's degree studies were asked to analyse four purposefully designed and structurally identical text-image-vignettes in an online questionnaire, leading to n=688 data sets (see Wirth et al., 2023 for more details about the vignette design). For this study, vignettes as representations of practice with reduced complexity (compared to videos or even real classroom situations) where chosen to fit the beginning stage of pre-service teachers emerging expertise. In advance, all participants have attended a one-semester course covering the following topics: teaching and learning basic concepts of numbers and operations in early arithmetic education. They were provided with subject-specific didactic knowledge about learners and understanding-oriented instructional methods and had been given ample opportunities to apply this knowledge when analyzing vignettes, tasks, and textbook material. Each vignette started with a short introductory description of a classroom situation (Figure 1).

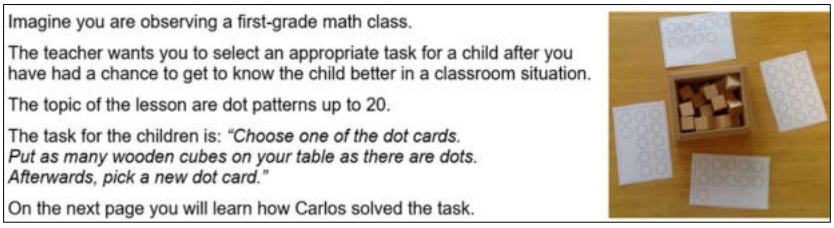

Figure 1: Introductory description of a classroom situation

The introduction was followed by information about one child's approach to solve an arithmetic task (Figure 2). The open-ended question for the participants as a stimulus to form a diagnostic judgment was: *Based on the information available, what can you say about Carlos's mathematical thinking related to his number concepts?*

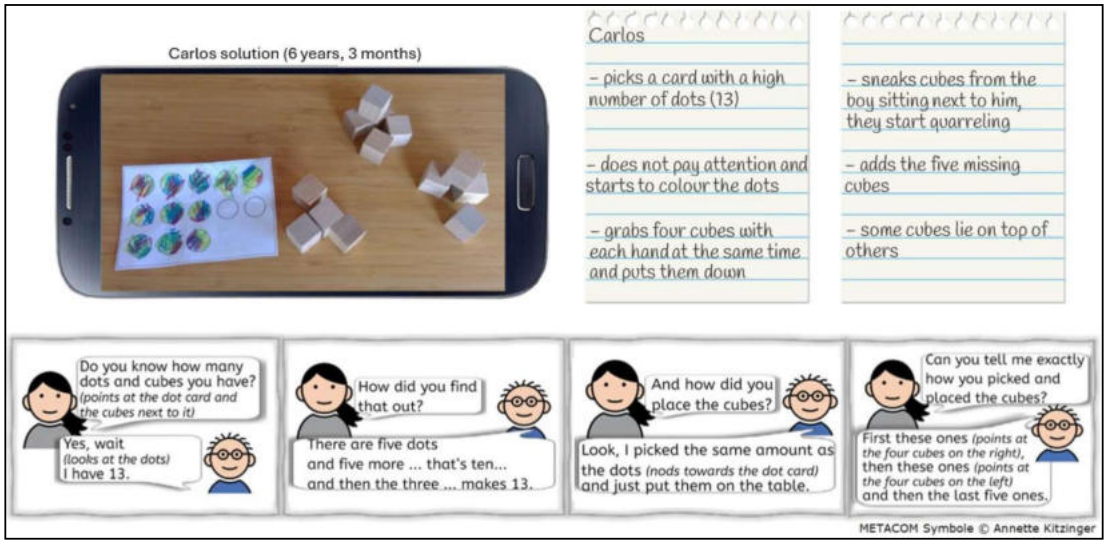

Figure 2: Information for diagnostic judgment

Finally, three suggestions for tasks were provided (Figure 3) and the partly open-ended stimulus for participants' instructional decision-making was: *Now please decide on a task that you would like Carlos to do next. Give reasons for your choice.*

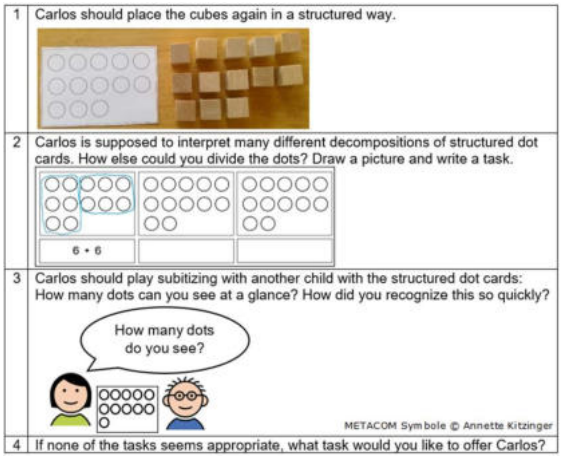

Figure 3: Task suggestions for instructional decision-making

While working on the vignettes, the two tasks: forming a diagnostic judgment and making an instructional decision could only be worked on one after the other, as it was not possible to go back to a previous page in the online questionnaire.

In order to develop an expert-rating for data analysis, all parts of the vignettes (here: Figures 1 to 3) were analysed by 14 researchers in the field of primary mathematics education regarding diagnosing and fostering children's basic concepts of numbers and operations. For example, the expert norm developed for the "Carlos vignette" was:

> Carlos recognised the structure of the dot pattern as 5+5+3 and captured the quantity without counting. He then used a different decomposition of the number 13 (4+4+5) to place the dice. Carlos can therefore recognise the structured amount of dots through conceptual subitizing and

has a solid part-whole concept. Task 2 would be an adequate choice for Carlos to work on next to strengthen his already existing understanding of part-whole relations.

Based on each expert norm deductive-inductive coding guidelines (Rädiker & Kuckartz, 2019) were created for the qualitative content analysis of pre-service teachers' 1) diagnostic judgments and 2) instructional decisions. 30% of the data was double coded by two independent raters and the interrater agreement according to Landis & Koch (1977) was at least substantial and more often almost perfect. Disagreements were systematically discussed regarding consequences for the coding guidelines. The coding was then used for a four-level categorical classification of the diagnostic judgment quality as well as the quality of the instructional decision (Figure 4).

To analyse the interplay between diagnostic judgment formation and instructional decision-making, chi-square tests according to Pearson were carried out in SPSS. The preconditions were met for all tests carried out, as the data originated from a random sample, the measurements were independent from each other and all expected frequencies per cell were greater than five. If a correlation was found, the effect size was calculated using Cramer's V and interpreted according to Cohen (1988).

Findings

Figure 4 presents an overview of the findings related to pre-service teachers' diagnostic judgment quality and the quality of their instructional decisions (related to research questions 1 and 2). The frequencies of diagnostic judgment quality indicate that pre-service teachers' judgments including precise statements about a child's approach and specific statements about underlying concepts are rather exceptions (code "4" in Figure 4 left). In a majority, pre-service teachers' judgments include only imprecise statements of a child's approach or unspecific statements of underlying concepts (code "2" in Figure 4 left). Here is a sample answer:

> "Carlos number concept could have improvements; he understands it to some extent as he is able to calculate the dots on the sheet by making 5+5+3."

In this judgment the statement about Carlos' approach is imprecise, because it is left unclear how Carlos captured the amount of dots and there is no statement about the way in which he placed the dice. Additionally, the statement of concepts he is capable of is rather general.

The frequencies of quality of instructional decision indicate that although follow-up tasks were suggested only in 55% of the cases an adaptive task was chosen. Moreover, only in 38% of cases the reason for the choice of an adaptive task was also adaptive regarding the children's existing concepts.

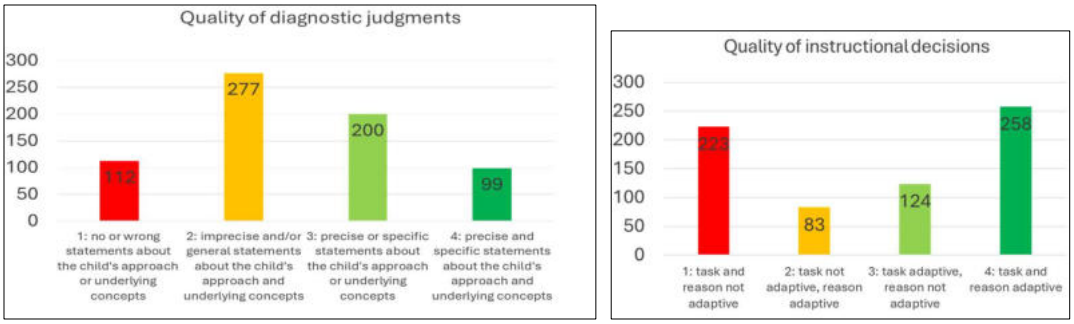

Figure 4: Frequencies of PSTs' diagnostic judgment quality and quality of instructional decisions

Figure 5 presents an overview of the findings related to the interplay of pre-service teachers' diagnostic judgment formation and instructional decision-making (research question 3). For example, in 47% (141) of judgments with higher quality and 30% (117) of the judgments of lower quality, an adaptive task with an adaptive reason for the choice was selected. A significant correlation was found between the diagnostic judgment quality and the quality of the instructional decision. The strength of the correlation lies between a small and medium range ($\chi^2(3, N=688)=29.430$, $p<.001$, $V=.207$).

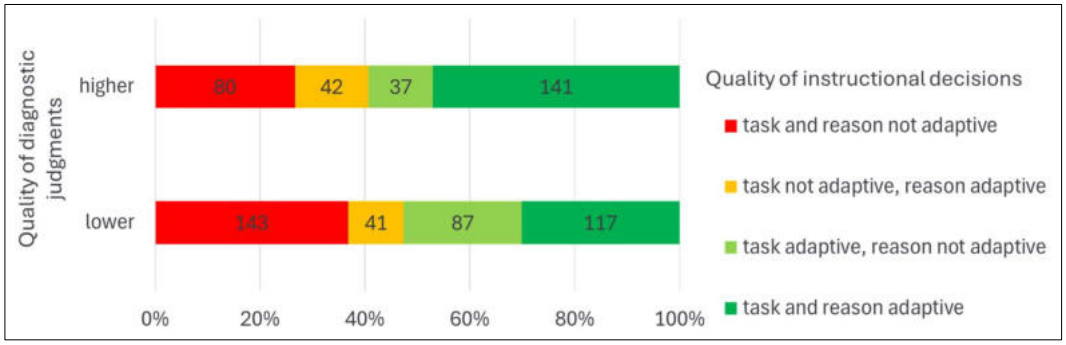

Figure 5: Frequencies of the interplay between pre-service teachers' judgment formation and instructional decision-making

Discussion and conclusions

Considering the limitations of the study (e.g., a limited representation of practice through a vignette-based instrument) the findings show that pre-service teachers' abilities to form precise diagnostic judgments about children's existing number and operation concepts and to make adaptive instructional decisions need to be fostered. Although a connection between pre-service teachers' abilities to form diagnostic judgments and to make an instructional decision could be found, the strength of the correlation was weak. The findings therefore strengthen the findings of Brandt (2022), Jacobs et al. (2010), Schulz (2014), and Streit et al. (2019) as well as the assumption of Moser Opitz (2022) that diagnostic judgment formation and instructional decision-making are separate processes.

Regarding the interplay between diagnostic judgment formation and instructional decision-making and the role of knowledge about understanding-oriented instructional methods the findings also strengthen van Ophuysen's (2010) assumption that it is reciprocal. For example, after being introduced to the task suggestions a considerable number of pre-service teachers were able to give an adaptive reason for their task choice although their diagnostic judgments were of lower quality. It can be assumed that these participants used the task suggestions as sources of information not only for the instructional decision but also for a final diagnostic decision. Maybe they noticed that no task was offered, which was adaptive to their previously formed diagnostic judgment. This may have triggered the diagnostic process again and even changed the diagnostic judgment. Due to the methodological design of the study, the participants had no opportunity to return to their previously formulated diagnostic judgments to adapt them during their instructional decision-making. They were only able to express their final diagnostic decision as the reason for their task choice. By this, the findings contribute to a better understanding of the interplay between pre-service teachers' diagnostic judgment formation and instructional decision-making in early arithmetic education and provide

important information for the education of future primary school mathematic teachers. Moreover, as the study is limited to the subgroup of pre-service teachers at an early stage of emerging expertise, the findings encourage further research about the interplay of diagnostic judgment formation and instructional decision-making of teachers in more advanced stages of expertise. To conclude, assessing and responding to children's mathematical thinking in an adaptive way is not an easy endeavor for pre-service teachers. Therefore, already in teacher education, future teachers should be provided with ample opportunities to develop diagnostic judgment abilities in connection with forming adaptive instructional decisions (Wirth et al., 2023) – an important prerequisite for all primary school children to be able to acquire a solid understanding of numbers and operations.

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Examining the lesson plan designs and reflections of pre-service elementary teachers regarding mathematics teaching

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Recognizing the importance of lesson planning in effective instruction, this study examines how pre-service elementary teachers design lesson plans and the challenges they face. The study involved 60 teacher candidates from a public university in Turkey and employed document analysis of lesson plans and reflective writings from a "Mathematics Teaching" course. Findings show that participants used varied strategies such as storytelling and real-life contexts to engage students. In the Explanation and Elaboration phases, they emphasized student-centered and collaborative activities. During evaluation, they incorporated both traditional and innovative tools including games and Web 2.0 applications. Participants also reported concerns related to classroom management, content transitions, and designing assessments. These reflections underscore the need for more structured support in teacher education programs.

Keywords: Lesson planning, pre-service elementary teachers, mathematics teaching.

Introduction

The quality of mathematics instruction is influenced by several factors, including teacher knowledge, instructional strategies, and the organization of the teaching process (Kahan et al., 2013). A critical yet often overlooked aspect of teaching is the preparation that occurs outside the classroom, specifically lesson planning. Lesson planning not only enhances instructional effectiveness (Rusznyak & Walton, 2011) but also plays a crucial role in the overall teaching-learning process, improving both time management and the efficiency of instruction (Krepf & König, 2022). The act of lesson planning is central to the teaching process, and research has shown that teachers face numerous challenges during this phase. These challenges, such as classroom management, transitioning between topics, and structuring complex mathematical concepts, can significantly influence the effectiveness of the lesson (Derri et al., 2014). These difficulties are particularly evident in the context of pre-service teachers, who are still developing their pedagogical skills. While many studies focus on the challenges teachers face during lesson planning, fewer have examined how structured instructional models, such as the 5E Model (Engagement, Exploration, Explanation, Elaboration, Evaluation), can help pre-service teachers navigate these challenges. The 5E Model has been widely recognized for its effectiveness in promoting student-centered teaching and providing a systematic approach to lesson planning. However, despite its potential, research on how pre-service teachers apply this model in real-world classroom settings remains limited. This study builds on the work of prior research by examining how pre-service elementary teachers design lesson plans for mathematics instruction using the 5E Model. The research also investigates the experiences and challenges they encounter during this process. By integrating findings from previous studies on lesson planning and instructional models, this research contributes to the growing body of literature on pre-service teacher preparation. Specifically, it aims to address gaps in understanding how structured instructional frameworks influence lesson design and how pre-service teachers manage the

complexities of lesson planning. The study is guided by two primary research questions: “How do pre-service elementary teachers design lesson plans for mathematics teaching according to the 5E Model?” and “What concerns and challenges do they face when designing these plans?” In this study, the 5E instructional model functioned not only as a pedagogical design tool but also as a framework for categorizing and interpreting the lesson planning practices of PSTs (Pre-Service Teachers). While its primary function was instructional, its structural stages provided an analytical lens through which reflections and lesson plan components could be examined systematically. This dual role of the model—as both a teaching scaffold and a methodological lens—offered deeper insight into the planning process, highlighting the multifaceted ways in which structured instructional models can support both practice and research. Theoretical frameworks such as the 5E Model provide a foundation for examining the intersection of pedagogical content knowledge, teacher anxiety, and the organizational aspects of lesson planning. By aligning this study with the broader literature on instructional design and teacher preparation, the research underscores the significance of supporting PSTs in overcoming the challenges they face during the lesson planning process. Furthermore, the study offers practical insights for teacher education programs, providing strategies for better equipping future educators to develop effective, student-centered lesson plans that align with evidence-based instructional models.

Methods

This study employs document analysis and was conducted with 60 PSTs enrolled in the third year of a primary education program at a public university in Turkey. Document analysis involves collecting, reviewing, questioning, and analyzing various written or electronic texts as the primary source of research data (Morgan, 2022). The data were collected from lesson plans prepared by the pre-service teachers, based on the 5E model, as part of the "Mathematics Teaching" course, prior to their sample teaching practices. The course was structured in two phases: theoretical and practical. During the first 8 weeks, pre-service teachers were provided with content related to elementary mathematics teaching (theoretical frameworks, lesson planning, teaching and learning approaches, etc.) by the mathematics teacher educator (MTE). Over the next eight weeks, during the first session of each week, the MTE presented an example of instructional design related to an elementary mathematics topic. In the second session, the teaching performance of the pre-service teachers was observed. Before conducting their teaching performance, each pre-service teacher submitted their prepared lesson plan electronically to the MTE. Additionally, the pre-service teachers also shared their reflective writings, in which they described their experiences while preparing the lesson plan, via electronic submission. All data were categorized according to the elementary mathematics learning domains (Numbers and Operations, Geometry, Data Analysis, Measurement). Lesson plans and reflective writings were randomly selected from each learning domain. This study includes the analysis of documents from 12 pre-service teachers. In the findings, each participant was assigned a number from 1 to 12. Pre-service teachers were coded as PSTx, where "x" represents the number assigned to each teacher candidate. The documents, including the lesson plans and reflective writings, were systematically examined and evaluated. The raw data were initially divided into meaningful units, then coded and categorized as explained by Patton (2002) to gain an understanding of the research problem. In this study, we analyzed the lesson plans of pre-service teachers by dividing them into three dimensions:

i) Engage, Explore, ii) Explain, Elaborate, and iii) Evaluate. Sub-codes related to the structure of each dimension were identified. Content analysis was conducted on the reflective writings of the pre-service teachers, and sub-codes were created for each category.

Findings

Structure of Pre-service Elementary Teachers' Lesson Plans

In this section, findings regarding the structure of the lesson plans designed by pre-service elementary teachers will be presented, organized by dimensions. The structure of each dimension is provided in Table 1 below.

Table 1: PSTs' lesson plans and their structure

Dimensions	Structure
Engagement and Exploration	Activating prior knowledge
	Relating to real life
	Gamification
	Storytelling
	Singing
	Exercises
	Assigning tasks
Explanation and Elaboration	Mathematical thinking activities (e.g., prediction, problem formulation, transformation, comparison)
	Student-centered activities
	Group work
	Alternative activities
Evaluation	Traditional assessment tools (e.g., tests, fill-in-the-blanks, matching)
	Game-based assessment
	Using Web 2.0 tools

Engagement and Exploration Dimension

At the beginning of the lesson, pre-service teachers aimed to activate prior knowledge by posing questions related to earlier topics. For example, PST1 described entering the class with a balance scale and a bathroom scale, asking students questions like: "What are these in my hands?" and "What do we use them for?" After briefly discussing the answers, PST1 would ask: "Do you remember? Last year, we measured our weights in class. Shall we see how much weight we've gained?" This approach aimed to engage students and refresh their previous knowledge about measurement before introducing new content, such as grams. Some prospective teachers used storytelling to explorate students for the lesson. PST6, for instance, entered the classroom with a basket of apples and asked the students to identify a picture (see Figure 1) from the story "Snow White and the Seven Dwarfs."

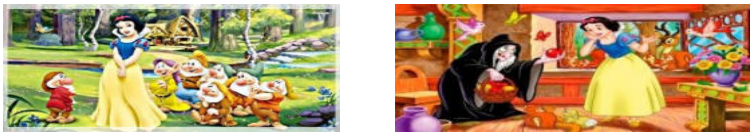

Figure 1: The image used in the story of PST 6

After recounting the story with the students, PST6 proposed changing the ending: "Let's say the witch is kind-hearted and gives Snow White a basket of apples." Each student was given an apple, and the teacher asked, "How many apples did everyone eat?" and "How many are left in the basket?" This led to a discussion about division, helping students recall a prior lesson while setting the stage for the new content.

Explanation and Elaboration Dimension

All pre-service teachers incorporated student-centered activities in the explanation and elaboration dimension of their lesson plans. While some activities were designed for individual work, group work was the predominant focus. Most lesson plans also included mathematical thinking activities such as estimation, problem formulation, transformation, and comparison. For example, PST 7 designed a lesson plan for a 4th-grade class on recognizing and modelling fractions (Ministry of Education, 2018). In this plan, students were given materials (Figure 2) and asked to complete a skeleton fish activity. Using three skewers labelled with the fractions 1/10, 1/2, and 1/5, students compared the fractions and arranged them from largest to smallest on a number line, represented by the skeleton fish. Additionally, the pre-service teacher included alternative activities for modelling fractions, such as group work using the puzzle technique. Students, divided into pairs (assuming a class size of 30), were tasked with assembling four puzzle pieces that corresponded to a specific fraction, its area model, its number line representation, and the type of fraction (Figure 2).

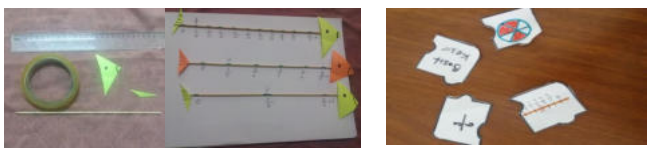

Figure 2: Example of task designed by PST 7 and Alternative task example designed by PST 7

Some pre-service teachers also designed activities to enhance students' problem-formulation skills. For instance, PST1's lesson plan for 3rd-grade students on measurement included an activity where

students, in pairs, were asked to create and solve two problems based on images (Figure 3) shown by the teacher. An example from PST1's plan is as follows:

Figure 3: Image of PST 1's problem-posing task

Problem: Aunt Hatice goes to the market and buys 550 grams of lentils, 300 grams of flour, and some red pepper. If she has a total of 1200 grams of goods, how much red pepper did she buy?

Solution: By adding the weight of the lentils and flour, and subtracting it from the total, we can find the weight of the red pepper. 550 + 300 = 850 grams. 1200 - 850 = 350 grams of red pepper.

Evaluation Dimension

Five of the pre-service teachers used traditional assessment tools in their lesson plans. They distributed individually prepared or pre-made worksheets to students, which included tools such as true/false, fill-in-the-blank, and multiple-choice questions. Some pre-service teachers, however, opted for game-based assessments. For example, in a lesson plan on place value for a 2nd-grade class, PST5 designed a BINGO game. Students picked a card (e.g., a pink one) and placed a LEGO piece on the corresponding number on their BINGO card if they correctly identified the number. The first student to correctly identify all the numbers and complete their BINGO card won the game (Figure 4 illustrates the BINGO setup).

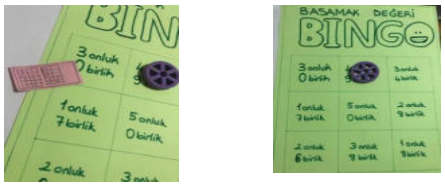

Figure 4: Game-based evaluation example of PST 5

Additionally, some pre-service teachers incorporated Web 2.0 tools in their assessments. For instance, in a 1st-grade lesson plan on geometric patterns, students were expected to distinguish between patterns and non-patterns, find the rule of the pattern, and complete missing parts of the pattern. PST12 used the LearningApps platform, where students took turns answering questions at the board. This interactive activity enabled the teacher to assess students' understanding of the topic (activity links: https://learningapps.org/watch?v=pzeji5qz524).

Experiences of Pre-service Elementary Teachers in Lesson Planning

Pre-service teachers who designed their lesson plans individually were asked to share their experiences alongside their plans. It was found that they faced several challenges and held certain concerns throughout the lesson planning process. These challenges and concerns are illustrated in Figures 6.

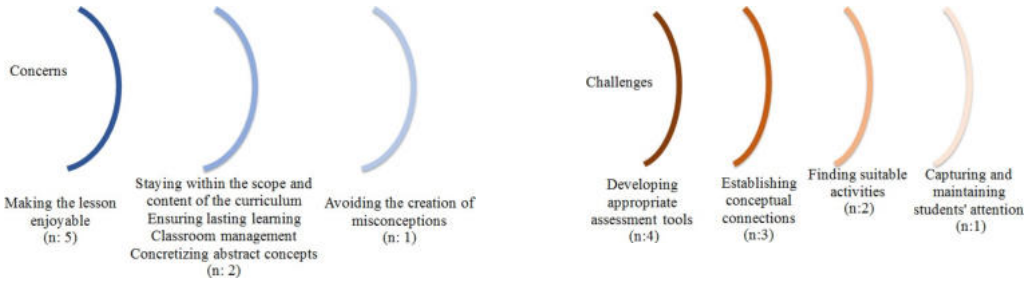

Figure 6: Concerns and challenges experienced by pre-service teacher during the lesson plan preparation process

When designing their lesson plans, pre-service teachers frequently expressed concerns about "how to make the lesson enjoyable." For example, PST6 shared: “I believe in learning through fun because my elementary years were filled with strict, sometimes threatening teachers. I never want to be like them. That’s why I aimed to create a plan where students aren’t afraid to speak up and are eager to share their ideas." Additionally, they mentioned concerns about staying within the scope of the curriculum, ensuring lasting learning, managing the classroom, and making abstract concepts concrete. Sample statements for these concerns are as follows. Staying within scope: “PST1: I noticed that many videos and activities I found online went beyond the learning objectives, so I did a lot of research to ensure I stayed on track.” Ensuring lasting learning: “PST7: I used multiple activities to explain fractions because I believe variety helps students retain information better.” Classroom management: “PST4: I might struggle with managing the class during my activities. The students could get too excited, and I’ll need to put extra effort into keeping them on track during group work.” Making abstract concepts concrete: “PST12: Since my target audience is 1st graders, I focused on using many visuals to make the lesson more concrete for them.” In addition to these concerns, the pre-service teachers also reflected on the challenges they faced while preparing their lesson plans. The most common difficulty was creating appropriate assessment tools. They also struggled with transitioning between mathematical concepts, finding suitable activities, and capturing students’ attention at the start of the lesson. Some examples include: Conceptual transitions: “PST10: After reviewing the frequency table and object graph, I struggled with how to introduce the bar graph. I wasn’t sure how to explain it, so I opted to transition through the object graph.”

Result & Discussion

This study offers significant insights into the lesson planning experiences of PSTs within the framework of the 5E Model, particularly in mathematics instruction. The findings reveal both the strategies employed by these PSTs and the challenges they encountered, each of which provides opportunities for deeper reflection on teacher education and professional growth. One of the key findings highlights the importance of the Engagement phase of the 5E Model, where PSTs used strategies such as storytelling and real-life connections to activate students’ prior knowledge and increase motivation. This aligns with the literature that emphasizes the role of engaging students early in the lesson to foster interest and a deeper understanding of the subject matter (Livy & Downton, 2018). By incorporating techniques that make lessons more relatable and enjoyable, PSTs aimed to create a positive learning environment, which is crucial for effective teaching. However, the

challenges faced during lesson planning—particularly in the areas of maintaining scope, ensuring conceptual transitions, and creating effective assessment tools—highlight areas where PSTs need more support. For instance, many PSTs expressed concerns about staying within the curriculum while still making the lesson enjoyable. As PST1 noted, the fear of going beyond the curriculum goals led to extensive research to ensure alignment with learning objectives. This finding is consistent with prior research, which suggests that staying within the curriculum while designing engaging activities is a common struggle for teacher candidates (Arias et al., 2017).

The findings also indicate that PSTs found it difficult to design assessments that accurately reflected student learning. While some relied on traditional methods, others explored more innovative approaches, such as game-based assessments and Web 2.0 tools. This diversity in assessment approaches reflects an ongoing shift in teaching practices, where traditional methods are complemented by interactive tools to engage students better and assess their learning in more dynamic ways (Matsumoto-Royo & Ramírez-Montoya, 2021). Teachers' ability to diversify their assessment strategies is crucial, as it allows them to meet the needs of students with different learning styles and preferences. Another critical challenge was the difficulty in making conceptual transitions between mathematical topics. As PST10 expressed, transitioning from one concept to another, such as moving from object graphs to bar graphs, requires thoughtful planning and creativity. This echoes the findings of Thompson (1984), who identified that teachers often struggle with creating smooth conceptual links between topics. Overcoming this challenge requires a deeper understanding of the interconnectedness of mathematical concepts and the ability to scaffold students' learning effectively. Importantly, these challenges should not be perceived solely as negative aspects; they also present opportunities for growth and improvement. Rather, they can be interpreted as reflective indicators of PTSs' active engagement in their professional development. The articulation of concerns and difficulties signifies a developmental process in which future teachers critically examine their practices, experiment with solutions, and cultivate pedagogical resilience. In several cases, participants were observed to spontaneously generate solutions to the challenges they identified, indicating not just awareness, but also adaptive professional behaviour.

As noted by Darling-Hammond and Bransford (2007), challenges in teaching often serve as catalysts for professional learning, pushing teachers to reflect on their practices and seek innovative solutions. Furthermore, the findings suggest that several pre-service teachers not only articulated concerns but also engaged in reflective practices that led them to devise their strategies and tentative solutions. For instance, while some participants voiced uncertainty about managing group dynamics or transitioning between abstract concepts, their reflective writings revealed subtle shifts in thinking that pointed toward practical adaptations. This duality reflects an important aspect of professional growth. It indicates that PSTs are not passive recipients of training but are actively shaping their learning through critical self-assessment and emergent problem-solving. This underscores the need for teacher education programs to recognize and build upon these spontaneous strategies as evidence of developing professional agency. Finally, the concerns and challenges expressed by pre-service teachers throughout the lesson planning process should not only be seen as developmental hurdles but also as valuable diagnostic tools for teacher education programs. These reflections, grounded in real-world classroom preparation experiences, offer genuine insight into areas where novice teachers

require support. Therefore, systematically collecting and analyzing such concerns can inform the development of more targeted, responsive, and context-specific professional development modules. Integrating PST's lived challenges into training designs ensures that these programs are aligned with the actual needs of future educators, enhancing both their preparedness and confidence.

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The Reflective Gaze: A Case Study on Eye-Tracking Technology's Role in Shaping Mathematics Teacher's professional identity

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Keywords: Eye tracking, mathematics education, teacher professional identity, reflective practice, professional development

Rationale

A teacher's professional identity development is a complex process influenced by various experiences and reflective practices. Teachers' professional identity comprises five interconnected components: knowledge, emotions, beliefs, confidence, and personal principles (Looney et al., 2017). Current research in mathematics education has established the value of eye-tracking methodology for understanding cognitive processes (Strohmaier et al., 2020). The relationship between visual attention and mathematical thinking has been well-documented (Schindler & Lilienthal, 2019), particularly in how eye movements reveal underlying cognitive strategies and problem-solving approaches. However, the potential role of eye-tracking in professional identity development still needs to be explored. This study takes a novel approach by integrating eye-tracking technology with reflective practices, offering a unique opportunity to make cognitive processes visible and concrete for teachers. This study, conducted as part of a master's program, combines these technological capabilities with Looney et al.'s teacher professional identity framework (2018) to create a unique method for supporting identity development through technology-enhanced reflection. This is especially important in mathematics education, where grasping cognitive processes is vital for effective teaching and assessment.

Research Aims and Methods

This case study examines how eye-tracking technology and stimulated recall interviews impact a mathematics teacher's professional identity development. The primary research question asks: How visualisation of and reflection on cognitive processes, captured through eye-tracking, influence various aspects of a teacher's identity. I analysed a single participant, Dana (pseudonym), an elementary mathematics teacher with eight years of experience, to examine the impact of eye-tracking on professional identity components as outlined by Looney et al. (2017). The eye-tracking session lasted 363 seconds and was part of a professional development framework. The research design used multiple methods: eye-tracking data during a geometric task with a Pupil Labs device, immediate recall interviews while watching recordings, written reflections on the experience's implications for teaching practice, and a follow-up interview after one week to assess sustained impact. Data analysis combined quantitative measures from eye-tracking data with thematic analysis of interviews and reflective writing. The interviews lasted approximately 45 minutes each, and analysis included both deductive categories based on Looney et al.'s (2017) framework and inductive coding to capture emergent themes, allowing exploration of how the experience impacted each.

Results

The eye-tracking data revealed complex patterns of problem-solving. During the 363-second solving period, Dana made a total of 1,493 fixations (4.11 fixations per second), with an overall entropy value of 4.178, indicating systematic exploration patterns. Though the active eye-tracking experience was relatively brief, the subsequent reflective processes—including stimulated recall, written reflection, and follow-up interviews—created an extended learning experience that facilitated deeper integration into professional practice. The initial 15 seconds 59 fixations (entropy of 3.21), suggesting intensive early problem space exploration. Dana's reflection demonstrated enhanced metacognitive awareness: "The experience sharpened all the theoretical knowledge we learned during the course", illustrating how abstract concepts became concrete through visualization. The data revealed an emotional trajectory throughout the experience. Initial anxiety was evident in statements such as "You are stressing me out", while later reflections showed professional growth: "Wow, amazing. You gave me therapy". Dana's beliefs about teaching and learning evolved through this experience. Her reflection, "It is important to remember the diversity in the classroom, that every brain is different", was supported by observed variations in her problem-solving approaches. Eye-tracking data evidenced strategic approaches and emphasized differentiated instruction. This objective data enhanced professional self-awareness. Dana stated, "I'm a planned person, always thinking ahead", reflecting her strategic planning patterns, a characteristic clearly visible in the sequential and organized patterns of eye movements observed during problem-solving, where she systematically scanned the problem space before focusing on solution paths. The follow-up interview showcased how Dana applied these insights in her classroom, illustrating that eye-tracking coupled with structured reflection, can foster changes in teaching strategies. The experience deepened core teaching principles, as shown in her statement, "The experience sharpened my need to adapt my teaching to each child". While a brief eye-tracking session alone will not transform professional identity, this study suggests that using such technology as a catalyst for reflection in a comprehensive professional development program can reveal cognitive processes usually invisible to teachers. These findings provide practical implications for educators, offering a new perspective on professional development and the benefits of technology-enhanced reflection in teaching and as a research method.

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Identifying, monitoring and fostering conceptual understanding of the arithmetic mean

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Keywords: Conceptual focus, representations, conversions, arithmetic mean.

Theoretical background

Conceptual focus is a crucial aspect of learning mathematics. In terms of instructional practice, conceptual focus involves *identifying* learning content, *monitoring* pupils' learning progress, and *fostering* pupils' understanding (Prediger et al., 2022). For the arithmetic mean, which is the subject of this study, the following aspects of understanding can be identified (Strauss & Bichler, 1988). The *aspect of compensating* relates to the determination of the arithmetic mean through redistribution. The *fair-share-aspect* concerns the equal distribution of the sum of the integers to the number of the integers. Furthermore, in the context of conceptual focus, *conversions* (Duval, 2006) play an important role, as they develop and make visible conceptual understanding by linking various representational registers (e.g., symbolic, iconic, verbal). Against the background, this exploratory study investigates the following research question: How do pre-service teachers demonstrate conceptual focus with regard to the aspects of the arithmetic mean through a text vignette?

Method

In order to answer the research question, 60 secondary level pre-service teachers in their 5th Bachelor semester at a German university worked on a text vignette as part of a lecture on the didactics of mathematics. Figure 1 presents the vignette, which is based on a pupil task that includes conversions of representational registers and addresses the three practices *identifying* learning content (1), *monitoring* pupils' learning progress (2), and *fostering* pupils' understanding (3).

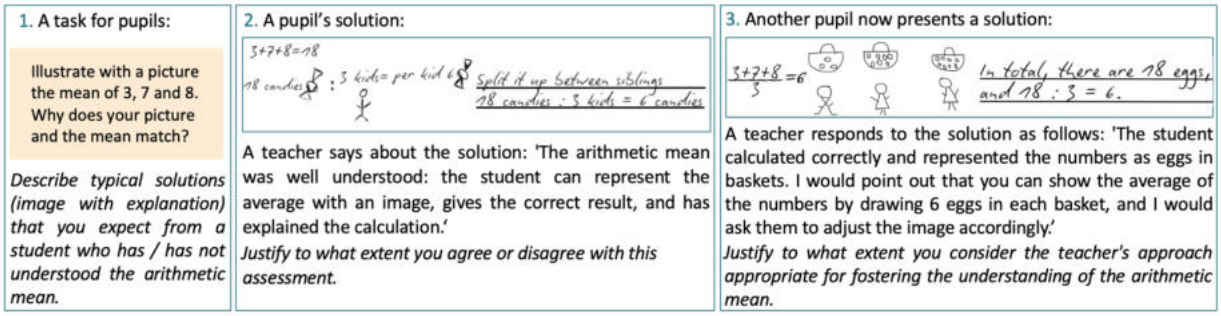

Figure 1: Vignette "Diagnosing and fostering conceptual understanding of the arithmetic mean"

Results

In the following, the answers of two pre-service teachers are quoted and analysed to contrast how conceptual focus can become visible through the vignette by referring to its three components:

Case 1

*Identifying: "*Aspect of compensation: Blocks are removed from the taller towers (larger numbers) until all towers are the same height." *Monitoring:* "The child understood the calculation but only decorated it with 'pictures.' It is not clear what exactly the quantity of 18 and the quantity of 3 mean and how the process leads to the result. […]" *Fostering:* "The pupil drew the initial situation correctly. […] The first drawing can be kept but should be supplemented by the teacher's suggested drawing" In this case, conceptual focus is demonstrated in the responses to *identifying* and *monitoring,* as they take into account both the conceptualisations of the arithmetic mean and the conversion of representational registers. In contrast, this is not reflected in the response related to *fostering*.

Case 2

*Identifying: "*I would draw the three numbers as stacks of building blocks with the number of blocks corresponding to the numbers, and then distribute the blocks so that the towers are of equal height*."* *Monitoring: "*I agree with the teacher. The arithmetic mean was only partially represented visually, but it was understood on a mathematical level." *Fostering: "*I also agree with the teacher here. The distribution of the eggs needs to be represented more clearly. However, the arithmetic mean was understood." In this case, the response to task 1 demonstrates conceptual focus based on the aspect of compensation. However, in the statements regarding *monitoring* and *fostering,* conceptual understanding is concluded from the correct calculation and the conversion of representational registers is not taken into account. Therefore, there is not conceptual focus here.

Conclusion and outlook

In summary, results from the analysis of the vignette suggest that identifying conceptualisations of the arithmetic mean does not imply monitoring and fostering conceptual understanding. Based on this, the poster introduces the vignette and illustrates the results to discuss the relationship between identifying, monitoring and fostering conceptualisations of the arithmetic mean.

Acknowledgment

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Teachers' discussions used to explore students' mathematical writing

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Keywords: Teacher collaboration, professional learning communities, students' mathematical writing

Introduction

Teachers' collaboration can be a strong means for developing the teaching of mathematics (Garet et al., 2001; Kennedy, 2016). It has been shown that discussions on students' thinking and learning tend to have a greater impact than those directed toward teachers' behaviours (Kennedy, 2016; Timperley, 2006). Furthermore, it is essential for the participating teachers to be able to identify with the group and that there is a common norm for the discussions (Karsenty et al., 2024). Adding to this is the importance of a suitable topic for a professional development, both in terms of coherence, and in terms of its relevance (Prediger et al., 2015). One such topic is students' mathematical writing. For example, there is a need to address how good mathematical writing can be described (Teledahl et al., 2023). In an ongoing project teachers' collegial discussions are used to explore students' mathematical writing. The research question guiding this text is how teachers' discussions can be modified to enhance the opportunities to explore students' mathematical writing. Twenty teachers at two schools of similar size participate in the project. Both schools include all grades from pre-school up to grade 6 with a student age range from 6 to 12 years old. Ten two-hour meetings have been conducted with discussions focused on students' mathematical writing. The arrangements for the discussions are explorative in the sense that the organization develops over time. The role of the researcher in the discussions is primarily to direct attention to students' mathematical writing. During each of the meetings, notes have been taken, which have been used in a strive to enhance the discussions without the interaction of an external mediator. Suggestions are tried continuously and related to an idea of good teacher discussions as something with a clear focus that allows teachers to describe as well as interpret happenings related to students' mathematical writing, and to put teaching into practice to create opportunities for students to develop their mathematical writing skills.

Results and discussion

In summary, three modifications have improved the discussions:

- To divide the teachers into sub-groups consisting of 3-4 teachers, teaching the same or close grades. The discussions were conducted alternately with the whole group and the sub-groups.
- To collectively work with designing activities to be implemented in the classrooms.
- To focus on the same mathematical topic and concepts, with similar relatable goals for all grades, and to include the same or similar questions.

These modifications had to do with two themes: coherence and the integration of students' actions and learning with the discussions. Consciously formed sub-groups made it easier for teachers from different schools to relate to each other and to identify with the group. The importance of a common

point of departure (Karsenty et al., 2024) was taken into consideration as the teachers could use their understanding of students of a specific age, making it possible to also relate the discussion to classroom work. At the same time, the importance of building a norm and point of departure (Karsenty et al., 2024) for the whole group was taken into consideration as activities and learning goals were designed to fit students in pre-school as well as grade 6. The use of the same task, with slight variations for different grades has been applied successfully before (Jäder & Wedman, 2023). The designed activities also served as a basis for a more active learning. Basing part of a discussion on students' thinking has been found to be functional for collegial learning (Kennedy, 2016; Timperley, 2006). A focus on students' learning may possibly also reduce pressure on teachers in the professional learning situation. Making each one of the teachers involved in the overall theme, but at the same time making them the expert on their own classroom and their own students in relation to the specific design of the activity, strengthened the possibility for active participation in the discussions (Garet et al., 2001). Overall, a conclusion is that teachers' discussions can develop exploratively through collaboration and joint decisions and may take advantage of the above-mentioned ideas.

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The interplay between mathematics teachers' professional development and classroom practice: A scoping review

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Keywords: Mathematics teachers' professional development, classroom practice, scoping review.

Introduction

In studies on mathematics teachers' professional development, effective principles for teachers' learning have been identified (Darling-Hammond et al., 2017; Desimone, 2009). However, other studies indicate that the impact of mathematics teachers' professional development on classroom practice is often limited (Pellegrini et al., 2021). This contradiction raises a query as to why mathematics teachers' professional development on the one hand is seen as promising for teachers' learning, but on the other hand, has a limited impact on classroom practice. To understand this contradiction, this study focuses on how and with what results previous studies have investigated the relationship between professional development and classroom practice. This is investigated through a scoping review guided by the following research questions: RQ1) How has the relationship between mathematics teachers' professional development and classroom practice been studied? RQ2) What are the results of studies on the interplay between mathematics teachers' professional development and classroom practice?

Method

This study was conducted as a scoping literature review (Peters et al., 2020). The databases used were ERIC and Web of Science. The search was structured around the following themes: professional development, classroom practice, impact, and mathematics (search string: "professional development" OR "teacher* learn*" OR "learning communit*" AND "classroom* practic*" OR "teach* practic*" OR "teach* prax*" OR "student outcome" OR "student* achiev*" OR "student* learn*" OR "chang* pra*" OR "teach* chang*" AND effect* OR implement* OR improv* OR innovation* AND mathematics). Only scientific, peer-reviewed articles published after 2009 in high-impact qualitative research journals were included. The articles were screened based on the following inclusion criteria: the study must involve some form of professional development, have a direct connection to the teacher's own classroom practice or impact on student learning, be focused on primary or secondary education, be empirical, and primarily focus on mathematics education. The screening process resulted in the inclusion of 150 articles. To answer RQ1, the articles were analyzed based on their research design, for example, experimental, longitudinal, comparative, or case study. To answer RQ2, and partially RQ1, the included articles were analyzed using coding in NVivo. An initial set of codes was established, while additional codes were developed during the analysis process. The codes were developed to capture key aspects such as the type of professional development program (e.g., lesson study, coaching, implementation of a tool or method), the study sample (large- or small-scale), the duration of the program (categorized as one year or less, one to three years, or more than four years), and the reported results. Results were coded as positive, none,

mixed or negative. Additionally, whether the results were reported from the student or teacher perspective was documented.

Results and discussion

Connected to RQ1, the results indicate that most studies included in this review are small-scale, involving fewer than a hundred participants, and that the majority have a timespan of one year or less. The results also show that experimental designs and case studies are the most common research designs. A notable gap in previous research appears to be the lack of longitudinal studies. To gain a deeper understanding of how changes in teachers' practices over time influence student learning, and how various forms of professional development for mathematics teachers impact teaching, longitudinal studies would be valuable. Regarding RQ2, the results from experimental and case studies reveal both positive and negative outcomes, but predominantly positive. Positive outcomes mean that the results show an improvement in student learning, or a development in the teacher's practice, and negative outcomes mean that no improvement was shown in either the students' results or the teacher's practice. Regarding study duration, studies with a longer duration report a higher proportion of negative results. This suggests that the effect of professional development diminishes over time. The results presented in the articles included in this review are primarily linked to teachers' practices. Fewer than half of the included studies report findings that directly measure student outcomes, and these measurements predominantly consist of pre-tests and post-tests. The interplay between mathematics teachers' professional development and classroom practices is rarely addressed in the reviewed literature. This lack of focus may contribute to the limited capacity of these studies to explain why professional development does not consistently yield the desired effects.

The poster will be organized into four sections. The first section will present the rationale for the study along with the research questions. The second section will outline the methods employed in the study. In the results section, diagrams will provide an overview of the distribution of various research designs, professional development approaches, and outcomes. By illustrating the interconnections between these aspects, the poster will then present a section highlighting identified gaps in the current research, providing implications and directions for future studies.

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Teacher noticing framework to examine multiple teaching skills: Analysis of teaching and lesson planning

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Keywords: Teacher noticing, mathematics education, analysis of teaching, lesson planning.

This study proposes a framework to examine teachers' analysis of teaching and lesson-planning skills and how these skills might mutually be the indicators for the other. The framework builds on teachers' noticing construct (Sherin, 2017) and extends its boundaries by using it to examine what teachers notice not only during teaching but also before teaching. This is important because by using the same framework it may be possible to understand the relationships between different teaching skills.

Teacher noticing can be defined as distinctive approaches teachers use to observe and interpret classroom events and instructional details. Some may argue that teacher noticing primarily refers to what teachers notice during teaching (Weyers et al., 2023). However, Sherin (2017) explained the scope of teacher noticing as "we should think of noticing as something that occurs when teachers prepare for and reflect on their teaching experiences" (p. 405). Choy et al. (2017) supported and argued that teachers should pay close attention to student thinking by using classroom artifacts and interpreting these artifacts mathematically at all points of a lesson—before, during, and after.

Many studies unpacked teacher noticing dimensions somewhat differently but in parallel to one another (Santagata & Yeh, 2016; Scheiner, 2016). For example, Kaiser et al. (2015) proposed teacher noticing as "(a) perceiving particular events in an instructional setting, (b) interpreting the perceived activities in the classroom, and (c) decision-making, either as anticipating a response to students' activities or as proposing alternative instructional strategies" (p. 374). Even though there are different terms for teacher noticing dimensions, they can mainly be listed as (1) attending, (2) interpreting, and (3) responding (Amador et al., 2017; Santagata & Yeh, 2016). Therefore, this framework suggests these teacher noticing dimensions (purple column in Figure 1) can be applied to multiple teaching skills to understand the possible relationships between these teaching skills.

Figure 1 summarizes how the noticing framework can be applied to examine both teachers' analysis of teaching and lesson planning skills by presenting which elements can be used to analyse these teaching skills and how these elements correspond in terms of teacher noticing dimensions situated in the literature. One conjecture can be teachers who are better at providing suggestions for improvement in an enacted teaching video might be more successful at foreseeing key learning moments of the lesson prior to instruction and anticipating students' thinking before and after these moments, as both skills correspond to the responding level of teacher noticing dimensions which is considered to be the highest level of noticing (Kaiser et al., 2015). Elements to analyse these two forms of teacher noticing (orange column in Figure 1) have been previously in research studies. From the lesson planning perspective, Choy et al., (2017) used *noticing* as an analytic framework to identify the lesson's key mathematical concepts and anticipate students' thinking. Within the analysis of the teaching video context, Santagata and Yeh (2016) included mathematics content, student thinking, pedagogical teaching moves, and suggestions for improvement in their rubric.

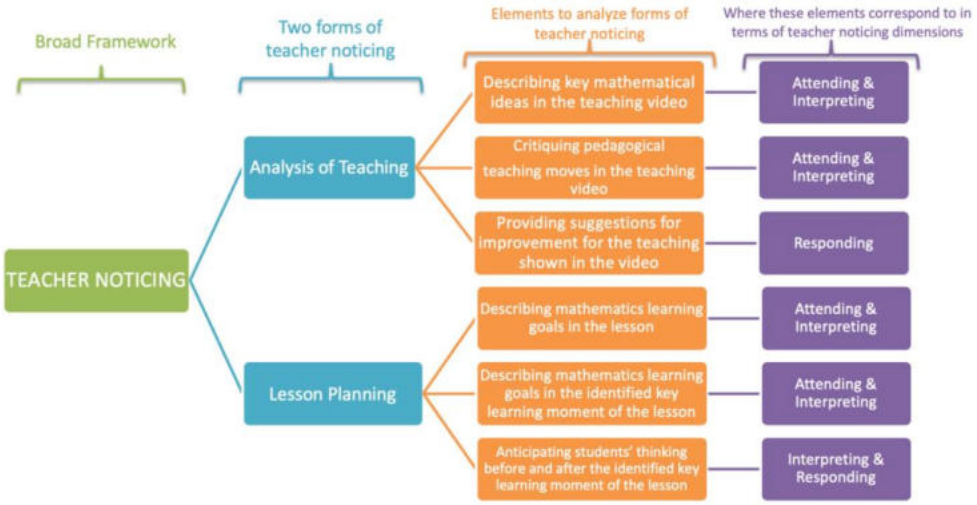

Figure 1: Teacher noticing framework to examine teachers' analysis of teaching and lesson planning skills

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Learning to teach problem solving using a four-step lesson structure

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Keywords: Problem solving, teacher education, teaching practice.

Introduction

Mathematical problem solving has recently been established as a key objective in Dutch primary education. Consequently, teaching problem solving has become a new subject in teacher education. At our institution, a course on problem solving is implemented at the end of the second year. We have initiated an ongoing design study to gain insights into how preservice teachers (PSTs) can develop teaching skills for teaching problem solving.

Theoretical Framework

In a literature review on the knowledge and skills necessary for teaching problem solving (Lit et al., 2025), we found that one of the most challenging skills for teachers is leading whole-class discussions. To address this challenge, Stein et al. (2008) developed five practices for orchestrating productive mathematical discussions: anticipating, monitoring, selecting, sequencing and connecting. These five practices were incorporated into our course design, alongside the lesson structure for teaching through problem solving proposed by Takahashi (2021), which is itself based on the problem-solving steps outlined by Polya (1945). As a result, we developed a four-step lesson structure specifically for teaching problem solving (see Table 1). The research question guiding this study is: How and to what extent can a four-step lesson structure assist preservice teachers (PSTs) in implementing problem-solving lessons in primary education?

Methods

Twenty PSTs participated in the study. During the course on problem-solving, they taught three non-routine problems to students aged 8-12 years. The PSTs documented their use of the four-step lesson structure and submitted video-recordings of their teaching sessions. The video-recordings were utilized for cross-validation to ensure the completeness and accuracy of the reports.

Preliminary Results

The implementation of our four-step lesson procedure by PSTs is detailed in Table 1.

Table 1: Steps in teaching problem-solving

Four-step lesson structure for teaching problem-solving	Results
1. Understanding the problem	All PSTs focused with their students on identifying the given information and the question posed in the problem. PSTs were motivated when they saw that most students began tackling the

	problem, even if the students were unfamiliar with non-routine problems.
2. Students solve the problem, while PSTs monitor, select, and sequence their approaches	PSTs reported being able to understand the approaches of at least half of their students. PSTs applied selection-criteria in line with the findings of Dunning (2022).
3. Whole class discussion	Nearly all PSTs asked two or three students with diverse approaches to present in class. They assisted these students in presenting effectively and ensured that their peers comprehended the presented material.
4. Connecting and looking back	Most PSTs reported difficulties with the last step, consistent with the observations of Gil et al. (2019). PSTs noted that children often do not generate reflections independently and PSTs struggled to find adequate questions to maintain student engagement.

Conclusion

The four-step lesson-structure for teaching problem solving is applied and appreciated by all PSTs. It helps PSTs focus on their students' problem-solving approaches and effectively present these in class. However, the step of connecting and looking back is found to be challenging to implement. This study will further analyse the questioning and noticing skills of PSTs. As an implication, we suggest that greater emphasis should be placed on learning problem solving within teacher education.

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Fostering argumentative dialogue in secondary math classrooms: Teacher development trajectories through task-based PD

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Keywords: Dialogic teaching, argumentative dialogue, task design, professional development.

Introduction, Methods, and Research Questions

Dialogic teaching, which encourages students to engage in collaborative problem-solving and critical discussions, is widely recognized as an effective approach in mathematics education. However, the systematic implementation of dialogic teaching in mathematics classrooms faces several challenges, including teacher-centered discourse dominance and lack of suitable tasks that invite discussion (Schwarz & Baker, 2017). While task-based professional development (PD) using expert-designed tasks has shown promise (Swan, 2007), pre-designed task sets may not always fit the specific needs of individual classrooms. Jones and Pepin (2016) suggest that teachers' active participation in selecting, adapting, or designing tasks can develop their agency in implementing new pedagogies. This study examines how a task-based PD program, focused on collaboratively adapting standard textbook tasks into more dialogic ones and enacting them, supports mathematics teachers in fostering argumentative dialogue in their classrooms. The study was conducted as part of DIALOGOS project, which aimed to foster dialogic teaching in mathematics, science and philosophy in Israel. Through multiple case study research (Stake, 2000), the study followed five mathematics teachers who participated in the DIALOGOS PD between 2018–2022. Data were collected through video recordings of PD meetings, documents (tasks, lesson plans, written reflections), personal communications, reflective interviews, observations, and audio recordings of dialogic lessons. The analysis was guided by Schoenfeld's (2011) ROG framework (Resources, Orientations, and Goals) for modeling teacher decision-making, extended to include Practices (P). Through iterative thematic analysis, 32 Professional Development Trajectories (PDTs) were identified — stories documenting specific changes in teachers' perspectives or practices related to dialogic teaching (averaging 6–7 PDTs per teacher). Each PDT captures a developmental shift from initial stance (e.g., "Upgrading dialogic tasks: from resistance to recognition of necessity") and is characterized by its area of development and dimension of change in PROG terms. The research questions are: (1) What areas of development and dimensions of change were observed in the PDTs of secondary school mathematics teachers who participated in a task-based PD aimed at encouraging argumentative dialogue in mathematics classrooms? (2) Which components of the PD program contributed to these PDTs?

Findings

Areas of Development and Dimensions of Change: Analysis revealed 32 PDTs, distributed among three main areas of development with inductively identified subgroups: Task Design (13 PDTs), including *Format selection and dialogic characterization of the task* (10), and *Independence in task design* (3); Approach to Dialogue (8 PDTs), including *Feasibility of independent dialogue among students* (1), *Developing distinction between dialogue types* (4), and *Defining new dialogic goals* (3); and Implementation of Dialogic Situations (11 PDTs), including *Small-group work* (4), *Dialogic*

instructions (2), *Teacher intervention* (3), and *Managing class discussion* (2). Changes occurred across all PROG dimensions, with Resources (50% of 32 PDTs) and Orientations (41%) being most frequent, followed by Practices (22%) and Goals (19%). However, the distribution of these changes varied across the three areas of development. Task Design showed dominant changes in Resources (77%) while Orientations influenced 38%. Approach to Dialogue had balanced changes across ROG dimensions (38% each). Implementation of Dialogic Situations emphasized Orientations and Practices equally (45% each), linking teachers' beliefs and actions in this area.

Key PD Components: Analysis revealed various PD activities' impact on teachers' PDTs: 1. Planning-implementation-reflection cycles were the most influential, affecting 23 PDTs across all areas; 2. Task Upgrade Workshop (influenced 6 PDTs): Provided teachers with strategies to transform standard tasks into dialogic ones by changing their formats; 3. Viewing classroom video recordings (9 PDTs): Encouraged reflection and sharing of practices among teachers; 4. A model for identifying dialogic triggers in tasks and potential dialogue types (8 PDTs): Offered a shared language and tools for selecting/designing tasks and planning/identify type of dialogue; 5. Personal support in designing dialogic lessons (7 PDTs): Provided tailored guidance in task adaptation and implementation.

Conclusion

This task-based PD, focusing on redesigning and implementing dialogic tasks, effectively promoted diverse PDTs across various areas of dialogic argumentation implementation. Identifying PDTs provides crucial insights into teachers' multifaceted growth, enabling more targeted PD design. The PD program combined well-established activities like video analysis and iterative planning-implementation-reflection cycles (Swan, 2007) with unique emphases on task formats and dialogue types. This approach revealed how different activities complemented each other by influencing various PROG dimensions. For instance, the Task Upgrade Workshop primarily affected Resources, while video viewing was more effective in changing Orientations. These findings highlight the importance of integrating diverse activities in PD programs to address different aspects of teacher development, contributing to more effective, tailored growth strategies in mathematics education.

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Teachers in noticing courses, how are they noticing and what do their educators notice about their discussions?

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Keywords: Teacher noticing, educator noticing, in-service teacher education.

Introduction

This study presents the initial findings of two years of a course on teacher noticing and in particular the perspectives of mathematics teacher educators (MTEs) on the important moments in this course. As part of teacher education at HU University of Applied Sciences Utrecht, we piloted this course for master's students, who are currently in-service teachers with teaching experience ranging from 4 to 20 years, aiming to enhance their teacher noticing skills. The participants contributed video segments from their own classrooms, examined and discussed these clips with their peers through the lens of teacher noticing, guided by an MTE. The discussions were recorded and the proficiency of the participants regarding their noticing skills were analysed using a measuring instrument based on existing frameworks as outlined in the methods section. In a second round of analysis MTEs identified what they considered to be important moments for learning to notice in these discussions. We therefore address the question Amador (2022, p. 169) poses "... *how* those who teach noticing to teachers actually notice..." by answering our research question: What moments do MTEs consider to be pivotal when teaching noticing in relation to what teachers notice?

Theoretical Background

According to Amador (2019), research on teacher noticing conceptualizes it as a discipline, an analytical tool for research or a framework for teaching. Our study takes a combined approach, a common practice among researchers (Amador, 2019), and studies three well-established aspects of teacher noticing (Stahnke et al., 2016): perception, interpretation, and decision-making.

Intervention

Participants were asked to bring video fragments from their classrooms, which they jointly viewed two consecutive times. The participants were prompted to note their observations regarding perception, interpretation, and decision-making. MTEs facilitated a group discussion highlighting different perspectives and providing connections with didactical concepts. Key ingredients of the intervention are working with video from the participants' classrooms, and a direct prompt for the participants to write down the three aspects of teacher noticing.

Methods

The group discussions were recorded and coded in ATLAS.ti, assigning one of three levels of perception, interpretation, and decision-making to the remarks made by the participants. Research data consists of transcripts with labels, where two coders discussed their initial differences until a final label was agreed upon. The development of this measuring instrument, which was based on a

synthesis of frameworks of van Es's (2011) and Jacobs et al. (2010), was introduced and detailed by the authors at CERME13. A second round of analysis was conducted, where the MTEs identified key moments in the discussions they considered to be fruitful or showed evidence of learning by the participants. These moments were compared to the labels of teacher noticing.

Results

MTEs identified fruitful moments in the group discussions corresponding to moments where participants show a certain level of noticing.

Table 1: Fruitful moments in the group discussions as identified by MTEs

Why the moment was fruitful according to MTEs?	Level of noticing of participants
"Questions helped to specify thinking"	Level 1: general, nonspecific, unconnected
"Participant set great example by story"	Level 3: specific, detailed, connected
"The participants had different interpretations of the same facts, the discussion encouraged them to think about their assumptions"	Mixed levels

The poster elaborates on these moments with transcripts of discussions and considers how these findings can be used to strengthen the development of noticing. The results provide insights for MTEs to notice opportune moments for learning when moderating discussions, and pointers for how a discussion may proceed matching the level of noticing the participants display.

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Contextualisation and complexity in online rehearsal of feedback: Insights from online mathematics teacher education

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Keywords: Mathematics teacher education, online, approximation of practice, rehearsals, role-playing.

Introduction

This proposal addresses the challenges of contextualising classroom teaching when using approximations of teaching practices in online mathematics teacher education (MTE) to prepare preservice teachers for ambitious classroom teaching.

A practice-based MTE programme is centred around key core practices in the teaching profession. By approximating these practices in simulated teaching scenarios, preservice teachers can learn core practices in MTE even when access to real classrooms and pupils is unavailable (Grossman et al., 2009). These approximations also provide a safe and low-stress environment where preservice teachers can rehearse teaching principles and strategies within the core practices without the demands of real classroom teaching. Previous research has shown that a practice-based approach in campus-based MTE better prepares pre-service teachers for ambitious mathematics instruction than traditional teacher education (e.g., Mancenido et al., 2023). Further, research on practice-based MTE advocates that teaching practices should be rehearsed in an environment that preserves the complexity of teaching, rather than in situations that simplify and focus on isolated aspects of teaching practice (e.g., Lampert et al., 2013). Although the number of preservice teachers pursuing their education online continues to grow, studies on online practice-based MTE are very limited. This raises questions about how complexity can be maintained and how rehearsals can be contextualised when preservice teachers practice in an online environment. This poster aims to highlight findings that focus on enhancing contextualisation and preserving complexity during online rehearsals in MTE.

Method

Educational design research focuses on iteratively developing theory as well as practical solutions to problems. The study reported in this poster is based on a design study on rehearsals in online MTE with data generation from three cycles to date. The design process is guided by theoretical underpinnings derived from previous research relevant to the study and informed by insights gained from reflective analysis after each iteration (McKenney & Reeves, 2019). Participating preservice mathematics teachers were recruited from an online MTE university course about assessment and feedback in secondary school. Each semester, two to three participants met online with a teacher educator (the author) to rehearse the core practice of eliciting and responding to pupils' thinking. The rehearsals were organised as role-plays with participants assigned the roles of teacher and pupil, while non-acting participants actively observed and reflected. A mathematics task with an incorrect pupil solution was shared with the participants on the screen before each role-play. The role-plays and later follow-up group discussions were recorded and transcribed. The insights presented in this poster are

based on the results of a thematic analysis of the transcripts, examined in relation to relevant prior research.

Findings

To address the need for contextualisation and maintained complexity in online rehearsals, this poster presents three strategies based on the findings.

Enhance the use of non-verbal forms of expression: Role-play activities revealed limitations in the digital environment for supporting non-verbal feedback, (e.g., pen-and-paper drawings could enhance feedback). To address this, digital annotation tools were more clearly introduced and the slides with mathematical tasks were reorganised to allow more space for digital drawings, thereby improving the incorporation of written feedback in online learning.

Present the mathematical tasks in advance: In mathematics teaching, the tasks pupils work on are usually not unfamiliar to the teacher. Therefore, allow participants to review the mathematical tasks prior to the online role-playing session. This ensures that the tasks do not come as a surprise when presented to the participants on the screen at the start of each new role-play during the online meeting.

Facilitate deep and varying mathematical conversations: To deepen the focus on mathematical didactics during role-plays, encourage the pupil role-player to challenge the teacher's feedback, prompting the teacher to consider different responses. Additionally, increasing variation in role-play scenarios, such as assigning observing participants the task of devising alternative misunderstandings, allows the teacher to rehearse various instructional strategies. This also helps prevent participants from becoming passive in the online environment when not assigned a teacher or pupil role. The researcher can actively participate as a pupil by introducing new challenges, further enriching the learning experience.

These strategies address the challenges of contextualising and preserving the complexity of classroom teaching when using approximations of teaching practices in online MTE. Additional approaches to better simulate real teaching situations in online rehearsals will be explored in my future research.

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Preservice teachers' awareness of their own developmental trajectories in primary mathematics teacher education

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Keywords: Primary school, teacher education, learning trajectories, preservice teachers.

Introduction

In the mathematics meetings at primary school teacher education most attention is devoted to the learning and developmental trajectories of primary school students (e.g., Van den Heuvel-Panhuizen, 2008). Preservice teachers (PSTs) are taught to notice and interpret primary school students' mathematical behaviour and ideas and thus 'place' them on particular spots in such a teaching and learning trajectory. Using such an idea, on where students are in their mathematical development, can help PSTs to decide about suitable next steps in their mathematics lessons. A topic to which much less attention is devoted, is PSTs' awareness of their own learning and developmental trajectories in mathematics teacher education. Evidently, PSTs develop their knowledge and skills in teacher education, both at the teacher education institute and in their practice schools. We distinguish three separate, albeit connected, trajectories in our primary school teacher education program: mathematics didactics, professional numeracy, and mathematical attitude (cf. Carillo-Yañez et al., 2018). Mathematical attitude is a multi-faceted concept concerning broadly two approaches: an appreciative stance towards mathematics and awareness of the ubiquity of mathematics in the outside world (Oonk & De Goeij, 2008). In this study, we investigated the awareness PSTs exhibit of these trajectories with the research question: How aware are PSTs of their developmental trajectories in mathematics in primary school teacher education?

Methods

To investigate PSTs' ideas about their developmental trajectories in mathematics at primary school teacher education we used an online questionnaire. For this, 58 PSTs in their first (n = 39) and in their second year (n = 19) at IPABO University of Applied Sciences, Amsterdam, responded. In the questionnaire, PSTs were asked 10 questions: three on their general impression of the learning process of mathematics at teacher education (*what have you learned, what do you still have to learn, how competent do you feel*) and seven on the specific learning trajectories (*do you recognize the learning trajectory [didactics; professional numeracy; mathematical attitude], can you provide an example of a teacher education activity, how important do you consider each learning trajectory*). We analyzed these data and categorized the open-ended answers using a deductive framework based on evidence for the three aforementioned learning trajectories. This analytical process is underway.

Results

PSTs in general expressed feeling rather competent about mathematics ($M = 7.01$, $SD = 1.4$ [min-max: 1-10]). In their explanations they provided two types of evidence for their competence, experiential ("because I have taught many lessons at primary school already" [didactics]) or more

test-related ("I successfully completed the WISCAT [the entry test for mathematics at primary teacher education]" [numeracy]). PSTs remarked that they learned most about mathematics didactics, but also have most to learn in that trajectory. They report, for example, to have learned how to analyze student work in mathematics and how best to explain several mathematical topics. Many PSTs mention that they developed their numeracy but have somewhat to learn still. In classes at teacher education, they did mathematical exercises which were discussed in the classroom. By participating "in these discussions I developed my own numeracy." The mathematical attitude trajectory was least recognized, PSTs mentioned not having learned much and also not recognizing it as something to be further developed. They did recognize it as being present in one of their first assignments: *Mathematics in the news* (cf. Stuber et al., 2021).

When asked to rank the different trajectories in importance, the didactics trajectory was deemed most important (34 times 1st place), followed by professional numeracy (18 times 1st place) and finally by mathematical attitude (7 times 1st place). When asked to provide examples for teacher education activities related to the different trajectories the PSTs struggled; about half of them were not able to mention any activities. When they did provide examples, they mentioned for example "analysing student work and discovering mathematical errors" (didactics), or "calculating in different number systems" (numeracy) or "an introduction with a game or open question or mathematical puzzles" (attitude).

Preliminary conclusion

PSTs do not show much awareness of their own learning trajectories. They do mention still needing some more knowledge, but what that consists of remains difficult to pinpoint. They rarely exhibit understanding of how activities in teacher education are related to the three separate learning trajectories. Nonetheless, the first- and second-year PSTs express feeling rather competent in teaching mathematics in primary school.

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TWG 19: Mathematics Teaching and Teacher Practice(s)

An introduction to TWG19: Mathematics Teaching and Teacher Practice(s)

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Introduction

Research on teaching is central to mathematics education, and TWG19 traces its history to CERME1. After CERME8, the single TWG, *From a Study of Teaching Practices to Issues in Teacher Education,* split into TWGs 18, 19, and 20. TWG19 has since sought to define its focus, build a supportive community, and advance scholarship. At CERME10, participants reflected on challenges in talking across papers. Papers were organised topically, e.g., on assessment or on lesson planning, but papers within groups seemed idiosyncratic, like apples and oranges. Theoretical foundations differed, as did contexts of schooling, policy, culture, and language. Moreover, purposes varied, e.g., describing common practice, responding to a new curriculum, or arguing for a particular theoretical orientation. In response, participants proposed developing a pool of shared data, so that analyses carried out on the same data could support thinking across papers. They also proposed organising papers using an underlying framing for teaching instead of surface level topics. Research on teaching and teacher practice(s) must contend with multiple, key domains of the work, yet any study will likely foreground one, while maintaining attention to others. As organisers, we have used the following foundational domains to group papers and stimulate cross-paper discussion.

- Consideration of **mathematics** and the central endeavour of extending the subject to students.
- Becoming acquainted with, relating to, and responding to **students** as people and learners.
- Organisation and **enactment** of design, interaction, and discourse of teaching and learning.
- Attending to broader social, cultural, and political **issues** that matter for teaching and learning, including imperatives of social justice.

At CERME14, we explored additional ways to enhance the work of the group: (i) designing sessions to be more inclusive; (ii) taking stock of past scholarship; and (iii) developing shared understanding of foundational efforts to conceptualise teaching. This paper reports on these initiatives, then reviews papers presented at CERME14, and reflects on our collective experience at CERME14.

Designing sessions to be inclusive and productive

Mindful of CERME's three Cs (communication, cooperation and collaboration) and in keeping with the paper acceptance policies introduced for CERME14, we continue to include and support early career researchers. Over half of the participants of TWG19 at CERME14 identified as such. Two

forms of support are crucial: personal and emotional support and substantive support for conducting and reporting on research. We are committed to thoughtful, constructive reviews of all submissions at all stages. This requires time and energy but provides essential support – for individual and collective learning about the nature and conduct of research and for helping participants feel at home. To help participants feel welcomed, we started the week by learning each other's names and hearing what each participant is passionate about related to research on teaching. In asking for feedback daily, reporting on what the participants said, and adjusting accordingly, they sensed that that their input mattered. To support authors' work, we kept presentations short (5 minutes each, for groups of 4 papers), then each author met with 4–6 colleagues to discuss their paper and provide feedback. We encouraged authors to provide minimal opening remarks about the status of the paper and the feedback they sought but to spend most of their time listening to the discussion without intervening. An organiser then commented on each of the 4 papers. After two rounds, with 8 papers discussed, the 4 authors in each group met to discuss their related papers (similarities, differences, and shared insights), while 4 authors, those not presenting that day, discussed similarities and differences between their papers and the papers presented. For some discussions, people chose in which group to participate; for others, we assigned groups, attentive to mixing and balancing participation. We provided similar structuring of posters and small group discussion of themes for the week. These design choices were meant to honour each participant's contributions, both as author and colleague.

Review of papers from previous proceedings

To gain a picture of ongoing work, we reviewed papers from proceedings since CERME9. We revisited the introductory papers to consider trends, used quantitative text analysis to compare trends in word use, and identified commonly used references. TWG19 has averaged 24 accepted papers (from 18 to 27). The introductory paper from CERME9 highlighted connections between teaching and learning, described classroom interactions as complex, and called for more specific definitions of teaching. Comparison of word use indicated that papers in CERME9 emphasised assessment, textbooks, and textbook tasks more than papers since. At CERME10, the introductory paper again reported concern about the clarity of terms and a need to develop more common language. Word use indicated a shift from *teachers* to *teaching*, with words such as *critical* and *reflection* appearing more frequently.

At CERME11, an entire session was devoted to papers using shared data and participants reported that this was helpful for thinking across different lines of inquiry. Additionally, a distinction between teaching as *activity* versus *work* was proposed, reflecting a shift in research on teaching from investigating the actions or moves that teachers make and their effects, to attending to issues, situations or problems that make teaching challenging. At CERME12 (a virtual conference) several papers used shared data, and the meaning of teaching was explicitly discussed. Analytic domains were given in the call for papers and used to organise papers. In comparing word use, *learners* and the *Mathematics Discourse in Instruction* framework were more prominent. The use of domains continued at CERME13, and participants discussed the often-implicit purposes of studies, attending to how purposes might relate to conceptions of teaching. Papers gave increased attention to *students* and more invisible aspects of teaching, such as *listening* and *noticing*.

In examining references, we were surprised to find few common references. Shared references to specific research on teaching did not appear in more than five of the papers each year. Most frequent were references to Franke et al.'s (2007) handbook chapter, Stein et al.'s (2008) paper on five practices for orchestrating productive discussions, Yackel and Cobb's (1996) paper on socio-mathematical norms, Ball et al.'s (2008) paper on mathematical knowledge for teaching, and Cohen et al.'s (2003) paper on the resources for instruction. Yet, the overall impression was that references were scattered, and common theoretical foundations were few.

Conceptualising teaching

Given participants' call for better ways to talk across the different lines of inquiry, at CERME14 we presented and discussed foundational literature. Participants used presented ideas to consider the nature of their own work and how it sits in relation to others' work.

Teaching is an under-theorised construct. One basic challenge lies in its ubiquity. As Lortie (1975) pointed out, nearly everyone in society "sees" teaching. Everydayness renders it unremarkable and taken-for-granted. More than this, Lortie describes an *apprenticeship of observation* in which students' years in classrooms lead not just to over-familiarity but to false impressions – because they see teaching from a biased, student-oriented perspective that has little access to teachers' intentions, planning, and reflection, instead relying heavily on imagination and projection. Related challenges include uncertainty, unnaturalness, and intricacy. As Ball and Forzani (2009, p. 499) argue:

> To listen to and watch others as closely as is required to probe their ideas carefully and to identify key understandings and misunderstandings, for example, requires closer attention to others than most individuals routinely accord to colleagues, friends, or even family members. To provoke discordant thinking or errors in logic and argument intentionally would seem odd if not downright irritating in many situations. And, few adults seek to learn about others' experiences and perspectives as systematically as teachers must.

In the face of these challenges, researchers have proposed variants of an instructional/didactical "triangle of interactions" with which teaching must contend. Goodchild and Sriraman (2012, p. 581) state that this commonplace "triangle in which student, teacher, and content form the vertices (or nodes) of a triangle is the classical trivium." Figure 1 offers three such variants.

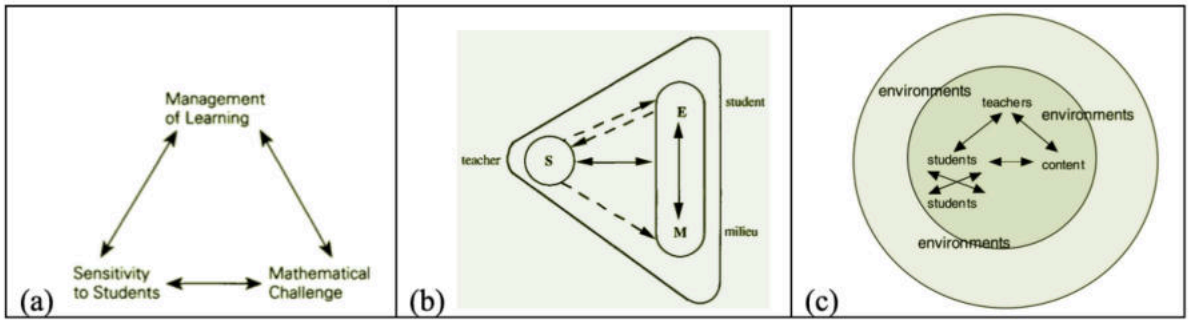

Figure 1: (a) Jaworski's (1994) teaching triad; (b) Brousseau's (1997) theory of didactical situations; (c) Cohen et al.'s (2003) visualisation of instruction as interaction

Cohen (2011) identifies teaching as a human improvement profession and intersects the three components, which he calls terrains of practice (acquaintance with students' knowledge, instructional discourse, and knowledge extension), with key social resources for doing the work (infrastructure of practice, mutual commitment to improvement, and work practitioners and clients do together).

Building on these ideas, scholars have sought to attend more fully to sociocultural complexity, oppression, and to the interplay of the micro and macro (Figure 2).

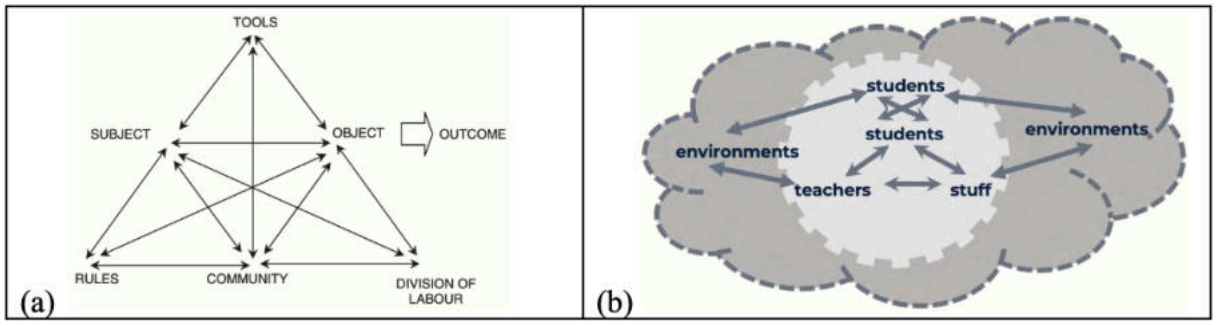

Figure 2: (a) Jaworski and Potari's (2009) expanded mediational triangle; (b) Ball's (2018) model of the discretionary spaces in teaching

Another important line of research on teaching has emerged from scholars from marginalised communities. They have approached the challenge of attending to sociocultural complexity and systemic oppression through a critical lens focused on academic success, asset-based attention to cultural competence, and advocacy for sociopolitical consciousness (e.g., Ladson-Billings, 2014).

In TWG19, as we investigate the layered, messy work of teaching, we continue to struggle to wrap our arms around it all and manage the challenges inherent in the work. At the same time, we find this to be an interesting and stimulating time to be involved in the study of teaching. The field continues to seek ways to disrupt long-standing patterns in which many students (in particular, women and racially and ethnically marginalized students) experience mathematics classrooms as unwelcoming, stressful, and exclusionary. It is also expanding awareness and approaches for addressing endemic challenges for doing, conceptualising, and studying teaching. As participants in TWG19, we find our interests and approaches vary widely, but we also find it useful to foster ways to talk across these and to see similarities and differences in our work more clearly.

Contributions of CERME14

Seventeen papers and four posters were accepted for presentation, with 26 participants from 15 countries. Germany had the largest representation, followed by Norway, Sweden, and Türkiye. Others were from Brazil, Czechia, Greece, Israel, Japan, Kosovo, Latvia, Netherlands, Spain, UK, and USA.

Table 1: Grouping of papers into domains

Domain	Paper	Count
Issues	Hartmann et al.	1
Mathematics	Chico et al.; France et al.; Lanting; Morina & Vondrova	4
Students	Hansen & Arnesen; Hoover & Ball; Mosvold et al.; Yasar et al.	4
Enactment	Häsel-Weide & Nührenbörger; McIvor et al.; Nama et al.; Öcal et al.; Ogawa & Miyakawa; Post & Sahin-Gür; Neufeld & Häsel-Weide; Wæge & Mosvold	8

The paper in the *Issues* domain explores the alignment of mathematics education teacher-researchers' attention to sustainability goals of the United Nations 2030 Agenda. Given the scope of the paper, it allowed participants to talk across their contexts during the introductory session.

Mathematics. Four papers foreground mathematics, highlighting the complexities of implementing student-centred and constructivist reforms aimed at developing students' understanding and problem-solving skills, and underscoring the need for effective teacher training and professional development. These studies emphasise the pivotal role of teachers' decision-making and the frequent divergence between planned interventions and actual classroom implementations. For instance, France et al. identify significant differences in Latvian teachers' performance when implementing a new mathematics curriculum, indicating varied professional development needs. Similarly, Morina and Vondrová found that while Kosovo teachers appreciate the SIMPLE approach for supporting word problem-solving, they face implementation challenges that require additional training. Lanting's theoretical paper proposes a variation of Polya's problem-solving method to foster analytical thinking in Dutch secondary students, emphasising mathematics as a language and highlighting the importance of structured group discussions and reflection. Finally, Öçal et al.'s study in Türkiye reveals that student teachers primarily select and sequence student-generated problems based on content and complexity, often overlooking incorrect or incomplete problems, further illustrating the complexities of effective pedagogical choices in practice.

Students. All four papers in this domain foreground the vital role of the teacher in attending to students' mathematical thinking across contexts. They collectively revealed the complexity involved in interpreting and building on student ideas to promote deeper mathematical reasoning, advocating a shift from deficit-based perspectives to valuing students' sense-making. Each study underscores the importance of reflective teaching practices and highlights the need for teacher education programmes to develop teachers' skills in attending, interpreting, and responding thoughtfully to a variety of student contributions. Hansen and Arnesen analyse a teacher's interactions with seventh graders solving a collaborative mathematics problem, showing that while the teacher aimed to foster generalised arguments, students struggled to follow the reasoning. Hoover and Ball focus on listening attentively to children's mathematical expressions through speech, gestures, and drawings, noting that traditional views of mathematics as solely correct or precise often obscure recognition of children's competence, especially among marginalised groups. Mosvold et al. propose a framework for monitoring student thinking, based on observing students' independent problem solving. Finally, Yasar et al. examine a preservice teacher's reflections on her noticing during one-on-one teaching, revealing emerging awareness and pedagogical decisions, though with some gaps in explanation. Together, these studies portray teaching as a complex, adaptive practice and call for enhanced preparation of teachers to effectively notice and respond to student thinking.

Enactment. Despite different foci in the papers foregrounding enactment, all explore opportunities to learn. Four of the papers examine the work of building inclusive environments. McIvor uses the concept of "status" from the theatre to discuss productive classroom discourses in which teachers work deliberately to minimise the status gap. Nama et al. focus on the tensions that teachers experience between challenging students mathematically and taking diverse learning needs into consideration. They underscore implications for teachers' professional development. Professional

development is also central in Neufeld and Häsel-Weide's paper, which argues that understanding of ways to support "struggling" children's learning of basic arithmetic skills is important for teacher development and professionalisation. In the context of professional development co-enactments, Wæge and Mosvold explore interaction between talk moves and other teaching moves. They find that the teacher they studied used different categories of teaching moves simultaneously and in interaction.

The other four enactment papers examine the complex role that teachers play in structuring classroom discourse, fostering inquiry, and adapting to diverse learner needs. Chico et al.'s study in Spain emphasises the importance of intra-mathematical connections, illustrating how teachers can support reasoning and communication by strategically utilising student contributions within a structured framework. Häsel-Weide and Nührenbörger explore inclusive classrooms in Germany, identifying discursive, content-related, and differentiated practices through long-term video-based research; their findings highlight the challenge of ensuring equitable participation, especially when certain students dominate interactions. Ogawa and Miyakawa's case study in Japan applies the *Anthropological Theory of the Didactic* to evaluate teacher preparation for inquiry-based learning, revealing the complexities of teacher decision-making and the need for training that fosters both teacher autonomy and structured collaboration. Lastly, Post and Şahin-Gür analyse teacher noticing within professional development contexts in Germany, indicating that although teachers often focus on moderation techniques, fewer attend to content-specific noticing, a gap that underscores the need for targeted professional development strategies. Collectively, these studies emphasise that while reforms encourage student-centred practices, their success hinges on teacher reflection, professional support, and thoughtful curriculum design.

Cross-paper themes. At CERME 14, several themes emerged from discussion across these papers. Hoover and Ball introduced the concept of *listening*, in ways that pay attention to students' mathematical productions with openness and curiosity, rather than evaluatively seeing whether students "get it". Other papers (e.g., Hansen's and Yasar's) use *noticing* for attending and reacting to students' contributions, and Mosvold et al. use *monitoring student responses* for the work of teaching that takes in and responds to student contributions. Although listening could be part of noticing and monitoring student responses, the specific focus on listening led to interesting reflections on a view of students as knowers or non-knowers and students' agency. For instance, listening and noticing might be thought of from preset ideas about mathematics and expected forms of behaviour or from an open stance that is attentive to alternatives.

Another theme discussed at the conference was *practice*. The name of the TWG includes practice, the concept is used in several papers and often heard in discussions. However, there is little uniform understanding. Some papers describe what they mean by practice, for example, Häsel-Weide and Nührenbörger state that a practice involves teachers' patterns of utterances and actions, as a shared phenomenon shaped by individual identity. Mosvold et al. examine the practice of monitoring student responses by unpacking the "work of monitoring". In the former, teaching is something teachers do; in the latter, it is core tasks to be done.

Teaching students who struggle to learn mathematics, as currently taught, was another theme of the discussions. A question that came up was how we can talk about this without labelling students.

Different ways of talking about these students, for example, students *with* difficulties, students *in* difficulties, or special educational *needs of* students, have consequences for mathematics teaching and for who owns and is responsible for the "problem". Participants expressed concern for equity and discussed how issues of language and clarity were implicated.

Reflections and conclusion

In the final session at CERME14, participants were invited to reflect on the week, identify insights, and look for themes. They recorded individual insights on a Padlet and then worked in groups to identify themes. They also responded to a survey. We discussed participants' reflections and identified three key ideas to consider in our ongoing support of TWG19.

Supportive atmosphere. During the week and at the end, participants expressed appreciation for the friendliness in the group, the design of the sessions, the structuring of discussions, the steady energy, and the overall sense of community. Several participants appreciated the attention given to pronouncing each other's names and opportunities to work in different groups with different people, some self-selected, some assigned. Looking to CERME15, participants suggested organising a dinner early in the week. They also suggested exploring ways to provide support between CERME conferences – meeting virtually in a small group to support writing, exploring potential paper collaborations for CERME15, sharing research artifacts and giving each other feedback, or discussing a professional practice, such as how to review a paper or read a foundational text.

Broadening perspectives. Participants in the group come with different perspectives and ideas about mathematics teaching and teacher practices. The group runs the risk of being too spread out for in-depth discussion, but this did not happen. With the added support, participants experienced the discussions as rewarding and the different perspectives as enriching. They found it helpful to hear and more deeply understand each other's thinking, perspectives, culture, and language and they appreciated how this contributed to a greater awareness of their own research. Reading papers, listening to presentations, and discussing research outside their immediate interest – but recognisably about mathematics teaching and teacher practices – helped to put their own research in a larger context and stimulate ideas for future studies.

Importance of wording. With so many different perspectives and studies situated in diverse literature, participants could see the need to clarify the meaning of central concepts. In addition to being more explicit about what we mean by *teaching*, we need to clarify what we mean by *listening*, *noticing*, and *practice*, and attend to ways language can be exclusionary. It is not an end in itself, and we do not need agreement on the meaning of core concepts, but we need to clarify, distinguish, and delineate our use of terms better as we communicate about our work.

In closing, we look forward to living into the three Cs and continuing to build a robust community engaged in crucial research on mathematics teaching, the heart and soul of mathematics education.

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Sustainability and mathematics education considering the 2030 Agenda: Understandings and practices of university professors in Portugal

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This text discusses the understanding and sustainability practices of university professors in mathematics education working in Portugal, considering the United Nations (UN) 2030 Agenda. The 2030 Agenda proposes 17 sustainable development goals to be achieved globally, based on three aspects of sustainability: economic, social, and environmental. We conducted semi-structured interviews with ten professors from six Portuguese universities between January 16 and April 10, 2024. Data analysis using content analysis techniques indicates that the practices developed by the interviewees are aligned with the economic, social, and environmental aspects of sustainability. We conclude that the professional activities of these university professors in mathematics education have contributed to implementing the UN's 2030 Agenda.

Keywords: Mathematics education, sustainable development goals, practices of professors.

Initial considerations, with a synthesis of the literature

Problems related to economic, social, and environmental aspects have plagued different countries worldwide. For example, in Brazil, the Amazon has seen a significant increase in deforestation since 2013, exacerbated by the weakening of environmental legislation and the expansion of agriculture (Messias et al., 2021). The populations of Portugal and the Czech Republic have suffered from high levels of indebtedness (Opletalova, 2015; Santiago, 2016), and the African continent has a disparity in economic development associated with social factors such as high poverty rates, informal work and low school enrollment rates (Messy & Monticone, 2012).

Issues like these have been the target of global concerns and movements, especially since the beginning of the 21st century, when the millennium development goals (MDGs) were proposed, challenging countries to end hunger and poverty (MDG 1) and reduce child mortality (MDG 3), for example. Subsequently, a new movement emerged with the need to continue and improve these goals, culminating in the document "Transforming Our World: The 2030 Agenda for Sustainable Development" (ONU BR, 2016).

The United Nations' 2030 Agenda, approved by representatives of 193 countries in September 2015, highlights 17 new goals, the sustainable development goals (SDGs), to be achieved globally by 2030. These goals are: eradicate poverty (SDG 1); eradicate hunger (SDG 2); quality health (SDG 3); quality education (SDG 4); gender equality (SDG 5); clean water and sanitation (SDG 6); affordable and clean energy (SDG 7); decent work and economic growth (SDG 8); industry, innovation, and infrastructure (SDG 9); reduce inequalities (SDG 10); sustainable cities and communities (SDG 11); responsible consumption and production (SDG 12); climate action (SDG 13); life below water (SDG

14); life on land (SDG 15); peace, justice, and strong institutions (SDG 16); and partnerships for the goals (SDG 17) (ONU BR, 2016).

These goals support the MDGs in the vision of sustainability already adopted, affirming the need to think about sustainable development in three aspects: social, economic, and environmental (ONU BR, 2016). However, despite the 2030 Agenda discussing various necessary contemporary elements and challenges, one of its problems is the inseparability and interdependence between the SDGs. Campos (2020, p. 40) exemplifies this criticism by highlighting that SDG 8 "has positive interactions, allowing for the reduction of poverty (SDG 1) and hunger (SDG 2), but also negative interactions such as increased consumption (SDG 12) and increased emissions (SDG 13)." Developing one SDG can thus impact other SDGs – positively or negatively. Additionally, we need to be aware that the formulation of the SDGs was carried out and is maintained by an intellectual and social elite, while many people need to be made aware of these goals to problematize them.

Through our discussions, studies, and publications, we have recognized the importance of advocating for quality education (SDG 4). Quality education helps raise awareness and empower individuals and plays a vital role in achieving other sustainable development goals, ultimately leading to a more just and equitable world across social, economic, and environmental domains. For instance, in mathematics education, concepts from ethnomathematics, such as education for citizenship and peace (D'Ambrosio, 2012), contribute to these discussions. Furthermore, critical mathematics education emphasizes the necessity of addressing fundamental conditions for acquiring knowledge, being mindful of social issues, inequalities, and oppression, and striving to make education a proactive agent for social change (Skovsmose, 2001, p. 101).

Previous discussions at the Congress of the European Society for Research in Mathematics Education (CERME) related these aspects. For example, Vorhölter and Siller (2023) report a program developed in Germany aimed at education for sustainable development and the role of mathematical modeling in this context. Furthermore, Fuertes-Prieto et al. (2023) reflect on including discussions about the SDGs in teacher training and incorporating sustainability into the curriculum alongside mathematics.

In other words, this research shows us that sustainability has been the object of analysis in mathematics education, but that we need to move forward in this area. The problems highlighted in the SDGs require global action, and we understand that one way is through education, putting them into discussion and thinking about practices that support the achievement of the objectives. As pointed out by Fuertes-Prieto et al. (2023), sustainability has been included in curricula, combined with mathematics, and we need to reflect on whether university teachers are aware of these issues and discussed with future basic education teachers. Still, for example, in Brazil's curriculum (Ministério da Educação, 2018), themes related to the SDGs, such as the environment, food education, human rights, health, gender and consumption are considered contemporary themes, which should permeate the mathematics subject. Vásquez et al. (2023) indicate that education for sustainable development is one of the ways to face the current crises, with mathematics education being a possibility to achieve this, although teachers have limited knowledge on the subject.

In this set of ideas, aiming to expand the relationships between mathematics education, sustainability, and the SDGs, we conducted research driven by the question: what relationships can be established

between sustainability and mathematics education based on knowledge and practices of university professors? Based on idealized actions and the results of the study conducted, in this text, we aim to discuss the understanding and sustainability practices of university professors in Mathematics Education working in Portugal in light of the United Nations' 2030 Agenda.

Methods

We conducted qualitative semi-structured interviews with university professors in mathematics education who work at universities in Portugal. When conducting semi-structured interviews, it is essential to have a prior script to guide the dialogue (Fiorentini & Lorenzato, 2012; Creswell, 2014).

We guided the interviews based on eight themes: i) education; ii) work; iii) projects and research lines; iv) sustainability; v) SDGs; vi) mathematics; vii) partnerships; and viii) indication of other projects and researchers. The first group of questions (i-iii) aimed to get to know the professors, their knowledge and practices, based on the exposure of their trajectories and experiences. The second part (iv-vi) focused on practices and knowledge about sustainability and mathematics education. Finally, the last questions (vii-viii) sought to address SDG 17.

We proceeded with research in Portuguese repositories and university websites and recommendations from research colleagues to invite university professors for the interviews. From the contacts made and the recommendations of new professors from the interviews conducted (by question viii), we totaled ten acceptances, whose synthesis we present in Table 1.

Table 1: Interviews conducted

Nº	*Date*	*Interview time*	*University*
1	January 16, 2024	30min 45s	Universidade de Lisboa
2	January 23, 2024	26min 29s	Universidade NOVA de Lisboa
3	January 26, 2024	1h 08min 56s	Universidade NOVA de Lisboa
4	January 30, 2024	41min 19s	Universidade de Lisboa
5	February 7, 2024	1h 08min 57s	Universidade Lusófona
6	February 12, 2024	49min 23s	Universidade Algarve
7	February 16, 2024	52min 29s	Escola Superior de Educação de Lisboa
8	March 1, 2024	35min 27s	Escola Superior de Educação de Coimbra
9	March 22, 2024	1h 01min 26s	Universidade de Lisboa
10	April 10, 2024	48min 17s	Universidade de Lisboa

The data generated from the interviews presented in Table 1 were analyzed using content analysis techniques (Bardin, 2016), following three phases: pre-analysis, material exploration, and treatment of results and inferences. In the pre-analysis phase, we transcribed the interviews with the help of the *Good Tape* software and then read and textualized them. Textualization consisted of organizing the transcription in text form, cleaning up language defects and word repetition.

In the material exploration, we identified the context units (data excerpts) and the registration units (themes related to the context units for categorization purposes). For example, in the interview with Professor 5, it was mentioned that: "my dear because the title is social and educational inclusion as a

social justice project. This title is a vision of inclusion issues, even in social and social terms, right?" We observed that this context unit was related to the social aspects highlighted in the 2030 Agenda. Therefore, we associated ourselves with this registration unit that we called "social sustainability".

In this text, we aim to discuss the understanding of sustainability of university professors in mathematics education working in Portugal. Considering the United Nations (UN) 2030 Agenda, we chose the registration units a priori since the 2030 Agenda is based on economic, social, and environmental sustainability. Finally, in the treatment of results and inferences phase, we proceeded with categorization through the aggregation/convergence of the recording and context units. The following section presents interview data on the three aspects of sustainability.

Presentation and discussion of data

By analyzing the interview texts using content analysis techniques, we observed the discourses of the university professors related to the three aspects of sustainability. Below, we briefly present some examples of the interviewees' statements (as similar ideas were repeated at other interviews) associated with the economic, social, and environmental sustainability of the 2030 Agenda.

Economic sustainability

The first aspect we highlight concerns the economic sustainability presented by the 2030 Agenda, which makes us reflect on issues related to an economically efficient and balanced world with coherent investments that support the other pillars of sustainability. Among the goals of SDG 17 – partnerships and means of implementation, it states, "assist developing countries in achieving long-term debt sustainability through coordinated policies aimed at fostering financing, debt relief, and debt restructuring [...]" (ONU BR, 2016, p. 40). The teacher-researchers in the research addressed practices and ideas inherent to economic sustainability, for example, by professors 3 and 8.

Professor 3: I am granted this second postdoc in environmental sciences because my sustainability process involves practical work with fishermen. And rethinking the fishing practice itself, right? I also revisited the materials I use, the way they fish, the importance of artisanal fishing, and how they need to engage with what is nowadays called sustainability.

Professor 8: Well, sustainability is one of the themes connected with financial education. When I work on the theme with my students, when we address the theme besides the economic issue, we always refer to the sustainability issue because, naturally, associated with the consumption of a product, we are, in a way, consuming more products, or more material goods from our planet. Therefore, when we address water consumption, healthy eating, and all these very connected issues, we can work on financial education, economic education, consumer education, and consumption sustainability. Health education related to healthy eating also comes with sustainability issues. Therefore, this allows us to work with students more reflectively, escaping what I usually call escaping mathematics, where the goal is to solve an exercise and see if it is right or wrong. No, we are on a path to, and the goal is that mathematics is a tool to help us reflect and be critical and informed citizens in different areas because we are forming, not students, but citizens, and citizens who will be consumers and will be on this planet for many years. This reflection type is associated with financial education, consumer education, and sustainability.

From these data, we can observe public investment in financing sustainability-related projects, as reported by Professor 3. The 2030 Agenda (2016, p. 25) indicates that "by 2020, substantially increase globally the number of scholarships available to developing countries, particularly least developed countries, small island developing states, and African countries, for higher education." Furthermore, there is the potential for discussions in financial education associated with sustainability, such as those related to consumption. Additionally, Professor 8 indicates work beyond numerical mathematics, where only the exercise and the result matter, a factor associated with the ideas of critical mathematics education, such as scenarios for investigation (Skovsmose, 2001).

Social sustainability

The social sustainability advocated by the 2030 Agenda highlights the need for and importance of thinking about opportunities and rights for all, focusing on people and society in general, where human rights are preserved and the less privileged have more opportunities to ascend socioeconomically. The document emphasizes pursuing a "world with equitable and universal access to quality education at all levels, healthcare, and social protection, where physical, mental, and social well-being are ensured" (ONU BR, 2016, p. 4). Ideas associated with social sustainability were explicitly presented by the research participants, such as Professor 1 and Professor 5, as follows.

> Professor 1: I am mainly at the university, contributing to the sustainability, let us say, of the planet, particularly in the domain of social sustainability, as it contributes to making society more coherent. Indeed, its members are better educated, especially regarding their domain of mathematics, but also, in a broader sense, when learning mathematics, we develop more globally. And therefore, from this perspective, there is undoubtedly a contribution. I contribute strongly to social sustainability, more directly in Portugal and through my work, travels, and interactions with people from other countries, particularly in Brazil, which I have visited many times. Many periods in Brazil... In various universities, working with teachers and students, especially postgraduate students from these universities, on topics that significantly contribute to social sustainability.
>
> Professor 5: My areas of intervention are in the master's degree in special education, where I teach a curricular unit, which I think has my, as I usually say, my face because the title is social and educational inclusion as a social justice project. This title is a vision of inclusion issues, even in social and societal terms, right? The issue of human rights encompasses all these fields: human rights, discrimination, and participation. This is the issue that Hannah Arendt brought us: the right to have rights, right? Therefore, it is an important issue. And that human rights also start at home, right? How we relate to those closest to us, listen to them, and express ourselves. Well, this is a curricular unit.

From the above, as Professor 1 indicates, mathematics has the potential to provide people with a global understanding of different aspects. Furthermore, Professor 5 reports practices related to social justice, involving the inclusion of the entire population and human rights. In this context, we highlight that SDG 16 aims to promote justice for all, and the vision of the 2030 Agenda (2016, p. 4) is associated with "a world of universal respect for human rights and human dignity, the rule of law, justice, equality, and non-discrimination; respect for race, ethnicity, and cultural diversity." For a socially sustainable world, ethnomathematics supports the discussion of education for citizenship and a world of peace, primarily through the appreciation of different knowledge and cultures (D'Ambrosio, 2012).

Environmental sustainability

The third and final aspect of sustainability in the 2030 Agenda, which concerns the environment, calls for respect and appreciation of natural ecosystems, where social and economic elements prosper alongside nature. Mentioning goals associated with SDG 12 – responsible consumption and production, for example, the 2030 Agenda (2016, p. 34) states the need to "encourage companies, huge and transnational companies, to adopt sustainable practices and to integrate sustainability information into their reporting cycle." The interviewees, especially professors 3, 7, and 9, revealed practices associated with environmental sustainability.

Professor 3: When we write, we write, right? Since I got here, SDG 14, the ocean one, all of mine now. It has to be, dear, right? When I did Urban Frontiers, there was no SDG yet, so that should be in SDG 4 easily. And one easily. How ridiculous, right? I will not eradicate anything, but... Also, there are spaces and paths for improvement. So, if I look now and need to categorize, I categorize, right? But it is not just that; it is also the economy. The environment, in general, is also in the neighborhood, right? Sustainability: I have 15, right? Anyway... If I go through my projects, if you want a straightforward answer, it is SDG 14, Smartfishing, Party Bridge. Party Bridge and Urban Frontiers, Party Bridge, too, right? SDG 4... I don't put Smartfishing in 4 because four don't fit informal ocean literacy. I didn't have any formal process. Or informal, I only had a non-formal process. Now... The others didn't, the others did, right? Party Bridge had classes here at the college, and the fishermen came to teach. For the master's degree, we prepared a lot of things beforehand. And... So, these are the two SDGs that permeate the most, and that is, I don't know if I needed to place them as sustainable; they were these two, too.

Professor 7: Mathematics in schools, environmental themes, or environmental aspects. Therefore. Interdisciplinary practices are based on research in initial teacher training. We considered that this chapter contributed to 4, therefore, quality education. 7, renewable and accessible energies. 11, sustainable cities and communities. 13, climate action. And 15, life on land. This is because the chapter encompasses work not only in this curricular unit that I mentioned. But also, others and other projects, namely from other teachers. They are also co-authors of this chapter from natural sciences. Then, we focused on group work that our future teachers had planned for this curricular unit. Indeed, the groups focus on a wide variety of environmental problems. Therefore, they touch on different aspects, from energy consumption to other issues. Therefore, there is an excellent diversity of themes. Others are more focused on the preservation of plant species.

Professor 9: Yes, it was interesting because, in that project, I mentioned statistical literacy and statistical reasoning; I don't know if it was by mere chance, but we explored some issues related to the environment. It wasn't intentional, but in the search for themes, it came up because it is natural for statistics to be a tool for interpreting reality. So, it came a bit from there, and it was even later, based on those materials we had created, that we developed a small project with colleagues from the area of the sciences, where we sought it was a very significant project for us... The theme was related to global warming issues. It was the theme we chose, bringing perspectives from science people, their knowledge, and mathematics, in this case, with statistics. This was a very significant project because they had the opportunity to work together over a semester.

By reporting the practices and results of a research project (Professor 3) and a published article (Professor 7), the teachers related their experiences to SDGs 11, 13, 14, and 15, mainly referring to the environmental dimension of the 2030 Agenda. Ideas like those these teachers expose help us problematize "the threat posed by climate change and environmental degradation" (ONU BR, 2016,

p. 11). Furthermore, Professor 9 indicates the potential of working with statistics to discuss aspects related to environmental sustainability, such as global warming. One way to address these issues is through mathematical modeling, as highlighted by Vorhölter and Siller (2023).

Final considerations

In this text, we discussed the understandings and sustainability practices of university professors in mathematics education working in Portugal in light of the United Nations' 2030 Agenda. The interviews, conducted with ten university professors in mathematics education from Portugal, indicated relationships with the three aspects of sustainability advocated by the 2030 Agenda: economic, social, and environmental.

Although this relationship between teachers' practices and understandings with the three aspects of sustainability is one of the results of this study, the participants did not carry out in-depth studies on the 2030 Agenda and the SDGs. During the interviews, they revealed that, although universities had begun demanding inclusion of the SDGs in subject plans and publications, training and opportune moments to actually learn about the 2030 Agenda's propositions were not provided.

We identified that practices in mathematics education have supported the promotion of the SDGs of the 2030 Agenda, as the excerpts from the interviews explicitly referred to various goals, such as SDGs 1, 4, 11, 13, 14, and 15. The interviewees revealed ideas concerning public investments in research related to sustainability (Professor 3); work with financial and statistical literacy to promote economic and environmental sustainability (professors 8 and 9); the potential of global knowledge to be explored from mathematics for social sustainability (Professor 1); as well as disciplines and publications related to sustainability (professors 5 and 7).

In conclusion, the findings of this study have significant implications for the field of mathematics education, particularly in the context of integrating sustainability into educational practices. By highlighting the alignment between mathematics education and the sustainable development goals of the 2030 Agenda, this research underscores the potential of mathematics as a transformative tool for fostering sustainability literacy among students. The study demonstrates that mathematics educators can play a crucial role in bridging the gap between abstract mathematical concepts and real-world sustainability challenges. By incorporating themes such as financial and statistical literacy, educators can equip students with the critical thinking skills necessary to analyze and address complex issues like climate change, resource management, and social equity. Furthermore, the study advocates for a curriculum that not only emphasizes technical proficiency but also encourages students to apply mathematical reasoning in evaluating and solving sustainability-related problems. This approach not only enriches the educational experience but also prepares students to become informed and responsible global citizens. As such, the integration of sustainability into mathematics education is not merely an academic exercise but a vital component of nurturing a generation capable of contributing to a sustainable future.

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Mathematics teacher practices to foster reasoning and communication through making connections

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We present preliminary results from an exploratory study aimed at identifying aspects of teaching practice that are relevant to establishing intra-mathematical connections related to processes in the classroom. A content analysis of both small and large group discussions in the classroom was conducted, which revealed two types of connections associated with the processes of reasoning and proof, and one type associated with the process of communication. The analysis of these connections has allowed for the identification of two aspects of teaching practice that facilitate the establishment of connections: using students' contributions to focus the discussion on a specific aspect of a process and repeating the same connection related to a process across different sessions and in various social settings in the classroom.

Keywords: Mathematics connections, elementary education, mathematical processes, teacher practice.

Introduction

There is a consensus within the educational community regarding the importance of establishing mathematical connections in the classroom to foster a deep understanding of mathematical ideas, concepts, and procedures (De Gamboa et al., 2023; Rodríguez-Nieto et al., 2023). This consensus is reflected in the curricula of various countries, such as the United States (NCTM, 2000) and Spain (LOMLOE, 2020), which highlight the necessity for students to develop key competencies, such as making connections, to promote the view of mathematics as an integrated whole. While students may make connections spontaneously, it cannot be assumed that these connections will occur without some form of intervention from the teacher. In this regard, the teacher's role in establishing connections in the classroom is crucial, not only in the type of tasks they propose to their students but also in the management of mathematical activity and the teaching practices they employ (Mosvold, 2008). However, the lack of specific models for how to introduce and establish relevant mathematical connections in the classroom hinders teachers from incorporating connections into their regular practice (De Gamboa et al., 2023). In this study, we examine how the teacher promotes the establishment of connections in the classroom.

Specifically, we are interested in a type of connection identified in previous studies (De Gamboa et al., 2023): intra-mathematical connections related to processes such as reasoning and proof or communication. We focus on studying these connections for two main reasons. Firstly, it is well-known that developing mathematical processes in the classroom is crucial to move away from teaching practices focused on the acquisition of symbols and techniques, which doesn`t focus on the

strategic and effective use of mathematical content in various contexts. Secondly, there is a need for further research into the teacher's role in establishing connections that support the development of processes (Jakobsen et al., 2013; De Gamboa et al., 2023). Thus, the aim of this communication is to identify some characteristics of teaching practice to promote mathematical connections related to processes, through a case study in a primary school in Spain.

Theoretical framework

In the field of Mathematics Education, there is a growing interest in the conceptualisation of connections, which has led various researchers to characterise different types of connections that can be established in the classroom (e.g. Businskas, 2008; De Gamboa et al., 2023; Dolores-Flores & García-García, 2017; Eli et al., 2011; Rodríguez-Nieto et al., 2023). Broadly speaking, a primary distinction can be noted (De Gamboa et al., 2023) between extra-mathematical connections, which occur when mathematics is linked to a phenomenon external to mathematics (real-life connections as discussed by Mosvold, 2008), and intra-mathematical connections, which occur when two mathematical objects are connected (Font et al., 2013). Within this second category, two types of connections can be further identified: conceptual connections, which emphasise specific aspects of a mathematical concept, and connections related to processes, which are the focus of this study. The latter emphasise a process that can be applicable to various mathematical concepts, such as the communication of mathematical ideas or the validation of results (De Gamboa et al., 2023). We understand mathematical processes as process standards (NCTM, 2000), that is, ways of acquiring and using mathematical content, ideas, and procedures across diverse contexts. In connections related to processes, a mathematical concept is linked with a particular aspect of a process. For instance, in the communication process, one might emphasise the clear and comprehensible communication of ideas to others, the use of precise mathematical language, or the organisation and consolidation of mathematical knowledge through oral or written communication. In the reasoning and proof process, emphasis might be placed on making and validating conjectures, developing mathematical arguments, or using strategically different types of reasoning.

In a classroom context, such as the one analysed in this study, mathematical connections emerge as a result of the interaction between the teacher and the students. We understand both intra and extra-mathematical connections as networks of more fundamental connections that arise from the interactions occurring within the classroom (De Gamboa et al., 2023). These fundamental connections can link two representations (Businskas, 2008; De Gamboa et al., 2020; Dolores-Flores & García-García, 2017; Rodríguez-Nieto et al., 2023); common aspects of different concepts (De Gamboa et al., 2023; Eli et al., 2011); a concept with a procedure (Businskas, 2008; Dolores-Flores & García-García, 2017; Rodríguez-Nieto et al., 2023) or two procedures (De Gamboa et al., 2023); and two propositions (Businskas, 2008; De Gamboa et al., 2023; Rodríguez-Nieto et al., 2023). For example, in the context of a classroom discussion, a connexion related to the reasoning and proof process can emerge when the teacher and students propose related arguments, using a range of representations, procedures, or propositions. Each of these relationships that arise in such utterances constitutes a fundamental connection, and when understood collectively, they form the connection related to reasoning and proof.

Methods

This study is situated within a qualitative and interpretive paradigm with the aim of exploring, describing, and explaining (McMillan & Schumacher, 2005) how connections are established in the mathematics classroom. As part of a broader research project focused on studying mathematical connections in the classroom, we designed a sequence of five sessions for students in the final year of Primary Education (11–12 years old). The learning goal of the sequence is the modelling of the relationship between the circumference of a circle and its diameter. This was conducted in a classroom at a school in Catalonia (Spain) over a period of two weeks at the end of the 2023–2024 academic year. The tasks, developed in collaboration with the classroom teacher, provide potentially rich contexts for fostering various connections in the classroom, incorporating both real-world contexts and work with diverse semiotic representations. Students worked in small groups and were required to complete individually the dossier containing the tasks. The teacher regularly encouraged discussion in both small and large groups, without any specific guidance from the researchers regarding the establishment of connections in the classroom. In this communication, we focus on analysing the data from the first two sessions, in which students explored the relationship between the length of a circle and its diameter. In the first session, students experimented with labelling bottles using paper and scissors. In the second session, they used the dynamic geometry software GeoGebra. The following is the statement of the first task.

Task 1. Marina owns a company that specialises in labelling bottles. A mineral water company has requested a proposal for labelling bottles of a new water brand they plan to launch. Marina has been asked to present three label options for three cylindrical bottles of different sizes: one with a diameter of 10 cm, another with a diameter of 20 cm, and the third with a diameter of 30 cm. The labels must meet the following requirements: 1) Each label must have a height of 4 cm, and 2) The labels must be adhesive and wrap fully around the bottle without overlapping (i.e., no part of the label should stick on top of another part).

1. How can Marina determine the length of the labels if she only knows the diameters of the bottles? Fill the table with the diameter of each bottle and the length of the label.
2. How many times does the diameter of the bottle fit into the length of the label? You could explain this with a drawing.
3. How would you express the relationship between the diameter of the bottle and the length of the label?

Diameter of Bottle (cm)	Length of Label (cm)
10 cm	
20 cm	
30 cm	

Figure 1: First task of the didactical sequence

The classroom sessions were recorded following a non-participant observation approach (Cohen et al., 2007). A content analysis (Krippendorf, 2018) of the discussions in both small and large groups involving the teacher and students was conducted in two phases. The first phase involved identifying and classifying the connections (De Gamboa et al., 2023) established in the classroom interactions between the teacher and students. The basic units of analysis are episodes defined by the collection of utterances that form each connection. In the second phase of the analysis, we focused on examining episodes where intra-mathematical connections related to processes were identified. This analysis helps to identify key aspects of each process (such as proving, verifying, and communicating effectively) and to explore relevant aspects of teaching practice in the establishment of these connections. In both phases, validation through expert triangulation (Flick, 2007) was crucial to

ensure the meaningful interpretation of the data and to minimise interpretative biases. First the data is analysed individually by each author, then we compare the results, discuss the cases in disagreement and write the agreements reached in joint meetings.

Results

In the analysis of the first two sessions, we identified a total of 27 episodes where mathematical connections were established. Of these, 10 episodes involved connections related to the processes of reasoning and proof, and communication, linking a mathematical concept with a fundamental aspect of one of these processes. Specifically, for reasoning and proof, connections were associated with the justification of mathematical statements (3 episodes) and the verification of conjectures or results (3 episodes). For communication, we identified 4 episodes that emphasised the clear written communication of mathematical ideas to others. Below, we present three episodes from the first session that are representative of each type of process-related connection identified in the data analysis, as well as aspects of teaching practice that foster the establishment of these connections.

Justifications of mathematical statements

In these connections related to the process of reasoning and proof, the teacher emphasises the necessity of justifying mathematical statements. An illustrative example of this is provided by the following episode, which occurs when the groups begin working on the first task (Figure 1), empirically measuring the diameter and the length of the circumference of each bottle. The connection emerges when the teacher approaches a group of students who, during their exploration of prior ideas, had proposed that the length of the circumference is the diameter multiplied by Pi (π).

Student1:	So, what do we need to discover? The number Pi, why do we multiply?
Teacher:	What is the relationship between Pi and the diameter?
Student2:	I mean, Pi...
Student1:	But I have to multiply to find out.
Teacher:	Yes, but if you need to multiply, it means they are related, right? Pi and a diameter. So, what is the relationship? Why Pi and not some other number?
Student2:	To find out the number Pi, you have to multiply.
Student1:	But Pi is 3.14...
Teacher:	Right, but why do I multiply by 3.14 and not by 2.65 or the number e, for example? What is the relationship?
Student2:	Between the diameter and Pi.
Student1:	Why do you multiply by Pi? Summary?
Teacher:	I think multiplying means doing something a certain number of times, right? So why do I do the base of the bottle 3.14 etc. times? Why not do it two or four times? Am I making myself clear?
Student1:	Yes, yes, yes.
Teacher:	Where does this Pi come from? Why is Pi the number that I have to use as many times as the diameter? Yes?
Student1:	Because they felt like it.
Teacher:	Nooo, mathematicians never come up with things just because. There is always a reason and a rationale.

Although the students are already familiar with the formula, the aim of this task is to give sense to the modelling process based on experimental data. The aim is for the students to use empirical measurements of the diameter and circumference of the bottles to argue the origin of Pi. Throughout the episode, the teacher recurrently poses questions related to the task that clarify and specify the

task's requirement: to justify the formula they already know. The teacher reformulates or follows up the students' contributions, focusing the dialogue on the need to uncover the reasoning underlying the use of Pi as a proportional constant between the diameter and the circumference length. By the end of the episode, Student1 reveals a view of mathematics in which the formula is seen as an arbitrary result that does not require justification. The teacher uses this contribution to emphasise that mathematics is never arbitrary and that there is always reasoning behind general rules. This highlights how the teacher directs the dialogue towards mathematical justification, fostering the connection between Pi as a proportionality constant (the relationship between diameter and circumference, expressed as a formula) and the notion of justification/proof in mathematics. Establishing this connection can help students new to using formulas to understand that these formulas are derived from mathematical reasoning that they can comprehend and replicate, as well as to recognise proof as an essential aspect of mathematics. In the second session, two additional episodes of this type occur, where the teacher reformulates, follows up, or questions the conjectures or partial solutions provided by the students to direct the discussion towards the need to justify the mathematical statements.

Validating results

In these connections related to reasoning and proof, the teacher emphasises the need to explore different cases before establishing a conjecture about an observed pattern. The following episode occurs when the students are working in groups on solving the first task (Figure 1). The teacher had recorded empirical measurements of the length and diameter of various bottles in a two-column table on the board. The episode begins as the students are engaged in finding the relationship between the diameter and the length.

Student1:	Teacher, when you multiply the radius by the diameter, it gives you more or less… It's that here we've approximated…
Teacher:	The radius by?
Student1:	By the diameter. The diameter gives you the length, the perimeter.
Teacher:	The radius by the diameter?
Student1:	Yes.
Teacher:	So, here it would be…
Student1:	3.25 times 6.5. Well, it gave me 21.1250.
Teacher:	But with just one operation, can we say this is a rule? Or could it be a coincidence?
Student1:	Okay, but…
Teacher:	No, no, I'm just asking, simply asking.

In this episode, the teacher questions whether a property or general rule can be deduced from a single case when Student 1 proposes an incorrect property for finding the length of the label by multiplying the radius by the diameter. The teacher does not focus on the validity of the property itself but rather on the validity of the reasoning used to deduce it. The student's reasoning is based on a single particular case where the radius is relatively close to Pi, leading to a length result that is near certain approximations previously shared on the board for that case. Thus, the teacher reformulates and questions Student1's contributions to highlight the distinction between a particular result and an observed pattern (conjecture), emphasising the need to test more than one case (operation).

This episode demonstrates a connection between an incorrect relationship involving the radius, diameter, and length, and the notion of inductive reasoning for establishing conjectures. Establishing

this connection can help students understand that if a property holds in a finite number of cases, it cannot be assumed to hold universally. In other words, before establishing a conjecture or accepting the validity of a property, it is advisable to check various cases to see if the property holds. In the other two episodes of this type, the teacher again uses students' contributions to focus the discussion on the reasoning employed to validate a conjecture or general rule, reiterating the notion of inductive reasoning for formulating conjectures or verifying results.

Written communication of mathematical ideas

In these connections related to the process of communication, the teacher emphasises that written communication must be comprehensible to others. We present the first episode of this type of connection identified in the data. This episode occurs when the students are formalising in writing the relationship between the diameter and the length of the circumference after a class discussion aimed at deducing this relationship from the results of the diameter and length measurements of different bottles, which had been recorded in a table on the board.

Student3:	Should we ask this question about our bottle or the bottle of ten, twenty, and thirty?
Teacher:	Nooo… Based on what we know right now, right? Regarding the bottle and the discussion we've just had, we can answer.
Student4:	But are we making a bottle?
Teacher:	You can make it understandable. If a person uses the explanation here, they should be able to reproduce exactly what you're explaining. Okay?

In this episode, the teacher guides a small group on how to communicate mathematical ideas in writing. In the first intervention, the teacher clarifies that the written response must be based on the class discussion surrounding the results from experimenting with various bottles. In the second intervention, the teacher instructs that the explanation should be understandable to anyone, such that they could replicate the problem-solving strategy used. This approach connects a classroom activity where a pattern is observed, identifying Pi as a proportionality constant between length and diameter, with the process of communication. Later, while the students continue working in groups, the teacher reviews a student's response that relies solely on a drawing and tells them that "it lacks text," meaning written explanation. Addressing the entire class, the teacher states, "Think about this; that is, your explanation should enable someone who has not solved this problem with us to solve it. Right? They should be able to reach the same conclusion we have". This episode demonstrates how the teacher promotes the communication of mathematical ideas as a narrative construction that should enable an outsider to understand and reproduce the activity's content. Establishing this connection helps students consolidate their mathematical thinking through clear written communication that is understandable to others. Such interventions are repeated throughout the two sessions, addressing various aspects of the tasks and involving interactions with individual students, small groups, and the entire classroom.

Concluding discussion

The identified connections in the first phase of the study refine the definition of connections related with processes proposed by De Gamboa et al. (2023). While De Gamboa et al. (2023) defined these as connections between a concept and a process, our study has identified that such connections can also occur between an incorrect or incomplete procedure and a process. Additionally, this study

documents some intra-mathematical connections related to communication and reasoning and proof, contributing to the existing literature that predominantly focuses on extra-mathematical connections or intra-mathematical connections between concepts, procedures, or representations. Documenting these connections is important for understanding the role of connections in the development of processes in mathematics teaching and learning, and for promoting the development of rigorous and complex reasoning and communication from an early stage.

In the second phase of the study, we observed how the teacher encourages the establishment of connections by leveraging students' contributions to focus the conversation on specific aspects of a process. Throughout the two sessions, and in relation to different aspects of the task, the teacher repeatedly highlighted the same connections related to the written communication of mathematical ideas; the justification of the relationship between length and diameter; or the verification of results obtained experimentally (inductive reasoning). The combination of these two characteristics of teaching practice, both in small group and whole class interactions, appears to be crucial in establishing connections in the classroom. The use of student's contributions to trigger the repetitions of the same connections facilitates the institutionalisation of this mathematical knowledge within the classroom. Although these are preliminary results that may be reinforced with the analysis of the additional data of the three last sessions, this experience exemplifies the importance of classroom management and teaching practices enacted by the teacher.

It must be assumed that if primary students are to stablish connections, their teachers must be active agents not only in the selection of connection-rich tasks, but also in the classroom management of these tasks to promote the establishment of mathematical connections. Therefore, this line of research should result in the design of formative tasks for preservice prospective teachers to develop specific teaching practices that facilitate the identification and the establishment of mathematical connections.

While most research on mathematical connections in the classroom focuses on classifying types of connections, this exploratory study points to the importance of analysing teaching practices that facilitate the establishment of specific types of connections. In this sense, as a future line, we ask whether teaching practices that promote connections related to processes are similar to practices that promote other types of connections. Additionally, we wonder if the model of teaching that teacher hold has an impact on the promotion of connections, and if so, how are they related.

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Identifying the needs of mathematics teachers during the implementation of new mathematics education goals

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Latvia, like many countries, is implementing a new mathematics curriculum emphasizing understanding and problem-solving. The success of these changes relies heavily on teachers. This study aims to explore the opportunities for students to understand mathematics at different grade levels and the teaching profiles of Latvian mathematics teachers. This research, conducted in rural and urban schools, used an exploratory mixed-method design, observing and analysing 78 mathematics lessons from the year 2022 to 2024. The study found significant differences in teaching performance across various criteria. Better performance in clarity of content and lesson design was compared to student cognitive activation and metacognition. This highlighted the need for targeted professional development to address the specific needs of teachers to best meet new curriculum goals.

Keywords: Teaching performance, lesson observation, professional development, understanding.

Introduction

Every country defines mathematical competencies for its youth and structures its education system to achieve these goals. Historically, these competencies emphasized basic mathematics skills. However, the digital age and global challenges have redefined goals for mathematics teaching and mathematical competence. They now emphasize data literacy and critical thinking to navigate personal and societal issues in the 21st century (OECD, 2023). The authors' country, Latvia, like many others, is experiencing educational reforms and curriculum change. In any reform movement, teachers are key agents in the delivery of the new curriculum. Therefore, the potential success of the reform is heavily dependent on them (Shimizu & Vithal, 2023). In Latvia, starting in 2020, a new mathematics curriculum was introduced. It emphasizes conceptual understanding, seeing and formulating connections, problem-solving, and the use of multiple representations. This implies that teachers must have a deep understanding of the mathematics that they are teaching. They must also understand how students learn mathematics, including a keen awareness of the individual mathematical development of their own students, and be able to select strategies that will enhance learning (Van de Walle et al., 2010). Given the significant variation in teaching performance levels in Latvia, both between different schools and within the same school, there are concerns that for some teachers, acquiring and applying the necessary skills to implement the new competency-based approach may be challenging. Additionally, some teachers may find it difficult to change their beliefs and attitudes to effectively adopt this new approach. This concern was highlighted by a recent study on the support teachers need to provide a 21st-century education (UL ICEI, 2023). It is crucial to evaluate the current teaching performance of mathematics teachers and determine the next steps to enhance their practice. There is limited research in Latvia on the cognitive depth of students'

classroom experiences. Therefore, this study contributes to identifying the needs of teachers and the necessary support.

Theoretical framework

It is a cognitive process in which the transformation of ideas or concepts takes place. The newly learned is constructed based on what was previously learned, forming a connection and being included in the existing knowledge structure (Ausubel, 1968; Van de Walle et al., 2010). When constructing meaning, the transformation of the student's initial concepts takes place (Resnick, 1987). When students transform their initial ideas into scientific ones, the formation of misconceptions is also possible. Students must learn mathematics with understanding, building on prior knowledge and experiences, which can be enhanced through classroom discourse (Sawada et al., 2000). Meaning is formed by reasoning, not by mechanical memorization as reproduction. That is why learning tasks should be meaningful - to give the opportunity to construct meaning of what is to be learned (Möller, 2019). Deep learning requires an understanding of what has been learned and the ability to transfer what has been learned from one context to another, employing higher-order thinking skills (Biggs & Collis, 2014). Mathematical proficiency encompasses more than just procedural understanding. It integrates conceptual understanding, strategic competence, adaptive reasoning, and productive disposition. Despite this, many teachers still prioritize procedural fluency, focusing on memorizing number facts and mastering traditional algorithms (Mhuirí & Neary, 2024). Respecting teachers' existing knowledge while enhancing their capacity for new teaching methods and acknowledging their work are key principles in curriculum reform (Shimizu & Vithal, 2023). Research (Richland et al., 2012) indicates that teachers' knowledge and beliefs about mathematics, including the importance of connections of mathematical concepts and representations, significantly impact their teaching practices. Teaching practices can be measured using rubrics, both by the teachers themselves and by experts observing the lessons (Namsone et al., 2021). Understanding the diverse needs of teachers can enhance professional development (PD) activities and ensure maximum benefit for all participants (Stevens et al., 2013). Accordingly, two research questions were stated: 1) What opportunities do students have to make sense of mathematics at different grade levels? 2) What teaching profiles are characteristic of Latvian mathematics teachers regarding the opportunities for deep learning in their lessons?

Method

This research was carried out in rural and urban schools. The researchers collaborated with municipality and school leaders. To answer the research questions, an exploratory mixed-method research design and content analysis were chosen. School mathematics lessons were observed from the year 2022 to 2024. A total of 60 lessons in Grades 1 to 6 and 18 lessons in Grades 7 to 12 were observed and transcribed by experienced and specially trained experts. The experts analysed the transcriptions. The authors used a previously developed and validated instrument – the Framework of Teacher Performance Assessment to Support Teaching 21st Century Skills (Namsone et al., 2021). The selected instrument is a category-criteria framework, consisting of 17 criteria that conceptualize teacher classroom practices that support the acquisition of 21st-century skills in the authors' country's

educational context. The criteria are structured in 6 categories (i.e., student self-regulation, student cognitive activation) and three domains (planning, teaching, and classroom environment) derived from teachers' everyday practice. The authors selected 4 categories and 5 criteria that are highly relevant to teaching mathematics for understanding (Table 1).

Table 1: Categories and criteria that are highly relevant to understanding mathematics

Category	Criteria
Learning content representation (CR)	6.1. Clarity of learning content
Instruction (INSTR)	5.1. Lesson design, 5.2. Teaching technique.
Student cognitive activation (SCA)	2.1. Learning tasks for deep learning, 2.2. Discussions for deep understanding
Metacognitive strategies (MET)	1.2. Promotion of metacognitive strategies.

Teaching practice in each lesson was compared with the level descriptions of corresponding criteria for an example, see Table 2. The theoretical framework and descriptions of performance levels have been previously validated. The analysis was carried out by filling in a form in which the teacher's performance was evaluated in 5 levels (scale 0, 1, 2, 3 and 3+): not observed, beginner, developing, proficient and expert, where level 3 is the desired or expected good practice and level 2 is sufficient for achieving the necessary goals.

Table 2: Example of a Performance level description: 2.2. Discussions for deep understanding

Level	Description
2.2. Discussions for deep understanding: Variety of questions asked by the teacher; possibility for various answers. Time to think. Conversational, thought-provoking environment. Number of students who ask questions; types of questions asked. Clarity of discussion procedure; teaching students to ask and answer.	
0 (Not observed)	The teacher introduces information and explains concepts. If questions are asked, they are closed. Students mainly listen, observe, and take notes.
1 (Beginner)	The teacher's monologue dominates. However, some elements of discourse can be observed – the teacher asks questions but expects specific answers. Questions are only used for checking knowledge. If students fail to provide prompt expected answers, the teacher answers them. Students answer questions and/or ask very few questions.
2 (Developing)	Alongside explaining and asking questions, the teacher creates a situation where students have to pose questions about the content of the lesson. If needed, the teacher instructs students on how to form questions and how to engage in conversation. The teacher tries to provide enough time for students to think over their answers; however, the time is often insufficient. Students feel free to ask questions and engage in conversation.
3 (Proficient)	The teacher asks diverse questions, including those where there is no single right answer, and asks for students' reasoning. The teacher uses questioning and different student responses to extend the discourse either frontally or in groups (pairs). If needed, the teacher rephrases student questions. The teacher gives students enough time to think during the discussion. Students have a clear understanding of the procedure of classroom discourse – how to ask and how to answer. Students ask diverse questions.
3 + (Expert)	With appropriate questions, the teacher creates situations to engage students and deepen their thinking.

	Students ask diverse questions, initiate and lead discussions, and are active and fully engaged in thinking.

For the identification of different teaching profiles, the following schema (Figure 1) was used (UL ICEI, 2023). It is based on the prioritization of the categories corresponding to teaching for making sense of mathematics. The schema for identifying mathematics teaching profiles is tailored and includes the essential criteria necessary to evaluate the opportunities provided in lessons that develop an understanding of mathematics. There were 6 teaching profiles identified, where the 1st profile with a high level of performance, characterized as professional teaching etc. (Čakāne et al., 2024). The identified mathematics teachers' teaching profiles correspond to teacher profiles identified through TALIS teacher survey data analysis – clarity of instruction, classroom management, cognitive activation, control over planning and teaching (OECD, 2020; Özdemir et al., 2023; Jang et al., 2023).

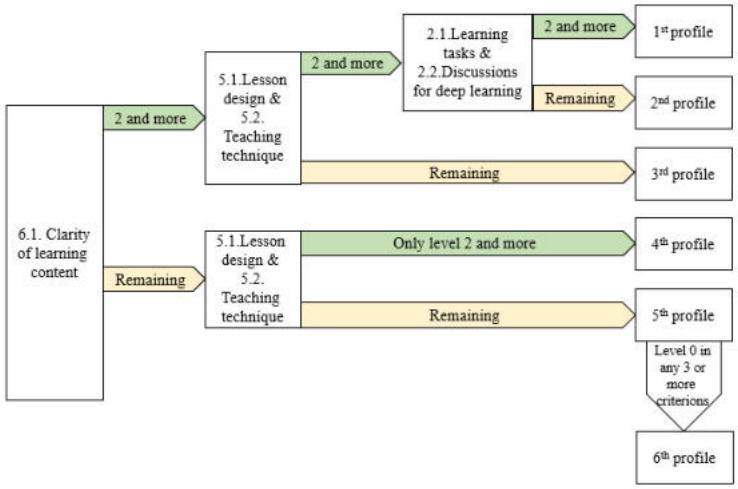

Figure 1: Schema for mathematics teachers teaching profile identification (using scale 0–3+)

Results

Results for Research Question 1

Table 3 shows the sample lessons' mean and median performance levels in the selected criteria, including data on teaching performance across five levels. It also indicates the minimal and the maximal observed performance distribution levels.

None of the teaching performances reached the desired competence level in all the selected criteria. Teaching mean performance levels in clarity of learning content, lesson design and teaching technique were overall better than in student cognitive activation and promotion of metacognition.

Analysis of the assessment of the six criteria shows significant differences in the opportunities provided to students between different grade levels. In 30% of Grade 1 to 6 and 73% of Grade 7 to 12 lessons, teachers selected appropriate learning content with appropriate representations, and efforts to develop conceptual development were evident (performance level 2 and above in criterion 6.1.). However, in 20% of Grade 1 to 6 and 6% of Grade 7 to 12 lessons, either the learning content was inappropriate or faulty and there were possibilities for students to form misconceptions. When

looking at the structure of the lessons – in 56% of Grade 1 to 6 and 78% of Grade 7 to 12 lessons, students' previous experiences were drawn upon to focus their attention, and tasks were consecutive

Table 3: Sample lessons in selected criteria characterizing teaching that promotes student conceptual understanding

Categories	CR		INSTR				SCA				MET	
Criteria/Results	6.1.		5.1		5.2.		2.1.		2.2.		1.2.	
Grade	1–6	7–12	1–6	7–12	1–6	7–12	1–6	7–12	1–6	7–12	1–6	7–12
Mean performance level	1,15	1,83	1,68	2,06	1,55	1,67	0,88	1,22	1,00	1,11	0,38	0,61
Median performance level	1	2	2	2	1	2	1	1	1	1	0	0,5
Performance level, 0,%	20	6	7	0	3	6	25	11	20	17	62	50
Performance level 1, %	49	22	36	22	48	39	61	56	59	61	34	39
Performance level 2, %	25	56	38	50	38	39	13	33	20	17	2	11
Performance level 3&3+, %	5	17	18	28	10	17	0	0	0	6	0	0

and necessary to reach the learning outcome (criterion 5.1.). In 86% of Grade 1 to 6 and 79% of Grade 7 to 12 lessons, most tasks required low-level thinking (performance level 0 and 1 in criterion 2.1.). In 79% of Grade 1 to 6 and 78% of Grade 7 to 12 lessons, the questions directed at students were mostly for checking their knowledge and not for seeing their reasoning and ideas (performance level 0 and 1 in criterion 2.2).

Results for Research Question 2

Following the previously shown schema (Figure 1), teaching was analysed, and profiling was conducted at six levels (Figure 2). Metacognition criterion (1.2.) was very low in both grade groups, it was used only as background information in the profiling of teachers.

The identified teaching profiles and median performance levels of each profile are presented in Figure 3. Highlighted are the criteria where differences between the two groups are observed.

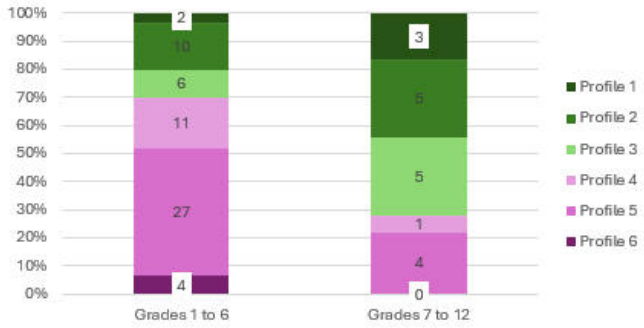

Figure 2: Teaching profile distribution

Grades 1 to 6

	6.1.	5.1.	5.2.	2.2.	2.1.	1.2.	N	%
Profile 1	2,5	3	2,5	2	2	0,5	2	3%
Profile 2	2	3	2	1	1	1	10	17%
Profile 3	2	2	1	1	1	1	6	10%
Profile 4	1	2	2	1	1	1	11	18%
Profile 5	1	1	1	1	1	0	27	45%
Profile 6	0	0,5	1	0,5	0	0	4	7%

Grades 7 to 12

	6.1.	5.1.	5.2.	2.2.	2.1.	1.2.	N	%
Profile 1	3	3	3	2	2	1	3	17%
Profile 2	2	2	2	1	1	0	5	28%
Profile 3	2	2	1	1	1	1	5	28%
Profile 4	0	2	2	1	0	0	1	6%
Profile 5	1	1,5	1	1	0,5	0	4	22%
Profile 6	-	-	-	-	-	-	0	0%

Figure 3: Profile distribution and median performance level

Differences were observed between primary school teachers (Grades 1 to 6) and middle and high school teachers (Grades 7 to 12). More teaching corresponded to the 4th profile (the teacher's basic activities would be performed skilfully if the teaching content were delivered proficiently; opportunities for learning to reason are only present in isolated cases) of primary school teachers than that of middle and high school teachers. Significantly more primary school lessons corresponded to the 5th profile (teacher's basic activities are skilful only in certain elements) and 6th profile (inadequate teaching of the curriculum content, i.e., students are not provided with opportunities to learn successfully) which shows weak subject knowledge and classroom management skills. In Grades 7–12, more lessons fit the 3rd profile, which can be described as theoretical teaching of mathematics, compared to primary schools. The content is represented adequately, but teacher`s basic activities lack proficiency.

Discussion and conclusion

The results show that the teaching of mathematics in the observed lessons, in terms of promoting students' understanding, varies. The study reveals that while teachers demonstrate proficiency in certain areas such as clarity of learning content and lesson design, there is a need for improvement in promoting student cognitive activation and metacognition. The observed variations in teaching performance across different grade levels and the presence of inappropriate learning content in some lessons highlight the importance of targeted PD and continuous support for teachers. Such considerations are important when designing PD as teachers may have different needs based on specific characteristics (Stevens et al., 2013). This means that teachers require various methodological support. However, Darling-Hammond et al. (2017) admit that at the system level teacher PD is often determined without understanding what teachers need. The authors conclude that for each of the teaching profiles, there are different needs for PD. For example, teachers that fit the 1st profile (characterized by a high level of performance, students having the opportunity to learn to think during lessons, and the teacher's basic activities in the classroom are effective) can either improve their knowledge about deep thinking and corresponding tasks as well as their skills in leading classroom discussions and the art of asking questions for deep understanding or work on teaching and improving student metacognitive skills. The priorities for the 2nd profile (characterized by effectively performed basic teaching activities; however, only isolated situations are observed in lessons where students have the opportunity to learn to reason) should be aligned with the school's established goals. Yet, there is a clear necessity to enhance student cognitive engagement during

lessons. Teachers in the 3rd profile (characterised by skilfully delivered teaching content; however, the teacher's basic activities are not always implemented effectively, and opportunities for students to learn to think are observed only in isolated situations) are characterized by a skilled representative of theoretical and algorithmic mathematics teaching. They need to focus on lesson planning and instructional strategies to develop a coherent lesson structure that promotes cognitive challenges for students. For teachers in the 4th profile, it is essential to deepen their understanding of mathematics pedagogy and didactics, emphasizing the use of various representations and the transitions between them, as their foundational teaching skills are adequate. Teachers in the 5th profile, particularly novice teachers, require mentorship or other forms of individualized support. If a teacher's performance aligns with the 5th or 6th profile, it is crucial to address their motivation for teaching and to conduct further research to understand the prevalence of such practices in schools.

Schools or municipalities can use the profiling schema from this research as a self-assessment instrument to determine their professional development needs. To determine more precisely the necessary support for teachers, the next step in the research could be to explore their beliefs.

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Teacher-student interaction in collaborative problem-solving: In-the-moment perspective on teacher noticing

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Many teachers use students' thinking but often miss opportunities to advance groups of students mathematically. In this study we analyse a teacher's interaction with four seventh graders from a Norwegian classroom through a noticing perspective. We highlight the teachers' noticing actions while engaging with students' shared reasoning during solving a mathematical problem, the Coin Problem. Through a noticing lens on observational data, we found that the teacher appeared to aim for students to develop more generalized arguments. Although the teacher built on their ideas, the challenge was that students struggled to follow his line of reasoning. Thus, we examine how the arguments constructed during the teacher's interventions did—or did not—become part of the group's shared reasoning. The results provide insights into how a teacher use students' in-the-moment arguments, possible reasons behind the chosen approach, and its consequences.

Keywords: Teacher-student interactions, collaborative problem-solving, teacher noticing.

Introduction

Research has emphasised that students can better understand mathematical ideas through meaning-making and by constructing their own mathematical knowledge. Thus, teachers should organise for classroom activities which can elicit students' different ideas, for students to respond and build on each other's ideas. Therefore, mathematical reasoning and small-group work are highlighted in research as activities influencing students' construction of mathematical ideas (Maher et al., 2018). Collaborative problem-solving (CPS) combines mathematical reasoning and group work. Thus, it is a complex, yet important, interaction for teachers to facilitate. Howe et al. (2007) found that students' most productive collaboration occurred when a teacher offered little intervention. In those situations, students proposed and defended ideas, and produced their own solution methods. However, teachers who are prone to little intervention with students, often withdraw from intervening at all (Stein et al., 2008). Some teachers will avoid telling students what to do or honouring students' work and progress, due to lack of guidance on what they could do to encourage mathematical reasoning and thinking (Stein et al., 2008). Consequentially, teachers miss opportunities to move a class, or a group of students, or an individual student, forward mathematically.

In our study, teaching is connected to how students are afforded opportunities for developing convincing mathematical arguments (Jeannotte & Kieran, 2017) for solving a problem together. We do not see teaching as defined by a single method or a set of methods, but rather by the interaction between a teacher and students, connected to a mathematical content (Jacobs & Spangler, 2017). Thus, teaching mathematics is a relational practice consisting of a complex interplay of interactional components aimed at enhancing students' mathematical reasoning. In an interaction between a teacher and students collaborating on a mathematics problem, what captures a teacher's focus

depends on various factors, such as students' understanding, the dynamics of the group, or aspects with the problem. In this study, we utilize teacher noticing as a frame to gain insights on teaching practices connected to students' mathematical reasoning. We ask: How does a teacher interact with students' collaborative problem-solving process seen through a lens of teacher noticing?

Theoretical background

Teacher noticing is often viewed as "focusing attention on and making sense of what students say or do before actually responding to them" (Jacobs & Spangler, 2017, p. 771). Recent frameworks operationalizing teacher noticing illuminate two themes: 1) consideration of the contextual aspects related to noticing, and 2) attending to and interpreting students' mathematical thinking (Amador & Weston, 2024). These perspectives highlight the importance of how teachers actively observe and interpret key moments in classrooms. In this process, known as professional vision, it is crucial for teachers to develop a critical awareness of their surroundings, enabling them to recognize particular situations, evaluate them carefully, and respond with purpose. By focusing on these elements, research into teacher noticing not only contributes to a deeper understanding of classroom dynamics but also holds the potential to enhance teacher professional development.

To study teacher noticing three different approaches are often utilized (Sherin et al., 2011): 1) teachers describe teaching of others and express what they notice, 2) teachers reflect upon their own teaching where they think about the lesson, or view video from a lesson on which they comment, and 3) researchers make inferences about what a teacher notices from video of teachers' actions in their classrooms. We study teacher noticing through the last perspective: emphasizing what and how a teacher attends to students' mathematical reasoning during teacher-student interactions. One approach to framing a teacher noticing study is through the framework attending, interpreting, and shaping from van Es and Sherin (2021). We did not have access to the teacher's thoughts and reflections. Thus, when we consider attending, we are restricted to the events the teacher focused on or disregarded (i.e., what the teacher ignored or did not attend to) in conversations with students. Considering the aspect shaping, we make inferences about "the integrated relationship between the information teachers gather from their observation and sense-making of students' thinking and their plans to respond in the moment" (van Es & Sherin, 2021, p. 23). In shaping situations teachers have the opportunity to attend to details of students' reasoning, develop interpretations of their claims, mathematical anchoring and justifications, as well as support with relevant mathematical instructions. To study students' arguments in such situations, we identified arguments by using Toulmin's model of argument (Toulmin, 1958). This process included looking for *claims* in the discussions, and then of how these claims were justified—from which *data* they were built and how the inference from data to claim were supported by a *warrant* and sometimes additional *backing*. Some arguments can also include a rebuttal, which is a counterargument.

To investigate how a teacher engages with and shape students' shared arguments, we study another aspect of CPS: students' processes for how they maintain and build shared understanding, which was identified after the three processes suggested by Roschelle and Teasley (1995): building, monitoring, and repairing. There are three discursive actions needed to construct and maintain a joint problem space: *building* (introducing and accepting new elements of an argument), *monitoring* (repeating or

explaining shared elements, asking clarifying questions, or identifying disagreements), and *repairing* (negotiating and discarding elements). Thus, we focus on how a teacher engages with and shapes students' shared argument.

To further unpack the complexities of studying what teachers attend to in teacher-student interactions, we needed frames to support relations between noticing and mathematics. Stockero (2021) claims to provide such a mechanism through the "Coding Framework for Attributes of Noticing". This framework focuses on three attributes that were used to identify what prospective mathematics teachers attended to in their noticing of their own lesson: *agent, specificity of mathematics*, and *focus*. We have applied the structure of the framework but modified it for use to study a teacher in action, where we do not have access to the teacher's reflections. Table 1 provides an overview of the modified categories. The *agent* category was who or what as the object of the noticing. We narrowed this category to focus on who interacted: teacher-students or teacher-student. This distinction identifies whether a teacher mainly gives the whole group or an individual student the attention in the conversation. The second attribute is *specificity of mathematics*, which was whether and how the mathematics was discussed. Stockero (2021) used this attribute to identify mathematical ideas that were noticed. Adapted to our noticing lens, specificity of mathematics is how a teacher attends to students' arguments. The final category, *focus*, concerned what about the students that was attended to, including their mathematical thinking as well as affective issues or general understanding (Stockero, 2021). In our focus category, we consider what the teacher, through interaction with the students, is focusing the student/students towards. As in Stockero's categorization, potential foci include various aspects of the problem-solving process, as well as more general, pedagogical actions.

Table 1: Noticing attributes of teachers in the moment interaction

Agent	Specificity of mathematics	Focus
Who are interacting? • Teacher-student (individual) • Teacher-students (pair/groups)	How are the students' arguments attended to by the teacher?	What does the teacher focus the student/students towards?

Methods

The data for this paper was collected in a Norwegian seventh-grade classroom (students aged 12-13), during the third out of five lessons designed for a larger intervention study (Klock, 2024). The aim of the intervention study was to develop materials for teaching reasoning and proving in primary school. Thus, the lesson studied here included a task that focused on how to foster students' argumentation. The students worked in small groups as part of the design, but the intervention did not particularly consider students' collaboration. The teacher in the current data did not follow any instructions or support particularly directed towards teacher actions. However, all teachers in the intervention study participated in task design, and the researchers aimed towards creating a shared understanding between teachers and researchers of reasoning and proving, mainly based on Jeannotte and Kieran (2017). Neither of the authors of this paper participated in the intervention, but the second author closely followed the intervention as it took place.

In this study, we consider students' work on a task called the Coin Problem: "In the central bank, they are working on streamlining their money production (…) They are considering that it might be enough with 3-krone coins and 5-krone coins. What prices can be paid if only these two types of coins exist?" The task was designed so that the answer to the task (*all* prices can be paid) was not obvious to a seventh-grader, and thus held the potential to foster students' need of convincing each other of the truth of a mathematical statement. We use data from one group: Asja, Olivia, Andrea, and Lars (pseudonyms). This group was chosen because they were engaged in the problem-solving process, yet it seemed that the teacher's interactions with the group did not provide sufficient support to move the group forward. The group worked with the task for approximately 30 minutes, during which their teacher walked around the classroom, conferring with students both with and without request. The group's work was videotaped and sound recorded. Our analyses are based on transcripts of the recordings, students' written work, and videotaped material.

For the analysis, we began by decomposing the entire transcript into a series of episodes, each episode including an argument that was constructed. The arguments were identified using Toulmin's model of argument (Toulmin, 1958). Next, we used Rochelle and Teasley's (1995) model of collaborative problem solving on the parts of the transcript where the teacher was not present. Combining the two analyses, we could track which arguments seemed to be shared in the group (did other students than the one who introduced the argument accept it, ask questions about it, or repeat it?) or whether it seemed to be the argument of a subset (usually a single student) only. As a third step, we used our operationalisation of teacher noticing (Table 1) to analyse the teacher's interventions with the group.

Results

We present the problem-solving process as a chronological narrative. "Lines" refer to line numbers in the transcript, and excerpts are translated into English. We have collected the episodes where arguments are constructed into longer segments relative to the teacher's interactions. Episodes involving group interactions that occurred without the teacher's presence are presented to comprehensively observe how the teacher's actions influence the students' co-construction of arguments.

Before the first teacher interaction (lines 1–38)

The group agreed early on that the task was pointless, and during the first 20 lines, no mathematical argument was attempted. From line 21, they constructed their first argument, focusing on prices that are multiples of 5 which they claimed can be paid by using 5 kr coins. Lars, Olivia, and Asja all contributed to this discussion, with Lars being the main builder and Asja being the most negative due to the task's apparent absurdity.

The first teacher interaction (lines 39–81)

The teacher's initial question "How are you doing?" was met by exclamations of how "idiotic" the task was from Asja and Lars. The *agent* in the teacher's attention was the whole group when he tried to focus their discussion towards the mathematical content of the task: "Is it possible? The question is not about whether it is practical or not, but this is the task, is it possible. How can you show it?" (line 43). After some protests (from Asja) and non-justified claims (from Asja, Lars and Olivia) the

teacher picked up one of their claims: "No? Why not, then?" (line 50) and engaged in a turn-taking dialogue with Olivia (lines 51-62), where Olivia presented the following argument: "Because if you're going to buy a bag that costs 2 kr, and you only got a 3 left! [Teacher: yes] Then you can't exchange 3 and get 1, because then…(inaudible)." Here, Olivia used "have 3 kr" as data to claim that you cannot pay 2 kr, by the warrant that it is impossible to pay 3 and get 1 back. The teacher noticed the *specificity of mathematics* of Olivia's argument, and that the data was invalid. He explained: "(…) you can bring as much as you want. And the shop has as much [exchange money] that you want (…) Is it possible to buy a bag for 2 kr?" Then the teacher *focused* Olivia's attention to support her in building a new argument, with the corrected assumptions as data:

57	Olivia:	Yes …
58	Teacher:	How, then?
59	Olivia:	If you have a five.
60	Teacher:	Yes, what happens then?
61	Olivia:	You exchange it and get a three back.
62	Teacher:	Yes, that's the answer. So now you know that 2 is possible. Right?

However, in this interaction, the *agent* in the teacher's noticing was Olivia, not the whole group. This is evident because of the turn-taking conversation between the teacher and Olivia. Next, he turned to the whole group to ask whether they could pay 1 kr, to which they answered "yes" or "no" without justifications. In line 73, the teacher said: "You can manage it with two fives, and you're to buy something that costs 1 kr." Subsequently, Asja said "Yes!" and Olivia said "Oh yeees! It works because nine is in the three-times-table." Although Olivia's warrant required further explanation, the teacher immediately concluded that it was correct, and further prompted them to go on to check for prices 3, 4 and so on, thus *focusing* the students on a given strategy. Then he left the group.

Between first and second teacher interaction (lines 82–168)

The group was left to check for price 7, which they concluded worked without giving or trying to provide a warrant. Similarly, they simply claimed that 8, 9, and 10 worked. In this discussion, Olivia was the main builder, suggesting a strategy "Should we just continue going up?", whereas Lars and Asja contributed by monitoring. In line 97, Andrea made her first contribution in the problem-solving process, suggesting that 11 worked because "11 times 5 is 55". This was not a correct argument, but Lars immediately followed up: "11 works! With 33! (…) 11 times 3 is 33.". Until line 129, this incorrect argument dominated the group's discussion, with Lars, Andrea and Asja in monitoring roles. Olivia, however, was sceptic to the argument. She tried several times to explain the mistake, thus taking a repairing role: "If you're to buy something for 11, and you pay with 33. Then you'll get 22 back. How are you going to do that?" (line 114). In line 129, Olivia demonstrated that you could pay 11 by paying 4 fives, then get 3 threes back. There was no response to Olivia's suggestion, but the group seemed to accept that 11 was payable. They continued half-heartedly with 12, 13 (Asja: 5+5+3), 14 (Olivia: "we take it with 4 fives…"), and so on. They were unsure about 19, but they claimed that all numbers until 22 worked. Olivia went to get the teacher (she did not state why).

The second teacher interaction (lines 169–204)

Olivia and Asja explained to the teacher that they had checked for all numbers up to 22. Andrea asked whether they needed to write down all of them. The following turn-taking occurred:

176	Teacher:	No, you don't need that, but it could be alright to do it up to 10. For example. Right? What numerical system do we have? What's the name of the numerical system we've got? Ten?
177	Lars:	…times table!
178	Teacher:	Base ten…
179	Lars:	Base ten…
180	Teacher:	…numeral system, right? So, if it goes from 1 to 10?
181	Lars:	So, it goes to 40, 50, 60, 70, 80, 90.
182	Teacher:	Yes, then it goes for all numbers.
183	Lars:	And 1000, 2000, 3000 …
184	Olivia:	So, that means that, three and five, everything goes, it just takes time.

Here, the *agents* of the teacher's noticing were the whole group, because the teacher did not address the questions and hints to a specific student, he directed it to everyone. The teacher did not, however, probe further into the *specificity of mathematics* of students' claims. Moreover, the *focus* was to build an argument or common understanding that you can pay all prices with the three and five coins. The *argument* constructed was because you can pay all sums from 1 to 10 kr (data), you can pay all prices (claim), warranted by the fact that we use the base ten numeral system. The data in this argument were the conclusions from the students' work, which they informed the teacher of in lines 169–175. The claim, however, was given by the teacher, although Olivia and Lars seemed to repeat the claim. The warrant of this argument needed support by further backing, e.g., *all numbers can be written as a sum of tens and ones, so if we can pay all numbers up to 10, we can use this to pay any number*, which was neither given nor asked for by the teacher. Next, the teacher asked the students how to pay 1000 kr, again causing the students to react to the task's absurdity: "Who goes around with 200 fives?" (Andrea, line 198). Then, the teacher gave the group a sheet on which they should write their final argument, repeated that a good argument should include drawings and explanations, and left.

After the second teacher interaction (lines 205–250)

In the first discussion following the teacher's interaction, Asja asked why 1 worked, and Lars repeated that 3 and 4 worked. Olivia tried to explain her previous argument that 1 worked but was interrupted. In the end, Olivia dictated what to write on the sheet, after refusing to do the writing herself. On their finished sheet it was written: "One works because: If you pay with two tens [sic] you get back three threes. That is the same as nine. It works for one to ten which means it works until infinity". The mistake of two tens instead of two fives is due to Olivia, which Asja copied onto the sheet, although Lars tried to correct the mistake. They did not discuss the final claim of the argument at all.

Discussion

In a process of creating a teacher-student interaction, a teacher makes way for attending to and interpreting student thinking (van Es & Sherin, 2021). Thus, untangling aspects of the process may provide insights on relations between process (who were the agents, and collaborative processes for building arguments) and the outcome (attending to and interpreting students' arguments, and support for their collaborative interaction). From the first interaction, the teacher noticed that the group was not focused on the problem, and his actions were to help the group concentrate on the problem. Further, he noticed Olivia's incorrect argument, that you cannot pay a price of 2 kroner, and supported her in correcting her argument. He asked the whole group if 1 was possible, and then he encouraged them to go on and systematically check for other numbers. The teacher's in-the-moment response

(van Es & Sherin, 2021) concerned a noteworthy mathematical aspect to attend to, which the teacher attempted to use for shaping the group's further discussion. However, Olivia's exchange-argument was not taken into the group's shared reasoning.

The strategy of testing for all numbers was the main component of the group's shared understanding of the problem. Additionally, the group seemed to agree that multiples of 3 and 5 could be paid. These ideas were monitored by all group members during the problem-solving process. Thus, students attempted to build and maintain a shared understanding (Rochelle & Teasley, 1995), which is important during CPS. Moreover, three group members asserted that they could pay a kr because 3·a or 5·a are multiples of 3 and 5, respectively, an incorrect argument which was rebutted by Olivia and thus never became a part of the group's shared understanding. Similarly, Olivia's arguments using exchanges (for example for 1, 11, and 16) were not part of the group's shared understanding, because none of the other group members ever repeated or probed into them. In the second interaction, the teacher was informed about the group's progress. He noticed their results, but did not ask them to elaborate on their shared understanding for why all prices up to 22 could be paid. This time, the response to the specificity of mathematics (base ten) was directed to the whole group. The teacher probably attempted progress towards generalising their observations: all numbers can be written as a sum of tens and ones. However, the mathematical argument he attended to did not shape a shared understanding between the students and him. Previously, and in that moment, the group's shared understanding was the strategy of testing for all numbers, starting at 1, which we interpret made the teacher's suggestion seem too advanced or out of reach. Thus, the teacher's final argument for why all prices could be paid, using properties of the numerical system, never became part of the group's shared understanding. Concerning the focus-category (Stockero, 2021), the teacher could have focused the students' thinking towards one another, rather than towards his solution strategy.

While it might seem that one, several, or all students in a group have understood a concept or solution strategy or achieved a satisfactory outcome, we believe that this perception can lead teachers to feel content with the group's progression. However, after the final teacher-student interaction, the students did not discuss their concluding argument. Moreover, they were not given the opportunity to bring together their own ideas for solving the problem, let alone review and connect the teacher's generalized suggestions. We see this as a 'challenge of closure' of both problem-solving and making a final argument in teacher-student interactions. Thus, this study sheds light on the usefulness to view the relation of what a teacher and students do, through the lens of teacher noticing. Since a teacher's professional vision is to actively observe and interpret classroom situations (cf. Goodwin, 1994, referred to in Amador & Weston, 2024), this study emphasizes the importance of developing professional skills that identify how students interact (agents and agency alike), how mathematical ideas are presented by a group, and how to respond purposefully promoting a shared understanding in the group. Such effort puts students' thinking in focus of the classroom interactions (Jacobs & Spangler, 2017). It could present one solution strategy to deal with the challenge of intervening with students' CPS, or the lack of it (Howe et al., 2007; Stein et al., 2008) by recognizing who and how arguments are collaboratively built and maintained in a group. Our suggestion is "focusing attention on" who's the agents in the group, and "making sense of what students say or do" about the specifics

of mathematics—their arguments (claims, data, warrants, rebuttals)—"before actually responding to them" (Jacobs & Spangler, 2017, p. 771).

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Practices in math discourses in inclusive primary school

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As a result of nationwide performance assessments, the promotion of basic skills in mathematics education is becoming increasingly important in the didactic debate in Germany. In order to make mathematics teaching a central place of support for all children, including those in difficulties in learning mathematics, it is necessary to design learning environments, that enhance the individual potential of each pupil. If mathematical learning is understood as a social-interactive process, the interaction between the teacher and the children is characterised by subject-specific and social practices that are established and shaped in the course of integrated support in the classroom. The focus of this article is on the reconstruction of these practices. They are analysed on the basis of videotaped classroom scenes, using interpretive classroom research, and critically reflected upon with regard to the possibilities and limitations of support.

Keywords: Inclusion, practices, discourse, participation, difference.

Introduction

Fostering basic skills in mathematics education is becoming increasingly important in the didactic debate in Germany as a result of nationwide performance assessments (Stanat et al., 2021). In this context, it is discussed how mathematics education can be organised in such a way that children with different abilities can learn together and receive the best support possible. Because of that, the question of how to organise inclusive teaching takes on a new relevance.

"Developing mathematical understanding requires that students have the opportunity to present problem solutions, make conjectures, talk about a variety of mathematical representation, explain their solution processes, prove why solutions work, and make explicit generalizations" (Frank et al., 2017, p. 230). Therefore, a learning environment must be offered that allows children with different competencies to improve themselves in collaboration with others, although, the practices in the classroom have to consider the opportunity for each child to engage in the ongoing discourse.

Our research therefore focuses on both areas. We develop learning environments for inclusive mathematics classrooms and investigate the interaction processes. In doing so, we ask for the mathematical practices that are established between teachers and students in inclusive situations where the teacher reflects on mathematical objects together with the inclusive learning group. Our contribution focuses on the social and content-related participation processes of learners who are in difficulties in mathematics learning.

Theoretical framework

According to the theoretical approaches of Steinbring (2005), mathematical learning is based on content-related social negotiation processes with focus on active explorations of mathematical

content and structures. The interactive negotiation of a common thematic meaning leads to the development of conceptual understanding at different individual levels. Accordingly, mathematical learning takes place through participation in collective mathematical constructions of meaning (Voigt, 1994). Learning mathematics in inclusive classes is always framed by the question of the extent to which the children, with their individual potential and approaches, have opportunities to participate socially and content relatedly in shared mathematics lessons with others.

In this sense, with view on the heterogeneity of an inclusive math class, there is a need for the design of learning materials for all children. A main goal of inclusive learning situations is "to involve all learners in quality learning experiences which empower them to become active participants in a more equitable system" (Scherer et al., 2016, p. 640). Häsel-Weide and Nührenbörger (2022) pointed out different guiding principles for the design of inclusive mathematics teaching. These design principles aim to "consider different potentials of the learners and to connect to them in a targeted way:

(1) accessibility to the common subject matter for all children,

(2) subject learning at different levels,

(3) active exploration of content connections and finally,

(4) initiation of common learning phases for all children with social negotiation processes of communicating, representing, and arguing" (Häsel-Weide & Nührenbörger, 2022, p. 4454).

From a praxeological point of view, mathematics education can be regarded as a social phenomenon and practices can be understood as a recurring pattern of how children and teachers act and speak across situations, through which the participants in the interaction take up and work on a specific teaching topic in relation to each other. Following Cobb and collages (2001, p. 121), "a practice is seen to be an emergent phenomenon rather than an already-established way of reasoning and communicating into which students are to be inducted".

In this sense, mathematical, subject-related practices are understood as ways of discursively dealing with subject-related questions that become established in classroom discourse, e.g. arguing mathematically or assigning subject-related meaning to signs and symbols (Cobb et al., 2001). Learning can be seen as increasingly competent participation in a practice (Lave & Wenger, 1991), and mathematical learning as a change in participation in subject-related practices (Heyd-Metzuyamin et al., 2019).

However, participation in subject-specific task-processing practices does not depend solely on the individual and their assets, but also on the way teaching discourse is understood as a discursive practice and its participatory possibilities. In the example of explaining, a distinction can be made between orchestrated explaining, in which several children explain knowledge under the guidance of the teacher, and solo explaining. The former is strongly guided by the teacher and follows a sequence described as initiation-response-evaluation (IRE), i.e. introduction by the teacher, response by the learners and subsequent evaluation by the teacher (Mehan, 1979; Voigt, 1994).

Design of the Study

In our project IGEL M (Inclusive practices in shared learning opportunities in mathematics) we develop existing learning environments further by conducting didactic teaching-learning experiments for inclusive learning settings and by analysing mathematical learning processes from a qualitative perspective (Häsel-Weide & Nührenbörger, 2022). We accompanied classes during their time in primary school and visited them 4–5 times each school year from 2018 to 2023. In cooperation between the teacher of the class and the authors of the paper we planned learning environments, selected material and developed further. Each subject was realised in two consecutive lessons. The lessons, carried out by the teacher, were video-graphed and used to a) answer the research questions as well as b) to reflect jointly the quality of the learning. The videos were qualitatively analysed by following an interactionist and epistemological perspective (Steinbring, 2005), focusing on mathematical, discursive and difference practices Our main question is: Which practices characterise the interaction between teachers and children with different assets? We distinguished between subject-related practices in discourse, discourse practices in mathematics and the relationship between.

The analyses follow five analytical steps: (1) Video scenes with episodes in which interpretive hypotheses can arise spontaneously according to the research question. (2) These are transcribed and (3) are paraphrased and interpreted in a turn-by-turn analysis by the participating researchers. (4) In discourse, a plausible typification of practices is elaborated by categorising patterns of interaction or activities. (5) The practices are compared with other analyses of other scenes and examined in a generalised way (Voigt, 1994).

Results

Result of the analyses are the reconstruction of content-related practices in discourse, discursive practices in the subject and practices of difference as well as their interwoven relationship (Häsel-Weide & Nührenbörger, 2022). In the following, they are characterised and illustrated by means of two contrasting examples. The examples are taken from a lesson in which the children were asked to find sequences of numbers that would lead to a target number (Figure 1, Steinbring, 2015).

Episode 1: Ilkay explains relations between numbers

When the teacher comes to Ilkay and Abdul, who are working together, she first asks how much the children have worked together on the task, emphasising that each child should be involved in the work process. She turns to Abdul and asks him, “Did you let Ilkay do something?” Abdul explains that they found the number sequences by sharing the sequences. Once the genesis of the number sequence is clarified in that way, the teacher asks the children to explain how they sorted the sequences (Figure 1). She poses the question directly to Ilkay, who explains:

1	Ilkay	Uh, this one here, calculating here, so here you always have one, two, three, four and so on
2	Teacher	Mmh.
3	Ilkay	And here we always count, wait (*4 sec. pause*).
4	Abdul	Ilkay, we always have this.
5	Ilkay	Here we always calculate minus one.
6	Teacher	Oh okay, right.

7	Ilkay	And here the middle number always remains the same.
8	Teacher	Aha.
9	Ilkay	And this, here.
10	Teacher	Mmh.
11	Ilkay	Here we always calculate plus one. And here, so these results are always.
12	Teacher	What was that again (*points at the target number of the first number sequence*)? How do you call that?
13	Ilkay	Forty-five.
14	Teacher	No, how do you call that this (*points at the target number of the first number sequence*)?
15	Abdul	(*points at the word "target number" on the worksheet*)
16	Ilkay	Ah, target number.
17	Teacher	Mmm.
18	Ilkay	That's always forty-five.
19	Teacher	That's amazing.

The discourse is one-sided between Ilkay and the teacher. The teacher gives Ilkay the space to demonstrate his technical knowledge and highlights his competences and positioned him as competent (Johnson et al., 2022), while Abdul can neither directly demonstrate his technical expertise nor receive recognition for his reticence. He takes part in the interaction when Ilkay shows difficulties and supports him by reminding him on of their procedure (turn 4) or pointing at the fitting word on the work sheet (turn 15). The teacher confirms the statements, and in some places a more precise explanation is requested. Overall, the teacher seems to be primarily concerned with gaining insight into Ilkay's findings. Ilkay's explanation focuses on the relationship between numbers in the same position. He describes for example, that in order to get to the number in the first position of the number sequence, you have to subtract 1. His description is correct, but remains mathematically superficial, describing the relationships between individual numbers without considering the opposing effects of the changes of the first and last number. The teacher strengthens the explaining practice with short confirmations and praises it at the end. Besides, she deepens his description by asking for the agreed terms (starting number, middle number, target number). To summarise, the impression arises that descriptive practice is established here with a focus on relationships between individual numbers using fixed terms.

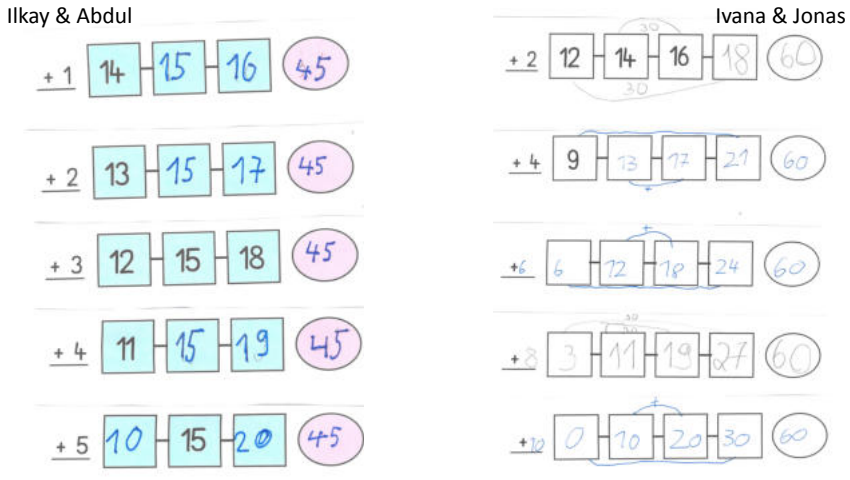

Figure 1: Sequences of numbers that are discussed in the episodes 1 and 2

Episode 2: Ivana and Jonas extend the border of natural number

The following episode is from the next day's lesson where the children are asked to find number sequences with an odd figure of numbers (Figure 1). Jonas and Ivana asked the teacher for help because they could not find more sequences:

1	Teacher	Mmm. Let's think about that, what is already given? What do you already know about this number sequence?
2	Jonas	Hmm, we know that this, uh, the result is always sixty (*points at the number sequences which are already filled in*).
3	Teacher	Why don't you fill that one in already? We're safe on that.
4	Ivana	(*writes 60 as the target, which adds plus 12*)
5	Teacher	Well done.
6	Jonas	And it has always been like this, here a plus, there a minus and always minus three here and plus three there.
7	Teacher	Mmh.
8	Ivana	And we did recognize something. We added this (*points at the number 18 of the first number sequence*) and this number *(points at the number 12*) and then it's thirty and if you add these (*points at the numbers 14 and 16*) then it's also thirty.

The episode shows that the children report their previous discoveries together, complementing each other without one child being addressed in particular. The order in which Jonas' mentions the changes between numbers (here a plus, there a minus) suggests that these findings were related. Ivana also describes relationships in a sequence, using concrete numbers that are general in nature. After the description of their discoveries, the teacher asks about the problem, without particularly much comment on the explanation. Jonas explains that they would then have to start the next sequence with the addition number 12 with a zero (Figure 1), because otherwise they would have to use a negative number and "That would not work". After the teacher brings this in question.

11	Teacher	Mmm. Why do you start from zero, when v you go down here (*points at the number sequence, which adds 12*)?
12	Jonas	Otherwise, you would need to start with a negative number and that wouldn't work.
13	Teacher	That wouldn't work? Why not? Where do you see that? Is it written anywhere, that you are not allowed to use negative numbers?
14	Jonas	(*shakes his head*)
15	Teacher	Which negative number would be fitting then?
16	Jonas	Minus three.
17	Teacher	Right, why don't you write that down?
18	Jonas	(*writes down -3 as starting number of the number sequence, which adds 12*)

After both children had been involved in the interaction on an equal footing at the beginning, an interaction develops here too, which primarily takes place between one child (Jonas) and the teacher. However, this seems to result from the course of interaction without Jonas being addressed exclusively. In turn 11, the teacher addresses both children. In terms of content, the argumentation makes it clear that the children have identified and used structures in the number sequences. After the formal border of the number set appears to have been fallen by the teacher's statement, they also have no difficulty finding the negative number. The discussion is not about the fact that or why there are numbers smaller than zero, but the socio-mathematical norm of working in the domain of natural numbers is overturned.

Comparing the two episodes

A comparison of the two scenes shows that the second scene is about extending the children's knowledge. The starting point of the interaction is a question from the children to the teacher, whereupon the teacher first obtains an overview of the content of the knowledge. The children practice an explanation that shows the understanding of relationships, supported by a corresponding presentation practice with arrows in the document. Erath (2017) describes this as an explanatory practice with which mathematical topics, representations or procedures are named, formulated, linked or even evaluated. This discussion, which is based on equal judgement, then leads to an extension of knowledge in a dialogue between the teacher and Jonas. In contrast, the interest in knowledge in the first scene is focused on the teacher, who wants to learn about a child's insights and addresses him accordingly. The findings that become visible in the argumentation practice are reinforced by demanding specialised terminology. Direct addressing allows children to show their competences, but at the same time it seems necessary to create space in the interaction to expand the insight. The interaction practices therefore appear to vary depending on the attributed capability of the children.

The content-related practices also differ, possibly depending on the competencies that the children show and the adaptive reaction of the teacher. However, there can also be a danger here if process-orientated practices are reinforced and no expansion is encouraged. It should be borne in mind that the selected scenes were small sections and that, of course, children with difficulties in particular can achieve very good individual performance in describing even the superficial level, which should then also be recognised.

Reconstructed practices in math discourse

Looking at the mathematical practices of explaining and representing more generally, the reconstructed practices can be located between two poles. Even, if the learning environments are following the idea of conceptual focus for all children, the reconstructed practices of explanation and representation do not necessarily reflect the understanding-orientated character. Instead, the interaction analyses show that the concrete discourse takes place between a rule-guiding and understanding-oriented character and also alternates between the two (Figure 2). In the learning support of a small group, i.e. in interactive situations between the teacher and some children, discourse practice support also moves between focusing on the child with difficulties in learning mathematics, who has been given the opportunity of professional and social participation in a special way at this moment, and opening up to the equal-based discursive participation of several children (Figure 2). This is also evident when the learning environment is designed for collaboration between children with different assets. This means that the teacher recognises and focuses on specific difficulties when supporting learning. A one-sided content and social support is established between one child and the teacher, while at the same time other children are not actively involved. If we look at the content of the discourse, we see patterns of interaction that are geared towards eliciting knowledge (Voigt, 1994), while discourses of extending and reflecting knowledge can be contrasted with this. These practices differ between different discourses in the same class and sometimes even change during the discourse.

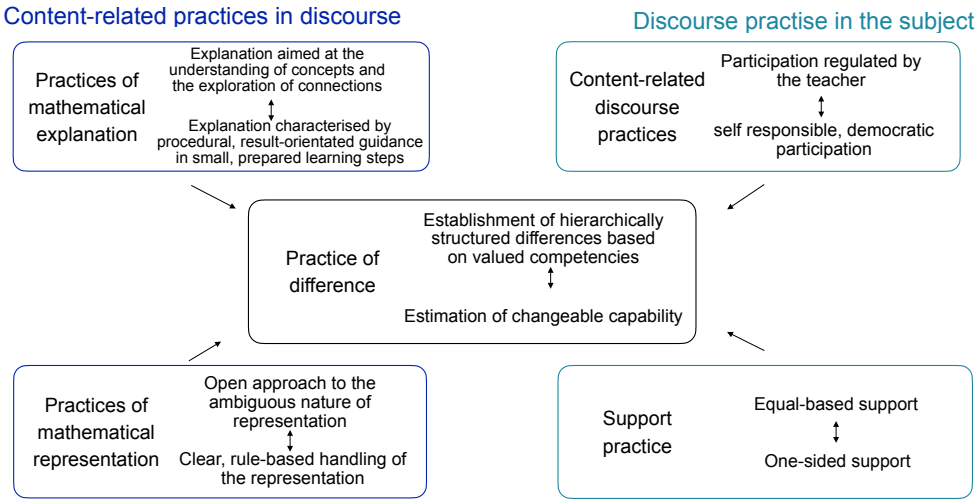

Figure 2: Practices in math discourse

Discussion and concluding remarks

The findings are not surprising in themselves, but reflect international findings on mathematics learning (Heyd-Metzuyamin et al., 2019; Moschovich, 2015). Of particular importance is the interplay between subject-specific and discursive practices, which in turn characterises differentiation practices.

In principle, it can be seen that the learning environment for exploring mathematical contexts offers the children a variety of approaches despite their different mathematical learning prerequisites and at the same time has the potential for interactive negotiation between the children. The children can communicate with each other and acquire knowledge collaboratively despite their different knowledge of the subject matter. This shows how individual math learning is woven into the practices. The practices of mathematical explanation and mathematical representation move in the field of tension between rather rule-guided and procedure-orientated interpretations of the content on the one hand and a rather open, ambiguous discourse about conceptual backgrounds and differences in interpretation on the other.

The mathematical practices establish themselves in connection with social practices of constructive support and discursive practices of negotiating knowledge (see Figure 2). The interplay of content and discursive practices creates differences that are revealed in the way in which participation in more rule-guided or understanding-orientated practices takes place and the extent to which this participation takes place as part of an equal discourse of all participants or as unilateral and initiated by the teacher and oriented towards a unilateral, dyadic discourse.

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The Work of Listening to and Hearing Children's Mathematical Competence

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This paper focuses on the work of attending to children's mathematical productions. Young children speak in their own terms, use gestures and drawings and have their own developed ideas and ways of thinking. The predominant construction of mathematics as a domain of correctness and precision, with a particular conception of proof and proving, impedes the openness, attention, and curiosity required to hear children's ideas. This pattern is more pronounced for children of historically marginalized identities, who are regularly positioned through deficit lenses. Drawing on data from classroom discussions, this paper analyzes the demands of listening to and hearing the competence in children's mathematics. The analyses contribute to developing more dynamic theoretical models of mathematical knowledge for teaching (MKT) in intersection with deliberate practice to advance equity.

Keywords: Teaching, mathematical knowledge for teaching, student thinking.

Introduction

No aspect of the work of teaching is more fundamental than that of listening to students (Ball, 1997; Schultz, 2003). It is what makes teaching difficult, but also what makes it fascinating. Many learners, especially those of historically marginalized identities, experience mathematics classrooms as unwelcoming, where they do not feel seen or heard. One common pattern is for teachers to listen evaluatively for whether students "get it." Attending to what is wrong or lacking in children's thinking reinforces their identities as "struggling." The brilliance of their mathematical observations, methods, and solutions goes unseen and unheard (Leonard & Martin, 2013; Willis, 2020).

This paper argues that advancing children's mathematical flourishing depends fundamentally on hearing, valuing, and using their ideas. Based on analysis of data from videorecorded interactions during mathematics lessons, we describe and analyze the nature and demands of listening across differences in the work of teaching elementary mathematics. These analyses further contribute to advancing more dynamic theoretical models of mathematical knowledge for teaching (MKT), theorizing them as critically intertwined with deliberate practice to advance equity.

Theoretical framework

Listening to students with the aim of hearing their ideas and being able to see how they make sense is foundational to supporting their learning. This study of listening in teaching is shaped by three intersecting theoretical lenses.

First is positioning theory (Davies & Harré, 1990) which shows how the construction of self is shaped through discourses that form storylines about the nature of competence and who is seen as competent. These discourses exist at both the micro-interactional level of day-to-day relationships in classrooms and through systems, historical patterns, and dominant narratives that operate at the macro level. These levels are intertwined and together construct the environments in which children's identities

and competence evolve and, specifically, in which Black and Brown learners are regularly pushed away from mathematical identification and success. These patterns are of long standing, rooted in the history of worldwide scientific racism (Darby & Rury, 2018) together with a pervasive orientation to "error" in mathematics teaching. Teaching that seeks to break with deficit orientations that concentrate on students' "misconceptions" and shortcomings depends on close listening that assumes that children are making sense for good reasons.

The second frame that shapes this research is how mathematics functions as a racialized, classist, and gendered space that has not welcomed a broad range of thinkers to develop the knowledge of the field. Proof and proving, central to the epistemology of the domain, center particular ways of knowing and value them over others (Dahlgren, 2024; Ladson-Billings, 2000). These ways of knowing have been predominantly the property of Western, white, and male mathematicians; ethnomathematics, a central focus for many mathematics education scholars (Ascher, 2017; D'Ambrosio, 1999; Powell & Frankenstein, 1997; Rosa et al., 2016), is, however, still often perceived as "folk" or primitive (Hottinger, 2018). Teachers' narrow perception of mathematics and the narrow representation of mathematics narrowly in curriculum materials combine to portray the domain as one in which only some people belong, and in which only some ways of knowing are valued. This shapes the work of hearing the things students say, making it challenging to notice or appreciate ideas consonant with other systems of knowing. Listening depends on an openness about what mathematics is and what it means to know it, making it possible to hear unconventional ideas and nonstandard approaches to reasoning.

The third theoretical frame underlying this research is that of the theory of mathematical knowledge for teaching (MKT). Hearing students' ideas, valuing their thinking, and being able to identify the mathematical competence in their ideas and their ways of explaining and representing them is mathematically demanding work. It requires a special kind of mathematical understanding, different from the mathematical expertise of mathematicians (Ball, 2017; Ball et al., 2008). This kind of understanding entails a shift from one's own ideas and ways of reasoning to being able to take up others' perspectives — in this case, the perspectives of children. Believing in the brilliance of Black children (Leonard & Martin, 2013) and that equity matters are essential. This sort of real listening requires both a commitment to understanding students' strengths and flexible MKT (Ball et al., 2008; Hill et al., 2005). As we examine the work of listening, we seek to identify the mathematical knowing that arises in the course of teachers' efforts to hear.

Method

This research uses a qualitative case approach, situated in a mathematics program for young learners that our research group conducts every summer. This program is designed to foster the participating children's mathematical flourishing by engaging them in challenging mathematics problems, in working collectively to develop ideas, approaches, explanations, and language, and in exercising agency and competence in the work. This program is a central component of our research laboratory because it enables us to closely examine student thinking, the work of teaching, and the role and use of mathematical tasks. Further, it affords the opportunity to study how mathematical knowledge for teaching is fundamentally intertwined with attention to equity in instruction. The mathematical

content of the instruction in the program includes work on fractions as well as some complex and unfamiliar mathematics problems. Mathematical practices and techniques include explaining, representing, proving complex claims, presenting in public, and listening to others' mathematical ideas attentively, respectfully, and critically.

The specific class on which this paper is based involved 30 grade 5 learners: 21 African American, 5 Latine, and 4 White children. In the program, the children worked on mathematics for 2½ hours each day for five days each across two weeks.

In order to study what is involved in listening to students' mathematical thinking, we meticulously recorded every class in the summer mathematics program and all individual conversations, using high quality video and audio recording equipment and techniques. We created detailed transcripts of classroom discourse, and additionally we represented aspects of embodiment, including how teachers and students moved in the space and their facial expressions. We paid careful attention to students' and teachers' language and the interplay in their mutual construction of meaning (McMahon, 2022; Sfard, 2009). Students' writing and drawings, e.g., in their class notebooks and on end-of-class checks ('exit tickets'), comprised further sources of evidence about students' ideas. We examined teachers' conversations with students about their work, seeking to understand their sense-making of these representations as a different form of listening to students.

To analyze the data, we created transcripts as a secondary record of classroom discourse, but our primary approaches rely on direct viewing of the video recordings so that we did not reduce the records to listening only to language (Sfard, 2021). Our analyses involved two stages. First, we examined the classroom interactions and discourse and identified spaces in which teachers' efforts to hear were apparent. These spaces were of different temporal lengths: what unified them as units of analysis was that they contained evidence of teachers working to understand, interpret, and take up students' contributions. Signs we looked for include teachers' questions and reframing of questions, students' responses, teachers' revoicing of students' ideas, and teachers' responses to student contributions. We indexed these and conducted open coding for elements of teachers' listening to students. Following McMahon (2022), we look for instances of teachers and students working to "get on the same page" in terms of mutual understanding, as well as moments of "talking past one another." After indexing the interactions during the class lessons, we conducted four rounds of coding based on our theoretical frames: (1) the degree of familiarity or novelty in the children's mathematical claims, questions, and statements; (2) the attunement of teachers' questions to students' statements; (3) teachers' apparent uptake, interpretation, and understanding of these contributions; (4) patterns in teachers' listening to students of different identities.

Findings

Based on our analysis, we identified six aspects of the work of listening to students that make visible the challenges of this crucial aspect of the work of teaching. Each entailed specialized understanding and uses of mathematical knowledge "in flight." Listening required both close attention to the actual words, representations, gestures, and tone of children's utterances, but also fluent mathematical interpreting and responding. It was also striking how much tentative hypothesizing about children's

meanings was involved and how the interpreting and responding were crucial to the dynamic checking of those hypotheses. The six aspects were:

1) Attending to children's talk as an intertwining of embodied communication, representations, and words. We identified many instances of children pointing, gesturing, using diagrams, and words in combination to express a mathematical idea or solution. For example, in explaining how they determined the fraction of this diagram that was shaded, a student pointed at the figure and said, "I saw *this* was the whole and then I counted 1, 2, 3, 4 and I knew it was ⅛."

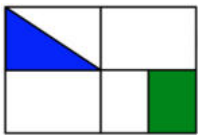

Figure 1: Fraction diagram for student discussion

The transcript of the student's words is incomplete as a representation of their statement. To fully hear what they are thinking requires watching where they point, seeing how they are counting 4, and what "this" refers to.

In another instance, a student showed a list they had written for equations for 10 and explained their strategy by indicating the list and saying, "I go up one and down one each time and that's how I know it is another one."

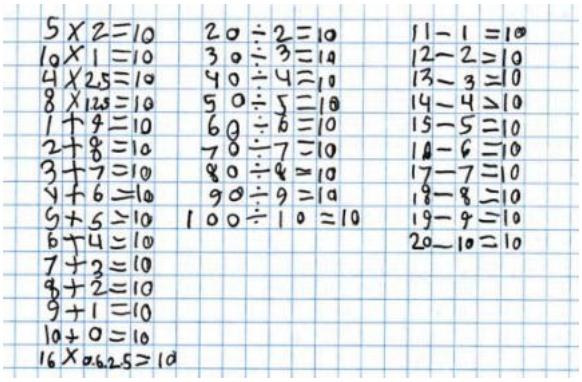

Figure 2: A student's list for equations for 10

We saw repeatedly that the work of listening required attending across more than words and interpreting the synchronized whole of children's "talk." The ubiquity of such "utterances" led us to avoid even enhanced transcripts for representing what students "said."

2) Suspending assumptions about children's use of language and probing for intentions and meanings. Children used language in ways that showed their sensible appropriation of words to describe something they are thinking, often using a term in a way that adults would not. Listening to students' ideas required a combination of mathematical knowing and an ability to hear and make sense of children's expressions and uses of language. This is a specialized kind of mathematical multilingualism necessary for teaching that includes the ways that young speakers use and invent language as they learn, combined with the differences in their out-of-school language contexts. We highlight here linguistic diversity related both to age and culture. One common example we saw

frequently in our data was children using the word "even" to refer to the idea of "equal" — e.g., when Chloe explains her classmate's reasoning for Figure 3, "Luis saw that the parts were not even so he made them even."

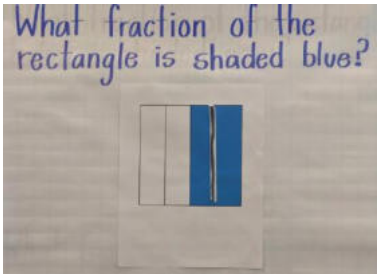

Figure 3: Fraction problem after Luis added a line to divide the single shaded blue rectangle in half

Children frequently used "boxes" for rectangles and said 0 was "not a number." In the course of explaining, they would use terms that made sense to them, and careful attention was needed for a teacher to hear their insight and not to immediately interpret them as wrong and interrupt their thinking. Using discretion to introduce a term is also part of this work, and we saw examples of refraining from pressing on language as well as using the opening to provide conventional language.

3) Thinking contextually and "historically" to construct possible hypotheses about how prior experiences with school mathematics might shape students' talk and thinking. As we coded the instances of teacher listening, there were examples of students saying things that reflected how they might have taken up an idea or an explanation in a previous grade in ways that distorted the math or reflected intermediate transition in their thinking. Examples included reading ½ as "one-twoth," a sensible generalization of the way other unit fractions are named, or saying that the 5 in 50 meant "fifty tens."

4) Posing questions that are understandable by students. We saw the regular need to probe or extend a student's idea, based on hearing and interpreting what they said. This complex work depended on, first, making sense of what the child said, and nimbly posing a question that opened an opportunity for the student to talk more, without leading or confusing their thinking. When Kassie, for example, argued that that "one and a half" of the rectangle was shaded, framing a question to follow up required quick thinking about what she was noticing:

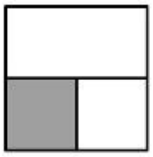

Figure 4: Fraction problem for Kassie's thinking

The work of following up a student's claim in ways that support their thinking was frequent and requires a quickness that is mathematically and pedagogically challenging.

(5) Managing the temptation to over-interpret out of a desire that students understand. Teaching is about helping students learn, and it is natural for teachers to hope that students

understand. The challenge we saw repeatedly was how to maintain both belief in a student's knowledge and simultaneously to be cautious and to probe so as not to assume common ground. One example was when a student, looking at the rectangle in Figure 4, said "I think it is NOT a fraction." This part of the work was, we identified, in tension with the second pattern above, suspending assumptions about students' use of language. "Not a fraction" might be a sensible way to make note of the fact that the parts are not equal. Or it could represent a conclusion that unless a figure is equally partitioned, it cannot be named with a fraction.

(6) Holding oneself accountable to students' talk and representations. Because students' expressions are often different in content, language, form, and even ideas, listening requires a conscious and deliberate effort to hear and make sense of what they are saying and showing. Further, it requires refraining from "taking over" students' utterances and restating them in accepted mathematical form and language. The tendency to revoice what students say often revealed a mis-hearing and an appropriation of their ideas to ones different from those they have expressed. In one example, a student named the number that the blue arrow is pointing to as "three fourths":

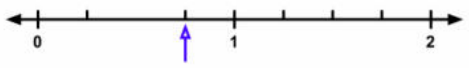

Figure 5: Blue-arrow fraction problem on a number line

Because this was the intended answer, an important next step in the work was to ask the student to explain their answer. Starting by pointing to the point labeled 0, the student counted, "One, two, three, four," pointing at each tick mark on the line. This explanation was a crucial contribution to the class's development of understanding about the number line because it surfaced the core idea that the interval from 0 to 1 constitutes the whole, and that one counts lengths, not tick marks. After the student explained their reasoning, it became the work for all the students to reach consensus about how to interpret this important mathematical object. If a teacher were to merely accept "three-fourths" without explanation, this mis-hearing would impede the mathematical work crucial to the content.

Although in this example, the student has an idea that might be seen as a misunderstanding, in fact it is part of the development of understanding of the number line that is crucial to emerge and be used in a class. When teachers do not recognize what students say, opportunities are lost. Many of our examples included cases where students had mathematically valid, if unexpected, ideas that, if unheard, would be missed opportunities to notice the competence of students' original mathematical thinking.

Discussion and implications

Across our analyses, three main themes stood out. One was the dynamic nature of using mathematical knowledge flexibly to hear students' language and other forms of expression. So much of the work of hearing students required listening "otherwise" (Lipari, 2009), stepping outside of one's own ways of thinking, yet depending on mathematical tools and sensibilities to make sense of others' meanings. Maintaining this duality in the effort to understand students' thinking was frequently evident and contributed to stretching the concept of MKT to be more about reasoning and using (Ball, 2017). A second theme was the importance of attending to racialized and gendered tropes about mathematical

competence and who is seen as competent. Actively working to disrupt patterns that value particular forms of talk and ways of presenting and expressing ideas was vital in listening to recognize their thinking and to hear their competence. The recurrence of this practice of hearing beyond normalized views of what counts as "knowing" and "understanding" required a broadening of conventional frames for competence. A third theme centered on language itself and reflected another crucial intersection of equity and mathematics. The learners in these data were linguistically diverse, including speakers of Black English and languages other than English. Further they spoke as children, not as adults. Our use of the term "multilingual" intentionally comprises the layering of child-language and ways of talking, as well as different linguistic identities and expression. We repeatedly documented the demands of listening multilingually tuned to mathematical ideas and reasoning.

These themes highlight the ways that listening mathematically comprises a combination of a special kind of ability to combine sensitivities to mathematical precision together with open attunement to young learners' own uses of language as they talk about mathematical ideas. It also entails a disposition to suspend the tendency to look for what is wrong, misunderstood, or incomplete that is often embedded in the very nature of mathematical analysis. Mathematical knowledge for teaching is fundamental to this work. Needed is a focus on supporting teachers to develop the fine-tuned capacity to listen mathematically. Important is to ensure opportunity to practice hearing, interpreting, probing, and responding to a variety of student talk and other forms of expression, beyond reading about students' conceptions or understanding. The dynamic nature of this work requires practice in real time with diverse students and varied contexts, and opportunities to identify threats to equity and practice with acting out commitments to listening "otherwise." Studies of ways to prepare and support teachers for this crucial work are an important next step to follow from this research.

Acknowledgment

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Polya's method variant: Developing students' analytical thinking

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In recent years, the level of basic mathematical skills as well as analytical thinking skills of secondary school students in the Netherlands has been on the decline, with consequences for the rest of these students' academic career. Polya's four step method for problem solving is a well-regarded and popular method throughout literature, which brings about the improvement of both students' attitudes towards mathematics and their (relational) understanding. In this paper a variation upon the method is proposed which aims to address the weaker aspects of Polya's method while enabling the students to develop their analytical thinking skills. In particular, the proposed method aims to promote the active use of mathematics as a language students can freely and easily express themselves in within the classroom so as to address the declining level of mastery within the subject mathematics.

Keywords: Polya's method, analytical thinking, systematic problem solving, mathematics teaching, reflection as a learning method.

Motivation

Thanks to the *Inspectie van het Onderwijs*, it has become clear over the last couple of years that the level for a number of basic (mathematical) skills of secondary school students in the Netherlands is too low. Especially critical reading, critical thinking and arithmetic, skills that play a role in almost every subject and that are important for developing analytical thinking skills of students, seem to remain very poorly developed. This often causes students to not understand what assignments require of them and to subsequently not be able to estimate or check the correctness of their own answers (Inspectie van het Onderwijs, 2022; Inspectie van het Onderwijs, 2023). The level of complexity and understanding that can be achieved for the subject of mathematics in particular suffers enormously from the low level of analytical thinking and reading comprehension (where, according to Prenger (2005), language knowledge and vocabulary also play an important role in the mathematical development of students). It is then becoming increasingly difficult to help students progress in their mathematical prowess: the students struggle with and remain stuck on the basics, leading them to learn in a purely instrumental way, using tricks and learning concepts by heart instead of really understanding the material and making it their own. This difficulty with basics results in repetition of what is supposed to be already acquired knowledge. This repetition then also becomes an increasingly significant part of the lesson, leaving little room for new topics, which must then be rushed through.

This decrease in mathematical knowledge is carried by the students throughout their entire academic career. In particular, in Inspectie van het Onderwijs (2022), it is reported that more and more secondary school students pass their final exams and proceed to tertiary education while the overall level of their analytical thinking and mathematical skills decreases. Students struggle then with the proposed mathematics courses in their further education, leading to high drop-out rates. In fact, correlation has been found to exist between the students' results in their final national high school exam, their GPA during their first year of university and their eventual graduation from a Bachelor

study programme (Hernandez-Martinez, 2016; Pinto & Koichu, 2023; Rach & Heinze, 2017; de Winter & Dodou, 2011). This knowledge gap between high school and university mathematics is a reoccurring issue which heavily impacts students and which is well documented across different countries and time periods (Higgins & Belward, 2009; Luk, 2005; Silius et al., 2011). The PISA 2022 study revealed for the EU in particular a trend in worsening educational performance and deeper inequality, with 30% of EU students not reaching a minimum proficiency level in mathematics (European Commission, Directorate-General for Education, Youth, Sport and Culture, 2024).

Students at the University of Twente have in past years had the opportunity to explore the advantages and disadvantages of Polya's four-step method for problem solving (Polya, 2004). Through this method, students were invited to complete assigned course work in a way that promotes understanding and mastering of the subject material. The method, with the addition of a reflection step (Mazur, 2016), was utilised by prof.dr. N.V. Litvak (Nelly) both at BSc (Bachelor of Science) and MSc (Master of Science) level for most of her courses (Litvak & Weedage, 2023). However, the introduction of the method brought about resistance from the students in spite of the clear benefits of utilising Polya's steps due to the amount of writing required, which the students were not used to.

The aim of this paper is to propose a problem solving method, inspired by Polya's method, to be introduced at the lower level of secondary school so as to provide students with a strong mathematical basis that they may trust upon throughout their academic career and to promote the development of analytical thinking from a young age. In particular, the proposed method is meant to bring forth the relational teaching and learning of mathematical topics and to bring about active discourse about mathematics through cooperation between students. Teaching within this method outlines the tasks of guiding the creation of an internal web through which students can place each new topic in relation to the ones already tackled, providing feedback as a form of personal development and assessment for the student, who may use this feedback to further grow and recognise both merits and demerits of their work and understanding, and promoting the cooperation among students. Teaching is here then understood as the fostering of the development of critical thinking among students and involves here also equipping the students with the understanding necessary to grade a fellow student and helping them develop mathematics as a (written) language students can clearly express themselves in.

In Heutinck (2022), a similar method to the one proposed in this work is introduced with the goal of replacing graded assessment with formative assessment that enables the development of relational understanding. The method proposed here has the final goal of being introduced as staple of the learning process within secondary school mathematics by which students regularly must use the method to complete exercises and develop a deeper understanding of the subject.

Theoretical framework

At the basis of this work sits the term **analytical thinking**, by which, based on the definition by Amer (2005), the ability is meant to be able to carry out the following thinking tasks upon a given problem:

- divide the given problem into main and sub-questions in order to be able to tackle the problem in a structured fashion;
- collect, organise and (possibly) visualise the data given in the problem;

- make connections between the given data of the problem;
- if possible, make connections between parts of the problem and own existing knowledge;
- make estimates for the answer required by the problem based on the given data;
- critically examine and analyse one's own process and the answer found so as to be able to indicate strengths, weaknesses and any present illogic;
- critically examine the process and the answer found by others so as to be able to indicate strengths, weaknesses and any present illogicality.

Someone portraying analytical thinking skills will, therefore, show interest in the workings behind processes and in the connections between subjects. This indicates that analytical thinking skills most likely develop in the presence of **relational understanding** (Skemp, 2006), understood as the ability:

- to understand how (mathematical) procedures, processes and rules work;
- to understand why certain methods may or may not work in certain situations;
- to understand why different areas of knowledge connect;
- to link new knowledge to existing knowledge.

Relational understanding is opposed to **instrumental understanding**, which can be understood as the application of rules without reason: while calculations can be performed, there is no understanding as to why the particular calculations should be carried out and why a certain method does or does not work. In particular, little to no connections are made between different pieces of knowledge, which makes memorisation rather difficult.

The idea of teaching analytical thinking skills to young students is not new. Morningside Academy, for example, has been implementing for years effective guidance in developing analytical thinking skills in the form of a variation on TAPS ('Think Aloud Paired Problem Solving') (Robbins, 2011), where students are taught to make use of 'private speech' (Berk, 1994) to guide themselves and classmates through new material and problems. Furthermore, analytical thinking is best developed through discussion and reasoning about the material, however, students hardly make use of such tools in the absence of the teacher (Murphy et al., 2014), meaning that thinking and sharing discoveries should be promoted in a different manner within the classroom.

The development of analytical thinking seems to be promoted by the use of modules regarding problem solving that includes the steps 'identify, define, explore, act, and look back' (Karenina et al., 2020), steps that are partially already present in Polya's method and which the method proposed in this paper aims to fully include. Additionally, previous research has shown the positive impact of Polya's method on students' attitude towards mathematics (Yapatang & Polyiem, 2022). In particular, Polya's method performs better than other educational methods (George & Enefu, 2019), is recommended as a classroom tool (Lee, 2017; Obiano & Parangat, 2023) and is even utilised as measure of mathematical thinking ability (Nurtamam & Jannah, 2024).

Proposed method

In Polya (2004), a method is described for solving problems systematically in four steps:

1. **Understanding the problem**: students are asked to read through the problem and to indicate in a clear manner what the question is asking of them and what (kind of) data is available to

them. Students are in this step then also invited to note down the interaction between the different pieces of data of the problem.

2. **Devising a plan**: requires students to use the data of the problem and known theory, situations and problems to come up with and write down a plan to be able to find a solution. The aim for students during this step should be to write down a sketch of the calculations needed to find the answer; it is not necessary to explain in extreme detail exactly what they are going to do. Students should in this step also write down any useful formulas as well as the theory and concepts that may be necessary for the assignment.
3. **Carrying out the plan**: students work out the problem based on their own drawn up plan.
4. **Looking back**: the purpose is for students to look back on the assignments so that they may learn from the process and better establish their knowledge. If the student has not found an answer, this step also serves as an opportunity to reflect on why their plan did not work and what the answer should have been.

While the method seems rather complete, it is the experience of the author and of many colleagues that step 2 very often poses a challenge to students and is experienced as unnecessarily difficult. Students, even university ones with much deeper knowledge of the mathematics necessary to complete the assignment, often get stuck at this step, causing them to either skip it or to abandon the assignment (Muniri & Choirudin, 2022). In addition, being able to reflect on one's own work is a skill that students learn best under the supervision of a teacher. Spending time within the classroom is a must to allow students to really develop this skill (Marzano & Miedema, 2018). For these reasons a new method is proposed.

The proposed method involves two separate stages: one outside of the classroom and one inside of it. During the outside stage, students are handed assignments to be solved by the time inside stage takes place. These assignments should be solved according to the following three steps:

1. **Preparation**: the students are asked to read through the assignment and to describe in own words what the goal of the task at hand is and what data is available to them. Students make in this step also an overview of the data and, if necessary, compile the data into a sketch to better describe the situation. In addition, students are here asked to indicate how they think the answer could be achieved. Some guiding questions may be provided to help the students along the way, such as: 'What pieces of information do you need to find the answer?'
2. **Calculations**: students are asked to follow up on step 1 by performing the calculations in an orderly manner. Students have to write out each calculation clearly by placing each new step on a new line and by indicating in a few words what they are doing.
3. **Explanation and Reflection**: students are asked to, by writing complete sentences, explain each step of their calculation. Guiding questions may again be provided to help students along the way such as: 'Why may you make this calculation' or 'Why did you take this step in your process and what did it help you achieve?'.
   If students are unable to reach a conclusion during this step, they are instead asked to explain what they were able to do and achieve (line by line) and to answer the following questions:
   - What have you achieved so far?
   - Where did you get stuck?

- Which pieces of information were you missing in order to continue?
- What did you expect the answer to be? Give an estimate and explain why you think this estimate could be correct.

By the beginning of the inside stage, students must have attempted to solve the given assignments and should bring their work to class to reflect upon their work and that of classmates.

During the inside stage, students are divided into groups and given time to discuss their solutions. This is a chance for students to actively talk about mathematics, their own process and to discover alternate routes that may lead to the same conclusion and learn the logic behind them. Additionally, students may in this step ask their peers about assignments they got stuck on and theory they did not understand. The teacher guides the discussion where needed and, by the end of the group discussion, solves the assignments on the board for all to see. Students must use this time to actively correct their own work and ask questions where necessary so that they may use what they learn for the final step: reflecting. At the end of the inside stage, students are handed a reflection sheet upon which they must indicate their understanding of the topics involved in the assignments (the list of topics is to be provided by the teacher), write down what type of mistakes they made and reflect on the qualities of their work. Students reflect here also on how to tackle the weaker parts of their knowledge.

Figure 1 visualises schematically the proposed method from beginning to end.

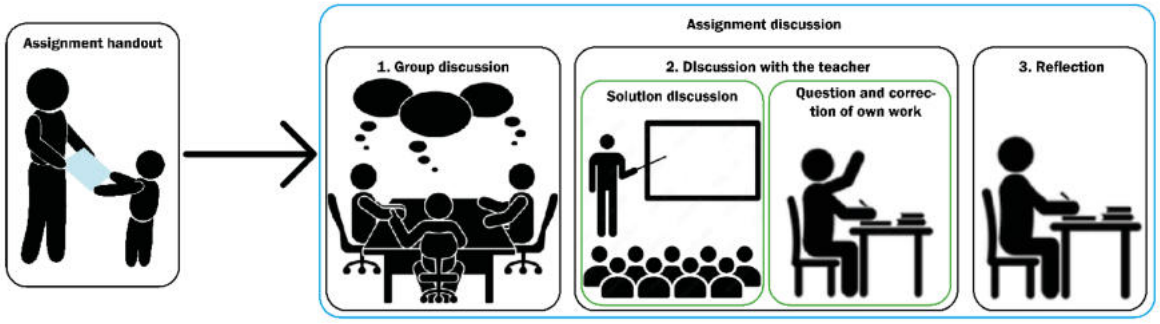

Figure 1: Schematic visualisation of the proposed method

Both stages should take place regularly, so as to invite students to often and consistently utilise mathematical terminology, logic and theory in the classroom. This way confidence in their mathematical skills may also be increased.

Compared to the standard Polya's method, the proposed approach has a number of theoretical advantages. The hurdle of having to set up a plan has been removed so that students may feel less discouraged to attempt the given assignment. Additionally, all the steps 'identify, define, explore, act, and look back' (Karenina et al., 2020) are properly addressed, with particular attention being given to the step 'look back', which now also lets students make use of 'private speech' (Berk, 1994) through having to explain their process in full sentences and through discussing their work with peers in a group discussion. Students are also given in this version of the method ample time and guidance within the classroom to learn how to discuss and reason about mathematics, effectively developing its use as a language with its own rules, demands and terminology, by which analytical thinking is then also better developed. The consistent use of the method throughout the students' secondary

school career may also prove to equip students with a valuable tool to tackle problems with: if a student feels stuck, they may refer back to the method, independent of the context.

Research and outlook

The proposed method was introduced within a K–8 class for a period of five weeks, during which students worked through one single chapter of their mathematics curriculum about Geometry. Three assignments were handed out to the students to work through according to the proposed method and, accordingly, three entire lesson hours were dedicated to the inside stage of the method. During the fifth week, the participating students had a test to conclude the chapter.

The implementation of this method during research was intensive, taking a lot of time out of the scheduled lesson time. This was due to the time constraints placed upon the research. A consistent implementation with few exercises per topic would benefit how time consuming the method can be.

Additionally, implementation in the class, due to the high amount of writing required by the method, was initially met with resistance. Preliminary results obtained in interviews with students reveal, however, that students later saw the benefits of the method and particularly appreciated being able to fall back on a clear plan whenever they would be stuck on an exercise.

Conclusion

The proposed method aims to be systematically introduced and applied within the subject of mathematics throughout secondary school (but may also be utilised within higher forms of education). The purpose is to use the method to help students develop strong analytical skills from a young age.

The proposed method improves upon aspects of Polya's method that, either according to literature or by own experience, may hinder the students' development of analytical skills and prevent, therefore, the development of mathematics as a language students may express themselves freely and easily in. Preliminary results already show a significant improvement in students' amount and quality of produced text in explanation to their steps when solving exercises.

The drawback to this implementation is the amount of classroom time required, which is a significant consideration when wanting to consistently utilise the method throughout secondary schools. For this reason, research on a greater scale and with greater consistency should be carried out as preliminary results do indicate that the method positively affects students: the produced text explaining students' steps in exercises increased both in quality and quantity. It would then also be of interest to compare the effects of the method across multiple countries, educational systems and schools.

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How expert mathematics teachers play 'status games'

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This paper argues that the concept of 'status', as understood by practitioners of improvised theatre, can be usefully deployed to illuminate some of the ways in which expert teachers of mathematics interact with their pupils. It suggests that this idea from the theatre can be thought of as a simplification of 'positioning theory', as it has been discussed in mathematics education literature and that it may offer a practical approach for classroom practitioners who wish to implement certain potentially productive classroom practices. The final section re-examines a set of observation and interview data from an earlier study to exemplify what 'status games' look like in the classroom.

Keywords: Secondary school mathematics, improvisation, expert teaching, positioning theory, interactions.

Introduction

The concept of 'status' as a tool for actors was developed by Keith Johnstone (1933–2033) when he was working as a teacher at Royal Court Theatre in London in the early 1960s. Faced with a group of actors who appeared unable to create 'convincingly ordinary' conversations, he gave them the instruction, "try to get your status just a little above or below your partner's", and in so doing, invented his first 'status game'. He reported that "the actors seemed to know exactly what I meant and the work was transformed…scenes became 'authentic', and actors seemed marvellously observant" (Johnstone, 1981, p. 33). Johnstone viewed status primarily as a practical tool for performers but also understood that it could be used to interpret social interactions, and it is the second – interpretive – application which, I will argue, captures elements of positioning theory. No attempt will be made to suggest some kind of equivalence between the complex, multidimensional structure of positioning theory, and the the much simpler notion of status. What will be argued is that the original purpose of status – as a tool for performers – has the potential to help teachers enact practices which the more complex framework of positioning theory points to as potentially useful in the classroom.

Theoretical background

According to Johnstone "status is a confusing term unless it's understood as something one *does*" (1981, p. 36) and to illustrate the point, offers the vignette in Table 1. To fully understand the exchange, it is important to know that 'Duchess' is the highest possible rank in the British aristocracy.

Table 1: adapted from Johnstone (1981, p. 36)

SPEAKER	SPEECH	STATUS CODING
1 TRAMP	'Ere! Where are you going?	***Tramp lowers the status of the Duchess***
2 DUCHESS	I'm sorry, I didn't quite catch . . .	***Duchess raises the status of the Tramp***
3 TRAMP	Are you deaf as well as blind?	***Tramp lowers the status of the Duchess*** further

Line (1) from the Tramp lowers the relative status of the Duchess in two ways: the content of what he says positions her as the transgressor (presumably she has intruded on his space in some way), and with his aggressive tone he positions himself someone with the right to make demands of her, in this this case, the right to demanding an explanation. This single line draws attention to what Johnstone identifies as a feature of nearly all status moves, what he calls "the 'see-saw' principle: 'I go up and you go down'" (1981, p. 37). There are exceptions to this principle which Johnstone addresses, however, in this particular exchange, the reciprocal nature of these moves is strongly evident. The way in which I have described line (1) suggests that the Tramp's words are status-lowering for the Duchess, while the tone is status-raising for himself, but it would have been equally plausible to switch these justifications around; what would not have changed is the overall effect, which is the change in their relative status. In (2) the Duchess, perhaps surprisingly given her title, seems to accept this reduced status, accepting her given position as the transgressor, and raising the Tramp still higher (or lowering herself further?) with an apology. By the end of this line, the normal social hierarchy has been reversed, and, at least in the context this interaction, the Tramp now has a higher status than the Duchess. The Tramp now consolidates his position in (3), lowering the Duchess still further by adding the list of her failings and increasing the status gap between them. Although it sometimes difficult to identify whether the Tramp's status is rising, the Duchess's is falling, or both are happening at once, what is clear is the direction in which the see-saw is moving, and for this reason, Johnstone tends to talk of 'status transactions' (e.g. Johnstone, 1981, p. 27), in preference to individual status moves.

Johnstone says a good deal about the dramatic impact of status, and status changes, suggesting that very large status gaps – particularly when they reverse normal expectations as in the example offered above – are often associated with comedy. He goes on to make the less obvious claim that when actors learn to use "minimum status gaps" (1981, p. 70) this can lead to more 'authentic' performances (1981, p. 33), and it is this idea of the 'reducing the gap' in relative status proves to be significant in the teaching episode analysed later.

There are intriguing echoes of status in van Langenhove and Harré's (1999) account of 'positioning theory', for example, their claim that "fluid positionings, not fixed roles, are used by people to cope with the situation they usually find themselves in" (1999, p. 17) resonates with Johnstone's discussion of "the difference between the status you are and the status you play (Johnstone, 1981, p. 36). Positioning theory sets out to "understand the dynamics of social episodes" (van Langenhove and Harré, 1999, p. 6) aligning with Johnstone's secondary purpose for considering 'status transactions' but digging little deeper, important differences soon emerge. Lagenhove and Harré identify three salient features of these episodes on which they base their analysis: the first of these is described as 'the moral rights and obligations' of the participants, the second is their conversational history and the third is what they actually say during the episode. Johnstone, as a theatre practitioner, is concerned with what is happening in the moment and has no real concern in the second of these features, the 'history' of the participants only being of interest to the extent that it is made explicit in the drama of the scene being played. Additionally, Lagenhove and Harré describe their framework as "tri-polar", consisting of "positions, storylines and relatively determinate speech acts" (van Langenhove & Harré, 1999, p. 18) which can be thought of as vertices of a triangle. The speech acts

of each individual impact on the positions of all the participants in the scene, and also with the over-arching storyline which interact dynamically in a much more complex way than in Johnstone's see-saw with a protagonist at each end.

Applying these ideas to the exchange between the Duchess and the Tramp recorded above, we have an initial 'moral positioning' of the two central characters implied by their names, which are really markers of traditional social rank. The scene takes place in the UK, where tradition also assigns certain rights and obligations to these ranks – the Duchess is wealthy and has high social status, and can therefore assume a good deal authority in most social situations, in contrast, the Tramp is poor and his minimal social standing confers very few rights, and an obligation to 'know his place'. If these social positions play out in the standard way, we might expect one of the following 'storylines': the Duchess ignores him and continues on her way, or she takes pity on his unfortunate state and makes some charitable gesture towards him. What these two storylines have in common is that they are determined by the action of the Duchess, but the story that actually unfolds is very different from either of these: instead of being about a powerful individual demonstrating either generosity or hard-heartedness, it is one of the underdog standing up to the powerful. This story is initiated by the Tramp's first 'determinate speech act' (1), which asserts his moral right to pass judgement on the Duchess and demands an apology. Johnstone's narrow focus on the relative status of the participants offers no insight into the way the storyline has changed beyond the general observation that 'audiences enjoy a contrast between the status played and the social status' (Johnstone, 1981, p. 36). From an analytical viewpoint, this is a significant omission.

There are, however, significant overlaps between the analysis provided by the two frameworks. The Tramp's 'determinate speech act' (1), as previously stated, positions the Duchess as the transgressor and the Tramp as victim. According to Lagenhove and Harré's framework, this is a 'first order positioning' move, which assigns new – and perhaps unexpected – rights and obligations on both participants, a change which is also captured by the 'see-saw' metaphor dscussed earlier. The Duchess could potentially respond in 'second order positioning' move, reasserting her social superiority and wealth – in Johnstone's terms, a status-raising move – and challenging the positioning proposed by the Tramp, but in fact, her words in (2) reinforce the position she has been given, which was also captured by the earlier analysis. When we reach the Tramp's words in (3) he repositions her again as guilty of further trangressions – in Johnstone's terms, lowering her status further. It is therefore argued that Johnstone's one-dimensional notion of status, while ignoring major features of positioning theory, does, at least in this instance, capture one important aspect of the relative positioning of the participants.

I now turn from the rather unlikely scenario of aristocrats arguing with tramps, to the more familiar setting of the mathematics classroom.

Drageset and Ell's (2024) review of literature around talk moves and discourse patterns in the mathematics classroom is concerned with the issue of equitable access to mathematical ideas for all learners. Using the lens of positioning theory, they analysed 44 separate studies to identify patterns of interaction which "can help us think about the relationships among teachers, students, and mathematics and the implications of these for equity in mathematics learning" (Drageset & Ell, 2024,

p. 354). In line with the tri-polar framework described earlier, they summarise their findings by identifying 5 archetypal 'teacher positions' grouped within three 'transcendent storylines', associating each position with particular types of 'communication actions' (see Table 2).

Table 2: adapted from Drageset and Ell (2024, p. 359)

STORYLINE	TEACHER POSITION	COMMUNICATION ACTIONS
Teachers as providers of insight	A teacher who tells	Telling to initiate student work Telling how to do Telling about connections
	A teacher who supports	Reducing complexity Assessing Progressing student thinking
Teachers as facilitators of learning	A teacher who uses student ideas to create learning	Providing access and sharing Pointing out Reasoning with a student idea
	A teacher who orchestrates	Managing Developing ideas Focusing on peer thinking
Teachers as participants in learning	A teacher who participates	Collaborating Acting in role

In their description of the first of these storylines – teachers as providers of insight – the authors state that both of the accompanying teacher positions "are based on a teacher who claims power and consequently positions students as subordinate" (Drageset & Ell, 2024, p. 362). It would seem reasonable to interpret this using the language of Johnstone as 'the teacher claiming high status relative to the pupil'. There are, however, two different teacher positions that are aligned with this storyline which are described in quite different ways. 'A teacher who tells' expects the pupil(s) to either listen passively, or to ask for further clarification where necessary, but it does not allow for the possibility of the pupil to offering ideas of their own. In contrast, 'a teacher who supports', while still adopting a high status position, does allow limited space for the pupil to contribute, perhaps by sharing their work for the teacher to verify – or potentially challenge – its correctness. To seriously engage with a pupil's work positions them as at least potentially being the possessor of useful knowledge, so a teacher who does this can, in Johnstone's terms, be described as raising the status of the pupil involved. I therefore argue that 'a teacher who supports' is working with a narrower status gap between themselves and their pupils than one who merely tells.

The third position – a teacher who uses student ideas to create learning – "describes a position where the learning should start with the student's ideas" (Drageset & Ell, 2024, p. 365). A teacher who places pupil ideas at the heart of the classroom discourse is significantly raising the status of those learners. The teacher being described here is not simply acknowledging the value of pupil thinking but using it as "a springboard to develop thinking together" (Drageset & Ell, 2024, p. 366), which represents a much larger increase in the status of pupils than the one bestowed by the 'teacher who supports' and narrows that status gap between teacher and pupils still further. The fourth position of 'a teacher who orchestrates' continues this pattern – the gap between teacher and pupils has now

narrowed to the point where the teacher retains the right to manage the discussion, but all of the content is now provided by the learners – and the final position of 'a teacher who participates' describes a situation where the status gap has disappeared.

The teacher positions in column two can therefore be seen a hierarchy of 'status gaps' between teacher and students: it is largest on the top row, where the teacher is 'telling' and the students are listening passively, and smallest at the bottom where the teacher us collaborating with their students as equals. This narrowing of the status gap corresponds to increasing levels of student involvement, suggesting that a teacher who wishes to increase the involvement of their pupils in classroom discourse should consider ways of reducing that gap.

Next, I describe an episode in secondary school classroom which shows a teacher managing their status, and the status gap between themselves and their pupils, to conduct a productive discussion.

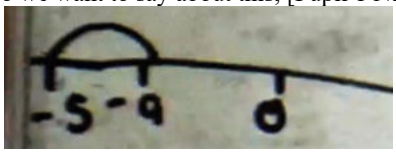
Data source

The interaction shown in Table 3 appears in McIvor et al. (2023) where it was used to illustrate a skilled mathematics teacher improvising in the classroom. The concept of 'improvisation' is peripheral to the discussion here, but it is helpful for the reader to understand that the dialogue recorded below was not part of the teacher's carefully planned exposition and emerged in response to many pupils revealing that they were unable to correctly draw a number line showing negative values. This had been addressed in detail in previous lessons, so the teacher had not prepared any material relevant to this content; everything she says and does in this episode is therefore 'improvised' in the everyday sense of 'made-up on the spot'. The mathematics teacher involved is a highly regarded practitioner teacher in her 12th year of professional practice. It was video recorded during a lesson with a group of 11–12-year-old learners which was the 5th in a sequence addressing the topic of directed numbers. The teacher was interviewed on the same day to discuss the decisions she made, and the videorecording used to remind her of key episodes. A full account of the data-collection procedure can be found in the original paper.

Analysis and discussion of a classroom interaction through the lens of status

Pupils had repeatedly failed to draw number lines which correctly showed the values -9 and -5 in relation to zero. The teacher collected three number lines drawn by different pupils on mini-whiteboards and the extract below describes the discussion of one (incorrect) representation.

Table 3: Number line discussion between teacher (T) and pupils (P1-P4) coded for status

	SPEECH/ACTIONS	STATUS TRANSACTION
1 T	What do we want to say about this, [Pupil 1 Name]? 	***Teacher claims high status*** by issuing a clear instruction ***Teacher raises the status of Pupil 1*** by addressing them by name and positioning them as the owner of valuable knowledge
2 P1	The 5 and the 9 should swap places because… because the number isn't… because it isn't…(*inaudible*)… like … on the positive side	***Pupil 1 raises their own status*** by providing an answer

3 T	(*Slowly, tentatively*) It doesn't go on the positive side… I think I know what you're trying to say to me. So, you're saying that the negative five and the negative nine should be the other way round.	***Teacher raises the status of Pupil 1*** further by reinterpreting their answer and designating it as correct
4 T	(*To class*) Why should they be the other way around? Think about the language that we're using with this.	***Teacher raises the status of all pupils*** by positioning them as knowledgeable
5 T	(T*o Pupil 2*) [Pupil 2 Name]?	***Teacher raises the status of Pupil 2*** by positioning them as knowledgeable
6 P2	Because negative nine isn't closer to zero than negative five	***Pupil 2 raises their own status*** by offering a correct, if convoluted answer
7 T	(*Slowly, unpicking what was said*) Yes, negative nine isn't closer to zero than…isn't closer to zero than negative five. Very nice.	***Teacher slightly raises the status of Pupil 2*** cautiously affirming their answer by repeating it
8 T	(*To class*) Anything… any other ways of phrasing that?	***Teacher raises the status of all pupils*** by positioning them as knowledgeable
9 T	(*Directly to Pupil 2*) That was really nice, actually.	***Teacher raises the status of pupil 2*** by repeating the earlier affirmation
10 T	(*To Pupil 3*) [Pupil 3 Name]?	***Teacher raises the status of pupil 3*** by positioning them as knowledgeable
11 P3	Negative nine is more negative than negative five.	***Pupil 3 raises their own status*** by offering a correct, clear answer
12 T	Okay, negative nine is more negative than negative five.	***Teacher raises the status of Pupil 3*** by affirming the answer immediately
13 T	So, they seem to be the other way round on our number line (*puts down board*)	***Teacher slightly raises her own status*** by starting to take control of narrative
14 P4	(*Eagerly holding up hand*)	***Pupil 4 tries to raise their own status*** by offering their own answer
15 T	(*To Pupil 4*) [Pupil 4 Name] what do you want to say?	***Teacher raises the status of Pupil 4*** positioning them as knowledgeable
16 P4	Negative 5 is closer to zero than negative nine	***Pupil 4 raises their own status*** by offering a correct, clear answer
17 T	Yeah, negative nine… negative five is closer to zero than negative nine.	***Teacher raises the status of Pupil 4*** by affirming their answer
18 T	So that's almost the same as what [Pupil Name 2] was saying.	***Teacher raises the status of Pupil 2*** by reaffirming their earlier answer
19 T	Okay, then… this one (*picks up next board*)	***Teacher raises her own status*** by taking charge of the agenda and moving on to the next example.

Having raised her status to bring the class together in (1), there are only two other occasions when the teacher raises herself. The first of these is in (13), when her statement "so they seem to be the other way round on our number line" combined with the act of placing the mini whiteboard back on the table suggest that she is taking control of the agenda (unsuccessfully, as it turns out), and the second in (18) when she decisively ends the current discussion and moves on to the next example. Nearly every other statement she makes can be seen as an attempt to raise the status of one or more pupils, with the effect of narrowing the gap between them and herself.

Her determination to achieve this status-raising objective is particularly evident when the responses from pupils are, from her point of view, 'sub-optimal' and the first example of this is (3), where the previous statement by a pupil has been so unclear that she is uncertain of whether or not it was correct. She resolves the confusion by creatively revoicing what the pupil has (perhaps) said, making a correct statement which she then ascribes to that pupil, raising their status in the process. A further example

of this is in (7) and (9) which record her developing a response to a technically correct, but hard-to-follow pupil explanation in (6). In (7) she first adopts the basic status-raising move of seriously engaging with what the pupil has said, in this case, by repeating some of the pupil's own words, then, having decided the pupil was correct, repeating it again and adding 'very nice.' After moving on to another pupil in (8), she seems to decide that her previous endorsement of Pupil 2's answer was insufficiently enthusiastic and returns to them in (9) to repeat the affirmation, raising their status once more, and further narrowing the status gap between herself and the pupil.

Perhaps the most surprising status-raising of a pupil is in (15) and (17). During the pre-lesson interview discussing what had been planned, this teacher had specifically referred to her desire to get the pupils to "use the language…[of]… becoming more positive, becoming more negative" so when she hears this language in (11) she has achieved her goal. This is reflected in the confident tone with she responds "okay" then repeats exactly what the pupil has said in (12). Since her first response to a pupil in (3) she has been working hard to minimise the status gap with pupils, or, in Drageset and Ell's terms, moving from 'reasoning with student ideas' in (3) to 'orchestrating' from (7) to (10). Having heard precisely the language she was hoping for in (11), she seems to have concluded that she can now move on to the next part of the lesson, and in (12) and (13) she starts to reassert her high status as 'a teacher who tells'. Unfortunately, Pupil 4 either fails to pick up on these signals, or simply ignores them, and makes a determined effort in (14) to be given space to contribute their own idea. The teacher's slightly offhand tone in (15) suggested that she had not intended to continue the dialogue, nevertheless she accepts the unsolicited offer and listens to what the pupil says in (16) before taking the opportunity raise their status in (16) and Pupil 2's status in (17). When asked in the post-lesson interview if she had kept the conversation going in order to hear as many different descriptions of the same error as possible, she replied:

> "Yes, but also giving them an appreciation of…like… all of their answers were right in in a different way, so it's like validating their explanations as well because they were correct in what they were saying."

Although she does not use the language of status herself, the teacher's desire to 'validate' what different pupils were saying suggests a commitment to recognising and publicly valuing pupil knowledge, and it therefore seems reasonable to describe her moves during this interaction as purposefully status-raising for pupils. The sequence from (15) to (18) also suggests that she is willing to prioritise raising the status of her pupils over 'getting on with the lesson'.

An important difference between Johnstone's example of the Duchess and the Tramp, and the classroom described above, is that in the latter case the 'see-saw principle' does not apply: when the teacher raises the status of her pupils by affirming their mathematical thinking, her own status does not fall. Johnstone acknowledges this possibility when he explains that "the exception… comes when you identify with the person being raised or lowered, when you sit on his end of the see-saw, so to speak" (Johnstone, 1981, p. 38). In the extract above, the teacher identifies with their pupils through an attitude of careful listening and the frequent use of 'we' and 'our', so they are able to raise the status of their pupils and narrow the status gap while maintaining overall authority in the room.

In contrast to the small number of occasions when the teacher raises her own status, the analysis above shows 12 separate instances of the teacher raising the status of learners. In terms of the language used in the title of this paper, the 'status game' being played by the teacher could be described as 'raise the status of pupils at every opportunity', and the consequence of applying this relatively simple rule was a near-constant narrowing of the status gap between teacher and pupils. It is interesting to recall Johnstone's observation that when he asked actors to minimise status gaps they 'authentic' and it may be that the most 'authentic' communication between pupils and teacher occur when such gaps are similarly minimised, or, in terms of positioning theory, when mathematics teachers 'aim for the bottom' of the list of teacher positions proposed by Drageset and Ell (2024). There is far too little evidence here to make any such claim, although the idea is an intriguing one.

Conclusion

I have argued that a mathematics teacher seeking to alter their position from one of "a teacher who tells" to one of "a teacher who uses student ideas" (Drageset & Ell, 2024, p. 359) could potentially make progress towards this goal by playing the Johnstone (1981) inspired game of 'minimise the status gap' between themselves and their pupils. I have further argued that this approach can be observed in the actions of at least some expert teachers of mathematics as they productively interact with their learners. In the final part of the discussion, I shared evidence that suggests some of these teachers justify those actions in in ways that resemble the ideas of Johnstone, although they may do so without using his language of status.

For teachers, there are two great advantages of status over positioning theory: first, it is simpler to understand, and second, status games, such as the one that was played by the teacher in the episode above, can be taught. The concept of status that has been outlined here was developed as a practical tool for performers to use in the rehearsal room and on stage, and over the last forty years, effective ways of training performers to manage their status have been devised and are used daily in acting classes around the world. If those approaches could be adapted to train mathematics teachers, it would offer the possibility of less experienced practitioners moving towards the kind of expert practice recorded above more quickly than if they were simply left to 'figure it out for themselves'.

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Implementing the SIMPLE approach: A study on teacher fidelity and perceptions in Kosovo classrooms

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This study explores the implementation of the SIMPLE approach, a constructivist-based intervention for enhancing word problem-solving skills through block models and metacognitive strategies in Kosovo's lower-secondary mathematics classrooms. Three teachers participated across three design-based research cycles, and data were collected via observations, interviews, and lesson recordings. Results show variation in fidelity, with one teacher closely aligning with the intervention, while others faced challenges adapting to pupil-centred teaching, block models and SIMPLE steps. Despite this, all teachers valued the approach for improving pupils' problem solving abilities, emphasising the need for thorough teacher training to support intervention success.

Keywords: Word problems, visual representation, block models, solving steps, teachers.

Introduction

Solving word problems (WPs) is a crucial skill that pupils need to develop, as it enables them, among other things, to apply mathematical concepts to real-life situations. However, solving WPs presents pupils with difficulties worldwide, including in Kosovo. Drawing on the literature on successful interventions focused on developing WP-solving skills, the first author developed a teaching intervention, the SIMPLE approach, aimed at lower-secondary pupils and embedded in socio-constructivism (Simon, 1995). Pupils are introduced to the block model method (Kho, 1987; Kaur, 2018) and to the model of solving steps fostering their metacognitive skills (Verschaffel et al., 2020).

The impact of the intervention on Kosovo pupils' WP-solving skills and the factors upon which reaching the desired goals depends are investigated within the first author's PhD thesis. This contribution focuses on only one factor: the teachers who implemented the proposed intervention in their lessons within the three design-based research cycles. We address the questions of how the teachers implemented the intervention after a short training, how they perceived the SIMPLE approach and its usefulness for their pupils' learning and how confident they were in applying it.

Theoretical background and literature

The rich literature focusing on the development of pupils' ability to solve WPs (Verschaffel et al., 2020) has informed the content of the proposed intervention in at least two ways. First, it focuses on visual representations, as a wealth of research shows promising results of using representations such as bar models in pupils' learning. Bar models visualise a situation model of the WP and mediate between verbal and symbolic representations (Kho, 1987; Kaur, 2018). Second, the intervention targets pupils' metacognitive skills, as awareness of one's cognitive processes when solving WPs improves achievement. We follow the work of others who developed scaffoldings for pupils to follow when solving WPs (e.g., CRIME [Careful reading; Recall possible strategies; Implement possible strategies; Monitor and Evaluation], Teong, 2003; SOLVE [Study the problem, Organise the facts, Line up a plan, Verify your plan with action and Evaluate your answer], Freeman-Green et al., 2015;

STAR [Search the problem; Translate the words into an equation in a picture form; Answer the problem; Review the solution], Özkubat et al., 2021]). None of the frameworks from the literature above includes an explicit step devoted to the visual component (bar model) and thus, they were modified and what I called the SIMPLE steps originated.

The intervention is embedded in the basic tenet of social constructivism (Simon, 1995), in which pupils are active participants in the learning process, and the teacher acts as a facilitator. It involves collaboration among pupils while solving WPs, as problem solving in groups is a rich learning experience and was found to support WP-solving skills (Ventistas et al., 2024). Nevertheless, much research shows that teachers find it often difficult to implement constructivist instruction. One of the reasons is that they do not accept that they are not the primary source of education, and often, the culture of the pupils in the classroom makes it difficult to comply with the constructivist philosophy (Windschitl, 2002). In this context, the concept of the didactic contract comes to the fore, an implicit agreement between teachers and pupils about their roles in learning (Brousseau, 1997). In a constructivist classroom, the contract involves pupil-centred learning, where pupils are encouraged to take responsibility for their learning and actively engage with the new material.

There is a difference between intervention-as-designed and intervention-as-implemented (Nelson et al., 2012). High fidelity is all the more difficult to achieve in teacher-led interventions, as they are the ones that deliver the intervention, and their understanding, commitment, and adaptability greatly influence its success. Thus, intervention studies involve the phase of teacher training. On the other hand, while strict adherence to the original design of an intervention is important, teachers often need to adapt to better meet their pupils' needs. These adaptations should not stray from the core components of the intervention to ensure that the intended outcomes are achieved (Carroll et al., 2007).

National context and research questions

The current Kosovo curriculum (MEST, 2016) aims to foster higher-order, critical and creative thinking, practical skills, and real-world competencies. It seeks to identify mathematics competencies that rely heavily on problem solving, focusing on contextual problems (WPs), and to improve the use of technological means, argumentation, reasoning, communication, and representation. However, lower-secondary mathematics textbooks in Kosovo provide little opportunity to support pupils in developing problem-solving strategies and higher-order thinking skills because most problems are routine and noncontextual (Berisha et al., 2013). The textbooks do not include visual representations for WPs, and teachers rarely, if at all, present visual representations when dealing with WPs in their lessons. Moreover, transmission whole-class work still prevails in mathematics lessons in Kosovo as teachers are only gradually adopting constructivist practices (Ismajli & Krasniqi, 2022). Thus, renegotiating the didactic contract from teacher-led to pupil-centred and involving group work are necessary prerequisites for implementing the intervention.

The study is guided by two research questions: RQ1. How did the teachers' implementations align with the three features of the SIMPLE approach? What were the causes of the misalignment? RQ2. How do teachers perceive the usefulness of the SIMPLE approach, and how confident are they in adopting it in WP teaching?

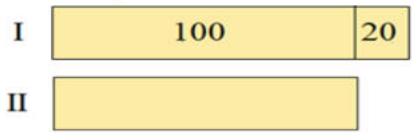
Methods

The current study is part of a design-based research project that aims to develop and refine a constructivist teaching approach for WPs, called the SIMPLE approach, in the context of Kosovo schools. The SIMPLE approach introduces pupils to the bar model method and metacognitive steps for solving WPs and is embedded in constructivism.

Overview of the SIMPLE approach

Bar model. The pupils receive two or three lessons in which types of bar models are introduced on examples, and WPs are solved with the help of bar models. Together, the pupils solve eight WPs, such as WP1_I, with the teacher's help or individually.

WP1_I. There were 120 visitors to Clara's art exhibit on the first day, which was 20 more than on the second day. How many visitors came to the exhibition on the two days?



$First\ day\ (D1) = 120$

$Second\ day\ (D2) = 120 - 20 = 100$

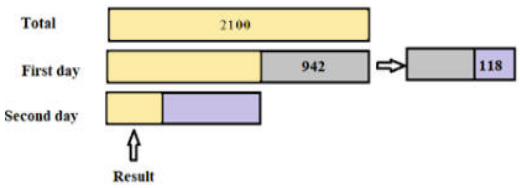
$Total = D1 + D2 = 120 + 100 = 220$

Figure 1: Bar model method for WP1_I

SIMPLE steps. In the following two lessons, the pupils are introduced to the SIMPLE steps: **S**tudy the problem, **I**nvolve the bar model method, **M**onitor the process, **P**repare/Present the solution, **L**ook over the solution again, and **E**valuate your answer (there is a similar acronym in Albanian). Pupils solve eight other WPs (such as WP2_I) explicitly targeting each step.

WP2_I: Of the 2,100 kilograms of potatoes in the warehouse, 942 kilograms were sold on the first day, whereas on the second day, 118 kilograms less than on the first day were sold. How many kilograms of potatoes are left in the warehouse?

Table 1: **Hypothetical solution**

S	*I have to find the kilograms of potatoes that are still unsold after two days. We know the number of those sold on the first day; we need to find for the second and then those left.*
I	*I draw blocks showing the kilogram of potatoes left on the first day. Then, another block is for those sold on the second day, and the third one belongs to those still unsold.* 
M	*Yes, I am on the right path since I know how much I must subtract to find the final answer.*
P	*Based on the blocks, we can do the calculations below:* $Day\ I = 2100 - 942 = 1158;\ \ Day\ II = 1158 - (942 - 118) = 158 - 824;\ \ Left = 334$
L	$334 + 824 + 942 = 2100$ / (*Left + Day I + Day II = Total*)
E	*The answer makes sense as we work backwards to arrive at the total kilograms of potatoes.*

Constructivist approach. Techniques promoting pupils' active involvement, problem solving, and reflection through group work and discussions with and among pupils are promoted.

Sample and data

Three lower secondary Grade 8 teachers implemented the intervention in three cycles of design-based research. They were willing to participate in the study, and their schedules enabled the first author to observe all the intervention lessons. The teachers were first instructed about the SIMPLE approach and provided with a detailed teacher guide. While a wealth of data has been gathered to evaluate all the research cycles, this study draws on the part of the data connected to teachers, namely, field notes made during the observations, audio recordings of lessons, teacher diaries, and semistructured interviews with teachers. The interviews and recordings were transcribed and analysed qualitatively.

First, in the transcribed recordings of lessons and the field notes, we identified episodes in which the teacher's implementation of the three features of the SIMPLE approach was visible. Second, these episodes were coded in terms of the alignment of the teacher's actions with the intended implementation. For example, whether the types of bar models were distinguished, how each of the SIMPLE steps was introduced, whether the teacher organised group work, how they managed discussions, and what support they provided to pupils. Third, the rest of the data were searched to triangulate information from the transcripts and the field notes and to find possible causes. For example, whether the teacher mentioned the coded episode in the diary or in the interview. Finally, all the data were revisited to find units where the teacher alluded to their perception of the SIMPLE approach and their confidence in its implementation.

Results

Alignment of the teachers' implementations with the SIMPLE approach as designed

Teacher 1: The first cycle was organised in School A in Gjilan and taught by a male teacher with 25 years of experience in teaching mathematics. He mentioned that WPs were part of his teaching and that he used them, especially with pupils participating in mathematics competitions. He claimed that he did not always use pupil-centred methods, as "pupils will misuse the situation and make unnecessary noise in the classroom". Before the intervention, the class covered the chapter on solving WPs via linear equations, and pupils were encouraged to solve WPs in such a way.

Next, we summarise how the teacher dealt with the three features of the SIMPLE approach.

The teacher solved WP1_I by drawing a bar model in front of the pupils. Then, he explained three types of bar models orally, e.g., by saying, "the part-whole type of bar model is when you connect two or more quantities to find the whole part". He mentioned a situation where "you have two books, and I have three more books, how many books in total are there?". No drawings accompanied the explanation. After that, he presented a WP similar to WP1_I for pupils to solve independently. The teacher drew the correct bar models for each WP solved during the lessons but never emphasised what type of bar model is associated with the particular WP and why. For example, for the problem "There are 550 books in total. On the first day, they sold 200 books, whereas on the second, they sold 50 more books. How many are left?" he drew a bar model for the total, then for the first day sold and for the second but did not refer to the types of bar models introduced earlier. The teacher introduced

the SIMPLE steps on the solution to WP2_I; he clearly distinguished each step and tried to make the pupils understand the importance of each one. Pupils worked mostly independently and could ask questions. However, we identified situations in which, after being asked if the solution was correct, the teacher said that the solution was not accurate, but he failed to associate it with the pupil's drawing and explained the mistake by pointing to the solution via an algebraic equation. Sometimes, the teacher felt unsure how to proceed with the lessons and asked the observer to help explain certain aspects of the SIMPLE approach to the pupils.

The data revealed several missed opportunities to initiate discussions and explore pupils' misunderstandings and ideas; for example, the teacher did not ask probing questions when pupils worked in pairs or discussed the homework solution and did not facilitate group work. During the interview, the teacher said, "They worked in pairs; if I had done group work, the class would have been messy."

Teacher 2: The second teacher was a woman from School B in the same city with 20 years of teaching experience. Based on our experience from the first cycle, more emphasis was placed on the aspects of the intervention in which Teacher 1 did not align with the teacher guide when instructing Teacher 2. Teacher 2 showed strong motivation and dedication in implementing the intervention to help pupils solve WPs. She agreed that the instructions provided in the guide fit with her teaching style and would not be challenging to implement.

The data showed that Teacher 2 adhered to the guide when introducing bar models and organising whole-class discussions in the classroom. She introduced the pupils to the SIMPLE steps one after the other by solving WP2_I on the blackboard and allowing them to practice it in other WPs. However, she focused on the step in which the bar model was to be made and did not give the same importance to all the SIMPLE steps as recommended by the teacher guide. Steps such as "evaluating your answer" or "monitoring the process" received much less attention. For example, for the "evaluate your answer" step, Teacher 2 said most of the time, "based on the result and the proof, the solution is correct" and told the pupils to write just "the solution is correct" or "the solution makes sense" without initiating a deeper discussion about evaluating the answer and monitoring the solving process. She probably did not ascribe much importance to these steps. She wrote in her diary: "…we do not need to ask pupils to write everything on paper if they are sure how to do it, to not take a lot of time while doing it". Second, as in the first cycle, the absence of group work was identified. In contrast to the guide, Teacher 2 allowed the pupils to choose whether they wanted to cooperate, and most pupils worked alone or with a friend sitting next to them.

Teacher 3: The third teacher was a female from School C with 12 years of experience teaching mathematics. Following the experience with Teacher 2, when Teacher 3 was instructed, all the steps in the SIMPLE steps were dealt with in-depth and group work was emphasised. The initial interview revealed that Teacher 3 tended to provide her pupils with an active learning environment in her lessons.

The data showed that Teacher 3 followed and implemented the teacher guide to a great extent. She adequately conveyed the lessons of the bar model method and the SIMPLE steps, organised group

work, and created a positive environment where pupils could raise questions and discuss solutions. The lack of time sometimes prevented Teacher 3 from providing feedback to all the groups.

The data confirmed that the constructivist underpinning of the intervention aligned with Teacher 3's usual style: "Implementing this approach was not hard, as I usually follow these rules in my teaching. I let pupils work together and do not interfere in their learning process, as sometimes they learn better from each other."

To summarise, the following differences between the intervention-as-implemented and the intervention-as-designed (Nelson et al., 2012) were identified:

- Introducing bar model types without visual illustrations (Teacher 1)
- Missing opportunities to initiate discussions and explore pupils' misunderstandings (Teacher 1)
- Failure to associate bar model types with the inner structure of the WP (Teacher 1)
- Failing to use visual representations to explain the solution; resorting to algebra (Teacher 1)
- Little group work (Teacher 1, Teacher 2)
- Failure to give adequate emphasis to all the SIMPLE steps (Teacher 2)

Teachers' perceptions of the SIMPLE approach

The teachers' perceptions of the new approach reflect the initial challenges and benefits observed in the classroom. Notably, during the interview, Teacher 1 stated that this approach, despite being challenging, was attractive and helpful for pupils. His diary entry indicates that, in his opinion, he and his pupils understood the SIMPLE approach well, but he feels that more time is needed for pupils to grasp it fully. Neither in the interview nor in the diary did Teacher 1 reflect on the situations in the lessons where he felt unsure of how to work with the new approach. He mainly appreciated that it was motivating for pupils and repeatedly expressed his intention to use it in the future.

Teacher 2 and Teacher 3 also benefited from the SIMPLE approach, in their opinion. For example, Teacher 2 said, "I liked it. After I did the intervention here, I also explained it to the 7th-grade class. We used it in some simple WPs, which helped them obtain a solution. […] I will continue to use it in mathematics classes."

For Teacher 3, it was easy to use the new method in the classroom: "No, it was not challenging at all. I think the block model made the new approach even easier to utilise, as it awakened curiosity and helped them [pupils] form the equation." She exhibited great faith in the new approach's potential to help pupils solve WPs and was enthusiastic about introducing it to pupils of other grades. For example, she brought three 9th-grade pupils about to participate in a mathematics competition to learn the SIMPLE approach, as she thought it would help them.

Discussion and Conclusion

This study investigated three teachers' practices while implementing the SIMPLE approach intervention. First, we focused on the fidelity of their implementation in terms of the content (introduction of bar models and SIMPLE steps) and pupil-centred teaching. For the former, the fidelity was high for Teacher 3, whereas Teacher 1 did not adequately present the types of bar models,

and Teacher 2 neglected some steps. While it was partially caused by the differences in the initial training of the teachers (each cycle informed the teacher's training in the next cycle), the teachers' teaching approach played an important role. Teacher 1's usual teaching style differed from that promoted by the SIMPLE approach, while it fit very well with Teacher 3 and partially with Teacher 2. The constructivist underpinning of the intervention went against Teacher 1's usual way of teaching. Teacher 2 did not initiate group work, and one can argue that group work is not an indispensable way of working within the intervention. Teacher 2 and Teacher 3 had to master only the bar model method and the SIMPLE steps, whereas Teacher 1 had to change his way of teaching substantially. The fact that we did not observe the teachers' regular teaching before the intervention and thus relied only on their descriptions is a limitation of our study.

Teacher 1's case revealed one of the most prominent challenges to implementation fidelity: the difficulty in adapting teachers' regular teaching to constructivist teaching (see also Windschitl, 2002). This is all the more important in Kosovo, as the transmission teaching method prevails.

The study's findings shed light on the importance of visual representation and metacognition in the solution of WPs, especially in the context where these factors are not usually used. Our data also highlight the necessity of thorough training before the intervention (Carroll et al., 2007). We underestimated Teacher 1's need to work deeper with the bar model method before presenting it to the pupils. Using visual representations for solving WPs is new for Kosovo mathematics lessons, as confirmed by the teachers involved in the intervention and the pupils we interviewed as part of the work on the PhD thesis. Thus, the length of the intervention should also be considered. While the first author's PhD thesis aims to evaluate a short-term intervention described in this contribution, the future use of the SIMPLE approach should be part of mathematics lessons in all grades.

Finally, regardless of the degree to which they adhered to active learning principles, all three teachers welcomed the new and could see benefits for pupils' ability to solve WPs. It remains to be seen how the intervention helped pupils solve WPs. It is performed within the first author's PhD thesis.

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Explicating the work of monitoring students' mathematical thinking

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Monitoring students' explorations is one of the practices of orchestrating mathematical discussions. Although studies have used different terms like observation, noticing, and listening and developed various frameworks, most studies have investigated empirically what teachers do, or the moves they use when monitoring students, and the focus is often on the mathematical ideas. This study takes a conceptual analytic approach to map out the terrain of what might be involved in the work of monitoring. We analyze an episode where a group of students explore a mathematical problem without any teacher interventions, and we identify plausible tasks of teaching that might be involved to map out the four domains of teaching that relate to mathematics, students, enactment, and issues.

Keywords: Work of teaching, domains of teaching, tasks of teaching, monitoring.

Introduction

In their paper on five practices for orchestrating productive mathematical discussions, Stein et al. (2008) present monitoring as a second practice of orchestrating discussions—following anticipation, which is carried out in the planning phase. Monitoring is thus the first practice of the interactive phase of teaching. Stein et al. describe this practice as monitoring the responses that students make when exploring tasks, and the practice of monitoring, when investigated in the research literature, thus tends to emphasize observations of students' mathematical contributions. Yet, anyone who has taught mathematics and observed students' explorations of tasks knows that the teaching practice of monitoring students involves much more than just attending to the mathematics that students put forth. It is a complex work that involves several challenges and decisions to be made. The present study aims at examining the practice of monitoring student responses when exploring a challenging task, to start mapping out the terrain of this slice of the complex work of teaching mathematics.

Theoretical background

Stein et al. (2008) state that the practice of monitoring student responses, "involves paying close attention to the mathematical thinking in which students engage as they work on a problem during the explore phase" (p. 326). Furthermore, they describe the goal of monitoring as identifying "the mathematical learning potential of particular strategies or representations used by the students" (p. 326). Although they are surely aware that teaching involves more than just attending to the mathematical contributions of the students, this is the aspect of the work of monitoring that Stein et al. (2008) highlight. In their effort to further investigate teachers' monitoring of students' groupwork, Ehrenfeld et al. (2020) emphasize monitoring of students' mathematical work, but they also discuss how monitoring involves attending to students' positioning and identity talk. Their framework of teacher moves includes how teachers approach the group (initiation), what they first say to the group (entry), what they attend to in their interactions with the group (focus), and the last thing they say

before leaving the group (exit). They also consider the patterns of participation in terms of whether the teacher interacts with the whole group or only parts of the group or individuals.

Sjöblom et al. (2023) apply Ehrenfeld et al.'s (2020) framework when they investigate the moves that some teachers made while noticing students' groupwork. In both studies, teachers' monitoring or noticing moves involved a lot of interactions with the students—including how teachers entered conversations with students, and how they used questions to initiate discussion or challenge students' mathematical thinking. Overall, Sjöblom et al. (2023) emphasize attending to the mathematical work of students, and this emphasis can also be found in other studies. For instance, Campbell and Yeo (2022) present a framework that emphasizes teachers' noticing of students' interactions around the mathematical content. They distinguish between the three elements of "attending to students' mathematical strategies, interpreting their understanding and deciding how to respond based on their understanding" (p. 492). In addition to these three aspects of noticing, their framework involves a mathematical dimension and a coordinated activity dimension. Jarry-Shore and Richardson (2024) study the role of teachers when noticing diverse kinds of struggle as students work on mathematical problems. They suggest that this aspect of teacher noticing involves attending to students' struggle, interpreting their struggle, and assessing whether students' struggle is productive or not. Whereas studies often refer to monitoring as teacher noticing, Hintz and Tyson (2015) unpack three types of teacher listening (evaluative, interpretive, and hermeneutic). Their focus is thus on what teachers attend to when engaging in various kinds of listening. Like in many other studies in this field, the examples Hintz and Tyson (2015) present and discuss in their paper almost exclusively focus on ways of attending (in terms of listening) to the mathematical content.

Where many study teaching in terms of what teachers do, or actions or moves that teachers make, our study takes a different approach. We follow Ball and Forzani (2009) who consider the work of teaching to be "the core tasks that teachers must execute to help pupils learn" (p. 497), and Hoover et al. (2014) who consider tasks of teaching to be decompositions of teaching into components that can be recognized by the professional community. Furthermore, Hoover et al. (2014) describe this decomposition as "identification of plausible conceptions of tasks of teaching around which professional consensus might be built" (p. 16). From this, we ask what constituent tasks of teaching that might plausibly be involved in the work of monitoring students' explorations, and what domains and sub-domains this slice of the work of teaching involves.

Methods

We explore data from the *Partners in Practice* project, an ongoing interdisciplinary professional development and research project. The project aims to establish a collaborative space for continuous development of teaching practice across subjects. Ten lower secondary mathematics teachers (teaching Grades 8–10) and three teacher educators were involved in the mathematics part of the project. In professional development sessions, the teachers solved mathematical problems and discussed a problem-solving approach to mathematics teaching. Teachers also tried out some of the ideas discussed in lessons with their own classes (students aged 13–16). We analyze data from one such lesson here, where a Grade 9 teacher tried out the "frame problem". The students were divided into groups of three, and each group got the problem on a sheet of paper (Figure 1).

We have an 8 × 8 region (light color), surrounded by a frame (dark color). The problem is to figure out a way to find the number of squares in the frame, without having to count them one by one. For each solution, try to create a mathematical expression that shows the number of squares in the frame. This figure has 10 squares along each side. What if there were 6, 40, 100, or *n* squares in the frame? Can you find a generic rule?

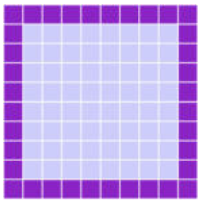

Figure 1: The "frame problem"

The teacher handed out the problem on paper and encouraged the students to find multiple solutions. We analyzed the discussion in a group of three students, as they began to share their ideas after having read the problem and considered it individually. The analysis followed four steps. First, we divided transcripts of the group discussion into units. A new unit was created whenever something new was going on in the conversation. These units were categorized by using the four teaching domains proposed in TWG19 (Mosvold et al., 2023). Second, inductive codes were developed to describe tasks of teaching that might be involved in each unit. In the third step, the tasks of teaching were sorted, and duplicates deleted. Lastly, tasks of teaching were grouped into the four domains, and sub-domains were identified inductively. Memos were written in all steps.

Unpacking domains of the work of monitoring

Below, we use the episode where a group of three girls (Anna, Berit, and Cecilie) explores the frame problem as a starting point for unpacking domains of the work of monitoring.

Mathematics

When students explore a mathematical problem, they are likely to suggest different solutions, strategies, or mathematical ideas, which teachers must attend to in their monitoring work. When analyzing this episode, we observe student contributions of various kinds—some contributions are expected, others are not; some contributions are mathematically correct, whereas others indicate misconceptions or are incomplete. The work of monitoring these diverse types of contributions might involve different tasks of teaching, which we will illustrate below.

There are several instances of strategies that teachers would likely have anticipated in the episode where the girls explore the frame problem. For instance, in the very beginning of the group conversation, Berit shares that she initially thought the problem was about finding the area of the inside square, but she has realized that is not the case. Next, she thinks, "what if it's 10, and it kind of becomes n, then it becomes like 10 minus, no n minus 2, times that again." When observing a student contribution like this, the teacher must attend to and interpret the student's thinking. For instance, Berit might be thinking about the enclosed region rather than the surrounding frame, or she might use the area of the whole and the inner area to find the number of squares in the frame, and this is something teachers might have anticipated. Cecilie then asks Berit to explain her thinking, and

Berit explains how she was initially thinking about the relationship between the outside frame and the side lengths of the inside square, saying "if there are 10 here, then there are 8 inside. If there are 6 here, then there are 4 inside." Cecilie agrees that there will always be two less on the inside, and Berit describes this as "n minus two times n minus two." For now, their mathematical ideas are incomplete, and their strategies do not yet help them solve the problem. After having agreed what the problem asks, Berit suggests that the frame consists of $10 + 9 + 9 + 8$ squares. This is a mathematically correct solution, and it is one that teachers would likely have expected. Berit adds that she initially "thought it was like ten times four, but it's not." This is also something teachers might have anticipated. Teachers who observe a group conversation like this must attend to the students' mathematical thinking. Monitoring also often involves interpreting students' thinking since the ideas and strategies that students propose are not always clear or explicitly formulated. During the monitoring phase, teachers must also consider if specific strategies are worth highlighting in a following strategy-sharing discussion or not.

Sometimes students produce unexpected mathematical ideas. After having spent some time pondering the pattern of $10 + 9 + 9 + 8$, realizing that the answer is 36 but not being convinced that this is a sufficient solution, Cecilie suddenly comes up with an idea. She remembers something they have learned in the previous semester about factorials, and she asks the others, "Didn't we learn something like that?" Not remembering what it is called, she tries to explain it in her own words: "The way it goes down, if you know what I mean. What is that sign called? So that it goes like this 10, 9, 8, 7, 6, 5, 4, 3..." Anna recalls that the sign is an exclamation mark, and the two of them spend some time discussing this idea. This is an idea that a teacher would not have expected to come up, and monitoring a discussion like this would both require attending to and interpreting the students' mathematical thinking. In addition, a situation like this would also require considering if this mathematical idea might be productive or not, and if it fits with the learning goal for the lesson. Anna then asks the others, "Is it possible to limit the exclamation mark?" This is an interesting mathematical idea, and it could have opened for exploring combinatorics and permutations. The contribution also involves a misunderstanding, as the students do not yet seem to recall that factorials involve multiplication rather than addition. Eventually, the students realize this and reject the idea, but this is an example of unexpected ideas that might be introduced when students explore problems, and that a teacher must attend to, interpret, and consider in terms of their mathematical potential.

Students

Monitoring students' explorations also involves becoming acquainted with the students as learners and people. We identify three sub-domains within this domain of work: attending to identity talk, talk that indicates uncertainty or doubt about own abilities, and several types of off-task behavior. We illustrate these sub-domains below and indicate tasks of teaching that might be entailed.

Students' explorations of mathematical problems often involve talk about how students view themselves. For instance, when Berit first starts sharing her initial idea about considering the area as $(n-2) \times (n-2)$, Anna comments, "I'm not a mathmagician, so I can't do this." A bit later she sighs, "I thought I was smart in math..." Some students might interpret their own struggling as an indication that they are not "smart." Such statements might develop into a more persistent identity

that might limit the student and her further progress, but it might also negatively influence the learning climate of the entire group. Teachers should therefore attend to this kind of identity talk in the mathematical discussions and consider approaching or even challenging them.

Sometimes students make comments or statements that indicate uncertainty or doubt about their own abilities. For instance, when the group is first stuck after realizing that the problem does not ask them to find the area of the interior square, Cecilie says: "Is it just us who are stupid, or is it like that... Or are we somehow supposed to know this?" A bit later in the conversation, she asks the others in her group: "Have other people figured out the answer?" There are several instances in the conversation where Cecilie makes comments like this, sometimes made with a chuckle, or in a low, whispering voice. Talk like this might indicate uncertainty or doubt about own abilities in mathematics, and this is something teachers should also attend to. Sometimes, it might be appropriate to engage the entire class in a conversation about uncertainty and doubt, highlighting that struggling and being stuck is common when approaching a mathematical problem, or when trying to solve a task that is unlike something they have worked on before.

Observing students' exploration of mathematical problems might also involve off-task behavior. For instance, after Berit has introduced her observation about the frame being $10 + 9 + 9 + 8$, Cecilie suddenly makes a comment about what day it is. Teachers must attend to off-task behavior in the work of monitoring, and they must consider if, when, and how to engage it. This leads to the next domain, as teachers' monitoring of this kind of behavior or talk might also involve considering what to do next, and how to organize and enact the interactions and discourse of teaching and learning.

Enactment

In the work of monitoring students' explorations, the domain of enactment is mostly about considering what to do and when. We identify five sub-domains of enactment.

The first sub-domain is to consider if, when and how to acknowledge students' contributions. Acknowledgment or praise can be given for several reasons, and teachers must consider this already as they monitor students' explorations. For instance, when Berit shares her first thoughts about the problem and how she initially believed that the problem asked about the interior square, a teacher might consider praising her for having the courage to share her misunderstanding and failed efforts. Also, when Cecilie follows up by asking Berit to explain how she has been thinking, the teacher might consider praising her for attending to another student's thinking, and for trying to understand it. Similarly, when Cecilie introduces the unexpected idea about factorials, or when Anna suggests "limiting" the exclamation mark, teachers might consider if these ideas should be acknowledged by presenting them to the entire class in a following strategy-sharing discussion.

Second, there are many instances where students make mathematical statements that are not immediately clear to a teacher, and the teacher might consider asking a clarifying question to elicit a student's thinking. For instance, when Cecilie suggests factorials, saying that "it goes like this 10, 9, 8, 7, 6, 5, 4, 3...", a teacher could consider asking where the student sees this sequence in the problem. When Anna and Cecilie start discussing the idea of factorials, Cecilie asks: "Is it possible to take *n* exclamation mark, and so on, since there is 8 in the middle, like that?" It is not immediately clear

what Cecilie is referring to by "8 in the middle" here. It could be the interior square, but a teacher might consider asking her to elaborate on this and explain how she thinks here.

A third sub-domain of enactment is to consider how long to let the students struggle and whether to interfere and help them along. Several instances in this group discussion indicate that the students are stuck or struggling. For instance, Berit's comment that her initial idea of $(n-2) \times (n-2)$ would not work because the frame is $10 + 9 + 9 + 8$ is followed by a long pause before Cecilie suddenly makes the comment about what day it is. This is an example of a situation where it might be tempting for a teacher to intervene, for instance by confirming that Berit's observation is correct, or by suggesting to explore if there are other ways of finding the number of squares in the frame.

Fourth, some instances would require teachers to attend to what students said and consider following up on this by engaging the group—or the entire class—in a conversation about meta-cognition or norms. For instance, when Cecilie asks if they are supposed to know this already, Anna comments that "it's a thinking task." This is something a teacher might consider noticing and coming back to in a discussion about what kind of task this is and strategies for approaching a task where there are no immediately available strategies. At another point, when the students are stuck, Berit goes back to reading the assignment aloud. This is another instance that a teacher could consider coming back to in a discussion about strategies for dealing with challenging tasks and emphasize that an initial part of the problem-solving sequence is to make sure you understand what the problem is.

A fifth sub-domain concerning enactment is to consider how to approach situations where students are dealing with uncertainty or doubt. For instance, when Cecilie and Anna engage in the idea of factorials, Cecilie reaches a point where she sighs, "I don't know, that's how it goes." There were many other instances in this conversation where students expressed doubt or uncertainty. In addition to considering such instances by attending to students' identity talk or consider making meta-comments about strategies for problem solving, teachers might consider letting the students stay in this state of uncertainty, or to encourage or nudge them to move forward.

Issues

As discussed by Hoover and Goffney (2019), students' explorations of mathematical problems might involve issues relating to race and gender, and situations where teachers must consider disruptive patterns of oppression. The situation analyzed in this paper is more limited in this sense, but we observe instances relating to two possible sub-domains here.

First, there are several instances where students make statements that might indicate potentially problematic social or cultural views about mathematics. For instance, Cecilie seems concerned that some of her ideas are not "math methods," which might indicate a view that mathematics is limited to application of approved methods or algorithms. Likewise, Cecilie's question about whether they are supposed to know this might indicate a social view of mathematics as a subject where you are only supposed to solve tasks that the teacher has already explained how to solve. Some of Anna's statements might also indicate social or cultural views about what it means to be "smart" in mathematics, and the teacher must consider if these are views that should be challenged somehow.

Second, there are instances where a student's behavior could be considered disruptive. For instance, when discussing the idea of factorials, Berit asks if factorials involve addition or multiplication, whereupon Anna says, "Oh yes," and she seems to realize that she has misunderstood this. After a pause, Cecilie starts laughing. When students start laughing in situations that involve failure or misunderstandings by other students, teachers should pay attention and consider if this is disruptive behavior that should be addressed or not. Since this situation involves three white Norwegian girls, other dynamics than race or gender might be at play. Still, these are considerations that teachers are faced with, already in the work of monitoring.

Concluding discussion

Monitoring students' exploration of mathematical problems is a challenging slice of the work of teaching. Some teachers seem to think that students' exploration or group work is time for the teacher to do something else, but there is important work involved in monitoring students' group discussions. Monitoring work is often invisible—the teacher might not say much or conduct any visible actions or moves—and our analysis indicates this. The potential tasks of teaching that might be entailed in monitoring typically contain verbs like *attending*, *interpreting,* or *considering*. These verbs, which correspond with the three categories of teacher noticing from Campbell and Yeo (2022), indicate that the work of teaching in this phase is mostly hidden, and it does not necessarily involve any verbal interaction or observable moves by the teacher. Whereas previous studies emphasize teachers' monitoring of students' mathematical explorations (e.g., Campbell & Yeo, 2022; Hintz & Tyson, 2015; Jarry-Shore & Richardson, 2024; Sjöblom et al., 2023), our study illustrates how the work of monitoring involves all four domains of teaching (cf. Mosvold et al., 2023). Like Ehrenfeld et al. (2020), we show how monitoring work also involves attending to students' identity talk, but we add the dimensions of considering further *enactment* and attending to broader social, cultural, and political *issues*. Our analysis highlights the complexity and involved nature of monitoring students' mathematical explorations.

Where other studies investigated monitoring by observing what teachers do, our study took a different approach. By considering a record of teaching—an episode where three Grade 9 students explore a mathematical problem—we have developed and tested ideas about possible tasks of teaching (cf. Hoover & Goffney, 2019). This follows Hoover et al. (2014), who highlight the idea of focusing on the tasks that teachers must deal with in their work—rather than what they do—and to develop plausible candidate descriptions of such tasks for further deliberation in the professional community. We acknowledge that our study is limited by the records we have analyzed. For instance, our data has limited potential to highlight issues of race and gender that might be involved, and we acknowledge that teacher monitoring can vary with different problems or tasks. Still, we suggest that our explication illustrates some important qualities as well as challenges that this work might entail, and we believe that our unpacking of this work might be useful for further studies of the work of monitoring students' mathematical thinking, and for mathematics teacher education. Our study also illustrates how the four proposed domains of teaching (cf. Mosvold et al., 2023) can be a useful analytic framework.

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Tensions in teachers' attempts to balance mathematical challenge with students' diverse learning needs

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We study how a group of six mathematics teachers in Greece balance mathematical challenge with students' diverse needs, focusing on the tensions they experienced and how they handled them during reflective discussions in a video club setting. Using an analytical model (the Teaching Triad) we characterize the teachers' tensions into three interrelated domains of teaching. Our findings reveal four categories of teachers' tensions: pedagogical, curricular/institutional, cultural/social, and epistemological. A range of actions that teachers propose or enact to handle these tensions were identified, such as fostering exploratory discussions, providing guidance through enabling and extending prompts, creating a safe and supportive environment, and encouraging small group work.

Keywords: Mathematical challenge, differentiation, teachers' tensions.

Introduction

A range of studies have highlighted the importance of engaging students in cognitively challenging work to enhance their mathematical learning and reasoning (Tomlinson, 2014). Compelling global evidence suggests that opportunities for differentiation play a crucial role in supporting mathematical learning for all students. This process involves aligning learning targets, tasks, activities, resources, and support with the unique needs, preferences, learning styles, and paces of individual learners (Tomlinson, 2014). Therefore, teachers must be supported in effectively differentiating their teaching practices within heterogeneous classrooms (Pozas et al., 2020). Research shows that mathematics teachers often encounter tensions when planning and enacting instruction (e.g., Stouraitis et al., 2017). The tensions teachers experience when balancing mathematical challenge with students' diverse needs remain underexplored in research (Potari et al., 2023). There are some studies that address teachers' perceptions of their students' mathematical abilities and the way teachers can engage them in high demanding tasks (e.g., Horn, 2007; Jackson et al., 2017). However, few studies (e.g. Vasilopoulou and Triantafillou, 2023) focus on the tensions teachers experience while they attempt to balance mathematically challenging teaching with all students' diverse needs. This study aims to contribute to this area by examining these tensions and how teachers manage them so as to achieve this balance. Our data source is reflective discussions in a video club setting (project EDUCATE, https://websites.ucy.ac.cy/educate/en/). The research questions are:

1. What tensions do teachers face while attempting to balance mathematical challenge with students' diverse needs in whole-class settings?
2. How do teachers handle the tensions they encounter?

Theoretical background

A number of studies have identified teaching strategies for balancing challenge with students' diverse needs such as: using tasks with a low entry point to allow all students to find a solution, while also

providing opportunities for advanced students to extend their thinking; using enabling or extending prompts based on students' performance; highlighting, synthesizing, and extending of important mathematical ideas while interacting with students; utilizing a variety of resources; and creating an inclusive and mathematically challenging learning environment (Charalambous et al., 2023; Psycharis et al., 2019; Sullivan et al,. 2016). These strategies can be seen as possible ways by which teachers attempts to manage the tensions involved in balancing mathematical challenge with students' diverse needs.

The Teaching Triad (TT) has been used as a research tool to study mathematics teaching and the balance between mathematical challenge with students' needs (Potari & Jaworski, 2002). The TT characterizes mathematics teaching through three domains: *Management of Learning (ML)* (the creation of the classroom learning environment)*, Sensitivity to Students (SS)* (awareness of students' emotional, cognitive, and social needs) *and Mathematical Challenge (MC*) (presenting challenges to promote mathematical thinking, and engaging students in meaningful activities). Research indicates that harmony between MC and SS indicates an effective learning environment. The TT has been extended to provide interpretations of the interactions between its three elements in the context of the activity of teaching considering the tensions that the teacher faces (Potari et al., 2023). This study uses the TT in its current development to study the activity of six teachers who attempt to balance mathematical challenge with students' diverse needs in the micro level of the classroom in relation to the tensions they experience and their handling at both the micro and macro levels of the teachers' activity.

Mathematics teachers often encounter tensions in their professional lives, particularly in planning and enacting teaching. These tensions can include pedagogical conflicts that arise during teaching-learning processes (Stouraitis et al., 2017) or conflicts related to teachers' expectations and perspectives on professional development (Yamagata-Lynch & Haudenschild, 2009). Herbst and Chazan (2012) provide a characterization of tensions through the construct of professional obligations. These include: (1) *Interpersonal tensions,* related to classroom interaction; (2) *Disciplinary tensions* related to the mathematical knowledge taught; (3) *Institutional tensions,* involving external obligations such as department policies; and (4) *Individual tensions,* stemming from teacher's attempts to consider and respect each student's unique physical and psychological traits. Horn's (2007) study illustrates how teachers often perceive a disconnect between their students' abilities (e.g., "fast kids," "slow kids," "lazy kids") and the rigorous curriculum they are expected to engage with. This perception often leads to tensions as teachers strive to balance the demands of high curriculum standards with the diverse learning needs of their students. From a similar perspective, Jackson et al. (2017) used the diagnostic and prognostic framework to analyze how teachers explain the sources of students' difficulties in mathematics (diagnostic framing) and how they describe their responses or solutions to these difficulties (prognostic framing). The study shows that many teachers do not view all students as capable of engaging in rigorous mathematical activity, often attributing difficulties to inherent student traits or external factors. This deficit-oriented perspective can further exacerbate the tensions teachers experience as they attempt to implement ambitious reforms while addressing the diverse needs of their students. The latter two studies implicitly address teachers' tensions, as their main focus is on teachers' views on students' capabilities. Only a limited number

of studies have specifically focused on teachers' tensions related to mathematically challenging teaching and students' diverse needs. For example, Vasilopoulou and Triantafillou (2023) explored teachers' tensions as they tried to create inclusive mathematics settings for both competent students and those who struggle with mathematics. They found that these tensions are related to selecting tasks that are both inclusive and challenging, teachers' personal perceptions of students' mathematical abilities, and institutional rules imposed by the educational system (e.g., limited teaching time). Our study contributes further to this direction by exploring teachers' tensions related to balancing mathematical challenge with students' diverse needs in teaching and learning as evidenced in teachers' reflective discussions in a video club setting.

Research context and participants

EDUCATE is a European professional development program with four partner countries (Cyprus, Greece, Ireland, and Portugal) addressing issues of cognitive activation and differentiation and supporting teachers in engaging all their students in challenging tasks. In this paper, we use data collected in Greece involving two teacher educators (TE1 and TE2), and six practicing secondary school mathematics teachers: T1 (Aleka), T2 (Aris), T3 (Apollon), T4 (Eli), T5 (Grigoris), T6 (Penelope) (all pseudonyms). The teachers were all degree qualified (e.g., master's degree, participation in several PD programs) with an average of 15 years of teaching experience.

The data collected comprised eight videotaped meetings. In this paper, we analyze and report results from two of these meetings. Meeting 2 focused on mathematical challenge and meeting 3 focused on differentiation. During these meetings, each teacher presented video clips that they selected from their own classrooms, highlighting what they considered significant in relation to the focus of the PD program. Participants then watched and discussed these clips through the lenses of mathematical challenge and differentiation. The data analysis aimed to explore the teachers' tensions related to balancing mathematical challenge with students' diverse needs in teaching and learning, as well as the potential or actions taken to handle these tensions. The discussion around each video clip that the teachers presented during each meeting was treated as a unit of analysis.

The analysis was conducted in three phases: **Phase 1:** We identified episodes within each unit that indicated teachers' tensions related to balancing mathematical challenge with students' diverse needs as well as potential or actual actions taken to handle these tensions. **Phase 2:** We coded the teachers' tensions identified in Phase 1 within each episode. Adopting grounded theory techniques and using the TT, we categorized these tensions into its three interrelated domains of teaching (ML, SS, MC). The tensions were interpreted in terms of the classroom interactions (micro level), and in the broader contexts (e.g., curricular, social) outside the classroom (macro level). **Phase 3:** We focused on identifying and analyzing the potential or actual actions aimed at handling these tensions. Actions discussed could be mentioned by the teachers themselves or their peers during the meetings. Grounded analysis was combined with theoretical ideas for balancing mathematical challenge with students' diverse needs. Below, we provide a brief example of the analysis:

Phase 1 involved identifying an illustrative episode (3rd meeting) demonstrating the tensions involved in balancing mathematical challenge with students' diverse needs as well as potential or actions taken to handle these tensions.

Penelope: I chose to design a lesson with an attempt to integrate some elements of differentiation. My aim initially was to use the basic tool concerning the property of the median of a triangle that divides it into two new ones with the same area … I struggled a lot to find a way to integrate differentiation. By inserting different exercises? How could I link differentiation to the management of the exercises in class? … I asked myself: What am I teaching? Areas. …So, I formulated the problem to have elements recognizable by students – plots of land, a square, etc.

Phase 2 involved two steps. **Step 1:** Coding the tension. Penelope's tension, as evident in the described episode, revolves around the challenge of balancing lesson planning and engaging all students with mathematical challenge. This tension arises from the need to effectively plan a lesson that caters to students with varying cognitive levels. We coded it as pedagogical tension related to lesson planning and student engagement. **Step 2:** Link to the TT: we categorized these tensions into three interrelated domains of the TT. Penelope's tension as evident in the episode revolves around the challenge of balancing lesson planning (ML) and engaging all students with mathematical challenge (MC). This tension arises from the aim to plan a lesson that caters to all students' varying needs (SS).

Phase 3 involved identifying and analyzing the potential or actual actions aimed at handling these tensions. To address this tension, Penelope decided to *introduce the mathematical concept within a real-world context*. By presenting data related to plots of land with geometric shapes she aimed to make the task more accessible to students.

Results

We first present the teachers' tensions. Next, we provide the teachers' actual or potential actions for handling them.

Teachers' tensions related to balancing mathematics challenge with students' diverse needs

Our analysis reveals four main categories of tensions (see Table 1): Pedagogical tensions; Curricular and institutional tensions; Cultural and social tensions; and Epistemological tensions.

Table 1: Teachers' tensions related to balancing mathematics challenge with students' diverse needs.

Tensions	Description	Example	Short analysis of the example
Pedagogical: Lesson planning/ enactment and Potential/actual student engagement	This tension arises when teachers attempt to balance the structured objectives of their lesson plans with the dynamic and unpredictable nature of student engagement during the lesson.	"I chose to design a lesson with an attempt to integrate some elements of differentiation. My aim initially was to use the property of the median of a triangle that divides it into two new ones with the same area … I struggled a lot to find a way to integrate differentiation…" [Penelope]	Penelope's tension revolves around balancing lesson planning (ML) and engaging all students with mathematical challenge (MC). The tension arises from catering to students with varying cognitive levels (SS).

Curricular and institutional: Curriculum and institutional demands and ML	This tension lies between balancing curriculum and institutional demands (i.e., mathematics inherent in the task, depth of content knowledge, program schedule, tools and resources) and ML (i.e., task planning, teaching strategies)	"I gave the students a task requiring them to find the sum and product of the roots of given equations, understand the concept of equivalent equations, and apply Vieta's formulas in various exercises... The equation had a negative discriminant... I struggled to discuss and explain it fully to students… due to limited time … The curriculum didn't cover this concept in-depth." [Grigoris]	Grigoris' tension arises from addressing advanced mathematical concepts that are not part of the curriculum. The limited time available to explain these concepts further compounded the issue, impacting his teaching (ML).
Cultural and social: Existing and new norms of classroom communication and potential/actual students' engagement	This tension arises when teachers struggle to create new norms of classroom communication facilitating mathematical challenge for all students.	"In the past, students' standing at the blackboard was a common classroom practice, ... However, as this tradition has declined, even capable students resist such activities… A student recently expressed reluctance to solve a problem on the board due to a lack of experience. How much should we push students out of their comfort zones..." [Aris]	Aris' tension arises from promoting new classroom norms (ML) and engaging students in challenging tasks (MC), while being sensitive to their emotional well-being and prior experiences (SS).
Cultural and social: Students' cultural background and potential/actual students' engagement	This tension arises when teachers struggle to balance students' diverse cultural backgrounds with fostering their engagement in mathematically challenging activity.	"I have a lot of foreigners… they have many problems … they struggle with the language… and in general I have children who are studying for half an hour at home [limited]… I also have a child in this class who is with Asperger syndrome… The challenge in this lesson is the way students developed an understanding of the triangular inequity with the use of sticks... " [Aleka]	Aleka's tension lies between planning lessons (ML) and fostering student engagement (MC) when dealing with diverse cultural and learning backgrounds (SS).

Epistemological : Nature of mathematical challenge	This tension arises from differing perspectives among teachers regarding the nature of the mathematical challenge.	"I would give them only the first task (asked students to find the Sum *S* and the product *P* of the roots of several equations) and somehow let's also put a parameter like gamma... then ask them to make a guess about the sum and the product …but in this case I could do only the first task… "[Grigoris]…"I wouldn't even ask them to find the roots… as initial questions we can find if the roots have the same or different signs without solving them... which is very challenging as an introductory task." [Apollon]	From Grigoris' point of view doing many exercises with demanding calculations is one way that students engage with mathematical challenge. For Apollon, students engage with mathematical challenge through inquiry and not through procedural tasks.

Teaching actions for handling the tensions

Our analysis reveals a multiplicity of actual/suggested actions aimed at handling the tensions aroused. These include: Making useful modification(s) to tasks so that students can productively address the task; Providing guidance (enabling and extending prompts); Utilizing manipulatives as a strategy for learning; Encouraging small group work; Encouraging different solution strategies; Introducing the mathematical concept with Real-World context; Fostering exploratory and conceptual discussion; Creating a safe and supportive environment; Using a smaller number of inquiry tasks rather than a set of exercises of different levels of difficulty. These strategies are essential for fostering an environment in which all students can engage meaningfully with challenging mathematical content. **To address curricular and institutional tensions**, fostering exploratory and conceptual discussions was the most frequently employed strategy. **In managing cultural and social tensions**, providing guidance through enabling and extending prompts, creating a safe and supportive environment, and encouraging small group work were highlighted. For instance, Aleka (see Aleka's tension in Table 1) used manipulatives, such as sticks, to enable further students' involvement while facilitating paired activities, addressing tensions arising from students' diverse cultural backgrounds and needs. For **pedagogical tensions,** a comprehensive range of strategies was suggested, including defining the lesson's theme and objectives and introducing mathematical concepts within real-world contexts. For example, Penelope (see Penelope's tension in Table 1) strategically focused on introducing mathematical concepts within real-world contexts, such as plots of land with geometric shapes, to enhance students' accessibility and engagement. **For epistemological tensions**, using a smaller number of inquiry tasks rather than a set of exercises with varied difficulty was suggested. For example, TE1 advised Grigoris (see Grigoris's tension in Table 1) that fewer, well-designed inquiry tasks are more effective for student learning than multiple procedural exercises.

Discussion and conclusion

This study contributes to the growing body of research on mathematics teachers' tensions. Previous studies have focused on tensions in teaching activities, such as pedagogical, curricular, and epistemological tensions (e.g., Horn, 2007; Potari et al., 2023). Our findings extend these studies by

employing the TT framework to explore the tensions teachers face when attempting to balance mathematical challenge with students' diverse needs at the micro level of the classroom in relation to the tensions they experience and their handling at both the micro and macro levels of the teachers' activity. In line with previous studies, this study identifies four key categories of tensions: pedagogical, curricular/institutional, cultural/social, and epistemological. Several aspects of the EDUCATE project allowed these tensions to be expressed. The discussions that took place during the meetings among teachers as they analyzed video clips from their own classrooms—viewed through the lenses of mathematical challenge and differentiation—emerged as a significant factor in alleviating these tensions. This collaborative approach aligns with existing research that highlights the benefits of mathematics teachers sharing and critiquing their experiences as part of professional development in video-club settings (van Es, 2012). Another aspect relates to the classroom environment and the students' diverse backgrounds and needs. Aleka's experiences with students from diverse cultural backgrounds underscored the cultural and social tensions inherent in her teaching practice. Her action of using manipulatives to simplify mathematical concepts exemplifies a culturally responsive teaching approach, which is supported by research emphasizing the importance of inclusivity in education (Tomlinson, 2014). In the case of Grigoris, curricular and institutional tensions arose from the constraints of the mathematics curriculum and the limited instructional time available. His actions to address complex mathematical concepts within the confines of the curriculum highlight the broader institutional pressures. This finding resonates with research that emphasizes the impact of external factors, including curriculum demands and institutional policies, on teachers' instructional decisions (Herbst & Chazan, 2012). While prior research has identified strategies for promoting differentiation and balancing mathematical challenge with students' diverse needs, such as using enabling and extending prompts (e.g., Charalambous et al., 2023; Sullivan et al., 2016), this study extends those findings. It highlights several strategies teachers employ to manage their tensions both at the micro level of classroom interactions and the macro level of their broader teaching activities. Overall, the teachers' actions to handle their tensions align with the aforementioned research that prompt teachers to balance mathematical challenge with students' diverse needs, such as using enabling and extending prompts (Charalambous et al., 2023). Further analysis of our data will also help us to identify emerging patterns of relationships between the tensions and the actions. The interplay of pedagogical, curricular, cultural, and epistemological tensions reveals the multifaceted nature of teaching mathematics in diverse classrooms. These insights underscore the necessity for ongoing professional development and support for teachers, enabling them to build an inclusive learning environment meeting all students' needs.

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Selecting and sequencing problems for classroom discussion in a problem-posing task: Insights from prospective elementary teachers

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This study aims to determine which problems and in what sequence prospective elementary teachers select for a problem-posing task to handle them in classrooms, as well as the factors that mobilize their selection. We presented eight problems to 40 participants, developed according to various criteria for a problem-posing situation. They were asked to select four of these problems and explain their reasons for their selections. The findings provide data on the participants' problem selections. Furthermore, we identified seven factors that influenced their reasons for problem selection, two of which—focusing on mathematical content and varying the complexity—were more emphasized.

Keywords: Problem posing, problem selection, prospective elementary teachers.

Introduction

The primary source of problems brought into the classroom is often textbooks or teachers; however, students themselves can also serve as sources of problems (Kilpatrick, 1987). This aspect implies the incorporation of two critical skills in the classroom: having students generate problems and/or reformulate existing ones (Silver, 1994), which, more broadly, pertains to problem posing. The endeavor to integrate problem posing into mathematics classrooms is valuable because it fosters students to become independent learners and encourages them to take greater responsibility on their learning processes (English, 1997).

The idea that students' own problems can serve as the starting point for instruction (NCTM, 2000) highlights the critical role of teachers in problem posing. For example, recent instructional frameworks on problem posing recommend selecting some of the problems generated by students and addressing them in classrooms from various perspectives (e.g., Cai, 2022). It would neither be feasible nor logical to discuss all the problems generated by students in classrooms. Consequently, teachers must select the problems they intend to address from among the student responses, determine the order in which to present them to optimize learning, and establish the criteria for analyzing these problems. Furthermore, learning through problem posing takes place while students pose problems or discuss some of these problems within the classroom (Zhang & Cai, 2021). Therefore, teachers' decisions for selecting and sequencing problems will influence the opportunities provided to students through problem posing. That is to say, the steps of selection and sequencing are critical not only for the integration of problem solving into mathematics instruction (Smith & Stein, 2011) but also for the effective integration of problem posing (Cai, 2022).

The present study aims to explore which problems, and in what sequence, prospective elementary teachers (PTs) select for a problem-posing task to be addressed in classrooms, along with the factors that mobilize their selections. The literature on problem posing emphasizes the need for more research on how teachers integrate problem posing into mathematics classrooms to more effectively. This study can contribute to deepening our understanding of the opportunities PTs intend to provide students in the context of problem posing, by focusing on the selection and sequencing of problems, which are critical aspects of their instruction. Furthermore, illustrating a critical aspect of PTs' instructional practices related to the problem posing can provide a valuable foundation for developing their knowledge and practice regarding the integration of problem posing into mathematics classrooms. Thus, this study holds the potential to contribute to both following perspectives outlined in the research trends on problem-posing studies focused on teachers: the teacher as manager of the dynamics of problem posing and the potential of problem posing for teacher education (Montes et al., 2024).

Theoretical Background

Theoretical frameworks have been developed regarding how problem posing can be integrated into classrooms. Cai (2022) proposed a task-based instructional model in which the teacher presents the problem-posing situation and prompt, the students pose problems, and (students and) teachers handle posed problems in the classroom. He proposed instructional practices for the final step of the model, which include analyze, select, sequence, and solve. In the analyze practice, the teacher examines the problems generated by the students from various perspectives, such as whether they are mathematically correct or not. In the select practice, the teacher decides which problems will be handled in the classroom. The teacher can make decisions based on criteria such as learning objectives and the difficulty level of the problems. Afterward, the teacher determines the sequence in which the selected problems will be handled, and the problems are solved in the order established through collaboration with the students.

A limited number of studies on problem posing, due to their classroom-based reports, offer some insights into teachers' pedagogical intentions regarding the selection and sequencing of problems. Zhang and Cai (2021) found that teachers tended to disregard problems that were irrelevant to the instructional objectives of the lesson, while considering the difficulty level for those that were relevant. Furthermore, teachers provided quick guidance to students during class discussions for easy problems; however, they either assigned more challenging problems as homework or indicated that these would be solved in the next lesson. Teachers showed a tendency to engage in discussions of moderately challenging problems. In a study investigating how problem posing is addressed in an elementary school classroom (first grade, 6- and 7-year-old students), Possamai and Allevato (2024) reported that the teacher encouraged students to share the problems they had created by reading them aloud to their peers in class. However, their findings revealed an absence of reflections on the structure of the problems and a lack of exploration of possible solutions. Montes et al. (2024) found that pre-service teachers' problem modification intentions were focused on deepening understanding of the mathematical content, varying the level of complexity, improving comprehension of the problem statement, and assessing understanding. Considering these findings within the context of our study, it suggests that the choices made by PTs may be influenced by factors such as the complexity

of the problems, the mathematical concepts and relationships they involve, and their effectiveness in evaluating student understanding.

Although the studies summarized above provide some insights regarding problem selection, they do not offer information on certain aspects, such as how in-service and pre-service teachers respond to mathematically incorrect student problems. From a behaviorist perspective, errors are considered detrimental, as they have the potential to reinforce incorrect responses (Santagata, 2005). Conversely, within the constructivist approach, they are regarded as an integral component of mathematics classrooms. As highlighted in Principles and Standards for School Mathematics document (NCTM, 2000), "mistakes are not viewed as dead ends but rather as potential avenues for learning" (p. 144). This study can also provide insights into how PTs address mathematically incorrect problems or those with insufficient data through their teaching practices related to problem posing.

Methods

Participants and the task

The study involved 40 PTs who were in their third year of an elementary school teacher training program at a state university in Türkiye. PTs must successfully complete a four-year teacher training program in order to be certified as elementary school teachers (teaching students aged 6 to 10 in grades one through four). The PTs were given one month to complete the assignment (Figure 1), and data collection occurred during the period in which the participants were enrolled in the Mathematics Teaching-II course.

We utilized the road-trip situation in the problem-posing task because it has been featured in numerous studies (e.g., Silver & Cai, 1996). The selection of the eight problems in the data collection tool was guided by the sample problems from the literature: (i) problems that were answered by doing operations with the existing data (P1, P3, and P4), (ii) problems whose answer was readily available in the existing data (P2), (iii) problems that were mathematically incorrect (P5), (iv) problems that were created by adding new conditions to the existing data (P6 and P7), and (v) problems that had insufficient data (P8). In this study, the data from the first two questions of the data collection tool were utilized to address the research questions (see Figure 1).

Data analysis

The frequency distribution of the types of problems selected, based on the order of responses, has been provided to illustrate the PTs' problem choices. Qualitative content analysis was used to determine which factors the PTs mobilized in their problem selections (Mayring, 2014). Two coders, who are also the authors of this study, identified a total of 24 codes for the reasons behind problem selection. Some of these codes were observed quite rarely in the sample, and in this context, codes that appeared in only one participant's problem were eliminated (e.g., assessing student mathematical knowledge, supporting classroom communication, and highlighting expectations from the problem). The final 21 codes are therefore listed in the findings section under seven categories.

Problem posing task: Mehmet, Ali, and Hasan drove their car back home after their vacation. Hasan drove 120 kilometers more than Ali did during the trip. Ali drove twice the distance that Mehmet drove. In this trip, Mehmet drove 80 km.

Dear students, based on this situation, create a problem for your classmates.

A teacher presents this task to one of 4th-grade mathematics classes and gives students 10 minutes to individually pose a problem based on the given situation. After that, the teacher aims to select and discuss some of the students' problems in the classroom. Below, you are presented with 8 problems of the students in this fictional lesson.

P1: How many km did Hasan drive?

P2: How many kilometers more did Hasan drive than Ali?

P3: How many kilometers did the three of them drive together?

P4: How many more kilometers did Hasan drive than Mehmet?

P5: How much longer is the total distance driven by Mehmet and Ali than the total distance driven by Hasan?

P6: If they refuel the car every 150 km during the trip, how many times do they need to refuel it in total?

P7: If the car consumes 2 TL of fuel per km during the trip, how much fuel do they spend in total?

P8: When they went on vacation together, Hasan drove 160 kilometers more than Ali did. Accordingly, how many kilometers did Hasan drive on the way to and from the vacation?

Accordingly, please answer the following questions:

1. If you were the teacher and wanted to choose four of the problems listed above to address in the classroom, which ones would you choose? Why?
2. In what order would you consider examining these problems during the lesson? Why?
3. Write a scenario for the teaching you plan to carry out, taking into account the problems you chosen and their order.
4. Explain the situations you focused on, considered important, or paid attention to while designing your scenario.

Figure 1: Data collection tool

Findings

Findings on the distribution of problems according to response order

In their first choices, PTs selected one of the options P1, P2, or P3 (as illustrated in Figure 2). The key characteristic of these problems is their focus on understanding the variables and their relationships in the situation. In this context, participants prioritized problems that would enable students to analyse the mathematical content within the problem-posing situation. Among these, P1 was the most frequently chosen. Participants highlighted several reasons for starting with such problems, including its potential to help students understand the mathematical relationships involved, contribute to solving subsequent problems, and be relatively easy for the initial attempt. For example, PT5 explained her selection as follows:

> [Referring to P1] The question clearly states how many kilometers only Mehmet drives. I adopt the working backwards strategy and try to find out how many kilometers Hasan, P1's answer, drives. In fact, while trying to find out how many kilometers Hasan drives, I have to find out how many kilometers Ali drives, so that I actually find out how many kilometers the people mentioned in the problem drive. By finding the unknown points in the problem, it will be easier for us to answer the questions that will come in the following stages.

For the second choice, participants' selections demonstrated a wider distribution, with a notable preference for problem types P3 and P4 among the six problem types. These problems are consistent with P1, which was predominantly chosen for the first choice, as solving P1 facilitates the solutions of these subsequent problems. Similar to their second choices, PTs' selections for their third and fourth problems were distributed across a broad spectrum (as shown in Figure 2). In these choices, P6 and P7 were particularly prominent. The primary characteristic of these two problems is their potential to expand mathematical understanding by introducing new data into the existing problem-posing situation.

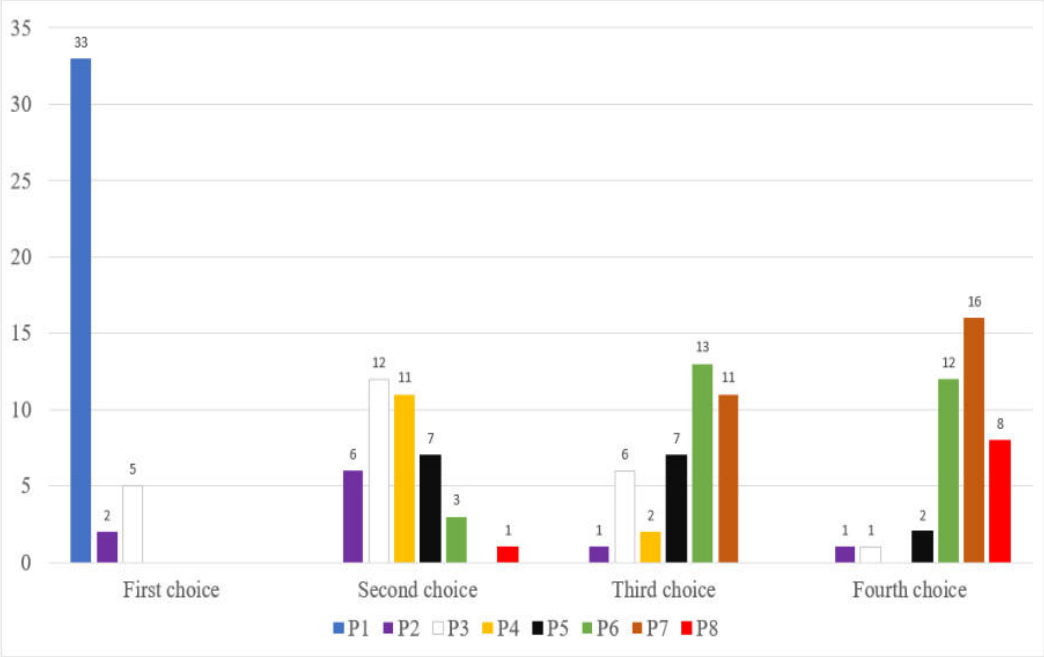

Figure 2: Distribution of problems based on response order

Given that PTs often highlighted the complexity of the problems as a rationale for their choices, they tended to introduce such problems in the third or fourth order in the classroom. PT36's explanation includes the following:

> I would choose R1, R3, R7 and R6, (respectively). The reason for my selection was to act according to the principle of simple to complex. At the beginning, I chose problems that were easy to understand and solve, and as I progressed, I wanted the problems to become more complex and difficult.

Another notable finding of the study is the low preference for analysing P5 and P8 in the classroom, with P5 being mathematically incorrect and P8 containing missing data. When all problems are grouped, P1-P4 and P6-P7 exhibit similar characteristics and were predominant across the sample. On the other hand, P5 appeared in two-fifths of the sample, while P8 was present in just over one-fifth. These results suggest that participants did not show a strong tendency to explore incorrect or incomplete problems within the learning environment. Additionally, when comparing the two types

of problems, incorrect problems were preferred over incomplete ones. Participants who selected the P5 and P8 for classroom discussion sought to illustrate to students both the error in the problem and that not every problem has a solution.

Moreover, some PTs failed to recognize the errors or missing data in the problems and instead justified their choices by emphasizing the skills required to solve the problem. For example, PT2's explanation includes the following:

> I would choose problem 5 as the third problem because I think that they will be able to solve this problem easily with the information in the second problem. Additionally, comparing among drivers will improve their comparison skills and addition-subtraction activities through this problem. That is; Mehmet = 80 km. Ali = 160 km. Hasan = 280 km. Mehmet + Hasan = 80 km +160 km = 240 km. 280 km-240 km = 40 km

According to another finding of the study, answer of the P2 is situated within the current problem-posing situation. Only one-fourth of the sample selected this problem for classroom discussion. Among them, only PT10 explicitly stated that she selected this problem to discuss the expectations from students in a problem-posing situation. A part of PT10's explanations are as follows:

> I would choose the problem "P2: How many kilometers more did Hasan drive than Ali?" Because the answer is given in the problem, I can show the students that this problem does not constitute a problem.

Distribution of reasons for PTs' problem choices

The reasons for problem choices were classified into seven categories: (i) focusing on mathematical content, (ii) considering reading comprehension skills, (iii) supporting association and representation skills, (iv) varying complexity, (v) motivation, (vi) supporting problem-solving skills, and (vii) considering instructional flow.

Within the first category, participants offered justifications related to enhancing students' arithmetic skills, supporting and reinforcing mathematical understanding, and focusing on identifying and addressing errors in problems. Five codes were identified within this category, appearing a total of 66 times across the sample. In the second category, a single code was identified. Two participants noted that the selected problem could allow for a focus on students' reading comprehension skills. An excerpt from PT1's explanation is as follows:

> P2: The reason I chose this problem is to measure students' reading comprehension skills. The answer to the problem is clearly stated in the story. It does not require additional operation. This provides convenience for students and increases their attention.

Under the third category, two codes were identified and appeared eight times in total. PTs noted that the problems they selected were relevant to daily life and that some problems allowed for the use of visual models as justifications for their selections. In the fourth category, varying complexity, six codes were identified and appeared 45 times in total. Participants provided explanations regarding the difficulty level of the problems, citing reasons for their selections such as increasing the complexity of problems in sequence, adding new data to make problems more challenging, ease of

solving, and complexity based on the types of arithmetic operations involved. In the fifth category, motivation, a single code was identified and appeared nine times in total. Participants emphasized that adjusting the difficulty level of problems according to the students' capabilities would enhance their interest in problem solving. PT25, who provided explanations related to both varying complexity and motivation, shared the following:

> The solution order would be 1, 2, 6 and 7 because I think it will be more understandable for students in terms of evaluating the problems in a sequence from easy to difficult and from simple to complex. I also think that solving easy problems first will be motivating for the students before moving on to other problems. When they cannot solve the first problem, they will have a negative attitude towards other problems.

In the sixth category, supporting problem-solving skills, three codes were identified and appeared 10 times in total. Participants indicated that the selected problems encourage the use of problem-solving strategies and help students understand that not every problem can be solved with the available data. Finally, under the category of considering the instructional flow, three codes were identified and appeared 34 times in total. Participants emphasized that certain problems act as prerequisites for others, and some problems assist in solving subsequent ones (Please refer to PT5's explanations in the first section of the findings).

Conclusions

In this study, the PTs tended to select problems with low mathematical complexity for the initial problems. In their subsequent problems, they attempted to make the problems more complex. Zhang and Cai (2021) found that teachers tend to assign problems they consider mathematically complex as homework, while they engage in classroom discussions on problems they perceive as moderately difficult level. Furthermore, our study suggests that PTs also consider incorporating complex problems into classrooms, yet the prevailing tendency is to address these in subsequent problems rather than in the initial ones. Moreover, the PTs did not frequently select problems with insufficient data, mathematical errors, or answers that were readily available in the data set, despite experimental studies showing that these problems are commonly seen in students' responses (e.g., Kar et al., 2024; Silver & Cai, 1996). Due to lack of knowledge on how to handle student problems— particularly those presented in textual form that do not involve any mathematical demands—teachers commonly limit their interaction by asking students to simply read their problems aloud (e.g., Possamai & Allevato, 2024). Our research highlights another aspect of classroom discourse that PTs neglect, namely, engagement with incorrect mathematical problems. When the instructional stance of the PTs is evaluated from behaviorist and constructivist perspectives (Santagata, 2005), they align more with the former in addressing errors, showing limitations in perceiving errors as "potential avenues for learning" (NCTM, 2000, p. 144).

The PTs provided multiple reasons for selecting problems and highlighted several reasons together for the selection of a single problem, indicating that they consider multiple variables when making decisions about problem selection. The problems were chosen primarily based on their mathematical content and complexity, as was to be expected. However, aspects such as developing a problem-solving culture, utilizing problems to assess students' knowledge, and affective factors had less of an

impact on the PTs' selection of problems. Our results highlight the strengths and weaknesses of PTs' preferences regarding the selection and sequencing when implementing problem-posing task in the classroom. Therefore, we believe our study could be helpful in the endeavour to enhance teachers' proficiency in integrating problem-posing into mathematics classrooms in various ways.

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Mathematics teacher practice in inquiry-based lessons: A case study in a Japanese upper secondary school

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The purpose of this paper is to advance the understanding of mathematics teacher practice characteristics in inquiry-based lessons. From the perspective of the anthropological theory of the didactic, we investigate the practices of mathematics teachers in the process of preparing lessons at a Japanese upper secondary school. Specifically, by regarding these practices as 'teacher inquiry' and analysing them with the notion of praxeology, we identify the types of tasks they were working on during their meetings and the concerns they encountered. As a result, we found that the following two types of teachers' practices are dominant in our case, which are not often carried out in the preparation of ordinary mathematics lessons: exploring teaching contents from multiple perspectives and designing teachers' supporting questions for students to promote their own inquiry.

Keywords: Mathematics teacher practice, Inquiry-based learning, Secondary schools, Anthropological theory of the didactic.

Introduction

Inquiry-Based Learning (IBL) has drawn considerable attention in international mathematics education research over the last two decades (Artigue & Blomhøj, 2013). IBL is a learning form modelled on researchers' works, in which students address questions as inquirers, using various resources (e.g., books, internet, PC) and support (e.g., teachers, colleagues) at their disposal. This learning form differs from ordinary mathematics lessons in that the mathematical knowledge to be learned is predetermined, and the available tools and support are limited.

Focusing on *teaching*, teacher practice in IBL lessons should also have characteristics that differ from those of ordinary lessons (Dorier & Maass, 2020). The teachers' roles in IBL lessons are not to explain mathematical concepts, provide problems, evaluate students' answers, and so forth. Therefore, IBL lessons require teachers to possess knowledge and skills specific to such forms. It is expected, for example, that teachers will have some experience with mathematical inquiry by themselves (Artigue & Blomhøj, 2013; Chevallard & Strømskag, 2022) and some skills to facilitate students' inquiries.

Given the above background, we are interested in the mathematics teacher expertise required to conduct such IBL lessons. In this study, as a preliminary phase, we focus on mathematics teacher practice *involved in* IBL lessons and aim to advance the understanding of its characteristics. In this paper, *teaching* is considered not only the teacher practice within the classroom but also outside it. Specifically, we investigate a specific case of a Japanese upper secondary school mathematics teacher who was engaged in the design and implementation of IBL lessons through one year, with the support of a researcher. By employing analytical tools developed within the *anthropological theory of the didactic* (ATD), we analysed the meetings for IBL lessons to shed light on the teachers' work and identified the characteristics of teachers' practices in this context.

Theoretical perspectives

This study adopts two conceptual tools developed within the ATD to investigate teachers' practices related to IBL lessons.

The first tool is *teacher inquiry*, which allows us to investigate teachers' practices in terms of the inquiry (Trouche et al., 2019; Miyakawa, 2022). In ATD, inquiry is an important object of study, and its process and structure are formulated as the notion of *Study and Research Paths* (SRP) (Chevallard, 2019). SRP is a series of inquiry activities that develop in a learning environment with all available resources (called *media* in ATD). The inquiry usually begins with a lively generative initial question (Q_0), new questions are derived progressively by pursuing Q_0, and it is further deepened by addressing these questions. This process is characterised as the *dialectic of questions and answers* and is a fundamental concept in SRP (in particular, questions). In fact, the design of an ordinary mathematics lesson begins with the initial question 'How to teach a particular mathematical content?', referring to all available resources such as textbooks and teaching resources and learning any necessary content for teaching as needed, and the final answer '$A^{\heartsuit}$: A design of the process of one lesson' is obtained. This process can be seen as SRP. Using this tool, the dynamic process of teachers' work can be described in terms of the questions and answers they handle during meetings.

The second tool is the notion of *praxeology*. In ATD, any human activity and knowledge can be modelled using praxeology (Chevallard, 2019). Praxeology comprises a *praxis* block that describes activities and a *logos* block that provides a theoretical background. The praxis block is described by *a type of tasks T*, which refers to the family of tasks or problems, and *a technique* τ, which refers to the method of solving the task. The logos block is described by *a technology* θ, which generates, justifies, and explains the technique, and *a theory* Θ, which does the same to the technology. Using this concept, a student's mathematical activity and a teacher's practice *inside* the classroom can be modelled respectively by a *mathematical praxeology* (MP) and a *didactic praxeology* (DP), and a teacher's practice *outside* the classroom by a *paradidactic praxeology* (Miyakawa, 2022). For example, the situation of solving a linear equation in middle school, like $2x + 3 = x - 5$, can be described as a MP, where T is to solve a given linear equation, τ is moving terms, θ is a property of the equation or an image of balance, and Θ is the theory of linear equation in middle school mathematics. We specifically use paradidactic praxeology to analyse teachers' practices in the preparation and reflection of IBL lessons.

Methods

In this study, the IBL lessons were conducted in a new subject called *Period for Inquiry-Based Cross-Disciplinary Study* at a Japanese public upper secondary school. This subject aims to nurture students' competence to conduct cross-disciplinary inquiry based on their knowledge of ordinary subjects. As part of a research project exploring the potential of mathematical inquiry within this subject based on SRP, a sequence of lessons was implemented for one year, consisting of 53 fifty-minute lessons. These lessons were collaboratively designed by a teacher (called Mr T) and a researcher (first author; called Supporter), both of which participated in this research project. Twenty-two Grade 12 students attended this class and were divided into seven groups of two to four students. Each group selected a

different theme: three were purely mathematical, and four were related to the application of mathematics. Mr T mainly led the lessons, while Supporter primarily observed them.

Meetings for the IBL lessons were organised for Mr T and Supporter before and after each lesson, which lasted from 15 min to 1 hour, totalling over 100 times. Mr T was a high school mathematics teacher with 10 years of teaching experience but less experience of teaching IBL lessons. Supporter was a PhD student who had some knowledge of didactic theory, such as ATD and SRP. The discussions during these meetings primarily focused on the design of slides for the next lesson and the reflection on each group's inquiry after the lesson. The discussions also progressed with reference to the students' activities, worksheets, and other materials used in the lessons.

In the analysis, we first created a *Q-A timeline* for each meeting (See Figure 1). In this study, the meetings were regarded as part of *teacher inquiry* (SRP), and we identified the questions and answers addressed by Mr T and Supporter in a chronological order. The initial question for pre-lesson meetings was 'How will the next lesson be?' while for post-lesson meetings, it was 'How was today's lesson?'. The discussions during the meetings were chronologically divided into stages according to the main questions. The activities in the meetings can be modelled using the concept of paradidactic praxeology. In this context, the main types of tasks were classified into four categories: *Exploration of MP*, *Evaluation/Selection of MP*, *Exploration of DP*, and *Evaluation/Selection of DP*. In other words, the main activity in the meetings can be seen as *investigating* MPs and DPs that will occur (or had already occurred) in the lessons. The answers at each stage were therefore identified and classified according to these four categories of the types of tasks in paradidactic praxeology.

After creating the Q-A timelines, we investigated the *concerns* encountered by Mr T and Supporter. We hypothesised that these concerns reveal mathematics teachers' practices that are both critical and unique to IBL lessons. Specifically, by focusing on the specific moments at which the concerns arose, we investigated what questions they raised and how they solved them.

Results

In this section, we outline the students' inquiries and provide the main questions that arose in the meetings, followed by an analysis of the post-lesson meeting in which some concerns were discussed.

Students' inquiries

In the first semester (20 lessons), students determined the inquiry themes in the first eight lessons and then started their inquiry in the remaining 12 lessons. In the second semester (23 lessons), students progressed with their inquiry and had opportunities to report their findings at the mid-term presentation as well as at the poster presentation organised at the later part. At the end of this semester, students began to work on their final reports and continued to work in the third semester (5 lessons).

In the second half of the first semester, the lessons comprised two phases: inquiry toward the mid-term in-class presentation (five lessons) and inquiry with teacher individual supervision (seven lessons). In the first phase, the inquiry was conducted over four lessons, including time for deciding on the themes, and each group gave an oral presentation in the final fifth lesson. In the second phase, each group had the opportunity to discuss the progress of their inquiry with Mr T during the lesson.

Main questions in the meetings (in the second half of the first semester)

Twenty-four meetings were conducted for 12 lessons in the second half of the first semester. The main questions addressed differed according to the pre- or post-lesson meetings. In the former, the questions repeatedly raised were as follows:

Q: How do we organise the students' activities in the lesson? (Exploration of DP)

Q: How do we support students to encourage them to advance their inquiries? (Exploration of DP)

These questions were related to *Exploration of DP*, in which the teacher's techniques and classroom interventions were investigated. Some activities were required as part of the school policy to be included: conducting the mid-term presentation (progress report) and the final presentation, and writing the final reports. The content and organisation of the lessons, excluding these activities, were left to the teacher's discretion.

Specifically, in the pre-lesson meetings held for the fourth and fifth lessons during the presentation phase, they worked on the question 'Q: How do we organise the session of the students' presentations?' In other pre-lesson meetings, the question 'Q: How do we organise the teacher supervision?' was also raised. In addition, in the pre-lesson meeting held for the last lesson during the supervision phase, as the summer break came after this phase, the question 'Q: How do we organise the inquiry during the summer break?' was discussed, and they decided to assign two tasks: summarising the results in the first semester and working on the questions provided by the teacher.

In contrast, the questions that were repeatedly raised in the post-lesson meetings were as follows:

Q: What are the possible directions for each group's inquiry? (Exploration of MP)

Q: What support should the teacher provide to promote these possibilities? (Exploration of DP)

Q: Was the teacher's support appropriate to promote their inquiry? (Evaluation/Selection of DP)

In the post-lesson meetings, after debriefing each group's activity and the teacher's support in the lesson, they often discussed the possible directions of each group's inquiry and the teacher's questions to promote these possibilities. Because each group worked on a different theme, much time was devoted to considering the teacher's support that would be more appropriate for each group's inquiry. The details of the specific discussion are described in the next section.

They also evaluated whether teacher support and lesson organisations were appropriate for advancing student inquiry. For example, during the presentation phase, in the post-lesson meeting after the presentation, they discussed the question 'Q: Is it necessary to set up a presentation session in every phase to provide comments?'. They found that the session was not effective because Mr T could not ask enough questions and provide suggestions to each group. As a result, they decided to implement teacher supervision in the next phase. In the post-lesson meeting during the supervision phase, in contrast, another question 'Q: Could we support other groups during the supervision of specific groups?' was addressed. They concluded that supporting only specific groups was not effective and decided that the first half of each lesson was used for the individual supervision (two groups per lesson), while the second half was that Mr T circulated in the classroom and provided occasional support to other groups. In this way, they improved their support and lesson organisations accordingly.

Post-lesson meeting of the fourth lesson in the presentation phase

Let us analyse the meeting discussion more precisely. A post-lesson meeting was held after the fourth lesson in the presentation phase. In the next fifth lesson, each group was expected to give a 5-min oral presentation on the progress of their inquiry. Up to the fourth lesson, for example, Group A had been working on the first question 'Q_0: How to calculate 20 decimal places of π by paper and pencil?' and tried to carry out an approximate calculation of π using the Ramanujan formula.

This post-lesson meeting was divided into 11 stages according to the main questions raised during the meeting (see Figure 1).

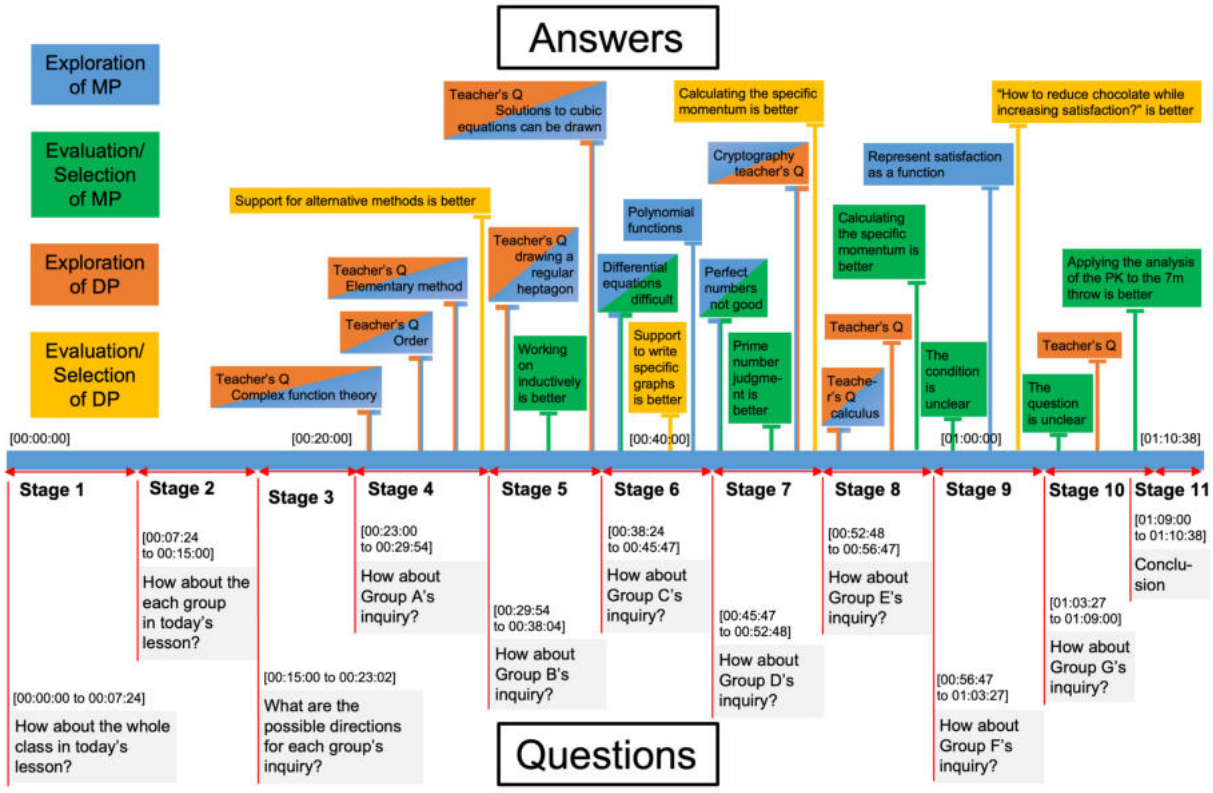

Figure 1: Q-A timeline for the post-lesson meeting of the fourth lesson in the presentation phase

In this meeting, Mr T and Supporter mainly discussed the students' work up to that point and directions for future inquiry after the presentation. In Stage 1 and 2, Mr T reported his overall remarks on the entire lesson and the progress of each group's inquiry. We did not provide any answers on the timeline because these were not directly related to the four categories.

In Stage 3, they discussed the comments/questions to ask each group in the next presentation. In this stage, Mr T expressed concerns about anticipating the possible directions of students' inquiry in response to the main question 'Q: What are the possible directions for each group's inquiry?' Supporter responded to this question by suggesting some ideas. In Stage 4, there was a seven-minute discussion about Group A's inquiry (calculation of π), they proceeded the following dialogue:

Supporter: When Group A addresses the question 'Why is the coefficient so strange?', the inquiry could go into university or graduate school mathematics content.
Mr T: Hmmm...
Supporter: Yes, well, it can go into an area related to complex function theory.
…
Supporter: One possibility would be to address the question of 'how quickly does it converge?' by working on the method that Group A is currently employing. Well, the inquiry will be about how it converges as n increases, and the order of magnitude involved in the question of "how do you judge how quickly it converges?" Well, I think that working on the question like this in the future would be one possibility.
…
Supporter: From a historical perspective, if Group A addresses the question 'How did people in the past, such as those who did Wasan (Japanese traditional mathematics), calculate?', we can expect that Group A will be able to find more, well, elementary methods.
Mr T: Ah.
Supporter: For example, methods of approximate calculation using 355/113.

They discussed three possible directions for future inquiry in Group A: complex function theory (the background theory of Ramanujan formula), order (speed) of convergence, and elementary calculation methods (e.g., a method of Japanese traditional mathematics, Wasan). While thinking about these possibilities, they also discussed the questions that Mr T could provide to Group A to promote in these directions (in Figure 1, these answers were represented as 'teacher's Q'). In this situation, different categories of answers are considered almost simultaneously, two of which are written as one answer on the Q-A timeline (in Stage 4, the answer of Exploration of MP occurs after Exploration of DP). At the end of Stage 4, they concluded their discussion on Group A with the following dialogue:

Supporter: From a historical perspective, there have been many investigations into how to improve the accuracy of approximations of π, so I think the method investigated by Group A is probably the best modern method to have a quick convergence.
Mr T: I think so too.
Supporter: However, there are various problems with this method, such as the difficulty of the calculations, so it might be good to support other methods and compare them with the current method.

Finally, they chose to direct Group A to the calculation of π using elementary methods because the Ramanujan formula was too difficult for them. This is related to the type of tasks in the Evaluation/Selection of DP. Similar discussions to those in Stage 4 were also held with the other groups from Stages 5 to 10. One may say that Mr T's main concern in this meeting was the possible directions for further inquiry, and much time was devoted to discussing them (two-thirds of the total).

Discussion and conclusion

In this section, we focus on two main concerns identified in the analysis of the post-lesson meeting: *possible directions for inquiry* and *supporting questions*.

As observed in the previous analysis, one of the main concerns in the meeting was to explore the mathematical content related to students' inquiry to identify directions for further developing those inquiry. In IBL or SRP, particularly in the inquiry subject implemented in Japan, the mathematical content or knowledge to be taught is *not predetermined*, unlike ordinary mathematics lessons. Students are therefore exposed to various topics as they progress through their inquiry. In such

circumstances, it is understandable that Mr T and Supporter were constantly investigating possible directions to provide students with appropriate support. In addition, it is noticeable that for ordinary mathematics lessons, teachers are also required to explore the mathematical knowledge before teaching (often called *kyozaikenkyu* in Japan), but this knowledge is predetermined in the curriculum, and teachers are familiar with it. These teachers' practices differ from those of IBL lessons. Therefore, *Exploring MP from multiple perspectives* should be one of the characteristics of teacher practice involved in IBL lessons. Furthermore, teachers may encounter unknown knowledge in such lessons (e.g., neither Mr T nor Supporter knew well Ramanujan formula). An inquirer's attitude is to behave as if knowledge is still to be discovered and conquered. In this case, this attitude, *procognitive attitude* (Chevallard, 2015), is critical in the design and reflection of IBL lessons.

Another important concern in the meetings was the design of supporting questions. This concern is closely related to the didactic gestures required for the lessons and their preparation. The teacher's role in IBL lessons is not to validate or summarise students' answers but to support students' inquiry so that they can deepen their own topics. One important type of tasks in DP is to provide *lively* and *generative* questions that students may be interested in and generate new questions to advance the inquiry, particularly when the inquiry is stagnating. As mentioned earlier, in SRP, questions play a critical role as a lever for advancing inquiry. In IBL lessons, teachers are therefore required to provide appropriate questions according to the progress of students' inquiry. Similarly, in ordinary mathematics lessons, teachers often provide students with questions, but these questions are supposed to be answered in a short time. The nature of such questions is, in fact, different from that of IBL lessons, and the same applies to their preparation work. Therefore, *designing supporting questions* should be one of the characteristics of teacher practice involved in IBL lessons.

However, these concerns arise because of the distinct nature of inquiry activities expected in the classroom, which markedly differ from ordinary mathematics lessons. As noted by Artigue & Blomhøj (2013), a critical aspect of mathematics teacher expertise in IBL is the first-hand experience of inquiry. In fact, when supervising PhD students (this is also an inquiry-based instruction), experienced researchers and university scholars do not necessarily explore questions and prepare supporting questions beforehand. Therefore, it is still questionable to what extent the characteristics of teachers' practices identified in this paper are general or whether they can be found under specific conditions. Furthermore, the roles of 'Supporter' in this case are not clear. Future research is required to analyse the nature of researchers' interventions in relation to mathematics teacher collaboration.

Acknowledgement

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What do teachers notice when reflecting on vignettes of discussion-leading practices? A topic-specific case study

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Discussion-leading practices are a central part of teachers' expertise and are substantially influenced by what teachers notice in the complexity of classroom discussions. With engaging teachers in reflecting on crucial content-specific aspects of classroom discussions, we aim to enhance their noticing and deepen their awareness. This paper presents a case study investigating what teachers notice when reflecting on a classroom discussion during a professional development course. The results indicate that teachers vary in the extent to which they reflect on content-specific aspects, such as concept element or representations.

Keywords: discussion-leading practices, *noticing*, *professional development* (*PD*).

Theoretical background

Design research agenda of decomposing discussion leading practices and teacher noticing

Mathematical learning opportunities can be created, for example, by well-designed tasks, however their effectiveness largely depends on teacher's enactment. Research thus concentrates on investigating and decomposing teaching practices (da Ponte & Chapman, 2006), particularly discussion-leading practices (Jacobs & Empson, 2016). Teaching practices constitute a central component of teachers' expertise (Prediger, 2019), enabling them to handle the "complex situational demands of subject-matter teaching" (p. 370). Qualitative decompositions of teaching practices, such as discussion-leading practices focusing on dealing with multiple representations (Post & Prediger, 2024), reveal variation in teachers' actions. These differences seem to be influenced by teachers' noticing, their ability to perceive, interpret, and respond to classroom events: For example, joint discussion of video clips from lessons of other participating teachers supports teachers' "ability to notice and utilize student thinking for instructional decisions" (Wallin & Amador, 2019 p. 515).

Research on teacher noticing emphasizes the importance of awareness and sensitization to specific aspects of teaching practices (Mason, 2011; Weyers et al., 2024). Sherin et al. (2011) describe noticing as a complex process "through which teachers manage the 'blooming, buzzing confusion of sensory data' with which they are faced" (p. 5). This process involves filters of perception and consists of two key components: "attending to particular events in an instructional setting" and "making sense of events in an instructional setting" (ibid., p. 5). For the second process of making sense, Sherin et al. (2011) highlight the relevance of categories: interpreting means "relating observed events to abstract categories and characterizing what they see in terms of familiar instructional episodes" (ibid., p. 5).

Many studies investigate the sub-processes of teacher noticing, such as how teachers perceive student thinking or classroom events, interpret them, decide how to respond, and subseqeuntly enact in classroom (Weyers et al., 2024). Rosaen et al. (2008) use the concept of noticing to investigate for three pre-service teachers how using videos can support pre-service teachers in reflecting on their

practice. Reflections based on video analysis compared to reflection based on memories support teachers to reflect more specifically and with a stronger focus on instruction and the students rather than on classroom management alone. Detailed analyses of the reflections showed that the pre-service teachers, for example, examined the impact of specific prompts in the conversation or the development of learners' thinking while analyzing their own classroom videos. The authors suggest that video analysis might support teachers in leading discussions, a possibility that requires further investigation. However, as van Es and Sherin (2002) emphasize, it is crucial to learn to focus on the relevant aspects while noticing.

This case study is part of a larger study that aims at investigating how teachers can be made more aware of discussion-leading practices for dealing with multiple representations for the case of conditional probabilities, and how they can be supported in establishing these practices. Previous studies have identified constituent elements of those practices using a topic-specific approach (Post & Prediger, 2024). This study aims to expand the research field by examining which aspects teachers actually notice when reflecting on their teaching practices.

Topic-specific approach for conceptualizing discussion-leading practices

Following Prediger et al. (2022), we conceptualize discussion-leading practices "as the teachers' steering trajectories on a content-specific navigation space" (p. 7). This includes in particular that the representations referred to and the relevant aspects of the content in view are key to capture the discussion leading practices (Post & Prediger, 2024). For instance, for the content of "interpreting statements on conditional probability", the part-whole relationship is a key conceptual element of probability statements: Analyses of learning processes and analyses of discussion-leading practices of different teachers (Post & Prediger, 2024) show that explicitly addressing the part-whole relationship, rather than focusing only the part and the whole separately, seem crucial for fostering a deeper understanding of conditional probabilities. Connecting representations consciously also plays a crucial role in promoting conceptual understanding. Understanding a complex statement such as "The probability that a buyer doesn't buy a bag on the condition that he doesn't buy a shirt", requires grasping this part-of-part structure: "All buyers who don't buy a bag and a shirt out of all buyers who don't buy a shirt."

Teachers' discussion-leading practices vary significantly in how explicitly they address the part-whole relationship and guide the discussion towards connecting the elements across different representations in the discussion (Post & Prediger, 2024). While some teachers focus on explicating the part-whole relationship in plenary discussions (*conceptual focus*), others limit their attention to the part and the whole separately or emphasize procedural steps for determining the correct probability (*procedural focus*). This study adopts the conceptual framwork of discussion-leading practices to analyze the aspects teachers address when reflecting on their practices. Particularly, the focus lies on whether teachers refer to *key concept elements, representations* and *connections for the topic in view*.

Research question

The primary goal of this study is to investigate what teachers notice when reflecting on discussion-leading practices, particulary in PD courses. We therefore ask: *Which aspects do teachers focus on*

when reflecting on vignettes of discussion-leading practices and their own practices? To what extent do teachers address the aspects of content-related discussion-leading practices?

Methods

This study is part of the larger professional development (PD) research project QuaMath (Prediger & Selter, 2024) in PD design research methodology (Cobb et al., 2017) and is applied, among others, on PD courses focusing on conceptually focused mathematics teaching.

In this paper, we examine teachers' noticing during a vignette-based activity conducted within a three-hour, synchronous online PD program. In this focused PD session centered on language-responsive teaching of conditional probability 26 German middle and high school mathematics in-service teachers participated. They had varying years of teaching experience and already some previous experience to the topic of language-responsive teaching. Before the activity, participants unpacked the mathematical topic of conditional probability into key concept elements and language demands. They also discussed the utility of a visual model and examined typical student misconceptions, laying the groundwork for deeper reflection during the vignette discussions.

The investigation specifically examined teachers' noticing during the vignette-based activity, informed by prior research on teacher noticing, which emphasizes the importance of identifying curcial aspects of teaching practices (Mason, 2011; van Es & Sherin, 2002).

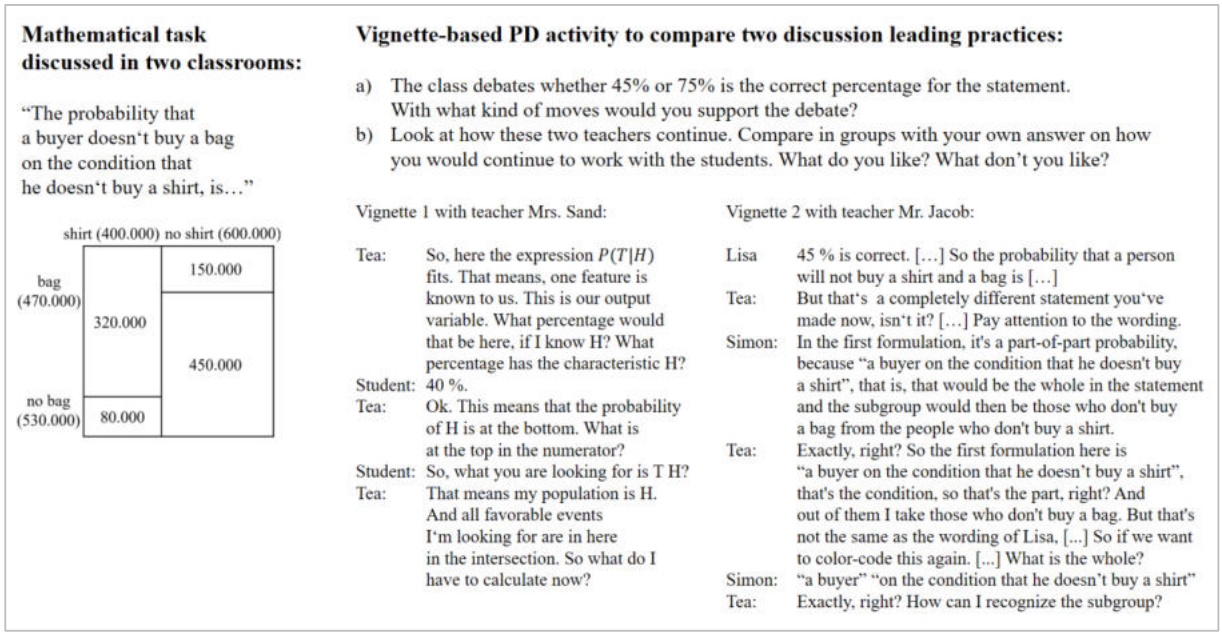

Figure 1: Vignette-based PD activity comparing the discussion-leading practices of two teachers

In part a) of the PD activity in Figure 1, participating teachers were asked to consider how they would continue working with students by documenting their own strategies (teaching moves). Part b) presented two short vignettes illustrating other teachers' discussion-leading practices. These vignettes were stylized for clarity from real transcripts of authentic discussion processes (Post & Prediger, 2024). The first vignette features Teacher Mrs. Sand, focusing on the procedure for determining the correct probability for a slightly different statement than in Figure 1. She examines the numerator

(the part), the denominator (the whole), and how to position both in the given formula. The second vignette, featuring Teacher Mr. Jacob, directs students' attention to the part-whole relationship and the connections between the given statements using meaning-related language to articulate this relationship.

The data corpus for this paper consists of two short transcripts from small group work in breakout rooms where participating teachers (Anna & Markus and Tobias, Melanie & Saskia) analyzed and compared the vignettes at the end of a three-hour PD session. The small group work was video-recorded and partly transcribed. The qualitative analysis of teachers' noticing used a deductive coding procedure. Main codes emerged based on whom the participants focused on in their observations: comments referring to *student* learning processes, *teachers' practices* in the vignettes or participants' *own practice*. To address the research question, we examined the topics participants addressed in relation to their practice: *cross-content moderation* or *content-related instructional actions* with a procedural or conceptual focus (operationalized by focusing on either the procedure for determining the correct probability or on explicating the part-whole relationship). Further, we analyzed in detail what aspects participants noticed about the content of teachers' or their own actions: addressed *concept-specific elements* (part-whole relationship, part, whole, ratio), *representations* (e.g., visual model, symbolic representation, given statement) and *connections* between concept elements and representations.

Empirical insights

The case of Anna and Markus

Anna and Markus begin their reflection by analyzing Mrs. Sand's teaching situation (as depicted in the first vignette in Figure 1):

7	Markus:	Here the teacher immediately specifies what is required […]?
[…]		
10	Anna:	[…] So Mrs. Sand would not have helped me as a student.
11	Markus:	Nope. Yes, especially if I specify it and then just show it like this. And what do you have to insert now? Well, […] that is far too tighly controlled.
[…]		
20	Anna:	So at the bottom she catches herself a bit, right? My basic population is H. […] So probably she shows that in the field. And the favorable ones are in there, in the part.
21	Markus:	Yes, but of course it's so very predetermined#
22	Anna:	#Yes, yes. Exactly.
23	Markus:	[…] that does not eliminate the misconception. […]

Focus of observations: In their reflection, Anna and Markus mainly focus on describing Mrs. Sand's actions. In Turn 23, they briefly address the students by considering that Mrs. Sand's approach may be inadequate for dealing with misconceptions. Overall, their observations center on evaluating here Mrs. Sand's teaching practices.

Aspects addressed: Markus observes that Mrs. Sand provides the formula und focuses on identifying the correct values (Turn 11). In Turn 20, however, Anna highlights the conclusion of the teaching situation, where Mrs. Sand addresses the whole and the part before asking the students to calculate. Anna supplements this by adding the term "part" and assuming that the teacher points to the

corresponding areas in the visual model. Anna and Markus mainly comment on content-related instructional actions, noticing Mrs. Sand's procedural focus. In Turn 11, Markus critiques the level of student involvement as "far too tightly controlled", reflecting on a moderation aspect closely linked to Mrs. Sand's instructional approach. Their reflections implicitly reveal that Mrs. Sand primarily addresses the part, the whole and the formula for determining the correct fraction. Anna appears to view the separate focus in the part and the whole positively, especially when combined with visually locating both values.

The group then focuses on the students briefly (for 3,5 minutes) before moving to the second vignette:

74/ 76	Anna:	I always find that so difficult, because in that situation I would have said: You somehow still do not agree. One says 45 percent, the others believe in 75 percent. Who can convince me of either of these solutions with the help of this area model? Explain to me again#
77	Markus:	#Yes, yes.
78	Anna:	How to find the numbers using this square. That you can tell me that again, I mean, that is the point of the thing, otherwise I would not need it.
79	Markus:	Yes, I would also pass the ball back again. And, above all, perhaps focus on the wording again. And the question is, is it part of the part or part of the whole? And to aim a little bit at which, how can I recognize that in this task?
80	Anna:	#Yes, yes.

Focus of observation: After analyzing the first vignette (Turn 7–28), the group shifts its attention to the students (Turn 29–69) and finally to their own teaching practices (Turn 74–80).

Aspects addressed: Anna proposes an alternative moderation strategy, suggesting that students should explain their reasoning again by using the visual model and contrasting their statements (Turn 74, 76). Markus builds on this idea, advocating for students to explain how the part-whole structure can be identified in the statement (Turn 79) and linking this to the visual model (Turn 81). By explicitly addressing the part-whole relationship as a key concept element and connecting it to the given statement and the visual model, Markus focuses on content-related actions with a strong conceptual focus/emphasis. In comparison, Anna primarily emphasizes moderating the discussion and engaging students, rather than focusing on content-specific elements/content-related actions. While she mentions the importance of linking representations, she does not explicitly address specific concept elements. Nevertheless, her approach encourages students to provide explanations, which could facilitate connections between representations and promote reflection on the underlying structure (Turns 74–78). Neither Anna nor Markus acknowledge that Mr. Jacob already addresses the part-whole relationship in the vignette.

The case of Tobias, Melanie and Saskia

This small group begins by reflecting on the teaching strategies (teaching moves) formulated by Tobias in the first part of the task. They then proceed to analyze the given classroom situations.

68	Melanie:	And Mrs. Sand is very much#
69	Saskia:	# in the formal language.
70	Melanie:	Yes, exactly. In procedures already # somehow, right?
71	Saskia:	# Yes. And Mr. Jacob starts with […] the ideas of percentages.
72	Melanie:	Do you know, what I have just noticed? Mr. Jacob needs a lot more text.
73	Tobias:	Yes.

74	Melanie:	So, and, um, well, I find this formulaic language very highbrow. […] But for students, it can be quite far away, depending on how long and how well they've been dealing with the topic. But Mr. Jacob takes a very long time to explain it. And I know that the longer I talk, the less they can take in.
[…]		
77	Saskia:	However, he basically remains in the image that the students themselves are currently thinking.
[…]		
80	Tobias:	[…] As I said, very fast with the formulas and perhaps over- so it can also be very good that this overwhelms one or the other, that you now immediately go into the fraction and immediately into procedures. Completely away from the picture.
81	Melanie:	Mhm.
82	Tobias:	But then immediately switches to, yes, formulaic language. I can also see that this could be difficult... Yes.

Focus of observations: In Turns 68–82, the participants' comments focus on the teaching practices depicted in the vignettes, with Melanie, Tobias and Saskia comparing Mrs. Sand and Mr. Jacob.

Aspects addressed: While comparing the two teachers, the participants highlight the content-related actions of both (Turns 68–71, 77–82). They note that Mrs. Sand primarily addresses procedures, while Mr. Jacob prioritizes ideas and student thinking. The group implicitly recognizes that Mrs. Sand focuses on the formula as a concept element. Tobias critiques the rapid transition to the formula and comments to the use of representations. However, the participants do not explicitly reflect on additional concept elements. Melanie shifts the discussion to cross-content moderation, focusing on the length of the vignettes.

The discussion continues:

83	Saskia:	The problem is, if the student is not clear about what is meant by this buyer on the condition that#
84	Tobias:	#Yes.
85	Saskia:	That is core of the conversation with Mr. Jacob. And if that's the crux of the matter, that they don't understand that, it doesn't help that he mentions it so often […]
86	Tobias:	Mhm.
87	Saskia:	That could be critical, that he stays in this picture, that there is no further change […]
[…]		
90	Tobias:	I mean, Jacob already justifies it a bit based on Lisa's statement.
91	Saskia:	Mhm.
92	Tobias:	No? That you confuse combined and conditional probability. That is what Lisa is doing here. […] omits the fact that we are only looking at people who do not buy a shirt […]
[…]		
94	Tobias:	[…] But I think it is nice that Lisa does not talk at all afterwards, only the other students. […]
[…]		
100	Tobias:	You might have to make sure that the conversation goes more in Lisa's direction and that you consider her misconception and take her with you again.

Focus of observation: In Turns 83–100, the group continues reflecting on the teachers' practices, with a brief focus on student processes in Turn 83, where Saskia identifies a central obstacle for students.

Aspects addressed: In the first part of the discussion (Turn 83–92), the comments primarily address Mr. Jacob's content-related actions. Saskia assesses Mr. Jacob's discussion-leading practices as insufficient, asserting that he merely repeats his own statements. Tobias provides a more refined view, noting that Mr. Jacob justifies his statements by reflecting Lisa's response and clarifies confusing formulations (Turn 90–92). By reflecting Mr. Jacob's teaching situation, the group observes that he focuses on explaining and comparing the statements. But they suggest that a more explicit explanation for comparing both formulations and a connection to representations could improve the teaching approach (Turn 81, 87). Despite these reflections, Tobias, Melanie and Saskia do not address specific concept elements or propose alternative approaches to consider concept elements. In Turns 94–100, the discussion shifts to moderation strategies, focusing on student involvement but neglecting content-related aspects.

Discussion and outlook

The participants' reflections primarily focused on the actions of the teachers in the vignettes, with occasional references to student learning processes. Notably, only Anna and Markus extended their reflections to include an analysis of their own teaching practices. Both groups identified the procedural focus in Mrs. Sand's teaching. However, Tobias, Melanie, and Saskia provided additional observations on the use of representations, discussing potential modifications to enhance instructional effectiveness. Their reflections on content-specific aspects, such as concept elements or connections between representations, were relatively superficial and often overshadowed by a focus on cross-content moderation. In contrast, Anna and Markus demonstrated a stronger focus on content-specific aspects, particularly the key concept elements of the part-whole relationship (Post & Prediger, 2024). Markus, in particular, highlighted the importance of connecting representations to conceptual structures, implicitly contrasting this approach with Mrs. Sand's procedural focus. This deeper engagement underpins the potential of structured reflection activities to enhance teachers' content-related noticing, a key factor in promoting conceptual understanding among students.

The analysis reveals that the extent to which participants notice and reflect on content-related aspects significantly affects the quality of their reflections, underscoring the importance of noticing critical elements like conceptual connections and representations (Mason, 2011; van Es & Sherin, 2002). Despite the study's small scope and specific focus, the findings align with research highlighting the role of content-related noticing in fostering conceptual understanding and effective teaching practices (Prediger et al., 2022). Building on these insights, future professional development programs should prioritize strategies that support teachers in deepening their noticing of content-related aspects, such as a focus on the meaning-related structure of the mathematical concept that is addressed in various representations. Structured video analysis and guided reflection, as suggested by Rosaen et al. (2008), could further enhance teachers' abilities to identify and address critical elements of their practices, ultimately improving student outcomes.

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Interactions between talk moves and other teaching moves in classroom discussions in ambitious mathematics teaching

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This study explores the relationship between talk moves and the categories of teaching moves of exploring, encouraging and connecting to formal notation. Data from nine classroom co-enactment sessions in a professional development program aiming to support Norwegian teachers in learning ambitious mathematics teaching were analysed. We present a case from one of the co-enactments to unpack the moves that the teachers used and the interaction between them. The findings reveal that the teacher used different categories of teaching moves simultaneously and in interaction. The teaching moves of exploring and connecting to formal notation were made in ongoing interaction between talk moves, student talk and representing students' strategies. The findings also show how talk moves can be used within different categories of teaching moves.

Keywords: Teaching moves, talk moves, professional development, ambitious mathematics teaching.

Introduction

Leading discussions is often considered a core or high-leverage practice (e.g., Ball & Forzani, 2009; Jacobs & Spangler, 2017), and researchers have identified different groups of moves that teachers can make to stimulate discussion. "Teaching moves" – a common umbrella term to describe what teachers do – can be defined on different levels, and with different goals (Jacobs & Spangler, 2017), and the connections between frameworks of such moves can be unclear. In ambitious mathematics teaching, for instance, researchers have identified sets of talk moves (e.g., Chapin et al. 2009; Kazemi & Hintz, 2014), which are specific moves, often defined on an utterance level (Michaels & O'Connor, 2015). In another framework, Jacobs and Empson (2016) specified four high-level categories of moves that teachers can make when interacting with students to support their development of mathematical thinking: 1) *ensuring* that students make sense of the problem, 2) *exploring* details of students' thinking, 3) *encouraging* students to consider other strategies, and 4) *connecting* students' thinking with other representations. Jacobs and Spangler (2017) suggest that future research on leading classroom discussions should aim at developing and adapting new instruments, while building on the strength of existing instruments. As a contribution to such efforts, the present study aims at exploring how the teaching moves identified by Jacobs and Empson (2016) might interact with talk moves, as identified by Chapin et al. (2009). We explore this by analysing data from the Mastering Ambitious Mathematics teaching (MAM) project.

Theory

Talk moves have been proposed as a foundation for facilitating equitable and productive mathematical discussions (Kazemi & Hintz, 2014; Murata et al., 2017). Chapin et al. (2009) identified five talk moves that promote student thinking and participation: revoicing, repeating, reasoning,

adding on, and wait time. Kazemi and Hintz (2014) added two more talk moves: turn and talk and revise thinking. Using talk moves productively in response to students' ideas and building on the diversity of students' thinking to guide their attention towards key mathematical ideas is challenging (Michaels & O'Conner, 2015). Murata et al. (2017) show how variations in use of talk moves might create different environments for students' learning and development of strategies. They suggest that "for talk moves to be used productively in mathematics classrooms teachers need to effectively break down, represent, and discuss student strategies at an appropriate grain size while connecting with other strategies" (p. 312). In our study, we explore how talk moves are used in interaction with the more high-level categories of teaching moved identified by Jacobs and Empson (2016).

Jacobs and Empson (2016) present four major categories of teaching moves that promote student thinking. The first category involves ensuring that the students understand the problem, and this move is often used during the launching phase of a problem or early in the interaction with students. The second category involves exploring details of the students' strategies, and highlights four subcategories: posing general starter questions to initiate whole class discussions or conversations with students; pressing the students for an explanation of specific parts of their problem-solving processes by asking situation-specific questions to elicit the details of students' thinking and thus make their explanations richer; linking the students' representations and the story context; and expanding the students' understanding. The third major category is about encouraging the students to consider other strategies and thus providing opportunities to compare different strategies and ways of reasoning. The fourth category emphasizes the importance of connecting students' thinking to symbolic notations and involves representing students' ideas in writing and making connections between their talk and representations (Jacobs & Empson, 2016). Ambitious mathematics teaching highlights students' emerging ideas and their mathematical sensemaking. Core principles include positioning students as sensemakers, providing equitable access to learning for all students, orienting students to each other's thinking, and seeing teaching as both intellectual work and a craft (e.g., Kazemi, 2017; Lampert et al., 2013). Being responsive to students' thinking and using their ideas in relation to mathematical goals is a complex and challenging part of ambitious mathematics teaching (Richards & Robertson, 2015). We explore how the teacher might use talk moves in interaction with different categories of teaching moves to facilitate equitable and productive discussions in mathematics, and we focus on the categories of exploring, encouraging and connecting (Jacobs & Empson, 2016), which are central to this work.

Methods

We analyse data from a group of seven Norwegian elementary teachers, who participated in the MAM project, where they collaboratively engaged in cycles of enactment and investigation to learn ambitious mathematics teaching (Wæge & Fauskanger, 2021). The cycles included six phases: 1) preparation; 2) collective analysis; 3) co-planning; 4) rehearsals; 5) classroom co-enactment where one teacher teaches an instructional activity with a group of students while the other teachers observe; and 6) collective analysis (Wæge & Fauskanger, 2021). We analysed data from the classroom co-enactment phase, to explore the moves they used when leading discussions.

We used *Sportscode* video-analysis software for analysis, and this allowed us to code the video data directly instead of analysing transcripts. Data consist of video recordings from nine four-hour sessions, all of which include the six phases described above. In our coding, we identified segments of data that were distinguished by "conceptual breaks" (Jacobs & Empson, 2016). An example of such a conceptual break is when the teacher switch from launching the problem to posing a question to explore students' thinking. This would constitute a conceptual break between two of the main categories in our codes. The size of the segments varied.

In our coding of the interactions that followed the launching of the problem, we used the following categories and codes from Jacobs and Empson's (2016) framework:

- Exploring details of students' thinking
    - Starter questions
    - Pressing for explanation
    - Linking students' representations
    - Expanding students' understanding
- Encouraging students to consider different strategies
    - Sharing other strategies
    - Making connections between other strategies
- Connecting strategies to formal notation
    - Representing students' strategies

To explore whether and how these teaching moves might interact with talk moves, we added talk moves as codes in each of the main categories of teacher moves.

Findings

In what follows, we report on a close analysis of the teaching moves from one of the co-enactments with students to exemplify our analysis process and to present a case that unpacks the teaching moves and illustrates the interaction between talk moves and different categories of teaching moves. The episodes are from an enactment of a *quick image*. The learning goal for students was to learn about the distributive property of multiplication ($a \times (b + c) = a \times b + a \times c$). The teacher, Anna, showed the students a quick image for a few seconds and then asked the students how many dots they saw (see Figure 2). The first episodes starts when Anna asks the students to share their thinking. Figure 1 shows the different teaching moves that Anna used in this episode. We see that Anna starts with *exploring* the students' strategies and *connecting* their strategies to formal notation. She then *encourages* the students to share other strategies, and this is followed by an *exploration* of several different strategies. Below, we analyse in more detail how the teacher engaged these different categories of teaching moves.

In the beginning of this episode, Anna is exploring details of students' thinking and we notice that there is an interaction between the teacher's use of *talk moves*, *representing* student ideas on the board, and *student talk*. We join the enactment as Anna asks the students a starter question about the number of dots (line 1), which is followed by *wait time* in which the teacher gives the students time to think before sharing their own ideas (line 2). One of the students, Lisa, then *shares her thinking*

(line 4). Anna then begins to draw on the quick image on the board, pausing to check with Lisa about the accuracy of the drawing. She is using the talk move of *revoicing* to make sure that she and the students in the class understand Lisa's ideas: "So, you were thinking, first you saw that it was five and that it was four? And when you got to see the picture for the second time, you saw that there were three fives and three fours?", whilst simultaneously *representing* the strategy on the board (line 5) (see Figure 1).

	Time	Student talk	Exploring	Connecting to formal notation	Encouraging
1	00.57–01.06		Starter question		
2	01.06–01.25		Wait time		
3	01.25–01.26		Call student		
4	01.26–01.36	Student talk			
5	01.36–01.58		Revoice	Representing	
6	02.11–02.15			Representing	
7	02.15–02.23			Wait time	
8	02.20–02.25		Starter question		
9	02.25–02.30		Wait time		
10	02.30–02.32		Call student		
11	02.23–02.33	Student talk			
12	03.29–03.39			Representing	
13	03.39–03.42			Reasoning	
14	03.44–03.46			Call student	
15	03.46–03.52	Student talk			
16	03.52–03.59			Reasoning	
17	03.59–04.00			Call student	
18	04.00–04.10	Student talk			
19	04.10–04.12			Revoice	
20	04.12–04.15				Other strategies
21	04.15–04.16				Call student
22	04.16–04.21	Student talk			
23	04.21–04.32		Revoice	Representing	
24	04.32–04.45		Revoice	Representing	

Figure 1: Teacher's use of teaching moves when sharing strategies

After representing Lisa's strategy on the quick image, Anna asks the students to generate a numerical expression to connect with Lisa's informal strategy and thus make connections between multiple representations (line 6). She gives the students a few seconds to think (*wait time*) (line 7), before she asks them if anyone will share their thinking (line 8). This is followed by another *wait time* (line 9).

One of the students, Martin, *shares his thinking* (line 11), suggesting they can represent it as four times three and five times three. After having discussed with her colleagues what she might do, Anna *represents* what the student said on the board: $(5 \times 3) + (4 \times 3)$ (line 12). She then uses the talk move of *reasoning* to ask the students whether this number sentence represents Lisa's strategy: "Is this (the numerical expression) the same as Lisa said?" (lines 13). One of students responds by saying "No, because you forgot the parentheses" (line 15). Anna agrees, and then she repeats the question (line 16). Another student responds by saying "yes" (line 18). Anna follows up his response by using the talk move of *revoicing*: So, you think it is the same? (line 19).

After *exploring* Lisa's strategy, Anna is *encouraging* the students to share other strategies: "Did anyone think differently?" (line 20). She then *explores* the details of the other strategies (see Figure 1), and similarly to the exploration of the first strategy, we notice that there is an interaction between *talk moves*, *student talk*, and *representing* students' thinking on the board.

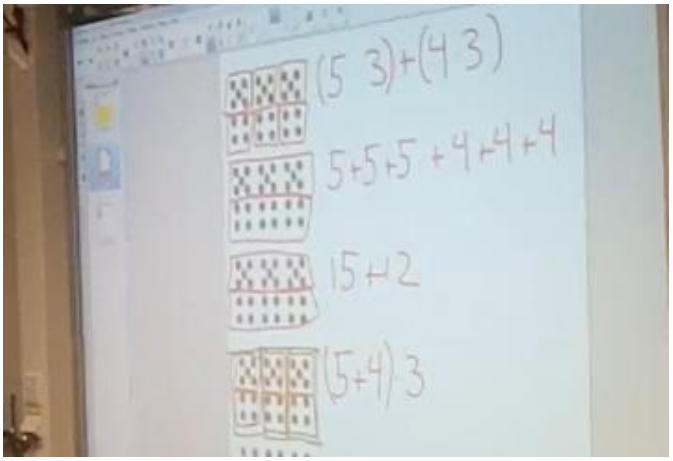

Figure 2: Student strategies for finding the number of dots

During this exploration, one of the students, Peter, presents a strategy that Anna and the other teachers had anticipated and wanted to highlight. He says that he added the five and the four, which equals nine, and then he multiplied nine by three (see Figure 2).

Later in the lesson, when the students had shared different strategies for finding the number of dots, Anna wanted the students to compare and make connections between two of the strategies: the one at the top, $(5 \times 3) + (4 \times 3)$, and the one at the bottom of the board, $(5 + 4) \times 3$ (see Figure 2).

The second episode starts when Anna asks the students if there is any connection between these two strategies. Figure 3 shows the different teaching moves that Anna used in this episode. We see that Anna *encourages* the students to make connections between strategies. She then *explores* the students' thinking about connections, before she *encourages* them to find more connections. Below, we analyse in detail how the teacher engaged these teaching moves.

In the beginning of this episode, Anna is *encouraging* the students to compare and make connections between the two strategies $(5 \times 3) + (4 \times 3)$ and $(5 + 4) \times 3$ and we notice that the teacher uses *talk moves* to give the students room to think and discuss possible connections. For instance, Anna starts with *asking about connections*: "Is there any connections between them (two strategies)?" (line 64). She then uses the talk move of *wait time* (line 65) before she asks them to *turn and talk* with their neighbour (line 65). Students then pair up and take turns explaining their thinking to a partner while the teacher (and observers) is listening and observing.

After the students have discussed their ideas with each other, Anne starts to *explore* their thinking, and we notice that there is an interaction between the teacher's use of *talk moves*, *student talk*, and the teacher *pressing* for explanations or justifications. Anna asks the students a *starter question*: "Is there anyone that would like to share what you talked about?" (line 67). Two of the students, Maria and Eli, *share their ideas*, explaining that the numbers are the same in the two numerical expressions,

but that there are two threes in the top one, while there are only one three in the lower one (line 69). Anna then *presses* the students to explain why (line 71) and gives them a few seconds to think (*wait time*) (line 71). The same two students, Maria and Eli, *follow up* her question by saying that on the top one there is five times three and four times three, and on the bottom one there is five plus four times three (line 73). Anna continues to *press* for more explanations and asks the students to discuss why we get the same answer for both strategies (line 74) and she then uses the talk move of *turn and talk* (line 75). One of the students, Tom, *gives an explanation* that is similar to what Anna and Eli said, emphasizing that the four and the five are added together in the one at the bottom (line 77). Anna summarises what the students have said (line 78). She then *encourages* them to find other connections: Do you see other connections?" (line 79) and uses the talk move of *wait time* (line 80).

	Time	Student talk	Exploring	Connecting to formal notation	Encouraging
64	10.05–10.35				Make connections
65	10.35–10.45				Wait time
66	10.45–11.40				Turn and talk
67	11.40–11.46		Starter question		
68	11.46–11.48		Call student		
69	11.48–12.03	Student talk			
70	12.03–12.07		Pressing		
71	12.07–12.14		Wait time		
72	12.14–12.16		Call student		
73	12.16–12–26	Student talk			
74	13.12–13.14		Pressing		
75	13.14–13.58		Turn and talk		
76	14.18–14.23		Call student		
77	14.23–14.45	Student talk			
78	14.45–15.04		Revoice		
79	15.04–15.06				Make connections
80	15.06–15.23				Wait time
81	15.23–15.30		Repeating		
82	15.30–15.33		Call student		
83	15.33–15.50	Student talk			
84	15.50–16.00		Revoice		

Figure 3: Teacher's use of teaching moves when connecting strategies

However, after having conferred with her colleagues, Anna goes back to Tom's explanation and asks the students if they can *repeat* what Tom said (line 81). One of the students, Heidi, then try to repeat what Tom said in her own words (line 83), and Anna uses the talk move of *revoicing* to make sure that she and the students understand his explanation (line 84).

Concluding discussion

From our analyses, we make two observations about categories of teacher moves and use of talk moves. First, we observe that there tends to be interactions between different categories of teaching moves. Although identifying individual teaching moves can be useful for research and education of teachers (Jacobs & Empson, 2016; Jacobs & Spangler, 2017), our analyses illustrate how teaching moves are not made in isolation but are used in ongoing strings of interaction. This corresponds with

how Cohen et al. (2003) describe teaching as interactions between teacher and students around some content. Although the discussion in the first example primarily aimed at exploring students' thinking, we saw how the teaching moves of exploring and connecting to formal notation were made in ongoing interactions between the teacher's use of talk moves, student talk, and representing students' strategies on the board. For instance, we saw how the teacher responded to the students' contributions by using the talk move of revoicing, in which she repeated what the student said and asked the student to verify if her revoicing was correct whilst simultaneously representing the strategy on the board. She stopped frequently to confirm with the students that she was representing their thinking accurately. Thus, the teacher used the two categories of teaching moves – exploring students' thinking and connecting it to formal notation – simultaneously and in interaction. We saw how the teacher never left the students as she revoiced and represented their ideas and asked them questions that allowed them to elaborate on their own strategies. With this approach, ownership of ideas remained with the students. This corresponds with how Murata et al. (2017) describe the dialogue as bouncing between the teacher and the students throughout the explanations, where the teacher breaks down and represents students' strategies at appropriate grain size. The second example, which was a sequence where the teacher aimed at encouraging students to make connections, illustrates the interaction between the teacher's use of talk moves, student talk, and the teacher pressing for explanations or justifications to highlight key mathematical ideas. We saw, for instance, how the teacher asked students to turn and talk after having pressed them to provide an explanation of a specific connection. Thus, she gave students space to explain and discuss their thinking.

Second, our analyses illustrate how talk moves can be used in different categories of teaching moves. For instance, the first episode illustrates how the teacher used the talk move of wait time when exploring the details of students' thinking. After having posed starter questions to invite students to share their thinking, she used the talk move of wait time to give students space to think and make sense of the questions. In the second episode we also observe how the teacher used specific talk moves within the category of encouraging students to compare and make connections between strategies. For instance, after the teacher posed a question about connections between specific strategies, she used wait time followed by turn and talk. Thus, she provided the students with space to make sense of it and to explain their thinking. By giving students time and space to engage with each other's ideas, she provided them with opportunities to connect their emergent ideas with learning goals and key mathematical ideas (Murata et al., 2017). We agree with Jacobs and Spangler (2017) that the promise of teaching moves hinges on their enactment, the learning goal, and the context in which they are enacted, and our analyses illustrate how the use of specific talk moves to enact the more overarching categories of teaching moves also depends on the context and the learning goal.

This study is limited by the scope of the episodes analysed. Yet, our study provides an existence proof that teaching moves are carried out in ongoing interactions, and that talk moves can be used to enact different categories of teaching moves – depending on the learning goal and the instructional context. Further research should be made to explore if there are other patterns of interactions among overarching and specific teaching moves, and if similar patterns can be found also in the move of ensuring that students make sense of the problem.

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Learning support practices in the fostering of basic arithmetic skills

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According to sociocultural theories, teaching consists of a collection of practices in which the participants interact and mathematical learning takes place. But which learning support practices do prospective teachers use when fostering children in difficulties in basic arithmetic skills? In this paper, the reconstruction of self-reported practices of learning support is presented as an example and preliminary results are discussed. The case selected as an example shows rather unproductive learning support by the prospective teacher with potential for further development into productive learning support. Further analyses of the underlying action-guiding beliefs are required.

Keywords: Teaching practices, prospective teachers, learning support, basic arithmetic skills.

Introduction

Central aim of German mathematics lessons at primary level (first to fourth grade) is to develop an understanding of basic mathematical concepts. The basic mathematical skills of understanding numbers, place value and operations are critical obstacles in the mathematical learning process. For example, the basic arithmetic skill "number sense" includes the idea of numbers as relations (Tubach & Nührenbörger, 2016). When learning these central concepts, particular difficulties can arise. Therefore, the children need targeted fostering and teachers who can organize this in a beneficial way. Crucial to fostering learners' skills is the professionalisation of teachers in demanding situations they must deal with (Prediger, 2019). Since both the disposition and the actions of the teachers are decisive (Clarke & Hollingsworth, 2002), a descriptive analysis of the practices and underlying beliefs could provide more insight into where there is a need for professionalisation and where the training of teachers can be targeted. This paper focuses on the teacher practices.

Key tasks for teachers are diagnosing the competencies and difficulties that arise as well as providing targeted teaching and fostering. In addition to selecting and adapting tasks, this also includes assisting the learner's learning process while working on the tasks and reflecting on and evaluating the learning process and the suitability of the activities (Prediger & Buró, 2021). We define the interaction that takes place in the assistance of the learning process while working on the tasks as learning support (Foster et al., 2023).

Theoretical Framework

Teachers realise teaching in so-called teaching practices, which we understand in the sense of Prediger (2019, p. 370) "as the recurrent patterns of teachers' utterances and actions." According to sociocultural theories, practices already exist in advance, but the individuum shapes its implementation of the practice (Cobb et al., 2001). People who learn practices, i.e. who are new to the community of participants in the practice, in turn influence the practice through their identity, which is why the practice continues to develop over time (Lave & Wenger, 1991). Hence, prospective teachers, as learners of teaching practices, should further develop and elaborate the learned teaching

practices and contribute their own way (Heyd-Metzuyamin et al., 2019). Teaching practices of learning support are among the key teaching practices, that need to be learned (Prediger & Buró, 2021). To foster students in their professionalisation process in the best way, it is necessary to first identify the productive practices of learning support.

To identify which learning support practices are productive, teaching practices and teacher actions are reconstructed in various studies and examined in terms of their impact on learners' performance and understanding (Prediger, 2019). Effective teaching practices of learning support are those that foster learners in controlling their own thinking process and gaining an awareness of their own activities (Cusi & Malara, 2013). These include practices that focus more on the meta-cognitive dimension rather than solely on the subject matter (Cusi & Malara, 2013). This is in line with the considerations of Foster et al. (2023), who divide professional learning support actions into those content-specific teacher actions that initiate high learning potential, such as stimulating reflection, providing further impulses or emphasizing central elements of understanding, and other content-specific teacher actions that are considered to be of lower quality in fostering the learning process, such as small-step guidance and an evaluation of the work status. Teachers are more likely to use practices and actions that are considered to be unproductive than productive ones (Foster et al., 2023).

This raises the question, of how teaching practices can be established that encourage active-discovery learning and further thinking and therefore are productive to prepare for the discovery of richer mathematical experiences in the classroom (Heyd-Metzuyanim et al., 2019). In their considerations on *situated learning*, Lave & Wenger (1991) conceptualise learning as increasingly competent participation in a practice. Heyd-Metzuyanim et al. (2019) transfer these considerations to teaching practices and describe the process of participation in teaching as initially imitative, rigid and process-orientated to explorative, flexible and goal-orientated. The change is supported by a bilinear approach: on the one hand through trying out a practice and reflecting on different learning outcomes, and on the other hand through the further development of one's own disposition (Clarke and Hollingsworth, 2002). To enable the teaching practices to be trialled, vignettes and complexity-reduced settings such as in teaching-learning labs are often used in teacher training in Germany (Fraefel, 2022). Reducing complexity here means simplifying the practical situation in comparison to later use at school, e.g. by teaching smaller groups or in a team. Regarding the learning of productive teaching practices, the focus is on the prospective teachers' learning to distinguish between practices with high and low potential, to try them out in practice and to reflect on their use in a theory-led way (Fraefel, 2022). One key point here is that the change of practices is slow and fragile and the prospective teachers first need to 'believe' and 'understand' how and why to teach in a certain way, before they can actually do it' (Heyd-Metzuyanim et al., 2019, p. 287).

While practices for supporting and fostering in inclusive teaching are already being researched at teacher level (i.e. Prediger & Buró, 2021), little is known about the practices of prospective teachers in fostering children in difficulties in basic arithmetic skills. Some studies use vignettes (i.e. Ulusoy, 2020) and therefore focus primarily on individual, hypothetical actions of the students. The suggestions for action made by the prospective teachers in Ulusoy's (2020) study aimed to improve the quality of the questions asked and the representations used by the teachers in the video vignettes, to use the learners' mistakes as a learning opportunity and to use correct technical language. However,

a good half of the prospective teachers did not formulate any suggestions for action for the first vignette, which assumes that formulating suggestions for action is very challenging for the prospective teachers (Ulusoy, 2020). In the few studies exploring prospective teachers learning support practices in real-life contexts, it was shown that prospective teachers established practices in which they gave learners time to think, considered and addressed learners' ideas, and followed up on learners' thinking (Santagata & Yeh, 2014; Sun & van Es, 2015). However, the correct answers and procedures of the learners were the primary focus of the prospective teachers (Sun & van Es, 2015).

There is a need for deeper insights into which practices prospective teachers use in situations in which they actually have to act ad-hoc and which disposition lies underneath. In this paper, we therefore focus on the following sub question of the project: Which practices of learning support of prospective teachers can be reconstructed in self-reports on fostering children in difficulties in basic arithmetic skills? We chose to focus on self-reported practices rather than the actual discursive practices during the fostering sessions, because they provide more insight into the underlying beliefs, which is a further central research interest of the larger study.

Methods

Sampling. The target group are prospective teachers studying to become teachers of special needs education in their second or third year of the bachelor program. They attended a seminar in which they fostered children in difficulties in basic arithmetic competencies as a team of two over the course of a semester. The aim is to develop productive practices in diagnosis and fostering. In line with the intended triad in teaching-learning labs, the planning, implementation and reflection of the fostering sessions, the individual fostering program is linked to the attendance of an accompanying seminar. There the prospective teachers deal with basic theories, plan and reflect their lessons and reflect video graphed situations of the fostering lesson jointly in a theory-based manner. In the summer semesters of 2022 and 2023, 24 prospective teachers attended the seminar and worked in twelve pairs, each fostering one child in difficulties in mathematics.

Data gathering. Two interviews were conducted to ascertain the practices of learning support from the prospective teachers' perspective as well as a possible development. The twelve pairs of prospective teachers were interviewed together after the second or third fostering sessions and after the seventh or eighth of nine fostering sessions (period of April to June) using self-developed semi-standardised interview guidelines. At the beginning, both prospective teachers were each asked to select a situation from the corresponding fostering session that they considered to be particularly decisive in the child's learning process so that individual actions and utterances can be reported as specifically as possible. To reveal how the actions and utterances emerged, the situations were focused on one after the other in the main phases of the interview with regard to the choice of situation, the child, their own actions and utterances and a reflection in accordance with the facets of professional teacher noticing. This resulted in a total of 24 interviews with 46 situations selected by the prospective teachers. The main phases of the interviews were transcribed in full.

Data analysis. To reconstruct the reported practices, a qualitative approach is required, which is why a structuring and summarizing procedure based on the qualitative content analysis was chosen. Because we conceptualise teacher practices of learning support by decomposing the practices into the

actions and utterances teachers use in relation to the actions and utterances of the children and the didactic subject matter, a category system of teachers' actions and utterances was created based on existing lists (Cusi & Malara, 2013; Foster et al., 2023), in the course of deductive-inductive category formation. The category system is shown in the analysis overview (Table 1). The data analysis was carried out in three steps:

- Step 1: In the main phase of an interview in the selected situation of a prospective teacher, the reported teacher actions and utterances were reconstructed in meaning sections and coded according to the categories. For the reported teacher actions and utterances that could not be assigned to a deductive category, an inductive category was developed. The categories *praise*, *restrain*, *linguistic role model* and *organisation* are developed inductively.
- Step 2: The assigned codes are displayed in the analysis overview according to the meaning sections, in which the turn section of the interview is noted
- Step 3: A comparative analysis of the analysis overviews was used to form types that represent the reported teaching practices

In this paper, we present an example of an analysis in the sense of the first step, which leads to the presentation of the analysis overview (Step 2, Table 1). The exemplary case is then assigned to a previously identified type (Step 3). All presented parts of transcripts were translated from German.

Case: Guided processing with minimal activation

In the following, the analysis of the main phase of the second interview with the prospective teachers Britta and Jessica, who do the fostering program with Jana (age 8), is presented using Britta's selected situation as an example.

The situation chosen by Britta arose in a mathematical game situation in which Britta and Jana were playing the game “Who has more?” (Tubach & Nührenbörger, 2016). The aim of the game is to find the difference between two numbers. To do this, both players roll the dice and place the corresponding number of dots in a vertical field (see Figure 1 using the example of 5 and 6 as the numbers rolled). The difference between the numbers is determined. The person with more tokens can keep the difference in tokens as points. The first person to have ten points is the winner. The prospective teachers extended the rules of the game so that the numbers are written down symbolically and the qualitative comparison is fixed with a larger/smaller/equal sign (see Figure 1).

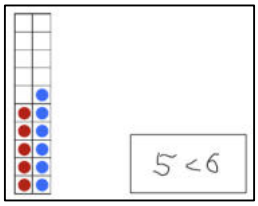

Figure 1: Reconstructed table setting in the situation reported by Britta

When asked to describe the situation in more detail and explain what Jana did, Britta reports:

21 Britta: So we played several rounds in a row. And um in some cases it worked out that way, then she rolled the dice, put her points there, then I rolled the dice, put my points there (.) then we put the note in front of her, where she then

always wrote it down directly. And then in one situation she also said, (..) oh I can't remember what exactly she rolled. But for example (*laughing*) I think there were five or six or so. Five is smaller than Six. And then she went straight for the one and (…) distributed one point

Britta reported that she *simulates the learning partner* by taking over the role of the opponent in the game. Parallel, she describes that she placed the piece of paper in front of Jana and thus implied the order in which Jana should complete the task. Through this *small-step guidance*, she provides Jana with a processing strategy.

The teacher's action of *small-step guidance* can be reconstructed in various actions reported by Britta. When asked what Britta herself did in the fostering session, she answers:

45 Britta: If she (.) didn't say directly on her own initiative, um, what she had written down, so (..) the number is smaller than the other or the number is larger than the other, then I said again yes, um, and what did you write down now?

For Britta, it seems crucial that Jana completes the task in full, which for her seems to include verbalising her own results or intermediate results. With her question, she explicitly asks Jana to verbalise the (interim) result. This focus on verbalisation is also evident in the following statement:

51 Britta: I then asked in any case, what is more now. Um, so yes, especially with questions, I somehow tried to get her to act and talk a bit.

It seems that Jana is not working on the task on her own initiative, but Britta encourages her to act and speak by asking questions. She reports that she asked the question "What is more now?" and in doing so once again provides a *small-step guidance* for the next processing step: After the two quantities have been displayed in the token field, Jana should first determine the qualitative difference. A third teacher action reported by Britta is shown in the following transcript excerpt:

63 Interviewer: How did you react to her actions and statements?
64 Britta: Mmm (*thinking*) um (..) yes, I would say that it was so reinforcing, because what she said was mostly right (*laughing*).
65 Interviewer: Mhm (*neutral*).
66 Britta: Um and then, yes, great, that's right, exactly, and then (.) we can carry on and yes, so I would say more in an appreciative and encouraging way, that you show her, yes what you're doing is exactly right and it can definitely go on like this.

Britta therefore reports that she *praised* Jana for correct interim results. This seems to be predominantly positive in the chosen situation, as Jana usually acts correctly. The following transcript excerpt shows what happens when Jana makes a mistake:

67 Interviewer: To what extent have her actions and utterances influenced your actions and utterances?
68 Britta: Mmm (*thinking*) (*pause of 7 sec.*) Um I don't think it was that strong in that situation, because she (.) mainly said the things that we would have expected, I would say. Um (…) except at the beginning when she said um, that if you have two and four and then four is the difference.
69 Interviewer: Mhm (*neutral*).
70 Britta: Um and that I then asked again in more detail and then, look again, where exactly do we have that it would be the same. Or how much would we have to take away so that it would be the same or something like that. Um in that situation it did influence the fact that we went into it more. But apart from that, she said a lot of things as we would have expected.

Jana determines the difference incorrectly. It can be interpreted from Britta's report that she seems to give Jana a *small-step guidance* to the correct result in two ways. (1) By first finding out to what point the quantities are equal and then determining by how much the quantities are different ("where exactly do we have that it would be equal"). (2) By taking away so much of the larger quantity until both quantities are equal and the quantity taken away makes up the difference ("how much would we have to take away to make it equal").

A deviation from the otherwise small-step approach becomes clear towards the end of the main phase:

89 Britta: Yes, sometimes um we also asked um, for example, um she rolled a five or something and then I asked which number would I have to meet to aim lower than yours, and she was able to tell us beforehand.

With Britta's reported question "Which number would I have to meet to aim lower than yours?", she gives a *further impulse* encouraging Jana to consider which numbers that can be met are smaller than her own. In this way, she encourages Jana to mentally determine the qualitative difference.

Britta's reported actions and utterances can be summarised as follows in the analysis overview:

Table 1: Analysis overview of teacher actions by Britta

Content-related with high learning potential							Other content-related learning support				Other
Further impulse	Emphasising central elements of understanding	Modelling	Stimulating reflection/ Evaluating of	Restrain	Linguistic role model	Simulating the learning partner	Evaluation work status	Small-step guidance	Errors or alternatives unused	Praise	Organisation
						T21		T21			
								T31			
								T45			
								T51			
										T66	
								T70			
T89											

The practice reported by Britta, is primarily located in the area of other content-related learning support. The accomplishment of the task through small-step guidance is supplemented with content-related teacher actions with high learning potential, such as in this case the simulation of the learning partner and the further impulse. Overall, the picture that emerges is of reported teacher actions that can be characterised as practice of guided processing with minimal activity. In terms of the didactic subject of quantity comparison in the game situation described, the teaching practice shows that a comparison of quantities is carried out by asking questions about the same quantity and checking the

symbolically noted numbers. This means that the comparison of quantities is first broken down into small steps and then initiated on a hypothetical level with stimulating questions.

Discussion and conclusion

The reported practice highlighted in the case study fits in with the previous findings on the learning support of prospective teachers, that the practices focus primarily on the procedure and correct response of learners (Sun & van Es, 2015). It seems as if the correct processing of the task is ensured at the beginning in a small-step procedure and therefore located primarily in the other, content-based learning support (Foster et al., 2023). This indicates that the focus is more on the correct result and procedure than on the child's understanding process. But building on this, the development of a conceptual understanding is stimulated by further impulses. It remains to be seen, whether these practices can be reconstructed in the learning support of other basic arithmetical skills.

The case analysis presented here originates from the initial development of the analysis method, so the preliminary results must be viewed with caution. So far, it has been shown that the analysis method can be used to reconstruct the prospective teacher's practice content based and to map well how the practice can be located in its productivity. The results presented here must be sharpened by further comparative analyses with behavioural patterns (presented in the analysis overviews) of other prospective teachers and placed in a larger context. However, the present study has the potential to deepen the insights into the practices of prospective teachers' learning support through the research in a real context and the data basis of the interviews and thus contributes new insights. Perspectively, it will become clear which other reported learning support practices of prospective teachers can be reconstructed. The results can provide a basis for evaluating and further developing the professionalisation of teaching practices of learning support. The first findings emphasise the need of opportunities for (prospective) teachers to test and reflect on their practices and to develop their teaching practices further.

One limitation of the study is certainly that it analyses reported practices that do not necessarily correspond to actual practices (Mullens & Gayler, 1999). Nevertheless, the self-reports provide insights into the action-guiding beliefs of the reporting persons, which is a central further interest of the research project.

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A Preservice Mathematics Teacher's Reflection on In-The-Moment Noticing of Students' Mathematical Thinking

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This study aims to investigate a preservice mathematics teacher's reflection on her in-the-moment noticing of students' mathematical thinking after her one-to-one teaching experience with a 7th-grade student based on pattern generalization activities. Following her pedagogical practice, the preservice teacher reflected on her in-the-moment noticing of students' mathematical thinking through a reflection paper and an interview. The analysis was based on her reflections. The preservice teacher attends to student thinking, such as student understanding, connections, and characteristics, as well as teacher action of disregarding some student understanding. Even though she provided some explanations for what she noticed, mostly her explanations were not well-grounded. She identified how she responded to student thinking by offering examples and materials, and asking questions. Moreover, the preservice teacher provided possible pedagogical decisions for future lessons.

Keywords: In-the-moment, teacher noticing, preservice teacher, student thinking, video case.

Introduction

In the literature, there is no consensus on the definition of teacher noticing; however, research studies regarding teacher noticing are united under the importance of noticing by stating that "noticing matters" (Schoenfeld, 2011, p. 223). Teacher noticing is considered an active perception process in which teachers intentionally choose the things that attract their attention (Erickson, 2011; Mason, 2011; Sherin & Star, 2011). As teacher noticing is considered as creating responsive interactions with students, responding to student thinking is not the end but creating other instances to attend and interpret (Jacobs et al., 2011; van Es & Sherin, 2021). Preservice teachers have challenges with attending, interpreting, and reflecting on students' thinking, and these skills are considered learnable and would be conveyed through teacher education programs (Jacobs et al., 2010; Sherin & van Es, 2005). Franke et al. (2001) stated that development in teachers' noticing skills through professional development programs would be sustained. In other words, preservice teachers would pursue their skills in future teaching careers. For developing teacher noticing, videos are commonly used tools since videos would provide "… access to teachers' in-the-moment noticing than has been available before" (Sherin et al., 2011b, p. 80). Teachers would identify missed opportunities or disregarded student thinking based on their in-the-moment responses. Since classrooms are a "blooming and buzzing" (Sherin et al., 2011a, p. 5) environment, preservice teachers struggle to identify and focus on student thinking in the classroom setting. One-to-one teaching was chosen to direct the preservice teacher's attention to student thinking and respond to students' thinking without concerning classroom management issues, thereby devoting her attention to various students' thinking that occurred simultaneously.

Pattern generalization is selected as context for preservice teacher's teaching practice since algebraic thinking and pattern generalization are recognized as key to mathematics (Kieran, 2022). Pattern generalization involves identifying and extending patterns and using different thinking strategies and mathematical language. Students have difficulties with pattern generalization, and teachers struggle to identify these difficulties (Girit, 2016). This study aimed to examine a preservice mathematics teacher's reflection on her in-the-moment noticing of student's mathematical thinking based on the one-to-one teaching regarding the pattern generalization concept since it provides a context for preservice teachers to notice various students' strategies, thinking, and difficulties.

In the literature, the study of Çatman-Aksoy and Işıksal Bostan (2023) focuses on the attention and interpretation of an in-service teacher in the context of geometry; however, this study extends it by including preservice teacher's responding skills and making student's algebraic thinking central focus of the study in terms of preservice teacher's noticing skills. Preservice teachers' challenges with noticing students' thinking are articulated in the literature, and this study would help to picture what the preservice teacher learned from her practice, which she would translate into the classroom in her future career.

Based on the purpose of this study, research questions are as follows: What aspects of student thinking did the preservice mathematics teacher attend to when she interacted with a 7th-grade student? How did she interpret what she attended to in the student's thinking in her reflection? How did she respond during her instruction? What kind of pedagogical decisions did she offer for future lessons?

Theoretical Framework

The commonly cited and used frameworks that rely on teacher noticing are the learning-to-notice student mathematical thinking framework by van Es and Sherin (2011) and the professional noticing of children's mathematical thinking framework by Jacobs et al. (2010). This research study focuses on how a preservice teacher responds to student thinking in addition to attending to and interpreting student thinking. Therefore, professional noticing of children's mathematical thinking framework is chosen since it addresses three connected skills: "(1) attending to children's strategies, (2) interpreting children's understandings, and (3) deciding how to respond based on children's understandings" (Jacobs et al., 2010, pp. 172–173). Furthermore, the analytic framework of Taylan (2017) and some categories from the study by Çatman-Aksoy and Işıksal Bostan (2023) are utilized, focusing on attention and interpretation skills, as they provide insight into the aspects of student thinking that the preservice teacher attended to and interpreted.

Methods

This research study requires an in-depth analysis of a preservice mathematics teacher's in-the-moment noticing of student's mathematical thinking; therefore, the case study method was adopted within qualitative research methodology (Yin, 2014). Concerning the framework of Jacobs et al. (2010), the unit of analysis is the preservice mathematics teacher's expressions regarding what she attends to, interprets, and how she responds to students' thinking.

Participants

This study's participants comprised one volunteer from the first author's PhD dissertation study. Mason (2011) articulated the awakening and developing of inner witness through intentional reflection as a purpose of noticing. Therefore, the participant's willingness to join this study and openness to develop her noticing skills and explain herself made her a participant in investigating the change in her inner witness towards student thinking in-depth. She was a senior undergraduate student in a mathematics teaching program at one of the state universities, and her sole experience with students through registration to practice teaching courses. Before participating in the study, she was informed about the research study and consented to join the study.

Data collection and analysis

The data collection of this part of the study was composed of the written and oral reflection of video recording of the preservice teacher's one-to-one application of the lesson plan with 7th-grade student. She participated in the larger study, which aimed to develop teacher's noticing of student thinking study, which covers five weeks of video case analysis sessions. She created a lesson plan and implemented it for one 7th-grade student in a public middle school in Türkiye. Her lesson plan was based on finding the general rule and the desired term of pattern generalizations. In the main activity, she used an ethnomathematical knitting model by stating a grandmother wants to knit a vest with a flower pattern for her granddaughter. The student drew the first few steps of the pattern and made a table to find the general rule of the pattern. The student was able to find a general rule but had difficulty naming variable and using different notations to represent the same variable. The main pattern generalization activity and the student's solution regarding this activity are demonstrated in Figure 1.

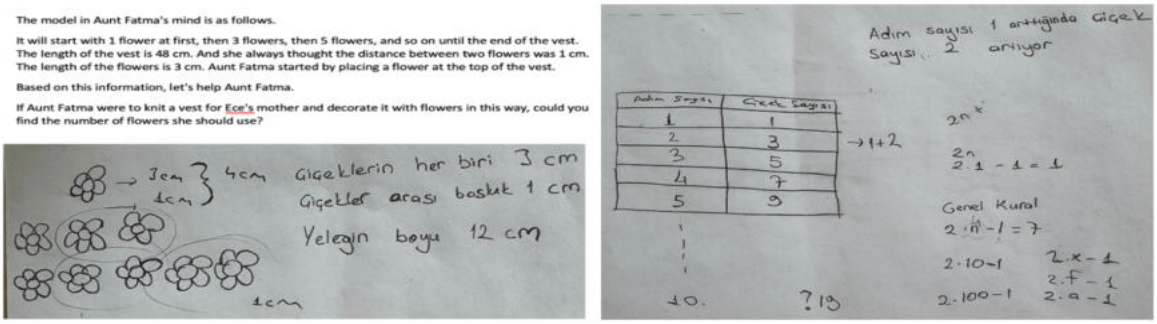

Figure 1: Pattern generalization activity example from the lesson plan and student's solution

The lesson was recorded by a camera facing the student and preservice teacher. The preservice teacher watched this video recording and wrote a reflection based on what she noticed, how she interpreted and reflected on student thinking in the moment, and what she would do if she had another chance to teach this lesson. She also reflected on these points in an interview conducted by the first author. The written reflection and the verbatim transcript of the interview with the preservice teacher compose the data of this study. The data was analyzed with MAXQDA (VERBI Software, 2024) based on Jacobs et al.'s framework (2010). For the attention component of the framework, student understanding, difficulty, characteristics, and connections occurred in the study, which is compatible

with the analytic framework of Taylan (2017). For the interpretation component, making connections between student and other students' thinking and teaching goals occurred in the data set, and these categories were adopted from Taylan (2017). For this component, making claims about student thinking without giving evidence is adopted from Çatman-Aksoy and Işıksal Bostan (2023). The explanation for these categories is provided in the table in the relevant section. Apart from these, the categories that emerged during the data analysis are provided in tables in the findings section.

Findings

Preservice Mathematics Teacher's Attention to Student Thinking

The preservice teacher pays attention primarily to student thinking. She noticed, both during her teaching and reflecting on her teaching, that the student had a conceptual understanding of the variable and that the variable referred to the number of steps in the task. During teaching and in her reflection, the preservice teacher noticed that the student was able to use step number and replace it as a variable, but the student was having difficulty using variable notation and finding the value asked for in the question when an inverse functional relationship was given. The preservice teacher also noticed that the student related the number of steps with the number of flowers by making mathematical connections to find the general rule of the pattern. Furthermore, she had difficulty understanding how to rename a variable. As in her explanation, the student was aware of the variable concept but was unable to call it a variable.

Another feature that the preservice teacher was able to notice is student characteristics. She stated in the interview that she noticed student's prejudice against mathematics. As a teacher action, she articulated that even though the student said she did not know how to name a variable, she was not able to attend to this student's understanding in-the-moment. The instances about student characteristics and teacher actions were only identified in her reflection after watching the video recording of her teaching. The examples of preservice teacher's attendance to student thinking, characteristics, and teacher action are provided in Table 1 with direct quotations from the interview and reflection paper.

Table 1: The instances that preservice mathematics teacher's attention to student thinking

Categories		Instances
Student Thinking	Student understanding (Taylan, 2017)	She said that n is the number of steps, and it is something that is constantly changing. She also knew that we should write that number instead of n, according to the number of steps asked. In other words, she is aware that it is a variable.
	Student connections (Taylan, 2017)	The student preferred to express the rule of the pattern by establishing a relationship between the number of steps and the number of flowers. This situation gave us the idea that the student understood the concept well.
	Student difficulty (Taylan, 2017)	I noticed that students can easily see and understand similar ideas, and after solving similar questions, they think a lot about when asked questions, which establishes an inverse relationship. For example, when finding the number of

		flowers in Step 7, the student could establish a relationship without much difficulty, but she thought more when there were seven flowers.
	Student lack of conceptual knowledge	I asked what n was, and she said the number of steps. I asked if the number of steps is fixed. She said no. It changes constantly; it becomes 1, 2, 3, 5, but she did not say the special name for it. She said there was none. I noticed this while watching the video.
Student Characteristics	Student characteristics (Taylan, 2017)	Generally speaking, the student was good. She was a bit prejudiced at first. She had a prejudice against herself; in other words, a prejudice against mathematics.
Teacher Action	Disregarding student thinking in-the-moment	That variable… when I asked what we call it, would we say something special, she said no directly, actually in there, I said that we call it variable. I didn't ask while teaching, but if it were right now, I would say, okay, this is something that changes; for example, this n is constantly changing.

Preservice Mathematics Teacher's Interpretation of Student Thinking

When the preservice teacher was asked to explain student thinking, she mainly did not explain or interpret student thinking without providing evidence. Even though she offered some explanations for what she noticed, mostly her explanations were not well-rounded based on the reasons behind her assumptions. She made some comments about relating the student she taught to the other students in the video club sessions that she attended and general comments about these students. In one instance, she stated how her expectations related to connecting ratio and pattern generalization were met by the student since the student was able to make mathematical connections between them. These instances are exemplified in Table 2 in the light of the preservice teacher's reflection on her teaching and student thinking.

Table 2: The instances that preservice mathematics teacher's interpretation of student thinking

Categories	Instances
Making claims about student thinking without giving evidence (Çatman-Aksoy & Işıksal Bostan, 2023) *The preservice teacher attends to student thinking but does not provide evidence*	When I asked the question in the problem situation, the student found the general rule by associating the steps and the number of flowers. In this case, I realized that the student had the right idea about how to find the general rule.
Use of evidence in making claims about student thinking (Taylan, 2017) *The preservice teacher interprets student thinking by providing evidence of a specific incident*	When we ask for the number of steps, she can find the number of flowers, but for example, when we ask which step has so many flowers, she thinks a little bit about it. In this sense, you had asked why this is the case, but it is actually a little more difficult to think in reverse. It is a little more challenging to establish a relationship.

Making connections between student thinking and other students' thinking from video case analysis sessions *The preservice teacher makes a connection between students' thinking*	I noticed that students can easily understand similar ideas after solving similar questions; they tend to think more about questions involving establishing an inverse relationship. For example, when finding the number of flowers in Step 7, she could establish a relationship without difficulty, but she thought a little when there were seven flowers.
Making connections between student thinking and teaching goals (Taylan, 2017) *The preservice teacher makes a connection between student's thinking and her teaching goals*	I wanted the student to move from ratio to pattern. The student was able to establish this relationship quickly. And we actually made a mathematical connection. The student realized that mathematics is not just a science consisting of formulas and rules, but that subjects can actually be related to each other by making connections.

Preservice Mathematics Teacher's Response to Student Thinking

During her teaching practice, the preservice teacher responded to student thinking by questioning, providing various examples, giving materials that would be used to facilitate student understanding, and providing multiple representations regarding pattern generalization. She acted towards any possible misconception about representing variables with different labels. She reflected on these instances in her reflection, as provided in Table 3. Furthermore, apart from how she responded to the student's thinking during her teaching, the preservice teacher provided possible pedagogical decisions that she would make for future lessons in her reflection on teaching. The examples of the actions she offered during the interview and in her reflection paper are provided in Table 3.

Table 3: The instances that preservice mathematics teacher's how to respond to student thinking

Categories	Instances
Questioning	The student knows that n is something that changes, but when I ask her if it has a special name, she says it doesn't. I would ask the student what name you would give to something that changes, since the concepts of variable and constant are essential.
Providing different examples	In the case of thinking in reverse, I would give different examples for this situation.
Providing materials to facilitate student discovery	I realized she had difficulty establishing an inverse relationship when I asked her which step is where we have 7 flowers. We developed our thoughts by thinking about this with the help of materials.
Using different representations	I first established a relationship between the amount of money accumulated without thinking about that 50. We had 50 liras at the beginning, didn't we? I added 50, 50+3, 50+6, 50+9. Then, together, we tried to find the rule for 3, 6, and 9 increases. I expressed it with a table so that she would see it better at the table, and she found that 3, 6, 9.
Taking actions towards preventing possible student misconceptions	For example, there is a variable, something that changes. She wrote 2n by herself, and we created a table. Then I stated that it could be 2x; it could be 2f, which is the first letter of my name, and 2a is the first letter of your name. It could be 2a-1 so that she would not think it is only expressed with that n, so that she would not have such a misconception.

Discussion

This research study examines a preservice mathematics teacher's in-the-moment noticing of students' mathematical thinking based on her reflection on students' mathematical thinking according to the video of her one-to-one teaching. She reflected on her teaching by identifying and interpreting student understanding and the difficulties students had during her teaching. Examining her teaching video provides a chance to consider the student's thinking that she missed in the moment of her teaching, as van Es and Sherin (2021) suggest the importance of things that teachers missed. In her reflection, she commented on student's difficulty that she missed in the moment. She informed her practice by recommending alternative decisions she would have made and recommended some pedagogical moves that she would take in future lessons toward developing her student thinking. While reflecting on her teaching, she had a chance to compare her student with other students she encountered during video case analysis in the dissertation study of the first author she attended. This would allow her to create a general picture of students' thinking and notice the different perspectives or incidents she would encounter while teaching. Even though the preservice teacher made interpretations of student's thinking, in most cases, she did not provide well-rounded evidence regarding her interpretations. Since the preservice teacher in this study had only one chance for one-to-one teaching, increasing preservice teachers' teaching practice would have allowed them to develop their noticing skills further and provide evidence for their interpretations by encountering various students' thinking. As Jacobs and colleagues (2010) stated noticing is a learnable skill in which the preservice teacher, as in this case, interpreted the things she noticed in the moment or the things she noticed while watching the video recording of her practice. Preservice teachers would learn about various students' thinking and how to notice it through armchair reflection (Erickson, 2011, p. 24). That would lead to development in their noticing abilities, which would translate into classroom teaching in their future career as this context provides an opportunity to practice their skills. As Sherin and van Es (2005) suggested, videos would contribute to noticing student thinking missed in the moment of teaching and develop preservice teachers' noticing skills.

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The implementation of a new curriculum in Greece: The case of learning and teaching statistics

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Keywords: Implementation, curriculum resources, statistics.

The study aims to investigate the way mathematics teachers enact a new curriculum in Greek schools. Trying to comply with current international perspectives about teaching and learning, the new curriculum dedicates a significant part to statistics, in contrast to the previous one. We focus on the relationships between influential factors, innovation enactment, and outcomes for the teachers. Our research question is: how the innovation enactment and the outcomes relate with teachers' understandings of the a) epistemological characteristics of statistics learning and b) ways to promote this learning, as both are underlined in the new curriculum? The influential factors can be traced in the tensions that the teachers experience. The importance of this study is highlighted by a gap in implementation research focusing on statistics.

Theoretical Framework

Teachers can use multiple material, cultural, and human resources in their design work. Among them, curriculum resources appear to have an important role in introducing reforms. The process of implementing a new curriculum has been the focus of researchers for the last decades. Koichu et al. (2021) addressed the relationship between implementation and resource by considering "implementation as a change-oriented process of endorsing an action plan based on a relatively well-defined resource (such as a research finding, a digital tool, a curriculum, a textbook, or an institutional policy)". In this paper our motivation lies mainly on one out of the five dimensions that Century and Cassata (2016) consider important to study implementation: we try to understand relationships between influential factors, innovation enactment, and outcomes. We also adopt the perspective of the "adaptation model" of implementation that emphasises the process of implementation and the conditions and contexts that affect it (Century & Cassata, 2016).

Today, one of the most crucial qualifications that a student should obtain while studying on secondary education is data literacy, which will lead to the ability to gasp how our data-driven world operates. Garfield and Chance (2000) stressed the basic elements of statistical thinking such as understanding the purpose, logic and process of statistical investigations and developing the mental skills to recognize, pose and answer questions that arise from the data. The new curriculum adopts those perspectives. All of the above create an unprecedented scene for secondary educators, requiring their preparation and familiarity in order to implement it effectively.

Methods

Our research is a case study of a group of teachers who worked together for a semester to design and enact lessons in their classrooms based on the new curriculum. In this paper we study a piloting process of six upper secondary mathematics teachers working in experimental/model schools. In this

project the group's discussions were focused on the teaching of statistics. The new curriculum shifts from a computational and procedural approach to a more creative, interpretational, inquiry based and problem-solving style. Data was gathered by recording four of the group's meetings held on approximately a monthly basis. We analysed their meetings using the Grounded Theory Method. Based on our data, we identified three main dimensions that emerged from the teachers' sayings and those led to the core of each episode's analysis. Those dimensions are about: i) epistemological and content specific issues, ii) teaching and learning issues and iii) emerging tensions during the process of implementation of the new curriculum.

Results and discussion

In the teachers' discussions we see epistemological issues emerging about teaching and learning statistics. Specifically, teachers are trying to make a decision about designing tasks that focus on a) computational procedures and b) interpretational statistical reasoning. Teacher 1 (1st meeting) proposes splitting the task into two parts so that in the first level students be introduced to statistics' basic calculations and then interpret the statistical concept involved. Teacher 2 (2nd meeting) proposes that teaching a statistics concept should firstly be introduced without focusing on a certain context and then be applied in a realistic context to produce interpretations. An effort to overcome the tension between the computational and interpretational dichotomy is being made by providing the needed formulas to the students to ease the computational procedures and focus on statistical conclusions. Furthermore, in meetings 3 and 4, while reflecting on their actions in class, teachers recognised that realistic context and statistical interpretations raised students' interest. This recognition highlights the teachers' struggling to coordinate the epistemological assumptions of the new curriculum with established perspectives and teaching approaches on statistics. Those epistemological issues appear as tensions. The factors involved are in the one hand, the new curriculum materials that provoke teachers to innovate and adapt their teaching into a new reality and in the other, their established perspectives and teaching approaches. These tensions are intertwined with institutional restrictions based on students' interests and time pressure.

In the implementation process of a new curriculum those epistemological issues concerning statistics need further study. This study will support our understandings of teaching and learning statistics and reinforce the implementation of the new curriculum.

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Digital poster available here

Support shifts in how teachers view their students' capabilities when co-teachers develop ambitious instructional practice

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Keywords: Co-teaching, ambitious instructional practice, ambitious teaching, coherent system, systematic literature review.

Introduction

Informed by decades of research, policy reforms and professional development programs aimed at raising educational standards have had only limited impact on teachers' instructional practice and student achievement. These results, from Sweden and many other countries, raise questions about critical features and coherence in teacher professional development programs. Also, the call for ambitious instructional practice in mathematics classrooms has intensified (Lindvall et al., 2022).

Ambitious instructional practice fosters problem-solving and reasoning skills and enhances a deeper understanding of mathematical concepts and procedural fluency. However, such ambitious goals are difficult to achieve without the critical features of teachers' knowledge and beliefs. A structure for teachers' learning and development toward these goals is also key to impacting teachers' instructional practice. Further, instructional materials, student assessments, and additional support for struggling students must also be coherent and align with the goals of ambitious instructional practice to impact student achievement (Cobb & Jackson, 2021).

Co-teaching has been recognized in response to classroom diversity and to address diverse student needs in inclusive educational settings, particularly in mathematics. It involves two or more educators working collaboratively in the same classroom and can manifest in different forms. Co-teachers may share the responsibility for planning, instructing, and assessing students, or one teacher may lead the instruction while the other is providing individual support to students as needed. Co-teaching can contribute to social and didactical inclusion and addresses all students' mathematical learning if it is flexibly organized (Gardesten, 2023).

Peer-reviewed academic articles examining co-teaching in Swedish mathematics classrooms are few to be found, however, several national reports describe successful implementation of co-teaching in Swedish schools. For example, findings indicate that co-teaching improves student outcomes and strengthens teachers' learning.

A hypothesis is that co-teaching, as an already implemented approach in many Swedish schools, may support coherent critical features to impact teachers' instructional practice and student achievement aligning with the call for ambitious instructional practice. This poster aims to present the results of a systematic literature review addressing the question *How may co-teaching support ambitious instructional practice?*

Method

The systematic literature review examined the relationship between co-teaching and ambitious instructional practice, providing a foundation for a forthcoming educational design research study.

The review followed the PRISMA (Preferred Reporting Items for Systematic Reviews and Meta-Analyses) guidelines to ensure a transparent and replicable process. The search was conducted in ERIC, Web of Science, and Teacher Reference Center focusing on peer-reviewed academic articles. Keywords were organized into three main blocks: "ambitious instructional practice", "co-teaching" and "mathematics". The search identified 15 articles that addressed the research question and were included in the review.

While analyzing the articles, a deductive approach was applied; hence, themes were specified in advance. The themes were conceptualized as coherent critical features to support ambitious instructional practice. Ambitious goals, teachers' knowledge and beliefs, quality instructional materials, additional support for students in need, and structured professional development for teachers must all work in harmony in a coherent system (Cobb & Jackson, 2021).

Results and discussion

The results will be more detailed and presented in the poster. Still, the hypothesis is somewhat confirmed: Co-teaching may support coherent critical features to impact teachers' instructional practice and student achievement aligning with the call for ambitious instructional practice. Co-teaching has been shown to improve teachers' instructional practice by combining different competencies in the classroom, which can help students at various levels achieve better outcomes. Proactive support for struggling students is often established in cases where the educational teacher and the special education teacher share the responsibility for planning, instructing, and assessing students. This contrasts with situations where the special educator takes the role of the second teacher only providing individual support to students as needed. The most important finding concerns the critical features of teachers' beliefs and ambitious goals. Co-teaching may support ambitious instructional practice as it includes teachers' learning and supports shifts in how teachers view their students' capabilities. Crucially, the belief that all students are capable and can succeed in mathematics must be shared by teachers (Cobb & Jackson, 2021).

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Teaching mathematically gifted students in secondary schools: A systematic literature review

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Keywords: Mathematically gifted, gifted education, learning environment, secondary schools.

School performance studies such as PISA show that the proportion of pupils with very good mathematical skills has decreased significantly in Germany over the last 10 years (Reinhold et al., 2019). This may be related to inadequate or inappropriate support for mathematically gifted pupils. While there are many suggestions for mathematical support programs in the literature, often based on practical experience, the empirical evidence regarding relevant design features of support programs still seems to be limited (Leikin, 2021), especially at secondary level.

Theoretical background

We understand *mathematical giftedness* as the individual potential to develop mathematical abilities (Ulm & Zehnder, 2020). Learning processes, which are often designed by teachers, are central to the realization of this potential in students. To this end, teachers design *learning environments* and define tasks, content or methods for the teaching and learning process (Ulm & Zehnder, 2020). Prediger et al. (2022) identify five principles for high-quality mathematics teaching (conceptual focus, cognitive demand, student focus and adaptivity, longitudinal coherence, and enhanced communication) that can be used to design learning environments. However, it is unclear to what extent they can be transferred to gifted education or, for example, have different relevance for it.

Research questions

In this study we would like to address the following questions: (1) What empirical evidence is already available on the design of teaching and learning processes for mathematically gifted students in secondary schools? (2) To what extent do these design features correspond to the five principles according to Prediger et al. (2022)?

Methods

The basis for answering the research questions is a systematic literature review (Newman & Gough, 2020). In a first step, a search for relevant articles, book contributions and monographs is carried out in four databases (ERIC, Fachportal Pädagogik, Scopus and Web of Science). The search terms include MATH*, GIFT*, PROMIS*, TEACH* and EDUCAT*. Subsequently, the titles and afterwards the abstracts as well as research questions and methods are screened, and inappropriate hits are sorted out using previously formulated exclusion criteria (e.g. “No reference to mathematical giftedness”). The remaining sources are read in full and appropriate hits are finally selected using inclusion criteria. Finally, suitable coding of the resulting sources, for example with regard to the investigated feature of a learning environment, allows the two research questions to be answered (see Figure 1 for an outline of the process of the systematic literature review).

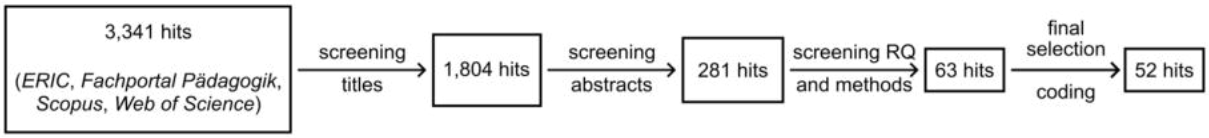

Figure 1: Process of the systematic literature review

Results

The previously described procedure of successive exclusion made it possible to reduce the 3,341 hits obtained in the first step to 52 sources, which were read in full and coded after the final selection, focusing on the characteristics of the design of promotional activities for the mathematically gifted (see Figure 1). The current review provides three key preliminary findings. (1) There is a large number of programs to support mathematically gifted students. These are often offered by schools or universities and have a variety of formats, such as summer camps or math clubs. In most cases, these are enrichment programs that take place alongside regular math lessons. (2) While there are many recommendations for promotion based on practical experience, i.e. anecdotal evidence, the small number of remaining hits shows that only few studies address specific features that should be considered when designing learning environments for mathematically gifted students. The majority of the reported features focus on partners in the learning process (e.g., the competences of the teacher or other students) and the mathematical tasks used. (3) Although the five principles by Prediger et al. (2022) are not explicitly mentioned, all but longitudinal coherence can be identified in the results. Their weighting and any necessary extension of the principles still need to be discussed.

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Help-seeking interactions in primary school mathematics classrooms

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Keywords: Subject-specific support, help-seeking, student-teacher interaction, primary school.

Introduction

One integral part of teaching consists in supporting students in their learning processes. Especially the art of supporting effectively constitutes a significant issue which concerns every teacher (Lozano Botellero et al., 2023). As Pianta (2016) states, the interaction between student and teacher entails a particular purpose or learning objective that preferably enhances the learner's performance.

In general, support situations can be initiated in different ways. When a student requests support, for example, by asking the teacher a question, this behaviour is referred to as *help-seeking* (e. g. Nelson-Le Gall, 1981; Newman, 2000). Being able to perceive one's own need for help is considered to be an important self-regulatory skill (Newman, 2000) since it represents a strategic move to keep the learning process going (Nelson-Le Gall, 1981). In order to potentially receive support which is adaptive to these needs, it is also necessary for the help-seeker to convey the difficulty accurately enough (Newman, 2000). However, a broader understanding of the help-seeking interaction can be obtained if not only the learner but also the teacher is taken into account (Davison et al., 2023). In these student-teacher interactions, the type of support teachers respond with is commonly distinguished into three separate domains (Lozano Botellero et al., 2023). The Teaching through Interactions (TTI) framework, for instance, includes the emotional, organisational as well as instructional domains and classifies the latter into general and content-specific instructional supports (Hamre et al., 2013). Even though research on support situations in secondary education exists, it mainly reports on general instructional support, whereas the focus on a specific subject is scarce (Lozano Botellero et al., 2023). Similar to secondary school, the small amount of investigations on teacher support in primary school does not include content-specific instructional support.

Methods

In an effort to gain more insights into the way teachers respond to content-related need for help, this study focuses on student-teacher interactions in which support is provided. Fyfe and Brown (2020) draw upon psychological as well as educational research stating that a learner's outcome on learning tasks is linked to, inter alia, teacher guidance. As the support of learners constitutes a typical classroom situation, it can be derived that teacher support pertains to teaching processes. Accordingly, teaching unfolds when interaction evolves. These perspectives are consistent with the posit of the TTI framework stating that "much of the effect of teachers . . . on student learning is located in the interactions that take place between teachers and students" (Hamre et al., 2013, p. 479).

Since help-seeking has usually been investigated to a lesser extent within the classroom (Davison et al., 2023), findings on support situations which are initiated by students who solicit help will be displayed on the poster. More precisely, it will be focused on the support teachers give students who demand content-related help while working on their tasks in a lesson. For this purpose, several

videotaped mathematics lessons in primary school, which are provided by the DFG Graduate School INTERFACH, will be analysed. The teachers who participated in this video study were supposed to conduct a lesson in which the students explored and worked with a generally well-known arithmetic problem format in Germany called *Zahlenmauer* (number wall). Considering that, a coding system was designed which comprises five primary domains of support, namely the content-related, strategic, organisational, affective and unspecific domains. Adjusted to working with Zahlenmauern, the content-related domain is further divided into various subject-specific categories. Based on the content-related request for help, the teacher's help-giving responses will be examined and presented. Up to now, it appears to be that teacher support mostly aligns with the learner's issue regarding the addressed domain. While these support situations usually start with providing broader hints, they conclude in more narrow indications. Only sometimes the teacher tends to change the subject-specific category so that previous steps of the task are revised or deeper processes are promoted.

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TWG 20: Mathematics Teacher Knowledge, Beliefs and Identity

Introduction to TWG20: Teachers' Knowledge, Beliefs and Identity

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This paper presents an overview of Theme Working Group 20 (TWG20) at the 14th Congress of the European Society for Research in Mathematics Education (CERME 14), with contributions and emerging themes from discussions on teachers' knowledge, beliefs, and identity. TWG20 featured 25 contributions from researchers across 15 countries and five continents. The overview reveals six cross-cutting themes that emerged from group discussions: the dual role of tasks in research and education, contextual considerations in teacher education, the centrality of mathematics in teacher education research, inclusion and special needs, theoretical framework diversity, and ethical considerations in research. This reflection aims to inform future research directions and enhance understanding of the complex interplay between teachers' knowledge, beliefs, and professional identity in mathematics education.

Keywords: Teacher knowledge, Teacher beliefs, Teacher identity, Teacher education, Mathematics Education

Overview of TWG20 in CERME 14

TWG20 at CERME 14 featured 25 contributions, including four poster presentations. Consistent with previous CERME conferences, the group demonstrated remarkable diversity, encompassing participants from 15 countries across five continents. This diversity provided an exceptional foundation for scientific exchange concerning mathematics teachers' knowledge, beliefs, and identity, whilst facilitating critical reflection on how sociocultural factors permeate research in these domains.

The group maintained a balanced composition between newcomers to CERME and experienced researchers, with approximately two-thirds being new to TWG20. Early career researchers represented about one-third of participants, contributing fresh perspectives to established research traditions within the group.

The working group used a structured approach across six thematic sessions. The opening session introduced TWG20's research heritage and outlined the conference structure. Each subsequent session was designed around transversal themes that served as focal points for reflection:

Session 1: Diverse theoretical perspectives in teacher education research

Session 2: Teachers' self-perception and professional identity

Session 3: Professional knowledge and its connections to broader educational concerns

Session 4: Methodological diversity, from intervention studies to large-scale research

Session 5: Problem-solving and problem-posing in teacher education

Session 6: Classroom inclusivity and formative task development

The closing session facilitated discussion of transversal themes identified by the TWG20 leadership team. Each session followed a consistent format: five-minute presentations followed by 30–40 minutes of small-group discussions, concluding with whole-group synthesis to share insights across subgroups.

Research on Knowledge, Beliefs and Identity in TWG20

This section characterises the research presented at CERME 14 concerning the three themes of the working group (knowledge, beliefs, identity). Each subsection examines the distinctive features of papers addressing the corresponding theme.

Mathematics Teachers' Knowledge

The nature and development of mathematics teachers' knowledge emerged as a recurring conference theme, with particular attention to how such knowledge is conceptualised, enacted, and supported in teacher education and professional learning. Rather than treating teacher knowledge as static or merely cognitive, discussions foregrounded its situated and dynamic character, arising from the interplay between disciplinary understanding, pedagogical reasoning, and specific teaching contexts.

Research on pre-service teachers (PSTs) revealed a significant disconnection between mathematical understanding and effective instructional application. Whilst PSTs often demonstrate confidence in their reasoning, they frequently encounter difficulties when adapting their knowledge to specific teaching situations. This suggests that apparent knowledge deficits more accurately represent a lack of access to knowledge-in-action, shaped by cognitive framing, varied pedagogical perspectives, and unexamined assumptions.

Consequently, teacher knowledge must be understood as inseparable from the capacity for reflection, critical distance from one's thinking, and engagement with multiple perspectives. This multifaceted understanding challenges traditional conceptualisations of teacher expertise.

Discussions drew upon established frameworks such as Mathematical Quality of Instruction (**Hill et al., 2008**) and Mathematics Teacher's Specialised Knowledge (Carrillo-Yañez et al., 2018). Whilst these frameworks aim to render teacher knowledge observable and comparable, participants noted inherent challenges in codifying teaching expertise—particularly that of experienced practitioners whose practices are guided by tacit understanding. The question of how to make implicit expertise visible and support its appropriation by novice teachers emerged as a critical concern.

Cultural and institutional factors were recognised as central to shaping both the nature of teacher knowledge and how it is valued and fostered. Comparisons between the Japanese model—characterised by practice-oriented approaches involving textbook writing and collaborative lesson

study—and Western contexts with more theory-driven orientations opened debates about the advantages and limitations of both perspectives.

Rather than positioning these approaches in opposition, discussions highlighted that cultural transposition should not be viewed as comparative exercise, but rather as an opportunity for critical reflection on one's educational system and thinking beyond conventional boundaries.

Ultimately, discussions invited a fundamental rethinking of what it means to 'know mathematics for teaching'. Teacher knowledge emerged not merely as content knowledge or pedagogical content knowledge, but as a complex, context-sensitive, and fundamentally human dimension of professional practice—one deeply entangled with motivation, beliefs, and identity. Recognising and supporting this complexity in both teacher education and professional development remains a central challenge for TWG20.

Mathematics Teachers' Beliefs

Consistent with previous CERME conferences, many TWG20 contributions focused on beliefs, typically examined in conjunction with knowledge and identity. Across various studies, researchers demonstrated shared interest in investigating how (prospective) teachers' beliefs influence their mathematics education practices, particularly in inclusive contexts.

A notable methodological similarity emerged in the widespread use of Likert-scale questionnaires for data collection, offering structured approaches to capturing teachers' beliefs across different contexts. Despite these methodological consistencies, studies demonstrated considerable diversity in focus and scope.

Research ranged from investigations of teachers' beliefs about assessment and standardised testing to content-specific areas such as word problem teaching or facilitating secondary-to-tertiary mathematics transitions. Some studies primarily mapped existing belief landscapes, providing comprehensive overviews of how educators conceptualise their roles and responsibilities. Others sought to broaden prospective teachers' beliefs through targeted interventions and professional development initiatives.

Studies revealed interesting divergences in theoretical frameworks and definitions. Whilst some research drew upon established definitions such as Philipp's (2007) interpretation of beliefs and frameworks like Ajzen's Theory of Planned Behaviour (1991), others did not explicitly adopt particular frameworks. This reflects ongoing field discussions about the need for shared definitions in beliefs research.

Future research considerations include questions of transferring findings into teacher education and practice, as well as ensuring clear participant identification. Given international teacher shortages, various countries have adopted different strategies: some education systems allow prospective teachers to begin teaching during training, whilst others offer alternative certification programmes for career changers. When evaluating teachers' beliefs, it is essential to consider that teachers may possess vastly different teaching experiences.

Collectively, TWG20 studies reinforce the understanding that teachers' beliefs constitute multifaceted, context-dependent constructs central to fostering rich and inclusive mathematical learning experiences. The observed divergences in research focus, conceptual frameworks, and cultural dimensions highlight the necessity for ongoing scientific dialogue and theoretical refinement.

Mathematics Teachers' Identity

Research on teacher identity revealed challenges similar to those in beliefs research. In some contexts, individuals enrolled in teacher education programmes discover during their studies that they do not wish to pursue teaching careers. These individuals are essentially students of teacher education rather than future teachers, raising the relevant question: *To what extent can we meaningfully speak of teacher identity in such cases?*

Conceptualising mathematics teachers' identity meaningfully remains complex. Whilst TWG20 expects papers to address mathematics teachers specifically—with findings not easily generalisable to other subject teachers—no robust theoretical framework deals explicitly with mathematics teacher identity. Consequently, authors often rely on general conceptualisations such as Sfard and Prusak's (2005) framework or avoid the term identity altogether, opting for constructs like self-perception that may not clearly align with identity as a construct.

This naturally raises questions about boundaries between beliefs, identity, and subjective theories. Whilst these constructs are not neatly separated in individuals' minds, research perspectives require careful attention to specific concepts being studied and consideration of how investigating particular constructs contributes to improving mathematics teaching and teacher development.

Papers in TWG20—consistent with previous years—seem to perpetuate the 'super-teacher' metaphor. These studies assess teachers' knowledge and beliefs, often implicitly conveying that teachers are failing if unable to respond correctly to researcher-selected tasks. Given that many researchers operate in dual roles as both researchers and educators, this perspective carries particular significance. Research should not foster impressions amongst students that they can become good teachers only upon possessing relevant knowledge and supportive beliefs.

Cross-cutting Issues

During TWG20, various cross-cutting themes emerged in discussions. Some have been recurrent in recent years, whilst others stem from more recent interests. We present six transversal axes of reflection, each with distinct nature and scope.

Tasks in Teacher Education Research

Most TWG20 contributions utilised tasks for teacher education in some capacity. Task usage represents a recurrent feature in teacher education research, typically serving dual roles as learning assignments and research instruments. However, despite this duality, research presentations commonly lack detailed explanations regarding design and development decisions.

This raises critical questions about how the dual nature of tasks—as both learning tools and research objects—is being articulated. Do tasks originate as learning tools subsequently used for research, or

vice versa? The former case raises methodological rigour concerns: to what extent can learning tools, used without explicit research adaptation, serve as adequate data collection instruments?

The term 'task' encompasses wide varieties of teacher assignments, yet more in-depth exploration of task nature seems necessary. This includes examining relationships to teachers' future classroom practice and intended task purposes. Such research would prove valuable both for structuring research processes in task-based teacher education and for organising teacher education according to expected professional activities.

Contextual Considerations

Context has been an implicit yet consistently highlighted element in recent TWG20 meetings. When using 'teacher education', we tend to assume speakers and listeners refer to the same concept—but this is not necessarily the case. 'Teacher education' carries multiple connotations heavily dependent on context.

Connotations are often shaped by national teacher education systems. Depending on country and targeted educational level, teacher education may be organised by discipline or follow generalist models where teachers train across subjects. Each context features distinctive structural characteristics, including those dedicated to mathematics content, mathematics education, or in-school training, as well as specific school-based training foci.

Implicit or explicit norms—frequently unwritten—govern educator profiles responsible for delivering different professional development types, significantly impacting teacher education. Mathematical-cultural aspects require consideration in certain studies. This ultimately can even affect the very nature of mathematical content. For example, two division algorithms are traditionally used in Europe—the French method and the Anglo-American long division method—while the partial quotients method has recently gained traction in some countries. Research that addresses this topic without specifying the algorithm in use can create communication barriers, particularly for researchers unfamiliar with the specific traditions or norms followed in different European contexts.

Increasing recognition exists for providing thorough descriptions of relevant contextual factors in each study to foster more effective and meaningful knowledge exchanges. Characterisations of context and its impact on mathematics teaching and teacher education (e.g., Brodie, 2025) can helpfully frame necessary reflections on this issue.

Mathematics in Mathematics Teacher Education Research

Recent TWG20 meetings have seen strengthening discussions regarding mathematics' role in presented research. Mathematics **education research** focused on teacher education gains meaning—in contrast to general education research—through specific connections to mathematics. This is often evident in teacher knowledge studies drawing upon precedents emphasising (pedagogical) content knowledge as distinguishing elements (e.g., Shulman, 1986; Ball et al., 2008; Carrillo et al., 2018).

However, when addressing teachers' beliefs or professional identity, the subject discipline's role in mathematics education research appears increasingly relegated to secondary positions in various studies. Among different ways mathematics can be approached in mathematics education

(Thanheiser, 2023), increasing numbers of studies treat it as contextual element, aiming to respond to primarily pedagogical issues equally applicable to any discipline.

This situation raises identity questions for mathematics education research, particularly teacher-focused research, regarding specialisation focus adopted in scholarly work. We believe it essential that, as a research field, we ask ourselves with each study: *Why is this mathematics education research? What mathematics is present in my research, and why is it relevant?*

Inclusion and Special Needs

During this TWG20, several researchers demonstrated evident interest in studies addressing situations involving particular sensitivity towards specific student populations, favouring their attention and classroom inclusion. This research often polarises around spectrum ends: students with learning difficulties and gifted students—those with cognitively-based special needs.

Group discussions highlighted the necessity of addressing inclusion from dual perspectives. First, preparing teachers for inclusive teaching practices responding not only to students with cognitive special needs but also those facing social exclusion or living in non-normative sociocultural contexts. Second, addressing inclusion within teacher education itself by reflecting on inclusion notions regarding prospective teachers' backgrounds and how mathematics teacher education can support future teachers from groups with various needs—cognitive, social, or otherwise.

Theoretical Frameworks: Evolution and Integration

TWG 20 has undergone significant evolution regarding theoretical framework usage. From CERME 9—where many studies aimed to underscore chosen frameworks' potential for educational research—to recent conferences where diverse constructs serve as means for studying different teaching aspects, there has been both focus shifts and framework employment manner changes.

Two theoretical models have become recurrent in recent TWG20 years, both centred on teacher knowledge: the MKT model (Ball et al., 2008) and the MTSK model (Carrillo et al., 2018). Whilst these models remained present in the most recent conference, wide varieties of theoretical constructs were represented in submitted works (including Anthropological Theory of the Didactic, TPACK, various belief frameworks, among others). This diversity enriches the group, fostering perspective exchanges.

We have observed developments toward networking between existing frameworks, aiming to complement their use in analysing diverse situations—from those specific to studied mathematical concepts to those allowing connections between constructs such as beliefs and knowledge, exemplified in Krišáková and Hubeňáková's (in this volume) work. Future conference contributions to TWG20 could consider adopting such theoretical articulation as foundations for understanding mathematics teachers' knowledge, beliefs, and professional identity in more complex and integrated ways.

Ethical Considerations in Mathematics Teacher Education Research

Ethics in mathematics teacher education typically does not receive direct attention. We may assume research is conducted ethically because we follow personal principles and values and established field

customs that presumably conform to ethical research standards. However, reflection on less evident aspects is necessary. These aspects are linked to a common circumstance: the often dual role we play as both teacher educators and researchers. This duality often involves us in research while simultaneously being involved in designing professional learning, yet there is insufficient reflection on our role within such research, how we might bias collected data, or how research process dynamics may limit teacher education effectiveness for given groups of prospective teachers.

This issue becomes particularly relevant when using methodological approaches involving control groups and intervention groups receiving different professional development—potentially denying control groups access to learning enhancements for research purposes. Another reflection point concerns informed consent related to the teacher educator-researcher duality. As educators of (prospective) teachers, hierarchical relationships exist with (prospective) teachers who may perceive educators' wishes to involve them in research activities as directives from authority figures. It is important to question whether informed consent is truly free or whether existing power dynamics exert implicit pressure on (prospective) teachers as potential research participants.

Finally, though sometimes related to educator-researcher duality, there is the issue of adopting deficit-based perspectives (Adiredja & Louie, 2020) in research. Studies focusing on shortcomings often overlook fundamental learning process characteristics and professional development importance, specifically building upon what learners already know and can do. Beyond this, deficit discourses imply researcher moral superiority (and their personal and sociocultural stances) toward studied future teachers, contrasting with needs to conduct research with them in support of their development.

Conducting research ethically involves commitment to treating teachers participating in our research as equals in the broadest sense of equality.

Conclusion

TWG20 at CERME 14 provided a rich forum for examining current research in mathematics teachers' knowledge, beliefs, and identity. The diversity of participants and contributions highlighted both the vitality of the field and the challenges that remain. The six cross-cutting themes identified through group discussions—tasks, context, mathematics centrality, inclusion, theoretical frameworks, and ethics—provide a roadmap for future research directions.

As the field continues to evolve, researchers need to remain attentive to the complex interplay between these themes and issues while maintaining focus on improving mathematics teaching and teacher development. The international and multicultural nature of TWG20 serves as a reminder that solutions must be culturally sensitive and contextually appropriate while advancing our collective understanding of mathematics teacher education.

The reflections presented here result from rich discussions among all TWG20 participants, whose contributions made this synthesis possible. Their diverse perspectives and collaborative spirit exemplify the best of international research cooperation in mathematics teacher education.

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Beliefs of elementary school teachers regarding the inclusion of special education students in mathematics lessons

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Considering that mathematics teachers' beliefs play a critical role in their practice, this study examines how professional development (PD) can impact elementary school mathematics teachers' beliefs regarding the inclusion of Special Educational Needs or Disabilities (SEND) students in mainstream classes. A mixed-methods approach was used in the study, combining questionnaires, reflections, and discussion. Involved were 34 teachers who participated in a 10-session PD program focused on teaching multiplication and division in inclusive classrooms. Findings indicated that the most notable positive change in teachers' beliefs was related to social integration, while obstacles appeared to outweigh their commitment to academic integration.

Keywords: Special educational needs, inclusive education, beliefs, professional development.

Introduction

Students with special education needs (e.g., learning disabilities) are prone to experience failure in mathematics, a field which requires generalization and abstract thinking, along with procedural competency (Geary, 2004). These students are characterized by having difficulties in cognitive skills such as working memory and automation of quick tasks (Willburger et al., 2008). Due to the increasing implementation of new laws that strive to integrate and include SEND students in mainstream classrooms (Blass, 2022), researchers have begun to study issues related integration and inclusion of Special Educational Needs or Disabilities (SEND) students in mathematics lessons (Lambert & Tan, 2019; Lindenskov & Lindhardt, 2020). The current study focuses on elementary school teachers' beliefs regarding the inclusion of SEND students in mainstream mathematics classes.

Background

Teachers' pedagogical knowledge and beliefs can impact the success of integration and inclusion (Brownell et al., 2010; Beswick, 2008; De Boer et al.,2011). Philipp (2007) defined beliefs as lenses that affect a person's views on some aspect of the world or as tendencies to act. Unlike knowledge, beliefs can be held at different levels. Beliefs are more cognitive than feelings and attitudes. Philipp (2007) emphasized that researchers, teacher educators, and professional developers are usually not only interested in measuring teachers' beliefs but also in changing them. Because teachers' beliefs influence their practice, having teachers reflect on their teaching can be a significant factor in supporting positive beliefs and practices. For example, Furinghetti & Morselli (2011) found that high school teachers' beliefs significantly influenced their approach to teaching mathematical proofs, affecting how they present and prioritize different aspects of proofs in the classroom. They emphasized the importance of raising teachers' awareness of their own beliefs and incorporating this understanding into professional development (PD) to enhance teaching practices.

Beswick (2008) determined that teachers develop professionally following professional learning experiences. Examining the beliefs of 22 mathematics teachers, she found that the teachers held

different beliefs regarding teaching mathematics to underachieving students compared to other students. The teachers believed that conceptual teaching was not suitable for underachieving students and that those students needed more basic mathematical skills to survive as well as some technical assistance in providing the correct answer. As a results, the curricula for underachieving students were narrower and less enriching and challenging compared to those for the mainstream students. However, after PD, there was encouraging evidence that teachers' beliefs regarding mathematics, mathematics teaching, and underachieving students could change. The teachers testified that they were motivated to improving their students' learning and were willing to change the teaching and learning processes in their classrooms.

Jordan and McGhie-Richmond (2010) found a connection between how teachers interact with their students and their beliefs regarding the abilities of integrated special education needs (SEND) students in their classes, as well as their role in fostering learning. About 25% of the teachers viewed disability as a fixed, internal condition that cannot be changed, blaming the student or their family for learning difficulties and spending less time with students with disabilities. In contrast, around 20% of the teachers saw disability as influenced by social and educational factors, taking greater responsibility for creating learning opportunities and investing more time in supporting students. This study incorporated PD through observations and interventions designed to enhance inclusive practices. It showed that teaching practices changed mostly among teachers who perceived disability as being shaped by social and educational influences.

In both studies presented, a PD program was conducted regarding teachers' beliefs about teaching students with difficulties. However, Jordan & McGhie-Richmond's (2010) research did not focus on mathematics, and Beswick's (2008) research focused on underachieving students, not those students characterized by special needs. Additionally, Beswick (2008) studied teachers with varying levels of seniority teaching mathematics in different settings, from kindergarten to high school. In a related article, Yada and colleagues (2022) noted that despite training and experience, changes in teachers' beliefs and attitudes towards inclusion, do not always occur.

The purpose of the current study is to examine the beliefs of elementary school teachers regarding the inclusion of SEND students in mathematics lessons before and after PD. The PD focused on the topic of multiplication and division in inclusive classes, operations relevant to students from second grade forward. Our research questions are: (1) What are the beliefs of mathematics teachers regarding the inclusion of SEND students in their classrooms, before and after PD? (2) What were the reasons that led some teachers to change their beliefs while others did not?

Methods

The research involved 34 elementary school mathematics teachers, with professional training in mathematics pedagogy and at least three years of experience teaching in inclusive classes. They were recruited through a city-wide registration system for teacher PD and had chosen to participate in this specific PD program entitled, "Inclusion and Integration of SEND Students in Multiplication and Division Lessons in Mainstream Classes."

The PD took place using the Zoom platform, and consisted of 10 sessions, each lasting three hours. The program content consisted of three parts: pedagogical content knowledge for teaching

multiplication and division, understanding the characteristics of SEND students, and implementing teaching practices suited for differentiated classrooms. During the PD, teachers worked in school teams to create teaching iterations for inclusion and integration in their mathematics classes. Additionally, the teachers experimented with various teaching practices and discussed their effectiveness. All sessions were videotaped and reviewed after each session to adjust the following sessions as various issues arose.

This research was conducted in accordance with the ethical guidelines as outlined by the Chief Scientist of the Ministry of Education and the Internal Review Board of the university. All participants signed an informed consent to participate in the study.

When researching beliefs to better understand beliefs, it is important to use different tools; quantitative tools provide a general picture, and qualitative tools provide depth about the beliefs. Thus, we used both quantitative questionnaires and observations as research tools. The beliefs questionnaire, given out during the first and last session of the PD, was based on Barnett's (2015) research. It included four statements (see Table 1) regarding teachers' beliefs about SEND students included in mathematics classes with a 1-9 Likert scale. Two mathematics researchers and two special education teachers validated the questionnaire.

All sessions of the PD program were videorecorded. During the second-to-last session, the teacher-educator asked participants the following two questions: “Do you feel you can successfully integrate SEND students in mathematics lessons? Who within your school community can facilitate the achievement of this objective?” These questions served as a catalyst for a vibrant discussion.

During the last session, participants reflected about teaching SEND students in mathematics lessons and were asked to comment in writing on the following two questions: (1) After the training, you were asked to try different assignments that integrate SEND students into mathematics classes. What insights have you gained about student characteristics from these experiences? Please provide details. (2) We spent five months together. During this period, have you noticed any changes in your teaching practices concerning the students in the integrated class or in general?

Qualitative analysis was carried out with the aim of understanding why some teachers changed their beliefs, while others did not. Towards this end, nine teachers were identified (see Table 2) for in-depth analysis.

Findings

Regarding the beliefs questionnaire, Cronbach's alphas showed high reliability for the four items, both before (0.824) and after (0.892) the PD program. Prior to the program, teachers were ambivalent regarding the inclusion of SEND students in mathematics classes (M=5.34, SD=1.71) and after the program there was a slight leaning towards being more positive (M=5.97, SD=1.75). The difference in their beliefs was nearly significant (t (33) =-1.960, p=0.058).

Taking a closer look, Table 1 shows results for individual items. The third statement (B3) showed the greatest increase and was the only item with a significant difference. In other words, after the program, teachers believed that SEND students integrated in mainstream classes would succeed better in society compared to SEND students studying in separate special education classes. However, when

it came to learning mathematics (items B1, B2, and B4), a change in beliefs was not found. We would like to note that teachers strongly believed in providing SEND students the opportunity to learn mathematics in mainstream classes.

Table 1: Changes in Individual Items from the Questionnaires Mean pre compared to Mean post

Statements	pre	post	t	df	p
B1. SEND students should be given the opportunity to study math with mainstream education students.	6.73	7.02	-0.635	33	0.530
B2. SEND students learn math best in mainstream classrooms that integrate them.	4.73	5.29	-1.569	33	0.126
B3. SEND students learning mathematics in mainstream classes have a greater chance of societal success than those learning in special education classes.	4.82	6.17	-3.206	33	0.003
B4. SEND students learning mathematics in mainstream classes have a better chance of success in mathematics than those learning in special education classes.	5.08	5.41	-0.787	33	0.437

Based on the questionnaires, we classified the teachers into nine categories according to their beliefs before and after the program (see Table 2). Teachers whose average response was less than 4.5 were classified as having a negative belief towards integration and inclusion. Those with an average response greater than 5.5 were classified as having a positive belief. Teachers with an average response between 4.5 and 5.5 (inclusive) were classified as neutral in their beliefs towards integration. The numbers in Table 2 indicate the number of participants classified in each category, and the letters represent specific teachers within those categories discussed further below.

Table 2: Frequencies of changes in teachers' beliefs

Post / Pre	Negative	Neutral	Positive
Negative	4 (RO)	2 (D)	4 (O)
Neutral	1 (M)	2 (L)	6 (RI)
Positive	- (A*)	2 (T)	13 (V)

According to the frequencies in Table 2, more than one-third of the participants (13) held positive beliefs about integration in the pre-program questionnaire and maintained those positive beliefs afterward. Furthermore, more than one-third of the participants (12) underwent a change in their beliefs about integration: 2 shifted from negative to neutral, 6 shifted from neutral to positive, and 4 moved from negative to positive beliefs. The specific teachers denoted by letters in Table 2 were

selected for further analysis based on their active participation and expression during the discussion in the second-to-last session and/or their reflective writing in the final session, expressing their reasons for being positive, neutral, or negative about integrating SEND students into mainstream classes. According to the results of the questionnaire, there were no teachers who were initially positive and then later became negative. Yet, we placed Participant A in Table 2 as going from positive to negative with an asterisk, because although her beliefs were positive in both the Pre and Post questionnaires, during the discussion (see below), she expressed opposition to academic inclusion, while supporting inclusion mainly from a social perspective.

To understand these results in more depth, and why little change seemed to occur, we analyzed the discussion which took place during the second-to-last session, and the teachers' reflections written during the last session.

From the questionnaire (item B3), we already noted that teachers believed inclusion could have social benefits for SEND students. This was expressed clearly during the discussion:

V: Our question as a society is not whether we are raising children who all have maximal proficiency in mathematics, but whether we ultimately want to accustom my children and yours that there are children with special needs in our society...

M: I think this conversation is sliding into two areas...there's a difference between seeing them as part of society, which is a very beautiful and correct thing to raise children and know to accept those who are different and know he's there, there is no debate among educators... But the second thing, the learning of mathematics and knowing from here how we deal with the situation? How can we advance them?...

RO: ... I understand that socially it's important, but if I want to advance them optimally, they need their small group.

During the meeting, challenges related to integration and inclusion were raised, along with discussions on how to address them. Below is an excerpt from that discussion:

T: In my class, it really depends on the child… The children I manage to advance are the ones where I can create cooperation with the parents. At school, it's mainly the principal who helps and supports when there are difficulties…

RO: I think there's no definitive answer; there are many factors... Everything we've learned and talked about requires resources from the teacher. If the teacher has other students she needs to help, it ultimately takes the time of the attention needed by the SEND students. In the end, there's one teacher who has to plan the 45 minutes optimally... We need to make a program that requires a budget and advances the children in a small group, and then integrate them into the class. This will advance them and create less frustration.

A*: I agree with RO. In my class, there are 12 integrated children... there's no way I can reach everyone, and there's no way I can advance with the material.

In the above exchange, we observe that the participants discussed the academic difficulties of integrating SEND students into mainstream classes. Some participants expressed frustration with these difficulties, and it was evident that these challenges hinder them from changing their beliefs about integration and inclusion. For example, T highlighted the academic challenges of integration and noted that a main factor is the cooperation of the students' parents. This participant showed a change from a positive to a neutral stance after the PD. Conversely, A* expressed great frustration and asserts later on that the education system failed by allowing parents the option to choose whether to integrate SEND students into mainstream classes. This is why we considered A* to be negative,

despite a positive trend shown in the pre and post-questionnaires. It may also be that A* is ambivalent. The conversation then turns to what can be done to cope with challenges.

RI: I don't quite agree... There are students whose difficulty is attention, and that's something we as teachers … can neutralize. And there are students whose difficulty is, for example, a girl in my class with dyscalculia, and it's true it's very difficult to integrate her into the mainstream class. However, I can integrate her into a group with children who will be good for her and help her...

A*: The children in my class are integrated into classwork and play, but I claim, on a general level, that the education system has failed in the last ten years when it gave parents the decision to integrate into a mainstream class...

RI: But that's the situation in the classes now; we can't change the education system. The question is what we can do within the existing framework.

V: It's really difficult and challenging, but it depends on the perspective we take. For a moment, if I put the teachers aside and look at the personal aspect, there's no doubt that integration is right for those children and for us as a society... I agree that it's difficult for teachers, and I also agree that the system doesn't give us enough tools; there isn't enough preparation, there isn't enough support... It's really necessary to train for it; it needs to be part of the training in colleges, and it's still not there.

RO: If my goal is for them to learn math and I want them to achieve in math, then I need to provide them with all the help they're supposed to get. Therefore, having them in my class means that they don't get all the help... because I don't have the ability.

Above, Participants suggest ways to address academic difficulties. For instance, participant RI, who shows a change from neutral to positive, mitigates challenges for students with attention disorders and dyscalculia through differentiated work groups tailored to each student. In contrast, RO, who remains negative, argues that the issue as systemic, he suggests advancing the students academically in a separate classroom and integrating them into the mainstream class during different lessons to promote social integration without hindering the teacher and the rest of the class.

Reflective writings from the final session provided additional insights and strategies for integrating and including SEND students. O, who shifted from a negative to a positive stance, wrote about the importance of individualized approaches to help each child succeed and especially stressed listening to the students:

> It is necessary to examine in each class for each child individually what is the optimal way to bring them to success... Following the PD, I learned to listen, examine processes, try to diversify, engage more, as well as deeply evaluate what can help and what does not.

M, who moved from neutral to negative, discussed the need to adjust the teaching pace and increase repetitive practice:

> I felt that I automatically had to lower my teaching pace, even my speaking pace, so as not to miss someone who can understand like everyone else, but at a slower pace... In addition, I felt there was a need for more practice exercises...

D, who transitioned from a negative to a neutral stance, emphasized the importance of paying more attention to students with special needs and finding appropriate learning methods for them:

> The training helped me to notice more and pay attention to the special needs students. Because I addressed them in various activities that were submission assignments, and I worked with them one on one... that there is a possibility to help them learn by seeking ways that suit them.

A recurring theme in the reflective writings was the acknowledgment of challenges alongside the variety of teaching methods introduced during the PD, enabling some teachers to overcome those challenges and move towards being more positive about inclusion.

Discussion and conclusions

Our first research question was: What are the beliefs of mathematics teachers regarding the inclusion of SEND students in their classrooms, before and after PD? The answer to this question is not clear cut and findings of this study highlight the complexities surrounding mathematics teachers' beliefs about the inclusion of SEND students in mainstream classrooms. Quantitative data revealed no statistically significant change in overall beliefs following the PD program, with teachers remaining ambivalent or only slightly more positive. The most substantial shift occurred regarding the societal benefits of inclusion, as reflected in a significant change in responses to Item B3. However, teachers' beliefs about the academic efficacy of inclusion remained unchanged, indicating persistent skepticism. These findings are consistent with Yada et al. (2022) and Bosse et al. (2016), who emphasize that teachers' beliefs about integration are relatively stable and resistant to short-term interventions. Qualitative analysis further underscored teachers' willingness to support social integration and equal opportunities for SEND students, even as they voiced concerns about the effectiveness of inclusion for academic progress.

In answer to the second research question, what were the reasons that led some teachers to change their beliefs while others did not, we first reflect on the PD program. A significant portion of time during the program was spent on enhancing teachers' knowledge for teaching multiplication and division, rather than expanding their understanding of what SEND students can achieve. This focus may have limited the program's overall impact on shifting beliefs. As Lambert and Tan (2019) suggest, emphasizing what SEND students can accomplish when provided the right opportunities, rather than focusing on their limitations, could have a more profound effect on teachers' attitudes. Teachers who reported positive shifts valued practical differentiation strategies introduced during the PD, but systemic obstacles like inadequate support and high academic standards continued to hinder change. Teachers who did not experience a belief shift often cited the challenge of meeting diverse student needs within a conventional classroom setting.

Future research could examine whether teachers' beliefs about the inclusion of SEND students differ across various mathematical topics, such as geometry, beyond the current focus on multiplication and division. It would also be valuable to explore whether the mode of professional development, online versus face-to-face, impacts program effectiveness and shapes teachers' beliefs about inclusive education. Although PD programs can lead to positive changes, fostering meaningful beliefs about academic inclusion may require more sustained interventions. Future initiatives could include initial teacher interviews to better tailor the program and emphasize the potential of SEND students. Additionally, ongoing qualitative research is essential to deepen the understanding of teachers' perspectives and develop PD programs that prepare teachers for diverse classrooms.

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How teacher education program experiences affect pre-service mathematics teachers' designated teacher identities?

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This study aims to explore the impact of experiences in a teacher education program on the designated teacher identities of pre-service mathematics teachers upon the completion of their final year. Participants were asked to share the three most influential experiences during the program that shaped their understanding of the teacher identity they aimed to develop or avoid. Analysis of the narratives written by the pre-service teachers revealed that 40.4% of the narratives (69) were related to developing didactical expertise, 35.7% (61) to pedagogical expertise, 20.5% (35) to self-referential characteristics (attributes not directly related to mathematics teaching), and 3.5% (6) to subject matter expertise. The narratives indicated that pre-service teachers' designated identities were primarily influenced by the behaviours and teaching methods of their instructors or the teachers they observed during their practicum.

Keywords: Designated identity, pre-service teachers, narrative, teacher education program.

Introduction

The process of teacher identity development often begins with observing teachers during the pre-service teachers' own years as students (Flores & Day, 2006). This development continues within teacher education programs through both cognitive and affective growth related to mathematics teaching (Van Zoest & Bohl, 2005). Studies focusing on the identity development process in teacher education programs are crucial for determining the extent to which these programs support the development of teacher identities aligned with national educational policies (Beauchamp & Thomas, 2009). Such studies are particularly significant when there is a discrepancy between pre-service teachers' prior educational experiences and those encountered in teacher education programs. In the context of this study, pre-service teachers primarily come from teacher-centered educational backgrounds but are expected to experience mathematics education in a more learner-centered manner through their program experiences. Therefore, this study focuses on the mathematics teacher identities that pre-service teachers aim to develop upon completing their teacher education program.

Theoretical background and related literature

Sfard and Prusak (2005) theorized identity from a narrative perspective and defined it as "collections of stories about persons or, more specifically, as those narratives about individuals that are reifying, endorsable, and significant" (p. 16). They conceptualized identity as two-dimensional, comprising actual identity, which consists of "narratives about the actual state of affairs," and designated identity, which consists of "narratives presenting a state of affairs that is expected to be the case, if not now then in the future" (Sfard & Prusak, 2005, p. 18). They described learning as the process of closing the gap between these two types of identity. Designated identities are crucial in guiding individuals' behaviours and determining the direction of learning (Sfard & Prusak, 2005).

The role of individuals' experiences in their social communities is significant in developing this type of identity. Previous studies have shown that individuals' designated identities are shaped by the expectations of their society and families (e.g., Lutovac & Kaasila, 2014; Sfard & Prusak, 2005) or by the expectations of their working communities (e.g., Haggarty & Postlethwaite, 2012). Consequently, the effect of pre-service teachers' experiences in teacher education programs on the development of their designated teacher identities becomes crucial. However, few studies specifically focus on designated mathematics teacher identity (Arslan & Haser, 2024). Strong designated identities are difficult to change (Sfard & Prusak, 2005) and can be decisive in pre-service teachers' careers, especially when aligned with educational policies, as they shape the effort teachers invest in achieving their goals (e.g., Arslan et al., 2022).

Research purpose

The purpose of this study is to investigate how the designated identities of pre-service mathematics teachers are shaped by their experiences upon completing four years in their teacher education program. In line with this purpose, the following research question is addressed: How do teacher education program experiences affect pre-service mathematics teachers' designated teacher identities?

Methods

Participants and context

The participants of this study were fourth-year pre-service mathematics teachers enrolled at a public university in Türkiye. The teacher education program spans four years, aiming to qualify them to teach mathematics in middle schools (grades 5-8). In the first year of the program, the curriculum is primarily focused on educational sciences, general culture, and pure mathematics courses. From the second year through to the end of the fourth year, pre-service teachers take various courses on mathematics teaching methods. These courses primarily focus on learner-centered approaches and activity-based learning in alignment with the national mathematics education policy. Pre-service teachers have their only teaching practicum experience in the final year, spending one full day (six hours) per week in schools each semester. While most of their time is dedicated to observation, they are responsible for teaching at least two lessons per semester.

Data collection and analysis

The data for this study were collected during the final week of the fourth year in the teacher education program. All 57 pre-service teachers in the program voluntarily participated in the study. The pre-service teachers were asked a single open-ended question: "Considering all your experiences in the teacher education program, what are the three experiences that have most influenced your desire to become or avoid becoming a certain type of teacher?" The pre-service teachers did not provide their names or any identifying information in their responses to this question. Therefore, it is assumed that they shared their honest views in their narratives.

The analysis of the data was conducted in two stages. In the first stage, it was examined whether the pre-service teacher's statement could be considered as reflecting their designated identity. In this regard, it was checked whether the designated identity statements (such as can, cannot, want, should,

have to, and must) as defined by Sfard and Prusak (2005), were present in the narratives. It was determined that there was a total of 171 narratives satisfying Sfard and Prusak's (2005) criteria for designated identity statements. In the second stage, it was identified which domain of teacher identity was present in these designated identity narratives. An analytical framework developed by Beijaard et al. (2000) and refined by Löfström et al. (2011) was used for this stage (see Table 1). Each designated identity narrative was coded for the domain it pertained to and the reasons influencing this effect. While this analytical framework may appear familiar due to the researcher's prior studies (e.g., Arslan, 2023; Arslan & Haser, 2024), the methodological limitation of the analysis being conducted by a single researcher should be acknowledged.

Table 1: Teacher identity domains used in data analysis

Domain	Description (focusing on…)	Example of designated identity narrative
Subject Matter Expertise (SME)	Teacher's knowledge of content	A teacher cannot have a limited knowledge of the content that they teach.
Didactical Expertise (DE)	Teacher's teaching methods specifically to support students' learning	I want to support my students' meaningful learning in mathematics by using appropriate methods.
Pedagogical Expertise (PE)	Teacher's relationship with students and role in their development as human being	I want to be a role model for my students through my behaviour.
Self-Referential (SR)	Teacher's characteristics and features that are not directly related to teaching (such as being smart, patient, amusing)	A teacher should be funny so that students are not afraid of them.

Findings

The analysis revealed that pre-service teachers' designated identities were affected by their teacher education program experiences mostly in didactical and pedagogical expertise domains. Narratives in these two domains comprised 76 percent (130 narratives out of total 171) of all narratives (see Figure 1).

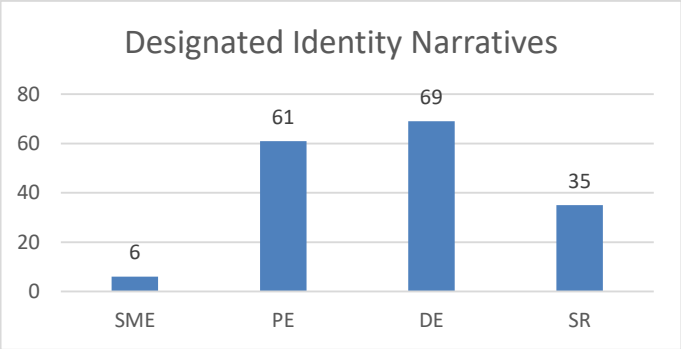

Figure 1: Analysis results of designated identity narratives

For the remaining narratives, self-referential characteristics were the most common category, while only a few narratives addressed the impact of the teacher education program on subject matter expertise. The details of the analysis for each category are provided below.

Didactical Expertise Narratives

Didactical expertise was the most frequently mentioned category, appearing in 69 of the pre-service teachers' designated identity narratives (40.4 percent). Pre-service teachers frequently referred to their experiences in the teacher education program that led them to adopt learner-centered mathematics teaching practices in their designated identities. These experiences were sometimes related to specific instances in which they participated in learner-centered activities and, at other times, to their general experiences throughout courses designed to introduce learner-centered methodologies to pre-service mathematics teachers.

> As a mathematics teacher, I want to teach in a way that allows students to learn by doing. In one of our courses, we went to a village school to do an activity with the students. I designed an educational game to teach the properties of geometrical shapes. Observing how much the students enjoyed and learned from the activity was amazing. It was the best day of my university life.

> I don't want to just lecture; I want to conduct activities that make the content meaningful for students. I had teachers, like [names], who showed me how a mathematical topic can be taught using different methods. They broadened my perspective on teaching with various activities and materials. They will always be my role models.

On the other hand, some pre-service teachers shared their negative experiences in courses where the instructor used teacher-centered methods, limiting active student participation. These pre-service teachers emphasized that they do not want to adopt this approach when they begin teaching.

> I don't want to teach by simply lecturing from the board. In one of our second-year courses, [course name], our teacher would just open the presentation on the board and read from it. This method did not help me learn anything. Without active participation, it is difficult to stay focused on the lesson. I never want to be that kind of teacher.

A significant portion of the experiences in this category pertained to the teaching practicum. Pre-service teachers highlighted positive outcomes from their observations and practical applications of learner-centered methods in mathematics. Conversely, they also described negative experiences associated with teacher-centered approaches.

> During my practicum, I observed two different teachers. The teacher I observed in the first semester constantly engaged students with activities and used various materials to make concepts concrete. However, the teacher in the second semester would just open the topic on the smart board and solve problems from there. The students in the first semester were much more engaged and learned better. That's the kind of teacher I want to be.

Pedagogical Expertise Narratives

In 61 narratives (35.7 percent), pre-service teachers discussed their experiences in the teacher education program that shaped their identities in terms of developing pedagogical expertise. They highlighted both positive and negative memories with their instructors and emphasized the importance of teachers developing close relationships with their students, which they aspire to achieve.

> As a mathematics teacher, I want to be someone whom my students feel comfortable with, can share their concerns with, and see as a friend. When I first came to university, I had a prejudice that instructors would be strict and that communication with them would be difficult. However, my own instructors were not like that; they treated me like a friend, which made me admire them. [...] During a period when my motivation was low, one of them called me to their office, discussed the reasons with me, and tried to help. The way my instructors, whom I nervously knocked on their doors, treated me like this was very valuable to me.

Similar to the narratives of didactical expertise, practicum experiences significantly influenced their designated identities in terms of their intentions regarding their relationships with future students. Based on their observations of student-teacher relationships during their practicum, they expressed their goal to develop close relationships with students throughout their teaching careers.

> I aspire to be a teacher who maintains strong, close relationships with students. This desire stems from my observations during practicum, where I had the opportunity to observe two different types of teachers. One teacher only communicated with students on an academic level. The other teacher, however, shared aspects of daily life, students' lives, and their own life outside of class. I never saw students eagerly anticipating the first teacher's classes, but they would be happy when the second teacher's class was about to start. I want my students to smile when coming to my class.

Self-Referential Narratives

Pre-service teachers focused on the self-referential characteristics that a teacher should have in 35 narratives (20.5 percent). These characteristics can be summarized as follows: being respectful towards the profession and students, having high motivation to teach, being fair, honest, energetic, clearly communicating expectations with students, and being open to professional development. Among these characteristics, the notion that a teacher should be fair emerged most prominently. This idea was influenced by various experiences within the teacher education program where teacher candidates perceived themselves as not being treated fairly. Based on these experiences, there was an emphasis on their desire not to become teachers who treat their students unfairly.

> I will be teaching at middle school level, and I believe fairness is crucial for this level. I want to treat my students fairly and equally. [...] One of my instructors at university [name], had favourite students. They received more attention and communication. It felt like the class was conducted primarily with those 3-5 students.

> I aspire to be a fair teacher who treats all my students equally. I have seen some instructors not act this way, which negatively affected me and decreased my interest in those classes. Therefore, I will never treat one student differently from others.

Another common self-referential characteristic was the aspiration to be a teacher eager for professional development to enhance student learning. This seemed to be influenced by their observations of some practicum teachers. Pre-service teachers positioned themselves differently from those teachers who did not demonstrate a commitment to ongoing professional development.

> I want to be a teacher who continually develops, especially in terms of educational technologies. My practicum teacher was not like this. When I used technology in my own lessons, it captured

the students' attention, and they were very engaged. Therefore, I believe a teacher should be open to self-development.

Subject Matter Expertise Narratives

Developing subject matter expertise was mentioned only six times (3.5 percent) in pre-service teachers' designated identity narratives. In these narratives, some pre-service teachers referred to the subject matter knowledge of their instructors from the teacher education program.

> I want to be very proficient in my content area. I can give two examples of such teachers at the university. When I was in their [names] classes, I could easily recognize that they truly knew what they were teaching. That made me admire them. A teacher should have a thorough understanding of the content they teach.

Some other pre-service teachers referred to their practicum teachers, both positively and negatively, regarding their subject matter expertise. Being able to answer all the questions from students was perceived as an indicator of subject matter expertise by these pre-service teachers. They believed that this capability influenced classroom management, based on their practicum observations.

> I want to be very knowledgeable in mathematics and be able to solve any kind of questions coming from students. [...] Our practicum teacher [name] was that kind of teacher. I observed that when you have such knowledge, students show respect to you and your lesson as well.

> I want to know every aspect of the mathematical content that I will teach and respond to any related questions that students ask. [...] S/he [Name] could not answer some questions coming from successful students. When you cannot answer the questions from students, they do not respect you. This negatively affected classroom management as well. Therefore, I want to be very proficient in mathematics.

Discussion

This study examined how pre-service mathematics teachers' designated teacher identities, the identities they aspire to develop or avoid, are influenced upon completing their teacher education programs. The findings of this study corroborate earlier research by demonstrating that developing didactical expertise is a primary component of pre-service mathematics teachers' designated identities (see Arslan & Haser, 2024). This result is anticipated, given that the teacher education program comprises numerous courses focused on learner-centered methods aimed at fostering students' meaningful learning in mathematics. Many pre-service teachers underscored their experiences within the program, expressing their aspiration to adopt these learner-centered approaches in their teaching. However, a notable number of narratives also highlighted the prevalence of traditional, teacher-centered methods observed among university instructors and practicum teachers. The pre-service teachers indicated their intention to avoid such approaches. Considering the objective of cultivating mathematics teacher identities aligned with learner-centered practices, the existence of conflicting teaching models within the program raises concerns about its structure and coherence.

It has been observed that the development of pedagogical expertise holds significant importance in the designated identity narratives of pre-service teachers. This focus may be influenced by their view that middle school years are crucial for students' behavioural development (Kasten et al., 2014). Additionally, the teacher employment policy suggests that their initial teaching assignments will probably be in schools with socio-economically disadvantaged student populations. This leads them to believe that these students will have a greater need for teachers who can build strong connections with them (Arslan & Haser, 2024). Some of the experiences that triggered this desire were related to their own positive or negative experiences with university instructors, where they positioned themselves as students, while others were based on their observations during their practicum in schools. Considering that establishing close relationships with students is an essential part of teacher identity development (Van Putten et al., 2014), these findings can be seen as desirable outcomes.

Emphasis on aspiring to be a fair teacher in narratives concerning self-referential characteristics has been notable. Particularly, these aspirations often stem from negative experiences in certain courses within the teacher education program, which may be viewed unfavourably. To prevent such situations in teacher education programs, pre-service teachers should be encouraged to freely express their experiences through anonymous narratives from the outset of their teacher education program, and necessary measures should be implemented. Otherwise, the feeling of being treated unfairly could potentially hinder their development of teacher identities throughout the program.

For effective mathematics teaching, it is evident that a deep understanding of mathematics is necessary. However, interestingly, only a few narratives were focused on developing subject matter expertise. This situation may stem from participants' perceptions that they already possess a profound knowledge of mathematics suitable for teaching at the middle school level. Nevertheless, some narratives indicated that pre-service teachers interpreted subject matter expertise not as understanding mathematics deeply and meaningfully, and establishing relationships between concepts, but rather as providing quick and accurate answers to students' questions. Therefore, future research could delve deeper into understanding what developing subject matter expertise means from the perspective of pre-service teachers.

Across all domains, narratives highlighted the significant impact of pre-service teachers' practicum experiences on their designated identities, reinforcing studies that emphasize the crucial role of practicum teachers in identity development (e.g., Arslan, 2023). Future research could further explore how both actual and designated identities evolve during the practicum, offering a broader perspective on teacher identity development. Furthermore, examining how actual and designated identities develop in practicum contexts where pre-service teachers have more opportunities for practice than those in this study could provide valuable insights into teacher identity development.

In summary, this study aimed to contribute to the literature by examining the impact of teacher education programs on the development of pre-service mathematics teachers' designated teacher identities. The analysis of pre-service teachers' retrospective experiences revealed that the dimensions of didactical and pedagogical expertise were most significantly influenced in their designated teacher identities. Future research could investigate the longitudinal development of pre-

service mathematics teachers' designated identities throughout their teacher education programs to gain deeper insights and inform necessary adjustments in these programs.

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Teacher knowledge and the incorporation of inclusivity in a primary level generalisation problem

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This case study analyses the carrying out of a primary level generalisation problem with a neurodiverse class of seven students (aged 9 to 12), one with Autism spectrum disorder, in a rural school. The lesson was videorecorded and subjected to a content analysis of the knowledge deployed. The results demonstrated that the teacher's specialised and pedagogical content knowledge were interconnected with the three inclusivity principles of the Universal Design for Learning framework. More specifically, we identified knowledge descriptors specifying teaching strategies for generalisation problems which were found to be closely related to inclusive practices for promoting comprehension, with one such connection especially strong.

Keywords: ASD, problem-solving, generalisation, inclusion, teachers' knowledge.

Introduction

This presentation draws on a doctoral thesis (Chico, 2024) which researches educational practices within the domain of problem-solving to support students with Autism spectrum disorder (ASD). In 2022, a problem-solving workshop for students with ASD was inaugurated, using teaching resources developed by TEACCH (Treatment and Education of Autistic and related Communication Handicapped Children). With the aim of translating the research results into classroom practice, we observed a series of problem-solving sessions in which primary teachers implemented problems from the workshop. This study focuses on one of these sessions, involving a generalisation problem. It analyses the specialised knowledge and principles of inclusivity demonstrated by the teacher, which we believe can contribute to an inclusive approach to mathematics teaching in teacher education.

Theoretical framework

We consider generalization as the process by which regularity is perceived over a series of instances, and hence involves establishing relationships between variables (Blanton, 2008). Primary school students use different registers to discover and express regularities (Blanton, 2008; Pinto & Cañadas, 2021), deploying different forms of thinking, from recursive to functional (Stephens et al., 2017). Students with ASD bring neurodiversity to the typical classroom, and their particular characteristics should be taken into account. Reviews of research into mathematics and ASD underline the important role of resources and structured teaching (Gevarter et al., 2016). Root et al. (2018) illustrate the benefits of concrete representations such as visual organizers in problem solving. Such proposals recognise the reliance on visual skills associated with ASD, as well as the difficulties related to executive functioning (organization, prediction and self-regulation), text comprehension and global processing (Root et al., 2018), which are key in problems involving generalisation.

Teacher education needs to equip prospective teachers with the tools to meet their responsibility to teach all their students across the spectrum of neurodiversity (Lambert, 2018), which means covering

not only knowledge of mathematical content and pedagogy, but also knowledge of inclusive practices and methodology. This study analyses the classroom practice of a teacher working with a neurodivergent pupil in order to identify his knowledge and the implications for inclusive teaching.

The Mathematics Teachers' Specialised Knowledge (MTSK) model (Carrillo et al., 2019) is used for analysing the teacher's knowledge. Problem solving, such as used in the lesson under analysis, mobilises mathematical knowledge (MK) of topics (KoT), connections between topics (KSM) and mathematical practices (KPM). At the same time, it involves the deployment of pedagogical content knowledge, which encompasses knowledge of teaching mathematics (KMT), the characteristics of learning (KFLM) and notions of learning standards (KMLS) (see Figure 1). For example, solving a problem involving generalisation might activate knowledge about different registers of representation (KoT), the sequencing of topics (KMLS), the value of examples in looking for regularities (KMT) and the use of heuristics as a tool for solving problems (KPM), among other aspects.

Domains	Subdomains	Categories	Subcategories
Mathematical knowledge	KoT	Phenomenology and applications	
		Definitions, properties and foundations	
		Registers of representation	
		Procedures	How to do something
			When to do something
			Why do something this way
			Characteristics of the result
	KSM	Connections based on increased complexity	
		Connections based on simplification	
		Transverse connections	
		Auxiliary connections	
	KPM	The practice of demonstrating	Doing demonstrations
			Methods and types of demonstrating
			Functions of demonstrating
		The practice of defining	Characteristics of definitions
		The practice of problem solving	Problem-solving strategies
		The role of mathematical language	Role of symbols
			Use of mathematical language

Domains	Subdomains	Categories	Subcategories
Pedagogical content knowledge	KMT	Theories of mathematics teaching	Personal
			Institutional
		Teaching resources (material and digital)	
		Strategies, techniques, tasks and examples	
	KFLM	Theories of mathematics learning	Personal
			Institutional
		Strengths and weaknesses in learning mathematics	
		Modes of student interaction with mathematical content	
		Emotional aspects of learning mathematics	
	KMLS	Expected learning outcomes	
		Expected level of conceptual and procedural development	
		Sequencing of topics	

Figure 1: MTSK domains, subdomains and categories (Escudero-Ávila, 2023)

At the same time, we draw on the principles of Universal Design for Learning (UDL) (CAST, 2018) in order to capture inclusive teaching. UDL is a framework of teaching practices which aim to optimise teaching and learning for all people (Meyer et al., 2014). It offers a range of suggestions for presenting information, motivating learners, and helping them to process and express their learning (across the three core principles of engagement, representation, and action and expression, Figure 2). TEACCH methodology (Schopler et al., 2013) is fully consistent with these principles.

UDL has been linked to Mathematics education (Lambert, 2018; Root et al., 2020), whereas this study emphasises the specialised knowledge and inclusive actions teachers require to foster inclusive mathematics in the classroom. The aims of the study were to: 1) identify possible connections between elements of specialised knowledge and the principles of UDL in a teacher's management of a generalisation problem solving task; and 2) obtain descriptors of enacted knowledge underpinning an inclusive approach to a problem-solving task involving generalisation.

ENGAGEMENT (E)	REPRESENTATION (R)	ACTION AND EXPRESSION (AE)
Recruiting Interest (**E.RI**)	Perception (**R.P**)	Physical Action (**AE.PA**)
E.RI.1. Optimize individual choice and autonomy **E.RI.2.** Optimize relevance, value, and authenticity **E. RI. 3.** Minimize distractions	**R.P. 1**. Offer ways of customizing the display of information **R.P. 2.** Offer alternatives for auditory information **R.P. 3.** Offer alternatives for visual information	**AE.PA.1**. Vary the methods for response and navigation **AE.PA.2.** Optimize access to tools and assistive technologies
Sustaining Effort & Persistence (**E.SEP**)	Language & Symbols (**R.LS**)	Expression & Communication (**AE.EC**)

E. SEP. 1. Heighten salience of goals and objectives **E. SEP. 2.** Vary demands and resources to optimize challenge **E. SEP. 3.** Foster collaboration and community **E. SEP. 4.** Increase mastery-oriented Feedback	**RLS. 1**. Clarify vocabulary and symbols **RLS. 2**. Clarify syntax and structure **RLS. 3.** Support decoding of text, mathematical notation, and symbols **RLS. 4**. Promote understanding across languages **RLS. 5**. Illustrate through multiple media	**AE.EC.1.** Use multiple media for communication **AE.EC.2.** Use multiple tools for construction and composition **AE.EC.3.** Build fluencies with graduated levels of support for practice and performance
Self-Regulation (**E. SR**)	Comprehension (**R.C**)	Executive Functions (**AE.EF**)
E. SR. 1. Promote expectations and beliefs that optimize motivation **E. SR. 2.** Facilitate personal coping skills and strategies **E. SR. 3.** Develop self-assessment and reflection	**R.C.1.** Activate or supply background knowledge **R.C.2.** Highlight patterns, critical features, big ideas, and relationships **R.C.3.**Guide information processing and visualization **R.C.4.** Maximize transfer and generalization	**AE.EF.1.** Guide appropriate goal-setting **AE.EF.2.** Support planning and strategy development **AE.EF.3.** Facilitate managing information and resources **AE.EF.4.** Enhance capacity for monitoring progress

Figure 2: Graphic organizer summarising the UDL principles, guidelines and checklist (CAST, 2018)

Methodology

The study took an exploratory-descriptive design, in which a novice teacher in a rural school, Andrea (pseudonym), was observed giving a lesson based on a generalisation problem. The class consisted of seven primary students (aged 9-12), among whom was included a pupil with ASD.

The lesson was thriven on the 'Archaeologist problem' (Figure 3), which involved generalising from two variables, the height of the steps and the number of boxes required, which were related according to the function f(x)=(x(x+1))/2. Some TEACCH resources (Figure 4) were provided as structured and visual support, considering ASD features (Root et al., 2018). Both the problem and resources had been tested previously in a workshop with students with ASD (aged 11-12) (Chico et al., 2023).

An archaeologist wants to clean the museum's dinosaur models. She doesn't have a ladder, but she has found a pile of boxes, all the same size, which she can stack to reach the dinosaurs. Each box is 1 metre high, and she can stack them as in the picture. How many boxes does she need to reach the height of the Velociraptor (2m), the Stegosaurus(4m), the Gigantoraptor (8m), and the T-Rex (10m)?

Figure 3: The Archaeologist Problem

Figure 4: TEACCH material: Eva foam pieces, organiser and Velcro stair pictograms

In the week leading up to the lesson, the researchers gave Andrea a teaching guide which included: (a) a brief description of the key characteristics of ASD pupils; (b) a list of the advantages and limitations of the TEACCH material provided; (c) guidelines for managing the task, including organising the students, using the resources, and the overall aims of the problem; (d) examples of the strategies employed by the pupils with ASD to tackle the task during the previous workshop. The task, with a flexible, open format, allowed Andrea for decision making and reflection on the students' understanding and needs (Coles & Brown, 2013), so as to relate the content on generalisation with didactic strategies. Therefore, teacher's knowledge and skills might emerge.

The lesson was video recorded and transcribed. This was then subjected to a content analysis (Krippendorff, 2018), paying special attention to the specialised knowledge mobilised by the teacher (according to the categories of Figure 1), and evidence of inclusivity (following Figure 2). This process involved identifying units of information (strings of text sharing semantic content) which provided evidence of the two focal points. This analysis was carried out separately by the first two named authors, who then met to agree on a single version, which was then scrutinised by the third author – experienced in applying MTSK – and the fourth – a specialist in ASD – before a consensus

was reached on the definitive analysis. With respect to the use of the MTSK model, when an element of knowledge pertaining to a particular category was identified, the researchers generated an appropriate descriptor to capture it (examples of these descriptors can be found in the results section).

Once all the units of information had been analysed, we quantified the co-occurrence of instances of specialised knowledge and inclusive practice. This was done by quantifying all units tagged by both frameworks. Hence, by way of example, if a given unit of information was tagged by two descriptors (say knowledge on *Examples for conjecture* and *Counterexamples*) pertaining to a category within the MTSK framework (say, KPM *demonstration*), each occurrence of an item from the UDL guidelines linked to that unit was counted twice — once for each descriptor: two connections between the MTSK category and the UDL checklist would be counted. All such connections were registered in an Excel file and then fed into the open source Sankeymatic programme to produce a Sankey flowchart (Figure 5), representing the co-occurrence and emergent relationships of MTSK and UDL.

Results and discussion

In line with the aims of this study, we will first consider the connections between the teacher's specialised knowledge and instances of the principles of inclusivity (Figure 5).

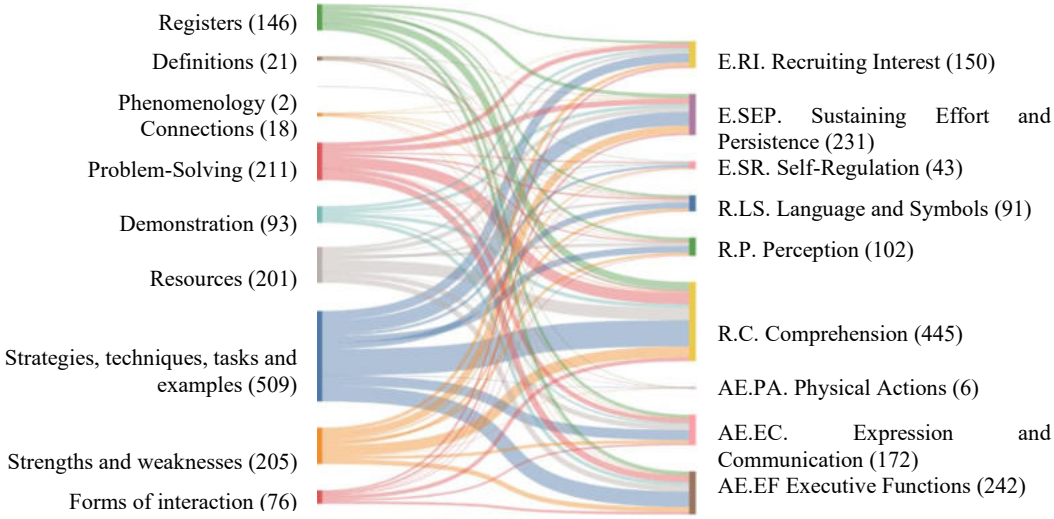

Figure 5: Flow diagram showing MTSK-UDL co-occurrence in Andrea's lesson

As shown in the flowchart, connections are established between categories within all the MTSK subdomains and the 9 UDL guidelines and checkpoints. In terms of the total number of connections in each case, the most frequently occurring categories in MTSK are *Strategies, techniques, tasks and examples*, followed by *Problem-solving strategies*, *Teaching resources*, and *Strengths and weaknesses in learning mathematics*. With respect to inclusive practice, the results indicate that C*omprehension* is the most frequent element within the UDL framework, followed by *Executive functions* and *Sustaining effort and persistence*. The strongest connection is that between Andrea's knowledge of *Strategies, techniques, tasks, and examples* (KMT) and the guideline *Comprehension* (UDL principle of *Representation*). The category *Strategies, techniques, tasks, and examples* also has strong connections with the guidelines concerned with *Executive functions* and *Expression and communication* (principle of *Action and Expression* within the UDL), and those concerning

Sustaining effort and persistence and *Recruiting interest* (within the UDL principle of *Engagement*). In like fashion, Andrea's knowledge of *Teaching resources* (again KMT) and the *Practice of problem solving* (KPM) show a significant connection with the UDL guidelines for *Comprehension*.

In order to consider the results regarding the second aim of this study, a more detailed analysis of the connections between the KMT categories and the UDL guidelines is offered below. The strongest interconnection between Andrea's specialised knowledge and his inclusive practices occurs between descriptors pertaining to *Strategies, techniques, tasks, and examples* (KMT) and the guideline *Comprehension* (UDL principle of *Representation*), the full range of which are shown in Table 1.

Hence, in the brief excerpt below (U170), for example, we can see Andrea's knowledge put into effect by suggesting that students work with smaller more manageable examples, a strategy aimed at encouraging them to formulate and/or check conjectures about the generalisation required for solving the problem. This strategy explores patterns, guides information processing and supports generalisation (R.C.2, R.C.3 and R.C.4, respectively) corresponding to *Comprehension* guidelines:

> Andrea: In the one with three, OK, the area of the square is three by three, only it's not a complete square. Three are missing. That actually matches the height you have searched for. So now I say: Let's see if there's a connection. (U170)

Immediately prior to this extract, Raúl had conjectured that the key to calculating the number of blocks required to reach four high was to consider the blocks as forming part of complete square. Andrea suggested that he consider a set of steps 3 blocks in height so as to put his conjecture to the test. In other words, Andrea suggested working with an example that was smaller but consecutive, enabling him to visualise a complementary set of steps. This strategy can be characterised by two other descriptors corresponding to the MTSK category *Strategies, techniques, tasks and examples*: first, focussing on contiguous terms to encourage the search for regularities; and second, using "complementary steps" to visualise the problem. These strategies support the pupil in his search for patterns and key aspects (R.C.2), help him to calibrate the mental model he needs to process the information (R.C.3) and so maximise generalisation from the results of the example (R.C.4).

A little later in the lesson, Andrea returns to Raúl's conjectures and paraphrases how he had realised that the pattern to calculate the total number of boxes required to construct the steps was not half the square of the corresponding height (U296):

> Andrea: [Indicating the drawing on the board] You realised that working out half of the area (8) doesn't give us the part that's not there. He said that the area of this square which is four by four, has an area of 16. 16! So he said that 16 into two, gives eight. Eight here and eight here [indicating the drawing on the board]. And then he said: No, there aren't eight here, because here there are 1, 2, 3, 4, 5 and 6. (U296)

This extract illustrates Andrea's knowledge of the strategy of paraphrasing in order to draw the class' attention to a pupil's solution and go over its details, and likewise his awareness of the capacity of visual representations to help pupils perceive the pattern when she returns to the complementary steps drawn on the board. In terms of the UDL guidelines, the strategies involve the guidelines *highlight critical features* (R.C.2) and *guide information processing and visualization* (R.C.3).

Andrea is aware that the pictorial register alone is unlikely to be enough for them to discover the pattern, and so recommends the students incorporate the numerical register after they have constructed their set of steps using EVA foam pieces. In this excerpt, Andrea aims to guide the pupils'

processing of the information, and to maximise their generalisations by measuring and annotating the figures they have put together:

> Andrea: OK, look, now I recommend that those who have made their figure, don't take the figure off your desk, and with a pencil, you can start writing here, on your desk, in pencil, either at the top or at the bottom, the number of metres it goes up to. (U100)

Finally, it is interesting to note that Andrea's understanding of the need to attempt non-contiguous terms in the sequence in order to encourage the students to go from recursive strategies to functional ones, is also interconnected with the final three checkpoints in the *Comprehension* guideline:

> Andrea: And if I put that here for you, OK? Alright, Antonio, are you seeing how it progresses? This one is two metres high and is made of three boxes. How many boxes do you need to make one 28 metres high? So, have a think about that. (U215)

Table 1: Descriptors of knowledge of Strategies, techniques, tasks, and examples (KMT) with connections to the UDL checklist of guidelines for Comprehension (principle of Representation)

Knowledge DESCRIPTOR	Checklist in the guidelines for Comprehension			
	RC.1: Activate or supply background knowledge	RC.2: Highlight patterns, critical features, big ideas, and relationships	RC.3: Guide information processing and visualization	RC.4: Maximize transfer and generalization
Importance of the understanding stage		x		
More manageable example to encourage establishing or checking of conjectures		x	x	x
Importance of the variable "complementary steps" to perceive regularity	x	x	x	x
Importance of the variable "boxes for adding" to perceive regularity		x	x	
Capacity of graphical representation to aid pattern perception	x	x	x	x
Need to incorporate numerical register to the pictorial so as to perceive pattern	x	x	x	x
Focus on contiguous terms to encourage search for regularity	x	x	x	x
Focus on contiguous terms to encourage recursive strategies to make generalisations		x	x	
Focus on non-contiguous terms to encourage functional strategies to make generalisations		x	x	x
Paraphrasing as a strategy for drawing group's attention to individual's solution	x	x	x	x
Paraphrasing as a strategy for reviewing a conjecture			x	x

Conclusion

This study of a teacher's lesson on a generalisation problem identified interconnections between their specialised knowledge and aspects of their inclusive approach. The first finding to underline is the large number of co-occurrences between instances of specialised knowledge and inclusion. The most frequently occurring strand of inclusive practices is that of *Representation*, and in particular the pertaining to the *Comprehension* guidelines, namely *Highlighting patterns, critical features, big ideas and relationships* (R.C.2), and *Guiding information processing and visualization* R.C.3).

Andrea's inclusive practices are particularly founded on Pedagogical content knowledge– something to be expected in a neurodiverse primary school lesson– through the categories *Teaching resources* and *Strategies, techniques, tasks and examples* (KMT), and *Strengths and weaknesses* (KFLM). Also, indications of knowledge about generalisation and patterns, mathematical demonstrations, and heuristics for problem solving are evidenced: *Registers of representation* and *Definitions* (KoT), *Problem solving* (KPM). Evidence of specialised knowledge on problem-solving, such as heuristic approaches and solving phases, align with expressions of pedagogical knowledge (Carrillo et al., 2019), concretely, through the numerous, diverse descriptors of *Strategies, techniques, tasks and example.* Previous research into pre-service teachers' specialised knowledge in a generalisation task (Pinto et al., 2023) and the archaeologist's problem (Chico et al., 2023), revealed higher levels of expected and mobilised knowledge in mathematics teaching (KMT) and learning features (KFLM).

This paper has specified how the connections were established by identifying descriptors within the category of *Strategies, techniques, tasks and examples* which corresponded to *Comprehension*, evidencing a link between specialized knowledge (MTSK) and inclusive actions (UDL). Indeed, the analysis has been developed regarding the whole Andrea's session, not only those interventions with the ASD student. Hence, inclusive mathematics teaching is naturally embedded within classroom practices, rather than supplementary measures for students with special needs. Considering the need for approaching inclusion in the ordinary classroom (Root et al., 2020), these findings might contribute to support the advancement of more inclusive teacher education in mathematics. Future research will explore concrete teacher actions permeated by MTSK and UDL perspectives.

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Pre-service teachers' professional development and mathematics large-scale assessment

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This research aims at investigating the knowledge and beliefs of a sample of Italian secondary school mathematics pre-service teachers (PSTs) regarding the INVALSI national large-scale tests. The units of analysis are the PSTs' answers to an online questionnaire, in which some items taken from the past INVALSI national tests are proposed and discussed individually by the participants, and the collected data are framed with the Mathematics Teacher Specialised Knowledge (MTSK) model (Carrillo-Yañez et al., 2018). The main purpose of the research is to identify the educational needs of the study sample, to then use them as a starting point for the construction of a model for teachers' professional development based on an innovative bottom-up approach. Results show evidence of beliefs and many different aspects of PSTs' knowledge, categorised in the Pedagogical Content Knowledge (PCK) domain of the MTSK model.

Keywords: Pre-service teacher education, Large-Scale Assessment (LSA), MTSK model.

Rationale

A new field of research has been emerging in recent years, aimed at investigating how to integrate results, methods, theoretical frameworks and findings from Large-Scale Assessment (LSA) into the actions of teachers, teacher educators and researchers, both in the school and classroom context and in the broader one of educational institutions (De Lange, 2007; Meinck et al., 2017). My research belongs to this framework and fits within the strand of research linking LSA with teacher professional development (Campbell & Levin, 2009, Ferretti & Santi, 2023). Several studies in this area have been carried out from macro-phenomena that emerged at the Italian national level about INVALSI LSA (Ferretti et al., 2022); starting from what is already known, the focus of my research is to investigate Italian pre-service teachers' (PSTs) knowledge and beliefs regarding national LSA to identify their educational needs, inspired by studies already conducted in teacher education contexts (Montes et al., 2019), with a view to contributing to the development of a new model of teacher education. My study is, indeed, part of a larger project, the Italian National Research Project "*Mathematics standardized assessment as a tool for teachers' professional development*", which involves several researchers from different Italian Universities and whose aim is to create a model for teachers' professional development, consisting of theoretical and operational tools, starting from the results collected from the Italian national assessment tests INVALSI. The INVALSI tests are administered nationally by the National Institute for the Evaluation of the Education and Training System; the administration involves students at all school levels (primary school grades 2 and 5, low secondary school grade 8, and high secondary school grades 10 and 13). The framework within which they are developed is consistent with the Italian Ministerial Curricular Guidelines and with the main results of research in mathematics education. All the information and data collected from the INVALSI tests are disseminated and made public on a regular basis and each school receives its own data processed and compared with different benchmarks. Having such information available, it is possible to use the data

provided by INVALSI as interpretative tools for some aspects of the mathematics teaching-learning processes specific to the Italian context, also and above all with a view to teacher professional development. In contrast to the traditional (top-down) impact of LSA, the innovative idea of the project in which my research is framed is to provide these instruments for a bottom-up approach, in which the results are used directly by pre- and in-service teachers, also with a view to integrate data from LSA with a formative perspective.

Thus, the research question this study aims to address is: *What are the knowledge and beliefs of pre-service Italian mathematics teachers regarding the national INVALSI standardized assessment tests?*

Theoretical framework

The main purpose of the Italian National Research Project to which my study belongs is creating a model for the teachers' professional development, inspiring the one outlined in Ferretti et al. (2020). The model aims to enhance specialized knowledge for teaching mathematics. To ensure an informed and conscious construction of the model and to meet the real training needs, a questionnaire was created. The design of the questionnaire and the analysis of the data are framed within the Mathematics Teacher Specialized Knowledge (MTSK) model (Carrillo-Yañez et al., 2018). In the MTSK model, mathematics teacher's knowledge consists of the two domains of Mathematical Knowledge (MK) and Pedagogical Content Knowledge (PCK), and the two areas are interrelated by the teachers' beliefs, both about mathematics and its teaching-learning processes. The main aim of the research phase addressed in this paper is investigating the knowledge and beliefs of Italian teachers regarding the mathematics teaching and learning process, rather than focusing strictly on the mathematical knowledge, so the data analyzed in this study are mainly framed within the subdomains of PCK. They are: *Knowledge of Mathematics Teaching - KMT* (concerning theoretical knowledge of mathematics teaching, i.e. strategies, resources, teaching theories, etc.); *Knowledge of Features of Learning Mathematics - KFLM* (knowing, in general, how students interact with mathematics while also considering affective aspects); *Knowledge of Mathematics Learning Standards - KMLS* (knowledge of the national indications for curricula, for any school grade).

Methods

The research tool used in the first phase of the study is a pilot questionnaire, declined into different versions containing similar questions but formulated according to the group of participants involved (pre- and in-service teachers) and the school level in which they are teaching/will be teaching (focusing on grade 5, 8 and 10). It contains questions on several INVALSI mathematics items, taken from the GESTINV database (www.gestinv.it) and chosen because they are indicative of national educational macro-phenomena, namely items where the percentage of correct answers is very low and the distractor choices can be framed within well-known constructs in mathematics education. There are also questions that can be defined as "transversal", aimed specifically at investigating the participants' perceived purpose and usefulness with respect to INVALSI tests and the impact of these tests on their teaching practice (or future teaching practice). The questionnaire was administered online by using a Google Form: in this way, it was possible to reach a larger and more varied sample of pre- and in-service teachers; regarding the PSTs, this was also important because it allowed us to reach participants from different universities, that were not influenced by the same mathematics

didactics courses attended. Moreover, the choice of this tool was also useful in the data analysis phase; in fact, in addition to the open questions, participants were also asked several closed-ended questions on a Likert scale: using the form, it is possible to have an immediate analysis of the participants' response rates.

Table 1 is an example of a part of the questionnaire for secondary school PSTs, whose responses will be discussed in the following sections.

Table 1: Question posed to PSTs and related to the item 15 of the 2011 INVALSI test, grade 10, proposed in the questionnaire

Consider the following INVALSI item proposed to the secondary classes of the Secondary School. **D15. Dividere un numero per 0,2 è lo stesso che moltiplicarlo per** ☐ A. $\frac{1}{5}$ ☐ B. $\frac{1}{2}$ ☐ C. 2 ☐ D. 5 Translation (by the author): "*Dividing a number by 0.2 is the same as multiplying it by…*"
1. In your opinion, what does this item predominantly evaluate?
2. The correct option is option D. What do you think has been the most popular wrong option nationwide? Why?
3. On a scale of 1 to 4, how consistent do you think the item is with the National Curricular Guidelines for High Schools and the Guidelines for Technical and Professional Institutes? (1 = not at all, 4 = completely)
4. Imagine you are teaching in a grade 10 class. On a scale of 1 to 10, how difficult do you think this item is for students? (1 = very easy, 10 = very difficult)
5. Imagine you are teaching in a grade 10 class. On a scale of 1 to 4, how influential do you think the wording of this item is in determining the difficulty of the item for students? Motivate your answer. (1 = not at all, 4 = completely)
6. Imagine you are teaching in a grade 10 class. On a scale of 1 to 4, how appropriate do you think the item is for assessing student learning? (1 = not at all, 4 = completely)

7. Would you use this item in the classroom? If yes, why and how? If not, why?

INVALSI item 15 was chosen because close to the Italian school curriculum: as stated in the National Guidelines, among the specific learning objectives, it is mentioned that "The student will develop their skills in calculation [...] with rational numbers both in fraction form and in decimal representation" (MIUR, 2010). Hence, the theoretical knowledge required for the correct resolution of this item involves knowledge of the concept of rational numbers and the required competence is about grasping and managing the various semiotic representation of rational numbers (Duval, 2006). In addition, it is a closed-answer item and, thanks to the GESTINV database, national percentages are available and these show that the item represents a macro phenomenon at national level (as described in the previous paragraph). The percentage of correct answers is very low (only 24.4% of students choose the correct option D) and the most chosen distractor is option A, with 46.1%, followed by option B with 20.2%. It can be interpreted that the students who answered A only focused on the numerical part of the problem and thus thought they only had to carry a treatment (in the sense of Duval) between the decimal representation of the number and its representation as a fraction of integers, without considering the operation verbally requested in the assignment. Those who have chosen option B also acted in the same way, but they also made a mistake in treatment.

Questions asked to PSTs allow us to collect information about the PCK sub-domains of MTSK model. Question 1 is aimed at revealing PSTs' cognition about the mathematical knowledge and competence that a student must have acquired to answer the item correctly; thus, highlighting if and what aspects of the KMT emerge from the sample. The purpose of question 2, on the other hand, is to explore which, if any, didactic constructs are used to interpret possible mistakes made by students and, in this case, we are within the KMT subdomain. Question 3 investigates another subdomain of PCK, i.e. KMLS; PSTs are asked whether the item is consistent with the National Guidelines or not, so we want to find out if they are informed about the national curricula. Questions 4 and 5 are still intended to bring out different aspects of KFLM; in particular, by asking for reasons about the choice of the score given on the Likert scale in question 5, it is also highlighted any beliefs of the PSTs concerning the students' learning processes. The last two questions are aimed at investigating what other aspects of the KMLS arise and, at the same time, what are the PSTs' beliefs about assessment and about their possible future teaching practices.

The teachers' answers were analysed differently depending on the type of request. Regarding the data collected from the open-ended questions, such as question 1, the units of analysis were divided into categories, to distinguish the different argument given by the PSTs to justify their answers. This approach allowed for an investigation of the aspects that actually emerged in relation to the considered knowledge domain. On the other hand, in the case of closed-ended questions with a Likert scale, such as question 3, the percentages collected and processed directly from the Forms platform were examined to deduce information regarding the knowledge domain of reference.

A total of 20 PSTs (all students enrolled in the master's degree program in Mathematics) completed the questionnaire for high secondary school and their answers are analysed in the next section.

Data collection and analysis

The answers to question 1 can be grouped into different categories depending on students' mathematical competences and knowledge PSTs consider necessary to be able to solve the problem. Most of the participants state that several skills (e.g. being able to carry out the treatment between the decimal and fractional representation of the number) and several knowledge (e.g. being able to carry out operations with rational numbers, with particular reference to division) are required. A few claim that the question mainly assesses only the ability to perform operations with rational numbers, while others identify the criticality only in the management of representations. Responses of this type can reveal an awareness of the theory of semiotic representation registers (Duval, 2006), even though no participant explicitly mentions it in their answers. In addition, consciousness of the intuitive "partitive division" model, which leads students to think that the quotient must be smaller than the dividend (Fischbein et al., 1985), emerges in several answers. An example is the following extract from a PST: "This item aims to make the student realize that a quotient is not necessarily less than the dividend, as one is used to with, for example, divisions between natural numbers." Even in these cases, however, there is no explicit reference to the didactic construct. It can be therefore identified in this data, as assumed a priori, some elements that can be framed into the KMT sub-domain of PCK. It comprehends knowledge (both personal and institutional) specific to the process of teaching mathematics, which stems primarily from research in mathematics education; these include aforementioned constructs implicitly highlighted in the PSTs' answers.

Regarding question 2, it is immediately noticeable that the majority of respondents (70%) state that the most chosen wrong answer among students is exactly the one chosen by the national sample (option A, selected by 46.1% of the students). The reasons given are various, but the main ones can be categorized into (in brackets, the number of PSTs who gave that motivation):

- Inattentive reading/distraction mistakes (2)
- Problems with the wording of the text: the treatment takes place between the two representations, but the operation is not completed because the words related to the division and multiplication operations are written in letters, while the students "give priority" only to the part of request where the numerical data appears (3)
- Misconception linked to the multiplication that increases (2)
- Difficulties in performing the operation correctly (6)

It is observed that, although most of the answers refer to mistakes strictly related to the procedure, other factors are linked to students' erroneous conceptions and/or the wording of the text. In the first case, we can frame those responses as evidence of the KMT subdomain within the MTSK model, because (as in the case of the previous question) the PSTs do not explicitly mention the mathematics teaching theories they are referring to, but they still show that they are aware of criticalities expressed by them (for instance, when they talk about "misconceptions", in the sense of Fischbein). On the other hand, assertions regarding the "problematic" formulation of the text can be framed within the KFLM subdomain. This type of knowledge, indeed, includes the ability to recognize which features of a specific content may be advantageous or disadvantageous to learning. From the PSTs', it emerges that the item's formulation is perceived as unfavourable for students' understanding of the task. There

is also a small percentage of PSTs (15%) who claims that the most chosen distractor is B, since when computing between decimal and fraction there is confusion between the numerical parts of the two representations (2 is present in both "0.2" and "1/2"). These responses can also be classified as KFLM due to the fact that they also refer explicitly to where students have difficulties and how they might act to solve the item, considering the possible obstacles.

Almost all the PSTs (95%) consider the item consistent with the National Curricular Guidelines for High Schools and the Guidelines for Technical and Professional Institutes; 25% of these consider themselves "in agreement" and 70% "very in agreement". This is evidence of a highly developed knowledge within the KMLS domain, which includes knowledge of what students should be able to know and do upon reaching a specific grade level, in accordance with the curriculum established in the given context. In our case, the PSTs show to know what the learning goals of grade 10 are.

Furthermore, the item is generally perceived as difficult for students: that 70% of PSTs choose 6 or more on the Likert scale. Additionally, 65% of them answer question 5 saying that the item's wording may have had a significant effect on its difficulty. The main reason given is that the words "dividing" and "multiplying" express two opposite concepts but are present in the same prompt, thus confusing the students. Compared to what was previously analyzed in the responses to question 2, this finding seems inconsistent. In fact, initially, few PSTs considered the wording of the text as a possible cause of mistake (only 3 among those who identified the most chosen distractor at the national level, and none of the others). After reflecting on question 5, however, the percentage of participants who contemplate it as a factor of difficulty, and consequently the likelihood of making mistakes, rose significantly. These responses can also be framed within the KFLM subdomain, as the PSTs demonstrated an awareness of how students might approach an item based on its phrasing, and how this factor could influence their learning process.

In addition, 80% of PSTs consider the item suitable for evaluating students' learning; this is probably because, being considered very consistent with the indications given at national level for school curricula, it can be considered a useful tool for confirming or not the achievement of the learning purposes foreseen by them. There is hence a sort of belief in the PSTs, which can be expressed as: "if a mathematical question/problem is consistent with national curricula then it can be useful in terms of assessment, regardless of its intrinsic characteristics" (which, in this case, are considered problematic for students by PSTs themselves, as seen in the previous answers). In agreement with this last observation, almost the same 80% says that they would use that item in the classroom: of those who would not use it, 3 out of 4 are those who did not consider it suitable for assessment, and therefore remain consistent with their idea. The main reasons listed for using the item in class are (in brackets, the number of PSTs who gave that motivation):

- overcoming possible misconceptions (3)
- identifying and correcting any lack and/or consolidating previous knowledge (5)
- reflecting on the reasoning needed for the solution (4)
- mastering the management of a number's representations (2).

Most PSTs would let students work individually on the item and then engage them in a collective discussion; alternatively, it would be used to design laboratory activities and/or group work. Others

would also set up an individual class activity on it and/or with the use of technology. Those who claim that they would not use the item during their future teaching practices, on the other hand, express as their main motivation its formulation, which makes it confusing for the students, and unsuitable for evaluating their learning processes. For example, a PST stated: "*I don't like to formulate questions that have tricky discursive texts, like double negations or things like that. That is, if I want to test whether they can translate a fraction into decimal, I ask only that; if instead I want to test if they can invert a fraction, I give them a numerically formulated text*.". Since the respondents to our questionnaire are PSTs who have mostly never had any class experience, their answer to these last questions about classroom practices can be categorised as beliefs they have on them.

Discussion and conclusion

The INVALSI tests can be a key tool for investigating the knowledge and beliefs of mathematics teachers: having the data from the national sample available, it is possible to use them to induce pre- and in-service teachers to think about some crucial aspects related to LSA resources. This is what was done in this research, in which findings suggest that a large part of the knowledge that emerged can be framed in the three subdomains of the PCK described by Carillo-Yanez et al. (2018), as hypothesized during the design phase of the questionnaire. At the same time, several beliefs about students' behaviour during the performance of mathematical tasks (e.g. inattention) and good teaching practices that a teacher should adopt during assessment (e.g. being as explicit as possible about the proposed requests) also arise.

In these terms, it is possible to identify if and what strengths and weaknesses stand out from the PSTs' answers, which can then be used as a starting point for the implementation of the teacher training programme addressed by the Italian National Research Project. For example, from the analysis conducted, it can be observed how important the wording of a mathematical question is considered (implicitly or not) for its correct understanding and performance by the students. Initially, few PSTs identified this as the main cause of students' mistakes, but when their attention was directed to the explicit question 5, the percentage of them claiming that the wording is deliberately misleading and/or can create confusion increased significantly. This must be considered during the design of the training model, in the sense that it would be appropriate to carry out activities aimed at enhancing this awareness, so that it develops as an established future teaching practice of mathematics teachers.

In future studies the intention is also to make a comparison between the responses of PSTs and in-service teachers to the same questionnaire. The aim is to analyse any similarities and/or differences: firstly, to see if and how the classroom experience can influence the development of the different knowledge areas of the MTSK model and, secondly, to identify further educational needs to be met by the teacher professional development model.

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Prospective mathematics teachers' specialised knowledge in response to a task on the density property of rational numbers

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This empirical study examines the knowledge demonstrated by two prospective mathematics teachers in their responses to a task focused on the density property of rational numbers within the real number system. The data are analysed using the Mathematics Teachers' Specialized Knowledge model. The findings reveal key aspects of the teachers' knowledge including their understanding of the density property, their teaching strategies for conveying this concept, and their insights into students' common understandings and misconceptions regarding this property. Furthermore, while not originally anticipated in the task design, the study identified other categories of knowledge. These include the expected level of conceptual development related to density, students' difficulties in grasping this property, and teaching theories. In conclusion, the potential of the task to enhance the knowledge of prospective mathematics teachers in relation to the property of density is highlighted.

Keywords: Specialised knowledge, Prospective mathematics teachers, Real numbers, Density property.

Introduction

Tasks for mathematics teacher training are currently of great interest to the research community due to the important role that they play in the development of knowledge and skills of prospective mathematics teachers. In this sense, according to Ribeiro et al. (2021), task for mathematics teacher training should address the development of specific knowledge required by prospective mathematics teachers. According to the author these tasks could be formulated based on dimensions or models of teacher knowledge. Among the different conceptualizations of teacher knowledge that are intrinsically connected to teaching and learning mathematics, in this work we used the Mathematics Teacher's Specialised Knowledge model (MTSK, Carrillo et al., 2018).

The design of tasks for teacher training based on the MTSK model represents a recently emerging area of research. Climent (2023) provides a comprehensive overview of studies in this field, outlining the characteristics of tasks specifically crafted to support mathematics teacher training. The tasks are organized into three primary groups. The first group encourages prospective teachers to engage in reflective practice by examining classroom scenarios, instructional interactions, or student responses to mathematical tasks. This reflective process is intended to foster prospective teachers deeper understanding of student thinking. The second group of tasks focuses on analysing various facets of teaching practice, including teacher-student interactions, classroom management, and the role of educational resources in teaching. These tasks aim to help prospective teachers to identify essential elements of practice, with an emphasis on offering alternative approaches to teaching. The final group utilizes video excerpts from mathematics classes to facilitate reflection among prospective mathematics teachers. By observing real-life teaching situations, prospective teachers are encouraged

to understand the complexity of classroom situations. In all cases, the MTSK framework provides the basis for both the design and implementation of these tasks.

Additionally, tasks for teacher training based on the MTSK model have primarily been developed within the context of primary education. Thus, further research is needed to investigate the application of MTSK-based tasks in the training of secondary mathematics teachers (Climent, 2023). In response to this gap, the present work proposes a teacher training task designed specifically for secondary education, focusing on the density property of rational numbers.

The density property of rational numbers in the real numbers (henceforth, density property) asserts that given two real numbers, a and b, with $a < b$, there exists a rational number q such that $a < q < b$. The density property implies that the successor of a rational number cannot be determined, distinguishing rational numbers from natural and integers numbers. Vamvakoussi et al. (2011) present several studies in which students of various ages mistakenly assert that there is a finite number of numbers between two rational numbers or incorrectly identify the successor of a specific rational number. Consequently, students' difficulties with the density property stem from their tendency to over-apply the discreteness of natural numbers to rational numbers (McMullen & Van Hoof, 2020). Similar difficulties are reported in the study by Merenluoto and Lehtinen (2004) regarding the understanding of the density property among mathematics teachers. Furthermore, the concept of infinity associated with the density property is introduced (albeit informally) in secondary school teaching: rational numbers are infinite because it is always possible to find another rational number between any two rational numbers (Montoro et al., 2016). However, the idea of an infinity of numbers between two values is counterintuitive and poses difficulties for students and mathematics teachers, as it requires an understanding of actual infinity, a notion that is cognitively and epistemologically complex.

In view of the difficulties that students and mathematics teachers face in grasping the density property, the present study aims to answer the following question: What specialised knowledge do prospective mathematics teachers demonstrate when engaging with a task focused on the density property of rational numbers?

The Mathematics Teachers' Specialised Knowledge

The MTSK model is proposed as a useful tool for understanding the knowledge that mathematics teachers possess or mobilize in their professional activity (Climent & Montes, 2022). The model is structured into two domains of knowledge, each one with its respective subdomains and categories. Additionally, the model describes a domain of teacher beliefs about mathematics, its teaching, and learning. In this work, we will focus on the domains of knowledge according to the descriptions provided by Carrillo et al. (2018) and Climent and Montes (2022).

Mathematical Knowledge (MK) characterizes teachers' knowledge of mathematical content, the structural nature of mathematics based on the idea of connections and the syntactic nature of mathematics. Knowledge of procedures, definitions, properties and their foundations, registers of representation, and phenomenology and applications of mathematical content, are grouped under Knowledge of Topics (KoT). Knowledge about connections between a topic and other mathematical

contents that could be based on increased complexity, based on simplification and transversal or auxiliary connections is included in the Knowledge of the Structure of Mathematics (KSM). The knowledge related to the construction, validation, and communication of mathematical knowledge is named Knowledge of the Practice of Mathematics (KPM) which includes knowledge of practices such as proving, defining, problem-solving, and understanding the role of mathematical language. Teachers' knowledge of the density property, which belongs to the KoT subdomain, is an example of MK.

Pedagogical Content Knowledge (PCK) is described by three sub-domains that characterise what the teacher knows about teaching and learning directly related to mathematics. Knowledge of mathematics teaching (KMT) encompasses three categories of knowledge: Theories of mathematics teaching includes both formal theories established in the research literature and personal theories derived from teaching experience; teaching resources includes both physical and digital resources used in the teaching process; strategies, techniques, tasks and examples includes a range of approaches to delivering mathematical content. The sub-domain Knowledge of Features of Learning Mathematics (KFLM) includes teachers' knowledge of formal and personal theories of mathematical learning; strengths and weaknesses that students exhibit in learning mathematics; understanding of the emotional aspects involved in learning mathematics; and knowledge of the ways in which students interact with mathematical content, recognising the typical and unconventional strategies that students develop in response to specific mathematical concepts, and the language that students use to articulate these concepts. The subdomain Knowledge of Mathematics Learning Standards (KMLS) includes knowledge of how to arrange topics within a course in relation to preceding or subsequent courses (sequencing of topics), knowledge of the mathematical content expected to be acquired within a course or educational level (expected learning outcomes), and knowledge of the depth at which mathematical content should be covered in a particular course or educational level (expected levels of conceptual or procedural development). Teachers' knowledge of students' difficulties in transforming finite decimals into fractions, which belongs to the KFLM subdomain, is an example of PCK.

Method

The data for this study were gathered from a written task proposed in a third-year course within a secondary mathematics teachers' training program at a university in Chile. During the first two years of the program, prospective teachers undergo mathematical coursework covering topics such as Calculus, Real Analysis, Algebra, and Geometry. Subsequently, in the third year, the curriculum focuses on teaching practices and didactics of mathematics encompassing the teaching and learning of numerical systems, functions, and geometry. The task was implemented at the end of one of the didactic courses with a total of 36 prospective teachers (henceforth PTs) as participants.

In this task, which is to be completed individually in written form, a real classroom situation taken from Zakaryan and Ribeiro (2016) is considered. The scenario presented in the written task depicts a mathematics teacher engaging with their eighth-grade students (aged 13-14). Upon posing the question, "Why is the set of rational numbers said to be dense?" most students responded that it is so

because it contains infinite numbers. Given the above situation, the PTs were asked: (1) interpreting the students' response and (2) providing an explanation they would give to correct this response.

In the context of the classroom, the teacher is referring to the density property in terms of the possibility of finding a rational number between any two of them and the impossibility of finding the successor of a rational number. Also, considering the analysis of teacher knowledge in this classroom situation developed by Zakaryan and Ribeiro (2016; 2019), it is expected that PTs provide answers to interpret students' response by referring to misconceptions that students might have about the density property (KFLM, *strengths' and weaknesses in learning mathematics*), students' understanding of density and infinity and the words that students use to talk about infinity (KFLM, *ways in which pupils interact with mathematical content*). For the explanation to correct students' response, it is expected that PTs refer to definitions and properties of the different numerical sets (N, Z and Q) (KoT, *definitions, properties and foundations*), expose counterexamples of infinite sets that are not dense in the real numbers (KPM, *proving practice*) and to propose examples or strategies for teaching the density property (KMT, *strategies and examples*). Additionally, other knowledge in both MK and PCK that could emerge in PTs' responses.

This work forms part of a broader research project that conducts an in-depth examination of the MTSK and the practices demonstrated by a group of PTs. These teachers were selected based on two criteria: the comprehensiveness of their responses and the explanatory power they displayed. Given the limited space and the objective of this work, we will focus on analyzing the responses provided by two of these participants, hereafter referred to as PT1 and PT4.

Results

When addressing to interpret the students' response, PT1 points out the following:

> PT1: The teacher indicates that the rational numbers are said to be "dense" this last word can be interpreted by the students as something quite large, therefore, they assure that this is because the rational numbers are infinite. Additionally, in eighth grade, students can recognize that the natural numbers are found, followed by the integers (composed of the natural numbers and negative numbers), later the rational numbers are found, where the natural numbers, integers, and fractions and decimals are located. Given the above, the students can affirm that the rational numbers correspond to a set that includes the largest number of numbers.

In this intervention, PT1 elucidates that the term "dense" is often interpreted by students as "quite large", yet it is also frequently associated with the concept of infinity. This results in students developing a relationship between density and infinity. This understanding of how students perceive the concept of density illustrates PT1's knowledge of *ways in which students interact with mathematical content* (KFLM). Additionally, PT1 refers to possible students' ideas of that rational numbers as "the largest number of numbers" reflects their tendency to focus on a limited range of numbers based on their existing knowledge. This expected understanding of numerical sets from eighth-grade students is based on PT1 awareness of the way in that rational numbers are introduce in secondary instruction. In other words, knowledge of what students should have learned in previous courses about numerical systems and to what extent, according to the national curriculum. PT1's knowledge of the *expected level of conceptual development* (KMLS) is identified.

Subsequently, to provide an explanation to correct the students' response, PT1 mentions the following:

> PT1: It is important to clarify to the students that the numbers that they know are infinite regardless of the numerical set in which they work. So, I would give the following devolution: Can you tell me about the last number in the set of natural numbers? Then, I would work with the students to build a scheme about the contention of numerical sets. Subsequently, I would ask the following question: Is there another number other than the rational ones? The idea is to find numbers like π or Euler's number, later I would highlight that there are also real numbers which contains the other numerical sets already mentioned.

In the above excerpt, we observe that PT1 presents a question that she identifies as a "devolution", a notion that is part of the Theory of Didactical Situations. PT1 mentions having studied this theory and the notion of devolution in one of their didactic training courses. In this sense, PT1 demonstrates her knowledge of this *formal teaching theory* (KMT). Additionally, PT1 mentions using a scheme to illustrate the concept of the density to students. In this sense, she demonstrates her knowledge of a *teaching strategy* (KMT) of density property.

In the case of PT4, when addressing to interpret the students' response, he states the following:

> PT4: Students should have related the word dense to quantities or cardinals. Then, when they hear that rational numbers are a dense set, they understand it to mean that rational numbers have many numbers or that it is a "large set" because it is infinite.

In the above intervention, PT4 suggests that students understand the word infinite as a quantity, synonymous with many and large. This insight into how students interpret the concept of density demonstrates PT4's knowledge of the *ways in which students interact with mathematical content* (KFLM). Continuing with his response to the task, PT4 refers to the explanations he would give to the students as follows:

> PT4: First, I would make them note that the set of rational numbers is not the only infinite set. The natural numbers, the integers, the irrational numbers are also infinite, and that is not why they are said to be dense in the reals. Then I would talk to them from a more historical-epistemological point of view about the fact that there are infinities that are greater than others and that, although it is difficult to understand, there was a mathematician named Cantor who proved this and that this is the reason why rational numbers are called dense in the reals.

According to the above, PT4 presents the naturals, integers and irrationals as infinite sets. He points out that being infinite does not imply being dense in the real numbers (KPM, *counterexamples*). However, he includes the irrational numbers in his statement without considering that these numbers do in fact fulfil the property. PT4 also recognizes the existence of distinct types of infinity and references proofs associated with the infinities of specific sets (KoT, *properties*). Following this idea, the PT4 refers to the process in which the infinity was constructed as mathematical object referring to the historical-epistemological point of view. The above, is a way to comprehend *students' difficulties* (KFLM) to understanding infinity based on the inherent complexity of the concept.

Discussion

When the two PTs interpret the response given by eighth-grade students to their teacher, knowledge of how students understand the property of density (KFLM) is demonstrated, as was expected in the

designing task. In this response, they coincide in mention students' interpretation of infinity as a quantity or very large number, which is acknowledged as one of the misconceptions students often have about infinity (Medina-Ibarra et al., 2019). In relation to the explanation provided by PTs to address the students' answers, the categories of knowledge about definitions (KoT), counterexamples (KPM) and teaching strategies (KMT) that were anticipated in the context of the designed task were identified.

PT1 demonstrates categories of knowledge was not part of the expectation of knowledge in the development of the task. For instance, the presence of KMLS underscores that learning standards, particularly the national curriculum, is considered as an organizing element of the teaching-learning process. In this context, it is important to note that while density may not be explicitly outlined in the Chilean curriculum as a property to be taught in eighth grade, it is implicitly present in the textbook tasks related to the representation of rational numbers on the number line. It is also noteworthy that PT1 refers to Theory of Didactical Situations (Brousseau, 2002) to address a classroom situation. This exposes a transition from theoretical comprehension to potential practical application. In this vein, PT1 serves as an example of the usefulness of didactic theories in teacher training, facilitating both reflective analysis and action within classroom contexts.

Additionally, PT1 elucidates a potential interpretation of students' comprehension of the infinite cardinality of rational numbers in relation to quantity and the inclusion of other numbers, which is consistent with the teaching strategy of employing a number-contention scheme to illustrate density. The ideas put forth by PT1 align with the usual conceptualisation of real numbers that is typically encountered in secondary education. This conceptualisation introduces the notion of real numbers as a means of encompassing numbers that extend beyond the set of rational numbers (Bergé, 2008). Nevertheless, we question whether this approach is appropriate for illustrating the density property and whether it is sufficient for students to recognise that density is not solely dependent on the infinity of sets.

On the other hand, PT4 demonstrated knowledge of students' difficulties despite this aspect of knowledge not being among the anticipated responses for the second question of the task. In this sense, he offers a potential explanation for the students' difficulties (KFLM) considering their understanding of infinity and associated theorems (KoT). The above relationship may be identified as an inter-domain relationship, as described by Delgado-Rebolledo and Zakaryan (2022). This type of relationship provides an illustration of the integrated character of MTSK (Carrillo et al., 2018) whereby teachers' knowledge is not comprised of isolated items.

Conclusions

The task presented in this work addresses teachers' knowledge of the density property, a property that poses difficulties in both teacher training and mathematics instruction. Crafted from a real-life classroom scenario, the task allows PTs to reflect on a context that closely resembles the challenges and experiences that they are likely to face in their professional teaching careers. This helps them to develop knowledge and teaching strategies that are directly applicable to real-world classroom settings. In this sense, we consider the importance of the design and implementation of training tasks aimed at helping PTs develop their knowledge and skills.

Developed with reference to the research of Zakaryan and Ribeiro (2016; 2019), this task serves as an example of how research findings can be effectively incorporated into teacher training. However, we consider that in future implementations of the task, it could be expanded to address other important aspects of density, such as the idea of infinity within the density property and the utilization of the number line and schemes to represent rational numbers. Also, additional questions could be added to the task to explore the knowledge of connections between the density property and the infinity with other mathematical concepts. This would provide a more comprehensive understanding of how prospective teachers approach these concepts.

On the other hand, the task served a dual purpose: it allowed PT to mobilize their MTSK from a teacher training perspective, while also provide an opportunity for inquiry on it from a research perspective. This aligns with the aim of other studies, as the work of Ribeiro et al. (2021). By combining both perspectives, we emphasize that when PTs respond to the task, they not only demonstrating the expected knowledge outlined in the task design but also demonstrate knowledge from other subdomains of their MTSK. The emergence of these different categories of knowledge in PTs responses highlights potential of the task in fostering the development of the MTSK of PTs.

Following the above, MTSK-based tasks would facilitate learning, promote critical thinking, and enhance PTs' ability to apply their specialized knowledge in real-world classroom scenarios. Other research has shown as through the performance of tasks, in-service mathematics teachers improve the use of their MTSK, particularly in terms of their KoT and KMT (Montes et al., 2021). However, more research is necessary to deepen our understanding of how PTs could develop their MTSK through the performance of training tasks. We intend explore this issue as a topic for further research.

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Pre-service teachers' knowledge of students' misconceptions and difficulties related to functional thinking: a German and Slovak comparison

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Functional thinking, as the ability to relate two or more quantities, is relevant throughout schooling and in everyday life. Regardless of its importance, student misconceptions and difficulties are empirically well documented. As part of their professional knowledge for teaching, pre-service teachers need to be aware of these misconceptions and difficulties in order to teach accordingly. The presented paper reports an intervention study investigating German and Slovak pre-service teachers' knowledge of what misconceptions and difficulties their future students might have. The intervention involved a university course for pre-service teachers designed in the context of the Erasmus+ project FunThink to support their future students' functional thinking. The results indicate a positive development in pre-service teachers' knowledge of students' misconceptions and difficulties in both countries but also indicate system-related differences between the two countries.

Keywords: Functional thinking, functions, misconceptions and difficulties, pre-service teachers, teacher knowledge.

Introduction

Functional thinking, known as a way of thinking that is typical for functions, describes a way of thinking in relationships, interdependencies and changes (Vollrath, 1989). In this manner, it plays a crucial role in everyday life and mathematics education. Despite its relevance, studies have empirically shown that students have many misconceptions and difficulties[1] (M&Ds) with this topic (e.g., Leinhardt et al., 1990; Sproesser et al., 2022). For mathematics teachers, it is important to be aware of these possible M&Ds students face as part of their mathematical knowledge that is required for effective teaching. With this knowledge in mind, teachers can plan their lessons accordingly, dealing adequately with such M&Ds and, hence, impacting their students' learning. In this context, one of the outputs of the Erasmus+ Project FunThink was, inter alia, designing a university course for pre-service teachers (PSTs) to develop their professional knowledge for teaching functional thinking.

The study presented in this article is embedded in a more extensive study investigating the PSTs' professional knowledge for teaching functional thinking and how it develops throughout this university course. This article focuses on one item of the used test instrument that addresses students' M&Ds from the German and Slovak PSTs' perspective. The research focuses on what M&Ds the PSTs know related to functional thinking before and after participating in the described course.

[1] Misconceptions and difficulties are used in line with Leinhardt et al.'s work (1990) which serves as a base for this study.

Theoretical Background

Functional thinking

Functional thinking can be described as thinking in relationships, interdependencies and changes (Vollrath, 1989). As the name and this description state, functional thinking is closely related to the topic of functions. To deal with functions in a meaningful way, students should acquire four aspects of functional thinking throughout schooling, namely the input-output aspect, the covariation aspect, the correspondence aspect, and the object aspect (Pittalis et al., 2020). In addition to these aspects, students need to be able to master algebraic prerequisites, to use and change between different representations, as well as to relate functions with real-world situations in the sense of modelling (Vollrath, 1989). In the following, we will describe how functional thinking is implemented in the German and Slovak curricula in order to equip students with the mentioned competencies.

In Germany, the concept of function is formally introduced in grade 7 or 8, depending on the type of school. Prior to this, different representations such as tables and graphs and informal tasks such as describing and continuing patterns are thematised in primary and lower secondary education. After the formal introduction of functions, different function classes are taught, starting with proportional and linear functions. Higher-order processes such as concatenating and differentiating functions are introduced in upper secondary education (Kultusministerkonferenz [KMK], 2022). In mathematics classrooms, the correspondence aspect is often emphasised, and the covariation aspect is somewhat neglected in teaching in Germany (Malle, 2000).

In Slovakia, the concept of function appears implicitly in grade 7 in the context of direct proportionality. In grade 9, the concept of linear function is continued with an emphasis on the input-output and correspondence aspects. The covariation aspect is not sufficiently covered or is missing in textbooks for lower secondary school (Krišáková & Slabý, 2022). Later, in upper secondary school, students learn the function definition and different classes of elementary functions, including their properties. Most Slovak teachers use older textbooks to teach functions because the new textbooks are methodologically different from the way teachers are used to teaching. This is one of the possible reasons why the correspondence and object-oriented aspects predominate in teaching.

The above descriptions indicate that teaching and learning functional thinking are complex and that, apparently, not all four aspects are evenly supported in (German and Slovak) mathematics classrooms. Besides other factors, such an unbalanced implementation of functional thinking aspects in the classroom can be a reason for students' M&Ds. In the next paragraphs, we will describe typical student M&Ds related to functional thinking.

Misconceptions and difficulties related to functional thinking

M&Ds among students are empirically well documented (e.g., Leinhardt et al., 1990; Sproesser et al., 2022). Some reasons for these M&Ds might be the abstract character of functions, the necessity to acquire the four aspects, multiple formal prerequisites such as understanding the variable concept, or the need to use and to change between different representations (inter alia but not restricted to the change between mathematical and situational representations when modelling real-world situations by mathematical functions). For the presented research, we use the work by Leinhardt and colleagues

(1990) as a base. Leinhardt et al. (1990) describe the following eight categories of M&Ds: What is and is not a function, linearity, continuous versus discrete graphs, representations of functions, concept of variable, notation, correspondence, relative reading and interpretation. Examples and brief descriptions for each category will be provided in the methods section of this paper.

Being aware of these possible M&Ds among their students is required for teachers to plan their lessons accordingly and to support student learning. Therefore, in the next section we will describe the connection of M&Ds and teachers' professional knowledge.

Professional knowledge for teaching functional thinking

It is well known that teachers have a decisive influence on their students' learning (e.g., Yoon et al., 2007). In this manner, PSTs need to develop comprehensive knowledge to teach their future students appropriately. As the present research relates to teacher knowledge on typical student M&Ds, we will demonstrate in the following exemplarily, how this knowledge is reflected in two models of teachers' professional knowledge that are well-recognized in mathematics education: For example, within the framework *Mathematical Knowledge for Teaching* (MKT), knowledge about typical M&Ds is described as a core element of the PCK subcategory *knowledge of content and students* (KCS) (Ball et al., 2008). Similarly, within the model *Mathematics Teachers' Specialised Knowledge* (MTSK), knowledge of students' M&Ds is part of the PCK subcategory *knowledge of features of learning mathematics* (KFLM) (Carrilo-Yañez et al., 2018).

Both models of teacher knowledge reflect the relevance for (future) teachers to know about typical M&Ds. However, research shows that teachers' knowledge of M&Ds is heterogeneous and often not sufficiently pronounced, what makes it difficult to prevent or adequately react to such M&Ds (Sproesser et al., 2020). Moreover, the study of Hubenakova et al. (2021) indicates that system-related particularities such as curricular requirements and typical culture-related teaching traditions can cause differences in teacher knowledge related to M&Ds. The project underlying this study aims at fostering German and Slovak PSTs' professional knowledge for teaching functional thinking by a university course. In this paper, we report and compare PSTs' corresponding knowledge and describe how it developed through participation in this course. The research question is: What M&Ds in the field of functions do German and Slovak PSTs know before and after participating in a course to support students' functional thinking? The corresponding research will be described in the following.

Method

Participants and context

The PSTs participating in this research chose the university course on supporting students' functional thinking as part of their lectures in didactics of mathematics. The course was developed within the FunThink project by German and Slovak partners inter alia and took place in the winter semester of 2022/23. It consisted of about 14 sessions (90 minutes each) focusing on the following topics: meaning and relevance of functional thinking, functional relationships and functions, aspects of functional thinking, the role of representations and representational changes, typical misconceptions and difficulties in the field of functional thinking (rather in the context of other topics such as "representational changes and corresponding difficulties"), development of functional thinking and

its embedding in the curriculum. In addition, the PSTs tried out, developed and reflected learning environments to foster students‘ functional thinking. Smaller variations between the implementation in the two countries were possible due to national requirements, but the general concept was identical.

In Germany, 14 PSTs of Ludwigsburg University of Education (11 female, 3 male), 22-34 years old, participated in this study. The course was offered as part of the master’s degree and participation was possible starting in the last semester of the bachelor’s degree. All participants are studying to become primary, secondary, or special education teachers and one of their main subjects is mathematics. Prior to this course, they all completed courses in mathematics and teaching mathematics.

In Slovakia, the study involved 13 PSTs from Pavol Jozef Šafárik University in Košice (9 female, 4 male) aged 22-23 years. All participants are first-year master's students and have a bachelor's degree in mathematics combined with another subject. They are studying to become teachers for lower and upper secondary schools. So far, they had courses in school mathematics and general teaching but with no pedagogical focus. Due to a country-specific adaption and contrarily to the German PSTs, the Slovak PSTs solved several content knowledge items related to functions before answering the PCK test presented below.

Test instrument, data collection, and coding scheme

The data was collected using a paper-pencil test in the first and last session of the course. Participation in the study was voluntary and not related to any grading. The test consisted of 8 to 17 items, again depending on local variations, and was developed within the FunThink project. The items focus on the PSTs’ knowledge of functional thinking and its teaching. The research presented in this paper focuses on the following item only: What learning difficulties and misconceptions do you expect when teaching functions? Use as many examples as possible to depict your answer.

For data analysis, a coding scheme was developed using qualitative content analysis, which allowed for both an inductive and deductive approach (Mayring, 2015). The deductively formed categories mainly focus on Leinhardt et al.’s (1990) classification of M&Ds. The PSTs’ answers were coded individually in each country, which led to modifications in the coding scheme. A joint discussion with colleagues from Germany and Slovakia, inter alia, resulted in the final coding scheme. To ensure consistency in coding between the countries, unclear examples were translated into English and discussed until consent was reached. The final coding scheme complemented by descriptions of PSTs responses (last row) is displayed in Figure 1 and will be briefly described in the following.

#	Code	Description	Example
1	F/N	No differentiation between what is and what is not a function	Misclassification of a graph as a function – assigning two x-values to one y-value.
2	LIN	**Linearity:** Overgeneralisation of properties of linear functions	Students connect two points by straight lines without paying attention if a linear function is appropriate.
3	1REP	**Representation of functions:** M&Ds related to one form of representation	Students are not able to draw a graph
4	LREP	**Linking representations:** M&Ds related to linking representations	Two formulas of functions and two graphs are given and students are not able to assign which graph belongs to which formula.
5	MOD	**Modelling:** M&Ds related to modelling	Students are not able to interpret graphs correctly (example, if they relate the shape of the glass to a graph of the function of the height of the water in the glass versus time when the water is poured evenly into the glass).

6	VAR	**Concept of variable:** M&Ds related to the concept of variable	Students are not able to understand how the values of variables can change in a given formula.
7	NT	**Notation:** M&Ds related to function notations	Lack of awareness that f(x) is another notation for y.
8	CvsD	**Continuous vs discrete graphs:** students cannot differentiate between continuous and discrete graphs	If the student draws a graph from the table, he does not connect the resulting points (e.g., pouring water into a glass of 100 ml - continuous graph).
9	1A	**Input-output aspect:** M&Ds related to the input-output aspect	Students are not able to assign input-output values.
10	2A	**Covariation aspect:** M&Ds related to the covariation aspect	Students are not able to determine from the table how x varies with f(x).
11	3A	**Correspondence aspect:** M&Ds related to the correspondence aspect	Students are not able to find the formula based on the provided data.
12	4A	**Object aspect:** M&Ds related to the object aspect	Student are mistaken about the types of functions if they are not indicated in their basic form.
13	GEN	**General:** M&Ds not related to the concept of functions	The student does not know how to start solving the problem.
14	Oth+	**Other meaningful answers:** M&Ds related to functional thinking that cannot be classified in any other category	Students do not know the rules for modifying expressions with powers, logarithms...
15	Oth-	**Meaningless or incomprehensible answers:** unclear answers	Misunderstanding of parents and their comments on student's learning.
16	BLK	**No answer or I do not know**	-

Figure 1: Coding scheme (adapted from Slabý & Semanišinová, accepted)

Categories 1, 2, 6, 7, and 8 are taken directly from Leinhardt et al. (1990) without modification. Categories 3 to 5 were derived from Leinhardt et al. (1990) adding further differentiation concerning dealing with representations. This differentiation includes changing representations, which causes difficulties for many students (Sproesser et al., 2022). Categories 9 to 12 are based on the four aspects of functional thinking as described by Pittalis et al. (2020). The last three categories (13-15) include rather general, other meaningful answers, and incomplete or incomprehensible answers. If a PST did not provide an answer or stated that they do not know possible M&Ds, code 16 was used.

Results and Discussion

The results will be presented for each country individually in the following. The numbers in the lowest line refer to the number of PSTs that mentioned each category; they do not relate to how often a PST mentioned one category. The numbers in the last column indicate how many meaningful categories (i.e., without "OTH-" and "BLK") a PST referred to in the pretest (grey) and in the posttest (black).

Figure 2 displays the German results. Most PSTs mentioned equally many or more M&Ds in the posttest than in the pretest. Only two PSTs mentioned less M&Ds in the posttest (GER4, GER14). The biggest changes can be observed for the categories of modeling (MOD) and representation of functions (1REP). This can be explained by the relevance of both these categories in connection to functional thinking (Vollrath, 1989) and a corresponding thematisation during the course.

Figure 3 indicates the responses provided by the Slovak PSTs. Most PSTs mentioned more M&Ds in the posttest compared to the pretest. Only one PST (SK9) mentioned equally many M&Ds in both tests and two PSTs (SK5 & SK8) mentioned fewer M&Ds in the posttest. Comparably big

improvements can be observed for the categories VAR, NT, and A4. Rather general and not function-related M&Ds answers decreased from pre- to posttest. In regard to the aspects of functional thinking, it can be observed that the PSTs focused only on the correspondence and object aspect (A3 & A4) in the pretest. In the posttest, these two aspects were also the dominant aspects, but A1 and A2 were also mentioned. This might be due to the previous training the PSTs underwent as described above.

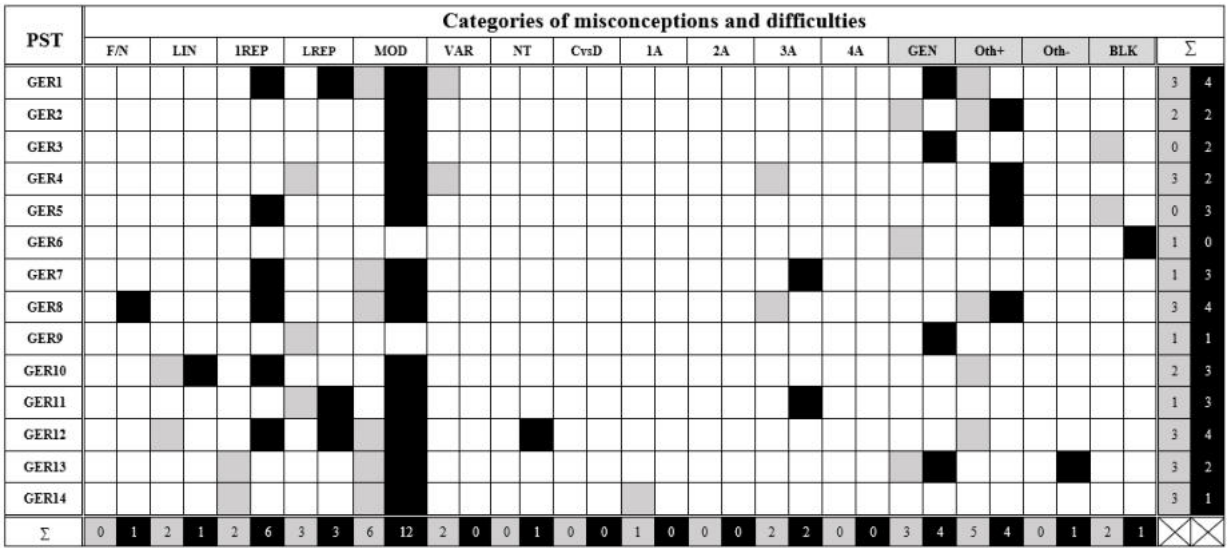

Figure 2: Mentioned M&Ds by German PSTs

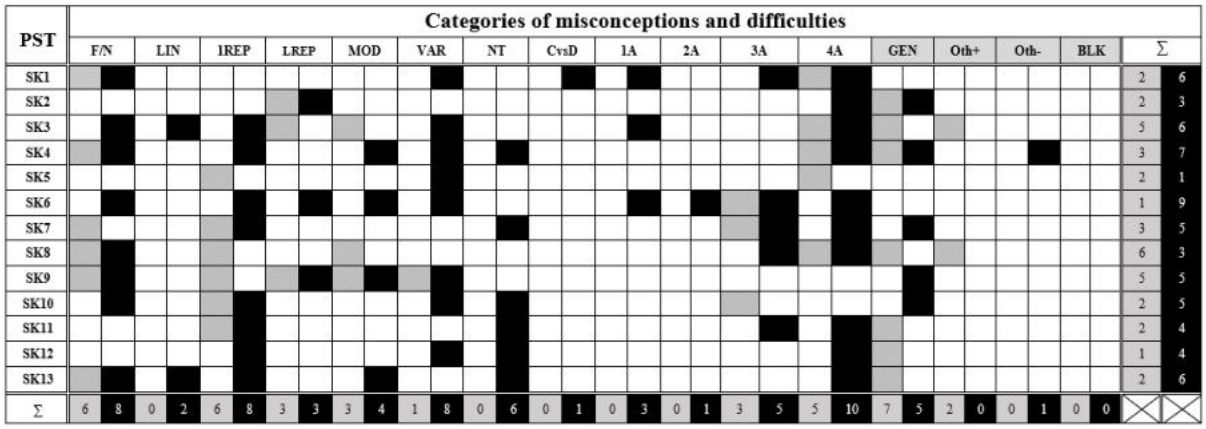

Figure 3: Mentioned M&Ds by Slovak PSTs

When comparing the German and Slovak results, it can be noted that in both countries the number of mentioned M&Ds increased from pre- to posttest. Concerning the individual categories, the German and Slovak PSTs focused mostly on different M&Ds. In the pretest, the dominant category observed for the German PSTs was Oth+, whereas in the posttest MOD was mentioned mostly. The Slovak PSTs mostly focused on F/N in the pretest and on F/N, VAR, NT and 4A in the posttest. A possible explanation for the differences between the countries might be the use of different project-related learning environments in the course that were chosen to best fit local curricula. The category 1REP was frequently mentioned in both countries. This might be because M&Ds related to representations are well documented (Sproesser et al., 2022). Yet, it appears surprising that changing representations

(LREP) was focused less often as this appears to be even more complex than mastering single representations. In total, the Slovak PSTs named more M&Ds in the pre- and posttest (36 & 65) compared to the German PSTs (28 & 36), even though the German sample included one PST more. This might be explained by the fact that the Slovak PSTs had to solve several content knowledge items before answering the PCK question on M&Ds, what might have posed difficulties for them and therefore possibly made them more sensitive for potential student problems. In this manner, it would be interesting to also have a closer look at previously attended university courses and the differences between the used learning environments to find further possible explanations. Limitations of this study are the relatively small sample size and slight variations between the German and the Slovak courses. In order to clarify if the found particularities and differences between German and Slovak participants are systematic, further research should include larger samples.

Conclusion

The results indicate that most PSTs' knowledge of students M&Ds with regard to functional thinking increased after participating in the described course in both countries which can be interpreted as a positive effect in regard to their professional knowledge (Ball et al., 2008; Carrillo-Yañez et al., 2018). This is even more remarkable since M&Ds were treated in the course rather secondary and in the context of other topics. Differences in the implementation of functional thinking in mathematics classrooms in Germany and Slovakia are obviously also reflected in PSTs' knowledge, which might also but not exclusively be moderated by differences in the learning environments and the university course. Hence, this study indicates that cross-country research is crucial, as it is obviously not possible to transfer the results from one country to another generally. Such country comparisons can also serve as inspirations in the sense of identifying and cumulating best practices of different countries, e.g., with regard to the implementation of functional thinking in the classroom.

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Exploring strategy flexibility and adaptivity in multidigit multiplication: a study with Austrian in-service primary teachers'

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Understanding teachers' strategy flexibility and adaptivity is crucial as their expertise shapes students' mathematical thinking and flexibility – key goals in primary education. This study examines the strategy repertoires and perspectives on adaptivity of 55 in-service primary teachers in Carinthia, Austria. Participants solved single-digit by two-digit multiplication problems and explained their reasoning for strategy selection. Results show a strong preference for decomposition strategies, with compensating and doubling used less frequently. On average, participants used 2.6 strategies. Teachers justified choosing the easiest strategy based on subjective criteria (e.g., familiarity, easy to understand, apply) rather than on task-based adaptivity. When evaluating a student's mathematically correct but non-adaptive strategy (5×13=5×20–5×7), responses varied: 38% considered it advantageous, justifying their evaluation from a pedagogical rather than mathematical view.

Keywords: Strategies, flexibility, adaptivity, multidigit multiplication, in-service primary teachers' knowledge.

Introduction

The development of strategy flexibility is an important goal in primary mathematics education, as emphasised in curricula and policy documents in Austria, Germany and worldwide (e.g. BMBWF, 2009). However, research highlights low levels of strategy flexibility among students, especially in traditional educational settings (e.g. Heinze et al., 2009; Torbeyns et al., 2017). Comparing strategies and exploring numerical relationships can significantly increase strategy flexibility (e.g. Green & Rathgeb-Schnierer, 2020; Greiler-Zauchner, 2022; Smedt et al., 2010). Teachers' expertise in strategy flexibility is crucial for fostering students' flexible mathematical thinking. Research highlights the significant impact of teacher knowledge on instructional quality and student achievement (Copur-Gencturk & Tolar, 2022; Hill et al., 2005; Kunter et al., 2013).

In this study, we investigate the strategy flexibility of Carinthian in-service primary teachers (IPTs) in single-digit by two-digit multiplication tasks. The study focuses on the two key constructs of strategy flexibility: first, the ability to know and use multiple strategies for these tasks, referred to as strategy repertoire. Second, we examine teachers' ability to select the most appropriate strategy for a given task, referred to as strategy adaptivity going beyond a purely task-based perspective by considering additional dimensions that may constitute strategy adaptivity for teachers.

Strategy repertoire in multidigit multiplication

Strategy flexibility, as defined above, involves the knowledge of multiple strategies. Depending on how numbers are handled, strategies can be categorised as (1) sequential strategies, (2) decomposition strategies and (3) varying strategies (Hickendorff et al., 2019). Sequential strategies involve repeated addition, e.g. 5×24=24+24+24+24+24. Efficiency increases with doubling, e.g. 24+24=48,

48+48=96, 96+24=120. Decomposition strategies use the distributive property of multiplication over addition and are classified according to how the factors are split. For example, 25×7 can be calculated as (20+5)×7=20×7+5×7 (by decades) or as 25×(4+3)=25×4+25×3 (by non-decades). Varying strategies adjust numbers and/or operations, including compensating, e.g. 19×8=(20–1)×8=20×8–1×8, or doubling and halving, e.g. 5×24=10×12 using factorization and the associative property. Another varying strategy exploits halving relationships, e.g. finding 5×24 by halving the product of 10×24.

Research on strategy repertoires and their use for multidigit multiplication among in-service and pre-service primary teachers (PPTs) is limited. A few studies have been conducted in different countries in and around Europe: Turkish IPTs predominantly preferred the standard algorithm for problems such as 15×19 and saw no need for alternative strategies, some of them even had difficulties understanding them (Aylar-Çankaya, 2020). Similar patterns were observed among Greek PPTs, with the majority relying on mental representations of the written algorithm for two-digit multiplication and only a small proportion using alternative strategies such as decomposition (Lemonidis et al., 2014). The study found that 44% of PPTs lacked strategy flexibility in the sense that they were unable to use different strategies appropriately depending on the multidigit multiplication problem. A study of Romanian PPTs found that more than a quarter of participants did not use any strategies beyond mentally following the standard algorithm, while more than a third used only one or two strategies, mainly decomposition (Baranyai et al., 2019). A study of Norwegian PPTs found that 59% of PPTs also used decomposition for mental multidigit multiplication (Månsson, 2023). Despite socio-cultural differences, a common trend shows reliance on standard algorithms and limited strategy flexibility, with decomposition as the main alternative.

Perspectives on strategy adaptivity

The concept of adaptivity as the ability to select the most appropriate strategy, is interpreted differently in the literature. Some researchers conceptualise adaptivity primarily from a task-based perspective, meaning that the appropriateness of a strategy is determined by the specific characteristics of the mathematical problem (e.g. Blöte et al., 2001; Heinze et al., 2009; Hickendorff, 2018). In the context of multidigit multiplication, this perspective favours strategies such as doubling for factors such as 4 or 8 and compensating when a factor is just below a multiple of ten (e.g. 19×8). Doubling and halving is particularly effective when one factor is even and the other is a convenient number, such as 5, 15, 25 or 50. Similarly, halving the product of 10 is advantageous when 5 is a factor. In contrast, decomposing by decades is often seen as a universal strategy, especially when no specific task characteristics suggest a more adaptive approach.

However, children do not always prefer task-specific strategies, even when they can use them. Instead, they tend to favour strategies with which they are familiar or feel more comfortable (e.g. Greiler-Zauchner, 2022). Further evidence suggests that task-specific strategies, although having the fewest and the easiest steps, are not necessarily easy for all students, as they require a strong understanding of numbers and numerical relationships (Hickendorff, 2022).

These considerations support a broader conceptualisation of adaptivity. In this sense, Verschaffel et al. (2009) define adaptivity as the ability to select and use the most appropriate solution strategy for

a given mathematical task, for a specific individual, in a particular sociocultural context. These different perspectives on adaptivity may lead to different justifications for why a particular strategy is considered most appropriate for an individual – a key focus of this study. In particular this study explores teachers' perspectives on adaptivity, which may also be influenced by pedagogical considerations.

Research question

Primary teachers' expertise in strategy flexibility and adaptivity has a significant impact on their teaching practices in arithmetic. Existing studies on IPTs' repertoires and the use of multiplication strategies have predominantly been conducted in specific geographical settings (e.g. Greece). Research is needed to see if these findings apply to other European countries, including Austria. Furthermore, teachers' perspectives on strategy adaptivity have been under-researched. Little is known about the criteria teachers use to judge the appropriateness of strategy choices. To address these research gaps, the study is guided by the following research questions:

1. What is the range and frequency of strategies used by Carinthian IPTs when solving single-digit by two-digit multiplication problems?
2. What criteria do Carinthian IPTs use to determine which multiplication strategy is the easiest?
3. What criteria do Carinthian IPTs use to evaluate the appropriateness of a student's chosen strategy for a given multiplication problem?

Method

An online questionnaire, designed as a self-assessment tool, was administered to 55 Carinthian IPTs in 2021 and 2022 (53 female, 2 male). The IPTs were volunteers enrolled in a professional development course for the promotion of mathematical competences, which aims to improve teachers' understanding of mental multiplication according to new curricula and standards and to provide them with validated teaching materials to support their classroom practice. The teaching experience of the 19 teachers who provided information ranged from 3 to 41 years. Data collection took place at the beginning of the first online session in autumn 2021 and 2022. The original questionnaire consisted of five questions assessing content and pedagogical content knowledge related to two-digit multiplication, with an estimated completion time of 30 minutes:

In Question 1a, participants were asked to solve the multiplication problems 17×4, 5×24 and 19×6 in various ways in order to assess their repertoire of strategies. The answers were analysed and categorised according to established strategy types (Greiler-Zauchner, 2022; Hickendorff et al., 2019). In Question 1b participants had to justify in a written report which of the strategies from Question 1a they found easiest and why. Their responses were categorised according to different dimensions of adaptivity (task-based and subject-based), with subject-based arguments further classified inductively based on the content of the responses. Question 2 presented a scenario in which a student *Verena* used the 5×20–5×7 strategy – inappropriate from a task-based perspective – to solve 5×13 for her homework. Participants evaluated Verena's choice of strategy in a written report. In the first round of analysis, responses were categorised according to whether they considered the strategy

to be inadequate, advantageous or neutral. The justifications within the *inadequate* category were further classified inductively based on their content.

Results

Question 1a: IPTs' repertoire of multiplication strategies

The multiplication strategies used by the 55 IPTs for the 17×4, 5×24 and 19×6 problems were categorized into sequential strategies (repeated addition and doubling), decomposition strategies (by decades and non-decades), varying strategies (compensating, halving the product of 10, doubling and halving), the standard algorithm and erroneous strategies. Figure 1 shows the frequency of the strategies used: Decomposing by decades was the most frequently used strategy, with 89% (49 participants) using 10×4+7×4 for 17×4 and 71% (39 participants) using 5×20+5×4 for 5×24. Compensating was common for 19×6, with 80% (44 participants) using 20×6–1×6.

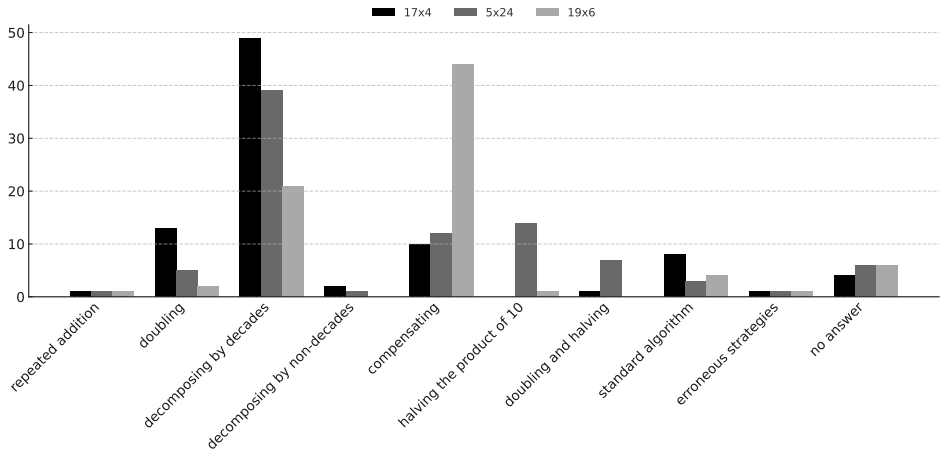

Figure 1: Frequency of multiplication strategies for 5×14, 4×16 and 19×6

Less common strategies included doubling, used by 24% (13 participants) for 17×4, halving the product of 10, used by 25% (14 participants) for 5×24 and doubling and halving, used by 13% (7 participants) for 5×24. A small number of participants used erroneous strategies (1 participant) or gave no answer (4–6 participants, depending on the problem). The standard algorithm was chosen by 3 to 8 participants, depending on the problem.

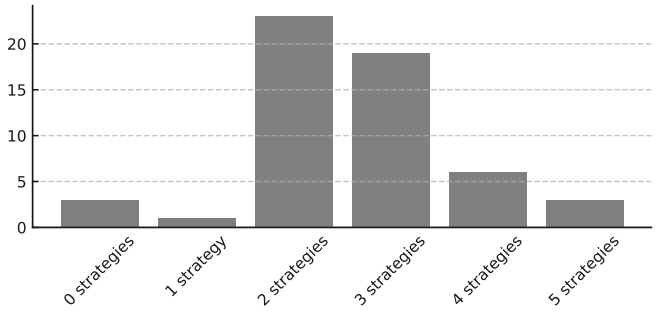

Figure 2: Number of different strategies used per person

Figure 2 shows the distribution of participants by the number of strategies used. The most common approach was the use of two strategies, observed in 42% (23 participants), with 16 favouring a combination of decomposing by decades and compensating. Three strategies were used by 35% (19 participants), most often adding either doubling or halving the product of 10 to the two main strategies. Fewer participants used four or more strategies (9 participants), while 5% (3 participants) provided no answer and 2% (1 participant) used only one strategy.

Question 1b: IPTs' justification of the easiest strategy

Question 1b asked participants to justify why they considered their chosen strategy the easiest. However, 11 participants mentioned a strategy as the easiest without giving a reason. In addition, 2 participants referred to number manipulation in their answers, which did not include a justification. 13 participants gave justifications for adaptivity, some with multiple responses (total: 19). When these justifications were categorised, five distinct reasoning patterns emerged. Table 1 provides an overview of these categories, along with translated examples of participants' responses.

Table 1: IPTs' justifications of the easiest strategy

Category	Reasoning/Example		Frequency (N=19)
Fast	IPT59:	For me, the second method [compensating] is the easiest because it's the fastest.	31.58% (6)
Familiar	IPT17:	In my opinion, strategy 1 [doubling] is the easiest because the kids are already familiar with doubling.	21.05% (4)
Easier to understand and apply	IPT15:	I find the method above [compensating] the easiest because I think it's simpler for kids who have trouble multiplying large numbers.	15.79% (3)
Frequently used	IPT58:	It is used most frequently.	15.79% (3)
Less prone to error	IPT36:	This strategy has less error potential.	15.79% (3)

Notably, none of the participants justified their choice based on task characteristics. Instead, the most frequently cited criterion was *fast*, mentioned by six participants (31.58%), who associated the easiest strategy with the one that could be performed the fastest. The rest of the answers were distributed among the categories of familiarity (4), easier to understand and apply (3), frequently used (3) and less error-prone (3). It is also worth noting that 53% (29 participants) did not answer this question. This may indicate that they overlooked the second part of the question.

Question 2: IPTs' evaluation of Verena's strategy choice for 5×13

In Question 2, the IPTs were asked to evaluate Verena's strategy for solving 5×13, which she approached by calculating 5×20–5×7. Although this strategy is mathematically correct, it is less appropriate compared to other strategies such as decomposing or halving the product of 10. The responses fell into three categories: (1) 54% (30 participants) found the strategy inappropriate, (2) 38% (21 participants) found it advantageous and (3) 4% (2 participants) were neutral. Two responses could not be categorized. Within the first category, the responses were further grouped according to the reasons given. 22 IPTs also suggested an alternative calculation strategy, most commonly

decomposition. Table 2 gives an overview of the categories and sub-categories and provides examples of the participants' responses, translated into English.

Table 2: IPTs' responses categorized with examples

Category	Reasoning/Example	Frequency (N=55)
Strategy considered inappropriate		
- due to the absence of relevant task features	IPT220: The multiplicand is closer to 10, so it would be more sensible for Verena to calculate as follows: 5×10=50, 5×3=15, 50+15=65.	11% (6)
- due to other reasons (e.g. error-prone, existence of easier methods)	IPT35: Too cumbersome with sources of error; subtraction is more prone to error here…	27% (15)
- without further explanation	IPT203: Correct, but cumbersome.	16% (9)
Strategy considered advantageous	IPT07: Verena calculated advantageously! Many children are not yet capable of this! IPT25: Excellent method; Verena thinks very complexly.	38% (21)
Neutral evaluation	IPT55: Creative method.	4% (2)
Answer not categorizable		4% (2)

Summary and Interpretation

This study investigates the strategy repertoires and adaptive reasoning of Austrian in-service primary teachers (IPTs) when solving single-digit by two-digit multiplication problems. The results indicate a strong preference for decomposition strategies, particularly those based on decades. Compensating was frequently used, especially for the 19×6 problem, where task-based features make it advantageous. However, other task-based strategies, such as doubling, halving the product of tens, or doubling and halving, were used less frequently, especially for the 17×4 and 5×24 problems. The standard algorithm was rarely used, probably because of the small numbers involved and the numerical characteristics of the problems. On average, participants used 2.6 different strategies, indicating a broader use of different strategies and less reliance on the standard algorithm compared to studies conducted in other countries, as discussed in the theoretical section.

Regarding the criteria used by the Carinthian IPTs to determine which multiplication strategy was the easiest, the results show that the teachers investigated did not focus on task-specific characteristics but justified their choice mainly on subjective criteria such as: quick, familiar, easier to understand and use, frequently used or less error-prone. The criteria used to evaluate the appropriateness of a student's chosen mathematically correct strategy, which does not follow the principles of task-based adaptivity (5×13 = 5×20–5×7), also illustrate the diversity of teachers' perspectives on adaptivity. Nevertheless, 38% of teachers found it advantageous, often citing its complexity or creativity and

appreciating the complex solution. In contrast, only 11% considered it inappropriate because of insufficient task relevance.

These findings show that IPTs' perspectives on adaptivity in these tasks are largely not aligned with a task-based view. Instead, their reasoning often reflects instructional and pedagogical considerations, such as encouraging student creativity, supporting higher order thinking, or minimising errors. This is an indication that teachers do not always justify their choices from a strictly mathematical perspective, which needs to be taken into account when interpreting the results. Given these tendencies, professional development programs should aim to strengthen teachers' task-based reasoning while also integrating subject-based and socio-cultural perspectives into discussions on strategy selection. The small sample size limits the generalizability of the findings and further research is needed to explore their applicability in different educational contexts.

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An investigation of TPACK using a 3D Techno-pedagogical integration matrix in the context of mathematics teaching

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This study uses a three-dimensional Techno-Pedagogical Integration Matrix (3D-TPIM) to re-conceptualize TPACK. Our case involved a senior pre-service mathematics teacher (PMT) in a teacher preparation program. The data consisted of a video recording of his lesson in a real classroom setting. The video was analyzed using the 3D-TPIM. The x-axis represents the sub-codes of the richness of the mathematics dimension of the Mathematical Quality of Instruction (MQI) framework. The y-axis represents levels of the MQI sub-codes. The z-axis represents the levels of technology integration using the SAMR (Substitution, Augmentation, Modification, and Redefinition) model. The findings indicated that the PMT used technology mainly at the augmentation and modification levels for linking representations and mathematical sense-making. We will discuss the theoretical and practical implications of our conceptualization of TPACK using the 3D-TPIM.

Keywords: SAMR model, Mathematical quality of instruction, 3D techno-pedagogical integration matrix, Pre-service mathematics teachers, TPACK.

Introduction

One of the critical issues in teacher education research has been the type of knowledge needed to successfully incorporate technology into the teaching of any subject matter and how this knowledge should be evaluated or developed. Mishra and Koehler (2006) call this *Technological Pedagogical Content Knowledge* (TPACK). In recent years, the framework has faced criticism for overlooking contextual factors (referred to as contextual knowledge, or XK, by Mishra, 2019, and XTPACK by Brianza et al., 2022) and for lacking a domain-specific focus in its approach (Voogt et al., 2013).

Building on our previous study (Akkoç & Hangül, 2023), this study also emphasizes TPACK as a domain-specific knowledge, treating it as a core area rather than focusing on the framework as a whole. We propose an alternative interpretation of TPACK tailored to mathematics and its pedagogy, moving away from a generalized model that downplays the importance of subject content. By integrating the Mathematical Quality of Instruction (MQI) framework, our conceptualization highlights the significance of the subject matter (mathematics). We aim to examine a PMT's TPACK through the lens of this new conceptual framework, as detailed below. Our research question is: "What is the PMT's TPACK through the lens of the 3D-TPIM?"

Conceptual Framework

The present study uses a three-dimensional Techno-Pedagogical Integration Matrix (3D-TPIM) to re-conceptualize TPACK. We developed this model to design the content of the Teaching Practicum I-II courses, analyze PMTs' TPACK, and monitor their development. The MQI sub-codes originally have levels. In our previous studies (Akkoç, Hangül & Çeziktürk, 2022; Akkoç & Hangül, 2023), we only included its categorical sub-codes without their levels and utilized a 2D-matrix: one of the axes

represented the richness of the mathematics dimension of the MQI framework which has six sub-codes: *mathematical sense-making, explanations, mathematical language, patterns and generalizations, linking between representations, and multiple procedures or solution methods* (Hill, 2014). The other axis represented the levels of technology integration using Puentedura's (2020) SAMR (*Substitution, Augmentation, Modification,* and *Redefinition*) model. This study includes the levels of the MQI adhering to its original version and formed a 3D-TPIM for a more in-depth examination. This current study adds the levels of the sub-codes of the richness of the mathematics dimension (*not present, low, mid, high*) as the third axis (See Figure 1). Utilizing this 3D-TPIM, we seek to examine the TPACK of a PMT in a real classroom setting. Below, we present our conceptual framework further.

Mathematical Quality of Instruction (MQI)

MQI is an observational rubric that provides a framework for analyzing mathematics instruction in five domains: *the richness of the mathematics, common core aligned student practices, errors and imprecision, working with students and mathematics,* and *classroom work is connected to mathematics* (Hill, 2014a). *The richness of the mathematics* dimension aims to determine the depth of the mathematics in the instruction within six sub-codes: *explanations, mathematical sense-making, linking between representations, patterns and generalizations, multiple procedures or solution methods,* and *mathematical language* (Hill, 2014b). The code of *explanations* focuses on the question of "why". For instance, why is an answer true/false, why is a solution method appropriate/inappropriate, and why does a procedure work/not (Hill, 2014b)? The code of *mathematical sense-making* is expressed to seize the connections between mathematical ideas (Hill, 2014b). The code of *linking between representations* examines how teachers connect between different representations of a mathematical concept (Hill, 2014b). *Patterns and generalizations* code is about whether a mathematical generalization is developed (Hill, 2014b). *Multiple procedures or solution methods* code is concerned with looking for multiple procedures of a problem type or multiple solution methods of a problem (Hill, 2014b). The code of *mathematical language* focuses on the teacher/students' fluency in mathematical language (Hill, 2014b). We used these six sub-codes on the *x*-axis and their levels (*not present, low, mid, high*) on the *y*-axis of the 3D-TPIM.

SAMR Model

SAMR model evaluates the level of technology integration in the classroom (Puentedura, 2010; 2020). It defines four sub-levels in two primary levels: *substitution* and *augmentation* (at the *enhancement* level), *modification* and *redefinition* (at the *transformation* level). Technology acts as a direct tool substitute with no functional change at the *substitution* level, while at the *augmentation* level, it acts as a direct tool substitute with functional change. The use of technology permits a substantial task redesign at the *modification* level. At the *redefinition* level, new tasks emerge that would not have been possible without technology. We also added not using technology as another level when PMT did *not use technology* for specific purposes. We represented these five levels on the *z*-axis of the 3D-TPIM.

Methods

We considered a single case study appropriate for an in-depth investigation of TPACK with a 3D-TPIM. Although the 3D-TPIM offers a quantitative approach by determining TPACK with three-dimensional coordinate points, how these points are determined needs to be examined qualitatively to increase the study's validity. The participant was a senior PMT (male) enrolled in a four-year mathematics teacher preparation program at a state university in Türkiye. We chose this PMT because he used technology in his teaching. He was a senior PMT, so he attended various courses such as technology-enhanced mathematics teaching. We collected data in the context of the Teaching Practicum II course. It was a 10th-grade class in a Science High School with twenty students. The data consists of a lesson on trigonometric functions using a modelling activity using GeoGebra. The classroom had a central screen, and students did not have their own devices to work with GeoGebra.

The 33-minute lesson video was divided into three segments following the MQI perspective. While determining the segments, the beginning and end of the teaching activities were considered. In this way, we ensured that each segment formed an integrity within itself. This study will only present the findings of Segment 1. We first determined whether the sub-codes (*mathematical sense-making, explanations, mathematical language, patterns and generalizations, linking between representations,* and *multiple procedures or solution methods*) of the MQI's *richness of the mathematics* component exist in the relevant segment. Then, sub-code levels (*not present, low, mid, high*) were determined. In the next stage, PMT's use of the technology was analyzed by considering the SAMR levels. Accordingly, in the 3D-TPIM, the *x*-axis shows the MQI-richness of the mathematics sub-codes, the *y*-axis shows their levels, and the *z*-axis shows SAMR levels (See Figure 1). We coded the segment independently based on the 3D-TPIM. The initial agreement between the first and second researchers was 0.67, but all disagreements were resolved through discussion to enhance validity.

Findings

This study only presents the findings from the analysis of Segment 1 (between 00.00 and 08.49 minutes of the video). In this segment, the learning goal that PMT focused on is as follows: Graphs of functions of the type $f(x) = a.cos(bx + c) + k$ and the effects of the coefficients on the graph are discussed.

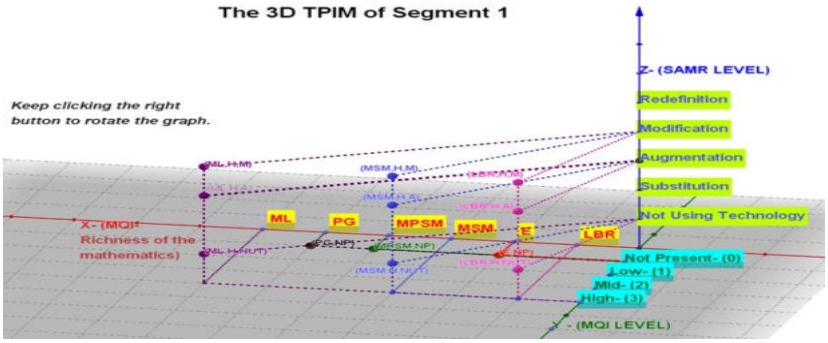


Figure 1: The 3D TPIM of Segment 1 (Click here for a dynamic version)

In line with this objective, worksheets containing different trigonometric functions and graphs were distributed to the students to evaluate their prior knowledge and to remind them about the subject. He

opened GeoGebra software on the smart board and showed the algebraic and graphical representations of functions on the worksheets. He showed many different functions, their transformations, and graphical representations with GeoGebra and explained them. As shown in Figure 1, the MQI codes detected in Segment 1 were *mathematical language*, *linking between representations*, and *mathematical sense-making.*

Mathematical language

During Segment 1, the PMT used mathematical language intensively and fluently and supported students in using mathematical language. For this reason, *mathematical language* was coded at the *'high'* level in this part of the lesson. During this process, PMT utilized either technology (at the *augmentation* and *modification* levels) or the worksheets distributed to the students. The findings are given as coordinates for the *mathematical language* sub-code in Figure 1 are *(ML, H, NUT), (ML, H, A),* and *(ML, H, M).*

The point *(Mathematical Language, High, Not Using Technology)* represents cases where the PMT used *mathematical language* at the *'high'* level without integrating technology into his teaching. While explaining the functions (such as $y = cos\,x$, $y = 1 + cos\,x$, $y = 2cos\,x$, $y = -cos\,x$, $y = cos\,3x$, $y = 1 + 2cos\,x$) and their graphs in the worksheets, PMT used mathematical language intensively by mentioning many different concepts fluently since he did not make any extra drawings on the board.

During his lesson, he used mathematical terminology such as "$cos\,x$ equation, translation, lower bound, symmetry, period, and trigonometric functions" correctly and fluently, using the worksheet in about 1.30 minutes. According to the MQI framework, the *mathematical language* sub-code is coded as *'high'* when the teacher uses mathematical language intensively in the discourse or makes the terminology more understandable to remind students of mathematical meanings (even if the teacher does not use language intensively) (see, Hill, 2014b).

The point *(Mathematical language, High, Augmentation)* represents cases where the PMT used *mathematical language* at the *'high'* level by integrating technology into his teaching at the *augmentation* level. Regarding the *mathematical language* sub-code, we observed that the PMT could use mathematical terminology intensively in a short time by explaining the algebraic and graphical representations of many different $cos\,x$ functions using GeoGebra. However, since he did not use the slider tool, we considered the technology used at the *augmentation* level of the SAMR model (see Figure 2a & 2b). In this context, the relevant section of the lesson showing the contribution of technology to the *mathematical language* sub-code is as follows:

> PMT: What if I multiply the $cos\,x$ function by 2? It would have to expand…Yes, the upper and lower intervals have expanded. So, what if I multiply x by a number? Its period should change (He drew the function $f(x) = 2cos(2x) + 1$ using GeoGebra). Yes, this time, I've shrunk it. What I'm actually changing here is the period.

As shown above, using GeoGebra, he obtained the function $f(x) = 2cos(2\,x) + 1$ and used mathematical terminology such as "period" to explain how the graph changed.

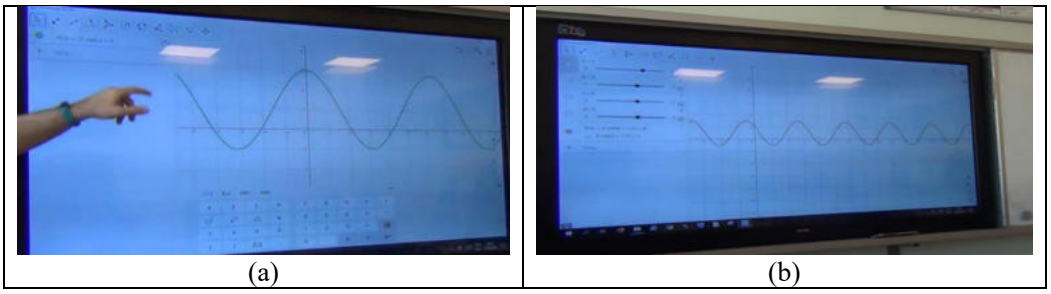
As explained above, we observed that PMT drew many different functions in GeoGebra (without using the slider tool) in a short time and used mathematical language intensively in his explanations

about them, reminding the students of the meaning of the concept of period to make the terminology more understandable. For this reason, *mathematical language* was coded at the *'high'* level, and this finding was shown in the 3D-TPIM as the point *(ML, H, A).* In the analyses of other sub-codes related to the use of technology, the graphical drawings made without using the slider tool in GeoGebra were similarly coded at the *augmentation* level.

The point *(Mathematical Language, High, Modification)* represents cases where the PMT used *mathematical language* at the *'high'* level by integrating technology into his teaching at the *modification* level. PMT expressed the function as $(f(x) = a.cos(bx + c) + d)$ using the slider tool in GeoGebra and obtained a different cosine function when he moved the slider. The task that PMT expects from his peers by using this feature is to make inferences about how the change in the coefficients of the cosine function changes its graph. In this process, similar to the dialogue above, the *mathematical language* reached at the *'high'* level. In this respect, in the analyses of other sub-codes related to the use of technology, obtaining different functions by using the slider tool in GeoGebra was coded as the *modification* level. For this reason, this finding is shown with the point *(ML, H, M)* in the 3D-TPIM. All these findings imply that technology contributes positively to the intensity of using mathematical language.

Linking between representations

Linking between representations, another code we identified in this part of the instruction, was also at the *'high'* level. The PMT used the periodic function's algebraic, verbal, and graphical representations and established the connection between them (see Figure 2a & 2b). Since more than one type of representation was used together and connections were established, the code level was determined to be *'high'* in this context. In this process, the PMT either did *not use technology* or used technology at the levels of *augmentation* or *modification*.



(a) (b)

Figure 2: A screenshot of GeoGebra activity

The point *(Linking Between Representations, High, Not Using Technology)* represents cases where the PMT linked different representations at the *'high'* level without integrating technology into his teaching. On the worksheets he distributed to his peers, he focused on algebraic and graphical representations of functions. He made connections between them through the graphical drawings on the worksheet.

The point *(Linking Between Representations, High, Augmentation)* represents cases where the PMT linked different representations at the *'high'* level by integrating technology into his teaching at the *augmentation* level. Using GeoGebra, he explained the cosine function by showing different types of

representations (algebraic, graphical). In the process in which he drew the graphs one by one without using the slider tool, the level of technology use was determined as *augmentation* (see Figure 2a).

The point *(Linking Between Representations, High, Modification)* represents cases where the PMT linked different representations at the *'high'* level by integrating technology into his teaching at the *modification* level. The use of technology was at the level of *modification* (see Figure 2b) in the process in which PMT used the slider tool in GeoGebra to help students make mathematical inferences from different functions and their graphs (e.g., concluding that the function $f(x) = cos\, x + 1$ shifted up by 1 unit compared to the original function). Here, since the PMT made explanations about how the change in the algebraic representation of the cosine function affected its graphical representation (see PMT's explanations in the (*MSM, H, M)* section for detailed information), *linking between representations* was coded as the *'high'* level in this part of the lesson. This finding was shown in the 3D-TPIM as the point *(LBR, H, M)*.

Mathematical sense-making

The last code in Segment 1 is *mathematical sense-making*. In this context, the PMT emphasized making sense of the cosine function, which is a periodic function, through its algebraic and graphical representations. In other words, since making sense of how the function's graphical representation changes as its algebraic representation changes continues throughout the segment, the level of this code was coded as *'high'*.

The point *(Mathematical Sense-making, High, Not Using Technology)* represents cases where the PMT focused on *mathematical sense-making* at the *'high'* level without integrating technology into his teaching. He targeted sense-making by concentrating on the relationships between cosine functions and their graphs on the worksheet. According to the MQI framework, the *mathematical sense-making* sub-code of this segment is coded as *'high'* when it comes to making sense of the relationships between mathematical ideas and their representations. While answering the questions in the worksheet, the PMT explained how changes in the algebraic representation of the $cos\, x$ function affect its graphical representation and how the period of the function changes. For this reason, *mathematical sense-making* was coded at the *'high'* level in this part of the lesson, and this finding was shown in the 3D-TPIM as the point *(MSM, H, NUT)*.

The point *(Mathematical Sense-Making, High, Augmentation)* represents cases where the PMT focused on *mathematical sense-making* at the *'high'* level by integrating technology into his teaching at the *augmentation* level. PMT sketched the graphs of cosine functions in GeoGebra without using the slider tool and expressed their relationship (see Figure 2a). PMT entered the $cos\, x$ function in GeoGebra and explained how the change in the algebraic representation of the function affects its period. Therefore, this part of the lesson coded as *mathematical sense-making* at the *'high'* level. The technology usage level was coded as *augmentation* since it explained the relationships of the functions obtained by applying different transformations to the cosine function without using sliders in GeoGebra (see Figure 2a). This finding is shown as the point *(MSM, H, A)* in the 3D-TPIM.

The point *(Mathematical Sense-Making, High, Modification)* represents cases where the PMT focused on *mathematical sense-making* at the *'high'* level by integrating technology into his teaching

at the *modification* level. PMT has shown how a change in the algebraic representation of the cosine function affects its graphical representation and the relationship between them through the slider tool in GeoGebra (see Figure 2b).

A sample of the expressions he used while making his drawings in Figure 2b is presented below:

> PMT: Let me have constants *a,b,c,d.* (In the meantime, he created four sliders between -5 and 5 in GeoGebra.) Let me retake the cosine function. (By writing $(f(x) = a.cos(bx + c) + d)$ I add a constant d. Look, as the number we multiply gets larger, it expands; as it gets smaller, it shrinks.

As understood above, PMT utilized technology at the *modification* level to help students make sense of periodic functions. This finding is shown in the 3D-TPIM as the point *(MSM, H, A).*

Discussion

This study aimed to examine PMT's TPACK in the context of MQI in a real classroom using the 3D-TPIM. The findings showed that the PMT used technology for *mathematical sense-making* and *linking between representations* at the *augmentation* and *modification* levels. The findings also revealed that the effective use of *mathematical language* supported technology-enhanced instruction. *Mathematical sense-making, mathematical language,* and *linking between representations* sub-codes were found to be at the *'high'* level when using technology. In addition, using technology at the *augmentation* and *modification* levels not only helped the PMT regarding the sub-codes mentioned but also increased the levels of these sub-codes. In this respect, we argue that using technology in teaching contributes to improving the mathematical quality of instruction.

The *substitution* and *redefinition* levels of the SAMR model, explanations, multiple procedures and solution methods, and patterns and generalizations sub-codes of MQI were not observed in the lesson segment. Analyzing the first 08.49 minutes of the instruction, which constitutes the introductory part, may have played a role in obtaining this finding. For example, the *"explanations"* sub-code was encountered in the second and third segments, where the participant elaborated on the concepts. The mentioned sub-codes and the *substitution* level were present in other segments. The lack of equipment such as tablets, mobile phones, and laptops in the hands of the students may have been a factor in the absence of the *redefinition* level. In using technology at the *redefinition* level, students need to experience the process of self-discovery through technology (Kihoza et al., 2016). The relevant literature shows that teachers have difficulty using technology at this level (Akkoç & Hangül, 2023).

This study proposed a 3D version of the TPIM that was initially two-dimensional (Akkoç et al., 2022; Akkoç & Hangül, 2023). With the third dimension, the z-axis (MQI sub-code levels), the revized matrix provided a more nuanced picture of TPACK. The levels of the MQI sub-codes were also used to operationalize the MQI framework following its original form to contribute to using both theoretical frameworks by demonstrating the impact of MQI in examining TPACK.

We suggest that the 3D-TPIM can be used to determine the development of pre-service teachers or teachers in intervention studies since it offers a quantitative approach. Similarly, teacher educators could systematically employ 3D-TPIM framework to analyze instructional practices, diagnose pedagogical challenges, and monitor teachers' or PMTs' professional growth over time. Finally, other

components of the MQI will be added to the *x*-axis of the 3D-TPIM to investigate lessons involving different PMTs or experienced teachers and covering different mathematical topics in future studies.

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Preservice teachers' self-perceptions about their knowledge for teaching

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Teacher training programs are crucial for supporting teacher candidates' professional development but candidates' beliefs, perceptions and motivations about teaching and learning are likely to influence how much they are benefited from the opportunities provided in those programs. In this paper, we aimed to explore how preservice mathematics teachers' self-perceptions about their knowledge for teaching varies with respect to some variables. We collected data from a total of 749 elementary and secondary mathematics preservice teachers from 10 universities via Self-perception about Teacher Knowledge scale. Preservice teachers' self-perceptions varied with respect to their universities, year at the program (junior vs senior students), teaching experiences and commitment to teach after graduating from their programs.

Keywords: Teacher knowledge, self-perception, teaching experience, becoming a teacher.

Introduction

Teacher training is one of the hot topics of educational specialists' and policymakers' agenda since many studies point out the fact that quality in education can be achieved and sustained by leveraging quality of teachers in terms of their knowledge and skills for effective teaching (Darling Hammond, 2020). The structure, coverage and dissemination of teacher training programs vary with respect to countries' vision, culture, needs, policies, economics and so forth, however, equipping candidates with knowledge of subject matter to be taught and subject-specific pedagogy of teaching and learning are common for any teacher training programs (Cochran-Smith et al., 2015; Kleickmann et al., 2013). For instance, mathematics teacher candidates are basically given opportunities to learn about and/or make practice about mathematics, how to teach mathematics and tailor their instruction according to readiness and needs of students in those teacher training programs.

For effective mathematics teaching, teachers should be equipped with strong mathematical and pedagogical knowledge for teaching accompanied with well-developed teaching skills. Ball and her colleagues (2008) described such knowledge as *mathematical knowledge for teaching* (MKT) as a composition of six knowledge types categorized under subject matter (SMK) and pedagogical content knowledge (PCK). In terms of SMK, they noted that a teacher should know about the fundamentals of mathematics, relationships between mathematical concepts, facts, and procedures as well as their connections with other disciplines and real life. They described PCK as knowing about effective and appropriate ways of teaching mathematics (knowledge of content and teaching (KCT)), how students learn mathematics including their difficulties and misconceptions in math (knowledge of content and students (KCS)) and coverage and aims of math curriculum identified for a specific grade level (curriculum knowledge (CK)). Studies revealed that many mathematics teacher education programs

somehow offer courses aiming to address and foster candidates' knowledge and skills about these components of MKT (Cochran-Smith et al., 2015).

Besides opportunities given at teacher training programs for developing professional knowledge and skills, teacher candidates' intrinsic and extrinsic motivations, beliefs and thoughts about teaching and learning are likely to influence their performance in those programs as well as after they begin to work as a teacher at schools (Baumert & Kunter, 2013; Charalambous, 2015; Fray & Gore, 2018). When preservice teachers are dedicated to be a teacher, have higher level of self-efficacy beliefs and perceptions about their teaching abilities, perceiving teaching as a valuable profession and so forth, they become more eager to learn how to be a *good* teacher and improve their teaching skills (e.g., Barış Pekmezci & Dağdeviren Ertaş, 2024; Fray & Gore, 2018). More specifically, preservice teachers feel more confident about their knowledge and skills as they do their practicum at schools or have some informal teaching experiences (Clark et al., 2015; Ferguson et al., 2023). However, studies revealed that preservice or in service teachers may lack of well-developed teacher knowledge and skills even though they possess high level of motivation for teaching and self-perceptions of themselves as a teacher (e.g., Odysseos & Pittalis, 2023). Therefore, in our study, we aimed to explore how preservice mathematics teachers perceive their knowledge for teaching and whether their perceptions are compatible with their actual performance on a short test consisted of items measuring their knowledge.

In this paper, however, we wanted to discuss how preservice teachers' (PSTs) self-perceptions about their knowledge for teaching, specifically their MKT, varied with respect to their teacher training programs, year at the program, overall academic achievement in the program, informal teaching experience and willingness to work as a teacher after completing the program. We would also discuss having teaching experience and commitment to become a teacher nourished PSTs' self-perception about which MKT component more. Moreover, we would talk about stimuli behind PSTs' choice of teaching as a profession.

Methods

Participants

In this study we used convenient sampling such that we asked our colleagues whether they would inform their junior (Year 3) and senior (Year 4) preservice teachers from elementary and/or secondary mathematics education program about the study. In Türkiye, elementary mathematics PSTs are trained to teach grades from 5 to 8 (middle school), while secondary mathematics PSTs are supposed to teach grades from 9 to 12 (high school). Both elementary (EMT) and secondary (SMT) mathematics teaching are 4-year bachelor programs administered under faculty of education. The candidates take common core mathematics and pedagogy courses in the first two-years of the program but later they begin to take area-specific departmental courses. They complete a total of 144 hours of practicum at relevant schools in their last year while taking some other courses at the university. We collected data from 749 PSTs (551 female, 198 male) from 10 universities such that three of them are ranked at top 10 public universities on teacher education according to Turkish Higher Education Council (HEC) ranking system which is based on students' scores obtained from university entrance exam (UEE) and two of them were foundation (private) universities offering elementary mathematics

teaching program (see www.yokatlas.gov.tr). Moreover, there are 12 universities offering SMT programs and 86 universities offering EMT programs such that UEE scores of SMT programs are generally higher than EMT programs in Türkiye.

Data collection and analysis

We revised our earlier *Self-perception about Teacher Knowledge* scale (Kilic, 2015) which consisted of 24 items with 5-point Likert type style. Based on our expertise, experiences in our earlier studies and relevant literature, we added four new items to strengthen the items having respectively lower item-loading in the previous study and asked participants to give a point between 1 and 10 (from least to most) to indicate how much they agreed about given statements. After eliminating cross-loading items, we obtained a 24-item scale with 5 factors (Chi-square/df=3.446, GFI=.905, CFI=.942, RMSEA=.061) and .91 as reliability coefficient. We named the factors as pedagogical knowledge (PK) (4 items), knowledge of content and teaching (KCT) (10 items), knowledge of content and students (KCS) (4 items), curriculum knowledge (CK) (2 items) and subject matter knowledge (SMK) (4 items) based on Ball and colleagues' MKT model (see Ball et al., 2008). Sample items from each factor is given below in Table 1.

Table 1: Sample items from self-perception about teacher knowledge scale

Item	Factor
I can use appropriate teaching strategies according to students' needs	PK
I can prepare tasks to support students' mathematical thinking skills	KCT
I have a repertoire of students' possible misconceptions in mathematics	KCS
I know about the order of topics with respect to grade levels identified in official math curriculum	CK
I have a strong content knowledge in mathematics	SMK

For ease of interpretation, we categorized PSTs' perceptions about their knowledge based on their total scores obtained from the scale as follows: A score between 210 and 240 refers to very high, 170 and 210 is high, 130 and 170 is moderate, 90 and 130 is low and below 90 is very low self-perception about teacher knowledge. We also used a 15-item form to collect some demographic information about participants including their GPA, teaching experiences and willingness to work as a teacher after graduation. Almost all items in the form, they were given options to select (e.g., their year in the program) which fitted to them and we used that data to classify the participants for further analysis. There were a few items asking them to write their answers (e.g., which grade levels they had teaching experiences) and we used that data to calculate frequencies of given answers.

Results

In this study, we applied *Self-perception about Teacher Knowledge* scale to elementary (EMT) and secondary (SMT) mathematics preservice teachers to collect data however we preferred to analyse some of the findings separately for further discussion. We initially looked at whether PSTs' perceptions varied with respect to which program they enrolled and we found that there was no significant difference between EMT and SMT preservice teachers' self-perceptions ($t_{747,.05}$=-1.022, p=.307). As seen in Table 2, both EMT and SMT mathematics preservice teachers' perceptions about

their knowledge was "high" according to our classification. Although there were some EMT preservice teachers with having low level of perception, almost all SMT preservice teachers had at least moderate level of self-perception about their teacher knowledge.

Table 2: Descriptive analysis of self-perception about teacher knowledge scale

	N	Min	Max	Mean	Std. dev.
EMT					
Year 3	336	109	236	176.05	25.2
Year 4	320	101	240	185.92	23.25
Total	**656**	**101**	**240**	**180.86**	**24.75**
SMT					
Year 3	44	131	215	181.27	20.30
Year 4	49	121	227	185.73	22.50
Total	**93**	**121**	**227**	**183.62**	**21.49**

We analysed how EMT and SMT preservice teachers' perceptions about their teacher knowledge varied with respect to their universities, the year in the program, academic achievement (GPA) and teaching experiences as shown in Table 3. We found that universities, year in the program and teaching experience led significant variance among EMT preservice teachers' self-perceptions but not for SMT preservice teachers'.

Table 3: Analysis of preservice teachers' self-perceptions about their teacher knowledge

		EMT					SMT				
		N	Mean	Std. dev.	F/t	p	N	Mean	Std. dev.	F/t	p
University	PubUniv1	113	181.66	24.96	5.739	<.000 η^2=.074	26	191.11	19.54	1.525	.214
	PubUniv2	62	184.05	22.20			20	180.65	20.35		
	PubUniv3	23	184.13	17.87			19	179.47	27.39		
	PubUniv4	64	173.81	27.22							
	PubUniv5	66	168.64	25.12							
	PubUniv6	105	190.89	21.57							
	PubUniv7	94	181.49	23.04			28	181.61	18.75		
	PubUniv8	65	179.91	23.95							
	PriUniv1	42	184.67	27.99							
	PriUniv2	22	166.64	22.16							
Class	Year 3	336	176.05	25.2	-5.203	<.000 d=.406	44	181.27	20.30	-1.000	.320
	Year 4	320	185.92	23.25			49	185.73	22.50		
GPA	0.00-1.99	12	183.92	26.37	2.843	.059	2	176.00	28.28	0.149	.861
	2.00-2.99	259	178.03	26.16			35	183.17	25.36		
	3.00-4.00	385	182.68	23.58			56	184.18	18.91		
Teaching experience	None	203	177.04	23.56	-2.659	.008 d=.225	13	186.92	25.71	0.595	.553
	Yes	453	182.58	25.10			80	183.09	20.86		

Notes: 1)"PubUniv" refers to public universities and "PriUniv" refers to foundation universities. 2) Each type of university is ranked according to Turkish HEC ranking system of universities in terms of students' UEE results. 3) SMT program are only offered in PubUniv1, PubUniv2, PubUniv3 and PubUniv7.

EMT preservice teachers who were from top 3 universities in our list indicated very close mean scores and their self-perceptions were significantly higher than PubUniv5 and PriUniv 2 according to post hoc analysis results. Moreover, the effect size of this difference can be accepted as at moderate level

(Muijs, 2011). Although SMT preservice teachers from the highest ranked public university revealed the highest mean score among all participants, their mean score did not vary significantly from other universities offering SMT program. In terms of the year at the program, senior preservice teachers were feeling more confident about their teacher knowledge as expected. The difference between EMT preservice teachers was significant with a modest level of effect size while it was not the case for SMT preservice teachers. We observed that as SMT preservice teachers' academic achievement score (GPA) got higher, they felt more confident in their knowledge but such difference was not significant as seen in Table 3.

Among 656 EMT preservice teachers 453 of them (69%), among 93 SMT preservice teachers 80 (86%) of them noted that they had some kind of teaching experiences in the form of tutoring (approx. 63%) or part time teaching at a private institution (approx. 8%) or being a volunteer in faculty-school collaboration programs (<1%) or taking a part in community service studies (approx. 4%). Almost half of EMT preservice teachers (approx. 51%) noted that they had experience with middle school students while 29% of them had experience with either both primary and middle school students or middle school and high school students. SMT preservice teachers mostly had experience with high school students (35%) or both middle school and high school students (40%).

We observed that EMT preservice teachers with teaching experience perceived themselves as possessing higher knowledge for teaching in comparison to ones who did not have such an experience while it was not the case, even conflicting, for SMT preservice teachers as seen in Table 3. However, the effect size of such difference between the groups was small. We further analysed on which component of MKT that teaching experiences led a variance between the groups. EMT preservice teachers with teaching experiences revealed higher self-conception about their CK ($t_{654, .05}$=2.643, p=.008, d=.223), KCS ($t_{654, .05}$=2.242, p=.025, d=.189), and SMK ($t_{654, .05}$=2.025, p=.043, d=.171) as well as PK ($t_{654, .05}$=2.626, p=.009, d=.222) with respect to ones who did not have such experiences. Although PSTs with teaching experiences noted higher self-perception about their KCT (M=7.87, SD=1.11), it was not significantly higher than from PSTs with no teaching experience (M=7.71, SD=1.13).

Because PSTs had high self-perceptions about their teacher knowledge, we wondered whether they wanted to work as a teacher after graduating from their programs. Approximately 91% of EMT and 90% of SMT preservice teachers declared that they wanted to work as a teacher. Those of EMT preservice teachers had higher self-perception about their teacher knowledge with respect to ones who did not want to ($t_{643, .05}$=2.264, p=.024, d=.336) where the effect size of such variance was at modest level. In terms of MKT components, EMT preservice teachers revealed higher self-conception about their KCS ($t_{643, .05}$=2.883, p=.004, d=.227) and KCT ($t_{643, .05}$=2.164, p=.024, d=.336) as well as PK ($t_{643, .05}$=2.170, p=.030, d=.323) with respect to ones who did not want to do teaching. We did not observe such a difference among SMT preservice teachers.

We also looked at the distribution of PSTs with respect to their universities, year at the program, GPA, and teaching experiences in terms of commitment to teach after completing their programs. Approximately one third of PSTs who wanted to work as a teacher were from top 3 universities in our sample while approx. half of them from the last 3 public universities. In line with this finding,

approx. 58% of PSTs who did not want to work as a teacher were from top 3 universities while 26% of them from the last 3 public universities. Moreover, both junior (approx. 90%) and senior (approx. 94%) PSTs were very dedicated to work as a teacher when they completed their program. Similarly, 95% of PSTs with GPA between 3.00-4.00 and 90% of preservice teachers with GPA between 2.00-3.00 expressed their desire to work as a teacher. Furthermore, approx. 72% of PSTs having some sort of teaching experiences noted that they wanted to work as a teacher when they accomplished their program.

We asked for stimuli behind PSTs' choice of working as a teacher or not when graduate from the program. Among the others, the most popular answers for working as a teacher were i) to like teaching as a profession (approx. 38%), ii) to had been enrolled to the program to become a teacher (approx. 14%) and iii) to like mathematics and kids (approx. 15%). The PSTs noted that they did not want to work as a teacher because i) they realized that teaching is not congruent to themselves, ii) teachers are getting low salaries and iii) they want to do another job (approx. 20% each).

Discussion

In this paper, we wanted to discuss how preservice mathematics teachers' self-perceptions about their teacher knowledge varied with respect to some factors including the program and universities they enrolled, year at the program, teaching experience, desire to work as a teacher after graduation. We found that being enrolled in EMT or SMT program did not make a difference in PSTs' perceptions but other factors led differences with statistically small or moderate level of effect sizes.

The preservice teachers of EMT programs from the highest-ranking universities revealed significantly higher self-perceptions about their knowledge for teaching. Those universities were the oldest universities with faculty of education in Türkiye whose graduates are accepted as well-qualified teachers or educators of the country. Therefore, PSTs might feel that their EMT or SMT programs are designed in a way to support their professional development and they would become well-qualified mathematics teachers in the field. Because those universities accept candidates who perform very well at university entrance exam, PSTs' self-conceptions about themselves might be valid since these PSTs' SMK and PCK tend to be higher than their peers in other universities (Kleickmann, 2013). From SMT preservice teachers' lenses, their teacher training programs and universities enabled them similar opportunities for development of their professional knowledge and skills as different from their EMT preservice teacher peers. Additionally, the number of participants from SMT programs was comparably lower than EMT programs which possibly had an effect on the results of statistical analysis, that is, even though there a difference between their mean scores, it was not significant. Furthermore, the reason behind the lowest mean obtained from EMT preservice teachers from recently established public university might be the PSTs' suspicious about the quality of their program or coverage of their courses or proficiency of their instructors or all. However, SMT participants were from old public universities which carry out the program for decades.

As expected, senior PSTs' self-perceptions about their knowledge for teaching was higher than junior PSTs because they had already taken more courses about teaching mathematics (Ferguson et al., 2023). However, PSTs' academic achievement (GPA) did not make any significant difference in their

self-perceptions. PSTs might think that no matter how they are performing in their courses, their programs are already designed to improve their professional knowledge and skills.

For EMT preservice teachers having a teaching experience led a significant difference in terms of self-perceptions while it was not the case for SMT preservice teachers. Seemingly, EMT preservice teachers thought that they learned from their practice more than theoretical knowledge provided in their programs (Ferguson et al., 2023) as opposed to studies showing that PSTs' self-perception about their teaching abilities gets lower as they have more practice in teaching (e.g., Clark et al., 2015) which probably was the case for SMT preservice teachers in our sample. On the other hand, as we expected, working closely with students enabled PSTs to perceive themselves as possessing the knowledge of mathematics they are teaching (SMK), coverage of the math curriculum (CK), how students learn mathematics and their possible difficulties and misconceptions (KCS) and appropriate teaching strategies and methods (PK) in comparison to PSTs with no teaching experience. Moreover, EMT preservice teachers with teaching experience desire to work as a teacher when they graduate more than PSTs with no teaching experience. They perceived that they were well-equipped in terms of KCS, KCT and PK which can be thought as critical components of PCK for effective teaching.

Most of PSTs aimed to work as a mathematics teacher when graduate from their programs because they were mostly intrinsically motivated to be teachers as noted by other studies (Fray & Gore, 2018). The PSTs stated that they like teaching, mathematics and kids and their training programs provide opportunities to improve their professional knowledge and skills. However, approx. one third of PSTs from top 3 universities declared their intention to work as a teacher when they complete the program while approx. 58% did not want to. Even though these universities were thought as the pioneers in teacher training, low salaries and facilities given for teachers in Türkiye might be the reason for changing their career plans for those PSTs (Fray & Gore, 2018). Indeed, job opportunities after graduating from teacher training programs is getting a serious treat for achieving and sustaining quality in teacher training and school education in long run in Türkiye.

Briefly, PSTs thought that their teacher training programs provided enough opportunities to learn about and make a practice for being a mathematics teacher and they wanted to do their job when they accomplish their programs. Moreover, they are learning from their informal teaching experiences which is likely to contribute to development of their PCK. Although PSTs' self-perceptions about their knowledge for teaching should be confirmed by data based on their actual performance on a knowledge test to discuss the quality of their teacher training programs, we can conclude that teacher training programs are providing similar opportunities for PSTs and candidates are mostly satisfied with those given opportunities. Thus, they want to work as a teacher when they graduate from their programs even though their salaries will be lower than other professions.

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The Theory of planned behavior as a tool for linking beliefs and knowledge within the MTSK model

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The Mathematics Teacher's Specialised Knowledge model (MTSK) is the inaugural model to incorporate teachers' beliefs alongside their knowledge. However, most studies utilising this model address either knowledge or beliefs in isolation. This study proposes a framework for elucidating the reciprocal relationship between them through the Theory of Planned Behaviour (TPB). In the paper, we demonstrate connections between the beliefs of pre-service teachers (PSTs) and three subdomains of the MTSK model: Knowledge of Topics (KoT), Knowledge of Mathematics Teaching (KMT), and Knowledge of Features of Learning Mathematics (KFLM) in the context of functional thinking development.

Keywords: Teachers' beliefs, teachers' knowledge, MTSK model, Theory of planned behaviour.

Introduction

A number of researchers have highlighted the interrelated nature of teachers' knowledge and beliefs, and their concomitant impact on the quality of teaching (e.g., Thompson, 1992; Charalambous, 2015). The frequently used model *Mathematics Knowledge for Teaching* (MKT) (Ball et al., 2008) was the subject of criticism precisely because it did not include the teacher's beliefs in any way. The newer model *Mathematics Teacher's Specialised Knowledge* (MTSK) (Carrillo-Yañez et al., 2018) sought to address this issue. However, even when beliefs have been incorporated into the model, only few papers (utilising the MTSK model) have explicitly discussed both knowledge and beliefs (e.g., Vasco & Climent, 2017).

One of the reasons for studying beliefs is to gain insight into their potential influence on human behavior. In this regard, the Theory of Planned Behavior (TPB) (developed by Icek Ajzen, 1991) may be a useful theoretical framework to consider, as it has been applied in a range of research areas and has demonstrated efficacy in linking beliefs and behavior, or the intention to engage in a specific behavior. It is also being employed with increasing frequency in the design and evaluation of interventions aimed at modifying behavior (Ajzen & Schmidt, 2020). According to Bosnjak et al. (2020, p. 352) "the TPB is one of the most applied theories in the social and behavioral sciences." This prompts the question of whether there are any connections to be made between the two models under discussion here, namely MTSK and TPB. The following research question is therefore posed: what are the connections between the various areas of MTSK knowledge and beliefs from the perspective of TPB? As the topic influences the content of teachers' knowledge and teachers' beliefs about teaching and learning such a topic (Carrillo-Yañez et al., 2018), we limit our response in this paper to functional thinking and its development.

This study was conducted as part of the FunThink Erasmus+ project, which aimed to enhance functional thinking from primary to upper secondary school. The project's objective was to develop

functional thinking, which can be defined as a mode of thinking that is oriented towards relations, dependencies, and changes. This involves the ability to identify, generalise, and abstract relations between co-varying quantities, to represent these relations through language, symbols, and graphs, and to utilise generalised representations in problem-solving situations (Pittalis et al., 2020).

Theoretical background

MTSK model

The MTSK model (Carrillo-Yañez et al., 2018) is an analytical tool designed for examining the complexity of teachers' knowledge. It consists of three main parts: *Mathematical Knowledge* (MK), *Pedagogical Content Knowledge* (PCK) and Beliefs. MK consists of the subdomains: *Knowledge of topics* (KoT), *Knowledge of the structure of mathematics* (KSM) and *Knowledge of practices in mathematics* (KPM). PCK includes three further subdomains: *Knowledge of mathematics teaching* (KMT), *Knowledge of features of learning mathematics* (KFLM) and *Knowledge of mathematics learning standards* (KMLS). Beliefs are divided into two types, *Beliefs about mathematics* and *Beliefs about mathematics teaching and learning*. They are placed at the centre of the model to emphasise the reciprocity between them and the other subdomains. Given the restricted length of the paper, we provide a brief overview of the three subdomains KoT, KMT, and KFLM, as we will use them to illustrate the connection between the "knowledge" subdomain of MTSK and beliefs.

KoT describes what a teacher should know about the content of the mathematics they teach. It comprises four categories: (1) *Definitions, properties and foundations* includes knowledge of the relationships between concepts and properties within the subject unit, (2) *Procedures* includes knowledge of different procedures for solving problems within the subject, including their critical evaluation in terms of conditions of use, (3) *Registers of representation* includes the ability to use different representations to discuss a concept and to solve problems, (4) *Phenomenology and applications* includes knowledge of the different contexts in which the topic can be applied.

KMT concerns knowledge about the teaching of mathematics and includes the following three categories: (1) *Theories of mathematics teaching* which can be applied to the design of learning opportunities, (2) *Teaching resources (physical and digital)* that can be used for specific mathematical content (3) *Strategies, techniques, tasks and examples* (their potential or limitations for teaching specific mathematical content).

KFLM focuses on understanding how mathematics is learned. KFLM contains four categories: (1) *Theories of mathematical learning* is knowledge of theories of the cognitive development of the pupil, (2) *Strengths and weaknesses in learning mathematics* that pupils have or might have in relation to the mathematical content, (3) *Ways pupils interact with mathematical conten*t is knowledge of pupils' strategies, and (4) *Emotional aspects of learning mathematics,* which includes awareness of the emotional and motivational aspects that influence pupils' engagement and success.

Theory of Planned Behavior

To conceptualize beliefs, we will use the Theory of Planned Behavior (TPB) (Ajzen, 1991). The TPB describes how do beliefs influence the *Intention* to conduct a behavior of interest, and subsequently

with consideration of *Actual (or Perceived) behavioral control*, they influence particular behavior (Figure 1).

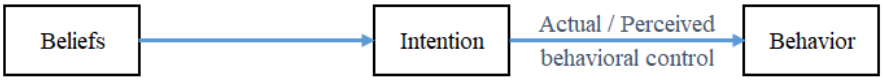

Figure 1: Simplified version of the TPB

The TPB is applicable if the behavior of interest can be defined in terms of its *target*, the *action* involved, the *context* in which it occurs, and the *time* frame. Each of these elements can be defined at varying levels of specificity or generality (Ajzen & Schmidt, 2020). For example, to study the implementation of inquiry-based learning in mathematics education may define the behavior such as "using inquiry-based learning to introduce the concept of a linear function".

Ajzen (1991, p. 189) assumes that "people can hold a great many beliefs about any given behavior, but they can attend to only a relatively small number at any given moment", these are *salient beliefs.* The TPB distinguishes between three categories of salient beliefs. Simply said, there are beliefs about the usefulness and/or pleasantness of the expected outcomes, beliefs about other people beliefs and beliefs about one's capacity to control the behavior. In the model, we can find them as *Behavioral beliefs*, *Normative beliefs* and *Control beliefs.* For the next analysis, we need to understand these categories. Figure 2 shows the structural TPB model as it is currently conceptualised.

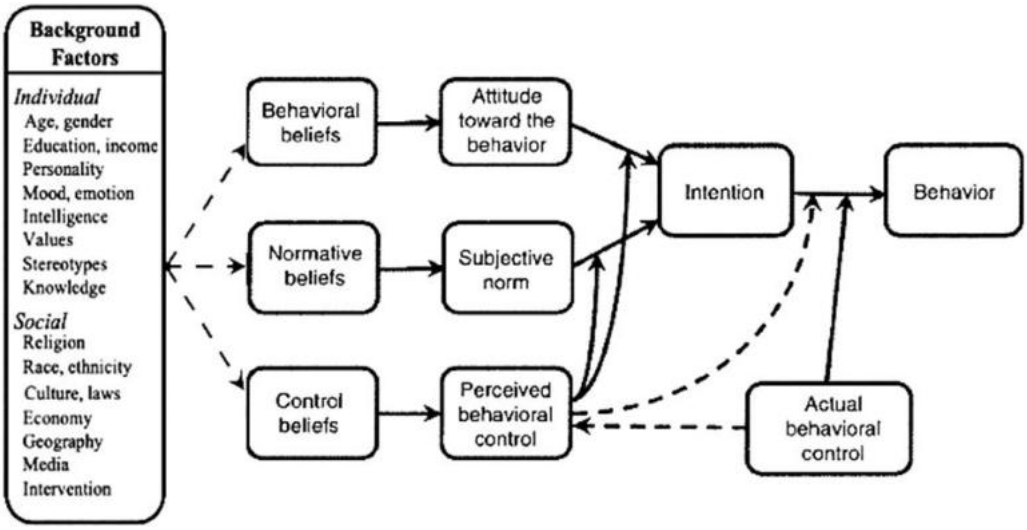

Figure 2: Theory of Planned Behavior with background factors (Ajzen, 2019)

As we can see, the core of the model is a causal chain of effects that begins with the formation of salient beliefs. A *behavioral belief* is the subjective probability that the behavior will produce a given outcome (*instrumental behavioral beliefs*) or experience (*experiential behavioral beliefs*). For example, the belief that if pupils discover something on their own, they understand it better. Behavioral beliefs are theorized to create positive or negative *attitudes toward behavior*. These beliefs answer the question "What do I think about a given behavior?". A *normative belief* is the subjective probability that a significant referent (e.g., leadership, parents, pupils) encourages performance of the behavior or is likely to perform the behavior him- or herself. Normative beliefs contribute to the

overall perceived social pressure to engage in the behavior (*subjective norm*). These beliefs answer the question "What do I think others think about a given behavior?". *Control beliefs* are related to the perceived presence of factors that may facilitate or impede performance of the behavior, such as required skills and abilities, availability or lack of time, money, and other resources. For example, the belief that I can't manage group setting in the classroom. Control beliefs answer the question "Can I perform a given behavior?". They determine the *perceived behavioral control* which refers to people's perceptions of their ability to perform a given behavior. It relates to a specific context and not to a general predisposition, so it usually varies in different situations. Perceived behavioral control is assumed to moderate the influence of attitude and subjective norm on behavioral intention. The *actual behavioral control* is assumed to moderate the effect of intention on behavior. Usually, knowledge about actual behavioral control is limited, therefore perceived behavioral control is used as a proxy to aid in the prediction of behavior (see Figure 2). As for background factors, they also influence intentions and behavior indirectly by their effects on salient beliefs. The dotted arrows in the diagram indicate that there is no necessary connection between background factors and beliefs. Whether or not a particular belief is influenced by a particular background factor is an empirical question (Ajzen & Schmidt, 2020).

Methods

One of the intellectual outputs of the FunThink Erasmus+ project is a course designed for pre-service teachers (PSTs). Data collection occurred at Pavol Jozef Šafárik University in Košice, Slovakia, during the autumn of 2023. This mandatory course consisted of 26 teaching hours, attended by 19 PSTs (grades 5-13), and was placed in the first semester of their master's studies. For them, it was the first course focused on the didactics of mathematics. The PSTs experienced different learning environments based on the principles of inquiry, digital tool use, embodiment and situatedness. They observed and reflected on classroom recordings, discussed pupils' solutions and potential misconceptions.

Each PST submitted two lesson plans: one at the beginning of the course (the pre-lesson plan) and another at the end of the course (the post-lesson plan). Both plans focused on introducing the concept of a linear function, which is part of the Slovak grade 9 curriculum. For the post-lesson plan, PSTs revised the pre-lesson plan by making necessary adjustments and adding comments to explain their changes. Additionally, the post-lesson plan was included in the final assessment. The aforementioned lesson plans were collected as data for our research purposes; however, they were not implemented in actual classroom settings. The pre-service teachers had no prior experience of teaching, although they had started participating in classroom observations in the autumn of 2023. Moreover, eight PSTs were interviewed at the beginning of the semester, and seven of them were also interviewed at the end. Participants were selected based on their pre-lesson plans. All participants were informed about the scientific and developmental objectives of the research.

The semi-structured interviews were transcribed and subsequently analysed together with the lesson plans using content analysis methods (Krippendorf, 1989), examining the PSTs' statements and comments to identify evidence of their knowledge and beliefs. The data were coded by both authors independently using Atlas.ti software. Various MTSK and TPB constructs were used as codes.

The coding was then compared. In this paper, we present data analyses from three PSTs where the clear connections between the models were visible.

Results

Our analysis of the interviews and lesson plans revealed several interesting links between knowledge and beliefs from the perspective of TPB. We describe them in the next paragraphs. The lesson plans reflect the behavioral *Intentions* of PSTs in introducing the concept of a linear function.

KFLM forming behavioral beliefs

Firstly, we identified multiple instances where the Knowledge of Features of Learning Mathematics led to the formation of a belief, which was subsequently integrated into the lesson plan. We will provide two examples.

Example 1: The course addressed the issue of misconceptions that can arise when the concept of a linear function is not adequately taught and learned. Student Nina illustrated how this knowledge, classified as KFLM, influenced her post-lesson plan.

> Nina: In the post-lesson plan, I tried to be more mindful of misconceptions that could immediately arise among the pupils. In the pre-lesson plan, I only started with a task that involved an increasing linear function. In the post-lesson plan, I incorporated a decreasing function as well, so that from the start, it was clear for pupils that not everything is increasing. [...] I hadn't really thought about that in the pre-lesson plan – that I should be careful there.

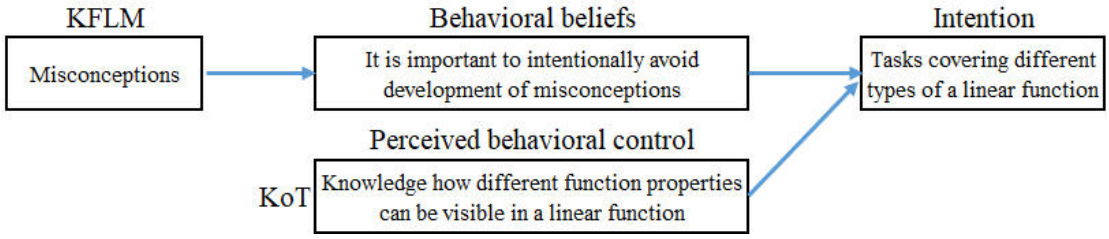

Figure 3: Beliefs as a knowledge (KoT) activator

As Figure 3 describes, Nina's statement indicates that the modification to the lesson plan was not solely a consequence of KFLM growth. She also provided an argumentation based on Knowledge of topics (KoT). Without knowledge about the different properties which linear functions can hold, it would not have been possible for her to choose appropriate tasks. Therefore, in this case, KoT can be considered a perceived behavioral control.

Example 2: During the course, PSTs were introduced to the concept of function as described by Pittalis et al. (2020), which also serves as a basis for developing pupils' functional thinking. Sára demonstrated how this knowledge was applied in the creation of her post-lesson plan.

> Sára: I really liked when we talked about those aspects of the concept of function. It was only then that I realized how that understanding was evolving. That in the beginning pupils seriously can't understand the function. Until they understand input-output, until they can find some kind of connection, [...] and that teaching the functions not only helps to learn the functions, but to learn, even to go through the whole process, those individual aspects.

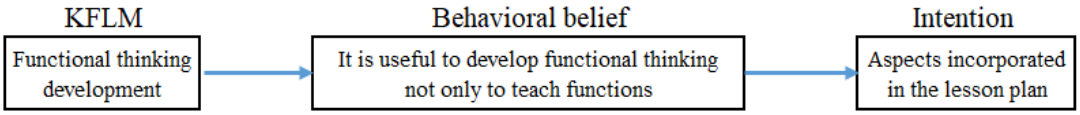

Figure 4: Knowledge (KFLM) as beliefs builder

From Sára's statement, we can infer that her decision to modify the lesson plan was informed not only by her understanding of the development of functional thinking but also by her belief that fostering functional thinking is a fundamental objective of mathematics education.

Limited content of knowledge as a perceived behavioral control

The next example is somewhat antithetical to the preceding Example 1. Here, PST articulated a behavioural belief that was not substantiated by the requisite knowledge. This contradiction resulted in a lesson plan that did not sufficiently meet the expectations for structure and goal fulfilment. It is from the beginning of the semester.

Example 3: Some of the PSTs, including Sára, were interviewed after they were present at the classroom observations. At these observations she could see inquiry-based teaching and she came up with the following thoughts:

Sára: I really liked it when I went to observe the class for the first time. Teacher gave group work, and it wasn't like she dictated the formula for combinations and said, "This is how it will be, apply it to the examples." That's what I experienced. But she gave the pupils three examples so they could discover the formula based on those examples. She called it inquiry-based learning, and it was such a shock for me, like, "Aha, there's a higher purpose," and she started talking about how she leads them to explore. Then they had that "Eureka" moment, realizing they had discovered something, and it wasn't just about memorizing the formula, but they came up with it themselves. […] I think I still don't fully understand what inquiry-based learning is, but I think the conclusion of my lesson plan is somewhat inquiry-based. Pupils can play and experiment there, and it's not just about receiving information, but they are also involved in the process of creating those functions.

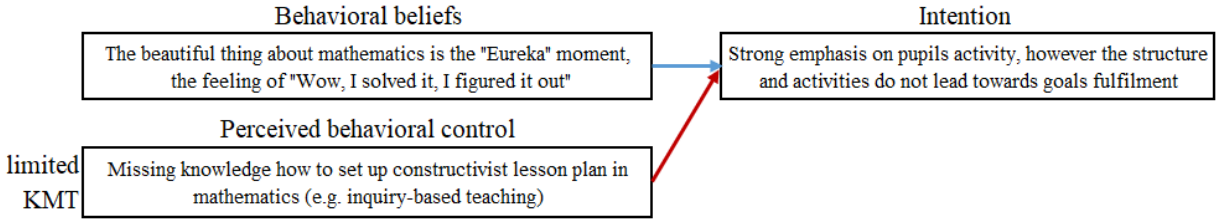

Figure 5: Limited knowledge (KMT) as a barrier of beliefs

As pinpointed in Figure 5, Sara's behavioral beliefs, also derived from an interview, led to a positive attitude toward constructivist teaching. Because she was unfamiliar with this type of teaching in mathematics, she was unable to develop a pre-lesson plan properly. Therefore, in this case, limited Knowledge of Mathematics Teaching (KMT) about the constructivist approach in mathematics, was perceived behavioral control.

Similarly, we found examples where limited KoT (real-life application of functions) restricted PST Jana from incorporating real-life problems into her lesson plan. Even though, her behavioral beliefs were quite clear about it.

Control beliefs in opposition with behavioral beliefs

Increased Knowledge of Mathematics Teaching (experience, examples and theory behind inquiry way of teaching) has led to the development of new salient beliefs. The following illustration demonstrates that these beliefs can potentially be in conflict and mutually constraining.

Example 4: The PSTs were exposed to the practice of inquiry-based teaching during the course and classroom observation. These experiences led to the formation of new behavioral beliefs, which subsequently resulted in the development of positive attitudes. However, in case of Jana, these positive attitudes appeared to be overridden by strong perceived behavioral control formed by her control belief. This perceived control regarding time pressure occurred for Jana in both interviews.

Jana: But I always have this feeling that anything that's somehow different or extra, like when we covered inquiry-based learning, it's great, it's nice, everything's good, but I'm not sure if it won't take up too much of my time. Like, if it might steal time from the lesson. I'm not sure if the allocated time is really suited for it.

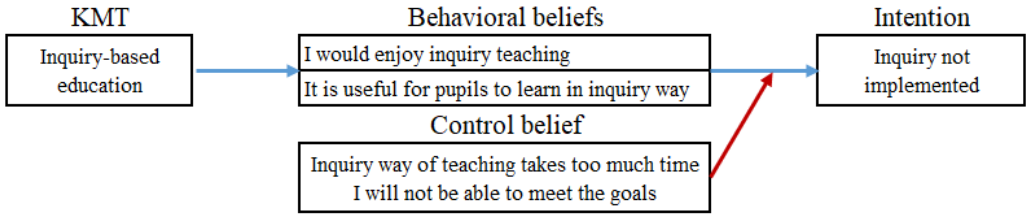

Figure 6: Control beliefs moderate behavioral beliefs

The other control beliefs that emerged from the other interviews were related to KMLS (a lack of experience with how topics are scaffolded) or KFLM (pupils' reactions and the specific needs of the particular class).

Discussion and implications for future research

Returning to our research question – *What are the connections between the various areas of MTSK knowledge and beliefs from the perspective of TPB?* – our findings reveal three distinct types of these links. Firstly, beliefs can prompt teachers to utilize the knowledge they already possess (Example 1). The second type is that knowledge can serve as a foundation for the formation of beliefs (Example 2). Another type of connection occurs when limited knowledge acts as a perceived behavioral control, preventing beliefs from being fully integrated into a lesson plan (Example 3). It is also worthy of note that Example 4 demonstrated how control beliefs moderate the influence of attitude on behavioral intention. Concerning normative beliefs, the PSTs only discussed the pupils, and did not mention other relevant referents. This could be a crucial factor influencing the resilience of newly acquired beliefs once they are put into practice, especially when parents, colleagues, and management come into play. These observations are deemed crucial for designing courses for PSTs, as further work to link the two models is believed to be beneficial.

Acknowledgment

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Promoting and unpacking pre-service teachers' thinking about students with exceptional difficulties in learning mathematics in the context of a university course

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In this paper, we present a qualitative study that unpacks pre-service teachers' thinking about students with exceptional difficulties in learning mathematics after they attended a specially designed university course on the topics of diagnosis and support with both theoretical input and practical application. The theoretical construct of subjective theories and the corresponding methodology was chosen for this study as it integrates knowledge and beliefs and thus provides the most detailed view of pre-service teachers' thinking. Our results show that pre-service teachers exhibit complex and adequate thinking about students with exceptional difficulties in learning mathematics after participating in the course, which contrasts with previous research on pre-service teachers. These findings provide a strong mandate for the large-scale implementation and research of similar courses to the one presented in this paper.

Keywords: Pre-service teachers' thinking, teacher education, learning difficulties, subjective theory.

Introduction

Previous studies have demonstrated that many pre-service teachers at the end of their university degree and thus at the end of their mathematics didactics training deviate greatly in their thinking about children with exceptional difficulties in learning mathematics from the desired perspectives and views on how to overcome the problems those students face (Larmann & Ludwig, 2023; Larmann & Ludwig, 2024). Often, the causes of exceptional difficulties are explained by purely personal factors and concrete actions for a more precise diagnosis of individual problems can hardly be named (Larmann & Ludwig, 2024; Gaidoschik et al., 2021). This is a major problem, as the number of children worldwide who are developing exceptional difficulties in learning mathematics is constantly increasing. The last Pisa study showed that, on average, 30% of children in Year 9 do not reach the minimum standards in mathematics, which represents an average increase of 10% since the previous survey (OECD, 2023).

Based on these findings, the Mathe.Kind project was launched at Goethe University Frankfurt, in the context of which a course for pre-service teachers was developed that is designed to prepare them for precisely this challenge and can be easily adapted by other teacher training institutions. With the present study, we now want to unpack the pre-service teachers thinking about students with exceptional difficulties in learning mathematics after participating in this course. Furthermore, we would like to know to what extent areas can be identified in which their thinking continues to deviate strongly from scientific theories on how to deal with children with exceptional difficulties in learning mathematics. To achieve this, we again utilize the theoretical construct of subjective theories to ultimately gain new indications for the adaptation of teacher education.

Exceptional difficulties in learning mathematics

As already mentioned, the OECD (2023) describes the problem that more and more students are no longer meeting the minimum standards in mathematics. These students have not been able to acquire basic mathematical skills over a longer period of time and are therefore severely disadvantaged in their ability to continue learning. By definition, they can thus be summarized under the term students with exceptional difficulties in learning mathematics (Gaidoschik et al., 2021), which, unlike the terms dyscalculia or learning disorder, extends its focus from the purely individual level to the student's entire learning process also including external influences from school and the socio-economic context in the consideration of the development of difficulties (Baiti et al., 2024). In most cases, these difficulties manifest themselves early on in basic mathematical competences. The understanding of natural numbers, the understanding of the decimal system and the understanding of arithmetic operations. Without an in-depth comprehension of these contents of primary school education, more advanced topics such as fractions or algebra cannot be understood and, in the best case, mastered through the use of familiarized strategies (Prediger et al., 2019). In order for teachers to face these challenges at primary and secondary level, they need a wide range of knowledge and an adequate thinking about causes, diagnostic possibilities and ways of preventive lesson design and support that enables them to provide assistance to affected students in overcoming their existing learning hurdles.

The Mathe.Kind university course

However, it seems that pre-service teachers largely do not have adequate ways of thinking about these topics (Larmann & Ludwig, 2024). For this reason, the Mathe.Kind project was established at the Goethe University Frankfurt with an associated university course, which could be chosen as an optional specialisation and is now offered in the second year for a maximum of 14 pre-service teachers per semester.

The underlying idea of the course is that pre-service secondary mathematics teachers meet once a week over the course of a full semester with a student with exceptional difficulties in learning mathematics from grade five to seven, identifying their individual problems and providing individualized support in a total of 12 sessions. The semester is organized in such a way that the course supervisor first contacts schools in the vicinity of the university and asks the available mathematics teachers to pass on an information and registration letter to parents of eligible children. The parents contacted can then register their children in the Mathe.Kind project, whereupon an approximately one-hour interview takes place between the course supervisor and the student in which the student's existing competence gaps are recorded and a decision for the inclusion into Mathe.Kind is made. For this purpose, the session is videotaped and then analysed by the course supervisor. Once all students participating in the project have been determined, two pre-service teachers are assigned to each student, thus forming a permanent support team for the rest of the semester. Before the support team meets for the first time, however, there are two full-day (8 hour) inputs for the pre-service teachers in which they are specifically prepared for the subject area of pedagogical diagnostics. Furthermore, they plan their own diagnostic interview with the student based on the protocol of the initial interview, which forms the basis for the first meeting with the child. The aim of this first

meeting is the detailed identification of existing competences that are important for the area of mathematics that is to be developed in the subsequent support and thus to obtain as precise a picture as possible of the child's mathematical foundation on which to build.

Before the weekly support program starts, the pre-service teachers receive a further one-day input, which provides an introduction to the topic of individual support. As every meeting of the support team is recorded on video, it is possible in the accompanying seminar during the semester to not only address other relevant topics in the area of support, such as the use of materials, language support or conversation training, but also to analyse video sequences from the support sessions in the plenum. This is always done using guiding questions that are thematically linked to the contents of the accompanying seminar. At the end of the semester, the pre-service teachers carry out a final diagnosis, the aim of which is to visualize the successes achieved by the child and to show which next steps need to be taken in order to close existing competence gaps. A final report is compiled on the basis of this diagnosis, which is presented to the respective child and their parents in an individual meeting.

Several factors that are crucial to change and promote ways of thinking were taken into account when designing this approach in the seminar. Firstly, it is important that the pre-service teachers' existing ideas about how to act are incorporated (Trevors, 2023; Dann, 1989), which is achieved by having them plan the diagnostic and support sessions independently. By subsequently reflecting on the procedure in the seminar and comparing it with the seminar content, a confrontation of the previous with new and more adequate ways of thinking and perspectives on students with exceptional difficulties in learning mathematics is then achieved (Trevors, 2023; Dann, 1989). It is important that this process of confrontation and ultimately change takes place in a practically relevant way, so that it becomes apparent that the new knowledge is better suited to overcome challenges, which is ensured by the continuous work in the support teams (Dann, 1989).

Subjective theories

The theoretical construct we have chosen to visualize pre-service teachers' thinking about students with exceptional difficulties in learning mathematics is that of subjective theories. Subjective theories are defined as the totality of a person's cognitions about a particular topic that contain at least implicit argumentation structures and fulfil the functions of explaining, predicting and guiding the person's behaviour and thus the functions of objective (scientific theories) for that person (Groeben & Scheele, 2000). The word totality allows us to include knowledge as well as attitudes or emotions into the understanding of those theories. If one follows this construct and transfers it to the educational context, this means that all professional actions of a teacher and thus also the interaction with students with exceptional difficulties in learning mathematics are based on patterns of thinking, knowledge and the structures organizing that knowledge (Dann, 1989). This construct differs from other constructs such as attitudes, beliefs or knowledge in the sense of TPACK or MTSK in that an integrated approach is chosen in which particular emphasis is placed on the connections between the contents of the various subjective theories. The more complex and adequate these argumentation structures and contents are, the more positive the effect on the success of teachers' actions (Dann, 1989), which is why a lack of fit between the subjective theories of pre-service teachers and teaching-relevant topics should give reason for adjustments to pre-service teacher trainings.

Research Questions

The following research questions arise from the above-mentioned motivation to unpack the thinking of pre-service teachers on how to deal with students with exceptional difficulties in learning mathematics:

RQ1: How do pre-service teachers' subjective theories on dealing with children with exceptional difficulties in learning mathematics who have attended a specialized training seminar on this issue look like?

RQ2: What areas in the subjective theories of those pre-service teachers can still be identified that differ from scientific research on students with exceptional difficulties in learning mathematics?

Methods

To answer the research questions, a qualitative interview study was conducted with four pre-service teachers who had taken part in the seminar course described above and who agreed to participate in the elaborate methodological structure. The methodology followed that of the Research Programme Subjective Theories (RST) (Groeben & Scheele, 2000), which was also used in the previous studies (Larmann & Ludwig, 2023; Larmann & Ludwig, 2024). The procedure is divided into four main steps, which we briefly present below, and which are described in the dialogue-consensus procedure of the RST (Groeben & Scheele, 2000). All interviews within the framework of the procedure were conducted by trained personnel who were not involved in the seminar in order to enable the interviewees to speak freely and without fear of negative judgement.

The first main step was a problem-centred semi-structured interview (Witzel, 2000), which aimed to capture the contents of the subjective theory. The interview was supported by a short questionnaire previously completed by the pre-service teacher, on the basis of which the introductory questions were generated. It is important in this step that the pre-service teacher can communicate their own personal view of the issue without the interviewer specifying exactly which aspects of the topic they should talk about. To this end, questions were asked about specific problem situations that may arise in the classroom. In the second main step, the interview was then analysed using a summative, qualitative content analysis (Mayring, 2022). For this purpose, the interview was first transcribed and divided into units of analyses. These units were then paraphrased and generalized in order to link them to didactic concepts (Mayring, 2022). The generalizations were then summarized in categories, which were then given a label that was as close as possible to the pre-service teacher's own phrasing. These labels were then written down on cards, the so-called content cards (Kindermann & Riegel, 2016). In the third main step, the subjective theories of the pre-service teachers were reconstructed with the help of a structure-formation-technique (Groeben & Scheele, 2000). For this purpose, the content cards are first validated, discarded or modified by the pre-service teacher and then related to each other using so-called structure cards. These structure cards symbolize different relationship structures between the content cards and thus represent the argumentation structures of the subjective theory. The result is a so-called structure image, which is a graphic representation of the subjective theory and is analysed in the fourth main step. The analysis was carried out in three sub-steps following the so-called Siegen variant (Kindermann & Riegel, 2016). Firstly, clusters were identified

that formed larger, related content areas in the subjective theory. These clusters were labelled and compiled into a rough cluster map. In the second step, these individual clusters were then analysed with their content and argumentation structures. This involved analysing not only the content and structure maps, but also the comments made by the pre-service teacher during the second interview on the individual content areas (Kindermann & Riegel, 2016). Finally, the clusters were placed in relation to each other, resulting in a detailed cluster map that depicts the subjective theory both descriptively and graphically (Kindermann & Riegel, 2016).

Results

In the following, we will briefly describe the subjective theories of the four participating pre-service teachers (PT 1 to PT 4) based on the detailed cluster maps. An exemplary detailed cluster map of PT 1 can be viewed in its entirety in Figure 1.

In his subjective theory, PT 1 describes dealing with children with exceptional difficulties as a chronological process from problems occurring in the classroom to diagnosis and support. Various causes for the development of difficulties are mentioned and a special focus is placed on a lack of competences in the mathematical content of the primary level. The diagnostic interview is the preferred method for identifying competence gaps and it is made clear that individualized support can only succeed on the basis of detailed diagnostics. Although individualized diagnostics are seen as a challenge, according to PT 1, there are even greater risks associated with not using them. Accordingly, PT 1 weighs up different forms of diagnosis. For individualized, but also supportive lessons, PT 1 refers to forms of teaching and materials that enable the respective student to discover new content themselves. To this end, lesson preparation and the teacher's own knowledge of methods and possible problems are regarded as fundamental. PT 2 attributes dyscalculia and mathematical difficulties to a lack of mathematical understanding and believes that targeted diagnosis and promotion of basic competences can help. For PT 2, the aim is to develop basic math concepts and strategies to enable long-term learning. A diagnosis-led and understanding-orientated approach is also mentioned for successful support. One of the main difficulties for PT 2 is reconciling large group sizes at school with individualized support. Poor teaching and a lack of support at home are identified as causes of persistent problems. PT 2 emphasizes the importance of teacher competence and in-service training.

PT 3 emphasizes in his subjective theory the importance of prerequisites such as attitude towards mathematics, basic ideas and prior knowledge that influence exceptional difficulties in learning mathematics. He takes a critical view of routine and memorized knowledge and demands that teachers take into account the heterogeneity in the classroom. A positive atmosphere during diagnosis, suitable tasks and careful evaluation are essential for him. He sees support as a long-term, individualized process in which cooperation with the students' parents is important. PT 3 sees the process as a challenge, but adequate support is at the centre of his considerations in the classroom. For PT 4, diagnosis and support are also central components of his subjective theory. He names various signs of difficulties, such as overwhelm, poor grades or disruptive behaviour. Quick intervention by the teacher is essential when these warning signs occur, as otherwise more long-term difficulties can arise. PT 4 emphasizes the importance of basic concepts and building on previous knowledge when providing support. Support should continue to be individually customized and not be limited to

current topics. For PT 4, open tasks and an error-friendly atmosphere are important for diagnosis. Cooperation with parents and strengthening the student's self-confidence in the classroom are also crucial for PT 4.

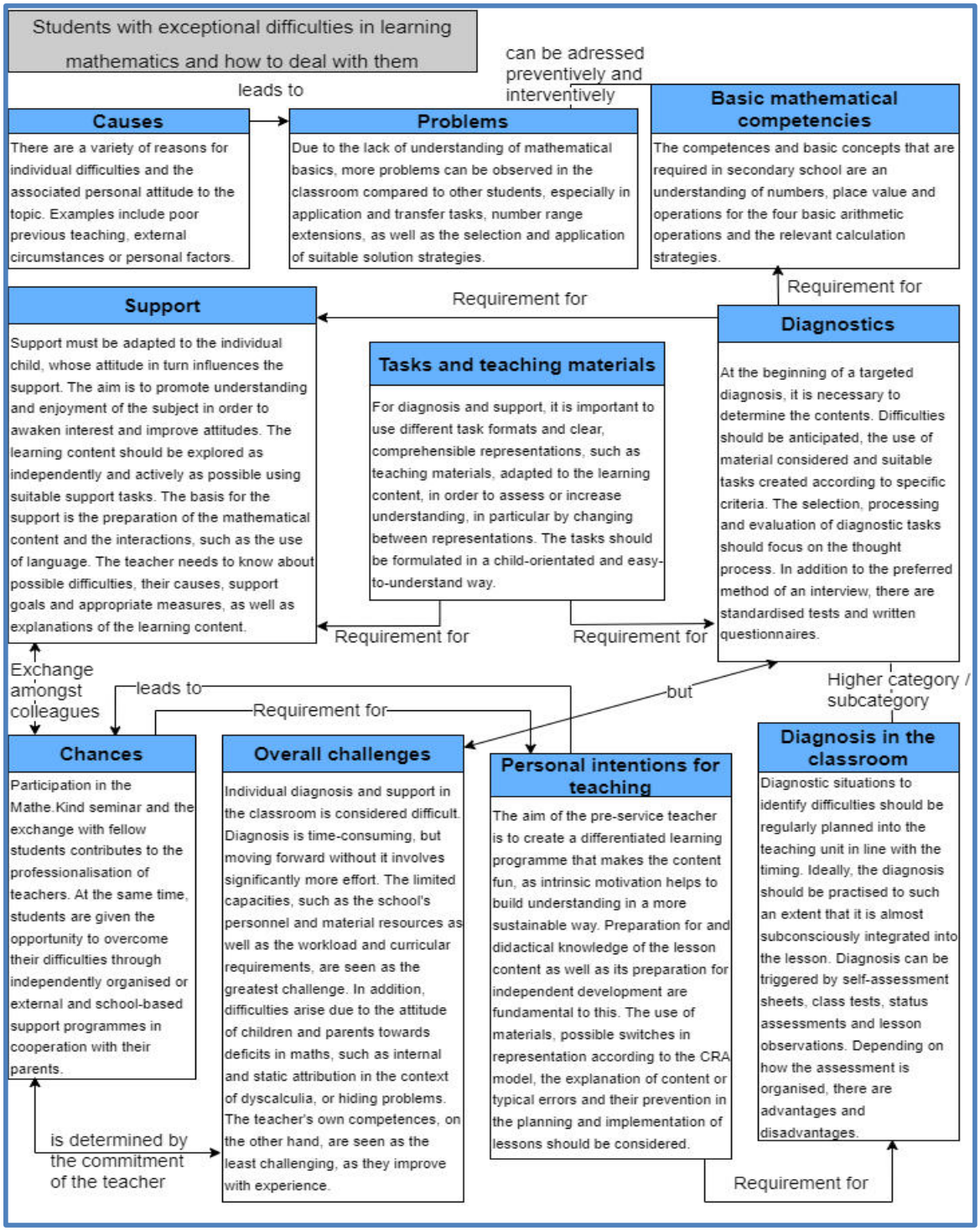

Figure 1: Exemplary detailed cluster map of the subjective theory of one of the participating pre-service teachers

Discussion and conclusion

The subjective theories of the pre-service teachers described above include many aspects that were not included in earlier studies (Larmann & Ludwig, 2024). When looking at causes, there is a strong focus on the mathematical content of the primary school level, which means that purely motivational and individual factors are pushed into the background. The social environment and previous schooling play a major role, which is why we can speak of a much broader way of thinking about the causes of exceptional difficulties in learning mathematics (Baiti et al., 2024). In all subjective theories, the area of diagnostics functions as a key focus alongside support. Within the clusters describing diagnostics, it is not just a list of possible methods, but a combination of possible approaches, materials and tasks needed for the diagnostic approaches as well as teacher behaviours that can help to reveal students' thinking on mathematical concepts. The focus on the diagnostic interview as the preferred method paired with less time-consuming methods for everyday teaching, such as observing students solving work assignments or making them complete short tests, speaks in favour of a tendency towards process-oriented diagnostics, which Gaidoschik et al. (2021) describe as desirable.

However, it is not only the contents of the subjective theories that can be perceived as positive. We also recognise a change in the argumentation structures between the different clusters. Support and diagnostics in particular are strongly interwoven in all of the subjective theories analysed and are mutually dependent. In general, diagnosis is seen as a necessary prerequisite for support, without which it is hardly possible to constantly adapt support to the individual needs of the child (Prediger et al., 2019). In addition, although all students recognise that dealing with students with exceptional difficulties in learning mathematics is challenging, they consider their diagnostic and support skills to be adequate, which was also not found in previous studies (Larmann & Ludwig, 2023; Larmann & Ludwig, 2024). It is also interesting that all pre-service teachers view the role of traditional extracurricular tutoring as critical, although this was neither part of the Mathe.Kind course nor discussed in any other way during the support sessions. They particularly criticised the lack of work on the students' actual problems and the fact that it is merely a treatment of symptoms, which may help in the short term but does not facilitate long-term learning. However, adequate extracurricular support is seen as desirable and helpful, which is also in line with the literature. (Gaidoschik et al., 2021).

In summary, it can be said that the pre-service teachers participating in the Mathe.Kind course have a predominantly adequate subjective theory of how to deal with students with exceptional difficulties in learning mathematics. With regard to our second research question, no areas could be identified in which the pre-service teachers' ways of thinking deviated strongly from scientific knowledge about dealing with those students. This may indicate that the described seminar can contribute to preparing pre-service teachers for corresponding tasks in everyday school life and to fostering their knowledge and thinking. In order to record the effect of the seminar on the subjective theories of the pre-service teachers more precisely, the next step will be to conduct a pre-test-post-test study with all course participants since these results can not be generalized based on the small sample size and the method.

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Inconsistencies in the process of posing problems for use in class by pre-service teachers

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This study is framed in a research project, in which we explore the professional competence of posing problems for use in class. Specifically, we analyse the processes involved in formulating and solving problems in the output of 72 pre-service primary teachers who were asked to pose a two-stage problem based on the multiplication of fractions. By means of a methodology based on constant comparison, three types of inconsistencies were identified: inconsistencies between problem statement and numerical solution; inconsistencies between problem statement and diagrammatic solution; and inconsistencies between numerical and diagrammatic solutions. The results suggest that the pre-service teachers tend to solve the mental image of the problem they think they have formulated rather than the actual problem given in the problem statement.

Keywords: Problem posing, Teacher education, Pre-service primary teacher, Inconsistency.

Introduction

It is widely recognised that problem solving should be both vehicle and objective of mathematics education (NCTM, 2020). Consequently, teachers need to be competent formulators of suitable classroom problems (Kilpatrick, 1987). Nevertheless, there are gaps in the research into problem posing which need to be addressed, especially with regard to the process of posing problems itself (Cai et al., 2022).

In this paper we note that during the process of devising problems for use in class there may occur a disconnect between the aimed for and the actual formulation. We consider such disconnects from a non-judgemental perspective, with the sole aim of advancing our understanding of the elements at play during this process. To this end, we carried out a study of the output of 72 pre-service teachers (PST) in a problem posing task involving the multiplication of fractions, which required them to formulate a problem, provide the solution, and explain the decisions made during the process. In the subsequent analysis of the responses, a number of the problems stood out for presenting a disconnect between what was intended and what was actually produced, which we refer to as inconsistencies. We describe a framework for categorising the inconsistencies identified during the problem posing process. The long-term goal of this work, extending beyond the scope of this paper, is to use these inconsistencies as pieces of evidence providing an insight into the mechanisms used by the pre-service teachers to devise, evaluate and vary problems, given the fundamental role that solving them plays in this process. Here, however, we limit ourselves to answering the following research question: What inconsistencies can be identified in the pre-service teachers' formulations of problems involving fractions?

Theoretical background

Problem posing by teachers and pre-service teachers has received considerable attention from the mathematics education research community (Crespo, 2003; Tichá & Hospesová, 2013; Montes et al., 2024). Notwithstanding, the processes involved in posing problems remain an under-researched area (Cai, et al. 2022), particularly with regard to the stages a problem goes through, and the heuristics the poser uses in developing their problem. In broad terms, the problem-posing process follows a cycle of formulation, solution, and revision of the problem statement, which goes through various iterations before being presented to the pupils. The cycle involves certain elements on the part of the teacher, such as mathematical and pedagogical knowledge (Tichá & Hospesová, 2013), which are directed towards the achievement of a specific teaching objective (Montes et al., 2024).

The cycle, with its constituent stages, has been studied by various researchers (e.g., Cruz, 2006; Baumanns & Rott, 2022). Cruz (2006), proposes a model in which the goal is an explicit element, making it especially interesting in terms of exploring the processes involved in the teacher's problem-posing. For their part, Baumanns and Rott (2022), identify five types of activities – situation analysis, variation, generation, problem-solving, and evaluation – which combine to generate a descriptive model of the stages involved. The model establishes the interrelation between the different activities which are carried out during the process, generating a series of sequential stages. In the course of each of the stages, the problem poser produces different outcomes (provisional problem statements and solutions, evaluations of the suitability of the problem, and so on). This paper focuses on the coherence of the outcomes, in particular the phraseology of the problem statement, and the stated solution, noting discrepancies that can occur between what is written in the problem statement and what is affirmed in the solutions provided.

Methods

This paper forms part of a larger project aiming to deepen our understanding of the processes involved in posing problems for use in class by pre-service primary teachers from an interpretative perspective. The study was conducted with a group of 72 pre-service primary teachers coming to the end of their second year, having followed a course in teaching numbers and arithmetic, including aspects of fractions. They were given a task to formulate a two-stage problem, one involving addition and the other multiplication (Figure 1). Participants were asked to give informed consent. Data was coded as PST 1.X, meaning X the number of student in alphabetical order. During the course, the pre-service teachers had practised problem posing for themselves, and the need to go through the process of solving the problems as a means of checking the problem statement had been reiterated on several occasions.

Devise a problem in which there is at least one addition and one multiplication of fractions.

a. What kind of fraction does the problem result in?

b. Solve the problem representing the fractions graphically.

c. Identify potential errors and points of difficulty for the pupils, in terms of both visual and arithmetic aspects.

Figure 1: Task presented to the pre-service teachers

The objective of the larger project entailed analysing the pre-service teachers' outputs in the search for key features of the problem-posing process. In the course of the analysis, various inconsistencies in this process became apparent, which offered a means of gaining insight into the features we were looking for. Given the limitations of space, this paper focuses solely on the inconsistencies, taking a non-judgemental position with the aim of illustrating critical points which can arise in the course of problem posing. Of particular importance is the recognition that although one might be in possession of certain items of knowledge, it does not necessarily follow that these will be deployed consistently.

Analysis of the inconsistencies using an emergent procedure based on the constant comparison method (Bryman, 2001), grouped them into three types, which were triangulated by external researchers. The three groups were: (i) inconsistency between the problem statement and the numerical solution; (ii) inconsistency between the problem statement and the diagrammatic solution; (iii) inconsistency between the numerical and the diagrammatic solutions. Examples of each type of inconsistency are given below in order to illustrate the inconsistency involved and suggest the possible causes.

Results

Table 1 presents a summary of the frequency of occurrence of the inconsistencies, from which it can be seen that their occurrence among the pre-service teachers' problems is very high.

Table 1: Quantitative summary of the occurrence of inconsistencies

	Absolute frequency	Percentage
Inconsistency between problem statement and numerical solution	28	38.9%
Inconsistency between problem statement and diagrammatic solution	33 (plus 26, without a graphical solution)	45.8% (plus 36,1% without a graphical solution)
Inconsistency between numerical and diagrammatic solutions	56	78%
No inconsistencies	16	22%

In the next section we give a detailed description of the key features and defining mathematical characteristics of examples from each of the three groups of inconsistencies. We will show the problems in Spanish, translated to English, in order to let the readers see the original information.

Inconsistency between the problem statement and the numerical solution

This group of inconsistencies are characterised by the disconnect between the problem statement and the numerical solution to the problem. A clear example of this kind of inconsistency is the response of pre-service teacher PST1.64, whose problem statement and numerical solution are given in Figure 2. He adds 7/12+11/12 to put all the slices together, and supplies a meaning for this fraction in terms of the part-whole relationship between the uniform slice and the total number of slices, although he fails to make explicit the correspondence between the total slices and the unitary item they represent, the cake. Subsequently, as seen in Figure 2, he expresses the idea of 'half' not through the natural idea of division into two, but through the arithmetic transformation of multiplying by a half, which he explains as "this is the way you obtain half of a fraction".

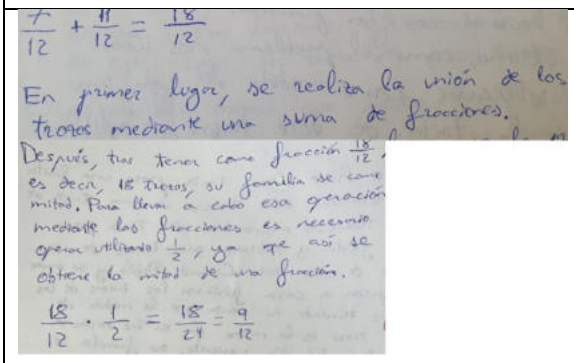
Problem: Pablo made a cake and afterwards there were 7 slices left out of 12. His mother made another cake and this time there were 11 slices left out of 12. When they got home, they put all the pieces of cake together, bearing in mind that they were all the same size. The next day, the family ate half of the slices. How many slices were left, bearing in mind the size of the cake is twelve slices?	
	7/12+11/12=18/12 First of all, the slices are joined together by adding the fractions. […] Then, after obtaining the fraction 18/12, in other words, 18 slices, the family ate half of them. To carry out this operation using fractions it is necessary to do the operation using ½, since this is the way you obtain half of a fraction. 18/12 x 1/2=18/24=9/12

Figure 2: Problem set by PST 1.64 with their solution (original with translation on right)

On the other hand, the PST's intention is to work with fractions, and in this regard, he clearly aims to meet the parameters of the task. However, he appears not to have thought through the nature of the problem he poses, which can be solved more directly with natural numbers. This illustrates a potential disconnect between the posed problem and the solution, in that the solution does not address the problem as it is actually posed, but rather the problem as it is interpreted through the filter of fractional thinking, thus introducing an overly complex approach to the problem.

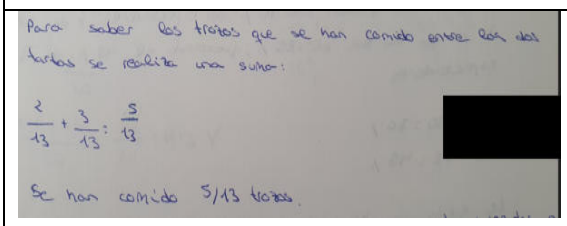
Problem: On Maria's birthday, there are two cakes. From the first one they eat 2/13 parts, and from the second 3/13 parts. How many pieces do they eat between the two cakes? If they had eaten twice as much, how many pieces would they have eaten?	
	In order to know how many pieces they eat between the two cakes, you use addition: 2/13+3/13=5/13 They eat 5/13 pieces

Figure 3: Problem set by PST 1.43 with their solution (original with translation on right)

In this problem (Figure 3), the pre-service teacher uses two terms, "parts" and "pieces", interchangeably to refer to a fraction. It can be assumed that she considers 2/13 parts in terms of a part-whole relationship in which the numerator represents the number of parts eaten and the denominator the total number of parts. In the problem statement, the questions ask about "pieces" that are eaten. Beyond the interchangeability of the terms, we can again see the arithmetical application of fractions in a context where it is mathematically unnecessary. The solution adds together two fractions to arrive at the conclusion that 5/13 pieces (of cake) have been eaten. This confirms the assumption above that a part-whole relationship underpins the problem, although, a literal interpretation of the phrase "5/13 pieces" takes the piece as the whole and a subdivision of the piece as part.

Inconsistency between the problem statement and the diagrammatic solution

This type of inconsistency involves a lack of connection or coherence between the problem statement and the visual representation of the solution to the problem. We illustrate this group of inconsistencies with the problem statement and visual representation by the pre-service teacher PST1.5 (Figure 4).

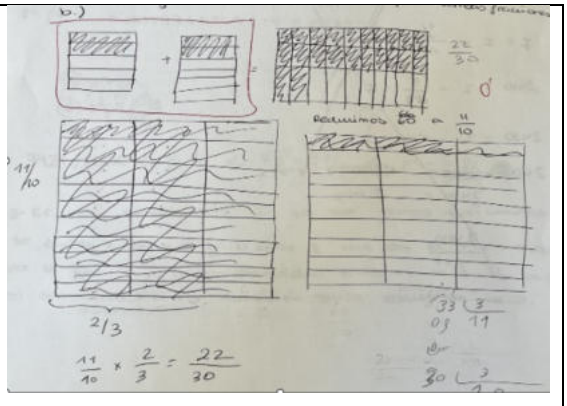
Problem: My grandfather bought $\frac{2}{5}$ of a cake, but as there were a lot of us he bought $\frac{2}{6}$ more of a cake. When he got home, he realised that he needed $\frac{2}{3}$ more of each total fraction of the previous amounts. What fraction of cake does my grandfather finally need?	

Figure 4: Problem set by PST1.5 with corresponding diagrammatic representation

We can see that this pre-service teacher first addresses the addition of fractions, in compliance with the task, with a inconsistency between the magnitude of each unit corresponding to the items to be added together as well as the unitary concept underlying the result. In other words, the setter does not seem to consider what wholes are being represented by the problem. This referential ambiguity is reinforced by the accompanying diagrammatic representation. Rather than present a visual representation of the problem and its solution, the PST draws two diagrams representing the fractions to be added, alongside a third representing the result. Underneath, she writes the fraction 66/60, without giving any clear indication of where it comes from. This fraction is then reduced to 11/10, this time without an accompanying diagram. In the second part of the problem, the PST carries out the multiplication of 11/10 x 2/3, and with it completes the solution, although the text of the problem statement, as written, seems to suggest that this should be added to the amount they already have. The treatment of this problem is very much like that of the first example, in which the pre-service teacher carries out the operations they had in mind to solve the problem rather than the actual problem they had formulated.

Inconsistency between the numerical and the diagrammatic solutions

In this type of inconsistency, there is a divergence in the reasoning underlying the pre-service teacher's solutions as represented in the numerical and visual registers, which in one extreme case results in two different solutions. In the example below (Figure 5), there are inconsistencies both between the problem statement and the two solutions, and between the solutions represented in the two registers, numerical and visual.

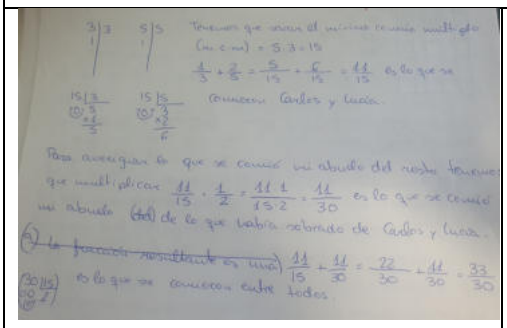
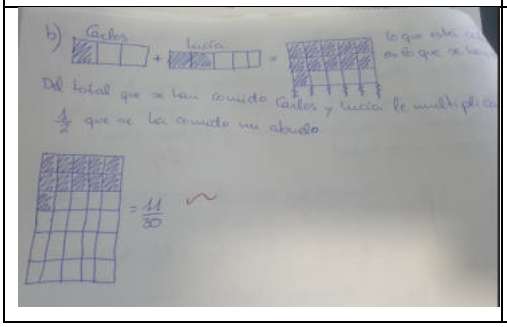
Problem: On my birthday, my friend Carlos ate 1/3 of the cake and my friend Lucía 2/5. How much did they eat between the two of them? Afterwards, my grandfather ate ½ of the rest. How much did they eat between all of them?	
	We need to find the least common multiple (l.c.m)=5x3=15. 1/3+2/3=5/15+6/15=11/15 is what Carlos and Lucía ate [Corresponding division sums included] To work out what my grandfather ate of the rest we need to multiply 11/15x1/2=11x/15x2=11/30 is what my grandfather ate of what Carlos and Lucía had left. 11/15+11/30=22/30+11/30=33/30 is what they ate between all of them
	[Diagram representing the addition of fractions] The shaded parts are what they ate The total of what Carlos and Lucía ate we multiply by ½, which my grandfather ate

Figure 5: Problem set by PST1.7 with corresponding numerical and visual representations

The numerical procedure is technically correct, in arithmetic terms, although the result, implying that more was eaten than there was originally, does not make sense. With respect to the accompanying diagram, it incurs the same inconsistency identified previously – the relative size of the parts representing the two fractions is not taken into account, and likewise the addition of these two amounts is carried out without reference to the actual elements of the problem to which they correspond. Observing the solutions provided in the two registers, numerical and visual, we can see that although in the numerical register the PST recognises the need to apply a procedure to make the size of the parts the same, he fails to do this in the visual register. Further, in the visual representation the pre-service teacher finishes the procedure with the product of 11/15 by ½ (which he does not explain, but simply provides a representation of the fraction, again with no reference to what it represents), whilst in the numerical representation the procedure finishes with the calculation of the total amount eaten.

Final reflections

Formulating problems for use in class requires the teacher to use the process of solving the problem posed (by the problem statement) as a means of revising the problem statement itself (Baumanns & Rott, 2022), or at least to think carefully about the solution the problem statement implies. The strategies required, and the stages through which the problem-posing process goes through require close investigation (Cai et al., 2022), as well as specific training. In particular, this study highlights three types of inconsistencies produced by pre-service teachers in the problem-posing process.

These inconsistencies suggest that pre-service teachers tend not to use problem-solving as a tool for evaluating the problems they devise. Indeed, we have found that in some cases the PSTs appear to solve the problem they thought they had formulated, but not the actual problem corresponding to the text of the problem statement (e.g., the first problem). In this respect, we consider that when they formulate a problem, pre-service teachers generate a "problem image" (in a similar sense that Tall and Vinner, 1981, refer to a concept image), and then solve the problem image rather than the stated problem. This is especially evident when pre-service teachers choose to tackle problems that can be solved using natural numbers by using structures associated with fractions.

Also worth noting is that the difference in occurrence of inconsistencies between the problem statement and the numerical resolution compared with those between the problem statement and the diagrammatic solution suggests that the mechanisms for connecting the problem statement with the solution as a means of reviewing the problem formulation might differ according to the register in question. Nevertheless, the aim of this study is not to identify areas where knowledge is lacking among pre-service teachers, but rather to illustrate the need to explore the finer details involved in the process of problem posing (Cai et al., 2022), which can feed into teacher education programmes and enable PSTs to develop the professional competence in formulating problems they need.

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Characteristics of teachers' lesson observations: A comparison of in-service and pre-service teachers' gestures

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This study focuses on lesson observation, which is a key teacher activity in a lesson study. Although previous studies have described the act of observation by particular individuals in particular contexts, there has been limited scientific discussion on teacher expertise that the lesson observation is based on, or how such expertise is formed. Based on the Anthropological Theory of the Didactic, this study addresses the research question: What praxeological equipment is used for lesson observations? Thus, questionnaire evidences for pre-service and in-service teachers were conducted and all praxeologies extracted from them were used as a reference model to analyse teachers' observational activities. In conclusion, the commonalities/differences in observation tendencies between pre-service and in-service teachers, according to years of teaching experience, as well as the relationship between mathematical content-related praxeology and other ones are discussed.

Keywords: Lesson observation, lesson study, anthropological theory of the didactic, praxeological equipment, praxeological reference model.

Introduction: *Problématique* on teachers' lesson observations

Lesson study has been widely known as the Japanese traditional approach to teachers' professional development process (e.g., Fujii, 2014; Takahashi & McDougal, 2016), and it is also disseminated internationally (e.g., Clivaz, 2015; Rasmussen, 2016). However, most of the research studies have focused on a teacher or a group of teachers who work as lesson practitioners through lesson study. Teachers participating in a lesson study can be viewed from two different positions. On the one hand, teachers take a position in which they engage in lesson design and lesson implementation. For this position, the infrastructure of the lesson study practice as the superstructure is considered a *lesson plan*. On the other hand, teachers take another position in which they engage in *observing the lesson* concerned. Post-lesson discussion is considered a sharing setting for both position teachers. Although lesson study has received growing attention from educational researchers, one of the underexplored areas is how the teachers observe and learn the lesson during lesson study. To better explain lesson study as a collaborative setting, it is worth including lesson observation as the object of study and then we explore a way to reveal teachers' activities and knowledge during the observation (Shinno & Yanagimoto, 2020). Some previous studies have described teachers' observation work focusing on particular individuals, in particular, contexts, as to 'what' and 'how' other teachers as such participants observed. Nevertheless, there has been inadequate discussion on the didactic or paradidactic gesture (Miyakawa & Winsløw, 2019) that work of lesson observation is.

It is noteworthy that traditionally, Japanese teachers' lesson observations have in most cases relied on training experience. Less experienced teachers tend to take a form similar to that of the Japanese

‘道 (*dou*)’ culture, which “means ‘way’ and it implies the training process to follow” (Hirabayashi, 2006, p. 57) to acquire ‘observational skills’ and ‘teaching skill’ similar to ‘veteran’ teachers by participating in many lesson study sessions. A similar situation can be identified in student teachers in pre-service teacher-training programmes. In lesson observation activities at school (including watching videos), most pre-service teachers only focus on superficial behaviours of practicing teacher and students in the classroom; for example, what the teacher and students said and wrote. We assume that lesson observation is a professional activity which may vary according to observers’ mathematical and didactic knowledge of ‘what’ and ‘how’ to observe. Thus, ‘observing a mathematics lesson’ is an activity beyond simply ‘seeing what’ but also ‘seeing why’. There are several educational studies on lesson observation in Japan, and these studies have described or prescribed ‘observation viewpoints’ and ‘observation methods’ for supporting teachers’ professional development. However, none have attempted to elucidate lesson observation as the object of study with theoretical underpinnings. To contribute to the existing studies in this field of research, this study attempts to adopt a scientific approach aiming to theoretical framing teachers’ observation activities as a phenomenon, that is interpreted as ‘human observation of the lesson’ within our theoretical framework: the Anthropological Theory of the Didactic (hereinafter ATD).

Theoretical framework, research question, and research context

Theoretical framework

Lesson study or classroom observation is an everyday practice in Japan, however, that may not always be the case in other countries. On the other hand, even in Japan, almost all cases of such phenomena have traditionally been described empirically, where it has not been clear who the observer is and by whom such acts of observation are described. Especially, lesson observation is not only a matter for the personal observer, but also for the institution concerned. Therefore, we begin by describing ‘lesson observation’ in terms of *cognitive algebra*, as used in ATD (Chevallard, 2019; Chevallard & Bosch, 2020). This is because such an approach allows us to better understand the object of study with the following questions. That is, the question arises as to who is responsible for the act of ‘observing a lesson’, who is the one when we say ‘he/she observed a lesson’, and by whom is the statement judged to be so? We use the notion of instance to clarify this ‘who’, as follows. In order to clarify these points, we describe lesson observation as didactic or paradidactic phenomena using the cognitive algebra.

The actions of one teacher implementing a lesson and other teacher(s) observing it are denoted as follows: An entity with position p in institution I is called an instance (I, p). The teacher instance (hereinafter $\hat{\jmath}$) expects the relationship $R(x, o)$ of x to the didactic stake o to be learned by the learner position (p) in the classroom institution (I) to approach the $R_I(p, o)$ (relationship that position p as a learner in I should have for o) is expected to approach:

$$\hat{\jmath} \vdash R(x, o) \cong R_I(p, o).$$

If the relation of x to o at the beginning of the lesson $(t = 0)$ and at the end of the lesson $(t = 1)$ are $R(x, o)_{t=0}$, $R'(x, o)_{t=1}$ respectively and $\bar{R} = R_I(p, o)$, the evaluation instance (hereinafter observer or $\hat{v}$) estimates the degree of attainment between $R(R')$ and $\overline{R}$, i.e. $\varphi(R, \overline{R})$ and $\varphi(R', \overline{R})$, as either:

$$\hat{v} \vdash \varphi\left(R, \overline{R}\right) \begin{matrix} < \\ \approx \\ > \end{matrix} \varphi(R', \overline{R}). \qquad (\#1)$$

Note that any observation by $\hat{v}$ will be determined by the reference instance (hereinafter $\hat{w}$):

$$\hat{w} \vdash [\hat{v} \vdash \varphi\left(R, \overline{R}\right) \begin{matrix} < \\ \approx \\ > \end{matrix} \varphi(R', \overline{R})]. \qquad (\#2)$$

When regarded in this way, the purpose of this study described above can be translated as follows: *when $\hat{v}$ observes a lesson, what praxeological equipment of $\hat{v}$ is judged to be used by $\hat{w}$?* (Q_0)

The praxeological equipment (hereinafter PE) is explained as follows. The ATD contains a theory of human action whose beginning point is the notion of a type of tasks T (and the notion of task t of type T, which is denoted by $t \in T$). The realisation of task $t \in T$ requires the use of a technique τ_T relating to T. The ordered pair made up of a type of tasks T and a technique τ relating to T is denoted by $[T/\tau]$ and is called the praxis block. Any technique requires an explanatory or supporting comment, called technology, denoted by θ. The technology θ is itself coupled with a justifying discourse at a higher level, which is the theory, denoted by Θ, of the technique τ. The ordered pair made up of the technology θ and the theory Θ is denoted by $[\theta/\Theta]$: it is the logos block. Any praxis block $[T/\tau]$ supposes a logos block $[\theta/\Theta]$ with which it forms a praxeology $\wp = [T/\tau/\theta/\Theta]$. Hence, the praxeological universe of $\hat{\imath}$ is defined by $\Omega^{\blacklozenge}(\hat{\imath}) \stackrel{\text{def}}{=} \{\wp / R(\hat{\imath}, \wp) \neq \emptyset\}$ and $\hat{\imath}$'s praxeological equipment by $\Gamma^{\blacklozenge}(\hat{\imath}) \stackrel{\text{def}}{=} \{(\wp / R(\hat{\imath}, \wp)) / \wp \in \Omega^{\blacklozenge}(\hat{\imath})\}$. (Extract from Chevallard & Bosch, 2020).

When the relations in #1 are denoted by $R(\hat{v}, \varphi)$ and the relations in #2 by $R(\hat{w}, R(\hat{v}, \varphi))$, the following are the key questions: Where do these relations emerge from? What are they born of? What is their genesis? These can be answered using the notion of praxeology.

Research question

The following research question (RQ) is formulated as a partial solution (first step) to Q_0.

RQ: *What PE is used for lesson observations in $\hat{v}$?*

RQ seeks to identify the PE ($\Gamma^{\blacklozenge}(\hat{v}) = \{\pi(\hat{v})_n | n \in \mathbb{N}\}$) that $\hat{v}$ uses to observe the lesson practice by $\hat{\jmath}$ (or $\hat{\jmath}$'s didactic gesture $\delta_{\hat{\jmath}}$), where $\hat{v}$ includes in-service and pre-service teachers. Thus, it attempts to identify the characteristics of the lesson observations of the two groups. This requires a praxeological reference model (cf. Corica, 2020) by $\hat{w}$. This research project can be translated as finally aiming to construct such a REM, but since this paper is a case study as part of the data resource for this purpose, we denote it as a PE of $\hat{w}$ ($\Gamma^{\blacklozenge}(\hat{w}) = \{\pi(\hat{w})_n | n \in \mathbb{N}\}$).

Research context

To address RQ, the following context is established in this study. The context is designed to compare the lesson observations of pre-service and in-service teachers, as both might carried out lesson observation activities differently.

For pre-service teachers, in a university class included in the teacher education programme, they watched a video of a lesson ('Written subtraction', primary school Grade 2) and were asked to write a report in which they answered the following questions.

Q_{pa}: What did you observe in the lesson?
Q_{pb}: Why did you observe it?
Q_{pc}: How did you observe the facts and what did you extrapolate?

However, in-service teachers (primary school) were not asked to observe a common lesson but were provided the following questions in which they were asked to reflect on the lesson studies in which they had participated.

Q_{ia}: What do you focus on when you observe a lesson?
Q_{ib}: Why do you intend to observe it?
Q_{ic}: How many years of teaching experience do you have?

It does not mean that such a setting is necessarily appropriate to address our research end. Rather, especially in the case of in-service teachers, they had already experienced many lesson studies, and we attempted to mine for their recognitions from such experiences.

Then, $\Gamma^{\blacklozenge}(\hat{w})$ denotes all the findings from these questionnaires, with which $\Gamma^{\blacklozenge}(\hat{v})$ is analysed.

Result

The $\pi(\hat{w})_n$ were extracted by organising the key words from the descriptions in the questionnaire survey. Both pre-service and in-service teachers responded with regard to more than one $\pi(\hat{w})_n$. Table 1 presents the results of the percentage (%) of the total number of responses for each category, divided by the number of pre-service teachers ($N = 37$) and in-service teachers ($N = 19$).

Table 1: Pre-service and in-service teacher observation items

$\Gamma^{\blacklozenge}(\hat{w})$	$\pi(\hat{w})_1$	$\pi(\hat{w})_2$	$\pi(\hat{w})_3$	$\pi(\hat{w})_4$	$\pi(\hat{w})_5$	$\pi(\hat{w})_6$	$\pi(\hat{w})_7$	$\pi(\hat{w})_8$	$\pi(\hat{w})_9$
	blackboard	things prepared	organisation	questioning	assistance	speaking style	interaction	appearance	mathematical content
Pre-service	16	7	51	21	28	2	26	26	14
In-service	21	0	16	68	47	0	26	79	26

Table 2 presents similar proportions contrasted by the years of experience of the teachers (Experienced ≥ 10 ($N = 14$), Non-experienced < 10 ($N = 5$)).

Table 2: Experienced and non-experienced teacher observation items

$\Gamma^{\blacklozenge}(\hat{w})$	$\pi(\hat{w})_1$	$\pi(\hat{w})_2$	$\pi(\hat{w})_3$	$\pi(\hat{w})_4$	$\pi(\hat{w})_5$	$\pi(\hat{w})_6$	$\pi(\hat{w})_7$	$\pi(\hat{w})_8$	$\pi(\hat{w})_9$
E ≥ 10	40		40	80	60		40	60	0
N < 10	14		7	64	43		21	86	36

The following are the provisions for each $\pi(\hat{w})_n$ based on the actual response descriptions. All these descriptions were expressed in Japanese and translated into English by the authors.

[$\pi(\hat{w})_1$] *Writing on a blackboard*: Regarding $\pi(\hat{w})_1$, both pre-service and in-service teachers commonly indicate to 'view the flow of the lesson at a glance'. While pre-service teachers tend to write abstract statements such as 'summarising diagrams and formulae to strengthen students' understanding', in-service teachers often describe clearly their intentions, such as 'how they organise

the lesson, including the objectives, summary and reflections' and 'whether they describe the student's thinking and the contents to be taught'.

[$\pi(\widehat{w})_2$] *Things to be prepared*: For $\pi(\widehat{w})_2$, responses were found only among pre-service teachers, and their attention was focused on the students' interest according to the pros and cons of the preparation.

[$\pi(\widehat{w})_3$] *Lesson organisation*: $\pi(\widehat{w})_3$ contains two different but related issues. One concerns time allocation or management and the other concerns the status of each lesson phase. With regard to the former, we found not only pre-service teachers' descriptions of, for instance, 'I was concerned about which stage in the lesson took more time', but also in-service teacher's description of 'I am not good at it, it is a challenge for me'. With regard to the latter, pre-service teachers were particularly involved in Japanese structured problem-solving lessons, answering questions such as 'how is teaching conducted according to individual differences?' and 'it takes time to refine and elaborate (*neriage*)'.

[$\pi(\widehat{w})_4$] *Teacher's questioning*: For $\pi(\widehat{w})_4$ as well, there are some differences in the way pre-service and in-service teachers are described, with in-service teachers generally being more specific in their descriptions. Pre-service teachers described 'how the teacher implements the lesson with a focus on dialogue with students', 'how the teacher supports the students' learning', and 'to understand the progress of the students' learning'. However, in-service teachers pay attention to 'student's inspiration, how they want to tell their friends, what they are thinking about', 'whether the students understand the questions to achieve the aim/whether they are trying to connect the students' ideas', and 'whether the questions are simple and easy for students to understand'.

[$\pi(\widehat{w})_5$] *Teacher's assistance*: For $\pi(\widehat{w})_5$, both pre-service and in-service teachers described relatively similar ways. Pre-service teachers provided answers such as 'to know how to respond to a wrong answer', 'what type of assistance is desirable in the context of understanding' and 'in a whole-class atmosphere [students] can present themselves or correct their trajectory and lead to an answer'. In-service teachers also need to ask themselves, 'what type of words of assistance, and how to connect them to the subsequent development? and what steps can be taken to raise the level of the activity?' and 'how is the students' goal picture envisioned?' Teacher's assistance has a special role in SPS. It may be a gesture specific to Japan. Therefore, it is distinguished from $\pi(\widehat{w})_6$ and $\pi(\widehat{w})_7$.

[$\pi(\widehat{w})_6$] *Teachers' speaking*: $\pi(\widehat{w})_6$ is found in only one pre-service teacher. The description includes 'tone and loudness of voice, strength of explanation and nomination', which have a highly pedagogical aspect.

[$\pi(\widehat{w})_7$] *Teacher and student interaction*: $\pi(\widehat{w})_7$ relates to the way teachers and students interact in lessons; all descriptions provided by pre-service teachers are of teacher-student interaction: 'whether there were any attempts to facilitate the smooth running of the lesson', 'she gave advice to children who seemed to be having difficulties', 'they rephrased or supplemented in easy-to-understand words', 'the teacher asked the students to pay attention', 'the students took the initiative by having them write on the blackboard and explain in front of everyone', 'active comments (by the students) make the lesson easier to understand and give them new perspectives', and 'a good response reduces afraid and increases speaking out'. In contrast, more than half of the descriptions of in-service teachers relate to

the way in which students interact with each other: 'I want to aim for lessons that strengthen learning through students' interaction', and 'whether they are learning and enhancing each other's peers, and whether their comments connect and strengthen each other's ideas'.

[$\pi(\widehat{w})_8$] *Students appearance*: For $\pi(\widehat{w})_8$, the pre-service teachers watched the video lesson, they were limited to the range of what they could capture on the screen. Consequently, their descriptions were limited to abstract statements such as 'did the learners actively participate?' However, in-service teachers reflectively described their direct lesson observations (lesson study); therefore, descriptions specifying concrete objects of observation (students' notes and utterances) were conspicuous. They are all regarded as a means of understanding the students' changes in the lesson as if they were lesson teacher themselves: 'are the students writing their thoughts, what types of notes are they taking?' In addition, some respondents paid attention to the students' facial expressions, with descriptions such as 'the moment their expressions change, I can feel how the students are feeling' and 'whether the students are engaged in the lesson in a lively way'.

[$\pi(\widehat{w})_9$] *Mathematical content*: The intent of the description of $\pi(\widehat{w})_9$ was different for pre-service and in-service teachers. For pre-service teachers, the reason for focusing on a mathematical topic is to improve the way of teaching: 'as simple an introduction as possible made it easier to engage with the lesson' or 'to facilitate students' understanding, [the teacher] attempted to repeat it many times and provide them the opportunity to expand on the problem and think about it'. In contrast, the intentions of in-service teachers are uniformly associated with the purpose of the lesson: 'Are the aims of the lesson being met, and are they adequate?' and 'what have the students learnt and what have they been able to do?'

Discussion and Conclusion

Comparison between in-service and pre-service teachers

When the pre-service and in-service teachers groups are combined, all praxeologies, $\pi(\widehat{w})_1 \sim \pi(\widehat{w})_9$, are described in some way, and as a whole, it can be said that they observe the lesson from the same perspective regardless of their teaching experience. However, many students used $\pi(\widehat{w})_3$, that is, *lesson organisation* (51%), in which they were interested in observing the allocation of time for each phase of the lesson, the learning process they spent the most time on, the smooth progress of the lesson, and the allocation of time to attract or keep students engaged in the lesson. They observed with interest and stated that they wanted to learn the skills they needed to become teachers. Furthermore, regarding $\pi(\widehat{w})_5$, that is, *teacher's assistance* (28%), which was used by several students, there were descriptions of how to respond to mistakes, to respond to individual differences, to proceed according to the lesson plan, to make everyone involved and interested in the lesson. Regarding $\pi(\widehat{w})_7$, that is, *teacher and student interaction* (26%), they observed how the teacher's behaviour was related to the students' learning, in particular how it led to creating an atmosphere of easy participation in the lesson and how a good response would encourage everyone to speak up without afraid of mistakes. In addition, $\pi(\widehat{w})_8$, that is, *students appearance* (26%), was described as an image of students who were motivated to learn, such as their attitude to expressing their thoughts and the ordinary state of the class.

Characteristics of teachers with less than 10 years of teaching experience

In-service teachers were asked to describe their teaching experience using a questionnaire. The results revealed that teachers with less than 10 years of teaching experience tended to differ from more experienced teachers. It was rather similar to the characteristics of the pre-service teachers, particularly $\pi(\widehat{w})_3$, they focused on the lesson structure and attempted to observe not only the time management but also the way the task was presented as a learning flow. Further, they described how they wanted to learn about the actual situation of the students as $\pi(\widehat{w})_8$ about their learning discipline or didactic contract (Brousseau, 1997), the way they talk, the type of thoughts they write, and whether they are proactively engaged in the lessons because they can observe a glimpse of their daily teaching. This is similar to pre-service teachers' descriptions of whether students concentrated on the lesson. Compared with those with more than 10 years' experience, although they focused on the same $\pi(\widehat{w})_4$, that is, *teacher's questioning*, teachers Es (≥10) described 'whether the questioning is to get closer to the aim', 'whether the questioning is aimed at strengthening the students' thinking', and 'whether the questions were designed to facilitate the students' thinking'. whereas teachers Ns (<10) described 'I want to know if the questions were simple and straightforward', 'I would like to know it because I am not good at', and 'I want to learn so that I would not talk for too long'. Thus, the Ns are aware of their own issues with lesson organisation and questioning in their daily lesson observations and are attempting to learn how to improve them by themselves.

The relationship between $\pi(\widehat{w})_9$ and other praxeologies

Pre-service teachers' descriptions were based on the fact that the questionnaire was administered after they had watched the actual lesson video, therefore, they wanted to know 'what type of teaching is appropriate for learning how to think?', 'what is the mathematical ability to provide logical explanations?', and 'what is the process for proceeding to applied problems?'. Their focus is on observing how the problem is solved and how the problem is developed, and there is minimal relationship between $\pi(\widehat{w})_9$ and other praxeologies. However, in-service teachers' descriptions related to $\pi(\widehat{w})_9$ include 'is the aim adequate?' and 'what is being learned?' together with 'is the board writing structured in such a way that the learning of the period can be viewed at a glance?', 'is the questioning adapted to the learning of the period and the actual situation of the students?', 'is the questioning adapted to the aims of the period?', 'how is the assistance provided according to the individual differences in the students' activities?' The respondents answered that they observe the mathematical content in relation to other praxeologies: $\pi(\widehat{w})_1$, $\pi(\widehat{w})_4$, $\pi(\widehat{w})_8$ or $\pi(\widehat{w})_5$.

Answer to research question (RQ) and theoretical open questions

To answer 'RQ: What PE is used for lesson observations in $\hat{v}$?', we set up an evaluation instance $\widehat{w}$ of $\hat{v}$ and PE of $\widehat{w}$, $\Gamma^{\blacklozenge}(\widehat{w}) = \{\pi(\widehat{w})_n | n \in \mathbb{N}\}$, as the praxeological reference model to identify PE of $\hat{v}$, $\Gamma^{\blacklozenge}(\hat{v}) = \{\pi(\hat{v})_n | n \in \mathbb{N}\}$. This way, the PE used for the lesson observation of $v_0 \in \hat{v}$ can be denoted from $\widehat{w}$'s viewpoint, e.g. $\Gamma^{\blacklozenge}(v_0) = \{\pi(\hat{v})_i | i \in \{1,2,5\}$. $\Gamma^{\blacklozenge}(\widehat{w})$ was extracted using a questionnaire survey and is therefore, as a matter of course, insufficient. However, we are convinced that the procedural or methodological validity of identifying $\Gamma^{\blacklozenge}(\hat{v})$ could be argued. Further theoretical work is required on how $\Gamma^{\blacklozenge}(\widehat{w})$ needs to be elaborated in the future, in particular for the judgement (⊢) in #2.

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Three-fold meta-didactical conflict. LSA and teacher education

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This research aims at investigating the presence of the three-fold meta-didactical conflict, comprising the relationship between teachers and large-scale mathematics assessments, in the context of Italian upper secondary school. The study involves both in-service and prospective secondary school teachers. The conflict is theoretically framed within the MTSK Model. The data collected show some inconsistencies with the national results and confirm the presence of the conflict in the population involved. This study contributes to the research line that uses data from large-scale assessment as tools for enhancing teacher education and improving teaching and learning processes.

Keywords: Type Teacher education, secondary school teachers, MTSK model, three-fold meta-didactical conflict, Large Scale Assessment.

Introduction

As highlighted by literature, from the macro-phenomena emerged through Large Scale Assessment - LSA, with an opportune and conscious integration in teacher education, it is possible to improve teachers' knowledge and skills (Campbell & Levin, 2009; Ferretti & Santi, 2023). On the one hand, the analysis of the results of LSA can be useful for teachers to reflect on teaching-learning processes, as well as on the possible strengths and weaknesses of specific teaching choices; on the other hand, precisely through the interpretation of macro-phenomena emerged in LSA by prospective and in-service teachers, information can be obtained on the possible weaknesses related to their specialized knowledge. As argued by Adler (2000), mathematics teacher education should focus on resources, understood as tools that enable them to refine mathematical activity in the classroom. In this perspective, macro-phenomena emerged in the INVALSI (National Institute for the Evaluation of the Education and Training System) standardized assessment can determine relevant observations on the specialized knowledge of teachers, to be able to subsequently construct a model of intervention for teacher education, that such as Mellone and colleagues claim is a "highly desirable" outcome (Mellone et al., 2017, p. 8). Our study moves within the strand of research that links large-scale assessments with teacher education and, in particular, is grounded in the Arzarello and Ferretti's construct of the *three-fold meta-didactical conflict* (2021). This conflict arises from the comparison between the context of a mathematics class and the general school context and, as the name suggests, consists of three components concerning the discussions on teaching and learning processes. The present study aims to explore this meta-didactical conflict and involves in-service and prospective secondary school mathematics teachers. The survey tool used is a questionnaire consisting of questions about eight selected 10th school grade Italian LSA INVALSI items. Each question allows for the investigation of the presence of the conflict's components consistently with the domains of the Mathematics Teacher Specialized Knowledge (MTSK) model (Carrillo-Yañez et al., 2018). As we will discuss in the following sections, teachers and prospective teachers' knowledge and beliefs are interpreted and framed with some sub-domains of the MTSK model.

Three-fold meta didactical conflict

The *three-fold meta-didactical conflict* is a construct consisting of three components and it arises from the comparison between the classroom reality and the national context. The contextualization of this meta-didactical conflict arises from the analogy between the small context of a mathematics classroom and the general national school context, which the research is considering. In the analysis of classroom interactions, it is common to observe what is referred to as "incommensurable discourse" (Sfard, 2008) between teacher and students: it often happens that - in the communicative processes of classroom teaching practices - students and teachers use the same words but with different meanings, without being aware of the differences. Conflicts are generated which, if not made explicit and overcome, can have serious consequences for the success of teaching/learning processes in the classroom. These incommensurable languages have also been identified between mathematics teachers and large-scale assessments and identify the so-called "*three-fold meta-didactical conflict*" (Arzarello & Ferretti, 2021). The conflict is meta-didactical as it concerns discourses on educational processes such as assessment, student skills and errors, etc. and not mathematical concepts as theoretical constructs. The three components that characterize the construct are the following ones:

(i) *the perception of the difficulty of the questions*. This component relates to the fact that teachers perceive the difficulties of the LSA mathematics test questions in a very different way to the national level data. The perception and, therefore, the estimation of item difficulty, is not part of the metrological validity of a test, but part of a broader issue that aims to investigate teachers' awareness of possible obstacles to correct student resolution of an item;

(ii) *the awareness of the causes of the errors*, this component concerns the interpretation of the causes of errors that are widespread nationwide and framed by researchers with some of the most internationally recognized constructs of mathematics education;

(iii) *the perception of the purpose of the questions*, a consequence of the two previous components, which concerns the contradictory way in which the teachers interpret the purpose of the questions in relation to their usefulness for teaching and, above all, learning goals.

On the one hand, the three components are specific and focused on targeted skills, while on the other hand, they closely intersect as a possible representations of the teacher's specialized knowledge. In line with Ferretti, Martignone and Viola (2023), we will interpret the data with reference to the three components of the meta-didactical conflict by relating them to the PCK sub-domains of the Carrillo MTSK Model.

The research objectives from the MTSK model perspective

The knowledge and beliefs of mathematics teachers bring into play during their profession is the focus of much recent research and most of these studies highlight how mathematics teachers' knowledge model is a useful tool for organizing also initial teacher education (e.g. Montes et al., 2013). Especially, from Shulman definition of the two macro-categories, Subject Matter Knowledge and Pedagogical Content Knowledge, several models have been developed, including the model by Carrillo-Yañez and colleagues (Shulman, 1986).

The Mathematics Teacher's Specialized Knowledge (MTSK) model, outlined by Carrillo-Yañez and colleagues in 2018, coordinates two broad areas of knowledge, Mathematical Knowledge (MK) and Pedagogical Content Knowledge (PCK), placing the teacher's beliefs at the center (Carrillo-Yañez et al., 2018). The first group is related to a mathematics teacher's own knowledge in terms of a scientific discipline within an educational context and is further divided into three sub-domains: *Knowledge of Topics* (KoT), *Knowledge of the Structure of Mathematics* (KSM), *Knowledge of Practices in Mathematics* (KPM). The PCK areas, that will be focused on in our study, represents knowledge related to the mathematical content in terms of teaching-learning processes and is composed by: *Knowledge of Mathematics Teaching* (KMT), which determines knowledge of mathematics teaching theories, resources, materials and technologies, but also knowledge related to strategies for introducing and representing content and concepts; *Knowledge of Features of Learning Mathematics* (KFLM), related to the ability to predict students' reactions: their detectable errors and difficulties on specific topics and, in general, their interactions with mathematics, taking into account all aspects, including affective ones; *Knowledge of Mathematics Learning Standards* (KMLS), i.e. knowledge of the national curriculum and international documents of all school levels. Beliefs about mathematics and its learning and teaching are at the core of the model, precisely to emphasize the reciprocity between beliefs and knowledge domains. *Beliefs on Maths* include all the beliefs and preconceptions that teachers have about mathematics, based on their own education and the *Beliefs on Maths Teaching and Learning* are built on teachers' personal and professional experiences (Carrillo-Yañez et al., 2018).

The connection between these facets of the teachers' specialized knowledge and reflections on macro-phenomena revealed in INVALSI tests to comprehend some of the critical aspects of teacher education is the basis within which the questionnaire and the entire research were developed. Our research aims to characterize Arzarello and Ferretti's *meta-didactical conflict* further, both by investigating its presence at secondary school level and also with prospective teachers, and by framing it theoretically with the MTSK Model.

Methods

The research involves a population of 62 interviewees, made up of 18 secondary school prospective teachers, including some master's degree students in mathematics on the teaching program and other PhD students in mathematics education and 44 in-service teachers, from new hires to teachers with more than fifteen years of experience. All data queries are selected from an online database, Gestinv (2024), where it is possible to search the INVALSI items from 2008 to the present. The survey was administered anonymously in a telematic way via Google Forms. The design of the questionnaire was guided by the identification of macro-phenomena that emerges with reference to INVALSI mathematics items for secondary school considered suitable for the investigation of the starting assumptions. Four items (D21 s.y. 2017, D15 s.y. 2011, D28 s.y. 2010, D5 s.y. 2016, Gestinv, 2024) make up the basis of the research on the first component of the *three-fold meta-didactical conflict*, of which two with a high percentage of incorrect answers (above 60%) and two with a low percentage of incorrect answers (below 55%) respectively. Therefore, regarding this component of the conflict, situations were proposed in reference to both questions with very low and very high percentages of correct answers at national level: this is another innovative element compared to the meta-didactical

conflict foundation research, which only investigated the first component with reference to questions with low correct answer percentages. The remaining four items (D22 s.y. 2011, D14 s.y. 2011, D14 s.y. 2013, D24 s.y. 2017, Gestinv, 2024) from the foundations of the survey for the second component and scored a very low percentage of correct answers at national level - less than 35%. Each question is followed by targeted, open-ended questions and closed-ended questions on a Likert scale of 1 to 4. The first closed-ended question "In your opinion, how difficult do you think the following item is for students at the end of the second class of the second-grade secondary school?" makes it possible to investigate the participants' perception of the students' difficulty of the items. This survey seems to relate to the concept of PCK in the MTSK model. The PCK required of teachers in identifying the correct level of difficulty considers both the knowledge inherent to teaching, KMT, and that one relating to the characteristics of learning, KFLM. The relationship between perception and knowledge of learning is fundamental to identify the appropriate level of difficulty that students perceive in the problem solving. The teacher must, therefore, be familiar with learning techniques and understand the way in which each student can tackle a problem. At the same time, knowledge of the transmission systems, with the possible difficulties that can be generated, is strongly related to the teacher's awareness. In the following figure (Figure 1), we show the representation of this priori analysis of the first component of the meta-didactical conflict involving the different sub-domains of the PCK.

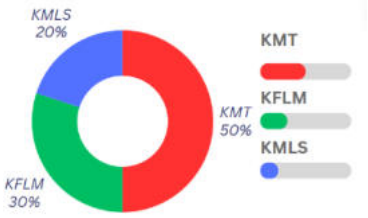

Figure 1: First component – PCK sub-domains

The open-ended question "In your opinion, which is the most chosen wrong option(s) and why?" is, on the other hand, useful for analyzing the level of awareness of the causes of errors. Also in this case the knowledge and beliefs involved concern all three sub-domains of the PCK. In particular, the factor relating to the learning process (KFLM) appears more relevant. This component seems to highlight the fundamental importance of the interpretative effort required of the teacher; the skill of the teacher to understand and to analyse the error brings into play different type of knowledge, as noted in this a priori analysis depicted by following graph.

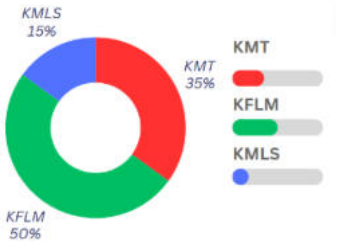

Figure 2: Second component – PCK sub-domains

The questions “On a scale of 1 to 4 how suitable do you think this item is for assessing the learning of students at the end of second class of second-grade secondary school?” (for prospective teachers and in-service teachers) and “On a scale of 1 to 4, how habitually do you use items like this in your assessment tests?” (for in-service teachers) are aimed at defining the rank of perception of the purpose of the items. This component arises directly from the two components described above. Indeed, to construct a definitive view on the usefulness of a problem for assessment purposes, an in-depth analysis is required, related both to the objective difficulty of the questions and to the subjective difficulties experienced by the students. Consequently, it is possible to frame the latter component in relation to the PCK. In this case, the KMLS seems to appear predominant. The teacher initially provides his/her own conception of the question by means of his/her specialized knowledge of teaching and learning. Subsequently, the teacher expresses his/her perception of the purpose of the problem, exclusively through the contextualisation of that in relation to the grade and to the student's curriculum.

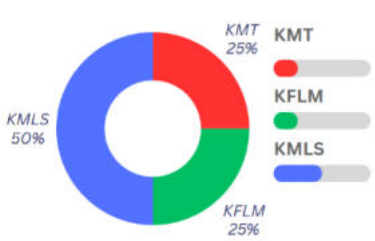

Figure 3: Third component – PCK sub-domains

In all components of the meta-didactical conflict, beliefs seems to play a role. The awareness of the causes of errors can be influenced by each teacher's personal conceptions of both mathematics and even more on teaching and learning; each teacher identifies with the student and provides reflections that may arise, in part, from their own experience and from their own beliefs. Especially if a teacher prefers a reproductive approach, they will have a different perception from those who implement a productive approach.

Data analysis and discussion

In this section we will analyse, by way of example, the results of questions about two INVALSI items that allow us to delve deeper the teachers’ perception of the difficulty and the awareness of the causes of the errors. This is part of a broader study on each item of this research where results are also obtained on the perception of the purpose. Following it is presented one of the considered items (Figure 4) about which teachers are asked to estimate the students’ difficulty in solving it without knowing the INVALSI results of the percentages of error. The item requires a *conversion* (Duval, 2006), the semiotic transformation of a representation from the graphic register to the numerical register. The percentage of the wrong answers is higher than 65% (Figure 5) and the most selected wrong options are answers A and D.

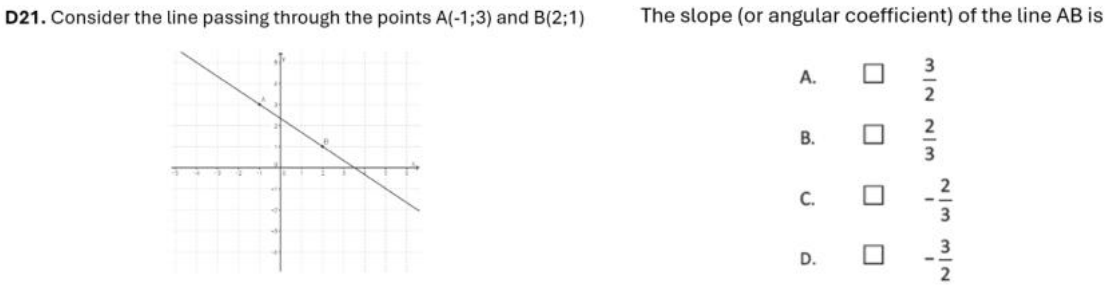

Figure 4: Item 21, grade 10 Mathematic INVALSI test 2017 (Translated by the authors, Gestinv, 2024)

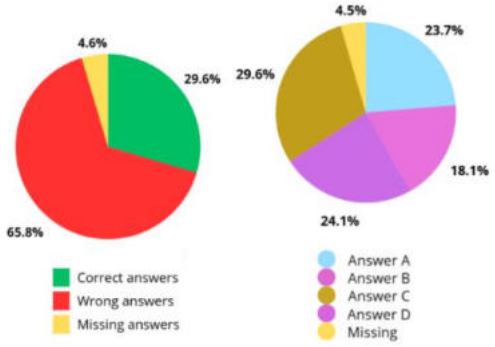

Figure 5: Percentage of choice of each option, item 21, grade 10 Mathematics INVALSI test 2017 (Gestinv, 2024)

Teachers are asked to estimate the degree of difficulty on a scale from 1 (*very easy*) to 4 (*very difficult*). In Figure 6 it is possible to see the answers of the population involved.

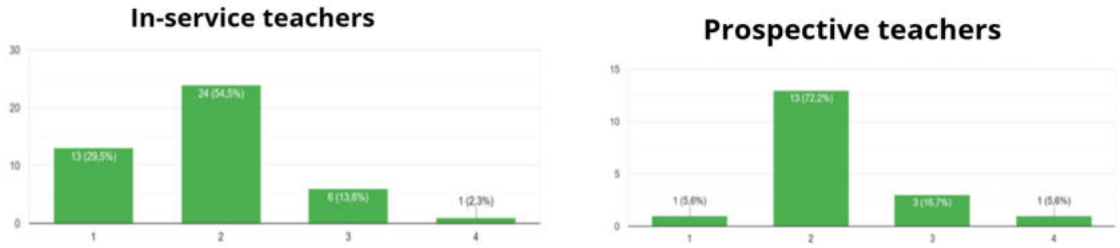

Figure 6: Percentage of choice - question on item 21 (Figure 4)

As highlighted in the Figure 6, the perception of the difficulty of the item appears in contrast with the national results (Figure 5): despite only about 29% of the students gave the correct answer, the majority of the population involved in the experiment (almost 85% of in-service teachers and almost 80% of prospective teachers) considered the question *easy* or *very easy*. This symbolises a possible criticality in the teachers' specialized knowledge, with reference to the PCK and, in particular, with regard to the KFLM sub-domain (Figure 1).

The other item considered (Figure 7) is about the awareness of the causes of the errors. In order to solve the item, it is necessary to carry out a *treatment* in Duval terms (2006), the semiotic transformation from a representation to another in the same numerical register.

D24. The expression $\left(a-\frac{1}{a}\right)^2$ can be written as

A. ☐ $a^2+\frac{1}{a^2}-2$

B. ☐ $\frac{a^4+1}{a^2}$

C. ☐ $a^2-\left(\frac{1}{a}\right)^2$

D. ☐ $\frac{a^2-2a+1}{a^2}$

Figure 7: Item 24, grade 10 Mathematic INVALSI test 2017 (Translated by the authors, Gestinv, 2024)

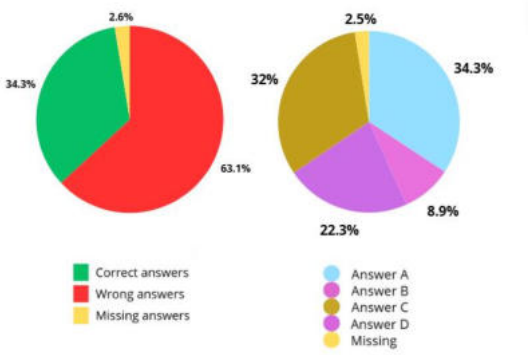

Figure 8: Percentage of choice of each option, item 24, grade 10 Mathematics INVALSI test 2017 (Gestinv, 2024)

In this case the percentages of the correct answers (34,3%) were reported on the questionnaire. Answers C and D respectively are selected by 32% and 22,3% of the national sample of students, showing difficulties in the management of treatments, in particular errors in splitting the square of the binomial in the first case and the failure of the lowest common multiple in the second case. In reference to the open-ended question about the choice of the most common wrong answer 50% of the population involved selects the answer C and about the 20% choose the answer B and D. The results about the erroneous options represent a possible critical element to reflect on regarding the teachers' specialized knowledge, with reference to PCK as in Figure 2.

Conclusion and further directions

In this paper we show some results of a broader work about the three-fold meta-didactical conflict on a population of in-service and prospective teachers. We frame the three components of the conflict with a priori analysis related to PCK sub-domains of the MTSK model, strengthened by the results obtained from the data related to the 10th grade INVALSI items considered. The answers collected represent inconsistencies between the perception of the population involved and the national students' results. In particular, this work not only confirmed the presence of the components of the three-fold meta-didactical conflict in the secondary school context considered but also defined some new characterizations. The data obtained are a possible starting point for some reflections on the teachers' specialized knowledge. The definition of the theoretical framework is currently in the process of refinement, facilitated by an expansion of the model, while also exploring the potential for a network of theories. The mainly further direction would be to contribute to the construction of an updated

model for teacher education that considers the difficulties of this conflict related to specialized knowledge.

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Mathematics teachers' identity in the prison context

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Within the Learning Mathematics in Prison project, we analysed the identity of mathematics teachers working in a prison context, with a specific focus on their motivations, goals and perceived role. Through the qualitative analysis of the interviews to four teachers working in different Italian school prisons, we derive that their professional identity is more strictly related to their role of educators in a prison context than to their role of mathematics teachers. This is also reflected in their perceived professional development needs, which are prevalently connected with the management of situations connected with the prison context rather than to the teaching of disciplinary contents.

Keywords: Mathematics education, prison education, teacher identity, teacher views.

Introduction and theoretical background

In this paper, we aim to explore the identity of mathematics teachers working within prison contexts, with a particular focus on their motivations and perceived roles. Teacher identity is a crucial concept in sociocultural research, understood as a dynamic and situated phenomenon constructed through discourse (Lerman, 2006; Sfard, 2008). Exploring teacher identities in prison settings is relevant as it sheds light on the pedagogical challenges and opportunities these environments present (Novek, 2019). Prison schools are institutions offering students (minors or adults) opportunities to obtain educational certifications and fostering both human and professional growth. Prison schools focus on lower secondary education and early years of upper secondary education. Collopy (2003) defines teacher identity as a "constellation of interconnected beliefs and knowledge about subject matter, teaching, and learning, as well as personal self-efficacy and orientation towards work and change" (p. 289). Expanding on this, Ponte and Chapman (2008) describe it as a teacher's professional self, reflecting a form of social identity. According to Holland et al. (2001), identities are closely linked to self-understandings that individuals articulate to themselves and others. Sfard and Prusak (2005) frame identities as "a collection of stories about persons" (p. 16), highlighting the role of communication in shaping and expressing one's identity. Bjuland et al. (2012) advocate for a narrative-based approach to analyzing teacher identity, emphasizing its construction within specific contexts (Hodgen, 2011). Narratives collected during interviews, as noted by Dalby (2017), provide authentic and credible insights into how teachers perceive and articulate their identities.

The notion of a stable identity trait contrasts with the socio-cultural view that sees social interactions as fundamental in shaping identity. In this framework, identity is constructed through narrative, making it socially situated, ever-changing, and uniquely created by the individual for each specific situation. As Holland et al. (2001) suggest, a narrative approach allows researchers to develop a more reliable analysis of identity, recognizing that it is uniquely constructed within the narrative context, by the individual, for the situation. In interview settings, therefore, the identity presented is

specifically constructed for the researcher, yet it maintains authenticity and credibility within this narrative framework.

Professional identity can be broadly understood as the aspect of identity that individuals develop in relation to their profession or occupation. Adopting a narrative perspective, in this case, means that teachers' personal accounts are essential for a comprehensive understanding of their professional identity (Dalby, 2017). This study focuses, in particular, on the professional identity of mathematics teachers working within prison contexts, a topic that, to the best of our knowledge, has not been specifically addressed in the literature.

Methodology

Data collection

The study has been conducted as a part of the PRIN projects LeMP (Learning Maths in Prison). This project aims to develop guidelines and educational materials for teaching mathematics in prison. It involves experts in Mathematics Education, Special Education, and Pedagogy from five different universities, and mathematics teachers working in prison schools in three different Italian regions.

We collected data by conducting individual semi-structured interviews, lasting about an hour each, with four mathematics teachers working in three prison schools, who participated on a voluntary basis. All the interviewees work with highly diverse groups of students, varying in age, educational, and cultural backgrounds, and characterized by high turnover rates. Caterina has a degree in Ecological Sciences, taught for 4 years in CPIA (Provincial Centers for Adult Education), and has now been teaching at the juvenile detention center for 4 years. Anna has a degree in Geological Sciences, with 21 years of experience in middle school, 4 years in CPIA, and two years in an adult prison. Salvo holds a degree in Biology, has a PhD and research experience in Neuroscience, with 2 years of experience in an adult prison. Camilla holds a degree in Marine Biology and has 20 years of experience in middle school and 4 years of experience in adult prison. None of them have received specific training for teaching, neither in a usual context nor in a prison context.

All the interviews were conducted online by two authors who took turns: one asked the questions, and the other took notes. They have been video recorded and fully transcribed. In drafting the questions of the interview, we chose to explore three predetermined categories: role of the school and the teacher in the prison context; teachers' goals and motivation for teaching in a prison context; teachers' perceived professional development (PD) needs. Our research interest was particularly focused on mathematics, so we investigated whether and how the role of mathematics emerged also from the answers to general questions. Otherwise, we prompted teachers to specify the answers in relation to mathematics.

In particular, the following questions constituted the plot of the semi-structured interview: *1)What led you to teach in prison? 2) In your opinion, what is the purpose of school in prison? 3) What is the role of the teacher in prison? 4) What significance do you think learning mathematics has for students in prison? 5) What is the main objective you set for yourself when teaching mathematics in prison? 6) What has been your educational and work path? 7) In what way, in your opinion, does your*

background influence your way of teaching? 8) Where would you like to continue your professional career in the future? 9) Do you feel you have any specific professional development needs for doing your job at your best?

Data analysis

We focus our analysis on teachers' answers to the questions of the interview related to the categories under scrutiny: role of the school and the teacher in the prison context; teachers' goals and motivation for teaching in a prison context; teachers'perceived PD needs.

We conducted a qualitative analysis of the collected data, categorizing the emerging themes based on qualitative content analysis principles (Cho & Lee, 2014). First, the three of us individually classified all responses according to the three pre-determined categories mentioned above and created new categories suitable for the data not fitting in with the pre-determined ones. In the second phase, we compared our findings to reach a consensus.

From our analysis, it emerged that the first two pre-determined categories (role of the school and the teacher in the prison context, and teachers' goals and motivation for teaching in a prison context) were very much intertwined, while a new category emerged: teachers' vision of mathematics. Even though a clear-cut division cannot be made between the first two categories, in the presentation of the results we divide them according to this criterion: in the first one we classified the themes prevalently related to the school in prison as an institution, while in the second one we classified the themes related to the single teachers, to their personal beliefs, goals and motivations.

Results

Role of the school and the teacher in the prison context

The teachers' declarations essentially focus on two values of the prison school, an educational one and an ethical one. The first value, mainly relevant for the teacher who worked with students in compulsory school age, is linked to the opportunity of providing continuity to the educational path undertaken before the detention (Caterina: "*the role of prison school is allowing students to continue their studies and facilitate the dialogue with the external schools*"). In this perspective, prison school is perceived as a hinge between the life before imprisonment and the next life, i.e. as an environment where students build conditions for a "*second chance*" (Camilla) once they leave prison. The teachers agree that it is very important to avoid school drop-out and "*do not miss the students again*" (Salvo), taking into account their previous experience of failure, not only in the scholastic field. According to Caterina, "*the relationship of trust with the teacher [...] is a glue for the students in prison, who had already had many failures".* The other value of prison school felt as a fundamental one by the teachers involved in the study is that concerning the "*personal redemption and social lift*" (Anna). The students have the opportunity of entering in touch with "*the beauty, the goodness of knowledge and human relationships*" (Camilla). In this sense the teacher is a figure guiding them to discover, to be curious and learn without being judged; he/she provides students with engaging challenges and offers support to overcome them, so fostering the regaining self-confidence and trust in the others. For example, Salvo declares that "*teaching mathematics in prison could give a great contribution for students: by overcoming [disciplinary] obstacles, they have the opportunity of finding themselves".*

Through the didactics, the prison school plays an ethical role: hopefully students find new impulses and interests, get used to collaboration avoiding conflicts and benefits from the exchange with the peers and the teacher. Consequently, students have the opportunity to find in school experience an opportunity for personal growth that goes beyond achieving disciplinary goals.

Motivation for teaching in a prison context

For all the teachers of our sample, entering the prison school system often occurs by chance, driven by curiosity, a desire to experiment, and the search for new stimuli.

A commonly shared experience among teachers is the need to open up to relationships, separating the incarcerated individual from the crime they have committed. This relationship requires empathy and brings forth a range of emotions, like compassion (Anna: "*When you're able to separate [...] the crime, which you obviously reject, from the person who committed it, who is a human being and therefore made a mistake, well, [...] in the end I connected with them and became super attached*"). The outcome of this interaction is an exchange, leaving teachers with something "*that stays with them*". Due to this openness, educators experience a strong sense of fulfilling a social role and feel that they can be important to their students. From one perspective, they aim to offer opportunities for change and personal growth, broadly guiding students and enhancing their self-confidence. However, they also feel the need to reaffirm their role as teachers and educators, distinct from psychologists and social workers.

All the interviewees set the goal to manage the reality of failure: an experience their students have often encountered, both academically and personally, and a potential outcome of the prison education process. Another recurring theme in the teachers' statements is the concept of challenge, seen as a motivation for self-improvement, both from the students' perspective - facing academic and cognitive obstacles - and from the teachers' perspective, as they navigate a context where engaging all students is essential yet difficult.

From the point of view of their motivations to teach in a prison context, some teachers are prompted by the willingness to follow their personal interests (curiosity about neuroscience, better empathy with older individuals, fascination with the prison context), while others focus on the benefit they can offer (expressing affection and love, knowing that "*you mean so much to them*"). All the interviewees express appreciation for the reduced bureaucratic and curricular constraints in prison education (fewer external pressures such as parental involvement or stringent assessments) and the smaller class sizes compared to standard schools. These aspects contribute to their desire to continue teaching in this environment.

Teachers' vision of mathematics

As a general observation, we can state that, even when responding to questions where a reference to mathematics would be expected, the teachers do not often speak about the subject in itself. Instead, they all mention the contribution that learning mathematics has on the imprisoned students' mindset. Camilla and Caterina refer to this in terms of the development of problem-solving skills. Caterina emphasizes "*developing reasoning ability, the capacity to find a solution*" and the importance of "*structuring the mathematics lesson as a question*". Anna and Salvo speak about fostering "*open-*

mindedness", focusing on logical structure and the ability to look beyond appearances. Salvo highlights the possibility for students to evaluate and choose among different procedures: "*I present them with different methods and then let them choose. I show them that switching to another method might have advantages, and then they decide*".

Concerning instrumental understanding or relational understanding of mathematics (Skemp, 1976), only Salvo and Camilla express themselves in such a way as to bring out one of the two approaches, while in the other interviews more aspects emerge that suggest the first approach (e.g. the reference to mathematical procedures and techniques), but also some aspects that suggest the other (e.g. a certain attention to diversifying representations). Terms such as "*operations*", "*calculations*", "*rules*", "*techniques*", and "*errors*" appear in most interviews, often marginally. However, Camilla frequently refers to these concepts throughout her interview, mentioning "*expressions*" eight times, "*calculations*" twice, "*rules*" three times, and "*fractions*" three times. All the interviewees mention the use of problems in their teaching practice. Camilla specifically mentions problems in the context of geometry, while the others attribute a more general and significant role to problem-solving. Salvo's interview also shows a desire to go beyond the execution of simple exercises, for example when he says: "*we don't want them to be simply self-sufficient in very simple mathematical tasks, [...] but we also want to get them used to opening their minds, to approaching slightly larger problems*".

All the teachers report - sometimes only after prompting - that their students, especially those with a migrant background, perform mathematical procedures and representations of mathematical content differently, due to previous educational paths, work experiences, or other factors. Viewing different procedures as an obstacle may indicate a procedural understanding (Skemp, 1976). Camilla remarks that "*when an expression has rules, you can't ... more or less the procedures are the same*".

Another recurring theme is the relationship between mathematics and language, primarily understood as formal and symbolic language. Camilla aims to "*show them that this formal language, in the end, makes sense*". This aspect is particularly significant for Salvo, who identifies a strong connection between mathematics and language, emphasizing them as human characteristics. He also raises linguistic challenges in definitions, especially in a multicultural context: "*Mathematical language is somewhat universal, but when concepts like perimeter, area, and volume appear in geometry, these are mountains for them [foreign students]*".

The connection between mathematics and reality is only partially recognized. Anna mentions this connection only concerning geometry: "*They like geometry because it has practical applications, helping them better understand reality*". Conversely, Camilla feels that this link is lacking in mathematics, except for topics functional to the sciences, attributing this as one of the main difficulties in teaching mathematics: "*If I could only teach science, I would feel much better in prison. And maybe just include a bit of mathematics when needed*".

Some teachers recognize mathematics as particularly important as a prerequisite for further studies. Anna notes: "*I see mathematics as laying the foundations to enable them to continue their studies*", while Camilla states: "*When I realize that someone in the class will continue studying, I try to make them engage with mathematics because it will be useful to them*".

All teachers address difficulties in learning mathematics, described in terms of "*effort*", "*obstacle*", and "*complexity*". However, according to Caterina and Salvo, this is a positive aspect in the prison context because it allows students to face challenges and gain satisfaction from overcoming them. In contrast to the deprivations of the prison environment, mathematics can offer a stimulating opportunity for mental escape. Caterina observes that "*it can provide opportunities to think, offering problems that help the student escape from reality, from daily life, and distance themselves from immediate problems*".

Overall, we can say that the teachers' perception of mathematics appears closely tied to the context in which they teach.

Declared professional development needs

From the responses to the question about the PD needs that the participants feel they have, a predominance of references to issues related to the prison context emerges. Far fewer are the references to PD needs related to specific didactic approaches to teaching mathematics.

Salvo refers to a need for PD on effective approaches for teaching mathematics to adults or individuals from different cultures ("*Professional development tailored to the type of individuals, meaning it should take into account their age, which [...] always involves a range of different ages. It should also consider, [...] the diversity of cultures and the different systems for using mathematics in various regions*"). Camilla emphasizes the need for PD to prepare more engaging math lessons that can capture the interest of students in prison, without using technologies forbidden in prison. She also mentions the need to learn to linguistically simplify the instructions for people who do not have a strong knowledge of Italian. These requests by Salvo and Camilla regard needs related to the prison context but also specifically referred to the disciplinary content.

In all other cases where this category emerges, the stated needs are related to issues of managing problematic situations at a relational level with or among the imprisoned students. This is likely because the teachers have not received any specific psychological training, which makes them feel unprepared to handle situations that significantly differ from those in an external school.

For example, Anna would like to learn more tools for emotionally handling relationships with women who are depressed because they are separated from their children. Caterina also expresses a need for PD to manage situations of low motivation to study among students who are depressed and, at times, drug-addicted. Caterina mentions a PD need related to conflict management and the dynamics that concern the lives of the students in prison. She also mentions the study of body language to understand what students do not express verbally ("*Methodologies… Yes, regarding disciplinary issues, but also, for example, being able to understand what is not said, such as aspects related to gestures*"). Finally, she highlights the need for language training, especially in Spanish and English, to conduct lessons according to the CLIL (Content and Language Integrated Learning) methodology.

The fact that the PD needs expressed by the teachers in our sample mainly concern issues related to the prison context seems consistent with an identity that is more connected to their role as educators of prisoner students rather than as teachers of a specific subject.

Concluding discussion

Our analysis allows us to note that two of the predetermined categories - role of the school and the teacher in the prison context, and teachers' goals and motivation for teaching in a prison context - are very much intertwined. This result suggests us that the identity of the teachers of our sample is strictly bound with the role that they attributed to school in a prison context, in general terms and not in relation to a specific discipline. These findings are aligned with those of Stevens (2015), who highlighted that, for the majority of the teachers in her study, the prevalent motivation to work in prison contexts was the willingness to work with "disadvantaged youth" and that "youth corrections is the perfect career for them" (p. 27).

This prevalent identity as educators in prison context is also confirmed in the analysis of their declared PD needs, which are more focused on managing prison-related situations rather than addressing discipline-specific issues in mathematics. This is also probably because they feel inadequate to fulfill a role that is more suited to a psychologist, as they lack training in this area. However, they recognize that the context demands it, and on the other hand, they appreciate taking on this role, which involves a form of affective rehabilitation of their students. These findings relate to the prison teachers' identities identified by Wright (2005). In particular, the teachers involved in our studies seem to have what Wright calls *settlers* or *cultural mediator* identities. Indeed, the first type of prison teacher "*makes commitment to stay [in prison] ... erasing attributes from the home culture that inhibits his or her integration in the new culture"* (p. 30); instead, the prison teachers who are cultural mediators "*make enormous efforts to pull the outside (the community, its norms and practices) into the school"* (p. 31). Coherently, the teachers' view of mathematics emerged from the interviews is primarily contextual, emphasizing what the subject can represent for incarcerated students in a broader sense, rather than its intrinsic disciplinary value.

From these results, we can conclude that the teachers' identity emerged in this study is more strongly aligned with their role as prison educators than with their role as mathematics teachers. This conclusion is similar to that drawn by Dalby (2017) in her study on teachers at adult vocational schools. Dalby found that they "often identified themselves more strongly with being a Further Education teacher than with being a mathematician" (p. 14).

We acknowledge that this study has limitations, including the small number of participants and the fact that none of them have a specific background in mathematics, holding degrees in other scientific fields. Future research aims to expand the study by including a larger sample of mathematics teachers working in prison contexts, possibly incorporating participants with a mathematics degree. This will allow us to explore potential differences in their views on mathematics, their specific teaching objectives, and the role they perceive mathematics to play for students in prison.

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Towards European measures for upper secondary teacher students' mathematical content knowledge

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We report on initial experiences with developing and trialling items to test school-relevant mathematical knowledge of prospective upper secondary level teachers in Europe. We give examples of how trials led us to reject or retain items, and our current rationales for developing and selecting items. Further perspectives and motivations for this line of research are outlined.

Keywords: Mathematical content knowledge, European upper secondary teacher education.

Introduction

Throughout the world, students' mathematical knowledge is routinely evaluated using written items, both in formative and summative assessment. A similar approach to evaluate teachers' mathematical knowledge is less common. It could be seen as sensitive or even offensive by teachers. A third case, that is somewhat intermediate, is that of pre-service teacher students. At universities, their mathematical knowledge is also commonly evaluated at written exams.

In an international context (with 17 countries), a comparative assessment of teacher students' knowledge was carried out by the TEDS-M study for prospective primary and lower-secondary level teachers of mathematics (see next section). A seed project (called "Knowledge for Teaching Upper secondary Mathematics", KTUM) was approved by the 4EU+ alliance to develop a similar approach within several European universities (located in Czech Republic, Denmark, France, and Italy), to develop a shared model of the mathematical content knowledge that is deemed relevant in the participating universities for students close to qualifying for upper secondary — as opposed to primary and lower secondary as in TEDS-M — teaching.

The motivations for the project were as follows:

- Within the EU, teachers who are qualified to teach in one country can teach in another: for linguistic and regulatory reasons, this is not (yet?) a common occurrence, but it shows the importance of knowing if the teachers get somehow similar or different "basic knowledge" and, potentially, of developing common European qualification standards in major and shared school disciplines like mathematics.
- The difference between national education systems is reflected in a wide array of organisations of pre-service teacher education. For example, in Denmark and Italy, most or all of the university study is in (pure) mathematics, although with some options to take courses in didactics; only later do those who wish to become teachers attend a professional course on pedagogy and practice. In France, after a generic bachelor's degree (typically in mathematics), future teachers take a master's degree specifically oriented towards the profession of teaching.

In the Czech Republic, both the bachelor's and master's degrees are oriented towards the profession, including the mathematics components.

- Moreover, different hiring processes — national competitive contests (Italy, France) vs. direct hiring by schools (Czech Republic, Denmark) — also contribute to different motivations and settings for developing profession-oriented knowledge.

It is natural to ask if these significant systemic differences are somehow reflected in students' capacity to cope with "basic knowledge" questions, directly related to the mathematics taught in secondary school. In this paper, we report on first attempts to develop and testing an inventory of such questions across countries in the European Union.

Theoretical Framework and Research Questions

There have been many attempts to develop measures of mathematics teacher knowledge, particularly in the US (e.g., Hill et al., 2008), but also internationally (Krainer et al., 2015). Such studies naturally seek to focus on elements of teacher knowledge that correlate with measures of efficient teaching, when controlling for other factors (for an overview, see Winsløw, to appear). To summarise, early results suggested that there is no simple correlation between the sheer amount of academic mathematics courses completed by teachers, and the results or knowledge gain of their students. However, deep knowledge of the mathematics to be taught, as well as knowledge about how to teach it, are promising categories in terms of their impact on teaching quality and student achievement. These categories of knowledge have been theorised in various ways, giving rise to categories such as MCK (mathematical content knowledge) and MPCK (mathematical pedagogical content knowledge), as used in the TEDS-M study (Tatto, 2015). Quantitative studies operationalise these categories in test inventories. Most studies have been focused on primary and lower secondary school teacher knowledge and education, and none has so far focused on MCK and MPCK resulting from upper secondary teacher education in the European context as we do in this paper. We used an adaptation of the subdomains of MCK from TEDS-M, summarised in Table 1, while including also some items which would be classified as MPCK in that study (meaning that the item specifically addresses teaching). In particular, the categories are adapted to the upper secondary level as shown in Table 1, while including also some items which would be classified as MPCK in that study (meaning that the item specifically addresses teaching).

Given these categories, we have worked with the following research questions, focused on methodology for developing test items corresponding to the categories:

1. What kinds of items are proposed by representatives of European universities, when asked to provide input to an MCK/MPCK test for future upper secondary school teachers?
2. How did students from the four universities do on them initially (according to category and country, are there any apparent differences)?

Concerning 1., the main sources of items have been exams given in teacher-oriented courses at the participating universities, subsequently adapted and revised for small-scale pilot experiments (cf. the next section) in order to provide first answers for 2. that could illustrate the kinds of result one might obtain with larger and more carefully sampled groups of respondents.

Table 1: MCK framework: subdomains and content areas (adapted from Tatto, 2013, p. 13)

Subdomain	Content Areas
Numbers	Number sets (e.g. rational); Induction; Number Theory
Geometry	Geometric shapes; Geometric measurements; Analytic Geometry
Algebra	Algebraic expressions; Algebraic equations; Vectors, Matrices
Functions	Equations; Algebraic and Transcendental functions
Data and Chance	Data processing and interpretation; Probability
Analysis	Continuity; Limits; Differential calculus; Integral calculus

Context and methods of KTUM

4EU+ is an alliance between eight European universities who share the goal "to create one comprehensive research-intensive European University through a new quality of cooperation in teaching, education, research and administration" (4EU+, 2024). Members from four participating universities — University of Milan, University of Copenhagen, Charles University Prague, University of Paris–Sorbonne — joined in a one-year programme (November 2023 to October 2024) called *Knowledge for Teaching Upper-secondary Mathematics* (KTUM) to explore the feasibility of sharing parts of the respective teacher training programmes, focusing on a definition of common framework of MCK and MPCK across the four programmes.

Each of the four universities offer a different model for pre-service teacher training; here we consider a typical path for a student to qualify for teaching mathematics at the upper-secondary level, noting that these are not necessarily the only paths for prospective teachers. In particular, in all the participating universities except Paris, it is also possible to study mathematics as a minor. All four universities follow the Bologna model, with a 3-year bachelor programme (B) followed by a 2-year master programme (M).

Milan U.: B and M in mathematics. A mathematics education track is available in M, but it is not compulsory. This is followed by a 60 ECTS course in pedagogy, mathematics education, school regulations, etc. Students qualify for teaching either mathematics or mathematics and physics.

Copenhagen U.: B and M in mathematics. The programme normally includes a minor, so students qualify for teaching two subjects. During their first year of teaching, teachers must pass a 20 ECTS nationally organised course on pedagogy, subject teaching, school regulations, etc.

Charles U., Prague: B and M in mathematics education. Periods of practicum are included in the programme.

Sorbonne U., Paris: B in Mathematics, followed by a M in mathematics education which includes practicum and elements of Pedagogy.

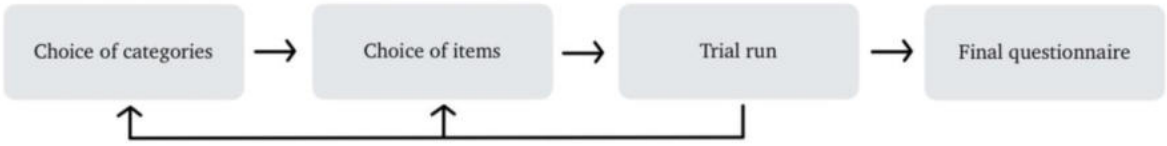

Figure 1: Development cycle of the questionnaire

The main objectives of KTUM were to map out the details of these paths, and then to develop a questionnaire to evaluate the knowledge of mathematics content (see next section) of students at the end of their master's programmes. The second step was not straightforward, and it relied both on the first step and on feedback from the trial runs of the questionnaire (Figure 1).

It was expected that the curricula in the four countries differ on points like whether or not synthetic geometry is emphasised or not. It surprised the authors of this paper to learn that, in the Czech Republic, differential and integral calculus is no longer part of the upper secondary curriculum. Nevertheless, since essentially all the concerned students in the Czech programme will also have taken courses in analysis, and analysis is a major topic in upper secondary school in the remaining three countries, it was agreed that this domain should be part of the questionnaire. As an example of more detailed differences, which affected our selection of items in the end, was whether teacher students could be expected to know the rational root theorem or not. The principle was to include only items that could be solved by knowledge taught in all four programs.

Each national group initially proposed 5-7 items based on their experience and practice of assessing MCK and MPCK. These items were mostly taken or adapted from exams in teacher-oriented courses, from national contests and standardised tests (INVALSI tests), etc. The KTUM team then met for three days in Prague, working in mixed groups (people from different universities) to discuss these first items, propose new items to cover missing areas or aspects (especially MPCK), and to fine tune the resulting inventory of items in terms of balance and formulation. Many items were discarded because some national group did not find them appropriate for their students.

The first phase was entirely conducted in English, being the only language mastered by all members of KTUM. The second phase required the items to be translated into the four relevant languages (Italian, Danish, French, Czech): since French is mastered by most members of KTUM, the French translation was used to double check translation issues. One example of these issues is the difference between "dot product" and "scalar product": in English, the latter typically refers to any symmetric bilinear form, while the former refers to the standard scalar product on $\mathbb{R}^n$. In, say, Italian, only the latter expression is used: "dot product" should then be translated as "prodotto scalare standard" (standard scalar product).

The group collaborated on a common LaTeX file hosted on OverLeaf, and special routines to deal with international text were developed by the first author: in order to minimise the risk of inconsistencies in the different localisations, it was deemed paramount that the mathematical part was not part of the localisation, and that the LaTeX machinery should be transparent enough to be usable by project members with little, or no, previous LaTeX experience (see Figure 2).

```
\en{Find} \it{Trova}\da{Bestem et tal}\cz{Najděte } \fr{Trouver  } $x$
\en{such that }\it{tale che }\da{ så }\cz{ takové, že }\fr{  pour que }
$\left(\sin(1/x)\right)^{-1}$
\en{has 128 digits}\da{har 128 cifre}\cz{má 128 číslic}\fr{s'écrive avec 128 chiffres}\it{abbia 128 cifre}.
```

Figure 2: Example of localised LaTeX code

At the end of the first phase, 14 items (most with several questions) were selected and administered to a group of 17 volunteer students: four from each of Milan, Prague and Paris, and five from Copenhagen. The students did the test in the same room in Milan, and after the test, a thorough follow up discussion with them was organised (also to let them know how to solve all items). The sample of students was far from being statistically significant or random as they were volunteers. All except the Milan students would presumably have been motivated by a paid trip to Italy. Still, the students' performance and assessment of the items was believed to be valuable for the further development of the item inventory.

Based on the first results and student input, nine of the fourteen initial items were selected (with some rearrangement and shortening). All questions were adapted to closed (multiple-choice) format, using the open answers from the first test as a source of (further) distractors. These questions were then pilot tested with 105 students (i.e., most students in the teacher training track, plus 17 Italian temporary teachers with a minor in Mathematics and enrolled in an alternative qualification programme) in Autumn.

Initial test items

As mentioned above, 14 items were selected for the first trial: 6 involved the domain Numbers, 7 Algebra, 3 Geometry, 2 Functions, 2 Data and Chance, 3 Analysis. Here, the total is greater than 14, since most items contain several questions which could be in different domains). Considering all questions, 17 involved the domain Numbers, 8 Geometry, 15 Algebra, 8 Functions, 3 Data and Chance, 6 Analysis. For the first trial test, students were given three hours. This allowed to experiment several items, mostly with open response questions. However, for a large-scale test, a shorter test time would be more realistic. Therefore, at any rate, some items or questions had to be dropped.

A main principle was to discard questions that were done correctly by all, or by none (or very few) of the students (so they had little or no discriminatory power). For example, in Figure 3, we have three apparently similar questions which test students' MPCK. The first one was selected by three (out of 17) students, the second was selected by half the students (IT: 1/4; DK: 0/5; CZ: 2/4; FR: 3/4), and the last one by none. We retained to two first options.

The second answer illustrates the mutual dependency of MCK and MPCK: during the follow-up discussion, we learned that the students who had not marked this exercise simply did not know the

relationship between logarithms and number of digits (MCK). So MCK clearly influences MPCK. The piece of MCK concerned is useful to circumvent difficult actual calculations and relates to intrinsic and historically crucial aspect of logarithms. The fact that Danish students did not know this might be related to the extensive use of computers in Danish upper secondary school. By contrast, that 3 of 4 French students marked the second exercise could be caused by the fact that French high school students are expected to solve exercises with as little help as possible from technical tools. Thus, differences in (relevance of) MPCK may also lead to differences in MCK.

Emma wants to select exercises that can be solved by using logarithms and their properties. Mark all the exercises that are relevant for her.

- ◯ Find x such that $(\sin(1/x))^{-1}$ has 128 digits.
- ◯ How many digits has 3^{280}?
- ◯ How many digits has the integer part of $\sin(3^{280})$?

Figure 3: A question on choosing exercises (extract)

We expected the questions in Figure 4 to be quite doable for master's level students in Mathematics or Mathematics Education, with the only difficulty being in the writing of a proper proof using one of the various possible arguments. All the Czech students answered the first question of item 13 correctly, while none of the French, one Italian and one Danish did so. However not a single student gave correct answers for the first question of item 14, and no student gave mathematically rigorous answers to the second questions. Both items were discarded.

13. For which natural numbers n does the equation $x^n + x + 1 = 0$ have at least one real solution? Explain your answer.
14. For what natural numbers n does the equation $x^n + x + 1 = 0$ have at least one rational solution? Explain your answer.

Figure 4: Questions that were too difficult

Initial test outcomes

We now analyse the data from the first pilot. The sample set is not yet large enough for any meaningful psychometric analysis (like Rasch models). We still consider rough trends in the shares of correct answers to the questions of some interest. In Table 2 we present the results by country and MCK subdomains (Cf. Table 1). For each subdomain D, the first column represents the percentage A_C^D of correct answers for students of country C; while the second column shifts the data in such a way that the average value in each column is zero and that the average results of different universities is zero: i.e., it represents the difference between A_C^D and the expected number, given the average number A^D of correct answers to the subdomain, the national average A_C, and the total average: $(A_C^D - A^D) + (A_C - A)$. This number provides a rough measure for areas in which the respondent groups over- or underperformed, given their overall performance.

We notice that the Italian students with a Maths minor performed significantly worse than all the other groups, despite having at the very least one year and a half of teaching experience, which hints

to a significant effect of majoring in Mathematics. It is interesting to notice that the Danish students, who are majoring in pure Mathematics, present results similar to the Czech ones, which are majoring in Maths Education.

We now ignore the Italian students with a Math minor: the variation in differential scores is much more contained in the shown pilot results than in the first trial, which should reflect a choice of items which do not discriminate on the difference in mathematical content in the four universities. Nevertheless, the Algebra subdomain show little variation on expected scores: this is an interesting factoid that could hint to the subdomain receiving a similar approach in different countries. The highest differential scores are presented by the Italian students in Analysis and by the French students in Data & Science: this could possibly follow from the previous attendance to classes focusing on the teaching, respectively, of Analysis and of Data & Chance.

Table 2: Results from the first trial test: raw data and difference from centred values.

	Size	Numbers		Geometry		Algebra		Functions		Data & Chance		Analysis		Total
IT	12	82%	*8%*	82%	*0%*	51%	*0%*	71%	*-7%*	56%	*0%*	71%	*9%*	**71%**
IT-M	17	47%	*0%*	51%	*-5%*	55%	*4%*	27%	*3%*	24%	*-5%*	34%	*-1%*	**44%**
DK	17	55%	*-8%*	70%	*-1%*	70%	*3%*	35%	*-4%*	45%	*1%*	55%	*4%*	**59%**
CZ	42	63%	*3%*	75%	*6%*	67%	*2%*	34%	*-3%*	37%	*-6%*	42%	*-7%*	**57%**
FR	33	67%	*-2%*	73%	*-5%*	70%	*-3%*	50%	*5%*	61%	*10%*	62%	*4%*	**66%**
Avg		**52%**		**64%**		**45%**		**60%**		**49%**		**36%**		**53%**

Plans and potentials for continuation

A new round of tests will need to be administered to (different) students in 2026; students minoring in Mathematics (where feasibile) should also be tested. If results were shown to be stable, a definitive item inventory to be employed at large scale, involving more universities and, possibly, more European countries.

In a similar way to TEDS-M, the relationship between the structure of teacher education programmes and results on the test can then be assessed on the short term. On the medium term, we could envision studying the connection to results in teaching (say, two years after graduation); while on the long term it will be possible to investigate the year-to-year performance gains of the teachers' pupils (controlling for other variables like socio-economic-cultural context of families and schools). The long-term goal of the project could include the establishment of an upper secondary level extension of TEDS-M, tailored to inform common policies and development in Europe.

Conclusion

The main outcome achieved so far consists in the possibility and methodology of developing items that can be used across different European programs and systems, by adaptation and trials that result in a test inventory with real discriminatory power and reflecting expert knowledge in the area. Of course, without demonstrating correlation with teaching quality in some sense, this consensus remains limited in value for policy making; however, experience from decades of effect studies gives some reason for optimism in this regard, as does the relative ease by which expert consensus was achieved, both on and within the categories of MCK shown in Table 1, and in regard to the composition of the second inventory of items that is now to be experimented.

Despite the sample size of respondents so far involved, the hypotheses thus generated are still of some interest, given that no similar evidence is available.

We also emphasise that the Czech students, who are the only one who are not enrolled in a pure mathematics degree, show results very similar to the Danish students who, on the other hand, are not offered a Maths education track. This, too, is an interesting factoid that should be explored in a statistically significant sample set and could inform policies regarding this major systemic difference. We also note that a major difference separating the Danish programme from the rest is the limited external motivation (like a national competition) and opportunity they have to attend courses focusing on MCK (and MPCK) closely related to upper secondary school mathematics.

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Perceptions of mathematical creativity: The case of one teacher in a special education class

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Teachers' perceptions of creativity can significantly shape their practice and influence students' creativity. This case study explores the perceptions of one special education teacher, examining the alignment between those perceptions and classroom practices. An interview revealed that the teacher focused on herself and how she expresses creativity, with less emphasis on the student's ability to express creativity. However, observations of her mathematics lessons showed that her practice had the potential to occasion mathematical creativity among her students.

Keywords: Teacher's perceptions, mathematical creativity, learning disorders.

Introduction

The ability to think creatively is inherent in every person, regardless of their cognitive abilities (OECD, 2017). In mathematics classrooms, encouraging mathematical creativity strengthens connections between mathematical topics and expands knowledge (Levenson et al., 2018). In this study, we presume that students with learning disorders (LD) have the potential to express mathematical creativity when provided with the right conditions, highlighting the importance of creating such opportunities. Because teachers play a key role in fostering creativity, their beliefs influence whether and how they do so (Levenson, 2013). Examining teachers' perspectives can reveal which approaches they adopt or should adopt to support students' mathematical creativity, especially considering that special education (SE) teachers do not necessarily receive specific training in mathematics. This study explores the perceptions of one SE teacher regarding mathematical creativity and how those perceptions are reflected during her mathematics lessons in her SE class.

Theoretical background

Creativity has been recognized as essential for success in the twenty-first century, and consequently, various aspects of fostering creativity among students are attracting interest. Within the field of mathematics education, several studies have examined expressions of mathematical creativity in different ages (Elgrably & Leikin, 2021) as well as tasks that encourage mathematical creativity (Levenson, 2013; Ron-Ezra & Levenson, 2023). Despite this, few studies have looked at mathematical creativity among students with LD.

Mathematical creativity is usually measured according to three components: *Fluency* refers to creating many ideas or solutions while working on a task. *Flexibility* involves approaching a task from different directions, producing solutions of various types, and associated with divergent thinking, which enables switching between different topics (Levenson et al., 2018). *Originality* refers to generating unique or innovative ideas (Elgrably & Leikin, 2021). Tasks that can encourage mathematical creativity may provide vague instructions that do not specify the method of solution,

enabling a variety of answers or solution strategies, and are called open-ended tasks (Levenson et al., 2018), or multiple-strategies tasks (Ron-Ezra & Levenson, 2023). Despite the importance of including such tasks in class, teachers tend to find it difficult to recognize their potential (Chiu, 2009). This study will examine if a SE teacher is aware of such tasks and teaching methods and if she implements such tasks in her SE class.

Mathematics teachers' perceptions affect their choice of teaching practices and the opportunities they create for their students. Likewise, teachers' perceptions regarding creativity can affect whether they choose tasks that encourage expressions of mathematical creativity and have implications for fostering creativity in the classroom (Levenson, 2013). Previous studies have investigated teachers' perceptions of creativity, not specifically related to mathematics. A review of such studies conducted by Andiliou & Murphy (2010) showed that teachers believe creativity can be developed, but often limit it to gifted students and the arts, struggling to identify less obvious creative students. Another study revealed that both general education and special education (SE) teachers consider themselves creative (Kasirer & Shnitzer-Meirovich, 2021). However, SE teachers prioritize fostering creativity, likely due to smaller class sizes and less pressure to meet standard curricula. They also display more originality in their teaching methods, tailored to students' unique needs (Kasirer & Shnitzer-Meirovich, 2021).

Mathematics teachers' perceptions of mathematical creativity are shaped by their views on the nature of mathematics. Some view it as a rigid, procedure-based subject, while others see it as fostering problem-solving and creativity. Pre-service primary teachers often believe that mathematical creativity involves teaching creatively with diverse resources and real-life examples (Bolden et al., 2010). Lev-Zamir and Leikin (2012) found that while elementary and middle school teachers hold positive perceptions of creativity in mathematics and related tasks, they do not always effectively implement these tasks to foster mathematical creativity in the classroom. High school teachers may recognize the importance of fostering mathematical creativity but are unsure how to achieve it in practice (Shriki & Lavi, 2012).

Lev-Zamir and Leikin (2012) noted that teachers hold two different conceptions of classroom mathematical creativity: a teacher-directed conception and a student-directed conception. The teacher-directed conception means that creativity is attributed to the teacher's work and the actions he takes (such as choosing tasks, conducting a conversation, or solving problems), which makes teaching creative (Lev-Zamir & Leikin, 2012). The student-directed conception of creativity refers to the teacher's perception of the mathematical creativity exhibited by the student while performing a task. A student's creativity can be manifested in asking questions, using a variety of strategies, developing his own methods for approaching problems, and exploring ideas (Bolden et al., 2010). Shriki and Lavi (2012) pointed out inconsistencies between teachers' claims and their actions. On the one hand, teachers claim that they develop mathematical creativity by encouraging students to solve problems in different ways; on the other hand, possibly because of time constraints, such practices are limited or non-existent. As for SE mathematics teachers, few studies have examined how their perceptions regarding mathematical creativity are reflected in their SE classes.

Consequently, the aim of the case study presented here is to examine the perceptions of one SE teacher regarding mathematical creativity and whether there is an alignment between the teacher's stated perceptions and the way in which mathematics lessons are conducted in her SE class.

Method

This study employs a case study methodology, allowing for an in-depth exploration of the teacher's perceptions of mathematical creativity and how her actions in the classroom encourage creativity among her students with LD. Through interviews and lesson observations, the study focuses on a specific situation, providing a detailed understanding of how one teacher fosters mathematical creativity in her classroom.

Niri (a pseudonym), the special education teacher in this study, has been teaching students with special needs for 19 years. While she received general training in special education, she had no specific training in mathematics. Her extensive experience, ability to engage students with mathematical tasks, and skill in selecting appropriate challenges made her a suitable focus for this study. The students in her class had been diagnosed with LDs and, due to significant gaps, mainly in reading, writing, and comprehension, as well as the need for support, were placed in a SE class. Yet, they followed the general education mathematics curriculum. According to Niri, their mathematical progress was slow, requiring many examples and repetitions, though not significantly below age expectations. All students and their parents consented to participate in the study, which was approved by the Chief Scientist of the Ministry of Education and the university's ethics committee.

The research tools included a semi-structured interview with Niri and observations of seven mathematics lessons in her class. The semi-structured interview with Niri took place three months after the beginning of the school year and was recorded and transcribed by the first author. The interview questions included: "How does creativity manifest itself during mathematics lessons?", "Is there a need to encourage mathematical creativity in a SE class?", "How can students with special needs be encouraged to express mathematical creativity?" In addition, Niri was presented with a task considered to encourage mathematical creativity because of its many solution paths (Figure 1). After presenting the task, Niri was asked: "Are you familiar with such tasks?" and "Can this type of task be used during a lesson in a SE class?"

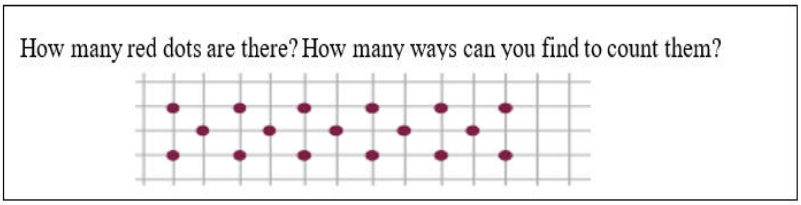

Figure 1: A creativity-promoting task (Bicer & Bicer, 2023)

Following the interview, seven observations were conducted during mathematics lessons where Niri taught the entire class. Each lesson was recorded using a stationary video camera at the back of the room, capturing the classroom setting, the teacher, and the students. The interview data were analyzed through repeated readings, identification of key concepts, and organization into themes (Patton, 2014). These themes were then examined through the "lens" of mathematical creativity, with

particular focus on Lev-Zamir and Leikin's (2012) distinction between "teacher-directed creativity" and "student-directed creativity." Out of the seven recorded lessons, two were selected for detailed analysis because they contained clear examples of student engagement and creativity. The analysis focused on the teacher's actions that encouraged creativity and examined how students expressed mathematical creativity, with an emphasis on fluency, flexibility, and originality.

Findings

This section presents Niri's perceptions of mathematical creativity and examines its expressions in her SE class by analyzing two mathematics lessons.

Niri's perspectives of creativity that arose from the interviews

The concept of mathematical creativity, as defined in the literature on mathematics teaching (e.g., in Elgrably & Leikin, 2021) was not familiar to Niri. She stated, "The definition of creativity is unclear; it isn't clear to me what you researchers are thinking about." Even when she was presented with a task thought to encourage expressions of mathematical creativity (Figure 1), she said that she was not familiar with such tasks and therefore does not give them to her students. She added: "It would have been nice if someone had given me things like that," demonstrating her willingness to experiment and expand her knowledge on encouraging mathematical creativity in the classroom. She continued, "A task like that would awaken children's curiosity and motivation." Without explicitly recognizing the potential of such a task to occasion mathematical creativity, Niri recognizes that such a task could attract the attention of students and make mathematics interesting for them.

Despite stating that she is not familiar with a formal definition of creativity, she does have implicit perceptions of what is meant by mathematical creativity. Niri explains that in her view, she demonstrates mathematical creativity by using mathematical games and instructional manipulatives for illustration purposes: "I use manipulatives. I use the computer. I offer them all the options available." Niri considers herself a creative teacher; in her opinion, the variety of activities she initiates during her lessons in the SE class are an expression of her creativity. She notes: "I look at it from the perspective of how I teach. I teach in all kinds of special ways and make the lessons creative." Her statements indicate a teacher-directed conception of creativity.

When discussing the subject of mathematics, Niri perceives creativity in mathematics when it is connected to other topics and everyday experiences. In her opinion, creativity in mathematics is manifested by finding connections between mathematical subjects and real-life examples. She described a history lesson in which she incorporated mathematics and said: "It is very important for me to include mathematics in all disciplines. I believe this is something that opens them (to new directions). This can also be defined as creativity." She also claims that her mathematics lessons are conducted creatively since she improvises and draws connections to everyday life. She adds: "I take a certain subject (in mathematics) and show them how it is in real life, which is also a type of creativity. I think outside the box and bring a lot of manipulatives and teaching aids, and I show them how they work in many ways." Once again, Niri emphasizes the importance of providing a variety of teaching methods during mathematics lessons.

Throughout the interview, Niri focuses on the ways in which she demonstrates creativity, and not on expressions of mathematical creativity among her students. Following Niri's descriptions of her teaching, the researcher tries to guide her, asking whether her students also display this variety: "So, the children can show it (their solutions) in several different ways?" Niri appears to dismiss this, saying: "Not necessarily," and goes back to focusing on creativity as a characteristic of the activities *she* implements in the SE class which make *her* teaching creative.

Niri does recognize that students in the present era are required to demonstrate creativity, which is reflected in their ability to express a different line of thought and to show a variety of solution strategies while using different representations (Transcription 1).

Transcription 1 – part of the interview with Niri: mathematical creativity in students

Researcher:	What would you consider an expression of mathematical creativity in a child?
Niri:	When he shows me a different way, when he thinks in a different way. When he calculates in a different way, he can use objects to show me another way to see the calculation, see the number.
....	
Niri:	I think you must have creativity in everything, because this is where the world is heading, not to robots, but rather, anyone who is more creative is more successful in the world in my opinion.
Researcher:	What does it mean, more creative?
Niri:	More original, innovative, brings new ideas to people around him and manages to see things in a different light and think outside the box.

The terms "original" and "innovative" are common in the field of mathematical creativity, but when Niri is asked to explain their meaning and provide a concrete example from the class, she is unable to do so. Subsequently, she states that she feels she does not give many tasks that encourage mathematical creativity, in which the students must work independently and demonstrate a range of appropriate solution strategies. When the researcher explains what is meant by fluency as a component of creativity, Niri's expresses reservations about students with special needs, saying that "In SE you will find it (fluency) very limited, especially in young students."

Mathematical creativity in the classroom

During Niri's mathematics lessons, expressions of mathematical creativity (fluency, flexibility, and originality) were observed among the students. In this part of the findings, we focus on two lessons: one centered around the "decimal system" and the other on "long addition and subtraction" and present examples of activities Niri carried out that had the potential to encourage creativity among her students, alongside examples of her students' expressions of mathematical creativity.

In both lessons, Niri used open-ended questions and instructions, allowing students the flexibility to provide answers in their own way. In the lesson on the decimal system, Niri wrote the number 792 on the board and asked the students to represent it. The open-ended instruction she gave was: "I'm writing a number on the board, and each of you will explain to the class, however, you like, what this number is." This instruction allows students to work independently in a variety of ways. The students start working immediately, apparently familiar with this type of instruction. At the class level, students demonstrate fluency by proposing different representations of the number 792, including an addition sentence (700 + 92 = 792), a verbal representation ("7 hundreds, 9 tens, and 2 units"), a

representation using base-ten blocks, and more. At the individual level, each student produces only one representation of the number, without switching between different representations. Moreover, the teacher does not encourage individual students to find more than one representation. Consequently, neither fluency nor flexibility are seen at the individual student level.

Similarly, in the lesson on long addition, Niri asks the students to represent the number 32. However, in this case, the open-ended question she presents (Transcription 2) stimulates a conversation that leads to a few answers. The students respond to the question and create together several ways of representation, which indicates fluency and flexibility at the student level as well.

Transcription 2 – a conversation between Niri and her students during the lesson on long addition and subtraction

Niri:	In order to explain what three tens mean, how can I draw three tens?
Student 1:	Circles.
Niri:	Circles? (starts drawing circles on the board)
Student 2:	Lines, lines (referring to a tens block which he calls a line).
Student 3:	Three (meaning, to draw three rectangles representing the three tens).
…	(Niri erases the circles and draws a rectangle divided into ten parts).
Student 2:	And you can also do units (meaning, to represent the number through units).

In this lesson, the teacher's open question invites the students to come up with a flow of ideas for representation. Student 1's idea was not accepted by the teacher. Student 2's idea, although not worded with the correct terminology, was understood to refer to the rectangle of the tens block, which looks like a line. Building on Student 2's response, Student 3 added that three of those "lines," meaning rectangles, would be necessary to represent 3 tens.

In both lessons, students use classroom resources to respond, while Niri encourages them to explore other manipulatives: "Is there another way? Something in the class to represent it?" She leaves the choice to the children without specifying an item, fostering flexibility and making students comfortable selecting tools that assist them. For instance, in a long addition lesson, the teacher invites two students to solve 32 + 77 on the board, while others use a teacher-designed navigation card to guide their solution or work in their notebooks. Meanwhile, one student, who struggles with writing, solves the problem using base-ten blocks.

Discussion and conclusions

The first research question sought to examine one SE teacher's perceptions of mathematical creativity. The teacher, Niri, appears unfamiliar with the concept of "mathematical creativity" as described in the research literature. Like other teachers (Chiu, 2009), she does not recognize tasks that are designed to occasion mathematical creativity. When the researcher explains to her the notion of fluency as it is used concerning creativity, she implies that her students' LDs and their young age could make it difficult for them. Regarding flexibility, although Niri does not use this term explicitly, her perception of mathematics as a subject that can be connected to other subjects and to everyday experiences, is in line with how researchers suggest promoting flexible thinking (Levenson et al., 2018). Finally, Niri specifically mentions original thought in relation to creativity but cannot recall an instance of original thinking in her SE mathematics class.

Niri demonstrates a teacher-directed conception of creativity (Lev-Zamir & Leikin, 2012). In her view, teaching methods such as using various means of representation, demonstrating different strategies, and creating connections to everyday life make her a creative teacher (Bolden et al., 2010). Yet, by demonstrating a variety of solutions, she is, although inadvertently, promoting mathematical creativity among her students (Levenson et al., 2018). In fact, she emphasizes that students can express themselves through different representations and suggests that this can lead to originality. In general, Niri considers creativity important and even mentions how students demonstrate creativity by showing her a "different way."

Regarding the relationship between Niri's perceptions of mathematical creativity and the way mathematics lessons were conducted in her SE class, a few differences were observed. While in the interview Niri claims to be unfamiliar with tasks that encourage mathematical creativity, the teaching methods she uses in class include elements that promote expressions of mathematical creativity (Kasirer & Shnitzer-Meirovich, 2021). A gap is evident between Niri's positive views on mathematical creativity and her pedagogical knowledge of how to implement tasks that encourage students to express creativity (Chiu, 2009). Niri does not mention using open-ended questions and instructions to encourage mathematical creativity among her students, but in practice, she incorporates this method into her lessons. Using open-ended questions and instructions (Levenson et al., 2018) allows students to choose the solution strategies that suit them (Ron-Ezra & Levenson, 2023), which aligns with researchers' views on flexibility and originality (Levenson et al., 2018). However, Niri does not encourage each student to solve a problem or represent a number in multiple ways. In the lessons observed, fluency and flexibility can be seen at the class level, but not at the individual level. Furthermore, Niri does not attempt to facilitate a discussion that might help students find connections between different representations, which could lead to collective creativity (Molad et al., 2020). Rather than examining mathematical creativity as a primary focus, the study identified creativity components within Niri's words and practices, focusing on behaviors and expressions instead of using creativity as the main analytical framework.

In conclusion, this case study highlights the gap between a special education teacher's willingness to promote mathematical creativity and her knowledge of how to do so. A key limitation is that students' creativity may be restricted because the teacher does not model fluency, flexibility, and originality in her task solutions. Additionally, the case study design limits generalizability, and it would have been insightful to observe whether the teacher herself demonstrates the expected creativity. While the findings cannot be generalized to all SE teachers, they provide insight into how conceptions of mathematics, creativity, and SE are interconnected. A key conclusion is the need for teacher training to encourage mathematical creativity by fostering personal engagement with tasks and self-expression.

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Teachers' beliefs regarding the knowledge they need to teach word problems in middle school

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This paper describes a part of research on several aspects of teaching word problem-solving. It involves 57 middle school mathematics teachers and focuses on teachers' beliefs regarding the knowledge they need to teach word problems: What do teachers believe is essential for them to know?. Teachers rated the importance of various knowledge components based on the framework by Deborah Ball et al. (2009), which includes common content knowledge, specialized content knowledge, knowledge of content and teaching, and knowledge of content and students. Teachers generally believe that specialized content knowledge and knowledge of content and students are the most essential for them. However, they prioritize different aspects within these components, considering some more important than others. These findings could influence teachers' guidance and training.

Keywords: teachers' knowledge, beliefs, word problems, middle school

Introduction and theoretical framework

Beliefs shape how people see the world and may influence their attitudes and performance (cross, 2009). Many studies have dealt with teachers' beliefs and suggested various definitions for this concept, such as perceptions, attitudes, opinions, worldviews, and more. They also refer to beliefs regarding different aspects of the mathematics teaching profession: beliefs about students, about teaching, about mathematics, about the role of learning mathematics, and more (Goldin et al., 2009). The present study is based on Voss' definition of beliefs as psychologically held understandings and assumptions about objects that are felt to be accurate and influence people's interactions with the world (Voss et al., 2016). Teachers' beliefs may affect their relationships with students and how they teach, i.e., their pedagogical decisions and overall effectiveness, ultimately shaping students' learning experiences (cross, 2009; 2015; Lau, 2021).

Teachers' beliefs about their required knowledge may influence their teaching, so it is essential to test it and design their training accordingly. This study addresses the knowledge components defined by Deborah Ball and colleagues (Ball et al., 2008), which defined four main knowledge components: Common Content Knowledge (CCK) is a type of mathematical knowledge required even for non-teachers. For example, knowing how to calculate and solve exercises and problems. Specialized Content Knowledge (SCK) is content knowledge unique to teaching. For example, knowing how to solve a problem in a variety of ways, knowing how to connect different representations to an exercise or a problem and understanding the theoretical principles behind them, and knowing how to examine an unconventional way of solution, i.e., whether it is correct and whether it will work in other cases. Knowledge of Content and Teaching (KCT) combines content knowledge with teaching. For example, knowing how to choose examples suitable for presenting a topic and examples ideal for deepening it, knowing different representations to teach a problem, and evaluating each representation's advantages and disadvantages for teaching the problem. Knowledge of Content and

Students (KCS) combines content and students' knowledge. For example, knowing common student errors and possible reasons for these errors, knowing whether a given example might interest and motivate students, and knowing what is easy and difficult for a specific population of students.

Some studies examined the relationship between teacher knowledge and teachers' beliefs. For example, Drageset (2010) examined teachers' beliefs related to what mathematics is (for example, "Mathematics means finding the correct answer to a problem") or what is essential for students to know (for example, "The ability to explain their answers"). He focused on CCK and SCK but not on what essential knowledge teachers should acquire, and the other elements of knowledge mentioned in Ball's theory were not tested. Also, he did not ask what is the most important when combining several aspects corresponding to the same knowledge component. Other researchers (Kim, 2018; Lui & Bonner, 2016; White et al., 2005; Yang et al., 2020) examined beliefs in three domains: general beliefs about mathematics, such as what is the nature of mathematics or what is the beauty in mathematics; beliefs about learning mathematics, such as what is essential in learning mathematics- the way to the solution or the correct answer; and beliefs about theories of mathematics teaching, such as constructivist, traditional and technological. These studies found a correlation between beliefs and knowledge. However, they did not systematically test beliefs regarding knowledge components defined by Ball (Ball et al., 2009). They did not test beliefs about the need for, or the importance of, each knowledge component.

Since knowledge is specific to each topic, a particular topic was chosen for this research: word problem-solving. Word problem solving is a central topic in mathematics curricula for all ages at all stages of education worldwide. Curricula and various studies emphasize the importance of problem-solving as a subject that connects everyday life problems with mathematical models, combining several concepts, developing a high level of thinking and creativity, and enabling the students to apply various skills they have learned. Also, there are many different strategies for solving word problems using multiple tools - arithmetic, algebraic, graphic, and more (Ministry of Education in Israel, 2013; NCTM, 2000; Verschaffel et al., 2020).

Despite the variety of problem-solving strategies, mathematical word problem-solving is considered a complex subject for students, which causes many errors and difficulties, which may be related to reading, calculating, algebra, and more (Pongsakdi et al., 2020). Given students' difficulties in this topic and the various teaching strategies, teachers' knowledge of this subject is fundamental. Hence, it is essential to test teachers' beliefs about what is important to know to teach the subject effectively. Knowing teachers' beliefs may lead to better preparation of guidance and professional development programs.

The research question is: What are the teachers' beliefs regarding the knowledge they need to teach word problems?

Methods section

Participants: Fifty-seven teachers teach mathematics in middle schools in Israel. All the teachers specialized in mathematics as part of their academic studies - 40 teachers with a bachelor's degree (B.Ed or B.A degree) and 17 teachers with master's degrees (M.Ed or M.A degree). Teachers' seniority varied from 1-2 years of experience to 25 years of experience: 21 teachers are novice

teachers, with seniority of up to 5 years, and 36 teachers are experienced teachers, with seniority of 10 or more years.

Research tool: An electronic questionnaire was built for this study. A preliminary study was conducted with 12 teachers who answered these questionnaires, and reliability was examined for each questionnaire. Cronbach's coefficient alpha was calculated using the pilot data, yielding an acceptable reliability score: $\alpha = .79$. The teachers received 14 statements in the questionnaire concerning the knowledge required for a math teacher to teach word problems. There were no time limits for answering the questionnaire, and it took a very short time—about 5 minutes. The statements addressed each of the four knowledge components according to Ball's framework. The teachers were asked to rate their beliefs regarding the degree of importance they attributed to each knowledge component that appeared in the statement. The rating was done on a Likert scale between 1 and 5, where 1 is "I believe it is not important at all to…." and 5 is "I believe it is important to….". Example for a statement related to CCK: "Know how to solve a word problem"; Example for statements related to SCK: "Know how to link different representations to a word problem" or "Understand the theoretical principles behind each representation"; Example for a statement related to KCT: "know how to choose appropriate examples for presenting the topic" or "know how to choose appropriate examples for deepening students' knowledge"; Example for a statement related to KCS: "know what common errors students make in solving a certain problem" or "estimate how difficult a given problem is for certain students". The statements appeared without classification (it was not written for each statement to which type of knowledge it belonged) and in random order, so not all the statements belonging to the same type of knowledge appeared in sequence. After the teachers rated their beliefs about the importance of the various knowledge components, they were asked to say which of the knowledge components they believed was the most important for mathematics teachers.

It is important to note: (A) The findings described in this paper are part of a wider study that examined teachers' knowledge, and the associated beliefs and self-efficacy. In the study, for each statement of beliefs, there was a corresponding statement of self-efficacy, and there was a question that tested teachers' knowledge. Therefore, only 14 statements were given, to avoid a questionnaire that was too long or tedious. (B) A key factor in solving word problems is the verbal aspect, which is one of the main factors in students' difficulties. Teachers were asked about students' difficulties in the study and were able to indicate these difficulties in the part that tested knowledge. In this part that deals with beliefs about the knowledge components, the types of difficulties that children have were not explicitly mentioned. Teachers were asked about the importance of knowledge that deals with students' difficulties without being limited to a specific difficulty. (C) Correlations were examined between the teachers' responses and their seniority, and between the teachers' responses and the training they received for teaching mathematics. Since no significant findings were found related to teacher training, and differences were also found between veteran and new teachers, I will not address the findings related to education and training.

Results

In the first part of the questionnaire, the teachers received 14 statements and rated each according to the importance they attribute to it. Each statement represents a specific aspect of a particular

knowledge component. Generally, all the statements were rated highly, ranging from 3 to 5. None of the teachers rated any statement with a level of importance between 1 and 2. According to the responses provided by the teachers (Table 1), the teachers prioritize certain statements over others within the same knowledge component. For instance, in Knowledge of Content and Teaching (KCT), teachers considered one statement the most important in the questionnaire. In contrast, another statement in KCT was rated as one of the five least essential statements.

Table 1: The number of teachers who gave each rating for each statement, the mean, and the standard deviation.

Knowledge component	Statement: "I believe it is important..."	No. of teachers that ranked each level			M (sd)
		3	4	5	
CCK	To know how to solve a problem	2	22	33	4.544 (0.559)
	To use terms and notations correctly	4	22	31	4.474 (0.525)
SCK	To know how to solve a problem in various ways using different representations	2	17	38	4.632 (0.546)
	To be able to evaluate the plausibility of students' claims	3	16	38	4.614 (0.58)
	To know how to link representations to underlying ideas	2	16	39	4.649 (0.541)
	To know how to modify tasks to be either easier or harder	8	13	36	4.491 (0.722)
KCT	To know which examples are most suitable to start teaching with	2	11	45	4.737 (0.509)
	To know which examples are most suitable to promote students' understanding of the content	1	22	34	4.579 (0.523)
	To be able to evaluate the instructional advantages and disadvantages of representations used to teach a specific problem	6	20	31	4.439 (0.67)
KCS	To know students' common errors and misconceptions	5	14	38	4.579 (0.642)
	To know the reason for students' common errors and misconceptions	2	20	35	4.579 (0.555)
	To be able to predict what students will find interesting and motivating	8	18	31	4.404 (0.716)

	To be able to anticipate whether students will find a problem easy or hard	1	21	35	4.596 (0.521)
	To be able to interpret students' emerging and incomplete thinking	5	16	36	4.544 (0.645)

Table 1 shows that the statement given the highest rating was one regarding KCT ("To know which examples are most suitable to start teaching with"). Other statements rated higher than the others are related to SCK ("To know how to link representations to underlying ideas"). Two statements were given the lowest rating: the first regarding KCS ("To be able to predict what students will find interesting and motivating") and the second regarding KCT ("To be able to evaluate the instructional advantages and disadvantages of representations used to teach a specific problem").

After the teachers rated each statement, they were asked to indicate which knowledge component they believed was the most important for mathematics teachers. The findings of this question (Table 2) show that teachers believed that the most essential knowledge component is KCS. The second knowledge component that they believed is essential is SCK.

Table 2: The number and percentage of teachers who rated each knowledge component as the most important one

Knowledge component	The statement	Believe this knowledge component is the most important	
		Number	percentage
CCK	Knowing how to solve a problem and use terms and notations correctly	2	3.51
SCK	Knowing to solve the same problem in different ways, linking the various representations, and knowing the meanings of the representations	21	36.85
KCT	Knowing how to choose examples and representations and knowing how to evaluate their advantages and disadvantages	12	21.05
KCS	Knowing what is easy or difficult for specific students, being able to anticipate student errors, and knowing the reasons for these errors	22	38.59

Table 2 shows that there is not one knowledge component the teachers believe is significantly more important than the others. It also indicates that the teachers believe the components of knowledge that are not common content knowledge are significantly more critical to their work, especially

specialized content knowledge and knowledge of content and students. The teachers seem to believe that the importance of these two knowledge components is significantly higher than the other two.

Examining the Pearson correlation coefficient between seniority and beliefs, it was found that there is no correlation between seniority and beliefs in the importance of specialized content knowledge or the belief in the importance of knowledge of content and students. However, a significant correlation was found regarding the other two knowledge components: experienced teachers were more prone to believe in the importance of common content knowledge ($r(55)=0.537$, $p<0.01$) and knowledge of content and teaching ($r(55)=0.265$, $p<0.05$).

Discussion and conclusions

This research focused on investigating the specific beliefs of middle school teachers related to the knowledge they require to teach word problems effectively. This study is particularly notable because it delves into the teachers' perspectives on the importance of different knowledge components outlined by Deborah Ball. What sets this research apart is that it examines the teachers' beliefs about these knowledge components in the specific context of word problems. Additionally, the study produced insightful and valuable findings.

The first meaningful finding is that teachers indicated that the most critical knowledge components were SCK and KCT. Previous studies found that teachers have partial knowledge of these knowledge components and lower self-efficacy regarding them (Schreiber & Filo, 2019; Visochansky & Schreiber, 2023). This may be due to past situations that the teachers had previously experienced in classrooms, where they had felt that they lacked the tools or knowledge necessary to deepen the student's understanding and problem-solving abilities. The fact that the teachers were aware of the importance of these knowledge components highlights the need to strengthen and expand their knowledge concerning these components and highlight the importance of teachers' training and guidance (Van-Inger et al., 2016). Because the teachers attribute importance to these components, they might be perceptible and willing to learn and improve their knowledge.

The second important finding is that teachers did not consider all aspects of a specific knowledge component equally important. Some aspects were highly rated in each knowledge component, while others were rated low. This suggests that further investigation is needed into the knowledge components and their different aspects as they are taught in teachers' training. Those involved in training or guiding teachers should examine, in each knowledge component, the aspects that teachers believe are less significant. If these aspects are indeed important, teachers must be shown their significance. If not, these aspects should be given less attention, allowing other aspects of the same knowledge component to take priority.

The third meaningful finding is the difference between novice and experienced teachers. The more experienced the teacher is, the more he recognizes the importance of common content knowledge and knowledge of content and teaching. This may stem from experience in situations in the classroom where the teacher discovered that he lacked content knowledge and realized that it is essential to have a thorough and good knowledge of the content. This also may impact teachers' professional development and guidance.

The study's limitations are its relatively limited scope, the small number of participants, only middle school, and the small number of questionnaire questions. It would be suggested that more factors, such as knowledge and self-efficacy in various mathematical topics, be tested thoroughly in primary school and high school and that more participants be obtained. Research done in the past testified that there was once a gap between teachers' beliefs and actual performance (Leatham, 2006; Lev-Zamir & Leikin, 2013). Therefore, it is essential to investigate in the future both teachers' beliefs regarding the elements of knowledge that are important to them and the knowledge they acquire when they come to teach a new or unfamiliar subject.

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Specialised mathematical knowledge in the scope of geometric isometric transformation translation and its impact on teacher's levels of Interpretive Knowledge

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We discuss the specialisation of mathematics teachers' knowledge in the scope of geometric isometric transformation translation, considering the Mathematics Teacher's Specialised Knowledge and Interpretative Knowledge. Students and teachers reveal difficulties in such topic which makes it a good subject for discussion. 14 mathematics teachers participated in a Professional Development context focusing on developing teacher's knowledge and here we focus on the knowledge (written, oral and gestural) revealed when discussing an Interpretative Task. The results pinpoint to connections between teachers mathematical specialised knowledge in the context of translation and the interpretation performed (IK levels).

Keywords: Mathematics Teacher's Specialised Knowledge, Interpretative Knowledge, Interpretative Task, geometric isometric transformation translation.

Introduction

It is expected that mathematics teachers' practices allow developing students' mathematical thinking and understanding the diversity of topics they have to deal with, and the connections among them. Such practice, starting from what the students know and how they know it, requires a specialised knowledge which is perceived here in the sense of the Mathematics Teacher's Specialised Knowledge – MTSK conceptualization (Carrillo et al., 2018) and the Interpretative Knowledge – IK (Jakobsen et al., 2014). Among the diversity of mathematical topics in which students reveal difficulties, we focus on Geometric Isometric Transformations and here we give a special look at translation considering the students' (e.g., Flores & Yanik, 2016) and teachers' (e.g., Gomes, 2012) difficulties. In that sense we address the following research question: which mathematical specialised knowledge do mathematics teachers reveal when participating in a Professional Development context and how is it related to the interpretation performed (IK levels)?

Teacher's Specialised and Interpretative knowledge

Both students and teachers reveal difficulties in geometric isometric transformation translation. Students' difficulties involve: (i) perform the translation (especially with polygonal figures and when the translation vector is parallel to one of its sides), considering the length of the vector as the space between the figure and the image (Godino & Ruíz, 2002); (ii) establish the distance between the corresponding points of the figure and the image during the process (Sünker & Zembat, 2012) as equivalent to the magnitude of the translation vector (Flores & Yanik, 2016). Teachers' difficulties include determining the magnitude of the translation vector mistakenly associating (as students do) the magnitude of the translation vector with the space between the figure and its image (Gomes, 2012).

Considering that teacher's knowledge is related to student's learning and results, understanding the specificities of teachers' knowledge is fundamental to contribute to design and implement Tasks for Teacher Education and Professional Development contexts leading to improve students' understanding, knowledge and, in consequence, results. In order to develop such student's knowledge and understanding we perceive as fundamental to assume as a starting point what the students know and how they know it so that we can "move" them to a state of knowing and for doing so it requires from teachers a specialised mathematical knowledge grounding which we name the Interpretative Knowledge (Jakobsen et al., 2014). Such mathematical specialised knowledge (MK) is perceived here as part of the Mathematics Teacher's Specialised Knowledge – MTSK (Carrillo et al., 2018).

The MK is comprised by three subdomains, and even considering that all of them are equally important, here we focus on Knowledge of Topics (KoT) which is composed by four categories: (i) procedures; (ii) definitions, properties and foundations; (iii) registers of representation; and (iv) phenomenology and applications. From these we address special attention to procedures and foundations.

Procedures (which we term as KoTp) includes teachers' knowledge on the different ways of doing mathematics to solve a problem or calculation, considering: how it is done; when it can or cannot be done; why it is done that way; and the characteristics of the result obtained (e.g., Carrillo et al., 2018). In the topic of translation, it includes knowing the different steps for adding the translation vector to "each point of the figure" to obtain the corresponding point in the image. Foundations (KoTf) refers to the knowledge of the set of fundamental mathematical attributes that mathematically support the existence of the topic and connect concepts (Reyes & Sosa, 2019). It is, therefore, a subset of properties and it includes, for example, knowing that the translation vector and its properties is one of the foundations of translation (in Portuguese we use three words – direção, sentido, magnitude).

The content of teacher's mathematical specialised knowledge grounds the interpretation performed to student's productions and, thus, the teacher's interpretative practices. Such practices are related with teachers' Interpretative Knowledge – IK (Jakobsen et al., 2014) which refers to a deep and wide mathematical knowledge that enables teachers to support students in building their mathematical knowledge starting from their own reasoning and productions, no matter how non-standard or incorrect they might be. IK completes the knowledge of typical errors or solution strategies, with the knowledge of possible source for typical or atypical error and the knowledge of possible use of errors (Di Martino et al., 2020). At the moment IK comprises three levels which are intertwined with the interpretative practices performed (Mellone et al., 2017): (i) Evaluative interpretation: a process through which the teacher determines congruence between student's productions and the mathematical scheme of correct answers (s)he has; (ii) Interpretation for the educational design: the manner in which the teacher designs educational steps based on the work produced by the students; (ii) Interpretation as research: teacher's willingness and ability to revise his/her mathematical formalization in order to ensure that it is coherent with students' productions (even when these seem in conflict with the traditional mathematics taught in school).

Teachers' Interpretative and Specialised knowledge does not naturally develop over the course of years of work experience, but it is something that requires a special attention from an educational

point of view (Ribeiro et al., 2013), and for doing so it is required to design and implement educational contexts with such an aim. Contributing for such development we have been developing Interpretative Tasks – IT (Mellone et al., 2020) aiming at assessing and developing teachers' IK, and for each IT we devise a research question, focusing on a specific mathematical topic – in this specific case the geometric isometric transformation translation. Each IK is comprised by three parts: Preliminary Part, with focus on teachers' mathematical specialised knowledge; Part I starts from a conceptualized task for students to develop their mathematical thinking and understanding on the topic (usually a problem), and a set of questions for teachers, where one of them is frequently related to "*anticipate possible correct and incorrect students' productions to the task and which mathematical knowledge grounds the associated reasoning*"; Part II considers, usually, some students productions to the task (written, videos, transcripts) and includes some questions requiring teachers to reveal their IK.

Context and method

This paper is part of a broad research project on teachers' knowledge and mathematical practices in which one of the focuses is Geometrical Thinking. Here we focus on interpretative and specialised knowledge of 14 mathematics teachers (upper and lower secondary) participating in the context of a professional development program (40 hours – nine meetings) aimed to access and develop teachers' IK and particularly in one IT on the space of geometric isometric transformation translation. Participating teachers had between one and 26 years of experience teaching mathematics at lower and upper secondary (students aged 11 to 18 in Brazil). All teachers had a degree on mathematics teacher education (four years). Data collection includes audio and video recordings of teachers' discussions in small groups (of three or four) and during the plenary discussion as well as teachers' productions to the IT. Here we focus on Part I of the IT. The task for students pursues the aim of developing students understanding the set of procedures to perform translation of polygonal and non-polygonal figures as well as some properties of translation. The first question to teachers is to solve the students' task without thinking in a teaching context. Part II includes three student's productions directing the focus of attention to the mathematical thinking grounding the student's errors and the teacher's knowledge to devise ways to overcome such difficulties. Due to space constrains, we present part of the IT on translation with the questions on which we focus our attention (Figure 1).

Teacher knowledge in the scope of translation

Preliminary Part

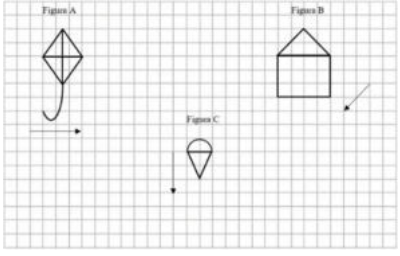
[...].

Part I

Task: Changing places…

(You should always explain your reasoning by describing the process you use to answer the question. For doing so you can use diagrams, words, calculations, ...)

1. Represent the figure (image) in each situation, according to the arrow:



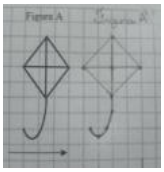
[...].

1. Consider the previous task:
   i) Solve the task by yourself (without thinking on a teaching context).

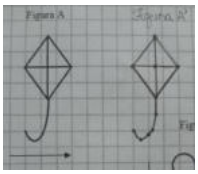
[...].

Part II

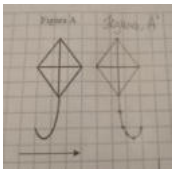
1. Teacher Josi implemented the previous task in her 7th grade classroom. She received different responses and as she was participating in one of the professional development contexts of CIEspMat she decided to take these productions to discuss them with the other participants.



Paula's production



Laura's production



Maria's production

a) For each of the students' productions, indicate whether you consider them mathematically correct (adequate) or not, justifying the mathematical reasoning shown.

[...].

Figure 1: Part of an Interpretative Task in the scope of translation

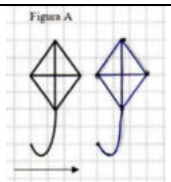
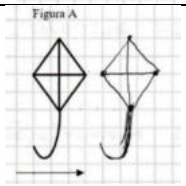
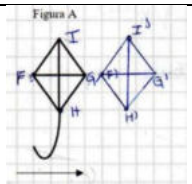
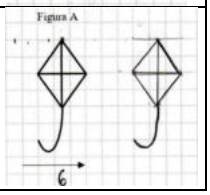
Each one of these productions were included as they allow discussing specific student's difficulties. Here, for space constraints, we focus on Laura's production because it has allowed a large span of knowledge revealed. Laura's production brings to front the students' difficulties in establishing the distance between each point of the figure and the corresponding point in image as equivalent to the magnitude of the translation vector. Preliminary Part was answered individually and Parts I and II (distributed separately), in groups of three or four (G1 to G4). After that, a collective discussion took place – having as a starting point the participants' production. Teachers' productions (written and oral) were transcribed, and the dialogues were complemented with teacher's gestures.

To identify teacher's specialised knowledge, the written productions were transcribed *ipsis litteris*, and for the analysis we used the MTSK categories (KoTpn and KoTfn are used to identify the knowledge revealed on procedures and foundations (*n* increases sequentially for each one separately). For IK, after transcribing the productions, we have identified (colored red) keywords associated to each level of IK (incorrect; partial correct; correct) considering the three categories presented by Mellone et al., (2017) and from the analysis a new category for the interpretative practice emerged (misinterpretation, corresponding to interpret as correct a mathematical incorrect student's production). For identification we used a set of symbols after the acronomous (*; **; []) where * is associated to an inadequate mathematical knowledge, ** when it is incomplete and [] absence of evidence of knowledge required to interpret the student production.

Analysis and discussion

From the four teachers' groups only two (G1 and G2) performed the translation correctly. G3 correctly performed the translation of the polygonal component of Figure A, but didn't provide the translation of the curved component. G4 performed an incorrect translation using six distance length measurement units for the translation vector magnitude instead of four (Table 1).

Table 1: Teachers' productions for the question "*Represent the figure (image) in each situation, according to the arrow*" focusing on Figure A

Group	Teachers' Productions			
G1		*G1*	*Transcription*	
	117	Aline	This point here must move four.	
	118	Joice	That's it.	
	119		[Aline counts according to the squares of the grid].	
	120	Aline	One, two, three, four.	
G2		*G2*	*Transcription*	
	166	Marcia	So, is this one going to come here?	
	167	Jorge	His vertex there.	
	168	Marcia	Here? One, two, three, four. Here more or less?	
	169	Jorge	That.	
	[…]			
	176	Jorge	That's it. You can now close the drawing above.	
G3		*G3*	*Transcription*	
	50	Igor	We can name the point, name each point	
	51	Igor	and place a "line".	
	52	Telma	You have to make the stitches.	
	53	Igor	And do A', B'.	
G4		*G4*	*Transcription*	
	42	Luana	I think that only the distance has to be the same,	
	43	Luana	if we are going to transfer the kite here, how many? Two	
	44	Luana	little squares? And answer and put it right here.	
	45	Luana	[Luana points to figure A and the place where its image	
	46	Luana	would be located on the task sheet].	

G1 reveals [117 – 120] knowing that the translation procedures consist of choosing crucial point on the figure and moving them, counting the sides of the squares on the grid according to the unit of distance measurement indicated by the vector magnitude, obtaining the corresponding image points (KoTp1**). This is a particular case of procedures for translation requiring to know the need to consider the three properties of vectors (direction, sense and magnitude) and adding the vector to fundamental points of the figure to obtain its corresponding image points [(KoTp2)].

Teachers in G2 employ similar procedures as G1, but explicitly chooses the vertices of the figure to be translated [167], so that they can later be joined, and the image is obtained [176]. This involves knowing that in polygons it is enough to translate the vertices, taking into account the translation vector and its properties, and then join them by straight line segments (KoTp3). Similarly to the first two groups, G3 [50 – 53], reveals knowing that for translating a figure it is enough to carry out the transformation of some of its points, naming the points [vertices] of the figure by Latin capital letters and their correspondents in the image using that same letter added by an apostrophe (KoTp4).

At G4 teachers reveal they know that it is necessary to maintain the distance between all points in the figure and the corresponding points in the image [42], but they incorrectly use six length units for the translation vector magnitude instead of four (KoTf1*), not establishing a relationship between this distance and the magnitude of the translation vector provided. Translation is a topic for students at grade 7 (12–13 years old) in the Brazilian context, but G4 performed the same students' common mistake, disregarding the magnitude of the translation vector (Flores & Yanik, 2016). It reveals an

inadequate knowledge on the three foundational properties of the translation vector – direction, sense and magnitude [(KoTf2)].

When interpreting Laura's production (Part II), G1 and G2 considered the production incorrect, while G3 and G4 assumed it was correct.

G1: Laura → incorrect, despite maintaining the size of the drawing, it did not correctly shift according to the vector.
G2: Laura → Mathematically there was misconception because she used the vector as the distance between the figure and the image and not as the translation of the corresponding points.
G3: Laura's production: Analysing the image, we noticed that the student used the limit on the right to start the entire transformation, (except) for this reason she performed the activity correctly.
G4: Laura's production: We consider Laura's production to be correct, as the student created the image with sense, direction and distance.

G1 and G2 carry out an evaluative interpretation (IK level 1). G1 identifies some indication of the student understanding (*despite maintaining the size of the drawing*) associated with knowing that in translation the dimensions of the figure in the image are maintained, and G2 focus on describing the error in the production (*used the vector as the distance (...) points*).

G3 and G4 provided an interpretation characterized as a misinterpretation (IK level 0), since they assume a production that is mathematically inadequate as correct – reveal similar difficulties as those of students (Gomes, 2012). Even if G3 identify one of the student's errors (*the student used (...) transformation*), revealing knowing to identify the students' mistake of associating the translation vector to the oriented segment that joins a point in the figure with a point closer to the image (Flores & Yanik, 2016), they interpret the production as correct. G4 does not consider the three properties of the vector (direction, sense and magnitude) and that needs to be considered in the translation of a figure, as they consider the student's production correct (*as the student (...) distance*); however, Laura did not use the magnitude of the vector provided, being this – not using all the translation vector properties – a common students' difficulty (Sünker & Zembat, 2012).

In Table 2 we associate the content descriptors of revealed knowledge to the interpretative practices.

Table 2: Synthesis of the interpretative practice carried out associated with the revealed knowledge and the adequate knowledge based on the evidence of the groups

Evidence	Revealed knowledge	Adequate knowledge	Interpretative practice and level of IK
G1: incorrect, despite maintaining the size of the drawing, it did not correctly shift according to the vector.	Know that during translation the dimensions of the figure in the image are preserved and that the properties of the translation vector must be considered to carry out this transformation.		Evaluative interpretation: Interpret as incorrect a student's production that, despite maintaining the dimensions of the figure in the image, does not move it according to the vector provided, indicating the error in the production. **(IK level 1)**
G2: Mathematically she had a misconception	Know that in translation it is the magnitude of the translation vector that		Evaluative interpretation: Interpret as incorrect a student's production that

because she used the vector as the distance between the figure and the image and not as the translation of the corresponding points.	determines the distance between each point in the figure and its corresponding point in the image.		considers the vector as the distance between the figure and the image and not as the distance measurement between the original and transformed points, making the error in the production explicit. **(IK level 1)**
G3: the student used the limit on the right to start the entire transformation, (except) for this reason she performed the activity correctly.	Know that students make the mistake of identifying the translation vector as the oriented segment that joins a point in the figure with a point closer to the image.	Know that performing the translation disregarding the magnitude of the translation vector as corresponding to the distance between each point in the figure and its corresponding point in the image is mathematically inadequate.	Misinterpretation: Interpreting as correct a student's production that is mathematically inadequate. The error consists of using the right-hand limit of a figure to apply the vector and start the translation. **(IK level 0)**
G4: Laura's production was correct, as the student created the image with sense, direction and distance.	Know that when translating a figure, it is necessary to consider the same sense, direction and distance to translate all its points and obtain the image.	Know that when translating a figure, it is necessary to consider the same direction, sense and distance to translate all its points and obtain the image, these properties being determined by the properties of the translation vector and that they all need to be considered.	Misinterpretation: Interpreting as correct a student's production that is mathematically inadequate and that obtains an image through translation, disregarding the magnitude of the translation vector provided. **(IK level 0)**

Some final comments

Developing teachers' IK is fundamental for promoting interpretative practices beyond evaluative, requiring, thus, developing teachers specialised mathematical knowledge in each mathematical topic, such as translation. We found that the content of teachers' MK impacts the type and level of interpretative practice and knowledge – low level of MK constrains teacher's interpretations. A new category emerged related with the so-called mistaken interpretation, which was associated to a level 0 of IK (it's important to notice that this does not mean absence of knowledge), as it does not refer to the "common" evaluative interpretation, which at least points out the student's error (level 1). The mistaken interpretation grounds on teacher's MK and reflects his/her insufficient mathematical specialised knowledge to interpret a mathematically incorrect student's production, and so, even with a minimal level of MK, teachers assume they can interpret the students' productions, independently of the level of mathematical elaboration of such production. In that sense one open question to further discuss refers to identify and define the content of specialised MK levels. Another further research path to follow, considering the PD context in which data has been gathered concerns to focus the attention on the IK development and on the factor sustaining such development – not only by solving IK tasks but also due to the context of preparing, implementing tasks to their students and selecting student's productions to elaborate IT for the PD colleagues.

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Preservice and inservice teachers' understanding of the concept of function: The case of motion problem

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This paper analyses the knowledge of inservice mathematics teachers (ISTs) and preservice mathematics teachers (PSTs) regarding the concept of function. The study is grounded in the Mathematics Teachers' Specialised Knowledge (MTSK) model (Carrillo et al., 2018) and involves 9 ISTs and 19 PSTs. A set of mathematical tasks served as the research tool, implemented through a three-part process: (1) individual task solving, (2) group task-based interviews, and (3) a joint task-based interview. This paper focuses on two motion problems, examining both written solutions and recorded discussions. The analysis reveals the specific knowledge mobilised by ISTs and PSTs when solving function-related mathematical tasks.

Keywords: Functions, inservice teachers, preservice teachers, mathematical tasks, motion problem.

Introduction

Since the 1980s, research interest in the knowledge and expertise of both PSTs and ISTs has grown significantly. Understanding and analysing their knowledge is essential for improving it. This paper focuses on functions and functional thinking, one of the most important areas of mathematics education. However, numerous studies have highlighted the persistent difficulties students face with functions (e.g., Ostermann et al., 2018; Leinhardt et al., 1990).

To analyse the specialised knowledge mobilised by PSTs, we chose the Mathematics Teacher's Specialised Knowledge (MTSK) model (Carrillo et al., 2018). This model adopts an interpretative approach, emphasising the understanding of knowledge as it is used or developed, rather than evaluating it. It consists of the domains Mathematical Knowledge (MK) and Pedagogical Content Knowledge (PCK), each further divided into subdomains that enable a detailed analysis of mathematics teachers' knowledge. The model also includes an important third domain, which brings together the teacher's beliefs about mathematics and the teaching and learning of mathematics.

We employed specific mathematical tasks as a research tool, recognising that while solving such tasks is central to student learning, it also fosters teachers' professional growth beyond mathematical knowledge (Zaslavsky & Leikin, 2004, p. 30). Some of our findings also stem from the international FunThink project, which shares the vision that strengthening functional thinking from primary to upper secondary education can enhance mathematics learning.

In this context, we pose the following research question: What specific knowledge do ISTs and PSTs mobilise when solving mathematical tasks related to the concept of function?

Theoretical background

Research suggests that tasks fostering functional thinking should engage learners in multiple solution approaches and help identify misconceptions (e.g., Zaslavsky & Leikin, 2004; Leinhardt et al., 1990). In movement-related tasks, for instance, common misconceptions involve continuity and linearity

(Leinhardt et al., 1990). Students often struggle to determine whether a graph should be represented as continuous or discrete, making errors in both directions – interpreting continuous data as discrete and vice versa. Their tendency to overgeneralise linearity may stem from learning linear functions first and extending their properties to other elementary functions. Ronda (2015) emphasizes that teaching functions should focus on recognising invariant properties across different representations and understanding their contextual advantages and limitations. Therefore, incorporating tasks that require transitions between multiple representations can be beneficial. Pittalis (2020) further highlights the importance of addressing the four key aspects of functions – input-output assignment, covariation, correspondence, and functions as mathematical objects – when designing tasks. In motion problems, in particular, covariational reasoning plays a crucial role in effective mathematical modelling (Thompson, 2011). These theoretical foundations guided the development of our research tool, which consists of a set of tasks, as detailed below.

The MTSK model was selected to analyse the specialised knowledge mobilised by ISTs and PSTs. As mentioned earlier, this model comprises two main domains: Mathematical Knowledge (MK) and Pedagogical Content Knowledge (PCK). This paper focuses on one subdomain of MK – Knowledge of Topics (KoT) – and two subdomains of PCK: Knowledge of Features of Learning Mathematics (KFLM) and Knowledge of Mathematics Teaching (KMT). These are examined through PSTs' approaches to solving mathematical tasks related to functions. This aligns with Zaslavsky & Leikin (2004), who argued that solving and discussing mathematical tasks is one of the most effective ways to reveal teachers' knowledge.

Method

As part of the FunThink project, we developed a course for PSTs focused on linear functions to enhance teachers' functional thinking. The course is available at www.funthink.eu. Our research involved four groups: two pilot groups (whose results are not reported here) and two research samples of PSTs and ISTs, whose results are analysed in this study.

The first research sample comprised ISTs who participated voluntarily as part of their professional development within the national IT Academy project. The training, conducted from April to June 2022, spanned five full-day sessions over two months. Nine middle school ISTs (two men and seven women) aged 28 to 52, each with at least four years of teaching experience, took part. All held a master's degree in mathematics combined with another science subject.

The second research sample included 19 PSTs (2 men and 17 women) aged 22–23, enrolled in the first year of a full-time master's programme at Pavol Jozef Šafárik University in Košice during the 2023/2024 academic year. All had completed a bachelor's degree in mathematics combined with another subject. As part of the compulsory course *Didactics of Mathematics*, they attended 26 lessons across 13 sessions (lasting 90 minutes). The course aimed to develop PSTs' specialised knowledge of functions according to the MTSK model, with a particular focus on functions and linear functions.

Our research was conducted using the Task-based interview method (Goldin, 2000). It consisted of three parts. In the first part, both research samples were given a set of tasks to complete at home and send electronically to the lectors. The set of tasks was chosen to cover equally the different

representations of functions, the connections between them, the 4 aspects of the concept of function, and the different actions that the task requires of the solver, as suggested by Leinhardt et al. (1990).

This paper focuses on Task 1 and Task 2 (see Figure 1) from the set of mathematical tasks. Both tasks are grounded in real-world contexts and, based on Leinhardt's (1990) classification of learner actions, primarily fall into the categories of construction and prediction tasks. Task 1 connects table and graph representations of a function, while Task 2 links. Task 1 is taken from a Slovak mathematics textbook for upper secondary schools as one of the introductory tasks to introduce the concept of function. It emphasizes the input-output aspect of functions. However, to connect the given table to its graphical representation, students must identify the type of change and predict the function's behaviour. This requires covariational reasoning, which is crucial for mathematical modelling, particularly in motion-related contexts (Thompson, 2011). Additionally, recognising invariant properties – such as the function's non-increasing nature, its non-elementary classification, and its continuity over the defined domain – includes the object aspect of function understanding. Task 2 is a standard mathematics problem commonly found in Slovak textbooks. An analysis of these two tasks through the lens of the MTSK model highlights specific areas of knowledge that they help develop (see Table 1, columns 1 and 2).

Task 1. In order for an artificial Earth satellite to move in a circular orbit, it must reach a certain minimum velocity, called the initial space velocity. The table below shows some pairs of the numerical value of the velocity v (km/s) as a function of the numerical value of the distance d (in hundreds of km) of the satellite from the Earth.

d	0,5	1	2	3	4	5	6	7
v	7,6	7,4	6,9	6,5	6,2	5,9	5,7	5,4

Construct the graph of the function given by the above table.

Task 2. Alex has bought a new car with 80 km on the odometer. However, this is about to change as he is going on a long journey tomorrow. Describe how (table, graph, equation) if his average speed is 70 km/h.

Figure 1: Task 1 and 2 from the set of tasks

In the second part of the research, ISTs and PSTs were asked to work firstly in groups of three (four) and discuss the tasks from the set, focusing on three questions:

1. Do you think this task is important for teaching mathematics? If yes, why? If no, why?

2. What are the advantages and disadvantages of this task?

3. Which task would be difficult for the students and why?

The questions above focus on the subdomains of KMT and KFLM of the MTSK model. However, in answering the questions, teachers were also concerned with KoT, so we included the KoT subdomain in our analysis as well. After the group discussions, we had a joint discussion with the ISTs and PSTs. All discussions were recorded.

After data collection, we first analysed the written solutions submitted by the ISTs and PSTs for Task 1 and 2. We then coded the group and joint discussions according to the observed knowledge within the MTSK model. Both authors coded the data separately. We then had checked each other's assigned codes and agreed on the final version of the coding.

Table 1: Knowledge mobilised by the Task 1 and 2

		PSTs	ISTs
K o T	Function definition	x	
	Domain and range of a function	x	x
	Definition of a linear function	x	x
	Properties of functions (monotonicity, continuity, boundedness...)	x	x
	Cartesian coordinate system and the points in it	x	x
	Representations of functions (graph, verbal description, table, formula)	x	x
	Linking function representations	x	x
	Function notation	x	x
	Applications of function (mathematical modelling, cross-curricular relations)	x	x
	Elementary functions	x	
	Formal definition of the function concept (notation; unambiguity...)		
K F L M	Difficulties in understanding the concept of function		x
	Difficulties in plotting points on Cartesian axes (function does not always pass through the origin	x	
	of the coordinate system; different scales of x-axis and y-axis...)	x	x
	Students' attitude towards the abstraction of the concept of function	x	
	Knowledge of misconceptions (generalisation of the properties of linear functions; interpretation of		
	a graph of a situation as a picture of the situation...)	x	x
	Variety of strategies in solving the problem (use of different aspects of the concept of function; use		
	of different representations)	x	x
	Potential of the task for teaching	x	x
	Students' emotional aspects to the task (reluctance to solve problems, dislike of mathematics,...)	x	
K M T	Logical consistency of definitions, procedures and tasks during the teaching process	x	x
	Use of digital technologies to teach functions		
	Use of everyday situations to teach functions		x
	Use of examples and counter examples to teach functions		
	Awareness of different ways of representing situations (table, graph, rule, verbal description)	x	x

Results

In Figure 2 we can see the different types of solutions that appeared for PSTs (see Figure 2, graphs A to D) and ISTs (see Figure 2, graphs E and F) to Task 1. All PSTs plotted points on the Cartesian coordinate system and connected the points. In eight cases the PSTs plotted a graph, which some PSTs described later in the discussion as a decreasing linear function (see Figure 2, graph A). 7 PSTs plotted a graph that visually looked more like an exponential decreasing function (see Figure 2, graph B). In graph C, we can see a typical misconception about the swapping of the x and y axes (Ostermann, 2018), which PST Simone made in several tasks. Only PST Dana (see Figure 2, graph D) solved the task in multiple ways with appropriate comments. On the other hand, two of the nine ISTs did not connect the points in the graph (see Figure 2, graph F) because they did not find the correct way to connect them, and 7 ISTs connected the points in the graph (see Figure 2, graph E).

If we look at the values in the table in Task 1, we see that the values do not decrease by the same constant. However, most PSTs have visualised this function as a linear function. We assume that they have not gone back to the table to link the two representations. In the discussion below, the PSTs discuss whether the points should be connected or not.

Episode 1. Discussion about the continuity of the graph of a function.

27	Simone:	I didn't know whether to connect the dots or not.	*[KoT]*
28	Mary:	I think the dots should be connected.	*[KoT]*

29	Simone:	But since it's about speed... I would probably connect the dots. It probably wouldn't make sense if it wasn't connected. *[KoT]*
30	Hannah:	It depends... that distance... I won't jump from distance one to distance two. This is also so intuitive for the students for sure. *[KoT]*
31	Simone:	By the fact that it is a movement in time... so it should be connected.
32	Lector:	And how do I know how to connect those two points side by side? *[KFLM]*
33	Mary:	The satellite could be going very fast and suddenly it would slow down. But, as we can see, we tend to draw it straight (a straight line). *[KoT]*

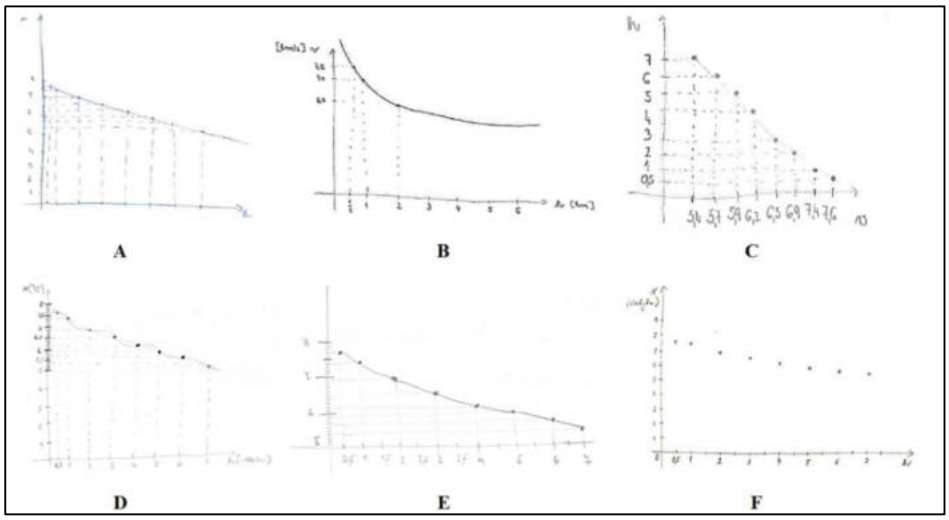

Figure 2: Different ISTs and PSTs solutions to Task 1

Dana is the only one who connected the points in two different ways, claiming in her solution that there are an infinite number of them (solutions) and stating in her comment that she cannot construct an unambiguous graph of the given function because of the information about uniform motion (see Figure 2, graph D). In the discussion, Mary adds her by mentioning that we tend to draw the graph as a linear function (replica 33), which is also her solution. Simone, who swapped the axes in her solution (see Figure 2, graph C), was unsure all the time (replica 27) whether or not to connect the dots in Task 1. In the following discussion, Rose admitted that she had not thought at all about how to connect the dots (replica 15). She connected the values and, thanks to an inaccurate sketch, produced a graph of a linear function. Dana even doubts whether it is a function or not (replica 18).

Episode 2. Discussion of the linearity of the function and whether it is a function.

15	Rose:	I think it is simple. I don't see anything complicated about it. And I think they would do exactly what I do... a straight line. *[KMT]* And they wouldn't think twice about it. Maybe... I don't know... but I don't think they would think about it. *[KFLM]*
16	Dana:	Even if you look at the table now, if you think about it, you can see that if this distance changes by 1 and the speed changes by 0.4, and here it's only 0.3. *[KFLM]* Now it shouldn't be linear. And there's also some rounding in there. *[KoT]*
17	Lector:	So, is it a graph of a function?
18	Dana:	Yeah... it's a function, isn't it? (uncertain) *[KoT]*

However, Dana adds that it cannot be a linear function, and although there may have been a lot of rounding, it is clear from the table that it is not an elementary function. Looking at the graph of the function, Hannah argues in replica 10 that even if we cannot describe the dependence by a rule, it

must be a function. Simone is still convinced that it must be an elementary function because the task assignment says that it is the graph of a function (replicas 12 and 14). She automatically associates the graph of the function with the elementary functions.

Episode 3. Discussion on whether this is some elementary function.

10	Hannah:	The task is good for uncovering misconceptions. Maybe with these non-linear functions they think it's not a function, I can't write a formula. But the dependence function is there, I just don't know how to describe it with this formula.	*[KFLM]* *[KMT]*
11	Lector:	That is, does the graph or table represent some elementary function?	
12	Lenka:	I think it's going to be something. But I don't know what.	*[KoT]*
13	Simone:	And I think again that it won't be any elementary function. I don't know.	*[KoT]*
14	Mary:	But… the assignment is: construct the graph of the function.	

In contrast to the PSTs, the two ISTs produced a solution where the points in the graph were not connected, outlining a discrete graph (see Figure 2, graph F). The seven ISTs in Task 1 sketched a continuous graph (see Figure 2, graph E), which in several cases visually resembled a linear function. The decision not to connect the dots stems from the following group discussion, where in replica 1 Greta claims that the function is "strangely decreasing". In replica 5 Greta is trying to make sense of at least some information, looking for something to help her, but she cannot make sense of her own thoughts. Replicas 6 and 7 show the ISTs' "problems" with the task, as they found no relationship (linear pattern) and thus no discernible pattern between the values in the table in Figure 2 of Task 1. In replica 8 it even seems that IST Viera sees only one solution to the task as she claims that the students would clearly connect the dots with a ruler.

Episode 4. Discussion on how to connect the dots.

1	Greta:	…that satellite. I didn't connect the dots, because this was so weird to me.	*[KoT]*
2	Vera:	I didn't connect dots because it didn't look like any elementary function. It was going down so strangely at the beginning.	*[KoT]*
3	Greta:	That's where the assignment took me into uncertainty. I can't really picture the movement or why it jumps like that.	*[KoT]*
4	Michael:	I connected the dots. It's not linear or anything, it's a stretched indirect proportionality, I think. But I've connected it.	*[KoT]*
5	Greta:	I don't think it's the proportionality, because...	*[KoT]*
6	Vera:	I didn't even know how to connect the dots.	*[KoT]*
7	Greta:	This is so unclear. I didn't know what to base it on.	*[KoT]*
8	Vera:	Students would be able to construct a graph, right? They would have played a bit with the ruler.	*[KFLM]*
9	Greta:	Basically, that's all they were supposed to do and nothing else.	*[KFLM]*
10	Vera:	The only problem would be... Connect the dots? Don't connect the dots?	*[KFLM]*
11	Greta:	I mean, the movement is continuous, but we don't know if...	
12	Vera:	In the beginning, there's a little curve.	
13	Greta:	The points are arranged irregularly; it jumps up and down. For me it's unclear. I would be insecure as a student, but also as a teacher.	*[KFLM]*

In the joint discussion, Jodie replies that she would definitely connect the dots because they are rounded values (the distance is in hundreds of kilometres per hour), and this can skew the measured data (replica 8). Emma argues that it is necessary to connect them because the satellite is not jumping from point to point, but it is a movement (replica 10).

Episode 5. Discussion about the continuity of the graph of a function.

6	Michael:	I connected the dots.	*[KoT]*
7	Patrick:	I also connected the dots.	*[KoT]*
8	Jodie:	Me too. Otherwise, it's in the hundreds of kilometres. I mean, there's already up and down with these units.	*[KoT]*
9	Vera:	Negligible rounding.	*[KoT]*
10	Emma:	But it (satellite) just doesn't jump from one point to another.	*[KoT]*

In contrast to Task 1, in Task 2 neither the PSTs nor the ISTs hesitated and, with the exception of two PSTs, drew a graph of the linear function, automatically assuming uniform motion of the car throughout the journey. They clearly connected the points with a half line and did not consider how the graph would behave between the points. IST Viera adds: "Sometimes we have to model things even though we know it won't be like that in everyday life". In this task, the ISTs were aware that we can't describe everything exactly with a graph, so it is necessary to adapt the graph of a function to a given situation and to discuss with the students why we do it this way. However, they didn't use this reasoning in the situation in Task 1, where they could not link it to a known elementary function.

Analysis of the PSTs' and ISTs' interviews subsequently revealed which knowledge was mobilised during the solution and subsequent discussion of Tasks 1 and 2 (see Table 1, columns 3 and 4).

Discussion and Conclusion

In order to answer the research question formulated above we will discuss which specific knowledge have been mobilised within two tasks dealing with motion situations. Concerning the KoT subdomain, we see that in both cases the ISTs and PSTs are mainly concerned with the continuity and properties of the Cartesian coordinate system within Task 1. The ISTs paid most attention to the continuity of the function, but one can often sense the uncertainty in their words. Two ISTs did not connect the points in the graph because they were not sure how to connect them. The ISTs tried to find out if their claims were correct and asked for solutions to the task. On the other hand, the PSTs were more confident in their statements. All PSTs connected the dots in the diagram. Dana suggested a number of ways to connect the dots, which another PST also considered. On the other hand, there were almost no doubts between ISTs and PSTs about whether and how to connect the dots in the situation described in Task 2.

In the KFLM subdomain, the more dominant student’s interaction with mathematical content. The ISTs tended to focus on how they would solve the task and only marginally mentioned the learners' problems with the task. In contrast, the PSTs focused their attention on the learner, apparently in preparation for their future profession. A strong aspect for both groups is motivation. Both groups perceived the tasks in terms of cross-curricular relationships. On the basis of their solutions to the task, they outlined a number of misconceptions that might be manifested in students.

From the perspective of the ISTs, the Task 1 is not important to teach because they think it hardly practise anything. They would use better tasks that give nice values to construct the graph of a function. The PSTs see the potential of the task to solve some misconceptions (linearity, generalising properties of a linear function, continuous vs. discrete graphs). They consider it an advantage to label the dependent and independent variables differently than the traditional labelling: the independent variable is labelled h instead of x, the dependent variable is labelled v instead of y. The ISTs showed knowledge of lesson organisation, reflecting on the time needed and the form of the students' work.

We think that the inclusion of these two tasks in the PST and IST courses is appropriate for teacher training. In our research sample, it helped to reveal misconceptions centred on connectedness and linearity. Discussing the tasks in groups and together has the potential to help participants to identify potential gaps in their knowledge and consequently to develop their specialized knowledge. Although this research has some limitations (e.g. the research sample was from one region and), we believe that such an analysis is an important starting point for designing professional development courses for ISTs and for PSTs in order to develop teachers' and thus their students' functional thinking.

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Impact of an academic intervention on prospective teachers' mathematical world views during their transition to university

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Numerous studies have examined the impact of mathematical world views on teachers' classroom practices in recent decades. A particular focus has been placed on the possibility of changing those beliefs. In light of these findings, this study explores the impact of an academic intervention on the mathematical world views of 30 prospective teachers during their first semester at university. Using a longitudinal design, participants' beliefs were evaluated via a standardized questionnaire at the beginning and end of their first semester. The quantitative analysis shows significant shifts. Dynamic mathematical beliefs increased, while static mathematical beliefs decreased. These findings suggest that targeted interventions during the transition from school to university can have an impact on the mathematical world view of prospective teachers.

Keywords: Mathematical world views, beliefs, teacher education, beliefs change

Introduction

The transition from school to university can be challenging for many students, including those who study mathematics. Therein, the new didactical approach provides students with more autonomy and self-regulated learning (Gueudet, 2008) and therefore may cause a shift from previous experiences (Geisler & Rolka, 2020). In particular, keeping in mind potential cultural differences, the emphasis on math application tasks with a strong focus on modelling (ibid.) and proofs (Vollstedt et al., 2014) at university level can be seen as examples of this change. High drop out and subject change rates illustrate the difficulties involved in this transition (Geisler & Rolka, 2020). Consequently, students in their first weeks at university require specialized support. Based on Geisler and Rolka's (2020) analysis of how beliefs affect the transition from school to university, this study focuses on how students develop their mathematical world views while attending a specific designed course during their first semester. This leads to the following research question:

How do prospective teachers' mathematical world views change as a result of a specific course focused on the transition from school mathematics to university mathematics during their first semester?

Theoretical Background

Beliefs

Over the last few decades, research on mathematics education has increasingly focused on (prospective) teachers' beliefs (Eichler et al., 2023; Philipp, 2007; Skott, 2015; Thompson, 1992). According to research conducted on mathematical beliefs, teachers' epistemological views influence students' motivation to learn, their learning strategies, and their performance (Beumann & Geisler, 2022; Eichler & Erens, 2015), along with their likelihood of dropping out or switching subjects (Geisler & Rolka, 2020).

According to Safrudiannur and Rott (2019, p. 2), there is no consensus regarding an explicit definition (Beswick, 2005, p. 39) of beliefs (see also Skott, 2015). This article uses Philipp's (2007) interpretation of beliefs, which he defines as “psychologically held understandings, premises, or propositions of the world that are thought to be true”. To analyze prospective teachers' beliefs, the present study utilizes the theoretical framework of "mathematical world views" by Grigutsch et al. (1998). The authors developed four aspects which categorize teachers' beliefs-cluster concerning the nature of mathematics and are known as a teacher’s “mathematical world view”. These four aspects can be defined as follows by Eichler et al. (2023, p. 1493):

- The formalist aspect represents beliefs implying that (school) mathematics is characterized by a strongly logical and formal approach.
- The schema aspect represents beliefs implying that (school) mathematics is a set of calculation rules and procedures to apply to routine tasks.
- The process-oriented aspect represents beliefs implying that (school) mathematics is a creative activity consisting of problem-solving using different and individual ways.
- The application-oriented aspect represents beliefs implying that the utility of mathematics for real-world problems is the main aspect of the nature of (school) mathematics.

According to Grigutsch et al. (1998), there is a significant correlation between formalist and schema aspects, or between process-oriented and application-oriented aspects. Thus, according to Viholainen et al., (2014), the first two aspects of mathematics are considered as "static", whereas the last two are considered as "dynamic" nature of mathematics.

Based on Geisler and Rolka (2020, p. 605) "[...] beliefs which do not fit to university mathematics [...]" can lead to dropping out of mathematical teacher training. Therefore, it is critical to clarify which beliefs of prospective teachers are beneficial as well as how to change hindering beliefs to enhance the likelihood of a successful study path. A study conducted by Geisler and Rolka (2020, p. 599) showed that dynamic beliefs of prospective mathematics teachers negatively predicted their intention to drop out. Moreover, dynamic views on mathematics, as well as on teaching and learning mathematics, are generally considered beneficial for learning (e.g., Ostergaard, 2024). The mechanisms through which teachers' beliefs change pose a challenge (Liljedahl et al., 2012). There is wide consensus that a long-term strong impact is required for sustained shifts in (teachers’) beliefs (e.g., Philipp, 2007; Levin, 2015). In this research approach, the transition from school to university is seen as such a strong impact that, accompanied by a specific lecture with a focus on the question "What is mathematics?", can lead to a change of beliefs.

Intervention

In order to implement the specified intervention, the mandatory lecture "Aspects of Mathematics" of teacher training program at the University of Vienna was selected because of its curricular requirements (https://ssc-mathematik.univie.ac.at/im-studium/studiengaenge/bed-uf-mathematik/curriculum-studienplan/)

In this lecture, which is based on the book "Panorama der Mathematik" by Loos et al. (2022) the guiding questions addressed were epistemological in nature (e.g., "What is mathematics?", "How do mathematicians acquire knowledge, and how do they know that they are correct?"). In addition, based on the explicit-reflexive methodology of belief teaching (see McComas et al., 2020), reflexive phases

were utilized to allow students to become aware of their mathematical world views and to question them based on the content discussed. As a result, the course consisted of a combination of lectures on specific topics and reflection/discussion-sections in small groups of approximately four prospective teachers.

Method Section

Sample

As part of a longitudinal study, 30 prospective teachers were surveyed with an online questionnaire regarding their mathematical world views within their first and last "Aspects of Mathematics" lecture.

Data Collection and Analysis

The longitudinal design was evaluated with quantitative analysis. Within the pre- and the post survey, the participants filled out an online questionnaire, which consist of 37 closed items on a five-point Likert-scale (5 = "totally agree"; 4 = "agree"; 3 = "neutral"; 2 = "disagree"; 1 = "totally disagree") taken directly from Grigutsch et al. (1998)'s questionnaire regarding the mathematical world view. In Table 1, sample items are presented for each of the four aspects, as well as the number of used items for each aspect and the internal consistency based on Cronbach's α values (pre-survey-participants). To determine whether the four mathematical world views were accepted or rejected, the (overall) arithmetic mean of the individual arithmetic means of the total values resulting from the four subcategories and their items embedded therein was calculated. The ordinal scaled data were interpreted as quasi-metric (Blömeke et al., 2008). This part of the analysis concludes with a comparison of the collected central tendencies with the Wilcoxon-test for related samples (longitudinal design).

SPSS-28 was used for this evaluation.

Table 1: Quantitative data collection and analysis

	Formalist aspect (F)	**Scheme aspect (S)**	**Process aspect (P)**	**Application aspect (A)**
Sample item	A fundamental component of mathematics is the application of logical rigor and precision, also known as objective thinking.	Mathematics consists of learning, understanding, and applying.	In most cases, in mathematics there is more than one way to solve a problem.	Mathematics helps to solve daily tasks and problems.
Number of items	10	8	9	10
Cronbach's α	0.738	0.720	0.739	0.768

Results

Cronbach's alpha values (Table 1) for all mathematical world views are higher than 0.7, which is considered sufficient in terms of internal consistency (Taber, 2018).

In Table 2, the descriptive statistics for each subcategory are presented. For the interpretation of the arithmetic means an equidistance scale was used which is derived from the Likert-scale range 1 to 5. Accordingly, we got: 1 to 1.8: strong disagreement (--); 1.8 to 2.6: disagreement (-); 2.6 to 3.4: neutral (~); 3.4 to 4.2: agreement (+); 4.2 to 5: strong agreement (++)

Table 2: Descriptive statistics of mathematical world views ($\overline{x}$ = arithmetic mean)

Sample		F	S	P	A
Pre $N = 30$	$\overline{x}$	3.68	3.43	3.82	4.09
	interpretation	+	+	+	+
	standard deviation	0.53	0.59	0.54	0.59
Post $N = 30$	$\overline{x}$	3.17	2.83	4.04	4.11
	interpretation	~	~	+	+
	standard deviation	0.71	0.78	0.57	0.46

These findings concerning the descriptive statistics of the longitudinal study show that the prospective teachers agree with all world views simultaneously within the pre-test. Participants at the end of their first semester agree only with the dynamic aspects of mathematics and have a neutral attitude towards the static view. Across all sample groups, the application-oriented aspect has the highest level of agreement, while the scheme-oriented aspect has the lowest one.

In light of the controversy regarding the use of parametric or non-parametric evaluation methods for evaluating Likert scales (De Winter & Dodou, 2010), and the presence of some outliers in the data, Wilcoxon-tests have been used to assess whether participants might have shifted their mathematical world views in response to their experiences during their first semester studying mathematics. The analysis of the longitudinal design (Wilcoxon-test) indicates that accept the application-oriented aspect, all other mathematical world views show (highly) significant changes with moderate ($0.3 < r < 0.5$) to strong ($r > 0.5$) effect sizes due to the scale of Cohen (1992) (F: $p < 0.001; r = 0.67$ / S: $p < 0.001; r = 0.67$ / P: $p = 0.022; r = 0.42$).

Discussion

Using a similar questionnaire (Laschke & Blömeke, 2013) to the one used in the present methodology, Geisler and Rolka (2020) examined whether students' beliefs could be interpreted as indicators of exam achievement and dropout risk as well as how students' beliefs about the nature of mathematics change during their transition from school to university. Their findings differ from those of this study: Geisler and Rolka (2020) analysed that prospective teachers' dynamic beliefs decline during their transition, whereas static ones remain stable. The present study evaluated significant changes and the trend of favoring dynamic aspects, which can be interpreted as positive, since dynamic world views are regarded as negative predictors of a student's intention to drop out (Geisler & Rolka, 2020). Kuklinski et al. (2018) conducted a similar study with the aim of increasing students' dynamic beliefs and found a weak decrease in static beliefs but no changes in dynamic views. There was no attempt to force the measured trend in favor of dynamic views on mathematics within the presented intervention. According to the theoretical background section of this paper, the purpose of the lecture in discussion was to facilitate the transition from school to university and to provide prospective teachers with the opportunity to reflect on their mathematical world views.

The measured output concerning beliefs can be viewed as a beneficial support for a successful transition from school mathematics to higher mathematics since it can be interpreted as providing benefits to learning (Ostergaard, 2024) and preventing students from dropping out.

Moreover, the measured significant changes in the mathematical world views are not consistent with similar evaluations at the University of Vienna. According to Woltron (2023), a quasi-longitudinal design over the entire course of study (12 semesters) reveals a stability of beliefs, especially within the mathematical world view. Additionally, this would accord with the common interpretation of beliefs as stable or slowly evolving (Geisler & Rolka, 2020). The present study illustrates that, in accordance with Liljedahl et al. (2007), it is possible to achieve significant changes in beliefs by designing learning environments that are embedded in specific situations that can facilitate a change in beliefs, i.e., which enable a strong impact (Eichler et al., 2023). Combining the mentioned particular situation of the transition from school to university with specific courses appears to facilitate the change in beliefs.

Conclusion

In spite of its strengths, this study has some limitations. Whenever Likert scales are used, there is always the possibility of favouring responses that are correlated with social desirability (Kreitchmann et al., 2019). Predefined (fictitious) contexts can mitigate this problem, but qualitative analysis is time-consuming and therefore not feasible for large sample sizes. It would be useful to conduct further studies to examine the change in students' mathematical world views due to special intervention within the first semester qualitatively by interviewing participants who attend the final exams and those who drop out earlier in the semester. Furthermore, a control group could be analyzed to gain a deeper understanding of how the lecture actually impacts mathematical world views. This missing criteria for an intervention study can also be seen as a limitation of the methodology. Students' transition from school to university is a special stage in their lives that is affected by a variety of factors. In this regard, it should be noted that significant changes in mathematical world views are

not solely attributable to the intervention mentioned above. In addition, all participants in this longitudinal study are volunteers. Students who drop out or are about to drop out are often the focus of research in the transition phase, where a volunteer sample appears to have a significant effect. During this particular phase of the study, volunteer sampling may create a bias towards higher-achieving students (Di Martino et al., 2023, p. 25).

The purpose of this study was to investigate how students' mathematical world views may change during the first semester as prospective teachers. An intervention in a specific situation (transition from school to university) resulted in a decrease in static beliefs and an increase in dynamic beliefs. These findings have implications for both university and school courses, both of which provide learners with the opportunity to broaden their understanding of this scientific discipline, keeping in mind cultural differences. Besides the approach described in the intervention, further methodologies for implementing such interventions can be found in the context of "Nature of Science" research (see McComas et al., 2020). In addition, the results show that, taking into consideration the limitations mentioned, the used method of combining special situations (e.g., transition, first practical experiences (Woltron & Götz, 2023), etc.) and specific intervention can be a possible approach to the difficulty of “changing beliefs”.

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Elementary preservice teachers' knowledge in creating alternative word problems: A guide for mathematics educators

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Keywords: Word problem, Mathematical knowledge for teaching, Preservice teachers.

Introduction

Hill et al. (2008) incorporated Shulman's (1987) pedagogical content knowledge ideas into a framework of mathematical knowledge for teaching. Ball et al. (2008) suggested that more work is needed to identify how different types of knowledge are utilized in the act of teaching mathematics. Examining aspects of knowledge that are integrated in the daily practice of teaching mathematics has important implications for developing and enhancing methods courses. Mathematics teachers engage in several kinds of work, including identifying, examining, selecting, and creating mathematical problems. Word problems (WPs) are a common type of math problem found in curriculum materials; additionally, there are decades of research on WPs, their importance for student learning, and the challenges in solving them (Pongsakdi et al., 2020). However, little is known about WPs through the lens of mathematics teacher knowledge. In this study, I examine elementary preservice teachers' (PSTs) knowledge in creating alternative WPs. An alternative WP maintains the same conceptual challenges as the original, so students succeed or struggle with both for identical reasons. Creating alternative WPs is a complex, multi-step process requiring PSTs to categorize problems, identify operations, examine numbers, and apply creativity.

Methods

Data was collected from eight PSTs enrolled in an elementary methods course in a public university in Kuwait. The PSTs completed a task in which they were instructed to create two alternative WPs for a given WP: one tailored for girls and another for boys, along with explaining their reasoning. The task was designed to serve two purposes: (1) to explore potential underlying gender biases, which is the focus of a separate paper, and (2) to evaluate the quality and originality of the WPs, which is the primary focus of this work. An example of a given WP is: *Fatima has 18 flowers and 4 vases. She want to put 6 flowers in each vase. Did she use all the vases? Explain your answer.* A total of 29 WPs were analyzed following Lincoln and Guba's (1985) constant comparison method. Allowing for a systematic comparison of the WPs characteristics to answer the question: What are the characteristics of WPs created by PSTs?

Results and discussion

I focused on mathematical and linguistic characteristics drawn from Daroczy et al.'s (2015) work.

Mathematical characteristics

In half of the cases, the PSTs were unable to select new numbers for their WPs. That said, maintaining the numbers as in the original WP did not always lead to having a solvable problem. For example,

PST 2.1 wrote: “Sara visited a cosmetics center and bought 36 lipsticks and 27 eye shadow palettes. Note that she has 140 KD. How much does she have left after buying what she needs?” Two of the numbers represent the quantity of the items, while the third represents the total amount of money, making the problem unsolvable, the error was having different units for the numbers. Unsolvable problems were also present due to having missing numbers, such as the total. Inattention to the property of numbers and to how computations are carried out reduced the difficulty level of the WPs compared to the original WPs. For example, having a one-digit number instead of a two-digit number, a straightforward addition without a carry over, and a division with no remainder.

Linguistic characteristics

All the PSTs created a context that differed from the original WPs. However, several linguistic errors were noted. Missing important verbal cues transformed WPs from having one solution into open WPs with multiple answers. Several PSTs did not use any of the words indicating a division problem. Others who used the words “distribute” or “share” failed to write that that was done equally. Another error was writing a story without a question. In other cases, the question did not match the context of the problem. PST 3.5 wrote, “You have four boxes and each contains four mixed cookies, and you share the cookies equally with 16 people in your family. How many rocks will each one of you get?” In conclusion, identifying aspects of PSTs’ knowledge in creating alternative WPs and reporting examples of their work and errors provide a guide for mathematics educators in preparing PSTs and enhances professional development experiences.

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Teachers' knowledge and beliefs about assessment in mathematics: initial evidence from a national survey of teachers' formative needs

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Keywords: standardised assessment, assessment, teachers beliefs, teachers knowledge

Introduction

This contribution aims to illustrate the first results of the National Project "*Mathematics standardized assessment as a tool for teachers' professional development*", that aims to develop theoretical and operative tools to utilize the results of mathematics national assessments INVALSI to support teachers' professional development. It also seeks to promote a formative use of INVALSI materials and related good practices in schools, both at the institutional and classroom levels. The impact of standardized assessment is traditionally top-down: survey results influence public opinion, which in turn affects policymakers who act on the system's structure at different levels; only at a final stage do surveys and their results have an impact on local teacher action. The innovative idea of our project is to provide tools for a bottom-up approach in which theoretical frameworks/test/results are used immediately by teachers, to implement formative assessment and individualized instruction. In recent decades, research has been conducted to create a bottom-up dynamic which starts with a conscious use of frameworks, released items and results by the teachers. Some conditions should be fulfilled, such as the connection between assessment and curriculum, which should be perceived by teachers, as assessment is a key component of their identity and practice (Hannula et al., 2016; Ferretti & Santi, 2023). Our research aligns within this line of thought, aiming to develop a model for pre- and in-service mathematics teachers based on the effective use of standardized assessments (Ferretti et al. 2020). As a preliminary step, an anonymous questionnaire was used to survey teachers' beliefs, knowledge and needs regarding assessment and standardized assessment. This investigation provides crucial insights for designing the model and serves as the starting point of our research.

The first stages and findings: the administration of the questionnaire

In the first phase of the project, we investigated mathematics teachers' knowledge and beliefs about standardized assessments, particularly INVALSI tests and data. This phase involved the design and administration of a research questionnaire, structured into six distinct versions, considering the possible differences between pre- and in-service teachers (e.g., in terms of training and experience), as well as the school level at which the participants teach or will teach (primary, low secondary, and high secondary school). A total of 131 teachers responded to it. The questionnaire is intended to examine three main macro-variables: the mathematical and teaching knowledge of the participants

(including how they interpret INVALSI items and their outcomes in relation to mathematical content); the affective factors related to teaching practice (such as the perception of their competence in mathematics and mathematics teaching, and how assessment is perceived); the beliefs and knowledge about the purposes and possible uses of standardized assessment in mathematics, formative assessment, and awareness of mistakes (e.g., the interpretation of the causes of mistakes identified in the macro-phenomena of standardized assessment) (Bolondi & Ferretti, 2021; Ferretti et al. 2023). The data collected highlight several phenomena: regardless of school level and experience, teachers seem unable to correctly interpret the most common mistakes, and, in some cases, there is also inconsistency between teachers' perceptions of item's coherence with the ministerial guidelines and their purpose in relation to the teaching practices they adopt. We consider these phenomena as teachers' formative needs, understood as the necessity for development to refine their knowledge and competencies in the context of standardized assessment. These results can therefore serve as a starting point for designing new activities to be integrated into the aforementioned professional development model for teachers.

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Alternatively certified mathematics teachers' perceptions, experiences, and commitment to continue teaching

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Keywords: Teacher certification, teacher motivation, teacher persistence, mixed methods.

Introduction and theoretical perspective

There is a need to study alternatively certified teachers (ACTs) to address a growing teacher shortage, as more teachers earn their credentials via alternative certification programs (Richter et al., 2022). In Germany and the U.S., researchers have explored connections between ACTs' intentions and commitment to continue teaching, job satisfaction, and experiences (McClintock, 2024; Richter et al., 2022). The current study focuses on alternatively certified mathematics teachers (ACMTs) in the U.S. and investigates their commitment to continue teaching. Research question: How do ACMTs' perceptions and experiences relate to their self-ratings of their commitment to continue teaching?

Situated Expectancy-Value (SE-V) theory (Eccles & Wigfield, 2020) and Job Demands-Resources (JD-R) theory (Bakker & Demerouti, 2017) are coordinated to frame the study. SE-V theory explains individuals' decisions about engaging in and persisting with an activity (e.g., teaching), and JD-R theory explains a dynamic system of job-related and personal factors that impact occupational well-being (e.g., burnout). Coordinating SE-V theory with JD-R theory helps explain aspects of being a teacher that affect ACTs' decision to persist in teaching or leave the profession (McClintock, 2024).

Instrument, population, data collection, and methods

The current mixed methods study examined ACMTs' commitment to continue teaching. Participants, 47 early career ACMTs, freely volunteered. The instrument, an electronic survey, included a self-rating question (0-100) and an open-ended question about commitment to continue teaching. No personally identifiable data were collected, and participants' rights were protected. Qualitative analyses included initial coding, modified holistic coding, and theming (Saldaña, 2013). ACMTs clustered into groups according to their self-rating level of commitment. Holistic profiles were developed for each group, thus qualitizing quantitative data (Onwuegbuzie & Leech, 2021).

Results

The clustering resulted in three distinct groups of ACMTs: high (90-100), medium (61-89), low (0-60). Holistic profiles for each group's commitment to continue teaching included a characterization of the group and a researcher generated first-person narrative from codes with highest frequencies.

High commitment profile (self-ratings between 90-100)

This group of 15 ACMTs is characterized by noting job affordances and personal resources. The researcher generated first-person narrative is: *I love teaching, and it is a great fit for me. I believe it's important for my community and the larger society. Teaching is fulfilling and allows me to work with kids. Doing something I'm good at is rewarding. There is no job I would rather have, and I'm excited for each new day.*

Medium commitment profile (self-ratings between 61-89)

This group of 18 ACMTs is characterized by noting job affordances, job limitations, personal resources, and job demands. The researcher generated first-person narrative is: *I'm not so sure teaching is the career for me. I see positives about teaching, and I've put a lot into becoming a teacher. I truly love teaching math. I like seeing my students learn, but the downsides of the job are tough. I feel conflicted about continuing to teach.*

Low commitment profile (self-ratings between 0-60)

This group of 14 ACMTs is characterized by noting job limitations, stress, workload, feeling neglected, job affordances, and an intention to leave teaching before retirement. The researcher generated first-person narrative is: *Teaching isn't the best fit for me. The negative aspects of the job far outweigh any positives. The pay is too low compared to the hours, and switching careers would allow me to earn more. Being a teacher is not what I thought it would be. I feel neglected, stressed, and overworked. I don't see myself doing this much longer.*

Discussion and conclusion

ACMTs' commitment to continue teaching connects to their valuations of teaching, experiences on the job, and personal resources. Results from the current study are consistent with results from a larger study (McClintock, 2024), involving 368 ACTs across different disciplines. In each study, the profiles represent roughly one-third of the participants suggesting there may be common challenges to retaining ACTs, regardless of discipline. Importantly, ACMTs' confliction about staying in the career connects to their commitment (e.g., loving to teach math may not be sufficient to stay).

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Developing prospective kindergarten and primary teachers' specialized knowledge through tasks for teacher education in the scope of operations with natural, integers and rational numbers

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Keywords: Tasks, teacher education, mathematics teachers specialized knowledge, numerical thinking.

Some initial ideas and discussions

Students' knowledge is frequently constrained to the mechanical execution of algorithms, often characterized by the mere application of predefined rules. This approach typically lacks a deeper comprehension of the mathematical problem faced, focusing instead on procedural proficiency of arithmetical algorithms or the retrieval of memorized procedure (Bessa & Costa, 2017). However, a deep understanding of arithmetical operations necessitates an appreciation of their inherent complexity and cognitive demands, supporting in this way the development of numerical understanding and meaning (Anghileri, 1989; Park & Nunes, 2001).

The National Common Curricular Base (BNCC) in Brazil (BRASIL, 2018), which serves as the guiding document for the national curriculum, stipulates that multiplication should be introduced starting in Grade 2 (for students aged 7–8). However, research indicates that foundational notions of multiplication can, and indeed should, be addressed as early as kindergarten. These discussions should extend beyond the view of multiplication as merely repeated addition of equal addends, encompassing its various conceptual meanings (Anghileri, 1989; Park & Nunes, 2001). To address the challenges teachers face in practice—such as the operational difficulties students often exhibit—it is critical to move beyond the mechanical application of rules and algorithms. Instead, emphasis should be placed on fostering mathematical reasoning. This involves teaching the four operations with a focus on understanding their meanings, integrating problem formulation and resolution, and attributing appropriate significance to each step of the algorithms that students are expected to master. To achieve this, it is essential for teachers to develop their specialized knowledge, as such expertise has been shown to have the greatest impact on students' comprehension and performance (Nye et al., 2004).

Given that teachers' practice is inherently rooted in the use of tasks (e.g., Mason & Johnston-Wilder, 2006), it follows that tasks should also serve as the foundational element in Teacher Education contexts for the development of teachers' professional knowledge. In this framework, tasks are considered essential both as tools and as frameworks for designing effective Teacher Education contexts. Recognizing the distinct nature of teachers' practices compared to students' learning processes, we adopt a specific structure for tasks in Teacher Education. This structure creates a meaningful context for the development of teachers' knowledge and practices, enabling them to engage in experiences that mirror those they are expected to provide for their students (Ribeiro, 2020). In order to contribute to the area of task design and the development of teacher's knowledge, this project aims to discuss the structure and characteristics of the Tasks for Teacher Education (TTE) (Ribeiro, Almeida & Mellone, 2021) and how these tasks influence the development of prospective kindergarten and primary teachers' knowledge, particularly in the area of arithmetical operations. The framework for this teachers' knowledge is grounded in the Mathematics Teacher's Specialized Knowledge (MTSK) model (Carrillo et al., 2018).

Context and method

Data has been collected in a course for thirty-nine prospective kindergarten and primary teachers (PPTs) through the implementation of an itinerary of eight TTE focusing on the four operations typically addressed in primary education. Data collection includes written productions to each TTE, narratives about their lived experiences and the expected changes in their future teaching practices, as well as audio and video recordings of all discussions with the prospective teachers. For the analysis, we are focusing on the content of the MTSK categories and on the core characteristics and properties of the different TTE that may maximize the development of PPT's specialized knowledge.

Some expected results

The expected outcomes are: understanding the influence of the TTF for developing prospective teachers' specialized knowledge, obtaining a set of specialized knowledge descriptors in the topic of operations, identifying levels of knowledge revealed to enquiry the content of such levels.

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TWG 21: Assessment in Mathematics Education

Introduction to the papers of TWG21: assessment in mathematics education

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Overview

The Topic Working Group 21 (TWG21) on Assessment in Mathematics Education was established at CERME 10 in Dublin (2017) to promote communication, cooperation, and collaboration among researchers interested in assessment across all educational levels and contexts. A central aim was to integrate insights from mathematics education research into assessment practices.

During the current conference, the working group discussions brought to light a variety of interconnected themes that reflect the richness and complexity of assessment in mathematics education. First, formative and summative assessment were framed as two sides of the same coin, each contributing in different but complementary ways to understanding and supporting student learning. The presentations reflected a wide range of methodological approaches - both quantitative and qualitative - demonstrating the diversity of perspectives and tools used in current research. The international composition of participants further enriched the dialogue, offering opportunities to learn from different national and cultural contexts.

A strong focus was placed on the role of student-teacher interaction within assessment processes, particularly regarding feedback. The integration and development of digital resources for assessment was also central, highlighting both their potential and the practical challenges they pose. Discussions emphasized the importance of teachers' professional knowledge in assessment, underlining the need for targeted teacher education. Participants also reflected on the types of mathematics that are assessed. Theoretical and curricular perspectives were a key to framing many of the contributions, with several participants examining the influence of theoretical frameworks and curriculum structures on assessment choices and interpretations. The distinction between quantitative and qualitative feedback was discussed in relation to its effect on students' engagement and learning. Finally, a critical theme concerned the implementation of research findings into practice, including how assessment results - particularly from diagnostic activities - can be used to both enhance teaching and support students' learning trajectories in meaningful and actionable ways.

The sessions

The first session began with an introductory moment devoted to mutual acquaintance among participants and an overview of the group's structure and organization. It was followed by two presentations focusing on the **use of technology in classroom assessment**, specifically from the perspective of teachers. Saralar-Aras and Tütüncü discussed how middle-school teachers in Turkey perceive the role of digital tools in enhancing classroom assessment practices. The qualitative study suggests perceived benefits, such as increased students' participation and efficient use of time, as well as teachers'access to detailed analyses of students' mistakes. At the same time, some challenges are reported, such as additional workload for preparation and limited access to technology. In particular, the use of technology for summative assessment purposes is still perceived as demanding. Chen, Bos, Doorman and Drijvers examine how mathematics teachers integrate digital technologies into formative assessment practices, drawing on the theoretical frameworks of instrumental orchestration and formative assessment with technology. The study is based on the analysis of 12 award-winning lesson videos from China, considered examples of best practice. The findings highlight that the effectiveness of technology largely depends on teachers' pedagogical choices. However, the use of exemplary lessons introduces a notable limitation, as they may not reflect the typical dynamics of everyday classroom practice. The subsequent discussion addressed the issue of what is needed by teachers to implement digital-assisted technology.

The second session showcased **how assessment can be used for various purposes**. The paper by Pilet and Grugeon-Allys examines how standardized assessment can be turned into a tool that helps education professionals to manage heterogeneous learning in the classroom. Using an example from middle school algebra, they show how mathematical didactics can provide a method for interpreting the results of standardized assessment that is complementary to statistical methods. In the paper by Haara, on the other hand, the context is the reform of the national 10th grade oral exam in Norway. The paper presents a practice-based model for teachers' professional development in which teachers participate in planning the implementation of the new exam. Finally, Donner and Lerchenberger study the multiple-choice test format. Using the dual-process theory, they examine how multiple-choice tasks might favour superficial solving strategies, and the authors propose recommendations for the design of such tasks.

The third session focused on **diverse approaches to enhance formative assessment practices**. The study by Lemmo, Cotroni, and Polidoro builds on critiques of using written mathematics tests as formative assessment tools and it examines how a primary school teacher uses mathematical competence during test design and analysis. The teacher is able to obtain meaningful feed-up, feedback, and feed-forward, which inform both her understanding of student progress and future instructional planning. Mikite, Burgmanis, France and Namsone evaluate the effectiveness of a multi-indicator scoring approach to assess mathematical literacy through a detailed breakdown of six tasks into 17 individual items. Rasch analyses confirms the method's validity and reliability, although improvements are needed in the person separation index for better differentiation. In a different context, Hankeln, Prediger, Kroehne, and Hahn explore the use of speech input as an alternative to typed responses in digital formative assessments to assess students' conceptual understanding. The researchers observe that spoken responses are longer and incorporate multiple arguments. Responses

are especially rich when students use everyday language, which improves access to their conceptual understanding.

The fourth session explored **how formative assessment practices are shaped and reinterpreted in diverse mathematics education contexts**. Kaplan-Can and Haser analyze preservice teachers' lesson plans and interviews, revealing a range of formative strategies alongside tensions between planning and anticipated enactment. Andersson, Palm, and Winberg focus on an experienced teacher receiving individualized support to implement formative assessment during independent student work, emphasizing the value of context-specific professional development. Manzoni, Regola, and Giberti examine assessment in prison schools, showing how this unique setting fosters personalized, student-centered practices that differ from traditional summative approaches. Collectively, the papers illuminate how context matters in the enactment of formative assessment, shaping both opportunities and challenges.

In session 5 we discussed three papers, concerning **how assessment tools can better capture and support students' mathematical learning across diverse educational contexts**. Drüke-Noe and Bauer analyze statewide exit exams in Germany and conclude that tasks without aids and their standard solutions effectively target all six nationally defined mathematical competences. Point distributions align with the aim of testing basic skills. The findings highlight how these exams convey normative expectations for both teaching and assessment. Through the Erasmus+ DiToM project, Brings and colleagues developed diagnostic screening tools based on a shared framework of key mathematical skills, enabling early identification of students at risk of falling behind in key mathematical skills across European countries. The framework is based on a criterion-referenced approach, allowing test development to align with diverse national curricula. Sango's study in South Africa explores how subscore data from high-stakes assessments can help predict first-year university students' performance in mathematics. It compares different scoring methods, such as IRT and number-correct, to evaluate their impact on subdomain ranking and performance variance. The findings inform better use of test data for academic support planning.

Session 6 comprised two papers and one poster, all referring to assessment at university level. The paper by Barana, Marchisio Conte, Roman, Sacchet, Vinerean and Wondmagegne deals with a binational study in which they experimented with computer-aided assessment systems. In formative group activities university students worked on graphical calculus tasks and received a digital graphical feedback. The results show beneficial effects on reasoning that in turn enabled students to perform well in abstract tasks, too. Ramö, Myyry, Häsä and Zi examine students' choices between self-assessments and an examination and asked them to give reasons for their choice. The students justify their respective choice with reference to perceived usefulness, psychological safety and affective attitudes. Other justifications refer to learning gains, the absence of stress or the accuracy of assessment and the students seem to appreciate such a choice. In their poster, Andreas Büchter and colleagues present the results of a longitudinal study with university students. The data was collected before and after the pandemic. The tasks test mainly conceptual or mainly procedural knowledge or a combination of both. The longitudinal results reveal a substantial decline in the percentage of tasks

that were solved correctly while at the same time the percentage of unprocessed tasks rose substantially.

Focus on individual perspective

A review of past TWG21 contributions reveals that the student perspective has received comparatively limited attention. This observation led us to devote some time in our sessions to examine collaboratively the opportunities and challenges of placing greater emphasis on the learner's point of view. To build a rich understanding of the issue, we started the discussion by asking the participants what *student perspective* meant to them. Among the various contributions, one respondent emphasized the importance of enabling learners to focus on their knowledge rather than their gaps ("Students should be able to demonstrate what they know as opposed to what they do not know") while another drew attention to the subjective and situated nature of the assessment experience ("Students' experience in the moment of assessment"). These examples illustrate the diverse ways in which the student perspective can be conceptualized within assessment discourse. The discussion highlighted the highly individual nature of students' experiences with assessment, underlining the complexity of designing effective practices. Particular emphasis was placed on the value of formative assessment, especially when it incorporates both cognitive and affective dimensions of student response, supports self-regulation, and provides feedback that is meaningful from the learner's point of view. It was noted that assessment can be perceived as judgmental or appreciative, as motivating or, conversely, demotivating. For this reason, students should be given the opportunity to demonstrate what they know, rather than being assessed for what they do not know. Assessment should provide fair and valid judgments and enable active student participation. Feedback, in turn, should help students to learn from the assessment process and draw meaningful inferences from the results.

Finally, participants were invited to work in small groups, each selecting a paper to revise from the students' perspective, with particular attention to the research question, rationale, and methodology. Following this activity, they were asked to reflect on the process of reinterpreting the paper through the lens of the student perspective. The outcomes of the discussion revealed a shared interest among participants in placing greater emphasis on the student perspective within research itself. This could involve initiating research from the student's point of view or collecting more student-related data, for instance, regarding their engagement in standardized tests or their emotional responses to assessment. The ultimate goal of this shift would be to promote a more balanced distribution of responsibility and agency between teachers (or other assessors) and students (or assessed individuals) within the assessment process.

Graphical tasks with graphical feedback on limits of functions to support abstract reasoning

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In the last few years, the use of automatic assessment has rapidly grown, especially in higher education. In Mathematics, computer-aided assessment (CAA) systems based on advanced computing environments make creating meaningful tasks to address higher-order thinking skills possible. This study focuses on graphical tasks with graphical feedback on limits and asymptotes of functions developed with a CAA system. They were experimented within formative group activities in two first-year university mathematics modules in Italy and Sweden. The answers to the final exam were analysed and compared with the group activity results to understand if the graphical tasks supported abstract reasoning during the exam. The results will be used to adjust the activities and improve the digital feedback provided.

Keywords: Asymptotes, computer-aided assessment, example-generation tasks, graphical assessment, mathematics education.

Introduction

In recent years, automatic assessment tools have seen significant and rapid growth, with a notable impact on higher education, given the need for scalable and efficient methods supporting remote and hybrid learning environments. In the context of Mathematics, computer-aided assessment (CAA) systems have emerged as powerful tools (Barana et al., 2021). These systems, often built on advanced computing environments, enable educators to create complex, meaningful tasks beyond basic calculations and memorisation. By incorporating sophisticated algorithms and interactive components, CAA systems allow for assessing higher-order thinking skills, such as problem-solving, critical thinking, abstract reasoning, and conceptual understanding (Fahlgren et al., 2021). This development is crucial in Mathematics education, where the ability to evaluate deeper cognitive processes is essential for fostering a more comprehensive understanding of the subject matter. With many CAA systems, it is possible to generate graphs and insert them into questions, answers, and feedback, so in turn to design tasks that allow students to interact with graphs, thus opening the way to different cognitive processes (Sangwin, 2015).

This study is framed within a joint research project between two universities, Karlstad University (KAU) in Sweden and University of Turin (UNITO) in Italy, about CAA in Mathematics (Fahlgren et al., 2024). The researchers developed digital tasks for formative assessment to enhance mathematical competences. During the academic years 2022/2023 and 2023/2024, tasks on calculus have been experimented within first-year university mathematics modules using graphical tasks involving limits and asymptotes of functions solved by students as group activities. The aim of this

paper is to show how the tasks were designed and to understand how working on similar tasks supported students' abstract reasoning.

Theoretical background

Computer-aided assessment is compelling in formative assessment (FA) since the enhanced Mathematical capabilities of CAA systems support using many FA strategies (Black & Wiliam, 2009). The possibility to personalise grading codes allows for the automatic grading of example-generation tasks, which ask students to provide examples of mathematical entities under certain conditions (Yerushalmy et al., 2017). Solving example-generation tasks is usually a non-routine problem-solving activity since the solution cannot be calculated through standard procedures (Watson & Mason, 2005). According to Watson and Mason (2005), the students' ability to provide different and varied examples that meet the given criteria is an index of their level of understanding. Thus, these tasks are a way to elicit evidence of students' deep understanding of mathematical entities.

CAA systems' algorithmic capabilities support the creation of graphs that can be included in the tasks. Graphs can depend on random parameters and vary at each attempt and for each student. This enables the design of graphical tasks, which are mathematics assessment items containing a graphic deemed integral to the solution process (Lowrie et al., 2011). Specifically, the graphic element contains information that needs to be decoded to solve the task. Graphical tasks are often used to foster students' sensemaking (Lowrie et al., 2012). When the topic of a graphical task is calculus, decoding the information from a graph involves analysing the representation of a function in a graphic register and translating the information into a different register (symbolic or numerical). This practice can help students make connections and enable abstract reasoning (Adu-Gyamfi et al., 2017). Abstract reasoning may be described as the ability to think conceptually, by understanding complex ideas and spotting patterns or relationships without having concrete examples to base on (Staub & Stern, 1997).

In CAA items, graphs can also play a relevant role in delivering feedback. For instance, in an example-generation task that asks students to write the expression of a function, showing the graph of one possible correct function is a way to provide implicit feedback (Sangwin et al., 2009). Students need to analyse the graphical feedback and decode the relevant information to try it again. Once again, the feedback analysis involves translating from one representation to another and vice versa. Where the solving process is more complex, the feedback could include a screencast and be proposed in the form of a silent video. Robinson et al. (2015) show that video feedback in the form of a screencast is effective according to Sadler's feedback definition (1989). In particular, it prompts action, which helps students close the gap between current and reference performance, thus developing mathematics understanding.

Method

This study aims to answer the following research question: *How does working on graphical tasks with graphical feedback on calculus in a formative group activity support abstract reasoning?* To do so, we set up international research involving students from two distinct post-secondary educational contexts: a mathematics module in the first year of a bachelor's degree in biotechnology at UNITO and a mathematics module in the first year of a bachelor's degree in engineering at KAU. In UNITO,

the module deals with basic calculus, linear algebra, and probability/statistics, while in KAU, the module only covers basic calculus. The international research group designed tasks through a CAA system that students in both countries could engage with during the course as group activities. This study is focused on one part of the group activities, which consisted of tasks about asymptotes: these results are a small part of a larger set of data under analysis. The tasks are described in Table 1. Items 1 and 3 were conceived as group tasks, while tasks 2 and 4 were individual: each student had different random parameters, which made the solution vary student by student; however, they were encouraged to discuss their solutions in groups. The group activities were eligible for UNITO students and compulsory for KAU students.

Table 1: Description of the group tasks about asymptotes

Task 1 and Task 3 – Group	**Task 2 and Task 4 – Individual**
An image represents the graph of a function. Students must use the graph to determine the analytical expression of the function. **Task 1** The image represents a function which has two vertical and one horizontal asymptotes. **Task 3** The image represents a function which has one vertical and one oblique asymptote.	Write two different functions such that they both have the following asymptotes: **Task 2** Two vertical asymptotes x=a and x=b, A horizontal asymptote y=c. **Task 4** A vertical asymptote x=a An oblique asymptote y=bx+c Note: a, b and c are random parameters
First feedback: Students can preview the graph of their response before submitting.	First feedback: Students can preview the graph of their response before submitting.
Second feedback: Students can verify their solution and, in case of an incorrect answer, watch a silent video feedback of the resolution process on another similar function.	Second feedback: The students in the group have different parameters, and they can discuss the solving process with their peers.

Tasks 1 and 3 are graphical tasks since students have to use information in the graphic register to solve them. The solution process includes translating the information into a symbolic register to provide the expression of the function. They are provided with implicit feedback that allows them to change their answer before submitting it, stimulating self-assessment. Moreover, silent video feedback emphasizes the abstract definitions for asymptotes and shows one possible solution process for those who failed the task. Tasks 2 and 4 are example-generation tasks conceived to encourage the expansion of students' example spaces by asking for two different examples, formulated using only the symbolic register; implicit feedback in the form of a graph is given before submission. The idea is that reasoning on the graphs in tasks 1 and 3 can support abstract reasoning in tasks 2 and 4.

Though different in the two modules, the final exam includes a common task about asymptotes. For UNITO students, it consists of one item among the following choices, where "ungiven" means that the asymptote(s) had to be of proper kinds and/or in the right number, but without requiring to satisfy a precise equation. The numbers in square brackets refer to how many students from KAU or from UNITO in the datasets had to answer to each choice:

Give an example of a function that has

- *(1) exactly 1 vertical asymptote and 1 horizontal asymptote* (ungiven) [220 KAU, 50 UNITO]
- *(2) the horizontal asymptotes y=a and the vertical asymptote x=b* [108 UNITO]
- *(3) at most one asymptote* (ungiven) [25 UNITO]
- *(4) exactly one vertical asymptote and one oblique asymptote* (ungiven) [14 UNITO]
- *(5) at least one asymptote* (ungiven) [4 UNITO]
- *(6) at least two vertical asymptotes x=a and x=b* [8 UNITO]

Note that an *asymptote* of a function is defined as *a line such that the distance between the function and the line approaches zero as one or both of the* x *or* y *coordinates tend to infinity*. This includes the possibility that a function coincides partly or completely with a horizontal or oblique asymptote.

During the exam sessions, students may be asked to justify their responses using limits. Similarly to tasks 2 and 4 of the group activities, the exam tasks did not include any graphs, so the required reasoning is purely abstract. In UNITO, the exam is delivered through the same CAA system used in the group activities, and items use random parameters *a* and *b* to provide variety in the students' exams. In KAU, the paper-and-pen based exam date is unique, so only Choice 1 was delivered.

Our datasets are based on KAU and UNITO students. For the UNITO ones, we considered the 209 students taking the exam between January 2023 and July 2024. For the sake of statistical significance, we mainly focused our analyses on choices 1–3, nevertheless taking into account choices 4–6 for interesting insights. For KAU students, we considered 220 of them taking the exam for a course in Mathematics in November 2022, all carrying out choice 1. For all students taking the exams, we analysed whether and how they worked on the group tasks about asymptotes to understand if the graphical group tasks with graphical feedback supported abstract reasoning during the exam.

We analysed the KAU and UNITO data quantitatively and qualitatively in the spirit of mixed-method research. Namely, the quantitative analysis involved relating the performance on asymptotes at the exam with either having taken the respective group activity (for UNITO data) or the outcome of the whole exam (for KAU data), and justifications. We categorised the justifications as entirely correct, generally correct with minor imprecisions (such as not splitting left/right or –inf/+inf limits, or saying that asymptotes can be computed only at the domain extrema), and incorrect. The qualitative analysis is devoted to the *kind* of functions students provided as answers, categorised as polynomial, homographic, higher-degree rational, and non-rational; some examples are explicitly presented to show peculiar alternatives.

Results

Let us start with the results relative to UNITO students. First, we want to establish whether taking the group activity about asymptotes, with FA including graphic tasks with graphical feedback, positively

influenced the final exam task results regarding the same topic. By considering choices 1–3, we detected a total of 102 students (56%) correctly answering the exam task, while 81 students (44%) answered incorrectly. If we subdivide them according to having taken the respective group activity or not, we found 74 correct (61%) and 47 incorrect (39%) answers among students who took the activity. In comparison, 28 correct (45%) and 34 incorrect (55%) answers were registered in relation to students who did not take the activity. The proportions between the two groups are quite different; this difference can be confirmed as statistically significant through a Fisher test for contingency tables, which yields a p-value of 0.043. This significance does not imply a causal relationship. However, it suggests that taking the group activity could have helped the students correctly solve the abstract task during the exam, given that several weeks (19–67 days; median: 35.5) passed between the activity and the exam.

We now analyse how correctly students justified their answers. By restricting our analysis to the sessions relative to choices 1–3 when we asked for justifications, we obtained a sample of 75 students relative to choices 1 and 3, of which 53 took the group activity and 22 did not. Only seven of them (all taking the activity) provided an entirely correct justification; a larger number justified their correct answer generally well but with either not splitting left/right or -inf/+inf limits (29/75: 20/53 and 9/22) or making other minor imprecisions (10/75: 7/53 and 3/22). A few students provided an incorrect justification to a correct answer (7/75: 6/53 and 1/22), while the remaining ones answered incorrectly (20/75: 13/53 and 7/22) or did not answer at all (two, both not taking the activity). We see that the best outcomes are associated with carrying out the graphical tasks with the graphical feedback in the group activities. At the same time, those who did not answer correctly gave it a try.

A subsequent analysis regards the *kind* of answers students gave when asked to construct functions with certain asymptotes. Depending on the precise task, we detected the following.

- In choice 1, most students answered with a homographic function (a ratio between a constant or a linear polynomial and a linear polynomial, not multiple of each other). However, two students provided correct answers using an irrational function 3/sqrt(x) and a higher-degree rational function $(x+2)/(x^3-1)$. Other students answered with non-rational functions such as ln(x) and arctan(x), but their answers were marked incorrect.
- In choice 2, most students answered with homographic functions, too. Other correct answers were either non-rational functions such as exp(-x)/(x-1)+3, -exp(x)/(x+1)+6, ln(x/(x+9))-9 (they also depended on which exact asymptotes were given), or higher-degree rational functions, possibly with a supplementary vertical asymptote (we did not ask for *only* $x=b$).
- In choice 3, students tended to answer with homographic functions, but in this case, their answers were marked incorrect since these functions have *two* asymptotes. The correct answers were mainly non-rational functions like ln(x) and exp(x), with a student providing the higher-degree rational function $x^3/(x-2)$. Notably, only one student chose the simplest function for this case: a polynomial. Indeed, constant and linear functions have one asymptote (themselves), while higher-degree polynomials have none. Students likely struggled with the idea of asymptotes coinciding with the function and missed that "at most one" includes "zero".
- Regarding choices 4–6, the requirement of exactly one vertical and one oblique (slant) asymptote was met by students with rational functions (second-degree numerator, linear

denominator), with the noteworthy exception of $(x^2+\text{sqrt}(x-2))/(x-3)$. The requirement of at least one asymptote brought again to homographic functions; requiring at least two vertical asymptotes, both given, was satisfied with rational functions (constant numerator, second-degree denominator). In this latter case, some incorrect answers were non-rational functions.

It can be noted that for choice 1, the functions $3/\text{sqrt}(x)$ and $(x+2)/(x^3-1)$ were both answered by students taking the group activity: all the others who answered correctly used homographic functions, showing less variety. This is also true for choice 3 and the functions $\ln(x)$, $\exp(x)$, $x^3/(x-2)$; the student answering the polynomial did not take the group activity, but non-polynomial answers, all responded by students taking the group activity, were varied. The situation is more balanced for choice 2, while little data was collected for choices 4–6. Data suggest that taking the activity might have allowed students to consider larger example spaces.

Let us now consider the results relative to KAU students. While we cannot directly relate having taken the group activity to the exam outcome, since it was a mandatory activity, we could put in relation the performance on the tasks on asymptotes with the outcome of the entire exam, where all the items concerned basic calculus and abstract reasoning. Indeed, this also allows us to detect the quota of students who were weak theoretically but were able to perform well in the exam on the asymptotes task. Of the 220 students considered, 125 (57%) managed the tasks on asymptotes well and passed the exam, but 16 of them (7%) would have failed the exam without the points for the asymptotes task, after rescaling the scores. This results in a total of 62 (28%) students performing well on asymptotes but failing the exam (46, 21%) or that would have failed it after rescaling, versus 109 (50%) managing asymptotes well and that would have passed the exam even by rescaling. About those not performing well on asymptotes, 8 (4%) passed the exam (all of them narrowly), and 41 (19%) failed the exam (with very low rescaled scores, showing low engagement in the whole course). Fisher's exact test yields a p-value of almost zero ($3.4 \cdot 10^{-9}$), while Cramer's V has a value of about 0.4, highlighting how being engaged well in tasks on asymptotes is related to having success on the exam, even by excluding the asymptotes task itself. This can indicate that participation in the group activity gave some moderately weak students more chances at the exam: working with graphical examples with automated feedback can improve even the abstract understanding needed in the written exam without computer support. Still, it remains more challenging to help very weak students.

If we extend our analysis to the justifications, 69 students provided an entirely correct one, with 5 of them (7%) failing the exam, 30 of them made minor imprecisions with 4 (13%) exam fails, 57 provided an incorrect justification to a correct answer with 31 (54%) exam fails. Thus, there is also a strong relation between having success on the exam and justifying correctly the answer on asymptotes beyond answering correctly. The ability to provide a theoretical explanation and connect it to the graphical representation suggests an improved abstract understanding. However, this improvement was not as evident in topics not addressed through group activities, as most students taking the course obtained only moderate grades in the exam, indicating a weaker understanding of other theoretical aspects. Regarding the kind of answers students gave, almost all the correct answers were expressed through homographic functions, but a student answered with the ratio of 2 third-degree polynomials. Incorrect answers included $\exp(x)$, $(x-1)^2 \cdot (y-3)^2$ (bivariate), and rational non-homographic functions.

Discussion and conclusions

The results provided some indications to answer the research question about *how working on graphical tasks with graphical feedback on calculus in a formative group activity supports abstract reasoning*. Indeed, the quantitative data can suggest that students performing the collaborative group activities concerning asymptotes were significantly more able to answer correctly and with proper justification to the respective exam question (UNITO group). In turn, students answering correctly and well justified to that exam question were significantly more able to pass a basic calculus exam, while that question was crucial for the exam success of several students (KAU group). At the same time, the qualitative data from the UNITO group show that students' answers at the exam resulted in rich example spaces, which were generally more varied for those performing the activities. This aligns with (Adu-Gyamfi et al., 2017), where connections and abstract reasoning are fostered by working on graphs in calculus. The results could suggest that the design of the group activities about asymptotes, which includes graphical tasks with graphical feedback, effectively conveyed a way of reasoning that enabled students to perform well in abstract tasks. The major limitation is that we cannot be sure these associations imply causation, and we cannot rule out possible other causes: for instance, the activities could have caught the attention mainly of already inclined students. As for future work, the first direction could be a more direct comparison between the two countries. Indeed, a relation between performance on asymptotes (either at the group activity or at the exam) and the whole exam would have been less significant for UNITO students since the course is broader than the KAU one. Conversely, for KAU students, given the compulsory activities, the performance on asymptotes at the exam can be related to their (formative) grade rather than whether they are being taken. Another direction could be administering analogous activities on different topics to activate diverse facets of abstract reasoning, thus overcoming the limitation of considering a specific topic (asymptotes). Lastly, the answers to the formative assessment tasks could be analysed to understand the students' reasoning and their common mistakes and misunderstandings. Based on this analysis, the tasks and feedback can be adjusted for higher effectiveness.

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Development of screening instruments for mathematics education: Theoretical Framework and item development

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The Erasmus+ funded project "Diagnostic Tool in Mathematics (DiToM)" addresses the need for national and international diagnostic instruments to assess mathematical key skills which are necessary to acquire competencies that build on them. Given the increasing proportion of countries struggling with decreasing students' performance in mathematics (e.g. France, Germany, Spain) (Mullis et al., 2020), the project focuses on identifying students at risk of being left behind in the following and advanced mathematical content. The project consortium consists of experts from seven European countries. The theoretical framework, i.e., the concept of mathematical key skills, forms the foundation of test development compatible with the similarities and differences in national curricula. Test development is criterion-based and involves creating an item pool and country-specific diagnostic screenings derived from it.

Keywords: DiToM, screening, mathematical key skills, criterion-referenced approach

Introduction

Results from national surveys in Germany indicate an increasingly high proportion (2016: 15.4%; 2018: 24.3%) of students failing to meet the minimum standards in mathematics (Stanat et al., 2019; Stanat et al., 2022). Similar tendencies and stagnant trends can be found for several European countries in the results of the international PISA studies (OECD, 2020). Such difficulties often stem from inadequate understanding of foundational mathematical concepts, leading to persistent issues in more advanced topics (Gaidoschik, 2019). Research by Ehlert et al. (2013) underscores the necessity of a solid grasp of basic arithmetic for successful progression in mathematics education. Humbach (2009) further emphasizes that more sophisticated mathematical understanding builds upon these fundamental skills, suggesting that basic arithmetic knowledge is a critical prerequisite for mastering more complex content (Humbach, 2009). Arithmetic skills are not only a prerequisite for further learning in mathematics but also serve as a significant predictor of success in the STEM field (Nakakoji & Wilson, 2014). In the context of school mathematics, a significant connection exists between arithmetic and algebra. Algebra, as it is commonly taught in schools, can be viewed as an generalization of arithmetic principles (Nydegger-Haas, 2016). Due to this close relationship, for instance, the German core curriculum, represents these topics under a combined content area titled *Arithmetic and Algebra* (MSB, 2022). Since these domains are closely intertwined, they are similarly

structured in many curricula across different countries. Errors made by students in algebra can often be attributed, at least in part, to faulty attempts to apply or adapt previously acquired knowledge from arithmetic to new situations (Kieran et al., 1990). There is a lack of national and international diagnostic instruments for continuous assessment at an early stage of the development of that basic content in arithmetic and algebra to effectively address the growing heterogeneity.

The aim of the paper is to build on this interface and to illustrate the current status of the project, i.e. how the DiToM project (Diagnostic Tool in Mathematics) meet the gap between the theoretical framework and the development of prototypical testitems. Therefore first, the DiToM project will be presented in its essential aspects. Following this, the theoretical framework of mathematical key skills will be examined and illustrated with concrete examples. Subsequently, the understanding of screening as part of a diagnostic process will be explained. In the methodology section, the process of item-shaping for one of the screening tests (grade 6) is stated. Finally, a summary will be provided, along with an overview of the next steps in the project.

The DiToM project

DiToM is a project that started in January 2023 as part of an Erasmus+ Cooperation Partnership (DAAD; KA220-HED-1194D587). The project consortium consists of Partners from the following seven countries: Germany as the coordinating partner, France, Sweden, Italy, Greece, Croatia and Spain. The project is structured into five work packages to achieve the project goals of test development, free availability of the tests via a website and dissemination of the tests. The tests are being developed for grade levels 0, 2, 4, 6 and 8, focusing on arithmetic and algebra and will be available in 2025. The official website of the project is: https://www.ditom.org.

The focus of the diagnostic tools is on arithmetic and algebra due to their indicative nature, particularly concerning the correlations in difficulties and vulnerabilities observed in learners' mathematical performance (Humbach, 2009). Additionally, the linear and hierarchical structure of these content areas (Ehlert et al., 2013), which is a common feature among participating partner countries and their curricula, further underscores the rationale for this emphasis. The intended purpose of the tests is to provide a binary risk assessment of learners' arithmetic and algebraic skills regarding “at risk” and “not at risk”.

Theoretical Framework

When developing diagnostic instruments, both the content to be assessed and the expectations placed on that content must be considered. Nowadays, the general agreed-upon expectation is that students develop operational knowledge. Operational knowledge is defined as "the capacity to draw on mathematical tools to deal with new and potentially problematic situations, rather than merely the capacity to reproduce procedures learnt in relatively stable contexts very similar to those in which they were taught" (UNESCO, 2012, p. 17). Niss and Højgaard (2019, p. 12) define mathematical competence as the "insightful readiness to act appropriately in response to all kinds of mathematical challenges pertaining to given situations". At the same time, a purely content-based description of competence, such as "being able to add two whole numbers", generally implies implicit competencies that are expected. These considerations require a transparent understanding of the terminologies of

mathematical competence and *competencies*, *mathematical concepts* and *mathematical skills* and more precise *mathematical key skills* to be articulated within the DiToM project. In the present project phase, the following understandings of the terms can be inferred: (1) *Mathematical competencies* are understood as the "insightful readiness to act appropriately in response to a specific sort of mathematical challenge in given situations" (Niss & Højgaard, 2019, p. 12). On the one hand, there are competencies of the first category that pertain to effective handling of posing and answering mathematical questions (e.g., mathematical thinking, mathematical problem-solving). On the other hand, competencies of the second category encompass abilities that are necessary for the conduction of mathematical activities (e.g., symbols and formalism competency, communication competency). (2) *Mathematical concepts* are understood as "a researcher's articulation of intended or inferred student knowledge of the logical necessity involved in a particular mathematical relationship" (Simon, 2017, p. 123). An example of an articulation of a concept could be "Understanding the addition of two-digit numbers" in which addition is limited to a specific type of task, involving the comprehension of the relationship between two-digit numbers and the addition operation. (3) *Mathematical skill*: Establishing a suitable link between mathematical competencies and mathematical concepts can evolve into mathematical skills. They can be seen as learned and applied abilities that lead learners to use mathematical knowledge effectively in some execution or performance. For instance, "mental calculations of the sum of two-digit numbers" or "articulations of the process of calculating two-digit numbers" can be seen as mathematical skills. (4) *Mathematical key skill*: A mathematical skill can be regarded as a mathematical key skill if the lack of such skill has the potential to make further learning difficult or even impossible (Gagné & Briggs, 1974). Lacking an internalization of mathematical concepts and the corresponding mathematical competencies, learners will be unable to, for instance, grasp and employ arithmetic algorithms or comprehend the principles behind decimal number notation. In the following, basic arithmetic operations are used as an example to illustrate the concept of mathematical skills and key skills. Basic arithmetic operations (addition, subtraction, multiplication, division) are elementary for dealing with mathematics (Humbach, 2009). Particular mathematical concepts are connected with the basic arithmetic operations: (1) Understanding of the operation (like "operation sense", "mental models"), (2) number facts, (3) calculation strategies, (4) understanding and use of the algorithms and (5) estimation. Comparing the DiToM countries, these concepts or part of these concepts are established in all of their curricula. The process students use to learn number intervals may be different for different grades, but the mathematical learning process takes these concepts into account. Based on these concepts, competencies are formulated - also in curricula - which indicate the learning progress expected at certain points in time. This can take the form of processes (e.g. students select and use appropriate mathematical rules and algorithms when working on problems) or content (e.g. students discover, use and describe operational properties (e.g. invertibility) and laws of arithmetic). Often, competencies are also described from both aspects. The mathematical skills describe the combination of concepts and competencies in this field of basic arithmetic operations concerning structuring, understanding and use of different situations in this field. For example, the use and explanation of structured addition, the use of basic operations in applied situations. Mathematical key skills in this

field focus on the skills that are necessary for the further learning process. For example, grade 2 students solve addition and subtraction tasks in the number range up to 100 using arithmetic laws and decomposition strategies orally or semi-orally (also using intermediate forms) in a reflective way. Consider the following tasks (for early learners):

- **Task 1:** 25 + 34 = ___ **Task 2:** 25 + 37 = ___

Task 1 involves addition without a tens transition and is considered a key skill for grade 3, reflecting the need for students to manage an expanding number range and increasingly complex operations. Task 2 introduces a tens transition, which further challenges students and illustrates a progression in mathematical complexity. This transition in the number range up to 100 is a mathematical key skill and therefore integrally related to the transfer to operations with more complex numbers. In the development of this screening test, it is essential to determine which skills are integral to mathematical key skills and which of them may hinder further learning progress.

Screening as part of a diagnostic process

The diagnostic tools developed within the DiToM project are specifically designed as screening instruments. These screening instruments aim to identify at-risk learners efficiently and with minimal resource expenditure, thereby targeting students who require additional support to advance their mathematical learning (Tröster, 2009). The design of these test instruments ensures that they can be administered to an entire class within a single lesson, making them both time-efficient and cost-effective. Screening procedures have the aim of identifying potential risks earlier than traditional methods, thus providing an early intervention opportunity (Breitenbach, 2020). These procedures are characterized by a relatively low depth of analysis, focusing instead on capturing early indicators of developing deficits (Tröster, 2009). As a result, they are well-suited for large groups, offering an economical means of assessment that minimizes student burden. The filtering capability of these tests allows for the identification of students who may require further diagnostic evaluation, thus establishing a basis for well-founded suspicions of risk (Walter, 2020). In the context of this project, the screenings concentrate on domain-specific evaluations of mathematical concepts within arithmetic and algebra. Early indicators for these screenings are defined based on the key skills identified due to the theoretical framework. The screenings follow a 2-year rhythm to match the curricula, which in most participating countries are structured in 2-year phases. The tests are designed for use at the beginning of the first grade to assess the entrance requirements for mathematics education and continue up to grade 8, before students obtain their first school-leaving certificate. These tests are administered at the end of each two-year cycle. The primary objective is to use these screenings as a preliminary diagnostic tool to pinpoint students at risk, thereby facilitating a more detailed and targeted follow-up diagnosis for those identified.

Design of prototypical test items

The objective of this paper is to offer a theory-based and transparent explanation of the process involved in transitioning from theoretical framework regarding mathematical skills and key skills to the development of items, by giving a practical example from a test for grade 6. This section outlines

the approach used to design the tests, detailing the steps from theoretical framework to the creation of prototypical test items.

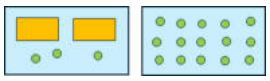
Criterion-referenced and norm-referenced approaches

In educational diagnostics, two primary approaches dominate the development of tests: norm-referenced and criterion-referenced approaches (Dunn et al., 2003). Norm-referenced assessments evaluate students in relation to their peers within a specific cohort, relying on a pre-determined distribution of performance. Such tests are commonly employed in contexts where placement or selection decisions are required (Burton, 2006). In contrast, the criterion-referenced approach assesses students' performance against a predefined set of criteria (Le Brun & Johnstone, 1994). The purpose of DiToM is to identify students at risk in mathematical key skills who require immediate support from their teachers. The diagnostic assessment in DiToM is conceptualized as a form of screening, aimed at identifying students at risk in acquiring key skills in arithmetic and algebra as outlined and defined in the theoretical framework. Consequently, the criterion-referenced approach (CRA) is most suitable for the objectives of the DiToM test. According to the CRA, learning is evaluated using a predefined set of criteria and corresponding performance standards, written descriptions of what students are expected to know and be able to do at a specific stage of education. For each key skill, these descriptions are organized into rubrics, the levels of which correspond to specific grades that may vary across countries. In this paper, the CRA is applied to the 6th grade test development as an example. The following methodological steps were undertaken: (1) Identifying relevant content: Relevant content was determined through didactic models of competency development and analyses of various national curricula. (2) Identifying corresponding mathematical skills and key skills: Mathematical skills and key skills were derived from the identified content. (3) Creation of prototypical items: Prototypical items were developed based on these key skills. (4) Item review and refinement: The prototypical items were reviewed within the consortium, contextually adapted and evaluated by a focus group of a minimum of two teachers from each participating country (n=24), leading to further refinements due to approval threshold of $\geq 80\%$ for the relevance of the item as a *key skill* (Lynn, 1986). (5) Trial of the items in order to obtain empiric data, leading to further refinements.

Exemplary execution of the test for grade 6

For the 6th grade test, three relevant content areas were identified: (1) algebra and pre-algebra, (2) proportionality and linearity and (3) technical calculations. These areas were selected based on a thorough analysis of the national curricula of the participating countries. The item review revealed that these domains cover the core mathematical content expected for 6th grade students in arithemtic and algebra. In the first step, specific mathematical skills were developed for each of these domains. These skills represent key mathematical competencies that students at this grade level are expected to acquire. In the second step, key skills were derived from these, which are considered necessary prior knowledge required for the successful development of the targeted skills (see Table 1). These key skills serve as the foundation for the development of test items and are closely aligned with the curricular expectations and competency standards of the participating countries. The absence of these key skills hinders the understanding of specific mathematical skills.

Table 1: Linking of mathematical key skills, mathematical skills and prototypical test items in the domain *Algebra and Prealgebra*

Macro Area	Lower skill (mathematical key skill)	Exemplaric item based on the key skill	Upper skill (mathematical skill)
Calculating	Calculating (numeric) - Recognize equality sign as an equivalence relation - Complete an addition or a multiplication	Find the missing numbers. a.) $3 \cdot \_\_\_ = 126$ b.) $\_\_ \cdot 4 = 172$ c.) $24 \cdot 6 = \_\_$	- Understanding equality - Understanding equivalence - Solving equations
Generalizing relationships	Generalizing a relation without necessarily producing a symbolic expression - Arguing or motivating an answer involving regularities	Which is the rule that can be used to continue this sequence of numbers? $256, 128, 64, 32, \_\_$ A. Substract 32 B. Substract 128 C. Divide by 4 D. Divide by 2	- Symbolic generalization of relations
Translating	Translating between different registers of representation (iconic, arithmetic, natural language…).	Tom follows the instrutions: The number 4 is added to 5. The result is multiplied by 8. What calculation can Tom use to obtain the result? A. $5 + 4 \cdot 8$ B. $(5 + 4) \cdot 8$ C. $5 + (4 \cdot 8)$ D. $5 \cdot 8 + 4$	- Translating data to symbolic representation - Understanding the structure of an expression
Problem solving and Modelling	Solving problems and modelling relationships between known and unknown quantities in an additive and/or multiplicative problem - Solve numerically a problem that can be modelled as a linear equation	The picture shows marbles, represented by green discs, and boxes, represented by orange rectangles. There are placed on two tables. Each box contains the same number of marbles. There are the same number of marbles on each table. How many marbles are there in each box? 	- Problem solving - Representing relationships between data and unknown

Thus, the test addresses the relevant mathematical domains and ensures that the items are adapted to country-specific curricular differences while maintaining consistent diagnostic validity across the entire student population. The focus group results led to revisions and a reduction of the prototypical item pool in clear cases (less than 50% agreement among experts on an item).

Summary and Perspectives

The next steps of the project involve finalizing the prototypical itempool of the screenings for all grades. Building upon this, the diagnostic tools will be developed, piloted and adjusted until mid 2025. In DiToM, the interrelation between mathematical competencies, concepts, skills and key skills are deemed essential. Key skills are necessary for learners to proficiently engage with associated mathematical content and subsequently build upon them to develop fundamental contents based on them. Additionally, the focus on the development of screening instruments and the emphasis on CRA were essential for the ongoing work. To develop screenings that can be used in all participating countries based on these initial decision-making processes, the following approach is being pursued: Mathematical competencies are derived from the literature and then enriched with mathematical concepts of the domain of arithmetic and algebra, which can be used to develop test items. This process results in a basic prototypical item pool. The binary categorization of the screenings allows for assigning learners' performance in terms of whether key skills have been acquired or not. Based on this pool of tasks, country-specific tests are developed according to the alignment with national curricula.

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Exploring instrumental orchestration practices for formative assessment through digital technology

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This study investigates how teachers orchestrate student learning by using technology as well as formative assessment practices in classroom settings, using instrumental orchestration and formative assessment with technology frameworks. Our research question is: how can teachers conduct formative assessment as part of their orchestration practices when using digital technology? This study analyses twelve videos from technology-rich mathematics classrooms in China. The results show frequencies and relations of orchestration and formative assessment practices observed in the videos, and that orchestration practices are predominantly teacher-led, involving supporting direct instruction and providing individualised feedback.

Keywords: Instrumental orchestration, formative assessment, digital technology.

Introduction

Digital technologies have greatly increased the possibilities and potential for mathematical teaching and learning as well as for formative assessment (Weigand et al., 2024). In the context of formative assessment, integrating technology not only provides teachers with immediate feedback on student progress, enabling real-time monitoring of understanding, but also aligns with pedagogical goals to make the formative assessment process more dynamic, responsive, and supportive for student learning trajectories (Dalby & Swan, 2019). However, Drijvers and his colleagues (2013) emphasise that despite the potential of digital technologies to enhance teaching and learning, their adoption by educators does not necessarily lead to improved outcomes. This suggests that simply having access to digital tools is not sufficient—what matters is how teachers use them in response to students' needs. The challenge for teachers in using technology lies not in the technology itself, but in understanding how it can be adapted to enhance student learning, highlighting the need for educators to carefully orchestrate and guide the students' learning processes.

Building on this view, while digital technologies offer new possibilities for formative assessment, their use in classrooms still relies on how teachers manage the integration of these technologies in classroom settings. To better understand this process, *Instrumental Orchestration Theory* offers a framework for analysing how teaching practices shape students' interaction with digital tools. This theory conceptualises a teacher's role in guiding students toward mastering tools and facilitating their learning process (Trouche, 2004). It emphasises how teachers can intentionally and systematically integrate digital technology into their teaching to support students' instrumental genesis, the dual process in which learners adapt tools to their needs while being influenced by the tools in their cognitive and practical activities. Teachers play an important role in orchestrating this process by designing learning environments that support the interaction between learners and tools, ensuring that instrumental genesis unfolds meaningfully (Drijvers et al., 2010). Panero and Aldon (2016) further argue that a teacher's orchestration skills, developed through the appropriation of technology use for formative assessment, are cultivated alongside the students' instrumental genesis. The relationship

between instrumental orchestration and formative assessment with technology could be synergistic, integrating digital technology to create a better way to evaluate and support student learning.

Research has explored digital technology's general capabilities, but few studies have examined the integration of digital technology into live formative assessment, highlighting a gap between ideal and practical implementations. For example, Børte and colleagues (2023) report that teachers struggle to adapt digital technology within their existing formative assessment practices, which not only results in a misalignment between digital technology and formative assessment but also creates difficulties in aligning digital technology with pedagogical methods. This misalignment leads to a missed opportunity to fully explore the potential of digital technology for student-centred learning. To address these challenges, this study investigates the relationship between digital technology, instrumental orchestration, and formative assessment in mathematics education, focusing on how formative assessment is integrated into orchestration practices when using digital technologies.

Theoretical Background

This study adopts the definition of formative assessment from Black and Wiliam, who describe it as an assessment that "elicits, interprets, and uses evidence about student achievement…to make decisions about the next steps in instruction" (2009, p. 9). This definition places emphasis on the continuous, responsive nature of assessment and highlights the teacher's role in adapting instruction based on student needs. In exploring how such processes can be supported through digital technology, this study draws on two theoretical perspectives:

Instrumental Orchestration Theory was developed to shed a light on how the integration of technology into whole-class instruction can be orchestrated in various ways. According to Drijvers and his colleagues (2010), teachers demonstrate technological tools during *Technical-demo*, explain on-screen activities to the whole class during *Explain-the-screen*, or foster discussions about on-screen content during *Discuss-the-screen*. Other orchestration types include highlighting relevant student work for whole class discussion as *Spot-and-show*, connecting digital activities to their representation in traditional mathematical formats as *Link-screen-board*, and asking *Sherpa* students to use technology to perform tasks guided by the teacher as *Sherpa-at-work* (Trouche, 2004). Furthermore, teachers' walking around to provide individual feedback that is referred to as *Work-and-walk-by* (Drijvers, 2011), and teachers' providing screen-based explanations and posing closed questions that is referred to as *Guide-and-explain*, or teachers' reliance solely on traditional board instruction without digital technology is termed *Board-instruction* (Drijvers et al., 2013).

Formative Assessment with Technology, drawing on the work of Dalby and Swan (2019), explores how digital tools can be integrated into formative assessment to support classroom teaching and learning. The framework outlines a circular process consisting of four interrelated stages: *Ask*, *Answer*, *Analyse*, and *Adapt*. The flow diagram (see Figure 1) shows how the four stages connect in practice and helps clarify how technology can be part of classroom formative assessment routines.

Taken together, these two frameworks offer complementary lenses: one emphasises how teachers manage digital tools during instruction, and the other focuses on how technology can support more dynamic and responsive assessment. This study connects these perspectives to examine how

formative assessment is shaped within technology-rich classrooms, and how orchestration practices contribute to that process.

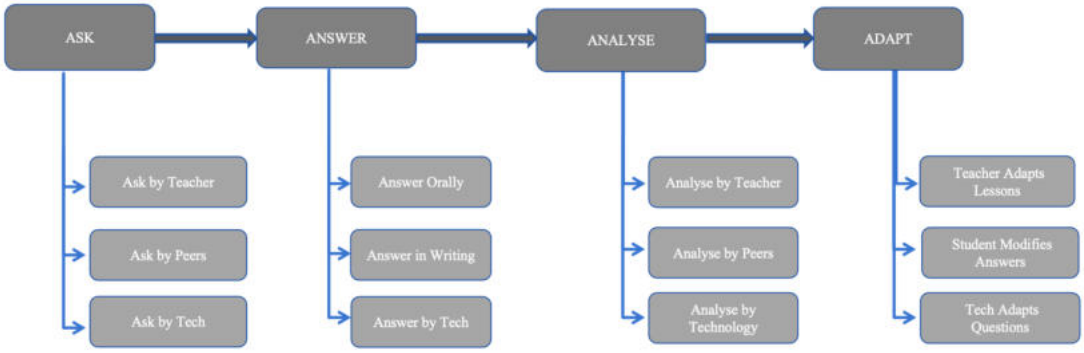

Figure 1: Formative assessment with technology, adapted from Dalby and Swan (2019)

Method

This study investigates through video analyses how secondary school mathematics teachers from Jiangsu Province, China, integrate digital technologies into their teaching practices, including interactive digital whiteboards, tablets, mathematical visualisation tools, software like Desmos and GeoGebra, and immersive technologies such as Augmented Reality. We choose twelve videos of award-winning lessons from the 2022 "*Jiangsu Province ICT-integrated High-Teaching Quality Course Competition*" for observation and analysis. These videos represent exemplary lessons, rather than typical classroom instruction, designed to demonstrate considered best practices in integrating technology within mathematics education. Thus, we expect that the videos could illustrate successful examples of technology-rich classroom practices. The research question of this study is: *How can teachers conduct formative assessment as part of their orchestration practices when using digital technology in mathematics classrooms?*

When analysing the data, we focus on understanding how the orchestration interventions employed by teachers provide them with opportunities to monitor the progress of their students and to assess their work. Initially, the identification of orchestration types was conducted using the instrumental orchestration framework (Drijvers, 2011; Drijvers et al., 2010; Drijvers et al., 2013). Subsequently, the analyses applied Dalby and Swan's (2019) framework to explore the teachers' formative assessment practices. Finally, we identified one significant event showing the potential of specific orchestration for formative assessment practice. The coding process was conducted via Atlas.TI, which is a qualitative data analysis software. The unit of analysis was defined as an *episode*, which includes the whole-class treatment of a single task within the video. These episodes were further divided into sub-episodes, if multiple orchestration types were observed in a sequence. Formative assessment instances were identified only when technology was explicitly used to evaluate student understanding, such as gathering evidence on student learning (e.g., digital polling for immediate feedback), analysing this evidence (e.g., reviewing digital submissions for accuracy), or adapting instruction based on technological results (e.g., adjusting tasks based on displayed responses).

Inter-coder reliability was established by having a second coder review three videos with predefined quotations. After an initial coding of one video, the coders reviewed discrepancies in their coding decisions. Through discussion, the coders resolved disagreements, such as whether an adaptation stage of the formative assessment process occurred during the lesson. After reaching a consensus on the coding, the second coder analysed the remaining two videos, resulting in a Krippendorff's alpha score of 0.695. This indicates moderate agreement, which is acceptable given the interpretive complexity of the videotaped classroom interactions.

Results

Table 1 provides information on the frequencies and the relationship between orchestration practices and formative assessment processes. The *Guide-and-explain* and *Work-and-walk-by* types are most prevalent, highlighting the teachers' preference for direct teaching with immediate feedback. The *Sherpa-at-work* and *Spot-and-show* types also frequently occur, though to a lesser extent. Other types are less commonly observed in the classroom settings, with some types such as *Technical-demo* and *Board-instruction* appearing only occasionally across the data (see Table 1, second column).

In the first stage of formative assessment, *Ask*, the orchestration types of *Guide-and-explain* and *Work-and-walk-by* prominently utilise teacher-led and technology-mediated questions. A notable number in Table 1 is the high engagement through technology, particularly in *Work-and-walk-by*, which shows the most frequent use of digital technology for questioning. We observed in the videos how during the *Answer* stage in orchestrations like *Sherpa-at-work* and *Work-and-walk-by* students are often invited to answer through digital technology. We noticed how the following orchestrations in the *Analyse* stage increased the depth of the analysis. *Guide-and-explain* is heavily utilised, with 51 instances of teacher-led analysis (see Table 1), demonstrating the teacher's central role in evaluating student responses. Similarly, *Spot-and-show* stands out with 17 instances of peer analysis, where student work becomes a focal point for whole-class discussion, enabling peer feedback. The *Adapt* stage, essential for completing the formative assessment loop, reveals fewer instances across orchestration types, indicating that teachers often implement the first three stages without consistently completing the adaptation phase. Notably, *Spot-and-show* emerges as a significant orchestration type in this stage, with 4 instances of prompting student rethinking and 8 instances of lesson plan adjustments based on whole-class discussion insights. However, a key observation in Table 1 is the zero instances of changes in technological resources, suggesting that digital technology may play a limited role in the adaptation process of formative assessment.

Based on the frequencies shown in Table 1, a small number of orchestration types, including *Guide-and-explain*, *Work-and-walk-by*, *Sherpa-at-work*, and *Spot-and*-show, appear more regularly and are often linked to key stages of the formative assessment process. These include teacher-led questioning (*ask*) and students' oral responses (*answer*), and in some cases, signs of teachers analysing student input and adjusting their actions (*analyse and adapt*). Other orchestration types, like *Technical-demo* and *Explain-the-screen*, are used less often and show little evidence of movement beyond the initial stages. The coding results reflect a general pattern of teacher-directed interaction with limited engagement in the full sequence of formative assessment processes.

Table 1: Frequencies of coded instrumental orchestration practices and formative assessment processes

FA practices ----- IO practices	Frequency	Ask			Answer			Analyse			Adapt		
		by peer	by teacher	by technology	in writing	orally	via technology	by peer	by teacher	by technology	changes in resources	changes to lesson plans	rethinking of students
Board-instruction	5	0	0	0	0	1	0	0	1	0	0	0	0
Discuss-the-screen	10	0	10	9	1	9	0	3	2	0	0	0	0
Explain-the-screen	10	0	0	0	0	0	0	0	5	1	0	2	0
Guide-and-explain	52	0	37	30	0	43	1	8	51	15	0	8	1
Link-screen-board	13	0	5	3	1	3	0	0	5	3	0	0	1
Sherpa-at-work	35	0	17	14	2	9	32	2	31	6	0	3	1
Spot-and - show	30	0	2	1	1	1	1	17	30	5	0	8	4
Technical-demo	8	0	0	0	0	0	0	0	0	0	0	0	0
Work-and-walk-by	52	0	45	50	24	2	27	1	20	1	0	1	0
Total	215	0	116	107	29	68	61	31	145	31	0	22	7

Illustrative Event

The following event was chosen because it vividly demonstrates how a specific instrumental orchestration practice uses digital technologies to enhance formative assessment, exemplifying the study's findings in a technology-rich classroom setting.

In Figure 2, the teacher employed the *Spot-and-show* orchestration type to select and display examples of student work that showcased a range of solution approaches, including typical mistakes and alternative strategies. By sharing these examples with the entire class, the teacher created an opportunity for students to critically compare different strategies, identify mistakes, and discuss the underlying reasoning behind them. The teacher supported this process by asking follow-up questions and helping students explain their reasoning clearly. When errors were identified, the teacher

prompted the original students who made these answers to revise their work and articulate their thought process. This iterative process of analysis, feedback, and revision fostered a collaborative learning environment in which peer discussion played an important role. As a result, students became more engaged and took a more active role in making sense of the mathematics.

Overall, the *Spot-and-show* orchestration type supported the complete formative assessment process by enabling the teacher to monitor student progress, promote peer feedback and discussion (*Analyse*), and guide students to reflect on their understanding (*Adapt*). This continuous cycle of feedback and reflection facilitated real-time identification and resolution of misconceptions, ultimately enhancing both individual and collective learning outcomes.

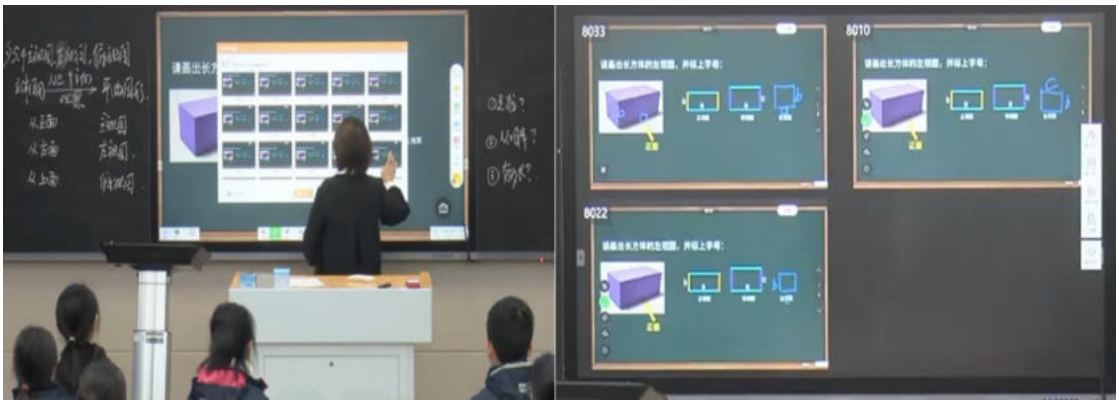

Figure 2: The screenshots of *Spot-and-show* orchestration type

Conclusion and Discussion

With this study, we investigated how teachers conduct formative assessment as part of their orchestration practices when using digital technology. The study has revealed how teachers integrated various orchestration practices at different stages of the formative assessment process.

Table 1 shows that in the *Ask* phase of formative assessment, *Work-and-walk-by* occurs most frequently. This orchestration uses technology-mediated questions to engage students. In the *Answer* phase, *Sherpa-at-work* actively involves individual *sherpa* students in solving tasks using digital technology, allowing them to showcase their problem-solving process to the entire class. In the subsequent *Analyse* stage of formative assessment, *Guide-and-explain* is used extensively in our sample videos for detailed examination of student work, demonstrating the teachers' central role in evaluating student responses and providing whole class feedback. This highlights that teachers typically lead the analysis process of formative assessment, consistent with prior findings by Panero and Aldon (2016). Although the *Adapt* phase is critical to completing the formative assessment loop, we observed that it is often not fully executed. The exception is *Spot-and-show,* where facilitating continuous observation, providing immediate feedback, fostering peer discussion, and encouraging reflection constitute a complete formative assessment process, as illustrated in the previous example. Finally, this study also shows that digital technology in our sample videos primarily facilitates interactions between the teacher and students, rather than direct student interaction with the digital technology itself. This outcome may, however, have been influenced by the choice of videos, which

focused more on whole-class practices, and the didactical configuration in the classroom. In addition, some orchestration types inherently provide more opportunities for students to engage directly with digital technologies.

This study also reveals that digital technology enables teachers to conduct prompt assessments, provide timely feedback, monitor student progress, and summarise student responses, which supports formative assessment, as noted by Dalby and Swan (2019). However, as shown in Table 1, the formative assessment process remains largely teacher initiated. This suggests a need to better balance teacher-led and student-driven contributions to formative assessment, highlighting the critical role of teachers in shaping formative assessment processes through their orchestration strategies, underscoring that the true value of digital technologies lies not only in their technical affordances but also in how they are integrated by teachers to enhance student engagement and support formative assessment (Cusi et al., 2017). However, how these orchestration strategies unfold in real-time classroom settings remains underexplored. Future research should investigate strategic orchestration of students' instrumental genesis by teachers, the collaborative design of digital technology between teachers and students, and the development of adaptive technologies for personalised feedback in automatic formative assessment (Børte et al., 2023; Panero & Aldon, 2016; Weigand et al., 2024).

The use of award-winning lessons as study samples introduces certain limitations of this study. Firstly, these lessons, designed to showcase best practices, may not represent typical classroom settings, which limits the generalisability of our findings to everyday teaching. Secondly, teachers and students might perform differently under competition conditions, potentially affecting the authenticity of interactions and technology use. Finally, this study found that the *Adapt* stage of formative assessment occurred less often, highlighting the need to explore its underpinning causes to better support the complete formative assessment process. Several practical factors may contribute to this pattern. Teachers often face tight schedules and feel pressure to stay aligned with predefined lesson plans, especially in high-stakes or demonstration contexts, making it difficult to make real-time instructional adjustments. Additionally, although formative assessment is generally recognised as important in mathematics education, *Adapt* is sometimes viewed more as a post-lesson reflection than as a process occurring during instruction. Furthermore, many teachers may lack access to professional development that supports teachers in using digital tools to inform instructional decisions, rather than solely for presenting content or collecting responses. Future research could first examine how the *Adapt* phase of formative assessment is enacted in regular, naturalistic, and more diverse classroom settings. This would help determine whether the patterns identified in this study extend beyond the current context and inform further inquiry into the factors that shape adaptation practices within formative assessment.

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Have you thought about short-cuts? – A recommendation for the design of assessment tasks

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The multiple-choice test format is used for formative and summative assessment as well as for mathematical competitions. A main focus of task design concerns the formulation of the wrong answer options, i.e. distractors. Findings from dual-process theory and about misleading problem-solving processes might influence the construction of distractors, as otherwise superficial strategies may hinder problem-solving approaches intended by the task designer(s). Within this paper, we bring together research results of our analysis of tasks of the international competition Mathematical Kangaroo *and the analysis of students' approaches on a particular task of the* Austrian Standardised Competence-Oriented Written School-Leaving Examination*. Based on that, a recommendation for the design of assessment tasks seems reasonable.*

Keywords: task design, multiple-choice, dual-process theory, competition, exit exams.

Introduction

The multiple-choice test format is used as part of the diagnosis of conceptions of students in several age levels on various mathematical subjects (e.g. Carlson et al., 2010; Akveld, 2023; Klingbeil et al., 2023) or even to analyze the mathematical content knowledge of ongoing teachers (e.g. Ball et al., 2008). Several tasks of national tests like the Dutch educational test (van Stiphout & Groenheiden, 2020) and school leaving examinations in countries like Norway (Smestad & Fossum, 2020) or Austria are formulated in multiple-choice format. This short list of examples is certainly not exhaustive. However, it already highlights the fact that the multiple-choice format is used for the purpose of formative as well as summative assessment for learners of several different stages of mathematical education and expertise.

Apart from mathematical assessment, several mathematical competitions consist of multiple-choice tasks. A main representative for this type of competition for students of all grades is the annual internationally conducted *Mathematical Kangaroo* in which more than 6 million students are taking part (AKSF, n.d.). The competition is held in six age levels with two grades being combined into one age level. The competition consists of several tasks for each age level, giving approximately 2,5-3 minutes on average per task. For choosing the right answer option, 3, 4 or 5 points are given for easy, medium or hard problems respectively. In contrast to most multiple-choice test in summative assessment, incorrect answers lead to a deduction of ¼ of achievable points of the specific task, so that the mathematical expectation is 0 for pure guessing (and in particular it is not positive). Therefore, it might be a good strategy not to choose an answer for some of the tasks within one competition. A competition's task for students of grades 9-10 is depicted in Table 1.

Tasks in multiple-choice format always consist of a so-called item stem, and answer options in which, in addition to the correct answer option, incorrect answer alternatives, so-called distractors, are also

given. The item stem generally consists of text and a question and sometimes (quite often) additionally an accompanying graphic.

Table 1: Task of the Mathematical Kangaroo (grades 9-10, hard task)

Some integers are written on a board, amongst them the number 2018. The sum of all these integers is 2018. The product of these integers is also 2018. Which of the following could be the number of integers written on the board?
(A) 2016 (B) 2017 (C) 2018 (D) 2019 (E) 2020

Designing high-quality multiple-choice tasks is challenging (Gierl et al., 2017). Within this paper, based on the dual-process theory, we focus on a particular property of some multiple-choice tasks, regarding both item stem and answer options, which might favour superficial solving-strategies. This property is extracted by task analysis of the Mathematical Kangaroo which is conducted by both authors. In order to justify our assumption that these findings of the analysis of competition tasks can be transferred to tasks designed for the purpose of assessment, our research aim is to investigate a possible link between this task characteristic and students' approaches on summative assessment tasks. To pursue our aim, we select a particular task from the Austrian school leaving examination which has this property and analyzed students' written descriptions on how they approached this task. Through this analysis, we highlight a little-discussed aspect of the creation of tasks and, in particular, of answer options for multiple-choice tasks.

Theoretical background

Dual-process theory

Within the dual-process theory, which is a discipline of cognitive psychology, two types of thought processes are differed: intuitive and analytical/reflective. A process is classified as Type 1 (intuitive) if it does not require any working memory resources, otherwise it is classified as a Type 2 process (reflective) (Evans and Stanovich, 2013). In other words, Type 2 processes require a certain amount of effort, whereas an intuitive solution may come to mind before a task has been fully analyzed. The latter is usually accompanied with great conviction of its correctness and with a feeling that it does not need to be varified (Van Dooren and Inglis, 2015). The phenomenon, that Type 1 processes might fail can be demonstrated by a task from the Cognitive Reflection Test (Frederick, 2005): 'A bat and a ball cost $1.10. The bat costs $1.00 more than the ball. How much does the ball cost?'. The intuitive answer is that the ball would cost 10 cents. However, this intuition is wrong. The salient feature of the task ('one dollar more') leads to a certain reaction ('the bat costs 1 dollar'), but precisely this reaction must be inhibited (in the sense of controlled) (De Neys et al., 2013). More generally spoken, a main "theoretical approach is one in which rapid autonomous processes (Type 1) are assumed to yield default responses unless intervened on by distinctive higher order reasoning processes (Type 2)" (Evans and Stanovich, 2013, p.223). However, it must be pointed out that the view of a fundamental superiority of Type 2 processes over error-prone Type 1 processes is wrong (ibid).

Within mathematical education research, among others, Vinner contributed to the explanation of misleading problem-solving processes by characterizing *pseudo-analytical thought processes*

(Vinner, 1997): Students identify aspects of a task that appear superficially relevant and then spontaneously execute typical solution approaches for these features. Vinner argues that one reason for applying these strategies is the fact that they sometimes lead to correct results. One example is the incorrect application of the so-called 'keyword strategy', which is an essential element of the construct 'superficiality' developed by Schlager (2020). "It seems that children are strongly inclined to start conducting a calculation that is explicitly shown in the problem (i.e. the subtraction) even when inhibiting that tendency and first considering which strategy to apply would be desirable" (Torbeyns et al., 2009, p.714). The existence of this strategy also serves as an explanation, why solving the bat and ball problem requires an analytical thought process.

Design of multiple-choice tasks

The basic principles according to which good multiple-choice tasks are created are a clear formulation of the item stem, as well as formulating meaningful distractors. What is meant by meaningful distractors differs depending on the purpose of the respective instrument. In order to having high diagnostic value, the distractors may represent typical errors or possible misconceptions of students within the specific topic at hand. Traditionally, experts create distractors by imitating possible approaches of students towards the tasks (Gierl et al., 2017). This strategy of developing distractors is based on the (in many cases justified) assumption that the selection of a distractor can additionally provide information about students' approaches, which is essential for accurate diagnosis of students' conceptions and errors (e.g. Klingbeil et al., 2023).

It is notable that by focusing exclusively on creating meaningful distractors, it is implicitly assumed that students approach the tasks analytically, with sufficient motivation and with clear focus on the task as a whole. However, alternative approaches, which could be explained by psychological phenomena (apart from pure guessing) which may lead to students' conducting undesired strategies are given remarkably little attention in current didactic discourse. For instance, neither content-related reflections nor evaluation of the plausibility of distractors can reason the following finding: By analyzing tasks of common intelligence matrix tests, Mittring and Rost (2008) demonstrate both empirically and by theoretical analysis, that the correct answer option of a considerable number of tasks within these instruments can be determined by simply comparing it with the given distractors and *without taking the item stem into account*.

Results

Analysis of particular tasks of the Mathematical Kangaroo

In Donner and Lerchenberger (2024), we analyzed the answer frequencies of 690 tasks of several years of the Austrian version of the Mathematical Kangaroo for all age levels from grade 3 onwards. In a first step, we extracted tasks in which a large proportion of participants chose one particularly frequently chosen distractor (at least 40% of all answers given), while at the same time this distractor was chosen at last twice as often as the right answer option. Surprisingly, 64 of all 690 tasks fulfill these requirements of containing one particularly frequently chosen distractor in combination with a quite unpopular right answer option. In a second step, we evaluated these tasks using qualitative content analysis (Kuckartz, 2018). It was conducted based on theoretical background of dual-process

theory and mathematical intuition, and by imitating possible approaches which may lead in a most directly manner to the specific particularly chosen distractor. We were able to determine 5 different categories for these distractors (for more details, see Donner and Lerchenberger, 2024). Two of the categories could be described as Type 1 processes, one of which is the category "AD" (*answer direct*). Within this paper, we will focus solely on this category. AD contains tasks, in which the particularly frequently distractor can be taken directly from the item stem. It appears 13 times in the analyzed subset of tasks and it occurs in almost all age levels of the competition. The task depicted in Table 1 is a representative of this category, as distractor (C) was chosen by 56,4% of all participants who gave an answer, whereas the right answer (B) just accounted for 12,7% of all answers given. (The remaining distractors (A), (D) and (E) were chosen by 11,2%, 12,3% and 7,4% of all answers given, respectively). A possible (and obvious) reason why distractor (C) 2018 was that attractive is that the number 2018 appears several times in the item stem.

A further, related finding emerged when analyzing tasks with particularly high solution frequencies. Lerchenberger and Donner (2024) found that several tasks that were solved exceptionally well by the Austrian participants share characteristics with tasks of the category *answer direct,* in a sense that the right answer option can already be found within the item stem. We assume that the reasoning of choosing/guessing the answer option is based on a superficial, intuitive Type 1 process in these cases, too. This assumption can be underlined by the fact that significantly more students decided to choose an answer option to tasks where an answer option is part of the item stem than tasks which were estimated to be of comparable difficulty but at which none of the distractors has this property.

Empirical insights in problem-solving processes of a task of the *Austrian Standardised Competence-Oriented Written School-Leaving Examination*

In order to justify our assumption that the findings from the tasks of the Kangaroo competition can be transferred to tasks designed for assessment, we were interested in investigating problem-solving processes of tasks from the standardized school leaving examination (grade 12) in Austria. For that reason, a task was extracted with the potential option for students to apply the strategy *answer direct*. Table 2 serves as an example of a task with this property: The correct solution is $k = 6$. Exactly the same number is depicted (in a completely different role) in the last line of the task's item stem.

Table 2: Task "Data set" from the Austrian Standardised Competence-Oriented Written School-Leaving Examination (Bundesministerium für Bildung, Wissenschaft und Forschung, 2019)

Below, you will see an ordered data set. One of the values is k where $k \in \mathbb{R}$.

1	2	3	5	k	8	8	8	9	10

Determine the value of k such that the mean of the whole data set has the value 6.

k = _____

It can be assumed that at least two variants of solving this task are intended. An algebraic approach corresponds to solving the following equation for k: $\frac{1+2+3+5+k+8+8+8+9+10}{10} = 6$. As an alternative,

one can argue that as the mean of the 9 values given in the data set is 6, the unknown value k must coincide with this mean in order not to change the mean of the whole data set.

As part of an unpublished pilot study of Lerchenberger (2023), the second author of this article collected data of 20 eleventh-grade students' written solutions as well as the written descriptions of their approaches to the task depicted in Table 2. In joint analysis of these documents, the two authors of this article found that among the eleven correct answers, both variants described above occurred. However, it also turned out that some students very quickly came up with the correct result completely differently, by just inserting the number $k = 6$ *because it was already there*. A typical student's answer to describing the approach to the task in a superficial manner is shown in Figure 1.

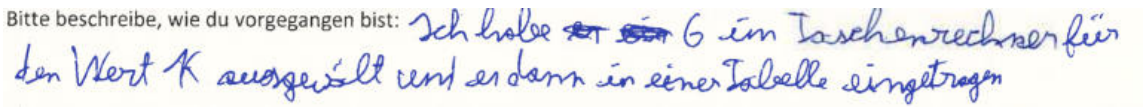

Figure 1: Student's approach to task depicted in Table 2. [*Please describe how you solved the problem: I selected 6 in the calculator for the value k and then entered it in a table*]

The student's argument depicted in Figure 1 describes the verification of the chosen number via calculator, but it does not give any hint why this number was chosen in the first place. It is important to emphasize that there is no plausible reason why exactly this number should be a good guess, as there are uncountably many real numbers in the interval [5,8]. A narrow reason, which corresponds to our findings described above, is that the number 6 was used because it was given in the item stem (*answer direct* in a sense that the solution can already be found within the task). This interpretation indicates the presence of this superficial strategy working on this task. The task designer(s) would have had a possibility to avoid enabling students to argue via this surely not intended pseudo-analytical thought process of obtaining the value for k by specifying a different arithmetic mean in the item stem. Further research is needed to answer the question, whether the solution frequency of the particular task is influenced by changing the mean and simultaneously reducing the amount of successful strategies. But in the form as it was used (see Table 2), it is highly doubtful whether students have to demonstrate the intended mathematical competence the task supposed to check.

Discussion

There are numerous recommendations for the process of creating sound multiple-choice tasks as well as suggestions for design criteria when creating multiple-choice tasks in general (e.g. Gierl et al., 2017) as well as for the purpose of formative or summative assessment in mathematics education. Thereby, findings of research on intuition and cognitive psychology hardly seem to be in focus of (didactical) reflection. The dual-process theory recommends that under certain conditions, intuitive solutions may come to mind before a task has been fully analyzed (Evans and Stanovich, 2013). Furthermore, the student may stay in a Type 1 process unless intervened on by distinctive higher order reasoning processes (Type 2). Pseudo-analytical thought processes are prime examples in which superficial task characteristics significantly influence intuitive approaches.

Following on from this theoretical basis and expanding the spectrum of possible misleading strategies in multiple-choice tasks, a task analysis of particularly frequently selected distractors in the Kangaroo of Mathematics leads to the assumption that strategies based on superficial task characteristics could possibly appear attractive for many students in numerous multiple-choice tasks (Donner and Lerchenberger, 2024; Lerchenberger and Donner, 2024). Besides others, a main superficial strategy may be to choose a distractor as answer, which is already part of the item stem (*answer direct*). This strategy is strongly related to the 'keyword strategy'. Our analysis of students' approaches on the task of the Austrian school exit examination points out that the phenomenon is not restricted to multiple-choice tasks. The latter result is consistent with empirical findings on students' use of superficial strategies when solving word problems (e.g., Schlager, 2020).

The findings of our study have to be seen in light of some limitations. The Mathematical Kangaroo is not created with the focus on mathematical assessment. Moreover, the specificity of the competition, that many tasks should be answered – but not exactly justified – in a very short time with the aim of achieving a high score, is different from a typical situation of assessment. However, we think that there are two reasons why the findings are of wider relevance for the purpose of mathematical assessment. First and foremost, the strategy *answer direct* can be applied independently of the content and intention of the (mathematical) multiple-choice task, it just depends on the structure of item stem and distractors. Second, the empirical results of problem-solving approaches of students dealing with the task of the Austrian school-leaving examination indicate, that the same kind of thoughts are relevant for task-designers of examination tasks, which only require students to give short answers (e.g. a number): Some students' strategies consist of pseudo-analytical thought processes like direct guessing based on information given in the item stem. As a next step, we want to qualitatively analyze students' problem-solving processes of competition tasks as well as assessment tasks by methods of thinking aloud as well as by eye-tracking method, to get an even more direct insight to the problem-solving processes than by written description of their approaches.

Summing up, it could be worthwhile to pay more attention on such phenomena of superficial strategies in task-design, as task characteristics which foster Type 1 processes, may hinder analytical approaches and therefore influence the diagnostic value of certain tasks. We recommend that if a distractor is an element (e.g. number) that is already mentioned as part of the item stem, it should be based on a conscious decision of the designer(s) and not happen out of carelessness. Moreover, we recommend that if superficial strategies like *answer direct* lead to the right solution of a (multiple-choice) task, the beneficial use of these kind of strategies should be omitted whenever students shall demonstrate higher-level competence than pseudo-analytical thought processes by solving the particular task. An additional question in order to designing sound distractors, that may expand the list which most likely already contains questions like 'What are common errors for solving this problem?' or 'What are the common misconceptions about this mathematical object?' is the following: 'Have I thought about short-cuts?'.

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How are different competences represented in standard solutions? An empirical study on statewide exit exams in mathematics in Germany at the end of upper secondary education

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Statewide exit exams in Germany are based on the national educational standards and can convey normative expectations with respect to teaching and to testing. This paper examines one part of these exams set at the end of upper secondary education in one German state (Hesse) in the years 2019 until 2023. Tasks set in this part are to be worked upon without any aids and can thus be considered to test basic competences. Analyses of the tasks, their standard solutions and the allocated points reveal that these tasks require all six competences as set in the educational standards and indicate that the numbers of points allocated reflect the approach to test basic competences.

Keywords: Statewide exit exams, mathematics, solutions, basic competences, backwash effect.

Introduction

"Assessment can play a significant role in improving the quality and relevance of the skills that are acquired at school" (Eurydice, 2012, p. 10). In various European countries and in addition to other forms of written assessment like class tests, students take central written exams at the end of a particular educational stage. In secondary education such exams are frequently taken at the end of upper secondary education (ISCED level 3) and in some countries additionally at the end of lower secondary education (ISCED level 2), which marks the end of compulsory schooling (see Drüke-Noe & Kühn, 2017, Kühn & Drüke-Noe, 2013). Usually at least Mathematics and the language of instruction are among the subjects tested. These exit exams are mostly compulsory for all students (Eurydice, 2009), which hints at the relevance of these exams. They are to test 'key competences' "which represent a combination of knowledge, skills and attitudes that are considered necessary" (Eurydice 2012, p. 7). 'Mathematical competence and basic competences in science and technology' is one of the eight key competences defined. These competences are either promoted by a national strategy or by applying large-scales initiatives (Eurydice, 2012) and are thus relevant in different forms of assessment. Across Europe, standardized tests are also supposed to focus on basic skills (Eurydice, 2012, p. 10).

In Germany, several conceptualizations describe basic skills at the end of compulsory schooling from a normative and/ or an empirical perspective (cf. overview in Drüke-Noe & Siller, 2019), but there is no general conceptualization that applies to the end of upper secondary education. For example Weber and colleagues (2023) define basic competences needed to begin university studies which require mathematics, but this conceptualization sets no framework for the design of exit exams.

Despite this lack of a theoretical framework for basic skills at the end of upper secondary education, this study examines for one German state (Hesse) a specific part of an exit exam at ISCED level 3

that can be assumed to explicitly address these skills (competences). This is done for two main reasons: one is the European-wide requirement to tackle low achievement in basic skills (mother tongue, mathematics and science) (Eurydice, 2012). A second one is that already in year 9 there is regular empirical evidence from standardized assessment that on average one quarter of all German students miss minimal educational standards (Kölm & Mahler, 2019; Pant et al., 2013) and thus lack basic skills; very similar findings come up in regular international studies like PISA. Though there is no comparable empirical evidence for upper secondary education, it is at least of interest to see which normative standards exit exams can set that are supposed to test basic competences. The German state of Hesse therefore serves as an example in this study.

Statewide exit exams in Germany at the end of upper secondary education (ISCED level 3)

In all 16 German states, written and oral examinations are obligatory parts of the general university entrance qualification (Allgemeine Hochschulreife, Abitur), which students acquire at the end of upper secondary education. As the educational system is federally organized by the 16 German states, specific exam regulations vary in all of them. As a consequence, since 2007 different national initiatives have been gradually implemented under the guidance of the Institute for Quality Development in Education (IQB) that aim at centralizing (parts of) the Abitur exam and its structure. Respective initiatives are only sketched here, though. One is the introduction of national educational standards in 2012 (see below). Another one is the development of a national pool of Abitur tasks (including their solutions and further documents) which has been going on since 2013/14.

The tasks in this pool are developed on the basis of the educational standards and thus cover five content areas (i.e. all three domains analysis, analytic geometry/ linear algebra and stochastics), six competences (K1: argumentation (arg), K2: problem solving (pro), K3: modelling (mod), K4: use of representations (use), K5: working technically (wor), K6: communication (com)) and three different levels of cognitive demand (low, medium, high). For each of the three domains the pool provides two different kinds of tasks: One is for an exam part without aids (part A). These are unrelated tasks, each of a so-called small size. The second kind for the exam part with aids (part B). These tasks are more extensive and divided into related sub-tasks. Since 2014 and to a yearly growing extent, an increasing number of states has been adopting tasks from this national pool and presented them identically or slightly modified in their final exams. The proportions of these tasks in a state's exam vary, however.

Statewide exit exams in one German state (Hesse)

The state of Hesse serves as an example in this study as in this state like in almost all other ones the exams are centrally set and thus based on the state curricula (or the core curricula) as well as on the educational standards (KMK, 2012) and specific regulations for the state Abitur. In Hesse, students take exit exams in five subjects, i.e. in their two advanced courses and additionally in three regular ones. The combination of subjects must meet certain requirements but Mathematics and German are always tested. The exams in both advanced courses and in one regular course are written ones. Only the other two subjects are not tested centrally. Both the written and the oral exams relate to the content of the four semesters in years 12 and 13. The exams must cover all three levels of cognitive demand and the focus is to be on the medium one. The total duration of an exam depends on the type of course (regular course: 255 min, advanced course: 300 min). The grade achieved is included in the overall

grade for the general university entrance qualification using a fixed calculation key. In Hesse, the written exit exam in mathematics consists of two parts, which are both compulsory and must be completed by all students: in regular courses, part 1 (45 min) tests all three domains and is to be solved without a calculator or a collection of formulas; this is why it is called aid-free (hilfsmittelfreier Teil). Since these tasks must be solved without any aids, they can provide information about the students' expected basic competences. Part 2 lasts further 210 min and only in this part students are allowed to choose from one of two calculus tasks and additionally either a task on linear algebra/ analytical geometry or one on stochastics (for details see Hessian Ministry of Culture, 2019). Parts 1 and 2 of the Hessian exam correspond with parts A and B of the national pool. Parts of the tasks in this exam are adopted from the pool, the rest is developed by exam designers in Hesse. Information on the origin of a single task is not publicly accessible. Officially, the Hessian Ministry of Culture finally holds responsible for the assembly of tasks in an exam, their standard solutions and the classifications according to the educational standards.

Marking exit exams – standard solutions

In this contribution we use the term standard solution to name the document which teachers use to mark the students' work and to assign points. The names for this document vary, though. Terms like example solution or model solution or range of expectations are known, too. The different names hint at the level of detail and rigidness of the specific document. A standard solution is to implement an assessment that is as objective as possible, and it also explicates the exam designers' normative expectations as regards a correct solution ideally expected by the students. All teachers are provided with the standard solution and are obliged to use it when marking the students' exams.

According to one of the accompanying documents set up by the IQB, a standard solution shows a possible solution for each subtask, only contains aspects that align with the educational standards, and highlights that correct solutions not explicitly named are equally to be accepted (IQB, 2020, p. 3). In Hesse, for each subtask this document contains the expected correct solution and the number of points to be awarded. Additionally, it contains two tables: one informs about the classification of each subtask as regards the competences needed to solve it. The other table shows per task (not per subtask) how the number of points is distributed per level of cognitive demand. Although teachers have to use this document, they still have some flexibility as in accordance with the regulations in IQB (2020) the standard solution in Hesse contains the following remark (Hesse, Abiturprüfung, regular course, 2019, standard solution, p. 1; translation by authors):

> The given solution specifically contains all essential aspects that have to be taken into account when working on the individual tasks. Based on experience, it shows those solutions that the examinees will usually apply. Solutions that deviate from the given ones but can be viewed as equivalent are also to be accepted.

Two teachers mark the students' work independently. The second teacher can either agree with the first teacher's assessment or submit an own one. If both results differ, a new and consistent assessment must be created together.

Due to the backwash effect of exit exams and also due to the fact that the application of a standard solution is mandatory, this document can set normative expectations as regards the marking of class

tests or as regards the distribution of points among tasks and their competences. Though there is some research on characteristics of tasks set in exit exams (see below), at least to our knowledge there is no research on standard solutions in such exams.

Research on statewide exit exams

International research on statewide exit exams addresses various perspectives like governance and accountability or national differences between these exams (Drüke-Noe & Kühn, 2017; Klein & van Ackeren, 2011) or the impact of such exams on either the selection of tasks for teaching (e.g. Cheng & Curtis, 2012; Neil, 2003) or the design of class tests (Drüke-Noe, 2023). In these contexts a direct or indirect backwash effect of exams on teaching and learning is frequently discussed (e.g. Biggs & Tang, 2011; Boud & Falchikov, 2006; Prodromou, 1995). Especially summative forms of assessment like exit exams can have this effect and thus exert a considerable influence on both teachers and students. These exams set standards as to what is to be achieved or considered important as for example research on design principles of class tests clearly indicates (Drüke-Noe, 2024).

As regards task characteristics in exit exams in mathematics at the end of lower secondary education, Kühn and Drüke-Noe (2013) identify a low level of cognitive demand of the tasks set at this level in most German states. Applying routines and using facts (working technically) is a dominant competence, whereas more complex ones, such as problem solving or argumentation, are far less needed to solve these tasks. For eight European countries, both authors come to very similar findings (Drüke-Noe & Kühn, 2017), i.e. only a limited spectrum of competences is tested and the tasks are mainly of a low cognitive demand. Frenken and colleagues (2024) analyse Abitur tasks (ISCED level 3) set in 2020 and 2021. They consider the cognitive potential with respect to mathematical competences and analyse the basic ideas involved as well as the digital tools used. The authors identify a generally low level of cognitive potential as regards argumentation, problem solving, modelling and communication. They also see hardly any differences between the tasks with or without different digital tools (computer algebra (CAS) and scientific calculator) and conclude that the potentials of CAS are apparently hardly used.

Methods

The following research question is addressed in this study: How do the points relate to competences (and are some competences somehow more relevant) and does the prevalence of individual competences change in the course of time?

Data material and data analysis

The sample includes all tasks set in part 1 of the Hessian exit exams (ISCED level 3) in regular courses (years 2019 until 2023) and the standard solutions, which indicate 20 points each year.

All analyses are based on subtasks. This is done for three reasons: one is that in most cases the solution informs on points per subtask and only rarely details this any further. A second one is that information on required competences is only globally given per subtask. A third one is that in line with regulations for this part these tasks are all of a small size only, which is why no further differentiation is made within a subtask.

Based on the standard solution only, the global information on required competences and the number of points assigned are combined to model their relation. Only in very few subtasks, the solution details further how to distribute the points within a subtask, which is then considered, too. To model the relation of competences to points, both researchers decide from a normative didactic perspective how to distribute the points among the competences of that subtask. This is done consensually by both authors taking into account which part of a subtask's processing and thus which competence(s) are in the focus of the cognitive activities. The example in Figure 1 is to illustrate this approach. The standard solution names three competences (argumentation, problem solving, working technically) and allocates 5 points. Since the main cognitive activities are problem solving and working technically, 2 points are assigned to each and only 1 point to argumentation.

Two points A(-6|8|1) and B(3|8|-5) and the equation of a line g with $g: \vec{x} = (-1 \quad 2 \quad 1) + r(2 \quad -3 \quad 1), r \in \mathbb{R}$ *are given. Confirm that* $\overline{AB}$ *and g intersect.*

Figure 1: Task 4, year 2019; translation by authors (layout of column vectors modified)

The numbers of competences and of points can differ either way. The number of points per competence agreed upon ranges from 0.5 (minimum) to 3 (maximum); units are no smaller than 0.5 points. This approach allows for a statistical evaluation and descriptive analysis of how points are distributed among the individual competences in the classifications underlying the (sub-)tasks.

Results

Each year four tasks are set that cover all domains. These tasks comprise 7 to 11 subtasks per year (2019: 7, 2020: 7, 2021: 8, 2022: 11, 2023: 8). Overall, the solutions indicate a range of one to three competences per subtask and a range of 1 to 5 points is allocated; the smallest unit is 1 point.

Table 1: Minimum and maximum number of points per competence

Competence	K1 (arg)	K2 (pro)	K3 (mod)	K4 (use)	K5 (wor)	K6 (com)
Minimum number of points; frequency	0.5; 6	0.5; 10	0.5; 3	0.5; 9	0.5; 8	0.5; 4
Maximum number of points; frequency	1.5; 2	2.0; 3	1.5; 3	3.0; 1	3.0; 1	1.0; 2

Based on our model, the data in Table 1 informs on the relation of points to competences. It shows the minimum and the maximum of points assigned per competence and additionally the frequency of these combinations. The minimum is 0.5 points for each of the six competences. This relation occurs least frequently for modelling and for communication, and it occurs most frequently for problem solving, use of representations and working technically. In all five years, 0.5 points per competence are allotted 40 times which amounts to 20 points out of a total of 100 points (total sum in five years). The highest numbers of points per competence differ: they range from 1 point (communication), which occurs twice, to 3 points (use of representations, working technically), which both occur once.

Figure 2 (left side) shows for each year the cumulative occurrence of the six competences. To do so, the points assigned to individual competences per subtask are added up per year. In the course of these years, problem solving and working technically are most strongly represented with 26.5 points each (out of the 5-year-total of 100 points). Together, both therefore account for over half of the points. The points vary from a minimum of 4.5 points (2020) to a maximum of 7 points (2019) for

problem solving and from a minimum of 3 points (2023) to 8 points (2019) for working technically. The two competences argumentation and communication are the least important ones with a cumulative score of 10 points and 4 points respectively. The prevalence of these two competences is relatively uniform, with argumentation ranging from 1 point to 2.5 points per year and communication from 0.5 points to 1 point. Modelling also accounts for a smaller share only with a cumulative 11 points, but the span ranges more widely from 0 points in 2019 to 4.5 points (i.e. just below a quarter of the total annual points) in 2022. The use of representations accounts for a total of 22 out of 100 points. Its span is the greatest and ranges from 2 points to 8 points in 2023.

In the course of time, the prevalence of argumentation, problem solving and communication is fairly constant, whereas that of working technically seems to be slightly decreasing. No clear time trends can be identified for the use of representations and modelling. The sum of points for both the use of representations and working technically is quite homogeneous over the years. It only varies between 8.5 and 11 points and either the one or the other competence contributes more to this sum.

Figure 2 (right side) illustrates the importance of competences per year measured by the number of points assigned to each of the six. Grouping the data accordingly (low: 0.5 or 1 points per competence, medium: 1.5 or 2 points, high: 2.5 or 3 points) and adding up the respective numbers of points per competence per year serves to qualify the relation of three levels of importance with reference to a yearly total of 20 points. Over the years, the proportion of competences of low importance ranges from 8.5 points (2023) to 16.5 points (2022) and reveals a fairly wide span. In 2022, for example, to a relatively great extent the subtasks are muchly characterized by a low importance of single competences, while in 2023 a total of 11.5 points is assigned to competences of medium and high importance (not be confused with levels of cognitive demand which are not considered in this study).

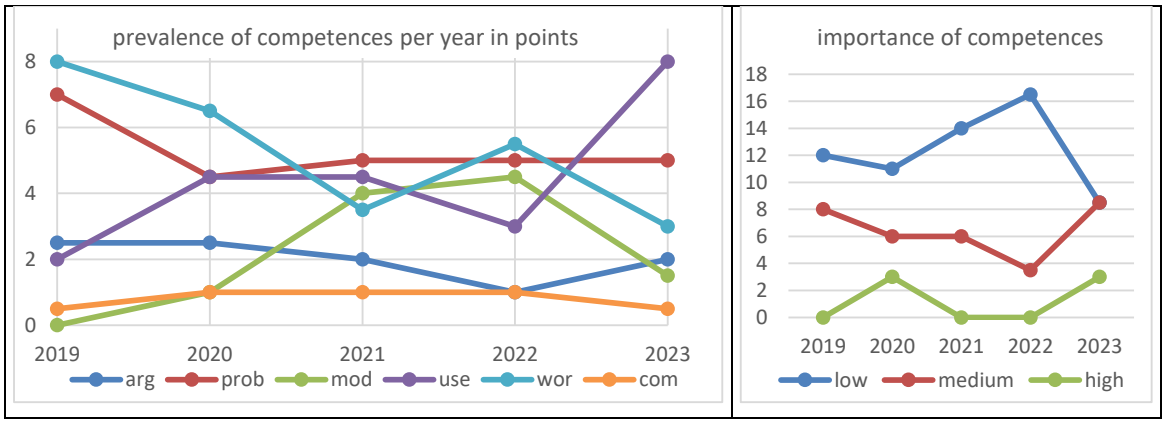

Figure 2: Prevalence (left) and importance (right) of competences per year

Conclusion, limitations and implications

The results show that part 1 of this exit exam tests a broad spectrum of all six competences and thus indicate a broad spectrum of tasks. In accordance with national regulations these tasks are short (small) as in a considerable proportion of subtasks a low number of points is assigned to each competence. This result supports the assumption that this exam part indeed addresses basic competences. The relative importance of the six competences varies: problem solving and working technically are rather important as the highest numbers of points are assigned to these two, while

argumentation and communication are of fairly low importance in all years. The importance of modelling varies but is generally low, which is in accordance with expectations as regards basic competences. The importance of the use of representations and working technically apparently interrelates and this surprising result requires further investigation.

These empirical results are subject to several limitations, though: firstly, all analyses rely on the exam designers' classification of competences given in the standard solutions. Secondly, information on the exam designers' qualifications is not accessible and we do not know how and why they assign points (Do they consider competences at all?, Why do they only assign full points?, Which other considerations influence the assignment of points (e.g. verbs like "Confirm"?)). Thirdly, we assume that classifications are valid over the five years. Since we assume a backwash effect from standard solutions and teachers must follow them, too, we refrained from newly classifying all tasks ourselves.

In further studies, it would be interesting to include information on the exam designers' intentions, to extend our approach to part 2 of the exam and to include the levels of cognitive demand to allow more detailed analyses. With respect to regular teachers and to understand the influence of standard solutions and their backwash effect better it would be interesting to examine how teachers really use them and if they consider the six competences at all when assessing students' work.

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Introduction of a practical element in oral examination of mathematics: Teachers learn first

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Revision of a national curriculum for compulsory school may bring along requests for revising structures affected by the revision, like for instance summative assessment practice for pupils at the end of compulsory school. This paper reports on use of a participation based model for teachers' professional development as foundation for a cooperative development project about moving from an oral exam to an oral-practical exam for pupils in mathematics at the end of 10th grade in the Norwegian school system. About 40 lower secondary school mathematics teachers from a Norwegian municipality were key representatives in the project. A self-study approach was applied, because of the opportunities it offers to describe and experience teacher development from the inside. The results describe how priorities made in compliance with the participation based model produced consequences for both teachers and pupils preparing for the up-coming oral-practical exam.

Keywords: Teacher learning, assessment, mathematics.

Introduction

As part of a revision of the Norwegian curriculum for compulsory school (1st grade – 13th grade) in 2020 it was decided by the *Norwegian Ministry of Education and Research* (KD) that the oral exam in mathematics (OM) that pupils might be drawn to conduct at the end of 10th grade (15-16 years old pupils), was to be extended with a practical element (UDIR, 2020a). Thereby the oral exam in mathematics was replaced by an oral-practical exam in mathematics (OPM). This replacing required an extension of the OM from consisting of a presentation of mathematics by the pupil related to a context presented 48 hours before the exam, and subject specific questions from the examinators, to also include *a practical element*. This change brought along distinctive and significant attention among both Norwegian educational researchers and mathematics teachers, made public both through publications (e.g. Haara & Taraldsen, 2020) and social medias. Documents regarding this change were published by the *Norwegian Directorate for Education and Training* (UDIR) (UDIR, 2020b; 2022), but this only to some extent made preconditions and consequences of a change from OM to OPM clear for school leaders in the municipalities and for the teachers supposed to organize and conduct this new, summative evaluation for tensed and exited pupils.

In October 2022 a municipality of about 25000 residents, with seven grade 10-schools and about 40 lower secondary school teachers of mathematics, made contact with the teacher education institution where the author of this paper works. The rationale for the municipality's request was the experienced need for a joint development of understanding of UDIR's guidelines for OPM within the municipality, and facilitation of the mathematics teachers' adjusting of priorities in directing and conducting an OPM instead of an OM. The municipality's project group, consisting of two teachers, two headmasters and the assistant head of education in the municipality, myself, and a colleague of mine from the teacher education institution, set two goals for what we identified as a cooperative

development project: To develop guidelines for the implementation of OPM, and to gain both a common understanding and confidence related to OPM, when it comes to preparation, implementation and assessment. Based on these goals, we developed and implemented a plan that we believed would meet such a request, making this municipality and of course the participating teachers in this municipality ready and set for conduction of the first, national round of OPM in June 2023.

We do know quite a lot about what makes teachers' professional learning both develop and persist (Guskey, 2002; Pedder & Darleen Opfer, 2013; Timperley et al., 2007; Wenger, 1998). This knowledge became core when making priorities for development and implementation of the cooperative development project, priorities that placed the teachers in a participative position through expectations of involvement in defining content and developing arenas for experience and learning in the cooperation. Inspiration for this was found in the participation-based intervention model developed by Haara et al. (2020), making the participating teachers' professional development their goal, and not someone else's objective. We do, however, know less about consequences of priorities made for a cooperative teacher learning approach when one aims to reach consensus about a vaguely defined concept (here: OPM). I therefore ask: *What were the priorities for the cooperative teacher learning approach regarding the introduction of OPM in a Norwegian municipality, and what consequences for both teachers and pupils preparing for OPM did the priorities bring along?*

Theoretical foundation

The participation-based intervention model for teachers' professional development developed by Haara et al. (2020) is founded on three conditions well-grounded in research on teachers' professional development (Pedder & Darleen Opfer, 2013; Timperley et al., 2007; Wenger, 1998; Wylie & Lyon, 2015). The model emphasizes involvement of the teachers in designing of the content, external representatives' direct involvement prior to and at joint gatherings, and providing of sufficient time in between joint gatherings to foster participation-based activity and progress at each school. Timperley et al. (2007) established about 20 years ago that teachers' learning outcomes tend to be better when they may influence the way competence development is organized than if frames and content are decided completely by externals. Important factors, according to Timperley et al. (2007), are that teachers must have the opportunity to participate in a community of learning peers, and to experience that they have actual impact on the progress of the developmental process. This aligns with the important features of learning communities, described by Wenger (1998), where one learns from and together with peers. In a study on implementation of formative assessment practice in mathematics Wylie and Lyon (2015) made experiences that operationalize the conclusions made by Timperley et al. (2007). The cooperative approach needs to balance between joint gatherings conducted by external representatives of significance with all the participating teachers attending, and workshops facilitated at each involved school. At these workshops in Wylie and Lyon's (2015) project the teachers had the opportunity to share experiences stemming from their try-outs regarding formative assessment practice, reflect on these experiences, and receive input and advices from their colleagues. A perspective on cooperative learning development that wants to be recognized as participatory-based, therefore needs to start where the participating teachers are, and leave no one behind. In such projects, according to Pedder and Darleen Opfer (2013), it becomes a collective rather than an individual responsibility to develop the practice.

Context and implementation

Each of the seven participating schools appointed a "resource teacher" who attended a Zoom-meeting where the project group presented the cooperative development project. This was followed by watching two short videos at each school that my colleague and I recorded as preparation for the teachers' project participation, where we kind of opened up Pandora's box on what the official guidelines from UDIR (2020a; 2020b; 2022) actually say, and what *practical* in OPM might be. In the videos we pointed out that it would be necessary to revise both organisation, preparation time (for the pupil), and actual conducting of the exam when including a practical element, and then entertained each other with various, plausible interpretations of what a practical element could be recognized as.

At the first joint gathering (December 2022) emphasis was put on content, form and organising of OPM, where my colleague and I orchestrated for the participating teachers' actual experiences both regarding understanding of what a practical element in OPM might be, organising of such an exam, and how a group exam organising might be possible if the pupils are familiar with collective working methods like for instance Liljedahl's (2020) *Building Thinking Classrooms* approach. This was followed by engaged discussions among the teachers and prolonged into cooperative work at each of the seven schools during the winter of 2023 where the mathematics teachers developed suggestions of organising and tasks for OPM. The teachers were encouraged to experiment with content and form, with an open mind towards making both good and bad experiences. All pupils that were eligible for OPM in June 2023 in the municipality took part in this experimental try-out at the school where they belonged. I observed some of these sessions at one school. After the try-outs at a school the teachers and the pupils evaluated and shared their experiences, and the teachers collected impressions related to the keywords *Done*, *Learned*, *Smart*. Based on this each school prepared a stand for sharing with teachers from the other six schools at the next joint gathering, towards the end of February 2023. The sharing of tasks and experiences at the joint gathering provided both bilateral discussions at each stand and multilateral discussions in a plenary session after the stand-based exchange of experiences was finished. Part three of this gathering was devoted to my colleague's and my orchestrating of discussion and reflection about what to emphasize when assessing a pupil who conducts an OPM. An important premiss for the assessing is UDIR's (2020b, pp. 4) explicit aim "for the pupil to have the opportunity to demonstrate competence in as large a part of the subject as possible based on what is possible within the exam format" (translated by me from Norwegian). The keyword in this is competence, and in what way competence might be shown by the pupil. In short the teachers concluded, for the locally developed OPM, first that the tasks mainly had to reflect the competence targets stated in the national curriculum for 10th grade. Second, in order for the pupil to be able demonstrate mathematical competence as extensively as possible, the teachers needed the opportunity to focus on mathematics emphasized in earlier grade levels than 10th grade as well. And third, it had to be based on a broad understanding of the term *practical*.

This joint gathering was followed by another round of try-outs at each school in early April 2023, where the teachers and pupils piloted an actual OPM for all the pupils in 10th grade. Then in June 2023 the municipality was informed by the County Governor's education branch that three of the schools were chosen for conducting of OPM. At the three schools 6, 12 and 18 pupils were drawn for this exam, respectively.

In September 2023 we had the third and final joint gathering. It focused on the experiences with the actual conducting of OPM, both pilots and actual exams, and reflections by the teachers on preparing for such an endeavor as moving from OM to OPM through a cooperative development project.

Method

The fact that the author of this paper was one of the participants involved in the reported study, places the paper within an action research perspective, narrowed down to have a self-study approach (Zeichner, 2007), since I was the sole producer of data. According to Loughran (2004: pp. 154) this is an approach to understand "from the inside out rather than the outside in". In the study reported in this paper, teachers who reflected on their own practice as teachers responsible for OM and future OPM with an aim of improving their own and others' practice, my colleague and I, were involved. Together, we tried to understand the roles both as a pupil and as a teacher from the inside and out, rather than from the outside and in. I prepared for a systematic investigation of practice, developing a personal narrative based on observations and onsite impressions made at the formal learning arenas of the project. This choice of approach served two main purposes. First, the investment in professional development with the participants, and enhanced understanding of how OPM may facilitate for pupils' display of mathematical competence in practice, requires analysis related to possible normative effects provided by the project. Second, the self-study approach was chosen because of the proximity I could accomplish as both a project participant, lecturer and as a researcher. Hence, in order to be able to address the research question in a trustworthy manner and offer mathematics teacher arguments for changing or not changing teacher practice, I made it a priority to be as close as possible to the actual activity, and thereby sacrificed some observational distance on the altar of involvement. This provided an intended and appreciated interaction between practice and research focus, with awareness to influence such a perspective may have on both the actual practice and the theoretical understanding of the practice.

The choice of methodical approach challenged my role as researcher because of the proximity to the production of data, and although my knowledge about the field of research is an important prerequisite for viable interpretations, I have strived to follow Feldman's (2003) criteria for securing validity and reliability in a self-study-based project. I do so far, and will in the forthcoming parts of the paper do this by: providing detailed descriptions of how data were produced and make explicit what counts as data, showing why the self-study approach was chosen with arguments for consequences of this choice together with what would have been consequences of a more distant researcher position, and finally being utterly aware that the self-study approach is a moral issue. Therefore I have been careful in my registration of observations, and relied on both the acknowledgement of the model applied for the cooperative development project and my knowledge about teachers' professional development and assessment in mathematics in my interpretative work.

The results are based on my narrative developed through participation in the project. Hence, this is not a regular, qualitative study primed to provide arguments for the reliability of empirical data. However, the study provides close contact with actual teacher learning experiences, as well as research perspectives on these experiences through the theoretical grounding of conditions for such learning. The teaching experiences related to the study and the reliance on data is difficult to replicate.

Tasks from conducted OPM will be easy enough to collect again and again, but it is impossible to reproduce qualitative observations and on-site impressions from these teachers' first meeting with development from directing and conducting OPM because nothing was filmed or taped. This may seem to be a rather unpredictable way to work in qualitative research, but this is not a paper about teachers' learning seen from the outside. It is a paper about teacher development opportunities experienced from the inside.

The three conditions given by Haara et al.'s (2020) model have produced the framework for providing observations and on-site impressions on which my narrative was developed. I do, however, find my narrative development through interpretation of observations and impressions to be inspired by the constant comparative analysis method (Glaser, 1965). My analysis has therefore consisted of four phases; placing and comparing of observations and impressions in categories, comparing of categories and their qualities, delimiting the interpretation and understanding based on a summary of the categories and their qualities, and description of the interpretation and understanding as a basis for discussing the paper's research question. Operationalized this means that new observations and impressions contributed to ensure that my interpretations were refined through an interaction between deduction and induction. New observations and impressions were seen in conjunction with prevailing interpretations and a growing body of studied literature, and influenced by my constantly revised preconception, provided revisions of the interpretations on which my narrative was first developed and then revised several times. My interpretations are based on four rounds of personal notes on observations and impressions: a first round immediately after the first gathering, with emphasis on organizational-, progress-, and result-based impressions, a second round after the second gathering, with emphasis on sharing of experiences from practical element try-outs at each school, a third round after observing OPM pilots with pupils at a school participating in the project, and a fourth round after the third gathering, when some of the teachers had gained actual experiences with directing and conducting OPM in June 2023.

Results

The first gathering revealed that in fact the vast majority of the teachers were unsure about the content of the term *practical*. The rest of the teachers, and these were mainly the youngest in the group, were silent. Some of the most experienced and profiled teachers tried to argue for the sustainability of the prevailing understanding of practice at OM, in order to seek similarity with OPM, like for instance that it would be better if there was a more drastic change that was made clear, because it was not so easy to look for adjustments in what one already was familiar with, or stating that it was common to bring something concrete like for instance dices, a tape measure and such, to the OM. On the other hand, some teachers identified that *something is rotten in the state of Denmark*, so to speak, and called in utter frustration for what might be reckoned as a possible shift of paradigm (Kuhn, 1996), through acknowledging the development project opportunity as a possibility to try something completely new, when hearing the attempts to adapt a new element to the prevailing understanding of what OM represents. The first gathering did not end in any consensus, but revealed agreement about the fact that change in the curriculum had brought along a necessary adjustment of the assessment system. The practical element was broadly acknowledged, but the discussions prolonged into the hallway after the gathering about what the element could be represented by, and arguments for what one

already was doing at OM. This kind of professional disagreement and bringing of social school culture to the surface, contributed to the legitimation of the cooperative development process, and highlighted that the teachers need to have content and form priorities at OPM that reflect the teaching that has been offered to the pupils. If consensus was reached at the first gathering, it was about these two issues.

The second gathering brought along experiences with reports about experimental change from previous practice in intermediate work that had been tried out since the first gathering: arrangements where the pupil had chosen a mathematical theme and then faced problems related to this theme in an examination room where lots of physical equipment were available for use, and group based arrangements with use of vertical boards, where pupils in pairs worked on exploring a mathematical relationship. Several arrangements, on the other hand, only represented shades of change from OM, by still assigning the pupil with a context-related topic for a preparation period, like for instance a trip to a remote cabin, a holiday to somewhere far, a prospect for a playground, etc., where the pupil had to create tasks by using mathematics from the curriculum and present them and their solutions at the exam. In this process, several schools left it for the pupils themselves to define *practical*. The discussions based on sharing of the newly acquired experiences saw the argumentation of difficultness in making significant changes for the up-coming spring once again being put forward, because one had worked with the pupils all the way from 8th grade and onwards with a standard OM in mind. However, something else seemed to find viable conditions for growth, as well. The cooperative development project opened for thinking differently and together about one's own teaching, and through the project's structure for possibilities regarding revision of teaching. The possibility of such revisions may, according to the participating teachers, over time affect how the practical element of OPM can be implemented.

The third and final gathering showed that the actual conduction of OPM in June 2023 at the three schools that were chosen for this exam, maintained a structure and priorities that where quite similar to what OM had contained the previous years. This could not be seen as a surprise, and the argumentation for this was to find in the teachers' arguments for forwarding of the established OM-policy found at the first gathering and to some extent at the second gathering. Hence, the third gathering showed that the first round of OPM appeared to be rather knowledge and product oriented, but the cooperative development group of teachers seemed to be ready to move in the direction of competence and process orientation for the future OPM, based on changed priorities in the teaching.

Discussion and conclusions

The necessity of making changes of one's practice may be a kind of painful experience. It is both common and understandable to strive for adjustment of new input to a prevailing policy (Glaser, 1965; Kuhn, 1996). Change may develop in a continuous manner, but sometimes a collective leap of some kind forces its way, as described by Kuhn (1996). A cooperative development project as the one elaborated on in this paper, based on a participation-based intervention model for teachers' professional development that prioritizes to start where the participating teachers are (Timperley et al., 2007), is a fundamentally democracy based project within a significant community of practice (Wenger, 1998) showing wide open that it is a collective responsibility to make the joint project effort

relevant and to influence both the actual practice and the theoretical understanding of the practice (Pedder & Darleen Opfer, 2013).

The first gathering saw resistance towards change rise to the surface through the voices of prominent and self-confident teachers. Righteously they advocated the signalized adjustment needs with the introduction of OPM distributed from UDIR (2020a; 2020b; 2022) to be adjusted to the prevailing content and form policy used in OM. On the other hand, well-founded and practice grounded input from external representatives of significance provided the participating teachers with sound examples of possibilities of change regarding both content and form priorities that proved the possibility of well-grounded change of practice. This seemed to enable confidence to grow and pave the way for arguments for change to meet the understandable arguments for standstill. This growth was nurtured, as in the study by Wylie and Lyon (2015), by the organized expectation that the teachers would try out specific attention to a practical element regarding both content and form in small scales at their own school. Such try-out experiences supplied teachers who advocated change with support in self-made practice based experiences. This support made it easier for them to challenge arguments from teachers who rather advocated adjustment of understanding of a practical element to the prevailing OM-policy. These discussions moved towards a negotiated consensus, where the joint group of teachers recognized it as an argument for moderate change that the 10th grade pupils had not been taught with OPM in mind, but with OM in mind. This seems like a pragmatic echo of an argument for conserving the prevailing policy, but is nevertheless an argument that brings the pupil into the equation of change. The teachers' desire for consensus seemed to solve this by settling for change of teaching practice when starting with a new group of 8th grade pupils, with OPM in mind. However, an actual joint implementation of such a change, starting from each teacher's starting point of course, is impossible to confirm because the teachers have already moved on from the initial starting point.

Hence, two consequences of the cooperative teacher learning approach are identified, providing implications for policymakers and future research. First, there was a registered movement towards a joint understanding of the practical element in mathematics. Second, a collective agreement of need for change of priorities in teaching of mathematics in order to reach a deliberate development from OM to OPM for the pupil was identified. This implies that the registered initial change in acknowledging the practical element and attitude towards teaching priorities will not be allowed to slide back to adjustment seeking defense of prevailing OM-policy contentment. Time will show how a policy regarding content and form in OPM will develop among teachers, and research ought to investigate what changed or not from OM to OPM, why this happened, and what measures that were taken to initiate changes requested by policy makers. A cooperative development approach that bases the teachers' learning on their participation in defining, developing and reaching milestones, deserves to be among the measures investigated.

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Students' reactions to speech input requests in a digital formative assessment of conceptual understanding

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Students' conceptual understanding can be assessed by eliciting their explanations and argumentations, e.g. in oral communication between student and teachers. In written formative assessments, however, many students avoid writing so that their responses offer only limited insights. This study explores to what extent digital formative assessments requesting speech input rather than typed responses could encourage students to provide more detailed and informative explanations and argumentations. In a qualitative study with 53 fifth and sixth graders, we observed that students produced longer responses, incorporating multiple arguments. These responses provided a more robust basis for assessing students' conceptual understanding. Additionally, we observed that these improvements were especially strong when students employed an everyday language register.

Keywords: Digital formative assessment, oral speech formats, conceptual understanding.

Challenges in assessing conceptual understanding

Although procedural skills and conceptual understanding are equally important learning goals in mathematics, many digital summative and formative assessments hold a strong procedural focus (Hoogland & Trout, 2018). Therefore, we investigate ways to enhance formative assessment (Black & Wiliam, 2009) of deep conceptual understanding. A prerequisite for this assessment is to make conceptual understanding visible.

Deep conceptual understanding includes the abilities to translate a concept between different (e.g., verbal, graphical, symbolical, or contextual) representations, to explain the meaning of particular concept elements and the connection between these representations, and to connect different concept elements in a wider network of elements e.g. for argumentations (Hiebert & Carpenter, 1992). These abilities tend to be more difficult to assess than procedural skills and shallow factual knowledge (Çekiç & Bakla, 2021), and can ideally be assessed in oral conversations between teacher and students (Clarke et al., 2002). Nevertheless, some digital formative assessment tools show that written assessment of these abilities is possible (Olsher et al., 2023), with carefully designed multiple-choice items and open items with longer answer formats, requesting students to write explanations of meanings or argumentations on connecting different representations. However, research documented the repeated challenge that students often provide only brief answers, lacking elaboration and detailed thinking processes, especially mathematically weaker students (Brennan et al., 2010). This heavily weakens the informativity of the assessment, as the students' thought processes remain hidden. Reasons for these limited brief answers can vary: Students might lack motivation for the time-consuming process of writing or typing extensive explanations. Alternatively, students might have

internalized implicit norms of what kind of response (and language register of that response) is expected. Research on the development of writing skills for students with disabilities show that speech input technology can have benefits for text quantity and quality (Almgren Bäck et al. 2024). This challenge occurs less in oral conversations in which students can express their ideas orally to a clearly determined recipient. With the new technological affordances of digital tools, this raises the following question: *To what extent can digital formative assessments requesting speech input rather than typed responses encourage students to provide more detailed and informative explanations and argumentations?* We report on an exploratory study in which fifth and sixth graders could give their answer via speech input (as a voice message or via dictating their answer) in formative assessment tools.

Speech input as assessment approach to support students' language production

When students' lack of motivation to give extensive answers might be due to the tediousness of writing or typing, it might be helpful to reduce this burden by allowing students to give their answer via speech input. Similar to mode effects that have been reported for the change from paper-and-pencil to computer-based assessment, we pursue the assumption that also the change from written to oral modes might make a difference in students' answers. This assumption is nurtured by findings from non-mathematical areas: Pereira et al. (2019) employed a ChatBot in a massive open online course where higher education students submitted voice messages. The authors found that 49 % of all answers showed a high degree of reasoning and were developed in depth, all while being produced spontaneously. Gavras et al. (2022) found that adults' oral answers (about political attitudes), also submitted via voice recording, were up to 40 % longer than written answers, were lexically richer and overall related to a broader range of topics. The authors remark that "Interestingly, in the oral answers, respondents articulate their question answering process, expressing how they came up with their answers, indicating a kind of think-aloud" (Gavras et al., 2022, p. 884). For some adults, speech input increased affective blocks due to anxiety to speak (Satar & Özdener, 2008). All findings refer to adults, no study was found about younger students.

Some indications can be found from learning (not assessment) situations with younger students (Schreiber & Klose, 2017): Fourth graders who were engaged in preparing mathematical audio-podcasts used substantially different language compared to writing, with longer answers. Their spontaneous audio-recordings were found to be longer, more in the everyday language (characterized by everyday phrases and less complex sentence structures) and to have more traits of a personal talk (e.g., with a clear addressee) than the final versions. By re-working on the script, the language use developed towards more academic language in more condensed sentences and more elaborate phrases. We anticipate that also in assessments, spontaneous oral answers might elicit longer and more everyday language use than written or well-planned oral answers.

Combining all observations mentioned above, the use of audio recordings bear promises related to the language register, the extend of answers and the enacted discourse practices. In our study of fifth and sixth graders' reaction to speech input requests in digital formative assessments, we focus on the mathematical topic of meanings multiplication (as counting in units) for which earlier studies revealed that students engage in explaining how different representations are connected and in

describing the underlying multiplicative unit structures is the main window to students' understanding (Lesh, 1979; Prediger et al., 2025). We explore how the language use changes with the offered oral or written modes.

Methods

Methods of data collection. For the explorative interview study, we used our formative assessment tool Mastering Math – Online-Check (Hankeln et al., 2025) and integrated speech input in two versions, by requesting to send a voice message (with students' possibility to listen it and submit a second corrected or extended message) or to dictate a response that was automatically transcribed, so that afterwards the text could be read and edited by the students, and compared it to 414 written responses collected in earlier studies. Oral data was collected in interview settings with 53 fifth and sixth graders. In a rotated design, each student worked on seven tasks on multiplication and division, some in voice message version, some in dictation version. This lead in total to 371 student responses. Figure 1 shows an example item on multiplication that assesses if student recognize the bundled units of three dice with two dots each, representing the multiplication 3 x 2. Before the assessment, an introductory item invited students to read aloud a fictive student's answer to a task on addition or subtraction, formulated in the everyday language to exemplify that spontaneous answers are welcome. After their work on the items, students were asked about their experiences with speech input, if they prefer typing or speech input and which of the two versions (voice message or dictation tool).

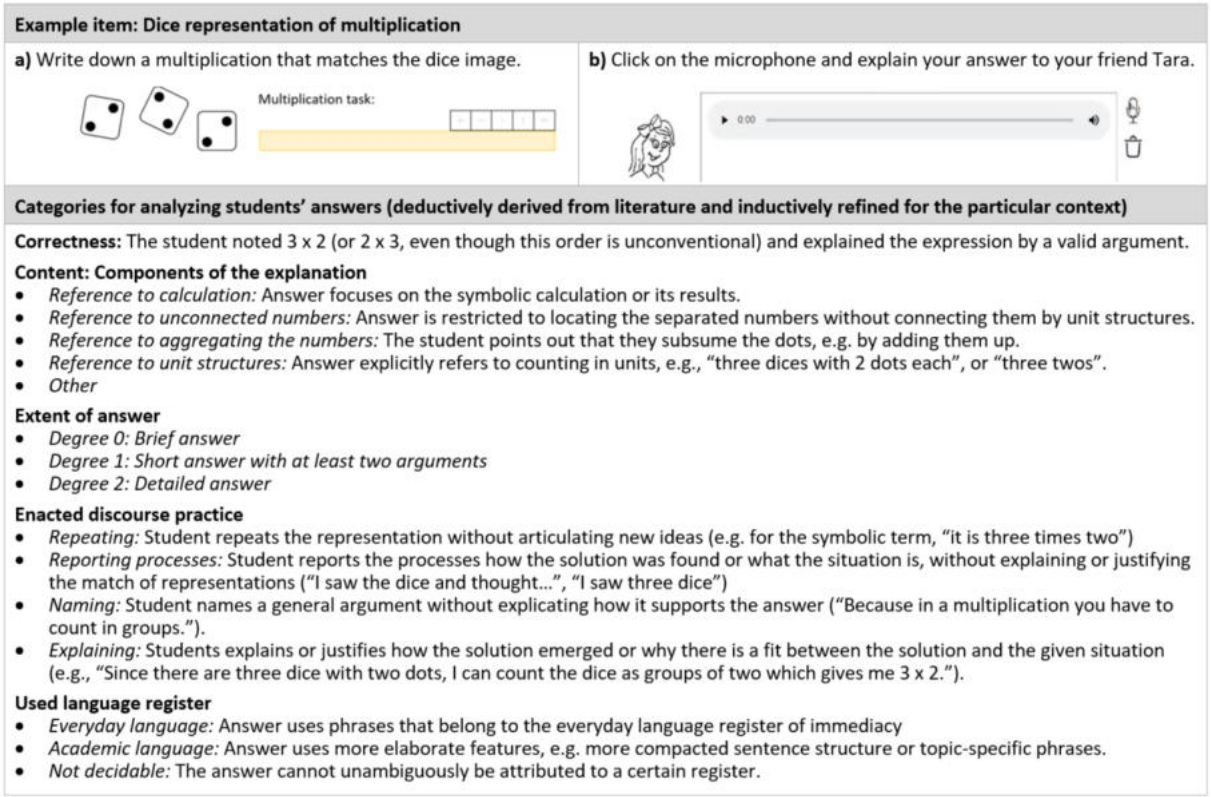

Figure 1: Example item on representing multiplication and analytic categories

Methods of qualitative data analysis. In this paper, we qualitatively analyse the students' oral answers. By comparing students' oral answers to written responses we had already collected in earlier studies, we generated hypotheses on typical characteristics that can later be quantitatively tested, also to compare the two versions of voice message and dictating (Hankeln et al., submitted). The qualitative analysis started with categories deductively derived from literature which were then inductively refined for the particular context by coding for each answer. The categories were (a) correctness, (b) components of the explanation, which are the different arguments that could be used in the answer, (c) extensiveness of the answer, (d) enacted discourse practices and (e) language register. Figure 1 details the categories for an exemplary item we use in this paper. While correctness is dichotomously coded with zero or one point, most categories have several codes with relevant distinctions.

Empirical findings

The qualitative analysis of students' responses led to five hypotheses on potential characteristics of students' speech input use that will be illustrated with one example each (and put the generated hypotheses to tests in a future quantitative analysis):

(1) Students' responses given via speech input seem to elicit more everyday language use than in written answers.

A phenomenon that emerged with the integration of speech input is the frequent occurrence of responses that were coded as everyday language, as Anna's explanation of her non-correct expression 2x2x2=8:

> Anna: So it's 2 times 2 times 2, because there are 3 dice with...2. And if you lump all the dice together, it would be 6, but because it's "times", it's 8. Because if you want to slap out 8 without multiplying, you would have to have 4 times 2
> [authors' remark: with 4 times 2, she refers to four dice with two dots each]

We could not see everyday phrases such as those underlined above in typed responses collected in earlier studies with the same items. With "lump together" and with "slap out without multiplying", Anna seems to refer to an additive process. She does not see the bundled structure of three units with two dots each but relates the representations to each other on a superficial level. Even though she recognizes the 3 as the number of dice, she uses every visible 2 in her expression and even proposes a different representation for another operation to get her result 8. Thus, this example (and various further cases in our data set) does not only plausibilize the hypothesis drawn from the literature (Schreiber & Klose, 2017), it illustrates that by inviting the students in oral models seem to go along with the invitation to express her understanding of the situation in her own language of comfort, and by this, more extended insights into the students' (in this case still flawed) understanding are gained.

(2) Students seem to address more components when they give their answers via speech input.

For explaining why their symbolic expression matches the given dice image in the example item, students mainly addressed four components for their explanations (listed in Figure 1): They either referred to the calculation only, the unconnected numbers, aggregating numbers additively, or the targeted unit structures. Tony, for example, noted the symbolic representation "2x2+2=6" in the dice item and justified his response with a detailed answer (with codes for components in brackets):

Tony: Because you see 3 dice, each with 2 dots on it *[unit structures*]. And then you could calculate, um, if you... then there are 2 dots. Then you could, um, count 1, with 1, 2, 3, 4, 5, 6 [*aggregating the numbers*]. And then you would also know that there are 6. Then you can count 2 times 2 plus 2. That would also be 6 [*calculation*].

When describing the situation, Tony makes use of conceptually rich phrases for articulating multiplicative units ("3 dice each with 2 dots on it"), without explicitly relating this observation to a multiplication. Instead, he focuses on the result as the most important indicator for the match between the representations. The disrupted language Tony uses and the omissions he makes are indicators for typical everyday language, with a think-aloud documentation of his solution process.

While the written answers addressed one or two of these components each, it is remarkable that like Tony, many oral explanations addressed more different components. From this observation we derive the hypothesis that inviting speech input might not only increase the extent of answers (with more detailed answers), but seem to increase their comprehensiveness by addressing more components.

(3) Extent of answers seems to be influenced by students' attitudes towards the input format.

In the end of the interviews, students expressed their approval or hesitation about speech input. Arguments against the speech input included disliking listening to own voices, but also that student felt it easier to sort their thoughts while writing or that the written response is less personal. Students who favoured the speech input, however, found speech input being less tedious, without requiring additional work to reformulate answers, and appreciated that their answer are more complete as they would tend to forget arguments while writing. Student seem to hold different views of the norms for this answer format. Marc for example explains his hesitation when using speech input as follows:

Interviewer: You're now telling me what you were thinking. Why do you think you can't say the same thing in the voice message?
Marc: Because that's not a solution. That's just saying what I think. [..] Simply thinking is not a solution, it should be a solution.
Interviewer: What would such a solution look like? So, what is a solution for you?
Marc: Something correct, not just what you think.

These remarks show the importance of the implicit norms with which a student measures quality of his or her answers. Even though it is the aim of the speech input format to grasp students' informal ways of thinking, Marc aims at giving a final, correct and perhaps compacted answer sentence instead of making his reasoning explicit. This aim is hard to achieve in a spontaneous oral communication.

A first qualitative analysis led us to hypothesize a connection between their attitudes and the extent of answers: Students who do not see the need to give sophisticated responses but engage in oral communication to a friend in everyday language, seem to feel free to express their reasoning more explicitly without being restricted to a purely academic register. Those who believe to be expected to give sophisticated answers seem to provide brief answers more often.

(4) Students' responses seem to engage in discursively richer discourse practices.

With the formal extent of answers and the number of addressed components, also the discursive quality seems to change with the speech input: While the short typed answers like "Because 3 x 2 equals 6" often were less informative, the more extensive responses via speech input also allow a

more reliable diagnosis which discourse practices a student applies. For example, Cara explains in her voice message in the dice item:

> Cara: I observed that there were 3 dice, and in each of these 3 there were 2 dots, and I always counted those 2 dots as 2 dots, and that resulted in 6, and then the task is to get 2 times 3, or also 3 times 2.

Cara shows a profound conceptual understanding of a multiplication and is able to express the bundled structure with her words. She successfully *reports* her processes of translating the image into a symbolic expression, yet does not *explain* the match between the symbolic and graphic representation. While we hypothesize that students seem to engage in discursively richer discourse practices in orally compared to written answered items, we also concede that the increase in extent allows for capturing the sophisticated differences of reporting a translation process and explaining the connection of representations. These kind of insights have substantial value for adaptive language-responsive teaching.

(5) The correctness of responses is not influenced by the input format.

As reported under (3), students reacted differently to the speech input. While some students kept the approach to formulate their answer carefully before starting the recording, some students started talking spontaneously. One could assume that this spontaneous talking without a second thought could have led to less correct solutions. However, we observed several times the phenomena that students started answering and while they were still recording their answer, they remarked a contradiction and changed their argumentation. Therefore, we assume that the correctness of the final answer does not change, however spontaneous misconceptions and fragile understandings become more visible as they are rarely documented in the written answers.

Discussion

The first results of this exploratory study seem to confirm the transferability of observations by Pereira et al. (2019) from adults to fifth and sixth graders that using speech input can lead to answers that are developed in more depth with a higher degree of reasoning. We add that the language used by students in this study often contained elements of everyday language. As students are rarely able to express especially intermediate states of conceptual understanding with rather formal academic language (complex sentence structures and sophisticated condensed phrases), this change opens new opportunities to gain access to students' individual understandings also in classroom assessments.

Like in Gavras et al. (2022), speech-input responses seem to address more components and often express how students came up with their answers, even when the question actually requested a different discourse practice. Important to note is that even if students gave more extensive answers in the speech input, the correctness of answers seems to be unchanged. From the diagnostic point of view, this would indicate that reducing the burden of typing an answer does not make an item easier, but students seem to make use of the time to integrate additional arguments, giving thus more insight into their net of representations and thereby allowing a more comprehensive diagnostic of students' understanding.

However, findings from a decade ago that speech input might generate hesitations due to anxiety to speak freely (Satar & Özdener, 2008) could still be valid for some students, even though this type of messages is nowadays much more present in our everyday communication via smartphone and messengers. Students like Marc clearly expressed their dislike of this format but justified it mostly with their insecurity of what norms apply to such kind of response to a test item. Therefore, a careful introduction and transparency of the expected input is needed for this item format.

As this study was of exploratory nature, the hypotheses generated still wait to be tested in a quantitative analysis for all the 372 responses. Additionally, several methodological questions are still open. Concerning this study, a qualitative interview-study was chosen to ensure a smooth proceeding even in the case of technical issues. The one on one setting however might have had an impact on the answers students have given. Therefore, a replication is needed to ensure ecological validity of the findings. In this study, eight students were taking part in the interviews at once in the same room. It still has to be evaluated how well the speech recognition functions with more students speaking simultaneously.

Another important aspect that has been left out so far lays in the evaluation of the voice inputs. Even though some teachers have expressed their interest to listen to students' audio files, transcribing and pre-coding the audio-files with the help of Large Language Models is desirable, especially since students' responses tend to get longer in speech inputs.

Conclusion

Innovations in the field of digital technologies open new opportunities to overcome problems in digital assessments (Olsher et al., 2023), in our study we focused on students' hesitation to write or type extensive answers about their individual understanding. The first observations reported and hypotheses generated in this paper give insights that using a speech input might bear potentials to bring orality into computer-based assessment to collect more extensive responses. As students show to take into account the recipient of their answers and use everyday language (Schreiber & Klose, 2017), their responses are somewhat closer to their thinking process than their written answers. The number of words in students' responses increases when using speech input, which is not only due to the repetition of words typical for oral modes, but also due to the fact that students take into account more components of an explanation. As their responses do not get more correct with higher extents, the responses seem to make visible flaws for example in the connection of different concept elements that were not accessible before. However, as the innovative format might break with internalised norms for example what counts as a mathematical response, speech input might also unsettle some students so that the introduction of this item format has also to be carefully designed.

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Preservice teachers' formative assessment strategies they intend to implement in a real classroom

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This study investigated formative assessment (FA) strategies that preservice mathematics teachers (PMTs) plan to implement in real classrooms. Twelve fourth-year PMTs who were enrolled in a four-year teacher education program for middle grades (grades 5-8) participated in the study. Lesson plans prepared by PMTs to implement in their practice teaching school and one-to-one interviews on these lesson plans constituted the data of the study. Findings showed that almost all participants included three or more FA strategies in their plans. The 12 PMTs addressed a total of 18 sub-strategies in their lesson plans. Teachers asking questions, making observations, and providing feedback in the form of clues or hints were the most mentioned FA sub-strategies. Other common strategies planned to be used in real classrooms were teacher and student review of the lesson.

Keywords: Formative assessment, preservice teachers, teacher education.

Introduction

Formative assessment, or assessment for learning, is a method of collecting information on students' progress in everyday classroom activities. While there are various definitions of FA, the definition provided by Black and Wiliam (2009) was referred to in this study since it encapsulates the essence of many other definitions. Black and Wiliam (2009) suggested that classroom practices can be formative when teachers, students, or their peers collect, analyze, and use evidence of student progress to make decisions about future instructional steps. These decisions are expected to be more effective or better grounded than those without such evidence.

This definition highlights the importance of involving all participants in the classroom—the teacher, the learner, and their peers—in the assessment process. Although teachers typically use evidence of student learning to guide their instructional decisions and enhance student performance, students also play a role by regulating their own thinking and learning to assess their current status and determine how to progress (Black & Wiliam, 2009). Additionally, the definition emphasizes that evidence of student learning should influence future instructional steps, which is the big idea of the FA strategies suggested by Wiliam and Thompson (2008).

Wiliam and Thompson (2008) introduced a model for FA, which identifies three key participants (the teacher, the learner, and peers) and three essential instructional processes: understanding where learners are in their learning, where they need to go, and how to get them there. The framework outlines five key strategies: (A) Clarifying and sharing learning intentions and criteria for success; (B) Engineering effective classroom discussions and other learning tasks that elicit evidence of student understanding; (C) Providing feedback that moves learners forward; (D) Activating students as instructional resources for one another; (E) Activating students as the owners of their own learning.

It also emphasizes a big idea: using evidence of student understanding to adapt instruction and to meet their needs (Wiliam, 2007).

Strategy A can be used at the start of the lesson, during coursework, and at the end of the lesson. For instance, teachers can set targets (Wiliam et al., 2004) and share student-friendly learning goals to clarify the lesson's aim (Wylie & Lyon, 2015). During coursework, providing expectations or success criteria prior to completing the assignment (Wylie & Lyon, 2015) and giving clear instructions to the students about the assignment and process (Moss & Brookhart, 2009) can also help students grasp what they need to do to achieve the assignment. To implement this strategy, teachers might focus on the learning objectives rather than the task itself. They can provide opportunities for students to evaluate their work using clearly defined criteria and encourage discussions about the quality of their work (Leahy et al., 2005; Wylie & Lyon, 2015). Teachers might also ask students to review the lesson content to check their understanding at the end of the lesson. Strategy B involves questioning and observing to create meaningful classroom discussions that reveal students' comprehension. Teachers can ask questions to stimulate students' thinking, identify misconceptions, and adjust instruction to address learning needs (Leahy et al., 2005). They can also ask hinge-point questions to check if students ready to move on during the coursework and/or range-finding questions to learn about students' previous knowledge (Wylie & Lyon, 2015). Questions to ask at the end to see if students have grasped the main idea can also provide evidence regarding students' learning (Wylie & Lyon, 2015). Evidence collected about students' work through class observations can also assist in creating effective classroom discussions. Providing feedback that advances students' learning (Strategy C), fostering student collaboration and participation in activities (Strategy D), and encouraging students to take charge of their own learning (Strategy E) are also crucial for FA (Moss & Brookhart, 2009). Teachers can give feedback by providing written or oral comments on students' work. Classroom activities that promote reflection on learning, such as writing or discussing their current understanding or evaluating each other's work, can support Strategy D and E (Wylie & Lyon, 2015).

Research showed that classroom practices involving FA can improve students' mathematics success (Palm et al., 2017). For this reason, teachers must know how to plan and implement FA, as they are expected to tailor their instruction to meet students' needs (Wiliam, 2007). However, research revealed that teachers fail to share assessment criteria with their students and do not use peer and self-assessment frequently (e.g., Guo & Xu, 2021). It is known that some social, educational, and cultural factors affect how to implement FA (e.g., Hume & Coll, 2009; Qin & Yi, 2021) and how to plan it. Lack of professional development in assessment is also one of the barriers to plan and integrate FA into teaching practice (DeLuca et al., 2012). Thus, teacher education programs need to train PMTs to increase their understanding of FA and to break their barriers to plan and implement assessment. On the other hand, teacher education models face challenges in effectively fostering the development of preservice teachers' understanding and assessment practices (DeLuca & Johnson, 2017). Therefore, it is worth investigating how preservice teachers plan FA that is crucial for students' learning.

Effective FA practice depends on well-designed strategies. This means teachers must carefully plan how to assess student learning and interpret the results in advance. Although there is documentation on teachers' approaches to FA, there is limited understanding of how these assessments are developed

and applied in practice (Ayala et al., 2008). Based on this, this study investigated which FA strategies PMTs plan to implement in real classrooms. Specifically, the study aimed to answer the research question: What are the FA strategies PMTs plan to implement in real classrooms? It is our intention that the findings will provide teacher education programs in mathematics with a documentation of preservice teachers' assessment plans and practices, which may further be utilized for improving these plans and practices in their future careers.

Methods

In order to answer the research question, the FA strategies PMTs integrated into their lesson plans to implement in middle school mathematics classrooms were investigated, and one-to-one interviews on these lesson plans were conducted with the participants.

The study involved 12 fourth-year preservice mathematics teachers (PMTs) enrolled in a four-year teacher education program for middle grades (grades 5-8). Participants were selected purposefully among 35 PMTs who had already taken a course on assessment in mathematics classrooms before the study, considering their diverse answers to a task involving an incomplete and improper lesson plan on equivalent fractions and a case where preservice teachers were asked to assume the role of an inservice mathematics teacher implementing this lesson plan. The lesson plan was improper for several reasons (see Kaplan, 2015). It was incomplete since any wording implying a FA strategy was excluded from the lesson plan. The task also involved four questions, asking PMTs about the strengths and weaknesses of the lesson plan and how to improve the lesson plan regarding to the deficiencies they found. Considering PMTs' responses to these questions, PMTs who did not mention any of the FA strategies (n=3), who referred to one of the strategies (n=5), and who mentioned at least two FA strategies (n=4) were asked to participate in the study to reach a diverse sample of participants. All participants volunteered for the study.

The study was conducted within the scope of the Practice Teaching course. As part of this course, PMTs were required to prepare and implement two lesson plans in their practice school, including an assessment. In this study, 12 lesson plans, one lesson plan by each PMT, were analyzed considering the FA strategies that Wiliam and Thompson (2008) proposed. After examining each lesson plan, semi-structured one-to-one interviews were conducted with the participants and audio recorded. The interview protocol consisted of 17 questions asking how PMTs prepared the lesson plan, PMTs' reasoning for including the strategies, and their intention to use them.

The lesson plans and interview transcripts constituted this study data. Data were analyzed through content analysis. Participants' statements in lesson plans and interview transcripts that implied FA were categorized under the five FA strategies (Wiliam & Thompson, 2008). Subsequently, these categories were further divided into sub-categories based on descriptions existing in the literature. The first author engaged in an iterative analysis process that required switching between data and codes while placing expressions into the sub-categories. The intercoder agreement was also utilized to ensure the categorization. Another researcher who conducts research on teacher assessment and an expert in qualitative research coded the lesson plans separately. Any disagreements between these researchers were discussed until a complete agreement. In this process, some descriptions were

combined under one code, or the same codes were placed under different FA strategies since they serve for several strategies. Some examples were provided in the findings section.

Findings

Nearly all participants incorporated three or more FA strategies into their plans, except one participant (P4), who included only two strategies (A and B). The FA strategies that PMTs intended to implement in their lessons can be categorized under the sub-strategies outlined in Table 1. Altogether, 12 preservice teachers included a total of 18 sub-strategies in their lesson plans.

Table 1: PMTs' planned FA strategies (In lesson plan + In interview only)

Strategy	Planned Sub-Strategy	Number of participants
A	Start of the lesson: making aim clear	(3+1)
	Coursework: summary of the topic	2
	Coursework: giving clear instructions	4
	End of the lesson: teacher review	9
	End of the lesson: student review	(7+1)
	End of the lesson: informing students about the next topic	3
B	Teacher asking questions ● Range-finding Qs ● Hinge-point Qs ● Qs that prompt students to think ● Qs at the end to see if students grasped the main idea (Qs, HW, journal or exit cards)	 11 2 (7+1) 11
	Observing students' work	(8+2)
	Sharing student-generated solutions	2
	Establishing discussion environment	4
	Deepening discussion to engage more students	4
C	Giving clues/help/hints	(7+2)
	Reflecting on the answer to the board	1
D	Peer assessment: assessing each other's work	(1+2)
	Peer Assessment: explaining solutions to the peers	6
	Working in collaboration	(2+1)
E	Reflection: student review of the lesson	(7+1)
	Reflection: writing journal	2

Preservice teachers who mentioned Strategy A included several sub-strategies for clarifying and sharing learning intentions. PMTs gave place to teacher review (n=9) and student review (n=7, with one noted in the interview) of the content in their lesson plans. Three participants indicated they would inform students about the next topic at the lesson's end, while two planned to conduct a lesson review during the coursework. Four participants emphasized the importance of providing clear instructions to keep students engaged in the activity. However, only three participants in their lesson plans and one in the interview intended to begin their lesson by clearly stating its aim. As shown in Table 1, none of the participants mentioned giving students a scoring rubric or marking criteria at any stage of the lesson.

Almost all participants referred to Strategy B related to creating discussion environment through various sub-strategies such as asking questions, observing students' work, sharing student-generated

solutions, and deepening discussions to involve more students. A common approach was using range-finding questions to assess students' prior knowledge, such as asking, "If I am p years old, what will my age be after 5 years?" or "If she has f flowers, how many flowers will she has if she multiplies the number by 5 and adds 6?" (P9), while introducing algebraic expressions. Additionally, questions at the end of the lesson were used to check students' understanding of key concepts, such as asking, "What is the definition of perimeter?" or "How do we calculate the perimeter of regular shapes?" (P12), after a lesson on the length of polygons. Eleven participants mentioned these types of questions. Some participants also included questions designed to prompt deeper thinking (n=7) and hinge-point questions (n=2) in their lesson plans. One participant, who did not include such questions in his lesson plan, discussed how he would encourage students to think critically during the interview. Eight preservice teachers indicated they would walk around the classroom and observe students during activities, while two participants who did not mention observations in their lesson plans highlighted their intent to do so during the interview. They emphasized that they planned to observe students working individually or in groups. Four participants noted that they would work to involve more students in discussions, while two participants stated that they would illustrate different solution approaches during class discussions (P5). Four others mentioned creating a discussion environment without detailing a specific strategy. However, during the interview, one participant (P1) mentioned she intended to foster discussions when students could express agreement or disagreement with their peers. Similarly, P5 explained that she wanted to share different student solutions to create a discussion environment by asking students for their opinions on their peers' solutions. Participants' comments suggested that these discussion environments could also serve as an opportunity for peer assessment, aligning with Strategy D (activating students as instructional resources for one another).

More than half of the participants (n=7) mentioned Strategy C (providing feedback that moves learners forward) in their lesson plans, with an additional two participants discussing it during the interview. They indicated they planned to provide hints or clues to students who needed assistance during the coursework. For example, P3 emphasized in her lesson plan that "the teacher walks around the classroom, and if a student is struggling to draw a rhombus, the teacher may offer clues and help." P3 was also the only participant who noted in her lesson plan that if students were unable to draw a rhombus, the teacher would ask them to look at the board and draw the shape while reminding them of its properties.

Eight participants mentioned Strategy D, with six referring to peer assessment and two indicating the creation of a collaborative learning environment in their lesson plans. Those who focused on peer assessment primarily included statements about asking students to explain their solutions to their classmates (n=6). Only one participant detailed a plan for students to assess their peers' solutions displayed on the board and to share whether they had different perspectives (P11). Two participants mentioning collaborative learning stated that students would work in pairs (P8) and in groups of three or four (P12). During the interview, another participant (P1) emphasized that if students struggled with solving questions, she would group them into pairs or small groups. Conversely, one participant preferred not to create groups, referring to the class size as "I cannot conduct a group activity in that

class. There are 35 students, so group work can lead to much noise. Besides, the classroom is not suitable for group work" (P2).

Participants who referred to Strategy E (activating students as the owners of their own learning) primarily focused on encouraging students to reflect on their current understanding of the topic. We categorized students' review of the lesson under both Strategy E and Strategy A, as having students review the lesson in their own words can help them reflect on what they know and identify gaps in their understanding. Additionally, two participants planned to ask students to write a journal about what they learned in class, which provides another way for students to reflect on their learning. Writing a journal could also offer valuable feedback for the teacher, helping to improve future lessons based on students' needs, which is the big idea of FA.

Conclusion

Findings showed that almost all participants included three or more FA strategies in their lesson plans. Considering the significant effect of FA on students' learning (Lee, et al., 2020), this is a promising finding. When the use of each strategy was examined individually, it was seen that PMTs planned to use mostly Strategy B and C in a real classroom. Teachers asking questions, making observations, and providing feedback by giving clues or hints were the most mentioned FA sub-strategies categorized as Strategy B and C. Preservice teachers mostly included range finding questions to learn about students' previous knowledge and questions at the end of the lesson to see if students grasped the main idea of the lesson in their lesson plans. Similar findings from previous research showed that teachers used questions to check students' understanding (Antoniou & James, 2014). Related to Strategy C, PMTs stated that they would provide feedback in the form of giving clues or hints to students when they need them, but they did not give examples of these situations. This raises a question about how prepared teacher candidates are for potential questions from students.

Although research suggested sharing lesson goals or objectives with students in a way that students can make sense of them at the start of the lesson (Wylie & Lyon, 2015), PMTs mostly preferred to use Strategy A at the end of the lesson through teacher and student reviews. On the other hand, previous studies demonstrated that preservice teachers were not be able to implement especially the end and assessment parts of their lesson plans due to various barriers such as lack of time or classroom management issues (Kaplan-Can & Haser, 2023). Therefore, although PMTs planned to use this strategy at the end of their lessons, they may not be able to implement it as they planned. The same situation applies to Strategy E. As seen in Table 1, PMTs included Strategy E in their lesson plans as student reflections at the end of the lesson. They did not include any sentence in their plans about students assessing their own work, which is one of the ways to activate students as the owners of their own learning (Wylie & Lyon, 2015). Hence, it can be anticipated that due to some barriers faced by PMTs as a result of their practical experience (Kaplan-Can & Haser, 2023), PMTs may not be able to implement Strategy E either in a real classroom. This situation needs to be investigated further. To clarify this, research could be conducted on how PMTs implement the lesson plans they developed.

Working in collaboration is one of the opportunities for students to share and criticize their ideas or solutions with their peers, so it can be considered as a FA sub-strategy (see Andersson et al., 2017).

However, only a few participants planned to form pairs or groups during the coursework. One of these PMTs justified why she preferred not to create groups by referring to the large class size. Hence, it can be concluded that the effect of barriers is also evident in using Strategy E. This study's findings also revealed that the strategies are not mutually exclusive. For instance, engineering effective classroom discussions that elicit evidence of student understanding (Strategy B) may also serve as an opportunity for peer assessment (Strategy D). Hence, future studies could explore the relationship between these five key strategies.

Findings suggest that PMTs have plans for FA in mathematics classrooms, but these plans should be diversified to meet the needs of diverse groups of students effectively. They may consider classroom size as an important factor in determining FA strategies. PMTs can be encouraged and guided to devise effective FA strategies that would include students in rather crowded classrooms. Interviews indicated that they may have additional intentions in the classroom that do not appear in their lesson plans. However, indicating these intentions in the lesson plans, such as providing hints or clues, can provide them with a thorough thinking of these intentions within the classroom and prepare them better for their practice. Teacher education programs in mathematics should emphasize the importance of FA for effective mathematics teaching and learning for the preservice teachers.

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Written tests through the lens of mathematical competence: how a teacher perceives feedback from test and data collected.

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Literature is critical with respect to the use of mathematics tests as a formative assessment method. The main criticism concerns the non-specific feedback provided and the failure to focus on how to move the teaching-learning process forward. In this contribution, we present a case study of a primary teacher who uses some suggestions to design and assess a mathematical written test by means of mathematical competence. We discuss the feedback the teacher perceives during the test design phase and the analysis of the data collected. The test design and the subsequent data provide the teacher with feed up, feed back and feed forward. Focussing on mathematical competence in the test design helps the teacher understand where and how she is going with her students and offers suggestion for future activities. Additionally, analysing the levels at which each competency appears in students' responses gives the teacher insights about activities to be undertaken.

Keywords: Mathematical written test, Feedback, Classroom assessment.

Introduction

In a previous contribution (Lemmo, 2023), we present some considerations concerning mathematical written tests as assessment methods that pursue curricular guidelines, educational goals and support the teaching and learning process. In detail, we present some suggestions for designing and evaluating mathematical tests aimed at formative assessment purposes. Consistent with the definition of *classroom assessment* (cf. De Lange, 1999), written tests could be a means for teachers to assess students' skills and understanding, using the outcomes to refine instruction and support the progression of student learning. Providing feedback that moves learners forward is one of the five strategies suggested by Black and Wiliam (2009) for activating effective formative practices. Feedback plays a central role by reducing the discrepancy between current and desired understandings with respect to an educational goal (Hattie & Timperley, 2007). Hattie and Timperley (2007) state that "Feedback has no effect in a vacuum" (p.82): with this sentence, they mean that teachers and their students should create learning contexts to which feedback is addressed, and dealing with tasks and tests should be one of these. In this contribution, we aim to investigate which kind of feedback our suggestions for designing and evaluating tests provide to a teacher. In particular, we present a case study of a primary school teacher who designs and evaluates a mathematics written test for her 4th grade students. Through an interview with the teacher during the design and evaluation phases, we show how and when the use of these suggestions results in effective feedback for her.

Theoretical background

Hattie and Timperley (2007) state that " "Tests" are one method used by teachers and students to address this question [How am I going?] and, […] often fail to convey feedback information that helps teachers and their students to know how they are going." (p.89). This concern lies in the

structure of written tests and the subsequent use of the data collected: tests are often composed of tasks in which students are asked to use procedural understanding, and the data consists in how well each task is performed and how many correct or incorrect answers there are.

In this section we explore some theoretical lenses with the aim of framing our suggestions for the design and evaluation of mathematical written tests. We start by describing the pedagogical lenses of powerful feedback, then we propose a shared definition of mathematical competence. Finally, we weave together pedagogical and mathematical theoretical lenses, explaining our suggestions for designing and evaluating mathematical tests through the lens of mathematical competencies.

The model of powerful feedback

Hattie and Timperley (2007) define feedback as "information provided by an agent (e.g., teacher, peer, book, parent, self, experience) regarding aspects of one's performance or understanding. […] Feedback thus is a "consequence of performance." (p. 1). The authors propose a *model of feedback*, outlining three questions to ask: (1) Where am I going? (What are the goals?), (2) How am I going? (What progress is being made toward the goal?), and (3) Where to next? (What activities need to be undertaken to make better progress?).

These questions correspond to *feed up* (1), *feed back* (2), and *feed forward* (3). Both teachers and students should seek answers to each of these three questions. The first question deals with educational goals: clarifying and sharing learning intentions and criteria for success to promote goal-directed action. In this perspective, feedback and educational goals are strictly related to each other in order to reduce the discrepancy between students' current understandings and goals. The second question concerns information connected to performance goals, for example in terms of success or failure on a specific task or part of it. Hattie and Timperley (2007) stress that feedback is effective when "it consists of information about progress, and/or about how to proceed" (p. 89) and is more effective when its focus is on correct rather than incorrect responses. From the teachers' point of view, it provides information that helps them understand how they (and their students) are going in terms of the criteria for success. Finally, the last question aims to provide information that leads to broader possibilities for learning (and teaching). For teachers, it means enhancing strategies and processes to work on the tasks, contents, situations, students' understanding and so on.

Mathematical competence

Dealing with tasks involves several and different mathematical competencies; for example, in determining the domain of a function or calculating the result of an expression: students have to know specific properties or theorems, use mathematical symbols and formalism, but they also have to deal with representations and undertake mathematical reasoning. Therefore, a test composed of a certain number of tasks allows for the collection of rich data in terms of competencies.

Niss and Højgaard (2019) define mathematical competence as being composed of two main components: *Posing and answering questions in and by means of mathematics* and *Handling the language, constructs and tools of mathematics,* further divided into four competencies each.

Handling the language, constructs and tools of mathematics is characterised by the ability to read, interpret, translate and use a wide range of representations (e.g. verbal, symbolic, tabular, graphic, or

visual) of mathematical objects, relationships and processes (*Mathematical representation competency*); the ability to relate to and deal with mathematical symbols, symbolic expressions and the formalisms that govern them (*Mathematical symbols and formalism competency*); engaging in written, oral, or gestural mathematical communication in different genres, styles and registers (*Mathematical communication competency*) and the ability to handle aids and tools for mathematical activity, which may be physical or non-physical (*Mathematical aids and tools competency*).

On the other hand, *Posing and answering questions in and by means of mathematics* involves posing and understanding generic mathematical questions, distinguishing between types of mathematical statements, and engaging in abstraction and generalization of concepts across varying contexts and domains (*Mathematical thinking competency*). Additionally, it requires the ability to pose and solve mathematical problems in a wide variety of mathematical domains and to critically analyse and evaluate one's own and others' solutions to problems (*Mathematical problem handling competency*) as well as the construction of models in extra-mathematical domains (*Mathematical modelling competency*). Finally, this component includes the analysis or production of arguments to justify mathematical statements, rigorous demonstration based on logical deduction from certain axioms, and similar processes (*Mathematical reasoning competency*).

Mathematical competence is expressed in various contexts and situations and requires a great variety of mathematical resources. Hence, each competency can never be fully or exhaustively possessed by individuals. Niss and Højgaard (2019) describe the mastery level of each competency through three dimensions: the *degree of coverage,* which is the extent to which all aspects framing the competency are part of the individual's baggage; the *radius of action,* which refers to the range of different circumstances and situations in which an individual can effectively apply the competency; and finally, the *technical level,* which indicates the degree of sophistication in the mathematical concepts, results, theories, and methods that the individual can utilize when demonstrating the competency.

Suggestions for designing and evaluating tests through the lens of mathematical competence

Lemmo (2023) describes some suggestions for outlining mathematical competence as a multidimensional and complex construct in a written test. In this perspective, both students and teachers could grasp the multi-faceted nature of teaching and learning mathematics.

As said before, a student needs to use different competencies with different mastery levels to solve a task; therefore, the teachers should highlight such aspects in both the *design* and *evaluation* phases (Drüke-Noe, 2023). In the design phase, teachers choose the tasks that compose the test; in the latter, they focus on the analysis and interpretation of the collected data. In both processes, teachers should focus on all the eight competencies with respect to different mastery levels. For example, they should propose more than one task (with different levels of complexity) for each competency to gather rich information and go beyond the results of the test and the number of right/wrong answers.

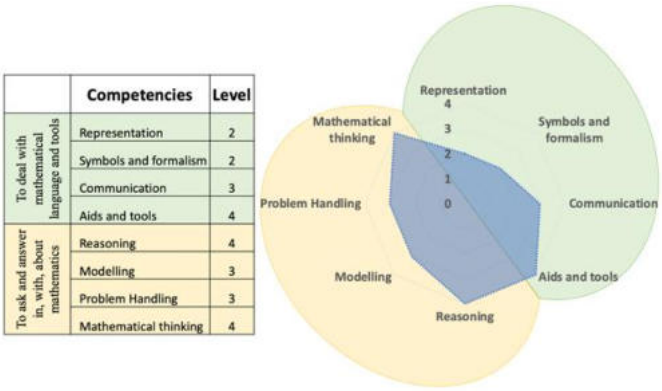
If teachers design tests composed of procedural tasks and they only provide a positive message when faced in correct results, this could be perceived by the students as a focus on procedural and not conceptual or relational aspects. This phenomenon is confirmed by Black and Wiliam (2009):

typically encourages superficial and rote learning, concentrating on recall of isolated details, usually items of knowledge which pupils soon forget . . . teachers do not generally review the assessment questions that they use and do not discuss them critically with peers, so there is little reflection on what is being assessed. (p. 17)

Conversely, the assessment message could change to favor mathematical processes and ideas, when teachers emphasize the competencies students employ. In this perspective, tests could provide feedback that enhances the processes used and the metacognitive attributes of each task.

Niss and Højgaard (2019) describe each competency in dept, paying attention to different tasks and assignment that relate to them. However, the three dimensions (degree of coverage; radius of action and technical level) are not explored in detail in their study. To better describe these dimensions, Lemmo (2023) translates them into 5 levels. Students at level 1 exhibit a limited space of work, as they depend on resources provided by the teacher, operate within a restricted range of familiar situations, and demonstrate a low technical level. In contrast, students with high competency (level 5) display a wide range of action across all situations, including unfamiliar ones. They possess an extensive space of work that reflects the use of personal resources, and they manage a refined technical level with autonomy and consistency.

Figure 1 illustrates what the teacher should compile and provide to the student during the test evaluation. The message given to the students focuses on the level of competencies they master in areas such as technical level, resources, and situations. The example in Figure 1 describes a student who exhibits an advanced level in asking and answering questions in, with, and about mathematics but remains at a basic level in using mathematical language and tools. The data presented in Figure 1 allow observation and interpretation of the student's actual situation in the test and could guide the student and the teacher in addressing situations that focus on representation competency in the future.



	Competencies	Level
To deal with mathematical language and tools	Representation	2
	Symbols and formalism	2
	Communication	3
	Aids and tools	4
To ask and answer in, with, about mathematics	Reasoning	4
	Modelling	3
	Problem Handling	3
	Mathematical thinking	4

Figure 1: Example of a student and teacher' s return form our suggestions (Lemmo, 2023, p. 4008)

This form of feedback encourages the selection of comprehensive tasks, emphasizing not the solution the student arrives at, but the approach taken to each task. Consequently, the emphasis on the number or severity of errors diminishes, shifting focus to how the student engages in solving tasks and the test as a whole. The suggestions seem to fit the model of powerful feedback. They seem coherent with feed up (1), feed back (2), and feed forward (3): each competency with its level shows both

students and teacher where and how they are going; further, reflecting on the different levels provides information about where to next.

In this contribution, we aim at investigating whether and how using mathematical competence in designing and evaluating written mathematical tests contributes to powerful feedback in terms of *feed up*, *feed back*, and *feed forward* from the teacher's perspective.

Methodology

We present the suggestions for designing and evaluating written mathematical tests in a teacher training program that involved five teachers in a primary school of a small city in central Italy (Lanciano, Chieti). Only one teacher was available to be observed and interviewed during the design and evaluation phases. We asked her to use the suggestions for designing and evaluating a mathematical test, we observed and interviewed her during both phases. In line with the aim of this contribution, we focus on the observation and interview related to the feedback she received from one written test and the data collected in one of her classes. We particularly observed the teacher at work on the evaluation phase of the test; during this observation, we asked her for more details about her choices and the feedback she received from the collected and evaluated data of the test.

The teacher has been teaching mathematics for 10 years, and she has a degree in Primary Education Sciences, which means that she attended courses and successfully passed exams in Docimology and Experimental Pedagogy. The class consists of 20 fourth-grade children, including 9 girls and 11 boys. According to the teacher, the class has a relaxed and collaborative mood, where the children have established a solid relationship with each other and with their mathematics teacher.

Results and discussion

Before the design and evaluation phases, we discussed about mathematical competence. The description of the 8 competencies was not immediately clear and easy for the teacher to use. She suggested decreasing the number of competencies to consider in order to make the design of the test simpler and the presentation of the results to the students clearer. In particular, she suggested grouping some competencies. In particular, she gathered: Symbols and formalism & Aids and tools, Reasoning & Mathematical thinking, and finally, Modelling & Problem solving. We asked her "what do you usually want to assess through a test and how you evaluate it?" The teacher answered that she usually focuses on mental calculation, confidence in moving through the different algorithms, and simple problem solving, and she evaluates test by assigning a score for each task.

In design process, the teacher did not reveal difficulties in proposing tasks. However, tasks involving Communication and Problem handling/Modelling were the only challenges for her. This provides an interesting feed up for her: she is not familiar with situations and tasks related to Communication and does not usually present her students problems requiring a high level of competency (in terms of unfamiliar situation). At the conclusion of the design process, during the interview, the teacher recognised that she had never structured a test in that way. She usually proposes tasks connected to familiar situation and so she has never proposed tasks requiring activation of high levels of competence. This awareness allowed the teacher to rethink her methodological choices in the past (feed up) in order to modify them in the future (feed forward). Regarding competencies, she decided

to promote more activities focused on communication and problem solving in the future. Furthermore, she also suggested to presenting unfamiliar situations to her students.

In evaluation process, the teacher showed concern. First, she was afraid of how time-consuming it would be. Looking at the whole test instead of evaluating each individual task seemed for her more difficult. In her experience, she used to evaluate tests using scores; the sum of tasks' scores determines the final mark. For example, she assigns scores corresponding to each correct task, then she evaluates the students' products counting the errors or missing answers she identifies; finally, she subtracts part of the points from the task's score. This procedure contrasts with our suggestion to evaluate tasks by analysing the competencies students demonstrate through the 5 levels. In the interview after the evaluation process, the teacher revealed an interest in processes rather than products, even though in the past she focused her attention on single tasks without offering her students an overview that extended beyond the specific test procedures and knowledge. The teacher stated that she usually does not provide written or articulated messages for students, but she communicates the final scores of each task and their sum. However, when she described the scores, she focused on the processes activated by the students and not simply on the outcome. In a different way, suggestions offer a description of the processes activated in the whole test by giving a more general perspective.

Without any suggestion from us, the teacher decided to average the levels of each competency her students demonstrated in the test (Figure 2). She stated that this information was useful for having a snapshot of her students; in this way, she gained a clear awareness of the situation in terms of competencies and their corresponding levels. With respect to Reasoning & Mathematical thinking, the class reached an intermediate level, demonstrating the ability to recall properties of mathematical concepts and to reason in mostly familiar situations. Considering Representation, Symbols and formalism & Aids and tools and Communication, the class ranked slightly above the basic level. An area for improvement is Problem Handling & Modelling, where students achieved the lowest level.

	Competencies	Level
To deal with mathematical language and tools	Representation	2
	Symbols and formalism & Aids and tools	2
	Communication	3
To ask and answer in, with, about mathematics	Reasoning & Mathematical thinking	5
	Problem Handling & Modelling	3

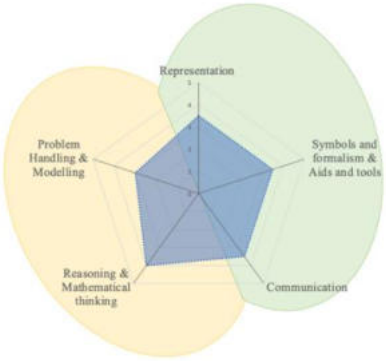


Figure 2: Average levels of class's competency

In the interview, the teacher stated that the graph representing of each student and the average was not enough: she could not understand how and where her students were distributed among the different competencies and their corresponding levels. For this reason, she independently constructed a new graph (Figure 3).

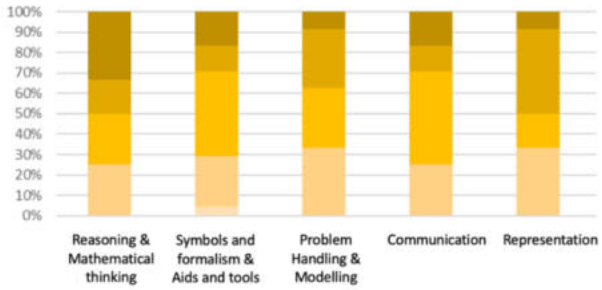


Figure 3: Levels of class for each competency

This graph enriches the feed back provided by the results. For example, although in Representation, Symbols and Formalism & Aids and Tools the class shows a similar level, in Representation there are several students with low level. Similarly, the average of Communication and Handling problems & Modelling is equal, but there are few students who reveal high levels in Handling problems & Modelling. On the other hand, concerning Reasoning & Mathematical Thinking, there is very high number of high-level students. Symbols and Formalism & Aids and Tools is the competency that most surprised the teacher because it is the only one that includes level 1 students. These data are interesting for increasing teacher awareness of students' situation and how to move class activities forward. She was surprised because the competencies on which to develop future activities were not the same as the design phase had led her to assume. Teacher stated that in the future she will work more on problem-solving and communication, also involving representations and mathematical symbols and tools. Furthermore, she will try to present students with more unfamiliar situations that require them to use personal resources to promote higher levels of competency.

Conclusions

In previous contribution (Lemmo, 2023), we offered suggestions for designing and evaluating a written mathematics test. These suggestions have been presented as a theoretical perspective; in this contribution, we aim to show how a primary school teacher used them and whether and how these suggestions contributed in terms of the feedback she received. From a general perspective, working with and interviewing the teacher showed that the suggestions provided feed up, feed back and feed forward. The design process provided feed up in terms of the activities she proposed to her students: she realised that she used to present familiar situations that required students to use resources provided by the teacher in communication and problem solving. The average of levels (Figure 2) and the graph of the levels for each competency (Figure 3) confirmed this consideration and provided feed back in terms of some specific competencies: Representation, Symbols and Formalism & Aids and Tools.

The suggestions used by the teacher and her personal adaptation seem to give her an overview of the class. In this regard, we did not imagine that our suggestions would prompt the teacher to customise the analysis tools, driven by her desire to know even more about her class in terms of competency levels (Figure 2 and 3). In addition, the teacher's suggestion to decrease the number of competencies to consider is interesting: she chose to merge the 8 competencies into 5. In further study, it will be interesting to understand in depth why this choice was made and whether it is more or less functional

in terms of feedback. This might suggest using different theoretical frameworks on mathematical competence, composed of a smaller number of components (e.g., Kilpatrick, 2001).

There are also some critical points: for example, exploring the suggestions with only one teacher does not allow for general and solid conclusions. To get a clear picture of the feedback, it is also essential to take into account the reaction of the class and to observe the teacher's choices over time. In addition, it would be interesting to discuss and describe the design and evaluation processes in detail. In this case, the test concerned numbers (multiplication between natural numbers); exploring other areas may provide further details to foster this research. The aim of this study was to analyse how a teacher used our suggestions and explore the information that their use gives back her in terms of feedback. We did not impose any particular mathematical content on the teacher in order to leave her freedom of choice and greater fluency in using the suggestions in both design and evaluation phases. This choice does not allow general conclusions on how to move the teaching-learning process forward. We do not delve into details about the difficulties/obstacles encountered by students and how the teacher will intervene in the future. At a later stage, we will focus on more specific aspects in order to provide pathways and training for moving students' learning forward.

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Evaluating a multi-indicator assessment scoring approach to support mathematical literacy teaching and learning

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This study examines the effectiveness of a scoring approach for mathematical literacy assessment by evaluating each response of the students using multiple indicators. This approach could be helpful in order to assess higher-order thinking skills and problem-solving in complex tasks. This method allows for detailed feedback and enables partly effective differentiation of student abilities. 122 students took a test with six items of varying complexity. Using this scoring approach, these six tasks form 17 individual items. A Rasch analysis confirms the reliability and validity, although improvements in the person separation index are necessary. The results highlight the method's potential for providing precise feedback, supporting tailored instruction and targeted interventions based on individual student performance, and it is essential when designing a national assessment. Further research is needed to explore the possibilities of applying this approach in other contexts.

Keywords: Mathematical literacy, Rasch analysis, assessment scoring approach, marking scheme.

Introduction

Latvia's national assessment system does not support the achievement of national objectives (Cross-Sectorial Coordination Center, 2020), as shown by the results of the Programme for International Student Assessment (PISA). The PISA 2022 results reveal that Latvian students perform above average in mathematics at the lower proficiency level but below average at the higher level (OECD, 2023b). Although individual student data could potentially enhance learning quality, it remains not used due to concerns regarding the ability of assessments to accurately differentiate between students' capabilities (Pestovs & Namsone, 2019). Previous research on exams has shown that they lack tasks with high cognitive demand, tasks with authentic contexts and tasks that require explanation (Mikite & France, 2023; Pestovs & Namsone, 2019). This type of exam gives the misleading illusion that completing several tasks with similar low cognitive demand indicates high performance. This is why the National Assessment in Primary School Education is being developed as a part of a support system for learning mathematical literacy. The new assessment aims to provide feedback at different levels of the education system – municipality, school, classroom and, most importantly, at student level. To measure higher-order thinking and problem-solving skills, students are often required to solve complex extended tasks. The challenge lies in designing assessments that evaluate these skills effectively, especially when the time for students to complete tasks is limited (Mertler, 2003). Time limits lead to using a small number of tasks, which results in low reliability and limits the ability to differentiate between students based on their performance (Bond & Fox, 2015). This pilot study aims to develop an assessment scoring approach for this kind of assessment where students solve complex problems. The new scoring approach evaluates a single student response using multiple indicators, treating each indicator as a separate item. The research question is whether this scoring scheme ensures both reliability and validity while allowing for the effective differentiation of students based

on their performance levels. This involves examining if the scoring method can consistently measure student responses and accurately reflect their varying abilities in mathematical literacy.

The concept of mathematical literacy is operationalised in various ways by different authors. The PISA study models three components of mathematical literacy, i.e. content, process, and context (OECD, 2023a). Another study identifies the following elements of mathematical literacy: (1) Mathematical knowledge, (2) Contexts, (3) Dispositions, (4) Tools, and (5) Critical orientations (Bolstad, 2020), building on research by Goos and colleagues on numeracy (Goos et al., 2014). It can be concluded that mathematical literacy is the ability to formulate problems, which requires you to understand a situation in multiple ways, for example, in texts, diagrams, and pictures. After that mathematical procedures need to be applied and solutions need to be interpreted based on the content and context of problems. Mathematical literacy determines the ability to succeed in private, social and professional areas. After graduation, citizens are expected to be creative and engaged, making non-routine judgments (OECD, 2023a). When developing an assessment, it is crucial to ensure that the essential components of mathematical literacy are retained so that the measurement accurately reflects what it intends to capture.

For assessing mathematical literacy, the two most common approaches are structured questions and selected-response questions (May et al., 2023). Selected-response (e.g., multiple choice, true/false, matching) questions are often chosen because they are easier and quicker to assess due to the use of dichotomous scoring. However, more comprehensive information on students' performance can be obtained by analysing students' responses to structured tasks (Mertler, 2003). These approaches naturally lead to two main scoring approaches – dichotomous scoring, which determines whether an answer is correct or incorrect, and partial credit, which allows for multiple levels of correctness. A national diagnostic evaluation should combine these two approaches to ensure the data obtained are reliable and valid, provide the most accurate feedback possible to support future learning, and are relatively easy and fast to score.

Method

This pilot study involved 122 students from three schools, differing in urbanisation level and average performance in national mathematics exams. This test was taken in the early semester of a 7th grade. The students completed the test in 60 minutes. The authors then scored the test using the assessment approach, that is, scoring one answer with multiple indicators. In this way, the six tasks form 17 items, of which 13 are dichotomous, and four are scored partial credit at three performance levels. The national assessment is in its initial stages of development, and this is the first pilot study.

In developing the national assessment, the authors have formulated 12 major indicators of mathematical literacy. They are grouped into three categories corresponding to the problem-solving steps: formulating, employing and thirdly interpreting and communicating. The test studied in this research measures four of the major indicators: (1) recognising a mathematical situation in given texts in different contexts, (2) creating or drawing objects, figures in space and on a plane, their arrangement on a plane, observing quantities and relationships, (3) calculating length, area and volume in real situations using properties of objects, figures in plane and space, (4) evaluating a mathematical solution and interpreting it.

The tasks were selected with varying complexity based on the framework developed before (Mikite et al., 2024). The framework consists of three dimensions: (1) the context of the situation in which the problem is given, (2) the complexity of the mathematical model, and (3) the strategic complexity connecting the context and the mathematical model. Each dimension has three possible levels of complexity. Six tasks with different levels of complexity were selected to form the student test.

The test specification is given in Table 1. The idea of the Q-matrix (Tatsuoka, 1983) is adopted in developing the specification. The columns represent the major indicators described above, and each row represents a task in the test. Instead of ones, as in a traditional Q-matrix, this table shows the corresponding minor indicators assessed in each task. For convenience, the levels of complexity in three dimensions are added on the left side of the table according to the framework in Table 1. D1, D2 and D3 stand for the three dimensions of complexity described above.

Table 1: Test specification

Level of complexity			Task	Indicator 1 (formulating)	Indicator 2 (employing)	Indicator 3 (employing)	Indicator 4 (interpreting and comunicating)
D1	D2	D3					
1	1	1	**1.**	1_1 Recognises a geometric solid in a picture	1_2 Understand what plane shapes form a solid in a picture		1_3 Provide an answer using text and drawing of figures
2	1	1	**2.**	2 Find the necessary information from a variety of sources			
1	2	2	**3.****	3_1 Understands a situation given in simple text and picture	3_2 Locates a geometric figure on a plan and subdivides the figure to calculate area	3_4 Calculate the area of a combined figure using the rectangular formula	
3	2	3	**4.**	4_1 Understands a situation given in a complex text that sets limitations *	4_2 Demonstrate an understanding of the circle and the relationship between its elements *		
2	3	3	**5.**	5_1 Understands the situation given in the text	5_2 Demonstrates a deep understanding of the positioning of spatial figures	5_3 Determine the largest number of small rectangular prisms contained in the largest *	5_4 Explains the solution
3	3	3	**6.**	6_1 Understand a new/unfamiliar situation in simple text and picture		6_2 Estimates the volume *	6_4 Justify the solution

** The partial credit scoring is used to assess these indicators.*

*** There is another indicator, 3_3 measured in this task. It is beyond these major indicators as it relates to measurement, which is a separate primary indicator in the construct.*

The assessment scoring approach analysed in this study is based on the concept that each minor indicator is evaluated separately. The item numbers associated with these indicators are displayed in

the table and are used for further analysis in the study. Task 5 in Figure 1 serves here as an example to illustrate the scoring.

The dimensions of a small calendar are 1 cm x 11 cm x 15 cm. What is the largest number of these calendars that can be sent in a box with dimensions 16 cm x 24 cm x 9 cm?

Figure 1: Task 5

A student's answer is analysed using four different indicators. The first is whether the student understands the situation and question – that smaller objects must be combined into a larger one. The second is if the student can draw or mentally rotate the given shapes and start to solve the given problem. The third indicator is rated partial credit. One point is assigned if the student finds the solution *9*, which is a valid solution but not the best. Two points are received in item 5_3 if the student finds the best solution, which is *18*. In 5_4, the student receives a point if he explains his solution.

To answer the research question of whether this scoring scheme ensures both reliability and validity while allowing for the effective differentiation of students based on their performance levels, we run Rasch analyses using Winsteps 4.6.2. We analysed reliability, person and item separation indices and general targeting using a Wright map. Local independence and unidimensionality were analysed using Winsteps Table 23 to validate this scoring approach. Rasch model fit statistics were analysed to evaluate the individual items, which in this case are assessments of individual indicators (Bond & Fox, 2015). To examine if the scoring approach can consistently measure student responses and accurately reflect their varying abilities in mathematical literacy, we analyse how students can be grouped based on their performance levels and the type of individualised feedback that can be provided to each student. This approach ensures the method can differentiate between students' abilities and offers precise feedback tailored to their specific needs.

Results

The overall reliability of the test demonstrates strong internal consistency, with a person reliability of 0.75 and an item reliability of 0.97, indicating that the test items are consistent and stable (Bond & Fox, 2015). The item separation index of 6.19 further confirms that the test items cover a wide range of difficulty levels, allowing for precise differentiation between them (Linacre, 2011). However, the person separation index of 1.73 suggests that the test is less effective in distinguishing between students of different ability levels. One of the reasons for the low separation index could be poor targeting.

The overall targeting between item difficulty and student ability is partially successful (Figure 2). There is a significant gap between the mean value of students and items. 67% of students are below 0 logits, which is the item mean. This indicates that the test was too difficult for this group of students, or in other words, lower-performing students needed more tasks to demonstrate their performance. There are 41 students (34% of all students) in the -1 to -3 logit range, but only two items (12% of all items). This described mismatch helps to explain the low person separation index observed in the results, but it does not directly address the scoring aspect, which is the primary focus of this study.

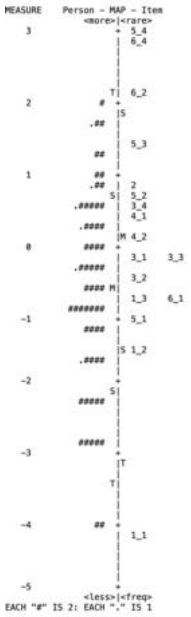

Figure 2: Wright map

A serious concern when assessing a single response with multiple indicators is whether local independence between these items is compromised. Table 2 shows the largest standardised residual correlations. All positive values are between 0.2 and 0.5, indicating that there are no significant problems that could seriously degrade the quality of the measurement (Linacre, 2011).

Table 2: Largest standardised residual correlations

Correlation	Item	Item
0.49	5_2	5_3
0.46	3_1	3_3
0.45	6_2	6_4
0.45	3_3	3_4
0.45	3_1	3_4
0.40	5_3	5_4
0.38	1_2	1_3
0.38	6_1	6_2
0.35	5_2	5_4
0.35	3_1	3_2

Correlation	Item	Item
0.31	5_1	5_2
0.30	4_2	4_1
0.26	5_1	5_1
-0.41	1_1	5_4
-0.35	3_4	4_2
-0.31	3_3	4_2
-0.29	3_2	5_2
-0.28	3_3	5_3
-0.26	3_4	6_1
-0.25	3_1	5_1

As the following criterion for evaluating measurement quality, dimensionality was assessed to verify that the test measures a single underlying construct. The Rasch model explains 43.5% of the total variance, while 56.5% remains unexplained. Notably, the eigenvalue for the first contrast is 2.75, which exceeds the critical threshold of 2, suggesting potential multidimensionality in the data. Furthermore, the second contrast's eigenvalue is 2.48, which also approaches the threshold, indicating that these contrasts may reflect a secondary dimension. Despite these findings, the percentage of variance explained by the first contrast (9.1%) and the second contrast (8.2%) is relatively low, suggesting that any multidimensionality is weak and may not significantly impact the unidimensionality assumption required by the Rasch model (Linacre, 2011).

Table 3 summarises the key item statistics, including item measures, Infit, Outfit, and Point-Measure correlations (PTMEASURE-AL). The Infit and Outfit mean square (MNSQ) values generally fall within acceptable ranges (0.5 to 1.5), indicating a good model fit for most items (Linacre, 2011). Additionally, the PTMEASURE-AL correlations, representing the correlation between each item's score and the overall ability measure, are positive and mostly strong. This suggests that the items are well-aligned with the measured construct and contribute effectively to differentiating between individuals based on their ability level.

Table 3: Item Measure, Infit, Outfit, and Point-Measure Correlation

ITEM	MEASURE	INFIT		OUTFIT		PTMEASUR-AL CORR
		MNSQ	ZSTD	MNSQ	ZSTD	
1_1	-4.27	1.13	0.52	2.12	1.43	0.18
1_2	-1.62	1.21	1.86	1.22	1.17	0.39
1_3	-0.92	0.99	-0.10	0.99	0.01	0.52
2	0.65	1.04	0.40	1.34	1.16	0.38
3_1	-0.42	0.82	-2.08	0.99	-0.03	0.60
3_2	-0.69	0.84	-1.78	0.73	-1.84	0.62
3_3	-0.42	0.96	-0.46	0.90	-0.58	0.53
3_4	0.44	0.83	-1.67	0.65	-1.49	0.57
4_1	0.19	1.05	0.39	1.06	0.44	0.50
4_2	-0.05	1.23	1.75	1.20	0.81	0.55
5_1	-1.19	1.01	0.10	0.99	0,00	0.52
5_2	0.54	0.93	-0.64	0.75	-0.92	0.50
5_3	1.19	1.11	0.64	0.91	-0.01	0.44
5_4	2.84	0.97	0.03	0.94	0.16	0.24
6_1	-0.92	1.05	0.53	1.06	0.41	0.49
6_2	1.98	0.90	-0.61	0.71	-0.93	0.51
6_4	2.67	1.01	0.15	0.86	0.03	0.22

In summary, the Rasch analysis indicates that the measurement is reliable and valid in terms of item performance, with high item reliability and item separation. This suggests that the scoring method itself is effective. However, the low person separation index highlights challenges in differentiating between students' abilities, pointing to the need for improved test alignment to better capture the range of student abilities.

Based on the wide range of item difficulty, it is possible to group items and set boundaries on the Wright map to define performance levels. The descriptions of each level are formed based on the content of the tasks within each group. However, if the person separation index is low, it may be more effective to provide feedback based on raw scores. This scoring approach enables the formulation of highly accurate and detailed feedback reflecting both correct aspects of student performance and areas for improvement (Bond & Fox, 2015).

Discussion

This method builds upon studies where such an approach, measuring one response using multiple indicators, was analysed using qualitative methods, such as Muliawanti & Kusuma (2021). Quantitative data analysis in this study is crucial in the context of national assessments, where the data must demonstrate both reliability and validity to ensure data-driven decisions about student performance and instructional support.

This approach not only provides precise feedback about what the student did correctly and where improvements are needed, but it is also easier to implement in practice. Most indicators can be assessed dichotomously, simplifying the evaluation process while maintaining accuracy. This makes the method particularly efficient for large-scale assessments, where rapid and reliable scoring is essential (Bond & Fox, 2015).

Future research should focus on increasing the test's ability to distinguish between students with different ability levels, as indicated by the person separation index. One way to achieve this is through better targeting, ensuring that the test aligns more closely with student ability levels. Additionally, two other factors limit the current study. First, the Rasch analysis was conducted on a relatively small sample, which may affect the stability of the estimates. Second, low student motivation or engagement during testing can lead to underperformance. If students are disengaged or do not take the test seriously, their responses may not reflect their true ability, resulting in poor person separation even with well-targeted items (Mislevy, 1996). Third, although mathematical literacy is part of the curriculum, it is unclear how explicitly teachers emphasize it in practice. Since mathematical literacy involves broader problem-solving skills than traditional mathematics, this gap may also contribute to mismatches between student ability and test performance (Bond & Fox, 2015).

To strengthen the current findings, additional studies in varied educational settings are essential, as the current results are limited to the specific context of this pilot. Conducting similar studies in diverse contexts would help to verify whether this scoring approach is consistently reliable and valid for broader applications, such as national assessments.

Conclusion

The scoring approach used in this study, where a single student response is assessed using multiple indicators, effectively measures mathematical literacy with high reliability and validity. This method enables detailed, individualised feedback that identifies student strengths and areas for improvement, which is crucial for supporting learning in national assessments. However, improving the person separation index would enhance the ability to differentiate students more precisely, allowing for a better understanding of their abilities. This allows educators to understand what students can and cannot do, providing evidence-based insights to adapt the curriculum and plan appropriate interventions.

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The contribution of mathematics didactics to interpreting the results of a standardized national assessment

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Standardized national and international assessments are widely used to evaluate student learning, but how can they be turned into tools for education professionals to help teachers better meet students' learning needs? This paper provides tools from mathematics didactics for interpreting, in a way that complements statistical methods for item responses, the performance scale of students' mathematical skills in a French national assessment. Our contribution consists in the didactic characterization of the assessment items and in relating the scale groups to an a priori defined level of mathematical activity. In this way, we characterize the knowledge mobilized by each group and rank them from one group to another in relation to the expectations of the school curricula. The analyses focus on school algebra at the end of middle school.

Keywords: Assessment, students' learning, school algebra.

Introduction

Standardized national and international assessments are widely used to evaluate students' mathematical knowledge and skills. However, they are rarely used by education professionals (inspectors, teacher trainers, teachers), especially in France, to help them manage heterogeneous learning more effectively in the classroom. It seems to us that they can provide interesting clues for managing heterogeneous learning in the classroom. To this end, we show that mathematical didactics can provide tools for interpreting the results of standardized assessments that are complementary to those of statistical methods used in such assessments.

In France, the Cycle des Évaluations Disciplinaires Réalisées sur Échantillon[1] (CEDRE) assessment of students' mathematical skills at the end of middle school in 2019 was designed to provide an overall view of students' knowledge and skills at the end of middle school in relation to the objectives set by the French curricula (Ninnin & Salles, 2020). Every six years, it is conducted by the Direction de l'Evaluation, de la Prospective et de la Performance (DEPP) using a statistically representative sample of the school population in metropolitan France.

In CEDRE, the statistical study of student responses leads to a six-group performance scale, determined by a psychometric approach using item response methods. The number of groups is set by the DEPP. The average score corresponds to the average performance of the students in the initial assessment sample (in 2008). The DEPP sets it by construction at 250 and the standard deviation at 50. This means that around two-thirds of students have a score between 200 and 300. The lowest

[1] https://www.education.gouv.fr/cycle-des-evaluations-disciplinaires-realisees-sur-echantillon-cedre-en-fin-d-ecole-et-fin-de-2870

groups on the scale include the scores of the 15% of students with the lowest scores. Those at the top are made up of the highest scores and include 10% of the students. In between, the scale is divided into three groups with equal score amplitudes. The performance scale is therefore developed *a posteriori*, i.e., on the basis of student responses. Because the item response model places student scores and item difficulty on the same scale, it determines the skills mastered by each group of students through their success on items that are characteristic of the group, i.e. items that students in the group pass with at least a 50% probability. This characterization of groups indicates the performance of students in terms of knowledge and skills, but in a very general way. It does not prioritize the learning outcomes of different groups with respect to the types of tasks in the same mathematical domain, and the knowledge and reasoning mobilized by students. In our view, this is essential if we want CEDRE results to be usable in the classroom to better manage the heterogeneity of students' learning. In addition, depending on the way an item is constructed, it can be passed through different procedures involving different types of knowledge that are more or less old in the curriculum. A detailed didactic analysis of the items is therefore necessary to identify the knowledge, properties and reasoning that can be mobilized to solve them and that can be taught at previous schooling levels. This paper addresses the following question: what kind of didactic analysis of the items and the students' results, in particular their difficulties, would help to link the performance scale groups to a hierarchy of knowledge as taught in the curricula? We illustrate this in the field of algebra.

Theoretical background

Didactic characterization of an item with the ATD

Each CEDRE item corresponds to a task. In the Anthropological Theory of Didactics (ATD), the mathematical activity involved in solving a task is modeled by two blocks: that of know-how, described by the type of task and the technique(s) enabling it to be carried out, and that of knowledge, described by the properties and the reasoning justifying these techniques (Chevallard and Bosch, 2020). Each assessment task is thus described *a priori* in terms of the type of task to which it belongs, the techniques targeted and possible, and their justification.

A task can be accomplished by one or more techniques, but these techniques do not all have the same scope (Chevallard and Bosch, 2020), i.e., their domain of validity. Depending on the characteristics of the task, they may or may not be effective. Some techniques can accomplish a part of tasks of the same type, while failing on another part. For example, the trial-and-error technique, used in primary school, is no longer effective in middle school for problems involving equations of the type $ax + b = cx + d$ with a, b, c, d integers and a and c non-zero integers and with a non-decimal rational solution. These problems, beyond the scope of the trial-and-error technique, require an algebraic technique aimed at solving a much wider range of problems.

Furthermore, a task is characterized by its complexity, that is, the notions it brings into play, their level of operationalization (Robert, 2003; Roditi & Salles, 2015), and the reasoning to be mobilized. In particular, this concerns the new or old nature of the mathematical notions involved, the fact that resolution steps or adaptations are left to the student, and the values of the didactic variables that characterize the task. For example, "develop the expression $3(x + 5)$" is less complex than "develop $2x(-5x + 3)$".

Didactic validity of an assessment

In addition to the psychometric validity of an assessment, we also investigate the didactic validity of an assessment (Grapin & Grugeon-Allys, 2018). It is based on an epistemological reference to the mathematical domain to be assessed, developed from research on didactics in the domain and on school curricula (for instance, for algebra, see Grugeon-Allys et al. (2022) and Grugeon-Allys & Pilet (2024)). The study of the didactic validity of an assessment focuses, on the one hand, at the local level, on the relevance and representativeness of each task (technique used and justification) in relation to what is being assessed. On the other hand, at the global level, it ensures that all the tasks cover the mathematical field in relation to the epistemological reference, i.e. that the types of tasks are sufficiently varied and that tasks of the same type are of different complexity. The validity is therefore based on an a priori analysis of the tasks, aimed at specifying the knowledge involved, inferring possible techniques and situating them in relation to the curriculum.

Some findings on school algebra

A large body of research links students' difficulties to the discontinuities (Kieran, 2007) that exist between arithmetic and algebraic thinking and to the specificity of algebraic semiotic practices. The Kieran's (2007) model of algebraic activity is based on three types of activity. Generative activity concerns the formation of algebraic objects (formulas, algebraic expressions, and equations) in order to solve problems (modelling, generalizing, proving) that requires translation between different registers of representation (algebraic scripts, numerical scripts, graphical representations, geometric figures, natural language). Transformational activity involves transforming expressions and equations in a way that preserves their equivalence. This may involve substitution, development, factoring, simplifying algebraic expressions and solving equations. Transformational activity is based on the structure of objects (sum, product). Global/meta-level activity involves solving modeling, generalization, proof, and equation problems. It leads to algebraic justifications and proofs as opposed to arithmetic reasoning.

These three activities call on the different status of letters (variables, unknowns, as opposed to label letters), equality as a relation of equivalence as opposed to equality "to perform," the complementary procedural and structural characteristics of the objects of algebra.

Three levels of mathematical activity

For each mathematical domain assessed, on the basis of the previous overview of school algebra research results, we (the project researchers) define *a priori* and hierarchize three levels of mathematical activity (Grugeon-Allys et al., 2022; Grugeon-Allys & Pilet, 2024). They inform the conceptualization level of the notions constructed by students. These levels are defined by the hierarchical use of notions, properties, and reasoning expected at the end of middle school in relation to epistemological breaks. They take into account the age of the notions according to the programs, the range of techniques and the complexity of the tasks. To make it easier to present these levels, we will start by presenting level A, which is the level expected by the educational institution, followed by level C, which is the lowest level, and finally the intermediate level B.

Level A corresponds to a mathematical activity that mobilizes expected knowledge and an appropriate level of conceptualization of concepts. For example, in algebra, Level A students mobilize algebra on their own to solve modeling, generalization, proof, and equation problems. They do so in cases that require reformulations between the statement and the register of algebraic language, cases we call semantic non-congruence. They transform (develop, factorize, reduce) algebraic expressions of expected complexity. They know how to mobilize properties (simple and double distributivity, conservation of an equality) and understand expressions in their procedural and structural aspects.

Level C corresponds to a mathematical activity that mobilizes old knowledge from primary school. It indicates a low level of conceptualization of notions, with a fragile entry into the construction of expected knowledge. In the case of algebra, for example, these students have not made the epistemological break between arithmetic and algebra. They often mobilize arithmetic techniques to solve algebraic problems, using examples to prove their point. They use techniques for transforming algebraic expressions and equations whose scope is limited to very simple cases, leading students to use them beyond their reach and to use incorrect rules ($3 + 2x \rightarrow 5x$). They understand algebraic expressions only in their procedural aspect.

Level B is intermediate and indicates a mathematical activity that mobilizes early middle school knowledge in simple, isolated tasks (Robert, 2003). Entry into the mathematical activity aimed at the end of middle school is in progress. For example, Level B students have negotiated the epistemological break between arithmetic and algebra, but in simple cases. They are able to mobilize algebra in some of the expected problems (modeling, proving, equating), often when the use of a letter is suggested to them and in cases that do not require a reformulation between the statement and the algebraic expression.

Method

In CEDRE 2019, the psychometric approach is to divide students into six groups on a performance scale. Each group of students is then *a posteriori* associated with characteristic items, i.e. items that the students in the group pass with a probability of at least 50%. We would like to extend the interpretation of the groups for algebra by associating each group on the achievement scale with a level of mathematical activity and by specifying what distinguishes one group from another in terms of knowledge and reasoning.

Our methods start with the *a priori* analysis of each of the 26 algebraic items according to the didactic tools introduced earlier (type of task, technique, complexity, scope of a technique). As this analysis indicates the complexity and possible resolution techniques, we can link *a priori* each task with the level of mathematical activity it requires. We then carry out *a posteriori* analyses based on the students' responses and the results of the psychometric analyses. Knowing the characteristic items of each group, we relate each group on the performance scale to a level of mathematical activity in algebra by considering the level to which most of the characteristic items belong. In addition, we also compare items within the same type of tasks, but performed by different groups, in order to identify breaks in learning between groups. For certain multiple-choice items, we compare the percentage of responses per distractor between groups. For error-related distractors, this analysis provides

information about the dominant error categories in each group in terms of students' constructed knowledge and conceptual development. These analyses allow us to classify the successful tasks and the knowledge required to solve them by group.

Results

Relating performance scale groups to levels of mathematical activity in algebra

Table 1 shows the results of the *a priori* analyses of the items (but without detailing them in this paper) and their relationship to the groups. In this table, each group has mastered the types of tasks and knowledge built up in the lower numbered groups.

Table 1: Performance scale groups and levels of mathematical activity

Group	Analysis of characteristic items in algebra of the group	Level
5	2 items of level A Tasks involving algebra as a tool for generalization and proof. Solving first-degree equations with rational solutions.	A
4	3 items of level A Modeling tasks in statements that are not semantically congruent with algebraic expressions. Calculation tasks with double distributivity and rewriting involving the denotation of algebraic expressions. Substitution tasks with relative numbers.	
3	16 items: 2 of level C and 14 of level B Modeling tasks in semantically congruent statements with algebraic expressions Tasks to prove that an assertion is true or false. Calculation tasks with simple distributivity, without rewriting and with structural aspect of algebraic expressions. Tasks to reduce simple first-degree expressions with rewriting.	B
2	5 items of level C Tasks involving the representation of a relationship by an algebraic expression or equation in very simple cases with semantically congruent registers. Calculation tasks with procedural aspect of algebraic expressions, allowing arithmetic practice (primary knowledge). Substitution tasks with whole numbers in simple algebraic expressions.	C

1	No item in algebra	
<1	No item in algebra	

Learning algebra remains as difficult as ever in middle school. Students in groups <1 and 1 did not pass any algebra items. This result is consistent with the knowledge and skills of students in these groups in the domain of "numbers and problem solving", also assessed in CEDRE. They were successful in items that mainly involved early primary numerical knowledge, such as the positional aspect of numbers (numeration). They have not sufficiently developed the different aspects of number (decimal aspect, different ways of writing the same number or numerical calculation, properties of numbers and operations, equality as an equivalence relation) to engage in algebra. Group 2 students have a better understanding of the equivalence of equality and the decimal aspect of a number. The three groups <1, 1 and 2 belong to activity level C and therefore have a mathematical activity that is expected in primary school.

More detailed analyses of three items characteristic of groups 2, 3 and 4

CEDRE 2019 includes three items (Table 2) relating to the same type of tasks *Representing a calculation program or its result by an algebraic expression.* Each is characteristic of one of groups 2, 3 or 4.

Table 2: Items succeeded in by group for *Representing a calculation program or its result by an algebraic expression*

Group 2	Group 3	Group 4
Program A: Select a number, square this number, add 2 Which of the following expressions always gives the same result as program A, whatever the value of the starting number x? ○ $(x+2)^2$ ○ x^2+2 ○ 2^2+x ○ $x+2$	Program: Select a number, square this number, multiply the result by 2, add 3 to the result We call a the number chosen at the start. Which formula is associated with this calculation program? ○ $a^2\times 2+3$ ○ $(a\times 2)^2+3$ ○ $a^2+2\times 3$ ○ $(a+2)^2\times 3$	The square of the product of 2.5 and x corresponds to the expression: ○ $2{,}5x^2$ ○ $(2{,}5+x)^2$ ○ $(2{,}5\times x)^2$ ○ $2\times 2{,}5x$

An *a priori* analysis of the items indicates that items in groups 2 and 3 differ in the complexity of the calculation program, which is related to the number of steps (two steps for group 2 and three steps for group 3), but the representations are semantically congruent in both cases. Items in groups 3 and 4 are distinguished by the involvement of procedural and/or structural aspects, semantic non-congruence between representational registers, and the nature of the numbers. In group 4, success

requires the mobilization of additional properties such as the structural aspect of an expression and operative priorities.

Table 3: Percentage of responses per distractor for group 4 item

Distractor	Group <1	Group 1	Group 2	Group 3	Group 4	Group 5
$2{,}5x^2$	30,00%	33,47%	36,07%	38,84%	19,42%	18,22%
$(2{,}5+x)^2$	23,72%	24,24%	17,04%	8,17%	5,62%	0%
$\mathbf{(2,5\times x)^2}$	**23,60%**	**28,97%**	**36,23%**	**49,38%**	**73,54%**	**80,70%**
$2\times 2{,}5x$	7,57%	7,19%	7,47%	2,98%	1,60%	1,36%
No answer	14,97%	6,31%	3,33%	0,43%	0%	0%

In addition, Table 3 also shows the percentages of distractors on the group 4 item. The non-response rate, below 15% for group <1, decreases to zero for groups 4 and 5. Although this item characterizes group 4, almost a quarter of students pass it from group <1 onwards, but errors are present in the majority of all groups below 4. Confusion between double and square (error $2\times 2{,}5x$) concerns around 7% of students in each group up to group 2. Confusion between addition and multiplication (error $(2{,}5+x)^2$) persists strongly up to group 2 and disappears by group 5. Students in group 3 are more likely to recognize multiplication, but without always taking into account the operative priority (error $2{,}5x^2$), which is recognized only by the majority from group 4.

Conclusion

The aim of this research is to interpret the six-group performance scale obtained from the psychometric analysis in a complementary way, in order to better identify the knowledge acquired at the end of middle school and the errors in each group, and to hierarchize them from one group to another as taught in the curricula. To this end, we have proposed methods based on epistemological and didactic contributions and applied it to the domain of algebra. The *a priori* analysis gives access to the conditions for item success and allows an *a posteriori* interpretation of the groups to which characteristic items. We then associate a level of mathematical activity for algebra with each group on the performance scale. One limitation of the study is that it focuses only on the domain of algebra and only on the CEDRE 2019 assessment. However, the approach seems transferable to other mathematical domains and other assessments, in particular the national diagnostic assessments at the beginning of the year offered at all school levels in France.

We hypothesize that this new characterization of groups will be a lever to make the CEDRE results more usable by trainers and inspectors in the service of teachers' practice and students' learning. With an appropriate transposition, carefully highlighting what distinguishes one group from another, this new characterization can be used in training courses to indicate what is a priority in regulating teaching and supporting students in their learning.

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Changing the assessment paradigm – What predicts students' choice of assessment method in an undergraduate mathematics course?

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In this paper, we investigated undergraduate students' perceptions on mathematics assessment by letting them choose the summative assessment method in a mathematics course and studying what predicts their choice. The participants (n=333) were given two choices for the summative assessment method of the course: they could choose whether their grade was determined by self-assessment or examination. Using deductive content analysis, we analysed the justifications students gave for their choices. Most common reasons for the students' choices were Perceived usefulness, Psychological safety and Affective attitude. A further inductive content analysis revealed that in these categories students valued learning gains, accuracy of assessment, absence of stress, and new experiences. The results give indications on what aspects should be considered when designing university mathematic assessment, especially when introducing new assessment methods to students.

Keywords: Self-assessment predictors, student perceptions, undergraduate mathematics.

Introduction

Assessment is changing in universities to reflect the needs and affordances presented by the modern society. For some time already, researchers have emphasised the need to increase student agency in assessment (Adie et al., 2018). Recently, the COVID-19 pandemic shook the academia as teachers were forced to find innovative methods to compensate for the lack of traditional classroom exams. One such method is self-assessment. Self-assessment has the benefit that it is independent of time, space and equipment constraints. It can also not be outsourced to another person. Moreover – and more importantly – practising self-assessment increases self-regulation and reflection skills (Panadero et al., 2017) and it is an important aspect of *authentic assessment*, as it is necessary for successful workplace performance and personal development (Ashford-Rowe et al., 2014).

Despite the benefits associated with alternative assessment methods, the academic community seems resistant to changing the prevailing testing and grading focused assessment practices, making self-assessment applications relatively rare at the course-level (Boud et al., 2016). Some studies have found that in mathematics, not only teachers but also students prefer traditional closed-book exams, as they find them the most accurate and trustworthy assessment method (Iannone & Simpson, 2017). Therefore, to change the assessment culture, one of the things we need to understand are the factors that influence students' attitudes towards assessment. As students' preferences may depend on the field they study and its traditional assessment regime (Iannone & Simpson, 2017), it is important to study student perceptions in different disciplines.

Self-assessment predictors and perceptions

Self-assessment has garnered increasing interest due to its promising impact on self-regulated learning (Panadero et al., 2017) and academic achievement (Yan et al., 2022). As self-assessment is a learning process that demands volitional effort and commitment, it is crucial to pinpoint the factors that could influence or predict self-assessment. This is particularly relevant in the context of digital learning environments where students are expected to be autonomous learners, responsible for conducting learning activities such as self-assessment independently and under their own control. A lack of comprehension regarding the potential facilitators and obstacles of self-assessment may impede the realisation and optimisation of its benefits.

A review by Yan et al. (2023) found that both individual and instructional factors play a role in shaping students' engagement with self-assessment. Individual factors refer to students' cognitive and psychological needs and include *Perceived usefulness*, *Affective attitude*, *Self-efficacy*, *Important others' opinions*, and *Psychological safety.* Instructional factors encompass how the teaching is organised. They comprise *Practice and training, External feedback, Use of instruments, Environmental support, and Self-assessment purpose*.

In this paper, we study self-assessment predictors of mathematics undergraduate students. In mathematics, assessment practices differ from many other fields. In Finland, the most common assessment method in school mathematics is a closed-book exam (Atjonen et al., 2019). Alternative assessment methods such as self-assessment and peer assessment are more common in other subjects than mathematics. This means that it is very likely that students entering university mathematics courses do not have any experience of alternative assessment methods in the context of mathematics. In what comes to university mathematics, Iannone and Simpson (2022) have found that assessment is still very traditional in the UK: exams dominate university mathematics assessment heavily.

Also, student perceptions on assessment are different in mathematics than in many other fields. In their review, Struyven et al. (2005) reported that, in general, higher education students think that alternative assessment methods enabled a better quality of learning and were fairer than traditional methods. However, their review did not investigate disciplinary differences in student perceptions. Iannone and Simpson (2015), on the other hand, have studied mathematics students in British universities and found that they prefer closed-book exams and other traditional assessment methods. When comparing mathematics students and education students in the UK, Iannone and Simpson (2017) found differences. Both student groups preferred assessment methods that differentiated students with respect to academic ability. However, education students thought that projects and dissertations achieved this purpose, whereas mathematics students thought closed book exams were better in this regard.

Based on the evidence above, disciplinary factors need to be taken into account when studying students' perceptions of assessment. In this study, the novelty is analysing self-assessment predictors in the context of mathematics and describing mathematics students' reasons for choosing between self-assessment and a more traditional option, the exam. Another new viewpoint in our study is the element of real choice: students must choose whether their grade is determined by self-assessment or

examination. We use the framework by Yan et al. (2023) to analyse what explains the students' choice. Our research questions are the following:

1. Which factors explain students' choice of assessment method (self-assessment or examination) in an undergraduate mathematics course?
2. How are these factors manifested in students' justifications for their choice of assessment method?

Context and participants

The participants of the study were undergraduate students in a Finnish research-intensive university. They took part in a first-year linear algebra course, which lasted 7 weeks and was worth 5 credits (ECTS). Typical major subjects among the students were mathematics (including mathematics education), computer science, statistics, economics, and physics. For most students, the course was their first university mathematics course. The teacher of the course was the first author of this paper.

During the course, all students were engaged in a structured self-assessment process in which they assessed their competencies, using a rubric designed by the teacher (see the DISA model; Häsä et al., 2021). They completed three self-assessment exercises, in which they assessed what skills they mastered. The rubric used in the self-assessment contained mathematical skills, such as "I know what a subspace spanned by one or two vectors looks like," and generic skills, such as "In my answers, I write complete and comprehensible sentences that others can understand." It should be noted that the students did not assess a product they had produced, but rather the competencies they had gained so far. Students were offered automated feedback on the accuracy of their self-assessments via the digital DISA tool (Häsä et al., 2021), which compared students' assessments against the weekly mathematical tasks they had completed during the course.

At the end of the course, the students' grades were determined by either self-assessment or exam. The students had to choose *at the beginning of the course*, which kind of summative assessment they wanted to take. If a student chose self-assessment for the summative assessment method, they completed one more self-assessment at the end of the course and gave their own grade. Students' assessments were checked against the mathematical tasks they had completed, using the same DISA tool that generated the feedback in the self-assessment exercises. If a student's self-assessment was significantly different from the work they had done, the teacher had a possibility to intervene. For students who chose the examination as the summative assessment method, the grade was determined by an exam along with bonus points earned from weekly tasks.

At the beginning of the course, the students were explained the different summative assessment options of the course. Because self-assessment in the context of mathematics was new to many students and all students would do self-assessment exercises, the teacher explained to the students what the benefits of self-assessment are. After this, the students chose the summative assessment method they wanted to take (exam of self-assessment) and gave a justification for the choice. However, the students could change their choice before the end of the course if they wanted to.

The participants of the study were the 333 students (86% of the course population) who gave their informed consent for using their responses in research purposes. The data consists of the justifications the students gave for their choice of assessment method ("On what basis did you make your choice?"). These justifications were non-anonymous and were collected in the second week of the course. The Moodle platform that was used in the teaching of the course was also used for data collection. Of the participants, 83% (n=275) chose self-assessment and 17% (n=58) examination.

Method

The students' answers were coded with the Atlas-ti programme for qualitative data analysis. The justifications ranged from a few words to several sentences, and each justification could include several reasons for the choice. Each reason formed one analysis unit. Altogether, 1147 analysis units were identified, which makes an average of 3.44 analysis units per student.

First, a deductive content analysis (Elo & Kyngäs, 2008) was performed based on the framework by Yan et al. (2023). The first and second author conducted the analysis by first familiarising themselves with the data, then discussing the interpretation of the categories given by the theoretical framework, and finally analysing one half of the data each. Since the students had made their choice of the assessment method at the beginning of the course, their justifications concerned individual factors rather than instructional factors. Therefore, the categories of the content analysis were chosen to be the individual factors of the theoretical framework: *Perceived usefulness*, *Affective attitude*, *Self-efficacy*, *Important others' opinions*, and *Psychological safety*. As the context of the study was quite specific, a new category, *Need of flexibility,* had to be created. Six analysis units were difficult to interpret and could not be coded. After the deductive analysis, an inductive content analysis (Elo & Kyngäs, 2008) was performed within each of the three largest categories, dividing them into subcategories. The inductive analysis was performed by the first author.

Results

The distribution of students' justifications into different categories is given in Table 1. Numbers reported in the table are students, and the percentages represent column-relative frequencies. The largest categories found in students' answers were Perceived usefulness, Affective attitude and Psychological safety. Of the students who chose self-assessment, 46% made a justification concerning Perceived usefulness, 27% concerning Psychological safety and 27% concerning Affective attitude. Among the students who chose the examination, Perceived usefulness was found in 17%, Psychological safety in 36% and Affective attitude in 24% of the answers. Since these three categories were large, a further inductive content analysis was applied to them.

Many of the students who reported Perceived usefulness anticipated that the chosen assessment method would improve their *learning*:

> I believe that self-assessment supports deeper learning more effectively, because often when studying for an exam, learning may remain superficial.
>
> Preparing for an exam requires reviewing topics and browsing through the subject area, which strengthens my skills and learning.

Table 1: Distribution of students' justifications for the choice of summative assessment method

Predictor		Student choice	
		Self-assessment (*n*=275)	Examination (*n*=58)
Perceived usefulness		127 (46%)	10 (17%)
	learning	52 (19%)	6 (10%)
	behaviour	30 (11%)	4 (6.9%)
	self-assessment skills	45 (16%)	0
	usefulness in general	13 (4.7%)	0
Psychological safety		75 (27%)	21 (36%)
	accurate measurement	45 (16%)	11 (19%)
	stress-free	39 (14%)	11 (19%)
Affective attitude		73 (27%)	14 (24%)
	interesting	60 (22%)	0
	pleasant	15 (5.5%)	14 (24%)
Self-assessment self-efficacy		20 (7.2%)	11 (29%)
Important others' opinions		4 (1.5%)	0
Need of flexibility		49 (18%)	0

Students also felt that the choice of assessment method would affect their *behaviour* during the course:

> I believe that by choosing self-assessment, I will invest more effort in the weekly assignments. This way, the learning process will be more consistent, and I will likely remember things better even after the course.
>
> I feel that taking an exam at the end of the course motivates me to work hard during the course more than self-assessment.

The students who chose self-assessment mentioned also that it improves their *self-assessment skills*, which would be useful in their studies or in working life. These justifications did not appear in the answers of those students who chose the examination. Finally, there were students whose justification was just a *general mention of usefulness*.

The second largest main category was Psychological safety. In this category, many students wanted the assessment method to *measure their skills accurately*:

> Self-assessment might better reflect the overall skills and work done throughout the course, an exam can sometimes go surprisingly poorly.
>
> I believe that taking an exam is a more reliable way to obtain information about the understanding of course topics than self-assessment.

Another common justification in this category was to choose an assessment method that felt *stress-free*. Many students found exams stressful: "Exam situations make me nervous, and I can't do my best work under a strict time limit." On the other hand, some students were more comfortable with exams than self-assessment: "Familiar and safe". Students also evaluated the risk level of the assessment method. For example, they wanted to make sure they would pass the course or receive a good grade.

The third largest main category was Affective attitude. In this category, students reported *interest* towards self-assessment: "I also think this kind of final assessment method sounds interesting, and I haven't tried it before." However, none of the students who chose the examination mentioned interest or curiosity towards it. In addition, students justified that either self-assessment or exam was a *pleasant* choice. They liked the assessment method, or it felt natural for them.

The remaining main categories were considerably smaller than the three biggest main categories, so they were not split into subcategories. Self-assessment self-efficacy included students' answers in which they stated that they either can or cannot assess their own skills. In the category Important others' opinions, students had been impacted by their teachers' motivational speech or more experienced students' opinions.

The category Need of flexibility was added to the framework and was specific to the context of this study. The self-assessment could be done online, during several days, but for the examination, students had to come to the campus at a specific time. Many students who chose self-assessment reported that they appreciated the flexibility of the assessment method.

Discussion

In this study, we let mathematics undergraduate students choose the summative assessment method of a course from two options: self-assessment and examination. The chosen method determined the student's grade. We analysed students' justifications for their choice using content analysis based on the framework by Yan and colleagues (2023).

Our results show that the usefulness of the assessment method was important to the students, in particular in what comes to learning gains, organising one's studying and learning skills that are relevant for life-long learning. This aligns with findings in other disciplines such as engineering education (Partanen et al., 2023). The students also valued self-assessment skills and felt they would be important to them in the future.

A second prominent category in students' answers concerned psychological safety. Students wanted to choose the assessment method that measured their skills better. This is in line with Iannone and Simpson (2017). It is notable that many students thought that self-assessment is more accurate than the examination. Regarding psychological safety, students also reported feeling stress and anxiety about exams and therefore chose self-assessment. On the other hand, there were students who were uncomfortable with self-assessment and chose examination instead.

Also, the affective factors played a major role in students' justifications. Many students were interested in trying a new assessment method. This is different from the studies of Iannone and Simpson (2015; 2017) who found that mathematics students prefer traditional assessment methods. Even though the studies cannot be compared directly – their students were in a mathematics programme whereas our students were merely students in a mathematics course – there are indications that something in our context encouraged students to have a positive attitude towards alternative assessment methods. It may be due to the way self-assessment was presented to them, or the Finnish education system in which grades do not play a major role. It was beyond the scope of this study to analyse the relationship of students' study programme and the justification they give for the choice of assessment method, but in the future, this would be worth investigating.

It is likely that not all students' justifications reflect their perceptions exactly. The students' answers were not anonymous, and they might have felt that certain kinds of answers were not appropriate. For example, it is likely that some students thought that they needed to give a rational justification and not mention their emotions. However, emotions have a central role in shaping students' perceptions on assessment (Crossman, 2007), and they should not be overlooked. In the future, interviews could provide a more nuanced and multidimensional insight into students' perceptions.

Our study sheds light on the aspects of assessment that are important to the students. This information can be used in organising assessment and communicating about it to the students – in particular, when encouraging students to try out a new form of assessment such as self-assessment. Our results indicate that teachers should emphasise the learning gains and relevance for working life, make sure that students feel that the assessment method is accurate and fair, reduce the stress experienced and offer flexibility in the assessment arrangements. However, one should be careful when generalising our results to different socio-cultural contexts.

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Using subscores from the high-stakes assessment to explain first-year performance in Mathematics courses

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The use of data analytics in predicting student success in first-year courses has been gaining momentum in South Africa. With that, various assessment data points are used to map out incoming students' academic profiles and help design curriculum support mechanisms. Scores on the high-stakes assessment subdomains could be used to explain the variance in course performance and suggest that subdomains with the greatest relative importance have the strongest association with the variance in performance in a course. Calculating subdomain scores (subscores) and establishing the meaningful weighting of each subdomain in a test is important. This study compares how different subscore calculation methods, such as IRT and number-correct, affect reporting and the ranking of subdomains in a relative importance analysis of a first-year mathematics cohort.

Keywords: Test scores, IRT, relative importance analysis, high-stakes tests, subscores.

Introduction

The use of data analytics in student success initiatives has been gaining momentum in South Africa. Various assessment data points, such as matric results in the National Senior Certificate (NSC) subjects and the National Benchmark Tests (NBT) scores, are used to understand incoming students' academic profiles and help design curriculum support mechanisms. The NBT is a high-stakes assessment which offers higher education institutions supplementary information to support the selection and placement of prospective students into suitable academic pathways. It assesses students' readiness for higher education studies in Academic Literacy (AL), Quantitative Literacy (QL) and Mathematics (MAT) domains. Students who take the MAT test typically aim to enrol in mathematically demanding disciplines, such as Engineering, Health Sciences, or Commerce, and most sit for the test during their final year of Grade 12. The MAT assessment probes their depth of understanding and knowledge of mathematical concepts essential for success in these fields. The test is 3 hours long and is developed by subject matter experts from participating universities (NBT, 2023).

In recent years, the decision was made to report on the proficiency of incoming first-year student cohorts at the test subdomain level. Understanding the AL, QL and MAT subdomain scores is considered valuable for identifying areas where prior learning may influence first-year performance. Subdomain scores are used to explore their relationship with historical course outcomes, and relative importance analysis (RIA) explains variance in course performance, highlighting the subdomains most strongly associated with that variance (DASS, 2024). One suggested intervention described by Naidoo (2023) integrates this information into the first few weeks of first-year courses to support concept development and address possible gaps identified by the NBT diagnostics.

This paper presents an initial investigation into the impact of calculating the subscores using the item-response theory (IRT) and the traditional number-correct method on the order of subdomains in the relative importance analysis for a first-year Engineering mathematics course.

Theoretical framework and research questions

NBT employ item response theory to estimate the test-taker's proficiency in Academic and Quantitative Literacy and Mathematics. These assessment domains, AL, QL and MAT, are each defined by a framework that outlines specific content areas, skills and abilities, organised into subdomains and clusters of items (NBT, 2023). The MAT test specification comprises items that are distributed across multiple competence areas, subdivided into different clusters, and categorised according to cognitive level. The NBT Mathematics domain is further divided into five subdomains, with each having an established weighting in a test blueprint (see Table 1).

Table 1: NBT Mathematics domain and subdomains

M1 - Algebraic processing (25% - 30%) = 15 scored items	Involves recognising and manipulating patterns, sequences, and algebraic expressions, solving exponential equations, and interpreting measurement-related problems.
M2 - Number sense (8% - 10%) = 4 scored items	Encompasses operations with various types of numbers, understanding number systems, performing calculations without the need for calculators, and making mathematical judgments.
M3 - Functions and graphs (20% - 25%) = 12 scored items	Focuses on understanding and analysing functions and their properties, interpreting and solving related graph problems, and applying principles of differential calculus.
M4 - Trigonometric functions and graphs (15% - 20%) = 9 scored items	Involves solving trigonometric equations, understanding trigonometric identities, and applying trigonometric concepts to two- and three-dimensional problems.
M5 - Geometric reasoning (20% - 25%) = 12 scored items	Includes understanding properties of shapes, calculating perimeter, area, and volume, and linking geometric properties with algebraic concepts.

The cognitive skills needed across the MAT subdomains are categorised into four cognitive levels: Knowing (recall, simple routine procedures; about 44% of the test), Performing routine procedures (all information is directly available; about 23% of the test), Performing complex procedures (a strategy is required to identify information; about 25% of the test), and Solving problems, reasoning and reflecting (complex contexts and procedures; about 8% of the test).

Multiple test forms are assembled and administered in each testing cycle, with a total of 52 scored items across five subdomains. The assumption is that, in addition to the overall mathematics ability measured by the MAT test, there are potentially some unique traits being measured by these subdomains. It was established that the data can be efficiently modelled with the three-parameter logistic (3PL) IRT model, whereby parameters of discrimination (a), difficulty (b) and pseudo-guessing (c) are estimated, which is appropriate for achievement-type tests where guessing can play an important role in examinee responses. The test forms are further linked by common items referred to as anchor items. The linking and equating process is performed based on a linear transformation using the Stocking & Lord method (1983), which implies rescaling the item parameters from the X form (current) to the Y form (base) (ASC, 2012).

Research questions:

- Given the current test framework, what can be observed about the reporting of subscores?
- How does the method of calculating subscores impact the ranking of subdomains in the relative importance analysis?

IRT-based subscores

There are numerous studies of different methods of subscore calculation and reporting in educational assessments, highlighting common issues related to test design and reliability. One notable concern is the total test length and the distribution of items across subdomains or sections of the test, which may impact the reliability of the obtained scores. A common method of calculating subdomain scores is by dividing the number of items answered correctly by the total number of items in each subdomain, which is used to report the percentage of correct responses. These indicators are then aggregated for a larger group of candidates to observe proficiency in the subdomain of interest. The problem associated with this method is often reported in the literature, with the main concern raised about the number of items in a test subdomain (Fan, 2016; Hambleton,1980; Kern, 2007; Provasnik et al., 2020). A lower number of items often leads to low reliability and could also make room for inaccurate interpretations, where the subdomain scores are used for making curriculum adjustment decisions. Looking at this from another angle, the raw (number-correct) scores are not necessarily accurate reflections of a test taker's ability, and more than a handful of items would be required to gain a more accurate understanding of a person's proficiency in a subdomain. These concerns could be partially elevated using a different method of calculating the subdomain scores, such as the Item Response Theory (IRT). One of the benefits of IRT is that it estimates the probability of a correct response on an item for a student of a particular ability estimate, which could address the concern of a low number of items in a test subdomain.

Many high-stakes assessments are not designed to be multidimensional, limiting their ability to provide an accurate representation of students' abilities in content areas defined by specific subdomains. The assessment instrument in this study exclusively consists of multiple-choice questions; thus, the data under consideration is binary, based on dichotomous item responses (correct or incorrect). Bejar (1980) notes that factor analysis of binary data can often be problematic and proposes the use of IRT for investigating test dimensionality. In this method, the assumption is made that item parameters should remain consistent between total-test-based estimates and content-area-based estimates. If the parameters differ, then the probability of passing the item will differ for a given level of achievement. This suggests that performance on a given item depends on performance in other items in the same test form.

When choosing a method of calculating the subscores, some studies have shown that IRT-based scores are better (or similar) predictors of performance than number-correct scores (Bock et al., 1997; Camilli & Dossey, 2019; Kreitchmann et al., 2024), and other studies argue against this claim (Ferrando & Chico, 2007; Kern, 2007). This divergence in findings underscores the importance of considering multiple perspectives when interpreting test results and making further inferences about student performance. There is agreement in the literature that a nuanced approach to subscores

calculation and reporting is necessary, and often, it is a combination and augmentation of different methods that leads to improved accuracy and reliability.

Methods

Two data sets of test scores were used in this study. The item responses of 3 354 examinees from a single administration of a MAT test form were used to estimate IRT parameters. These parameters were estimated in two different ways, and the item difficulty parameters were compared between the total-test-based estimates and content-area-based estimates. The subscores were then calculated from the total-test-based parameter estimates for a smaller data set (n = 144), and IRT-based subscores were compared to number-correct subscores. These subscores were used in a relative importance analysis to assess the contribution of MAT subdomains as predictor variables for final results in the first-semester mathematics course.

Step 1: Choice of parameter estimation method and reliability of subscores

The program Xcalibre Version 4.2 (Guyer & Thompson, 2011) was used to estimate item parameters for the normal approximation (D = 1.7) of the three-parameter logistic model. Xcalibre implements marginal maximum likelihood estimation of IRT item parameters. As the subscores on the MAT test are of interest to this study, some elements of Bejar's method were considered useful in evaluating item parameter estimation methods. The estimated *b*-parameters were compared across the test items between the total-test-based estimates and content-area-based estimates.

Step 2: Comparison of IRT-based and number-correct subscores

The number-correct subscores method involves calculating the percentage of items answered correctly within a specific subdomain. For example, if a student answers 6 out of 12 questions correctly in a geometric reasoning subdomain, their number-correct score (as a percentage) would be 50%. This method is straightforward but does not account for the difficulty of items or the underlying ability of the test-taker. Additionally, the possible number-correct scores in a subdomain with 4 items will always be 0%, 25%, 50%, 75%, or 100%. In contrast, IRT-based subscores utilise a probabilistic model to estimate a student's ability based on their responses to test items. IRT models consider item characteristics, such as difficulty and discrimination, allowing for a more nuanced understanding of a student's abilities. For instance, a student might perform well on difficult items, indicating a higher ability level than suggested by a simple number-correct score.

Step 3: Comparison of the impact on RIA and what could this mean in practice

When dealing with multiple predictors, a key question is which one is most important for predicting the outcome variable. Although correlation indicates the relationship between two variables, it is not effective with multiple predictors. A recommended approach is to calculate the "relative importance" of predictors. Relative importance package `realimpo` with the recommended metric `lmg` provides a decomposition of the model-explained variance into non-negative contributions (Groemping, 2006). The relative importance analysis based on the method by Lindeman, Merenda, and Gold (1980) is a variance decomposition of correlated predictor variables (each predictor's contribution to R-squared) that tells you how much of the variance in the model is explained by each predictor.

Results

In 3PL models, the distribution of *b*-parameters gives insight into the spread and range of item difficulties. Comparing *b*-parameters across test items between total-test-based and content-area-based estimates revealed differences, with the difficulty level in subdomains with fewer items estimated significantly higher in the content-area-based method (see Table 2). When scored using both parameter sets, the M2 subdomain, having the fewest items, showed a weaker correlation with the total MAT score: 0.5039 with content-area-based estimates compared to 0.6169 with total-test-based estimates. Thus, the total-test-based estimation method was preferred.

Table 2: Parameter estimates and MAT score correlations

	M1	M2	M3	M4	M5
Mean *b*-parameter estimates: Total-test-based	0.68353333	1.58425	0.64825	0.94233333	0.86733333
Mean *b*-parameter estimates: Content-area-based	0.746	2.759	0.73133333	1.10811111	1.00391667
MAT score correlation: Total-test-based	0.8992	0.6169	0.8797	0.8223	0.8542
MAT score correlation: Content-area-based	0.9002	0.5039	0.8844	0.8150	0.8426

A cohort of Engineering students in a first-year Mathematics course wrote this test at the start of the semester. The subscores were calculated (after equating to the base form) for a sample of 144 Engineering students in a first-year Mathematics course who remained enrolled in the course and received final results for the first semester. Table 3 provides an overview of the full test, including its subdomains (M1 to M5). The reliability of the full test (51 calibrated items, 1 item was removed due to a low discrimination parameter) is high, with a Cronbach's alpha of 0.876, indicating strong internal consistency, but the subdomains' reliability differs. Subdomain M1 has acceptable reliability (0.705), and subdomains M3 and M5 show moderate reliability, with alpha values of 0.621 and 0.657. However, subdomains M2 and M4 have very low alpha values (0.363 and 0.501, respectively), which could be due to the number of items associated with each subdomain (4 items and 8 items, respectively). Notably, M2 (0.609) and M4 (0.700) also show relatively lower correlations with the total test score compared to the other subdomains.

Table 3: Cronbach's Alpha for the Full Test and Test subdomains

Test	Full Test	M1	M2	M3	M4	M5
Items	51	15	4	12	8	12
Alpha	0.876	0.705	0.363	0.621	0.501	0.657
Full Test Scores Correlations	1	0.848	0.609	0.810	0.700	0.832

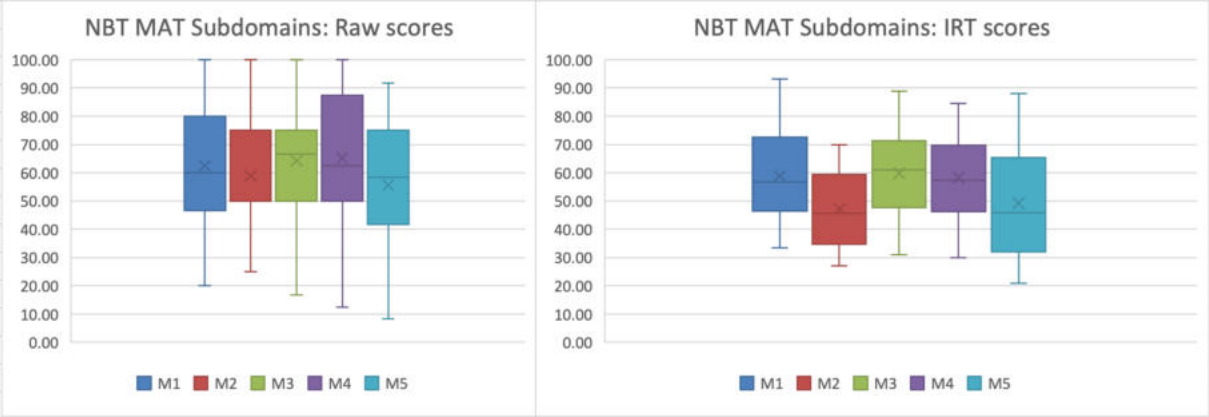

Figure 1: Distribution of scores within subdomains, n = 144

The distribution of scores within subdomains and the relative importance analysis were done on a smaller sample (n = 144) of first-year students in a mathematics course who had their final first-semester course scores. This is a first-year Engineering calculus course and includes concepts such as Functions, Limits, Continuity and Differentiability.

Score distributions within each subdomain are shown in Figure 1. IRT appears to offer a more normalised scoring method across subdomains, reducing the variability across different subdomains, while raw scores (number-correct) capture the direct, unadjusted performance. The IRT scores distribution within subdomains shows that it takes into account a person's ability in all items on the test and provides an estimate of their ability within a specific subdomain. In simpler terms, even if a student answers no questions correctly in a subdomain with fewer items (e.g., Number sense with a

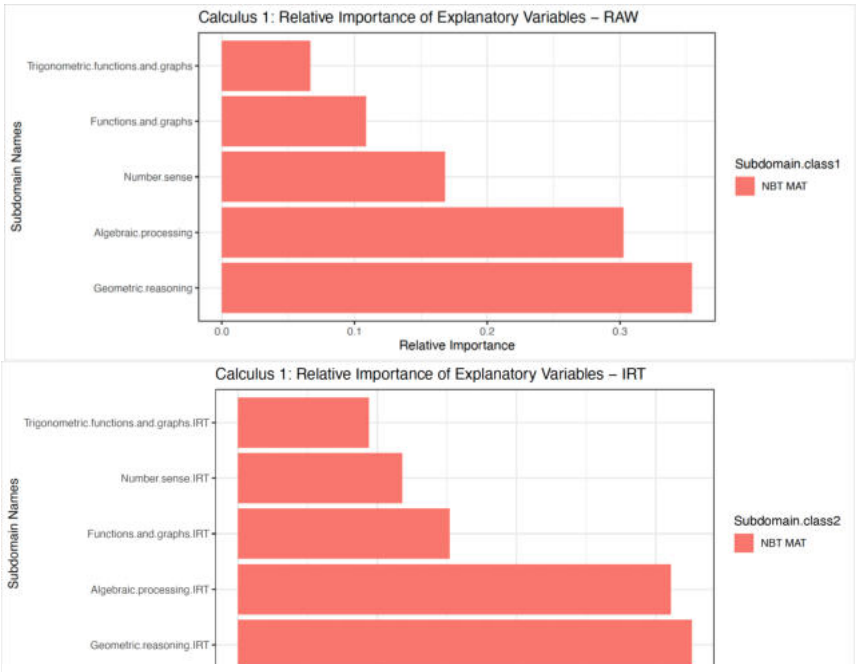

Figure 2: Relative Importance based on raw subscores and IRT, n = 144

total of 4 items in the test), it doesn't necessarily mean that the student has zero proficiency in that subdomain. We can assume that the level of proficiency is low, but not non-existent.

The relative importance is shown in Figure 2, which shows that geometric reasoning comes as the strongest of the five subdomains, closely followed by algebraic processing. Geometric reasoning involves higher-order thinking, and its cognitive complexity could potentially serve as a proxy for performance in first-year calculus. It is expected that algebraic processing is one of the strongest predictor variables; however, it might be surprising that functions and trigonometry are two of the lowest subdomains. In the case of IRT-based subscores, the number sense subdomain gave way to the functions and graphs subdomain.

- RAW (number-correct): Total R-squared is 21.5% - meaning that 21.5% of the variation of grades in this course is explained by NBT MAT.
- IRT: Total R-squared is 20.7% - meaning that 20.7% of the variation of grades in this course is explained by NBT MAT.

RIA on IRT-based subscores and number-correct scores do not show very different pictures, but a slight change in the order of the two subdomains. In both models, geometric reasoning is important, but it is statistically significant in the raw score model (p-value = 0.0396) and only marginally significant (p-value = 0.0704) in the IRT model. Other subdomains (Number sense, Functions and graphs, and Trigonometric functions and graphs) were insignificant (p-values > 0.05) in both methods, with only minor changes in their coefficients.

The IRT method may offer improved measurement precision, but in this case, it has not drastically altered the overall interpretation of subdomain importance. Geometric reasoning had the highest relative importance indicator, and students' scores in this cohort showed greater variability in this subdomain (see Figure 1).

Concluding remarks and further research

With the decision to use subdomain scores for further interpretation, more efforts should be made to improve the accuracy and significance of predictions. While the relative importance output in this study of a course cohort was similar, a larger data set will be necessary to understand the impact and differences in subdomain score calculations. It will also be important to understand the effect of predictors on student performance when integrated with intervention strategies.

The use of other models, for example, the score augmentation method of supplementing subdomains with a low number of test items, can provide scope for more accurate subscore estimates. The least stable subdomains could be supplemented with additional information, such as performance in Quantitative Literacy items that are relevant to Mathematics items, particularly in MAT subdomains with fewer items (M2 and M4). Additionally, the ideal weighting of each subdomain (the number of items) in the test may need refinement to improve subscore reliability. A relative importance analysis based on cognitive levels, rather than (or in addition to) subdomains, which often show high correlations and overlap in the skills and concepts tested, could also provide a new direction for improving reporting to better inform intervention design.

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The role of digital tools in enhancing classroom assessment practices

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As educational technology becomes increasingly embedded in classrooms, it is crucial to understand how these tools are utilized for both formative and summative aspects of teachers' assessments in the classroom. Through a combination of surveys and semi-structured interviews, the study probes teachers' experiences with technology-enhanced assessments. The findings highlight the benefits of digital tools, such as increased student motivation, immediate feedback, and improved visualization of mathematical concepts. However, the study also reveals the challenges teachers face, including technical difficulties, inequities in access to technology, and the added workload associated with using these tools. The study contributes to a deeper understanding of how digital tools can enhance assessment practices in middle school mathematics, offering insights into the opportunities and obstacles faced by teachers in Türkiye.

Keywords: Formative assessment, summative assessment, technology-enhanced classrooms

Introduction

Digital tools have increasingly become integral to educational environments, particularly in mathematics classrooms (Hillmayr et al., 2020, Moliner et al., 2022). These tools offer various advantages that significantly enhance classroom assessment practices (see Saralar-Aras, 2022). In middle school mathematics, tools like dynamic geometry software, graphing calculators, and online platforms provide teachers with innovative ways to assess student learning.

Formative assessments, which function as improving student learning through adjusting learning environments, are enhanced through the use of digital tools by providing immediate feedback, allowing for real-time adjustments to instruction, and facilitating a more personalized learning experience (Cosi et al., 2020). Teachers can use these tools to identify student strengths and weaknesses more efficiently, enabling a targeted approach to instruction. Digital platforms also allow for continuous monitoring of student progress, making formative assessments more dynamic and responsive to individual student needs. Summative assessments, traditionally used to evaluate student learning at the end of an instructional period (Brookhart, 2001), are also transformed by digital tools (Newhouse, 2014). These tools can automate scoring, offer diverse question types, and provide more detailed analytics, allowing teachers to better understand student outcomes. Furthermore, the visualization capabilities of digital tools help students get better understanding of tasks which facilitates more accurate assessment of their knowledge. However, while digital tools offer numerous benefits for assessment, they also come with challenges (Onasanya & Ajamu, 2024). Issues such as inequity in access to technology, technical difficulties, and the potential for misuse must be addressed to fully harness the potential of digital tools in classroom assessments.

Literature review

Introduction to educational technology in mathematics education

The integration of digital tools in mathematics education has significantly transformed teaching and assessment practices over the past few decades. As early as the 1990s, educational researchers began to explore the potential of technology to enhance learning, with a particular focus on its applications

in mathematics (Kaput, 1992). Today, educational technologies such as dynamic geometry software, graphing calculators, and online platforms have become integral components of middle school mathematics classrooms. These tools provide interactive and accessible ways to deepen students' understanding of mathematical concepts while offering new avenues for both formative and summative assessments.

Formative assessment and digital tools

Formative assessment is an ongoing process used by teachers to monitor student learning, provide feedback, and adjust instruction to improve student understanding and performance throughout a lesson or unit (Cosi et al., 2020). It is a critical component of effective teaching, providing teachers with ongoing insights into student learning and guiding instructional decisions (Black & Wiliam, 2018). Digital tools have been shown to enhance formative assessment practices by enabling immediate feedback, which is essential for adjusting teaching strategies in real-time (Shute, 2008). In addition, dynamic geometry software like GeoGebra allows students to visualize and manipulate geometric concepts, facilitating deeper engagement and understanding (Hohenwarter & Jones, 2007). Teachers can use these tools to create interactive tasks that make students' thought processes visible, aiding in assessment by providing insights into students' reasoning and understanding. By observing students' actions and choices within the software, teachers can gather valuable data that reflects student comprehension, thereby enhancing formative assessment.

Research suggests that when teachers use digital tools for formative assessments, they can more effectively identify student strengths and areas for improvement (Nicol & Macfarlane-Dick, 2006). Furthermore, online platforms and educational apps enable continuous monitoring of student progress, allowing for a more personalized approach to learning. This capability aligns with the growing emphasis on differentiated instruction in mathematics education, where teachers tailor their teaching strategies to meet the diverse needs of students (Tomlinson, 2014).

Summative assessment and digital tools

Summative assessments, aimed at evaluating student learning at the conclusion of instructional periods, have been notably transformed by digital tools. Traditionally confined to paper-based tests, which are often time-intensive to administer and score (Brookhart, 2001), summative assessments now benefit from digital tools that offer automated scoring, diverse question formats, and in-depth analytics, enabling nuanced insights into student performance (Onasanya & Ajamu, 2024). For example, tools like graphing calculators and software such as GeoGebra empower students to engage with complex mathematical problems, allowing educators to assess higher-order thinking skills that are difficult to capture through traditional assessments.

Moreover, integrating digital tools in summative assessments fosters alignment with learning objectives, enabling teachers to design tasks that more authentically mirror real-world applications of mathematics (Hillmayr et al., 2020). The visualization capabilities of these tools aid in deepening students' comprehension of abstract mathematical concepts, thus enhancing the accuracy of knowledge assessments (Clements & Sarama, 2011). Consequently, digital summative assessments not only streamline scoring and expand task diversity but also contribute to a more meaningful evaluation of student understanding.

Challenges in the use of digital tools for assessment

While digital tools offer numerous benefits for formative and summative assessments, their use also presents several challenges. One of the primary concerns is inequity in access to technology, which can exacerbate existing educational inequalities. In middle school mathematics classrooms, not all students may have access to the necessary devices or internet connectivity, creating barriers to effective assessment (Warschauer & Matuchniak, 2010). Additionally, technical difficulties, such as software glitches or connectivity issues, can disrupt the assessment process, leading to frustration for both teachers and students (Inan & Lowther, 2010).

Another challenge is the potential for digital tools to increase the workload for teachers. The implementation of technology-enhanced assessments requires significant time and effort, including learning how to use the tools, creating digital assessment tasks, and interpreting the data generated by these tools (Ertmer & Ottenbreit-Leftwich, 2010). Moreover, managing large groups of students in a technology-enhanced environment can be difficult, particularly when trying to ensure that all students are actively engaged and on task (Sawyer, 2006).

In the context of middle school mathematics in Türkiye, the role of digital tools in enhancing classroom assessment practices is multifaceted (Cumhur & Çam, 2021; Özpınar, 2020). It involves balancing the benefits of immediate feedback and personalized learning with the challenges of accessibility and teacher preparedness. As digital tools continue to evolve, their role in classroom assessments will likely expand, offering new opportunities to enhance student learning outcomes. In this context, we looked at the following research question: How do middle school mathematics teachers in Türkiye perceive the role of digital tools in enhancing classroom assessment practices?

Methods

Research design

This study was designed as a qualitative research project with some quantitative elements and conducted in two phases. In the first phase, 16 middle school mathematics teachers completed an online survey. Participants were purposively selected to ensure diversity in regional representations across Türkiye and included experienced teachers with at least five years of teaching experience. 13 teachers had over 10 years of experience while 3 had between 6 to 10 years.

The survey aimed to gain comprehensive insights into teachers' use of technology for assessment purposes focusing on five aspects: (1) the digital tools used for assessment; (2) examples of their preferred practices; (3) challenges and limitations faced; (4) professional development support received and (5) teachers' anticipations for future development of technology in assessment. It involved both close-ended and open-ended questions. To show the idea of it, a sample question from the survey was "Could you share an example of an effective and/or innovative classroom practice where you used digital technologies for assessment?". The close-ended questions gathered demographic information, identified the type of digital tools used, and assessed the frequency of their use, and inquired about professional development sessions attended that can assist their use of technology for assessment. The open-ended questions asked teachers to provide detailed responses

about their preferred practices, the challenges they encountered and their anticipations regarding the future technologies in assessment. The survey was administered electronically via Google Forms.

In the second phase, 6 teachers were selected for semi-structured interviews from the 16 who completed the survey. Selection was based on two criteria: first, these teachers demonstrated more frequent or varied responses compared to others; second, they represented diversity in gender and geographical area they teach in. The interview protocol was carefully designed to gain deeper insights into their experiences with digital tools in classroom assessment. The interviews were audio-recorded and transcribed for qualitative analysis.

Data analysis

A thematic analysis (Braun & Clarke, 2022) was employed to identify recurring patterns in terms of the opportunities and challenges in teachers' integration of digital technologies into classroom assessment practices. As the first step, teachers' reported assessment practices were categorised based on whether they related to formative or summative aspects, using the "implementation by teacher" component of the classroom assessment model proposed by Black and Wiliam (2018, p. 556). The formative aspects were categorised according to the five formative assessment strategies outlined by Wiliam and Thompson (2007, p. 63): "(1) clarifying and sharing learning intentions and criteria for success, (2) engineering effective classroom discussions and tasks that elicit evidence of learning, (3) providing feedback that moves learners forward, (4) activating students as instructional resources for one another, and (5) activating students as the owners of their own learning". The summative aspects were categorised according to teachers' aim of evaluating what students have learned typically by assigning a final grade, score, or a combination of both (Black and Wiliam, 2018). Subsequently, the recurring patterns in the benefits and challenges associated with each type of assessment were identified.

Findings

This section presents the role of digital technologies in enhancing the participant teachers' classroom assessment practices, focusing on both formative and summative aspects. Teachers' responses underscored the complexity of the issue, linking benefits and challenges to technology itself, its use in assessment more broadly, or specifically in mathematics assessment. Before going into detail, it is useful to first outline the technologies used by the participant teachers which were technological tools for classroom assessment, including online test makers/platforms (Quizz, Kahoot, Google Forms, Padlet, and Canva), mathematics software packages (GeoGebra and Cabri), digital books and a learning management system provided by the Ministry of National Education (namely, EBA).

The findings suggest that digital technologies significantly enhance formative assessment in mathematics lessons, despite some challenges. Teachers demonstrated adaptability in addressing issues such as students' limited access to technological devices and internet connectivity during formative assessments. They found it easier to switch to alternative plans and blend digital tools with traditional teaching techniques for formative purposes. However, the impact of digital technologies on summative assessment practices appears more limited due to the challenges involved. Issues such as unequal access to technological devices and reliable internet made it difficult for teachers to fully

integrate digital tools into summative assessments like digital tests and coursework projects that require the use of digital technologies. Although teachers were thoroughly probed about their use of digital tools for both formative and summative assessments, their responses were more detailed regarding formative practices. When discussing the integration of digital tools, teachers provided more comprehensive insights into the opportunities within formative assessment, while they viewed potential benefits for summative assessment as less feasible under current conditions. The following two sub-sections will present an overview of the opportunities and challenges reported by teachers.

Benefits of the use of digital technologies for effective classroom assessment

Teachers identified several opportunities that digital tools offer for enhancing classroom assessment practices. The recurring opportunities identified in teachers' responses include increased student participation, efficient use of time, easier visualization of abstract mathematical concepts, convenient access to a wide range of mathematics tasks, and provision of detailed analytics. In Table 1, the first column presents these opportunities, the second column includes teachers' responses as evidence to support these identified patterns and the third column presents the aspects of classroom assessment that were linked to these practices.

Table 1: Teacher-reported opportunities of digital technologies in classroom assessment practices

The benefits	Example supporting evidence	Link to the elements of classroom assessment
Increased student participation	Digital tools help create a stress-free learning environment. Digital games increase students' motivation. Digital tools make it easy to capture students' attention. Digital games and online competitions facilitate peer feedback.	Providing feedback that moves learners forward Activating students as instructional resources for one another Activating students as the owners of their own learning
Efficient use of time	Digital tools provide immediate feedback. Digital technologies facilitate immediate access to a variety of mathematics tasks. Digital tools enable automated marking.	Engineering effective classroom discussions and tasks that elicit evidence of learning Providing feedback that moves learners forward Summative aspects
Easier visualization of mathematical concepts	Digital tools enhance the visualization of abstract mathematics concepts such as absolute value, and geometrical concepts like 3D shapes.	Engineering effective classroom discussions and tasks that elicit evidence of learning Summative aspects
Convenient access to a wide range of assessment tasks	Digital tools provide interactive mathematics tasks. Digital tools and online resources facilitate project-based learning.	Clarifying and sharing learning intentions and criteria for success Engineering effective classroom discussions and tasks that elicit evidence of learning Summative aspects
Provision of detailed analytics	Digital tools provide detailed analysis of students' mistakes.	Formative use of summative tests Summative aspects

Challenges with the use of digital technologies for classroom assessment

While digital tools offer various opportunities to enhance classroom assessment practices, teachers also reported several challenges in using them. These recurring challenges, along with example teacher responses as supporting evidence, are outlined in Table 2.

Table 2: Teacher-reported challenges of using digital tools in classroom assessment practices

The challenges	Example supporting evidence	Link to the elements of classroom assessment
Additional workload	Teachers may need extra preparation to learn how to use specific digital technologies. Developing digital tools that align with learning intentions can be time-consuming. Additional instructions might be required for students and parents. Teachers might need contingency plans for unexpected technical issues.	Summative aspects All five formative assessment strategies
Limited access	Some useful software requires payment. Students might have limited access to technological devices and reliable internet connections.	Summative aspects All five formative assessment strategies
Limited features of existing tools and resources	Readily accessible digital or online materials may not meet the specific needs of students. Available tasks might provide limited opportunities for open-ended and long-answer questions, which are necessary to assess higher-order learning.	Summative aspect All five formative assessment strategies

These challenges can potentially limit the implementation of formative assessment strategies and the effectiveness of teachers' summative assessments. However, teachers seem to manage them more effectively in formative contexts through flexible adaptations. The limited flexibility in using digital technologies for summative assessment appears to stem from the specific demands of summative assessment, which require a higher level of accuracy and fairness in obtaining and interpreting students' scores or grades.

For instance, during interviews, two teachers expressed a preference for digital assessments due to their convenience of scoring exams. However, they noted that this is not feasible under current conditions. One teacher highlighted issues with students' access to technological devices and the potential for unstable internet connections at school. She suggested that allowing students to complete tests outside of school, where they might have better access, could lead to inequities and make it difficult to monitor for cheating. Another teacher pointed out the challenges related to the types of questions that digital technology can automatically assess. He mentioned that relying on digital tests for summative assessment would predominantly limit him to short-answer and multiple-choice questions, which could undermine a comprehensive interpretation of students' learning.

Discussion

The findings suggest that integrating digital technologies has significant potential to enhance teachers' classroom assessment practices, both in formative and summative aspects. Consistent with existing literature, the results indicate that teachers perceive digital technologies as enhancing feedback practices by facilitating immediate feedback (Shute, 2008), visualizing abstract mathematical concepts (Hohenwarter & Jones, 2007), and effectively eliciting student learning while motivating active engagement (Nicol & Macfarlane-Dick, 2006). However, despite the literature highlighting the benefits of digital technologies for summative assessment (Onasanya & Ajamu, 2024), it appears that teachers in this study may have been hesitant to adopt these technologies for summative purposes, potentially due to concerns about meeting the rigid requirements for accurate and fair grading. This difference between teachers' tendency to use digital technology seems to stem from the issues with students' equal access to digital technologies (Warschauer & Matuchniak, 2010) and technical difficulties (Inan & Lowther, 2010).

Conclusion and suggestions for future research

This study has demonstrated that digital technologies offer significant benefits for enhancing classroom assessment practices. However, the challenges associated with their integration must be acknowledged. Ensuring that all students have access to adequate technology is particularly important. Additionally, advancements in educational technology should focus on developing more valid and reliable assessment tools. Future research could explore the design of digital tools that better address teachers' needs for more effective assessment practices and conduct large-scale trials of these tools.

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Professional development support helping teachers overcome difficulties of implementing formative assessment

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Keywords: Formative assessment, elicitation, feedback, professional development.

Background

Research has shown the potential of formative assessment (FA) to enhance student achievement in mathematics (e.g. Palm et al., 2017). However, professional development (PD) programs often struggle to effectively support teachers in implementing high-quality FA (Anders et al., 2022). To better understand difficulties in implementing FA and how to overcome such difficulties, we followed one experienced primary school teacher, who, after completing a comprehensive school-based PD, received individual support to implement FA in the common context (e.g., Hiebert et al., 2003) where the teacher provides help to students working individually on mathematics tasks. Implementing FA in this context involves: (1) eliciting evidence of student knowledge and skills, (2) interpreting and making inferences about student learning needs, and (3) providing feedback tailored to those needs.

Our goal was to understand why a highly engaged and experienced mathematics teacher, despite supportive PD, struggled to implement desired changes. Additionally, we sought to explore strategies to help teachers overcome difficulties, contributing knowledge about essential PD features for effective FA implementation.

Methods

Classroom observations revealed that the teachers in the initial PD did not consistently use the FA practices they had reported. Consequently, we offered additional individualized PD. Elsa (a pseudonym) volunteered for this study and the extended PD. The individual PD followed a cyclical process. Elsa audio-recorded her interactions with students during 2–3 lessons and then shared them. The researchers analyzed the recordings and formulated feedback to Elsa, including examples from the recordings. This feedback was discussed in a digital meeting between Elsa and a researcher. Elsa then used the feedback to refine her FA practices during the following 3–4 weeks, initiating a new cycle of co-operation. Five cycles, and thus five digital meetings, were conducted. All meetings were recorded, transcribed, and analyzed to identify barriers (i.e., difficulties and their causes) to FA implementation and strategies to address them. Using thematic analysis (Braun & Clarke, 2006), difficulties, causes, and potential strategies were categorized and their relationships illustrated. Classroom recordings were also used to track changes in FA practices over the course of the PD.

Results

The classroom recordings revealed that one key area for improvement was Elsa's feedback to students, which was often not based on the students' thinking. She did not consistently gather sufficient information about individual student learning needs to provide targeted feedback.

Two main difficulties were identified: (1) Encouraging students to share their thinking until sufficient information is received, and (2) Helping students use feedback autonomously. Each difficulty stemmed from different causes. For example, students were reluctant to share their thinking because they did not see its purpose and feared negative comments from peers.

Our study underscores the importance of having first hand and sufficient information about the individual teacher's needs when providing PD support. Such information, combined with discussions of difficulties and targeted strategies, emerged as key components of successful PD. The iterative, cyclical structure of support also proved crucial, as challenges cannot always be solved immediately.

One simplified example of this iterative process involved Elsa's struggle to encourage students to share their thinking. Elsa noted that students did not see the purpose of doing so. Together, Elsa and the researcher discussed strategies to help students understand the purpose of sharing their thinking. Over time, students would be able to see benefits of sharing their thinking by experience the advantages of receiving individualized feedback based on their expressed needs. However, to accelerate this understanding, Elsa would explain why their input was necessary and consequences of not having information about their thinking. She would also provide specific encouraging feedback when students shared valuable information (e.g., "That's a great way of thinking because..."). Eventually, Elsa would provide feedback showing students how their actions led to success (e.g., "This went well because you..."). However, this example simplifies the complexity of implementing FA in the classroom. In reality, Elsa struggled to overcome many of the difficulties.

Discussion

The additional PD provided more valid and detailed information about the teacher's FA progress than the previous school-based PD. This information made it possible to provide support better tailored to the teacher's needs, which helped the teacher to overcome some of her difficulties of implementation. Thus, in addition to identifying difficulties, and their reasons, in implementing FA, the study shows how FA characteristics of the PD are important for its success. A key question for future research is how to scale up this approach to larger groups of teachers bearing in mind that assessing teachers' needs and providing efficient feedback is not an easy endeavor.

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Effects of the COVID-19 pandemic on mathematics achievement: An investigation in university math pre-courses

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Keywords: Mathematics Achievement, Achievement Test, Secondary School Mathematics, Pre-Course Mathematics, COVID-19 Pandemic

Introduction and background

The COVID-19 pandemic has led to some radical changes in the framework conditions for teaching and learning mathematics (Chan et al., 2021). The meta-analysis by Betthäuser et al. (2023) shows that learning outcomes in math and reading decreased worldwide in schools during the COVID-19 pandemic and how this trend varies across different countries. Results of large-scale assessments substantiate these findings for primary and lower secondary schools in Germany (grade 3: Gasteiger et al., 2023; grade 5: Schult et al., 2022; grade 9: Lewalter et al., 2023). Measures taken to contain the COVID-19 pandemic are regarded as one reason for this decrease: "The results of our study can therefore be taken as an indication that school closures and distance learning during the pandemic did indeed have an impact on students' mathematics achievement" (Gasteiger et al., 2023).

However, there is hardly any relevant data available for upper secondary level in Germany, especially for secondary-tertiary transition. Since subject-specific prior knowledge is an important part of the learning background of first-year university students, we investigated how the math performance of pre-course attendees differed before and after the COVID-19 pandemic.

Design of the study

For our study, we used an achievement test for math pre-courses with 32 tasks that we developed back in 2016. The tasks are based on the current math curriculum of the federal state of North Rhine-Westphalia, which is valid until 2025. Of the 32 tasks, 7 tasks relate to content for lower secondary level and 25 tasks to content for upper secondary level. All tasks are relevant for university math courses. Regarding the distinction between conceptual and procedural knowledge (Rittle-Johnson et al., 2001), the tasks contain different requirements, namely *mainly conceptual knowledge*, *mainly procedural knowledge*, or a *combination of both* (for examples, see Table 1).

Table 1: Examples for the different requirements

Conceptual knowledge	"Specify a function f with $\int_{-1}^{3} f(x)dx = 0$."
Procedural knowledge	"Simplify the following algebraic expression: $(6a^3b)^2 \cdot \left(\frac{1}{3}b^2z^2\right)^3$."
Combination of both	"Given is the function f with $f(x) = e^{\frac{1}{2}x}$. Specify the antiderivative of f."

The test was conducted without changes at the University of Duisburg-Essen before the COVID-19 pandemic, in the years 2017 to 2019, and after the COVID-19 pandemic, in 2023, in each case at the beginning of the math pre-courses in September.

Main results and outlook

The results show that there are practically no differences between the years 2017 to 2019, but significant differences between the years before the pandemic and 2023 (see Table 2).

Table 2: Average results for all attendees (32 tasks)

	2017	2018	2019	Ø 2017-2019	2023
correct	48 %	46 %	52 %	49 %	35 %
false	32 %	35 %	28 %	32 %	32 %
unprocessed	20 %	19 %	20 %	20 %	33 %
n	95	70	49	69	69

While an average of around 16 out of 32 tasks were solved correctly in the years 2017 to 2019, only around 11 tasks were solved correctly in 2023. At the same time, the quota of unprocessed tasks rose from an average of 20% to an average of 33%. Further analyses have not revealed any topics or types of requirements to be primarily responsible for the decline in performance.

The Faculty of Mathematics at the University of Duisburg-Essen is, therefore, intensively discussing how lectures can be adapted to the changing prior knowledge of first-year students.

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How can the prison context change teachers' perspective and practices about mathematic assessment?

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Keywords: Assessment, prison context, teacher perspective, teacher practice

Rationale and theoretical framework

Black and Wiliam (2009) use the term 'assessment' to refer to all those practices, involving both teachers and students, that provide information that can be used to modify teaching and learning activities. Assessment can be differentiated by its purposes: formative assessment (FA), or assessment for learning, and summative assessment (SA) or assessment of learning (Black & Wiliam, 2009). Generally, in ordinary Italian classrooms there is a preponderance of SA (Amado & Morselli, 2023); teachers are frequently not aware of the distinction between FA and SA and assess students' learning through written and oral tests which involve a single student called to the blackboard, performing in front of the teacher and peers (Pastore et al., 2019). The role of the context in which the entire learning-teaching process takes place is crucial to better understand teachers' perspectives and practices regarding assessment (Wong et al., 2016). This is particularly true in the multi-complexity of a prison school context, which is characterised by: (i) heterogeneous classes in terms of age (from 14 to 70 years old), socio-cultural and linguistic background, literacy and personal experiences; continue turnover of students (usually up to 10 students); (ii) frequent experiences of failure encountered by students in their educational paths, in particular in mathematics, and (iii) the fact that "evaluation" in the sense of "judgment" permeates any prison setting (Decembrotto & Maffia, 2024). Furthermore, there is a lack of literature on the issue of mathematics education in the prison context and, even more, on the specific issue regarding assessment. We present the preliminary results derived from the "Learning Math in Prison" project (LeMP) with the aim to highlight if and how, the prison context can overturn some of the teachers' perspectives and practices related to assessment in mathematics compared to those of teachers' in ordinary schools.

Data and methods

Data collected during the exploration phase of the LeMP project were qualitatively analysed to shed light on teachers' perspectives on assessment in mathematics. No specific questions on assessment were proposed thus all teachers' reflections emerged spontaneously. We consider data gathered through a digital platform collecting teachers' reflections, a focus group and individual teachers' interviews. The 4 mathematics secondary school teachers involved from 3 Italian prisons had different experiences in the prison context, none of them had a specific training for teaching in prison. All data were coded independently by all three authors considering the components of the FA (Black & Williams, 2009), then questionable cases were shared and discussed.

Preliminary results and further steps

The teachers were aware of frequent students' experiences of failure in their previous school path, especially in mathematics, and they stated that this leads students to low levels of self-efficacy.

Teachers interviewed stated that assessment in prison schools differs from that in ordinary schools. Particularly, two spontaneous overturns of perspectives and practices related to assessment emerged from teachers' comments: (i) FA as almost frequently used with respect to SA; (ii) Oral tests as a practice promoting FA and personal motivation. Although teachers did not explicitly refer to FA, several aspects in their reflections can be identified as FA practices. The prison context and in particular the need to consider the frequent variation in school attendance, the classes' heterogeneity, the frequent turnover of students and their detention conditions, allows teachers to disengage from the logic of grades and led them to personalise the assessment process by considering personal starting point and needs. The data collected also revealed that, in the prison context, the practice of calling a student to the blackboard, as usually teachers do in Italian oral tests, can be a virtuous student-centred practice as far as it is *'an emersive approach [...]in which students want to emerge'* and, as one teacher stated, as one teacher stated, the student recognises himself as effective. Oral tests allow teachers to observe students' understanding and provide them with feedback, serving as implicit assessment and enhancing their self-efficacy in mathematics. These identified overturns are two examples of how spontaneously the prison school context allows mathematics teachers to create a relaxed climate without the anxiety of grades and to promote and personalise assessment based on students' educational path and motivation. These preliminary reflections will be deepened in the next phases of the project: the other aspects of FA might be present in teachers' practices but did not emerge in data collected. Further information related to mathematics assessment practices in the prison context will be collected through the analysis of teachers' learning units design and their implementation.

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TWG 22: Curricular Resources and Task Design in Mathematics Education

An introduction to TWG22:
Curricular resources and task design in mathematics education

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At CERME14, TWG22 continued to integrate macro-level research on curricular resources with micro-level research focusing on more specific units of curriculum design, namely tasks. The sessions covered several key aspects: the analysis of (digital) textbooks and curricular resources to examine their impact within broader curricular systems and to compare educational systems; the design of tasks and resources (including digital resources) across different contexts and for various purposes; and empirical studies investigating students' and teachers' interactions with curricular resources and textbooks, with particular attention to their perspectives and experiences. Throughout the sessions, participants highlighted numerous connections between TWG22 and topics explored in other thematic working groups, emphasising the potential for further collaboration.

Keywords: Curricular resources, task design, task implementation, textbook research, educational systems.

Scope and focus of the working group

TWG22 takes into consideration an interconnected dual perspective on research on curricular resources and task design in mathematics education. On one hand, at a macro level, curriculum resources – both digital and print – are the object of work for teachers, lecturers and students in various formal and informal learning contexts. These resources are chosen, adapted, (re)designed and used, by teachers and instructors, to work - individually and collaboratively - on planning lessons, organising teaching, and assessing student learning progress. Such resources, including educative curriculum materials, are often central in professional development activities. In these contexts, mathematics teachers work together with education researchers to create or transform print, digital, or hybrid resources, building their design skills and pedagogical knowledge in the process.

On the other hand, at a micro level, curriculum resources mainly consist of tasks taken from textbooks or other sources. The way tasks are presented, their sequence, and how teachers implement them in class can influence their cognitive demand, shaping students' perceptions of mathematics. This affects students' chances to make connections within mathematics and to develop concepts, skills, and mathematical habits of mind. Research literature highlights that tasks are fundamental to effective teaching. Recently, there has been a growing interest in studying task design features (including features of tasks in digital environments) that encourage specific mathematical activities, methods to analyse task affordances and principles for implementing tasks effectively. Furthermore, involving

students in designing tasks can prompt them to reflect on their own knowledge, understanding, and practices.

The following aspects were discussed in the TWG:

- Analysis of (digital) textbooks and curricular resources to explore how they operate within and shape broader curricular systems, and how these resources and textbooks provide a lens for studying and comparing educational systems.
- Task and resource design in various contexts (e.g., schools, teacher training) and for different purposes, with particular attention to digital environments and AI.
- Empirical research on students' interactions with curricular resources and textbooks, focusing on their perspectives and experiences.
- Empirical research on teachers' use of curricular resources and other materials specifically designed to support their practice (e.g., lesson plans, teaching guides).

In the next section, the papers presented during the conference are introduced following the order and themes in which they were grouped during the sessions.

Issues addressed during the sessions

An issue discussed in TWG22 was teachers' use of curricular resources and other materials specifically designed to support their practice. In particular, two papers focused on task design for teacher education and professional development:

- Lorenzo Pollani, Laura Branchetti & Francesca Morselli: Design principles for epistemological dépaysement of pre-service teachers within interdisciplinarity between mathematics and physics
- Patrick Ebers: Noticing of teacher students on an explorative task to exemplify the potential of using technology

Pollani and colleagues propose a theoretical paper aimed at presenting a framework for fostering epistemological dépaysement (Jahnke et al., 2002) – i.e., the critical reflection that emerges from epistemological discomfort – in pre-service teacher education focused on mathematics-physics interdisciplinarity. The authors define design principles for tasks that promote such reflection and exemplify them with a textbook analysis task. The goal is to develop interdisciplinary teacher expertise by encouraging pre-service teachers' critical reflections on their assumptions and disciplinary perspectives.

The study by Ebers explores how a video case showing two students using digital tools to solve an open-ended mathematics task can support teacher education by developing pre-service teachers' noticing skills. The analysis of 44 pre-service teachers' reflections revealed that, while many identified surface-level difficulties, few recognised deeper mathematical or tool-specific challenges, such as the special case explored by the students in the video. The findings highlight the importance of well-designed resources in preparing teachers to interpret student thinking and integrate technology meaningfully into mathematics teaching.

Another issue was how textbooks (including digital ones) operate within and shape curricular systems and how they could provide a lens for studying and comparing educational systems. Three papers were focused on this issue:

- Kevin Larkin, Silke Ladel, Ulrich Kortenkamp, Mellony Graven, Lise Westaway, Pamela Vale, Kym Fry, Tammy Booysen & Dinah-Marie Wiedenhöfer: Place value concepts as presented in German and Australian student workbooks
- Magdalini Lada, Iveta Kohanova & Birgit Pepin: Examining the paths of reasoning presented in the topic of 'patterns' in Norwegian mathematics textbooks
- Dubravka Glasnović Gracin, Goran Trupčević & Katarina Širanović: Exploring the use of digital textbooks in classrooms: focusing on automated feedback

These three contributions demonstrate that various formats of textbooks are being examined in mathematics education, all of which can be analysed through document analysis methods.

Larkin and colleagues compare how place value concepts are represented in German and Australian student workbooks. For this purpose, 102 tasks are analysed and categorised according to different concepts and subconcepts. The study reveals interesting differences between the textbooks, some of which can be explained by the numeracy range prescribed in the respective curricula; however, this does not account for all variations.

The paper by Lada and colleagues investigates two traditional mathematics textbooks used in Norway. Here, the focus is on the paths of reasoning in the topic of patterns as presented in the textbooks, shaped by national guidelines. Notably, the textbooks differ in their chosen approach – numerical versus figural. One reason identified for these differences lies in the ambiguous formulations found in the curriculum documents.

Glasnović Gracin and colleagues analyse the use of digital textbooks with a particular focus on automated feedback. This involves examining both the types of tasks already implemented and how the provided automated feedback is dealt with during classroom practice. It is noteworthy that teachers are required to consistently check the correctness of this feedback and decide how to respond to it in their teaching.

In the overarching discussion, it became clear that the criteria for selecting textbooks for classroom use can vary significantly. These criteria may be set by administrative authorities or be adjusted according to new educational guidelines.

The issue of design (of digital environments, of resources, of tasks, of units within textbooks) represents the focus of different papers. The following papers discuss the issue of the design of digital environments and resources (including AI) adopting different perspectives:

- Carlotta Soldano, Sonia Palha, Sara Bagossi, Cristina Sabena, Anders Bouwer, Erwin M. van Crasbeek & Myrto Karavakou: Digital game-based mathematics curriculum resources: methodological reflections on collaborative prototyping
- Sylvia Patricia Van Borkulo, Peter Boon & Johan Jeuring: Learning goals in view: Design of an intelligent environment for arranging and monitoring secondary mathematics education

- Burcu Nur Baştürk & Menekşe Seden Tapan Broutin: CHATGPT as a resource for pre-service mathematics teachers to design their lesson plans and lesson scenarios
- Francesco Contel & Annalisa Cusi: Students' perspectives on GPT-4's supported problem solving

Soldano and colleagues' paper examines the collaborative design process by analysing two game-based prototypes and illustrating their development through feedback and revisions. The goal of the study was to develop a methodological tool that monitors how collaborative prototyping affects the evolution of prototypes. To examine this evolution, the authors qualitatively analysed the data in these prototyping design documents using a deductive approach, namely, their educational, technological, and mathematical aspects. The MDA (Mechanics-Dynamics-Aesthetics) framework (Hunicke et al., 2004) was used to track design evolution, highlighting how collaborative processes may improve the educational potential and adaptability of digital tools for learning.

The poster by Van Borkulo and colleagues introduces the design of an AI-supported environment for secondary mathematics education, focused on aligning teaching materials with curriculum goals. The system incorporates design principles (flexibility and customisation, alignment with learning goals and their graphical representations, and performance) with the aim of creating an intelligent environment for arranging and monitoring secondary mathematics education processes.

The poster by Baştürk and colleagues explores how and for what purposes pre-service mathematics teachers in Turkey use ChatGPT to design lesson plans and scenarios. Using the Documentational Approach to Didactics (Gueudet & Trouche, 2009), the authors examined ChatGPT as a resource. Ten participants, with no prior experience with ChatGPT, used the tool over an eight-week period. The study found that ChatGPT supported them in acquiring pedagogical knowledge, validating ideas, and simulating student responses. Various "usage schemes" emerged, revealing how teachers integrated AI in lesson preparation through prompts and content co-construction.

Contel and Cusi investigate how a 15-year-old student employed GPT-4 for support during problem-solving. Using the instrumental genesis approach, the study examined the student's metacognitive processes and evolving interaction patterns with the chatbot. The analysis highlights clusters of themes that reveal the complexity of the instrumental genesis processes in human-LLM interactions, suggesting that this complexity should be considered in the design of suitable tasks involving LLMs.

Another sub-theme in design was the investigation of the features of tasks and curricular resources that promote specific aspects of mathematical thinking. The following five papers and posters elaborate on this issue:

- Hyman Bass: Task designs to support Connection-oriented mathematical thinking
- Yosep Dwi Kristanto & Zsolt Lavicza: Down-to-earth mathematics: transforming curriculum and teaching on function transformations
- James Gray: Prime Times: a game to develop multiplicative reasoning
- Jana Višňovská & José Luis Cortina: Conceptualisation of Number in Curricular Resources: Does it Matter?

- Nataly Pincheira, Paula López, Marta Gual & Ángel Alsina: A proposal of tasks to integrate sustainability in the early ages from mathematics education

Bass's paper proposes a cross-topic perspective on mathematical tasks, called connection-oriented mathematical thinking. This perspective utilises tasks that promote students' skills to interact with the connections between seemingly unrelated mathematical entities. These interactions include noticing, articulating, and explaining these connections. Suggesting the practicing connections framework, the paper demonstrates core concepts and practices of the domain; representations used within the domain; and applications outside the domain. By constructing a connection network between the different entities, the students engage and promote their ability to connect and communicate mathematical ideas.

Kristanto and Lvicza propose that teaching function transformations can benefit by promoting productive meanings to transformations and integrating mathematics with other domains. By combining Duality, Necessity, and Repeated Reasoning (Harel, 2008) and emergent graphical shape thinking as a boundary object, the study introduces a digital task sequence on the topic of climate science. The findings show how students' envisioning of a graph in terms of what is made (a trace) and how it is made (covarying quantities) supports their construction of productive meanings of function transformations.

Višňovská and Cortina explore the manifestation of continuous quantities in the development of the notion of numbers in the Australian elementary school curriculum. This analysis demonstrates how certain ideas can be restricted in the way that national curriculum documents progress in the learning and development of mathematical concepts. The analysis captured different treatments of numbers in terms of continuous quantity across the different topics and discusses reasons and implications.

The poster by Gray introduced the design and implementation of a multi-stage visual multiplication game. The cyclic design process was motivated by the intention to move students from recognising prime factors not only as a string of numbers, but also transformational and relational aspects in multiplicative reasoning. The game was experimented with in-service teachers and led to meaningful discussions but also revealed additional challenges in using opportunities to identify the greatest common divisors that promote a new design iteration.

The poster by Pincheira and colleagues introduced the design tasks for integrating sustainability into mathematics education. The study proposes a series of tasks that enable teachers to play an effective role as agents of social change by integrating sustainability in early age mathematics education. By introducing mathematical topics such as data analysis, algebra, and geometry in contexts such as food production or house construction, the tasks demonstrate a possible bridging between mathematics and the environment.

The teachers' use of textbooks, curricular resources and other materials, such as lesson plans and teaching guides was discussed by the following papers:

- Per Nilsson, Simon Sjölund, Jannika Lindvall & Andreas Ryve: Empowering mathematics instruction: Variation theory, lesson plan, and instructional quality

- Marcus Gustafsson: Upper-secondary school mathematics teachers' use of a research-based teacher guide
- Marzia Garzetti: Teacher Perspectives on Textbooks and Their Role in Implementing the Mathematics Laboratory Approach in Italy

Nilsson and colleagues' paper explores how a teacher with low mathematical knowledge for teaching, in a class with average socioeconomic status, manages to establish high instructional quality with the support of a lesson plan based on variation theory (VT) (Marton, 2015). The structured yet flexible plan supported students' understanding of fractions by guiding them through critical aspects of the object of learning. The teacher adapted the lesson in real time to student responses, demonstrating how VT can scaffold effective instruction even in complex contexts. The findings suggest that well-designed educational resources grounded in VT can potentially reduce the relative impact of the teacher compared to teacher-level factors and student-level factors.

The study presented in Gustafsson's paper explores how Swedish upper-secondary mathematics teachers use a research-based teacher guide on quadratic equations as a resource in planning teaching. The teachers' resource use is described in terms of perceived affordances and constraints and of the mobilisation of this resource into designed instruction. Using surveys and interviews with six teachers, the study identifies factors that influenced their mobilisation of different guide components.

Garzetti investigates how Italian lower secondary mathematics teachers use textbooks to implement the mathematics laboratory approach promoted by national curriculum guidelines. The results of a questionnaire completed by 609 teachers highlight that, while textbooks are widely used, mainly for exercises, they are rarely employed to support laboratory-based activities and promote collaborative and argumentative practices. Many teachers supplement textbooks with online materials or self-designed activities, especially for laboratory work. Satisfaction with textbooks increases with teaching experience, but a lack of ready-to-use resources is a common barrier, especially for less experienced teachers. The findings highlight a gap between policy intentions and classroom practices, to be further explored through qualitative follow-up studies.

The following four papers and posters relate to the theme of empirical research on students' interactions with curricular resources and specific environment, considering both the students' and the designers' perspective:

- Cristina Poli: With great freedom comes great responsibility: exploring potentials and pitfalls of using open environments through a discourse analysis
- Caterina Papa & Agnese I. Telloni: Use and perception of differentiated scaffolding by primary school students involved in mathematical tasks
- Annika Volt, Kairit Tammets & Mart Laanpere: Flexible Instructional Trajectories for Learners' Self-Regulation in 9th Grade Algebra
- Sara Gagliani Caputo, Laura Branchetti & Annalisa Cusi: Synergies and dissonances between designers' and students' perspectives on key features of digital mathematical discussion

All four contributions entail intervention studies and relate to either digital or paper and pencil environments. Together, they cover secondary and elementary school in Italy and Estonia and draw on qualitative and quantitative analysis. They address features such as personalised learning, scaffolding, online discussion, and open-ended assignments. Collectively, these studies showcase the importance of studying student experiences when designing specific instruction resources.

The poster by Poli explores, drawing on a commognitive lens (Lavie et al., 2019), grade 10 Italian students' and tutors' engagement with open-ended assignments in a digital environment. The findings suggest that while open-ended assignments can add with various tasks, they also come with demands on teachers to facilitate student learning.

Papa and Telloni study how 18 grade 3 students in Italy use and perceive differentiated scaffolding in the form of hints and prompts to a sequence of four geometry tasks. From field notes during the intervention study and student answers on a survey, the authors identified four user profiles, reflecting various levels of awareness and ways of turning the hints and prompts into action (effective use of scaffolding elements, immediate use of scaffolding elements without generalisation, using scaffolding ineffectively, recognising difficulties but refusing to use feedback). The authors conclude that a major challenge is to find ways to enable students to take advantage of the scaffolding.

Volt and colleagues investigate how 129 grade 9 Estonian students perceive a digital environment, designed to learn algebra in flexible and personalised ways. They identify four factors that seem to affect students' learning experiences (perceived efficiency, shared control, cognitive load, and importance of own role) as well as three student use profiles (independent learners, habitual learners, learners in difficulty).

Gagliani Caputo, Branchetti, and Cusi study how 63 grade 9 Italian students perceive digitally facilitated mathematical discussions. Central features of the digital mathematical discussions are its written and asynchronous nature. Comparing students' perspectives with the ones from the designers has allowed the authors to see points of alignment and divergence, which is argued to be helpful in further design cycles.

Possible directions for future research

Discussions within TWG22 highlighted the extensive connections with research from other TWGs. The papers presented were not confined to a single school level, covering the use and development of resources from early education through to university. They also addressed a wide range of mathematical topics and were carried out in various educational contexts. Moreover, curricular resources discussed were not limited to traditional print formats but included digital tools, games, and real-world contexts. Many other interdisciplinary connections were evident as well. Moving forward, we aim to explore various ways to foster interactions among different TWGs, both during and between conferences, to support these shared interests. For TWG22 specifically, discussions identified the following directions for future research:

- Focus on integrating physical resources (tasks, teacher guides, lesson plans, (digital) textbooks, AI as a resource, etc.) with human resources to better achieve educational goals;

- Deepen the study of coding schemes and representations (e.g., illustrations and diagrams) to strengthen the validity of task and resource analysis methodologies;
- Investigate collaborative task design involving communities of interest, where one actor is a digital environment (e.g., ChatGPT);
- Explore the types of support users need, considering their different dispositions, characteristics, and contexts;
- Study of curriculum features that cut across educational contexts and features that are unique to specific educational systems.

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Task designs to support connection-oriented mathematical thinking

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Connection-oriented mathematical thinking refers to the skills and disposition to notice, articulate, and use deep, often surprising, connections between mathematical entities that may, on the surface seem unrelated. This paper presents and analyzes some instructional designs meant to provide opportunities for students to develop this mathematical disposition and practice.

Keywords: Mathematical connections, task design, problem-solving.

Introduction

> *Science is built up with facts, as a house is with stones. But a collection of facts is no more a science than a heap of stones is a house*. Henri Poincaré

Mathematical connections are widely discussed, illustrated, and promoted, both in the research literature and in curriculum materials (e.g., Carlson & Rasmussen, 2008; Cuoco, 2005). Even at the frontiers of mathematics research, some of the most dramatic breakthroughs have been the discovery and articulation of a surprising and deep connection between seemingly distant parts of mathematics (See Greer & Harel, 1998). Making connections, or sensing, articulating, and using them, is a significant expression of mathematical thinking. This is related to what Hatano and Inagaki (1986) call *adaptive expertise*, in contrast with *routine expertise*. Routine experts have a lot of knowledge which they can apply well in familiar circumstances. In contrast, adaptive experts can flexibly apply their knowledge in a wide range of contexts, both familiar and novel. 21st century skills increasingly call for adaptive expertise.

Connections between two entities that are different can be of different kinds. A connection may be measured by some combination of: the degree of separation between the domains or contexts of the entities; their conceptual distance; the substantive depth and significance of the connection; and the extent to which the connection was hidden or unexpected. By some combination of such measures, one can distinguish "far" connections from those that are "near" and easily seen, to borrow terminology from transfer theory in cognitive psychology (Barnett & Ceci, 2002). This paper focuses on *far connections between mathematical problems*. But they could be other kinds of mathematical entities, for example, mathematical domains, theorems, methods, mathematical representations, or skills. This is distinct from the important and well-developed area of mathematical modeling, in which a real-world context is represented by and is therefore connected to a mathematical model (e.g., Arseven, 2015; Dunn & Marshman, 2019).

In many discussions of connections, the primary attention is on the entities compared, and the connection is treated as an aid in understanding them. In transfer theory for example, a connection is treated as a resource to mobilize knowledge of one entity in order to understand another entity to which the connection links it. In contrast, my primary focus is on learning to perceive the connection itself, its nature and significance, even when both entities are separately well understood.

Although mathematics instruction can present interesting mathematical connections, this paper offers some instructional designs to prompt students themselves to notice and make such connections.

Specifically, the aim of the project reported here, is a contribution to the study of what I term "connection-oriented mathematical thinking," and of how it might be supported by instructional designs.

Conceptual framework

Because mathematics instruction is often organized around solving problems, I have initially framed this project in terms of mathematical connections between different mathematical problems. Given two problems, A and B, I ask, in what way are they mathematically related, possibly even "the same," and what is nonetheless importantly different about them (typically the context)? I am mainly interested in the case when the difference is striking, yet the "sameness" is substantial, deep, and perhaps subtle or surprising. It is the existence and nature of this "sameness" that constitutes the connection on which this project focuses.

To capture this kind of sameness I use the disciplinary concept of *mathematical structure* (see Bourbaki, 1950). Broadly speaking, a mathematical structure consists of a set of objects together with some relations among subsets of them, together with some types of operations that can be performed on them. Common examples are groups, vector spaces, function spaces, geometric objects, and various combinatorial structures. For structures of the same species, there is a notion of structure preserving transformation between them, including the case of isomorphism (Greer & Harel, 1998). To unify many cases, these structures often have abstract axiomatic formulations.

In the above context, problems A and B may have underlying structures of the same species, for example structures on their solution spaces. This is already a significant connection. There may further be a structure preserving transformation from one to the other; that strengthens the connection. The transformation may even be an isomorphism, in which case we can say that A and B are structurally the same (Bassok & Holyoak, 1989; Simon & Hays, 1976). Examples are given below.

This paper proposes and analyzes instructional designs aimed at supporting student learning to notice, articulate, and use such connections, what I call "connection-oriented mathematical thinking" (COMT). Fries et al. (2021) assert that connectedness is key to the coherent mental schemas that underlie deep understanding and transferable skills. The development of COMT, like other kinds of learning, is gradual and is not fully mastered at one point in time.

Although some of the problems are challenging, I have crafted them to be conceptually connected to fundamental mathematical ideas, and to call for basic conceptual insight, not speed, extensive factual knowledge, technical virtuosity, or ingenious invention. My premise is that COMT can be made meaningfully and developmentally accessible to all students.

My instructional designs draw upon the *practicing connections framework* Fries et al. (2021). This framework emphasizes attention to three types of connections: core concepts/principles and practices of the domain; key representations used for communication and thinking within the domain; and applications outside the domain. For example, in arithmetic, the core principles include the "rules of arithmetic," the basic properties of arithmetic operations, and their conservation as the number systems expand across the grades, and into algebra, ultimately becoming the field axioms. In geometry, core ideas include symmetry and its dynamic meaning in terms of transformations. Another is similarity, viewed as geometric rescaling, and how this affects linear, area, and other kinds of geometric measure. Multiple representations of a mathematical construct are important but should

be coupled with detailed understanding of what each best affords, and of the explicit correspondences showing how they each represent the same thing.

For instruction, my design differs from Fries et al. (2021) in one important regard. They apply their framework to a statistics curriculum for community college students, recognizing the challenge of even noticing deep connections, so they propose that such connections be made explicit to students, and then practiced. The designs reported here however, propose scaffolded discovery of connections, situated in various mathematical domains across the K-16 span, with examples of gradually increasing complexity across grade levels, with the goal of developing fluency incrementally.

Literature review

The idea of making connections is not new to curricula and research in mathematics education. Much of the connection-related literature focuses on generative features of individual problem types (Kondratieva, 2011; Leikin & Levav-Waynberg, 2008; Sally & Sally, 2000; Steel et al., 2000).

Scholars have also developed several types of non-routinely connected problem sequences. One is a problem-based curriculum, in which development of the main ideas and principles of the subject domain are presented through problems, to be resolved through substantial student engagement. A post-secondary example is the "Moore method" (Jones, 2012). Another was developed by Hungarian educators Tamas Varga and Lajos Posa (Katona, 2020; Matzal, 2020).

In general, problems having the same answer may be a weak signal of mathematical connection. Exceptionally, the Catalan numbers (Bass, 2023b; Stanley, 2015) provide the common answer, for substantial reasons, to a remarkable variety of problems. Still another type of connection is a collection of diverse, seemingly unrelated problems that can, after perhaps unrelated kinds of mathematical work, all be reduced to one core problem (Bass, 2023a; Usiskin, 1968). An example will be provided.

The main type of connection studied here is for problems whose underlying structure is common, or closely related. An important longitudinal instructional example is reported in Maher and Martino (2014), in which a student, over grades 1-6, developed the underlying structure of the problems of counting all bi-colored unifix towers of given height. Another is Sriraman (2004), in which a student distills the "pigeon-hole principle" from his solution, over time, of a number of its applications.

In the literature on expertise connected knowledge has been discussed in terms of schemata (Schoenfeld, 1992). From the survey of Weber et al. (2006), "Experts can categorize problems based on their underlying mathematical structure, sometimes after reading only the first few words of the problem."

Design and analysis, illustrated in one example

What distinguishes the work reported here from the literature briefly surveyed above, is the present focus on problems for which there is an identifiable structure that is demonstrably common to all of the problems, yet the commonality of that structure may be both unexpected and not easily discerned, even by experts. Moreover, a mathematical structure of one problem and that of another may be "proximal" yet, for subtle reasons, not being identical. An early expression of this focus is evoked in Greer and Harel (1998). This, and the examples above of Maher and Martino and of Sriraman, come closest to the perspective offered here.

The products of this project, still in evolving development, are: (a) a specific approach to the construction of collections of structurally connected problems, with deep connections that are often far and unexpected; (b) explication and refinement of the concept of "common structure"; and (c) designs of instructional formats to stimulate and support connection-oriented mathematical thinking (COMT). It is worth noting in (a) that, when the connections are deep, the problems themselves need not be. In order to reify these complex ideas, I now present and analyze an example.

The basic artifacts of this work are structurally connected problem sets, so here is one, called the "3-permutation problem set."

A. What are all three-digit numbers that you can make using each of the digits 1, 2, 3, and using each digit only once?

B. What are all three-letter English words that you can make using each of the letters, a, e, t and using each letter only once?

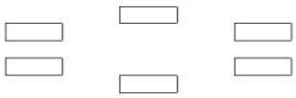
C. You are watching Arisha, Brianna, and Carmen on a merry-go-round. At each moment you see them in some order (left, middle, right). As the merry-go-round turns, what are all the different orders in which you see them?

D. If Arisha, Brianna, and Carmen have a race, and there are no ties, what are all possible outcomes: first, second, third?

E. In a 3 x 3 grid square, color three of the nine (unit) squares blue, in such a way that there is exactly one blue unit square in each row and in each column. What are all ways of doing this?

F. What are all the symmetries of an equilateral triangle?

Discernment format (intuitive): Put each problem letter in one of these boxes.



Put letters in the same box if the problems are structurally the same. If two letters are in different boxes but the problems are closely related structurally, connect the boxes with a line. Provide an explanation of each choice in your "network."

Connected structure format: Identify a mathematical structure underlying each problem. Indicate and justify when these structures are the same, and, if not, whether and how they are significantly related.

Figure 1: The 3-permutation problem set.

Two related dimensions of design are visible here. The *mathematical dimension* is the problem set itself, and its salient features. What is involved in constructing such sets? Mathematically, the problems should be diverse, yet they should involve some common mathematical structure — in this case the set of permutations of three objects — yet not all in the same way.

The *pedagogical dimension* is the framing of the task presented to the learners; two versions are offered here. My work has focused on college students, although the ideas could be adapted for broader use. Several things are worth noting. First, students are not explicitly asked to solve the problems. Indeed, parts of the task can be performed short of solving all of the problems. Second,

the work is intended to be done collaboratively in small groups and over an extended time, more than a single class period. Third, this is a novel and complex kind of mathematical work, so initiation was designed to make less rigorous, more intuitive demands on students, to open up for discussion the complexity and likely unanticipated subtleties involved. I hypothesized that the connected structure format is best used only after practice with the more accessible discernment format.

What does the underlying structure of a problem mean? And what does it mean for two problems to have the same underlying structure? The absence of definition in the example above is not an oversight. *Generic* answers to the first question seem to be unhelpful. Yet one can give a persuasive answer in particular cases, and its details emerge only in answering the second question, which takes the form of a transformation from one structure to another, or in linking both to some overarching structure. It appears to be more intuitively meaningful to identify a structure that two problems share than to discern and articulate *the* underlying structure of a single problem.

Consider the number *five*. Where is it? Can you point to it? You can hold up five fingers to exemplify five. And to show that another set also does so, you can list its members, i.e. match them with your five fingers. So, we can concretely demonstrate that two things both represent five, but *five* itself is an abstract concept naming a sameness shared by all five-element sets.

Faced with two problems, the learner might discern something similar about them, for example their solution spaces, and a transformation between them, and the structure (new, and yet un-named) emerges as the set of features, deemed to be fundamental to the problems, preserved by this transformation. This is the hoped-for discussion that could emerge from learner responses to the prompts above.

Consider, finally, an analysis of the problem set above. A defensible "connection network" is:

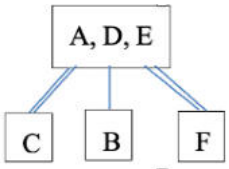


Figure 2: Defensible network for the 3-permutation problem set.

Clearly each of the solution spaces of A (digits) and D (racers) consist of all six permutations of three objects. The same, less obviously, is true of E (three blue squares). One can, for example, match the solution spaces of D and E by interpreting the columns of the (3x3)-grid square as the running lanes of the three racers, and blue squares as the runner positions at the finish line. The solution space of B (letters) consists of only five of the permutations of {a, e, t}. For problem F, a symmetry of the triangle is determined by the permutation it performs on the three vertices. Again, all six permutations occur, but this is not a priori clear, as in the case of A and E; one has to show that each permutation on performed by a geometric symmetry. Similarly, the solution space of C (merry-go-round) is a set of permutation of the three girls. It turns out that all six permutations occur, but this again requires some significant geometric reasoning.

Some people may question then why C and F are separated from A, D, E, since their solution spaces are all the same. One way to clarify this is to consider the problem set for 4, or n, in place of 3. Then the number of solutions remains n! for A, D, and E, but it is 2n for F, and n(n-1) for C, and with very

different arguments in the latter two cases. These numbers agree only when n = 3. The instructional design calls for this elaboration to n = 4 to be brought to learners' attention to prompt reflection on their previously constructed connection networks for the case n = 3.

Note that, despite the subtlety of some of these connections, these problems themselves are all reasonably accessible, say to middle school students.

Conclusions and next steps

This project was inspired by the well-recognized importance of mathematical connections, and it is directed at the least developed kinds of deep connections, those that express some of the larger scale coherence of mathematical thinking as a whole. Its conception was crafted in light of the large literature on mathematical connections; informal and cursory observations of some curricular materials; and the structures and norms of disciplinary mathematics. The designed products, reported here, are bi-partite, consisting of connected-structure problem sets, together with designs for their instructional use. While such designs may, at first sight, seem to be simply curriculum materials, they are distinctive in some ways worth noting. First, they consist of carefully curated sets of problems designed to support the teaching of COMT, itself an important and sometimes overlooked mathematical disposition and practice. Second, the problem sets are accompanied by a specific instructional design, situated in collective group work. Third, construction of the problem sets themselves was challenging and called for some novel skills. To advance this work, it will be important to develop more of them, and more diverse examples. Finally, the designs demand more than usual instructional time and multi-leveled developmental structure.

The designs discussed here were developed at first informally in a variety of my undergraduate mathematics courses, including one for secondary mathematics teachers. In those contexts, I have informally experimented with the designs, with results that informed their development.

This project is a first step in a longer-term agenda. Next steps for its advancement are:

1) More empirical evidence that connection-oriented mathematical thinking (COMT) is a broadly needed skill for students, especially those in STEM.
2) Well documented and analyzed enactments of the kinds of designs presented here, to determine to what extent they support the learning of COMT.
3) Development of measures of COMT, which is a co-requisite of 2).

Consistent with the findings of Maher and Martino (1996) and Weber et al., (2006), the learning of COMT takes time to mature. And so, one cannot expect polished student performance from fresh exposure to these complex ideas. Serious ambitions in this direction must be comfortable with modest initial outcomes, yet remain bold in aspiration. Even if enacted, COMT would not always be a deliberate goal of instruction. But, outside those dedicated episodes, the residual effect should be development of what might be called, "*connection-sensitive mathematical thinking*."

Some have argued that, under the influence of technology, notably AI, many work environments will be changing radically and rapidly (National Academies of Science, Engineering, and Medicine, 2024). And that this will call for adaptive knowledge and skills to enable continued functioning in less familiar environments. While this thinking remains speculative, it offers some potential support for the broad importance of COMT.

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Students' perspectives on GPT-4's supported problem solving

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In this paper we present an exploratory study aimed at investigating upper secondary school students' perspectives on the role that GPT-4 could play in scaffolding their problem solving processes. Data have been collected by means of semi-structured interviews developed before, during and after students' interaction with GPT-4. The data analysis, developed by the IPA methodology, is presented by focusing on a paradigmatic example of a 15-year-old student. The results of the analysis highlight clusters of themes that allow us to shed light on the complexity of the instrumental genesis processes that characterize human-Large Language Models (LLM) interaction, suggesting that this complexity should be taken into account in the design of suitable tasks involving LLM.

Keywords: GPT-4, students' perspective, instrumental genesis, problem solving

Introduction

Problem solving (PS) is a significant part of worldwide mathematics curricula. In an educational and research context that is increasingly affected by the presence of Large Language Models (LLMs), mathematical PS activities mediated by this technology represent a starting point both for empirical exploration and theoretical conceptualization of a broader class of AI assisted mathematical teaching and learning situations. Effectively approaching problems requires the necessary involvement of metacognitive processes (Schoenfeld, 1992). Given the inherently linguistic dimension of metacognitive processes, these could be effectively scaffolded by LLM-based chatbots, due to the capabilities of real-time and human-like text generation that LLMs possess. However, a number of concerns have been identified: LLMs produce either fictional or non-factual information, they lack understanding of natural and symbolic language at the semantic level, with specific implications in the field of mathematics (Frieder et al., 2024). To use LLMs for mathematics tutoring, it is crucial to improve their mathematical skills and the quality of their feedback (e.g., Schorcht et al., 2024). This suggests the need of addressing these concerns by developing research studies aimed, at the practical level, to identify criteria for the design of teaching interventions based on mathematical tasks involving the use of LLMs, and, at the theoretical level, to clarify the status of LLMs and of the interactions mediated by them in relation to existing frameworks in the context of Mathematics Education with digital resources (DRs). It is precisely at the level of the theoretical foundations that we position our contribution: following the assumption that LLMs can be impactful if used as teaching resources, we believe that to better understand which phenomena occur in human-LLM interactions is a prerequisite to inform effectively the methods for the design of tasks mediated by this technology. The research documented here is part of a broader study focusing on the role of LLMs in supporting students' metacognitive processes in problem solving (Contel & Cusi, 2024).

Theoretical Framework

The *instrumental approach to mathematics education* emphasizes the role of artifacts as essential components of learning processes and proposes to consider an instrument as a living entity throughout

the students' activity when facing mathematical problems. At the heart of this framework lies the notion of *instrumental genesis* introduced by Rabardel (2002), which is the process of developing an instrument from the interaction between a subject and an artifact, with a specific set of goals in mind. This relation is so defined in dialectical terms: the instrument develops from a given artifact, but also the artifact itself could be transformed by the instrumental genesis; drawing from Vergnaud's concept of scheme (1998), Rabardel introduces the concept of utilization scheme as "the unchanging organization of the subject's behavior for a class of situations, both in terms of action and symbolic activity" (Rabardel 2002, p. 78). An instrument therefore is defined as a material or symbolic artifact with one or more associated utilization schemes, resulting either from an autonomous construction specific to the subject, or from an appropriation of schemes formed outside of him/her. In the context of dynamic technological environments the notion of instrumental genesis has been extended, by Hegedus and Moreno-Armella (2010), to the notions of simultaneous co-actions between a user's use of a tool and a software environment's use of a tool, interpreting processes of utilization of the tool as the feedback and reactions of a user. In the authors' words:

"For instrumentation, we additionally define it as how co-actions with a tool shape the user's actions and understanding of the use of such a tool within, and with respect to, an environment. Instrumentalization is extended to how the tool is shaped by the user (user's knowledge) and the environment, i.e., when the tool is manipulated by environmental factors following a user-input." (Hegedus & Moreno-Armella, 2010, p.30).

Following a previous research (Contel & Cusi, 2024), we interpret the user (the student) and the digital environment with which he/she interacts (the GPT-4 environment) as alternately playing the roles of actors and re-actors in performing actions. According to this interpretation, the utilization schemes developed by the students in their interaction with GPT-4 influence the way in which GPT-4 re-acts and, at the same time, the development of these utilization schemes is influenced by the way in which GPT-4 scaffolds the students' problem solving processes. In this paper, we investigate students' perspectives on the instrumental genesis, by addressing the following research questions, which respectively focus on the instrumentation (RQ1) and instrumentalization processes (RQ2):

RQ1. How do students interpret the influence GPT-4 has in their problem solving processes?

RQ2. How do students interpret their use of GPT-4 as a tool to support the problem solving process?

Analytical framework and research methodology

The exploratory study that involved 19 upper secondary school volunteer students (grades 9 and 10). Students worked individually on two tasks focused on the use of algebraic language as a tool to construct proofs in arithmetic (see "Results" on page 4), with GPT-4[1] as the only means of support. GPT-4 was pre-prompted by the researchers with the following instructions, selected after careful examination of its behavior in previous simulations:

[1] GPT-4 is one version of the popular chatbot ChatGPT developed by OpenAI. For further info about the different models: https://chatgpt.com/

We want to use GPT–4 to guide 15- to 16-year-old students in conjecturing and proving in arithmetic through the use of algebraic language. GPT–4 should act as a tutor, without giving direct suggestions or confirmations, but only by asking questions that stimulate students' thinking. At the end, it should invite the student to revise the results obtained in relation to the initial objective. It would be better to ask one question at a time. Each question should be as concise as possible, using language that the user can understand, without losing rigour at the mathematical level.

A semi-structured interview was conducted before and after students' work on the two tasks. Each session lasted about 100-120 minutes. The data collected were the written notes taken by the students, the chats, the transcripts of the audio- and video-recordings of students' interaction with GPT-4 and of the semi-structured interviews.

We carried out a twofold analysis of the collected data: (1) analysis of the video-recordings of the interactions between the students and GPT-4 to highlight the characteristics of the PS processes that are developed; (2) analysis of the students' interviews to identify themes emerging from their reflections on the influence of GPT-4 and to their use of GPT-4 to support their PS processes. Level (1) of analysis was developed by referring to Schoenfeld's (1992) analytical model of PS protocol analysis. It takes into account different processes observable in different environments (e.g. with the mediation of a DR, like in this study): analysis, exploration, planning, implementation, verification. We created timeline diagrams to show the processes developed by students and the time spent on each process throughout the PS activity. Moreover, following previous findings (Contel & Cusi, 2024), the diagrams were enriched highlighting the occurrence of specific monitoring processes that are relevant in the characterization of human-LLM interaction during PS. We included 3 types of visual indicators of the moments when students asked for help activating a monitoring process; process-oriented monitoring (in Figures 1 and 2, dashed red line), conversation-oriented monitoring (dashed green line) and requests for confirmation after procedures are implemented (dashed blue line). In these diagrams it is not possible to look directly at GPT4's interventions, but, coherently as our interpretation of it as a re-actor, the change in the PS phase is started by a student's request. The co-occurrence of PS phase shifts and monitoring process activation in our interpretation shows an impactful interaction.

Interviews (2) were analyzed with the Interpretative Phenomenological Analysis (IPA) approach (Smith & Osborne, 2003). IPA aims to understand the complexity of meanings rather than to measure their frequency and it does so through a sustained examination of the text in order to identify emerging themes from the interviewed person's perspective, which are subsequently categorized into clusters; clusters are then organized in a narrative account of the interview. Starting from the findings of the previous phase we looked for instances of alignment and misalignment between the student's perspectives and the characteristics of the observed interactions according to the protocol analysis.

Results

We present the data analysis methodology by focusing on a single example, the case of Jack, a grade 9 student who was 15 years old at the time of the study. He described himself as "familiar with ChatGPT, I use it almost every day", although not specifically in the context of mathematical PS. His

previous experience with ChatGPT is the main reason why we decided to focus on his case. Previous experience drives his awareness of his own instrumentation and instrumentalisation processes; these processes are subjective, but Jack's acts and reflections focus on aspects which we consider to be relevant for the instrumental genesis process in a general way. Generality should be read here in theoretical and analytical terms rather than being interpreted as a statement about the sample derived from a statistically accurate data analysis. Jack's case is proposed as a paradigmatic one because of his "expert user" perspective and for his ability to take into account many different factors that were mentioned separately by different students.

Jack was asked to solve the following tasks through the support of GPT-4:

T1: Consider a natural number. Write down the difference between its square and the square of its antecedent. What regularities do you observe? Consider other numerical examples to help you develop your conjectures. Can you prove your conjectures?

T2: Consider a natural number. Write down the difference between its cube and the number itself. What regularities do you observe? Consider other numerical examples to help you develop your conjectures. Can you prove your conjectures?

Figures 1 and 2 show the results of the protocol analysis (level 1) relative respectively to T1 and T2. Figure 1 shows two distinct cycles of Exploration – Planning – Implementation (E-P-I); although the second E-P-I cycle starts with "Analysis", we consider the two patterns to be equivalent since Analysis and Exploration phases are not completely distinguishable. The resolution process in Figure 2 consists of two distinct macro-phases which lead quite straightforwardly to a verified solution; however, the task should not be interpreted as worked in a routinely manner: transitions between phases occur in general in correspondence of Jack's questions to GPT-4 or of answers to the previously received questions. This suggests that GPT-4's support is impactful in driving the overall process. This is testified by the fact that the two key transitions between the two cycles and before the verification phase occur after Jack's request for conversation-oriented monitoring (dashed green lines). We interpret these as moments of effective co-action between the chatbot and the student: Jack is able to steer the chatbot's help towards the desired direction, while GPT-4 re-acts positively to the student's prompt.

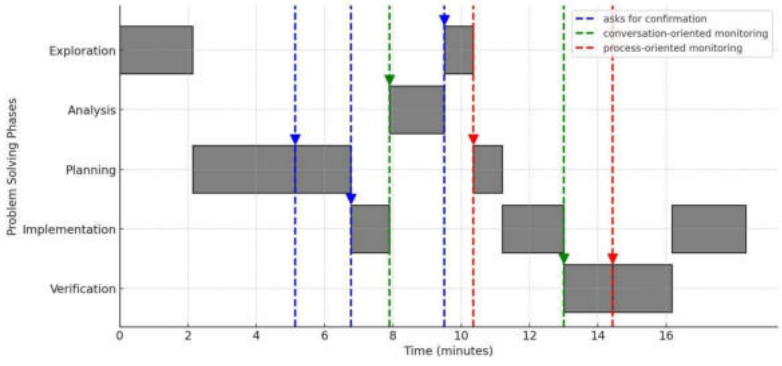

Figure 1: Timeline diagram of T1 problem solving process

Figure 2 highlights that the PS process related to resolution of T2 is more complex: this process is highly non-linear and the interaction lasts almost twice as long as T1. Pattern E-P-I, suggesting initial linearity in Jack's actions, is abandoned after the first cycle, leading to a phase characterized by jumps between Exploration and Planning.

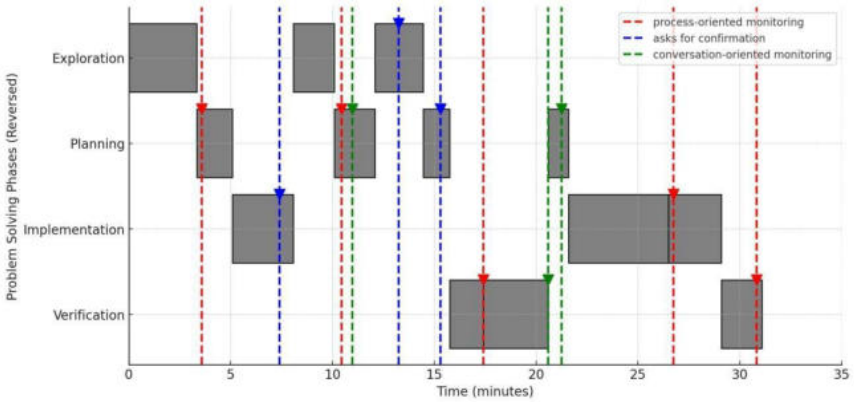

Figure 2: Timeline diagram of T2 problem solving process

The impasse is overcome at minutes 20-21, with the following conversation (in Figure 3, the two subsequent vertical dashed green lines):

1. Jack: This way seems too complex to me, are there other simpler ways?
2. GPT-4: I see, let's explore another way that might be simpler. A good strategy might be to look directly at the algebraic expansion of the difference between the cube of a number and the number itself. In other words, examine the expansion of $n^3 - n$. The question to consider is: Can you algebraically decompose the expression $n^3 - n$ in a way that will reveal a regularity or a pattern? Hint: You might want to consider the fact that $n^3 - n$ can be written as $n(n^2 - 1)$, and then examine the $n^2 - 1$ term more closely.
3. Jack: Don't tell me too much.
4. GPT-4: You're right, let's proceed step by step. Let's start with the first part of the decomposition: can you decompose the expression $n^3 - n$ into a product of simpler factors?
5. Jack: n x (n+1) x (n-1)

Jack's explicit request for a different kind of help (line 1) is a turning point, confirming the importance of metacognitive processes in the management of the DR; however, GPT-4's intervention is not aligned to Jack's will and needs, prompting the student to execute an order and partly giving him the answer, more than stimulating his autonomous cognitive and metacognitive processes.

The IPA analysis (level 2) provides insights into these phenomena: 3 clusters of themes emerged from the semi-structured interview. The first cluster of themes includes the support Jack has received by GPT-4, suggesting a possible answer to RQ1. GPT-4's help is seen as *substantial* from many different perspectives: the feedback is timely ("One thing is that this is immediate, i.e. you can ask

him[2] straight away and he gives you the answer straight away.") and being guided by many questions and prompts is helpful. He recognizes that this kind of support is somehow overly assertive ("Maybe, I don't know, in another case I would have done a different process."), having made him focus on something he would have not done. At the same time, Jack shows to be aware that what he received was not always the desired support: he evidences the difference between being helped with reasoning and being helped with the solution ("then when he had corrected me and I had asked 'what did I do wrong?' he had helped me more with the solution, that is he had told me *more than I actually wanted to know*"). The second cluster groups themes around the difficulties that Jack faced and the way GPT-4 contributed to them, suggesting further ideas to answer to RQ1, but also providing issues to bridge Jack's perspectives on the instrumentation (RQ1) and the instrumentalization processes (RQ2). A major theme is the ambiguity of language in case of misunderstandings. Jack reads many of the problematic situations he had to face through the lens of communication; he provides examples of words or constructs used by the chatbot which prevented him from focusing on some aspects of the problem, while driving him into others:

> "My block was more about the language used: 'How could you use algebra to prove your conjecture?' I thought I had to do something more complicated[...] reading the text, I would have done it, then he, by saying this, sent me into a bit of confusion just what to do, how to proceed […] he uses specific terms, exactly, and also expressions. But for someone like me who is not an expert, it is more difficult."

The other major theme is GPT-4's limitations: Jack is perplexed by its computational and symbolic capabilities, claiming that "you gotta be careful with its calculations, Chat might correct you and then state that he made a mistake himself". Jack gives much importance to the communicational issues, almost never mentioning mathematical difficulties.

The third cluster, which concerns the ways Jack tried to deal with the difficulties characterized through the second cluster, provides ideas to answer to RQ2. In relation to the issue of communication and language, he shows contrasting reflections: on the one hand, he points out that GPT-4 is adaptive since it may follow language and instructions, if properly prompted; on the other hand, Jack reveals that his behaviors were influenced by his previous experience with the chatbot:

> "so maybe it should be re-specified here 'I want to know but don't give me the solution of the final calculation'. So it should always be specified what you actually want and what you don't want, even if it has been said before."

He claims that an autonomously devised strategy is fundamental to get the support he wants:

> "One point I have made is that first it is always preferable to do the things you think for yourself, that is without the help of what he tells you that can be confusing […] Because, for example, I got

[2] Italian is a language with two categories of grammatical gender: masculine and feminine. Since there is no neutral form, in Jack's case ChatGPT is described as in masculine terms (some people use the feminine, because AI is feminine, as "intelligence"). We chose to translate coherently with this, emphasizing the student's original discourse.

> stuck before because I didn't know what to do, maybe thinking first, even if wrong, and then telling him what you have thought; first do all the reasoning, even if you want to do it algebraically, then present it in some way and see what he tells you, so whether right, wrong or whatever, instead of starting from scratch with the problem."

Jack confirms this stance in several moments; we interpret this as a personal initial utilization scheme, which helps the student to accomplish new results. Seen from the co-action perspective, we read in Jack's words a kind of synergy that brings understanding: "When we changed course, then he started from what we had said [...] *then there was the union* between what I had seen and what he wanted me to do, then there I understood." A connection is being recognized between his monitoring and the chatbot's desired behavior. Deeply linked to this last theme is that of trust and control: Jack feels in control of the situation as long as he understands the math involved;

> "In this case I trusted him, because we always remain in things I understand, that is, when there was the road I couldn't understand what he made me do and why he made me do it, there I wouldn't trust him because I just didn't understand. [...] What I have seen is that with what he gives you, and clearly with the skills you have to have, because if you do that exercise you have to have the skills, *united* you get to the result much more easily."

Discussion

With reference to RQ1, we propose evidence of the significance of GPT-4's impact in driving the overall PS process: the protocol analysis confirms the potential of this technology to interact with students. The user's perspective of this interaction as a "union" of forces suggests that this might happen in a synergic fashion, in line with the conceptualization of co-action between user and environment (Hegedus & Moreno-Armella, 2009). The correspondence between PS phase transitions and monitoring processes activated by Jack suggests that his metacognitive skills and knowledge of the artifact emerge as crucial factors, which strongly influence his interaction with the conversational agent. However, while the support is effective, the LLM does not allow for an autonomous reflection during PS: co-action, to be properly developed, requires full control of the user over the resource. Jack's reflection about control provides a bridge to RQ2: we interpret the need for control as the need to develop an effective utilization scheme, to steer the chatbot towards the desired goal. Jack shows awareness of the limitations such as the possibility of hallucinations or conversational loops leading to dead ends; it is also remarkable the connection he draws between his trust in GPT-4's support and his ability to control the underlying mathematical meanings. This awareness is reflected in the development of reflections and the performance of actions meant to manage these sources of issues. The user's reflection on its own instrumentalization processes is evident in the management of the PS process, that becomes the management of the LLM, in order to receive a support perceived as effective and aligned with the user's aim.

Our results highlight the need to reconceptualize the notion of co-action between a user and an environment (Hegedus & Moreno-Armella, 2010) in the context of human-LLM interaction, since the GPT-4 environment does not allow the environment's response to be uniquely associated with the user's action, despite the users' efforts and despite the back-and-forth dialogical structure of co-action.

Flexibility is a required feature of utilization schemes in users' interactions with an artifact, because, as Rabardel (2002) points out, utilization schemes are constantly evolving. What we see in this case, however, is something more: complex tasks such as those tested in our study can hardly be handled with a fixed approach that can be associated with a scheme that becomes stable over time. This has implications for future studies about the design of LLM-mediated mathematical tasks. More research is due to better understand the complexity of human-LLM interactions also to address the limitations of this study: the specific nature of the instrumental genesis process requires longer and more systematic observation of the students' activities; Future studies will also address the issue of prompt engineering, as this feature has a significant impact on the overall behavior of the chatbot and consequently the instrumental genesis process.

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Noticing of teacher students on an explorative task to exemplify the potential of using technology

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The potential of technology unfolds above all in the context of explorative tasks. This represents a major challenge for teaching, for which teacher students must be prepared. Using an exemplary task within a video case, 44 teacher students were asked to create an appropriate answer to the ideas of the students in the video case. The study shows that pupil's difficulties are noted at the levels of subject content and use of the medium, but that the pupil's thought processes are not examined. As a result, there is often a lack of in-depth diagnosis of the causes. The role of the technology for the learning process of all learners in this task is underestimated in many cases.

Keywords: Task design, open-ended approach, technology, video case, teacher noticing.

Introduction

Teachers' reactions in the classroom are based on what they notice. For this reason, (teacher) noticing has become one of the most important components of teaching expertise (Gaudin & Chaliès, 2015). The training of this skill is very often done by video cases. The presented study deals with the construction and implementation of a video case with a special focus on students' subject-specific learning using digital tools while working on an open-ended task. Since tasks play a central role in teaching mathematics (Jones & Pepin, 2016), the use of a good task is a basic prerequisite for the construction of the video case, as first own design experiments showed. For this reason, this article first presents the task used and the video case used. Learning opportunities for student teachers or professional development courses can be derived from the analysis of the video case. The use of the case is structured with four questions. In this study, the answers of 44 teacher students to these four questions are analyzed and related to the learning opportunities described. The aim is to further optimize the case and generate findings for use in teacher training and professional development.

Theoretical Background

Research on task design has been growing since the early 2000s (Kieran et al., 2015). In the study presented here, the focus is on design as implementation (Kieran et al., 2015) for teaching with technology. The presented task stands as a good exemplary for dealing with teaching with technology and is shown in a video case to be used for pre-service teacher training and professional development. The statement "We can no longer consider that teaching mathematics with technology is just an option." (Artigue, 2013, p. 2) still holds true as ICILS 2018 shows (Eickelmann et al., 2019). Tasks must therefore be used that demonstrate the power of digital tools for subject-specific learning. Research reviews have shown that the potential of technology for learning mathematics can unfold primarily in the context of open exploratory tasks (e.g. Drijvers et al., 2016). However, this is a major challenge for teachers, for which they must be prepared. But there is often a lack of time and no chance for detailed observations of individual pupil's work. Video cases therefore offer a good opportunity to look at pupil's work in detail and without pressure to act.

The aims of using cases in teacher training therefore lie in the training of teacher noticing (Santagata et al., 2021). Case based learning offers many opportunities for teachers to learn by example or to follow students' thinking as they work on tasks. Videos are ideally suited to depicting the complexity of teaching, so that noticing can be practiced without time pressure and the need to act (Staal Jenset et al., 2024). Case analysis in pairs or small groups enables a change of perspective that is not as rich without other people. The aim of video case work is therefore in particular to sharpen perception and to consider alternatives for teaching activities (Santagata et al., 2021). These are just some of the benefits that Staal Jenset et al. (2024) highlight as findings from the three major literature reviews on teacher noticing. The research related to teacher noticing is becoming increasingly diverse. In particular, studies dealing with mathematics are very present. Videos that focus on teachers' actions are frequently used. Comparative analyses between the noticing of pre-service and in-service teachers are also presented. Only a few studies focus on student thinking. In this context, Jacobs et al. (2010) focus in their study the professional noticing on children's mathematical thinking in grades K-4. A study on teacher noticing, which deals with the use of digital tools for subject-related learning, was not found in comprehensive research.

In the present study, a video case is to be created as a medium for teacher training, which offers prospective teachers the opportunity to observe the processing of a task with digital tools, to understand the pupils' thinking processes and to consider what actions the teacher can take to support the pupils. The guiding research questions of the study are: (RQ1) What do teacher students notice in pupils' interactions solving an explorative task with technology? and (RQ2) What do teacher students learn for their own professionalization when working with the video case?

Methods

The study presented here has already been preceded by several design experiments with video cases and the testing of a video case for the same task in the context of different training courses (see e.g. Thurm et al., 2022). The insights gained into the quality of the task and the type of video recording of the pupils are important foundations for the implementation of this study. To construct the video case of Lukas and Robert, a sequence was selected from existing video material in which an interesting interplay of digital tool use and subject-specific learning can be seen. The selected sequence was analyzed by the author and a master's thesis student in order to find out how the pupils work on the task with a function plotter. In a mutual exchange, difficulties were identified at a subject-specific level and at the level of media use. This led to the formulation of four questions that were used for the student teachers' task and to structure their case analysis. Mathematics student teachers for lower secondary level in Germany from the second semester were asked to analyze the video case about the work of two pupils on an open-ended task. The teacher students were already familiar with the pupils' task through their own work a week before the data were collected. 44 teacher students submitted written responses to the following four questions:

1. What are the subject-specific difficulties?
2. What difficulties are there in using the medium?
3. Where do you see a need for support?
4. What insights have you gained for your lessons?

When analyzing the data, it was investigated which subject-related difficulties and which difficulties in media use were noticed by the teacher students. With the third question, the prospective teachers were asked to put themselves in the role of the teaching teacher so that it becomes clear how they would react and where they see the reasons for difficulties. The chosen focus makes clear which aspect is particularly important to the student teachers and where they would start with their support. The last question is intended to initiate a reflection process relating to their own teaching activities. The student teachers' responses were analyzed using qualitative content analysis (Kuckartz, 2012) and then compared with the categories already developed from the analysis during case construction.

The pupils' task and the video case

The task set can first be found in Drijvers (1994). There are various German translations that do not contain literal wording, but wording that is understandable for pupils. However, the task has retained its character. It has been used for many years in the context of courses at the University of Duisburg-Essen and since 2019 as part of various research projects (e.g. Thurm et al., 2022). Figure 1 shows the translated task as it was presented to Lukas and Robert.

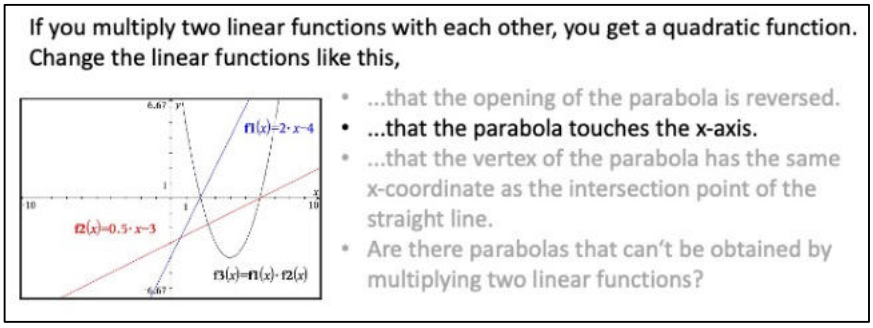

Figure 1: Pupils' task (Thurm et al., 2022, p. 4; highlighted the relevant subtask for this study)

The task is initially an unusual task and yet it can be anchored at different points in the curriculum. Lukas and Robert have already worked with quadratic functions in their lessons. For them, this is an opportunity for productive practice. The solution for this task is, that the parabola must have a double zero. To achieve this, the zeroes of the two linear functions must be identical. Based on the graphic in the task (see Figure 1), you could set $f_2 = 0.5 \cdot x - 1$ and would thus have the double zero at $x = 2$. From another sub-study, in which written solutions of pupils from a grammar school in Berlin were evaluated, it was found that many solutions were just one example or contained an example with a generalization or consisted purely of the general solution. In some cases, the pupils indicated the possibility $f_1(x) = -x$ and $f_2(x) = x$, which solves the first three subtasks at once. This shows the range of what can be expected when pupils solve the problem. The task offers the potential for the advantages of using the tool to become clear. Algebraic solutions to the task are possible, but they cannot be followed by the pupils. The tool offers the possibility of finding a different approach to the solution through the dynamic visualization. This includes the linking of graphic and symbolic representation, so that changes become visible, and the generation of examples through static representations which facilitates the work of developing and testing hypotheses. The task is therefore particularly suitable for the aim of the study. The pupils can use the tool to make rich mathematical

explorations that can be used to analyze the thought processes and the interplay of subject-specific learning with the media and the interaction of the pupils with these resources.

The video case Lukas and Robert shows two pupils from a tenth-grade comprehensive school in North Rhine-Westphalia, Germany. With the help of a function plotter, they are working on the presented task, which is an open-ended approach (Shimada, 1997). By varying the parameters (slope and y-intercept) of the linear functions, the parabola should touch the x-axis. In the video case the pupils Lukas and Robert have already completed the first subtask and are working on the second subtask. Although the task has been used frequently and the range of expected solutions is very broad, Lukas and Robert generate a surprising special case. Based on their knowledge that the parabola $h(x) = x^2$ touches the x-axis, they draw up the following plan: They want to "remove" the values in front of the x in order to generate an expression with x^2 without a prefactor when multiplying. They do not take into account in their explanations that the coefficient 1 in the product with x does not have to be written down. Both linear functions now have a slope of 1 and their graphs are parallel to each other in the coordinate system and the resulting parabola does not yet touch the x-axis (see Figure 2).

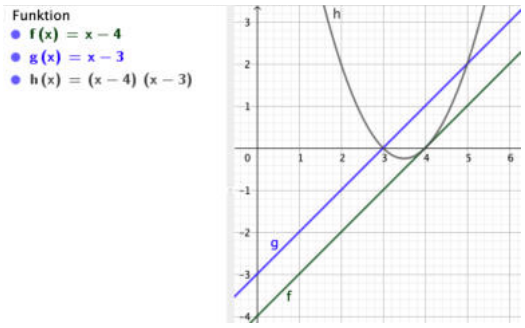

Figure 2: Reconstructed screenshot from the case of Lukas and Robert

By considering this special case, Lukas and Robert are confronted with an additional difficulty as the y-axis intercept and the zero point now have the same value in terms of magnitude (see Figure 2). They now change one of the two y-axis intercepts step by step, reflect on the resulting graph and think again about which value they want to try next. Due to the slope 1, incorrect interpretations of the parameters are made. This procedure can be assessed as systematic trial and error. The systematic variation is shown by the fact that they first consider which y-axis segment they want to change and only one. To do this, they always first say what they want to try, then comment on the result and then change the selected parameter again. In the end, they manage to find the one correct solution that was still possible for their special case with variation of the one parameter:

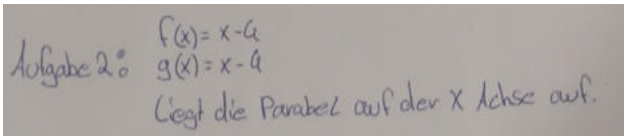

Figure 3: Written solution of Lukas and Robert

The role of the teacher in the video case is supportive. She tries to help the pupils to understand the task and explains the dependence of the parabola on the linear functions. The language chosen is not technically correct. This can also be taken into account when analyzing the case. It becomes clear

from working on the task that Lukas and Robert construct a special case that only allows the solution of equal linear functions. They find this solution and write it down. They do not reflect on their result and see no need to look for further solutions. The case analysis revealed three important levels (subject-specific skills, media use and the role of the teacher) of learning opportunities of the case, that guided the questions for the student teachers. The identified levels are shown in Table 1.

Table 1: Identified learning opportunities of the case

subject-specific skills	(a) the misconception that the coefficient for the slope is 1, even if there is no number
	(b) the confusion of the y-intercept with the zero due to the fact that the slope is 1
	(c) difficulties with linguistic correctness and the use of technical terms
media use	(a) general uncertainties during input, which are also linked to the requirements of the task
	(b) opportunities that a slider might have had advantages over manual input
	(c) a potential when switching between symbolic and graphical representation
role of the teacher	(a) correctness of used language
	(b) she could offer more help related to the specific problems

Results

To answer RQ1, the teacher students' responses to question 1 and 2 were connected. The subject-specific difficulties mentioned by the teacher students can be roughly divided into two areas: a) basic subject content and b) difficulties in processing. The statements relating to a) include comments on the meaning or confusion of the parameters. Here, students do not mention any causes such as the gradient with the value 1. In fact, no one refers to this special case. This lack of specific focus on mathematics is also described by Jacobs et al. (2010), who state that much more general pedagogical aspects are considered by pre-service teacher. It is surprising because the special case can be seen as the cause of the confusion between the value of the zero point and the parameter for the y-intercept in the case of Lukas and Robert. The confusion of the terms "y-value, f(x), x/y-axis" is mentioned and suggests that the use of correct terminology seems important to the students. Interestingly, it becomes clear at various points that the teacher students also show weaknesses in the use of terminology. Each word 'parameter' and 'factor' is only used once in each of the 44 answers, but words such as 'component' or 'element' are used instead. Four teacher students mentioned the confusion of axes. One person made it clear that this could just be a slip of the tongue and referred to a point in the video. The fact that Lukas and Robert construct a special case by generating parallel straight lines through their input is also not mentioned by the students as a difficulty or hurdle. The difficulties in working on the tasks are identified as understanding the task with the dependency of the functions and the resulting initial problems in varying the parameters. Despite the teacher's support in the video case, the teacher students very often see further difficulties, that the pupils do not know how to move the functions, as the biggest problem. Unfortunately, the teacher students also

do not recognize a systematic approach when trying things out. In 7 out of 44 answers, the procedure is described as pure trial and error.

The teacher students' answers to the question about difficulties in tool using can be divided into four categories: a) problems with input or shifting, b) undirected use in the form of trial and error, c) emphasis on skills and no difficulties and d) comments independent of the medium, based on the processing of the task or subject-specific principles. In 15 of the 44 responses, general problems in the use of tools are seen and these are attributed to a lack of routine. A total of eight of the responses mention the specific situation in which Lukas and Robert want to enter x^2 and are not familiar with the notation x^2. Three answers mention that Lukas and Robert do not know how to move the straight line. However, a drag mode is not available in this tool. One teacher student says: “It would be very helpful for the pupils if they could move the straight lines.” This is one of the few in-depth explanations of tool usage. Three of the 44 responses contain statements on b) undirected use in the form of trial and error and random insertion of numbers. One teacher student explains this by the openness of the tool, that there are too many possibilities for variations. A total of nine statements can be assigned to category c) emphasizing skills and no difficulties. Many answers remain very superficial. Twelve statements can be assigned to category d) media-independent comments. Of these, seven answers are to be understood in such a way that the teacher students indirectly reproduce the task and name this as a difficulty in using the tool. In addition, four times subject-specific basics are mentioned that lead to difficulties in completing the task.

One teacher students’ answers are highlighting an algebraic solution as optimal. It is stated that the medium tempts the pupils to solve the task geometrically. Already the answer to the first question is, that the pupils must solve the task algebraically. From the third answer it can be inferred that the pupils must be informed about the lure of the medium and that they should be led to an algebraic approach. This shows how fixated this teacher student is on their own solution path.

For answering RQ2, teacher students’ answers to question 3 and 4 were analyzed. The responses on the perceived need for support reflect the previously described levels of professional foundations and tool use. Methodological approaches were also mentioned. These include process-related and content-related support e.g. in a way that the pupils’ explorations are more structured, or the graphical representation should be analyzed in more detail. Four answers state that the task needs to be explained in more detail so that the dependencies are better understood. Content-related support should be given to pupils in the subject-specific basics of linear and quadratic functions and the meaning of parameters. Two answers are interesting which similarly describe that the pupils must first know the solution in order to then set this on the tool. This changes the role of the tool in such a way that it only serves to represent the solution and not to find it. This shows that the potential of the task for independent exploration of the relationships is not recognized or is assessed as non-existent by the teacher students. Another teacher student mentions that there are also solutions where the linear equations are not identical. In a discussion on teacher PD level, it could be mentioned that this can be used methodically so that Lukas and Robert can extend the solution they have found.

The final question on the insights that teacher students take back to their own future lessons is intended to provide information on which aspects cause them to change their way of thinking and

thus perhaps find their way into their lessons. For the case work, it is important that each specific case should contain something generalizable, which is meaningful for their future teaching (Staal Jenset et al., 2024). Unsurprisingly, the answers can again be divided into the areas of subject-specific basics, in particular the explicit repetition of individual content, and the use of the tool. The focus here is on explanations and assistance. The use of the tool is often praised, and it is emphasized that work with it should be intensified. The pupils' work with the tool is described as pure trial and error, which according to some should be better guided, and it would be better not to lose sight of the pupils so that not too much time is lost. At the same time, others positively emphasize that the time for trial and error is particularly important so that the pupils can deal with the mathematics independently. Fortunately, the latter opinion predominates in the teacher students' responses.

The following opinions regard to the task. Five out of 44 responses refer to the fact that the task needs to be explained better, and that pupils' questions need to be clarified before the task is worked on. Three times, the open exploration of the pupils and the associated character of the task was praised. One teacher student even highlights that they would like to use this kind of task themselves. It was emphasized that solving the task as a team was particularly helpful, also to stimulate discussions.

Conclusion

The teacher students perceive the nature of the task as the greatest challenge for the pupils. In addition to the recommendation to better guide the explorations with the tool or to prepare them with structure-like tasks, it is important to them that the pupils are given time to try things out. There is a difference between the perception that trying things out is aimless and praise for persistent exploration. A step-by-step instruction for exploring the relationships would be easier to grasp, particularly through dynamic visualization. Instructions on how to use a slider would be a simple means of using the tool. But it should always be clear that the tool character of the medium is retained, with which the pupils make their explorations and is not used solely for presentation their solution. In a final discussion at teacher PD level, it can be made clear why Lukas and Robert's exploration can be described as systematic experimentation. Perhaps this also changes teacher students' attitude towards the potential of the tool when working on the task. Different approaches include explicit input of parameters, sliders or, in some tools, a drag mode. Those possibilities once again show the potential of this task.

For their own future lessons, teacher students learn how difficult it is to observe the work of all pupils. It is therefore important to create the best possible conditions for independent work. To this end, they state that they would like to discuss tasks thoroughly and use tools continuously in lessons. Step-by-step introduction of various commands and tip cards should support the use of tools in order to reduce barriers to operation. Concluding discussions are intended to bundle the pupils' explorations and uncover difficulties. No one in the study rejected this type of task. The teacher students have become aware of the challenges and even noted that this way of teaching can be a long-term goal.

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Synergies and dissonances between designers' and students' perspectives on key features of digital mathematical discussion

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In this paper, we showcase students' perspectives on their experiences with Digital Mathematical Discussion (DMD) and we compare them with the designers' perspectives. After presenting the design of the DMD and the key features of its design in terms of the timing and communication modalities, we report an analysis of the responses given by grade 9 students through online questionnaires following their experience with DMD. We aim at looking for insights into students' interpretations and assessments of how a DMD is structured and operates, potentially revealing areas of strength and opportunities for improvement in future redesign.

Keywords: Digital mathematical discussion, asynchrony, written communication.

Introduction and theoretical background

In recent years, digital technologies have played an increasingly central role in educational environments, particularly by offering new opportunities for extending collaboration and communication beyond the traditional classroom setting (Ball & Barzel, 2018). According to Ball and Barzel (2018), there is still significant, understudied potential in utilising social networks and other communication technologies to connect students learning mathematics from different locations. Moreover, when considering communication through technology, the nature of technology use transforms how students engage with, learn, and think about mathematics (Ball & Barzel, 2018). In this perspective, Digital Mathematical Discussion (DMD) has been introduced to allow students to collaborate, discuss, and co-construct mathematical knowledge also outside the physical space of the classroom (Gagliani Caputo et al., 2023). DMD is framed within mathematical discussion (Bartolini Bussi, 1991) and asynchronous discussion (Wang, 2023). Specifically, DMD relies on the concept of mathematical balance discussion, in which students socialise and evaluate strategies set up in their individual or small-group work on a mathematical problem (Bartolini Bussi, 1991), and it is realised in asynchronous modality with support of specific digital environments that facilitate its unfolding. The discussion is guided by the teacher organising and sequencing excerpts from students' solutions for comparing similar or contrasting strategies (Cusi et al., 2017). The teacher orchestrates the discussion acting as a model of aware and effective attitudes and behaviours (Cusi & Malara, 2013) aiming at making thinking visible to students and at fostering metacognitive reflections.

In this contribution, we consider students' and designers' perspectives on the main features of DMD with the aim of highlighting synergies or dissonances between them. When referring to features, we consider Trouche's (2004, p. 290, footnote 6) distinction between tools' constraint (obliging or impeaching the user), enablement (making the user able to do something), potentiality (virtually opening possibilities), and affordance (favouring particular gestures). In the case of the features of DMD, we interpret Trouche's definitions referring to *constraints* as the norms and rules that shape

how the discussion unfolds, to *enablement* as the conditions or resources that allow participants to contribute effectively in the discussion, to *potentialities* as the range of possibilities that the discussion opens up for participants, and to *affordances* as the features that favour certain types of involvement in the discussion.

Design of Digital Mathematical Discussion and its key features

DMD has been designed by the authors (Gagliani Caputo et al., 2023) and consists of three phases:

1. The first phase involves students in small-group work (4-5 students per group) for developing strategies for solving mathematical problems. The mathematical problems can vary depending on the objectives the teacher aims to achieve through the discussion. This phase takes place over some days (3-4 days in our study) within chats on an instant messaging platform (e.g., WhatsApp). The teacher is present in the group chats but only intervenes if explicitly requested by students. At the end of this phase, one student from each group submits the collective solution to the teacher via an institutional platform (e.g., Google Classroom).
2. The second phase is an asynchronous whole-class discussion orchestrated by the teacher with the aim to guide students in analysing, comparing, and reflecting on the strategies proposed by the groups in the first phase (Cusi et al., 2017). The discussion occurs over multiple days (3-4 days in this study) on a collaborative web platform (Padlet). The teacher structures columns in the Padlet asking questions on selected excerpts from the group solutions (Gagliani Caputo et al., 2023). Students and the teacher can add posts containing text, images, or attachments, and can insert comments and react to posts through upvotes or downvotes.
3. The third phase, following the whole-class discussion on Padlet, is a one-hour in-person discussion in the classroom to summarise and consolidate what emerged on previous phases.

The first two phases of DMD are conducted online and asynchronously as homework over several days. Students are asked to make all thought processes explicit, including problem-solving steps, doubts, new ideas, difficulties, and individual explorations, so that all participants can see the reasoning. Moreover, they are asked to interact on the activity only through the digital environment provided for the discussion (chat and Padlet) taking advantage of the technical features provided by the environments (e.g., sending pictures or attaching documents) and to maintain communication in written form. The digital environment was selected by the designers through an analysis to ensure it closely matched the desired settings for the DMD and could adequately support its implementation.

In this paper, we focus exclusively on the first two phases of DMD, as we are interested in investigating asynchronous modality and written communication. The asynchronous nature of DMD allows participants to engage with discussions at their own pace, affording flexibility that enables students to reflect, process information, and contribute thoughtfully without the constraints of synchronous timing (Wang, 2023). This flexibility is particularly beneficial for accommodating diverse learning styles and schedules, as students can revisit discussion threads, formulate responses over time, and manage their commitments. Similarly, written communication supports clarity and permanence, and provides a transcript of the discussion that can be referenced later (Wang, 2023). Moreover, as highlighted by Santos and Semana (2015), effective communication through writing is crucial for developing a deeper understanding of mathematics. Writing aids students in clarifying and refining their ideas, and in presenting and justifying their problem-solving methods formally.

Asynchrony and written communication can be characterised as key features of DMD as presented in Table 1. The characterisation is based on the design of DMD presented above and introduces the designers' perspective on asynchrony and written communication as key features of DMD.

Table 1: Designers' characterisation of asynchrony and written communication as features of DMD

	Constraints	**Enablement**	**Potentialities**	**Affordances**
Asynchrony	It is required not to interact with others in synchronous modality Time to submit contributions and to refresh to see new contributions	The duration of the discussion is set over multiple days Engagement of participants who feel less comfortable in real-time interaction	Exploration of concepts over time Introduction of ideas that open up new directions for the discussion	Flexibility in participation
Written communication	Impossibility of exploring functionalities that do not involve written communication Lack of body language for non-verbal communication	Making thinking processes explicit for both the one writing a contribution and those who read it Engagement of participants who feel writing as support for developing their reasoning or reflecting on others' contributions	Control over the whole discussion thanks to the availability of its complete transcript and consequent potential activation of metacognitive reflections on the structure of the discussion and on the links between its parts	Encourage structured contributions Encourage the use of mathematical symbols

Context, research question and methods

The study presented in this paper is part of the 2nd and 3rd cycles of a wider design-based research project (Cobb et al., 2003). These cycles involved two Italian science-oriented high schools and took place between February and April in 2023 and 2024. The experiments were conducted with two grade 9 classes and their teachers for each cycle. The mathematical problems students faced during the activities were focused on using algebraic language as a tool to construct proofs in arithmetic.

In this paper, we aim to investigate synergies and dissonances between designers' and students' perspectives on asynchrony and written communication as key features of DMD, in order to highlight relevant aspects of DMD design that should be reconsidered for re-design. The study is guided by the following research question: How do students perceive asynchrony and written communication as key features of DMD in their articulation as constraints, enablement, potentialities and affordances?

In order to answer the research question, we collected students' responses to an online written questionnaire with open-ended questions that highlighted their reflections on the role of the asynchronous modality and written communication in DMD after completing the four activities of the experimentation. The questions aimed to investigate students' perspectives on the challenges they

encountered during the asynchronous phases of the DMD, the usefulness of having access to their peers' written messages, the benefits and weaknesses of working asynchronously, and their reflections on the differences between working in-person and in the different digital environments involved in the DMD. Specifically, to gather students' perspectives on the asynchronous phases of DMD, their reflections were analysed with a focus on the perceived usefulness of having time to read and respond to messages, as well as the value they attributed to accessing written contributions from group members. Although the latter aspect focuses specifically on the availability of peers' written contributions during group work, students' answers were general enough to allow us to analyse their broader opinions about the written communication modality in DMD. We collected 65 students' responses to the questionnaire and we analysed only the 63 responses containing answers to both questions. The analysis of students' answers to the questionnaire is conducted through a thematic analysis. The analysis is conducted reading all students' answers and grouping them according to common themes emerging from the content of their answers with respect to asynchrony and written communication in DMD. The emerging themes are articulated as constraints, enablement, potentialities and affordances recognised by students. In particular, a theme concerns constraints if it highlights students' perceived difficulties or limitations, enablement if it refers to what actually allows them to participate in the DMD, potentialities if it concerns new possibilities DMD offers to students, and affordances if it relates to what students experience in the DMD in relation to their modes of involvement.

Analysis and results

The analysis of the answers given by students to the questionnaire allow us to identify themes of students' perceptions concerning the asynchronous modality and the written communication in DMD as constraints, enablement, potentialities, and affordances. In this section we present the results of the analysis that consist in the identification of themes based on students' reflections and the percentage of answers pertaining to each theme (the total percentage does not add up to 100% because each response may fall into multiple themes).

Concerning the asynchronous modality of DMD, four perceived themes emerged in students' answers: (1) flexibility in work management (22% of answers); (2) availability of time to reflect on one's contribution (60% of answers); (3) availability of time to reflect on others' contributions (18% of answers); (4) lack of pressure to respond immediately (3% of answers).

Students' answers in theme 1 concern how the asynchronous modality supports them at the organisational level both as enablement and affordance. Indeed, they noted that the flexibility in time allows them to effectively taking part in the discussion (enablement) but also that it shapes their time of interaction within the discussion (affordance), as they could choose when to participate in discussions without restrictions on their peers' timing ("I didn't have any problems coordinating with others, and I could write in the group whenever I had time"), and manage their participation alongside other school's activities ("I was able to organise myself better with the rest of my homework") and personal commitments ("With my personal commitments, I wouldn't have always been able to write"). Themes 2 and 3 show students' exploitation of the potentiality identified by the designers of exploring concepts over time. Students reported that the possibility to take their time allowed them

to reason more thoroughly ("[taking my own time] gave me the opportunity to have all the time to reason") both revisiting their own reasoning multiple times before submission ("one can review their ideas more calmly") and reading and understanding others' contributions more easily due to the ample time available ("I had the opportunity to read and understand others' contributions more easily, given that I had a lot of time available [for the activity]"). Theme 4 includes students' interventions who state explicitly that they "didn't have the pressure to respond immediately". Even though theme 4 was identified in only a few students' contributions, it is still relevant to be considered, as it highlights a potentiality of DMD recognized by the students but only partially addressed in the design. With respect to asynchrony, students were generally positive in their comments, nevertheless some have identified some constraints. For theme 1, students noted that the freedom to organise their timing sometimes led to procrastination and close proximity to deadlines ("You could work on the activity whenever you wanted, but sometimes, there was a tendency to always postpone the work, thus delaying it until almost the deadline"), or to forgetting to respond ("Sometimes they [peers' contributions] would arrive while I was working on other assignments, and I needed to focus, and then I would forget to respond"), or losing the line of reasoning ("If you take time to respond you risk losing the line of reasoning, which in turn results in wasting more time").

Regarding the use of written communication, three themes were identified. According to students, written communication in DMD supports: (A) read and reflect on others' reasoning while discussing (25% of answers); (B) compare others' reasoning with one's own (34% of answers); (C) re-read old contributions (42% of answers).

In theme A, students reflected on the enablement of written communication for accessing others' contributions more easily within the discussion. Examples include statements such as "[having access to others' written contributions] was helpful for comparing and seeing the different types of solutions" and "I was able to better understand what my classmates wanted to communicate". In theme B, students noted that having access to written contributions enabled them to revise their own opinions ("I was able to have direct comparisons between my opinions and those of my classmates, and on many aspects that I thought were clear, I changed my mind") and to find new insights ("[others' written contributions] were tools for gaining insights, but it was also challenging to read them and sometimes not understand them"). Theme C concerns students' exploitation of the potentiality identified by the designers concerning the availability of the transcript of the discussion. Students highlighted the possibility to re-read contributions once the discussion has moved on or finished to gain a better understanding of others' interventions ("since they [others' contributions] were written, they could be reviewed multiple times and reflected upon more deeply"), revisit one's own reasoning ("[having access to written contributions was useful] to revisit my reasoning even after days"), avoid forgetting important points during the discussion ("I didn't risk forgetting them [others' contributions]"), recover from missed parts of the conversation ("It was useful because I sometimes lost track of the conversation when others were writing in my absence, but by re-reading the messages, I was still able to understand what had been discussed"), or reconstruct the entire work ("It was easier to trace the development of the work"). However, few students expressed a perceived constraint that was not completely considered in the design, that the challenge to deal with long written contributions ("Having so much written in a single message is really annoying").

Students' comments focus also on combined features concerning asynchrony and written communication, such as the possibility to use the written contributions at their own pace to develop individual reasoning, and the availability of the written transcript of the entire workflow and discussion to all participants at any time. Regarding the first aspect, some students highlighted a potentiality of written communication, stressing that taking their own time to reflect on their own contribution was possible because they were writing their reasoning and that it wouldn't have been possible otherwise, such as using vocal messages ("before sending any message, there is time to review and correct any possible errors (which is not the case when sending a voice message)") or calls ("it was useful because I could respond when I had time. Therefore, it was more convenient than a phone call and offered more flexibility."). Regarding the second aspect, students shared their views on the availability of transcripts in response to the question "Do you think having access to the messages (both on chats and Padlet) even after the activities ended was helpful? Why? Did you find yourself going back to re-read messages (both on chats and Padlet) after the activities ended?". Forty students said that having access to the transcripts was beneficial and that they used them for various reasons: to clarify initially unclear issues ("I found myself going back to re-read the messages at the end of the activity to try to reflect on and understand some questions that were initially unclear"), to support subsequent class work ("In my opinion, keeping the messages was useful because, for example, while we were discussing in class, we could review our written reasoning from the chat"), to review the entire activity ("The messages show the progression of the entire activity, allowing one to 'retrace' it"), or to review parts of the activity when transitioning between phases of the discussion ("I found myself re-reading the passages when we moved from the chat to Padlet to understand our reasoning" or "[it was helpful] to compile the summary of the discussion that I then submitted on Classroom"). On the other hand, 29 students reported that they did not take advantage of the opportunity to revisit the messages. Many of them (21) do not recognise the enablement and affordances of written communication considered in the design, as they declare that having the transcripts was not useful at all ("I did not revisit the messages after the end of the activity because, once the posts were read, the reasoning was understood and memorised, making it unnecessary to go back"). Other 4 students, instead, acknowledged the potential future utility of having all contributions saved ("I did not review the messages after the end of an activity, but it is useful to have them saved in case they might be needed in the future"), even if at the moment they experienced DMD they have not recognised the availability of the transcript of the discussion as a potentiality.

Discussion and conclusion

The analysis of students' answers to the online questionnaire following their experience with DMD allow us to answer the research question by characterising their reflections on the asynchronous modality and the written communication in DMD and highlighting alignment and divergence in how the key features of DMD were characterised by designers. Students' responses reveal a more complex interaction with these features, introducing new insights that were only partially anticipated by the designers. Concerning asynchrony, for designers, flexibility (theme 1) is seen primarily as an affordance, allowing participants to engage in the discussion at their own pace. This feature aligns with students' perspectives, but according to them, flexibility functions as both an enablement and an affordance. It enables participation by allowing students to engage in the discussion according to their

own schedules, making it possible for them to manage their time effectively and participate despite other commitments. This aligns with the designers' intentions but goes further as students perceive it as actively shaping their interaction with the discussion. However, this flexibility also introduces a constraint that was not fully considered by designers: some students experience a loss of control over their work, leading to procrastination, forgotten responses, and difficulties maintaining the flow of reasoning. This reveals a dissonance between the designers' idea of flexibility and the practical challenges students face when this flexibility is not well managed. Moreover, designers emphasised the potentiality of asynchrony for exploring concepts over time. This potentiality is exploited by students, as reflected in their recognition of the benefits of having time to reflect on both their own and others' contributions (themes 2 and 3). Students describe how this extended time allows them to carefully reason, revisit, and refine their ideas before submission, as well as to understand others' contributions more deeply. This alignment between designers' intentions and students' experiences suggests that the potential of asynchrony to enhance reflective thinking is being effectively realised in practice. A key difference between students' and designers' perspectives emerges around the emotional dimension of participation, particularly the lack of pressure to respond immediately (theme 4). While designers considered the emotional aspects of asynchrony only partially, they did not specifically identify this as a significant feature. However, for students, the reduced pressure is perceived as a new potentiality of asynchronous discussions. This emotional benefit, which allows students to engage without the stress of immediate responses, fosters a more relaxed and thoughtful approach to participation, in line with Wang's (2023) vision of online discussion as a "relaxing environment" (p. 3) in which students can participate at their convenience. This aspect adds a new dimension to the designers' understanding of asynchrony and suggests that the emotional impact of flexible timing should be given more consideration in future design iterations.

Concerning written communication, designers identify it as a key affordance that facilitates reflective and structured engagement and thinking within discussions (as in Wang, 2023), but students' experiences introduce additional insights and challenges that were not fully anticipated. Designers characterise the ability to read and reflect on others' contributions (theme A) as an enablement, intending to make thinking processes more explicit. This aligns with students' perspectives, as they view this feature as an enablement that allows them to easily access others' reasoning within the discussion. Also concerning theme B, students' and designers' perspective align since written communication serves as an enablement that supports participants in comparing others' reasoning with their own, fostering deeper understanding and self-reflection. Themes A and B also align with the use of written communication experienced by Santos and Semana (2015) to introduce and justify reasonings in a clearer way to themselves and to others. Designers identify the potentiality of re-reading old contributions (theme C) as a feature of written communication, allowing participants to engage in metacognitive reflection. This potentiality is actively exploited by students, who highlight the advantages of being able to re-read contributions once the discussion has moved on or concluded. However, some students perceive a constraint in managing long written contributions, finding them challenging to navigate, an issue that was not fully considered in the design. This suggests a need for more user-friendly ways to manage extensive written content in future redesign.

Overall, the comparison reveals that while there are significant synergies between designers' intentions and students' experiences, some dissonances and new insights also arise. Students highlight new potentialities regarding the emotional benefits of reduced response pressure, but also new constraints, suggesting that further refinements could enhance the usability and effectiveness of asynchrony and written communication in DMD. Concerning the potentiality of having a transcript of the discussion identified by designers, just some students used it for further work even if most of them recognised the importance of having it for possible future uses. The ability to go back, review, and rethink is an attitude promoted through discussion and students should be taught how to exploit the potentiality of having at disposal the transcript of the discussion. In future experimentations this can be realised asking students to re-read and comment on some excerpts of their previous asynchronous discussion from the transcripts.

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Teacher perspectives on textbooks and their role in implementing the mathematics laboratory approach in Italy

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This study examines the use of mathematics textbooks by Italian lower secondary school teachers, focusing on their role in implementing the mathematics laboratory approach of the 2012 National Guidelines. Through a questionnaire completed by 609 teachers, the results show that textbooks are mainly used for exercises, with limited use for promoting laboratory-based, interactive learning. The results provide valuable insights into the role of textbooks for teachers and their relationship to the implementation of the mathematics laboratory. In addition, this study lays the groundwork for more systematic research into how the intended curriculum is implemented in practice.

Keywords: Curricular resources, Instructional change, Role of mathematics textbooks, Laboratory of mathematics.

Introduction

This research addresses the broad issue of curriculum translation from policy to practice. Although many European countries have introduced guidelines emphasizing active, competence-based learning, the implementation of these reforms in day-to-day teaching - particularly through key curriculum resources such as textbooks - often proves difficult (Rezat et al., 2021). The Italian context offers a particularly insightful case study to explore the role of textbooks. The 2012 National Guidelines for the Curriculum of Early Childhood and the First Cycle of Education (MIUR, 2012, own translation) advocate an instructional change towards a laboratory approach to mathematics that promotes exploration and collaborative problem solving reflecting similar shifts in other European education systems. However, unlike some contexts where textbook selection is centrally regulated, Italian teachers enjoy considerable autonomy in choosing the materials they use. This combination of policy-driven instructional change and teacher-driven resource selection makes Italy particularly well-suited to understand the dynamics influencing the use and role of textbooks in the relationship between policy and practice. By focusing on textbooks as the primary dimension of investigation, this study deliberately narrows its scope. Other relevant factors, such as teacher training, available infrastructure and classroom dynamics, while important, fall outside the scope of this work. Through a questionnaire administered to secondary mathematics teachers, the research explores how textbooks are used and adapted to support the laboratory approach. By providing descriptive insights into textbook use and its alignment with policy goals, these findings lay the groundwork for subsequent, more in-depth qualitative explorations - through planned focus groups - that aim to enrich understanding of the relevant dimension of resource use that influences the relationship between policy and practice.

Theoretical framework

Travers and Westbury (1989) distinguish between intended, implemented and attained curriculum, where intended curriculum is what is prescribed in official guidelines, implemented curriculum is what happens in the classroom and attained curriculum is the results achieved by students. As outlined

by Rezat et al. (2021), curriculum resources, that can be textbooks, multimedia resources or other teaching materials, are considered instruments of educational change. They mediate between the theoretical aims of curriculum documents, the intended curriculum, and the practical realities of classroom teaching, the implemented curriculum. These resources can either facilitate or hinder the implementation of educational reforms. Furthermore, they can be adapted to embed new pedagogical ideas and thus become objects of change themselves (Rezat et al., 2021). Rezat and colleagues (2021) highlight three key areas that are influenced by curriculum resources: mathematical content, instructional practices and vision of mathematics. This study focuses specifically on how textbooks affect instructional practices.

The role of textbooks as educational resources has been extensively explored across various educational contexts. Fan and colleagues (2021) highlight that the usage of textbooks can vary significantly among different groups of teachers, influenced by factors such as experience, training, and national context. They also highlight that the organization of teaching, instructional strategies, and the selection of explanations and exercises are heavily influenced by textbooks. Haggarty and Pepin (2017) discovered that less experienced or non-specialist teachers tend to rely more heavily on textbooks due to a lack of time for lesson planning, confirming that the teacher's level of experience is a critical factor in how textbooks are utilized as educational resources. However, literature reports conflicting results about the role of textbooks as facilitators or barriers in the teaching process. While some studies, like that by González-Martín et al. (2018), indicate that textbooks can be useful tools for sequencing and presenting content, others, like Matić & Gracin (2021), have shown that they can also limit the adoption of more innovative and interactive instructional approaches. In the Italian context, the investigation of textbooks as curriculum resources and teachers' perspectives on them remains underexplored, except for a few studies, mainly focused on primary schools (e.g. Sbaragli, 2020).

To contextualize this research, it is necessary to make a brief reference to the Italian school structure. In Italy, lower secondary school, serves as the second cycle of compulsory education and accommodates students aged 10 to 14 (grades 6 to 8). It lasts three years and is open to all pupils who have completed primary school. The National Guidelines (2012) outline the general learning objectives and competences expected at the end of the cycle. In particular, the national indications emphasise the importance of promoting the mathematics laboratory, described as

> […] both as a physical place and as a moment in which the pupil is active, formulates his own hypotheses and checks their consequences, plans and experiments, discusses and argues his choices, learns to collect data, negotiates and constructs meanings, brings the construction of personal and collective knowledge to temporary conclusions and new openings. (MIUR, 2012, p. 49, own translation)

The laboratory approach is defined in this research as an instructional practice, using the terminology of Rezat and colleagues (2021). Italian textbooks are not subject to any institutional review or approval process for use in schools; they are selected directly by the teachers of each school. However, they sequence the learning objectives and activities across the three school years and are owned by both students and teachers. Textbooks usually divide each topic into two parts, one to

introduce the theory and the other to practice, which typically consists of exercises focused on repetition and reinforcement.

Given the significant role of textbooks in structuring educational content and teacher practices, alongside the National Guidelines' emphasis on the mathematics laboratory as a central instructional practice, the teacher's perspective on the resources at their disposal for the laboratory are fundamental in determining how effectively they can implement the intended curriculum. Despite the guidelines advocating for an active, laboratory approach, there is limited understanding of how these recommendations are actualized in classroom practices through the mediation of textbooks. This gap raises the questions: (RQ1) How do teachers integrate textbooks into their instructional practices, and which specific textbook features or characteristics do they find valuable? (RQ2) To what extent do teachers find textbooks useful in implementing the mathematics laboratory? Acknowledging this viewpoint can shed light on the ways in which textbooks influence the implementation of the intended curriculum.

Methods

To quantitatively assess the variables of interest, a questionnaire was developed and distributed to a large sample of 2332 secondary school teachers in January 2024. This paper specifically focuses on the analysis of responses from 609 lower secondary school teachers, providing insights into this subgroup's educational practices and perspectives in relation to textbooks and the mathematics laboratory. The questionnaire consists of 14 questions, divided into 31 items, designed to analyse the use of textbooks, their role in promoting the laboratory approach and the resources available to teachers for such educational activities.

Each section of the questionnaire addresses different aspects of educational practices. The initial questions (Q1 – Q3) gather background information about the respondents.

The second section of the questionnaire concerns the adoption and use of the textbook (Questions Q4-Q7): question Q4 is an open-ended question that identifies the textbook adopted by the teachers; question Q5 a multiple-choice question exploring the utilization of the teacher’s guide associated with the textbook; Question Q6 examines the frequency of textbook usage in the classroom for five specific activities, selected based on the work of Sbaragli et al. (2020) and Fan et al. (2021). Respondents are asked to specify how often they engage in each of these activities. Question Q7 is a multiple-choice question that quantifies the overall frequency of textbook usage at school.

The third section of the questionnaire deals with the Evaluation of textbook characteristics (Questions Q8-Q9): question Q8.1 - Q8.10 employ a Likert scale to collect teachers’ opinions on ten different textbook attributes. Items were defined looking at Sbaragli and colleagues (2020) categorization of teachers answer to open ended questions about book satisfaction. Question Q9 requests a holistic evaluation of satisfaction with the textbook, rated on a scale from 1 to 10.

The last section of the questionnaire is dedicated to the mathematics laboratory (Questions Q10-Q14)**:** questions Q10 and Q11 assess the frequency at which open-ended problems are presented and laboratory activities are conducted, respectively. Question Q12 is a multiple-choice question that asks teachers to indicate the resources they use as a reference for designing and conducting lab activities, such as textbooks, websites, etc. Question Q13 is an open-ended question that invites teachers to

provide specific details about the resources they use for lab activities. Question Q14 utilizes a Likert scale to measure teachers' satisfaction with the resources available for mathematics laboratories and their accessibility. At the end of the questionnaire the participant is asked to leave an e-mail if he/she is available for a follow-up interview.

The questionnaire was distributed electronically to ensure wide reach. Teachers from various educational institutions across different regions participated, enhancing the generalizability of the findings. Participation was voluntary. Data from the questionnaire were analysed using statistical techniques suitable for the distribution of the data and the research questions addressed.

Results and discussion

The sample includes mainly experienced teachers: 43.8% of the respondents have more than 15 years of teaching experience and including those with more than 10 years of experience reaches 58.6% of the total. The respondents come from the different Italian regions, with a predominance for Northern Italy, where the number of schools is higher. Regarding the adoption and use of the textbook, in secondary schools, the daily use of textbook is widespread: 507 out of 609 teachers, 83.3%, say they use it every lesson, 79, 13%, use it weekly, 12, 2%, use it once a month and only 11, 1.8%, never use it (question Q7). However, it appears that the use of the book (question Q6, Figure 1) is mainly for exercises, with 95.6% of the teachers stating that they often or always use the book for exercises. The textbook therefore seems to be seen more as an exercise book than as a reference book for theory or for introducing topics, which is confirmed by the fact that about 70% of the teachers say that they never or hardly ever use the book to introduce or review content. Regarding mathematics laboratory, 53% of the respondents say that they often or always carry out the laboratories suggested in the book. 62.5% of the teachers rarely or never use the multimedia content of the textbooks. From this question, it appears that there is a general tendency to prioritize the use of the textbook for exercises, with less emphasis on its use as a reference for theoretical content or for supporting laboratory activities.

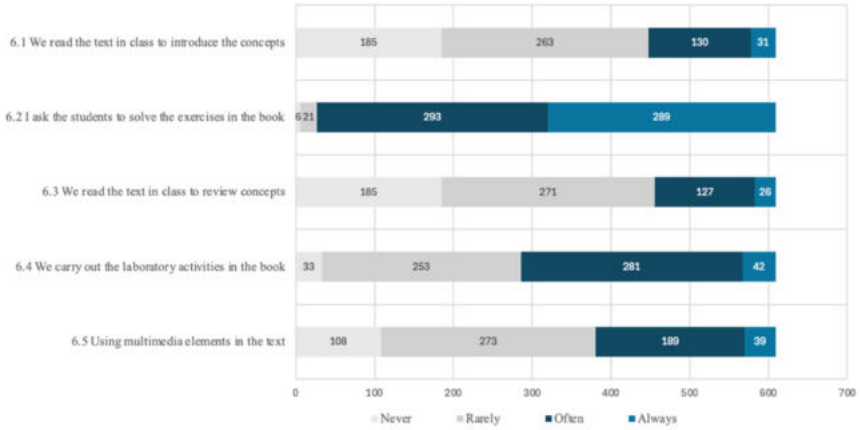

Figure 1: Answers distribution to question Q6 on the use of the textbook at school.

The evaluation of textbook characteristics indicates that, although the book is not typically used to introduce theory or serve as a handbook, it is generally recognized as having the qualities of a good handbook. According to responses to question Q8 (Figure 2), 83.7% of teachers consider it well-structured (item 8.1), 78.1% view it favorably as a tool for lesson planning (item 8.2), and 87.2% find that the instructions for the activities are clear (item 8.5). These results suggest that the limited use of

the book as a textbook is not due to dissatisfaction with its content, but rather to conscious teaching choices. Another interesting aspect concerns the perspective on the laboratory activities included in the textbooks. 73.5% of the teachers recognise in the book the presence of laboratory and problem situations useful for the development of mathematical competences, but the agreement drops to 61.1% for the promotion of argumentative practices and to 50.9% for group work. The textbook appears to be less effective also in areas such as multimedia integration (49.8% agree on item 8.7) and the provision of real-world modelling activities respectively (57% of teachers agree on item 8.10). This suggests that although the textbook is well-regarded for supporting individual activities and exercises—as evidenced by the generally positive responses in question Q8—it may not provide sufficient resources for more interactive and collaborative teaching. The data also implies that the concept of 'laboratory' perceived by the respondents might focus more on the mathematical content and task than on instructional aspects such as argumentative practices or peer collaboration and confrontation. Furthermore, it is possible that teachers are supplementing the textbook with their own methods to implement these interactive aspects of laboratory work, indicating a possible adaptive use of the textbook's content to fit more collaborative educational models.

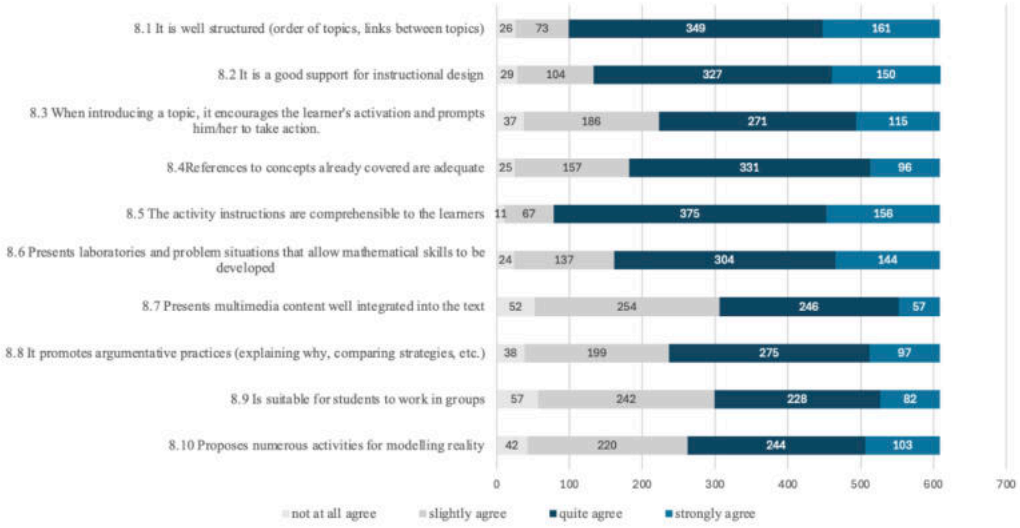

Figure 2: Answers distribution to question Q8. Respondents are asked to agree to statements on the textbook they use.

The analysis of satisfaction with textbook usage across different years of teaching experience reveals a clear trend of increasing satisfaction correlating with longer teaching tenure. The mean satisfaction scores demonstrate this progression, with teachers having 1-5 years of experience reporting an average satisfaction of 6.62, which gradually rises to 7.59 for those with more than 15 years of experience. This upward trend in satisfaction was further validated using the Kruskal-Wallis test, a non-parametric method selected due to the violation of normality assumptions in the data, as indicated by the Shapiro-Wilk test ($p < .001$). The Kruskal-Wallis test highlighted statistically significant differences in textbook satisfaction among groups of teachers with different years of experience ($\chi^2 = 35.4$, $df = 3$, $p < .001$). This result suggests that perspectives on the quality and effectiveness of the textbook vary significantly with the accumulated experience of the teachers. These findings provide evidence that the duration of teaching experience impacts teachers' satisfaction with textbooks, with

more experienced teachers generally reporting higher satisfaction. This could reflect greater familiarity with the textbook's content and features or possibly a more developed skill set in integrating the textbook effectively into their teaching practices.

Regarding the mathematics laboratory, among the respondents, 59.1% reported regularly proposing open problems—problems without a single, predetermined outcome—at least monthly: 25.9% weekly and 33.2% monthly (Figure 3). This indicates that proposing open problems is a common practice among the participating teachers. Furthermore, a significant 80% of respondents engage in laboratory at least once a month, with 51.6% conducting these activities weekly (Figure 3). This highlights the prevalence of laboratory teaching among the study's participants.

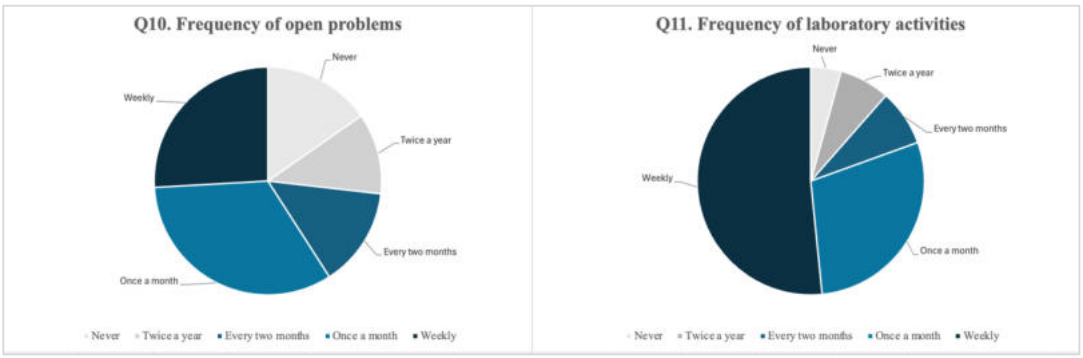

Figure 3: On the left the distribution of answers for question Q10, on the right question Q11.

However, the relationship between conducting laboratory activities and proposing open problems is not strictly one-to-one, as it is made visible comparing the two graphs in Figure 3. For instance, among the 314 teachers who conduct weekly laboratory activities, 39% also propose open problems weekly. Conversely, 35% teachers engage in weekly laboratory activities but only present open problems monthly. Additionally, 8% teachers propose open problems every two months, 7% biannually, and 11% never propose open problems despite weekly laboratory activities. This highlights a possible disconnect between the national indications ‘definition of a laboratory in mathematics and the teachers’ definition of a laboratory. There may be a disconnect between the concepts of open-ended problems and laboratory activities.

Another significant finding concerns the resources used by teachers for mathematics laboratory (Q12). 28.6% of teachers say that they use the textbook as their main reference for these activities. However, a similar percentage, 28.6% of teachers, rely mainly on online resources, while 26.3% design the activities themselves. This suggests that although the textbook is used for laboratory activities, teachers tend to look elsewhere for additional materials and inspiration, perhaps due to a perceived incompleteness or inadequacy of the content in the books themselves, or the need to adapt the resources to meet specific classroom needs. Question Q14 shows that 67.1% of teachers indicated that they spend a lot of time searching for suitable materials for laboratories (Figure 4, item 14.1), with an even higher percentage (76%) of teachers with less than 5 years' experience. These data highlight that materials for mathematics laboratory are not always readily available and accessible, especially for less experienced teachers, who may find it more difficult to adapt or design appropriate activities without external support. To analyse the factors influencing the performance of laboratory activities by teachers, the responses to question Q11, which inquired about the frequency of the

mathematics laboratory, were transformed into a dichotomous variable. This variable categorizes teachers into two groups: those conducting mathematics laboratory at least once a month and those conducting them less frequently. The correlation between this dichotomous variable and the responses to items in question Q14 was then examined. The analysis revealed statistically significant correlations ($p < .001$) with items 14.1 and 14.3, indicating that the frequency of laboratory activities is associated with the availability of adequate resources.

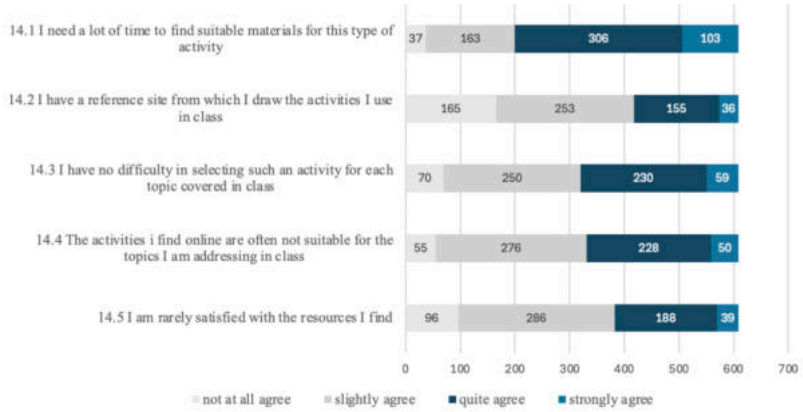

Figure 4: Question Q14. Respondents are asked to agree to statements on laboratory resources.

The observed differences may reflect variations in instructional choices or the availability of suitable materials. Teachers who infrequently conduct such activities might be constrained by the absence of appropriate resources or might choose different teaching methodologies for other pedagogical reasons.

Conclusions

This study underscores the important role textbooks play in Italian lower secondary mathematics education, particularly in relation to the implementation of the mathematics laboratory approach advocated by the National Guidelines. Drawing on the theoretical framework proposed by Rezat et al. (2021), which positions curricular resources as mediators between the intended and implemented curriculum, the findings allow to answer question RQ1, highlighting how textbooks are integral to teaching and learning practices, being used in every lesson by 83.2% of respondents. However, their use tends to be concentrated on exercises rather than fostering laboratory-based activities or promoting collaborative and argumentative practices. As noted in the literature (Fan et al., 2021), textbooks heavily influence lesson planning and the organization of classroom activities. This study supports these findings, revealing that 95.6% of teachers rely on textbooks primarily for exercises. However, a substantial portion of teachers report using external resources for laboratory activities, with 28.6% using online materials and 26.3% creating their own resources. Only 53% of teachers report regularly using laboratory activities found in textbooks, reflecting the mixed findings in the literature regarding the role of textbooks in supporting innovative practices (González-Martín et al., 2018). Teacher experience also seems to influence textbook satisfaction, aligning with the findings of Haggarty and Pepin study (2017). This may be due to greater familiarity with additional resources or different teaching practices, as teachers with over 15 years of experience began teaching before the National Guidelines were introduced and may employ distinct methods. The answer to RQ2

emerges from several trends observed in the data: teachers who infrequently conduct laboratory activities may be constrained by the lack of appropriate resources, as suggested by responses to question Q12 and Q14. Although textbooks are used for laboratory activities, teachers often turn to other sources for supplementary materials (Q12), which can be time-consuming (Q14). Focus groups will be conducted to explore the reasons behind these trends. Key areas of investigation include understanding how teachers interpret the concept of a mathematics laboratory, as there appears to be a gap between the national guidelines and classroom practice. Additionally, the focus groups will examine why many teachers supplement or replace textbooks with external resources for laboratory activities. Lastly, the influence of professional experience on textbook use and perspective will be analysed, particularly the differences between experienced and less experienced teachers. In conclusion, while textbooks remain essential tools for structuring mathematics education, their current use does not fully support the implementation of the mathematics laboratory approach, particularly in its collaborative and argumentative aspects. This article presents only the initial findings and mainly descriptive analyses of the questionnaire; further in-depth study is still required to explore these issues more comprehensively.

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Exploring the use of digital textbooks in classrooms: focusing on automated feedback

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Automated feedback is a significant potential of digital tasks in e-textbooks that supports learning. This paper examines the types of automated feedback present in digital textbook tasks in Croatia and how they are utilized in the teaching process. The study focused on the geometry chapters of the three most widely used mathematics textbook sets in Croatia for grades 5 to 8, as well as a multiple case study involving six middle-grade teachers. The findings reveal that automated feedback is most prevalent in simple digital tasks, primarily in the form of verification, while the exploration tasks often provide hints. However, in classroom observations, these digital textbook tasks were mostly used in whole-class teaching, limiting the full potential of automated feedback.

Keywords: Digital textbooks, automated feedback, use of resources, mathematics education.

Introduction

Over the past several decades, digital textbooks have become important resources in mathematics classrooms which can introduce new potentials such as dynamics, automated feedback, and personalization (Choppin et al., 2014; Glasnović Gracin & Krišto, 2022). These new digital features have the potential to transform mathematics education by enhancing the understanding of mathematical concepts and their relationships.

Feedback, a crucial component of teaching and learning, is defined as "information about the gap between the actual level and the reference level of a system parameter which is used to alter the gap in some way" (Ramaprasad, 1983, p. 4). Schute (2008, p. 154) described formative feedback as "information communicated to the learner that is intended to modify his or her thinking or behaviour for the purpose of improving learning". Van Der Kleij et al. (2015) classified feedback types into knowledge of results (verifying the response correctness), knowledge of correct response (providing the correct response), and elaborated feedback (providing additional information, such as strategic hints, etc.). Findings from their meta-analysis (Van Der Kleij et al., 2015) showed that elaborated feedback is more effective for learning than the other two categories. This categorization also applies to the digital environment and automated feedback. Automated feedback on student solutions is one of the most prevalent digital features of e-textbooks, with the potential to support learning and change and improve instruction (Rezat, 2021). Therefore, we found it important to investigate the types of automated feedback present in digital textbook tasks and how they are used in classroom practice.

The use of digital textbooks, along with their content and features, has the potential to transform educational processes and "bring about new educational dynamics" (Pepin et al., 2017, p. 799). As noted by Remillard (2012), teachers develop a *mode of engagement* with resources over time, shaped by their experiences, which influences how they approach a new (digital) textbook. Consequently, teachers' established practices in using printed textbooks may also shape how they interpret and apply automated feedback in digital resources. Furthermore, the use of textbooks in classrooms inherently

involves questions on their authority, including the teacher's role as a mediator between textbooks and students (Pepin & Haggarty, 2001). Since school textbooks represent authorized versions of knowledge, they carry a certain authority. This extends to automated feedback provided by digital textbooks, whose authority remains an underexplored area of study.

These considerations highlight important questions regarding the design and implementation of automated feedback in school settings. To our knowledge, the relationship between the (educative) features of automated feedback in digital textbooks and its use has not yet been the subject of research. Investigating these issues could contribute to a deeper understanding of the role of automated feedback as a tool for educational transformation and improvement. In our study, we explored these questions within the Croatian context.

In Croatia, textbooks can be printed and electronic, and are mandatory educational resources approved by the state board. The Ministry of Science and Education's (2019) Textbook Standard outlines specific requirements for digital textbooks, including dynamics, simulations, and various types of interactions. The student-content interaction in digital textbooks serves both for evaluation (interactive quizzes with feedback) and learning (didactic games, etc.). This aspect of the Textbook Standard highlights the need for automated feedback in digital textbooks and the following research questions: (1) What types of automated feedback appear in digital textbooks? (2) How is automated feedback from digital textbooks utilized in classroom practice and why?

Theoretical framework

Shute (2008) classified feedback types according to their complexity, including categories such as no feedback, verification, correct response, try again, error flagging, elaborated feedback, attribute isolation, topic contingent, response contingent, hints, misconceptions, and informative tutoring. Building on Schute's (2008) classification, Glasnović Gracin and Krišto (2022) developed a framework for analysing tasks in digital textbooks (see Table 1). This framework forms the foundation for the study presented in this paper.

Table 1: Automated feedback types (Schute, 2008; Glasnović Gracin & Krišto, 2022)

Feedback type (code)	Description
No feedback (F1)	A student responds in a digital space, but no feedback is provided
Verification (F2)	Returning a simple right/wrong response
Correct response (IF1)	Provides the correct answer, with no additional information
Try again (IF2)	Trying until the correct answer is given
Error flagging (IF3)	Highlighting incorrect answers in the task, but without offering the correct answer
Elaborated (IF4)	Explaining why a specific response was correct or incorrect

Hints/cues/prompts (IF5)	Guiding students in the right direction by giving examples or recommendations on what to do next, for instance, but the correct answer is not explicitly given
Other	Other types of automated feedback

According to Van Der Kleij et al.'s (2015) classification mentioned in the introduction, feedback types of verification, try again, and error flagging (Table 1) are categorized as feedback on the knowledge of results, while elaborations and hints/cues/prompts belong to elaborated feedback.

Regarding the use of automated feedback in classrooms, we applied the types of teacher resource utilization identified by Brown and Edelson (2003). These include offloading (where teachers heavily rely on curriculum materials such as textbooks), adapting (where teachers use and modify parts of the curriculum materials), and improvising (where teachers do not depend on textbook content). Based on Brown's Design Capacity for Enactment Framework, Pepin et al. (2017) developed a model for teacher design capacity while interacting with digital resources. It comprises the goal of design (resource accordance with the curriculum and the teacher's aims), design principles (robust and flexible concepts of teaching when digital resources are used), and reflection-in-action (teacher decisions during the instruction). Flexible principles may refer to accepting and using educative features of automated feedback in the classroom.

Methods

In this study, which is part of a broader investigation into the analysis and use of digital textbooks, a mixed-method approach was employed. To address the first research question, document analysis was conducted, focusing on the digital versions of the three most widely used mathematics textbook sets in Croatia for grades 5 to 8. The analysis included all digital tasks within the geometry chapters (1,381 tasks in digital textbook set A, 1,250 in set B, and 363 in set C). The instrument for analysing these digital tasks was based on the framework presented in Table 1, with the addition of a new feedback type found during coding, Worked solution (code WS), where the automatic feedback provides an elaborated solution. The codes for feedback types are not mutually exclusive. The coding process was carried out by the second author, while the first author analysed 10% of the items, achieving a high level of agreement in coding (99%). The coded data were then analysed using relative frequencies for each type of feedback.

To answer the second research question, a multiple case study was conducted involving six middle-grade mathematics teachers. This study included observations of 29 geometry lessons and supporting semi-structured interviews. The observations included the following questions: *Is automatic feedback used in this activity? If so, describe what types are used and in what way*. The interviews aimed to gain a deeper understanding of the participants' views, experiences, and reasons for utilizing automatic feedback. Key questions included: *In your opinion, what are the benefits of using digital textbooks and automated feedback in geometry education? What challenges and drawbacks do you encounter? Please explain.* The observation notes and transcribed interviews were analysed using qualitative content analysis, with the resulting categories subsequently grouped into larger thematic clusters.

Results

Automated feedback in digital textbook tasks

The tasks found in the geometry chapters of digital textbooks were categorised into those requiring a closed answer (simple tasks) and those designed for exploration activities. The results indicate that the number of closed tasks with feedback (Table 2) greatly surpasses the number of exploration activities (Table 3). In closed tasks, automated feedback is predominantly utilized for basic performance checks (verification, error flagging), while largely overlooking formative assessment and the transformation of learning within a digital environment. While other types of feedback are rarely present in digital textbooks A and B, textbook C offers knowledge of correct response (IF1) as feedback and provides some hints for solving the closed tasks. Excluding verification, closed tasks rarely provided a combination of more than one type of feedback (1,1%).

Table 2: Automated feedback in tasks with closed answer

Dig. textbook	No feedb. (F1)	Verification (F2)	IF1	IF2	IF3	IF4	IF5	WS	Other
A (1270)	0.1%	99.9%	0.4%	1.7%	34.0%		2.2%	0.2%	0.1%
B (1187)	0.3%	99.7%	0.3%	0.3%	48.1%		1.1%	0.1%	
C (257)	1.2%	98.8%	17.9%		33.9%	0.8%	11.3%	0.4%	0.4%

Textbook tasks involving exploration activities (Table 3), primarily made in GeoGebra, had a focus on the student's exploration within the applet rather than providing a specific answer to a question. As a result, verification as a form of feedback is less common compared to Table 2. Conversely, these tasks frequently offer students hints (IF5) and occasionally provide worked solutions (WS) and correct answers (IF1). Moreover, hints are often given together with correct answers and worked solution. These findings imply more educative features of automated feedback.

Table 3: Automated feedback in tasks involving exploration activities

Dig. textbook	No feedb. (F1)	Verification (F2)	IF1	IF2	IF3	IF4	IF5	WS	Other
A (111)	7,2%	21.6%	10.8%		1.8%	0.9%	53.2%	20.7%	0.9%
B (63)	6,3%	6.3%	30.2%	1.6%			55.6%	30.2%	
C (106)	5,7%	12.3%	78.3%				87.7%	2.8%	

These results offer a deeper understanding of the educational potential of automated feedback in analysed digital textbooks. The focus is primarily on simple knowledge of results, which is inadequate for fostering the comprehensive development of geometric thinking. However, in tasks involving exploratory activities, feedback plays a supportive role, aiding students in developing a deeper understanding of geometric concepts.

Use of automated feedback in the classroom

The 29 observed lessons included altogether 150 activities with digital textbooks, 44 of which included using tasks with automated feedback. Other activities involved using textbook content that did not provide interactions in the digital space, but merely static images (PDF of a printed textbook). The observations identified two types of automated feedback usage in the observed classrooms: (1) direct use of automated feedback from the digital textbook tasks (offload), and (2) modified use of automated feedback (adaptations).

Direct use of automated feedback. This type of use occurs when, after answering a question, the feedback button is clicked. In the observed lessons, this approach was employed by students during individual work (16 activities) and by teachers or students in frontal instruction (13 activities). It is important to note that all activities related to students obtaining individual feedback occurred during a single lesson, where the students were brought to a computer classroom and each student sat at their own PC workstation (T1L2A1–16 means: teacher T1, lesson 2, activities 1–16):

> Observation T1L2A1–16: Students individually work with a digital textbook on the topics of supplementary and alternate angles. They complete each activity and solve all the tasks from these two topics, utilizing the automatic feedback provided. To log in, they use their student IDs. The teacher walks around the classroom, helping and engaging in discussions with the students.

Teacher T1 explained that these activities provided each student with automated feedback, enabling them to practice at their own pace. This approach can be interpreted as a form of didactical flexibility in her use of digital textbooks. Still, the teacher expressed satisfaction with the feedback which primarily offered *knowledge of results* at both the task and lesson levels. She noted, “they receive feedback on whether their answers are correct or not. (...) Additionally, they are shown the total percentage of all tasks they have completed”. Satisfaction with knowledge of results implies robust and traditional principles in her teaching design while using automatic feedback.

Thirteen observed activities involved teachers presenting the digital tasks to the entire class on the smartboard. In this approach, automated feedback was utilized as part of the frontal teaching.

> Observation T6L2A3: Direct teaching to the entire class. The teacher presented a GeoGebra applet from the digital textbook (topic: area of a parallelogram) to demonstrate an animation that explains how to derive the formula for the area of a parallelogram using the area of a rectangle. Discussion with students throughout the activity. The applet included buttons that provided feedback in the form of hints and the final area formula. The teacher clicked on these buttons and used hints to drive discussion. Additionally, the teacher derived the formula on the board.

Here, automated feedback had elaborative elements used in the classroom as a part of interactive discussion (flexible principles in teacher’s design). Still, the robust principles refer to the teacher’s

control over the process because she decided to click on the feedback buttons, not the students, because "I always try to make it an interaction between me and them so that we come to a solution together" (T6). The other reason participating teachers clicked on feedback in the frontal setting was the limited availability of PCs for all students.

> T5: If we had an IT classroom at our disposal like we used to, then each student could try a lot of things (including feedback) on their computers.

Furthermore, observations of teacher T3's lessons revealed an interesting issue related to students' use of automated feedback. Namely, the students entered the correct solution (estimation) but became confused by the negative automated feedback:

> Observation T3L1A6: The teacher used a GeoGebra applet from the digital textbook (topic: finding a central-symmetric point). Two students came to the smartboard and, using their finger, estimated the position of the symmetric point. Automated feedback was used (by students), but because the GeoGebra applet was set to a very high accuracy level, the student received an "incorrect" response, even though the estimation was satisfactory. The teacher encouraged them by saying their estimations were well done.

This illustrates a problematic aspect of automated feedback and serves as an example of *reflection-in-action* regarding feedback. In such cases, the teacher creates connections or guiding actions that may not be immediately apparent but emerge and take shape during the teaching process. Within this context, the feedback provided directly by the teacher holds greater authority over the students than the automated feedback they receive after completing the task.

Modified use of automated feedback from tasks. In 15 cases, the use of automated feedback from digital textbooks was modified. In some observed activities, the digital tasks were presented to the class, but the automated feedback was ignored, as teachers decided to provide feedback personally. This is an example of teachers prioritizing their authority over automated feedback, sometimes overlooking its potential and instead relying on familiar practices as robust principles. For example:

> Observation T2L1A2: A quiz from the digital textbook is displayed on the classroom board. Students work on the quiz tasks in their notebooks. Afterward, they exchange notebooks in pairs and read each task aloud along with their answers. The teacher types in the answers into the quiz via laptop and provides feedback on the screen, deliberately ignoring the automated feedback. At the end, each student counts their correct answers, and the teacher types the scores in her Excel table, followed by a discussion about the students' achievements.

In the interviews, teachers offered several reasons for preferring to provide feedback themselves rather than relying on automated feedback. These included the ability to offer better explanations and facilitate further discussion (T1), the presence of incorrect answers in the automated feedback (T5), and technical issues with the smartboard that prevented the use of interaction buttons required for automated feedback (T2).

The observations also included several activities where teachers combined automated feedback with their own feedback, depending on the student's correct or incorrect responses (reflection-in-action):

> Observation T2L3A6: The teacher introduces a practice task from the digital textbook to the class (topic: supplementary and alternate angles, 11 computer-generated subtasks). She selects a student to provide the angle magnitude, then she inputs the response. If the answer is correct, the teacher provides feedback herself, disregarding the automatic feedback. However, when a student gave an incorrect answer, she allowed the automated feedback to deliver the response.

In this instance, the output "incorrect" provided by the automated feedback acted as both an authority and a catalyst for classroom discussion (elaborated feedback with the teacher).

Discussion and conclusions

The study presented in this paper investigated the extent to which automated feedback is incorporated into tasks within digital textbooks and how it is utilised in mathematics instruction. The study found that automated feedback is indeed present in Croatian digital textbooks, primarily as simple verification, while elaborations are more commonly found in exploration activities. The use of automated feedback in observed lessons revealed both flexibility (using automated feedback from digital textbooks) and robust design principles (teachers' control and providing their own feedback). Sometimes teachers disregarded automated feedback because of the frontal teaching, issues with IT, or because it did not provide elaborated information, so further classroom discussion was needed.

Although automated feedback is recognised as a digital potential that can support learning and enhance instructional practices by taking responsibility from the teachers, so they are not the only ones providing feedback (Rezat, 2021), the results of this study suggest that it was not fully utilized in this way. At times, participating teachers chose to disregard automated feedback and provide their own when they believed it could not fully realise its potential. This may be related to a strong tradition in Croatia of whole-class instruction and the use of printed textbooks in mathematics education (Glasnović Gracin & Krišto, 2022). The influence of automated feedback was shaped not only by the available technology but also by the robust pedagogical intentions and personal preferences of the participants, who ultimately determined whether and how to incorporate feedback into their instruction. Previous studies suggest that teachers often adhere to one or more familiar activity formats in the classroom and prefer using materials that do not require completely new approaches to activities (e.g., Monaghan, 2004). Their mode of engagement (Remillard, 2012) with printed textbooks may have influenced the design principles in their lessons.

These findings highlight the issue of the *authority* of digital textbooks and their potential, such as automated feedback, and the authority of a teacher as a mediator between textbooks and students (Pepin & Haggarty, 2001). In our study, the presence of automated feedback in digital textbooks led to a kind of "battle of authorities" during the observed activities: the participating teachers chose to disregard the automated feedback in favour of providing their own feedback to students or combining them within reflection-in-action. Additionally, instances where the automated feedback was incorrect necessitated teacher intervention. It seems that the full pedagogical potential of automated feedback should be further examined and described, including an examination of the students' perspectives, such as their perceptions of automated feedback compared to feedback provided by teachers.

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Upper-secondary school mathematics teachers' use of a topic-centered research-based teacher guide

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This paper reports on an exploratory study in which upper-secondary school mathematics teachers used an experimental research-based teacher guide for the topic of quadratic equations as a supplement in their planning and teaching practice. A survey and semi-structured interviews were used as data-generating instruments. Building on data from six teachers, a map of the use was constructed qualitatively. From this map, categories of affordances and constraints perceived by teachers in relation to the teacher guide content features, and types of mobilizations were constructed. The main finding on this use is that lesson activities that were perceived as having a variety of affordances were mobilized to a greater extent than other features. The role of research and context for this use is discussed.

Keywords: Teacher guide, research-based, upper-secondary, quadratic equations

Introduction

Teachers need access to relevant research to plan research-based mathematics instruction. However, Swedish teachers' access to such findings is limited, as few teachers have the habit or know-how to find research and schools are typically not organized to support teachers' planning practice (Nordgren et al., 2019). The notion of knowledge brokering by providing research summaries (e.g., Rycroft-Smith & Stylianides, 2022) could be a fruitful way to help teachers gain access to research. If centered around curriculum topics, such summaries could constitute curriculum resources (CR). This paper aims to contribute to research by combining research on design, mathematics, and user studies of CR (Rezat, 2024), through exploring how research summaries for a specific mathematical topic designed into CR could be used by teachers in their planning practice.

When planning and teaching, teachers select, modify and create new resources (Trouche et al., 2020). Through this use, they perceive and mobilize the resources available to them to design instruction (Brown, 2009). Most Swedish teachers, despite their autonomy in choice and use of resources, conform culturally to the common classroom practice of a teacher-led instruction, the Swedish *genomgång* (e.g., Andrews & Larsson, 2017), followed by individual work in the textbook (Boesen et al., 2014). For upper-secondary school, the student textbook still plays a major role in planning and teaching, and teacher guides are typically not used, despite availability of other resources. (Gustafsson et al., 2024). The prevalence of research and external support in planning in this teacher practice can therefore be assumed to be low.

This paper presents the first cycle of a design research study (Bakker, 2018) within a larger project aiming at understanding how research-based teacher guides could be designed to support teachers in their planning and teaching practice. This exploratory phase investigates how Swedish upper-secondary mathematics teachers interact with an experimental research-based teacher guide on the

topic of quadratic equations (QEs), made available within their usual practice. The aim is to describe how Swedish upper-secondary mathematics teachers within their cultural context perceive and mobilize content from a research-based teacher guide.

Theoretical framing

In this study, planning teaching is in accordance with Brown (2009) a design activity, in which teachers perceive and mobilize resources in designing instructional activities – using their pedagogic design capacity. The focus in this paper is not to evaluate different teachers' capacities, but to use the terms perceiving and mobilizing as a lens for describing teachers' resource use. Elements of perceiving CR include reading, interpreting and evaluating, both in preparation for teaching and in enactment (Remillard, 2019), and this study focus on preparation in planning. When a teacher perceives an affordance within a resource, this may lead to a mobilization of this resource into designed instruction, while a perceived constraint, both within a resource and in the context, can hinder such mobilization. Elements of mobilization can be seen in terms of different degrees of artifact appropriation, which Brown (2009) termed offload, adapt and improvise. The meaning of these terms, stemming from socio-cultural roots, corresponds to the amount of agency in deciding on instruction that is given to an artifact, from using an artifact as it is (offload), modifying the suggested design (adapt), or to fully deviate from the original intended use to create something new (improvise). A teacher guide can be seen as an artifact, with the intention to enable action in a specific way. Even though the teacher guide in this study is designed with some intentions (see Method), it is here the teachers' perceived affordances and constraints using the guide as a resource in planning teaching that are in focus, and not the fidelity in following the intentions of the designer.

Method

Six Swedish upper-secondary teachers (T1-T6), who received a designed teacher guide for QEs prior to planning and teaching the topic, were surveyed and interviewed on their use. They all had at least three years teaching experience and had taught the topic before. Geographically, they were spread out over four different schools from three different cities. Before part-taking, they were informed that the guide was based on research on the teaching and learning of QEs, including types of content that could be supporting for teachers. Part-taking meant to use as much or as little (even nothing) within their normal planning and teaching practice.

The questionnaire included qualitative open answer items, as well as fixed items that for example let teachers grade the extent to which they had used different features, how they had used these, the time spent working with the guide, as well as rate the usefulness and how they valued different features. An interview guide for semi-structured interviews, organized around the different content features, and adapted to the outcome of the individual teacher's use was constructed, aiming to obtain insights regarding reasons to why the guide and its different features was perceived and mobilized in the specific ways.

The content of the experimental research-based teacher guide

The teacher guide was experimental, designed for this project by the author, with the intention to support teachers by informing and giving suggestions, rather than to impose any particular type of

teaching or classroom practice. It consisted of three main types of content features: an expository text on the teaching and learning QEs, one document with lesson activities, and one document with supplementary mathematical tasks. These types of content were chosen as research on educative features of CR (e.g., Davis et al., 2017) and findings from the context from upper-secondary school teachers planning (Gustafsson et al., 2024) indicated that these might be useful for teachers. The notion of 'research-based' therefore acts on at least two levels: the material contains a research summary and the content features are informed by research.

The expository text contained a research summary synthesized from a literature review on the teaching and learning of QEs. The summary included mostly qualitative findings on important mathematical concepts for learning QEs, common student (mis)conceptions and mistakes with possible reasons and implications for teaching. Sub-sequent sections contained teacher narratives from experience, aligning with the research summary, and final suggestions of how to handle different issues in teaching. Included was also a description of the Swedish curriculum context for QEs, including the curricula standards, how QEs are treated in three common curriculum materials and how it was historically tested in national exams.

The first appendix consisted of four lesson activities (A1-A4), all aiming to address some of the issues raised from the expository texts. Each activity had a text describing the rationale of the activity to provide transparency for the teacher, as well as suggestions on how to modify the activities, in line with educative features (e.g., Davis et al., 2017). A1 and A4 aimed to build student conceptual understanding, by students sorting and grouping different mathematical objects, while also opening up for different types of procedural exercises. Support for teachers summarizing the issues in whole-class discussions were included. A1 addressed student difficulties of differentiating between mathematical expressions, equations and functions, identifying if these were quadratic or linear in nature, and offering practice in rewriting sorted QEs into equivalent forms. A4 addressed awareness of how different types of solution strategies are useful for different types of QEs. A2 had the same aim as A4, and provided equations to solve after first consulting with a student colleague. A3 aimed at discovering the relationship between the discriminant and the number of solutions to a QE, through a suggested pathway. A2 and A3 were in a sense more traditional, and less elaborate in their design.

The second appendix contained a collection of mathematical tasks to use without explicit guidance, previous national exam tasks on QEs from publicly available tests (NT), tasks constructed around fictional student solutions with common mistakes (CM), challenging tasks for students (CT), a quick reference for different solution types of QEs (QR), and tasks in the form of exit tickets (ET).

Analytic procedure

The data material consisted of the survey answers and audio-recorded transcribed interviews (~60 minutes each) of each of the six participating teachers. Using a qualitative content analysis approach (Kuckartz, 2019), the answers from the questionnaire were used to create an initial map on the use and evaluation of the guide's content features. From the semi-structured interviews, reasons of use or not use, and ways of perceiving was generated. Being an explorative study, the interviews were quite open in letting teachers talk about what was important to them. Expressions regarding the content features therefore emerged in various places in the interviews. Different features also got different

focus from different teachers. Passages from the transcriptions (as well as the occurring open-ended questionnaire answers) about the different content features were coded deductively. This updated the initial map. For each part of the explored map, types of reasons and use were created as categories of perceptions or mobilizations. These categories were not necessarily exhaustive, but representatives of some patterns. A choice of using offload and adapt (Brown, 2009) as categories for use of tasks and activities deductively was made. Finally, the map was adjusted by going through all transcripts, until a reasonably accurate map was obtained (presented in Figure 1). From the map, the categories for reasons were interpreted in terms of teachers' perceived affordances and constraints and the uses in terms of mobilizations.

Findings

All the teachers stated they normally use student textbooks, colleagues and official documents in terms of the curriculum and the national exams as their main resources for designing instruction. None of the teachers expressed having used a teacher guide before. They all refer to their genomgång in describing parts of their practice, and through the interviews there were several expressions about allocating lesson time for students' individual work.

The interaction with the expository texts

As seen in Figure 1, the research summary impacted teacher the most of the expository texts (1). This was emphasized by the positive evaluations and that the issues mentioned by teachers in interviews, stemmed mainly from this section.

The majority of presented categories of reasons for evaluations as important (2) corresponded to perceived affordances related to research or to teacher preferences. First, research played a big role, corresponding to the guide being perceived as research-based, and that teachers trusted it due to the institutional setting of a researcher from a university providing it. This was connected to the guide being perceived as well-written and accessible, and that findings were summarized (quotes translated from Swedish by author):

T1: I think that if one gets a well-packed and well-presented material, I trust that there is some research. \...\ I think I have to trust what implications someone has found

T3: I have trusted this [guide] more than if I just had found one, considering it is research-based. \...\ I trust that those who have constructed the teacher guide also have done a good survey of the research

Second, reasons within teacher preferences reflected value in that the texts were focused on students and learning, that they resonated with teacher experience, and contained previous teacher experience. The main perceived constraint was that the expository text was too long for teachers to read given the time constraints of their planning situation (3). All teachers read the research summary, but few read further into the expository text. In particular, the section containing the connection to Swedish curriculum was expressed as redundant (4); it did not add any value as teachers were already familiarized with the curriculum of QEs.

The mobilizing influence of the expository texts on teachers from reading, interpreting and evaluating the texts can be divided into two main categories. One category (5) was that the guide enabled teachers to focus on important issues in planning, exemplified by mentions of picking up on common

student mistakes or misconceptions, or the important student skill of being able to choose suitable solution methods based on the identification of type of QE. Even though the issues mentioned by teachers covered many of the issues presented in the expository text, some issues were not mentioned at all. A second category (6) captured the confirmation teachers expressed regarding their professional experience, as their own experience was in line with the issues presented in the teacher guide. Reading it was expressed as a way to reactivate and be reminded of what they already knew, but also to add small new insights:

T1: For me, it confirmed much I had experienced and I was strengthened by it
T5: It is a security for me that when I read it, it confirms things I already know, but also that I can learn some things

The focus on important issues made teachers plan specific parts of teaching, such as their genomgång, or the use of activities or tasks. As can be noted in Figure 1, the issues mentioned by teachers being picked up in the expository texts (7) to a large extent coincide with teachers using activities or tasks that targeted those particular issues, especially for A1, A2, A4 and CM.

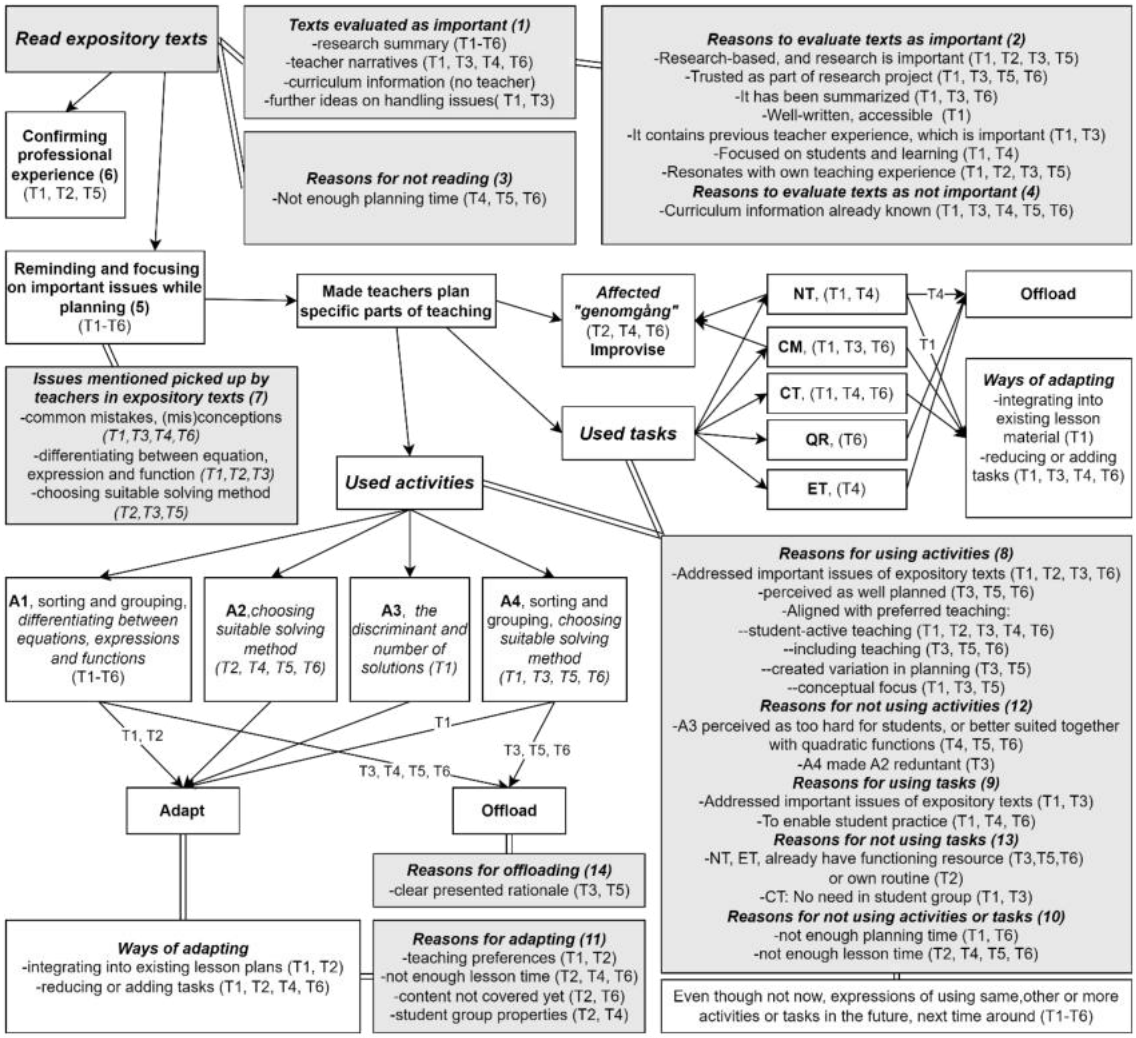

Figure 1. Aspects of teacher perceptions (grey) and mobilization (white) of the content features.

The interaction with activities and tasks

Activities that were student-centered (A1 and A4) were used to a greater extent than other activities or tasks. The majority of presented categories of reasons for use (8) corresponded to perceived affordances that were related to usefulness or to teaching preferences. First, the activities were perceived as well-planned, and a way to handle the issues teachers picked up from the expository texts. Second, they aligned with ideas on how teachers wanted to teach. For instance, having conceptual focus, students being active instead of teachers talking, students discovering instead of teachers telling, and every student being able to part-take:

> T3: One reason was that I was made aware that there is a need in students in general, after reading the teacher guide.. \..\ I also like how the activity is built, so that no matter if you are a high- or low achieving student you can contribute. That is something I think is problematic in other situations when one wants students to work together or in activities. That is also a reason why I used that activity. Also, I rather let the students be active, than listening to me, so it aligns good with my vision of teaching. That's why I used it.

Usefulness could also be seen an affordance in enabling students' practice (9), through national exam (NT) or common mistakes (CM) tasks being solved and discussed in genomgång, which can be seen as a form of improvising. They were also put in worksheets, as also in the case for exit tickets (ET), challenging tasks (CT) and the quick reference (QR).

Perceived constraints in relation to using activities and tasks were related to time, students, or the teaching context. First, lack of lesson time and planning time were categories of reasons for not using tasks and activities at all (10), or adapting by removing mathematical tasks to make the activity shorter to fit lesson-time constraints (11). Not using activities (A2 and A3) were related to that the aim of A2 overlapped with A4, so given lesson-time constraints both could not be used, and A4 was perceived as the better (12). Second, categories such as the lack of need in the student group to use CT (13), or that not using A3 was due to that it was deemed too hard for students (12) showed student-related constraints. Third, that A3 was expressed to be better suited together with the topic of quadratic functions, as the number of solutions could be better connected to the graphical view of x-intercepts, or that content was adapted because content was not yet covered yet (11) was example of constraints within the teaching context. Related to this, already having a functioning resource made teachers not use tasks, being more of a "non-affordance" than a constraint, expressed as reasons for not using NT and ET (13). Even though these constraints existed in the teachers' current planning situation, expressions were made that activities and tasks that were not used now were to be used in the future.

In terms of mobilization, the majority of the teachers reported planning and enacting activities (especially A1 and A4) in line with the description in the teacher guide – offloading, relating to that the activity had a clear presented rationale (14). Some adaptation took place, besides adaptations already mentioned in relation to constraints, with small modifications being organizational, such as organization in pairs, or in groups of four, to fit different classroom management practice. One teacher (T1) adapted activities and tasks to a large extent, by integrating it into existing lesson plan documents that the teacher already had, for example by cutting and pasting tasks. The general use of the tasks could be seen as a mix of offloading, adapting and improvising, as they did not have the same detail of suggested use as the activities.

Discussion

In this study, it was found that issues from the research summary that was also targeted in the activities were mobilized to a larger extent than the issues without a "solution". The main perceived affordances by teachers in their interaction with the teacher guide seems to be the proposed lesson activities, for reasons that they were well-planned, handling an important mathematical learning issue from the expository text, had conceptual focus, and were student-active and student-inclusive. The role of the research summary in the expository text, besides being perceived as valuable for teachers, acted as a way to prime teachers on certain issues. The main expressions made by teachers on these issues were to a larger extent overlapping with the issues handled in the proposed lesson activities. This indicates that as a whole, the lingering ideas from teachers when reporting in the survey and interviews post-teaching and planning, are the issues targeted both by the research summary and the activities. A research summary in itself is therefore not necessarily enough to mobilize action for teachers, if one wants to design a teacher guide resource that induces change of practice. Which, without such a specific intention, the guide in this paper to some extent did.

The teachers in this study can be assumed to be representatives of the Swedish cultural context for upper-secondary mathematics teachers in terms of the reported common textbook use, genomgång and individual student practice (Boesen et al., 2014). However, when presented a teacher guide, all teachers were able to perceive some affordances leading to mobilizing some content features into different kinds of instruction. Even though there were individual differences in offloading and adapting behavior, reflecting individual teachers' pedagogical design capacity (Brown, 2009), teachers utilized support for lesson design, mainly through the proposed activities. This can be seen as filling a gap for which teachers not already had an external resource (see Gustafsson et al., 2024). Using a teacher guide, even though it is a new practice, did therefore not seem to be culturally challenging, except for the required planning time. Despite their teaching context, these teachers seemed to have aims and goals in line with more reform-mathematics education (Boesen et al, 2014). Presenting and supporting teachers with proposals of lesson activities, intended to handle important issues from research, with a clear rationale, made teachers perceive those activities as important and allocate time for them.

Planning time affected teachers in that they did not have the time to read the entire guide; the majority of teachers stated they did not have the time to read everything, so they only read the things that "came first" in the guide, and the rationale for activities they planned on using. Available lesson time seemed to be a larger constraint than the available time for planning, as it in many cases seemed to be the reason for not mobilizing certain activities at all, or adapting them to take shorter time. Consequently, when we think about time as a constraint for teachers, we need to separate planning and lesson time.

This study can be seen as a systemic approach to the mathematics, the teachers, and the CR (Rezat, 2024). The findings also contribute to the field of teachers' use of research summaries (e.g., Rycroft-Smith & Stylianides, 2022). In this study, the role of research acted in several ways. First, the fact that the teacher guide summarized research findings on the topic of QEs was expressed as valuable by most teachers, as it was perceived as an affordance to access findings in a digestible way and

trusted from the researcher backing. Second, as the research summary findings for many of the teachers aligned with their teaching experience, teachers expressed a feeling of being strengthened in their professional experience through the reading of the teacher guide. Third, the perceived link between the research summary findings and the goals of the activities seemed to act motivational for mobilizing that specific content feature.

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Down-to-earth mathematics: transforming curriculum and teaching on function transformations

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The present study proposes and evaluates a digital task sequence aimed at transforming the curriculum and teaching of function transformations by promoting productive meanings of these transformations and integrating mathematics with other domains, specifically climate science. The findings of this study show how students' envisioning of a graph in terms of what is made (a trace) and how it is made (covarying quantities) supports their construction of productive meanings of function transformations. Furthermore, this study also highlights challenges experienced by students during the learning process, which can be used to improve future instruction.

Keywords: boundary crossing, DNR, emergent graphical shape thinking, function transformations.

Introduction

Function transformations are often taught by first introducing a parent function and treating other functions as transformations of this parent shape. While this approach can enhance students' fluency in applying function transformations, researchers (Moore & Thompson, 2015) argue that it may not fully support students' understanding of what they are transforming—specifically, representations of covarying quantities. This study proposes and evaluates a digital task sequence aimed at transforming the curriculum and teaching of function transformations by promoting productive meanings of these transformations and integrating mathematics with other domains, specifically climate science. The task sequence incorporates emergent graphical shape thinking (EGST) as a boundary object, providing students with opportunities to navigate between the domains of mathematics and climate science within a DNR-based instructional context.

In the following sections, we first describe the theoretical frameworks that informed the design of the digital task sequence. We then detail the structure and rationale behind the task sequence, illustrating its role in supporting students' understanding of function transformations as transformations of covarying quantities.

Positioning EGST within DNR and Boundary Crossing Frameworks

Our design integrates two frameworks: DNR-based instruction in mathematics (Harel, 2008) and boundary crossing (Akkerman & Bakker, 2011), along with its associated learning mechanism, particularly reflection. We draw on DNR's principles of duality, necessity, and repeated reasoning to inform the design of our digital task sequence. Drawing on the principle of duality, the task sequence focuses on both developing meanings of function transformations and using EGST as a way of thinking. Drawing on the principle of necessity, we designed the task sequence to include problems that ignite students' intellectual needs. Drawing on the principle of repeated reasoning, we structured the task sequence to provide opportunities for students to use EGST repeatedly throughout the tasks.

Building on the boundary crossing framework, we design the task sequence to support students in reflecting, through taking and making perspectives on both mathematics and climate science, in order to develop meanings of function transformations. We hypothesize that EGST can serve as a boundary object, facilitating students' perspective-taking and perspective-making across the domains of mathematics and climate science. This aligns with previous studies showing that graphical reasoning is an important tool for understanding both mathematics and science concepts (Paoletti et al., 2024; Stephens, 2024).

From both frameworks, EGST plays an important role: as a way of thinking to construct meanings of function transformations and as a boundary object bridging the domains of mathematics and climate science. We refer EGST as "envisioning a graph in terms of what is made (a trace) and how it is made (covarying quantities)" (Moore & Thompson, 2015). Paoletti et al. (2023) argue that for students to engage in EGST, they must engage in both situational and graphical reasoning. We integrate these forms of thinking and reasoning throughout our digital task sequence, which is described in the following section.

Digital task sequence

Our digital task sequence is titled "Down-To-Earth Mathematics: How Hot Is Our Earth?". The phrase "down-to-earth mathematics" was chosen for two reasons: first, to make mathematics more relatable and realistic for students, and second, to use the context of the Earth to raise awareness about the real and urgent challenges related to global temperature trends. We designed the task sequence in Desmos, and it is publicly accessible at https://s.id/down2earth-math.

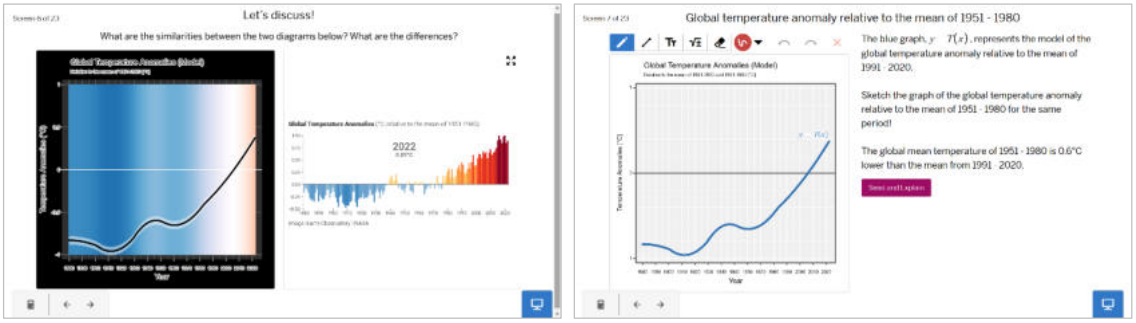


Figure 1: Sample screens from Task VT

The task sequence guides students in using perspectives from climate science to interpret global temperature anomalies over time, and then making mathematical perspectives to construct meanings of function transformations from contextualized problem situations. This understanding of function transformations is subsequently applied to taking mathematical perspectives on function transformations, allowing students to make connections in climate science and predict future global temperature anomalies. This approach highlights the opportunities for students to reflect on both mathematics and climate science through the processes of perspective-taking and perspective-making. Building on this approach, the blueprint of the task sequence is outlined in Table 1.

Table 1: Blueprint for the digital task sequence

Task	Description
MT	**Modeling global temperature anomalies.** Students watch a video animation showing global temperature anomalies (relative to the 1991–2020 average) from 1880 to 2022, featuring both a heatmap and warming stripes. They record what they notice and wonder about the animation. Then, they sketch a graph to model the global temperature anomalies over time.
VT	**Building meanings of vertical translation.** Students compare and contrast a model based on the 1991–2020 average with another graphical representation using a different baseline (1951–1980 average) for global temperature anomalies. Using the information that the 1951–1980 average is 0.6°C lower than the 1991–2020 average (see Figure 1), they then sketch a graph modeling global temperature anomalies relative to the 1951–1980 average. Finally, students relate the resulting model to the previous one to develop an understanding of vertical translation, both visually and algebraically.
HT	**Building meanings of horizontal translation.** Students are presented with a problem of "seeing the y-axis," requiring them to modify the previous model so that it starts from $x = 0$ (the domain of the previous model is from 1980 to 2022). They watch an animation illustrating the construction of the new model alongside the previous one (see Figure 2). Finally, they connect the new model to the previous one to develop a visual and algebraic understanding of horizontal translation.
VS	**Building meanings of vertical shrinking/stretching.** Students sketch a graph of a new model where global temperature anomalies are measured in °F, based on the previous model in °C. They then connect the new model to the previous one, developing a visual and algebraic understanding of vertical stretching/shrinking.
HS	**Building meanings of horizontal shrinking/stretching.** Students watch a video animation demonstrating a strategy for adjusting the x-axis scale to match the y-axis scale. The video illustrates the construction of a new graph alongside the previous model. Students then connect the new model to the previous one, developing a visual and algebraic understanding of horizontal stretching and shrinking.
PT	**Predicting future global temperature anomalies.** Students predict the trend of future global temperature anomalies by extending the graph of the previous model. They then observe the intersection of the graph with $x = 2.25$. Using their understanding of function transformations, they identify the year corresponding to $x = 2.25$.

The design of the task sequence presented in Table 1 follows DNR's instructional principles of duality, necessity, and repeated reasoning (Harel, 2008). First, it addresses both the meanings of function transformations and the use of EGST as a way of thinking. EGST is incorporated in two ways: (a) by asking students to sketch a graph based on a given situation (MT, VT, and VS, see Figure 1), or (b) by having students watch an animation of the graph's construction (HT and HS, see Figure 2). Second, the task sequence creates intellectual needs for students to develop their understanding of function transformations: (a) changing the baseline of temperature anomalies for vertical translations; (b) "seeing the y-axis" problem for horizontal translations; (c) converting temperature units for

vertical stretching/shrinking; and (d) adjusting the x-axis scale for horizontal stretching/shrinking. Finally, throughout the task sequence, EGST is repeatedly incorporated to build students' fluency.

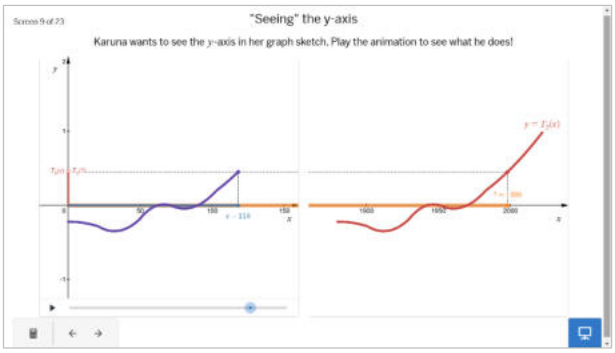


Figure 2: A sample screen from Task HT

Research questions

While the DNR and boundary crossing frameworks inform the design of the task sequence, the EGST framework plays a crucial role in guiding the design. Therefore, this study narrows its focus to investigating the opportunities the task sequence provides in promoting EGST to support students in constructing meanings of function transformations. The research questions guiding this evaluation are: How, and to what extent, do students develop their meanings of function transformations through the use of EGST?

Methods

This study is part of a larger design research project aimed at transforming curriculum and instruction related to change and relationships. Specifically, it presents a lesson case showcasing the implementation of a digital task sequence on function transformations. The participants were 22 first-year undergraduate students enrolled in an algebra and trigonometry course at a private university in Yogyakarta, Indonesia.

Prior to these lessons, the students had already engaged with EGST in interpreting and constructing graphs, as reported elsewhere (Kristanto et al., 2024). The lessons were conducted in two sessions in early October 2023. The first author, who also served as the teacher-researcher, was the regular course instructor. At the beginning of the first session, the teacher explained temperature anomalies, including their definition and how they are calculated, as this was an essential prerequisite for the tasks. Students worked in groups of two or three, resulting in nine groups. In the first session, they completed Tasks MT, VT, and HT; in the second session, they worked on Tasks VS, HS, and PT. During the sessions, the teacher facilitated productive mathematics discussions by allowing students to work in groups while monitoring their progress, purposefully selecting and sequencing groups' work for presentation, and connecting students' approaches and the underlying mathematics (Stein et al., 2008).

Students' work on the digital platform Desmos was recorded. Additionally, the teacher's Desmos dashboard was recorded to provide supplementary data on students' progress. To analyze how students developed their meanings of function transformations, we coded their works and video

recordings of the teacher's dashboard using the EGST framework (Paoletti et al., 2023). The codes comprised situational quantitative and covariational reasoning (MS), reasoning with graphical representations of covarying quantities (MR), and emergent graphical shape thinking (ME). To assess the extent of their understanding, we focused on evaluating students' answers to Tasks VT, HT, VS, and HS, as these tasks align directly with the research questions guiding this study. Using the rubric presented in Table 2, we analyzed their responses to these tasks, where students were asked to (1) represent their new model algebraically and (2) explain their reasoning.

Table 2: Rubric

Mark allocated	Description for algebraic representation	Description for reasoning
0	No answer or incorrect solution.	No answer or incorrect solution.
1	Correct solution.	The answer demonstrates a conceptual understanding of the problem, as reflected in the mathematical approach used. However, no evidence links the reasoning to the problem's context exists.
2	–	The answer demonstrates a complete understanding of the problem, is accurate, and includes clear reasoning that is well-connected to the context of the problem.

Results

We begin by qualitatively examining how students develop their meanings of function transformations. Next, we quantitatively assess the extent to which they have developed these meanings.

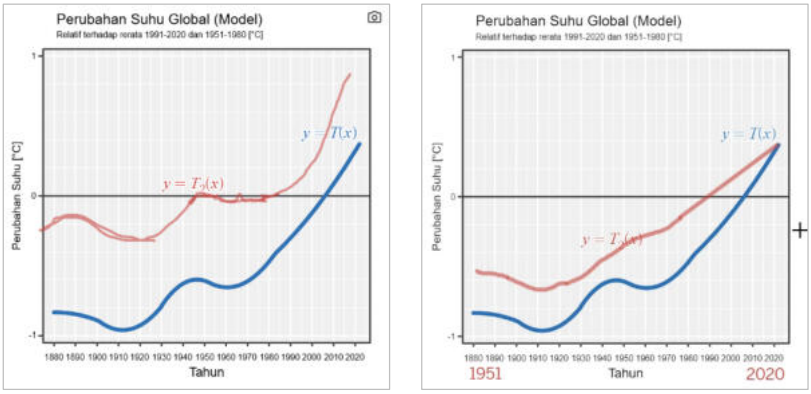
How students develop their meanings of function transformations

Nearly all groups accurately sketched continuous graphs when tasked with modeling annual global temperature anomalies in Task MT. Only one group, Group 9, plotted discrete points but followed the correct trend. This suggests that all groups demonstrated meanings of MS and MR, while one group did not engage in ME as they did not account for all possible points.

In Task VT, six groups adopted a strategy to shift the graph upwards by 0.6 units when the baseline was changed from 1951–1980 to 1991–2020 in their initial sketch. The reasoning from these groups was similar. One group explained their approach as follows:

> Due to the difference in the average global temperature anomalies, the period from 1950 to 1980 differs by 0.6 degrees from the average global temperature changes between 1991 and 2020. This causes the entire temperature comparison graph to shift upwards, indicating that temperatures are getting hotter (Group 7, translated from Indonesian).

Group 7's reasoning indicates engagement with MS, MR, and ME. They appeared to conceive each point of the previous model (with the 1991–2020 baseline) as representing a co-ordination between years and global temperature anomalies, treating each point as a multiplicative object. Using this meaning, they sketched a new model (with the 1951–1980 baseline) by shifting all points of the previous model upward by 0.6, see Figure 3(a). This demonstrates that they developed a meaning of what the y-coordinates in each model represent—specifically, the global temperature anomalies based on the 1991–2020 and 1951–1980 baselines, respectively.



(a) (b)

Figure 3: Sketches from two groups for Task VT

Group 1 employed a different strategy by initially plotting the graph from 1951 to 1980 and then extending it to cover 1880 to 2022, as shown in Figure 3(b). This group likely had an incomplete meaning of the baseline in calculating temperature anomalies. Two other groups drew a new graph by shifting the old one downwards. Their rationale for this strategy was the same: the average temperature from 1951–1980 was 0.6°C lower than from 1991–2020. This suggests that the groups lacked a comprehensive meaning of ME, particularly regarding the concept of temperature anomaly and its relationship to the baseline.

In Task HT, all groups arrived at the correct answer, $T_3(x) = T_2(x + 1880)$. One group used the explanation of a "shifted reference point" as follows:

> This is because Karuna assumes that the year 1880 represents zero, so the intervals for subsequent years start from 1880 as the reference point (Group 5, translated from Indonesian).

For Task VS, while all groups successfully sketched the transformation graph, only a few accurately explained their strategies. For example, one group explained it as follows:

> If we assume that 0°C corresponds to 0°F, then if Celsius increases to 1°C, Fahrenheit will rise by 9/5 or 1.8 units. This means the graph shifts approximately 1.8 units downward (if Celsius is negative) or upward (if Celsius is positive) (Group 9, translated from Indonesian).

In Task HS, eight groups wrote $T_5(x) = T_4(71x)$, but only three groups provided accurate reasoning. One group explained it as follows:

The graph of $T_5(x)$ is derived from the graph of $T_4(x)$ by compressing the input (x) by a factor of 71 (Group 4, translated from Indonesian).

In summary, students' meanings in MS, MR, and ME supported their development of function transformation meanings. Students who understood function transformations as transformations of covarying quantities envisioned each point on the graph as a multiplicative object, which helped them relate it to the new graph. For example, in Task HS, they observed that each y-coordinate in the new model corresponded to the y-coordinate at 71 times the x-coordinate in the previous model.

To what extent did students develop meanings of function transformations?

Figure 4 shows the distribution of group scores across four tasks (VT, HT, VS, and HS). The students performed best in Task VT, where most groups scored 2, indicating they made correct algebraic work with sound mathematical reasoning but did not connect their reasoning to the context of changing the temperature anomaly baseline. In contrast, Task VS saw lower scores, with most groups receiving a score of 1. While these groups demonstrated correct algebraic work, they did not provide reasoning to explain their approach.

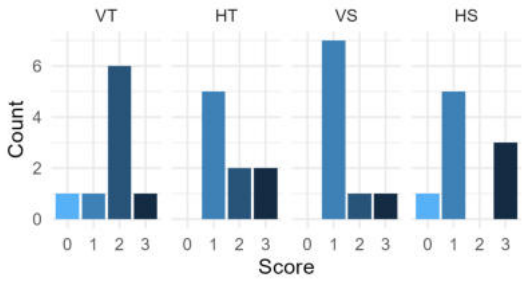


Figure 4: Frequency of group scores across tasks, with scores ranging from 0 to 3

Discussion and conclusion

The findings of this study highlight the potential of the proposed digital task sequence in supporting students in developing productive meanings of function transformations. The qualitative data analysis revealed that EGST supports students in understanding the relationship between a function and the result of its transformation, both algebraically and visually. The results of the quantitative data analysis further reinforce this finding. The findings of this study reveal that some students still demonstrated an incomplete understanding of the baseline used in temperature anomaly calculations. These misconceptions hindered their ability to fully develop mathematical concepts, particularly concerning vertical shifts in functions. This insight highlights areas for improvement in refining the digital task sequence for future iterations. Additionally, an interesting case emerged with Group 1. Their sketch in Task MT provided evidence of their meanings of MS, MR, and ME. However, their incomplete understanding of a critical quantity, i.e. temperature anomaly, prevented them from succeeding in Task VT, which required a proper understanding of this concept.

In refining our task sequence, we plan to incorporate a strategy from cognitive load theory to minimize the extraneous cognitive load associated with understanding the concept of temperature anomaly (Chen et al., 2023). Specifically, we will redesign the sequence to include an initial phase

where students learn how to calculate temperature anomalies using a given baseline, followed by opportunities to practice these calculations independently. We hypothesize that this approach will help reduce their cognitive load when working on Task VT.

While this study focuses on how students construct meanings of function transformations through EGST, it does not extensively analyze how students navigate disciplinary boundaries between mathematics and climate science. Future research could explore how integrating real-world contexts, such as climate change, fosters or hinders students' meanings of function transformations.

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Examining the paths of reasoning presented in the topic of 'patterns' in Norwegian mathematics textbooks

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This paper presents the findings of a textbook analysis of two commonly used Norwegian mathematics textbooks for grades 8 and 9. We examined how these textbooks operationalize the national curriculum (LK20)'s competence aims related to the generalization of patterns. The analysis revealed two main differences: whereas Book A has a strong numerical focus, Book B adopts a more figural approach. Additionally, Book B incorporates reasoning processes that encourage proving through generic examples, whereas Book A involves only processes related to making generalizations. These differences appear to stem from the lack of clear conceptualization of mathematical reasoning in LK20, which has led to varied interpretations in textbook design. Clearer guidance on reasoning in the curriculum could help textbook developers, teachers, and professional development providers align instructional practices with curriculum aims.

Keywords: Textbook analysis, paths of reasoning, patterns, generalization, curriculum change

Introduction

Despite the fast-evolving development of digital resources in mathematics education (Pepin et al., 2024), textbooks remain the most frequently utilized resource by mathematics teachers across several European countries, including Norway (Çakıroğlu et al., 2023). Designed to help teachers organize their teaching in terms of both pedagogy and content, textbooks play a key role in shaping what school mathematics becomes for teachers and students (Johansson, 2003). At the time of mathematics curriculum reforms, especially when new topics or perspectives on mathematics are introduced, textbooks play an even more crucial role for teachers, who may lack experience or knowledge on how to approach these new topics. Over the past five decades, Norway has undergone six major mathematics curriculum revisions. The most recent update, the 2020 curriculum (LK20), introduced a competence-based structure for grades 2–10, specifying 10 to 13 competence aims per grade. Each aim consists of a verb indicating a specific competence (e.g., "explore," "describe," "argue") and a noun phrase outlining a particular mathematical domain (e.g., "relationships between the length of the sides in triangles") (Directorate of Education, 2019, p. 14). Alongside these aims, LK20 introduced six process-oriented *core elements* that emphasize how students should learn and engage with mathematical concepts. These core elements act as overarching "umbrellas" that apply across all topics and grades. One such core element, *Reasoning and Argumentation*, focuses on developing students' ability to evaluate mathematical reasoning and justify their conclusions. Processes related to mathematical reasoning (as defined by Jeannotte and Kieran, 2017) are also integrated into other core elements (such as *Abstraction and Generalization*), reflecting LK20's increased focus on reasoning compared to previous curricula. However, as Valenta and Enge (2020) highlighted, the competence aims often remain general and lack detailed explanations or examples. For instance, they noted that "it is not made explicit in the curriculum whether what is arrived at should be validated" (p. 14), and called for an examination of "how proof-related competencies are addressed in textbooks

and schools" (p. 16). Vague competence aim formulations can create a gap between the curriculum and textbooks, posing challenges for teachers in interpreting and effectively implementing these aims (Ochieng & Kamina, 2022). Given LK20's focus on reasoning and the call of Valenta and Enge (2020), it is important to examine how textbooks, as one of the primary resources for teachers, support this aspect of the curriculum. To be able to provide a detailed and focused analysis, we have chosen to concentrate on the topic of 'patterns', due to their potential to engage pupils in multiple reasoning and argumentation processes (Orton, 2005). In LK20, competence aims related to patterns are listed among the aims for the 8th and 9th grade:

- For 8th grade: "describe and generalize patterns in one's own words and algebraically",
- For 9th grade: "describe, explain, and present structures and progressions in geometric and numerical patterns" (Directorate of Education, 2019, p. 12 - 13).

Given the pivotal role textbooks play in 'translating' curricular policies, such as the emphasis on reasoning, into classroom practices, they are regarded as the potentially implemented curriculum (Valverde et al., 2002). Our study is guided by the following research question: *How do two commonly used mathematics textbook series operationalize the Norwegian National Curriculum's competency aims related to the generalization of patterns?*

Theoretical background

In this section we explain how we understand the different terms used: patterns; and generalization.

Mathematics is often described as the study of patterns. The term 'pattern' is not a well-defined mathematical concept and has been associated with different meanings. As a result, curriculum developers and textbook authors have adopted diverse approaches regarding the topic (Carraher et al., 2008). While a complete classification does not appear to exist, a variety of types of patterns, have been used at different grade levels and mathematical topic areas (Orton, 2005; Rivera, 2013).

Tasks involving generalization of patterns are rich in reasoning processes. Jeannotte and Kieran (2017) include "identifying a pattern" and "generalizing" in those mathematical reasoning processes related to the search for similarities and differences, and Stylianides (2008) places identification of patterns under the broader category of "making mathematical generalizations". Not all generalizations are related to mathematical reasoning. Stylianides (2008) relates *structural generalizations* of definite patterns to mathematical reasoning, as opposed to *empirical* ones. Similarly, Radford (2006) distinguishes *naive induction,* a trial-and-error process through which a pupil guesses a rule, from *generalization.* He defines *algebraic generalization* as "grasping a commonality noticed on some elements of a sequence S, being aware that this commonality applies to all the terms of S and being able to use it to provide a direct expression of whatever term of S" (p. 4). Note that the definition is formulated particularly for sequences and in what follows we consider this type of patterns that comprises the main body of the relevant literature.

Grasping the commonality, or in other words identifying the pattern, entails discovering the rule under which the sequence is built. This rule can be either *recursive* or *explicit* (Lannin et al., 2006). In addition, when a number sequence is illustrated with figures, the generalizing process can be either *figural* or *numerical* (Rivera & Becker, 2005). Last, by focusing on how students express generality,

Radford (2006) identified three types of generalizations: *factual, contextual,* and *symbolic*. The first two are pre-symbolic generalizations with the difference that while factual generalizations operate in the concrete level, enabling the solver to express a particular term of the sequence (e.g. figure nr 100), contextual generalizations belong to a higher, abstract level, from where the solver is able to view and discuss the general term of the sequence (*the* figure).

The choices students make in identifying, generalizing and expressing patterns have consequences for succeeding in a task. Lannin et al. (2006) found that students had difficulties moving from recursive rules to explicit formulas, whilst Rivera and Becker (2005) concluded that figural mode leads to more successful generalizations, so teachers should use activities that de-emphasize numerical generalizations. Examining how these reasoning processes are treated in textbook tasks enables us to identify (a) the *paths of reasoning* promoted, regarding generalization of patterns, and (b) possible differences in the operationalization of the related competence aims between different textbooks. Our paths of reasoning include processes from the whole spectrum of mathematical reasoning processes (Zeanotte & Kierann, 2017) and are therefore an extension of Even and Silverman's (2024) 'paths of justification' that include only processes of mathematical reasoning related to validating.

Methods

The Norwegian education system is centralized, with the Ministry of Education developing the national curriculum. However, schools and teachers can freely choose textbooks, as there is no government approval process for instructional materials, allowing for local flexibility and autonomy in how the curriculum is taught. Given our focus on competency aims related to the generalization of patterns, we selected 8th and 9th grade mathematics textbooks from two of the largest publishing houses with the most commonly used textbooks: Aschehoug (referred to as Book A in this study) and Cappelen Damm (referred to as Book B, with Book B9 denoting the 9th grade textbook). Both books were updated according to LK20.

The textbook analysis was restricted to the textbooks' sections related to the competence aims regarding generalization of patterns. These were identified by the explicit mention of the term 'pattern' in the title and/or in the learning goal(s), whenever those were listed. We initially classified inductively all the patterns found in the relevant sections according to their type. The resulting categories are summarized in Table 1. Then, we investigated the reasoning processes connected to each task involving a pattern, identifying its path of reasoning. This was conducted in three steps: a) identifying the parts of the task related to *identification* of pattern, *generalization* of pattern, and *expression* of the general term, as in Radford´s (2006) definition; b) further categorizing *identification* as either *recursive* or *explicit* (Lannin et al., 2006)*, generalization* as either *numerical or figural* (Rivera & Becker, 2005)*,* and *expression* as either *factual, contextual* or *symbolic* (Radford, 2006); and c) categorizing the processes related to proving as either *generic example* or *demonstration* (Stylianides, 2008). The tasks were coded independently and differences in coding were resolved by discussion leading to agreement.

Findings

The various types of patterns that were found in the textbooks are described in Table 1, along with the frequency they occurred. The largest category, *sequence,* consists of sequences of mathematical objects which are either numbers, or figures, the latter being further divided into *repeating* and *growing*. Figures 1 and 2 provide solved examples involving figural growing patterns. An example of a pattern occurring within *natural number properties* is presented in the task shown in Figure 3. Few tasks discuss patterns in *arrays,* like symmetries in the Pascal triangle or the multiplication table. Last, a pattern occurring when investigating a problem posed verbally, such as modelling of a real-life situation, was coded a *word problem*. Note that although patterns in the last three categories may also be expressed sequentially, it is the solver who must identify the regularity in the task to produce the sequence, and they are therefore coded separately.

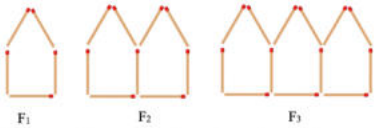
Table 1. Types and frequencies of patterns included in the textbooks

Type of pattern	Frequency			
	Book A8	Book A9	Book B8	Book B9
Sequence	**43**	**23**	**28**	**23**
Number sequence	35	0	7	0
Figural sequence	6	23	21	23
Repeating	0	6	0	0
Growing	6	17	21	23
Natural number property	**8**	**0**	**1**	**12**
Array	**1**	**0**	**0**	**2**
Word problem	**1**	**0**	**0**	**2**

While most of the patterns offered in both textbook series are sequences, differences occur both in the type of sequence and in the ways generalization is dealt with. In particular, Book A includes mostly number sequences in grade 8, while Book B only a few in grade 8. In addition, Book A contains a few repeating figural sequences, while this type is absent from both grades in Book B.

In Book A the process of pattern generalization is introduced gradually. The tasks initially involve only identification of a pattern, which is most often recursive, and then progress to tasks that require algebraic generalization of the identified patterns. The generalizations are mainly numerical and typically result in a symbolic expression. Only one task is categorized as inviting for a contextual generalization. The symbolic expression is sometimes used to determine a far term in the sequence. This path of reasoning *identification/recursive → generalization/numerical → expression/symbolic* is illustrated in the solved example in Figure 1.

Look at the pattern the figures form when you answer the task.



a) Draw figure number 4, and state how many matches this figure has.
b) Explore the pattern and find how many matches each new figure increases by.
c) Find a formula that describes the number of matches in the n-th figure.
d) Use the formula and find the number of matches in figure number 10.

Solution

a) There are 17 matches in the fourth figure.
b) The increase is 4 matches for each figure.
c) The total number of matches increases by 4, and the first figure has 1 more match. Therefore, the number of total matches in figure number 1 can be written as: 4 . 1 + 1. The total number in figure number 2 can be written as: 4 . 2 + 1.
The formula then becomes: $F_n = 4n + 1$
d) We insert $n = 10$ and calculate.
$F_n = 4n + 1$
$F_{10} = 4(10) + 1 = 41$
There are 41 matches in the tenth figure in the pattern.

Figure 1. Solved example with typical reasoning path in Book A9

A near generalization (Rivera, 2013) is asked for in a) (Figure 1), solved by drawing and counting. The open question "explore the pattern," which could allow for various strategies, is narrowed by focusing on how each new figure is built from the previous one, leading to the discovery of the *recursive* rule. Parts a) and b) were therefore coded as *identification of pattern/recursive.* In c) the *symbolic* expression is generated from the *generalization* of the numeric expressions of the first two terms, without using the figures in the reasoning. The mode is therefore *numerical*, and this part is coded as *generalization/numerical.* Part d) of this task is not coded in terms of reasoning paths as it is a mere application of the formula that the solver is instructed to perform.

Book B, on the other hand, offers mostly figural sequences that gradually increase in complexity. The grade 8 textbook presents various strategies and includes tasks related to all types of identifications, generalizations and expressions. While there is no clear dominant reasoning path followed, there appears to be a tendency towards a figural mode of generalization. This is evident in tasks where pupils are asked to describe figures and the use of distinct colors to highlight parts of the figures. Additionally, symbolic expressions are often produced after factual or contextual ones. Textbook B9 focuses on breaking down complex figures into simpler components. Then the symbolic expressions of the parts are combined to construct the one for the original figure. This approach is demonstrated in the solved example in Figure 2, where the path of reasoning is: *identification/explicit* → *generalization/figural* → *expression/contextual* → *expression/symbolic*.

To divide figures into smaller units

[…] The orange chips illustrate the rectangular numbers. The number of orange chips in figure nr. n is thus $n.(n+1)$.
The blue chips illustrate "the previous triangular number". The total number of blue chips in figure nr. 2 is the triangular number 1, the total number of blue chips in figure nr. 3 is the triangular number 2, etc.
We therefore replace n with n-1 in the algebraic expression for the triangular numbers. The number of blue chips in figure n is thus $\frac{(n-1)n}{2}$.
The number of chips in figure nr. n is $n.(n+1) + \frac{(n-1)n}{2} = [\ldots] = \frac{3n^2+n}{2}$

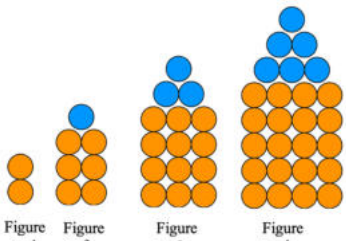


Figure 2. Solved example from Book B9

The process of identification of this pattern starts by decomposing it into two simpler ones, where each one is handled separately, and their formulas are in the end composed to produce the symbolic expression of the original pattern. The rule observed in the identification of both simpler patterns is *explicit* and while for the case of the rectangle numbers, the formula can be used as it is, in the case of the triangle numbers it needs to be adapted as the general term contains "the previous triangular number", a formulation that can be linked to a *contextual* level of expression. The statement is supported by exemplifying it using the two first terms of the sequence. The contextual expression is then used to describe the *symbolic expression* of the blue part of the pattern. The different colors used for the two parts of the figures are also used in the two parts of the formula helping pupils understand the connection between the figural pattern and its symbolic expression. This technique is commonly used in this textbook illustrating the focus it has on the *figural mode* of generalization.

The next significant category of patterns found in both textbooks contains patterns related to *natural number properties* and operations between them. The properties investigated are similar in the two books, but in contrast to Book A Book B moves beyond making mathematical generalizations and invites the pupils to engage with proving and in particular providing generic examples. A typical task is discussed in Figure 3, with the following path of reasoning: *identification* → *generalization* → *proving/generic example*.

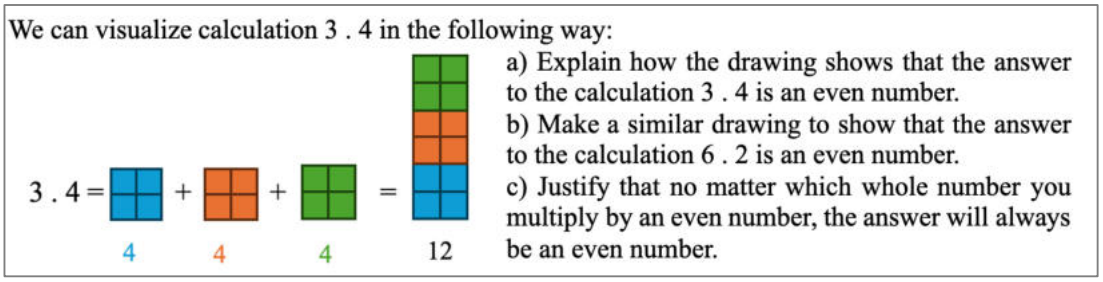


Figure 3. Natural number properties task from Book B9

The pupils have already seen earlier in the textbook that any even number can be represented as a tower of pairs of blocks, and any odd number as a tower of pairs of blocks plus an additional one at the top. The pattern that they must identify and generalize here is the visual argument that will support the statement in c). The cases a) and b) are therefore used to identify and generalize this pattern. The argument is coded as generic as it seems natural here for the pupils to use the particular cases to argue for the general. We note that the same task is included in Book A8, formulated as an open question prompting the pupils to explore products of odd and even numbers and identify the patterns.

Discussion and conclusions

In this study, we explored how two commonly used mathematics textbook series in Norway operationalize the competency aims related to the generalization of patterns that are outlined in the LK20 curriculum. Our findings highlight two main differences in how these textbooks approach pattern generalization, which may have significant implications for how reasoning and argumentation are fostered in the classroom. The first one regards the strong numerical focus Book A has, manifested by the choice of patterns, which are mostly number patterns in grade 8, and the mode of generalization, which is mostly numerical, compared to the more figural approach Book B

adopts. The second one is related to the reasoning processes promoted by the textbooks in the sections about generalization of patterns. In Book B, patterns occurring as natural number properties and operations between them are used to guide the students in proving using generic examples, while in Book A tasks centered around this type of pattern involve only processes related to making mathematical generalizations (Stylianides, 2008). We contend that these differences may be a result of the vaguely formulated competence aims, which, in addition, include terms that have been associated with different meanings in the mathematics education literature. It appears that there is no clear conceptualization of mathematical reasoning in LK20, with reasoning processes spread in different core elements. However, our study has limitations, the tool used to study reasoning processes is topic-specific and would need to be adapted for broader application across different mathematical content areas. Further research could extend the analysis to other mathematical topics, different textbook series or involve classroom studies.

As a general conclusion, we argue that, although there has been a significant change towards including processes of reasoning and argumentation in the recent LK20 curriculum (compared to the previous national curriculum), the competency aims with respect to reasoning were left rather vague, as also suggested by Valenta and Enge (2020). This is not an uncommon feature of national curricula, but it leaves the challenge for textbook authors and teachers to interpret the competency aims and to operationalize them in the classroom. Given the role of textbooks as the potentially implemented curriculum (Valverde et al., 2002), textbook developers, teachers, and professional development providers would benefit from a clearer conceptualization and understanding of reasoning and argumentation processes in the curriculum. This would help teachers plan and enact lessons aligned with these processes, enhancing the effectiveness of reforms.

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Place value concepts as presented in German and Australian student workbooks

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In this paper we investigate how place value concepts are presented in selected German and Australian workbooks used by Year One students. Place value is an important concept that provides the foundation for further mathematical learning. Analysis of student workbooks is limited, especially in relation to the topic of place value. Student workbooks are a useful indicator of the opportunities for learning that students are provided. A cross-country analysis of the similarities and differences in the extent to which various concepts are addressed is useful to identify gaps that need to be addressed in student materials, and in the work we do as teacher educators.

Keywords: Place value, comparative textbook analysis, mathematics workbooks, role of zero.

Introduction

A poorly developed concept of place value is one of the main reasons for difficulties in mathematics. A substantial body of evidence identifies that understanding place value is central to the success of students in primary school mathematics (e.g., Kortenkamp & Ladel, 2013; Yuan et al., 2019). However, research also suggests that students struggle to master place value concepts (e.g., Yuan et al., 2019). Place value involves the learning of several "big ideas" (Van de Walle, 1998) and fundamental to these is the integration of early experiences of counting, grouping, and bundling amounts in tens (and in tens of tens, and in tens of tens of tens, and so on). Building upon the prerequisite knowledge of counting, grouping, bundling, and part-whole, is the concept of positional notation, which comprises the sub concepts of i) the value of the place (that is the value of the bundle unit); ii) the value of digits (based both on the symbol and on its position), and iii) the value of the number (that is the application of the part-whole concept to each of the digits in a number) (Ladel et al., 2023).

Given that textbooks and student workbooks[1] frame the teaching activities and learning opportunities provided to students (Sievert et al., 2021), and have an enormous influence on teaching and learning, we now provide a short analysis of research regarding their use.

Literature regarding textbook/workbook analysis

Textbooks have drawn considerable research attention and are regarded as the main resource used by teachers and students (Rezat, 2022). According to Jäder et al. (2020) more than 50% of students in secondary school in countries such as Australia, Finland, Singapore, Sweden, South Africa, and Canada are taught in classrooms where the textbook is the basis of instruction. We regard workbooks as a mediator between the intended and the implemented curriculum. In taking this position, we are

[1] In this article textbooks refer to sources of tasks which teachers can modify/adapt, and students answer elsewhere, and workbooks present tasks for students that are completed as prescribed.

not arguing that student workbooks and textbooks *should* be a key component of children's experience of mathematics but do acknowledge that these artifacts *are* a part of many children's mathematics learning experiences and are therefore worthy of analysis. Our view is that an analysis of student workbooks will get us closer to the implemented curriculum than if we had analyzed textbooks, as the student workbooks more closely indicate the tasks that the students will complete. Comparisons between countries (see Jäder et al., 2020) can assist in determining the quality of textbooks in their individual countries.

Our review of recent literature indicates that there has not been an analysis of how place value is presented in textbooks or student workbooks; therefore, the aim of this article is to present our analysis of Year One student workbooks from Germany and Australia regarding place value. Although in our larger project we have reviewed workbooks from four countries (Germany, South Africa, Japan and Australia), for the CERME audience we felt it most appropriate, and within the scope of the paper, to only discuss the German and Australian workbooks. Importantly, these two workbooks provide a good contrast regarding how place value is taught. Our research question is

What is the extent and nature of place value coverage evident in tasks in the selected German and Australian workbooks?

Methods

Methodologically, textbook analysis is a form of content analysis, whereby researchers make replicable and valid inferences from the text (in our case student workbook examples) they are evaluating. By its nature, this will involve making inferences based on the content and the contexts in which the texts were written, with the goal being to assess the likely impact of the text on the students. Following van Zanten and van den Heuvel-Panhuizen (2018), in this research the unit of analysis was any task that demanded a response from the student. Where we could see that a task included significant subtasks (that comprised work at about the level of a normal task in that workbook), these subtasks were treated as a task and coded accordingly. We acknowledge that there is no guarantee that any task, if it is completed at all, will be completed in the order presented and thus "all textbook analyses, irrespective of the approach adopted, are probabilistic" (p. 218).

Selection of workbooks

Both Germany and Australia have a system of Federated States (Länder) or Territories, each with responsibility for education and this results in textbooks/workbooks within each country varying greatly; indeed, in the Australian context there are workbooks written specifically for one state (e.g., the series NSW Targeting Mathematics workbook is advertised as being "fully aligned to NSW K-10 Syllabus"). Given this context, it was not possible to determine the most popular workbook in the Australian context. The selected workbook was chosen because i) it is published by a reputable educational provider in the Australian/New Zealand market; ii) is representative of the types of workbooks that students in Australia use; and iii) the entire student activity book could be accessed online. In Germany the three biggest publishers of workbooks were asked which mathematics workbook for primary school had the highest usage in Germany and based on their response, a workbook was chosen for analysis. Although there is variability between different workbooks in different countries, we are confident that the two workbooks chosen are indicative of the work that

students in these two countries are learning. As much of the conceptual work in place value occurs in the early years, we have chosen to evaluate Year One workbooks for students in Year One, where students in both countries are first introduced to place value concepts.

Analysis process

In the first step of coding the data, the team met and reviewed a sample of tasks from each workbook to ensure that we would evaluate place value prerequisite knowledge and place value sub concepts presented in the workbooks in a consistent manner. Each task was mapped to key place value concepts (See Ladel et al., 2023), including pre-concepts (counting, (decimal) part-whole, grouping and bundling) and place value sub concepts (bundling [Base Ten], the decimal part-whole concept, value of the place and positional notation) as these are critical for flexible place value understanding (Ladel & Kortenkamp, 2014). For this analysis we excluded the prerequisite knowledge of counting single ones, cardinal number, and part-whole as these aren't exclusive to the development of place value understanding. In addition, as we are only interested in this article in Base Ten - we also excluded the concepts of "different possibilities to group due to the group size" or "different possibilities to bundle due to bundle size" as these concepts are only relevant if we are using a range of bases.

Grouping and bundling are related yet different concepts with the difference largely a matter of what language is used to describe the sets that can be formed. While both concepts require groups of amounts, bundling results in a group of objects being recognised as a new object with its own identifier (i.e., name or label). For example, ten objects are initially referred to as a group of ten ones but then are bundled and labelled as one ten. Likewise, a group of ten tens can be bundled again and labelled as one hundred. Maximal grouping and maximal bundling occur when no further groups or bundles can be formed. For example, the number 123 can be represented as 12 bundles of ten and three ones or one hundred and 23 ones; however, these examples are not maximally bundled as the 12 tens can be bundled again into one hundred and two tens or the 23 ones can be bundled again into two tens and three ones. We require maximal bundling when representing amounts using standard notation.

In the second step of coding, as with the approach used by Sievert et al. (2021), each workbook was analyzed by two pairs of team members and workbook tasks that were considered to potentially develop place value understanding were identified. Tasks that passed this initial screening were then assessed to determine whether the place value concepts were present. In the final step of coding, all authors met to resolve the coding of any contentious tasks.

Scoring of items.

Every task in each workbook that was considered as potentially developing place value knowledge was coded as either a "0" (not explicitly developed) or a "1" (explicitly developed). As a task could be coded against multiple concepts, the total number of concepts being developed in each workbook is greater than the number of tasks. This scoring process led to the analysis of each task yielding a sequence of ones or zeros, according to the presence or absence of each concept in that task.

Results

We examined the workbooks in terms of four clusters – decimal part-whole, grouping, bundling or place value sub concepts. The data indicated that decimal part-whole was developed in every place value task (as might be expected) and, as this finding does not add any new knowledge, we have not presented this data. The data for the other three clusters is presented in Tables 1, 2, and 3.

Table 1: Distribution of Grouping prerequisite knowledge within Workbooks

Country	*Australia (N of tasks = 36)*		*Germany (N of tasks = 66)*		*Combined (N of tasks = 102)*	
Concepts	*N of tasks developing concept*	*% of tasks developing concept*	*N of tasks developing concept*	*% of tasks developing concept*	*N of tasks developing concept*	*% of tasks developing concept*
Counting groups of ten and single ones	15	40.54	10	15.15	25	24.51
Maximal grouping	10	27.03	4	6.06	14	13.73
Grouping Cluster	*25*		*14*		*41*	

Table 2: Distribution of Bundling prerequisite knowledge within Workbooks

Country	*Australia (N of tasks = 36)*		*Germany (N of tasks = 66)*		*Combined (N of tasks = 102)*	
Concepts	*N of tasks developing concept*	*% of tasks developing concept*	*N of tasks developing concept*	*% of tasks developing concept*	*N of tasks developing concept*	*% of tasks developing concept*
Counting bundles of ten and single ones	19	52.78	11	16.67	30	29.41
Maximal bundling	19	52.78	9	13.64	28	27.45
Bundling Cluster	*38*		*20*		*58*	

Table 3: Distribution of Place Value sub concepts within Workbooks

Country	*Australia (N of tasks = 36)*		*Germany (N of tasks = 66)*		*Combined (N of tasks = 102)*	
Concepts	*N of tasks developing concept*	*% of tasks developing concept*	*N of tasks developing concept*	*% of tasks developing concept*	*N of tasks developing concept*	*% of tasks developing concept*
Value of place	33	91.67	56	84.85	89	87.25
Standard Partitioning	32	88.89	49	74.24	81	79.41
Role of zero	12	33.33	59	89.39	71	69.61
Value of digit	29	80.56	18	27.27	47	46.08
Value of number	29	80.56	24	36.36	53	51.96
Place Value sub concepts Cluster	*135*		*206*		*341*	

As the workbooks only provide numbers up to 100 in Australia and up to 20 in Germany, there is no opportunity for the related concepts of repeated grouping or repeated bundling (i.e., ten groups of ten make one larger group of one hundred or ten tens are equal to one hundred). Consequently, these two

concepts were not included in Table 1. Although it is possible to demonstrate different possibilities to group or bundle with numbers less than 100 (e.g. 25 objects could be grouped as 10 objects, 10 objects and 5 single objects or 10 objects and 15 single objects or bundled as 2 tens and five ones or 1 ten and 15 ones - i.e., non-standard partitioning), there were no examples of this flexibility of bundling in the Australian workbook. In the German context, it is not possible to demonstrate different possibilities to group or bundle with numbers less than 20, and consequently these two related concepts were not included in Table 2. In our view the lack of opportunities for repeated grouping or bundling limits developing place value understanding and is a direct consequence of curriculum design, which limits students' initial exposure in Australia (Year 1) and Germany (up to Year 2) to numbers to less than 100. This is compounded in the German context by the design of the Year 1 workbook, which only provides students with place value opportunities up to 20 (the only exception being the two final pages that include working with tens up to 100).

Discussion, limitations, and conclusion

In this section we discuss the impact of workbook design decisions on how students learn place value. It is important to note here that we are not attempting to determine whether one workbook is better than the other, as to do so would be to ignore a whole range of other factors that impact upon how students learn place value. Rather we discuss how the decisions regarding what content appears in each workbook offer advantages and disadvantages for learning place value. Both workbooks have interesting, but different, ways of scaffolding students' progress. In the Australian workbook, "call out text bubbles" direct the students to notice conceptual and/or mathematical connections. In the example below (Figure 1; left), the callout text (in green) encourages students to move beyond grouping ten objects and five single objects and ten objects and three single objects to the bundling of 10 objects into 1 group of ten by noticing that the first digit in each number is a 1 (i.e., one group of ten). In the German workbook, students are accompanied by a frog called Mini, and a dog called Max, who accompany their learning (See Fig 1; right). The structure is always the same as Mini and Max scaffold each activity by completing a sample task and discussing it. Analysing these prompts were necessary for accurate coding as they indicated whether a task was a place value task or not.

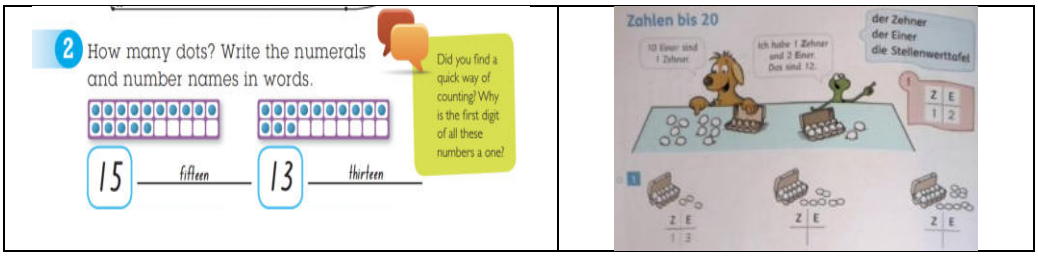

Fig 1. Examples of scaffolding in Australian (left) and German (right) workbooks

We now turn to a discussion of the descriptive statistics with the combined data suggesting a wide variability in the number of opportunities that students have to develop pre-requisite place value knowledge as compared to the number of opportunities for the development of place value sub concepts. Overall, the 102 tasks in the two workbooks offer 440 opportunities for students to engage with the place value concepts identified earlier. By far the least developed concepts are those related to grouping tens and single ones/maximal grouping (41 instances) and bundling tens and single

ones/maximal bundling (58 instances) with less than 30 percent of tasks across both workbooks developing these four concepts. This is most likely influenced by the number ranges - up to 100 (Australia) and up to 20 (Germany) - as these ranges limit opportunity for working in groups or bundles of tens. Given that bundling and unbundling is a fundamental prerequisite to understanding place value, we view this as a missed opportunity for students. The five place value sub concepts (value of place, standard partitioning, role of zero, value of number and value of digit) are more strongly represented (341 instances) with the first four of the five sub concepts being developed in over 50% of the tasks. Most regularly presented to students are the concepts of value of place and standard partitioning (87% and 79%).

Of importance to us is how place value knowledge is developed respectively in each workbook. There are two important comments to note here. Firstly, the German curriculum allows for students in Year One to be working in the number range up to 100, and it is therefore a publishing decision to limit the number range to 20. Secondly, as we indicated earlier, we are not making a judgment on whether one workbook is better for student learning as we have no empirical performance data to make such a claim. An initial difference between the workbooks is the relative weighting of concepts developed per task. Although the German workbook presented close to double the number of tasks (66 vs 36) and overall, more opportunity to engage with concepts (236 vs 198), the tasks in the Australian workbook included an average of 5.50 concepts per task, as opposed to 3.58 concepts per task in the German workbook. Given the connected nature of place value concepts, this might provide additional scope for Australian teachers to highlight the relationship between place value prerequisite knowledge and/or place value sub concepts when several of them are present in the one task.

In terms of the specific emphasis on individual concepts, the pattern of development differs markedly. In relation to Tables 1 and 2, the Australian workbook provides students with greater opportunity to develop two concepts related to grouping, namely counting in groups of ten and single ones (40.54% of workbook tasks include this concept) and maximal grouping (27.03% of tasks) when compared with the German workbook with (15.15% and 6.06 %) respectively for these concepts. Likewise, the Australian workbook provides greater opportunity for the related concepts, counting bundles of ten and single ones (Australia 52.78%, Germany 16.67%) and maximal bundling (Australia 52.78% and 13.64%). As we have described, grouping and bundling are very similar concepts and so we might expect that the relationship between the percentages of the concepts is similar, and this is indeed the case (counting in groups of tens/bundles of tens: Australia 40.54% & 52.78% and Germany 15.15% & 16.67% respectively). However, this relationship does not hold true when grouping and bundling (maximal grouping/bundling: Australia 27.03% & 52.78% and Germany 6.06% & 13.64%), where in both workbooks the opportunities to maximally group are less than to maximally bundle. The difference in allocation of tasks between maximal grouping and maximal bundling reflects the emphasis in both workbooks in representing amounts in the standard partition, whereby the focus is on naming the number of tens or ones in each place, which requires maximal bundling rather than maximal grouping. The lower percentage of tasks in the German workbook in both cases is again a consequence of the design decision to only use numbers up to 20.

There are also noticeable differences in the opportunities for students to develop place value sub concepts (Table 3), and here we focus on three, firstly the related concepts of the value of digit and

the value of number and then the separate concept of the role of zero. In relation to the two concepts - value of digit and value of number, our data suggests that these two concepts are closely connected in tasks in both workbooks (Australia - 80.56% & 80.56%; Germany - 27.27% and 36.36%). This data indicates that i) both concepts share the same or similar emphasis respective to each other in the separate workbooks; and ii) they are less often presented to students in the German workbook. The second observation is again accounted for by the design decision of only going to 20, which offers less variability in representing the value of digit, as any digit other than 1 only has one value, that is, students only ever encounter the digits 2-9 in the ones and thus encounter less tasks overall as the tasks would become repetitive; or the value of number, as it is likely the case that the ordinal nature of numbers is used by students to complete tasks. It is likely easier for students to just count, and not see these numbers as parts of ten and ones (e.g., 12 is just one more than 11 or 16 is one more than 17). This could be considered as a missed opportunity for Year One students in Germany to develop their early place value knowledge.

A final point of interest for us involves the emphasis on the role of zero, and we come at this point from two perspectives. Firstly, both workbooks depart somewhat from their respective national curriculums. In our earlier research into the respective curriculums (see Westaway et al., 2023) we discussed the lack of emphasis on the role of zero and identified this as a limitation of both. However, contrary to the curriculum, in both workbooks, the place value role of zero is emphasised. We see this as a strength but acknowledge that this could be problematic for teachers who are tasked with delivering the national curriculum. Secondly, there is a marked difference in emphasis between the two workbooks in highlighting the special nature of the zero (Australia - 33.33%; Germany 89.39%). A possible explanation is that there is a strong emphasis in the German workbook on combinations bridging through ten, highlighting the special nature of a 10 (See Fig 2.).

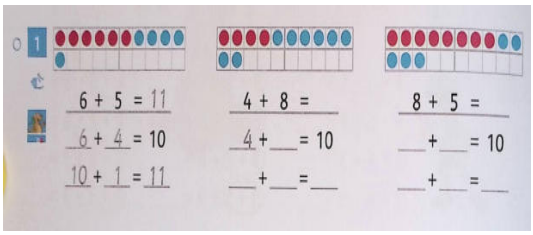

Fig 2. Examples from German workbook supporting knowledge of the role of zero

In many examples such as these, the students encounter that the symbol 10 means one ten and zero ones and this understanding is supported by the use of the ten frames that again indicate a bundle of tens and some, or indeed no, number of ones.

Conclusion

This conference paper has presented a component of a larger program of research where we investigated workbooks from four countries. It serves as an initial attempt to understand the implications of workbook design on how students are presented with tasks developing place value concepts. We argue that only offering initial exposure to small number ranges (i.e. up to 100) can limit opportunities for students to develop the important prerequisite knowledge related to repeated grouping and bundling of tens, as well as opportunities to investigate the value of digits within a

number. Given the increasing reliance by teachers on prepackaged materials, it is important that we better understand how place value concepts should be presented to students. In terms of progressing our work further, we have learned that we need to develop a mechanism for examining more closely the pattern of how prerequisite knowledge and place value sub concepts are presented together in tasks, beyond what we could say in this paper regarding the density of concepts per task. Pedagogically, it might be appropriate in our ongoing research to compare workbooks across number ranges (i.e. up to 100) rather than across age ranges where curriculums differ between countries.

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Empowering mathematics instruction: Variation theory, lesson plan, and instructional quality

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This paper presents a case study that aims to explore how a teacher with low mathematical knowledge for teaching, in a class with an average level of students' socioeconomic status, manages to establish high instructional quality in the lesson with the support of a lesson plan that draws on variation theory. The analysis shows how patterns of variation, such as contrast, generalisation, and fusion helped students in an upper primary school class to discern critical aspects of fractions. The study highlights the need for lesson plans that provide teachers with structure while supporting their flexibility, to modify instruction based on real-time student feedback and needs.

Keywords: Variation theory, lesson plan, high-quality mathematics instruction, fractions.

Introduction and aim of the study

The quality of mathematical instruction is influenced by a complex interplay of various factors that together shape the effectiveness of teaching and the learning outcomes of students. Much effort has been put into investigating factors that concern the individual teacher, such as teachers' MKT (Ball et al., 2008) and/or views of what constitutes high-quality mathematics instruction (Munter, 2014). Another group of studies focuses on student characteristics, showing that what constitutes instructional quality (in terms of positive effects on student achievement) depends on factors such as students' age and socioeconomic status (SES) (Broer et al., 2019). A third group of studies brings up the role of educational resources for the quality of mathematical instruction, both theoretically (Remillard & Heck, 2014) and in effect studies (Jackson & Makarin, 2018). Teaching based on well-structured teaching materials can potentially reduce the relative impact of the teacher compared to other factors within a classroom learning community (Shechtman et al., 2010). In this study, we will focus on educational resources, while also taking into account factors at the teacher and student level. More specifically, we explore how a teacher with low MKT, in a class with an average level of students' SES, manages to establish high instructional quality in the lesson with the support of a lesson plan that draws on variation theory (VT) (Marton, 2015).

Educational resources constitute an important link between curriculum and teaching practice. The teacher and the educational resource, such as a textbook, form "a dynamic inter-relationship in which both are active participants and contributors to instructional outcomes" (Charalambous & Hill, 2012, p. 449). However, for instructional material to be the educational resource it is intended to be it must provide structure and guidelines for the teacher (Charalambous & Hill, 2012) and questions and instructions should be close to everyday life in the classroom so that the teacher can immediately see its relevance and implement them in their classroom practice (Davis & Krajcik, 2005). Indeed, there is a risk that a lesson plan can instrumentalise a teacher's work in the mathematics classroom. Strongly structured lesson plans

may leave little room for teacher autonomy, restricting their ability to adapt instruction based on students' immediate needs, interests, or understanding (Remillard, 2005). Hiebert and Morris (2012) argue that, while structured lesson plans can offer consistency, they often limit teachers' ability to modify instruction based on real-time student feedback and needs. The hypothesis of the present study is that a lesson plan that follows the principles of VT (Marton, 2015) can balance between structure and flexibility, noting the importance of allowing teachers to make instructional decisions based on student interactions and understanding.

Variation theory as an instructional design principle

The VT has proven accessible and useful for teachers to design and implement well-organised and structured lessons that provide students the opportunity to engage with and explore important ideas in different mathematical areas (Kullberg et al., 2017; Mason, 2017; Ryve et al., 2013). The theory emphasises variation as a necessary condition for learners to be able to discern new aspects of an object of learning (Kullberg et al., 2017). VT entails *holism*: People learn through experiencing differences and similarities. Nothing has one aspect only: "We cannot grasp just one meaning; it takes at least two. We can only find a new meaning through the difference between meanings. The difference between meanings thus becomes visible against a background of invariance in other respects" (Marton, 2015, p. xi).

VT spells out two conditions of learning: 1) that learning entails discerning necessary aspects of the object of learning and 2) that discernment cannot happen without the learner having experienced variation (Kullberg et al., 2017). For instance, it would not be possible to notice the colour red if there was just one colour. One must be exposed to variations in colours, such as green, blue or yellow to experience something as red or, to make sense of the concept colour(s) in general. It is the structure of an exercise as a whole, not the individual items, that promotes mathematical sense-making. Tasks that are carefully sequenced are generally likely to result in progress in ways that unstructured sets of tasks do not. In the words of variation theory, "to make the (critical) aspect possible to discern, the teacher must open up the aspect as a dimension of variation (Kullberg et al., 2017, p. 560). The variation theory describes three patterns of variation for structuring teaching that promotes discernment, i.e., learning (Marton, 2015):

Contrast: To experience something, we need to compare it with something else. For instance, in order to understand that triangles are geometrical objects with three vertices, we must experience geometrical objects that have not three vertices, such as a rectangle or any other geometric shape (Kullberg et al., 2017).

Generalisation: To fully understand what constitutes a triangle, we must also experience varying appearances of a triangle. Generally expressed, generalisation means that it is necessary for us to grasp the focused aspect of a phenomenon (e.g. features defining a triangle) and separate it from irrelevant features (e.g., the thickness or colours of the lines to illustrate a triangle). In teaching practice, this means that different (irrelevant) aspects are systematically varied while keeping the focused aspect invariant.

Fusion: If there are several critical aspects of a phenomenon that a learner has to take into consideration at the same time, they must all be experienced simultaneously. In the case of a

triangle, this could imply the simultaneous discernment of three vertices, that each vertex corresponds to an angle, and that the sum of the three angles is 180°.

Method

We follow the methodology of a case study and, particularly, that of a paradigmatic case (Flyvbjerg, 2006). In case study methodology, a paradigmatic case serves as an exemplar to illustrate a particular phenomenon in a clear and representative manner. In the present study, we are interested in how an educational resource (a lesson plan) that follows the principles of variation theory can support a teacher in establishing a lesson in mathematics with high instructional quality. To increase the probability of detecting this, we wanted to reduce other factors that could also explain the quality of instruction. Therefore, the selection of the lesson was based on the fact that a) the educational resources that informed the lesson showed principle from variation theory, b) the instructional quality was high, c) teacher level factors (e.g., MKT; Training in VT) were low, and d) student level factors (e.g., SES) were at or below average.

Concerning a), the lesson in this study is conducted in the context of a national-scale professional development program (PDP) for mathematics teachers in Sweden – Boost for Mathematics (BfM). During the PDP teachers, among other things, study PD materials (e.g., texts, video) available on a digital platform and, together with their colleagues at the respective schools, plan a teaching activity (usually a lesson to conduct). The lesson analysed was such a teaching activity, focusing on fractions. The PD materials, forming the basis of the teachers' joint work, did not explicitly articulate VT. That the lesson plan follows VT principles comes from our reading of the PD material and from the observation of how sequences of tasks were presented on the pre-planned PowerPoint, through which the teacher led the teaching. So, when referring to the lesson plan in the present analysis, we refer to the organised sequence (plan) of tasks through which the teacher led the teaching.

Concerning b)-d), it is important to note that our pragmatic case is part of a dataset for a larger research project that involves a large data set, including video recordings of ~170 mathematics lessons conducted during BfM as well as teacher interviews (~50 teachers). Some of these data have been coded within the research project and serve as a basis for the selection of our case. For example, concerning b) instructional quality, different aspects of instructional quality of the mathematical lessons have been rated using the UTeach Observation Protocol (UTOP) (Walkington & Marder, 2018), an observation instrument that has been shown to have high correlation between observation score and student achievement (Kane & Staiger, 2012) and that rests on dimensions of instructional quality corresponding with ambitious mathematics instruction (Walkington & Marder, 2018). The pragmatic case is a lesson that was known to be of high quality (UTOP-score in the upper percentile of the sample). Concerning, c) teacher level factors, teachers' MKT has been captured using interview questions derived from Empson and Junk (2004), and teachers' views of what constitutes high-quality mathematics instruction has been captured using an established survey (Swan, 2006). The teacher in our case had low MKT (MKT-score in the lower percentile of the sample) as well as a view of high-quality mathematics instruction that leans more toward viewing the teachers' role as either 'chalk and talk' or as a facilitator, rather than a knowledgeable other. Finally, concerning d) student-level factors,

students' SES (measured in terms of the proportion of students who have at least one parent with tertiary education), was in line with the average value for our sample.

UTOP captures a wide range of mathematical quality of instruction and to provide a detailed analysis of all these dimensions is beyond the space limitation of the present paper. In this paper, we focus on how a lesson plan that follows the principles of variation theory can enhance the quality of instruction according to the dimensions Lesson Structure and Mathematics Content and their relationship. Lesson Structure assesses the teachers' plans for the organisation of the lesson and includes items such as the sequencing of the lesson, the resources employed, and the opportunities and/or methods used to gauge student understanding. Mathematics Content assesses the quality of the mathematics content enacted during the lesson and includes items regarding, for example, the teachers' communication of the content and the content significance.

We analysed the lesson through the lens of VT (Marton, 2015). We first discerned episodes where aspects of fractions were enacted during the lesson. We then took a closer look into these episodes. In particular, we examined how the aspects came through in patterns of variations, mediated by the lesson plan, through which the teacher led the lesson.

Analysis and Results

The sequencing of activities kept students engaged in the content, had a clear sense of purpose, and allowed students to engage with mathematic concepts for most of the class period. There were, however, some activities in the lesson that appeared a bit difficult for the students to follow. The lesson included several opportunities and/or methods for the teacher to gauge student understanding by using multiple examples. Most of the time, the teacher made clear attempts to involve all students in the lesson, and several attempts were made for formative instruction by displaying different student solutions. The teacher used appropriate digital (Smartboard) and non-digital (small whiteboards) resources for interaction and visualisation, with some missed opportunities in using the resources. The teacher frequently used questioning techniques to encourage and maintain participation. Next, we elaborate on how the enacted lesson plan, following principles of VT, provided structures to the lesson to achieve these qualities. All names referred to are synonyms.

The overall object of learning was the part-whole relationship of fractions. The object was structured in two related critical aspects, constituting the foreground of the lesson structure. One was that fractions can be ordered according to variations in the numerator when the denominator is kept constant, and the other was that fractions can be ordered according to variations in the denominator when the numerator is kept constant. These two aspects were captured in a number string of fractions (Figure 1) where the students were asked to suggest a fraction that was smaller than, but as close to 1 as possible. The number string was structured according to a pattern of contrast *within* pairs of fractions, where the denominator is given in the first fraction and the numerator is given in the second fraction in each of six pairs of fractions. The number string was also structured according to a pattern of generalisation, taking place *between* each pair of fractions. That is, the contrast between the fractions with an eight in

the denominator and an eight in the numerator was generalised in the subsequent five pairs of fractions.

Before the lesson turned into working with this number string, the teacher articulated, ”We will try to find a fraction that is as close to one as possible, and then we must first ask: What is one then when you have fractions?” Hence, she moved the lesson forward by contrasting close to one, with one. One student voiced, “One whole”, which the teacher confirmed by revoicing.

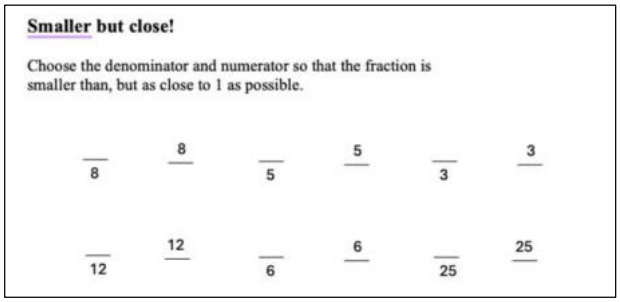

Figure 1: Number string on smaller but close

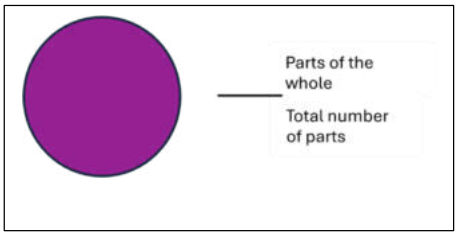

Figure 2: Circular disc (text translated)

The teacher switched the display on the smartboard to a circular disc filled in purple, together with a fraction line to the right of the circular disc (Figure 2). The teacher used the display to prompt students to investigate and make sense of how to express 1 as a fraction. The prompts were structured in a pattern of generalisation, where she asked students to add the numerator and the denominator in a pre-planned sequence of filled circular discs. The form and the size of the disc were kept constant and, so was the symbolic representation of the fraction form. What varied was the number of the equal parts a disc was divided into. This variation pattern turned the students to focus on the number of parts making up a disc and how to link this to a symbolic representation of a fraction. However, the variation was also important for making explicit to the teacher the difficulty some students had in making sense of a filled disc in the form of a fraction. After two introductory examples, the teacher showed a filled disc divided into three equal parts. She then asked the students to answer on their small whiteboards what the numerator and denominator should be. Most students answered 3/3 but, some students answered 1/3 and some 3/1. The teacher selected one board of each and showed them to the class. The selection of answers was not random but followed variation theory. The teacher used 3/1 and 1/3 to support the learning of 3/3. In other words, she helped students recognise the link between the disc and 3/3 by noting why the disc cannot be linked to either of the other two fractions. In achieving this, she utilised the possibility for visualisation on Smartboard. In practice, she pulled two of the thirds away from the disc, to prompt the students to make sense of 1/3. It was easier for students to link 1/3 to the Smartboard display, now visualising the circle sector 1/3, and, in the means of variation theory, by understanding 1/3 the students were in a better position to understand what it is not.

The teacher turned the lesson back to the number string (Figure 1) and the task to find a fraction smaller than but as close as possible to 1, given the numerator or the denominator. The teacher constantly used the former activity, making sense of a 1 in the form of a fraction, as a questioning technique, based on the pattern of contrast, to support students in discerning a fraction smaller than 1. What then contributed to the quality of the lesson was that she deviated

from the pre-planned structure of the number string. Instead of dealing with one pair at a time, the class first worked through all the fractions with the denominator being given. In terms of variation theory, instead of contrasting the fractions within each pair, she developed a pattern of generalisation along the fractions with given denominators. She kept as constant that the denominator was given through all six fractions. The values of the denominators varied. This helped the students to see the pattern that the numerator should be one number less than the denominator for the fraction to be smaller than, but as close to 1 as possible. When the last fraction of the six first fractions was done (24/25), the teacher asked; "What pattern can we see here?" Several students raised their hands and Lucas answered, "That it is one more in the denominator". The teacher revoiced, "That it is always one more in the denominator". The use of "always" can be seen as an act, adding to the pattern of generalisation. What then added to the high quality of the lesson, was that Lucas filled in the teachers' re-voicing by, "or one lesser in the numerator". This act by Lucas added to the quality in two respects. Firstly, it confirmed that several students discerned the connection between the numerator and the denominator in fractions. Second, it contributed to enacting the variation pattern fusion; two ways of discerning the pattern were enacted simultaneously.

Next, the lesson turned to the string with the numerators being given. The teacher asked, "So, how are we supposed to think here [pointing to the Smartboard]?". Peter replied, "One more". The contrast within each pair of fractions was made explicit on the Smartboard. For instance, the eight was kept constant in the first pair of fractions, while its position as a denominator and a nominator varied. The teacher completed the number string of fractions, asking different students to voice what denominator to add to each fraction. The completion of the number string ran smoothly.

Discussion

The present study shows how a teacher with low MKT (Ball et al., 2008), in a class with an average level of students' SES (Broer et al., 2019), manages to establish high instructional quality in the lesson with the support of a lesson plan that draws on VT (Marton, 2015). Teachers' MKT and students' SES are not objects of study. Instead, we refer to these two contextual factors to provide for generalisations from the case study. For instance, a case-to-case generalisation builds on the assumption that, if something worked or happened this way here, it might also work or happen in a similar situation elsewhere. Teachers' MKT and students' SES can play a key role in determining the similarities between instructional contexts. Moreover, the combination of a teacher's low MKT and students with an average SES suggests that the instruction is not reliant on exceptional conditions. This indicates that many teachers, guided by the theory of variation, could achieve similar teaching outcomes.

One of the key outcomes observed in this study is the balance between structure and flexibility provided by the VT principles. Overly structured lesson plans might inhibit teacher adaptability (Hiebert & Morris, 2012; Remillard, 2005). In the present case, the structured lesson plan gave the teacher a clear framework to follow, guiding students through critical aspects of the object of learning—fractions—while still allowing room for teacher autonomy. The teacher could respond to real-time student needs by deviating from the pre-planned sequence in meaningful ways, such as focusing on generalising patterns before addressing individual pairs of fractions.

However, the analysis also calls for further investigations on the balance between structure and flexibility, connected both to VT and to the teacher's design capacity (Remillard & Heck, 2014). We observed some students found it hard to make sense of a filled disc in the form of a fraction. In all cases, the teacher started with a filled disc. Acting flexibly, she could have tried to start with a blank disc and then fill up its parts, linked to corresponding fractions, until the entire disc was filled. On the one hand, the educational resource (the Smartboard) allowed for such flexibility. On the other hand, however, it was not part of the pre-planned sequence of tasks to start with a blank disc, which potentially restricted the teacher from making proper changes to real-time students' needs.

While the study focused on a single case, the findings offer valuable insights for broader educational practices. The analysis suggests that a lesson plan following VT can potentially reduce the relative impact of the teacher compared to teacher-level factors (e.g., MKT) and student-level factors (e.g., SES) in a classroom learning community (cf., Shechtman et al., 2010). This raises important implications for teacher education and professional development, particularly in designing resources that allow for a combination of structured guidance and instructional flexibility for all classrooms.

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Use and perception of differentiated scaffolding by primary school students involved in mathematical tasks

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In this paper, we focus on scaffolding elements provided in a sequence of four geometrical tasks to support primary school students' activity and overcome potential difficulties. Using a qualitative method, we analyze 18 3rd-grade students' performance and their answers to a survey aimed at investigating their use and perception of scaffolding elements, and their impact on the subsequent activity. In particular, we focus on the reasons preventing students from proper use of scaffolding elements, intended as a form of formative feedback. The results of the data analysis allowed us to identify four distinct categories of students' behaviours regarding the use, the perception, and the impact of the scaffolding elements on the subsequent work. These categories reflect different ways of interiorizing and translating the acquired information into action.

Keywords: Feedback, scaffolding, primary school students, task design.

Introduction

Scaffolding is a fundamental part of the teaching-learning process. Providing scaffolding means giving students suitable support to solve problems, perform tasks, or achieve formative goals beyond their unassisted efforts (Bakker et al., 2015). The effectiveness of a scaffolding action can be defined in terms of responsiveness (Koole & Elbers, 2014), i.e. based on the extent to which it benefits the receiver. This is related to the scaffolding intended as a form of formative feedback (Shute, 2008), i.e. as a piece of information clarifying the students where they are in the learning process, where they are going, and what steps should be taken to go forward. As highlighted by some research, feedback needs to be understood and actively used by students to be effective, and this is not always the case (Jonsson, 2012; Withey, 2013). Although many studies have been devoted to the effective use of scaffolding and feedback by secondary school and university students (Cusi & Telloni, 2019; Winstone et al., 2017), there is little research on this topic in primary school (Antoniou & James, 2014; Hargreaves & Flutter, 2019). This study should contribute to this lack of research, focusing on the use and perception of scaffolding by primary school students involved in mathematical tasks.

Theoretical framework

Scaffolding is described as a process where an expert supports a learner in completing a task. A particular feature of scaffolding with respect to other forms of support is that it should be progressively faded, i.e. gradually reduced and possibly eliminated so that students' competencies and autonomy increase (Bakker et al., 2015; Sharma & Hannafin, 2007). In this study, we focus on scaffolding elements we designed and provided within mathematical tasks as forms of formative feedback (Shute, 2008). In particular, the scaffolding elements are intended as triggers of

facilitative feedback, i.e. not providing an answer to the tasks, but guiding the students to organize the solving strategy. In tune with Hattie and Timperley (2007), these elements should be easily accessible but not immediately visible, to allow children to think about possible solutions before recurring help. Moreover, they should contain a limited quantity of information, preferably content-specific (Shute, 2008). Scaffolding actions and the triggered feedback can be regarded as two-way communicative events (Nicol, 2010). Consequently, their effectiveness depends on the fluency of the dialogue and, specifically, on the students' capacity to participate in it (Winstone et al., 2017), deciding when seeking scaffolding elements to support their activity. Relying on the mentioned studies about the design of scaffolding that generates effective feedback (Alessio & Telloni, 2024; Nicol, 2010; Sharma & Hannafin, 2007; Shute, 2008; Winstone et al., 2017), we structured mathematical tasks with scaffolding elements such that: a) take into account the specific difficulties afflicting the learning of the topic at stake; b) are of different kinds, content-specific and envisaged the differentiation of cues so that different learning needs could be supported; c) are easily accessible and not immediately visible; d) are student-driven, i.e. the students could receive them only if they decide to require these elements; e) contain a limited quantity of information to provide a sustainable cognitive load; f) are provided informally; g) progressively faded. Although some research highlighted a generally positive perception of scaffolding as a form of formative feedback by students, with different preferences for different kinds of feedback (Hargreaves & Flutter, 2019), unfortunately, they often do not take advantage of it. This could depend on cognitive, metacognitive, or affective factors within the learning (Jonsson, 2012; Rakoczy et al., 2013). Winstone et al. (2017) identified four main barriers preventing students from effectively using feedback: a) difficulties in decoding feedback content and purpose; b) poor knowledge of suitable strategies and theoretical elements addressed by the feedback; c) sense of disempowerment and difficulty in translating feedback in actions; d) lack of receptiveness and engagement. In particular, students often lack awareness of their difficulties and monitoring control, so meta-scaffolding is needed (Pea, 2004). Within wider research about the use of scaffolding as a form of formative feedback by students at different grades (Alessio & Telloni, 2024; Cusi & Telloni, 2019), in this study, we investigated if and how primary school students choose to use specific scaffolding elements aimed at supporting their solving process of mathematical tasks.

Context and method

The participants in the study are 18 third-grade primary school students, attending a school located in a semi-rural area in central Italy. The class has been chosen for the students' differences in learning rhythms, typical approaches to solving tasks, and their sense of self-efficacy in mathematics; indeed, all these aspects could influence the use and perception of scaffolding elements. The authors designed and developed a sequence of four geometrical tasks of increasing difficulty, provided with scaffolding elements, also called *hints*, to support the students' work, and *queries* to guide students in choosing the most suitable hint for their possible difficulty. After the students solved the sequence of tasks, they answered a semi-structured survey about their use and perception of the hints. In particular, the participants were invited to explain if and how they used the hints, what suggestions they considered more useful and what improvements they would

suggest to structure more effective feedback. Moreover, concerning perception, students were asked how the presence of potential help influenced their solving process. One of the authors was in the classroom as an observer during the activity. The specific research questions addressed in this study are: 1) Did students use the hints in case of difficulty? How did they use them?, and 2) What factors prevented students from requesting hints and/or taking advantage of the feedback they received?

The data collected are the field notes of the author who was present in the classroom, the students' answers to the survey, and their solutions to the submitted tasks. Using a qualitative approach (Cohen, 2017), the authors made a cross-analysis of the data. They individually compared the field notes with the students' productions, highlighting the moments in which the students were in difficulty, their choice of using the hints or not in those moments, the consistency between the kind of difficulty and the chosen hint, and the effects of the use of the hints on the subsequent work. Moreover, the authors individually analyzed the students' answers to the survey, identifying and coding the excerpts concerning the students' motivation for using or not the hints and their perception of the provided support. This allowed the researchers to define categories of behaviours, in tune with the model elaborated by Winstone et al. (2017), based on the factors preventing students from properly seeking, using the hints, and acting upon the related feedback. In the following we describe the design of the tasks with the hints and queries provided.

Design of the tasks and the scaffolding elements

The authors designed a sequence of four tasks to be solved in a paper and pencil environment, concerning elementary geometry, three multiple-choice and one open-ended. Each task is provided with hints addressing different aspects, i.e. a) theoretical concepts; b) strategical aspects, possibly counter-examples; c) guiding questions to foster the procedural steps. All the hints are student-driven. Indeed, they are provided by a system of folded and glued paper: the hints are glued on the back of the sheet, they remain hidden, and the paper is folded until students opt to access them. The choice to recur to hints subsumes the students' will to invest resources and time in decrypting feedback, processing information, and modifying the solving strategy accordingly. The hints are preceded by specific queries, fostering the students' monitoring of the solving process and guiding the choice of the hints, as a form of meta-scaffolding (Pea, 2004).

Design of the four tasks and description of the survey

The first task of the sequence requires students to recognize the definition of "polygon" by selecting one or more possible options: a) shape bounded with straight sides, b) shape bounded with lines, c) shape bounded with closed lines, d) shape with equal sides. The right complementary options are a) and c). Specific hints are provided, associated with the single options, and aimed to give feedback based on counter-examples. For instance, the hint associated with the first option shows a rectangle with a missing line and the question "Is this a polygon or is something missing?". The second task shows a series of five shapes and requests to find the isoperimetric ones. Two of them are the same polygon but rotated, other two were shaped differently but isoperimetric. Different hints are provided. The first one, introduced by the query "I do not remember what the perimeter is", is theoretical and gives the definition of "perimeter"; the second hint, introduced by the query "How

can I find the length of the sides?", is procedural, and shows how to find a side. The third and fourth hints are strategic and have been designed considering the possible difficulties the students could encounter. The queries "Do similar shapes have the same perimeter?" and "Can very different shapes have the same perimeter?" introduced the hints, respectively. In the former case, two similar shapes with different sizes are shown and students are encouraged to compare their perimeters by measuring their sides. The latter case, based on the same logic, shows two different shapes with the same perimeter. The third problem proposes a narrative context in which four characters determine the area of a rectangle (drawn on graph paper) using different strategies. The numerical result is the same, but three strategies are correct, and one is wrong. The first character groups the inner squares of the rectangle horizontally and then adds the obtained number as many times as the rows are. Similarly the fourth character, but grouping the squares vertically. The second character uses the classic formula $b \times h$. The third character adds the measure of the sides, using a wrong strategy. Students had to determine which options were correct and explain the solving strategy used (open-ended task). Four different hints were provided. The hint linked to the wrong resolution is based on some guiding questions that should help students identify procedural mistakes and reject the flawed approach. The hints for the correct solving processes invite students to think about how the area of the rectangle is obtainable using various groupings. The open-ended question takes the analysis to a deeper level because students had to effectively explain the method used by someone else to solve the problem. We designed this task to understand if the hints have been not just consulted but also understood and used by the students. The last task asks students to verify if the shapes presented, a rectangle and a square, are isoperimetric or equivalent. The queries are similar to the previous tasks and should guide the students in choosing the most suitable hint. According to the idea of fading scaffolding, the hints are less orienting, also to verify if the information provided with the previous scaffolding elements was kept in mind by students after completing the task.

In the survey the first questions are: 1) "What do you think of the opportunity of using suggestions to solve problems, and 2) "You have used them because…". They should allow us to understand if students consider the hints useful, to what aim they decided to use them, and if the feedback provided made them change their strategies. The other questions are oriented to identify the barriers students encountered in understanding and using feedback, as well as their perception of the effectiveness of the provided hints. The third question is open-ended: 3) "Which problem do you think was the most difficult?". The fourth 4) "For that problem, you would have had…" is a closed question that gives students three possible options: a) a simpler explanation, as I didn't understand the hint; b) more information: the hint was useful, but after reading it I still didn't know what to do; c) no changes: the hint was useful.

Results and discussion

In this section we report the frequencies of students' correct answers to the tasks and their uses of the hints, relying on the field notes of the researcher present in the classroom and the students' productions. For the first task, about 67% of the students found the solution and 50% of them used hints; 22% chose only one of the two correct answers, half of whom used all the available hints; 11% did not solve the problem, despite having consulted all the available hints. Solving the second

task, about 22% of the students identified the correct solutions, 75% of them used the hints; about 66% identified only one couple of isoperimetric shapes, 54% of them recurred to hints; finally, 11% did not find the right answers, 67% of them consulted the hints. Both groups of students, those who could not solve the problem and those who found the correct answer, used theoretical hints. In the third task, all the students identified at least one correct strategy, but only 22% of them were able to motivate their answers; 75% of these students used the hints and answered the guiding questions aimed at reconstructing the resolution strategy. About 28% of the sample gave a partially correct explanation, the totality of them recurred to hints. For the fourth task, all the students identified the correct answer, and only 22% of them used hints. The theoretical hints were the most used, by 75% of students, whilst 25% of students used the procedural ones. This last problem was the most complex because students had to apply the concepts of area and perimeter and also compare the data. Instead, only a few students required hints. This can be due to the fact that the theoretical references needed for the solution were also provided in the previous problems, and students were able to transfer them. The analysis of the collected data allowed us to identify four categories of students' behaviours for their use and perception of the scaffolding elements and the impact of the related feedback on the subsequent work. This allowed us to answer our research questions.

Effective use of scaffolding elements

About 17% of student**s** suitably required the hints, took advantage of the related feedback, and applied the suggestions provided to an effective solving strategy. These students did not make a generic use of hints but sought them according to the specific difficulty encountered. Moreover, they were able to accommodate new information in their strategies. An example is L.: he provided correct answers to the first, second, and fourth problems without using hints; solving the third problem, he fruitfully used all the hints and correctly explained all the strategies to find the area of the rectangle. In the survey, L. declared that the third problem was the most difficult and that he used the hints "*to solve doubts about the resolution method used by the characters and to understand their reasoning*". Students whose behaviour is in this category are high-achievers. They maximized the efficacy of hints and feedback since they exploited the new information to (re)organize their solving process.

Immediate use of scaffolding elements without generalization

44% of students made effective use of feedback, but only at an immediate level, without taking advantage of the received information for the subsequent tasks. These students are those who typically perform correctly standard tasks, but struggle with non-routine ones. As an example, we present the case of M. Solving the third task, M. properly motivated the first strategy, after using the hint, arguing that "*[the character] formed groups vertically and carried out the operation*", but was not able to extend the same strategy to other answers. Indeed, M. did not recur to other hints, but he was not able to correctly explain the other correct strategies. As in Winstone et al. (2017), M. seemed to exploit the feedback to overcome the procedural difficulty, rather than grasping its long-term potential. This suggests that the activation of M. remained blocked at the level of recognizing his learning needs. Although the feedback was useful and exploited to give the correct answer, M.

struggled to apply it to other cases. With respect to the research question (2), the factors preventing M.'s effective use of hints can be linked to a lack of active thinking and self-regulation in monitoring the process and the results provided. Coherently, M. declared in the survey that "*the hints were useful to give the correct answer"*, highlighting the immediate effect she expected.

Using scaffolding elements ineffectively

About 28% of students identified specific difficulties encountered in solving the tasks and decided to recur to hints. However, they were unable to use the information obtained to reorganize their solving process. In these cases, the consulted hints were not understood or were not effectively applied to a resolution strategy. Specifically, investigating the factors preventing the proper use of the hints, it emerged that "*the hints were not clear*". C. is a student whose behaviour is in this category; she is not familiar with the Italian language. She used all the hints provided in all the tasks, but in the first and the third ones she was not able to transfer the information obtained into an effective solving strategy. In the survey, C. required a simpler language. In tune with Winstone et al. (2017, p. 2030), a barrier to the effective use of feedback can be linked to "*decoding the feedback and the academic jargon*". For students whose behaviour is in this category, who are low-achievers, more orienting feedback could be useful, based on binding guiding questions that enable them to reconstruct the process and transfer the hint to an actual strategy.

Recognising difficulties but refusing to use feedback

About 11% of students chose to not use feedback, even if they often required additional help from the teacher when they faced the tasks. Our analysis of the factors preventing the students' use of hints reveals that this resistance could be linked to the fact that feedback is expected as not helpful, and students do not want to invest cognitive resources. This behaviour was displayed, among the others, by some students with a high self-efficacy sense. This behaviour is in tune with Winstone et al. (2017, p. 2038), i.e. "students' agency in using feedback can be impeded by a sense of helplessness, and by unrealistic expectations about how apparent and immediate the results of their efforts should be". As an example, E. did not use the hints, even if he asked for the teacher's advice to solve the tasks. When the teacher suggested he refer to the provided hints, he refused. In the survey, he declared "*I did not use any hints, I did not need them, I was not in difficulty"*. E.'s choice could also have been influenced by the "social cost" of seeking help, in fact as in Hattie and Timperley (2017, p. 96), "many students do not seek help because of perceived threats to self-esteem or social embarrassment". In this respect, we noticed that E. was sitting next to a competent classmate who did not need to use scaffolding elements during the test. Consequently, as in Winstone et al. (2017, p. 2037), "social influence or distraction associated with collecting feedback in public can prevent them engaging" and discourage students from using feedback even if it is needed.

Conclusive discussion

Taking responsibility for using feedback effectively is a fundamental skill that supports the development of self-regulation (Nicol & Macfarlane-Dick, 2006). In this paper, most of the participants understood that the scaffolding elements and the provided feedback are intended to help

them, but effective results can be gained only by acting upon the feedback. The analysis of the student's use of provided scaffolding allowed us to identify four categories of behaviours, corresponding to different levels of awareness and ways to convert information into action. These categories, especially the last three, are related to Winstone et al.'s (2017) classification of barriers preventing effective use of feedback. Students in the first category understood when it was necessary to recur to hints, converted the information obtained into an effective solving strategy, and acted upon the feedback. They showed awareness and mastery of the importance of feedback and its proper use. Students in the second category recognized their difficulty and consequently chose to recur to hints. However, their use of feedback remained blocked at the immediate level and was oriented to overcome a contingent difficulty. They immediately transferred the feedback into action, without extending the information to other similar cases. In tune with Winstone et al. (2017), this behaviour is linked to poor knowledge of appropriate strategies. Students in the third category consulted the hints but were not able to properly use the information obtained and translate it into action. In tune with Winstone et al. (2017), this behaviour reveals a lack of understanding of the jargon used, and of awareness of the potential usefulness of feedback. Students whose behaviour is in the last category refused to use the hints even perceiving to be in difficulty. This behaviour can be linked to a sense of "learned helplessness", by an unrealistic expectation of how the results of feedback should be, or due to social pressure. Notwithstanding the small number of participants, the results suggest that the challenge is to enable the students to take advantage of the feedback. Therefore, a potential redesign of the tasks or the methodology could consist of making feedback more accessible and communicative for students, helping them to transform information into strategies and overcome the sense of learned helplessness and unrealistic expectations of immediate pay-offs (Winstone et al., 2017). Further research is needed to deepen how to design effective scaffolding elements and improve students' sensitivity to their key role in fostering learning.

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Design principles for epistemological *dépaysement* in pre-service teachers focused on interdisciplinarity mathematics and physics

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In this theoretical paper, we develop a conceptual framework for promoting epistemological dépaysement in pre-service teacher education activities focused on interdisciplinarity between mathematics and physics. This framework is then operationalized by providing design principles for tasks. Finally, we present a specific example of a task involving textbook analysis to show how this approach might work in practice.

Keywords: Interdisciplinary approach, preservice teacher education, teacher expertise, textbooks.

Focus and rationale of the paper

This theoretical paper concerns the design of tasks for mathematics teacher education in the perspective of interdisciplinarity between mathematics and physics.

This study contributes to the body of research on the use of tasks in mathematics teacher education. Such tasks may vary widely in terms of learning goals and content. For instance, Zaslavsky (2007) discusses different uses of mathematical tasks in mathematics teacher education, where teachers may: perform the mathematical activity; analyze, compare, or create tasks for students; and analyze students' solutions to mathematical tasks. Other types of tasks promote teachers' reflections on the teaching practice and their interaction with peers and tasks, to highlight connections within mathematics, between mathematics and everyday experience, or between their own experiences as learners and teachers. Cusi and Morselli (2022), for instance, provide examples of narrative tasks where pre-service teachers, after being provided with theoretical tools concerning teaching-learning processes and the roles played by the teacher during classroom discussions, design tasks for students and create fictional classroom discussions focused on the same tasks, by referring to the theoretical tools. The comparison among peers and the discussions led by the teacher educator aims at promoting a change of perspective, from university students to future teachers, and at fostering the appreciation of theoretical lenses not just as a support for creating discussions but also as relevant guidelines for their future practice as teachers.

In our case, the learning goal is to develop expertise in an interdisciplinary perspective (Akkerman & Bakker, 2011; Erduran & Dagher, 2014). Numerous studies on interdisciplinarity emphasize the importance of becoming aware of perspectives from other disciplines (e.g., see the literature reviewed in Satanassi et al., 2023), which requires the ability to critically reflect on and discuss one's own perspective without treating it as absolute. Accordingly, we look for tasks that promote such a reflection, as we consider it part of the expertise. We draw the idea of *epistemological dépaysement*

from research on the use of history of mathematics and we study whether it is possible to exploit *dépaysement* in the not yet explored field of interdisciplinarity. Originally, the concept refers to a "cultural shock" (Jahnke et al., 2002, p. 293) that prompts critical reflection on one's epistemological assumptions about mathematics, which often remain implicit. Making these assumptions explicit leads to heightened awareness, which is crucial for effective mathematics teaching, especially in pre-service teacher education. We argue that tasks that promote epistemological *dépaysement* can create valuable learning opportunities for teachers and in turn develop their expertise, especially in the perspective of interdisciplinarity. Therefore, designing such tasks becomes a pressing priority. Our research question is: What design principles can be identified for teacher education tasks that promote epistemological *dépaysement* concerning interdisciplinarity between mathematics and physics?

We address the research question by grounding in studies on interdisciplinarity, teacher education as developing expertise, and the use of the history of mathematics in teacher education. In the subsequent paragraphs, we sketch a conceptual framework that frames *dépaysement* in interdisciplinarity and propose design principles for teacher education tasks aimed at promoting such *dépaysement*. We then propose a task that exploits textbook analysis as an instantiation of these principles. Textbooks are often recognized as instruments of change for practice (Rezat et al., 2021). In our case, we explore their potential are carriers of *dépaysement*, when inserted in carefully designed tasks.

Interdisciplinarity between mathematics and physics

Interdisciplinarity between mathematics and physics is a very broad and complex concept that was addressed by research with several different approaches. In this paper, we consider a framework for interdisciplinarity developed within the European Erasmus+ project IDENTITIES (n. 2019-1IT02-KA203-063184) that encompasses the Family Resemblance Approach (FRA; Erduran & Dagher, 2014), and the constructs of boundary, boundary objects and boundary people (Akkerman & Bakker, 2011). Specifically, "a *boundary* can be seen as a sociocultural difference leading to discontinuity in action or interaction" (p. 133). The two sides of a boundary can be linked through artifacts known as *boundary objects*. Additionally, *boundary people* are individuals who identify as members of a community and are recognized as such but whose knowledge, practices, and interests possibly extend beyond that community. These individuals can cross outside the boundary and introduce new languages and challenges. Inspired by Akkerman and Bakker, Pollani et al. (2024) describe *discipline-based communities* as communities characterized by expertise in a discipline (e.g., scientists or mathematics teacher educators). Coherently, they describe the *boundary between disciplines* as an encounter where members of a discipline-based community interact with (boundary) objects (e.g., physics textbooks for mathematics pre-service teachers, as in Pollani & Branchetti, 2022). The paradoxical nature of boundary that simultaneously belongs to both sides and neither carries not only ambiguities and risks but also opportunities. Pollani and Branchetti (2022) reflect on the learning potential of boundaries for increasing awareness by introducing the construct of disciplinary or interdisciplinary learning potential.

To reflect on the single disciplines not by separating what belongs to one discipline and what does not, Satanassi et al. (2023) propose adopting the FRA (Erduran & Dagher, 2014). In the FRA, the *epistemic core of a discipline* consists of *aims and values* (such as abstraction, objectivity,

consistency, etc.), *practices* (e.g., collecting data, making experiments, proving, etc.), *methods and methodological rules* (like inductive/deductive reasoning), and *knowledge* (like the laws of dynamics, the theorem of mean value, etc.). One common assumption is that interdisciplinarity is not merely the additive juxtaposition of disciplines. When disciplinary knowledge is analyzed critically considering all these dimensions, often it is possible to find differences in the way the same problem is addressed by different members of the same community. This heterogeneity might concern any component of the epistemic core of the disciplines and is at the basis of the scientific debates. To turn this richness into a productive complexity in education, it is necessary to expose learners to different voices and ask them to position themselves critically in such debates. Levrini et al. (2015) introduce two design principles for teaching aimed at fostering appropriation of thermodynamics and promote critical thinking at the epistemological level: *multi-perspectiveness* asserts that the same physics content should be analyzed from two different perspectives (the macroscopic and microscopic approaches to thermodynamics), each one characterized by a specific approach to the content. *Multidimensionality* asserts that the two approaches should be analyzed and compared according to different dimensions involved in physics (i.e., for their conceptual, experimental, and formal implications) but also for their philosophical-epistemological peculiarities. These principles were exploited also in the teaching of other physical topics and to trigger boundary crossing mechanisms in interdisciplinary activities involving mathematics and physics pre-service teachers (Satanassi et al., 2023).

Teacher education as developing teacher expertise

Teacher education refers to the process of fostering the transition from novice to expert, as described by Mason (2002), who states that "The mark of an expert is that they are sensitised to notice things which novices overlook" (p. 1). According to Mason, "noticing is triggered by a disturbance. […] It involves making a distinction between foreground and background, stressing some features and consequently ignoring others" (p. 148). Disturbances can manifest in various forms, such as "a surprise remark in a lesson, a particularly poor showing on a test, something said by a colleague, something asserted in a journal or book, or a moment of insight. Whatever it is, something startles me out of my current habits" (p. 10). Mason emphasizes the importance of disturbance in fostering teacher noticing, which, in turn, develops teacher expertise. He also suggests various ways to provoke disturbances, one example involves books. According to Viennot and Décamp (2018), a key characteristic of expert teachers is their engagement in intellectual dynamics in first person and their ability to critically discuss the knowledge at hand. They also highlight the value of examining this characteristic when working with textual resources designed to explain concepts, such as textbooks. Furthermore, Viennot (2001) introduces the notion of *critical detail* to describe aspects that have the potential to trigger, and deserve, critical discussion. From Viennot and Décamp's (2018) perspective, recognizing and thoroughly analyzing such critical details in textual resources can support the development of teacher expertise. In our opinion, teachers can be disturbed by various aspects they consider critical. However, we argue that an interesting kind of disturbance for developing an expertise in an interdisciplinary perspective would involve epistemological issues. Indeed, interdisciplinary teaching requires teachers, that in principle have a strong preparation on one discipline, to be prepared also to address epistemological issues emerging within the disciplines and at the boundary between different disciplinary epistemologies.

Epistemological *dépaysement* through historical sources

The concept of epistemological "*dépaysement*" emerges in research on the integration of the history of mathematics in mathematics education and in mathematics teacher education (Jahnke et al., 2002). It is a widely accepted hypothesis that reading historical texts allows for a critical examination of epistemological certainties and challenges the image of mathematics as a fixed, universal, and eternal discipline (Barbin, 1997). History is "dépaysante" because it provokes "étonnement" (Barbin, 1997, p. 21). This happens "through making the familiar unfamiliar" (Jahnke et al., 2002, p. 292). The need to engage with history is fulfilled by reading historical texts in a suitable, non-teleological way. These texts, through the intervention of a distance or difference, either cultural or geographical, between the readers and authors – both temporal and spatial – create a "cultural shock" (Jahnke et al., 2002, p. 293). This shock has the effect of altering the image of mathematics and reshaping the relationship that students and teachers may have with mathematical knowledge (Barbin, 1997). The epistemological *dépaysement* can then be interpreted as a kind of disturbance in the sense of Mason since it results in startling out of one own's current habits. In addition, it is intrinsically epistemological. Although this concept has been extensively discussed, it has not yet been clearly operationalized. Moreover, its application in interdisciplinary research and its actual role in the teacher education remain unexplored.

Conceptual framework for epistemological *dépaysement*

We propose conceptualizing the term *dépaysement* in teacher education as a complex process that involves multiple phases and agents. The primary agent in this process is the teacher experiencing it, in our case, the pre-service teacher. Based on the literature on teacher education and on epistemological *dépaysement* in history of mathematics, this process can be understood as experiencing a disturbance (Mason, 2002) or as being metaphorically placed in a foreign and unfamiliar country (Jahnke et al., 2002). From an etymological point of view, the French term "dépaysement" is formed by three components: a prefix, a root and a suffix. The prefix "de-" gives the meaning of removal, departure or separation. The root "pays" means "country", and the suffix "-ment" to form the substantive. To further clarify and unpack this concept, we provide a metaphor that helps illustrate the dynamics more clearly. In the context of electricity, imagine two plates with a potential difference generating an electric field. Not all objects within this field experience an electric force – electrically neutral objects, for instance, will not. In this situation, we can identify a *prompt for dépaysement* which is the electric field. Its potentiality for making experience *dépaysement* from one own's country can be discussed a priori. Using Mason's words, the country represents the "background" to which the disturbance is created. In this metaphorical sense, "country" can also represent one's beliefs, assumptions, habits, or previous experiences, even those of which one may be unaware. However, this is just a potential, and the actual experience depends on the background hold by the pre-service teachers. On the other hand, there could be observable behaviours due to the prompts, such as feelings of discomfort, wonder, confusion, withdrawal, questioning, vertigo, or a sense of the abyss. This distinction serves the purpose of refining our research aim in this paper, which concerns the *prompts for dépaysement*, since we are interested in the features of tasks to promote *dépaysement*, that is to make one leave his country.

The FRA helps clarifying the concept of *epistemological* prompt for *dépaysement* in this context. Specifically, a prompt for *dépaysement* is *epistemological* in nature within interdisciplinarity between mathematics and physics when the force driving one away from their own country is related to epistemological aspects. These aspects can pertain to the aims and values, methods, practices, or knowledge of a discipline. For instance, the way a proof is presented (or not presented), which may reflect the methods and practices of mathematics, can serve as a prompt for *dépaysement*.

Design principles for tasks that promote epistemological *dépaysement*

We draw the following design principles for tasks that promote epistemological *dépaysement*, which acts as instruction to generate the electric field but without guaranteeing any change, even unstable: P1. Set up a task where multiple epistemologically different voices are presented. These voices can come from various resources, as well as from discussions with individuals who have different educational backgrounds (Levrini et al., 2015). This aligns with Mason (2002)'s notion of creating a background and a foreground. In this case, they are between one own's familiar country and the incoming voices. The space between these two creates a distance or boundary, which may prompt *dépaysement* and carry a learning potential. The differences between these voices should reflect epistemological differences that prompt epistemological *dépaysement*, the feeling of unfamiliarity (Jahnke et al., 2002). This idea draws from Akkerman and Bakker's work, as well as the FRA framework. P2. Formulate the task requirement to foster reflection, encouraging pre-service teachers to write or discuss about how they interpret the content they are working on, and eventually articulate and explain their ways of thinking. This principle is grounded on the work of Viennot and Décamp (2018), which emphasizes the importance of critically discussing the knowledge at stake. This principle is linked with the previous one, as it is expected that the tension between the background and foreground each pre-service teacher experiences may foster reflection. P3. Encourage discussion, interactions and reactions to what other pre-service teachers but also teacher educators say. This principle derives from conceptualizing interactions as crucial for creating the boundary between disciplines, but also for reinforcing the disturbance. This is in line with Mason's recommendation that the disturbance is triggered by "something said by a colleague" (Mason, 2002, p. 10). Discussion, interactions and reactions are also in line with the idea of critically discussing the knowledge (Viennot & Décamp, 2018) rather accepting quick and easy answers as absolute.

Applying design principles: description of an example of task

In this section, we present a task for pre-service mathematics teacher education, which represents an instantiation of the previously established design principles (P1-P3) for including prompts for pre-service teachers epistemological *dépaysement* within interdisciplinarity. As previously outlined, the task exploits textbooks as resources. The resources are two excerpts from secondary school physics textbooks, both addressing the motion of projectiles (Ruffo, 2014, pp. 96–97; Walker, 2010, pp. 79–80; translated excerpts here https://shorturl.at/NnBE0). Pre-service teachers are asked to reflect first individually, then in small groups, and finally in a whole-class discussion, which includes teacher educators. The written assignment is as follows: "Read the two excerpts provided and then try to answer the following questions: 1. Is there any aspect that stands out for you? Why? 2. Which one of

the texts would you use in your class? Why? 3. Which one of the texts do you feel most comfortable with? Why?".

In alignment with the design principle of including reflective tasks (P2), three questions are encompassed. The first one prompts pre-service teachers to critically reflect on aspects that "stand out" to them. The use of this phrasal verb recalls the idea of a dynamics between background and foreground, as described by Mason (2002). This is an application of P1. Questions 2 and 3 encourage reflections related to a sense of familiarity, to the background. It is important to note that our goal is not to infer pre-service teachers' beliefs, although we acknowledge that these are part of the background from which we promote *dépaysement* and increased awareness. The additional request to explain their answers is intended to make pre-service teachers more aware of their own perspectives. We expect that this heightening in awareness will also be fostered through the collective discussions, both in small group and then as a whole class, aligning with P3.

Also, the choice of the topic is in line with P1. The motion of projectiles, indeed, was chosen due to its relevance in the co-evolution of mathematics and physics (see for example, Bagaglini et al., 2021; Branchetti et al., 2023). According to Bagaglini et al. (2021), physics textbooks bring an epistemological perspective on mathematics and physics, which often remains implicit. In our case, the two textbooks bring two epistemological perspectives (for more details, see Bagaglini et al., 2021), as already explored in previous works (see Bagaglini et al., 2021; Branchetti et al., 2023; Pollani et al., 2024). Bagaglini et al. (2021) compare Walker (2010) to other textbooks and underline that it is epistemologically richer because it more closely aligns with the historically development of the two disciplines. Once selected the excerpt from Walker, we sought another resource that offers a voice epistemologically distinct from Walker's. In line with P1, Ruffo (2014) was selected because it contrasts with Walker from an epistemological point of view. This contrast exposes pre-service teachers to multiple ways of presenting the same scientific concept. Both Branchetti et al. (2023) and Pollani et al. (2024) explore epistemological differences that may emerge as critical details and as prompts for *dépaysement*. The explored details are related to the use of examples and of representations. The contrasting use of these elements in the two excerpts is a prompt to creating an environment where pre-service teachers can experience epistemological *dépaysement*. Hence, the two excerpts represent two voices, as advocated by P1.

The choice of proposing two excerpts from two textbooks is also functional to promote *dépaysement* in *any* pre-service teacher. Indeed, we must consider that each pre-service teacher has an own voice; it is possible that their voices align with Walker's, and this fact may not create the desired epistemological tension needed for *dépaysement*. If a pre-service teacher's voice perfectly aligns with the text's voice it becomes harder to experience *dépaysement*, as pre-service teachers' current habits (Mason, 2002) or what they consider familiar (Jahnke et al., 2002) are not challenged. The choice of proposing two excerpts, then, is also functional to maximize the possibility of creating such tension and thus promoting *dépaysement*.

Discussion and conclusions

This contribution concerns developing expertise in an interdisciplinary perspective between mathematics and physics as a learning goal for mathematics teacher education. We argued that this

expertise can be developed by promoting *dépaysement* in the perspective of interdisciplinarity. We proposed task design principles for the aim of promoting such *dépaysement* and identified the construct of the epistemological prompt for *dépaysement* to fulfill this aim. Such elements of the task may provide experiences of disturbance, that may enhance teacher noticing, which in turn develops teacher expertise by shifting from novice towards expert, as advocated by Mason (2002). However, we do not expect a single experience of *dépaysement*, such as the task described above, to transform pre-service teachers into experts by itself. Instead, we believe that teacher education aimed at promoting interdisciplinarity should provide multiple opportunities for pre-service teachers to experience *dépaysement*, enabling them to become aware of and critically examine their perspectives. Unfortunately, this process is far from linear or stable. We are interested in how to identify and describe events in which a *dépaysement* happens. Further research is needed to analyze the factors that either stabilize a new perspective or lead to a regression to previous ones. We are currently reflecting on what evidence can inform the design and the effective implementation.

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Digital game-based mathematics curriculum resources: Methodological reflections on collaborative prototyping

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In the TransEET project, experts from various fields collaborate to design STEAM curricular resources. This collaboration is built on the exchange of prototypes within communities whose members possess diverse expertise. This paper reflects on the collaborative design process by analyzing two game-based prototypes, Carrots and Radishes and Geometry Deconstruction, showcasing their evolution through feedback and revisions. The aim is to develop a methodological tool that tracks the impact of the collaborative prototyping process on the prototypes' evolution. Since the prototypes are digital games, their evolution will be demonstrated by analyzing different game layers through a game-based framework.

Keywords: Games, prototypes, community of interest, collaborative design processes.

Introduction

Collaborative work between various stakeholders and communities in the context of mathematics task design has garnered increasing attention (Margolinas, 2013; Olsher et al., 2023). However, less focus has been placed on studying collaboration within heterogeneous teams working to develop innovative educational curricular resources, particularly involving the use of existing and emerging technologies, to drive dynamic educational transformation that meets the needs of the 21st century. The increasing attention on interdisciplinarity and transdisciplinarity and the always more sophisticated digital competencies requested bring to the fore the need for professionals with different expertise and backgrounds, and so belonging to different communities of practice (Wenger et al., 2002), to work together. Such a process should also foster the creativity of the actors involved (Essonnier et al., 2024). Even though the common interest may be clear and well defined, e.g. the design and development of a digital prototype, challenges emerge, both concerning the interlacing of different practices and the necessity to communicate and understand each other when the specific terminology used may not be understandable to the co-workers.

One of the main objectives of the project of the European Research Executive Agency 'TransEET' (Transforming Education with Emerging Technologies), is to investigate this collaborative work and suggest a methodology to approach it. Indeed, in TransEET (https://transeet.eu/), individuals with diverse expertise and academic backgrounds—including mathematics education, computer science, and arts education—from universities in Greece, Italy, the Netherlands, Austria, and the UK have formed *Communities of Interest* (CoI) (Fischer, 2001; Kynigos & Daskolia, 2021). These CoIs collaboratively co-design digital prototypes that support the integration of STEAM disciplines

(Science, Technology, Engineering, Art and Mathematics). This paper reports on the collaborative development process of game-based prototypes with mathematical content, which were co-designed in a CoI around digital games in STEAM, as a case study. Several studies, such as Byun and Joung (2018), highlight the effectiveness of digital games in mathematics education by demonstrating their potential to foster students' interest, motivate goal-setting, and encourage persistence through goal-oriented activities. In addition, games also provide opportunities for students to make critical decisions in order to solve problems, which can lead to the emergence of problem-solving strategies and process skills such as imagination and creativity

The collaborative development process takes place within the aforementioned CoI, which is composed of mathematics educators, game designers, and technology experts. The process is shaped by key elements of the TransEET project, which are briefly described in the following sections. The first two sections present the working methodology within TransEET, focusing on prototype production. The subsequent section outlines the methodology used to analyze the prototypes' evolution. Then an exemplification of its application is applied to two game prototypes, whose different versions are documented by TransEET members. Drawing on the TransEET experience, the conclusions share methodological reflections on the design process of educational games.

The TransEET project and its Co.F.E. method

The TransEET researchers and practitioners belonging to different communities of practice are engaged in CoIs (Fischer, 2001), each one of them focused on a specific and shared thematic interest. Members of CoIs are engaged in a *Flow of Rapid Prototypes* (FRP), a continuous process of creation and exchange of prototypes inside a CoI. According to existing literature, typically developed outside Mathematics Education (e.g., Syafril et al., 2021), the term *prototype* refers to the digital artefact itself, while the term *prototyping* is used to refer to the process of its development. Within the CoIs such a process develops according to the well-established phases of prototyping which are: design and development of a prototype, feedback and evaluation of the prototype, and redesign. Getting feedback during the development process, before waiting to have the final product, typically allows the designers to save time and energy. In this case, the goal of the FRP is not to have perfect digital tools, but to stimulate the process of creation and the creativity of the members involved (Essonier et al., 2023). Therefore, the term *rapid* in the acronym FRP remarks that such prototypes should not be considered as finished tools, but as artifacts with enough technical readiness to be shared, discussed, and exchanged within a CoI. While engaging in such a productive and creative process, the members of a CoI interact with other members of the CoI and develop and acquire new skills external to their initial background. We may say that they evolve into *actors with hybrid Expertise*.
The three aspects described so far, namely CoIs, FRP, and actors with hybrid Expertise, constitute the pillars of the working methodology within the TransEET project: the so-called Co.F.E method. In the present study we focus on the FRP, in particular on the flow of evolution of two prototypes.

FRP within TransEET

The FRP within a CoI is expressed through the exchange of digital prototypes accompanied by a document called *Prototyping Design (PD) document*, which contains a description of the prototype

elaborated by the designers/developers, as well as its reviews produced by other members of the CoIs. If the format allows it, the digital prototypes are collected in a shared Google Drive folder, otherwise exemplary videos or codes are shared within the community. In both cases, the prototypes are presented and discussed by their designers during the CoI meetings and project meetings. Then they are tried out and reviewed by 1-10 TransEET members who volunteer to engage in such a reviewing process. This leads to the production of 2 to 4 reviews of the prototype. Reviewers can have access to the prototype's code in certain cases (in this case the prototype is called *authorable,* meaning that anyone can modify it and generate a new version) while other prototypes require more technical knowledge to work on the code (hence the prototype are only *partially authorable*). Most prototypes within the CoI are open source to foster accessibility and collaboration. The reviews are strongly influenced by the reviewers' expertise hence different reviewers can focus on different aspects of the prototype. Examples of expertise within the CoI are mathematics, education, game design, game based learning and software engineering. The reviews are strongly influenced by the reviewers' expertise hence different reviewers can focus on different aspects of the prototype. Reviews are written into the PD document, based on which the designers can prepare a new version of the prototype, which triggers a new review within a new PD document and eventually a new version of the prototype. Each PD document refers to a single version of the prototype. The collection of different PD documents referring to different versions of the same initial prototype allows us to follow its evolution from different points of view. More in detail, the PD document is structured in two main parts as follows: 1) the *presentation* of the prototype and 2) all its subsequent *reviews*.

The *presentation* of the prototype is divided into four subsections containing the identikit of the prototype, its detailed description, an example of an educational activity and some screenshots of the prototype. The *reviews* of the prototype contain strengths and weaknesses of the prototype, possible extensions or changes, an example of activity flow with the modified version of the prototype and possible integration of emerging technologies if not yet present. The names of the designers and reviewers are also recorded in the PD documents.

Method

The PD documents produced during the FRP process track the design changes and evolution of the prototypes. To investigate this evolution, we qualitatively analyzed the data contained in these PD documents using a deductive approach. This involved identifying aspects related to three main categories considered in the PD documents: educational, technological, and mathematical aspects. We then observed how the FRP process influenced the evolution of the prototypes. Since the prototypes discussed in this paper are digital games, we employed a framework from game design literature to analyze the evolution of game-related elements within the three categories mentioned above. The framework, known as MDA (Hunicke et al., 2004), consists of three elements. The mechanics (M) of a game refer to its fundamental components, such as rules, goals, and players' available actions. The dynamics (D) refer to the behaviors that emerge from applying these mechanics and player interactions during gameplay. The aesthetics (A) involve the player's subjective experience, including elements like emotion, enjoyment, sensation, fantasy, narrative, challenge, fellowship, discovery, expression, and submission. In our case, aesthetics will include also and in a primary way the player's mathematical experience, considered as a subjective experience. When a

game is played, the mechanics generate its dynamics, which in turn evoke a particular aesthetic. These three aspects should be considered together and do not follow a specific order.

In this paper, we apply the MDA framework to analyze the features of two prototypes and their evolution throughout the FRP process, as documented in the PD records. Our research interest lies at a methodological level, specifically in the use of PD documents as tools to support the collaborative design of mathematics digital game prototypes. The prototypes, described below, were selected for their contrasting characteristics, representing the variety of prototypes that emerged within the CoI. Their FRP processes are marked by an initial version, at least one review, and a second version. For both prototypes, we collected the digital files and corresponding PD documents for each version.

Two examples of evolution of game-based prototypes

Prototype 1: Carrots and Radishes

Carrots and Radishes (CR) is a game meant to stimulate the learning of multiplicative relations between quantities (specifically, those of carrots and radishes).

Figure 1: On the left: first version of CR prototype. On the right: an example of falling object

CR has been developed by means of *SorBET* (Sorting Based on Educational Technology)[1], a digital tool that enables the development of decision-making games in which players have to classify objects under time-critical conditions. These games help develop essential cognitive skills such as set operations, class encapsulation, and property discrimination, making them versatile tools across any learning context. The content of these games can be tailored to various domains, and they are supported by fully authorable tools that allow educators to modify existing game structures or create new ones from scratch. In CR, a game co-developed by the first and third author, figures representing a certain number of carrots and a certain number of radishes fall on the screen (an example is provided in Figure 1, right side) and must be moved to boxes labelled as: one third, half, twice, or three times, depending on how they are depicted in the various images (Figure 1, left side). The game is connected with mathematics representations, in particular, it requires conversions between different registers of representations, in the sense of Duval (2017), and specifically from graphical representations of quantities to verbal expressions of the relations between the same quantities. Boxes and pictures have been chosen in a way that the same relation between carrots and radishes can be described through a multiplicative relation or a ratio.

[1] http://etl.ppp.uoa.gr/sorbet/

The player's *aesthetic* experience is influenced by the challenge of moving the falling objects aligned with the boxes representing the object's category, so that it can fall inside. The *dynamics* of the game are fast-paced: players feel the time pressure in decision making because they lose a point if the object does not fall in the correct box. SorBET allows the user to modify the time of falling objects by acting on the setting hence the teacher (or player) can diminish the time pressure and therefore also the challenge. Moreover it is possible to modify the falling objects increasing or decreasing the mathematical challenge. The core game *mechanics* in CR is ordering the objects; other mechanics are: objects are falling from the sky, the player needs to drag the objects in the direction of one box; there is a limited amount of time to drag an object. The falling objects the student sees are randomly picked from a set of pictures uploaded by the designer. The labels of the boxes are chosen by the designer. Advanced players who recognize that the same object can belong to more than one category can randomly choose between the boxes. At the end of the entire game, the choices made during that play can be shown to players.

The following table reports some comments made by the CoI members within the prototyping PD document and the analysis using the MDA framework (text in italic). The comments are organised according to the three rows: mathematical, technological and educational aspects.

Table 1: Analysis of the FRP of Carrots and Radishes prototype

CR	Version 1	Review 1	Version 2
Mathematical aspects	The relations among quantities in the falling objects is expressed through deployed radishes and carrots. The labels of boxes express the multiple/ratio relation in words.	"The use of mathematical symbolism is missing (fractions, decimals and percentages)."	*Aesthetics: use of mathematical language and symbols to express the multiple and ratio/relation.* *Dynamic: some representations express the same relation adopting different registers*
Technological aspects	Move the figures to place them in the correct box.	"We could create different levels to strengthen the game approach. With AR or VR students could move images with their hands."	*The reviewer's comments did not provoke any change in the dynamic (levels of difficulty) and in the aesthetic (moving images using hands)*
Educational aspects	Ratio and multiple of quantities expressed as quantities of vegetables (carrots and radishes).	"We could change/add the boxes to display more relations."	*The reviewer's comments did not provoke any change in aesthetics of the prototype.*

Regarding the *mathematical aspect*, Review 1 suggested a change in the prototype that influences the game's *aesthetics*, as it affects the mathematical experience students can have while playing.

Specifically, the suggestion involves introducing mathematical language and symbols into the prototype. The revision of *technical aspects* impacts the *dynamic* of the game, recommending the addition of different levels of difficulty and the *aesthetic* aspect, as the mathematical experience changes when students move objects with their hands instead of dragging them with a mouse. Finally, the revision of the *educational aspects* is meant to the game's *dynamics* by encouraging players to think about multiple relationships at the same time.

Prototype 2: Geometry Deconstruction

Geometry Deconstruction (GD) is an educational Augmented Reality (AR) game prototype designed to facilitate the learning of surface area and volume formulas at primary education and lower secondary education. In GD, players interact with a variety of three-dimensional objects such as cubes, cuboids, cylinders, spheres, cones and various types of prisms and pyramids. Utilising AR technology, players must select the faces of these objects and assign the appropriate mathematical formulas to calculate their surface areas. This interactive approach aids in reinforcing geometric concepts by connecting visual and spatial recognition with mathematical reasoning, providing immediate feedback to enhance learning outcomes. The game is co-developed by Patrick Tibboel and researchers from the Smart Education Lab at the AUAS (Tibboel, 2024).

In the GD the *aesthetic* experience is centered around exploring and trying out. The player has to move around the object to be able to select surfaces, which gives more physical interaction. This is possible because the AR object stays stationary in place, which allows the player to walk around it. The *dynamics* of the game are slow-paced; there is no time pressure in the game. The players can take as much time as they need to explore the geometrical figures and think about the connection between their shape and the analytical formula. The various geometrical figures have been grouped into four difficulty levels. The player can select which difficulty they want to start with and play from there. The core game *mechanics* in GD are exploring geometrical figures by moving around an object and deconstructing it, by associating the right formula for calculating the area of a particular surface area of the object. The object the student sees is randomly picked from a set. The student selects a surface of the geometrical object, and chooses a formula for the selected surface area. Advanced players who recognize that multiple sides are identical are able to select multiple surfaces at once instead of just one. After a confirmation step, the system checks whether the answer was correct. If it was correct, the selected surface is highlighted briefly and removed from the figure. If not, feedback is given to indicate the answer was wrong, and the student has to try again.

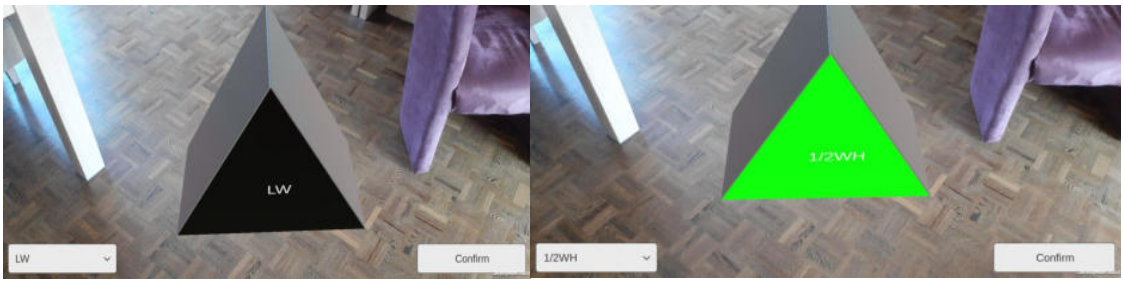

Figure 2: Left is a wrong answer and right the correct answer

Table 2: Analysis of the FRP of Geometry Deconstruction

GD	Version 1	Review 1	Version 2
Mathematical aspects	Formulas for the area of surfaces are connected to 3D images of geometrical objects	"Maybe you can do something with volume, e.g., select the right formula for the volume to make the whole object disappear?" "You could add labels like L & H for variables such as length and height."	*Mechanics: Formula for volume added, as step after calculating areas.* *Aesthetic: Variable labels added to figure and explanations of labels added.* Selected formula remains visible on the surface.
Technological aspects	Randomize what object the students get. Geometrical surface disappears when the correct formula has been used.	Order of objects sometimes included repetition or too difficult objects to start with. Feedback after correct/incorrect answer was not always clear to players	*Dynamics: Geometrical objects grouped into four difficulty levels* *Aesthetics: Enhanced visual feedback; color adjusted for accessibility and preferences*
Educational aspects	Markerless-based AR used to prevent need for preparing markers, allowing more freedom; User placement of object	"Walking around to change perspective helps to distinguish the object." "I liked walking around the figure - different perspectives and 3D experience"	*Aesthetics: Audio feedback (e.g., applause) added after answer to complement visual feedback*

The revision concerned all three aspects of the MDA framework and specific suggestions were made to improve the mechanics, dynamics and aesthetics of the game. The second version of the prototype included a better didactical approach to learning geometry: the formula for volume is added, as step after calculating areas and grouping the objects by levels. Moreover aspects such as visual and audio feedback are expected to improve the overall learning experience.

Concluding remarks

This paper shares some methodological reflections on the collaborative design process of digital educational games drawing on the experience of the TransEEt project. In particular, we investigated the collaborative prototyping processes of two digital games with mathematical content undergoing the FRP. For this analytical purpose we used a table with rows that reflect the nature of the prototypes—technological, educational, and specifically focused on mathematics education. Through the PD documents, we tracked the modifications made to the prototypes. In the CR game, the changes primarily affected the dynamics and aesthetics of the game. In GD the changes impacted all three aspects. These cases demonstrate that the collaborative rapid prototyping process significantly influences the quality of the prototypes, particularly in terms of their potential for facilitating

mathematics learning. Rather than focusing on defining patterns of affected aspects, we believe it is more urgent to develop a methodological tool for collaborative prototyping. In this sense, the PD seems a promising and accessible methodological tool to support work within a CoI. So far, the PD and prototype reviews have only been shared within the TransEEt project. Making them public could help gather broader feedback on their potential.

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Learning goals in view: Design of an intelligent environment for arranging and monitoring secondary mathematics education

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Keywords: Secondary mathematics education, learning goals, intelligent environment, curriculum mapping, teacher autonomy, visual representation.

Introduction

In secondary mathematics education, the integration of self-developed materials and innovative resources, such as online instructional videos and digital practice environments, is becoming increasingly common (Engelbrecht & Borba, 2024; Inganah et al., 2023; Kennisrotonde, 2019; Purnomo et al., 2019). However, ensuring these materials align with curriculum goals introduces several challenges. Our project addresses these challenges by developing an AI-based environment that maps resources to learning goals, visualizes the interrelations between learning goals, and tracks student progress. The system enables flexible curriculum adaptation, offering tools for teachers to integrate diverse resources. Key innovations include graphical learning goal representations and AI-supported material mapping. Initial efforts focus on the topic of differentiation in upper-secondary mathematics, with principles applicable across subjects.

System Design Principles

The intelligent system will be developed by Numworx (https://www.numworx.nl/en), a company affiliated with Utrecht University. Numworx provides a digital math environment with online exercises and lessons for secondary and higher education. Their software supports mathematics education through comprehensive lesson materials and interactive assignments, visual development tools, and flexible classroom management.

Our system will be based on the following design principles:

1. Flexibility and customization: Teachers can integrate materials into existing curricula.
2. Alignment with learning goals: AI aids in linking materials to curriculum objectives.
3. Graphical representation of learning objectives: Graphical representations in the form of a graph show learning goal dependencies, facilitating informed decisions on curriculum adaptation.
4. Performance monitoring: Detailed student models guide targeted support.

Following these design principles, we aim to create an intelligent environment for arranging and monitoring secondary mathematics education. Teachers are empowered with actionable insights, and students benefit from a tailored learning experience.

Research Questions

1. How does the environment support linking teaching materials to learning objectives?
2. How do teachers and students utilize this environment?

3. How can student data enhance teaching materials quality?

Methods

A three-phase approach will guide the research: (1) Conceptual design of learning goals, (2) Development through intervention studies, and (3) Validation in classroom settings. Methods like Bayesian Knowledge Tracing (Yudelson et al., 2013) will create and update student models.

Outlook

The project seeks to empower teachers and foster personalized learning, offering a flexible framework for educational innovation.

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Conceptualisation of number in curricular resources: Does it matter?

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We analyse one national curriculum framework by searching for presence and role of continuous quantities in how the notion of number is developed. We justify this focus drawing on number conceptualisations that allow students to encounter fractions and whole numbers as the same kind of number. We demonstrate that, in the analysed curriculum, attention to continuous quantities is highly restricted. We explain that this curricular stance has a historical connotation and argue that its perpetuation prevents the curriculum from providing a more adequate support for teaching and learning number. This curricular stance more broadly counteracts research and improvement efforts, which typically share a different view of what mathematics is and what it needs to be in school.

Keywords: Curriculum, number, length measurement, continuous quantity, design research.

Analysis context and motivation

In this paper, we invite readers to consider an analysis of a national curriculum framework, one currently used in Australia, as a case on which we highlight some of the 'forces' that shape the space of design, research, and use of teaching resources, including but not limited to textbooks and tasks.

Our motivation to analyse a national curriculum framework emerged as part of our ongoing endeavours to understand the demand that researcher-produced teaching resources present for teachers who would want to use these resources in their classrooms (cf. Cobb & Jackson, 2015). To contextualise our approach to the presented analysis, we briefly overview our work with the *Fraction as Measure* instructional sequence (Cortina et al., 2014), which we initially developed in classroom design studies (Cobb et al., 2003) conducted in under-resourced classrooms in Mexico. The sequence was subsequently trialled in classrooms in multiple countries, with consistent, cautiously optimistic results (e.g., Vale & Graven, 2018, discuss a trial in South Africa).

The sequencing of mathematical ideas in the designed resource proved key to its viability in classroom teaching. However, its misalignment with the content sequencing in local curricular framework made it unethical for us to trial the resource in Australian primary classrooms, where teachers are typically accountable for closely adhering to the sequencing outlined in the curriculum.

Later we became aware of additional differences, both less obvious and more impactful than the sequencing of the content alone: Our work in fractions and proportionality heavily built on students' reasoning with continuous quantities in the context of length measurement. However, students often did not have experiences in reasoning with continuous quantities in whole number contexts, which we were hoping to leverage. The notion of whole numbers as something that is produced multiplicatively via an iteration of a unit was foreign to many. We started noticing that tasks where whole number arithmetic scenarios would be situated in contexts that involved comparison,

measurement of, or reasoning with continuous quantities were rare in both Australian and Mexican textbooks. This oriented us to explore the institutional contexts, where we hoped our sequence could be used. The curriculum framework analysis presented here is part of those explorations.

We situate the overview of method of analysis in our design research work. The aim to teach fractions well in schools highlighted to us the importance of what is included in conceptualisations of number in curricula, which then guided our choices of analysis categories and directions.

Fraction as Measure sequence

The conceptualisation of number that guides our fraction design work aligns closely with the mutually consistent conceptualisations put forth by Davydov, Bass, and Freudenthal. According to Davydov (1990, p. 196), number is "a particular case of the representation of a *general relationship* of quantities, where one of them is taken as a measure and is computing the other". Bass (2018, p. 466), building on Davydov's work, clarifies that number "arises from *measuring* one quantity by another, taken to be the 'unit': How 'much' (or many) of the unit is needed to constitute the given quantity?"

In these statements, "quantities" exist in the physical, material world, and they are something that can be compared. As a result of such comparison, a quantity can be the same as, or bigger than, or smaller than that to which it is compared. By treating one quantity as a reference unit, number emerges as a tool that makes further organisation of comparisons among multiple quantities possible. While some quantities in the physical world are discrete and some are continuous, only measurement of continuous quantities produces whole numbers and fractions as numbers that are experientially alike. Bass highlighted the need for such alikeness stating that "the measure pathway [unlike the counting, discrete pathway] … makes available from the beginning the continuous number line as a coherent geometric environment in which all numbers of school mathematics eventually reside" (p. 467).

The view of fraction as a comparison (of two magnitudes) is central to Freudenthal's (1983) didactical phenomenology of number. From the two types of situations in which fraction comparisons arise, Freudenthal highlighted *fraction as comparer* situations—where two physically separate quantities are compared, with one of them serving as a reference unit—as being neglected and mathematically more robust than the current approaches based on *fraction as fracturer* (equipartition) situations.

In the designed sequence, fractions are introduced through *fraction as comparer* situations, following students' experiences with whole number measurement in context of continuous quantities. Fractions are introduced in response to a need for increased precision and sophistication in length measurement, the evolution of which is simulated in the classroom. For instance, unit fractions are defined in reference to the previously used measurement unit (*the stick*), when new units of a smaller size (sub-units) are needed (e.g., a rod corresponding to a sub-unit known as *a third* would be of such length that the stick measured with this rod measures three, see Figure 1).

Figure 1: Three iterations of *a third* (dark rod) are the same length as the stick (i.e., reference unit)

As a result, students come to reason with fractions as representing quantities (amounts, sizes), and in terms of actions on these quantities. Fraction reasoning emerges as being mathematically consistent with whole number reasoning, and fractions emerge as numbers (e.g., Bass, 2018).

We came to appreciate the importance of viewing number as arising from the measurement of continuous quantities for teaching and learning and felt that this was not a conceptualisation broadly available to teachers. This oriented our initial analysis towards documenting how AC:M includes number, measurement, unit, counting, and continuous and discrete quantity.

Analysed curriculum framework

We analysed Version 9 of the Australian Curriculum: Mathematics (AC:M), Foundation through Year 6 (Australian Curriculum, Assessment and Reporting Agency [ACARA], n.d.). This version came to use in 2024 as the first notable modification of the national curriculum introduced in 2012. Foundation year precedes Year 1. Students start Foundation when between 4.5 and 5.5 years old.

In each year, AC:M includes six content strands (number, algebra, measurement, space, statistics, probability). For each year level, the document provides two overview sections, *Year level description* and *Achievement standard*, each about 250 words. These are followed by *Content descriptions* and *Content elaborations* for each of the six content strands, at about 3500 words per year level.

Method

We conducted the analysis of AC:M to document *where* continuous quantities were included, and *what kinds* of reasoning with these quantities were described or implied. Doing this, we aimed to highlight how AC:M developed numbers beyond whole numbers. First, the two authors red through AC:M document, outlining the gradated coding categories to capture treatment of number in terms of continuous quantity: from absence of continuous quantity (Counting Number), through continuous quantity included implicitly, with or without focus on counting (Counting Number Plus, Potentially Continuous, respectively), to continuous quantity included overtly (Continuous), and included without references to number (Measurement Literacy, see Table 1).

Table 1. Coding scheme

Code	Description	Example
Counting Number	Overt references to order, counting, and discrete quantity	Naming discrete things to be counted or used in tasks (e.g., beads, people)
Counting Number Plus	Instances of focus on counting, which could (to some extent, or inadvertently) facilitate quantitative sense of discrete collections	Size of the arrangement with dots in subitising activities (the space the arrangement takes), ten frames, frequency graphs in statistics, calendars as a means to compare duration in days, months
Potentially Continuous	Potential to position numbers as accounting for quantitative	Claims/tools where interpretation of continuous quantity can be made, even though it is not

	differences around us (i.e., grounding in measurement of quantity)	suggested (e.g., number line, even if not used to represent accumulation of quantity)
Continuous	Overt references to continuous quantity in relation to number	Quantification of comparisons of relative size
Measurement Literacy	Measurement without link to conceptualisation of number	Named attributes to be measured, standardised units of measurement, "uniform units" (sic)

Second, as the *Year level description* and *Achievement standard* sections appeared to offer a more generous codes for quantity, these were independently coded by the authors, the resulting codes were compared, and the differences (fewer than 10% of the codes) reconciled. Third, the coded documents were used to create a narrative overview of each year level and finalise Table 1 Descriptions and Examples. Verifications and rebuttals of the developed claims were sought in the *Content* sections, including checking for elements that appeared to be missing from the year level overviews.

We report a narrative of the findings based on the claims that could be robustly supported with data. Where possible, we include illustrations. While we restrict our findings to number and measurement sections, we note that the character of the document was consistent across the remaining sections.

Finding 1: Different kinds of numbers

Overall, AC:M conceptualises number in the early years as a *counting number only*. For all practical purposes, number is equated to the cardinality of discrete collection of objects. The extent to which this was the case was striking to us. For example, between Foundation and Year 3, AC:M names 77 different countable things in *Content elaborations*. These include counters, dots, dice, but also pencils, tigers, bolts, or windows in a train carriage. In contrast, there are only three *elaborations* in the same range, where an overt reference to a continuous quantity is made in relation to number:

(1) use of "a lump of playdough" and "a cup of water" to indicate half (Year 2 Number),
(2) "estimat[e] the height of a tree by comparing it to the height of their friend and quot[e] the result as 'the tree is about 3 times as tall [as my friend]'" (Year 3 Measurement), and
(3) markings on measurement tools (a ruler, measuring jugs, scales) interpreted as "the beginning and end of each unit" with no meaning for "unit" given (Year 3 Measurement).

While words such as *quantity* and *comparison* occasionally appear, their meaning is not clarified. The interpretation most consistent with the rest of the analysis is that these words refer to discrete quantities, and to comparisons of numerical order or one-to-one mappings, that is, to additive comparisons. Foundation year contains words quantity, quantifying, comparison, sameness, difference. Some of these refer to qualitative comparisons that are important at this level (e.g., same and different colour) but as comparisons of length or size remain a possibility, they were coded as Potentially Continuous. From Year 1, students seem to be expected to know and use numbers, and the discourse shifts away from quantities to numbers: *numbers* are to be partitioned, compared (i.e., ordered), give rise to patterns, and are equated with their symbolic form: having increasingly more digits every year. Phrases "quantify collections" and "quantify sets of objects" in AC:M describe

subitizing, counting, and one-to-one correspondence (Foundation), and skip counting and partitioning (Year 1). This suggests that "enumerate" (assign a number) would be a more accurate word to use. After Year 1, "quantification" is no longer present.

From Year 4, word "numbers" refers to whole numbers only. For instance, statement "Students use their proficiency … to … multiply and divide numbers efficiently" (Year 4) does not refer to fractions or decimals, because multiplication and division with those had not yet been introduced. The most reasonable conclusion is that fractions and decimals are, indeed, not considered to be numbers. When this problem is later resolved, it is not by admission of fractions and decimals among numbers. Instead, it is done by an introduction of a new umbrella notion of *rational numbers*, which subsume whole numbers, fractions, and decimals by an unspecified mechanism: "[students] expand the repertoire of numbers they work with to include rational numbers and the use of (sic) integers" (Year 6). This unification is considered as a mere renaming, but it is indeed a unification of mathematical objects that—from students' perspective—do not share their origin, meaning, or operations.

Returning to fractions, AC:M does not offer a way in which to conceptualise fractions as numbers at the time of their introduction. When it first introduces fractions (Year 2) the overwhelming—if not the only—meaning of number is created through cardinality and counting. We know that fractions cannot be meaningfully defined as numbers through counting at this level of schooling (Bass, 2018).

Finding 2: Measurement only as measurement literacy

With the two exceptions noted above, whenever shape, space, and measurement sections of AC:M attend to continuous quantities, they pursue aims of establishing what we refer to as a basic *measurement literacy*. This focus includes familiarisation of students with the attribute to be compared or measured, and the introduction of informal and formal tools with which the attribute could be compared or measured (including assigning a number to it). The focus of these themes does not include mathematising of measurement, such as through exploring the construction of measurement tools (e.g., creating a measurement strip to speed up measuring, and coming to view it as a record of accumulation of length, Stephan et al., 2003).

Attempting to guide some explorations of measurement, AC:M makes references to "uniform informal units" (Year 1), "uniform units" (Year 2), progressing to introducing uniformly segmented "conventional instruments" with metric units (Year 4). These descriptions rely on solely additive imagery, and miss the multiplicative imagery that measurement affords. The specification that the units used are to be "uniform" suggests that to determine whether a desk fits through a door frame, the width of the desk and that of the door opening are to be covered with, say, toothpicks to determine whether the door opening is large enough. This, of course, is only meaningful if one uses toothpicks of the same length. However, unit of length in this situation is not 'the uniform toothpicks'. The unit is the length of *a toothpick*, and the measurement is the iterative process in which the number of times (N) that the toothpick length can be stacked up to match the relevant dimension of the desk is being determined. This gives rise to a potential that the resulting length will be viewed in relation to the length of the toothpick as being N-times as long, a multiplicative comparison of the two lengths. While the initial measurement comparison certainly can be enacted with multiple toothpicks of the same length, the mathematically important conceptualisation of notions of measurement unit and

measurement as the iteration of this unit are not currently included in AC:M. Similarly, the notion of unit fraction as a unit through which other lengths are established multiplicatively, is also omitted.

Finding 3: Nature of mathematics

The final observation relates to how views of what mathematics is shape opinions on *what mathematics should be learned* (Borba & Skovsmose, 1997; Clements et al., 2012; Sfard, 1998). Is mathematics a collection of important facts and calculations? Is it a collection of specific disciplinary practices? Is mathematics *natural*, out there to be discovered? Is it a human practice to be invented and re-invented based on a perceived need?

From Year 2 on, AC:M overtly positions mathematics as something that is outside of, and separate from, the (students') world. Mathematics, instead, is presented as something brought to students so that they could model the world with it: "mathematics can be used to investigate things they [students] are curious about, to solve practical problems and model everyday situations" (Year 2). Instead of positioning students' experiences as something from which they could, when adequately supported, make mathematics (i.e., mathematise phenomena to give rise to new ideas and ways of organising the world, Freudenthal, 1981), AC:M encourages teachers to guide students' recognition of the opposite relationship. Mathematics is to be seen as helping students understand their everyday experiences, which would, presumably, not be understandable to them without mathematics. It is never clarified in AC:M where this kind of essential mathematics comes from.

For example, base-10 place value system is presented as a numerical pattern to be learned for the purposes of increasing one's calculational fluency (e.g., "manipulate numbers," Year 3). It is not presented as a human invention that encapsulates increasing effectiveness in both measurement and calculation, in response to a need for such effectiveness (Bass, 2018). Similarly, *mathematics* is discussed as something to be represented: students are to "become aware of the importance of context and purpose … [in] how they choose to represent mathematics" (Year 4). This evokes an image that is very different from a classroom practice where *problem situations* are to be represented and organised, with increasing sophistication, thus giving rise to new insights about the situation as well as to new mathematical ideas that made such insights possible. The discourse employed in the content overviews indicates how mathematics is what is being given to students to be learned, represented, and applied. It is not something to be created or produced in the classroom, and the purposes for and the ways of its creation are not to be explored in classroom activities.

Discussion and elaboration of findings

To summarise, AC:M develops counting number which emerges from enumeration of discrete objects, meaning of number is not anchored in measurement of continuous quantities, and mathematics is learned so it can be applied to make sense of the world. To argue that this summary is of broader significance, two additional clarifications are needed: (1) Australia's curriculum is not unusual, and (2) the notions for which AC:M was analysed are not solely of importance in relation to authors' pet research-based resource on fractions. Our final aim in this paper is to address these points.

The choices that currently organise AC:M are at play in other national curricula (e.g., Mexico, analysis under development). These choices are consistent with the broad curricular framing changes

characteristic of New Math reform (Kilpatrick, 2012), a period when many mathematics curricula transitioned from being organised by classical mathematical theory to modern mathematical theory. Within classical approaches, mathematics was a description of reality. The objects that mathematics studied (e.g., numbers) found their source and their reference in reality (e.g., in the measurement of continuous magnitudes). The shift to modern mathematics (where set theory became a foundation of formal axiomatics) came about as "[m]athematicians began to reflect more profoundly *about their own constructions and activities*, rather than about the givens of the external world" (Otte, 2007, p. 247, italics ours). In modern mathematics, axioms only refer to reality through object attributes: They define *properties* of number sets (e.g., existence of an additive inverse, commutativity of multiplication) but the meaning of numbers is no longer a concern of mathematics as a discipline.

The change towards set theoretical abstraction benefited the discipline. But thanks to the intentional separation from reality, modern mathematics could only ground school mathematics in abstraction (Freudenthal, 1979). Given that numbers were now defined in relation to *sets* (i.e., collections of objects and operations that satisfy specific conditions), their source or reference was discrete. Whole number, now a cardinality of a set, became tightly linked to counting. Inheriting a set theory definition of rational numbers, fraction became a pair of whole numbers. The tools for reasoning with numbers that were previously essential—quantity, units, and measurement—were no longer central to the notion of number in mathematical theory and thus were no longer central in school. Measurement was separated from number; algebraic manipulations were brought in to organise operations and reasoning with numbers, replacing reasoning with quantity; abstract mathematics had to be taught to students first, then illustrated by examples, and then applied to reality (Chambris & Visnovska, 2022).

A common belief about the New Math reform is that it proved unsuccessful, and the changes were later reverted. Considering our findings, Kilpatrick's (2012, p. 569) note that "(i)n no country did school mathematics return to where it had been before the New Math movement began" reads as a call for a more thorough exploration of what exactly was changed. May some of the widespread 'learning difficulties' we face today be a consequence of those past changes? The omission of a conceptualisation capable of supporting students to understand fractions—an area where many face difficulties—seems, indeed, problematic. Where this omission was accepted, it had time to propagate through teaching resources, through what teachers teach, and through what researchers notice.

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Flexible instructional trajectories for learners' self-regulation in 9th Grade Algebra

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Personalised learning has become increasingly prominent in educational policies as it has shown some promises in research as one of the potential means to increase students' motivation to learn difficult subjects such as mathematics. Systemic and scalable approach to personalisation requires introducing more flexibility and non-linearity in designing our curricula, learning resources and instructions. This paper reflects on an empirical study carried out in Eduflex project that aimed to develop a concept of learner-centred learning through flexible instructional trajectory. The study was conducted in an authentic classroom situation with 156 9th grade students using a flexible instructional trajectory with a set of interactive learning resources targeting the 9th grade algebra learning outcomes in Estonian national mathematics curriculum.

Keywords: Flexible learning, flexible learning path, personalised learning, instructional trajectory, self-regulation.

Introduction

Both educational policies in Estonia (Education Strategy 2021-2035, 2020) and international research results (e.g. Shemshack & Spector, 2020) support effectiveness of learner-centred approaches to teaching and learning, where the digital technologies have a potential to play an important role. While research demonstrates that the use of technology in learning can increase student motivation (Rovai et al., 2007), there are also examples to the contrary (Tang & Hsu, 2014) or studies showing that the use of technology does not directly benefit or harm learning process (Müller & Mildenberger, 2021). Therefore, the quality of the learning resources, interaction between students and teacher and the mode of feedback should be more important than the format of learning (Müller & Mildenberger, 2021). In order to provide large groups of students with effective, personalised and flexible learning instructions where student does not become "programmed" by predefined computer algorithms, the new approach is needed to designing the digital learning resources, scaffolding, and learning activities where control over learning remains on the side of humans - learners and teachers. To effectively design flexible instructional trajectories, it is essential to understand how different factors (including learners' characteristics, perceptions and the instructional design of learning activities) impact learning outcomes. Therefore, the following research question was formulated:

1) How do learners' perceptions of flexible learning activities vary in the context of flexible learning trajectories that incorporate digital learning resources in middle school algebra?

This paper presents findings from an intervention study where 9th-grade students engaged with a flexible instructional trajectory designed within an interactive digital learning environment. The study aimed to explore how the use of these resources supports achieving learning outcomes in algebra, focusing on learner diversity in perceptions and experiences.

Literature review and theoretical background

When referring to flexible ways of learning, three important attributes are mentioned: time, place and the mode of learning (Hammersley et al., 2013). These features require students to be able to orient themselves in their choices and to recognise the situations and factors that trigger their learning. In a technology-enriched learning environment (TEL), students need to organise and regulate their learning more autonomously, as there might be limited external guidance (Pelikan et al., 2021). The use of self-regulated learning strategies, such as planning, goal setting and self-monitoring, is therefore seen as critical for successful learning outcomes (Bol & Garner, 2011). Also, it requires from student the ability to cope with a moderate cognitive load, as self-regulation and cognitive load have interplay to each other (Seufert, 2018). To reduce the cognitive load for student, it is important to provide consistent structures, support, and affirmations that allow the student to know what to expect and how to free up space for more challenging tasks (Darling-Hammond et al., 2019).

Since the level of instruction in a TEL environment can be limited and unguided, learning often leads to suboptimal outcomes (Kirschner et al., 2006), it is important to design the learning process with a strong didactic rationale. For reducing the cognitive load, the opportunity is seen to provide the student with a sequence of tasks from easier to more complex (van Merriënboer et al., 2003). Task complexity can be controlled by task design: e.g. number of element interactivity, number of distinct acts and concepts used to perform a task, the number of distinct hints to be processed for performing acts, the relations among acts (Chen et al., 2023).

Research methodology

The following provides an overview of the context of the survey, key concepts, survey design, data collection and analysis.

The context and key concepts of the study

The research on flexible learning was part of the EduFlex project funded by the Estonian Ministry of Education and Research guided by the aim to develop the concept of flexible and personalised learning to provide a methodology and tools for analysing and supporting flexible teaching practices.

In our research, we have decided to split the generic term of flexible learning paths in two distinct parts: (1) instructional trajectory that belongs to the realm of instructional design and (2) learning path that belongs to the field of learning analytics. An instructional trajectory is a forward-looking plan for teaching and learning designed by a teacher or instructional designer on the basis of a domain model and curriculum of the given subject. It contains expected learning outcomes and related instructional episodes that follow a predefined order of activities based on a domain model, providing a coherent progression, as well as tools and techniques for teaching, learning and assessment (Volt et al., 2025). Flexibility is designed into trajectories by adding non-mandatory tasks, possibilities for selection of type or complexity level and pace of learning tasks, guided metacognitive prompts for reflection, analysis, repetition and recalling, visualisation of basic learning analytics to support data-informed decision making. The learning path represents a digital footprint of the learner's interaction with digital learning resources: it documents in a machine-readable way the learner's actions, choices

and outcomes that can be used for learning analytics and visualised for teacher and learner to support their personalised control over the learning process (Volt et al., 2025).

Research design

This study follows the design-based research (DBR) approach, which is applicable to educational research and for designing TEL environments (Wang & Hannafin, 2005). Study was informed by a previously conducted pilot-study in 2023 to validate the initial model of instructional trajectory. Therefore, this study is a second cycle of DBR where, building on the research that preceded this intervention, we developed characteristics of a flexible learning environment that could better support students. In this phase of the study, we examined the flexible delivery mode of the created instructional trajectory in order to investigate the features of flexible learning and the perceived effect of these characteristics on students. To this purpose, instructional trajectories for the 9th grade algebra learning outcomes and an instructional delivery mode were developed. Learning resources and instructional trajectories were designed to cover two learning outcomes from national curriculum:

1) learner cancels and expands an algebraic fraction; adds, subtracts, multiplies and divides algebraic fractions, including adding and subtracting both like and unlike fractions;
2) learner simplifies rational expressions which includes maximum two operations.

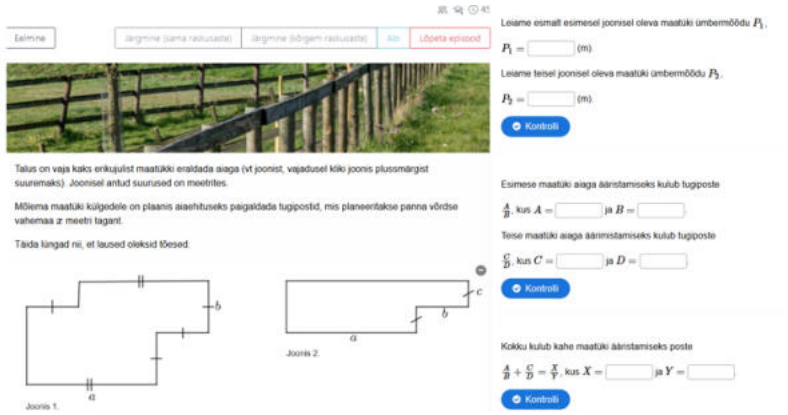

Figure 1: An example of a student's view on an instructional trajectory: task display with options to view supporting material, hints, and navigate to a more complex or less complex task

The creation of instructional trajectory is based on a domain model in a form of a Bayesian network derived from didactic principles and developmental psychology, which describes in a machine-readable way the anticipated cognitive development trajectory of the learner, presenting the units of knowledge (the units of conceptual and procedural knowledge), along with their interconnections (prerequisite, kind-of and part-of relationships). To design our pedagogical approach, we relied on the principals of the 4C/ID (Four-Component Instructional Design Model) framework (van Merriënboer & Kirschner, 2007) and Merrill's (2002) key learning principles. The learning resources were presented to the student at increasing levels of complexity and in an intuitive easy-to-use interface, where the student had the freedom to decide whether to increase or decrease the complexity of the tasks or to remain at the same level of complexity (in Figure 1). Some of the tasks in the trajectory were compulsory for the student, while some were optional and their completion depended

solely on the student's free initiative. The learning resources include multiple-choice, fill-in-the-blanks, branching scenario and interactive video content types in the design of tasks and supporting materials.

Intervention, data collection and analysis

Convenience sample of teachers, who were already active users of digital learning resources, was used to conduct the study. Five teachers from four schools participated in the study, involving a total of 156 9th-grade students. Data from 129 students were included in the analysis, comprising 56 boys, 64 girls, and 6 students who chose not to specify their gender.

The students participated in the study under pseudonyms: teachers assigned the students' usernames at their discretion, and the researchers were able to associate the username with the student's gender and the data collected on the student's track record from the learning path. Intervention was designed as part of students' normal classroom situation in math. For all the students, the topic learnt was novel, and the teachers taught the topic according to their own schedules, i.e. some teachers started at the beginning of December 2023 and some teachers started at the end of January 2024. The duration of the study ranged from one month to one and a half months, depending on the teacher. In addition (to completing the digital learning trajectory with students), the teachers conducted in between also practice lessons of their own choice and frequency to consolidate what they had learnt when using the instructional trajectory. Depending on the teacher, the practice lessons were either collaborative or individual task-solving lessons.

The environment students used was a closed H5P-Drupal based learning environment, where students logged in with their username to interact with digital learning resources, as well as to compare interactions with the data obtained from the questionnaire responses. To understand students' learning experiences, a survey was distributed in the end of the intervention. In designing the survey, we used existing and validated instruments, and also designed new items to answer our research question. Survey integrated following blocks: perceived efficiency (Tammets et al, 2021); shared classroom control (Eshel & Kohavi, 2003); cognitive load and technology acceptance (Hwang et al., 2013). Additionally, questions were created to understand how students perceived their own role in flexible learning scenarios (for instance the student was asked to evaluate a claim: *Even when the topic was boring or difficult, I made an effort to complete the tasks*).

Students answered the questionnaire after the intervention ended and evaluated items using a 1-5 Likert scale. A follow-up questionnaire was completed by 146 students, including 129 students whose responses to the follow-up questionnaire could be matched with the data from the learning path and who had given their consent for their data to be used in the study. The data were analysed using confirmatory factor analysis and cluster analysis to examine which groups of students emerged based on the identified factors.

Results

In the following, we examine the results of which factors influenced students' perceived experience of flexible learning and which groups of students emerged from the given experience. To perform the analysis, we used Jamovi v. 2.6 to conduct a confirmatory factor analysis (CFA) with the aim of

exploring the factors that are important determinants when using a flexible instructional trajectory and conducted a K-means cluster analysis to explore emerging student profiles.

Factors affecting students' learning experiences in flexible learning scenario

According to CFA findings, we focused on four main factors: perceived efficiency, shared control, cognitive load and the importance of one's role. The empirical data showed a good fit with this 4-factor model: CMIN=221, CMIN/df=1.534, CFI=.945, TLI=.935, SRMR=.0687, RMSEA=.0645.

Table 1: Confirmatory factor analysis

Factor	Indicator and standard deviation of factor loading			
Perceived efficiency	TAJT_1	0.884	TAJT_2	0.865
	TAJT_3	0.767	TAJT_4	0.764
	TAJT_6	0.713		
Shared control	JAG_KT_4	0.559	JAG_KT_5	0.556
	JAG_KT_6	0.597	JAG_KT_7	0.601
	JAG_KT_8	0.647		
Cognitive load	KOG_KOORM_1	0.791	KOG_KOORM_2	0.837
	KOG_KOORM_3	0.896	KOG_KOORM_4	0.767
	KOG_KOORM_6	0.685	KOG_KOORM_7	0.867
	KOG_KOORM_8	0.784		
Importance of learner's own role	ISE_3	0.698	ISE_4	0.761

While this represents a preliminary analysis, demanding further investigation, the identified factors highlight key influences on learning in flexible, TEL environments. These factors include cognitive load, which reflects the mental effort required to navigate and engage with the learning environment; the perceived importance of students taking ownership of their learning, emphasizing their active role in the process; the shared control between teachers and students in the classroom, illustrating the collaborative dynamics essential for flexible learning; and students' perceptions of how the flexible learning environment supports their individual learning and skill development. Together, these elements underscore the complexity of designing effective flexible learning environments and the need for a balanced approach that considers both student autonomy and teacher guidance.

Students' profiles in flexible learning scenarios

Next, we sought to understand whether there are differences between students' experiences, as it can be assumed that not all students are exposed to flexible learning in the same way. Therefore, a K-means cluster analysis was carried out to investigate which clusters of students emerge when four

factors influencing the learning are considered: perceived efficiency, shared control, cognitive load and the importance of one's role in the flexible learning process.

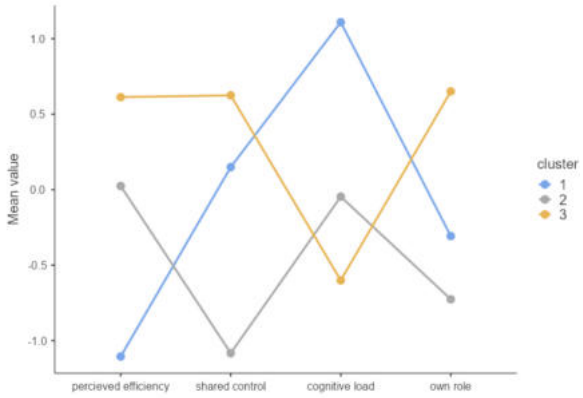

Figure 2: Plot of means across the clusters derived from aggregation of items in factors

Figure 2 shows that three clusters emerged: Independent learners (N=58, cluster 3), Habitual learners (N=38, cluster 2) and Struggling learners (N=33, cluster 1).

Independent learners had the most to gain from flexible learning. They were autonomous and perceived their learning as effective: students felt the freedom to decide their own pace, perceived the teacher as a support rather than a controlling factor and perceived setting for learning to be effective. Compared to other groups of students, independent learners had the lowest cognitive load, and students in this group perceived the importance of their own role as the most important among peers.

Habitual learners perceived the learning experience to be somewhat less effective, but still effective, compared to independent learners. At the same time, however, students in this group did not perceive the importance of their own role in the learning process and saw the teacher as more responsible for guiding learning than themselves, their peers or the technology. Habitual students perceived their role in directing their own learning to be low, felt a lack of control over the pace of their own learning and teacher was not perceived as a valuable resource of help. Although the cognitive load of this group is moderate and closer to the perceived cognitive load of the first group than that of the third group, these students do not see opportunities to implement self-regulatory strategies. Rather, these students seem to be used to the traditional rhythm of the classroom, where they have fewer opportunities to decide for themselves.

Learners in difficulty perceived compared to other groups of students' cognitive load as the highest and also perceived learning opportunities as inefficient. Although these students' perception of shared control is closer to that of the first group, learners in difficulty do not perceive the importance of their role as high and perceive the importance of their role rather close to the students in the habitual learners group.

Conclusion

The aim of this study was to investigate how students perceive and engage with a flexible learning environment designed for 9th grade algebra, particularly in terms of the perceived efficiency, shared control, perceived cognitive load and importance of their own role in the learning process. The

findings reveal that students' perceptions of the flexible learning environment varied significantly across the identified groups, with distinct differences in perceived efficiency, shared control, cognitive load, and recognition of their own role in learning. As self-regulated learning strategies are essential for effective learning (Bol & Garner, 2011) and student autonomy plays a crucial role in technology-enriched learning environments (Pelikan et al., 2021), the results suggest that students who value their role in learning and demonstrate autonomy are better equipped to thrive in flexible learning settings.

To enhance learning outcomes, it is crucial to reduce cognitive load by providing consistent structures, targeted support, and affirmations, thereby freeing cognitive resources for more complex tasks (Darling-Hammond et al., 2019). Since the flexible instructional trajectory was designed based on the mentioned design principles, some students still perceived the learning process as inefficient and struggled to recognize their role in learning. This raises important questions about how such students interact with and make choices in the learning environment. Similarly, the findings prompt investigation into why habitual learners did not perceive flexibility as an opportunity to make meaningful choices. Future research will focus on examining learning analytics data collected from students' interactions comparing the behaviours and choices of students in different clusters (e.g. independent vs struggling student) to improve design requirements for flexible learning features.

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ChatGPT as a resource for pre-service mathematics teachers to design their lesson plans and lesson scenarios

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Keywords: ChatGPT, documentational approach to didactics, lesson plan, lesson scenario.

Introduction

Generative AI tools like ChatGPT have found various purposes in education, especially among students, as in other fields (Albadarin et al., 2024). While students use generative AI tools for education purposes, it is crucial to examine how teachers will shape their lessons with the knowledge that this tool is around their students. For teachers, knowing and using these tools are important, because they can only understand the nature of these tools when they use them. Using these tools can shape the professional activities of pre-service teachers and in-service teachers. The investigation of how the use of generative AI tools by teachers and pre-service teachers influences their professional activities is a crucial research topic that can provide valuable insights of its impact on the teaching. In this context, the aim of this study is to explore how and for what purposes pre-service mathematics teachers (PSTs) collaborate with ChatGPT to develop lesson plans and lesson scenarios, with particular attention to the usage purposes and usage schemes.

Theoretical Framework

In the full version of this research, since PSTs mentioned ChatGPT with their other resources such as digital resources and textbooks, we decided to analyse the usage schemes of all the resources with the Documentational Approach of Didactics (Herein after DAD) (Gueudet & Trouche, 2009). However, the usage of ChatGPT as a resource was interesting. Mentioning the resource as a concept, DAD explains the resource as a special concept more than a tool. For instance, teachers' drawings on the blackboard can be a resource (a tool), however students' responses and teacher's explanations can be a resource, too. In this case, the conversations between PSTs and ChatGPT can be accepted as resources.

Method

This paper is a part of a bigger study which was designed as a case study and was conducted during the spring semester of the 2022-2023 academic year. The participants are chosen purposefully from a classroom with 25 PSTs enrolled in the second year of a Middle School Mathematics Teaching program in a state university in Turkey. Ten PSTs who did not use ChatGPT before for educational purposes are chosen for this study. In addition, the participants have attended theoretical lessons about teaching mathematics but they did not have any practical teaching experience. Thus, it is aimed that they feel the need to use ChatGPT as a requirement of social interaction in their plans and scenarios. The study lasted for eight weeks, with sessions of 2 hours per week.

Table 1: Data collection process

Weeks	Actions	Data Collection Tools
Week 1	Access to ChatGPT and Initiation	No data collected
	Choice of teaching subject for lesson planning	
Week 2 – Week 7	Interactions with ChatGPT to prepare lesson plans and lesson scenarios	Screenshots of participants' conversations with ChatGPT
		Participants' diaries
Week 8	Finalization	Participants' lesson plans and lesson scenarios

The data collection tools used included participants' diaries filled out when interacting with ChatGPT, screenshots of participants' conversations with ChatGPT, video recordings of the sessions that served for the monitoring of PSTs' progress where they mainly discussed teaching concepts or situations they found difficult or were hesitant about, and lesson plans and scenarios prepared in collaboration with ChatGPT.

Data Analysis

First, the data is coded for "usage purposes" after several readings of the transcripts of PSTs' diaries, screenshots and video recordings in addition to lesson plans and scenarios. Each data first analysed separately, then used to check each other's reliability. Secondly, the data is coded for revealing "usage schemes", so all data is analysed with the aim to find the stable organizations for similar situations.

Results

In this study, we have resulted that PSTs used ChatGPT for the purposes of (i)obtaining theoretical and pedagogical knowledge, (ii)consulting for ideas and validation, (iii)simulating the students' behaviours or responses. In addition, PSTs used ChatGPT for these (possible) usage schemes (i) asking for simulations, (ii) asking for preparing the lesson, (iii) asking for using suitable materials.

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Prime Times: a game to develop multiplicative reasoning

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Keywords: Number theory, expressions, educational games, task design.

Introduction

Even though number theory has its roots in arithmetic and is closely related to algebra, its didactics are less well researched. It is common to refer to the whole of the relations involving multiplication in the natural numbers as its multiplicative structure (Brown et al., 2002). The proposed poster will introduce and illustrate *Prime Times*, a game I am developing which provides a learning environment rich in representations of the multiplicative structure of the natural numbers, and is part of a project addressing the research question: How can the design of a game develop students' multiplicative reasoning? This proposal will report on the theoretical development of the game's design principles.

Theoretical framework and previous studies

From a synthesis of the well-developed research field of algebra education, Tondorf and Prediger (2022) extracted three typical perspectives adopted by students when considering algebraic expressions: operational, transformational, and relational. In an *operational* perspective, expressions encode a calculation and thus the student will focus on the surface syntax of the operations to perform the calculation. In a *transformational* perspective, expressions are seen as sets of symbols that can be transformed by use of rules. In a *relational* perspective, expressions are connected in a web of relations and representations that give them meaning.

Through the lens of this theoretical framework, research shows that there is a dominance of operational approaches in students' reasoning. From an analysis of preservice elementary teachers interview responses, Brown et al. (2002) conclude that they "believe that it is common for students to see prime factorizations as processes. Rather than treating a number in prime-factored form as a single number, students often perceive it only as a string of numbers, providing a set of instructions for obtaining the intended number" (p. 79).

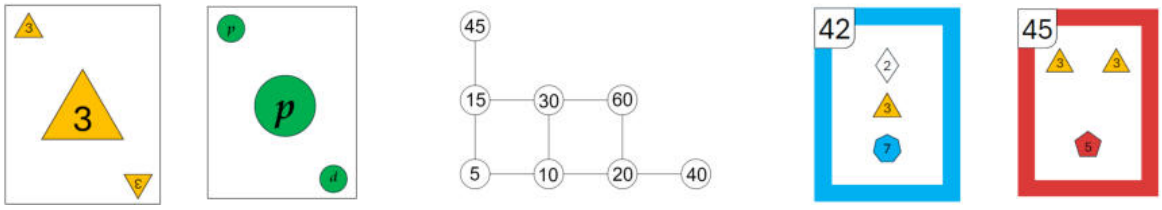

Figure 1: Examples of playing cards (left), part of playing board (centre), and objective cards (right)

Later theoretical studies have suggested ways to promote relational perspectives using representations of the multiplicative structure. Dias (2005) reports on using a lattice representation of factors (e.g. see figure 1, centre) to give intuitive explanations of the greatest common divisor (GCD) and least common multiple (LCM) with preservice teachers. Kiss (2020) proposes using cards with primes on them as a manipulative, and to have learners construct numbers multiplicatively (to introduce the idea

of the fundamental theorem of arithmetic) and to discover relationships with the GCD. There are no empirical studies of these representations in classroom settings, and the present study addresses that need with a possible aim to further develop a sequence of lessons based on these representations.

Method and design principles

This project will employ design research with cycles of conducting and analysing design experiments. A preliminary cycle with 18 in-service middle school teachers attending a professional development course was conducted to gain initial feedback, and to recruit classrooms for further cycles. Subsequence cycles will be performed with eighth-grade students because of curriculum suitability.

The game is played on a board with a lattice representation of divisibility of the natural numbers up to 64 (excluding primes greater than 32) and makes use of two sets of cards: playing cards for the primes two, three, five and seven, and wild prime cards marked with a p (multiple copies of each); and objective cards for each number on the board. During play, the players use the playing cards to form prime factorisations and place counters on the board. The aim is cover as many of the numbers corresponding to their objective cards as possible. Games last about 30 minutes and in the design experiments students played pair against pair. This led to discussion of strategies which is seen as an integral part of the game's learning environment. The game play was developed according to the following principles:

- From the findings of Tondorf and Prediger (2022), there is learning potential through promoting relational and transformational perspectives and their interaction. Through multiple representations, play invites relational perspectives. The playing cards can also be used to multiply or divide a number covered by a player's counter thereby encouraging a transformational perspective for prime factorisations.
- In addition, to counter the dominance of operational perspectives, the lattice and goal card representations offer transformational and relational perspectives that are less cognitively demanding than the operational alternatives.
- Because there are multiple goal cards visible at any time, play advantages covering the GCD or LCM of the objective cards, or an already covered number and an objective card.
- Rules should be simple as possible so that valuable class time is not used to learn them.

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A proposal of tasks to integrate sustainability in the early ages from mathematics education

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Keywords: Sustainability, education for sustainable development, mathematics tasks, primary education.

Education for sustainable development

Since the first international calls on sustainability to address social, environmental and economic crises began to emerge several decades ago Education for Sustainable Development (ESD) has been gradually incorporated into all educational stages, to promote the development of sustainability skills (e.g., Tilbury, 2011; UNESCO, 2017). To establish a strategy as united as possible, UNESCO (2017) initially suggested 17 Sustainable Development Goals (SDGs) linked to economic growth, social inclusion and environmental protection.

This initiative has already begun to generate various research results in recent years on the knowledge and beliefs of pre-service and/or in-service teachers in various stages of integrating sustainability into mathematics education (e.g., Coles and Helliwell, 2023; Roe et al., 2018). Alsina (2023) notes that, currently, it does not seem to make much sense to train teachers to teach mathematics if doing so does not have any effect on the great challenges and transformations of our day. Therefore, according to this author, both in the initial and ongoing training of mathematics teachers, the necessary knowledge and tools should be provided so that teachers can play an effective role as agents of social change. From this perspective, our aim is to propose a series of task to enable teachers to play an effective role as agents of social change by integrating sustainability in early childhood mathematics education.

Tasks for integrating sustainability from mathematics education

Below are three tasks that integrate sustainability from mathematics education and discuss the National Council of Teachers of Mathematics [NCTM] (2000) standards involved and their link to the SDGs (UNESCO, 2017).

Task 1: Can we run out of bread? *Description:* Every day in classrooms a lot of leftover bread is thrown away and on a global scale it is a problem called food waste. By becoming aware of the processes involved in the production of this food, it is possible to change consumption habits and assess whether bread can run out and what can be done to prevent this from happening. *Data Analysis and Probability:* Propose questions and collect data about themselves and their environment; Data representation using concrete objects, drawings and graphs; *Measurement:* Understand how to measure using non-standard and standard units. *SDGs:* SDG 2 Zero Hunger; SDG 3 Good Health and Well-being; SDG 12 Responsible Consumption and Production.

Task 2: What do we find on the beach? *Description:* Increasingly, different elements are found on beaches, some of which can pollute the environment. Based on the question posed, different elements

are identified and classified, and their origin and the consequences of human action are reflected upon. *Algebra:* Select, classify, and order objects by size, quantity, and other properties. *SDGs:* SDG 14 Life Below Water; SDG 13 Climate Action.

Task 3: Houses around the world *Description:* Through the discovery of houses around the world, students will learn how to describe a house. In addition, they will understand that each country has different houses depending on resources and climate. *Algebra:* Select, classify, and order objects by size, quantity, and other properties; Describe qualitative changes; *Geometry:* Recognize, construct, draw, compare and classify two- and three-dimensional shapes. *SDGs:* SDG 10 Reduced inequalities; SDG 11 Sustainable cities and communities.

Final considerations

The analysis of the mathematical tasks has shown that it is possible to establish bridges between education for sustainability and mathematics education from an early age. Furthermore, the treatment of the tasks makes it possible to address the various standards of the mathematics curriculum (NCTM, 2000). The mathematical task proposal that has been presented follows this approach, with the purpose of guiding teachers working in the first years of primary school in the design of authentic proposals and contextualized in sustainability. Therefore, it is necessary to offer teachers tools that allow them to develop students' motivation and ability to understand, interpret and critically evaluate, that is, to lead students, in the long term, to conscious decision-making in order to create a more sustainable world (UNESCO, 2017).

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With great freedom comes great responsibility: Exploring potentials and pitfalls of using open environments through a discourse analysis

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Keywords: Task design, open environments, digital technologies, teaching, discourse analysis.

Motivation and theoretical framework

Many studies grounded in socio-constructivist approaches highlight that open-ended assignments in open learning environments promote student learning and participation in mathematical discourse. (Baccaglini-Frank et al., 2020). Aligning with this perspective, the DynaMat project has designed and implemented sequences of mathematical activities[1] with interactive digital environments with Italian 10th-grade students (https://www.carme.center/progetto-dynamat/). These activities were designed to promote exploration and encourage students to develop various solution strategies. However, despite these intentions, the author, who participated in the project as a tutor for one of the students, frequently encountered challenges in helping the student to make sense of the activities, and she felt the student's answers did not align with her expectations. Therefore, in this poster proposal, we aim to explore the potential of such activities in open digital environments in terms of the discourse they allow to create, focusing on the tutor's role in these situations.

To capture the aspects of the mathematical discourse emerging during the activities, we use a commognitive lens (Lavie et al., 2019). Within this framework, a *task situation* is defined as any setting in which a person feels obliged to act, often created by a teacher's assignment or question. The student's interpretation of the task situation, i.e. what they consider that needs to be done, is defined as the *task*, while the *procedure* is what the student does in response to the task they interpret.

The research aim is to study the role of the tutors in the context of DynaMat activities by analysing students' and tutors' discourses to identify the tasks emerging through the interaction. More precisely, we ask the following questions: How do the students' tasks change in the context of open-ended assignments with digital environments? How does the tutor manage these changes?

Methods

In this poster we will present an example of an activity involving a GeoGebra artifact designed for the dynamic exploration of functions, called *dynagraph* (Antonini & Lisarelli, 2021). In this activity, students manipulate a slider representing the independent variable along a numbered line, while parallel lines show sliders for the dependent variables, defined by mathematical expressions input by the students. The assignment that sets the task situation in this case is: "By entering any expression you like, make sure that one of the sliders is always to the right of the other." The goal is to engage students in exploring relationships between functions by observing the movements and relative

[1] We use the word "activity" to refer to the interaction with a digital artifact to accomplish a given assignment. We avoid using the word "task" to prevent confusion with the commognitive construct that we introduce later.

positions of the sliders while developing a mathematical discourse on how expressions should be formulated to achieve the desired relationship. Two tutoring sessions in which this activity was done were video-recorded and the discourse of students and tutors was analysed, identifying the tasks by looking at the procedures implemented. The discourse analysis will be illustrated on the poster.

Findings

The first episode involves a tutor, Giulia, with two students, Irma and Sonia. From the discourse analysis of this episode, it emerged that, after accomplishing the initial task situation, the students shifted to new tasks, such as finding alternative solutions or generalizing their findings. This led them to implement various procedures that achieved the goal of the activity and encouraged the tutor to propose new task situations.

Another episode involves the author interacting with a student named Carlo. From the analysis of this episode, a *communicational gap* (Sfard, 2023) was identified: the student's task differed from the tutor's task. This is evident from Carlo's continued use of the expression "ahead" instead of "to the right" to describe the relative position of the sliders. Hence, the tutor shifted her task, engaging in a discussion on the relationships between the expressions "greater than", "to the right", and "ahead" on the number line. However, this still did not help the student reach the goal of the activity.

Discussion and conclusions

These examples highlight how open-ended assignments can generate a wealth of diverse tasks, while also leaving significant responsibility to the tutor to manage these situations. Our broader goal is to develop teacher guides that can support educators in exploiting the richness of open digital environments to align with mathematical and educational goals while offering suggestions on how to adapt the activities to meet the specific needs of their students.

Acknowledgment

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TWG 23: Implementation of Research Findings in Mathematics Education

An introduction to TWG23: Implementation of research findings in mathematics education

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Introduction

This introduction outlines the thirteen contributions presented at TWG23 during CERME14. The content of the contributions revolves around seven pivotal themes, to be elaborated on below. Furthermore, three thematic discussions took place, emphasizing the following key points: Core Elements of Innovation and Fidelity of Implementation; The Life Cycle of Implementation and Criteria for Success; and Conflicting Goals, Perspectives, and Theories in Implementation. The thematic discussions in TWG23 at CERME14 were not explicitly related to any particular presentation but instead concerned a holistic view of the implementation research field.

Contributions to the TWG23 at CERME14

The TWG23 at CERME14 was led by Ola Helenius, Boris Koichu, Linda Marie Ahl, and Julie Vangsøe Færch. The TWG23 accepted 11 papers and 2 posters, addressing different aspects of implementation research in mathematics education. The authors of the contributions came from Denmark, Israel, Sweden, the Faroe Islands, and the United Kingdom. The contributions were organized into thematic categories:

1. Reporting guidelines for implementation research
2. Design principles
3. Fidelity and adaptations
4. Scaling issues
5. Replication
6. The role of Habits
7. Pre-, present-, and post-implementation issues

Reporting guidelines for implementation research

The session began with a poster by David Nordqvist, Tina Åkegård, Ola Helenius, and Linda Marie Ahl, exploring what should be reported in implementation research. Hill, Lynch, and Polanin (2024) recently highlighted that inconsistent reporting of classroom interventions impedes knowledge accumulation. They recommend structured guidelines that detail aspects such as intervention content, teacher preparation, and implementation support. Nordqvist and colleagues tested this framework by applying it to "Algebra Monday," a program aiming to strengthen foundational algebra across grades. Using a cascading scaling model, the program trains lead teachers, who in turn train others within their municipality. The researchers found Hill et al.'s guidelines to be broadly applicable, although they lacked elements specific to implementation and scaling. They concluded that the framework is promising but needs to be extended to

capture scaling design processes.

Design principles

Julia Tsygan presented a study on implementing the MUSK model to enhance students' mathematical communication. Instead of the typical fidelity-versus-adaptation lens, the study used an integrity framework, examining how core elements were preserved while adapting to local needs. Using the Professional Obligations model (Herbst & Chazan, 2012), it explored how institutional and practical demands shaped decisions. Adaptations, such as feedback cycles and explicit reasoning instruction, made the model more sensible, although they also revealed tensions between innovation design and school conditions. The study emphasized the need for alignment with professional norms and assessment structures.

In the Faroe Islands, a project used design research to address declining PISA performance. With support from the Ministry of Education, the initiative applies iterative, classroom-based principles to bridge theory and practice. Building on the work of Century and Cassata (2016) and Koichu et al. (2021), this framework views innovation as an ecological disruption and promotes teacher collaboration through Communities of Practice. Project MAC targets mathematical modeling and digital tool use through three guiding principles: problem formulation, digital tool integration, and teacher collaboration. It operates across three schools, emphasizing iterative cycles and combined evaluations to inform broader application.

Fidelity and adaptations

Hanna Palmér, Jessica Elofsson, and Camilla Björklund explored fidelity and adaptation in scaling a preschool math intervention. Using frameworks from Century and Cassata (2016) and Coburn (2003), they evaluated how key elements were preserved or adapted. Coburn's dimensions—depth, sustainability, spread, and ownership—guided the analysis. Though early results were promising, scaling raised challenges in maintaining integrity. Productive adaptations occurred when teachers understood and applied core concepts in a flexible manner. Rigid use or superficial changes often limited effectiveness. The authors noted that local contexts, like Sweden's emphasis on preschool teacher autonomy, significantly influence outcomes.

Mark Boylan's contribution examined fidelity through two lenses: design and theory. In mathematics education, where teacher agency is vital, striking a balance between these is essential for effective scaling. Using the Maths Excellence programs as a case, Boylan showed how different components may require different fidelity approaches, aligning core principles with contextual flexibility.

Scaling issues

Uffe Thomas Jankvist, Linda Marie Ahl, Johan Prytz, and Iresha Ratnayake presented a review showing that implementation success varies by scale. Small-scale projects benefit from direct links between developers and practitioners. Medium-scale efforts rely on structured support and voluntary participation. Large-scale initiatives often succeed when focusing on curriculum rather than pedagogy, as altering teaching practices proves to be more challenging. Scaling pedagogical change requires deep, context-specific support rather than top-down mandates.

Tomas Højgaard discussed the Danish Ishøj Project, which aimed to embed competency-based teaching through school-based seminars and teacher collaboration. While the initiative reached all K-9 math

teachers in the municipality and showed progress, it faced challenges such as limited supervisor expertise, leadership gaps, and constrained resources. Effective scaling, the study concludes, needs strong support systems and leadership commitment.

The MIST (Middle-School Mathematics and the Institutional Setting of Teaching) project, presented by Johan Prytz, Iresha Ratnayake, Linda Marie Ahl, and Uffe Thomas Jankvist, used a Design-Based Implementation Research approach. MIST's Theory of Action focused on leveraging district leadership to enhance teaching quality. The project emphasized structured communication and accountability, with district leaders translating feedback into professional development strategies. MIST's success depended on systemic alignment, suggesting that replicating it requires similar leadership capacity and infrastructure.

Julie Vangsøe Færch presented the only replication study, which examined young students' proportional reasoning. The discussion centered on the role of replication in validating findings, understanding contextual limits, and exploring how adaptations affect outcomes. The overlap between replication and implementation research was acknowledged, with both focusing on variability and context.

Habits and Routines

Andrew Grant explored how interleaved practice challenges teachers' planning habits. Although not unique to mathematics education, this practice necessitates adjustments in how teachers structure lessons. For successful implementation, habit formation strategies must be integrated into professional routines, ensuring that innovations become an integral part of daily teaching.

Pre-, Present-, and Post-implementation projects

Morten Misfeldt and colleagues investigated the integration of programming into math education, which remains a challenge. Tools like Scratch can introduce unfamiliar content that clashes with standard curricula. The team distinguished between Computational Thinking (CT), which aligns with existing structures, and Computational Literacy (CL), which requires broader pedagogical change. Based on Piagetian theory, CT is seen as assimilation, while CL reflects accommodation. The study illustrated how each reshapes geometry learning differently.

Jason Cooper and Boris Koichu discussed post-implementation sustainability. In Israel, some teachers continued to engage in problem-solving after formal incentives ended, motivated by enjoyment. Although classroom integration remained limited, their ongoing involvement suggested increased agency and readiness for future innovation.

Finally, Cecilie Carlsen Bach and colleagues introduced a poster on Technology Comprehension (TC) in math education. Using speculative tarot cards, they encouraged teachers to imagine future digital learning scenarios. This participatory approach aimed to foster ownership and align educational innovations with the perspectives of both teachers and students, countering top-down impositions.

Emerging Themes in Educational Implementation Research

One strength of the CERME conference, particularly within smaller discussion groups like TWG23, is the time available for deep dialogue. This summary captures overarching discussions not tied to individual papers but relevant to the broader field of implementation research.

Core Elements of Innovation and Fidelity of Implementation

There is increasing agreement that innovations should clearly define their core components. These "non-negotiables" support fidelity but must still allow space for context-sensitive adaptations. Determining how specific these elements should be remains a challenge, especially as implementation moves from research to practice. Definitions of *productive adaptation* differ, and while overarching concepts are applicable, local adaptations vary widely. Teachers often carry the burden of interpreting innovations, so involving them in co-design from the outset was identified as key to balancing clarity and flexibility.

The Life Cycle of Implementation and Criteria for Success

Implementation efforts typically unfold across dynamic life cycles influenced by institutional backing, funding, and stakeholder engagement. Our discussions raised questions about how to define success, whether in terms of short-term project outcomes or longer-term, systemic change. Readiness for change, both at the individual and institutional levels, was considered essential but challenging to sustain over time. Success criteria should include fidelity of implementation, stakeholder learning, and sustainability, in addition to student learning outcomes.

Conflicting Goals, Perspectives, and Theories in Implementation

The group acknowledged that differing goals and perspectives often shape implementation efforts. At the practitioner level, clarity on shared strategies or theories of change is usually lacking. Similarly, within the research community, debates continue about how much conceptual agreement is necessary to advance the field. This diversity can be productive, but may also create barriers for newcomers. A significant theme was the inherently political nature of educational settings, in contrast to more technical fields. Ignoring these dynamics risks oversimplifying implementation. Researchers were encouraged to address policy, power, and ideology explicitly, as exemplified in Mark Boylan's work.

Future Themes and Research Directions

The group identified key future directions that may be explored further at CERME15:

- Specifying core elements across phases and contexts
- Defining and evaluating success and failure
- Understanding transitions between small-, medium-, and large-scale implementations
- Revisiting foundational literature to develop updated theoretical tools

The discussions reinforced CERME's spirit of collaboration and thoughtful critique. Building implementation research within mathematics education requires shared commitment, open dialogue, and a culture that supports both rigor and inclusivity.

The implementation of mathematics education programmes: fidelity to design and fidelity to theory

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Considering fidelity and adaptation are important to the implementation of mathematics education programmes for designers, evaluators, and users. Researchers sometimes view fidelity and adaptation as opposites, where fidelity refers to faithfully implementing processes, and adaptation deviates from this. However, for some mathematics education programmes this is a false dichotomy. An expanded understanding of fidelity can address this in cases where fidelity to programme theory or causal assumptions is important rather than fidelity to planned implementation or design. The differences between these two forms of fidelity and the value of considering both are illustrated by considering a set of four programmes, each with multiple components, that aim to improve outcomes and progression for disadvantaged pupils in England.

Keywords: Fidelity, adaptation, implementation.

Introduction

The concept of fidelity is central to the study and evaluation of implementation in education generally and important in mathematics education (Munter et al., 2016). In the study of implementation in education, fidelity carries both technical and common language meanings and is also used in various ways in mathematics education (Jankvist et al., 2022). Adaptation is also important to innovation in mathematics education for designers, evaluators, and users, including when implementing research findings. Fidelity and adaptation are sometimes viewed as opposites. However, clarifying different meanings of fidelity dispels this false dichotomy. In this paper, I distinguish between fidelity to design and fidelity to theory.

Fidelity to design is the common meaning used in the study of implementation and means adherence to a programme's design, structure, and processes: doing what was intended and planned for (O'Donnell, 2008). Attention to fidelity to design allows for the identification of essential features of innovations and ensures these are implemented when programmes are replicated. Where the programme is not implemented as planned, attending to fidelity brings a focus to key moderating influences and contextual factors. However, where innovations are being developed, adaptation will be inevitable and desirable to develop an effective innovation. This will often be the case in early applications of research findings or in design research phases in many mathematics education innovations, but also when replicating or scaling to understand contextual influences (Cai et al., 2020). Further, there are features of mathematics education curricula, pedagogy, and professional development where flexibility, agency, and an adaptive orientation are essential to intended practice**.** In such cases, what is important is fidelity to the underpinning theory of how activity will lead to outcome, the core ideas of the innovation (Cai et al., 2020; Jankvist et al., 2022), and echoes arguments for flexibility within fidelity (Kendall & Frank, 2018).

These two meanings may not cover all cases. In some cases, where very high levels of adaptation may be desirable or acceptable, for example, in pilot or exploratory stages of innovations, a further form of fidelity may be relevant. This is fidelity to programme goals. McNeill et al. (2018) report an example in science education where they contrast fidelity to procedure with fidelity to the goal—in the case they consider the goal of developing scientific argumentation. It is not clear in such cases whether it is then meaningful to consider activity as implementation, and so I note this third possible meaning of fidelity but do not consider it further in this paper.

In the next two sections, I develop the argument for distinguishing the two forms of fidelity. I consider fidelity in implementation science in general and then briefly summarise discussion of fidelity in mathematics education, drawing on examples from mathematics education innovations. Following this, I illustrate the application of the two forms of fidelity through an example of four programmes that aim to improve outcomes and progression for disadvantaged pupils in England.

Fidelity and implementation

As stated, fidelity is a key concept in implementation science, and proponents of its importance have developed various models of different aspects of fidelity and how frameworks to guide its measurement are based to varying extents on conceptual theorising or reviews of how fidelity is used and applied by researchers and others (see Gage et al., 2020 for examples). These frameworks are concerned with the common meaning of fidelity as implementation of the designed programme. For the purposes of distinguishing between the implementation of design and the implementation of theory, and given the limits of this paper, I illustrate with components of an early influential model of fidelity (Dane & Schneider, 1998). In summary in this model, fidelity consists of a combination of:

- adherence of delivery to the prescribed programme
- exposure of recipients to the programme activities
- quality of delivery
- responsiveness of participants
- differentiation of participant experience by planned activities only.

Later developments of models may have different or additional components, but in common with the above features, the focus is on the relationship between the planned activities in a programme and implemented activities.

In the study of implementation, there are different views of the value of fidelity, with two positions that can broadly be summarised as pro-fidelity and pro-adaptation perspectives (Century & Cassata, 2016). A pro-fidelity perspective considers it important that effective practices should be implemented without deviation and that any changes should be made in a design phase. In this view, users should be encouraged and supported to implement with fidelity. A pro-adaptation view considers that high fidelity of implementation may be hard to achieve in practice and innovation users will respond to context, and that those implementing innovations are able to add effective strategies or improve the innovation. Such adaptations can mean innovations can be more widely applied beyond any one specific context, and this can support sustainability (Century & Cassata, 2016). For

new research-informed innovations, adaptation in use aligns with a different model of the relationship between researchers and designers and practitioners in which there is collaboration and co-design. Practitioners are positioned as agentic rather than end users.

Underlying the pro-adaptation perspective is an emphasis on the underpinning causal assumptions for a programme or its component parts or the programme theory (Ahl et al., 2022; Munter et al., 2016). 'Theory' is used in multiple ways with nuanced meanings, including when considering programme implementation (Helenius et al., 2023). It is often used as part of a compound phrase, 'Theory of Change', but this term too has different interpretations, often being used to mean the visual representation of programmes casual structure (Boylan, 2023; Chen, 2012). Other terms are used instead of or alongside 'theory' to refer to the same or similar phenomena, such as determinants (Chen, 2012; Helenius et al., 2023). though often used with reference to implementation and contextual determinants (McLoughlin et al., 2021). Others use 'mechanism' to identify causal relationships (Lewis et al., 2018), often drawing on realist social theory (Coldwell & Maxwell, 2018). Conversely, statisticians may more commonly consider 'assumptions' in inferential models (Colnet et al., 2024). However, regardless of language used, these different terms draw attention to how designed programme processes and structure may cause intended outcomes. Thus, fidelity to the casual relationships of a programme or a component is fidelity to programme theory.

From a pro-adaptation perspective, considering fidelity to theory as a separate construct is helpful in addressing what adaptations are acceptable or desirable for each specific innovation or component. Acceptable adaptations support the enactment of the programme theory, and desirable adaptations are those that help to amplify the programme theory mechanisms. In cases where programmes are in a mature stage of development, including when core components and essential fidelity are established, then fidelity to design will often include fidelity to theory, as the theory is embedded in the programme design, process, and structure.

Mathematics education and fidelity

Turning to mathematics education specifically, it is important to consider implementation in connection with the specifics of the discipline and the mathematics education ecology (Koichu et al., 2021). In mathematics education, specifically, adaptation is likely and may be desirable and intended (see, Ahl et al., 2022, for an example). In mathematics education, a central concern is for the interplay between mathematical content, teacher activity, the learner, and the learning environment. While the conceptualisation of this interaction varies in different paradigms, it is significant in arguably all traditions in mathematics education and underscores the significance of responsive adaptation of practice. Adaptation is particularly important in pedagogical approaches that stress the importance of both teacher and learner agency in the learning process. Beyond individual classrooms, similarly adaptive practice is important to the implementation of mathematical education programmes.

For some mathematics education practices, fidelity and adaptation are a false dichotomy. This is because fidelity to the underlying principles of the programme or activity may require adaptation of materials or practice. I take as an example the implementation and evaluation of a professional and curriculum development programme in England for lower secondary pupils—Realistic Maths Education (RME) (Demack et al., 2022). In the RME programme, one aim was for teachers to move

towards greater use of models and representations in teaching than their previous practice. Teachers received curriculum and teaching resources, along with professional development, to encourage the use of bar models, ratio tables, and double number lines. It was thought that if teachers could choose and use different models in different ways, it would first help students learn more and then help students learn how to choose the right models to help them solve problems. Given this, although teacher use of programme materials was important (fidelity to design), the fact that these were used flexibly and with adaptation was also important (fidelity to theory).

A typology of three types of professional learning frequently found in innovations provides a heuristic to identify whether fidelity to design or fidelity to theory, or both, are important specifically for mathematics innovations that include professional development (Boylan & Demack, 2018). Technical professional learning aims for teachers to use new techniques or use new materials in specific ways. Here fidelity to design is most relevant. Pedagogical professional learning focuses on growth and change as a professional through reflective and agentic practice. Here fidelity to theory is most relevant. Curriculum professional development combines professional learning with the use of research-informed curriculum materials. Here, as in the RME programme, it is likely that both forms of fidelity should be considered.

Regardless of the forms of professional learning or the value in the typology, for any given programme or component of a programme, three possibilities are possible in terms of the importance of the two different forms of fidelity (for instances where fidelity is important at all):

- fidelity to design
- fidelity to theory
- fidelity to design and fidelity to theory

The MX programmes as an illustration

I now consider an example to illustrate the application of the different forms of fidelity in practice. This example is a set of four programmes – the MX programmes - funded by the Maths Excellence Fund (n.d.) in England, informed by research on relative attainment of students across their education experience in relation to their demographic characteristics (Noyes et al., 2023). The programmes aim to support disadvantaged students to attain highly at 16 and for more to progress to A-level and university mathematics, with 46 secondary schools participating. The National Foundation for Education Research is evaluating the four programmes, and I advise on the application of Theory of Change to the evaluation. The four programmes are led by several types of organisations.

Each programme is comprised of multiple components (see Table 1), with some types of components common across programmes and some specific to individual programmes. The MX programmes' overall meta-theory posits that addressing systemic and long-standing inequalities in mathematical outcomes requires multiple programme components that can have an additive impact.

Some components involve activities similar to activities often reported on in mathematics education research: parental engagement, teacher training and development, classroom technology use, tutoring, and mentoring. Others are more specific to the programme or to the context in England:

- tracking and monitoring target groups of pupils and collecting data on outcomes

- In-school champions who coordinate and encourage participation
- An extended, more challenging mathematics curriculum for 11- to 14-year-olds
- Additional supplementary maths qualification for 16-year-olds
- Y6-Y7 transition to support pupils' mathematical learning as they move from primary (elementary) to secondary (junior high school) phases
- Maths enrichment through after-school maths clubs or mathematics events at universities
- Interschool competitions between local schools inspired by maths Olympiad competitions.

Fidelity of design and theory in implementation and evaluation

Table 1 summarises the different components for the four programmes and, for each programme, posits whether fidelity to design (d), fidelity to theory (t), or fidelity to design and theory (d & t) is most relevant. Empty cells show that the component is not part of a programme. These ascriptions are an interpretation informed by my analysis of programme implementation plans and logic models and then checked with the team responsible for oversight of the four programmes for the funder. For some components, expected outputs from activities are identified, for example, the number of students participating in additional coaching sessions. For others, the programme design indicates that activities will be developed and modified on specific sites.

We can attribute fidelity to the design, if there are specific indicators of adherence to the design, such as the number of students participating in specific activities with defined processes and structures. Ib the MX programmes, pupil tracking systems most fully exemplify this. Components that demonstrate fidelity to theory are those that allow for significant flexibility in implementation. So, for example, in three programmes, an in-school champion will be identified, but what they will do is not predetermined. In one programme, the in-school champion will be a new member of staff, working closely with the target groups of students undertaking activities that are specified in other programme components. Two types of components are considered to have both fidelity to design and relevance to theory. Some activities have plans for the number of students participating or the number of sessions offered, but programme teams will develop or tailor the details in practice to suit identified needs. Alternatively, the components feature both pre-designed subcomponents and ones that will be developed.

Table 1: Types of fidelity relevant to selected programme components

Programme components	United learning	East Maths Community	MEI	Co-op Academies
Tracking and monitoring	d	d	d	d
In school champion	t	t	t	d & t
Parental engagement	t	d		t
Teacher training and development	d		d & t	d & t

Programme components	United learning	East Maths Community	MEI	Co-op Academies
Community learning hubs		t		
Classroom technology use				t
Extended KS3 maths curriculum	d		t	
Additional maths qualifications	d	d	d & t	d
Tutoring	d	d & t	d & t	d & t
Mentoring			d	d & t
Y6-Y7 transition	t			
Maths enrichment	d & t	d & t	t	d & t
Inter-school competitions		t		

Identifying forms of fidelity that are most relevant to components supports Theory of Change development and evaluation activity. Where fidelity to design is relevant, it is important to consider appropriate fidelity measures (O'Donnell, 2008). When fidelity to theory is considered important, fidelity measures might not be suitable. Instead, when developing Theory of Change models and evaluation activity, the underpinning programme theory relevant to the component under consideration should be examined.

Conclusion

I have considered two forms of fidelity: fidelity to design and fidelity to theory. Fidelity to design corresponds to common meanings of implementation fidelity. Fidelity to theory is important, particularly for programmes in early stages of development where essential components are not yet identified.

It is also important for programmes in which adaptation, agency, and flexibility of mathematics teachers are central to programme aims and important to intended outcomes. Considering fidelity to programme theory. provides a means to assess whether adaptations are potentially useful additions to programmes or should be viewed as a failure of implementation and supports the articulation of causal theory. This may be particularly important when replicating or scaling innovations (Aguilar et al., 2023; Cai et al., 2020). The distinction between these two types of fidelity potentially has wider applicability beyond mathematics education.

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Post-funding implementation—Attending to teachers' readiness

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Our goal is this article is to contemplate what might happen with mathematics education innovations at a post-implementation stage, when funding for professional accreditation of the participating teachers terminates. To this end, we attended to our experience with running a voluntary mathematics club for teachers that features mathematical problem solving, and analysed one of the meetings of the club in some detail. We found that of four processes under the umbrella of implementation as defined by Koichu et al. (2021), traces of three processes—constructing agency, changing communication, adapting the innovation—can be identified in the club discussions. Furthermore, we provide evidence in support of a hypothesis that these processes can be related to the development of the teachers' readiness to implement the innovation. We conclude by a call to further explore anchoring and sustainability of R&D projects having limited funding periods.

Keywords: Implementation, mathematics teacher knowledge for teaching problem solving, post-implementation, readiness to implement, voluntary professional development.

Introduction

Design and implementation of research-based innovations is often stipulated by funding that the innovation proponents are capable of securing; part of the funding is sometimes designated to support professional-development (PD) accreditation for participating teachers. Not surprisingly, the sustainability of a funded implementation project is challenged when funding ends. This phase of a project brings to mind the term post-implementation, which Misfeldt et al. (2023) define (in one sense of the term) as "attendance to anchoring and sustainability" (p. 5) when adoption and dissemination of an innovation may continue until the innovation ceases to disrupt the system, possibly through general adoption or general rejection.

Misfeldt et al. (2023) do not explicitly suggest that post-implementation begins when funding ends. While this may be a natural assumption, we will argue that at least in some cases, processes identified by Koichu et al. (2021) under the term "implementation" can continue after funding ends, though in different shapes. In particular, the funded stage of an implementation may rely on incentives that do not align with the nature of the innovation as communicated to the funding agencies. For example, a common practice in PD is to require teachers to try out some innovative pedagogy and to report on their experience. This requirement can be justified, for instance, by the need to enhance mutual learning within a community of practice (Graven, 2004). When teachers comply, it is tempting to believe that they do so while recognizing opportunities to learn as planned by the proponents of the innovation, yet it is possible that they are merely doing what is required for professional accreditation. It seems possible that external motivation used to encourage teachers to enact an innovation when funding is available may in fact hinder the construction of teachers' agency over it—a central process of implementation according to Koichu et al. (2021).

We have often asked ourselves what implementation might look like without teachers' external motivation to acquire accreditation. An opportunity to investigate this occurred in 2023–2024. This

was a challenging school year in Israel, fraught with political and military unrest. Teachers, many of whom had experienced personal trauma or loss, were struggling to find ways to engage students who had many things on their minds other than learning mathematics (Koichu, 2023). In late October 2023 we initiated an online mathematics club for middle and high school teachers where we introduced an educational innovation—non-standard problem-based activities—that we felt might engage distracted teachers and students.

Our running of the mathematics club drew significantly on resources developed in the context of a funded R&D program. The 2023–2024 school year was the fourth and final year of a project called Raising the Bar in Mathematics Classrooms (RBMC, for details see Cooper & Koichu, 2021; Koichu et al., 2022). The goal of the project was to encourage grades 8–10 teachers to organize some of their lessons around problem solving (PS), supporting them in this practice in various ways. A collection of thirty problems had been developed and piloted for this project, and teachers were required—for professional accreditation—to enact PS in their classes at least three times throughout the year, and to submit online reflective reports on the lessons. In each PD session teachers worked on a problem and participated in a discussion that attended to both mathematical and pedagogical aspects of the problem (e.g., strategies and solutions, anticipated student approaches and errors, how and when to present the problem, which classes it is appropriate for, possible adaptations, etc). Additionally, teachers shared and discussed their enactment experiences with their peers. As funding for the project was nearing its end, we had been giving much thought to its sustainability. We realized that without the incentive of professional accreditation, which relied on the project's funding, it was by no means certain that teachers would continue to implement PS. In this sense we regard the voluntary (i.e., not for credit) club as a kind of testing ground for investigating models of post-implementation.

Many of the 2023–2024 club meetings were organized around problems from the RBMC collection, and the structure of the meetings drew on experience that we had gained from working with teachers in RBMC. We encouraged teachers to make use of these problems, but they were under no obligation to do so and were offered no accreditation for their participation in the club. Nearly 200 teachers participated in at least some of the club meetings in October and November. As teaching conditions gradually improved in most parts of Israel, less teachers attended each session, yet a core of approximately 16 teachers continued to attend regularly throughout the year and attended most of the 30 club meetings (see Cooper & Badarne, 2024, for details).

In January 2024 we conducted interviews with some of the regular and intermittent participants in the mathematics club (Cooper & Badarne, 2024), and found that the teachers claimed to be participating primarily for the sake of the mathematical activity, which they enjoyed, and that while they considered the problems appropriate for their students, none of them had enacted any of the problems in their classrooms. Our initial response was disappointment, first and foremost with respect to the club itself—we had not met the goal we had set ourselves of supporting implementation of PS activities in classrooms. Our findings also raised concerns regarding prospects of sustainability for the RBMC project. The club members' reluctance to enact problems in the classrooms raised concerns that teachers who had begun to implement the PS innovation might neglect this practice without the incentive of accreditation.

When we had gotten over our initial disappointment, we asked ourselves whether these teachers' voluntary participation in a PS club may nonetheless contribute indirectly to the implementation of

this innovation. While becoming a proficient problem solver is not a sufficient condition for teachers to enact PS in their classes (Chapman, 2015), we have long believed that it is a necessary condition and have organized the PD activities of RBMC around teachers' PS experiences. However, to date we have not explicated in what ways participating in PS may contribute to implementation. It is this that we wish to attend to in the current article.

Theoretical framing

Implementation occurs in interaction between the innovation, its proponents and its adapters, and, according to Koichu et al. (2021), the proponents have ultimate agency over the innovation at the beginning of the process. During the implementation, the adapters should come to experience at least some of the following four processes: "(1) constructing agency over the innovation, (2) gradually changing within-community communication or across-community communication, (3) gradually changing practice so that it accommodates the innovation, (4) adapting the innovation to their needs and aspirations" (Koichu et al., 2021, p. 986). Koichu et al. (2022) proposed an analytical framework—*problem-solving implementation chain*—for investigating implementation of PS as a sequence of interactions between actors en route from proponents to adapters in which adaptations are negotiated. For example, proponents of a problem-based innovation and teachers may interact in a PD setting and negotiate adaptations for the teachers' particular educational contexts and needs. Such negotiation will often include what should change, for what reasons and purposes, and what is lost and gained as a consequence. Adaptation may continue to take place in the teachers' individual planning, and later in their enactments.

A tacit assumption in our work to date has been that interactions along the implementation chain take place with the ultimate goal of classroom enactment with students. Thus, investigations of interactions between proponents and adapters in PD has focused on the third and fourth of the four implementation process mentioned above, namely, *gradually changing practice* and *adapting the innovation*. We now wish to challenge this tacit assumption and consider interactions between proponents and adapters that are not around immediate changes in practice and the negotiation of adaptation. In principle, such interactions might contribute to other processes of implementation.

We suggest that our attention should be on the development of teachers' readiness to implement PS, which Zaks and Koichu (2022) have taken to encompass their inclination to enact as supported by relevant knowledge. Zaks (2023) singled out the anticipation of students' responses in PS activity as an important facet of supporting teachers' knowledge for enacting PS and has found five characteristics that constitute the thickness of teachers' anticipation: specificity, detail of mathematical moves, organizational aspects, first person solver's voice, and variety.

Developing readiness to enact has imperfect overlap with the notion of implementation—not all aspects of implementation can be conceptualized as building readiness, and vice versa. However, there is significant overlap. In particular, development of agency (one of the implementation sub-processes) may contribute to an inclination to enact (aspect of readiness), and the development of supporting knowledge (aspect of readiness) can contribute to changes in discourse (one of the implementation sub-processes). In this article we are concerned with this (hypothetical) relationship, and ask accordingly: How may teachers' participation in a voluntary mathematic club that features

PS contribute to changes in agency over the problems considered in the club, in teachers' discourse on PS instruction and in their readiness to enact PS in their classes?

Data sources and analysis

Given the exploratory nature of the above research question and space constraints, we selected one meeting from a later stage of the club (February 2024), by which time there was a core of 16 regular participants. The meeting was selected as typical regarding its structure and interactions. The meeting, led by the first author, took place over Zoom and was recorded. The meeting was transcribed, and relevant parts were translated from Hebrew by the authors. In the data we sought exchanges that could be interpreted as manifesting processes of implementation (constructing agency, changing communication, adapting the innovation) and/or as contributing to teachers' readiness (inclination to enact supported by knowledge of how to enact).

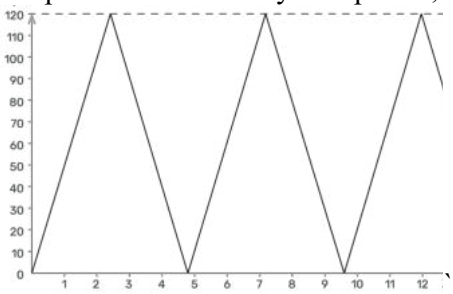
Findings and Discussion

We begin by presenting four excerpts that concern a problem presented in Figure 1, and then discuss these excerpts in terms of the above–mentioned processes and notions.

Out-and-back Trains

The distance between two cities is 120 km. A train line runs back and forth between them all day long (without intermediate stops). The train's maximum speed is 50 km/hr and it can transport up to 900 passengers. At high season the number of passengers wishing to travel on this line is greater than its capacity, and the management is looking for ways to increase the capacity.

1. How many passengers per hour can the train transport between the two cities?
2. The management intends to purchase a new train that can travel up to 120 km/hr and can hold up to 300 passengers. There is a single rail between the two cities so the trains cannot pass each other. In your opinion, should they replace the old train with a new one?



3. Inspector Gabriel had a suggestion: "Each station has a number of platforms, and they can be used to operate the two trains concurrently". Plan a timetable that will allow this. You may use the provided graph, which describes the old train's schedule during part of the day. How many passengers per hour can be transported according to your suggestion?
4. The management agreed to double one section of the track between the two cities to allow the trains to pass each other. Inspector Doron suggested: "If we double the track midway between the two cities, we can build a timetable that will allow us to transport up to 600 passengers per hour". Explain how this is possible.
5. Inspector Hila commented: "I have an idea for some other place between the cities where we can double the track instead of midway, and this will increase the number of passengers per hour even more". Suggest such an alternative place, plan a schedule and calculate how many passengers per hour it will allow. Compare your solution to Doron's.

Figure 1: The problem that teachers worked on and discusses in the club

Excerpt A: Seeking clarifications regarding Question 1 (see Figure 1)

1	Teacher R:	Can I ask a clarification question? How many passengers per hour, does that mean that a train needs to arrive at a destination once every hour? Because the first train to leave will not arrive in one hour.
2	Facilitator:	How long does it take to get from one city to the other?
3	Teacher R:	Two hours and a bit.
4	Facilitator:	And it transports 900 passengers.
5	Teacher R:	It's the same train that runs back and forth?
6	Facilitator:	Yes
7	Teacher R:	There aren't two in opposite directions?

Excerpt B: Features of realistic literacy problems (with respect to Question 2)

1	Facilitator:	The question was, *in your opinion* should they replace the old train. This invites all sorts of answers. It's true that it's less passengers per hour, but each passenger has a shorter ride. Whether that's beneficial or not…
2	Teacher V:	The question is what the goal of the management is.
3	Teacher R:	And who benefits from buying a second train.
4	Teacher G:	Perhaps you can charge twice as much for the faster train.
5	Teacher R:	And maintenance costs [for a new train] are lower too.
6	Facilitator:	One of the things I really like about literacy problems is that they invite discussions that are more interesting than you might think… You need to construct a mathematical model for "beneficial", make some assumptions, like… beneficial is measured in profit for the rail company, construct a formula for "beneficial" that is not given in the problem and justify answers based on this… Sometimes a slightly non-standard question is asked, and suddenly [students who don't normally participate] have something to say.

Excerpt C: A teacher shares her work related to Question 3

1	Teacher R:	I don't mind sharing, but I did something quite simple [shows a drawing, see Figure 2].
2	Facilitator:	It's best to start with the least sophisticated and move on to more sophisticated solutions.
3	Teacher R:	The fast train needs to wait.
4	Facilitator	Where? For what?...
5	Teacher R:	For the other train to pass.
6	Facilitator:	How can it pass?
7	Teacher R:	Ok, cancel my solution.

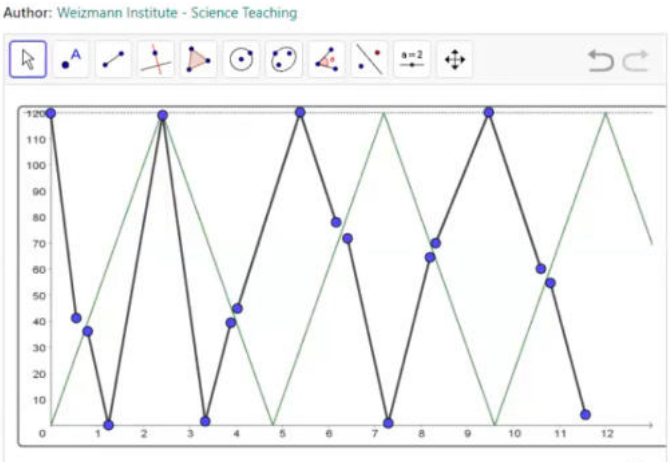


Figure 2: A teacher's flawed solution to question 3

8	Facilitator:	Save your solution. Who knows, maybe there will be a question for which this is a good answer… This is excellent. If I'd known this was your solution, I'd have chosen to begin with it and ask what's wrong with it.

Excerpt D: Teachers proposing adaptations of Question 3

1	Facilitator:	In motion problems, if someone walks part of the time and rests part of the time, and we want to know the average speed, the rest lowers the average speed.
2	Teacher V:	That's right, but when you ask a student what average speed is… [he] understands what average speed is, here [passengers per hour] it's just a calculation.
3	Teacher G:	For me it's more important that students be able to do this [work out passengers per hour] than calculate average speed. This is the use of mathematics in real life…[but] I would add a twist. The problem is there is no single hour of the day when I can say that 500 passengers travelled. If the question were how many passengers per *day* it would be less abstract.
4	Facilitator:	You can adapt the problem. I certainly don't know which is preferable… Sometimes you need to try something, help students if they need it, and do it differently next time.

Readiness to enact: Developing thick anticipations by tacitly simulating students' actions

Often problem solvers ask clarification questions that they could have answered on their own (e.g., line #1 in excerpt A), or present solutions that are flawed (e.g., excerpt C). When the problem solvers are experienced mathematics teacher, one might wonder what purpose the questions and hasty solutions serve. In our experience, teachers in PD often take on behaviours that are typical of their students. We believe that this is a kind of transference (Klein, 2018), a process studied in psychanalysis where an individual's feelings and actions, which are inappropriate in the present context (e.g., PD), are a repetition of significant situations from other contexts (e.g., teachers' classroom experiences with students). However, it is not the psychological interpretation of this phenomenon that is important, but rather its affordances in PD. We posit that such questions are often teachers' implicit anticipations of students' questions. Making these questions public serves the purpose of sharing thick anticipations (Zaks, 2023) of possible responses that are 1. Specific; 2. Presented in first person voices; 3. Varied. Furthermore, the facilitator's responses (excerpt A) offer detailed teaching moves to respond to such questions. This phenomenon was repeated by many teachers, and we present a second example:

Teacher G:	Are trains permitted to stop on the way?
Facilitator:	I don't know. It's a question you need to direct to yourself at this point.

Implementation: Facilitator promoting change in discourse around innovation and its merits

Preceding excerpt B, a teacher replied to Question 2 that the old train should not be replaced because the new one transported less passengers per hour. The facilitator led three teachers to consider alternative answers based on different mathematical models of "beneficial". Making these answers public may help teachers anticipate productive responses from their students, which may increase their inclination to enact. Later (line #6) the facilitator presented some key components of the modelling cycle and discussed merits of the innovation that are not obvious, about the multiplicity of correct answers and the opportunities for all students to participate. In this he was promoting a process of implementation by leading a change in teachers' discourse on PS. Furthermore, teachers often shy away from flawed student solutions for fear that revealing the flaws will discourage these students. In excerpt C the flawed teacher-solution allowed the facilitator to explicate some pedagogical tips:

How to capitalize on errors while valuing the quality of thinking that led to them (line #8), and how to sequence a discussion based on students' work (beginning with incomplete or flawed work).

Implementation: Teachers considering adaptations and constructing agency

Excerpt D was triggered by a teacher's difficulty to deal with units of passengers per hour. We deem it significant that the difficulty served as a springboard to consider adaptations (passengers per day instead of per hour). Of note is that even though the teachers had no intention of enacting the problem in their classes, they spoke of the activity with respect to imagined students (lines #2, #3).

Summary and contribution

In this article, our goal has been to begin contemplating what might happen with mathematics education innovations at a post-implementation stage (Misfeldt et al., 2023) denoted by termination of funding for professional accreditation of the participating teachers. For this, we attended to our experience with running a voluntary mathematics club for mathematics teachers, whose operation relied on experience gained and on instructional materials developed during the funding period.

We have found that of four processes under the umbrella of implementation as defined by Koichu et al. (2021), traces of three processes—constructing agency, changing communication, and adapting the innovation—can be identified in the club discussions. This finding sheds light on the specificity of anchoring and sustainability associated with the post-implementation notion (Misfeldt et al., 2023). Furthermore, we have provided evidence in support of a hypothesis that these implementation–related processes are related to the development of the teachers' *readiness* to implement the innovation (Zaks & Koichu, 2022; Zaks, 2023) even though the circumstances prevent them from so doing in proximity to the club meetings. Namely, we found that the teachers' anticipations of what might happen with the problem of interest in a classroom were specific, varied and alluding to students' first-person voices. To this end, we attribute value to teachers' intensions to implement, and their developing experience as problem solvers and potential enactors of problems that were not easy for them to solve, even when there is no actual implementation in classrooms. We realize of course that this stance is speculative and needs to be tested by following individual teachers' future classroom practices.

Speaking more generally, we suggest that impact of R&D projects having limited periods of funding should be further explored not only in terms of the innovation dissemination with students but also in terms of its effect on teachers' knowledge, worldviews and affiliation with communities of practice (see. Cooper & Koichu, 2021, for elaboration of the notion of *impact*). Needless to say, our study is only a small step on the way to reveal diverse phenomena of post-implementation. Additional research in this direction is desperately needed.

Acknowledgment

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The role of habits in a theory of change: Lessons from an implementation of interleaved practice

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There has been a growing call for an explicit theory of change to be used within mathematics education implementation research to improve both the coherence of the implementation and its evaluation. This paper uses a pilot study into the implementation of interleaved practice to investigate the role of habits and habit forming within its theory of change. The results showed that the teachers relied heavily on personal and easily applied routines in order to implement interleaved practice. It is recommended that for future iterations, the role of routines is understood within the theory of change with clearer strategies explicitly shared to aid the routine forming and ultimately habit formation. It is argued that the steps towards habit formation would benefit from being explicitly considered in any theory of change.

Keywords: Implementation research, theory of change, habit formation, interleaved practice.

Introduction

There is a call within implementation research to explicitly use a *theory of change* shared between all the stakeholders in order to both guide the process of implementation and evaluate its success (Jankvist et al., 2021; Century & Cassata, 2016). In this paper, we address the role of teachers' habits and routines in a theory of change. In an overview of professional development research, Kennedy (2016) explains when teachers undertake professional learning, they are not simply assimilating new ideas but abandoning an old practice, which has guided them in the past and is likely to be habitual.

The habitual nature of teaching is highlighted by Hobbiss et al. (2021), who emphasise the insensitivity of habits to goals or rewards. This seems important in implementation research, in particular given the difficulty in sustaining a reform after an initial introduction, which leads Coburn (2003) to give sustainability as one of four key dimensions needed for lasting change. In order to understand the efficacy of professional development, Kennedy (2016) questions the traditional process model, calling for more focus on strategies and theories of action. An example in this vein is Lindvall et al. (2022), who analysed the Swedish program of Boost for Mathematics and concluded that the lack of effect of the program could in part be down to the overemphasis of pedagogies of investigation (what to know) at the expense of pedagogies of enactment (what to do).

This paper aims to build on the understanding of *theory of change* by investigating the role that habits play when an innovation in mathematics education is implemented. In our case, the innovation concerns interleaved practice.

Interleaved practice, and its implementation, is a suitable case for several reasons. Interleaved practice in mathematics teaching concerns the sequencing of tasks and exercises, which is an essential part of mathematics teachers' habits and routines. Moreover, it has a growing experimental research base, but there are numerous barriers to implementing the innovation of interleaved practice, see below.

The research question is: How can *routines* and *habits* inform a *theory of change* in the case of implementing interleaved practice?

Theory

This project is within the implementation research tradition of investigating the interaction between an innovation and the real-life settings of its implementation, where an innovation is a practice that is "new to the user" (Century & Cassata, 2014, p. 82). As such implementation research examines "the ways innovations are used in practice, the impact of contextual factors on enactment, and the extent to which particular enactments have strong relationships to desired outcomes" (Century & Cassata, 2014, p. 98). A theory of change has been described as "a systematic and cumulative study of the links between activities, outcomes, and contexts of the initiative" (Connell & Kubisch, 1998) and can be shared amongst all the stakeholders for the effective implementation and evaluation of the innovation (Jankvist et al., 2021). Jankvist and colleagues argue that sharing the desired processes that lead to successful outcomes for the pupils can increase the coherence of the implementation through all its stages as well as enabling a more accurate theory-based evaluation of the implementation, that is how and why it did or did not succeed in its goals. In this paper, we focus on how habits relate to the links within a theory of change.

Teaching is complex with a high degree of repetition which leads to the need to automate tasks in order to reduce the cognitive load for the teacher (Feldon, 2007). Habits are how we characterise this automation and are understood to be structured behaviours which are triggered by environmental cues and are often insensitive to goals or rewards, particularly under stress (Hobbiss et al., 2021). Habits can offer an explanation for the levelling off of teacher performance, where after initial rapid improvement, the rate of improvement decreases or levels off entirely which coincides with habitual behaviours (Hobbiss et al., 2021). The automaticity is double edged as although it can make changing practice more challenging given habits' automaticity despite competing goals, it also reduces the cognitive load for a teacher (Feldon, 2007) and thus make a challenging profession easier. Habits form through repeated actions in response to contextual cues, but deliberate choices can hinder this habit formation (Wood & Rünger, 2016).

In this paper we use interleaved practice as case study to discuss the role of habits in a theory of change.

Interleaved practice

Interleaved practice in its broadest sense defines the practice undertaken by pupils, where the questions answered are rotated between different concepts and is best understood in comparison to the traditional blocked practice where all the questions on one exercise use the same concept and are often very similar in style. For example, a traditional blocked approach to teaching Pythagoras' theorem might have a first practice lesson with plain triangles and substituting the side lengths into the formula and solving; the following lesson would be a set of word problems where the pupils have to construct the triangles themselves before substituting the values into the formula and solving. In both sessions the pupils are practicing the execution of a single strategy (Rohrer et al., 2014). In contrast interleaved practice includes a mix of questions which require different strategies forcing the

pupil to both choose a strategy and practice its execution. For example Mayfield and Chase (2002) asked pupils to choose the correct arithmetic strategy to use from different alternatives when manipulating algebraic exponents and then execute that strategy, while Rohrer et al. (2014) used four superficially different types of problems.

The empirical research on interleaved practice has shown fairly consistent results, with the learners often initially performing better with blocked practice, but this quickly reverses with blocked learners not retaining their knowledge (Mielicki & Wiley, 2022; Rohrer, Dedrick, Hartwig & Cheung, 2020; Rohrer & Taylor, 2007). For example in Rohrer and Taylor's (2007) comparison of interleaved practice (mixers) and blocked practice (blockers), the blockers performed better in the practice with 89% accuracy compared the mixers' 60%. These results were reversed in a test a week later with the blockers at 20% and mixers 63%. In summary, within mathematics education interleaved practice seems to promotes long-term retention over short-term performance.

Teachers, who participated in a trial of interleaved practice, were positive towards the use of interleaved practice (Rohrer, Dedrick, Hartwig & Cheung, 2020). In a survey the majority reported that they believed that it had a positive effect on pupil outcomes and would be easy to assign without changing their teaching routines, even if it took a slightly longer time and the students found it a little harder.

Interleaved practice was initially concieved as an inductive category learning tool, where similar but contrasting concepts are used to foster learning. The interleaving effect was ascribed to either the contrast between the examples (discriminative-contrast hypothesis) or the increased attention paid to the examples when compared to the later problems in blocked practice (attention-attenuation hypothesis) (Firth et al., 2021). However mathematics education research has expanded the concept to include the consolidation of learning with the efficacy of this practice put down to the required conscious strategy choice combined with the known advantages of spacing and retrieval practice (Rohrer et al., 2014). Both inductive and consolidating interleaved practice could be described as a desirable difficulty (Bjork & Bjork, 2019), where the process of forgetting and remembering supports better long-term retention over short-term perfomance.

In all the experiments on interleaved practice researchers have created resources with the questions either blocked or systematically interleaved. For interleaving to become implemented widely, teachers need to have access to such resources, however there is currently a scarcity of interleaved practice in premade resources and textbooks in particular (Rohrer, Dedrick & Hartwig, 2020), meaning that teachers need to create their own. The need for teachers to generate the material brings the challenges of writing resources faithful to interleaving as a mechanism, how to disturb teacher habits to fulfill this process, and finally whether this disturbance overburdens teachers' time and energy. A theory of change which addresses these issues can only aide the implementation process.

Methodology

This paper is based on a pilot study into the implementation of interleaved practice in mathematics education. Six teachers from two Swedish lower secondary school mathematics departments were given a half day's training session, which included a theoretical introduction to understand the

advantages of desirable difficulties and the benefits of interleaved practice over blocked practice, followed by a brief discussion on how it could be implemented in a real-life classroom. The teachers had a mixture of experience from newly qualified to 10 years of experience. To track the implementation and potential habit formation the teachers' use of interleaved practice was monitored over a 2-month period in weekly 6-question questionnaires before interviews were undertaken to understand the mechanisms of change. The interviews were transcribed and then thematically coded in order to understand the usage of interleaved practice, the process of implementation, and potential repeated actions on the road to habitual use.

Results

All the teachers reported that they understood the theoretical advantages of interleaved practice and could relate to the problems of blocked practice as their earlier usage of blocked practice often led to short-term success which was not sustained in the medium or long term. However, this understanding did not lead to an easy or immediate implementation of interleaved practice, rather they all developed their own individual routines in order to embed it into their own teaching. These easily repeated actions were unique in both how they created the resources and how they implemented them in their regular lesson structure.

Newly qualified Teacher 1 described their creation process as follows:

> I was creating worksheets in the beginning but now that I'm doing relationship and change, I'm picking up questions myself. And so, I'm not trying to reinvent the wheel anymore. In the last few days, last few weeks what I did was I picked up summaries from all the [textbook] chapters. We have a test coming up and it's a big one, like some number, some algebra for A-level questions and then relationship and change, the whole chapter. So now I'm using different summaries and it's more flowing through the practice now.

The teacher described how they eventually found a more efficient and sustainable routine to create their resources. The sense of before and after finding their routine was very clear in their weekly usage of interleaved practice. In the weeks before changing the method 36% (n=14) of their lessons used interleaved practice, while afterwards this increased to 73% (n=11) of the lessons.

A second newly qualified teacher at the same school used a different routine for resource creation; they simply started using the textbook more. Although the textbook is split into sections and chapters where each section is predominately concerning a single concept, the questions within a section each addressed its concept from a different angle, for example question 1 might be a pair of simple calculations, question 2 using it in a different context, while question 3 could be slightly harder calculations and so on. This differs markedly from the teacher's previous practice of chiefly giving worksheets on one concept with repetitive problems which only practiced of the execution of that single type of problem. It is noteworthy that the intervention was only partially successful for this teacher as they did stop using blocked practice but instead of interleaved practice, they employed a form variation theory where the same concept is tackled from different angles (Kullberg et al., 2017) as opposed to different concepts in interleaving.

Within the same school the final teacher, with 10 years of experience, used a third routine. They first looked at previous presentations on the topic and the exercises in the textbook, making notes on these before adding ideas from both challenging topics covered earlier in the course and topics which are popular in the national test. Although the first steps here do not differ radically from the first teacher, this teacher then employs their pedagogical experience of teaching the course before and knowledge of the national test, tools unavailable to a newly qualified teacher, to create their own questions, stating "I think if you know the curriculum very well, for us in Sweden it's not a very complicated thing." Similar to the first teacher, this teacher took some time before finding a process that worked for them.

In summary, all the teachers used a unique routine, which they found achievable based on their own situation, experience and old routines. With some teachers this came quicker, while others took time to find a routine which was workable for them.

In addition to the routines in creating the resources there were also individual approaches to incorporating the interleaved practice in their own lessons. Generally, each teacher's individual standard lesson structure did not change, with the exception that the previous blocked practice part had been swapped out for interleaved practice. For example, the second teacher continued to have a single-question warm-up, which was reviewed as a class, before either introducing the new concept and then practice, or direct to practice. The difference now was that the practice now came in a more varied form as opposed to before they "just wasted time by answering four pages of the same question". However, it was commonly reported that this exchange of blocked practice to interleaved practice caused the practice itself to take more time, but they all assimilated this into their standard lesson pattern by adjusting the timings of their typical lesson structure, as they appreciated the advantages for the pupils' long-term learning that they gained from practicing a little longer.

Conclusion – habits in a theory of change

To understand the role of routines and habits in a theory of change we first discuss their role in the pilot, before considering its design and finally proposing a theory of change for the subsequent iterations of the project. For all the teachers in the study to implement interleaved practice, it meant not only enacting a new idea, but also abandoning a prior approach, which they did by replacing an old routine with a new one. A shared understanding that blocked practice did not deliver long term learning was not sufficient for the teachers to start using interleaved practice. To regularly implement it in their teaching they needed to translate the theory into an easily replicable routine. Each teacher had an individual journey to create their process, depending on their experience, current situation and practice, and inclination. However, the commonality amongst all the new routines was the ease of implementation for that teacher within their current situation. Although it was too early to define this routine as a new habit, a habit is formed by repeated actions in response to contextual cues (Wood & Rünger, 2016) and each teacher had found an easily repeatable routine to create interleaved practice when cued to plan the lesson.

When examining the Swedish program Boost for Mathematics, Lindvall et al. (2022) suggest that one of the reasons for its lack of effect on pupil learning was the greater reliance on the pedagogies of investigation (what to know) with less focus on the pedagogies of enactment (what to do). The pilot

study spent most of its instruction explaining the theoretical grounding of interleaved practice and so also had a heavy dependence on pedagogies of investigation whilst being light on the pedagogies of enactment. Despite this, all the teachers were motivated to change their practice, however they did this in diverse ways with all of them finding a defined routine to make the change. When describing their journeys to their new routines, they did not always find it straightforward to create one which fitted the individual circumstances. To aid the teachers in the process of finding an efficient routine, it could be beneficial to present clearer strategies to go with the theory behind the change, thus both guiding their choices of routines and reducing the difficulty in forming a routine.

When introducing these strategies, the experience and situation of each teacher needs to be considered. In the pilot study the third teacher stated how their pedagogical knowledge of the curriculum eased the creation of the resources. Clearly not all teachers have this knowledge and so the chances of success of this strategy depends on the individual teacher's experience and knowledge. To present this to a new teacher could risk either poor implementation, due to insufficient knowledge, or create unrealistic expectations, increasing the risk of not implementing the innovation. There needs to be a balance to the strategies presented which does not overburden teachers with unrealistic strategies for their situation nor presents a simplified version for those who have the greater knowledge or skills. In other words, the strategies need to be appropriate to the teachers and their teaching experience.

By understanding the implementation of interleaved practice in terms of creating new routines and ultimately habits it is possible to describe a theory of change as follows: A theoretical introduction to interleaved practice forms the teachers' understanding and desire for change; within the teacher's standard lesson structures, there must be the opportunity to replace blocked with interleaved practice; a range of appropriate strategies are introduced; each teacher needs to establish a realistic routine either from the strategies suggested, by adjusting one of these, or by creating their own; this personalised routine can straightforwardly replace the old blocked practice and eventually become habitual; and finally the pupils now undertake a more challenging form of practice and this consequently leads to more effective long-term learning for the pupil.

Discussion – the conclusions in a wider context

When considering the implications of this paper for theories of change in general it is worth considering two of Century and Cassata's (2014) five factors affecting implementations, namely the innovation itself and the individual users involved. For the implementation of interleaved practice, the innovation itself is unusual as the largest required change is in teachers' planning, which occurs outside the classroom. After substituting in the interleaved practice, it required only minor adjustments to the norms of lessons in terms of the type and length of practice, these were reported to be overcome quickly. Such adjustments depend also on the characteristics of the individual users, in this case the inclination of the teacher and the pupils who in the study quickly accepted it. In comparison, an innovation that changes a teacher's in-lesson behaviour, for example how a teacher asks questions, can be seen as much more challenging, given it is a far more dynamic environment and potentially involves a larger change to the norms of the classroom. Furthermore, habitual behaviour is stronger under stress (Hobbiss et al., 2021) and so the potentially higher stress levels in

a teaching classroom implies that habitual behaviour is more likely. This automaticity does have the benefit for the teacher of easing their cognitive load of the teacher even though it simultaneously hinders change and leads to potentially suboptimal default actions (Feldon, 2007). Given this, it could be argued that in the more dynamic in-class environment, addressing the role of habits in any theory of change is even more important.

Kennedy (2016) describes how when a teacher implements a new skill, they have to abandon a prior approach, and when this understanding is combined with teacher automaticity it implies that changing habits is core to the implementation of any innovation. It also has the potential to explain how often an innovation is not sustained after the initial intervention (Coburn, 2003); if the new practice is not made a habit, then under stress the old habit will return even if the motivation is to change. Thus we argue that a theory of change must incorporate an implicit strategy to facilitate the innovation becoming habitual. This understanding of individuals as habitual beings adds a dimension to the characteristics of individual users (Century & Cassata, 2016) and any innovation would benefit from having a plan within its theory of change of how it is going to either replace an old habit with a new one, or how it will instigate a new habit.

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Teaching for mathematical competence: "Stumbling blocks" in teacher in-service training at scale

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In this paper, I analyse a developmental process in which every mathematics teacher in the municipality of Ishøj in Denmark took part in a three-year in-service teacher training programme. The purpose was to learn to plan, organise, conduct, and assess teaching explicitly aimed at students' development of mathematical competencies. Following a description of the project, the core of the paper is an analysis of the efforts made by and collaboration between stakeholders at various levels of the developmental process. The analysis is structured around the identification of three "stumble blocks" for succeeding with the developmental endeavour.

Keywords: Mathematical competencies, in-service teacher training, development at scale, stumbling blocks.

Introduction

In the course of the latest decades, mathematics teaching in many parts of the world has been subject to curriculum reforms representing a shift from subject matter learning goals related to mathematical concepts and procedures towards learning goals described as combinations of mathematical subject matter and more mastery-oriented ambitions. This overall development in the conceptualisation of learning goals for mathematics education is well covered in the research literature. Many reports focusing on specific countries document this shift (Niss et al., 2016).

Such curriculum reforms create a need for reconsidering many other issues related to educational development. One of these issues is what it takes to teach the new mastery-oriented ambitions. In this paper, I focus on the education of practising mathematics teachers to meet this challenge, guided by the following research question: *What challenges and ways forward can be identified when conducting in-service teacher training at scale in competency-oriented mathematics teaching?*

I address this question by describing and analysing an attempt to conduct such teacher training as part of the Danish *Ishøj Project*. The approaches applied in this project was an attempt to upscale results from previous developmental research projects also focusing on competency-oriented mathematics teaching. Hence, the paper reflects a scaling (Coburn, 2003) and an implementation perspective (Century & Cassata, 2016) on mathematics education research and development, the latter in two different ways: The written curriculum is sought to be implemented as a pivotal part of the operational curriculum (Akker, 2013), and results from previous studies is sought implemented and further developed in new settings.

In what follows, I begin by describing the methodological approach, the core of the Danish competency framework, and the key elements of the Ishøj Project. Then I analyse the project guided by the research question (cf. a Danish version of this analysis in Højgaard & Winther, 2021), before finishing off with some developmental recommendations.

Methodological approach

Methodologically, I rely on what Lincoln and Guba (1985) call human instruments, i.e., that the researcher "elects to use him- or herself as well as other humans as the primary data-gathering instruments" (p. 39). In this case, these instruments are me and the municipal mathematics consultant in the Ishøj Project, throughout this paper referred to as "we". Initially, when applying for funding of the project, I designed the developmental approach to be used. Subsequently, when the municipal consultant was hired, the two of us jointly facilitated the developmental processes in the project and adapted the approach taken along the way, cf. the description below. Hence, we were there throughout the project, discussed it continuously with each other and with the other participants, made all the major decisions about the way things were done, and therefore know the rationale for and reality of what happened.

The above imply that in this paper, the Ishøj Project is used as a case study of the type that Merriam (1998, p. xiii) describes as "an intensive, holistic description and analysis of a bounded phenomenon […]." For two reasons, it also resembles a critical case, "defined as having strategic importance in relation to the general problem" (Flyvbjerg, 2006, p. 229). Firstly, Denmark can be considered something of a best-case scenario for exploring competency-oriented teaching because many circumstances have aligned to make such a teaching approach both possible and wanted, cf. the description of the local context below. Secondly, the main ambition behind the didactical design of the Ishøj Project was an attempt to systematically facilitate the incorporation of mathematical competencies as a key element in mathematics education for grades k-9 in Denmark. As one of the national experts in such an endeavour, I was quite naturally and enthusiastically the driving force behind this design, and my ambition was only met with support and joint efforts by the other key players in the project. Hence, The Ishøj Project was arguably a best-case attempt to facilitate competency-oriented mathematics teaching by means of in-service teacher training, and as such critical for exploring this endeavour. Not as a here-is-how-to-do-it-description, of course not, no such thing exists when it comes to mathematics education. Rather, the project is a critical case for in-service teacher training facilitation of the students' development of mathematical competencies in the sense of being grounded on a very thoroughly developed and research-based design and conducted under very favourable times and conditions, locally as well as nationally.

Exploring the KOM framework in a Danish context

In Denmark, the mastery-oriented curricular ambitions are described by means of a set of mathematical competencies (Højgaard & Sølberg, 2023). The scaffolding of this idea was the hub of the so-called KOM Project (KOM is an acronym for "competencies and mathematics learning" in Danish), conducted in the years 2000-2002 with Mogens Niss as the director and me as the academic secretary. The key steps in the project's analysis were—cf. Niss og Højgaard (2019)

- to move from a general understanding of the concept competence, conceptualised as someone's insightful readiness to act appropriately in response to the challenges of given situations,

- to a focus on a mathematical competency, conceptualised as someone's insightful readiness to act appropriately in response to a specific sort of mathematical challenge in given situations,
- and then identify, explicitly formulate and exemplify a set of mathematical competencies that can be agreed upon as independent dimensions spanning what it means to master mathematics.

In Denmark, the publication of the KOM report (Niss & Jensen, 2002) initiated a developmental approach, where the mastery-oriented curricular ambitions are described by means of a set of mathematical competencies. In 2009, the set of mathematical competencies proposed in the KOM report became a fundamental part of the curriculum for compulsory mathematics education (grades K-9) in Denmark (Undervisningsministeriet, 2009)—and this remains the case in the current curriculum (Børne- og Undervisningsministeriet, 2019). This process of curricular development is thoroughly described and analysed in Højgaard and Sølberg (2023). Hence, Denmark has been and still is a good place to study competency-oriented mathematics teaching head-on by establishing "naturalistic laboratories" where teachers of good will join in on experimenting with this approach. Together with colleagues, among them not least several excellent teachers, I have done so in four longitudinal, 3-10 year long case studies over the last 25 years.

The Allerød Project was a developmental research project conducted in the years 1998-2003. A central part of the project was experimental teaching at an upper secondary school in the years 2000-2002. It involved a class of 25 students, their mathematics teacher and me as the PhD student who initiated, framed, and followed the experimental teaching processes (Jensen, 2007).

KOMPIS is an acronym (meaning "buddy" in Swedish) for KOmpetenceMål i PraksIS, Danish for "competency objectives in practice". It was another longitudinal developmental research project designed to test and document a strong focus on competencies in the teaching of mathematics, Danish, and science in lower secondary school. For each of these three school subjects, a group of 3-6 teachers (from the four participating schools in the same Danish municipality), one of their classes of 20-25 students, a teacher educator and a researcher framed and carried out experimental teaching in the years 2009-2012 (Højgaard & Sølberg, 2019). I was the mathematics education researcher in the project and followed the experimental teaching processes particularly closely in one of the mathematics classrooms.

In the ongoing *Hvalsø Project*, the results and approaches from the KOMPIS Project are being used and further developed when following the mathematics teaching in one class of 20-25 students in grades 1-9 (2016-2025). As opposed to the Allerød and KOMPIS Projects, it is a design feature of the Hvalsø Project to conduct classroom experiments focusing on all the mathematical competencies described in the KOM report (Niss & Jensen, 2002) one by one, and to iteratively carry out such a "competency tour de force" at both lower primary (grades 1-3), upper primary (4-6), and lower secondary level (7-9). Hence, the project investigates and offers examples of teaching experiments focusing on the same mathematical competency (at least) three times over the course of nine grades, and with a focus on and interviews with the same students followed consistently over the years.

The Ishøj Project was conducted in the years 2014-2017, where the Danish municipality Ishøj ran a project involving all their students and all their teachers of science and mathematics in the compulsory

school system, i.e. grades K-9. The aim of the project was to develop the competencies of the students in mathematics and the various science subjects by means of developing the qualifications of the teachers involved. The hub of this development was experiments carried out by the teachers in their own classrooms, framed by bi-monthly school-based seminars for the entire group of mathematics or science teachers at each of the six schools in the municipality (Højgaard & Winther, 2021; Winther, 2017). I was responsible for the mathematics part of the project, explicitly based on the experiences and results from the KOMPIS Project. The main part of this effort was to run the more than 100 seminars (six seminars at each school each of the three year) for the group of 5-15 mathematics teachers at each school.

Collaboration between participants at different levels

Organizationally, the Ishøj Project can be characterised as a municipally anchored, school-based experiment on the development of teacher competencies. The project was established, framed and administered by the school administration at Ishøj municipality, but the developmental process needed to respect the differences between the six participating schools, and therefore had the individual school as an organizational focal point. Thus, there were effectively six parallel developmental projects in the project. Both during the process and now in retrospect, this framing of the project initiated two kinds of considerations and subsequent analysis regarding the conditions for the desired developmental processes: One about the importance of the effort from and cooperation between participants at different levels, and another about how differently such collaborations was exercised at the different schools.

The developmental process–a cooperative utopia

In the analysis of these considerations, I distinguish between the following types of key participants: The students, the individual mathematics teacher, the team of mathematics teachers at a school, the school-based so-called mathematics supervisor, the school leadership, the municipal mathematics consultant, and the municipal school leadership. In utopia, their mutual roles and cooperation can be described as follows:

- *The students* develop solid mathematical competencies because *the teacher* is good at meaningful competency-oriented mathematics teaching.
- *The teacher* is–among other things–good at it because *the mathematics supervisor and/or the municipal mathematics consultant* has taught him or her how that can be done and inspired him or her to remain committed to the developmental process.
- *The teacher feels* supported when it comes to preparation for and evaluation of the teaching, among other things because *the mathematics team* at the school is a well-functioning academic collective facilitating the teachers' development of their individual teaching practices.
- *The mathematics team* is well-functioning because–among other things–the *mathematics supervisor* puts him-/herself in front of a well-established and systematically facilitated collaborative culture within the team.
- *The mathematics supervisor* considers him-/herself well framed and supported because–among other things–*the municipal mathematics consultant* puts him-/herself in front of a well-

established and systematically facilitated collaborative culture among the mathematics supervisors at the different schools.

- *The mathematics supervisor* succeeds in managing the collaborative culture because–among other things–*the school leadership* show commitment in this and establish positive expectations about and room for it.
- *The school leadership* succeed in establishing positive expectations about and room for a developmentally oriented culture with a shared sense og direction among the mathematics teachers because–among other things–*the municipal school leadership* show commitment in this and establish positive expectations about and room for it.
- *The municipal mathematics consultant* keeps the pot boiling by communicating with and mediating between the various levels of development from the teachers to *the municipal school leadership*, among other things because the latter has chosen to employ the consultant with such a communication and mediation need in mind.

With this cooperative utopia as a frame of reference, I will now describe three experienced developmental stumbling blocks. In each case, this is accompanied by a description of what the municipal consultant and I believe are the main problematique behind the stumble stone, and by a description of what we believe may be possible ways forward.

Developmental stumbling block 1: Supervisors responsibilities and qualifications

Qualificational problematique: The municipal consultant and I had the impression that most of the participating mathematics teachers found it meaningful but difficult to teach competency-oriented–both because goal-oriented teaching in itself is demanding and because mathematical competencies are complex ambitions to pursue.

Experienced stumbling block: Mathematics supervisors are typically not qualified to teach competency-oriented mathematics teaching, simply because there is nothing even close to this in the one-year diploma program for mathematics supervisors in Denmark. Consequently, these supervisors rarely want to teach competency-oriented mathematics teaching, which, for example, was the case for all the mathematics supervisors in the Ishøj Project.

Possible solutions: The most direct answer to this challenge is to strive for the mathematics supervisor in the schools typically having the type of competency-oriented in-service education that the Ishøj Project points out is needed. A key element in this will be to develop a new educational program for the supervisors. Alternatively, as was the case in the Ishøj Project, a municipal mathematics consultant can be hired to help bridge the qualification gap. As a last resort, one can hire an external consultant to take care of teaching the supervisors, which was my role in the Ishøj Project. However, in a long-term development perspective it is the least attractive solution, partly because external contributors, as was the case in the Ishøj Project, are often only temporarily involved, partly because the external resources available are far from sufficient for a nationwide effort.

Developmental stumbling block 2: Managerial responsibilities and qualifications

Managerial problematique: In some of the schools, a few teachers opposed development and common direction, and as a result the mathematics team was not homogeneous and well-functioning.

Experienced stumbling block: The school leadership forward the problem to the mathematics supervisor, who, however, has neither formal nor actual managerial competence to handle it, and does not want to take up this role among mathematics teacher colleagues. As we experienced it, mathematics supervisors are typically driven by a subject-specific didactical interest and a desire for a collective collegial approach to developmental processes, not by managerial ambitions.

Necessary resort: The school leadership need to take responsibility for handling such individual obstructional approaches and more generally engage in and provide support and resources for the mathematics supervisor's effort to lead the developmental process of the mathematics team. At some schools, we found this to be largely the case. Among other things this was consistently manifested by the school leader closely following the developmental process and often being present in parts of the associated seminars, and therefore by self-examination being able to observe some mathematics teachers turning their back to it. In other schools the leadership was completely invisible in the development process.

In one school we experienced a remarkable change of attitude during the three years of the project. Two very experienced teachers at the school explicitly told us and the rest of the teachers in the mathematics group that they had no interest in the change of teaching approach that was on the agenda in the Ishøj Project, based on a "don't fix it if it isn't broken"-approach to their teaching. The school principal during the first year of the project had been at the school for many years and had established a tradition of not interfering more than absolutely necessary with the work of the teachers in general and the old and experienced teachers, with whom he had worked for decades, in particular. He explicitly told us this when confronted with the problems arising from the non-participation of the two experienced teachers, which left the mathematics supervisor quite frustrated. After the first year, that school principal retired, and his successor expressed a markedly different approach from day one: At a meeting about the problem, she told us and the mathematics supervisor that she would not accept anyone turning their bag to a decided joint effort, and that the supervisor should come to her if the problem remained. She then showed up at the subsequent project seminar and—following an appraisal of the project and how she looked forward to getting to know more about it—said the same thing to the whole group. The two experienced teachers somehow got the message, which paved the way for a complete change of atmosphere at the seminars, and the school changed from being the most reluctant to the most positive among the six schools.

Developmental stumbling block 3: Time and energy for development

Resource-wise problematique: Competence development of one's own practice requires time and energy for commitment over a longer period with the same focus, and many teachers and mathematics supervisors do not feel that they have these resources.

Experienced stumbling block: We experienced school leaderships forwarding the time and resource challenge to the individual teacher, justified by not wanting to interfere with the teachers' management of their time, which has been one of the burning issues in continuation of a Danish school reform in 2014. This well-meant approach caused a lot of frustration among most of the teachers affected by it and implied a completely individual approach to the preparation of teaching. In some cases, the problem was exacerbated by the leadership taking too many developmental initiatives at the same time, in the initial phase of the Ishøj Project dictated by the municipal school administration.

Necessary resort: The school leadership must actively help direct the developmental efforts by creating space for and the consequent expectation of these efforts, and the municipal school leadership must provide space for and support such a managerial focus. In the Ishøj Project, the approach within the municipal school leadership and within all the school communities developed in this direction during the three years of the project. A key element in this development was the previously mentioned, systematically introduced (by the municipal consultant and me) and managerially supported follow-up organizational meetings in continuation of each seminar for the whole mathematics team.

Developmental perspectives

The analysis of the efforts from and the cooperation between participants at different levels invites for the following developmental recommendations:

- Make sure you have, develop, or get a mathematics supervisor who is competent and would like to take on the role as *the person responsible for teaching the mathematics team competency-oriented mathematics teaching*—by conducting the teaching him-/herself and/or by collaborating with someone else who can and will carry out this task.
- *The school leadership must support and enable the supervisors to succeed,* firstly by dealing with possible individual leadership problems standing in the way of a collaborative approach in the mathematics team, secondly by engaging in and providing support and resources for the mathematics supervisor's effort to facilitate the developmental process of the mathematics team.
- *The school leadership must actively support a shared sense of direction in the developmental work* by providing room for—and the consequential expectations of—the developmental effort, and *the municipal school leadership must provide room for and support of such a managerial focus.*

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Implementing PISA research to practice: a design research approach to enhance mathematical literacy

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This paper provides a theoretical exploration of how design research methodology can translate insights from PISA assessments and implementation research into actionable classroom practices in the Faroe Islands. In response to declining PISA results, the study develops humble design principles aimed at enhancing mathematical formulation skills and the use of digital tools, while integrating Communities of Practice (CoPs) into the design to foster teacher collaboration, professional development, and sustainable implementation. This collaborative approach is designed to positively influence the ecological disruption described in implementation research, facilitating the modification and integration of interventions in alignment with the designers' goals. The findings aim to support national scaling but also to provide broader insights into design research implementation processes in mathematics education.

Keywords: PISA research; implementation research; design research

Introduction

When the Faroe Islands[1] joined the PISA survey for the first time in 2005/2006, the results had a profound impact on both the Faroese education system and the broader community. The performance of students in mathematics, reading, and science was notably lower than that of any other Nordic nation (Egelund, 2006). In the following PISA cycles, Faroese results improved steadily (Højsted et al., 2016; Olsen et al., 2019) until the latest edition of the PISA assessment in 2022, in which results again plummeted (Olsen et al., 2023). Following the results of the most recent PISA assessment, various stakeholders have expressed concerns about the overall state of the education system and the application of research findings into practice (e.g. Olsen, 2024). With the support of the Faroese Ministry of Education, a group of researchers are working on bringing the accumulated research knowledge about Faroese primary school students provided by PISA into practice.

Effectively addressing these challenges demands a methodology that bridges theoretical frameworks with practical classroom application. Design research (DR) (Bakker, 2018), provides such a pathway, offering a systematic, iterative approach to developing and refining teaching interventions that are both theoretically sound and adaptable to local educational contexts.

DR has established itself as a common methodology in educational research, distinguished by its dual focus on advancing both practical solutions and theoretical insights. Conducted *in* and *for* practice, DR ensures that theoretical constructs remain connected to real-world learning environments (Bakker, 2018).

[1] The Faroe Islands is a self-governing nation within the Kingdom of Denmark, with a population of around 54,000 people (Hagstovan, 2024).

A central outcome of DR is the development of *design principles*, which are theoretically informed guidelines that emerge through empirical testing and iterative refinement. These principles act as a bridge between theory and practice, providing predictive insights that evolve from concrete classroom experiences (Prediger, 2019). In their early stages, when principles are primarily theoretical and remain untested, they are often referred to as *humble*, an acknowledgment of their preliminary status and the need for validation through further implementation (Prediger, 2019).

This paper is a theoretical exploration of how DR can strengthen the implementation of educational interventions based on PISA evidence into Faroese classrooms. Using notions from implementation research literature, the paper highlights the role of communities of practice in fostering teacher collaboration, in order to attain a sustainable innovation (Koichu et al., 2021). The guiding research question for this study is:

How can design research methodology support the implementation of PISA based educational sequences in Faroese mathematics classroom practice?

Emphasizing this question allows for an exploration of DR's role in bridging theoretical insights from PISA with actionable classroom practices.

In the next section, relevant notions on implementation research and communities of practice are presented. This is followed by an elaboration of DR methodology and of the proposed humble design principles for the intervention using selected results from the analysis of Faroese PISA data. Afterwards, the outlines of an early-stage research project utilizing these design principles are described as well as the next steps of scaling the project. Finally, concluding remarks are given on the use of DR in the early-stage project and more generally in implementation research.

Implementation research

According to Century and Cassata (2016), implementation research focuses on two primary aspects: understanding the characteristics of educational **innovations** and identifying the surrounding conditions or contextual factors that shape its enactment. The notion of innovation has been conceptualized in multiple ways. Rogers (2003) defined innovation as any idea, method, or object that is perceived as new by its adopters, regardless of its broader societal familiarity. Jankvist and colleagues (2019) proposed that innovations in educational settings should be recognized as changes in practice that are valued by key stakeholders such as teachers, students, or educational policymakers.

Koichu et al. (2021) offered a nuanced perspective of implementation research by conceptualizing it as an *ecological disruption* within the educational system. This disruption occurs as innovations are gradually adopted and endorsed in response to identified challenges or needs. The implementation process, as described by Koichu et al. (2021), involves two primary roles: the proponents of the innovation, who introduce and guide the change, and the adapters, who modify and integrate the innovation into their practices. Over time, adapters engage in four interdependent processes:

1. Developing a sense of ownership or agency regarding the innovation,
2. Adjusting communication practices within and between communities,

3. Modifying classroom practices to align with the innovation, and
4. Tailoring the innovation to address specific contextual needs.

Koichu et al. (2021) description gives rise to the perspective that successful implementation of innovations in education may be fostered by including a focus on teacher collaboration and professional development. Communities of Practice (CoPs), as conceptualized by Wenger (1998), offer a framework for understanding how practitioners learn and grow collectively. CoPs are groups of individuals who share a common interest or concern and work together to improve their practice. They can provide a space for collaboration and knowledge sharing, which can foster sustainable educational change. Hodges and Jong (2014) describe how school-based communities of practice supported an implementation process by providing opportunities for teachers to reflect on and refine their instructional decisions. In contrast, the absence of such communities hindered the effective integration of specific educational materials.

In this study, Koichu et al.'s (2021) perspective and the notion of CoPs are included into the design of the intervention.

Design research

DR is a well-established methodology in educational research, emphasizing the importance of conducting studies both in practice and for practice (Bakker, 2018). Its primary focus is to address real-world educational challenges while simultaneously contributing to practical and theoretical advancements (Bakker, 2018). DR at its core focuses on the creation of educational materials, such as tasks or instructional activities. Typically, this approach involves iterative cycles consisting of three phases: (i) the preparation and design of educational materials, (ii) testing these materials through teaching experiments, and (iii) retrospective analysis of data collected during the teaching experiments. Through repeated testing and refinement of educational materials, DR facilitates both the enhancement of design and the development of new theoretical insights.

In DR, the goal is to advance both practical applications and theoretical understanding. A key outcome of DR studies is often the formulation of design principles, which serve as guiding frameworks grounded in both practice and theoretical perspectives. These principles bridge the gap between research and practice by being informed by real-world implementation while contributing to broader theoretical insights (Prediger, 2019). However, when design principles are still in their early stages, having been developed theoretically but not yet tested or refined through practice, they are referred to as humble principles (Prediger, 2019).

Design principles may have the following structure:

"If you want to design intervention X [for the purpose/function Y in context Z], then you are best advised to give that intervention the characteristics A, B, and C […], and to do that via procedures K, L, and M […], because of arguments P, Q, and R" (van den Akker, 1999, p. 9).

Proposing humble design principles

In order to develop humble design principles, previous analysis of Faroese students' performance in PISA was used, particularly analysis of the 2022 results (Olsen et al., 2023), highlighting several areas requiring targeted intervention.

Faroese students scored significantly below the OECD average and Danish peers in mathematical modeling processes, which are formulating, employing, and interpreting (OECD, 2023). The greatest challenge lies in the process of *formulating*, where students struggle to identify and translate real-world problems into mathematical terms. This gap suggests that current instructional approaches do not adequately emphasize the full modeling cycle, particularly the initial and final stages that connect mathematics to real-life contexts (Olsen et al., 2023). Faroese students reported minimal use of *digital tools* in mathematics lessons, with 40% indicating "never or almost never" using them, compared to just 3% in Denmark. Similarly, 45% lacked experience with spreadsheets, compared to 10% of Danish students. This underuse of technology, including the underuse of spreadsheets, limits students' ability to solve data-rich problems (Olsen et al., 2023). Given the increased emphasis on digital tools in the latest PISA cycle, students' unfamiliarity with technology places them at a disadvantage when tackling tasks that require analysing, organizing, and interpreting data.

Communities of practice (CoPs) can provide essential support, enabling teachers to co-design tasks, share experiences, and reflect on instructional challenges, and to adjusting communication practices within and between communities. Such collaboration may fosters teacher ownership and ensure the effective and sustained implementation. Integrating CoPs into the design may affect the ecological disruption in a favorable manner, i.e. support the modification and integration of the innovation in alignment with the intention of the proponents of the innovation.

Using the structure proposed by van den Akker (1999), three ***humble design principles*** A, B, and C are proposed, followed by procedures K, L, and M, as well as the arguments P, Q and R:

If you want to design an educational intervention to enhance students' ability to mathematically formulate real-world situations and effectively use digital tools [to improve mathematical modelling competency and technology use in Faroese mathematics classrooms], then you are best advised to include the following characteristics:

A: *Support students' mathematical formulation process*

Provide tasks and scaffolding to help students identify mathematical concepts, variables, and relationships. Use structured prompts and realistic, context-based tasks to connect mathematics with real-world situations.

B: *Integrate digital tools for data analysis and problem solving*

Incorporate tasks using spreadsheets and other digital tools to analyse, organize, and visualize data, in order to interpret, and draw conclusions. Engage students with real-world data to enhance their reasoning and efficiency through technology.

C: *Foster collaboration and professional development through communities of practice*

Organize collaborative mechanisms where teachers achieve professional development, share experiences and reflect on the implementation of teaching sequences. Provide structured opportunities for co-designing, adapting, and refining instructional tasks. Support the adjustment of communication practices within and between communities.

To achieve this, follow these procedures:

K: Break down complex problems into smaller, manageable steps to guide students progressively through the formulation process. Provide examples of successful formulation processes as models for students to analyse. Encourage classroom discussions where students reflect on and articulate their thought processes during mathematical formulation.

L: Develop tasks that require students to input, manipulate, and analyse data within digital tools like spreadsheets. Include teacher-guided activities that demonstrate how digital tools can simplify mathematical processes.

M: Schedule regular meetings before, during, and after the implementation of teaching sequences to facilitate ongoing dialogue and reflection. Offer professional development resources to support teachers in integrating mathematical modelling and digital tools into their instruction. Involve teachers in the co-design and adaptation of tasks to align with their specific classroom needs.

This is important because:

P: Students often struggle to recognize mathematics in real-world contexts, which hinders their ability to initiate the modelling process. Providing structured support helps students independently identify and frame mathematical problems, while context-based tasks increase engagement and demonstrate real-world relevance

Q: Limited experience with digital tools, places Faroese students at a disadvantage when solving data-rich problems, such as those posed in the PISA mathematics assessment. Integrating technology aligns with modern mathematical practices, prepares students for technology-driven problem solving, and fosters deeper understanding of data analysis and interpretation.

R: Collaboration through communities of practice supports teacher ownership, improves instructional practices, and supports a sustained implementation by shifting the modification and integration of the intervention towards an alignment with the intentions of the designer.

Project MAC (Mathematics in action)

The proposed humble design principles are the foundation of a design intervention currently in progress in the Faroe Islands. The intervention is taking place in three Faroese lower secondary schools, and it consists of the design of two teaching sequences per year for grade 7 (sequence 1a and 1b), grade 8 (sequence 2a and 2b), and grade 9 (sequence 3a and 3b). Each teaching sequence spans three weeks (15–20 school hours). Sequences 1a, 2a, and 3a focus on mathematical modelling, especially on the process of *formulating*, while 1b, 2b, and 3b include the use of digital tools in this work. The iterative process of design, testing and refinement takes place during a 4-year time period. A community of practice (CoP) is established between mathematics teachers in grades 7-9 at the participating schools.

School A functions as a pilot school that is one year ahead of schools B and C in terms of the implementation process, see table 1.

Table 1: Timetable of implemented teaching sequences and BC´s in project MAC

Timetable project MAC		School A	School B	School C
2024	Autumn	1a		
2025	Spring	1b - **BC1**	**BC1**	**BC1**
	Autumn	1a, 2a	1a	1a
2026	Spring	1b, 2b – **BC2**	1b – **BC2**	1b – **BC2**
	Autumn	1a, 2a, 3a	1a, 2a	1a, 2a
2027	Spring	1b, 2b, 3b – **BC3**	1b, 2b – **BC3**	1b, 2b – **BC3**
	Autumn	1a, 2a, 3a	1a, 2a, 3a	1a, 2a, 3a
2028	Spring	1b, 2b, 3b – **BC4**	1b, 2b, 3b – **BC4**	1b, 2b, 3b – **BC4**

Prior to the introduction of each teaching sequence, the involved teachers participate in a professional development module developed specifically for the teaching sequence. Once during and once after each implementation period, the local CoP is activated in an evaluation and reflection session. In addition, every spring there is a *between communities* (BC) seminar, fostering cross community communication, collaboration, and sharing of experiences.

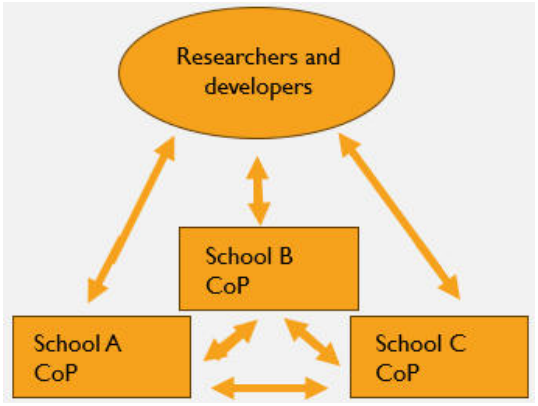

Figure 1. Supporting CoPs within and between communities

Figure 2. Task booklet cover from sequence 1a

The early-stage research project has begun in School A, with the establishment of a CoP and the first iteration of testing of sequence 1a.

The effectiveness of the intervention will be measured using both qualitative (interview with teachers and school principals) and quantitative methods (student questionnaires and tests), with a key measure being the PISA 2029 results. The full participation of the Faroese student population in the PISA

assessment enables rigorous comparisons and accurate tracking of progress in the three selected schools, leveraging the unique small-scale dataset for precise evaluation.

The project aims to scale up the intervention to the entire Faroese education system, contingent on the successful implementation of the initial phases. With funding and support from the Faroese Ministry of Education, the refined teaching sequences and professional development modules will be extended to all schools nationwide.

Concluding remarks

The paper has highlighted how DR can provide a systematic and iterative framework to support the design of educational interventions aiming to alleviate identified deficiencies in mathematical literacy. By bridging theory, practice, and teacher agency, this study addresses the specific challenges within the Faroese educational context while also offering broader insights into the application of DR and CoPs in implementation processes within mathematics education. The integration of iterative design cycles, teacher collaboration, and targeted interventions demonstrates how DR may drive meaningful educational change by aligning humble design principles with classroom realities. CoPs provide support by enabling teachers to co-design tasks, share experiences, and reflect on instructional challenges, while also facilitating adjustments to communication practices within and between communities. Such collaboration fosters teacher ownership and ensures the effective and sustained implementation of interventions. By integrating CoPs into the design, the ecological disruption described in implementation research may be influenced in a favorable manner, supporting the adaptation and integration of innovations in alignment with the intentions of the designer.

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What Works? Implementing Innovations on Small, Medium, and Large Scale

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Based on a literature review, this paper investigates the challenges and strategies involved in scaling educational innovations in mathematics education. The study focuses on small, medium, and large-scale implementations, highlighting the necessity of maintaining core innovation ideas, the determinants, during adaptation processes. It highlights that small-scale implementations succeed due to close contact between developers and users. In contrast, medium and large-scale implementations face complexities due to reduced developer involvement and varied material supports. The study emphasizes the critical role of voluntarism and sustained support in medium-scale projects. Furthermore, it questions the feasibility of large-scale reforms, particularly those requiring shifts in teaching practices, without significant resource allocation and structural support.

Keywords: Mathematics education, innovations, implementation, scale.

Introduction

In mathematics education, the successful implementation of innovations is often marked by adaptation challenges within diverse educational contexts. These challenges are particularly pronounced when scaling from small, localized projects to broader, more extensive implementations. Innovations in education, while necessary for addressing persistent issues such as the gap between curricular goals and student outcomes, require careful consideration of the context in which they are implemented.

Drawing on insights from a systematic literature review of implementation research in mathematics education (Ahl et al., 2023), where the results reported here can be considered to be serendipity findings outside but close to the second research question asked in the review "Which innovations from research in mathematics education have received such attention that they have been implemented in teaching practice and reported in implementation research?" (Ahl et al., 2023, p. 139). This paper narrows down on the question and asks: *What are the characteristics of innovations implemented successfully on small, medium, and large scales, respectively?* We use our insights about scaling at different levels, where we have identified characteristics of studies that can be classified as medium-scaled projects concerning the contact, material, and organizational factors for the implementation (Aguilar et al., 2023) to categorize the characteristics in relation to scale. The novelty of this paper is thus that we provide insights into the practicalities and challenges of educational reform for different innovations across varying implementation scales.

IR theoretical constructs applied

We depart from a pro-adaptability perspective (Century & Cassata, 2016); that is, we recognize that any innovation will be adapted to the educational context in which it is implemented. Adaptations are

necessary and expected, but they must be made without losing the innovation's core ideas (Ahl et al., 2022). The core ideas can be described in a set of norms called determinants (Chen, 2012). The determinants must continue to apply when the innovation is adapted to the context. If the determinants are lost, the idea with the innovation is lost.

To categorize the studies in our review, we adopt a classification framework for small, medium, and large-scale studies, as presented by Aguilar et al. (2023). Our understanding of scaling is aligned with Coburn's (2003) definition. Coburn outlines four critical dimensions necessary for effectively scaling educational reforms: depth, which entails profound changes in classroom practices and teacher beliefs beyond surface-level modifications; sustainability, which focuses on the enduring nature of reforms despite evolving circumstances; spread, which involves both the expansion of reforms to additional schools and their deep integration into existing systems; and the shift in reform ownership, which emphasizes transferring control from external parties to local educators and institutions to ensure that reforms are ingrained and maintained internally.

To move away from a model where the innovation designer continues to play a central role in enacting innovation, we consider three key factors: the interaction between developers and users, the nature of material support, and the organizational structure (Aguilar et al., 2023). These elements help distinguish between small-scale and large-scale projects. Typically, small-scale projects involve direct developer involvement, less sophisticated material support, and a central role for developers within the organizational framework. In contrast, large-scale projects generally feature limited direct contact between developers and users, more advanced material supports that enable independent operation, and organizational structures that do not centralize the developers' roles.

An emerging concept is the medium-scale implementation project, combining features of small-scale and large-scale projects (Aguilar et al., 2023). In medium-scale projects, responsibility for adapting the product shifts to the user, with developers potentially serving as training multipliers to support broader dissemination. Various scenarios for medium-scale projects exist, such as leveraging existing resources to scale up within a specific locality or using trained personnel to launch new projects elsewhere, possibly maintaining close contact to ensure implementation fidelity. As illustrated in Table 1, these examples demonstrate the diverse nature of medium-scale projects, which bridge small and large-scale efforts.

Table 1: Classification of implementation projects according to the contact, material, and organizational factors at different scales (adapted from Aguilar et al. 2023, pp. 18-20)

	The contact factor	**The material factor**	**The organizational factor**
Small-scale implementation projects	Participants in the project have direct contact with the innovation designers.	The material support structures for participating in the project are not yet so developed that it is possible to start working on the project	The organizational support structure is set up in a way that involves the developers in a central position

		without direct contact with the innovation developers.	
Medium-scale implementation projects,	Participants in the project have direct contact with the innovation designers or other experts assigned by the developers, mainly for monitoring purposes.	The material support structures for participating in the project have been developed so that it is possible to start working on the project without direct contact with the innovation developers. These structures might be complemented with locally produced instructions. When new teachers start working on the project, the material support structures for participating must be accompanied by expert developer support.	The organizational support structure includes experienced implementation teachers as advisors in a position where fidelity to the implementation ideas can be overseen and enforced. Organizations that do not involve developers in central positions but use developers or persons assigned by the developers as experts can set up an organizational support structure.
Large-scale implementation projects	Most participants in the project have no, or at most, sporadic direct contact with the innovation designers.	The material support structures for participating in the project have been developed so that it is possible to start working on the project without direct contact with the innovation developers.	Organizations that do not involve developers in central positions can establish an organizational support structure.

Method

The literature review commenced by utilizing the ERIC database as the primary source. The structure of our review adhered to the Preferred Reporting Items for Systematic Reviews and Meta-Analyses (PRISMA) guidelines (Moher et al., 2009). We searched for manuscripts containing the term "implement*" within the title and/or abstract across the top 20 journals in our sample, as identified by Williams and Leatham (2017). To ensure thoroughness and that no relevant articles were missed, we replicated the search for "implement*" in the title and/or abstract directly on the websites of each of these 20 journals. This process resulted in identifying 1,093 peer-reviewed articles that met our inclusion criteria. The selection process followed PRISMA's four key steps: (1) identification, (2) screening, (3) eligibility, and (4) inclusion (Moher et al., 2009). The literature search was conducted on February 4-5, 2021. We conducted an additional follow-up round where we incorporated articles that met our search criteria from the newly established journal *Implementation and Replication Studies in Mathematics Education (IRME)*, as well as from a special issue on implementation in *ZDM—Mathematics Education* that was published after our initial systematic search.

We revisited our search across the 20 journals initially included in our review to identify any new studies published between February 4, 2021, and August 26, 2022. In May 2023, we repeated this search for IRME and the 20 journals, focusing on articles published after August 26, 2022, with an official publication date of 2022. This approach was intended to ensure we covered complete publication years rather than partially including articles published in 2023. The review process management was facilitated by the software Covidence.

Each article underwent a two-stage screening process and was reviewed by two independent reviewers. Papers were included based on their alignment with Century and Cassata's (2016) definition of Implementation Research (IR). According to this definition, innovation refers to the practical application of ideas derived from research to enhance education. The final sample included 103 papers.

Data extracted from these papers included general information such as the authors, title, purpose statements, country of the study, research questions, methods, target group, and results. Specific information regarding the implementations covered aspects such as the type of innovation in mathematics education, the goals (whether short-term or long-term), the phase of implementation being studied, the stakeholders responsible for the implementation, and factors identified as influencing the outcomes. Due to space limitations, it is impossible to include 103 references. We refer to Ahl et al. (2023) for complete information on the review methodology and a full reference list. This paper refers to articles showcasing trends we have identified in our review.

Results

Implementing innovations at small scale

Innovations implemented on a small scale generally work and positively affect teachers' and students' learning of the content and/or teaching model the innovation addresses. The close contact with innovation designers ensures that the teachers' adaptations align with the determinants. We have chosen Stylianides and Stylianides (2009) as an illustrative case for this category. The authors develop and implement an instructional sequence designed to help students recognize the limitations of empirical arguments in validating mathematical generalizations and the need for proofs. It is based on a 4-year design experiment in an undergraduate mathematics course, focusing on cognitive conflict to achieve learning goals. The first author, G. Stylianides, conducted the last research cycle reported in the article, describing a close match between the classroom community's hypothetical and actual learning trajectories. The innovation designers developed and implemented the innovation and ensured that the adaptations made in the design process aligned with the determinants.

Implementing innovations at a medium scale

Innovations implemented on a medium scale are generally successful when participation in the project is voluntary. We have chosen the TRIUMPHS project (Barnett et al., 2022) as an illustrative case for successful medium-scale projects based on voluntary participation. The TRIUMPHS project integrates primary historical sources into undergraduate mathematics instruction through Primary Source Projects (PSPs), which combine historical materials with curricular resources to enhance students' understanding. These PSPs offer flexible, open-source content for instructors and are

designed to replace traditional textbooks. The success of the TRIUMPHS project was significantly influenced by its innovative structure, adaptable materials, detailed instructor guidelines, and ongoing support from experts assigned by the developers. As Aguilar et al. (2023) highlight, the strong collaboration between creators and users, which evolved into a co-production model, is a key factor in TRIUMPHS' success alongside the high adaptability of the materials.

Often, medium-scale projects based on something other than voluntarism have mixed outcomes regarding innovation implementation (Alarcon & Stoudt, 1997; Beswick & Jones, 2011). While some teachers find sufficient support for innovation enactment, even though innovation designers are not in direct contact or have little direct contact, the support is insufficient for others. Not surprisingly, it seems that teachers who do not voluntarily participate in the implementation project generally need more support than teachers who do participate voluntarily.

Implementing innovations at large scale

When it comes to large-scale innovations, we see that what works is changing the content of curricula and the sequencing of tasks in teaching materials as long as it does not require changes to teaching practice (Prytz et al., 2022). Both Stylianides and Stylianides (2009) and Barnett et al. (2022) implementation projects had not worked to implement on a large scale in their current project organization, according to Aguilar et al. (2023). For Stylianides and Stylianides (2009), the close contact with the innovation designs is impossible to maintain on a larger scale. For Barnett et al. (2022), the voluntariness factor is difficult to maintain on a large scale. Having established this, we want to convey what else we found does not work concerning large-scale projects.

Implementing so-called Process Standards on a large scale is very difficult. Process Standards mean that the innovation aligns with or is inspired by the National Council of Teachers of Mathematics (NCTM) recommendations for K-12 education, outlined in the *Principles and Standards for School Mathematics* (NCTM, 1989). These standards emphasize problem-solving, reasoning and proving, communication, representation, and making connections, reflecting a shift from teaching mathematical facts to focusing on the processes involved in doing mathematics. Overall, it appears difficult for teachers to change their routines. Even when teachers say they are positive about reforming their teaching practice, they fall back to the more traditional teaching they are already used to. We illustrate this phenomenon with Stickles (2011). She found three broad categories in her study of 50 elementary school teachers enacting a new Standards-based curriculum. The vast majority, 80%, are categorized as *Growing reformists*. They are positive about a changed teaching practice but do not succeed in thoroughly changing it. There are some superficial changes, but the determinates (Chen, 2012) of the innovations are not implemented with depth (Coburn, 2003), so-called benign positive appraisal (Gregoire, 2003).

Reformed teaching in line with NCTM Standards is also of great interest in teacher training courses. A significant research issue in teacher education is the disconnect between the practices endorsed by university credentialing programs and the methods teachers use in K-12 classrooms (Gainsburg, 2012). Specifically, mathematics credential programs often advocate for reform-oriented practices, yet mathematics instruction continues to follow traditional methods. Various explanations have been suggested for the limited adoption of university-endorsed teaching practices, including new teachers'

challenges in navigating the 'two worlds' of university and school, the complexity of reform-based teaching, and the shortcomings of the conventional teacher-preparation model, which emphasizes general pedagogy before specific instructional techniques.

Conclusions and hypotheses

In Aguilar et al., 2023, we coined the term medium scale from the size of the project organization. In this paper, we change the starting point and consider the characteristics of the innovation in relation to the scale of the implementation. For small-scale implementations of innovations, close contact with the innovation designers ensures that the teachers receive sufficient support to implement the innovation without adaptions violating the determinants. However, the critical question for all mathematics education research is how to plan and scale successful innovations so that they reach large groups of teachers and students. We hypothesize that if small-scale studies are used in a pilot phase before a more significant scale-up, authentic situations with representatives of the large-scale target group are needed to try out the innovation. The innovation designers need to create a small-scale authentic situation with support for the teachers similar to the support planned for the large-scale implementation if the small-scale project is to provide information on whether the teachers can implement the innovation without making adaptations that conflict with the determinants.

We doubt that close contact with designers who teach models to teachers, as in the project by Stylianides and Stylianides (2009), can work on a medium or large scale because it requires, by definition, a distance from innovation designers.

In the medium-scale implementation, where contact with the innovation designers is carried out through facilitators who mediate between innovation designers and the teachers' enactment, voluntariness is a success factor. While teachers who are optimistic about the innovation and the changes in practice required succeed well even when the contact with innovation designers is a step away, teachers who are not initially optimistic about the innovation need more support for enactment than is usually offered on a medium scale. However, it is also possible that positive teachers in medium-scale projects exhibit what Gregoire (2003) calls benign positive appraisal, which means that only superficial changes to teaching practice occur. The teacher is convinced that the teaching is entirely in line with the innovation, which puts obstacles in the way of change. Despite the danger of benign positive appraisal, we hypothesize that teachers with a well-functioning practice can handle a lower degree of support than those who already struggle with their teaching routines.

For large-scale projects targeting all teachers in a district or country, we hypothesize that the only innovations with prospects of success are innovations that concern changes in subject content and the order in which mathematical content is taught (without demands for changed practice). These innovations can be implemented in detail through curricula.

Innovations aiming to implement process Standards do not seem possible to implement with significant effect at scale through curricula alone. We hypothesize that the kind of support and contact with innovation designers described for small and medium-sized implementation projects is required. This, in turn, means that it is very resource-intensive to implement innovations that require changes

to teachers' routines. However, if the goal is to implement reformed teaching, teacher education must also work toward changing practices.

Regarding teacher training, our review has shown that future teachers are told that teaching must be done through problem-solving, classroom discussions, reasoning, and inquiry-based practices. However, they rarely get to practice these practices. We hypothesize that we will not see an impact of Standards-based teaching practices if the students are not taught according to these principles and are allowed to practice these teaching models during their education. Then, we can mix up the existing staff of teachers with recent graduates who can carry out Standards-based teaching. These may also function as support for the large majority of teachers who are positive about change but fall back to traditional routines for teaching.

In conclusion, after delving into 103 articles on implementation projects, we cannot emphasize enough how difficult it is for teachers to change their classroom routines regardless of good intentions to change practice. Gaps may exist between the teacher's intentions and her routines, most of which are too automatic to be controlled (Sfard, 2023). The consequence of the strength of teachers' established routines is that determinants are often lost in favor of adaptations that align with the teacher's established routines rather than the innovation determinants.

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Implementability of programming and computing education: Accommodative and assimilative approaches to turtle geometry

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This paper explores the challenges and opportunities of integrating programming and computing into mathematics education through two approaches: computational thinking (CT) and computational literacy (CL). Using a turtle geometry activity as a case study, we compare their potentials and challenges in relation to supporting implementation efforts. We introduce a distinction between assimilation and accommodation to illustrate how CT and CL differ. The results shows that CT adds new methods to existing practices (assimilation), while CL transforms mathematical concepts (accommodation). In that sense CT aligns with current practices, while CL requires a deeper conceptual shift. We discuss the challenges of implementing programming and computing in mathematics and the value of differentiating between assimilative and accommodative innovations.

Keywords: Computational thinking, computational literacy, turtle geometry, implementation.

Introduction

In recent years, there has been an increased emphasis on integrating programming and computing in mathematics teaching, a shift that has proven challenging to implement effectively (Tamborg et al., 2024). While computing holds potential to enrich the learning experience, its integration often encounters significant obstacles. For instance, tools like Scratch, which can be used to draw geometrical figures, tend to confuse students by requiring an understanding of external rather than internal angles, thus complicating rather than clarifying mathematical concepts (e.g., Elicer & Tamborg, 2023). This has led to critical voices arguing that computing adds little to the existing curriculum (Kilhamn et al., 2021). The value of using computational methods for working with tasks that are already part of the traditional mathematics curriculum is in large a matter of perspective. Some even contend that computing is not inherently mathematical and therefore should not be a part of mathematics instruction (Kilhamn et al., 2021). Moreover, the competency gap in relation to computing education further hinders its effective implementation (Bocconi et al., 2022; Wilson et al., 2010).

One potential reason for these challenges is that current efforts to incorporate programming and computing into mathematics are unclear about disciplinary stability and boundaries. Often, we assume that both school mathematics and programming/computing are stable and well-defined. But for Seymour Papert—one of the earliest and most influential proponents of programming and

computing in relation to mathematics teaching, the entire idea is to challenge and transform a flawed "paper and pencil mathematics" (Papert, 1980).

In this paper, we use the Piagetian concepts of assimilation and accommodation to explore different approaches to integrating programming and computing into mathematics, and how these approaches influence the implementability of this effort. Specifically, we investigate the difference between viewing programming as merely adding new and foreign phenomena into the mathematics classroom and viewing programming and computation as something that alters the underlying concepts and phenomena of classical mathematics. We look at two frameworks/concepts that view the interplay between programming and mathematics in slightly different terms: Computational Thinking and Computational Literacy (CL). CT emphasizes that certain new practices related to decomposition and algorithmic thinking are incorporated into mathematical problem-solving practices, whereas CL emphasizes the development of new ways of thinking and understanding through computational practices. Based on this, we ask in which ways do CT and CL potentials for implementation differ? In which ways do CT and CL differ regarding implementability? By analysing these two frameworks in the context of a classical mathematics activity—turtle geometry—we seek to contribute to the discussion of implementation and implementability, by showing how a difference in framing and theoretical understanding can greatly matter to the way a specific type of activity is received in mathematics education practice.

Computational thinking and computational literacy

Originally introduced by Papert (1980) in the book *Mindstorms* from 1980 and revitalized by Wing (2006) in the influential short paper from 2006, *computational thinking* (CT) emphasizes "thinking like a computer scientist" through abstraction decomposition and problem-solving. In recent years, research on CT in mathematics and science teaching has expanded, resulting in multiple definitions such as the framework of Weintrop et al. (2016) and of Pérez (2018). Weintrop et al. (2016) focus on four practices: data practices, modelling and simulation practices, computational problem-solving practices, and systems thinking practices. Pérez (2018) focuses on integrating computational thinking (CT) dispositions—tolerance for ambiguity, persistence, and collaboration—into mathematics learning.

Before Wing's (2006) paper, and long before the broader push for CT as part of the school system, diSessa (2000) introduced the concept of computational literacy, focusing on the broader transformations to cognitive and social structures that computational technologies provide, rather than just on developing technical skills and thinking like a computer scientist. Later, diSessa (2018) defines literacy as a significant cultural achievement, and views CL as a new form of literacy that emerges when digital technologies reshape how people think and learn (diSessa, 2018). Unlike Wing (2006), diSessa (2018) argues that computational literacy should not have a specific affinity to computer science and computer scientists; instead, it should be seen as an important way of *being* and *learning*, adaptable to various disciplines and contexts.

Implementation and Implementability

According to Koichu et al. (2021), implementation in mathematics education can be conceptualised as a "disruption to a particular mathematics education system, through the gradual endorsement of innovation in conjunction with an action plan" (Koichu et al., 2021, p. 986). Further, Koichu et al. (2021) argue that implementation in mathematics education takes place in an interaction between the proponents and adapters of an innovation (Misfeldt et al., 2023). This interactive approach to understanding implementation highlights the importance of the ideas, beliefs and conceptualisations that all involved stakeholders have about the purpose and nature of the innovation. This also includes how much an innovation is invited into a reshaping of the practice it is intended to support. Sometimes implementation of innovations can lead to radical changes to practices that can be hard to grasp from a perspective as deeply embedded in an existing practice (Misfeldt & Aguilar, 2024). We do see the introduction of programming and computing in mathematics teaching as very unclear in relation to how disruptive this innovation is, and CL and CT views the transformative potential in very different ways.

With a metaphor from Piagetian psychology, the integration of programming and computing into school mathematics can be seen as either an *assimilative* or *accommodativ*e process (von Glasersfeld, 1995). Assimilation refers to incorporating new information connected to existing knowledge without altering these. On the other hand, *accommodation* is the process in which new information requires a restructuring of previous knowledge. We use this distinction to describe how CT and CL interact with the establishedpractice of mathematics teaching. Assimilative approaches adapt computing content to fit within our pre-existing understanding of mathematics, preserving the discipline as we (as well as teachers and students) know it. In contrast, accommodation describes a situation where new ideas necessitate a fundamental rethinking of school mathematics. These concepts have previously been used to discuss technology adoption in organizations (Reinking et al., 2000), highlighting that accommodative changes are often much more difficult to reach than assimilative. However, accommodative approaches are sometimes necessary to reach the benefit from radical innovation.

Context and case

At the Danish Knowledge Centre for Digital Technology Comprehension, we explore how children and young people can critically and constructively co-create the digital technologies shaping their lives. Our ambition is to ensure everyone, regardless of gender or background, understand digital technology to participate as democratic citizens in a digitized society. The TC curriculum includes digital empowerment, digital design, computational thinking, and technological agency, with a focus on both integrating TC into subjects like mathematics and developing TC as a topic on its own.

Turtle geometry activities are designed to integrate elements of both geometric understanding and basic programming in primary school mathematics education, specifically aimed at students in grades 7–8. This type of activity was originally created by Seymour Papert, who used a digital "turtle" to help learners explore triangles and angles. By programming the turtle with commands like `FORWARD 50` and `RIGHT 120`, students can draw equilateral triangles and experiment with different angles to create various types of triangles. This hands-on approach deepens understanding

of geometric properties, such as angle sums and symmetry, while also developing problem-solving skills and creativity in line with Papert's constructionist learning philosophy.

Through this activity, students can apply their geometric knowledge, first in a physical context and then translate this understanding into a Scratch programming framework, providing them with deeper insights into both mathematical and computational concepts.

Results

One way of looking at turtle geometry activities is as a way of accessing geometry through algorithmic, step-by-step actions. This aligns with a CT approach. Students work with geometric figures such as triangles by following sequential movements based on established mathematical geometry knowledge/concepts, such as angles and lines, and these concepts remain stable and unchanged. This part of the activity is assimilative, as both the mathematics and CT competences are familiar and unchanged. Students apply their existing knowledge of geometry and algorithmic thinking in a new context, but their fundamental understanding remains intact.

But one could also tell another story about turtle geometry. A geometry building on movements (such as *forward* and *turn*) rather than gestalts (such as *lines* and *polygons*) is, in a sense, a *different* geometry. Building curves from inner movements, using recursion and iteration as key building blocks leads to different geometrical insights that are difficult to see from the perspective of computational thinking.

When the perspective on the activity shifts to a CL perspective, students are introduced to a curve and movement-based approach to geometry using Scratch. In contrast to looking at this through the lens of CT, where geometric concepts are applied in a familiar and stable form, the Scratch infrastructure enables a new way of interacting with geometry. Students now work with a sprite that has an orientation in space, and they define geometric figures based on vector-based understanding, where direction and movement in space are central. This vector- and movement-based geometry represents a change in how students understand and manipulate geometric phenomena. Although mathematical concepts such as angles remain the same, this digital approach requires students to restructure their cognitive schemas to integrate the new technology and the understanding of vector-based geometry. Therefore, this part of the activity is accommodative.

One activity that highlights the difference between classical geometry and turtle geometry is finding the sum of the angles in polygons. In classical geometry, the sum of the angles in a triangle is always 180°. As the number of sides in a polygon increases, the sum of the interior angles also increases. This can be proven by adding a triangle to an existing polygon: for example, a quadrilateral (4 sides) has an angle sum of 360° because it can be formed by adding a triangle to a triangle. To form a pentagon (5 sides), another triangle is added, increasing the angle sum by 180°, and so on.

In turtle geometry, instead of counting the interior angles of a polygon, we measure the change in direction as we move around the shape. This approach leads to a remarkably simple result: all polygons, regardless of the number of sides, have a total angle sum of 360°. This is because the total change in direction for a complete loop around the shape is equivalent to a full circle, or 360°.

In our own work, we have also seen this phenomenon and the problems it can lead to. We examined the work of an expert teacher, who co-designed and implemented an innovative task integrating block programming into a sixth-grade geometry lesson (Elicer & Tamborg, 2023). While the teacher effectively adapted the innovation to align with her expertise and pedagogical values, the outcomes were mixed, highlighting challenges in combining computing and programming with mathematics teaching. Despite her efforts to balance change and continuity, her revisions sometimes led to unintended learning outcomes, suggesting a need for an integrated expertise in both programming and mathematics education (Geraniou et al., 2024).

From a CT perspective, the task draws on familiar school-level geometry content and competences from CT on algorithmic stepwise actions and part/whole relations. Both mathematics and CT are stable and unchanged. The task of the teacher would be to articulate and make sense of the interdisciplinary relations between CT and mathematics. From a CL perspective, the task introduces curve-based geometry. Contrary to the CT perspective, the scratch infrastructure provides a way of interacting with geometry based on a sprite with an orientation in space. Students define outer and not inner angles of geometrical figures.

The different approaches to programming and computing in mathematics teaching and learning described in the case above show that the CT here acts in an *assimilative* manner respecting the nature of school mathematics. Furthermore, few new mathematical insights are provided when working with turtle geometry, but it allows students to enact CT in a mathematical context. On the contrary, the CL approach challenges the existing school mathematics allowing students to develop entirely new perspectives on geometry. However, the latter is difficult. Especially at scale (Aguilar et al., 2023).

Discussion: Difficulties implementing programming and computing

To understand the difficulties in implementing computing and the differences between a CL and a CT approach, we will describe a historical development and show an example from the ongoing work implementing computing and programming.

A historical account of computing and programming can be seen in two waves (Misfeldt, 2022). The first wave in the 1980s saw the integration of technology into classrooms, particularly in mathematics education, promoting creative approaches. Abelson and diSessa (1980) and Papert (1980) were driving figures in this movement. However, interest waned by the 1990s due to mixed results and a shift toward task-oriented software use.

Currently, a second wave of technology education is underway. Beginning in the mid-2000s, rather actively building on CT and Wing's (2006) work. This second wave, unlike the first, is driven more by political decisions, requiring significant educational capacity building and consideration of the balance between STEM career preparation, equity, and citizenship.

In brief, the first wave of computing and programming in mathematics teaching was in large shaped by the progressive and radical ideas by Papert and diSessa. And in that sense, this approach was based on a less developed version of what is now referred to as CL, whereas the second wave was very directly related to CT. And history is a hard judge, CL did not work, but CT did.

From the above description, it becomes evident that the way computing and programming are framed within mathematics teaching significantly impacts their integration into the curriculum. A focus on computational literacy and a radical reform of school mathematics may appeal strongly to a small group of scholars and educators who are already aligned with these ideas but are likely to fall short in engaging a larger group of mainstream teachers. Table 1 below illustrates the differences in CT and CL in terms of who are likely to be proponents of the two innovations, adaptors/adopters, the ownership one could expect, and the type of transformation the innovation will lead to. In contrast, when we frame computational thinking in terms of problem-solving and competence development while emphasizing its relevance for the labour market, it appears to offer a more accessible approach for mainstream mathematics teachers. Hence, this perspective may facilitate broader acceptance and smoother integration of programming into the standard mathematics curriculum.

When applying the theoretical framework by Koichu et al. (2021) on implementation to the context described, we see clearly why this has been the case and why it might not need to be like this. The framework highlights that the implementation of innovations in mathematics education, such as programming and computing, depends on stakeholders' beliefs and perceptions of the innovation's purpose and potential for disrupting existing practices. In Table 1 we compare the two approaches on parameters relevant for implementation according to Koichu et al. (2021).

Table 1: Proponents, adaptors, adopters, ownership and transformation as key differences in the implementability of computing and programming in mathematics education depending on if you build upon a CT or a CL style of thinking

	Proponents	**Adaptors and adopters**	**Ownership**	**Transformation**
CT	Political system and industry	Teachers Teacher trainers and educational institutions	Mainstream – high among many	Low
CL	Scholars	Teachers and non-profit organizations	High among few	High

The table shows how the different factors draw in the same direction. Apart from being assimilative toward school mathematics, our initial impression is that the CT approach also has more influential proponents and a much broader base of ownership than the CL approach (Bocconi et al., 2022). This full understanding of the implementation ecology around CL and CT will require further research, but Table 1 suggests that several factors influence this.

Conclusion

In this paper we have introduced the Piagetian concepts of assimilation and accommodation to explain the difficulties that transformative educational ideas can meet in implementation. We have further discussed how CT and CL as two approaches to computing and programming exemplified this difference. To do that, we have analyzed the case of turtle geometry and shown that CT acts in an

assimilative fashion towards school mathematics, adding a new topic but stabilizing central elements such as problem-solving and algebra. On the other hand, CL acts in a much more accommodative fashion, questioning and transforming key elements of school mathematics. More research is needed to understand the actual reasons why CT is more successful than CL in mainstream implementation, but this paper has shed light on the transformative nature as one factor in the question of implementability.

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Success in implementation in early mathematics education - Fidelity and/or adaptation?

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In this paper, the tension between fidelity and adaptation is elaborated on within the context of a large-scale implementation of a preschool mathematics innovation. We elaborate on how fidelity and adaptation can be understood in relation to depth, sustainability, spread, and shift in ownership. The results show that fidelity may hinder ownership while adaptations, when they are productive, may indicate depth, spread, sustainability, and ownership and thus be empirical evidence of successful implementation. The challenge in implementation research, then, becomes to clarify what adaptations that may be productive as they do not distort the core elements of the innovation. These results must, however, be seen in the context of Swedish preschool, where preschool teachers are qualified and expected to make productive adaptations in relation to the children they teach.

Keywords: Implementation, fidelity, adaptation, early mathematics, preschool.

Introduction

According to Century and Cassata (2016), implementation research is the endeavour to understand whether and how educational efforts to implement an innovation accomplish their goals. This paper focuses on how we in implementation research know if we have accomplished our goals. To elaborate on this, we use as an empirical example a study in which preschool mathematics education developed within one study was scaled up in a subsequent study. The first study was an intervention where mathematics education for the youngest preschoolers was developed in collaboration between two researchers and three preschool teachers. This study showed both qualitative and quantitative developments in the use of numbers by the children in the intervention group compared to those in the control group (Palmér & Björklund, 2024). Based on these positive results, a new study was initiated with the aim of implementing the innovation large scale. However, implementation research has demonstrated that successful interventions are not easily transferred to a larger community of teachers and that all interventions undergo changes when implemented in new settings (Century & Cassata, 2016).

In the large-scale implementation, Coburn's (2003) framework was used. This framework was developed as an alternative to implementation studies that focus on the number of teachers, schools, or districts involved rather than on the quality of the implementation. The argument for developing an alternative framework was that numbers provide limited insight into the nature of the envisioned or enacted changes, as well as the degree to which any observed changes are sustained. Coburn developed a framework comprising four interrelated dimensions: depth, sustainability, spread, and shift in ownership.

Regardless of the framework used in implementation research, adaptation and fidelity are two important notions. Fidelity implies that implementations should adhere closely to the original model, while adaptation considers adjustments of innovations essential to reproducing positive outcomes from one context to another (Century & Cassata, 2016). In this paper, we build on Coburn's (2003) framework to elaborate on tensions between fidelity and adaptation with reference to the large-scale implementation of an innovation in preschool mathematics education.We ask the following research question: How can fidelity and adaptation be understood in relation to depth, sustainability, spread, and shift in ownership in a large-scale implementation of preschool mathematics education? Elaborating on this question will contribute with knowledge on how to know if implementation is successful.

Literature review

Based on a review of implementation research, Century and Cassata (2016) conclude that improving education requires processes for changing individuals, organizations, and systems. In line with this, Maass et al. (2019) emphasize that instructional improvement involves substantial teacher learning, which requires systematic support.

> […] implementation research [is the] systematic inquiry regarding innovations enacted in controlled settings or in ordinary practice, the factors that influence innovation enactment, and the relationships between innovations, influential factors, and outcomes (Century & Cassata, 2016, p. 170).

Implementation can be conducted and studied at a micro level with a few teachers (e.g., from one school), at a meso level with several teachers (e.g., from different schools within a region), or at a macro level (e.g., a district or nation) (Maass et al., 2019). However, this division of scale may emphasize a focus on quantities instead of qualities. Coburn (2003) argues that definitions of scale have traditionally had a restricted scope, focusing on increasing the number of schools or teachers reached by an implementation rather than on the depth of change achieved through it. In line with this, Century and Cassata (2016) emphasize that studies that isolate, decontextualize, and simplify issues regarding the complexity of education decrease the applicability of the results from implementation research. Coburn (2003) offers an alternative definition of scale that places less emphasis on quantities and more on the nature and quality of the implementation. This definition emphasizes a focus on depth (the nature and quality of change), sustainability (the persistence of change over time), spread (within each setting), and ownership (the autonomy of teachers).

In implementation, not all elements of an innovation are regarded as equal; some are considered core components (the essential and indispensable elements of the intervention), serving as key contributors to outcomes of interest and primary mechanisms for change (Century & Cassata, 2016). Therefore, to succeed with implementation, we need to identify which elements of the intervention being essential to replicate – and for how long and with what intensity they should be implemented – and which elements can be modified or eliminated entirely (Coburn, 2003). This is in turn connected to fidelity and adaptation. Jankvist et al. (2022) conclude that the notion of fidelity, both inside and outside of mathematics education research, is diverse, in terms of both definition and measurement aspects. For example, they highlight that fidelity may be considered in relation to the pedagogy, the

content, and/or the cognitive dimensions of an innovation. Regardless of the dimension in focus, according to Durlak (2010), fidelity refers to the degree of correspondence between the intended and actual outcome of implementation (Durlak, 2010). Measuring fidelity of implementation is, according to Century and Cassata (2016), based on the view that once an innovation is found to be efficacious, future implementations should not deviate from the established model. Adaptation, on the other hand, refers to the nature and degree of changes made to the original innovation; the focus is on the reasons for adaptation and the relationships between adaptation and innovation outcomes (Durlak, 2010). While some argue that adaptation undermines fidelity and thus must be avoided, others argue that a blend of fidelity and adaptation can yield positive results (Century & Cassata, 2016). Since educational contexts differ both within and between regions and countries, it is important to understand how innovations can be adapted to local conditions. However, adaptations must be in line with the core components of the innovation (Maass et al., 2019). When adaptations respond to the demands of the context while remaining consistent with the core components of the innovation, Debarger et al. (2013) refer to these as productive adaptations.

The study

Maass et al. (2019) emphasize that scaling up means different things in different countries due to variations in school systems. For example, the presence of a national curriculum, the content of that curriculum, and the education of teachers have an impact on implementation. The large-scale implementation focused on in this paper was conducted in Swedish preschools. In Sweden, approximately 87% of all children aged 1-5 are enrolled in preschool (Swedish National Agency for Education, 2023). Two professional groups work in the preschool: preschool teachers, who have a three-year university education, and childminders, who have a three-year high school education. The national curriculum emphasizes both care and learning. Sometimes the term *educare* is used to emphasize that learning must always be seen in relation to care (Bass & Good, 2004). Although the curriculum does include mathematical goals, these are not achievement goals but goals to strive towards. While the curriculum determines the experiences children should be offered in preschool, it does not dictate how these experiences should be implemented, allowing preschool teachers a great deal of autonomy in their teaching practice.

The implementation focused on in this paper was an innovation developed within this context. In a combined research-development project conducted in collaboration between researchers and preschool teachers (financed by the Swedish Institute for Educational Research, grant no. 2018-00014), mathematics education aimed at providing the youngest preschoolers the best conditions for learning mathematics was developed and tried out in authentic preschool settings (see for example Björklund & Palmér, 2022; Palmér & Björklund, 2023). The aim was partly to investigate how ongoing preschool activities could become the starting point for mathematics education, giving toddlers the opportunity to distinguish necessary aspects of numbers, and partly to investigate how such activities can advance toddlers' development of numerical skills. In the study, teaching activities were developed in accordance with three design principles (Palmér & Björklund, 2023). First, the context of the activities ought to be based on children's experiences, needs, and interests, and they should be familiar so that the children can participate in, relate to, and reason about the content based on their previous social and cultural experiences. At the same time, in line with variation theory

(Marton, 2015), interference from irrelevant aspects ought to be minimized. Second, the activities ought to make it possible for the children to discern essential aspects of numbers (i.e., representations, cardinality, ordinality, and part–whole relations). Third, the activities ought to allow the children to express different ways of understanding and allow a variety of experiences and expressions. The intervention was successful, indicating quantitative as well as qualitative differences in how children in the intervention group and control group used numbers (Palmér & Björklund, 2024). Based on these positive results, a subsequent study on how the successful intervention could be implemented on a large scale was initiated (financed by the Swedish Institute for Educational Research, grant no. 2022-00010).

The first step in the subsequent study was to identify the essential and indispensable elements of the intervention, that is, to identify the core components of the innovation to be implemented. These core components were to comprise the content of a guiding material to be used in the implementation (Century & Cassata, 2016; Coburn, 2003). The guiding material was developed in collaboration between researchers and the same preschool teachers involved in the initial research-development project. The innovation was described in the guiding material as consisting of five parts, where each part focused on one of the core components: part 1: mathematizing; part 2: contrast and generalization; part 3: representations; part 4: cardinality; and part 5: part-whole relationship of numbers. Each part consisted of one text to read, one video illustrating the content of the text in authentic teaching situations, questions to discuss with colleagues, one scripted activity to be carried out with children, and finally, one activity to be planned and carried out by the preschool teacher.

The subsequent study was conducted in line with educational design research, which implies implementation in cycles, where new preschools are successively included in an increasingly broader implementation. In the first cycle, the guiding material was implemented in two preschools. After that, in the second cycle, six new preschools were involved, and finally, in the third cycle all preschools from one municipality implemented the guiding material. The data generated were individual interviews (12), focus group interviews (28), and video documentations (219) from the teaching at the preschools. Also, delayed interviews (after 6 and 12 months) were conducted, and video documentations of teaching were generated. The interviews focused on the guiding material and how it was used and experienced by the preschool teachers. The video documentations made visible how the theoretically driven new forms of instruction were adopted by the preschool teachers. These video documentations were also used during the interviews as a form of stimulated recall.

These empirical data were analysed with a focus on whether the core components were visible in the teaching and, if not, how the guiding material needed to be developed. The analysis was grounded in depth, sustainability, spread, and shift in ownership in accordance with Coburn (2003). Based on the findings from each cycle, the guiding material was revised.

Depth, sustainability, spread, and shift in ownership

The question to be explored in this paper is how fidelity and adaptation can be understood in relation to depth, sustainability, spread, and shift in ownership in a large-scale implementation of preschool mathematics education. First, in this section we describe how depth, sustainability, spread, and shift in ownership became visible (or not) in the large-scale implementation of the preschool mathematics

innovation. After that, we elaborate on how fidelity and adaptation can be understood in relation to these four interrelated dimensions.

According to Coburn (2003), depth implies "deep change" (p. 4), which means that the change goes beyond surface structures or procedures such as materials, classroom organization, or the addition of specific activities. Deep change includes change of teachers' norms, views on the subject, and pedagogical principles as well as their assumptions about how students learn.

In our analysis, we searched for depth in both interviews and video documentations. However, since we did not have data on the teaching in the preschools before the implementation, the focus of the analysis was not on the nature and quality of change but instead on the nature and quality of the core components of the innovation. Depth was particularly visible in the video documentations of the activities planned by the preschool teachers themselves, along with their reflections on these video documentations during the interviews. This is in line with Coburn (2003), who argues that capturing depth requires in-depth interviewing and classroom observation. In the interviews, we searched for statements about the content, pedagogical principles, or young children's learning that aligned with the core components of the innovation. In the video documentations, we observed whether the core elements were visible in the teaching. We found that, regardless of whether the scripted activities were conducted in line with the innovation, the core components of the innovation were sometimes visible in the teachers' own planned activities and sometimes not. We interpreted this as an indication of whether the preschool teachers' learning went beyond the surface (i.e., they were able to plan their own activities in line with the core components) or if they remained superficial (i.e., they were not able to plan own activities in line with the core components).

The second element of scale is sustainability as implementation must last over time and not fall into disuse. This was investigated through the delayed interview and video documentations of teaching based on the same logic as above. According to Coburn, depth plays an important role in sustainability as teachers with a deep understanding of an innovation are better able to respond to new demands and may also continue deepening their understanding of the innovation over time. However, we found this hard to investigate as the delayed interview and video documentations of teaching were not carried out unprepared. The teachers themselves chose what teaching to document on video, and they knew when they would be interviewed and what the interview would be about. Thus, for example, they could have read the guiding material again. Therefore, the documented teaching and what was said in the interview may not show true sustainability of the intervention. What the delayed interview and video documentations could show was non-sustainability; that is, they could show whether and when the intervention was not visible. To investigate sustainability, observations would probably provide more reliable data.

When Coburn discusses spread, she refers not only to expansion outward to more schools and classrooms but also to spread within schools and classrooms. This can become visible if teachers begin to draw on innovation beyond specific innovation-related activities. In our study, this was apparent in the video documentations and interviews. Similarity between the scripted activities and the teacher-planned activities (when core components are kept) may indicate spread. For example, a scripted activity that focuses on ordinality and that uses a song about elephants might be changed so

that the song is instead about cats. Such a change might not indicate spread in the same way as changing the scripted activity into a completely different one but still with ordinality as the core components. Further, in the interviews, how the preschool teachers talked about the innovation may indicate spread, for example, if they talked about the innovation in connection to situations and activities other than specific innovation-related activities.

The final dimension of scale from Coburn is ownership. This implies that the ownership of the innovation is no longer external but internal, which means that teachers have the capacity to sustain, spread, and deepen the core components themselves. Similar to sustainability, this can be studied through the delayed interview and video documentations of teaching. Thus, the same pros and cons of the study's data apply.

Fidelity and adaptation

How do fidelity and adaptation relate to the above? Coburn (2003) highlights that there is a tension between ownership and fidelity, particularly if the implementation strongly emphasizes fidelity. In line with this, we quite quickly concluded that pure fidelity to the scripted teaching was not something to aspire to. This is because when the teachers were conducting an activity from the guiding material exactly in line with the script, the teaching became artificial and disconnected from the children's experiences, needs, and interests, which thus contradicted the core components of the innovation. Therefore, some adaptations were needed. We also realized that all included preschool teachers made adaptations to the intervention, but not all adaptations were productive (Debarger et al., 2013).

Adaptations were observed in both the scripted teaching and the teachers' own planned activities. In the interviews, the preschool teachers talked about their adaptations. When focusing on productive adaptations we realized that these could be understood as indicating depth and spread; that is, to be able to make productive adaptations and to transfer these adaptations between teaching activities, the teachers had to really understand the core components of the innovation. Productive adaptations were also visible when following the preschool teachers over time, where some of their own planned activities developed in line with the different parts of the guiding material while others did not. This was interpreted in terms of sustainability. Further, ownership was apparent in the interviews when the preschool teachers talked about how they made productive adaptations but without distorting the core components. Similarly, Coburn (2003) writes that ownership exists when the decisions about what and when to adapt are made by the preschool teachers themselves rather than imposed by the implementers.

Discussion

The question explored in this paper is how fidelity and adaptation can be understood in relation to depth, sustainability, spread, and shift in ownership (Coburn, 2003) in a large-scale implementation of a preschool mathematics innovation. To summarize, we found that adaptations, when they are productive, may indicate depth, spread, sustainability, and ownership and thus be empirical evidence of successful implementation. This insight gave us a new lens in the analysis of the implementation of the innovation, that is, it prompted us to search for productive adaptations. To understand which adaptations are productive, we found that the core components of the innovation needed to be clearly

identified and documented. In our case, that entailed clearly documenting valid signs of the core components of mathematizing, contrast and generalization, representations, cardinality, and part-whole relationship of numbers. The more precisely these valid signs are documented beforehand, the more precisely productive adaptations can be separated from non-productive ones. And, if identified, non-productive adaptations can be addressed in the guiding material as examples of how not to teach the innovation.

The positive view on productive adaptations presented in this paper must, however, be understood within its national educational context (Maass et al., 2019), that is, in connection to the Swedish preschool. As mentioned, Swedish preschool teachers have a three-year university education and a significant degree of autonomy in their teaching practice. Thus, they are both qualified and expected to make adaptations in relation to the children in their preschool why productive adaptions of core components can be considered within reach if understanding the core components. Similarly, productive adaptations have been found in other implementation studies involving Swedish teachers. For example, in a study by Ahl et al. (2022), the two studied teachers attended to the intended organization of the mathematical content, but their attendance to the intended classroom interaction patterns differed. In our study, no explicit interaction pattern was promoted; instead, it emphasized mathematizing, which entails making mathematics necessary for the situation from the child's perspective. We found this to be the hardest core component of the innovation to implement. Thus, like the teachers in the study by Ahl et al. (2022), the teachers in our study more easily implemented the mathematical content than the context in which this content was to be taught. This may indicate that the three dimensions of pedagogy, content, and cognitive dimensions of an innovation often connected to fidelity (Jankvist et al., 2022) could also be addressed in connection to productive adaptations.

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Building on success: The possibilities to reproduce the MIST project in different educational settings?

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The Middle-school Mathematics and the Institutional Setting of Teaching (MIST) project was a successful development project conducted across four US school districts. Central to the project's contribution is the development of a Theory of Action, which provides actionable strategies for professional development in mathematics instruction. In this paper, we analyze the MIST project through the lens of implementation research, focusing on key factors that influence its success. Our material consists of semi-structured interviews with people involved in MIST. The analysis suggests that while the MIST project has the potential to be reproduced in other contexts, specific contextual factors must be carefully considered to ensure successful adaptation and implementation.

Keywords: MIST project, implementation research, innovation, influential factors.

Introduction

The Middle-school Mathematics and the Institutional Setting of Teaching (MIST) project, conducted in the US from 2007 to 2015, stands as a notable successful development initiative. The research team produced numerous publications, including a book, to disseminate their findings. Their book, *Systems for Instructional Improvement: Creating Coherence from the Classroom to the District Office*, represents the project's culmination in the field of mathematics education. In this work, the authors present a Theory of Action (ToA), which offers "…actionable recommendations about potentially productive instructional improvement strategies." (Cobb et al., 2018, p. 7). The MIST researchers themselves have described their project as a Design-based Implementation Research (DBIR) initiative (e.g., Cobb et al., 2013). Given the success of MIST, it is highly relevant to develop methods for reproducing it.

In this paper, we examine the MIST project from a traditional Implementation Research (IR) perspective. We aim to gain insights into the possibilities of reproducing the MIST project, that is, implementing it in different contexts. An important step in this process is understanding the nature of MIST—the object to be reproduced and implemented in a specific context. To achieve this, semi-structured interviews were made with key people involved in MIST. Based on IR theory, concerning four types of factors influencing the successful adoption of innovations, we posed a research question: What are the characteristics of MIST in relation to:

- the innovation itself,
- the users,
- the organizational setting,
- and the external environment?

These four categories and their theoretical background are explained in the next section. Following are sections covering the MIST project (background information), our method, and our results and findings. The final section, titled Discussion and final remarks, addresses the possibilities of reproducing the MIST project based on our results and findings.

Theoretical constructs

In this section, we present a literature review on implementation research and outline the theoretical construct we used for data analysis.

Implementation research

Century and Cassata (2016) offer "a working definition for implementation research as a systematic inquiry into the enactment of innovations within controlled environments or typical practices, exploring the factors that influence such enactments and the interconnections between innovations, influencing factors, and outcomes" (p. 170). According to them innovation is the object of implementation (Century & Cassata, 2016). Century and Cassata (2016) define "innovations as programs, interventions, technologies, processes, approaches, methods, strategies, or policies that involve a change (e.g., in behavior or practice) for the individuals (end users) enacting them" (p. 170). According to Rogers (2003), innovation is "an idea, practice, or object that is perceived as new by an individual or other unit of adoption" (p. 12); what matters in human behavior is not whether an innovation is objectively new but whether it feels new to the individual. Jankvist et al. (2021) argue that innovation needs to be novel for the stakeholders, not necessarily for the end users. In all three definitions, there is a common core of what is innovation, which we agree with. We argue, however, that Jankvist et al.'s (2021) concept of multiple users is more suitable for understanding the innovative features of MIST. We return to the issue at the end of the paper.

Exploring implementation research reveals the key factors influencing the successful adoption of innovations, as highlighted by Century and Cassata (2016). These factors span four areas: the innovation itself, the users, the organizational setting, and the external environment. The specific characteristics of these factors vary depending on the context, such as healthcare, sports, manufacturing, or, in this case, educational institutions, with a focus on school systems. Key factors related to the innovation include its adaptability, relevance, and alignment with operational practices. Educators, especially teachers, play a crucial role as change agents in schools. Their understanding of innovation shapes their effectiveness in implementing it, as well as their expertise in relevant domains, prior experiences, organizational skills, classroom management, and personal traits such as beliefs, attitudes, motivation, self-efficacy, and openness to new approaches (Century & Cassata, 2016). Organizational factors influencing implementation arise from decisions made by educational institution stakeholders, including those related to class sizes, resource allocation, infrastructure, scheduling, and the broader organizational structure that guides instructors. These factors also encompass administrative procedures, management practices, and policy decisions specific to the innovation. On the other hand, external environmental factors encompass opportunities and constraints that are beyond the institution's control, such as economic conditions, infrastructure limitations, and shifts in political priorities. Examples include poor network connectivity hindering instructor development and budget changes resulting from shifts in political leadership.

About the Middle-school Mathematics and the Institutional Setting of Teaching (MIST) project

The MIST project was an eight-year initiative conducted between 2007 and 2015 in collaboration with four urban school districts in the United States. This was some years after the No Child Left Behind Act had been established in 2002. MIST was implemented in two distinct phases. All four districts participated during the first phase, from 2007 to 2011. Two of the original four districts continued their involvement in the second phase, which took place from 2011 to 2015.

The primary goal of MIST was to support the development of ambitious, inquiry-based instructional practices among middle-grade mathematics teachers in ethnically diverse urban districts. The initiative sought to develop a Theory of Action that could be broadly applicable to Research-Practice Partnerships across various academic fields (Cobb et al., 2021). The project was conducted as a design-based implementation research (DBIR) study, spanning four cycles. Like other DBIR studies, MIST had two key objectives. First, to provide timely feedback to district leaders on their instructional strategies, offering recommendations for improvement. Second, to develop a scalable, generalizable Theory of Action for instructional improvement in mathematics (Henrick et al., 2015).

Prior to partnering with the districts, the research team formulated preliminary conjectures based on relevant literature in mathematics education, teacher development, and educational policy and leadership. These conjectures addressed various elements of instructional support, including instructional materials, resources, professional development sessions, teacher collaboration, school instructional leadership, and district leadership (Cobb & Jackson, 2021). The study was structured to allow researchers to examine, refine, and expand upon these conjectures iteratively. After each cycle, recommendations were provided to district leaders to enhance their instructional strategies. By the end of the project, the research team had developed a comprehensive Theory of Action (ToA) aimed at supporting system-wide improvements in mathematics instruction. The ToA created through the MIST project consists of three key components: a coherent instructional system (key components similar to NCTM Common Core Standards) designed to improve teachers' instructional practices, the instructional leadership of school leaders in mathematics, and district leaders' efforts to build school-level capacity for instructional improvement (Cobb et al., 2018).

Method

We conducted semi-structured interviews, each lasting 60 minutes, via Zoom with four key people (KPs) of the MIST project and one district leader from each of the four districts that participated in the MIST project. The key people are the members of the research team of the project. The interviews were conducted in November 2023. The interview protocol consisted of six themes, each accompanied by probing questions designed to elicit comprehensive information through the deliberate use of open-ended questions. The six themes were background, planning of the MIST project, implementation of the MIST project, evaluation, identification of supporting and limiting factors of influence, and the role of mathematics education theories. The interviews were transcribed using Zoom's transcription feature and manually reviewed for clarity. For this paper, we analyzed the responses to the interview questions using influential factors as the theoretical construct. We coded the transcripts to identify the influential factors: the innovation itself, end users, organizational

setting, and external environment factors. Additionally, we referenced the project's final output (Cobb et al., 2018) alongside the interview transcripts to gather precise data figures. We used the pseudonyms KP# for KPs and DL for the district leader for anonymity.

Focusing on influential factors, we found that KP1 and KP2 provided the most relevant responses, likely due to their integral roles throughout the project. In contrast, KP3, an applied statistician, and KP4, an expert in professional learning communities, contributed specialized expertise. While all four KPs and the DL provided valuable data, KPs 1 and 2 offered the most pertinent insights.

Results

In this section, we present the results of our analysis of the MIST project, focusing on the influential factors suggested by Century and Cassata (2016).

Innovation

A central focus of Implementation Research (IR) is innovation. However, the nature of the innovation in the MIST project is somewhat ambiguous. When we reviewed the publications associated with the MIST project, one key question emerged: What is the core innovation of the MIST project? As a result, we specifically included this question — What is the core innovation in the MIST project? — in our interviews. KP1's immediate response was, 'I don't know if we had a core innovation.' According to KP1, the MIST project differs from typical implementation research initiatives.

> KP1: I would say… In the work after MIST, it was implementation research. I wouldn't classify MIST implementation research.

KP1 added:

> KP1: One thing I want to clarify … I've got a sense from reading the questions that it was, so you had this idea on how to make things better. What was it [the innovation]? And did it work? Sort of thing? That's not what the MIST project was about. It was about learning. … what I viewed the project was about, because it was funded as a research project, not as a make the world better project, was to figure out what it takes to improve the quality of math teaching and learning, going all the way from the classroom to system leadership. And to figure that out.

KP1 further described the intention of the project to provide details as to why it is not an ordinary IR project:

> KP1: Sadly, in the United States, principals have typically given a very active role in leading instructional improvement. Whether that's a good idea is something that's up for debate… But it's deeply institutionalized here; that's not going to change. So, the question then becomes what should principals who don't have a background in mathematics, let alone in ambitious or inquiry into math teaching be doing, specifically doing to facilitate, encourage, contribute to improving the quality of math teaching, and learning. …

KP2 emphasized the creation of their innovation rather than its implementation, stating: "In terms of innovation, I believe one of our achievements is the development of a theory of action that addresses classroom dynamics on such a scale." This view is shared by KP1, who also highlighted their contribution through the book that presents the Theory of Action. However, KP1 expressed uncertainty about whether this truly constitutes an innovation. KP1:

KP1: And I would say what came out of it. I think you know that at the end there we wrote, we tried to put everything together in a book. What I would say that book, if it makes a contribution, …Is just it gives us a portrait or a model of what a high-functioning district can look like and should look like if you are aiming for rigorous learning goals and ambitious and equitable instructional practices in math. …what high-quality coaching or what high-quality principal instructional leadership looks like. So, I do not know if that counts as an innovation. But that's what I view as our contribution from that work.

As a design study, however, the MIST project began with "… an initial, tentative, and eminently revisable theory of action by formulating conjectures about both supports for teachers' learning and accountability relations that press them to improve their practices." (Henrick et al., 2015. p. 502). This preliminary ToA was developed based on the existing literature in mathematics education, learning sciences, educational leadership and educational policy (Henrick et al., 2015).

End Users

The MIST project engaged various stakeholders as users, including district leaders, principals, coaches, and teachers. The researchers collected data through interviews with these individuals and by video recording several of the teachers' lessons. The transcripts from the interviews and the video recordings were analyzed to produce a document containing recommendations for the district leaders. KP2 detailed the role of school principals in implementing the project.

KP2: Principals, I would say, we interviewed them every January. And we were analyzing what they were saying about how it is that the instructional improvement strategies were playing. Hang out in their schools. The support they felt they had, what they were accountable for, [and] the challenges they were facing. That was all part of the interview. … So, we would produce those feedback reports that would go to the district leadership in May. But then, in August, in most of the districts, we also had a session where we shared that feedback with principals and got their reactions. So, in that sense, we were getting again more information from them in terms of did what we say resonates with them or not. What did it mean for their practice? We were involved and then co-designing and co-leading professional development for the principals. So, they were an integral part of the project.

KPs described the involvement of both district leaders and teachers in the process of data collection.

KP2: In that first phase, especially at the beginning, … we would interview the leadership in the fall so that we could document what their theory of action was. And then, we were generating/collecting data to find out. How [or] in what ways were those instructional improvement strategies playing out as the leaders had intended or not? And so that was through the interview data. It was also through observation of teachers' teaching. Assessment assessing teachers' mathematical knowledge for teaching was that improving over time, there were the survey measures that KP3 is referring to.

KP2 further described:

KP2: In terms of evaluating teachers' enactment, that was primarily through each year, each teacher who participated in this study, there was a video recording of 2 days of instruction where what they were asked to teach a lesson organized around a cognitively demanding or rigorous task that included a whole class discussion of students' ideas. … So, we use that to evaluate teachers' instruction and changes in instruction over time. The teachers also took the mathematical knowledge for

teaching assessment out of the University of Michigan. We use that to assess teachers' mathematical knowledge for teaching and any change in that over time.

There are four distinct types of users involved in the MIST project: district leaders, principals, coaches, and teachers. Although the MIST project collected data from four types of stakeholders, the recommendations were directed toward the district leaders. Thus, the primary users of these recommendations were the district leaders.

Organizational setting

The US school districts have unique organizational settings. The researchers of the MIST project demonstrated an understanding of these settings and designed and executed the project with careful consideration to enhance teachers' professional development (PD) opportunities. They did not interfere with the selection of curriculum resources or materials but instead provided recommendations to improve the strategies that district leaders incorporated into teacher professional development. As per KP2:

KP2: So, I would say the design was very much informed and responsive to the challenges that were arising as we engaged with the districts. We did not prepare curricula for teachers. The districts had selected curriculum. We were not involved in any curriculum design in terms of what happened in the classroom. What we were involved in, I would say, is curricular for the professional development of teachers. We contributed to that, but not the classroom curriculum design.

In summary, the intention of the MIST project researchers was "to support districts in making it happen" (KP1), where 'it' refers to the plans each district had for providing support for teacher PD.

Aligned with the organizational settings of school districts, the MIST project's organizational structure is also distinct (see Figure 1). In Figure 1, arrows labeled with 1 represent the PD initiatives conducted by district leaders (DL). The diagram illustrates the flow of PD support from the DL to the principals (P in the diagram), then to the coaches (C in the diagram), and finally to the teachers (T). Data were collected from all the stakeholders—DL, P, C, and T—denoted as 2 in the diagram. Recommendations were subsequently provided to the district leaders, labeled as 3.

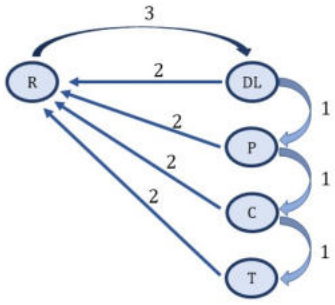

Figure 1: Organizational structure of the MIST project

External Environment

As mentioned, the content of MIST followed the general trend concerning Common Core Standards. The project can be seen as a response to the federal No Child Left Behind Act. However, in other

respects, the influences of the external environment appear small. The political and administration authorities within the districts seemed to have had minimal impact on the MIST project. During the interview, we asked a probing question:

R: Did politicians or school administrators have any influence on the project design in some way, or did they interfere in some way?

KP2: Within the districts, I would say we worked most closely with people who are in district leadership positions, so the curriculum specialist for mathematics would support all of the schools in the district. The principal supervisors, so the people who were in charge of the actual principals are the school leaders. [they are] the academic officers who would oversee all of what was happening for curriculum and instruction, not just math, but all of the content areas. That tended to be the level at which we worked, and they very much shaped the design. So, particularly moving into the latter part of phase one and what phase 2 became, was very much shaped by conversations with people at that level. … And then in terms of other politicians like, for example, school board members. They were not. … I don't think they were involved at all. I know that, for example, the chief academic officer or the superintendent would sometimes share what they were learning with a school board, but the board wasn't shaping what was happening in the project. It was very much, I would say, kind of the bounds were at the district leadership. … I don't think we had any cases of that.

The conclusion regarding minimal influence adds to the finding about the MIST's organization (Figure 1). Information in MIST was not only circulated; it appears to have been a closed circulation.

Discussion and Final Remarks

Despite the challenges faced by district leaders during implementation, 67% of the recommendations made by the project team were attempted by these district leaders (as noted by KP1 during the interview). This indicates a degree of success, suggesting that the MIST project may be worth reproducing. Consequently, the paper focuses on evaluating the project from an IR perspective and exploring how it might be adapted for other contexts. From an IR perspective, the project does not represent a traditional form of innovation. Instead, it developed an innovation—a Theory of Action—that district leaders can use in designing and implementing PD programs for teachers. Thus, the primary target audience for the ToA is district leaders or others responsible for designing teacher PD programs. The research team developed the ToA based on insights gained from various districts.

We suggest that the primary users of the ToA should be district leaders or those responsible for organizing PD programs, with teachers as end-users. Thus, we find Jankvist et al.'s (2021) definition of innovation, which emphasizes change for several users, to be more relevant, especially with respect to reproducing MIST, which requires the existence of an equivalent to district leaders (primary users) in terms of responsibilities and powers. It is not sufficient to have teachers as the only end users.

Regarding organizational settings, we observed that the structure of the US school system is complex, involving numerous different stakeholders. Similarly, the organizational structure of the MIST project involved several stakeholders (see Figure 1). This indicates that the researchers played a vital role in collecting data from various stakeholders and delivering recommendations, integrating theory from existing research with practical insights. Our analysis suggests that this role should be considered when reproducing the MIST project. Thus, if the innovation of MIST—the Theory of

Action (ToA)—is to be reproduced, a variety of stakeholders with different areas of expertise must be involved to ensure the necessary circular flow of information, a dynamic essential to the ToA.

Lastly, the design of the MIST project and the Theory of Action were closely aligned with the structure of the US school district system. Much of the information in the circular motion could flow through existing channels between different bureaucratic levels—district leaders, principals, and teachers—who held formal responsibilities and authority. This also involved an established culture of information sharing, including the types of information that teachers felt they could or should share with researchers, which would eventually reach their district leaders. It is important to note that the US school system, one of the most established and oldest systems of mass education in the world, operated within a democratic society and had long integrated educational research. Hence, we cannot assume that information sharing would function as effectively in countries with younger or less developed school systems. We argue that the innovation of MIST, particularly its Theory of Action, is deeply tied to the specific characteristics of the US school system and its culture. Therefore, when considering a reproduction of the MIST project in a different context, it is crucial to account for the organizational structure of the local school system as a key influencing factor.

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Research-designed, teacher-refined: making sense of teachers' adaptations of an instructional innovation

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How do secondary school mathematics teachers adapt a research-based instructional model designed to improve students' mathematical communication skills? In this case study involving a team of four experienced teachers, data from a school year of regular team meetings was analyzed using a hybrid inductive-deductive approach, combining the deductive application of the Professional Obligations framework with inductive analysis of emergent themes. The findings reveal that teachers made productive adaptations based on three main clusters of factors: individual student needs, institutional constraints, and didactic concerns about the model itself.

Keywords: Mathematics, Program implementation, Design research

Introduction

To study the transposition of a research model from the theoretical or controlled research context to real-world educational settings, both researchers and implementers often consider implementation along a spectrum ranging from pro-fidelity to pro-adaptation stances (Century & Cassata, 2016). This unidimensional approach focuses primarily on the degree of adherence to the original model design. In this paper, we instead adopt the more nuanced integrity approach to implementation (Meland & Brion-Meisels, 2023). This approach posits that implementation should be evaluated not only in terms of fidelity versus adaptation but also in terms of which parts of the intervention are adapted and how. The integrity approach thus allows for a more comprehensive assessment of implementation processes. In this context, integrity represents

> ...the degree to which an intervention was implemented maintaining its core active ingredients, while authentically and fully integrating the assets and needs of the local community. (Meland & Brion-Meisels, 2023, p. 2)

When informed by the needs teachers see in their community or the challenges they encounter during implementation attempts, changes in the teaching model may be necessary or beneficial. When this is the case, one may follow DeBarger (2013) and call these adaptations *productive*.

Practitioners' decisions about whether and how to adapt an innovation during implementation are influenced by a complex interplay of factors. Research across various sectors, including healthcare and education, has identified several key categories of these influencing factors. In the healthcare sector, Moore et al. (2013) found that logistical factors, particularly time constraints and resource availability, play a crucial role in implementation decisions. They also noted that philosophical factors, such as the perceived alignment of the innovation with established cultural values, exert some influence, albeit to a lesser extent.

In mathematics education, Handal and Bobis (2004) found similar patterns with domain-specific nuances. Their study on implementing thematically organized mathematics teaching confirmed the

significance of logistical constraints like time and resources. However, they also highlighted philosophical factors unique to educational settings. Teachers perceived misalignment between the thematic approach and institutional assessment practices, hindering implementation. Additionally, teachers questioned the didactical soundness of thematic organization over traditional topic-based approaches. These concerns, rooted in personal experience and established praxis, may significantly influence which aspects of an intervention teachers implement, adapt, or abandon.

Here, we investigate the implementation of a specific innovation: the MUSK[1] teaching model for developing students' mathematical communication. The teaching model is founded on three core ingredients (Teledahl et al., 2023): the model separates problem-solving from the formal communication of the solution, the model includes spending time discussing the merits of student-produced communications, and the model posits an imaginary or actual absent peer as the intended audience, encouraging students to write clear, convincing solutions for a peer who is unfamiliar with the problem. These principles have been operationalized into four possible teaching sequences, each spanning two lessons with 5 to 7 distinct phases (see Figure 1 for one such sequence).

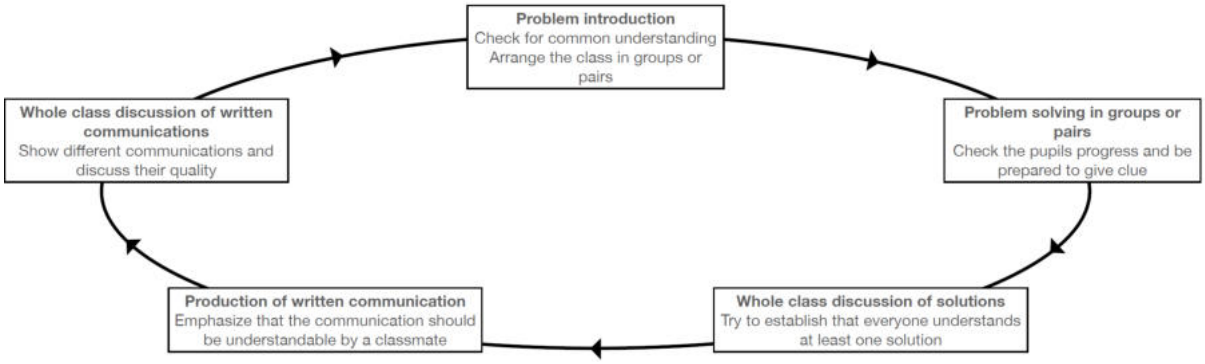

Figure 1: one version of the MUSK teaching model

In this paper, we are interested in how and why teachers adapt the MUSK model. An analysis of the changes teachers made or intended to make may reveal how teachers navigate potential tensions between the model and the teachers' specific context.

The teachers' regular documented meetings over a full school year provide a *useful heuristic case study* (Thomas, 2011) for examining adaptation processes in implementation of researcher-designed educational innovations. Heuristic case studies are representative of some general phenomenon and help to identify relationships between factors within those phenomena. In Thomas' (2011) terminology, the *subject* (the example that is being studied) of the case study is the group of teachers' implementation of MUSK. The *object* (that which the case serves to explicate) is the set of adaptation processes made by teachers implementing a research-designed innovation.

This paper thus seeks to answer: What contextual factors influence teachers' adaptations of research-based teaching models?

[1] "Modell för Undervisning av Skriftlig Kommunikation" [Model for Teaching Written Communication]

Background Research

In analyzing how teachers implement and adapt the core components of MUSK to their local contexts, we draw upon the Professional Obligations framework (Herbst & Chazan, 2012). The Professional Obligations framework posits that teachers' actions and choices in the classroom are shaped by four key obligations: disciplinary, individual, interpersonal, and institutional. The disciplinary obligation refers to the teacher's responsibility to accurately represent mathematics as a discipline. The individual obligation pertains to addressing the needs and characteristics of each student. The interpersonal obligation involves managing classroom dynamics and relationships, while the institutional obligation encompasses the teacher's responsibilities to the school, district, and broader educational system.

Research using the Professional Obligations framework has illuminated the complex nature of mathematics teaching and the tensions that arise between different obligations. For instance, Webel and Platt (2015) examined how institutional pressures, such as standardized testing, influence teachers' instructional decisions and potentially conflict with their perceived duties to individual students or the discipline. In the context of implementing the MUSK model, we anticipate that teachers might experience tensions between, for example, the model's emphasis on communication (aligning with the disciplinary obligation) and assessment demands (part of the institutional obligation).

When adopting a research-based teaching model, teachers could be said to face a fifth obligation: fidelity towards the core components of the teaching model. This obligation is distinct yet intertwined with the other four, influencing how they are fulfilled. The case study seeks to explore whether teachers perceive such a new obligation, and, if they do, how they integrate it with the others.

Method and Results

This section presents the method of analysis together with the results. The results are incorporated here to illustrate the analytical process and to avoid redundancy that would occur if a separate Results section were provided.

The four teachers, all working in secondary school at an international school in Stockholm, Sweden, came from four different national educational backgrounds (only in Western Europe and the U.S.A., however) and had a minimum of 10 years of teaching experience. The teachers met regularly, roughly every two or three weeks, during one school year and had in total 11 meetings of 90 minutes each. Each meeting was documented using careful meeting notes, most often but not exclusively by one of the participating teachers who was also the team leader and is the author of this paper.

The meeting notes were analyzed using a hybrid of inductive and deductive content analysis (Fereday & Muir-Cochrane, 2006). A hybrid approach allows one to approach the data deductively, based on themes that stem from prior research such as the Professional Obligations framework, while also allowing for unforeseen themes to emerge, in our case themes pertaining to the MUSK instructional model. The process was sequential, following the sequence outlined in Fereday and Muir-Cochrane (2006), and is explained below.

First, two coding manuals were developed. One focused on the challenges teachers perceived and adaptations that they suggested or made to the teaching model. Challenges were defined as discrepancies between desired and actual outcomes, identified in the text where teachers expressed complaints about unexpected events, competing priorities, or lack of resources or competency. Adaptations were changes to the given teaching model, recognized in suggestions or reports of instructional choices not explicitly stated in the original instructions. The second coding manual was based on the Professional Obligations framework, comprising four categories. Disciplinary obligations were identified in references to mathematical norms, reasoning validity, and notation. Individual obligations were recognized in mentions of personal needs, behaviours, and differentiation strategies. Interpersonal obligations were recognized in references to group behaviour, spatial orientation, and student-teacher interactions. Lastly, institutional obligations were identified in mentions of assessment demands, regulations, value systems, and resource availability.

In the second step, the coding manual was tested on a sample of data from one meeting in October 2023. It was straightforward to identify themes from both coding manuals, so the codes were deemed to fit well. However, this initial coding process revealed the importance of tacit knowledge shared among the teachers. Actions like assigning homework or reusing test questions were often motivated by multiple factors, including time constraints, assessment alignment, and individualized student support. These motives were rarely explicitly stated, as they are tacitly understood by the group. A brief follow-up interview with the teachers confirmed that these modifications were intended to save time by addressing multiple objectives simultaneously.

In the third step, raw data theme related to MUSK implementation were extracted. For example, an extract from the meeting notes was this:

> M7 work - simplifying tricky algebraic expression - interesting distinction between arbitrary / conventional rules, and deduced and therefore valid or invalid. What about 1m = m?

The extracted raw data theme of this extract became: “distinction made between arbitrary / conventional and deduced rules”.

The fourth step involved coding extracted raw data themes using Challenge and Adaptation themes, followed by Professional Obligations themes. Coding was guided by the manual while allowing for inductive elements. Lower-order themes were identified within each theme. For instance, *Time* and *Assessment demands* were categorized under the *Institutional obligation* theme. Inductive coding was applied to data not fitting the Professional Obligations framework. For example, teachers’ uncertainties about the MUSK model’s task-suitability were coded under the inductively derived theme *Didactic concerns about MUSK*, which involves teachers’ considerations of instructional choices within the model. This theme emerges from questions about language precision, individual vs. group text production, and MUSK instruction specifics. It also includes efforts to refine the model by addressing task suitability.

The fifth step was to connect Challenges and Adaptations with the Professional obligations and the inductively arrived-at themes. I explored code co-occurrence of challenges and adaptations with the four professional obligations, and with the inductively extracted themes, as seen in Table 6.

Table 6: Challenges and Adaptation connected to each of the Professional Obligations and the inductively found theme of didactic concerns.

Theme	Challenges	Adaptations
Disciplinary	Students struggle with valid justifications in their written solutions	Explicit teaching of mathematical reasoning
	Student choice of representations may not be consistent with mathematical norms	Explicit teaching of mathematical notation
Individual	Students do not participate equally in analyzing communication	Asking students to give feedback to peers
	Students have different individual needs in communication and mathematical reasoning	Instructing students to set improvement goals based on feedback
Interpersonal	Students do not participate equally in groups, with some being less active than others	Assigning roles in group work
Institutional	Students prefer teachers to set standards	
	Students do not transfer new communication skills to assessment situations	Including assessment of communication in summative assessments more frequently than before Communicating assessment expectations to students
	Limited time for lengthy problem-solving and communication exercises	Giving problem-solving or writing as homework, reusing tasks that students have previously solved
Didactic concerns about MUSK	Cohesive group norms do not form	Documenting and returning to lists of good qualities identified by students
	Repetitive discussions showing lack of progress	
	Tension between students', teachers', and assessment instruments' expectations	
	Some problem-solving tasks are not useful for MUSK	Choosing problems that elicit a variety of communication approaches
	How to handle students' incorrect solutions	Including incorrect solutions in discussions
		Using MUSK with tasks other than problem-solving

The teachers found adaptations that helped them meet most but not all of the encountered challenges. No adaptations were found that would remedy the sense of repetitive discussions nor the existence of tensions between individual students', teachers' and assessment instruments' interpretations of quality. Additionally, one adaptation (using MUSK with other kinds of tasks than it was designed for) stemmed from an opportunity rather than a challenge. For the most part, however, adaptations could be understood as attempts to address challenges.

In the final step, the connections found between the codes were thematically clustered as seen in Table 7.

Table 7: Higher-order clusters of themes

Cluster	Challenges	Adaptations
Individual student needs	Students vary in readiness, ability, and engagement.	Teachers use peer- and self-feedback to tailor tasks for individual students Teachers teach those skills students are lacking
Institutional constraints	Time constraints make it difficult to devote as much time to MUSK tasks as is necessary Assessment demands specify forms of communication not necessarily aligned with student opinions.	Teachers reuse tasks from other contexts and assign homework rather than spend class time Teachers tell students about assessment demands and ask students to evaluate based on these
Didactic concerns about MUSK	Teachers perceive the MUSK model as lacking in some areas.	Teachers develop the MUSK model and apply it to novel tasks

Finally, a member check was carried out by emailing the teachers tables 6 and 7, together with the descriptive text before and after the tables, and asking whether they feel that this analysis captures their experience of the implementation process. All agreed that it did.

Discussion

The adaptations made to the MUSK teaching model are possible to understand through the concept of programme integrity (Meland & Brion-Meisels, 2023). The core elements, separating communication from problem-solving, positing an absent peer, and orchestrating student discussions of qualities of communication, were preserved. Peripheral elements, such as the kinds of problems students worked on and how student opinions were collected and documented, were adapted in response to challenges teachers encountered.

In addition to changing peripheral elements, teachers adapted the model by introducing new elements: the use of feedback and rewriting (to ensure individual development) and explicit teaching of necessary reasoning and notation skills. The adaptations could therefore be said to be of two types: changes to elements included in the model and adding new elements to the model.

The findings of this study are coherent with Moore et al. (2013) and Handal and Bobis (2004) because logistical concerns (such as time) and institutional concerns (such as assessment) were important influences on teachers' adaptations in this as well as in the earlier studies. In particular, assessment constraints stood out as especially important in this study and in Handal and Bobis (2004). Didactic considerations also played a role in both studies. Teachers used their professional judgement to decide what parts to implement as intended by the model designer, and what parts to adapt or abandon. It is, however, difficult to gauge the influence of philosophical factors in this study, since the teachers agreed to implement the MUSK model precisely because it fit so well with the student-centered teaching and learning of the IB programmes and the participating teachers' preferences.

From the Professional Obligations framework, adaptations were made based on each of the four obligations, but to varying extents. MUSK has student discussion as a core element, so the Interpersonal professional obligation was met with little need for adaptation. Other obligations, such as towards individual student needs and growth, are not as central in the MUSK model, and teachers consequently made more adaptations in those areas. Finally, the data contained themes pertaining to an obligation towards the instructional model itself. Didactic concerns about the instructions led to multiple adaptations of the model.

The teachers' adaptations of MUSK could be said to be productive (DeBarger, 2013) in the sense that they make the model more practical within the context in which MUSK was implemented. Because this study did not measure the impact on students' communication development, we cannot say whether the adaptations made the model more effective. Practicality is, however, a necessary condition for effectiveness, because teachers must find it meaningful and realistic to implement a teaching model for it to be used at all and, ultimately, for it to have any effect on student learning.

A reliability check was intended but not carried out in the third step of the data analysis because the concise nature of the meeting notes made it difficult for someone outside of the group of teachers to understand the notes. This limitation was mitigated by carrying out a member check which is an established method for assessing the credibility of findings (Cutcliffe & McKenna, 2002).

This study suggests some tentative recommendations for design and implementation research. To avoid difficult challenges that give rise to potentially unwanted adaptations, innovations should not conflict with teachers' professional obligations. Ideally, an innovation should serve multiple purposes simultaneously. Finally, teachers and students care about assessment. The innovation is more likely to be implemented with programme integrity if it can be made to align with assessment expectations.

Conclusion

This study demonstrates how teachers tailored the MUSK model to their specific context, guided by their professional obligations and concerns about the model's didactic soundness. Based primarily on the contextual factors of heterogeneous classes and institutional constraints on time and assessment

criteria, they adapted the model to better suit individual student needs in a timely way while preparing students for success on assessments. The Professional Obligations framework, with the additional obligation to the MUSK model, proved valuable in analyzing teachers' implementation choices.

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Young students' proportional judgements: A replication study

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This paper describes a conceptual replication of a set of tasks developed by Spinillo and Bryant, incorporating extensions in terms of population, time period, and classroom implementation. Despite notable methodological differences in how the tasks were implemented in the two studies, students' reasoning patterns remain largely similar to those observed in 1991. While the new study allowed students to collaborate in small groups, unlike the individual work in the original study, the analysis reveals that this collaborative approach only serves to support the initial findings.

Keywords: Replication study, proportional judgement, first- and second-order comparisons.

Introduction

Previous research has concluded that children have significant difficulty performing proportional reasoning (see, for example, Spinillo & Bryant, 1991). Experiments on proportional reasoning typically involve tasks in which children assess relative size by examining relations between relations, also known as second-order relations. According to Spinillo and Bryant (1991), prior research in proportional reasoning had attributed students' difficulties to their inability to compare two proportions. The authors sought to explore whether students' challenges might not stem from second-order comparisons but rather from insufficient understanding of the two underlying first-order relations, such as assessing how full a glass is. They approached this by selecting first-order relations that children as young as 4 years old can understand, specifically *greater than* elations. By designing tasks in which the only proportional reasoning needed was based on *greater than* comparisons, they aimed to determine whether children aged 4–8 years could make proportional judgements. Their argument was that if a child could identify whether one figure contained more blue than grey, remember this information, and later distinguish it from another figure containing more grey than blue, the child would have successfully executed proportional discrimination.

Spinillo and Bryant (1991) concluded that cases where one figure is divided vertically and the other horizontally are especially crucial for determining whether students are genuinely making second-order comparisons – typically by comparing part–part (ratio) relations – or if their reasoning depends on perceptual judgement. Notably, the concept of half plays a critical role in these judgements, as it represents a key boundary in children's proportional judgements, making comparisons easier when ratios are on opposite sides of a half rather than on the same side (Spinillo & Bryant, 1991).

To determine whether the findings of the original study can be generalised to a different context, it can be essential to conduct a replication study. This paper undertakes a conceptual replication (Aguilar, 2020) of Spinillo and Bryant's (1991) research on proportional judgements, extending it in terms of population, time frame, and classroom implementation. By examining whether the original results hold true for Danish students more than three decades later and within a distinct educational environment, this study aimed to assess the transferability of Spinillo and Bryant's (1991) findings. Additionally, the replication study was intended to gain insights into the strategies employed by

Danish students when engaging with tasks involving proportional judgement. Consequently, the research questions guiding this investigation were as follows:

1. What strategies do Danish second-grade students use when solving tasks related to proportional judgement, and how do these strategies compare to those used by 8-year-old students in Spinillo and Bryant's (1991) study?
2. What insights into students' proportional judgements can be gained through a conceptual replication study of Spinillo and Bryant's tasks, conducted with a different population in a different setting?

Theoretical constructs related to replication studies

Replication studies play a crucial role in verifying the reliability and validity of original findings, thereby enhancing the credibility of a field. By adhering to rigorous scientific methods and transparent research practices, replication efforts demonstrate a commitment to the robustness of social science research. These studies contribute to building a cumulative and dependable body of evidence, which is essential for the development of sound theories and the formulation of effective practices and policies. Moreover, conducting replications in different settings, populations, or time periods can uncover variations in social phenomena, thus providing valuable insights into the complex and context-dependent nature of human behaviour and social processes (Schoenfeld, 2018; Thanheiser, 2018).

This paper presents a conceptual replication, defined as a replication of a test, hypothesis, result, or, as in this case, a task, but with the use of different methods (Aguilar, 2020). Additionally, the study qualifies as an extension study, as the replication was conducted within a different population (Danish rather than American) and at a different time (the year 2023 and not 1991). This approach allows for an examination of whether the findings of the original study hold true outside the context in which they were reported (Aguilar, 2020).

The original study

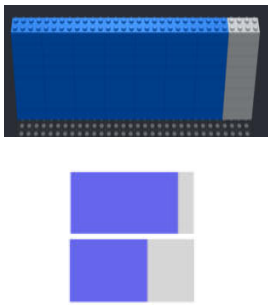
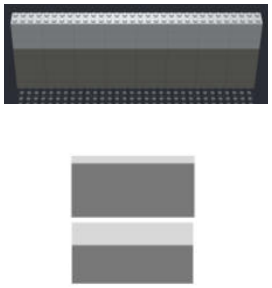
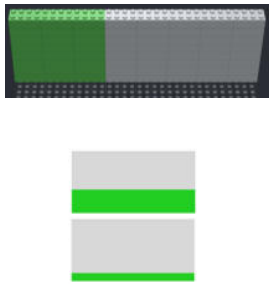
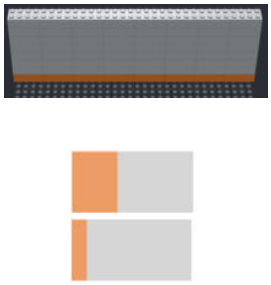
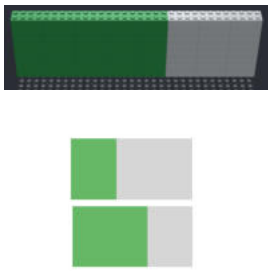
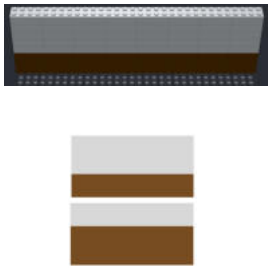
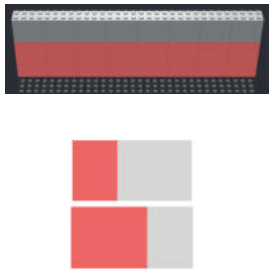
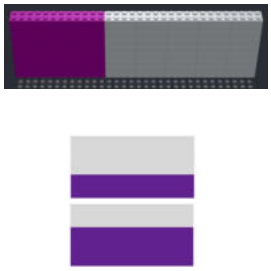
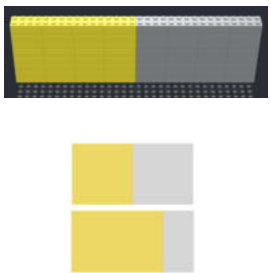
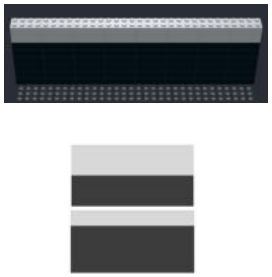
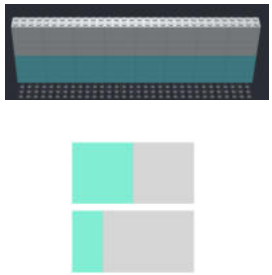
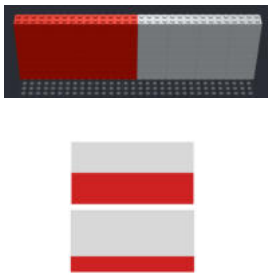
The theoretical foundation for this paper is based on Spinillo and Bryant's (1991) study, which investigated the development of proportional judgements in children aged 4–7 years as described in the introduction. In the original study, the students were shown one figure – the comparison figure – and then asked which of two boxes had the same amount of grey to coloured as the comparison figure (see Table 1). The authors hypothesised that proportional judgement that crosses the half boundary should be easy, and those that do not should be difficult for children. Furthermore, the authors predicted that children would solve proportional judgement problems on the basis of part–part relations, e.g. by comparing one part of the whole with another part of the whole.

Method

In designing the tasks for the experiments in Spinillo and Bryant (1991), care was taken to ensure that children could not base their judgements on the absolute size or shape of the figures. Therefore, the comparison picture was smaller than the two boxes, and the boxes were not subdivided into bricks, so counting was not an option either. The authors included two kinds of comparisons: cross-half and within-half comparisons. While the cross-half comparisons entailed fewer coloured bricks than grey

bricks in one box ($\frac{3}{8}$) and more coloured than grey in the other ($\frac{5}{8}$) (or the other way around), the within-half comparisons had either more coloured than grey in both boxes ($\frac{5}{8}$ vs. $\frac{7}{8}$) or less coloured than grey in both boxes ($\frac{1}{8}$ vs. $\frac{3}{8}$). Furthermore, the authors discriminated if the boxes were in the same or a different orientation than the comparison picture. For each of the four combinations of comparisons and orientations, two tasks were administered to the children in experiment 1. In experiment 2, four tasks were included: two using half comparisons in the same orientation, and two using half comparisons in a different orientation (see Table 1).

Table 1. Presentation of the 12 tasks

	Same orientation		Different orientation	
Within-half comparison				
Cross-half comparison				
Half comparison				

Results

The results of the two experiments showed that even though 4–5-year-olds used the *greater than* relation, they did not use the half as a boundary as older children did. The older children were, however, more likely to use the half boundary in tasks involving a different orientation than they were in those depicting the same orientation. There were no significant differences between the

within- and cross-half conditions in the same orientation. Many more 6–7-year-old children scored above chance level in the cross-half compared to the within-half comparisons. In particular, the 7-year-olds found the cross-half comparisons easier than the within-half comparisons when the orientation differed. There were no significant differences in children's answers when comparing cross-half and half comparisons. The authors concluded that some proportional judgements are made by children from the age of 6 years who discriminate between more than and less than half, thereby indicating that children's initial understanding of proportion involves both simple first-order relations and relations between relations (second-order relations). Children's ability to compare part–part relations was reflected in the justifications the children gave for their answers. Spinillo and Bryant (1991) identified five categories of justifications. While the first two categories did not involve any comparative relations, and the third category involved justifications using single relations, the fourth and fifth categories involved relations between relations. In either of the two latter categories, orientation had no effect on the justifications used by children at any age. However, the fourth category was generally seen in cross-half comparisons, while the fifth category was largely seen in within-half comparisons. In this paper, no distinction will be made between categories 4 and 5.

The replication study

Method

The 12 tasks (see Table 1) were administered to 20 students (one second-grade classroom) in Denmark in January 2023. The students were 8–9 years old. The first task was introduced to the class by their regular mathematics teacher. That task was answered collectively by the students with support from the teacher. The comparison figure was shown in Lego-building blocks, and a picture of the figure was shown on the blackboard together with the two boxes. The students were then split into groups of 2–3, randomly chosen by a picker wheel. All groups were handed cards with comparison pictures and cards with boxes, first for tasks 1–8 and, later in the lesson, for tasks 9–12. The students were then asked to choose the box where the proportion of coloured to grey was the same as in the comparison picture and give a justification to the other group members.

One camera filmed the students' group work during the lesson. The author and an assistant teacher participated in the classroom, answered questions if the students had any, encouraged the students to argue for their choice of box, and took notes of the students' discussions. The excerpts used in this paper comprise a mixture of these notes and transcripts from the videorecording.

Methodological deviations from the original study

This replication study introduced several key differences from the original research, reflecting both an extension of the original study and a conceptual replication. One significant extension was the shift in population and timing: the original study was conducted with children aged 4–8 years in the United States, while this study involved Danish students aged 8–9 years and was carried out 30 years later. Although the Danish students had not received formal instruction on fractions before the lesson, their age group differed from that of the original study (as children aged 4–7 years did not take part).

Another important difference lies in the method of task administration. In the original study, each child was individually given one task at a time. In contrast, this study involved students working in a

whole-class setting where they collaborated in small groups to complete several tasks. This shift from individual to group work represents a conceptual replication, as it explores the same tasks through a different approach.

Additionally, the materials used in this study were adapted from the original. While the original study used blue and white boxes, the current study modified the materials to enhance clarity. The white parts of the boxes were changed to light grey to distinguish them more clearly from the background, and each of the 12 tasks featured different colours instead of the consistent blue used previously. These changes were made to facilitate the students' ability to reference specific tasks while being video-recorded, thereby improving the data collection process.

Results

In order to analyse the students' justifications, all of the justifications in the transcripts were colour-coded based on the four types of justification identified by Spinillo and Bryant (1991):

1. No comparative relation: mention colour without quantification (red)
2. No comparative relation: mention of quantity but no comparison (orange)
3. First-order (single) relation: comparison of quantities of both colours in one box only (part–whole comparison) or comparison between boxes using only one colour (green)
4. Second-order relation (relation between relation): comparison between colours within and between relations across the picture and the box (blue)

All the colour-coded justifications were arranged in a table corresponding to the tasks in Table 1, as can be seen in Table 2. Only justifications where the specific task was clearly identified in the students' justifications are included in this table.

The first four tasks involved two boxes where the proportion of the coloured section fell on the same side of half (within-half comparisons). Since the student groups quickly agreed on the correct box with minimal discussion, this indicates that they found it relatively easy to identify the correct option. The students predominantly used terms such as *more*, *most*, and *less* in their reasoning. None of the students used the term half in their justifications. When possible, they used the comparison figure to measure against, for instance by rotating the comparison figure, especially if they were asked to justify their choice. In tasks with different orientations, many students attempted to rotate the figure. The next four tasks given to the students involved cross-half comparisons. In these tasks, 2–3 students used the term half to argue which box had as much coloured as the comparison image. In the final four tasks, one box had exactly half its area coloured, while the other either had more or less than half coloured. Here, more students employed the term half in their reasoning.

The analysis revealed that in tasks in both *Same* and *Different orientation*, students described the size of the different parts in the boxes by positive degrees such as *much*, *thick*, or *thin*, and by using comparatives such as more/less of, *thicker* or *thinner*, or *equally big*. The use of positive degrees is however only seen in *Within-half comparisons*, whereas students' only used comparatives or the term half in the *Cross-half comparisons*. In the *Half comparisons*, however, the use of positive degrees was repeatedly observed. In Table 2, positive degrees are marked in *italics*, while comparatives are marked in **bold**. Use of the term half is <u>underlined</u>.

Table 2. Colour-coded student justifications

	Same orientation		Different orientation	
Within-half comparison	"The one on the left, because the one with the *small* grey piece resembles the LEGO figure the most. The other one has **more** grey."	"You can see that there are **more** lines on the LEGO figure, so it matchesthe one with the **most** grey."	"This one is really *thin* . The other one is a bit **thicker**. You can see on the LEGO figure that the green is *thick* , so it matches best with 3a."	"It's because there's **more** grey compared to orange [4b], and there's also a lot of grey compared to the orange on this one [figure 4]. Here [4a], there's **more** than on that one [4b]."
	"I think it's the one on the right, because there is **too much** grey on the other one."	"It's because, look. Here there's a very *thin* line [2a], but here there's a much **thicker** line [2b], just like in the picture [figure 2]."	"Because, there's **more** green here [3a] than here [3b]. And there's **less** grey [3a], because it's not just like this [rotates figure 3 to match with 3b]. I can't explain it."	"I think it's because there's **more** grey here, og **less** orange here [4b]"
Cross-half comparison	"It's easy. They're the **same** size."		"That's way too little. Then it would be less than half."	"I think this one matches with this one because if you turn it this way, they would be the **same** lenght."
	"I think it's 5b because the green is the **same** lenght."			"Shall I tell you why? It's because here there's **more** purple than grey, and here there's **more** grey than purple. Can you see it?
Half comparison	"I think it's this one. Look. Half is grey. Half is yellow." "I think it's the other way around. Look, if we flip it, then it's half here."	I think this one matches with that one. This one has *a bit* of grey and *a lot* of black, and here it's also *a bit* of grey and *a lot* of black	"I think it fits here [11a] because there's an **equal** amount of green and grey"	"So, for example, it could be that there's *a bit* of red and *a lot* of grey here, and here red and grey are **equal**, and it's the same here, so it matches well with this one."
	"The grey one has *a bit* of colour. The yellow one has *a lot* ."			"*A bit* of red and *a lot* of grey. Here, it's **equal**"
	E1: "I think that one [9] matches there [points] because there's an **equal** amount of grey and yellow, and it's the same here, but not there."			
	E2: "Here there's *a lot* of yellow, and here there's **mostly** grey."			

While none of the students' justifications were identified as belonging to the first type of justification, types 2–4 were frequently seen. For tasks in the category *Same orientation*, most student answers contained either single relations (category 2) or first-order comparisons (category 3), whereas justifications related to tasks in the category *Different orientation* were based on either first-order (category 3) or second-order (category 4) relations. In the last category, more students used first-order comparisons when the tasks were related to within-half comparisons, while more second-order justifications were seen in cross-half and half comparisons. *Half* was used in one of the justifications related to a task in *Cross-half comparison*, *Different orientation*, but was otherwise only identified in *Half comparison* in *Same orientation*.

Based on the student responses in Table 2, it is not clear whether they found tasks in the *Same orientation* easier or more difficult than those in *Different orientation*. However, several student quotes from the video recordings indicate that tasks in the *Different orientation* category were perceived as more challenging:

Student 1: I thought 1 was easy because the box looked like the one in Lego. Number 3 was the hardest because it's facing the other way.
Student 2: I thought 5, for example, was easier than 3 because in 3, it's facing the wrong way. In 5, it's just the same.
Student 3: 11 is the hardest because the green line [11b] looks like the green line on the Lego figure, but when you turn it, there's an equal amount of green and grey, so it's that one [11a].

The third student response also highlights that students found it easier to compare the boxes with the comparison figure by rotating one of them, despite the figure appearing significantly different. However, this did not seem to confuse the students. In a dialogue with a classmate, one student articulated that even when the figure was rotated, the proportion of coloured sections remained the same, as the student perceived the coloured sections as a liquid that can adapt to another form.

Student: And if you think of this as water, it just flows down here, so it doesn't matter.

Discussion and conclusion

The findings indicate that the Danish students found it relatively easy to make within-half comparisons. Unlike in cross-half comparisons, where a significant number of students referenced half in their justifications, students did not use half as a boundary in their reasoning for *Within-half comparisons*. The use of half and *equal parts* was more prevalent in *Half comparisons*. Additionally, the students' justifications revealed that they perceived tasks with the *Same orientation* as easier than those with a *Different orientation*. The concept of half was used as a justification in *Different orientation* tasks for *Cross-half comparisons*, but not in the *Same orientation*. Conversely, in *Half comparisons*, half was used in the *Same orientation*, while equal was used in *Different orientations*.

Unlike the original study, this one did not conduct a comprehensive count of all students' responses across all tasks. Therefore, it is not possible to determine whether Danish students found *Cross-half comparisons* easier or more difficult than *Within-half comparisons* to the same extent as reported in the original study. However, it is possible to compare the findings regarding the older students in the original study with the Danish students, where the ages overlap. The results in this study are therefore not an indication that an equal proportion of students found cross-half comparisons easier than within-half comparisons. Instead, this study explored whether the same types of justifications observed in the original study were also present in a different population at a later time, and whether students who had not yet received formal instruction in fractions would use half as a boundary when making proportional judgements.

Given the different methodological approach used in this study, this conceptual replication (Aguilar, 2020) offers new insights that were not captured in the original study, where the children worked one-on-one with a teacher. In this study, the students worked together in a setting that more closely resembled their regular mathematics classroom, allowing them to learn from one another and to provide justifications to their peers. This collaborative environment, however, became problematic

in tasks where the students quickly reached a consensus, making further justification unnecessary. Many students were willing to justify their choices when prompted, but particularly in *Within-half comparison* tasks, where they could not use half as a boundary, it became difficult for them to articulate their reasons for choosing a particular box. This finding underscores the importance of the half boundary in students' proportional judgements (Spinillo & Bryant, 1991), indicating that students in this age group are able to distinguish between more than and less than half.

One finding not reported in the original study is that when students were given tasks to be completed in small groups, they attempted to shift the orientation so that the boxes and the comparison picture had the same orientation, even though this made the boxes look significantly different. A key point that supports the notion that students were indeed using part–part relations in such cases is that they focused on the relative amount of the coloured part in one figure compared to the coloured part in the other. Even though the figures looked markedly different, several students in the present study employed the idea that the coloured part were like a liquid that could shift within the box, so that even if the boxes no longer appeared identical, a certain proportion of the box would remain coloured. Although this was not observed in the original study due to the different methodology, it supports Spinillo and Bryant's (1991) argument that students can solve proportional judgements based on part–part relations.

The conceptual replication described in this paper largely supports the original findings despite the methodological differences and extensions (Aguilar, 2020) made: namely, that the tasks were given to Danish students as part of classroom instruction more than 30 years later. To make a closer comparison with the original study's results, it would be relevant to conduct the study with younger students, as was done in the original study. Additionally, more systematic data collection of all students' responses to all tasks, ideally involving a larger population, would be beneficial. Moreover, this analysis did not discriminate between categories 4 and 5 in the analysis, as it was challenging to differentiate between the two codes when analysing students' answers.

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Critical speculative implementation discussion: Future scenarios on technology and mathematics in schools

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Keywords: Technology comprehension, implementation, tarot cards, future scenarios.

Implementing technology comprehension into mathematics

Implementing technology comprehension (TC) into mathematics provides students with prerequisites to participate in a digitalized society and to create and act using digital technologies (DT) (Danish Ministry of Education, 2019). Yet, research on TC and computational thinking (CT) highlights the challenges of introducing new subject areas (e.g., Benton et al., 2017; Slot et al., 2021). In Denmark, a national center for digital TC in K-12 education investigates how to develop TC as a new subject and how to implement TC into the mathematics and Danish (L1) curriculum. This poster presents a novel approach aiming to foster discussions with mathematics teachers in design-based research. Drawing on Bayne and Ross (2024), we developed 12 "tarot cards" that offer speculative scenarios related to technology and education, aiming to support teachers in shaping their ideas for implementing TC into mathematics.

Tarot cards presenting future scenarios of technology and education

Our 12 cards were heavily inspired by Baynes and Ross' (2024) cards. However, we were also inspired by trends in mathematics education research. For instance, Card 2 concerns CT (Wing, 2006). In August 2024, the tarot cards were tested in a workshop with three teachers from primary and lower secondary schools at the beginning of a design-based research process. During the workshop, the teachers evaluated the cards by discussing whether the scenarios were "dangerous" or "cool," and "realistic" or "unrealistic." At the end of the workshop, the teachers created their own vision cards to outline their hopes when implementing TC in mathematics. See Figure 1. Four of the cards are presented in shortened versions below:

Card 1: Back to the Analog Past: *What if the school became completely analog again?* We need to go back to the time when DT didn't have a place in schools. Students' learning and well-being were much better before the use of DT.

Card 2: A Necessity for the Job Market: *How can we prepare students for the job market of the future?* The school must help educate the next generation of the workforce. Therefore, we need to focus on developing students' technological abilities and computational thinking.

Card 4: AI Academy: *Why shouldn't students' learning just be taken over by artificial intelligence?* There is no longer a need for trained teachers in schools. In school, there are adults to take care of the children, but all learning is controlled by DT, particularly AI.

Card 10: Computer science is the new mathematics: *Why shouldn't mathematics be replaced by informatics?* Mathematics, as we know it today, must be replaced by informatics. Thus, our students must become more technically competent!

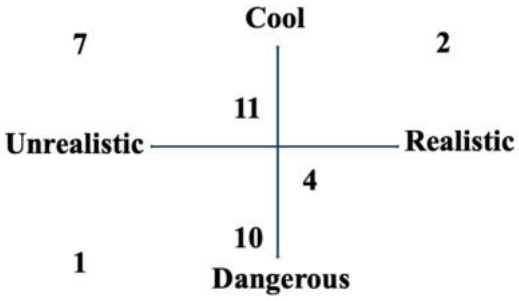

Figure 1: An overview of how the teachers valued Cards 1, 2, 4, 7, 10, and 11

The tarot cards were used in discussions before we designed teaching aiming to implement TC in mathematics. We hypothesize that implementing our tarot cards may ease the implementation of TC as teachers become aware of their aims for implementing TC into mathematics. Potentially, discussions involving our cards could lead to a stronger relationship between teaching aims, and how the teaching is carried out.

At the CERME conference, we bring a printed A0 high-format poster as well as our tarot cards.

Acknowledgment

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Using structured reporting guidelines for intervention research to set up a program theory: The case of Algebra Monday

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Keywords: Implementation, Algebra, Scaling.

Introduction

In a recent working paper by Hill, Lynch and Polanin (2024) the authors argue that a reason for the educational research field's lack of ability to build an accumulated evidence base is the inconsistent reporting of critical facets of classroom interventions. Building on research reviews as well as on other analytical work, the authors suggest a set of structured guidelines for reporting intervention research concerning the five dimensions: Program provider and intervention content for students; Program content for teachers; Program professional development; Program curriculum materials; Program support for implementation, as well as a specific category for Setting-Level Resources, Contexts, and Barriers. Within each dimension, several specific descriptive categories are suggested.

This poster proposal and the associated poster has the purpose of bringing the discussion of such guidelines into the field of implementation research in mathematics education. As a background for such a discussion, we apply the suggested guidelines to an implementation case, the case of *Algebra Monday* (described further below) and compare with established ways of discussing implementation variables, as for example described by Coburn's scaling factors (2003) and the role of the innovation, user, organization and external environment as described by Century and Cassata (2016). A goal with the poster and the discussion we hope it can spark, is to examine if the framework Hill et al. suggests is helpful when designing a program theory (Chen, 2012) in addition to being useful when reporting research.

Background

The *Algebra Monday* program is based on the premise that all mathematics teachers from grades 1 to 9, as well as those teaching upper secondary courses, engage with content related to algebra and structured arithmetic once a week, following a specific whole-class instructional design. It is developed in the municipality where the two first authors are working. Using conclusions from a review of implementation research (Ahl et al. 2023) we have decided on a cascading scaling type of design, where we in the planning, designing and piloting phases work with 3 teachers for grades 1-3, 4-6, 7-9, and upper secondary school respectively. Together with these groups we will develop and pilot curriculum materials for Algebra Monday, as well as a professional development design, where the pilot teachers will act as PD leaders in accordance with the medium-scale deliberate multiplier design suggested by Aguilar et al. (2023). This means that Algebra Monday is a complex enough case for examining the feasibility of using the reporting guideline scheme (Hill et al. 2024) as a tool for planning and design. It concerns specific content for students and teachers, a PD component and a

curriculum material component, both developed within the project, as well as support for implementation.

Poster format

The poster contains a description of the background of the Algebra Monday project and brief introduction to the concepts of innovation, user, organization and external environment (Century & Cassata, 2016) as well as the four scale factors described by Coburn (2003). The main part of the poster will be a description of Algebra Monday following the table structure used by Hill, Lynch and Polanin (2024, appendix). In a conclusion section we compare what aspects of the implementation of Algebra Monday that are captured by the structured reporting framework relative to typical implementation aspects described by Century and Cassata (2016), Coburn (2003) and Aguilar et al. (2023).

The main contribution to implementation research will be a suggestion that the reporting guidelines is not only worthwhile when reporting a researched project but can also serve as a planning and design tool. Since the planning and design stages of implementation projects are under-reported in the literature (Ahl et al. 2023), the poster can serve as an inspiration on how to report the planning and piloting stages, using Algebra Monday as a case.

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TWG 24: Representations in Mathematics Teaching, Learning and Research

TWG24@CERME14: Representations in Mathematics Teaching, Learning and Research – an overview

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This paper explores the discussions and findings from Thematic Working Group 24 (TWG24) at the CERME14 conference, focusing on the role of representations in mathematics education across teaching, learning, and research. We examine how representations function as cognitive and communicative tools in various educational contexts and explore theoretical perspectives that inform the use and study of representations. Through a synthesis of the discussions and presentations, we highlight emerging themes, key challenges, and future directions in research and practice.

Keywords: Internal vs. external representations, Multimodal representations, Representations, Realizations, Semiotic registers of representation, Representations of data.

Introduction

Mathematical representations play a crucial role in teaching, learning, and research. They enable learners to visualize, conceptualize, and communicate mathematical ideas effectively. The CERME Thematic Working Group 24 (TWG24) *Representations in Mathematics Teaching, Learning and Research* brought together researchers and educators to discuss theoretical perspectives, pedagogical practices, and methodological challenges associated with representations in mathematics education. This paper synthesizes the group's discussions, presenting key insights into how representations function in classroom instruction, student learning, and research methodologies.

The group consisted of 21 participants and reviewed 17 research papers addressing diverse aspects of representations. The sessions followed a structured format, including short presentations, critical peer discussions, and collective analysis. The discussions centred on how representations support mathematical reasoning, problem-solving, and conceptual understanding in different learning environments.

We organize the following discussion in three main sections: representations in teaching and learning, representations in mathematics research, and theoretical perspectives used.

Representations in teaching and learning

Several papers introduced and discussed the role of representations in mathematics education, arguing that representations are fundamental tools serving multiple pedagogical purposes. These include facilitating communication of mathematical ideas, particularly in inclusive settings (e.g., Angeloni et al.); illustrating mathematical structures, making abstract concepts more accessible (e.g., Copperman et al.; Kadan-Tabaja; Lisarelli et al.); supporting problem-solving and modeling, allowing

students to express and refine their reasoning (e.g., Cosci; Lemrich; Lobnig; Köhne; Wendelborg et al.).

Authors presented various examples of representations in classroom practice, including several innovative approaches to using representations, such as bodily representations. For example, physical movement was used to explore line symmetry, engaging learners in embodied cognition (see Hölzer & Vogel). Sonified representations in the form of sound-based function representations were introduced to support visually impaired learners and emphasize covariation perspectives in function learning (see Manolino et al.; Piroi et al.). Graphical representations, including explorations of the role of graphs, dynagraphs, and symbolic representations in understanding functions were also examined. This research emphasized the need for seamless transitions between different forms (see Aviram et al.; Poli & Macchioni; Lisarelli et al.).

Some contributions presented studies about the roles of representations of mathematical objects, such as expressions, linear equations, functions or fractions, created in digital environments (e.g., Aviram et al.; Lisarelli et al.; Manolino et al.; Poli & Macchioni), also focusing on potential and pitfalls of their affordances including the feedback offered (e.g., Köhne; Kadan-Tabaja).

Our working group also discussed representations in teacher decision-making: Recognizing that different representations highlight different mathematical properties, how can educators select and sequence representations to optimize student learning? (see Weingarden).

Representations in mathematics research

In response to our novel call for papers which explore how representations are used by researchers to analyze and present their data, some contributions discussed the use of representations in research methodologies. Indeed, representations are integral to data collection, analysis, and interpretation in mathematics education research. The following key methodologies were discussed: eye-tracking analysis, gesture-based coding, and discourse analysis. Two papers analyzed eye movements when solving non-linear geometry problems, identifying fixation points, gaze patterns, and areas of attention (see Schroeder & Götze; Volbers et al.). In another contribution to our working group, color-coded visual representations were used to segment and categorize gestural and verbal interactions in video data (see Hölzer & Vogel). Finally, realization trees were employed to track the evolution of mathematical discourse in classroom and research settings (see Weingarden).

A common challenge in research is the static nature of representations in academic papers, which limits the ability to capture dynamic, multimodal mathematical interactions. The group discussed the potential of video-based representations, interactive data visualizations, and multimodal transcripts to improve research communication. While these ideas raise new technical and ethical issues related to the communication of research, they could support research audiences in accessing the more lively and mobile role of representations in both learning and researching.

Theoretical perspectives on representations

To understand the cognitive and communicative functions of representations, studies by various participants made use of a variety of theoretical frameworks, which were then carefully discussed and examined by the group to better understand the perspectives and insights provided by each. Such

discussions focused especially around; 1) commognition (Sfard, 2008), which defines representations as realizations of mathematical objects (e.g., Aviram et al.; Baccaglini-Frank et al., 2022); 2) Duval's semiotic registers (Duval, 2006), which focuses on the transitions between different representational systems (e.g., Wendelborg et al.) and Bruner's modes of representations (e.g., Hölzer & Vogel); 3) Rabardel's Instrumental Approach (Rabardel, 2002), which views representations as tools that mediate mathematical thinking (e.g., Kadan-Tabaja); 4) Peirce's diagrammatic reasoning (Peirce, 1965), which examines the role of diagrams in mathematical reasoning (e.g., Piroi et al.), 5) Goldin's internal versus external representations (Goldin & Shteingold, 2001), which differentiates between internal cognitive structures and external mathematical expressions (e.g., Wendelborg et al.), and 6) Fischbein's intuitive models (Fischbein, 1987), which explores how intuitive mental representations shape mathematical learning (e.g., Cooperman et al.). Other papers developed theoretical frameworks driven by concepts drawn from mathematics education literature. For example, productive irritations (Nührenbörger & Schwarzkopf, 2015, as cited in Schroeder & Götze); Tall and Vinner's (1981) notions of concept image and concept definition (in Lobnig); Polya's (1973) conceptualisation of problems, as opposed to routine tasks (in Volbers et al.).

Our discussions highlighted the need to bridge different theoretical perspectives, particularly in understanding how learners transition between representations and how external representations support internal cognitive processes. However, the efforts to align, interlink, and differentiate between the theoretical concepts that we use to conceptualise thinking, learning, and interacting can be quite challenging (e.g., Hölzer & Vogel). A diverse set of theoretical approaches is important to broaden our research questions and methods related to questions of representation. But the special languages of these theories can silo our research findings, making them meaningful to only those who speak a specific theoretical language. Especially because papers tackle a wide range of mathematical and pedagogical issues, finding common theoretical ground can be important. Similarly, working to understand tensions between specific theoretical approaches drives our work to become both more complex and precise.

Conclusion and future directions

In response TWG24 emphasized the central role of representations in mathematics education, both in practice and research. Key takeaways include the following:

1) The importance of promoting flexible transitions between different representations to enhance mathematical learning,

2) The need for multimodal representations, such as auditory, gestural, and interactive tools, especially because this approach supports the inclusion of a wider group of learners,

3) The current limitations in representing dynamic mathematical interactions in research, and the desire to explore the potential of interactive and video-based representations to address these issues,

4) An interest in understanding how the range of theoretical approaches used to study representation in mathematics and mathematics education can be aligned or debated,

5) An interest in investigating how researchers use representation, not only to communicate results, but also as a part of the research process.

Future research should focus on developing new representational tools and further investigating how representations mediate learning and reasoning in diverse educational contexts. By continuing to explore the intersection of theory, practice, and technology, we can advance both mathematics pedagogy and research methodologies.

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Boxes and matches as a model for introducing variables in a sign language-oriented mathematics class

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Sign languages possess characteristics that differ from those of spoken languages and these differences can influence mathematics learning and teaching. In a signed discourse, concepts are typically organised syntagmatically, with a topic in the centre and other topics around it. The study presented here investigates how the "shell aspect" of a variable can become the topic in the centre of the mathematical discourse about the introduction of the variables concept in a sign language. To explore how the students handle mathematical sign language communication, a learning environment was designed and realised with a small group of students. The result is a learning environment that has been shown to be an effective means of introducing the concept of variables in a sign language.

Keywords: Sign Language, algebra, variables, equations.

Introduction

Signed and spoken languages have a different modality: visuo-gestural in sign languages and audio-verbal in spoken languages. Sign languages offer a different kind of access to mathematical concepts (Krause & Wille, 2021). Therefore, modality-related differences should be considered in mathematics teaching that employs a sign language (Grote et al., 2018). For example, Healy et al. (2021) states that "the ways in which movement is incorporated into the signs […] perhaps serves as an effective means of enacting meanings for" (p. 153) algebraic symbolism. This is consistent with the preference for simultaneous signing of information (Krause, 2023). Dessbesel et al. (2023), in a study on algebraic thinking in deaf education, stated that sign languages need to be valued not only in the translation of content from a spoken language to a sign language, but also in communication and search for strategies to achieve learning. Their study, however, does not focus directly on variables, which is what will be done here. Indeed, the study presented in this paper focuses explicitly on introducing variables in a sign language-oriented mathematics class. Linguistic tools are used to develop a learning environment for sign language-oriented teaching to introduce the concept of variables. Such an approach has already been used by Nordheimer et al. (2024), who adapted linguistic tools in geometrical learning contexts to develop learning environments in German sign language. In the study presented here, the learning environment – understood as a teaching and learning offer (Barton, 2023) concerning different methods, learning materials and media (Reinmann & Mandl, 2001) – was developed, discussed and is available in Austrian sign language (ÖGS).

Theoretical framework

Sign languages in the mathematics class – some essential properties

Sign languages organize content differently than spoken languages (Grote, 2010). It has been observed that deaf signers tend to initiate communication with a particular content aspect and give

further contextual information so that all the aspects are in relation to each other (simultaneous-circular structure). In a spoken language, content is typically organised in hierarchical manner starting with something general (sequential-linear structure) (Grote et al., 2018). Therefore, the conceptual development that is typically based on the characteristics of spoken languages often does not match the conceptual understanding of signing students (Krause & Wille, 2021). Healy & Fernandes (2014) explored algebraic expressions of deaf students signing about the notion of variable in the context of solving tasks about constructing sequences. One of their conclusions was that there is a need "to attend more closely to the practice of communicating mathematics" (Healy & Fernandes, 2014, p. 55) in sign language. When considering communication in sign languages, a linguistic distinction between syntagmatic and paradigmatic relation is central. A paradigmatic relation is a hierarchical relation with terms and sub-terms. For example, "car" is a sub-term of "vehicle". A syntagmatic relation, however, exists when terms are usually used together – i.e., "car" and "road". Terms can also be signed simultaneously, whereas in spoken languages terms are always spoken linearly. Grote (2010) stated "the use of sign languages has an effect on the strength of syntagmatic relations in the conceptual system, i.e. on conceptualization" (p. 317). A reason is that *simultaneity* is typical for sign languages and is only possible for terms that are syntagmatically related. Preferring simultaneity tends to strengthen semantic relations between syntagmatically related concepts. The study presented here focuses on syntagmatic relations because of their relevance to concept for learning. In addition, *centring*, which is another property of sign languages, is considered in this paper. This means that a topic is placed at the centre of a discourse and a syntagmatic context is built around it. Such a context can be created by changing the perspective, switching from one specific concept to another to describe it in more detail. In the mathematics class, this would mean that a specific topic should be placed at the centre, around which other "knowledge units […] are placed" (Grote et al., 2018, p. 430).

Aspects of variables in Austrian Sign Language

Fig. 1a Fig. 1b Fig. 1c Fig. 1d Fig. 1e Fig. 1f Fig. 1g

Fig. 1h Fig. 1i Fig. 1j Fig. 1k Fig. 1l

Figure 1: EMPTY$_1$ (a), EMPTY$_2$ (b), NOT-YET (c), CONTENT$_1$ (d), TO-SUBSTITUTE$_2$ (e), IX-No:THREE (f), X-No:THREE (g), TWO-TERMS$_1$ (h) , DO-NOT-KNOW (i), CONTENT$_2$ (j) (Angeloni &Hausch, accepted), BROSHURE-STACK (k) (Angeloni, 2023, p. 531), TWO-TERMS$_2$ (l)

Variables can be considered in different ways. The following section considers the semantic aspects of variables according to Malle (1993) and Wille (2008) and the results of previous studies in the

context of sign languages in relation to the aspect. According to the *object aspect* (Malle, 1993), a variable represents an unknown number. A variable under this aspect can be signed by the signs EMPTY_1 (Figure 1a) and EMPTY_2 (Figure 1b). Both signs express "that the variable does not contain a number" (Angeloni & Hausch, accepted) – the number is unknown. The sign NOT-YET (Figure 1c) conveys that the number is temporally unknown (Angeloni & Hausch, accepted). There are other signs that can be used to sign a variable under the object aspect. They depend on the iconic properties of the reference object. The sign in Figure 1k is an example of this. It expresses, with the movement of the right hand up and down, that the number of brochures on a stack is unknown. According to the *substitution aspect*, a variable acts as a placeholder for numbers (Malle, 1993). For example, this can be addressed by the sign in Figure 1e: a value from a table is substituted for the variable. The information about the table and the equation were previously placed in the signing space with the corresponding signs. According to the *calculus aspect*, a variable is a symbol with which one may operate according to certain rules (Malle, 1993). Under the *shell aspect*, a variable is like "a cover or a box for the number but it is still here" (Wille, 2008, pp. 422–423). Signs in relation to this aspect are CONTENT_1 (Figure 1d) and CONTENT_2 (Figure 1j). Both signs express "that a variable is like a container that can contain a number" (Angeloni & Hausch, accepted). Moreover, the shell aspect of variables can be also identified in the signs in the Figures 1f and 1g. The sign in Figure 1f is used to point to the location of the term in the sign language space where three is to be substituted for x, and, sometimes, to point to the individual locations in the sign language space where x was previously placed. The shell aspect is visible in the Figure 1g with the sign No:THREE behind the sign X: "[…] X acts as a placeholder into which the number [three] is inserted and remains" (Angeloni & Hausch, accepted). The research of Angeloni & Wille (submitted) explores a possible syntagmatic framework between the signs presented here and other signs about variables and operations with them, and consequently, between the different aspects of variables. In the sign language communication studied in Angeloni & Wille (submitted), the shell aspect was often addressed, with the signs presented above, to sign about a variable from the perspective of the other variable aspects – such as in the following phrase from the study: […] CONTENT_1 X_1 X_1 NOTHING_1 NUMBER EMPTY_2 CONTENT_1 PLUS_1 No:THREE *(the content of x is not a number, it is empty, but we calculate plus three)*. Here, the perspective changes repeatedly from the shell aspect (the variable has a content) to the object aspect (the content is unknown) and the calculus aspect (we calculate with the symbol x for the variable). So, the shell aspect is centred in the network and syntagmatically connected to the other aspects.

Research interest

For introducing the concept of variables, the following questions arise: *RQ 1) How can the shell aspect be centred in a learning environment to introduce variables in sign-language-oriented teaching? RQ 2) How do students handle mathematical sign language communication in such a learning environment?*

Methods

A group of three deaf secondary school students (aged 12 to 14) took part in a series of 50-minute mathematics lessons about variables. The lessons addressed the aspects of variables described above. Two students were in the 8th grade, so they should already have been familiar with the topic of

variables. One student was in the 6th grade and, therefore, should have more basic knowledge – such as setting up "simple" linear terms. The regular mathematics class of these students was spoken language-oriented and they were accompanied by a special education teacher who summarizes the content of the lessons for these students. Students' knowledge about variables was not previously tested so as not to interfere with their participation in the regular class. The first two lessons of this series included the introduction of variables. According to the aim of the learning environment (RQ 1), the focus was on the shell aspect of variables. To address this aspect, matchboxes were used to represent variables and matches were used to represent numbers. Each box has a colour so that different variables can be distinguished and there must always be the same number of matches in the boxes of the same colour. This representation can address the object aspect of variables: if the box is closed, we do not know how many matches are in the box. The number of matches can also be changed. Therefore, a box can stand for any (positive integer) number. With these representations, variables can be introduced without notations like x or y. Boxes and matches can be strung in a row to form an expression. Two such expressions can be set equal and form an equation, here called a "box-equation". The shell aspect can be directly addressed: a box is still here when matches are added. Therefore, as described below, the learning environment is based on "boxes and matches". The participant should have achieved the following learning objectives by the end of the two lessons: 1) they are able to explain that a variable can represent an unknown number using appropriate signs – such as CONTENT$_1$, NOT-YET and EMPTY$_1$; 2) they can transform box-equations into linear equations with two variables; 3) they can find solutions of box-equations with two variables by substitution with numbers. To investigate question RQ 2, the students solved the tasks set within the learning environments according to the principle of "think, pair and share". In this way, they should become active in signing. The students were interviewed individually at the end of the series of lesson to obtain further information about their knowledge of mathematical signs and feedback on the lessons. The interviews included questions such as: Was the topic new to you? What signs did you know before? Did you understand the signs in the lesson videos? The lessons and the interviews were held in ÖGS and were videotaped. The video data (from the lessons and interviews) and the written students' solutions to the tasks were analysed using qualitive content analysis (Mayring, 2014).

Findings

The first section addresses the research question RQ 1 about a learning environment that focuses on the shell aspect of variables. The research question RQ 2 is discussed in the second section.

A learning environment for introducing variables in sign-language-oriented teaching

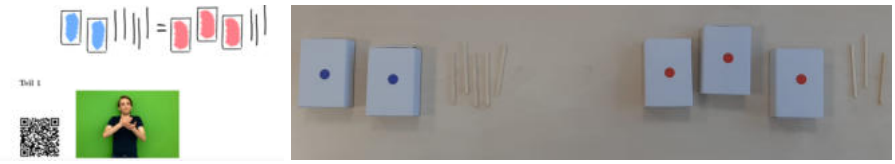

Figure 2: Exercise 1, part 1 of the worksheet (left) and an arrangement of boxes and matches (right)

The learning environment includes a worksheet, blue and red boxes with matches, has a duration of 90 to 100 minutes and it is designed to teach in ÖGS. In this learning environment the concept of variables is introduced from the perspective of the shell aspect by using the sign CONTENT$_1$ (Figure

1d). To centre the shell aspect, a syntagmatic context should be constructed by changing the perspective. Therefore, the perspective changes multiple times during the learning environment, for example to the object aspect, and, every time, the specific signs for the variable aspects presented above are used. The worksheet contains two tasks which are given as videos in ÖGS. It also contains an explanatory video in ÖGS. To watch the videos, the students scan the QR-codes. Each student is equipped with a tablet with internet access, blue and red boxes, matches, paper and pencils. The signs that are signed in the videos are the results of previous studies (Angeloni & Hausch, accepted). The first task consists of three parts and addresses the learning objectives 1 and 3 as follows: 1) The box-equation in Figure 2 is presented in the video: "There are boxes without content and there are two colours – red and blue boxes. There are also a lot of matches. Place the boxes and matches on the table as follows". The box-equation in Figure 2 is now signed. The students must set up this box-equation as in Figure 2 and compare it with the picture above. The signs CONTENT_1 and EMPTY_1 are signed in the video. At this point, the shell aspect is addressed by CONTENT_1 and the object aspect of the variable represented by a box is addressed by EMPTY_1. 2) The question is signed: "How many matches are there in a blue or red box?". The information that the sum number on the left and the sum number on the right of the box-equation must be the same is given by the equal symbol on the picture (Figure 2, left). Nevertheless, the teacher can sign it using the sign in Figure 3d. According to the "think, pair and share" principle, the students should think about a possible solution and discuss them with each other. 3) Finally, the students are asked to set each other similar tasks and solve them.

Fig. 3a Fig. 3b Fig. 3c Fig. 3d Fig. 3e

Figure 3: Signs NOT-A-NUMBER (a), EQUAL_3 (b), TIMES (c), RESULT_3 (d), PLUS_1 (e)

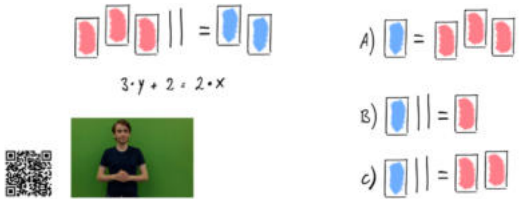

Figure 4: Explanatory video (left) and exercise 2 (right)

In the explanatory video, learning objectives 1 and 2 are addressed by explaining the relationship between box-equations and algebraic equations. The content of the video is the following: The box-equation in Figure 4 (left) is first signed. After that, the signer signs that we do not know how many matches are in a blue box, therefore we need to set x for the unknown number. This occurs using CONTENT_1 (Figure 1d) and the sign in Figure 1i. The location in the sign space is relevant to express the relation between the blue box and the variable x: the box is placed on the right and the variable is placed on the left. The box-equation is no longer present. The relationship between x and the blue box is signed once again. The signer explains, additionally, that "the content of x is not a number. It is empty." The signs COTENT_1, NOT-A-NUMBER (Figure 3a) and EMPTY_1 (Figure 1a) are signed

here. The perspective moves to the object aspect and the variable is described in more detail, but the syntagmatic relation to the shell aspect is still there. Indeed, the signs CONTENT_1 and EMPTY_1 occur in the same sign phrase and at the same place. Similarly, the signer introduces the variable y, related to the unknown number of matches in a red box. The box-equation is taken back and signed once again: its left (resp. right) side is placed on the left (resp. right) side of the sign space. This is relevant in the next step, when the equation is introduced with the question: IX.left TOTAL HOW-MUCH IX.right TOTAL HOW-MUCH WRITE WHAT *(What do we write for the total number on the left and the total number on the right side?)*. The signer signs now the equation $3 \cdot x + 2 = 2 \cdot x$. EQUAL_3 (Figure 3b) is used for the equal symbol, TIMES_1 (Figure 3c) resp. PLUS_1 (Figure 3e) for the multiplication resp. addition symbol and TWO-TERMS_2 (Figure 1h) is signed for the equation. The explanatory video, with the mentioned signs, addresses learning objective 1. The explanation of the transformation of a box-equation into a linear one addresses learning objective 2. This is also addressed by the second task (Figure 4, right), which consists of two parts described as follows: 1) First, the picture in Figure 4 (right) is signed above in the sign space. The students are then asked to transform the box-equations into equations with x and y. The sign TWO-TERMS_2 (Figure 1l) for "equation" is used and signed below in the sign space "under" the box equation. 2) The students must transform the box-equations that they have created in the first task. The learning environment presented here centres the shell aspect of variables and can be used to introduce the concept of variables in a sign-language-oriented class. This answers to the first research question.

The handling of mathematical sign language communication in the learning environment

Fig. 5a Fig. 5b Fig. 5c Fig. 5d Fig. 5e Fig. 5f

Figure 5: Signs from the data of the lessons

Regarding the signs signed by the students, EQUAL_4 (Figure 5a) was not signed in the videos or by the teacher, was used introduced by the students themselves for the equal symbol. EQUAL_1 (Figure 5b) for "equal" or "it is equal to" was never used to sign the equal symbol. The equality (resp. the inequality) of two terms was signed on the vertical axis of the sign space (Figure 5e) with the sign in Figure 5c resp. 5d. CONTENT_1, CONTENT_2, and EMPTY_1 were also signed by the students, as explained in the videos of the worksheet. Moreover, the students used the construct in Figure 1f, but also with other numbers, to express the substitution. This occurred in both representations of variables with boxes or letters. It was also observed that the students expressed the substitution by placing the number sign at the place where the variable was in the sign space. Regarding the use of the sign space, the students often signed in the vertical plane (Figure 5f), even when the referred object was on the horizontal plane. For example, they resigned the picture with the boxes and matches – which they drew on the paper – on the vertical plane. In summary, the students achieved most of the learning objectives at the end of the two lessons: 1) they signed CONTENT_1, CONTENT_2 and EMPTY_1 while communicating about variables, but they did not explicitly explain that a variable represents an

unknown number or that numbers can be "put" into a variable; 2) they found different solutions of the box-equations by adding some matches in the boxes; 3) they transformed the box-equations into equations with x and y. Additionally, task 2 was extended: each student invented a box-equation, signed it to others and they transformed it into an equation with x and y. The signing student placed the terms of the box-equation in the same way as in the videos of the worksheet and as the teachers had signed during the lessons, i.e., the left (rep. the right) term of the equation was signed on the left (resp. right). In the interviews the students' main comments were that the task videos of the lessons were too complicated at first, but understandable. They also said that they were not familiar with all the signs in use and that they would like to have more content in ÖGS.

Discussion

The learning environment centres the shell aspect for introducing variables in a sign-language-oriented class, constructing a syntagmatic network with the other variable aspects. However, the construction of the syntagmatic network is not yet complete with this learning environment. The occurrence of EQUAL_4 indicates that the possibilities of signing about variables and operating with them should be further investigated. This should be considered in research, in teaching mathematics, and in designing learning environments for a sign-language-oriented classes. The students have achieved most of the learning objectives. With the deliberate choice of signs from the syntagmatic network, the explanatory video addresses learning objective 1. Such videos are important for sign language-oriented students to be able to find mathematical information in ÖGS outside the mathematics class. Moreover, it can contribute to the development of technical sign language by the students. Therefore, the representation with "boxes and matches" can be a suitable way to introduce the concept of variables and to centre the shell aspect. The symbolic representations, such as letters, has been partially implemented in task 2 of the worksheet, but it will be considered in more detail in further studies. These studies will also explore learning environments that focus on other variable aspects. Bimodal-bilingual mathematics teaching should also be considered more intensively, as this can be particularly important for teaching signing and non-signing students. The representation with boxes and matches and the centring of the shell aspect can be used by both groups of students.

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A Pair of algebra beginners working in a spreadsheet environment: Commognitive perspective

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The purpose of the current study is to identify the mechanisms by which a spreadsheet environment can facilitate learning. We examined the work of a pair of seventh-grade low achieving students as they engaged in a mathematical activity in a spreadsheet environment. Using the commognitive theory, we analysed the routines the students enacted, seeking for changes in their performance. We found that at the beginning of the process the students did not same between the realizations given in the task and performed the routine in a ritual way. As the process went on, they developed saming between realizations and performed the routine in an explorative way, meaning there was a de-ritualization process. We conclude that working in a spreadsheet environment can strengthen developing of saming between realizations and this could be one of the mechanisms by which a spreadsheet fosters students learning.

Keywords*: De-ritualization process, commognition, saming between realizations.*

Introduction and Literature Review

Studies have documented obstacles students are facing while taking their first steps in entering the algebraic discourse at the beginning of middle school (e.g. Rojano, 2002), especially for low achieving students. This transition from arithmetic to algebra is particularly important due to the central role algebra plays in middle school and beyond. One approach that was suggested to help students enter the algebraic discourse is learning in a spreadsheet environment (e.g. Tabach et al., 2013). Spreadsheets are marked by a dominance of numerical representation. This representation is strongly linked with students' arithmetic discourse, which was developed in elementary school mathematics lessons. Indeed, it is possible to type numbers into spreadsheet cells. Yet, early on students learn that they can write symbolic expressions that make use of another cell, and the spreadsheet will present the number which is the result of the expression calculated for a certain value of the variable. When dragging the expressions to other cells, the spreadsheet presents the appropriate numerical values in these cells. In that sense, working with a spreadsheet has the potential to encourage students to use symbolic representations, as was found in several studies (e.g. Tabach & Friedlander, 2008). Yet, less is known about the specific mechanisms that seem to promote students' learning in a spreadsheet environment. The current study explores the learning processes that occur when working in a multi-representational spreadsheet environment, aiming to uncover mechanisms that facilitate algebra learning. To achieve our aim, we adopted the commognitive theory (Sfard, 2008)[1].

[1] In the commognitive theory we talk about realizations and not representations. We use that word from now on.

The commognitive theory

The commognitive theory (Sfard, 2008) views mathematics as a discourse, with distinct characteristics. A change in any of these characteristics is evidence for learning. Routines are one of the discourse's characteristics. Routine consists of a sequence of actions performed in response to a prompt. A routine is the combination of task and procedure (Lavie et al., 2019). The task consists of what the person performing the routine perceives as her objective in a specific context, while the procedure consists of the series of actions that she enacts to accomplish that task. A routine can be process-oriented (ritual), that is performed according to social expectations (e.g. the teacher), or product-oriented (explorative), that is performed to achieve an outcome. These two types of routines are on a continuum, and the transition from process-oriented to product-oriented routine is called a de-ritualization process, and it is a learning process.

One characteristic of product-oriented routines is objectification. Objectification is the extent to which the narratives are about abstract mathematical rather than concrete objects (Lavie et al., 2019). The process of objectification involves, among other things, saming (Sfard, 2008). Saming is an act of calling different things by the same name, or associating one name with many realizations (Sfard, 2008). It stands at the heart of the objectification process since mathematical objects are often the result of multiple processes being samed and reified. For example, "three" is a result of saming multiple processes such as counting to three, making sets of three objects of various kinds, and more. These processes are denoted by various realizations (e.g. '3', 'III', 'three'). To objectify "three", young children need to become familiar with the various processes that lead to endorsing the narrative (e.g. "there are three things here"). This process of objectification continues as students are introduced to increasingly sophisticated mathematical discourses. For example, Weingarden et al. (2019) showed the usefulness of classroom discussion that relates to various realizations of an algebraic pattern, and the ways by which such exposure can encourage saming, and through that, increase explorative participation. However, previous studies of saming processes did not involve technological environments.

In the current study we look for evidence for saming between realizations in a spreadsheet environment and examine if and how it can facilitate learning. In the context of pairs of beginning algebra low achieving students, working in a spreadsheet environment on tasks that require saming between realizations, we ask – what mechanisms of de-ritualization can be identified in students' performed procedures? Due to space restriction, we detailed one mechanism that emerged from the work of one pair of students working on one task.

Methods

This study is part of a larger project exploring how spreadsheet-based tasks might aid beginning algebra low achieving students with developing their mathematical discourse. The study involved five pairs of seventh-grade students selected based on their teachers' recommendations as low achieving students. In this case study we focus on Ido and Uri (pseudonyms), who demonstrate in their work a change in the way they connect between realizations. The students participated in ten, one-hour long meetings with the first author, in an out of school setting once every two weeks. During these meetings, they engaged in specially designed spreadsheet activities that aligned with the

seventh-grade mathematics curriculum. The researcher refrained from intervening unless the students were unable to proceed. All sessions were recorded with two cameras (one capturing the screen, the other directed at the students and the computer). They were then transcribed verbatim.

The main task in the activity on which the analysis of this paper focuses (based on Resnick & Tabach, 1999) was to compare weekly allowances of four children given in different realizations: table (Dina), graph (Moshon), symbolic expression (Danny) and verbal (Yoni) (see Figure 1). The students had to build a table in a spreadsheet, one column for each child's allowance, by writing an expression that represents the allowance. For that, the students needed to define a column for the week's number (expectedly column A) and write an expression that makes use of that column. We focus here on the expression the students wrote for Moshon's allowance. They had to translate the graphical realization to a symbolic realization and write an expression such as $5 \cdot A1 + 30$ (A1 is the first cell in column A in the spreadsheet).

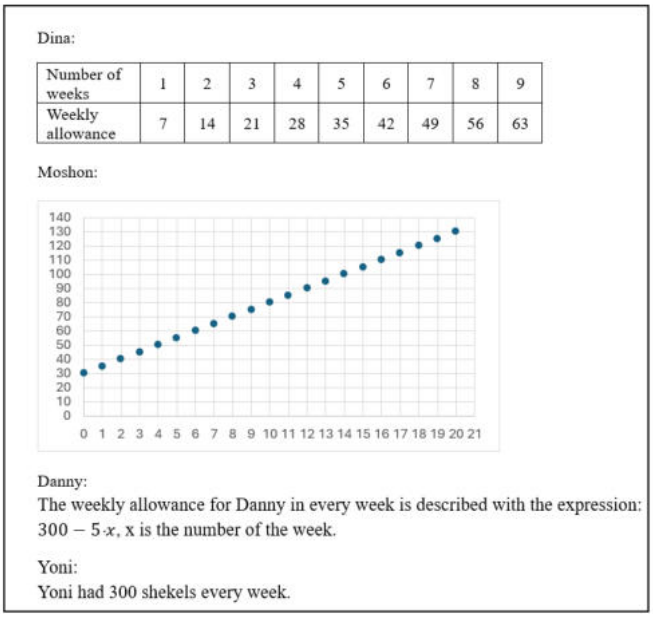

Figure 1: Weekly allowances for Dina, Moshon, Danny and Yoni

We analysed the data based on the commognitive theory. In what we report below, we examined the routines the students performed and looked for changes in the routines' performance. Specifically, we sought evidence for saming between realizations of Moshon's allowances, by searching for evidence that the students see two realizations as representing the same thing.

Findings

In this section we show growth of the routines the students enacted, when initially Ido and Uri did not same between realizations in the activity and performed the routines ritually, and at the end of the process they samed between the realizations and this is evidence of de-ritualization.

At the beginning of the process, Ido and Uri wrote an expression (symbolic realization) in the spreadsheet for Moshon's allowance, given in a graph. They first created a column (A) for the number of the week, wrote the numbers 1 and 2 in the first two cells of column A and dragged the numbers

until number 20. Then they wrote the expression *A2+5* (the number 1 is written in cell A2) for the first week's allowance. Ido explained the expression: "because every week 5 is added". After dragging the expression down the column, the numbers 6, 7, 8, 9 etc. appeared (see Figure 2). The students saw the numbers and that they did not match the statement that every week Moshon had 5 shekels more, and decided their expression was wrong. Then they tried to fix it. The routine the students enacted for fixing the expression is presented in Episode 1.

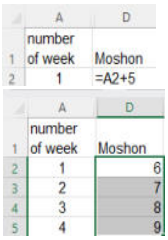

Figure 2: First expression for Moshon's allowance

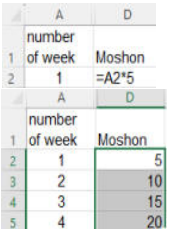

Figure 3: Screenshot for Episode 1

Episode 1 – Fixing the expression for Moshon's allowance

174	Ido:	I think we are wrong. One second.
175	Uri:	So multiple 5.
176	Ido:	Why?
177	Uri:	Let's try. I think. *(writes the expression* $A2 * 5$*, see Figure 3)*.
178	Ido:	Like every week he has 5 more.
179	Uri:	Yes *(presses the "enter" key and drags the column, see Figure 3)*.
180	Ido:	So…
181	Uri:	Here *(says when he sees the results of the dragging)*.
182	Ido:	Why?
183	Uri:	Here, 5, 10, 15.
184	Ido:	Oh, yes, I got it.

The routine presented in Episode 1 has ritual aspects: after they rejected their first expression, Uri suggested to correct the expression by multiplying the number of the week by the number 5 [175], probably because Ido mentioned that every week Moshon had 5 shekels more. Uri could not explain why ("Let's try. I think") [177], but Ido repeated his reasoning [178]. Uri agreed, pressed the "enter" key and dragged the column. The numbers 5, 10, 15, etc. appeared in the column. Ido was not sure the numeric realization is correct [182], but when Uri read the numbers [183], Ido accepted that [184]. The students continued with the activity, so we conclude they were pleased with the expression they wrote.

By accepting the A*5 formula as correct, Ido and Uri showed they were not saming the graphical realization for Moshon's allowance (Figure 1) and the numerical realization that was created from dragging the symbolic realization they entered the spreadsheet (the expression A2*5) - since the numbers in the table did not agree with the numbers in the graph. For example, according to the graph, in the first week Moshon had 35 shekels, while the spreadsheet showed he had 5 shekels.

During the next tasks a change happened. The students were requested to "find on the table you created all the weeks in which the allowances of two children are identical". Ido and Uri marked

Moshon and Danny on the table that had 150 shekels in week number 30. Then they were presented with four graphs (see Figure 4), one for each child's allowance, and were asked to match each graph to a child and to mark on the graph the dots of two children having the same allowance in the same week. Ido immediately said: "Let's check week number 30" (he meant in the graph), and they saw that this week Moshon and Danny had different allowances, contrary to what they marked at the table. Ido said: "Maybe we were wrong". This statement leads us to conclude that at this point Ido started saming between the graphical realization and the symbolic/numerical realization and thought they should match. The students then created a new task for themselves – try to understand why the result was not as they expected; this move to focusing on the *product* of their procedure is evidence for a de-ritualization process.

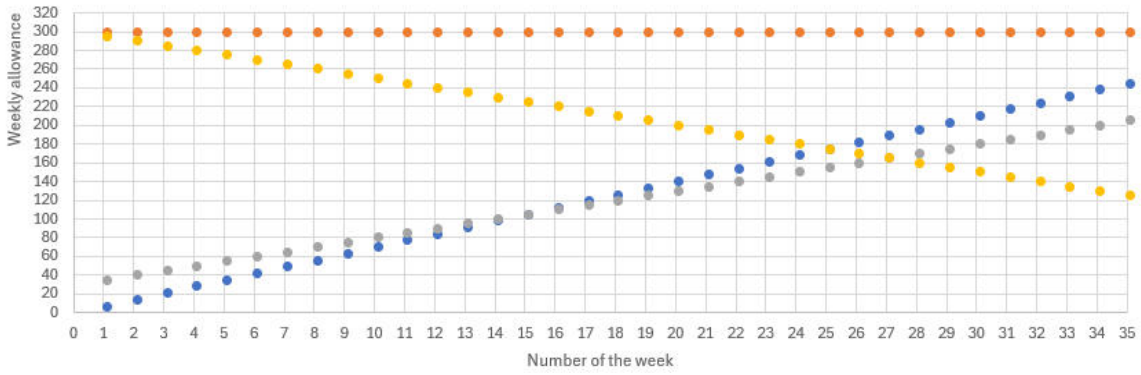

Figure 4: Graphs of allowances

After checking the intersection points in the graph and trying to find them in the spreadsheet table, the students concluded that the expression they entered into the spreadsheet to calculate Moshon's allowance was wrong. Their attempt to fix the expression is presented in Episode 2.

Episode 2 – Correcting the expression for Moshon's allowance

427	Ido:	You said you wanted to start from 35. 35 and…35...
428	Uri:	Plus
429	Ido:	No, let me think. 35 and then every week, 35, write equals 35
430	Uri:	Well…
431	Ido:	35 *(Uri writes* "*=35*"*)*
432	Uri:	Like, plus 5
433	Ido:	No, plus the number of the week times 5. No.
434	Uri:	What? I think we should do like that *(deletes the 35)*
435	Ido:	Plus the number of the weeks,
436	Uri:	I think we should do *(marks the cell A2, the number of the first week)* multiple or plus *(writes the expression A2+35, see Figure 5a, presses the* "*enter*" *key and sees the number 36 in the cell)*. Oh, right, I know why.
…		
439	Ido:	plus 35, plus 35 plus 5.
440	Uri:	multiply 35 *(writes the expression A2 ∗ 35, see Figure 5b)*.
441	Ido:	no, why? Why?
442:	Uri:	Look *(presses the* "*enter*" *key and they see the number 35 in the cell, Uri starts dragging the column)*.
443	Ido:	Oh, I got you.
444	Uri:	Look *(after he drags the column, see Figure 5c)*.

445	Ido:	No, it's not good. *The students write different expressions that lead to an error response from the spreadsheet:* "$= 35 + 5 = A2$", "$= A2 = 35 - 1$". *Then the researcher intervenes.*
465	Researcher:	How much does he get every week?
466	Ido:	5.
467	Researcher:	What does it mean, what should be in the expression?
468	Ido:	multiply 5. *The discussion continues and with mediation of the researcher, Ido writes the expression* "$35 + 5 * A2$"*, they press the* "*enter*" *key and see the number 40 in the first cell. Ido decides to write the number 35 in the first cell and to write an expression only from the second cell.*
484	Ido:	Number of the weeks *(writes the expression* "$5 * A3$"*, presses the* "*enter" key, and they see the number 10 in the cell. Ido deletes the expression). After that they write the expression* "$35 + 5 * A3$" *(see Figure 5d). They observe the number 45 in the cell and change the expression to* "$35 + 5 * A2$" *(see Figure 5e). They observe the number 40 in the cell and drag the column.*

5a

	A	D
1	number of week	Moshon
2	1	=A2+35

5b

	A	D
1	number of week	Moshon
2	1	=A2*35

5c

	A	D
1	number of week	Moshon
2	1	35
3	2	70
4	3	105
5	4	140

5d

	A	D
1	number of week	Moshon
2	1	35
3	2	=35+5*A3

5e

	A	D
1	number of week	Moshon
2	1	35
3	2	=35+5*A2

Figure 5: Screenshots for Episode 2; 5a Screenshot in line 436; 5b Screenshot in line 440; 5c Screenshot in line 444; 5d Screenshot in line 484; 5e Screenshot in line 484

In Episode 2 Ido and Uri tried to write a symbolic expression for Moshon's allowance: In line 427 they said that the first number should be 35, not like in the previous expression that gave the number 5 in the first cell. They cancelled the expression "$A2 + 35$" (cell A2 contained the number 1) because it gave 36 and not 35 [436]. This is evidence for saming between the realizations of the graph and the symbolic expression of Moshon's allowance. They wrote the expression "A2*35" [440], that gave the number 35 in the first cell. When Uri saw that number, he started dragging the column, meaning he was pleased with this number. But when they saw the result of the dragging (the numbers 70, 105, 140, etc.), Ido said "it's not good" [445]. This is another evidence for the saming they made – each number should be bigger by 5 from the previous number. In all those attempts to write the symbolic expression, they samed between the symbolic realization and the graphical realization – they wanted the expression to create the number 35 in the first cell of the column [427], and every number after that to be bigger by 5 (they said "plus 5" twice [432, 439]), similarly to the graph. They disqualified every expression they wrote that did not match one of those conditions. After 8 procedures they succeeded in writing the correct expression. The iterative corrections were all due to the saming they made between the graphical and numerical realizations. These quick iterations were made possible by the spreadsheet realizing the algebraic formula quickly and efficiently, enabling the students to concentrate on the product of the formula (or algebraic realization) rather than on calculations of the numerical values.

Discussion

We examined the work of a pair of seventh-grade students, Ido and Uri, while handling activity in a spreadsheet environment that involves different realizations. We showed a change in the way the students enacted their routines. Initially they worked ritually with no saming between the realizations of Moshon's allowances that were involved in the activity. They translate graphical realization into wrong symbolic realization and did not see the gap between these realizations. But as they continued working with the spreadsheet, they developed saming between graphical and numerical realizations of Moshon's allowances, and as a result, identified their wrong expression. In the process of changing the expression (the symbolic realization), they tried writing many expressions, comparing each one of them to the graphical realization. This means they performed the routine exploratively. This movement, from ritual routine to explorative routine, is a de-ritualization process.

We suggest that saming between realizations is one mechanism by which a spreadsheet can help students enter algebraic discourse. Working in a spreadsheet environment encourages the development of saming between realizations due to several factors. The task, which is in the focus of this analysis, required the students to interpret a few different realizations and to move from one to the other. Also, working in a spreadsheet makes it is easy to create numerical values, by dragging an expression. Beginning algebra students feel it is more convenient to work with numbers rather than symbolic expressions. Another reason is that when changing a symbolic expression, the spreadsheet automatically changes the related numerical values (makes all the computations). In that, it gives the students the opportunity to easily same between three different realizations: the algebraic, the numerical and the graphical. Saming is a challenging process that takes time to develop (Sfard, 2008). It seems that the possibility the spreadsheet allows, to move easily from symbolic realization to numerical realization, encourages developing the saming process.

This paper demonstrates the effectiveness of the commognitive theory in analyzing and identifying learning process in a technological environment. The de-ritualization process' characteristics were examined also in other studies (e.g. Nachlieli & Tabach, 2022), but they were not examined in a technological environment. Here we showed that one characteristic – saming, which is part of the objectification characteristic – is relevant also for a de-ritualization process that takes place in technological environment.

Baccaglini-Frank (2021) showed that technological environment can foster explorative participation in mathematical discourse. In this study we go one step further in understanding the contribution of the technology, in this case – spreadsheet, for learning, by suggesting one mechanism by which this may happen – saming of different realizations. More research is needed on other de-ritualization's characteristics, looking for more such mechanisms.

Acknowledgment

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Emergence and Manifestations of the Analogy Between Algebraic and Graphical Models

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Abstract

This research investigates the emergence and manifestation of analogies between algebraic and graphical models in the context of solving systems of equations. Drawing on Fischbein's theory of analogical reasoning, which underscores the significance of graphical representations in understanding functions, the study examines students' performance using both models. Through tasks specifically designed to require the use of both algebraic and graphical models, the study highlights when and how students establish connections between these methods, revealing that many struggle with transitioning between them. Findings suggest that while necessity can trigger the use of analogies, mastery of both models and the development of strong mental images of graphs are crucial for facilitating these connections. The paper concludes by discussing the educational implications, proposing that the ability to link different models is a teachable skill that can improve mathematical problem-solving.

Keywords: Analogical Models, System of Equations, Graphs, Solution, Mental Image.

Introduction

This paper presents partial findings from an ongoing study investigating how students construct mathematical knowledge through analogies, guided by Fischbein's (1987) theoretical framework of analogical models. Fischbein defines an intra-mathematical analogy (of the second kind) as an analogy between a symbolic (algebraic) model and an intuitive (often graphical) model. A key example is the graphical representation of functions based on the fundamental isomorphism between numbers and figures.

Mathematics educators emphasize the significance of mastering multiple representations in algebra, particularly the ability to navigate both algebraic and graphical models. However, research shows that students often struggle to connect these forms of representation when solving equations. Despite awareness of this issue, many students find it challenging to transition between algebraic and graphical models.

This study investigates the moments when students form analogies between algebraic and graphical models while solving a system of equations, focusing on the knowledge and tendencies that support this cognitive shift. We aim to reveal how these analogies between models manifest and determine if specific prerequisites are needed for their emergence, providing valuable insights for educators and researchers in mathematics education. In this context, "emergence" refers to the initial, internal formation of connections between the models within students' cognitive processes—an intangible, invisible event. Conversely, "manifestation" is the observable application of these analogies, evident as students employ these mental connections in their problem-solving methods.

Theoretical Background

The importance of multiple representations in algebraic thinking has been well-documented in mathematics education, with research showing that proficiency in advanced mathematics depends on students' ability to navigate various semantic representations (Duval, 2017; Heinze, 2009). While solving equations involves both algebraic and graphical models, many students struggle to establish connections between them (Friedlander & Tabach, 2001; Star & Newton, 2009; Bieda & Nathan, 2009). Knuth (2000) noted that students often have only a superficial mastery of algebraic-graphical connections, particularly when solving routine problems. Similarly, Testa and Catena (2022) observed that high school students found algebraic representations more challenging to interpret than other forms.

We hypothesize that these difficulties arise from personal reflections of representations in students' minds—cognitive models—so these difficulties have a cognitive nature. To explore this, we turn to analogical reasoning in mathematics. Polya (1954) demonstrated the value of analogies in mathematical thinking, highlighting their accessibility to both experts and novices. Fischbein (1987) stressed the importance of educating students to recognize similarities, identify isomorphisms, and describe common structures in mathematics. He wrote, "much more attention should be given to educating students sensibility for similarities, the ability to identify isomorphisms and describe common structures." (1987, p 210). Hofstadter and Sander (2013) further reinforce the view of analogy as a core mechanism of thought.

Our study builds on Fischbein's theory of intuitive models, which suggests that students use intuitive representations to make abstract ideas more understandable. Following Fischbein, we describe a model (system B) as analogical to another (system A) if they share a certain isomorphism, allowing solutions in A to be reflected in B and vice versa. Fischbein posits that the transformation from symbolic to intuitive representations facilitates productive thinking. The nature of these intuitive models is sensorial; objects within them can be perceived and manipulated much like concrete reality. This enables them to serve as platforms for generating hypotheses. The results of mental operations performed within these models can then be reflected in the original symbolic models, enhancing understanding through a dynamic isomorphic relationship. Fischbein particularly highlights the graphic representation of functions as a significant type of intra-mathematical analogy due to its isomorphism between numbers and figures.

This study aims to examine the emergence and manifestation of analogies between algebraic and graphical models during students' problem-solving processes, specifically in the context of solving a system of equations. Our objective is to identify when and how students naturally form these analogies and the subsequent impact on their problem-solving strategies. This is an observational inquiry, not a pedagogical intervention.

Research questions

1. How does the analogy between the algebraic and graphical models manifest in students' problem-solving processes when solving systems of equations?
2. What conditions foster the emergence of analogies between algebraic and graphical models?

Method

Task design

The task (Figure 1) was designed to facilitate students' formation of analogies between algebraic and graphical models. It features two systems of equations: the first is solvable using both models. In contrast, the second necessitates graphical reasoning, as it cannot be resolved through standard algebraic methods alone and must incorporate the graphical model for resolution.

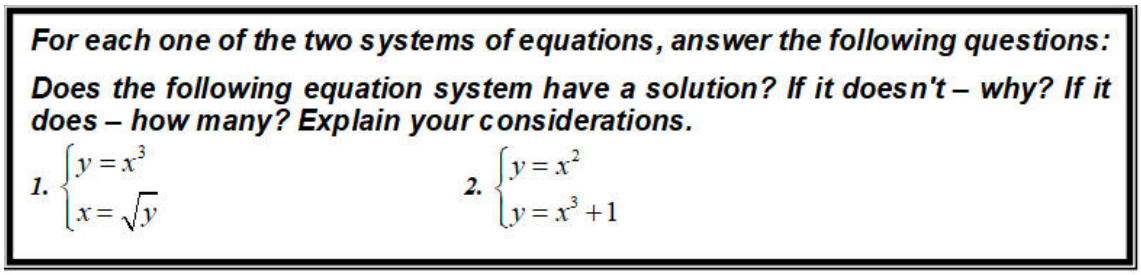

Figure 1. The task

Research population

Twelve pairs of student volunteers participated in the study, encompassing secondary school students from both regular and advanced mathematics levels and first-year engineering students. According to the Israeli mathematics curriculum, all participants were versed in the relevant mathematical concepts for this study.

Research description

The study was organized as a two-hour clinical interview (for each pair) in laboratory conditions. The students' work was videoed and transcribed to analyze their learning process. The interviews were conducted as semi-structured interviews. The interviewer was instructed to minimize interference in the interviewees' discussions. Instead, the interviewer was to use structured, pre-designed hints to facilitate the emergence of analogies between representations should the students require assistance. Examples of these hints are detailed in the findings section.

Data analysis

We define the "emergence" of an analogy between models as an internal cognitive event where students form connections between different models, inferred from observable actions such as explicitly referencing or applying a term from one model while solving a problem with another. Fischbein highlights the intrinsic properties of these models, noting, "An intuitive model is, by its very nature, of a sensorial kind. It may be perceived, represented, or manipulated like any other concrete reality" (p. 121). In addition, Fischbein states, "Using a model means thinking productively in terms of the model. A mental model is a reasoning device. It inspires solving strategies, suggests solutions, and often confirms the coherence and meaningfulness of solutions obtained by means of the original" (p. 124). These quotes elucidate the various manifestations of productive thinking, which include students' use of sensorial language and gestures, their application of the model to generate hypotheses, and their leveraging of insights from the graphical model to draw conclusions in the algebraic model.

Findings and Analysis

In our study of 12 student pairs, an analogy between the algebraic and graphical models emerged in six pairs. Interestingly, this analogy did not surface in interviews with regular-level students nor in two pairs of advanced-level mathematics students. These students initially approached the problem algebraically; upon encountering difficulties, they concluded that a solution was non-existent. Even when hints were provided—urging a shift to alternative methods—they persisted with algebraic strategies without success. It was not until the interviewer explicitly suggested the use of the graphical model that they considered this approach, though they failed to see its relevance or to apply it. Conversely, for the other six pairs, the analogy emerged first and was then actively used to form a link between the algebraic and graphical models.

The following sections explore how the analogy between models uniquely manifested among three pairs of students. We will focus in-depth on two pairs, analyzing interview excerpts to uncover different aspects of the analogy's impact. For one pair, we will demonstrate how the interviewer's hints catalyzed the emergence of the analogy. For another, we will illustrate how the analogy's manifestation contributed to solving the system of equations, addressing our first research question. Additionally, the third pair briefly demonstrates another instance of the analogy's emergence. Further exploration of what specifically triggered the analogy for each pair will lead us to identify potential prerequisites for its emergence, thus answering the second research question.

Individual Cases

When analyzing the emergence of analogies, we treat the student pair as a unit, regardless of who introduced the analogy. We observed that once one student suggested the analogy, their partner cooperated even if it wasn't their original idea.

Sam and Talya, first-year engineering students, attempted various algebraic methods to solve the system of equations during their problem-solving session but could not find a solution, leading them to conclude that no solution existed. The interviewer then questioned their reasoning.

Sam:	We didn't find a solution through algebraic methods.
Interviewer:	You didn't find a solution, algebraically, you're saying now, yes?
Talya:	Correct.
Interviewer:	Algebraically?

The interviewer repeatedly emphasized "algebraically," hinting at other possible methods for finding a solution. Despite this, Talya and Sam missed these cues. The interviewer then used a pre-prepared hint and explicitly suggested considering alternative representations.

Interviewer:	Consider the equations $y = x^2$ and $y = x^3$. Could there be another representation for these equations?

Talya and Sam's fixation on algebraic methods limited their exploration of alternatives

Talya:	With t, usually like... (substituting an expression with a new variable- t)
Sam:	...no, but it doesn't work here.

The interviewer continued to guide them with an example of a linear equation, which could inspire a connection to graphical representations:

Interviewer: For instance, if the equation is y=2x, how can it be represented other than as an algebraic expression?

Talya: x equals half of y?

Interviewer: True, That's still algebraic. Is there a different tool we could use?

Finally, the analogy between the linear function and its graphical representation emerged for Sam, who gestured with his arm elevated in a straight line to illustrate.

Sam: Could it be a graph? Like graphing y = 2x as a function.

Following their new insight, they proceeded to draw the graphs.

Talya's and Sam's eventual success in solving the system underscores their proficiency in algebraic manipulations and familiarity with graph functions. Despite this, they required numerous hints before the analogy between the models became clear. The turning point appeared to be the equation y=2x, which Sam recognized as a linear function. This equation evoked a mental image of a straight line rising across a coordinate system—an image Sam revisited when the interviewer asked to consider a different form of representation. It seems that they lacked such a mental image of the functions $y=x^2$ or $y=x^3$, as the analogy didn't arise when the interviewer suggested another representation for them. Perhaps their mental images for $y=x^2$ and $y=x^3$ were not vivid enough to spontaneously trigger the analogy between the models.

Ron and Effy, advanced-level high school students, encountered a similar challenge: not reaching an algebraic solution. After unsuccessful algebraic attempts, they reached the equation $x^2 = -\frac{1}{x-1}$.

Ron: …I got $x^2 = -\frac{1}{x-1}$, so the function x^2 is a parabola…

Effy: It's as if we have two functions…

Ron: Oh, but we can also look at these: x^2 and $x^3 + 1$

Ron visualized x^2 as a parabola—a concept he could immediately and easily graph. He hesitated with the graph of the other function, acknowledging his lesser familiarity but confident he could delineate it if needed. Once he articulated his visualization of the parabola, the connection between the models became clear, enabling him to extend this approach to the original functions, x^2, and x^3+1, which he found simpler to graph. For Ron and Effy, the analogy emerged after they exhausted their algebraic skills but without any prompting from the interviewer.

Yoni and Nir, first-year engineering students, immediately related the problem to typical questions about the intersection of a straight-line y=k with a function, noting that identifying the existence of intersection points didn't necessarily require finding their exact locations. From the outset, the analogy between the algebraic and graphical models was apparent, exemplifying the benefits of using both models constructively. Yoni preferred the graphical approach, solving the first system of equations graphically despite the availability of algebraic solutions. Upon confronting the second system, both Yoni and Nir found algebraic methods inadequate and shifted to the graphical model. Their work exemplifies how they kept both models active in their awareness, using the graphical model to think productively across models. Here are some examples illustrating their approach:

Yoni and Nir identified two intersection points for the first system at (0,0) and (1,1). Yoni then sketched the graphs of x^2 and x^3+1, noting no intersections in the first quadrant. Nir probed further into the range x>1:

Nir: Who says they won't meet later when $x>1$?

Yoni: Will they meet later? Excellent question. x^3 increases faster than x^2, so they aren‘t likely to meet. This means that if at this point it is above it, from now on, they won't meet, x^3 flees away.

Yoni describes the relationships between the graphs using everyday language, employing terms such as 'meet,' 'flee away,' and 'one function is above the other.' This choice of words indicates his sensorial use of the graphical model. This behavior aligns with Fischbein's description of an intuitive model, which he characterizes as "of a sensorial kind, which can be perceived, represented, or manipulated like any other concrete reality." (Fischbein, 1987, p. 121)

Nir then considered the range $x<0$, where an intersection point was evident, implicating a solution:

Nir: On the left-hand side,

Yoni: There is one intersection point.

Nir: How can you find it?

Yoni: I don't need to find it. The question is if there is a solution, not what the solution is, so we conclude that there is one solution in the second quadrant.

Nir and Yoni agreed that identifying an intersection point in the graphical model, even without precise coordinates, suggests a solution in the algebraic model. This observation supports Fischbein’s second claim about the contribution of analogies to productive thinking: "(b) By providing an intuitively accessible structure, mental operations which would be difficult in relation to the original system are facilitated. The products of these mental operations can then be interpreted and evaluated in terms of the original system" (Fischbein, 1987, p. 142)

Nir then evaluated the interval $0<x<1$, using numeric analysis to assess solution feasibility:

Nir: Between 0 and 1, every number is a fraction; squaring it keeps it less than one. So x^3+1 is more than one, indicating no intersection in this range.

Nir's algebraic considerations led him to conclude that no intersection points existed between the graphs within the domain of $0<x<1$ in the graphical model, confirming the absence of solutions to the system of equations in the algebraic model for this range. This type of analytic thinking was a consistent feature throughout their problem-solving process. Their approach involved simultaneous considerations of graphical and algebraic elements; for instance, Yoni observed that x^3 increases faster than x^2 for $x>1$—an algebraic observation, which he coupled with the graphical statement 'it is above it.' Furthermore, even without precise coordinates, their confidence in the existence of an algebraic solution was evident upon identifying an intersection point in the second quadrant.

The solution process of Nir and Yoni exemplifies how they constructively used the analogy between the models, in line with Fischbein's description: "Analogies contribute in the following ways: (a) As a heuristic means of generating hypotheses." (Fischbein, 1987, p. 142) They effectively utilized the graphical model to hypothesize about potential solutions. This model enabled them to focus on specific ranges, facilitating a segmental analysis of intersections that would not have emerged solely from algebraic reasoning. This strategic use of both models underscores the productive application of analogies in enhancing mathematical understanding and problem-solving capabilities.

Discussion and Conclusions

As outlined in the methods section, our study involved two tasks: the first could use algebraic or graphical methods, and the second resisted algebraic solutions. We hypothesized that the limitations of algebraic methods in the second task would catalyze the emergence of an analogy between symbolic and graphical models. The results showed a spectrum in the analogy's emergence. For some students, particularly at a regular level, the analogy remained elusive despite hints and explicit instructions. For others, like Sam and Talya, the analogy materialized following interviewer prompting. For Effy and Ron, it appeared only after recognizing the futility of algebraic solutions, without external cues. Conversely, Nir and Yoni preferred the graphical approach from the start, consistently opting for it even when algebraic methods were available. This variability raises key questions: Are students like Nir and Yoni unique in their approach? Is their preference for graphical methods a distinct personal trait, or does it suggest a teachable skill—a robust linkage between models that could be cultivated in other learners?

Conditions for the Emergence of the Analogy

Our findings identify key factors influencing the emergence of the analogy between algebraic and graphical models:

1. Mastery of Skills in Each Model: Proficiency in both algebraic and graphical models is essential. Students lacking graphical skills often failed to utilize the analogy effectively as the graphical model was absent from their toolkit. For example, regular-level students with limited graphing ability struggled to leverage graphs for problem-solving despite direct prompts from the interviewer.

2. **Necessity:** We hypothesized that necessity would catalyze the analogy's emergence, but results varied. Some students, like Ron and Effy, driven by necessity, adopted new strategies beyond algebra, fostering the emergence of the analogy between the algebraic and graphical models. Others stuck to algebraic methods without success, preventing the analogy's emergence. Meanwhile, Nir and Yoni experienced spontaneous analogy emergence without a pressing need for alternative approaches.

3. Mental Imagery: Strong mental images of equations played a significant role. For Sam and Talya, it was the image of a straight line from the linear equation; for Ron and Effy, the parabola from $y=x^2$. Yoni had vivid images of functions like x^2 and x^3. The data indicates an inverse relationship between necessity and mental imagery: stronger mental images reduce the need for necessity in analogy emergence. For instance, Nir and Yoni quickly formed analogies without external prompts, whereas Effy and Ron and Sam and Talya required exhaustive attempts and multiple hints, respectively.

Additional Influences on the Emergence of the Analogy

Beyond primary conditions, we found that students' beliefs about solutions significantly influence the analogy's emergence. For instance, Sam and Talya insist that legitimate solutions must have precise coordinates, which restricts their acceptance of valid non-algebraic solutions.

Sam: But how can we achieve this result algebraically? There must always be an algebraic way to achieve the solution because it is math!

Talya: If I used only algebraic methods, I would conclude that there was no solution.

This belief that algebraic solutions are inherently more legitimate or accurate may deter students from recognizing or accepting solutions that emerge through graphical models, thereby influencing the likelihood and manner in which the analogy between models manifests.

Implications for Teaching

The results show that all engineering and half of the advanced-level high school students encountered the analogy between algebraic and graphical models with varied immediacy. This variability may arise from personal visualization tendencies, which help some effectively utilize the graphical model. The analogy was more prevalent among higher-level students, suggesting it's a skill developable through training, not merely innate talent. Educators should emphasize building strong mental images of functions to promote analogy development, enhancing students' problem-solving capabilities.

Acknowledgment

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The different functions of illustrations in solving word problems

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Word problems are a topic of great relevance in mathematics education. Research on this topic has developed in several directions, classifying, and investigating the functions, and effects of illustrations on word problem solving, just to name a few. The findings of these studies are somewhat inconsistent concerning the impact of different types of illustrations on problem solving processes. This paper describes a study that investigated the utilization and interpretation of diverse types of illustrations by lower secondary school students during the resolution of word problems. The findings suggest that a range of factors influence students' consideration and interpretation of illustrations, thereby raising questions about the possibility of a priori and univocally classifying the functional role of the illustrations that flank word problems.

Keywords: Word problem, problem solving, functions of illustrations, use of the illustrations.

Introduction

In mathematics education, a word problem is a text that describes a situation familiar to the student and poses a quantitative question, an answer to which can be inferred from the information (usually quantitative) conveyed by the text (Greer et al., 2002). Word problems are an extremely relevant topic, especially for primary and secondary schools, as they allow students to apply mathematics to solve a problem within a context. The interest in this topic is also evident in the importance placed on it by institutions around the world (Eurydice, 2011; OECD, 2017), which consider competence in solving this type of problem as one of the key objectives of mathematics education. International research on this topic has developed in different directions. Some of these strands of research have investigated the effects of different word problem formulations on the solutions of students across various grade levels, focusing on the authenticity of the questions, enriching the narrative context, introducing specific warnings (Dewolf et al., 2017) and placing different types of illustrations alongside the problem (Dewolf et al., 2014; Elia & Philippou, 2004). Research about illustrations is crucial to teaching and learning in general, and specifically in mathematics. The relevance of the topic, specifically in the context of problem solving activities, is highlighted by the PISA 2022 Mathematics Framework (2023). This framework highlights the ability to interpret information presented in graphical form and/or diagrams, as mentioned in its definition of mathematical literacy. Research on illustrations in problem solving activities has focused on classifying their function for the resolution and their effects on students' performance. In particular, a relevant study in the area of illustrations in relation to word problems is that of Elia and Philippou (2004), which introduces a taxonomic model for illustrations in terms of the functions they perform during the problem solving activity, distinguishing them into decorative, representational, informational and organizational illustrations. The authors find an effect on problem-solving activity for all types of pictures except decorative pictures. Subsequent research on this topic, on the other hand, has produced conflicting results, confirming the non-significant effect of decorative illustrations, but also refuting the positive

effect highlighted by Elia and Philippou in 2004 for representational illustrations (Agathangelou et al., 2008; Dewolf et al., 2014; 2017), finding that students tend to ignore this type of illustration (Dewolf et al., 2015) and that informational illustrations also have, in some cases, negative effects on problem-solving activity (Gagatsis & Elia, 2004; Elia et al., 2007). Further research is therefore needed on the role that different types of illustrations can play in problem solving, to clarify which of them are observed, manipulated, or used as a source of information, and under what conditions this occurs. This paper presents the findings of a study investigating the use of the different types of illustrations identified by Elia and Philippou**,** by sixth, seventh and eighth grade students**,** during the problem-solving process. This study constituted a part of a larger research project whose aim was to investigate the effect of such illustrations on the problem-solving process.

Theoretical framework

Research on the use of illustrations in educational contexts is part of the broader study of multimodal learning, specifically the combined understanding of different forms of representation, such as text and illustration. Schnotz (2005) outlines an integrative model of text and picture comprehension (ITPC model) and discusses, on the basis of experimental data, the potential of his model to predict the conditions under which the relationship between text and picture supports or hinders learning. The ITCP model predicts that the combination of a text and an image can help students build a mental model and thus improve learning if the two representations are semantically correlated (coherence condition) and if they are presented in close spatial and temporal proximity (contiguity condition). The study by Elia and Philippou (2004) focuses specifically on word problem illustrations, which are intended as pictorial representations of the context of the problem. The authors classify pictures according to their function in relation to the problem-solving task, based on the categorisation formulated by Carney and Levin (2002). *Decorative*, if the illustration is completely unrelated to the context described by the problem; *representational*, if it represents part or all the narrative contexts; *organizational*, if it is structured in such a way as to provide a direction for thinking or working to produce a drawing that supports the solution process; *informational* if it contains relevant information not included in the text of the question. This qualitative study of 8 students found significant effects for representational, informational, and organizational illustrations, and showed that representational illustrations had a positive effect on problem solving for sixth grade students. Elia et al. (2007) investigated the same types of illustrations in solving one-step problems of additive structures by first, second and third grade students. The authors conclude that students essentially overlooked the presence of the decorative image, while they performed poorly on problems involving informational illustrations. Research on the same types of representations for one-step mathematical problems of additive structure was conducted by Agathangelou et al. (2008). Their results show that the presence of decorative and representational pictures did not affect students' performance.

The effect of representational illustrations is also investigated in the studies conducted by Dewolf et al. (2014; 2015; 2017), considering word problems with realistic modelling complexity. The research team investigates the influence of representational illustrations on the production of realistic responses by fifth and sixth grade students to problems that require contextual considerations for solving. The underlying hypothesis is that appropriate representational illustration can provide a rich pictorial description of the problem situation, supporting the construction of a more detailed mental

model of the context, and thus, in conclusion, prevent students from solving the problem by mechanically applying a mathematical operation inferred from data or keywords in the text, but encourage them to consider the context of reality.

The research teams conducted three studies. The first study (Dewolf et al. 2014) examined the effect of illustrations and warnings on the percentage of realistic answers, finding no significant impact.

To investigate why the illustrations did not have a positive effect on the realistic solution of word problems, the authors set up a second study (Dewolf et al. 2015) in which they observed the solvers' eye movements during the task using an eye tracker. The results of this second study show that the students involved ignored the displayed illustrations in almost all cases, although representational illustrations were observed significantly more often than decorative ones. Furthermore, even when students were prompted to look at the illustrations by being shown only the illustration for five seconds before presenting the text of the question, the percentage of realistic answers did not change significantly. The third study (Dewolf et al. 2017) tested the effectiveness of enriched illustrations and explicit warnings, but again, no significant improvement in realistic answers was observed. The three studies, particularly the second, found that illustrations were actually excluded from the students' solving process. The authors hypothesized that students' beliefs about word problems and the nature of illustrations may have hindered their ability to use visual information effectively. They suggest that future research should explore more detailed illustrations.

Our study moves in this direction, by investigating the use of different types of illustrations during the problem-solving process of mathematical word problem. Specifically, the following research question guided the study: To what extent do lower secondary students employ illustrations when solving mathematical word problems? How do lower secondary students utilize different types of illustrations when solving mathematical word problems, in terms of observation, manipulation, and extraction of information?

The following text will discuss how, in some cases, students can pay close attention to illustrations and the information they contain when solving problems, formulating hypotheses about the factors that led to the careful use of illustrations in this case study.

Methods

The study involved 254 students from three school years (sixth, seventh and eighth) in northern Italy. Guided by the working hypothesis that variations in the presentation of the questions could be reflected in the student's answers, five tasks were selected from the database of the national tests of the National Institute for the Evaluation of the Education and Training System (INVALSI), which were considered significant from a content point of view, with a clear reference to the real context and a low rate of correct answers. All problems were illustrated, and at the same time the functional role of the illustrations was classified in the light of the theoretical framework, all illustrations were discussed before approval, but not pre-tested. Different versions have been produced for four of these, distinguished by the type of illustration flanking the word problem. In this article, due to space limitations, only two of the problems involved in the study, problems 4 and 5, will be presented. The problem 4 was presented to 33 sixth graders, in the format shown in Figure 1, with the illustration

placed centrally and immediately below the text. The illustration shows part of the context described by the text, depicting three characters walking through a classroom, and was therefore classified as representational according to Elia and Philippou (2004).

Mario, Renato and Fabio measure the length of their classroom in steps.
Mario counts 26 steps, Renato 30 and Fabio 28. Who has the longest step?

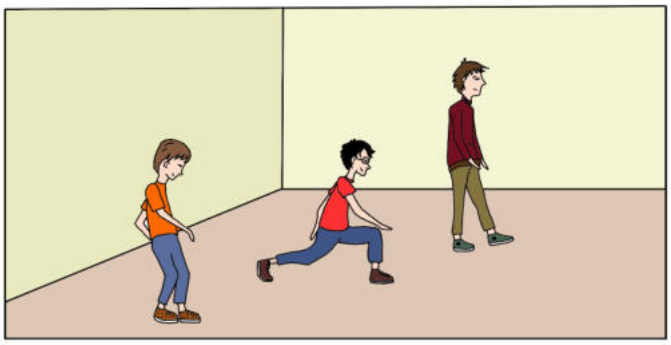

Figure 1: Problem 4

The problem 5 was presented to 79 sixth and seventh graders in three versions, which differed in the type of illustration. Figure 2 shows one of the versions presented to the students. During the stage of problem making the illustration was considered organizational, since it depicts all the pills allowing students to group them by drawing on it, thus supporting the solution procedure.

A medicine is sold in boxes of 28 divisible pills like the one in the picture. Each pill contains 20 mg. Piero's grandmother needs to take 30 mg of this medicine every day for a month. For how many days can Piero's grandmother take her daily dose of the medicine using just one box?
Write down how you found the answer and then report the result

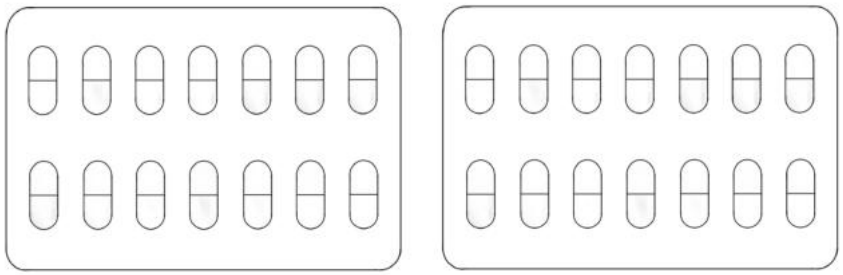

Figure 2: Problem 5

In addition, both problems were accompanied by the instruction: 'Report the result and explain the reasoning you used to answer, specifying whether you used the illustration or not'. This indication was also orally repeated when the problem was presented to the students.

Data consists of students' written productions, audio recordings of class discussions, and individual interviews with students. Students were given a maximum of one hour to solve one problem. The data was qualitatively analysed by searching for any references to the illustrations in the answers, and relating them to the information gathered from the discussions and interviews conducted with the students, with the aim of highlighting and interpreting the phenomena in focus, such as the use of the illustration, the resolution strategies that emerged, and the interaction between these aspects.

Data analysis

The qualitative analysis allows us to observe different uses of the illustrations. In particular, it was observed that the students paid close attention to all the illustrations considered in the study. Moreover, in some cases, students derived additional information from the illustrations and used this information in the solution procedure, even though, in the authors' conception, these pictures should have had other functions, such as helping to understand the context (representational function) or supporting the solution procedure (organizational function).

The qualitative analysis of the data relating to problem 4 shows that 17 out of 33 students (52%) pay attention to the illustration and that of these 11 students draw information from it that is essential for their own reasoning and solution procedure. Specifically, the students derive information from three distinct aspects of the illustration. Firstly, from the order of the characters depicted. In this case, the students associate a name with the characters in the illustration by assuming that the order in which the characters are depicted in the illustration is the same as the order in which the characters are introduced in the text, and then they reason about the illustration to reach an answer to the problem question.

For example, Maurizio writes: "[...] Fabio has a normal stride compared to Renato who does it on purpose to lengthen his foot, so I say Fabio". Maurizio claims to have thought about the illustration and seems to have considered the way in which the characters are depicted in the illustration, in the act of taking unnatural steps. In this type of answer, the numerical data in the text are not mentioned in the resolution and do not seem to have been involved in the reasoning.

Another type of information extracted from the illustration concerns the spatial arrangement of the characters. Some answers seem to relate to the illustration in a different way to the information drawn from it, the detail being taken into account is not the order of the figures but rather their arrangement in space. This is exemplified by the explanations of some students who reported that they had noticed that one character had walked a greater distance than the others. Matilde, for example, answers that it is Fabio who has the longest stride because "it is clear from the drawing that he is also the one who is in front of Renato", denoting that she had paid attention to the aspect of the arrangement of the characters in the illustration. Although it is not clear how she arrived at her answer, she may have first assigned names to the characters using the above strategy to then identify which of them had moved a greater distance.

Finally, other aspects of the illustration that served as sources of information were the height of the characters and the fact that they were depicted in the act of forcing their stride unnaturally. Carolina, for example, concluded that "none of them has the longest stride" because "Renato does it on purpose to make it so wide". During the discussion phases, the height of the characters also came up for consideration, as described by Luca, who recounts some of the confusion felt during his resolution: "I honestly didn't know where to turn, because Mario should be the tallest one, and the one with the fewest steps, so the one with the longest step. But if we look at the drawing, we can see that one boy is putting much more distance into the steps than both Mario (he points to the boy on the right in the illustration, who actually appears to be the tallest) and this other boy (he points to the character on the left)".

What was observed in the administration of this problem is a conspicuous variety and amount of information that was drawn from the illustration during the resolution. It was noted that 11 out of 33 students interpreted the illustration as informational, i.e. an illustration containing information relevant to the solution and not in the problem's text.

Concerning problem 5, 6 out of 45 students gave the answer "9 days", exactly half of the expected answer of "18 days". These students reported that they had looked at the illustration and inferred the number of tablets in a box, having interpreted the illustration as showing two boxes. Alberto, for example, writes: "Piero's grandmother has to take a box of 14 tablets, and if she takes half a tablet + half a tablet, she should finish it in 9 days".

In this case, the quantity of tablets was explicitly stated in the text. However, these students utilized the data extracted from the illustration to solve the problem.

Discussion

These observations call for a re-evaluation of the classification made when designing the task. The illustration of the problem 4, for example, was conceived as representational, taking into account the expected answer, because it represented part of the context described by the text of the problem. Similarly, in problem 5, in addition to the organisational function, an informational role for the illustration is evident, as students interpreted the two blisters shown as two different boxes. Significantly, these students chose to use the illustration rather than the text to get the data for their own reasoning, confirming an informational interpretation of the illustration.

It will also be necessary to reconsider the possibility of investigating the use of certain types of illustrations by students, where the latter attribute a different functional role to the illustrations than that assumed a priori by the researchers through the model used. We, therefore, question whether it is possible to classify illustrations of word problems unambiguously and a priori in terms of their function for solution, ignoring the interpretations given by the solvers themselves. In both above cases discussed, what was observed differs from what was obtained by Dewolf's (2014; 2015; 2017) research group in terms of the attention paid to the illustrations and the use made by students in solving word problems, specifically, our data shows that in some contexts students seem to attribute great importance to illustrations. Furthermore, it was observed that students drew additional information from the illustrations, attributing to them an informational function in Elia and Philippou's (2004) framework, even though the authors' intention was for these illustrations to have a representational and organizational function.

Conclusions

In light of the observed results, it can be posited that, under certain conditions, students may interpret the functional role of an illustration, flanking a word problem, differently from what can be said through Elia and Philippou's model, for example by interpreting the representational illustration as informational and thus extracting information from it that will be involved in the reasoning and solution. An informational interpretation is also accompanied by a certain focus on illustrations. The latter aspect seems to differ from the findings of previous studies on this topic, as the research by Dewolf et al. (2015, 2017). It can be hypothesised that these observations can be interpreted in the

light of conditions relating to the classroom context, the structure of the problem and the illustrations themselves and students' beliefs. In particular, students' interpretations of the function of illustrations could be related to the context in which the problem solving activity takes place.

This context is determined by factors such as the relevance attributed to the illustration, directly or indirectly, by the text of the problem and question, by the structure of the illustration itself, its size, its position in relation to the text, or by the researcher during the delivery and presentation of the problem. For example, the prompt accompanying the problems: "Report the result and explain the reasoning you used to answer it, indicating whether or not you used the illustration" may have drawn the students' attention to the illustration and contributed to the idea that it was fundamental to the solution. In this case, the students may have been looking for information in the illustration that could be used in the reasoning. This observation does not seem to be consistent with the findings of Dewolf et al. (2017), who do not observe a significant effect of warnings on attention to illustrations. It can be hypothesized that this discrepancy in results can be interpreted in light of the contextual aspects discussed in this paragraph, which may have convinced students that observing the illustration can be useful, to the point of interpreting it as informational and thus looking for information useful for the solution.

A second type of factor that can be used to interpret these results relates to the contribution of the illustrations to the narrative context. Specifically, it could be hypothesized that if the relationship between text and illustration is strong, and thus the illustration faithfully and realistically represents the context of the problem text, students may be inclined to pay attention to it. Factors that may influence the representativeness of illustrations include references to them in the problem text itself, the presence of decorative details, whether it is a drawing or a photograph, and whether it is schematic in order to emphasize certain aspects depicted (such as the spatial order of the characters in Figure 1). Finally, factors such as students' beliefs about the role, and importance of an illustration within a problem and more general beliefs about the problem solving task, seem to play a decisive role.

In conclusion, our results show mostly an informational use by students of all the types of illustrations considered, from which data and information not foreseen in the formulation of the questions were drawn and subsequently used in problem solving. This suggests that is not possible to do a univocal and a priori classification of the illustrations accompanying the word problems since contingent and contextual factors could influence the students' interpretation of the function of the illustrations. The factors that seem to influence students' attention to illustrations and interpretations of their functional role seem to be the context in which the problem-solving activity takes place and the relationship between illustration and text in constructing the narrative context. Particularly in cases where the illustration realistically represents the content of the text, students may be led to draw information from it for the solution procedure. The results of this study suggest that further research is needed to investigate the role of the factors proposed here and new factors in determining students' interpretation and use of the illustrations that accompany word problems.

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Learning mathematics with body-centred and object-centred movement: Expanding on Bruner's enactive mode of representation

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This paper focuses on considerations for expanding the known modes of representation (according to Bruner) with the mode of movement. Our expansion is based on an examination of the modes of representation described by Bruner – which are widespread in (mathematics) lessons – and their significance for cognitive development and learning processes. In particular, we clarify the difference between internal and external representations. We illustrate the extension of Bruner's model of representation modes through a content-analytical examination of a learning environment, revealing the potential of the movement mode of representation for the mathematical learning process.

Keywords: Elementary school mathematics, mathematics activities, modes of representation, body-centred movement, object-centred movement.

Introduction

Supporting learning processes and thus accompanying the development of students' mathematical skills is a central aspect of teachers' professional competencies. There is a long tradition in mathematics education research of exploring the use of different representations for learning mathematics. Already in the 1960s, Bruner (1967) distinguishes between three modes of representation: iconic representations, enactive representations and representations in words or language (symbolic representation). These representations are significant in cognitive development of learners and are later (in adulthood) used in combination for learning processes. Each mode has its own way of representing cognitive processes and their results (Bruner, 1971). These three modes of representation are often used in everyday school life to justify decisions in the design of mathematics lessons (Bruner, 1974). In this context the paper focuses on extending the enactive mode of representation through body movement. This extension appears to make sense, since Bruner's reception initially seems to refer exclusively to object manipulations/actions with the enactive mode of representation, although Bruner also considers movements in space in the individual modelling of the experienced world.

In the designing of mathematics teaching and learning large motor movements of the whole body with and without material arrangements as well as small motor movements without material (including – among others – gestures) seem to be rather neglected and not explicitly used in school practice. Indeed, although gestures and large motor movements are currently being researched (e.g., Huth, 2021; Radünz & Benölken, 2021), they are still hardly considered in the design of learning environments in the school context.

In this paper, the following overarching research question will be pursued using an example: *How can the modes of representation formulated by Bruner be further differentiated regarding to body*

movement and the movement of objects in space (large motor), and how can the relevance of such an extension be illustrated?

Theoretical Framework

According to Bruner (2009), children develop knowledge in an active and social process. For learning activation, Bruner (1967) recommends offering students different access to learning content for absorption and understanding. In this case, it can be assumed that the term 'representation' refers to external representations. These are "text, pictures, graphs, tables, sound and music" (Schnotz and Kürschner (2008, p. 175) that can be used flexibly by learners. This medial specification is also addressed by Bruner (1971). Internal representations, meanwhile, are also referred to as "mental images" (Goldin & Shteingold, 2001, p. 5), and these "multiple internal representations interact with multiple external representations" (Schnotz & Kürschner, 2008, p. 176).

> Accordingly, there is not a one-to-one relationship between forms of external representations and forms of internal representation. On the one hand, a text as an external descriptive representation results internally both in a descriptive and a depictive representation. On the other hand, a picture or diagram as an external depicture representation results internally both in a depictive and a descriptive representation. (Schnotz & Kürschner, 2008, p. 180)

Overall, the difference between external and internal is less discussed, as Bruner rarely makes the distinction explicitly; this leads to differences in interpretation. However, Bruner's work certainly reveals differences between the internal and external representations. For example, he writes: "But the child's symbolic world does not make a clear separation between internal motives and feelings on the one hand and external reality on the other" (Bruner, 2009, p. 34).

> We cannot, of course, observe anyone's internal representations directly. Rather, we make inferences about students' internal representations on the basis of their interaction with, discourse about, or production of external representations. Skilled teachers do this almost automatically, paying attention to their students' words, written work, use of manipulative materials or use calculators and computers as they try to understand individual conceptions and misconceptions. (Goldin & Shteingold, 2001, pp. 5–6)

These explanations show that a distinction should be made between internal and external representations, because they mean different kinds of representations: internal representations refer to the internal conception of a mathematical content, while external representations refer to the externally visible representation of a mathematical content (for example by manipulating/sorting/arranging blocks).

Further models of modes of representation are being developed in mathematics education research. For example, Lesh's translation model names "five forms of mathematical representation" (Johnson, 2018, p. 2): "pictures", "manipulatives", "written symbols", "real-life situations", and "verbal symbols" (Johnson, 2018, pp. 2–3). Learners interactively use different mathematical representations to develop mathematical concepts. This requires "representational fluency" (Johnson, 2018, p. 2), which describes the ability to use different representations and to flexibly switch between them. Johnson (2018) also describes the change of representation as a central element in learning.

In this paper, we further differentiate the enactive mode of representation based on Bruner. Bruner (1967) describes situations from everyday life, such as playing tennis, skiing, or cycling; in these cases, it is difficult to verbally describe the exact actions, and images only model partial aspects of the movements. This means that learning processes for these situations can only be represented through actions:

> We know many things for which we have no imagery and no words, and they are very hard to teach to anybody by the use of either words or diagrams and pictures. […] We have come to talk about the first form of representation as enactive […]. Enactive representation is based, it seems, upon learning of responses and forms of habituation. (Bruner, 1967, as cited in Lotz, 2022, p. 20)

Bruner (1971) also considers the tying of a knot to be enactive. One knows the knot through the usual process of tying it, but the representation in words or in a picture never succeeds completely; how the knot is actually tied can only be partially represented. The representation of the knot tying therefore takes place in the medium of the action.

Regarding mathematical learning, Bruner (1974) describes activities, such as investigating quadratic equations by assembling wooden blocks, which can be assigned to the enactive mode of representation. Bruner (1974) also mentions the creation of equations by working on the beam balance as an example of enactive work. According to Bruner, in such situations, enactive constructive work is a useful introduction to a topic.

Looking at the implementation of the enactive mode of representation in school life, it seems that small motor movements with materials are predominant. Here at most, the objects are moved in the plane on the table. In most cases, object manipulation with the hands takes place (here, referred to as small motor object-centred movement). These object manipulations could be compared with Bruner's description of knot tying or building with wooden blocks in connection with equations. Movements of objects in three-dimensional space (here, referred to as large motor object-centred movement) are rarely observed in mathematics lessons. Bruner's research does not describe large motor object-centred movements in relation to mathematical learning.

The skiing movements described by Bruner refer to movements of the body. During skiing, limbs are moved in relation to the trunk, and the trunk is also tilted in different ways in space (here, referred to as small motor body-centred movement). At the same time, skiing also involves spatial movement: the body as a whole is moved through space (here, referred to as large motor body-centred movement). We can infer that Bruner had already considered including the body in the individual modelling of the physical experience, but he did not extent this idea to the mathematical world.

The body-movement-inclusive view of enactivity is supported by embodied cognition theory, which highlights the relevance of body movement for (mathematical) learning and thought processes. According to Tran et al. (2017), learners' actions and how they think are directly interrelated. Both gestures and manipulation of material, as well as body movement, support mathematical learning processes (Tran et al., 2017). Ottmar et al. (2019) distinguish movement between "body-centred", "body-in-action", "object-centred", "object-in action" and "environment centred".

A differentiation of the enactive mode of representation becomes clear here. In the context of mathematical learning, the switch between the different modes of representation is also important (Bruner, 1967). This is not only a vertical change, from one mode of representation to another (for example, a change from the iconic to the symbolic mode of representation), but also a horizontal one (for example, switching from one iconic representation to another iconic representation) (Lotz, 2022).

In summary, the following can be said regarding movement as an enactive representation: a distinction can be made between object-centred movements (material-supported) and body-centred movements. In object-centred movements, objects are manipulated in the plane and in space. In body-centred movements, the body itself is manipulated and moved in space. Both modes of representation (object-centred and body-centred) can involve small motor or large motor movements. With small motor movements, the movement space is small (small movements in the plane); with large motor movements, the movement space expands (large movements in space) (see Table 1).

Table 1: A comparison of object-centred and body-centred movement

Object-centred movement	Body-centred movement
Small motor activities on and with materials: for example, placing symmetrical figures with tiles, also known as action.	Small motor movements with your own body (without moving in space): for example, bending both arms symmetrically at right angles pointing upwards. Here is an overlap with gestures.
Large motor movements on and with materials, and the dimension of movement in space is added: for example, placing symmetrical arrangements with baskets and chairs in the room.	Large motor movements with your own body, and the dimension of movement in space is added: for example, the symmetrical movement of two people with arm movement.

Methods

To better describe the differentiation of the enactive mode of representation, we observe the working of the young learners on symmetry problems. These symmetry problems are presented to the students in a learning environment that enables an open approach to a learning subject and gives learners the space to work independently on a mathematical problem.

This analysis of this exemplary problem-solving process focuses on the following research question: *Which large motor movements were used in the learning environment for the topic of symmetry?*

A team of university students in a seminar for the primary school teaching degree program developed the learning environment on the topic of "mathematical learning from a multimodal perspective with a focus on movement". The environment was tested with primary school children (third graders, about nine years old).

The university students designed learning environments that focused on using the children's own body movements in line with the seminar's overall theme. In the learning environment on the topic of 'symmetry', an empty room and various objects, such as chairs and boxes, are used as the starting

point. Two children are involved in the learning environment. The university students divide the learning environment into different sequences, which are initiated by problems and impulses. The following sequence order is chosen: 1. objects (chairs, baskets, bottles, etc.) are used to create as much chaos as possible in an empty room; 2. the children organize the objects; 3. the room is divided into two fictive rooms, which are designed in the same way with the existing objects (the word symmetry is not explicitly used during the implementation); 4. the wall between the two rooms (i.e., the axis of symmetry) is marked with adhesive tape; 5. the two children enact symmetrical body movements; 6. the children participate in an symmetrically designed racket play.

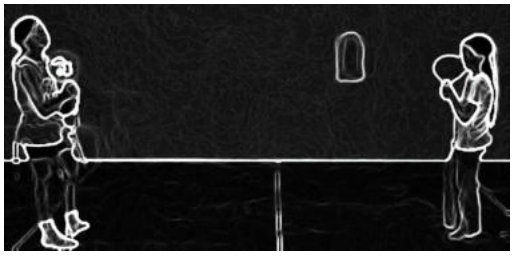

Figure 1: Sequence 6 – Racket play

For the analysis, we use the methods of the qualitative content analysis according to Mayring (2014). The category system was developed according to this study's research aim and consists of main categories and subcategories derived from the types of movement described in the literature (Ottmar et al., 2019) and the authors' expansion of the enactive representation mode (see Table 1). The system was s also expanded inductively by working with the video. The types of movement described in Table 1 serve as main categories that are differentiated into subcategories: object-centred movement – large motor, object-centred movement – small motor, body-centred movement – small motor, body-centred movement – large motor, symbolic mode (spoken language), gesture (deictic), and iconic mode. The movement categories arise from Bruner's differentiated enactive mode. The categories symbolic and iconic also originate from Bruner´s theory about modes of representation. The 'Gestures' category has also been included to categorise deictic gestures in particular. Gestures, which are described as movements of the limbs, partly show small motor movement. Gestures are divided into iconic, metaphorical, deictic, and beat gestures (McNeill, 1992). Consequently, gestures can also have an iconic character, as they can symbolise or visualise something. In this case, they were categorised as iconic. As soon as a movement of the limbs was not connected to language and the movement alone carried meaning, it was categorised as a movement category. The coding unit comprises 10 seconds. One code is assigned per time unit.

Analysis

The coding was carried out separately for each person (G1 – girl 1, G2 – girl 2): a code bar was created for each child (see Figures 2 and 3).

In the first step of the analysis, the code bars are described in terms of their modes. Sequence 1–4: Here, the large motor, object-centred movement is mainly expressed by the two learners distributing the large objects (symmetrically) in the room. Sequence 5: Here, both small and large motor body-centred movement is expressed – small motor, because individual body parts are placed

symmetrically in relation to each other, and large motor, by moving one's own body symmetrically in relation to another person. Sequence 6: Both body-centred and object-centred movements occur here. These tend to be large motor movements, as the children move around the room when passing the ball and use the racket to handle an object.

Two parts with frequent code changes are selected for presentation of results: sequences 4 and 6. These sequences indicate a lively change of representation and possibly the potential for intensive learning processes. Sequence 4 shows that objects are moved around the room, and positions are sought in accordance to find a symmetrical objects-arrangement in the room. The code bar shows (see Figure 2) that in the subsequent observation of the object constellation (iconic mode), linguistic negotiations take place, which in turn lead to a continuation of the arrangement of the object constellation. This alternation between large motor object movements and linguistic negotiation processes suggests that large motor object movements require a still image as an intermediate stop for orientation. On the contrary when objects are moved on the table, such an orientation or verification can be carried out permanently. At the same time, still images (taking distance from the objects-arrangement and view it from the outside) encourage descriptions of object constellations to a greater extent. The movement of objects in space (large motor object-centred movement) requires more spatial orientation (in relation to objects and people) than small motor object-centred movements.

Figure 2: Sequence 4 – Attaching an adhesive tape as an axis of symmetry

The code bar for the sequence 6 shows an intensive alternation between large and small motor object-centred movements and large and small motor body-centred movements, as well as longer speech sequences. This is partly due to the game and partly due to the task of making symmetrical movements during the game. These symmetrical movements are countered by the individual preference in games to strike objects with either the right or left hand. Learners can experience the change in axial symmetry orientation in a special way through the inconsistency between the task assignments and their individual hand preferences during movement. The linguistic phases enable an exchange about this inconsistency and can stimulate processes through the change of representation.

Figure 3: Sequence 6 – Racket play

Conclusion

The theoretical analysis of Bruner's modes of representation shows the importance of the distinction between internal and external representations. The external perspective on representations provides information on how learning opportunities can be presented and learning processes can be supported, while the internal perspective on representations focuses on individual learning processes and the development of mathematical concepts. This connection can be reconstructed in the multimodal expressions of the learners (see sequence 6).

Furthermore, the analysis of the example reported here shows that selected mathematical topics can be implemented in learning environments involving large motor movements. These movements can initiate cognitive activation and a cognitive irritation in the learners, which can lead to a differentiation of mathematical concepts. This becomes clear in sequence 6, where the constraints of axial symmetry conflict with individual preferences. This irritation causes a bodily experience of the properties of axial symmetry. In this context, it can be presumed that the immediacy of mathematical experience through the integration of the body enables different approaches to mathematical concepts than those involving small motor movements with objects (horizontal change within the enactive mode – big and small one), which can only be experienced at a distance through observation.

We can also advance the hypothesis that the physical expression can show mathematical concepts that cannot initially be expressed by learners in language. This has already been showed in the range of gestures (McNeill, 1992). In our example, the property of non-orientation is physically experienced as something that is not right, i.e., does not correspond to the individual movement with the racket. In this situation, a mathematical interpretation and the resulting mathematical property of axial symmetry are not explicitly stated, i.e., they are not expressed in words.

In this context, body-centred movements are frequently accompanied by speech, as is evident in the analysed learning environment. This shows that the fleeting body-centred movements – large and small like gestures (Huth, 2021) – are often paired with an intensive verbalization of the experience, enabling a communicative continuation. On the other hand, the joint discussion requires a argumentation that makes individual experiences accessible to others. In summary, considering the Bruner's enactive mode as merely ‘actions’ is insufficient, because mathematics can also be experienced in other physical ways.

The project MoMaMat (Moving Makes Maths) aims to investigate this assumption (Hölzer & Vogel, in press). In particular, we intend to pursued the following research question: What knowledge (concepts) of the learners can be reconstructed in the large motor movements compared to small motor movements?

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Visual feedback in example-eliciting tasks: from artifact to instrument

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This study examines how the use of visual feedback is integrated into example-eliciting tasks that requires students to reflect on them, transform from an artifact to an instrument on their learning process. By analyzing the interactions of middle school students (ages 12-13) with these tasks and feedback, we track their progression from procedural use to deeper conceptual understanding of comparing fractions. Episodes involving students Ali, Noa, and Ameer demonstrate how visual feedback supports their self-reflection processes and enhances comprehension of mathematical relationships. The findings underscore the need for designing learning environments where visual feedback fosters conceptual insight and generalization.

Keywords: Fractions, visual representation, artifact, instrument.

Introduction and theoretical background

Visual representation is crucial in mathematics education as it serves as a bridge between abstract concepts and concrete understanding. A significant advantage of a visual representation is that it supports cognitive processes which are essential for effective problem solving in mathematics and lead to deeper conceptual understanding. Additionally, these artifacts foster self-reflection, enabling learners to identify and correct misconceptions more effectively (Taylor et al., 2021).

Based on Olsher and Lavie (2021), task design is crucial for assessing generalization processes. Their research emphasized that well-constructed tasks should incorporate visual representations to support and enhance the generalization process. Visual representations such as diagrams and graphs help learners abstract and apply mathematical concepts beyond specific instances, facilitating a deeper understanding of the underlying principles. By engaging with tasks that integrate visual representations, users can develop more robust strategies for recognizing mathematical patterns and relationships. The study highlights that tasks designed with these visual elements not only aid in problem solving but also encourage students to generalize their knowledge across different contexts, ultimately enhancing their ability to transfer mathematical ideas to novel situations. Visual representations become particularly powerful when they are integrated to formative assessment processes. When visual feedback is used in tasks in which students generate examples, it can transform from a simple artifact into a dynamic instrument that enhances learning.

According to Rabardel, instrumental genesis involves learners' reflection on and adaptation of their use of artifacts. Artifacts are defined as "anything which human beings create by the transformation of nature and of themselves' (Wartofsky, 1979, p. 14) and are incorporated in human activity in order to support problem-solving. As students refine their use of artifacts, their cognitive processes are also enhanced. This relationship underscores how adapting artifacts fosters deeper learning and artifact proficiency, which Rabardel (2002) defines as an instrument. An instrument is 'a composite entity made up of an artifact component (an artifact, a fraction of an artifact or a set of artifacts) and a scheme component (one or more utilization schemes, often linked to more general action schemes'

(p. 86)." In other words, although artifacts are objects themselves, instruments are broader constructions that encompass both these objects and the ways they are used.

Rezat (2021) examined how automated feedback, initially a mere artifact in the form of digital responses and hints provided by textbooks, evolves into a critical instrument for fostering conceptual development in primary students. The study highlights that, while automated feedback itself is an artifact—essentially a technological feature designed to offer immediate responses—it becomes an instrument when integrated into the learning process. This transformation occurs as students interact with the feedback, using it not only to correct errors but also to deepen their understanding of mathematical concepts. By analyzing how students engage with and use this feedback, Rezat demonstrated that automated responses can support conceptual development when they are effectively incorporated into instructional strategies. Thus, automated feedback, originally a passive artifact, has become an active component of the learning environment, aiding students in applying and generalizing mathematical principles more effectively.

Despite the recognized importance of visual representations, there is a notable gap in how these representations are used as feedback in example-eliciting tasks (EETs) on assessment platforms. Although existing research has focused on the design and implementation of visual artifacts, there has been limited exploration of how these artifacts function as feedback instruments that can influence students' problem-solving strategies. Specifically, there is a need to understand how visual representations, used as feedback in EETs transform from mere artifacts to integral instruments that drive learning and engagement in formative assessment platforms.

This study addresses this gap by investigating how visual feedback in EETs can evolve from simple artifacts to crucial instruments within the learning process. It explores how middle school students engage with visual feedback during tasks to compare fractions, and examines how this engagement affects their problem-solving process and overall mathematical understanding.

Our research question was: How does visual feedback in example-eliciting tasks for comparing fractions transition from an artifact to an instrument?

Methodology

Participants and data collection. To answer this research question, we designed activities that contained EETs when students were asked to construct fractions with specific requirements using an interactive diagram (ID). Seventy-five students participated in the overarching project. In this study, we focused on three episodes of three students, aged 12-13 years from Israeli middle school, while working individually on two tasks and asked them to think aloud. The interviewer's role was to help them with technical issues and repeatedly ask them to explain what they did and why. Before the experiment, students participated in a preparatory session that included five tasks completed in 30 minutes, during which they experimented with the platform, the ID, and the types of questions asked to become familiar with them. Data were collected by videotaping students' work on their activities. Students were selected based on their parents' agreement to participate in the study.

Data analysis. This study applies Rabardel's (2002) Instrumental Genesis Theory, which explains how learners transform external tools into cognitive instruments for mathematical understanding, to

analyze students' interactions with visual representation feedback. Following the framework of Baccaglini-Frank et al. (2024). The analysis examines students' transition through three phases: Initial Engagement (Artifact Use), where they manipulate the tool procedurally without deep reflection; the Instrumentation Phase, where they begin shaping the tool to support reasoning; and the Final Instrumented Scheme, where they internalize the tool as a conceptual instrument for problem-solving. Data were analyzed through video recordings, screen captures, and student reflection.

The tasks in this study asked the students to construct fractions by dragging points on the Cartesian coordinate system (the left side of Figure 1) and to self-reflect (the right side of Figure 2) on given statements that are mathematical characteristics. These characteristics are predefined mathematical properties of the fractions that the student chose to satisfy the requirements of the tasks (e.g., the fraction is less than one) and the relationship between the properties of this fraction and the properties of another fraction (e.g., both fractions have the same denominator). The student was asked to construct examples that had as few characteristics as possible while still meeting the requirements of the tasks.

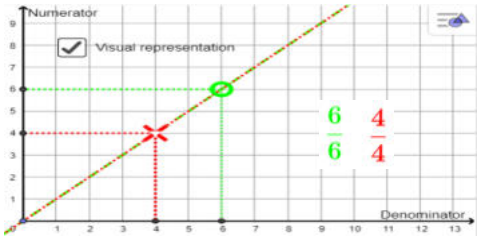
Construction part	**Reflection part**
	☑ 1. The fractions that you chose are equivalent
	☑ 2. The visual representation ray crosses all points at the same time
	☑ 3. You chose a fraction that is less than one
	☑ 4. The fraction is an expansion or reduction of the green fraction
	☑ 5. You added the same integer to both the numerator and the denominator of the fraction to create an equivalent fraction

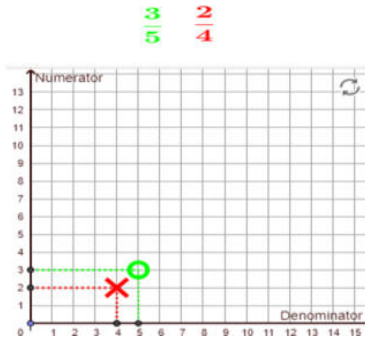
Figure 1. Construction and self-reflection parts of task 1

In the interactive diagram (ID) (Figure. 1), which is used in this study, the fractions are represented in the Cartesian coordinate system by a point whose vertical coordinate is the numerator and whose horizontal coordinate is the denominator (Arnon et al., 2001). All the equivalent fractions are represented by a straight ray passing through the origin, which appears automatically when the fractions are equivalent. The origin and points on the vertical axis do not represent any fractions. Points that exist on a ray with a greater slope represent a greater fraction (Figure 1).

The first task (Kadan-Tabaja & Yerushalmy, 2025), "Choose two equivalent fractions by dragging the X (red) and O (green) points. To the right of the ID are five statements that can help characterize the two fractions that you have created. Check for each characteristic present in the example before submission. *Try to submit two fractions in a way that your submission should comply with as few statements as possible.* Submit three such examples, as different as possible from each other." In this task, we focused on statement 2, which is related to visual representation. Applet for illustration: https://www.geogebra.org/m/fmfvcdjr.

The second task (Kadan-Tabaja & Yerushalmy, 2024) (Figure 2), "Choose two fractions, the O (green) one greater than the X (red) one, which applies *as few statements as possible* from the given

set of statements. Mark the statements from the set of statements that exist in your example and answer the task requirements." In this task, the ID automatically presented a fraction that was greater, and visual representations were reflected for each tagged statement as a construction in the ID. Statement 6 should be included in every example that satisfies these requirements. In this task, the ID automatically identifies the fraction that is greater. Applet for illustration: https://www.geogebra.org/m/xypss2f6.



The green fraction is greater than the red one

- ☐ 1. The red fraction is smaller than one whole, and the green fraction is greater
- ☐ 2. The red fraction is smaller than one half, and the green fraction is greater
- ☐ 3. The green fraction has the same denominator as the red one
- ☐ 4. The green fraction has the same numerator as the red one
- ☐ 5. The numerator and the denominator of the green fraction are greater than the numerator and the denominator of the red fraction, respectively
- ☐ 6. The ray of the green fraction is above the ray of the red fraction
- ☐ 7. The denominator of the green fraction is a multiple of an integer of the denominator of the red fraction

Figure 2. The interactive diagram of Task 2

Results

This paper presents an analysis of the three episodes of Ali, Noa, and Ameer. These cases were selected because they demonstrated the typical use of feedback as an instrument when working with visual representations in EETs.

Episode 1: Ali's discovery of the relation between the ray and the equivalence fractions

Ali worked on an equivalent fraction task using an ID. This task required him to construct three examples of equivalent fractions as different as possible. Initially, Ali applied the expanding method by multiplying both the numerator and denominator by the same integer (e.g., $\frac{2x2}{5x2} = \frac{4}{10}$) to generate equivalent fractions. He then dragged points on the ID to visually represent these fractions (e.g., $\frac{2}{5} = \frac{4}{10}$). At this stage, Ali treated the visual representation as an artifact using a diagram to check correctness rather than exploring the deeper mathematical relationships involved.

As he activated the ray representation to verify its presence in his example, he noticed that two points representing equivalent fractions always formed a straight ray. While reflecting on his example through the given statements, he attempted to create equivalent fractions where they answered the requirement of the task with as few characteristics as possible. In other words, he tried to have equivalent fractions in which this characteristic was not present, but failed. This repeated failure prompted deeper reflection, leading him to recognize that any two equivalent fractions must lie on the same ray. This realization marked his shift from procedural engagement to conceptual reasoning.

Ali internalized visual feedback, realizing that any fraction along the same ray was equivalent to the given fraction. Testing this insight, he chose $\frac{1}{4}$ and moved to another point along the same ray, confirming that fractions such as $\frac{4}{16}$ and $\frac{6}{24}$ were also equivalent. By the end of the task, Ali no longer

relied solely on numerical calculations but instead used visual representation as an instrument to identify and generate equivalent fractions.

His concluding statement, "I can move the point anywhere along this ray, and it will always give me fractions that are equivalent to the chosen fraction," demonstrates his conceptual understanding. This episode exemplifies the students' final instrumented scheme, demonstrating how visual feedback facilitates the transition from artifact-based interaction to generalizable mathematical insight.

Episode: Noa's discovery of slope and fraction value

Noa worked on the comparing fractions task using ID. She used "the same denominator" method to construct fractions, ensuring that when the numerator was greater, the fraction was also greater (e.g., $\frac{4}{5}>\frac{3}{5}$).

At the beginning of the episode, Noa interacted with visual representations of fractions and their corresponding rays, treating them as artifacts—a tool for verifying correctness, rather than exploring deeper mathematical relationships. While experimenting with tagging statements with as few characteristics as possible, she adjusted the points on the interactive diagram without a structured method. However, when she tagged statement 6, she observed that the ray representing the greater fraction was consistently positioned above the ray of the smaller fraction.

Noa began engaging in instrumentalization, shaping the tool to match her cognitive processes. She systematically placed points on rays with different slopes and compared the corresponding fractions such as of $\frac{3}{5}$ and $\frac{4}{5}$. This process led her to a fundamental insight: "The higher the slope of the ray, the greater the fraction." This marks a shift from procedural manipulation to conceptual understanding of fraction comparison using slopes.

By the end of the task, Noa confidently predicted the fraction values based on ray steepness: "If the ray is steeper, the fraction is greater." This transformation indicates that she had moved from an artifact-based interaction to an instrumented scheme, where she internalized visual representation as a conceptual tool. Her concluding statement, "I can tell which fraction is greater just by looking at the slope of the ray," exemplifies her development of a generalizable mathematical strategy.

Episode: Ameer's use of visual representation as an artifact-to-instrument shift

Ameer was assigned the task of constructing a fraction greater than another, while tagging a set of provided statements with as few characteristics as possible (Task 2). Initially, he manipulated the interactive diagram without a clear strategy, merely confirming fraction comparisons. His use of the tool was procedural, activating and deactivating statements and dragging points without systematically analyzing their effects. At this stage, visual representation serves as an artifact rather than an instrument for deeper understanding.

As Ameer continued, visual representation transitioned from simple verification to active reasoning. He tagged all statements activating visual representations, and then carefully dragged points to find counterexamples for the given characteristics. He began recognizing critical fraction properties such as the fact that the ray of the greater fraction was always positioned above the ray of the smaller fraction. He also realized that the denominator and numerator values alone do not determine the fraction magnitude, marking a key conceptual breakthrough.

Ameer refined his strategy by systematically eliminating incorrect fraction possibilities and leveraging visual feedback to construct mathematical generalizations. He realized that he could systematically find counterexamples and construct fractions that satisfied the task conditions by focusing on the slope of the rays. He pointed to the black area on the diagram (Figure 3) and explained, "The greater fractions should be in this area because the ray that crosses the O point is above the one that crosses the X point."

His final realization—that fraction comparisons can be determined by analyzing the ray position—demonstrated his shift to an instrumented scheme. His ability to generalize across multiple scenarios shows that he uses visual representations as a cognitive tool, rather than just a procedural aid.

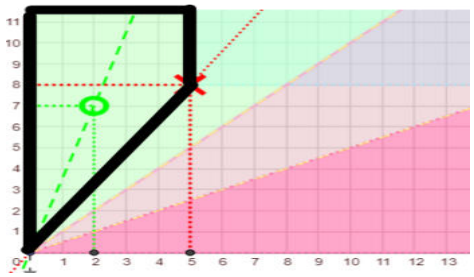


Figure 3: One possibility of the example that meets the requirement of Task 2

Table 1 summarizes the three episodes, illustrating how Ali, Noa, and Ameer transitioned from artifact-based interaction to instrumented understanding through visual feedback in Example-Eliciting Tasks (EETs).

Table 1: Summary of the three episodes

Student	Initial Engagement (Artifact Use)	Instrumentation Phase	Final Instrumented Scheme
Ali	Uses expansion method without recognizing ray significance	Notices that equivalent fractions always form a single ray	Uses rays as an instrument to generalize fraction equivalence
Noa	Uses ray steepness as a verification tool	Notices pattern between fraction values and slopes	Generalizes that steeper rays indicate greater fractions
Ameer	Tests conditions by activating/deactivating statements	Eliminates incorrect fraction properties using visual feedback	Uses systematic reasoning to construct valid fraction comparisons

Discussion

The episodes presented in this study illustrate how students transition from using visual representations as artifacts to employing them as instruments for deeper mathematical understanding. The interactive diagrams (IDs) provided not only support for procedural engagement but also guide learners toward conceptual reflection and generalization, demonstrating the critical role of feedback in promoting cognitive processes.

Our findings align with broader literature that underscores the essential role of visual representations and dynamic interactions in learning environments. Olsher and Lavie (2021) explored how dragging

in dynamic settings revealed user intentions and cognitive processes. For example, Ali initially used an ID to verify the equivalent fractions. He recognized that equivalent fractions align on the same ray through dragging points and received visual feedback. This transformation of visual representation from an artifact to an instrument allowed him to generalize his understanding of equivalence, reflecting Yerushalmy's (2005) assertion that interactive artifacts facilitate the comprehension of complex concepts by making them more manipulable.

In Noa's episode, her interaction with the ID led to an understanding of the relationship between the ray slope and fraction values. Initially focused on checking correctness, Noa's reflection on visual feedback revealed that a steeper slope indicates a greater fraction. This shift from procedural use to conceptual insight exemplifies the cognitive advances described by Arzarello et al. (2002), where dragging supports conjecture development by dynamically testing the geometric properties. Noa's ability to generalize the relationship between slope and fraction value demonstrates the educational potential of dragging, as noted by Leung and Lee (2013), in refining problem-solving strategies and uncovering geometric relationships.

Ameer's episode further illustrates this transition from artifact to instrument. Initially, Ameer used visual feedback to systematically test different fractions. As he engaged more deeply, he recognized the correlation between the slope of the ray and the fraction value, leading to a more generalized understanding of fraction relationships. This progression echoes Baccaglini-Frank and Mariotti's (2010) description of dragging as a means of conjecture making in which continuous movement aids in forming and verifying hypotheses. Ameer's refined strategy of eliminating incorrect options and narrowing choices highlights how visual feedback evolved into an instrument for logical reasoning and problem-solving.

The cognitive benefits observed in these episodes are supported by Robutti (2010) and Koedinger and Corbett (2006), who emphasized the role of visual representations in enhancing cognitive processes. The EETs in this study align with this goal by integrating visual representations designed to promote conceptual insight (Olsher & Lavie, 2021).

Furthermore, the findings of this study align with those of Baccaglini-Frank et al. (2024), who examined the transition from tools to instruments in mathematical learning environments. This study demonstrated how students develop instrumented schemes by integrating visual feedback with their conceptual understanding, reinforcing the notion that dynamic tools facilitate deeper mathematical insights.

In conclusion, this study demonstrates how dynamic visual representations, particularly when paired with dragging in EETs, facilitate a shift from procedural to conceptual understanding. The automatic visual feedback provided by IDs plays a key role in fostering generalization, allowing students to internalize mathematical concepts more effectively. This shift from artifact to instrument underscores the importance of designing learning environments that support not only interaction with mathematical objects but also reveal the cognitive processes underlying problem-solving strategies. As illustrated by the episodes involving Ali, Noa, and Ameer, these interactive artifacts can effectively guide students toward deeper mathematical insights and enhance their overall learning processes.

Note: This study and the associated writing are part of the author's Ph.D. research, titled *Automated analysis of students' interaction with elements of an assessment platform: A metacognitive perspective on comparing fractions*, conducted at the University of Haifa under the supervision of Prof. Michal Yerushalmy.

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Representational affordances of addition in the app Rechenfeld: Pitfall or potential? An object-integrating perspective on interaction

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In the past years, virtual manipulatives have moved into focus in teaching mathematics. Their programmability offers new possibilities in representing mathematical objects, but the affordances of a virtual manipulative might run the risk of turning into pitfalls. The paper aims to investigate and better understand the processes by which an affordance effects the development of an interaction and thus the resulting learning process. To reconstruct the interactional process of an exemplary sequence where a first-grade student is confronted with the affordances of the virtual manipulative in presenting an addition term, an object-integrating interaction analysis is used. The analysis reconstructs how a potential pitfall for the students' understanding of addition emerges.

Keywords: Manipulative materials, interaction process analysis, mathematical concepts, tablet computers, elementary school students.

Introduction

Manipulatives are widely used for teaching mathematics. In this paper, I adopt the definition of virtual manipulatives proposed by Moyer-Packenham and Bolyard (2016):

> an interactive, technology-enabled visual representation of a dynamic mathematical object, including all of the programmable features that allow it to be manipulated, that presents opportunities for constructing mathematical knowledge. (Moyer-Packenham & Bolyard, 2016, p. 13)

The programmer of a virtual manipulative usually wants to transport a certain mathematical idea by intentionally designing action possibilities and restrictions (Larkin et al., 2018). Such action possibilities and their relation to the agent (in this papers case, the child) are constitutive to the concept of affordance as used in this (Bulamaqui & Dong, 2014). Constraints, limitations or restrictions are regarded as affordances of a virtual manipulative, too (Tucker, 2016). If the virtual manipulative is used as intended, the representational affordances can contribute to the learners' construction of an internal representation of a mathematical object that is close to what the programmer had in mind. Tucker (2016) states that although the same affordance is accessed, the outcomes can vary among students. Thus, affordances may affect the construction of varied internal representations that not necessarily follow the programmer's idea of the mathematical object.

Affordances in terms of limitations of actions are usually meant "to restrict the students' externalizing actions to support the internalization of specific properties of the objects in consideration" (Ladel & Kortenkamp, 2013, p. 3). For example, in a study by Manches et al. (2010), students' strategies on partitioning were compared in a physical environment and in a digital environment where in the latter, restrictions to the possible actions were an integral part. Due to the restrictions, children used much more compensation strategies, whereas children in the physical condition used commutative strategies (Manches et al., 2010). The results showed that changing the action a child could perform

led to “a significant difference in how children identified solutions” (Manches et al., 2010, p. 638). Soury-Lavergne (2016) described how the limitation of actions on a digital tool, the e-pascaline, can convey a more appropriate image representation of the mathematical object than its analogue counterpart. Despite these positive experiences, we have to take a careful look at how the representational affordances are understood by the children using the virtual manipulative. That is because “virtual manipulatives always contain and reflect the experience and the knowledge of the programmer or designer” (Ladel & Kortenkamp, 2016, p. 28), but that does not certainly imply that children develop the intended ideas. Considering this, we need a better understanding of how precisely an affordance influences the mathematical learning process or – during that process – might turn into a pitfall.

The regarded data derived from a study that takes a closer look at the interaction of a child, a teacher and a virtual manipulative. The focus of the study is to examine the interactional roles of these participants especially in moments where confusion is interactionally displayed, regarding how mathematical meaning is negotiated in the interactional process. Special to this study is the perspective on the virtual manipulative not as a material, but as a participant in the interaction. In this article, we will take a look at an exemplary scene of a student interacting with a virtual manipulative and its intended affordances. The scene serves as an example of a course of interaction between a student and a virtual manipulative where a moment of discontent was indicated by the student. Since this interactional phenomenon occurred similarly in several of the conducted interviews, it requested closer examination.

Rechenfeld (Urff, 2024) is a tool to foster understanding of the operations of addition and subtraction from first to third grade. It allows many possible actions, but due to this papers’ focus, only those relevant to the analysis will be introduced. As the mathematical object, which refers to the “concept, content or mathematical process […] targeted by the app” (Larkin et al., 2018, p. 68) in the regarded sequence is addition up to twenty, the surface’s settings are defined accordingly. By touch-gestures tapping the corresponding area, the child can make the virtual manipulative place red (left hand side, representing the summand) or blue (right hand side, representing the addend) counters in a 20-frame. It is possible to either place one, five, or ten counters at the same time. Whilst the counters are being placed, the place value of the corresponding number in the addition term that is represented below the field changes linked simultaneously.

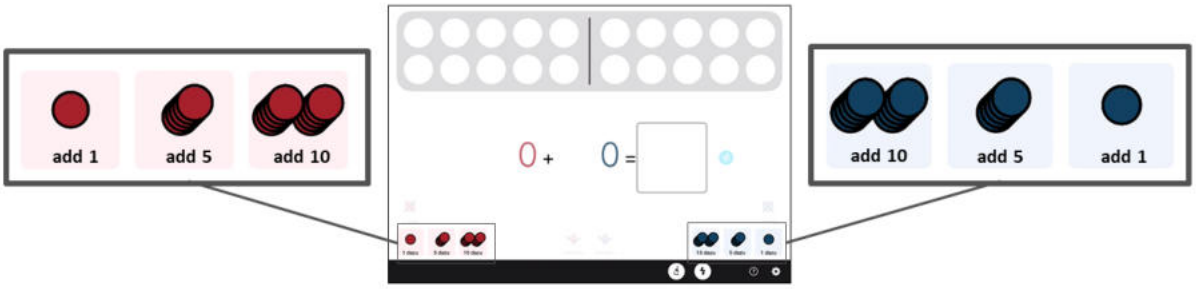


Figure 1: Screenshot of the virtual manipulative Rechenfeld

As the summand corresponds to the quantity of red and the addend to the quantity of blue counters, Rechenfeld only understands addition represented by two quantities of different colours as an

addition. This affordance is intended by the programmer: the virtual manipulative is programmed to only show the requested symbolic addition term if it's generated through the corresponding tapping of blue *and* red counters. This issue is going to be the topic of the analysed sequence.

Theoretical Framework

The exemplary sequence will be regarded from an interactionist perspective on learning (Schütte et al., 2019). This is a constructivist perspective aiming to describe how learning takes place in interaction. Deriving from symbolic interactionism, the collective negotiation of meaning is seen as constitutive part of the learning process (Schütte et al., 2019). Following the thematic development of the interaction, one can gain information about how the individual notion of the mathematical object – which is the mathematical idea in question, e.g., addition – is being constructed by the actors during the interaction. The mathematical object itself does not contain meaning, but the meaning has to be given to it by the individuals. "According to the interactionist perspective, individual processes of interpretation are developed, transformed and stabilised in the exchange with other individuals" (Schütte et al., 2019, p. 105). Mathematical learning – in contrast to the use or reproduction of existing knowledge – from an interactionist perspective takes place during the ongoing negotiation of meaning in an interaction (Schütte et al., 2019).

Because in this paper the virtual manipulative plays a central role in the process of negotiation of mathematical meaning, it is of interest to examine the role of the virtual manipulative in the interaction in close detail. Latour (2005) offers a sociologic perspective on objects (by object, Latour here refers to e.g., the virtual manipulative, not to the mathematical object as described above) by stating that "objects too have agency" (Latour, 2005, p. 63) and can actively participate in an ongoing interaction by offering for example (new) thematical contributions to a process or by taking over parts of an argument (Fetzer, 2013). Fetzer (2013) incorporates this idea to "follow the actor" – the object-actor – by focusing on its actions and roles in the interaction process. Therefore, Fetzer combines Latour's sociology of objects with the interactional theory. Thus, the virtual manipulative is no longer only talked about and used for argumentation, but it is regarded as participant in the interaction. Through its representational affordances, the virtual manipulative (partly) transports the programmer's construction of meaning of the mathematical object.

Methods

The regarded sequence derived from an interview study in which an interviewer conducted semi-structured clinical interviews. During the 30-minute interview, the first-graders worked with the virtual manipulative *Rechenfeld* (Urff, 2024) on a tablet device. The children were asked to solve addition tasks (e.g., 5+1, 2+6, 10+5) using the virtual manipulative and to explain their actions and thoughts.

The theory presented above implies the methodological approach of the interaction analysis, which is used to analyse an exemplary sequence. The aim of the interaction analysis is to investigate how and at what moments the virtual manipulative's (restrictive) representational affordances come into play and how children react and interact with them. Thereby, I want to investigate how an affordance can affect the interactional process of negotiating mathematical meaning.

The interaction analysis is combined with the object-integrating perspective (Fetzer, 2013) to closely follow the actors through the interaction process. The turn-by-turn interaction analysis is a micro-sociological, sequence-analytical method that stems from the tradition of interpretative social research (Schütte et al., 2019). Turns can be actions of the student, the interviewer, or the virtual manipulative – or two of them taking a combined turn. Every single action is interpreted as a turn to previous actions with the analyst generating as many alternative interpretations as possible. The interpretations are reviewed during the following interaction with the final reconstruction being presented in the results section. The object-integrating interaction analysis (Fetzer, 2013; Fetzer & Bräuer, 2023) based on Latour (2005) regards not only humans, but also objects as actors (Fetzer & Tiedemann, 2018):

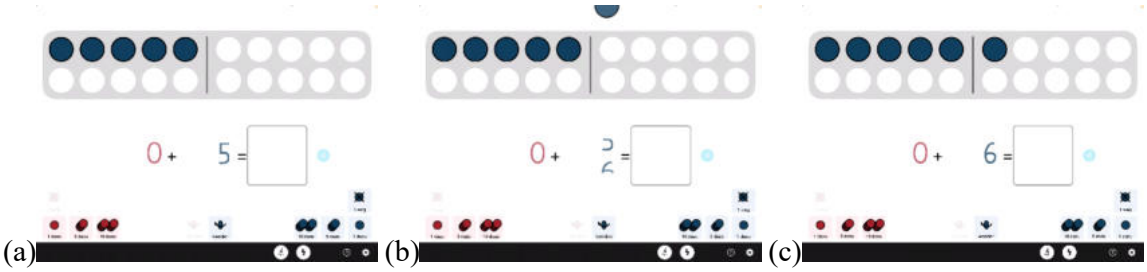
> An object's opportunity to become an active participant in the course of action is strongly connected to human-actors' interpretations and perceptions. It is only in moments of networking, that their acting becomes observable. When inter-acting with other actors, objects traces render perceivable. Accordingly, the basic idea of analysis is to grasp the acting of objects indirectly. (Fetzer, 2013, p. 7)

To investigate the object's action, specifically the turns following on an object's turn are relevant. The response of the student to the object's actions provides information about how the object participates in the interaction process (Fetzer, 2013).

Results

The following transcript is translated from German to English. Actions are transcribed in italics. Figure 1 and Figure 2 illustrate the virtual manipulative's turns.

Interviewer:	How would you calculate 5+1 with it?
Tablet:	[Figure 1]
Student + Tablet:	So without dots? *Points with one hand over the 20-frame [Figure 1].*
Interviewer:	No, with dots.
Student + Tablet:	Then I have to write the task down there. *Taps add 5 blue [Figure 1].*
Tablet:	*5 blue counters slide from the top into the 20-frame. The addend changes to 5.*
Student + Tablet:	*Looks at the surface [Figure 2a]. Taps add 1 blue.*
Tablet:	*One blue counter slides into the 20-frame [Figure 2b]. The addend changes to 6 [Figure 2c].*
Student + Tablet:	*Looks at the surface [Figure 2c].* Aaaah bother! Bumms. I did it wrong. Wait- *Taps the broom.*



(a) (b) (c)

Figure 2: Three screenshots (a), (b), (c) of Rechenfeld

The presented scene took place right after an explorative period in which the student discovered the functions of the different touch areas. The student was asked to "calculate 5+1 with it", by which the

interviewer referred to the virtual manipulative presented on the tablet screen. The virtual manipulative offered the surface (Figure 1) including the affordances the student had discovered so far: tapping on the touch-areas results in the virtual manipulative placing a certain number of counters in the 20-frame and changing summand and addend accordingly. The student reacted to the prior turns by asking “So, without dots?”, pointing over the 20-frame. The interviewer answered that she wanted her to use the “dots”, picking up the wording used by the student. By that, she likely moved the students focus onto the virtual manipulative’s iconic representation. The student then assumed that she had to perform the task “there”. Right afterwards, she starts tapping the add-5-blue area, so she begins to work on the first step of the task. The tablet, as programmed, places 5 blue counters in the 20-frame as an iconic offer and changes the addend of the term from 0 to 5 as a symbolic offer. The student looks at the surface and proceeds with another action. That indicates that she is satisfied with the offer(s) she got from the tablet. Now it would be interesting to know if she checked for the offered iconic representation of the counters in the 20-field, for the offered symbolic representation or for both. It can be assumed that she looks at the iconic representation out of two reasons. Firstly, she was requested to do so. Secondly, the symbolic representation – if regarded and negotiated – could already have provided her with an indication that the virtual manipulative was not going to represent her input in the manner she had anticipated: not the summand altered – as she might have expected –, but the addend did. She continues by tapping the add 1 blue area, presumably to perform the addend. The virtual manipulative answers by sliding one blue counter into the 20-frame, where then, in total, six blue counters are presented as an iconic offer. At the same time, the addend of the symbolic term changes from 5 to 6 and thereby offers a change to the symbolic term that can be negotiated. It is possible that offer containing the animation of the counter sliding into the 20-frame still satisfied the child (as well as the total amount of 6 counters, assuming the student had already solved the task). Also, the six counters presented may be regarded as appropriate. As the animation and the total amount may align with the child’s expectations, it is unlikely that they would cause her irritation. Therefore, it can be inferred that her negotiation of the symbolic term is the source of her discontent. The student looks at the surface, followed by an articulation of dissatisfaction: “Aaaah bother! Bumms” and a verbalization of acknowledgement of herself having done something wrong. By “wait”, she presumably wants to stop the interviewer from interfering. As she moves on to the next action, she already has an idea of how to proceed, which she probably got from the virtual manipulative’s offer.

Discussion: potential and pitfall?

Following the interaction, it is obvious that despite the virtual manipulative’s different offers (enactive: animation of counters, iconic: counters in 20-field, symbolic: addition term), the student’s attention is directed towards a particular form, in this case, the iconic one. Turn by turn, the virtual manipulative reveals its affordances (e. g. counters are placed in the 20-field by tapping, the different units of the additional term are informed by the amount of counters, the addend is informed by the amount of blue counters, placing further blue counters increases numerical value of the addend corresponding to the amount of counters placed, no change occurs to the summands numerical value if no red counter is placed) in presenting an addition term, but it takes the student until the last turn to realize them. The role of the virtual manipulative that can be reconstructed in the interactional

process changes significantly due to how the child reads its offers. Indeed, as long as the child seems to be focused on the iconic representation, the virtual manipulative's role is to offer an iconic representation. This offer suits the child's expectations enough, so the interactional process continues until the point where the virtual manipulative's symbolic offer (addition term) is considered by the child and the interaction is disrupted. This emphasises that the manipulative's offers only change the interaction when they do not correspond to the student's expectations. The student does not seem to read the symbolic term as an offer to interpret but rather as a correction to her own actions. This becomes evident when the student instantly seems to abandon the previous interpretation (placing five counters first and one counter afterwards corresponds to the term 5+1) as soon as she regards the symbolic term (0+6). There is no evidence that a discursive negotiation process took place, instead, the virtual manipulative's offer is accepted as right while her own process is exposed as wrong in contrast. Accordingly, the student directly assumes that she was the one who made a mistake. This may be attributed to experiences that convey the notion that digital devices are invariably accurate. Another possible explanation might be coming from previous experiences implying that digital devices cannot be corrected by the user. This hinders the student from questioning the virtual manipulative's offers – what can make certain affordances of virtual manipulatives even more powerful.

Of interest is that the virtual manipulative's affordances give rise to the assumption that there is only one single "correct" way to present an addition term: using red counters for the first and blue counters for the addend. Through its offers, the virtual manipulative gives the child clues to its intended affordance. Following the interaction, the student in the end might not only have understood how to use the virtual manipulative properly to solve the given task, but the interaction might also have a narrowing influence on her (former) construct of addition. Solving the task, the student shows a concept of addition that contains the notion that the total of identical units represents one summand, no matter if they appear simultaneously or one after the other. When she regarded the virtual manipulatives offer, her solution process was no longer negotiated as 5+1, it was considered "wrong" for representing 5+1. The conflicting experience made in the sequence can have an influence on her construct of addition. Whether or not that experience is relevant for the student depends on various factors, one of which is the stability of her previous experiences.

In this example, we can see that the virtual manipulative has the potential to disrupt the current interactional process. The meaning of addition seems defined in a sufficiently aligned way by all participants for the first eight turns. When the virtual manipulative's offer does not suit the student's construction anymore, the student starts to attune her construction of meaning to an extent that happens to satisfy all participants of the interaction. Because such processes of negotiating or attuning of meaning from an interactional perspective are "constituent of learning processes" (Schütte et al., 2019, p. 103), it becomes inevitable to consider that the virtual manipulative not only disrupted the current interactional process, but also the student's construction of the mathematical object in question. By trying to negotiate a meaning that is suitable to the virtual manipulative's intended affordances as well as the student's ideas, as the virtual manipulative cannot adjust its offers, the student needed to adjust her notion to the offer. To maintain a functioning interaction afterwards, the student needs to align her actions to the affordances of the virtual manipulative – from now on,

representing an addition term by using red and blue counters instead of only blue ones – and thus, she might also adjust her notions: notion is being modified during the processes of negotiation of mathematical meaning in the interaction (Schütte et al., 2019).

Concluding remarks

Affordances, as we learned before, might hold the potential to guide student's ways of doing math. However, if left unaddressed, they may also lead to an unreflective acceptance of an intended affordance as part of a notion of a mathematical object, applicable in a broader range of contexts. In that, the affordance effecting the present interaction might also influence further situations: if the child activates this subjective domain of experience, containing their modified notion of addition, and therefore struggles to mathematically make sense of a new mathematical situation, the learning process might be disrupted.

This example provides insight into the importance of regarding the role of the virtual manipulative in the interaction, especially how its affordances come to bear during the interaction of offers from the virtual manipulative and how they are interpreted by the other participants. In order to take a closer look at the probably emerging pitfall, the object-integrating interaction analysis was conducted. By retracing the interaction of the subject and the virtual manipulative, including especially the intended affordances that the virtual manipulative consists of, it becomes visible how intended affordances can guide and narrow students' actions, meaning making processes and thus, can influence the student's learning of mathematics.

As "differences in actions resulting from the representational medium may lead to differences in the ideas fostered" (Manches et al., 2010, p. 624), we have to start following those actions and how they seem to be perceived by the child. Teachers have to be aware of virtual manipulatives' representational affordances – with their possibly structuring and/or restricting properties – and their potential pitfalls' influences on learning processes and take them into account when teaching.

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Interpretation and representation of fractions in problem solving

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This exploratory study investigates the dominant conceptions of fractions in problem-solving contexts using various interpretations and representations of fractions. For this purpose, we proposed six problems for pre-service teachers and students to solve. We examine the strategies and the representational registers that they use to solve these problems. The results show that pre-service teachers tend to use solving strategies that align closely with the anticipated interpretation of the problem, while students tend to rely more on the part-whole strategies. We demonstrate that several conceptions coexist, but the part-whole interpretation is the dominant conception among participants. Students mainly use symbolic representations.

Keywords: Fractions, dominant conception, problem solving, representation, mathematics education.

Introduction and conceptual framework

Teaching and learning fractions are considered one of the most challenging areas of primary school mathematics. Several studies have suggested that one of the main factors contributing to this complexity is that understanding fractions requires grasping multiple interpretations and representations (Behr et al., 1983; Charalambous & Pitta-Pantazi, 2007). Behr et al. (1983) identified five interpretations of fractions. The part-whole interpretation expresses the relationship between a whole divided into equal parts and one or more of those parts. The measure interpretation expresses the measure of a quantity in terms of a unit of measurement. The operator interpretation operates on a set in a multiplicative way (increasing, decreasing). The quotient interpretation expresses the result of a division. The ratio interpretation expresses the relationship between two quantities of the same or different nature. Authors suggest that these interpretations can be associated with operations (for example, addition related to measure), equivalence (related to ratio), and problem-solving. If some students do not recognise or understand a particular interpretation, they may have difficulty in solving specific problems (Charalambous & Pitta-Pantazi, 2007). Furthermore, fractions can be represented with a variety of signs depending on the context, such as words, numbers, or geometric shapes. These signs are not the mathematical object itself; rather, they are representations of mathematical objects, facilitating its accessibility (Duval, 2006). Based on the research of Duval (2006), four types of representation registers are identified: the symbolic, the verbal, the visual, and the number line. The symbolic register includes symbolic notation and writing (e.g., $\frac{1}{2}$, which in the symbolic register can also be 5/10 (fractional writing), 0.5 (decimal writing), or $1 \div 2$ (divisional writing), or 50%). The verbal register uses natural language (e.g., “half”, “one out of two”). The visual register includes drawings and figures (e.g., a disc is divided into two equal parts, and one part is taken). The number line register is a point on the number line (e.g., for ½, the point exactly between 0 and 1).

In the field of mathematics education, the concept of *conception* emerged from empirical work highlighting the stability of student errors that hinder learning and focusing on an individual's

construction of concepts. Brousseau (1987) points out that a conception is an instance of the student's knowledge in a situation, identified by a dual approach: an epistemological analysis of the concept and a study of the individual's knowledge of the concept. He gives examples of conceptions of fractions (measure and ratio) and shows that they can coexist. In cognitive and developmental psychology, a conception is a set of principles and mental representations that guide the interpretation of concepts and individual behaviour. Conceptions allow for the prediction of individual behaviours. They are qualified as intuitive when they are based on prior knowledge and persist after learning at school (Fischbein et al., 1985; Gvozdic & Sander, 2018). Teachers remain influenced by their conceptions and may interpret students' errors in this way (e.g., for conceptions of division, Tirosh & Graeber, 1991).

The current study

Research shows that some conceptions resist instruction and continue to influence individuals, even after the concept has been formally taught in school (for a review, see Gvozdic and Sander, 2018). Studies suggest that people rely on these conceptions to build their knowledge (Brousseau, 1987), which is particularly relevant in the understanding of fractions (Lemrich et al., 2023). Conceptions provide a means of predicting and explaining errors, particularly those related to the whole numbers bias (Ni & Zhou, 2005). In this study, the individuals' conceptions are characterised by the mobilisation of interpretations that can be related to problem-solving strategies (Charalambous & Pitta-Pantazi, 2007). We refine some of our results by also considering representations, which facilitate the accessibility of an aspect of the mathematical object (Duval, 2006).

At CERME13, in TWG24 (Lemrich et al., 2023), our aim was to identify the intuitive conceptions of pre-service teachers using an open-ended questionnaire. We showed that the intuitive conceptions of pre-service teachers were the part-whole interpretation. We have also assumed that the quotient interpretation is a conception most likely related to teaching fractions. Overall the symbolic and visual registers were the most used. In this research, we are interested in extending these initial results by exploring dominant conceptions in problem solving and contrasting pre-service teachers' and students' conceptions. Research has already examined the success of teachers and students when they are asked to solve fraction problems linked to a specific interpretation (e.g., Charalambous & Pitta-Pantazi, 2007; Lamon, 2006). However, to our knowledge, no study has ever shown, by analysing solving strategies, that a fractions problem may indeed be tied to a specific interpretation and that some solving strategies are dominant in several contexts. Firstly, we aim to confirm the link between a problem and a specific interpretation. For that purpose, we identify a priori a specific interpretation for each problem and then compare it with actual solving problems strategies of participants. We assume that adults, who are more experienced than students, should use the most expert and direct solving strategies that are close to the identified interpretation. Secondly, we want to find out whether there is a dominant conception from which problem-solving strategies are derived. Building on previous research (Charalambous & Pitta-Pantazi, 2007; Lemrich et al., 2023), we hypothesise that although some adults may use part-whole strategies in some contexts, students are more likely to mobilise them systematically to solve problems, as well as symbolic and visual (circular) registers.

Methods

We collected data from 206 pre-service teachers and 69 sixth graders (average age 11) during the 2023-2024 academic year. The pre-service teachers were in university training to obtain tenure in the French-speaking part of Switzerland for either primary (103) or secondary (103) schools. Sixth graders were studying fractions for the second year.

We created a questionnaire consisting of six problems to be solved (see Figure 1). Each problem was selected based on its proximity to one of the fraction interpretations, as suggested by previous works (e.g., Charalambous & Pitta-Pantazi, 2007; Lamon, 2006). For each problem, we chose fractions that were sufficiently familiar to students, allowing them to represent them in different ways. We also varied the contexts. For the interpretation of quotients, we chose to present the same problem twice, changing the size of the fraction and using one proper and one improper fraction. In the given material, all fractions are written in the form $\frac{a}{b}$.

Part-whole	Andrea celebrates his/her birthday, his/her cake is cut in 12 parts, Andrea eats 5 of them. What fraction of the cake was eaten?
Measure	I'm taking part in a bike race. Each stage represents 1/4 of the race. How many stages must I complete to reach 3/4 of the race?
Operator	There are 120 children in the school playground, and 3/4 of them are playing ball. How many children are playing ball?
Quotient	There are 5 people and 6 chocolate bars. Each person receives the same quantity of chocolate. How much will each person get? And the same problem with 8 people and 3 chocolate bars.
Ratio	A chocolate drink is made by mixing chocolate and milk. Alix's drink has 2 parts chocolate to 3 parts milk. Charlie's drink has 1 part chocolate to 2 parts milk. Who has the most chocolatey drink?

Figure 1: Problems proposed to participants linked to interpretation

For each problem, we defined a priori solving strategies close to each interpretation (between two and four, see for example Figure 2). To determine which strategy is closest to which interpretation, we referred to research (e.g., Behr et al., 1983; Charalambous & Pitta-Pantazi, 2007). Part-whole strategies involve dividing a whole into a certain number of parts and taking a specific quantity. Measure strategies involve considering the size of a part as a subunit and repeating it a certain number of times. Quotient strategies involve dividing the number of units by a given number. Operator strategies involve applying a fraction to a continuous or discrete whole. Ratio strategies involve determining the relationship between two distinct quantities.

For the *part-whole* problem, we defined 3 a priori solving strategies.	
Part-whole	The cake is cut into 12 slices, so I take 5 slices to make 5/12.
Measure	The cake is cut into 12 pieces, each piece is 1/12. 5 slices have been eaten, which means 5×1/12=5/12.
Quotient	I do 5 divided by 12, which means 5/12.
For the *operator* problem, we defined 4 a priori solving strategies.	
Part-whole	I split the 120 children into 4 groups. That's 30 kids per group. And there are 3 groups of children playing ball, making 90 children.
Measure	3/4 is 3×1/4, 1/4 of 120 is 120/4=30, 3/4 is therefore 3×30=90
Quotient	I do 3/4 × 120=90
Operator	I divide 3 by 4 and multiply by 120, which means (3÷4)×120=90.

Figure 2: Example of solving strategies linked to interpretation proposed to participants

To collect data, pre-service teachers received problem-solving tasks in the form of multiple-choice questions. For each problem, they first solved the problem and gave the answer, then chose the solving strategy (see Figure 2) that was closest to their own. This step was necessary to check that they were answering the problems correctly and to understand the strategies used. Given the simplicity of the problems, we opted for multiple-choice questions, assuming that they would provide an answer without detailing their strategy. The students solved the problems using a pen and a paper questionnaire. They were explicitly asked to document their strategy, which gave access to the representations and strategies used to characterise the conceptions.

With the aim of avoiding that the order of the problems would influence the results, the problems were proposed in four different orders, taking care not to pair the two quotient problems too close together. For each problem type and for each participant, we first checked whether the problems were solved correctly (correct, incorrect, no answer). If the answer was correct, we evaluated the strategy used. For the pre-service teachers, because of the multiple-choice format, the information was in a spreadsheet. For the students, we used MaxQDA software to code the responses and then extracted the data into a spreadsheet. As they did not respond to a multiple-choice test, we had to evaluate correctness and score the strategies they used. These strategies were sometimes the same as those found in the a priori analysis, so they were coded using the same indicators (i.e. strategies related to interpretation) as the pre-service teachers. Students sometimes left no trace of their solving strategy. In these cases we coded their answer in the category without justification. Occasionally, students used strategies that we had not considered in our a priori analysis. For example, some students solved the 3/8 quotient problem by adding an arbitrary (but relevant) number of squares to each bar. They managed to solve the problem by distributing the squares to each individual tree. These strategies were coded in the category other.

To analyse the representation registers mobilised by the students for each strategy, we used four registers: the symbolic, the verbal, the visual and the number line registers. For each problem, we coded the registers used among the correct answers. We did not include incorrect answers in our coding process because the remnants left by the students were not consistently usable (partially crossed out or erased writing, incomplete drawings or writing, etc.). We did not have access to the representations for the pre-service teachers through the multiple-choice questions.

Results

To identify the most successful problems, we first looked at those that were solved correctly (see Table 1, the percentages therefore reflect the correct answers). Among the pre-service teachers, the most successful problems are part-whole (91%), measure (95%) and operator (92%). The same sucess rates hold for students, although they are slightly less successful on the operator problem (37%). Participants are less successful overall on quotient and ratio problems. Their success rate is close to 70% for pre-service teachers and 30% for students.

To confirm whether a problem could be associated with a particular interpretation, we categorised the strategies used by the pre-service teacher (see Table 2). For all the problems (except for the operator), the pre-service teachers tended to use the strategies closest to the type of problem proposed (see Table 2, grey cells). However, for each problem, a few pre-service teachers also used part-whole

as a solving strategy. Other procedures closely related to other interpretations were also used, but to a much lesser extent.

Table 1: Percentage of correct answers by problem type and participants group

	Part-whole	Measure	Operator	Quotient 6/5	Quotient 3/8	Ratio
Pre-service teachers (206)	91%	95%	92%	75%	73%	71%
6th Grade students (69)	61%	67%	37%	31%	26%	26%

Table 2: Percentage of response strategies mobilised, for each problem, for the pre-service teachers (206), among correct answers

	Strategies close to					Other procedures
	part-whole	measure	operator	quotient	ratio	
Part-whole	73%	18%		9%		
Measure	45%	55%				
Operator	41%	24%	30%	3%		
Quotient 6/5	5%	13%		83%		
Quotient 3/8	9%			91%		
Ratio	20%				68%	12%

In order to determine which conceptions were most frequently used by the students for each problem type (see Table 3), we categorised the strategies used by the students. For all problems (except ratio), students used the part-whole strategy more often than any other possible strategy, which they hardly used at all (see Table 3, grey cells).

Table 3: Percentage of response strategies mobilised, for each problem, for the 6th grade students (69) among correct answers

	Strategies close to					Other procedures	Without justification
	part-whole	measure	operator	quotient	ratio		
Part-whole	52%						48%
Measure	24%	9%				2%	65%
Operator	73%	8%		4%			15%
Quotient 6/5	50%	5%		45%			
Quotient 3/8	39%			22%		28%	11%
Ratio	28%				28%	6%	39%

To examine the representation registers used by the students for each solving strategy, we used four registers: the symbolic, the verbal, the visual and the number line registers. Symbolic register is the most used of all the strategy. In the case of the part-whole strategy, 60% of the students used at least one symbolic register, 31% used the visual register, 8% used the verbal register and only one student used the number line. For the quotient strategy, most students (94%) use symbolic register and the remaining 6% use verbal register. For the measure strategy, 83% of the students use the symbolic register and 17% use the verbal register. For the ratio strategy, 63% of the students use the symbolic register and 38% the verbal register. The operator strategy is never used by the students.

Discussion

Our first aim was to confirm the link between a problem and a specific interpretation based on the solving strategies used. Our results show that pre-service teachers were more likely to use solving strategies that were closest to the type of problem proposed. This confirms that the proposed problems are aligned with the expected interpretation. Therefore, it confirms that the participants interpreted some problems more closely to some interpretations than others, without ignoring the fact that they can be solved in several ways, using several interpretations. Our second aim was to find out whether there is a dominant conception from which problem-solving strategies are derived. Although most pre-service teachers use the solving strategy close to the interpretation identified, some of them used the part-whole solving strategy, indicating that this interpretation is widespread, despite the possibility of using more expert and direct strategies. On the contrary, students make more use of the part-whole strategy for all problems, compared to the other strategies, which they hardly use at all. Ratio and quotient strategies are not yet part of the curriculum, but students should already have encountered the operator and measure strategies. These results show that the use of solving strategies derived from the part-whole interpretation is the most common for both pre-service teachers and students, used in all situations. This interpretation manifests itself in different situations, it persists for most participants, hence the dominant aspect. Consequently, we characterise the use of the part-whole interpretation as a dominant conception. To check whether this knowledge is acquired outside the classroom and whether it is also an intuitive conception, it would be necessary to interview students who have not yet been taught fractions. This study is currently in progress.

To refine our findings and enhance the characterisation of the conception, we aimed to identify the representations used to guide the students problem-solving processes. The results suggest that regardless of the strategy used, they tend to rely more on the symbolic register than on the visual register. We assume that the students, unused to solving problems with fractions, thought that they had to approach these problems in the same way as with whole numbers, using operations and the symbolic register. The representations available to them are then fractional and decimal writing, which they have just discovered in their curriculum (introduced in 5th grade in the Swiss curriculum). We hypothesised that the visual register would be mobilised in students' answers for all strategies. However, this hypothesis was not verified. We assume that some representations are more spontaneous than others, but their use is strongly influenced by the context of solving problems and that further studies (including adults) are needed to verify these results.

For this reason, and in view of the results of previous study (Charalambous & Pitta-Pantazi, 2007; Lemrich et al., 2023), we qualify the dominant conception of fractions as part-whole interpretation with symbolic or visual representation. Indeed, while the dominant conception is associated with the part-whole interpretation, the other conceptions associated with other interpretations should not be overlooked in teaching with an aim to promote a deeper mastery of the concept of fraction. According to Brousseau (1987), the different conceptions must coexist and be articulated to form the conceptual field of the mathematical concept. This highlights the complexity and richness of mathematical understanding. For a comprehensive understanding of the representation used, it would be interesting to review all the tasks of the previous study (Lemrich et al., 2023) on the use of representational registers, with the aim of integrating its findings with the present study. To this end, in a forthcoming study, we will combine the two questionnaires in order to obtain a better characterisation of conceptions through spontaneous responses and problem solving, with adults and students who have or have not had previous instruction. In this way, the intuitive aspect of the conceptions can be examined.

While the results of this study provide valuable insights into dominant conceptions of fractions in problem-solving contexts, it is important to acknowledge a few limitations that may affect the generalisation and interpretation of the results. Firstly, the problems proposed in the questionnaire are not of the same nature. For the operator and measure the fraction is given and the participants must operate with it. The other problems (part-whole, ratio and quotient) consist of finding the fractions involved. This might explain why the operator and measure problems are almost as successful as the part-whole problem for pre-service teachers. In addition, the choice of numbers also makes it easier to solve the problem using alternative (e.g. for the quotient, adding an arbitrary but relevant number of pizza slices) or erroneous (e.g. for the ratio, referring only to the quantity of chocolate and not taking the quantity of milk into account) strategies to the one derived from the hypothesised interpretation, which may explain the greater success with these problems. Secondly, the difference in the method of data collection between the two groups of participants may influence the comparison. Although the participants solved the same problems, the teachers used a multiple-choice format while the students worked on paper and pencil. The criteria for defining the students' strategies were based on those proposed to the teachers, which were based on an a priori analysis of possible solutions. We considered the changes necessary in order to access the strategies used by the pre-service teachers, but any change of format might imply a bias in the comparison of the groups and affect our results.

Conclusion

For each problems, this study has shown that for adults, with the exception of the operator, the solving strategies employed were indeed consistent with the intended interpretation. The results also highlighted the prominence of the part-whole interpretation, from which the most common solving strategy for solving these problems is derived. Some of the problems proposed were close to the part-whole interpretation, others were further distant. Given this and the persistent use of this strategy over time, this suggests that the dominant conception of fractions is the part-whole one. Other ones, which could be described as secondary, coexist and need to be mastered, but are not the most spontaneously mobilised. A question then arises: what are the most promising ways of intervening in the classroom?

Given the strong anchoring of conceptions in various contexts, building on them seems inevitable. Therefore, students need to be guided on how to evolve from one interpretation to another in order to avoid an overemphasis on the part-whole strategy and to enhance the capacity to adaptively switch from an interpretation to another when such a switch makes a strategy more adequate than another (Scheibling-Sève et al., 2022). This work will be carried out in a pilot study in a primary school.

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From multiple discourses to algebraic discourse

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This study is framed within a commognitive perspective and is based on the theoretical stance that subsuming is a key discursive process for relating different representations, providing an alternative way of learning compared to memorizing meaningless procedures. We designed activities in digital learning environments to promote this subsuming process in the context of school algebra. We analyze the story of Aldo and Giulio, two high school students, showing how this process can feature and how algebraic discourse can emerge as a possible subsuming discourse.

Keywords: Algebraic discourse, commognition, digital learning environments, equations.

Introduction and theoretical background

A large body of research into school algebra has highlighted several key challenges and strategies for improving teaching and learning. Classical studies in mathematics education show that students frequently rely on memorized procedures and this results in well documented problematics as a procedural view of the equal sign or difficulties with the letter-symbolic form of equations (e.g., Carpenter et al., 2005; Kieran, 2022). In a different perspective, Kieran and Drijvers (2006), along with Nachlieli and Tabach (2012), emphasize the importance of learners coming to consider different mathematical representations as being the "same" object. An implication for task design is to use multiple representations to make algebraic objects such as variables, expressions, and equations more meaningful to students. In this study, we draw on these two strands of research that we see as related, since a way to give meaning to the memorized procedures might be to associate the involved objects with multiple representations and relationships among them. Within the broader project this study is situated in (DynaMat), multiple dynamic explorative objects (DEOs) are designed for representing the relationships between two algebraic expressions of the same variable. The students participating in the project are involved in sequences of activities requiring for an explorative participation with these DEOs. We embrace the commognitive lens (Sfard, 2008) for analyzing students' interactions with the DEOs, and, through this analysis gaining insights into their learning of school algebra. This theory has proven effective in studying communication processes at different school levels (e.g., see, the special issue by Cooper and Kontorovich, 2021).

The processes of saming and subsuming in the Theory of Commognition

According to the Theory of Commognition, mathematical objects are discursive objects and this leads to mathematical communication involving continuous transitions from *signifiers* to other entities that are called *realizations*. In particular, "signifiers are words or symbols that function as nouns in utterances of discourse participants whereas the term realization of a signifier S refers to a perceptually accessible object that may be operated upon in the attempt to produce or substantiate narratives about S" (Sfard, 2008, p. 154). For example, the word 'equation' can have the role of signifier in a discourse and its possible realization can be a symbolic writing as $6+x=2x+1$ or a graph

of two intersecting lines. A peculiarity of mathematical signifiers is that each of them has several possible realizations such that every *endorsed narrative* - i.e. every narrative labeled as true by the community of discursants (Sfard, 2008, p. 134) - about a signifier can be translated into an endorsed narrative about one of its realizations. In these terms, the learning process may be expressed as the creation of new signifiers by collapsing different realizations into one, that may give rise to the creation of new mathematical objects. An important role in this complex process is played by the discursive act of *saming* which is described by Sfard (2008, p.170) as "assigning [the signifier] to a number of things that, so far, have not been considered as in any way 'the same' but are mutually replaceable in a certain closed set of narratives". An example of saming can be seen when an algebraic expression, e.g. $6+x=2x+1$, and a scale in balance are considered as realization of the 'same object' and they are all-in-one addressed by the word 'equation' (Weingarden et al., 2024).

Commognition describes the development of mathematical discourse as a recursive expansion in which new discourses integrate previous ones, either by merging familiar discourses or by creating a meta-discourse that encompasses older ones. More specifically, Sfard (2008) defines *subsuming* the process through which existing discourses are incorporated into newly developed discourses and it is possible to distinguish between two situations. The first one occurs when a discourse is subsumed by its meta-discourse. For example, arithmetic discourse was subsumed centuries ago by the algebraic one, which in commognition is seen as "metaarithmetic, or more precisely, as the unification of arithmetic with its own metadiscourse" (Sfard, 2008, p. 120), that is the mathematical discourse on arithmetical relations and processes (Caspi & Sfard, 2012). The second occurs when two discourses are subsumed by a discourse containing an isomorphic reflection of these two discourses. For example, the discourse on algebraic expressions and the discourse on graphs were both subsumed, during the age of Descartes, by the discourse on functions in which an algebraic expression is an isomorphic reflection of a graph. The echoing of historical processes in students' individualization of mathematics discourses was described in a study by Caspi and Sfard (2012), which found that students' formal algebraic discourse was built on the foundations of their informal meta-arithmetic discourse. In the next section we describe how this theoretical lens informed our study and offered insights into students' integration of separate discourses into a cohesive mathematical understanding.

Discourse about balancing mobiles and discourse about moving arrows

Within the DynaMat project we developed DEOs as realizations of mathematical equations and inequalities, both designed in the software GeoGebra. We realized the relationships between two expressions of the same variable in terms of changing weights of the two sides of a mobile and of reciprocal movements of moving arrows (examples are described in Lisarelli et al., 2024; Weingarden et al., 2024). We have discussed in the previous section how different branches of discourse can be seen as subsuming each other. In our study, we initially supported students' development of a discourse about balancing mobiles and of a discourse about moving arrows by engaging them in activities involving separately the two different realizations. Then, one heuristic we used was to design task situations that cannot be fully addressed in just one of these discourses. For instance, we designed a DEO in which a mobile and some moving arrows are simultaneously embedded and connected (Figure 1).

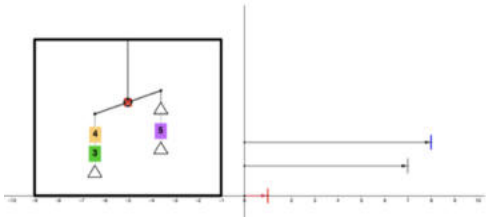


Figure 1: Screenshot of the DEO embedding a balancing mobile and moving arrows

The value of the white triangles in the mobile corresponds to the length of the red arrow below on the right of the screen. The mobile hangs to the side in which the total value of the weights is bigger or, in the case both sides have the same value, it stays balanced. On the right there are three arrows. Two of them, ending respectively with a blue and grey tick mark, correspond to the sides of the mobile, since their length is the same of the value of each side of the mobile. Finally, the length of the red arrow on the numbered line, ending with a red tick mark, as said, defines the value of the white triangles on the mobile (and consequently, also the values of the two sides of the mobile and the lengths of the other two arrows). The red tick mark can be dragged along its line, changing the length of the red arrow. In response to this dragging all the other objects of the DEO move accordingly: the slope of the mobile changes in function of the new weights and the arrows change their length to their new corresponding value. The task situation requires students to drag the red tick mark and describe what happens in the DEO.

Our hypothesis was that students, interacting with this DEO, may produce a broader, subsuming discourse to describe simultaneously all the objects in the DEO and their relationships. Algebraic discourse offers a possible form of subsuming discourse, allowing to express almost everything that can be said in the discourse about the mobile and in the discourse about the arrows, with their own signifiers (Figure 2 shows an adaptation of the diagram in Sfard, 2008, p.175). We observe that the protagonists of this subsuming discourse are expected to be the products of a saming process. For instance, the reciprocal positions of two arrows and of the two sides of a mobile as in Figure 1 can be seen as 'the same relationship', that can be expressed by the inequality '$7+x>2x+5$'.

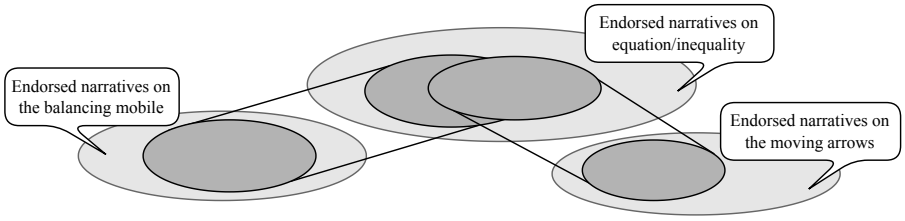


Figure 2: Algebraic discourse subsumes discourses about balancing mobiles and moving arrows

Research objective

In this study we aim to investigate to what extent the discourse developed by students in their interaction with a DEO in which both a mobile and arrows are embedded, subsumes the discourse about the balancing mobile and the discourse about the moving arrows.

Methods

Data collection occurred in an out-of-school learning center with 13 10th grade low achieving students (according to themselves and their teachers), volunteering from different Italian high schools. Participants attended five 2-hours sessions, during which DEOs-based activities were proposed by a researcher, tutoring during the sessions. Data were collected in the form of video recordings, students' written productions, and screen recordings of the tablets.

This paper focuses on Aldo and Giulio's activities after having worked with us with DEOs involving separately mobiles or moving arrows. During the last two sessions, they were given explorative activities aimed at fostering a discourse subsuming the previously constructed discourses about the balancing mobile and the moving arrows. The tutor's questions to students seek to promote saming between the different realizations in the DEO (e.g., "What does this arrow/this side of the mobile correspond to?"). We analyzed data from these two sessions, looking for the possible presence of a subsuming discourse. According to what discussed previously, we operationalized this objective by focusing on the realizations involved and on instances of saming narratives in students' discourse.

The story of Aldo and Giulio

During a preliminary individual interview, both Aldo and Giulio showed difficulties in dealing with algebraic expressions. Giulio, for instance, when asked to say if there are solutions for the equation *13-a=13+11* said, without writing anything, "To find them [the solutions], one must bring all the integer numbers to one side and the unknown to the other, so it would be thirteen plus thirteen minus eleven equals minus *a*... and it would be fifteen but in this case there are no solutions, I think it is impossible […] because fifteen cannot result in minus *a*". Aldo, instead, when asked the same question for the equation *3+a=a+3,* said "it says that three plus *a* equals *a* plus three, that is the two sums are equals […] I don't think it has solutions […] because if *a* goes to the left it becomes *a* squared and so the three goes to the right and becomes minus three and so that ones on the right simplify end so there is no number. So *a* squared equals zero, which has no sense." Both Aldo and Giulio's discourses seem focused on performing procedures and neither of them relates the outcome of these procedures with the signifier "solution of an equation". We do not know how linear equations were specifically addressed in school by their teacher; however, given the common practices in Italy at this level, the two students were certainly exposed to procedures for solving linear equations. Then they participated together in our activities. We report on two episodes where they are the protagonists. For each episode, after describing the task situation, we analyze a short excerpt of the transcript.

Episode 1

During the 4th session, Aldo and Giulio were given, for the first time, the DEO above described in which a mobile and some moving arrows are simultaneously embedded (Figure 1). The task situation was: "What do you notice in this file? Drag the red tick mark and describe what happens".

The following excerpt starts after the students have explored the DEO for approximately one minute.

1	Aldo:	So, that if the triangle is two the mobile stays balanced
2	Tutor:	How can you say if the triangle is two?
3	Aldo:	Eh because we moved *x* onto two and both the arrows are on the nine
4	Tutor:	Ok. And these two arrows?

5	Aldo:	Yes, I mean they meet only at two
6	Tutor	They meet at two, ok
7	Aldo:	But if the triangle is bigger than two, the right arrow will be always bigger
8	Giulio:	Yes
9	Tutor:	Wait, the right arrow
10	Aldo:	The arrow below
11	Giulio:	I mean the arrow below represents the right side of the mobile
12	Tutor:	Ok
13	Aldo:	While if it is less than two the arrow above or the left side of the mobile will be always bigger
14	Tutor:	Ok so explain to me better … the relationships, I don't know, because you already told me something
15	Aldo:	So, if *x* equals two the mobile stays balanced
16	Tutor:	What is *x*?
17	Giulio:	*x* is the triangle
[…]		
21	Aldo:	If a triangle is worth two the mobile stays balanced while if the triangle is bigger than two, two *x* plus five is bigger than seven *x*.
22	Tutor:	Ok, and here *[points to the arrows]* what happens?"
23	Aldo	Ehm the right side so five plus two *x* will be bigger… than seven plus *x*, if *x* is bigger than two. While if *x* is less than two, seven [plus] *x* will be always bigger than two *x* plus five"

At the tutor's request of synthesizing what they found, they write: "If $\Delta=2$ the mobile is balanced and the two arrows will be on the same position, while if $\Delta>2$, $2x+5>7+x$, while if $\Delta<2$, $7+x>2x+5$".

Giulio and Aldo's discourse is characterized by a series of interesting elements. First of all, we can see a rich intertwining of narratives referring to the mobile (e.g., [1]; [15]), to the arrows (e.g., [3]; [5]; [10]) or to both, blending together terms of the two objects (e.g., [7]; [11]; [13]). Aldo's utterance "if the triangle is bigger than two, the right arrow will be always bigger" [7] well exemplifies this blending: the triangle is an object of the mobile but, here it is linked with the arrow's behavior. Moreover, Aldo talks about "the right arrow" and when asked by the tutor he explains that "I mean the arrow below represents the right side of the mobile" [11]. This can be considered as a meta-discourse because it is a narrative referring the relations about words used in narratives about mobile and words used in narratives about arrows. Moreover, with the "right arrow" he has condensed into the same expression both the "below arrow" and the "right side of the mobile". A second interesting element is that this discursive intertwining is also marked by several saming narratives. Some of them are explicit, such as "the arrow below represents the right side of the mobile" [11], others implicit in the students' discourse, as "if the triangle is two…because we moved *x* onto two" [1-3] where the object whose value is two is, in the same narrative, the triangle and the red arrow. Another saming between the triangle and the red arrow features in students' discourse through the use of the term *x*. It is introduced autonomously by the students to refer both to the red arrow ("we moved *x* onto two", [3]) and to the triangle ("*x* is the triangle" [17]). In other words, the term *x* allows the students to refer to the 'same object' that the triangle and the red arrow realize. Finally, as we can see in the last part of the excerpt, the students exploit the result of their saming processes to describe in a condensed way the DEO's behavior. In turns [21] and [23] their discourse is about the relationships between two algebraic expressions of the variable *x* and it could refer, simultaneously, to the mobile and to the arrows. In other terms, these narratives form a discourse which subsumes the two previous discourses. Evidence of this subsuming discourse is also shown in the written inscription they produced at the

end, in which separate descriptions of what happens to the mobile and the arrows are provided for the equality, but then, only symbolic notation with x and Δ is employed for the other cases.

Episode 2

During the 5th session, Aldo and Giulio were given a DEO similar to the previous one, with the difference that the two sides of the mobile are realized by algebraic expressions, depending on x, within blue and grey boxes (Figure 3). These ones can be defined by means of an input field on the top right of the screen. The same expressions also define the length of the two arrows on the right. As in the DEO of the previous episode, all the objects move accordingly when the red tick mark is dragged. The task situation was: "Insert expressions so that: One tick mark indicates 'x+5'; The other tick mark indicates '$2x$+1'. What do you think will happen when dragging the red tick mark?"

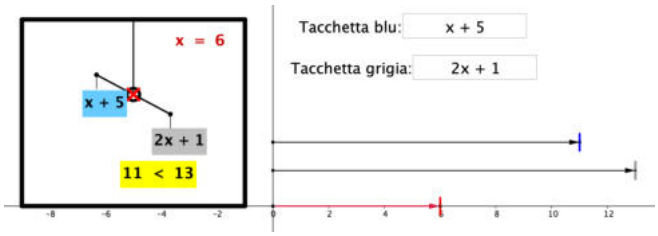
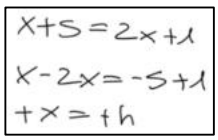


Figure 3: Screenshot of the DEO involved in episode 2

The following excerpt starts after the students have inserted in the input fields the two given expressions and before they could drag the red arrow to interact with the DEO.

1	Giulio:	To find out the point where they are balanced, we could find the x … and we could solve it like an equation *[writes 'x+5=2x+1']*
[The tutor asks Giulio to explain what he meant]		
4	Giulio:	I mean, to find out x that would be the value for which the tick marks meet and so the mobile stays balanced we could make an equation
5	Tutor:	Ok. How do you write this equation?
6	Giulio:	x plus five equals two x plus one, that would be when the mobile is balanced
7	Giulio:	*[solves the equation, see Figure to the right]* So x would be four
8	Tutor:	Ok, please repeat me, we wrote an equation, why Aldo?
9	Aldo	Because, I mean, we wrote this equation as if these two numbers, that is, if these two…if these two parts were balanced and then we did, I mean, we found how much an x is worth
10	Tutor	Ok, how much an x is worth…
11	Aldo	I mean, how much the unknown is worth, that is, this is x, I mean, this x is the one that makes both of them balanced, it makes the mobile balanced



Giulio focuses immediately his prediction on the equality between the two expressions. He writes and solves an equation to determine the value of x, which he uses for expressing "the value for which the tick marks meet and so the mobile stays balanced" ([4]). Here we have an instance of saming that is made explicit by the term "so". Moreover, this approach, relying on the use of algebraic discourse, reflects the student's attempt to integrate the discourse about the mobile and the one about the arrows. Similar to the previous episode, the algebraic discourse, now in the form of an equation, subsumes the other discourses. In turn [9], Aldo articulates his narrative about equations and the procedure of solving equation. He explains that we write the equation as if the system was balanced, and then find

the value of *x* that satisfies this specific state, that he addresses with the term "unknown" ([11]). The students' discourse about the equation, on the one hand encompasses the different algebraic expressions as realizations of DEO's objects, and the equality as a realization of the balancing and alignment states. Moreover, it also includes a saming between the number '4' in the inscription for the last step of the resolution procedure ('*x*=4') and the value of the unknown for which balance is assured, offering in this way a realization for the object 'solution of an equation'.

Discussion and conclusions

The story of Aldo and Giulio offered an example of how well-designed DEOs can foster students' participation in algebraic discourse. This general result confirms other studies claiming on how working exploratively with multiple realizations of algebraic signifiers can be an effective entry point into algebraic discourse for students in respect to the memorization of procedures which often are only treated as symbols on the paper without realizations (e.g., Baccaglini-Frank, 2021). More specifically, this study highlighted how our activities, designed to offer simultaneously multiple realizations, as the mobile and the arrows, can promote students' development of an algebraic discourse subsuming multiple discourse. The conducted analysis on the realizations addressed in the students' discourse and on their saming narratives allowed us to highlight something more, which is *how* this algebraic discourse could feature during the activity. It emerged that Aldo and Giulio's discourse was characterized by a rich intertwining of different narratives, some of them referring to one realization (the mobile or the arrows) and some others referring simultaneously to them both. In this latter case, it happened that students blended terms referring to the mobile with terms referring to the arrows (e.g., "the right arrow" of episode 1), they translated narratives about the mobile into narratives about the arrows, or they produce meta-discourse on the relations about mobile and arrows (e.g., [11] of episode 1). This intertwining characterized students' saming processes. For instance, Aldo and Giulio started referring to the triangle and the red arrow, or one side of the mobile and one arrow, as being the same thing. Note how the saming did not only include objects (referred by nouns and symbols), but also the operations involved (referred by verbs). Indeed, moving the red arrow corresponded to giving a weight to the triangle, and this action would have its counterpart in algebraic discourse in assigning a value to *x.* As a result of these saming processes, algebraic objects appeared in students' discourse (e.g., *x*) providing the protagonists for a subsuming discourse. Moreover, as we saw in the episode 2, the emergence of an algebraic discourse as a form of subsuming allowed students to give meaning to the outcome of the memorized procedure for solving equations. In the analysis we observed that a saming occurred between the number 4 in the inscription '*x*=4' and the value of the "unknown" for which the mobile would stay balanced (and the arrows would be aligned). In this way, the DEO offered also realizations for the object "solution of an equation", that in Aldo's and Giulio's initial interview was just the end of the procedure.

In conclusion, the story of Aldo and Giulio offers valuable insights for both research and educational practice. Specifically, it contributes to the literature on challenges related to the letter-symbolic representation of equations and inequalities (e.g., Carpenter et al., 2005; Kieran, 2022). It underscores the importance of designing didactical approaches aimed at fostering students' learning to talk about multiple DEOs as realizing the same signifier. In this way, it offers thought-provoking perspectives for educators, who often focus their efforts on procedural approaches to solving equations. Shifting

the emphasis toward fostering familiarity with multiple realizations and the connections between these and memorized procedures could open new doors for students into algebraic discourse.

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Conceptions of converting units of measurement of length and mass: Two case studies

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The present study deals with students' conceptions of conversion of units of measurement. The study examines the existing literature and focuses on two cases of fourth-grade students in Austrian primary schools: the approaches of David, who employs a procedural method for the majority of the time, and Luca, who uses the conception of refining and coarsening units to explain unit conversions. This study is part of an ongoing dissertation project, which is designed as an exploratory study and uses semi-structured one-to-one interviews. The findings confirm that the conceptions outlined in the literature are observable in students' approaches to unit conversion.

Keywords: Conversion of units of measurement, units of length and mass, basic ideas, conceptions, internal representations.

Introduction and theoretical background

Measures is one of the four formulated mathematical areas of competence outlined in the Austrian primary school curriculum "Lehrpläne der Volksschule und der Sonderschulen" (2024) and is regarded as one of the most challenging topics in primary school mathematics (Nührenbörger, 2002). Some studies (e.g., Programme for International Student Assessment) also deal with students' and teachers' individual conceptions of measures in general (e.g., Hagena, 2019) as well as with mathematical competences for measures which also include the topic of converting units of measurement. The conversion of units of measurement is a particular challenge for students (e.g., Lassnitzer & Gaidoschik, 2019). And yet, an understanding of unit conversion is an important basis for solving application tasks in STEM subjects.

A way to conceptualise understanding of a mathematical concept is inferring and describing the internal representations that students develop (Goldin & Shteingold, 2001). These internal representations, also referred to as mental representations (e.g., Kosslyn, 1980) or conceptions (e.g., Piaget, 1954), take various forms, including verbal/syntactic, imagistic, formal notational, strategic and heuristic representations (Goldin & Shteingold, 2001). Since these mental representations cannot be observed directly, they must be inferred by interpreting students' externally expressed and produced representations (Goldin & Shteingold, 2001). However, these external representations may not perfectly align with the internal conceptions of a person (Mainali, 2021). It is important to note that conceptions can be either consciously or unconsciously effective (vom Hofe & Roth, 2023), and since only conscious processes are accessible for verbalization (Jääskeläinen, 2010), it may be difficult for students to articulate them (Goldin & Shteingold, 2001). The term "representations" is often intended to include both external formats and internal cognitive structures (Mainali, 2021). In

this paper, “mental representations” are referred to as “conceptions”, while “external representations” are simply termed “representations”.

Research on students’ conceptions has a long tradition in several countries, which has led to the development of different notions to describe conceptions. For example, Tall and Vinner (1981) developed the theoretical constructs of *concept image* and *concept definition*, where a concept image includes all cognitive structures associated with a given concept. This includes all mental representations, as well as related properties and processes. In German-speaking mathematics education, the term conception is primarily used within the theoretical construct of *basic ideas* (in German: *Grundvorstellungen*). In their work, vom Hofe and Blum distinguish two main types of conceptions: individual conceptions, which provide information about student’s “individual images and explanatory models” (vom Hofe & Blum, 2016, p. 232), and adequate mathematical conceptions called “basic ideas”, which enable students to achieve “adequate conceptual understanding” (vom Hofe & Blum, 2016, p. 231). These basic ideas interpret a context or its mathematical core in typical applications or diagrammatic representations. An example from primary school mathematics is the addition of natural numbers. Several different basic ideas can be associated with this operation, each corresponding to a different application. For instance, in one situation, an initial amount is increased by a further amount resulting in a new total, as when a child has €3 and is given €4 more (vom Hofe & Roth, 2023). In another, two separate amounts are considered together without any change and the question is how much they have in total, for example, when one child has €3 and another has €4 (vom Hofe & Roth, 2023). Each of these typical applications highlights a particular way of interpreting addition. Developing such basic ideas becomes a goal of teaching and learning mathematics. The first step is to consider which basic ideas are useful for teaching certain content or procedures, followed by the development of suitable learning contexts that activate the learners' prior knowledge and lead them to the new content (vom Hofe & Roth, 2023). In this way, learners can grasp the learning context, build up basic ideas and, after appropriate practice and application phases, understand the term or procedure (vom Hofe & Roth, 2023). In short, “basic ideas […] describe relationships between mathematical structures, individual-psychological processes, and subject-related contexts” (vom Hofe & Blum, 2016, p. 231).

For the conversion of units of measurement, there are only a few suggestions for conceptions or basic ideas. For example, Franke and Ruwisch (2010) describe the conversion of units of measurement as refining and coarsening units of measurement. In the metric system, units of measurement can be refined or coarsened simply by multiplying and dividing by a power of ten. To help children understand the relationships between the units of measurement, Franke and Ruwisch (2019) recommend demonstrating these relationships with concrete objects. To exemplify this, they describe the refining a unit of measurement using a one-meter strip of paper that can be divided into centimetres. To demonstrate the coarsening of a unit, they suggest, as an example, walking one kilometre and marking each meter with chalk.

Gaidoschik (2024) writes that if you know the relations between the units, unit conversion can become a kind of algorithm that you have to follow. For example, for metric units of measurement, it is only necessary to shift digits within the metric place value table. Moreover, he states that such an algorithm for unit conversion is easy to understand if you have an adequate basic idea of how to

convert. Gaidoschik (2024) suggests the idea of "measuring the same size twice" as a basic idea for the conversion of units of measurement, finally represented as an equation. In this case, the left side of the equation represents the result of a measurement in the first unit, while the right side represents the result of a second measurement in the second unit. He exemplifies this with the task 5000 cm = ___ m. The conception that to the left of the equals sign is the result of measuring a length in centimetres, and to the right is the result of measuring the same length in meters, would be helpful in solving this task.

Both Franke and Ruwisch's as well as Gaidoschik's suggestions describe possible adequate conceptions for converting units of measurement into smaller or larger units. However, the theories to conceptions mentioned above not only consider adequate conceptions, but also emphasize the importance of considering learner's individual conceptions.

Research question

This paper focus on whether the two theory-based suggestions for adequate conceptions "refining and coarsening" and "measuring twice" are recognizable in unit conversions of students. In more detail, the study reported on in this paper intend to address the following research question: Which of the two theory-based conceptions for unit conversion "refining and coarsening" and "measuring twice" can be identified in the unit conversion approaches for length and mass of two fourth-grade students?

My research focuses on the measures of length and mass due to their common metric structure. Of the six relevant measurement domains in the primary school curriculum in Austria, time is not structured metrically, area is a derived quantity, money requires consideration of decimal notation, and capacity includes only the two units l and ml that are relevant at the primary level. Consequently, for the investigation of unit conversion, the metric-based measurement domains of length and mass offer an appropriate basis.

Methods

The two cases presented are part of an ongoing doctoral project, in which I study the conceptions that children demonstrate when converting units of measurement length and mass. The research is designed as an exploratory study, as this topic has not yet been sufficiently investigated in German- and English-speaking countries. Children's conceptions are collected using an identical set of tasks in semi-structured (Roulston & Choi, 2018) one-to-one interviews. An interview guide is used in the interviews. It was piloted with 15 third-grade students in January and February 2024 to identify the setting (individual, pair or group interview), types of tasks, questions and wording that best support children in expressing their conceptions of unit conversion. To date, 34 primary school children in the fourth grade from six different classes in three different schools in Austria have been interviewed by me. Fourth grade was selected as the target group because, by the end of third grade, students in Austria are typically introduced to all relevant units of length and mass covered in primary school curriculum. The interviews took place in May and June 2024. The plan is to conduct further interviews in the 2024/25 school year. The cases are selected using the theoretical sampling method (Schreier, 2018). Case selection is ended as soon as no new insights can be gained from the interviews

and data saturation is reached (Schwandt, 2007), meaning that the same conceptions are repeatedly observed without offering additional perspectives.

During the semi-structured one-to-one interviews, each lasting about half an hour, students are encouraged to think aloud to identify their underlying conceptions. Before the interview begins, students are informed that the interview is not a test, that there are no right or wrong answers, and that sharing their thoughts through speaking, writing, drawing, or acting is helpful. The interviewer uses gentle prompts (e.g., "What are you thinking now?") and positive reinforcement to support students' efforts to express their thoughts. Paper and pencils are provided for notetaking, drawing, and calculations. One task also provides materials as measuring devices, e.g. measuring tapes of different length, rulers, a beam scale and a digital scale with various weights. This allows students to express their conceptions in spoken words, enactively, iconically, or symbolically. The interviewer listens carefully and asks non-suggestive questions, such as: How do you know that? How do you imagine it? What do you think about…? and Why…? to explore the student' conceptions and understanding of unit conversion. It should be noted that, since only conscious processes can be verbalized (Jääskeläinen, 2010), identifying children's conceptions is challenging, as they often struggle to articulate them. As a result, interviews often provide only fragmented explanations, making it difficult to formulate definitive conclusions about students' conceptions of unit conversion.

At the beginning, the interview tasks for primary school children are very open to ensure minimal influence on the children. For instance, they are simply asked: "What do you understand by converting units of measurement?". As the interview progresses, the tasks become successively more specific. The following are examples of such tasks:

- Solving specific conversion tasks, such as: The interviewer shows the card "7000 cm" and asks how many meters that is.
- Providing explanations for younger children: The interviewer asks how the student would explain to a younger child what converting means.
- Answering questions like "Is it true that 9 mm is the same as 9 000 meters?".

The aim of these tasks is to give children lots of opportunities to formulate their thoughts and to talk about their conceptions of the topic. At the end of the interviews, stimulus materials such as scales and measuring tapes are provided to give children further reasons to talk and act.

The data collected through the interviews consists of verbal statements as well as notes, drawings, and activities with measuring devices. To capture both verbal expressions and non-verbal actions, the interviews are videotaped. The resulting data are then transcribed (in the case of spoken language), interpreted, analysed, coded, and thematically categorized using qualitative content analysis according to Kuckartz (2019). In the context of this study, as a first step, the interviews are coded with two theory-based categories for conceptions of unit conversion: "refining and coarsening" and "measuring twice".

These two students were selected because in some parts of the interview the two adequate conceptions for unit conversion proposed in the literature, "refining and coarsening" and "measuring twice", were recognizable.

Case 1: David

The interviewer tries to find out about David's conceptions about converting units of measurement.

In the first phase of the interview:

Interviewer: What do you understand by converting?
David: […] I always just imagine the table, for both measurements.

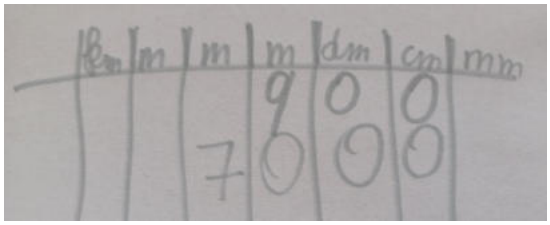

Figure 1: metric place value table for conversion

For most of the interview, David uses the metric place value table as his primary tool for converting units of measurement. On rare occasions, he converts smaller units to larger units by simply deleting zeros. It seems that David understands unit conversion as a memorized procedure in which he fills in numbers in the table, moving either from left to right or right to left and filling in zeros, depending on the task. For example:

David: I actually always have the table in my head and then I imagine it in front of me and then I just fill in the numbers.

Later:

Interviewer: […] If I say convert 7,000 centimetres into meters. How many meters is that?
David: Zero, zero, zero, seven. [David enters the number 0 in the columns cm, dm, m (right column) and the number 7 at m (middle column).]
Interviewer: How do you know? How did you decide where to start writing?
David: At the centimetres. […] Or because you enter in meters first. So, if you enter the meters, you start in kilometres – meters, where the number is and then go to the back. If I also have a number to convert to the front, I go from the back to the front.

Or later with another example:

David: I simply entered the 800 next to the meters in the "How many centimetres are 800 meters?" and then just dragged the zeros backward to the centimetres.

As already mentioned, most of the time he uses a fixed plan, like an algorithm for converting units of measurement without formulating any other conceptions. Once there was an approach to the idea of "refining and coarsening" referring to the gram as a "part" of a kilogram:

Interviewer: Exactly, but how do you explain the gram, for example?
David: The gram? I, the gram is the least, so, underneath. You can just say a part.

Similarly, an approach to the conception "measuring twice" was recognizable one time:

David: If you had a measure, you would probably have to say, because if there were only grams: Please 60,000 grams of concrete bags or something. That's why it's easier with the units of measurement, just 600, 600, 60 kilograms of concrete, something like that.

In summary, David seems to successfully solve conversion tasks by using the same procedure most of the time, by filling in numbers in a metric place value conversion table, either mentally or on paper. His conception of unit conversion seems to be an algorithm, expressed by filling in numbers in the metric place value table. On two occasions, he mentions the conceptions for unit conversion "refining and coarsening" as well as "measuring twice" as described in the literature.

Case 2: Luca

Most of the time, Luca converts units by multiplying and dividing by a conversion factor:

Luca: How many kilometres are 30, 300,000, 300,000 meters? [The task 300,000 : 1,000 was solved before by long division. His result was 300.] I did 300,000. […] 300,000 divided by 100.
Interviewer: I see. You divided that now?
Luca: Divided by 1,000.

His explanations of unit conversions correspond to the conception of "refining and coarsening", especially the conception of "refining" units. He formulates it as "fitting" smaller units into larger ones. For example:

Luca: Converting is something like how many centimetres fit into one meter, for example.

About a minute later:

Interviewer: Okay. And how do you imagine it when you say you can fit 100 centimetres into one meter? How do you imagine that exactly?
Luca: Yes, I think, uh, uh, I think in such a way that I actually see a meter in front of me. And then I would mark the centimetres like this.

By referring to the concrete activity of marking smaller units within larger ones, his conception of unit conversion as "refining" units is emphasized.

When asked what exactly he understands by converting units of measurement, he explains it by describing how a kilogram on a beam scale weighs the same as 1,000 grams:

Luca: 1,000 grams just fit into one kilogram.
Interviewer: Okay. So, 1,000 grams and one kilogram.
Luca: Yes.
Interviewer: How do you imagine that again? [...]
Luca: I often imagine it with weights. One gram weight. One kilo weight. And yes, this is then, this is easier for me then when I…, I take 1,000 one-gram weights and then I take a kilo weight, on a scale or something.

Later in the interview, with a beam scale and weights of one kilogram, one dekagram, and one gram on the table, he actually shows what he explained earlier:

Luca: So, I have a scale [Luca takes the kilogram weight in his hand] and now, that's [Luca takes the gram weight in his hand] one gram. If I put a kilogram in there [Luca puts the kilogram weight in a bowl of the beam scale] [...] If I now put 1,000 of the little things, [Luca puts the gram-weight in the other bowl] then it's the same weight. We don't have 1,000 there now.

This explanation does not correspond to any of the theoretical conceptions discussed above. It does not involve refining or coarsening a unit, because the weight of a kilogram is not divided into smaller parts, nor does it involve measuring an object twice. Perhaps it can be interpreted as an extreme case

of the conception "measuring twice", in which the weight of a kilogram can be considered as an object that is measured twice, once with a kilogram and once with 1,000 grams, or defined as a separate category.

In summary, Luca seems to have in mind the conception of "refining and coarsening" units when talking about unit conversion or solving conversion tasks. Also, his choice of presenting his way of solving these tasks by multiplying and dividing the measure by the conversion factor underlines his conception about unit conversion. In addition, he (mentally) refines concrete objects or uses measuring devices as a representation for unit conversion.

Conclusion

The conceptions "measuring twice" proposed by Gaidoschik (2024) and "refining and coarsening" proposed by Franke and Ruwisch (2020) are intended as instructional recommendations for teachers aiming to support students' understanding of unit conversion. The cases of Luca and David show that some fourth-grade students in Austria have already developed such conceptions and are able to apply them effectively. David's approach to unit conversion appears to be predominantly a memorized procedure, but the conception of "refining and coarsening" as well as the conception "measuring twice" were recognizable only one time each during the interview. In contrast, Luca seems to consequently use the conception of "refining and coarsening" units when converting units of measurement, which he expresses in various parts of the interview. The presence of these theoretically grounded conceptions in students' successful strategies suggests that they have practical value for classroom teaching. However, the diversity of students' conceptions also highlights the need for further research to identify and categorize additional conceptions beyond those described in this paper. The cases of David and Luca demonstrated that this is a challenge.

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Navigating function graphs with audio: an investigation into the narratives of a blind expert using Audiofunctions

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This study examines Audiofunctions (AF), a multimodal interactive digital artifact designed to represent graphs of real-valued functions of a real variable. AF displays function graphs and uses sonification to represent the coordinates of a moving point as the independent variable changes. The paper discusses key theoretical considerations underlying the design of AF, related to a covariational perspective on functions and function graphs. Using a commognitive perspective, it analyses the case of an adult blind user, an expert in mathematics, in interaction with AF. The study investigates the narratives she produces, the function realizations involved other than the auditory one, and how they are related to the function characteristics, providing insights into the potential impact of AF.

Keywords: Function graphs, multimodality, inclusion, covariation, visual impairments.

Introduction

The notion of inclusive education, aimed at providing high-quality education for all students, regardless of disabilities, gained attention in last decades (e.g., Ainscow, 1997). Nevertheless, ensuring the right to education for all faces persistent challenges, such as improving accessibility of materials and content. Despite advancement of assistive technologies, while accessible formats such as haptic or audio options are available for digital textual content (e.g., braille display – a device translating text written on screen in Braille characters accessible to user by touch – or screen readers), there is still research going on to find a general solution for accessible graphs and formulas (e.g., Shahira & Lijiya, 2021). Moreover, a lack of educational research on use of these tools remains, even if some studies showed that using sonification for teaching-learning spatial concepts seems promising (e.g., Damsma, 2024). This paper focuses on the issue of ensuring sustainable accessibility for function, a fundamental concept in scientific education. In functions teaching-learning, interpreting information embodied in function graphs or, conversely, using available information about an assigned function to graph it, is a crucial competence (e.g., Baccaglini-Frank et al., 2023). Access to function graphs' information is essential for understanding functional relationship between dependent and independent variables, but function graphs pose accessibility challenges due to their graphical nature. In this context, Laboratory "S. Polin" of the University of Turin is developing since 2014, and now with the Erasmus+ project "Sonairgraphs", a software solution called *Audiofunctions* (AF), publicly available since 2019 as a web-based application designed to generate accessible single-variable function graphs through use of sound (https://audiofunctions.netlify.app/). In these graphs, at visual level, the cartesian function graph is displayed, with a bullet point moving on it depending on the position of its coordinates; at auditory level, a sound is produced whose characteristics depend on the position of the bullet point on screen (e.g., move mouse pointer wearing headphones to hear *f(x)=sinx*, linked at: https://linkshort.accesee.it/sinx). Therefore, sonification in AF allows visually impaired students to access functions by "discovering" gradually their behaviour through sound.

Next to accessibility issue, literature underscores learning difficulties encountered by students, with or without impairments, in making sense of function graphs. Antonini et al. (2019) emphasise that in literature students show fewer difficulties when they deal with "numerical evaluation or prescribed algebraic manipulation, but they tend to encounter difficulties in analysing or even describing a function's behaviour by looking at its Cartesian graph" (p. 2). Function graphs are often perceived by students as static representations "in which the variations, and the relation between variations – *covariation* – related to functional dependency remain hidden" (Antonini et al., 2019, p. 2, our emphasis). In this paper, we address the ability to envision the relationship between simultaneous variation of values of independent and dependent variables. Prior research (e.g., Thompson & Carlson, 2017) highlighted that covariational perspective on quantities dependence was characteristic of first conceptualisations of function in mathematics' history, even if it was subsequently abandoned. However, Thompson and Carlson (2017) argue that "continuous variation and covariation seem to be epistemologically necessary for students and teachers to develop robust conceptions of functions" (p. 35). In other words, familiarising students with function graphs alone does not guarantee covariational reasoning skills' development, essential for understanding function concept. While ensuring access to this content for all students is critical, it may not be sufficient to foster meaningful learning. It could be worth studying whether AF can be exploited not only as a tool for ensuring accessibility, but as a tool to support development of covariational reasoning for *all students*, since its sonification offers a dynamic but temporally sequential mode of access to function graphs (see Audiofunctions' design paragraph). For sighted students, this dynamic representation contrasts with the static, comprehensive, visual graph. Sound dynamism is replicated at visual level through bullet point's movement, allowing to see the visual graph as a dynamic time-related evolving trajectory. This would make AF suitable for designing inclusive learning activities.

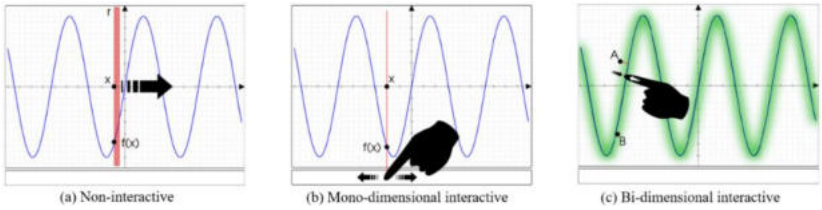
Theoretical Framework

We take a *commognitive* perspective (Sfard, 2008) consistent with previous studies on interaction with digital artefacts investigating development of meaningful narratives in the context of functions and variables (e.g., Baccaglini-Frank, 2021; Antonini et al., 2019). In this perspective, the term "discourse" is intended as a form of communication characterized by specific features, among which we find specialised vocabulary (e.g., terms used to name mathematical objects, as "function") and unique *visual mediators* (such as numerals, symbols, or graphs). Constructing mathematical objects involves a *saming* process, i.e. naming seemingly unrelated things that are seen as equivalent in certain contexts (e.g., "linear function" can refer to *"3x"*, a straight line, or a table of paired couples of numbers). These things become different *realizations* of the same mathematical object. A mathematical object is thus to be understood as a purely discursive object that, in a given discourse, is constituted by the set of its realisations. We emphasize here that realizations of mathematical objects are defined by Sfard without specifying their perceptual nature (Maffia et al., 2023), hence potentially including also auditory mediators. "[Mathematical] discourses are made distinct […] by the form and outcomes of their processes, that is, the *routines* and *endorsed narratives* that they produce" (Sfard, 2008, p. 161). In mathematical discourse, experts "interpret most task-situations as requiring a (re)formulation and endorsement of a particular type of mathematical narrative" (Sfard, 2020, p. 96). "*Narrative* is any sequence of utterances framed as a description of objects, of relations between objects, or of processes with or by objects, that is subject to *endorsement* or rejection with

the help of discourse-specific substantiation procedures" (Sfard, 2008, p. 134). Baccaglini-Frank et al. (2023) discuss how experts talk about function behaviour starting from its graph, emphasizing their exploitation of a covariational perspective that enables them to make sense of the function beyond what they "just" perceive with sight. Different realizations of the same signifier in a discourse are related precisely thanks to discourse rules allowing the translation of endorsed narratives about a realisation into ones about the other. AF offers multiple realizations of the same function, involving visual mediators (algebraic expression, Cartesian graph) and an auditory mediator (sonification). The latter, covariational and dynamic, provides a unique representation for visually impaired users, who can integrate it with other, more common but usually non-dynamic, representations (e.g., tactile ones). In this paper, we want to focus on how a blind expert recognise AF' sonification as a realization of function. Then, we address the following research questions: What other realizations of the function are present in the expert's discourse dealing with AF? How does she translate her endorsed narrative on function into new narrative on sonification?

Audiofunctions' design

The first version of AF, *Audiofunctions.web*, was thoroughly investigated in technical aspects, usability, and accessibility (see Ahmetovic et al., 2019). Its sonification technique maps function related information to sound features: if we name $P=(x, f(x))$ the moving point, sonification conveys the current position of x mapping the range on x-axis from left to right with auditory *panning* (i.e. an x visually closer to the left graph bound is heard more by the left ear), and the value of $f(x)$ mapping the range on y-axis with sound *pitch* (the perceived sound frequency). Pitch is central to sonification, the human auditory system is sensitive to its variations (Dubus & Bresin, 2013). In this way, exploring function from left to right, users can "hear" the displayed portion of graph as the covariation of sound panning and pitch. Originally emerging as a tablet application, the prototype of Audiofunctions.web (Bernareggi, et al., 2016) underwent evaluation by comparing the interaction experience of blind users with respect to their accustomed use of tactile drawings.



(a) Non-interactive (b) Mono-dimensional interactive (c) Bi-dimensional interactive

Figure 1. Function exploration modes (Bernareggi et al., 2016, p. 8)

Users freely engaged with three available interaction modes (see Figure 1a,b,c respectively): *non-interactive mode*, where pressing "b" key the software sonifies automatically the entire function, varying x from left graph bound to right one; *mono-dimensional interactive mode*, where, in a dynamic exploration, sonification is reproduced as users move with keyboard arrows or slide the pointer along a horizontal line; *bi-dimensional interactive mode*, where spatial relationships can be grasped by difference in volume since that the highest volume occurs on function line. Bernareggi et al. (2016) indicate Audiofunctions.web as more efficient in terms of response time compared to tactile methods. This highlights the software as a valuable tool for enhancing information accessibility. AF is the latest version developed to specifically facilitate its educational implementation. Among other enhancements, AF implements two distinct types of sonification: *continuous sonification* with clarinet and sine wave timbres, and *discrete sonification* with guitar timbre. There are two key

differences between them: *pitch density* and *threshold passing*. Pitch density refers to the fact that in discrete sonification a predetermined number of notes is established *a priori*, then the displayed *y*-axis range is partitioned into a corresponding number of subintervals of equal width and each interval is then associated with a guitar musical note. In continuous sonification, instead, the notes are distributed continuously along the displayed *y*-axis range. The threshold passing is a sonification strategy that leverages the lower pitch density in the discrete sonification: instead of sonifying a value on the *y*-axis (as in continuous sonification), it sonifies the threshold passing from one of the subintervals on the *y*-axis to another. The *x*-axis is also discretised into subintervals $[x_1, x_2]$, … , $[x_{n-1}, x_n]$, with each position x_i mapped to the note of the *y*-axis subinterval where $f(x_i)$ falls. However, if x_i and x_j are two subsequent positions sonified by AF with guitar, j is the first index greater than i where the note associated to the subinterval containing $f(x_j)$ differs from that of $f(x_i)$. Therefore, users hear a pause between two successively produced notes, since all the x_t comprised between x_i and x_j do not give place to any sound by AF. AF' design is grounded in explicit educational intentionalities, consistent with principles of a well-designed digital environment (Baccaglini-Frank, 2021). This approach, informed by literature on mathematical content sense-making, is crucial for ensuring that technology not only provides accessibility but also supports users in achieving autonomy and fosters cognitive engagement. AF' design in the continuous sonification mode is inspired by a strand of research by Italian researchers, focusing on the use of digital artifacts, called *dynagraphs,* where two ticks moving along one or two straight lines represent the independent and dependent variable (e.g., Baccaglini-Frank at al., 2023; Antonini & Lisarelli, 2021; Antonini et al., 2019). In many experiments the two ticks have a functional relationship unknown to users, who can act directly only upon the tick representing independent variable, thereby indirectly affecting the movement of the tick representing dependent variable. Baccaglini-Frank et al. (2023) emphasise this asymmetry in dragging ticks, arguing that it supports students' development of a covariational view of variables' functional dependency, that they later recognise in Cartesian function graphs. Antonini and Lisarelli (2021) discuss a task sequence, where Cartesian function graph is generated as trajectory of a point of coordinates *(x, f(x)),* dragging the tick representing *x*. Considering AF in mono-dimensional interactive mode (Figure 1b), AF mirrors aforementioned asymmetry between variables, since users can directly act on the independent variable, changing the moving point position, and thus enabling users to interpret function graph as trajectory of that point. AF differs from dynagraphs in the fact that trajectory is multisensorially rendered maintaining visible the static Cartesian representation.

Methods

To answer our question, we carried out some interviews with adults with visual impairments, experts in scientific disciplines, to collect data about their ways of interacting with AF when exploring different kinds of functions. Since our aim was to portray blind users' point of view, we opted for an exploratory approach and a case study design, similar to previous studies concerning blind solvers (e.g., Maffia et al., 2023; Miragliotta et al., 2023). Despite their familiarity with function concepts, participants were unaccustomed to AF ways to represent functions. In this paper, for reason of space, we examine the case of Miriam (pseudonym), a woman in her thirties who works as a business analyst and graduated with an honour degree in mathematics. She has been blind since birth and does not use speech synthesis but reads Braille. She knew that she was going to experiment a software for representing functions but did not know AF; she was used to their algebraic expression and to their

tactile graph. The interview was conducted through an online remote calling; the interviewer was the first author, who shared through the chat of the videocall platform links related to different functions already setted in AF, provided one-by-one during the conversation. For interacting with her computer, Miriam could use tools as screen reader, keyboard, or braille display. The interview was semi-structured: it involved a first phase where she was introduced to the software, and she was asked to freely explore AF setted with some non-standard functions. This was done to favour a focus on the description of the qualitative characteristics of the function behaviour rather than on an attempt to guess its algebraic formula. In a second phase, then, Miriam was asked to explore: different constant functions (e.g., $y=0$ and $y=3$); linear functions with different slopes (e.g., $y=x+3$, $y=\frac{1}{2}(x+3)$, $y=-x+3$); functions having similar behaviour with respect to some properties (e.g., $y=x^4$, $y=x^2$, that are decreasing for $x<0$ and increasing for $x>0$, have a point of minimum for $x=0$ and tend to infinity for x going to $+infinity$ and $-infinity$); functions whose domain is not the whole set of real numbers (e.g., $y=\sqrt{x}$). The displayed x-axis and y-axis ranges for all the functions were pre-determined and set as follows: $-16<x<16$; $-9<y<9$. The choice of functions with similar behaviour was aimed at understanding aspects of sonification used by Miriam to distinguish similar functions from each other. The interview was recorded and transcribed. Accordingly with previous studies exploiting a commognitive perspective (e.g., Baccaglini-Frank, 2021), Miriams' discourse was analysed by looking for words related to different function realizations (e.g., algebraic, visuo-spatial, etc.) and their characteristics, and for words related to transitions between different kinds of realizations.

Data analysis

Miriam conducted the interview using the braille display to modify sound settings or switch functions, and using keyboard keys to explore function graphs. In Table 1 an excerpt is reported where Miriam explores graphs of the constant functions $f(x)=0$.

Table 1. Miriam considerations on $f(x)=0$ graph (available at: https://linkshort.accesee.it/afy0; wearing headphones is recommended, and use "i" key to switch from discrete to continuous sonification and viceversa)

Line	What is said	What is done
1		**Miriam** presses the "b" key, and AF plays a single note with the guitar timbre while the red dot moves from left to right along the entire line [graph of the function $f(x)=0$].
2	**Miriam**: But how? Is it already over?	
3	**Interviewer**: Yes. I mean, it's not true that it's over, but it has a strange "thing" here; you should...	**Miriam**, acting on the settings slide bar, changes the timbre from guitar to clarinet.
4	**Miriam**: ...it's all at the same pitch.	**Miriam** presses the right arrow key several times. AF plays, for each key press and while the red dot moves step by step, the same note previously reproduced.
[...]		
11	**Miriam**: It's all a horizontal function.	
12	**Interviewer**: How did you figure it out?	
13	**Miriam**: The sound was always the same.	

The first interaction mode used by Miriam here is non-interactive, using the "b" key, with guitar timbre; she appears confused by what she hears ([2]), and decides to switch the timbre to the clarinet (continuous sonification). She needed the change in sonification to recognise that the pitch was constant ([4]), together with a change in the interaction mode, from non-interactive to mono-dimensional. Continuous sonification in a mono-dimensional interaction mode probably made her experience a multimodal interaction to analyse the function graph, involving her panning-related and

proprioceptive strategies using arrow keys to orientate herself on the graph along the *x*-axis, and her auditory perception of sound pitch. In fact, it's worth noting that Miriam's discourse passes from a narrative about the constant pitch she hears ([4]) – aspect of the auditory realization of $f(x)=0$ – to a narrative about function through an adjective pertaining to discourse on *spatial* level ("a horizontal function" in [11]). When asked why, she states again her perception of constant pitch (narrative about sound in [13]). Miriam's implicit association of the constant sound pitch (one dimensional) with the overall two dimensional shape of the cartesian graph of a class of functions means that she is interpreting pitch as dependent variable and left-right direction as independent variable. Moreover, she juxtaposes an adjective describing the cartesian graph at spatial level ("horizontal") with the signifier "functions", which indicates identification of the function with its graph, that becomes a spatial realization of the function for her. In Table 2 Miriam interacts with non-constant functions.

Table 2. Miriam considerations on the graphs of $f(x)=x+3$, $f(x)=\frac{1}{2}(x+3)$ and $f(x)=-x+3$ (available respectively at: https://linkshort.accesee.it/x3, https://linkshort.accesee.it/12x3, https://linkshort.accesee.it/3x)

Line	What is said	What is done
38	**Miriam**: This one is diagonal! Going upwards. It starts from the bottom and goes up to the top.	The $f(x)=x+3$ graph is displayed. The selected timbre is guitar. **Miriam** presses the "b" key. AF plays a complete scale of notes while the red dot moves from left to right along the line
39	**Interviewer**: Ok. Switch the function again so we can hear the next one, and then you can tell me about the differences.	
[...]		When the $f(x)=\frac{1}{2}(x+3)$ graph is displayed, **Miriam** presses the "b" key. AF plays a complete scale of notes, narrower than the previous one and with longer pauses between the notes, while the red dot moves along the line
44	**Miriam**: It has a lower slope.	
45	**Interviewer**: It has a lower slope. And can you tell me anything else?	**Miriam** plays the function with "b" again. And then she thinks for a while in silence.
46	**Miriam**: No.	
47	**Interviewer**: No. How do you perceive the difference in slope?	
48	**Miriam**: It's a slope that lasts longer. It rises less steeply.	
49	**Interviewer**: Ok. Switch the function again.	When $f(x)=-x+3$ graph is displayed, **Miriam** presses "b" key. AF plays the complete scale of notes sonified for function $f(x)=x+3$ but starting from the higher note to the lower one, while the red dot moves from left to right along the line
50	**Miriam**: It is a descending line. I mean [she seems to correct herself], with a negative slope ("coefficiente angolare" in Italian).	

Despite her return to non-interactive mode with guitar timbre, Miriam keeps speaking about what she hears using words pertaining to discourse on a spatial level (e.g., "diagonal", "upwards", "bottom", "top" in [38]). Miriam uses terms related to vertical dimension to describe sound behaviour ("upwards", "bottom", "top" in her narrative in [38]) but the overall function behaviour is summed in an adjective about two-dimensional space ("diagonal", in [38]). We interpret this as a sign that Miriam is relating pitch with some other variable, maybe time or left-right dimension. Moreover, the dynamic connotation of verbs such as "going upwards", "starts from... and goes up...", "rises", shows that she is associating the dynamic pitch changes with a trajectory on the plane – similarly to endorsed narratives of sighted experts reading static functions graphs (Baccaglini-Frank et al. 2023). In [48] she uses expressions related to time ("lasts longer") and speed ("rises less steeply"), that perhaps refer

to discrete sonification features: the shorter the pause between two successive sonified notes, and the further apart they are on scale, the steeper the function is. When she switches to $f(x)=-x+3$, she introduces the term "slope" ([44]), which pertains again to discourse on a two-dimensional space. In [50] Miriam passes to the formal Italian term for line slope, a shift that we interpret as a way to make her discourse more formal, since she is aware that she is talking to a mathematician. Therefore, we infer that probably here she is implicitly referring, for the first time, to algebraic function realization.

Discussion and conclusions

The interview with Miriam, expert in mathematics, confirms that AF emerges as a valuable tool, providing a comprehensive qualitative overview of functions' behaviour. Miriam seems to implicitly refer to functions realizations she was used to when she was a student: her recurring to words related to two-dimensional space seems to be connected to her knowledge of the cartesian function graphs – developed thanks to the use of tactile graphs. Talking about the function slope, she seems to implicitly refer also to the function algebraic expression. She passes mainly from auditory realization of AF to cartesian graph, in analysed extracts, often linking sonification with the overall two-dimensional shape of function graph, and describing both the sound behavior and the cartesian graph in terms of dynamic variations (e.g., in [38]), a fact that shows Miriam's mastery in covariationally reading the function and interpreting the variation of the sound pitch in dependence on time or on shifts in the left-right dimension. This recalls what is shown by the interviews with experts in Baccaglini-Frank et al. (2023) but in a reverse perspective: sighted experts can read the graph, a static image, as a dynamic trajectory of a moving point. Miriam, otherwise, is able to read a dynamic sound realization of function in AF as a two-dimensional visuo-spatial configuration, "synthesizing" overall sound behaviour. In this perspective, AF seems a potential tool for developing a covariational view of function graphs for students with and without visual impairments, since it allows to work on the connections between a dynamic, covariational function realization and a static synthetic realization. It should be clarified whether and how AF should be used combined with other function realizations, such as tactile graphs, for coordinating auditory mediators with visuo-spatial ones. There is more to understand about how people with visual impairments perceive haptic/auditory information, but Miriam's case suggests that, through AF, users can qualitatively grasp some function features, such as points on the graph where the function is higher compared to other points. Users can compare functions in terms of their growth or decline rates, leveraging the concept of speed of discrete sonification. Combined interaction of discrete and continuous sonification can offer users an initial embodied conceptualisation of derivative, through an auditory completion process when listening to two successive notes. Users can also grasp functions monotony, even if it may not be always easy to distinguish between functions with similar behaviours (about $y=x^4$, Miriam commented "It goes very fast […] at the beginning is at infinity then goes down for a moment and goes up and back to infinity […] But how do I know if it's a V, or if it's like a basin?"). Further studies are needed in this direction.

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Supporting diagrammatic reasoning for users with visual impairments with NoVAGraphS

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This paper presents a case study on how individuals with visual impairments interact with mathematical diagrams using NoVABOT, a dialogue-based system developed for the NoVAGraphS project. The study focuses on the experience of a blind participant navigating function diagrams represented as Euler-Venn sets. Through verbal interaction with NoVABOT, the participant explored key characteristics of the diagrams, demonstrating how the system can support the development of diagrammatic reasoning by guiding users through a process of inquiry and reflection. The findings illustrate how NoVABOT enables users with visual impairments to access and interpret graphical information, fostering deeper engagement with mathematical content. This case study highlights the potential of dialogue systems to bridge accessibility gaps in STEM education.

Keywords: Diagrams, functions, inclusion, visual impairments.

Introduction

In recent years, various assistive tools have been adopted by individuals with visual impairments, but these technologies have significant limitations in non-purely textual contexts, and they can restrict user autonomy (Bernareggi et al., 2016). Accessing structured graphical representations, such as tables, graphs and diagrams, presents particular challenges for at least three reasons. First, these images are designed to be perceived through the visual channel (Heller, 2002). Second, common non-visual representations, such as tactile drawings (e.g., Miragliotta et al., 2023), can be cumbersome and require expensive equipment, with overlapping tactile labels potentially reducing readability. Printed drawings lack interactivity, plastic sheets demand manual dexterity, and digital boards, while offering greater interactivity, still require fine motor skills and do not support permanent storage. Sonification approaches provide an alternative but face challenges, including the difficulty of transposing content into sound, the need for extensive training for non-spontaneous metaphoric content-sound association (Manolino et al., 2024), and cognitive demands when interpreting images through audio (Flowers, 2005). Third, creating effective non-visual representations of structured images requires an in-depth understanding of the domain and non-visual exploration techniques.

In the following section, we introduce a chatbot developed by a multidisciplinary team from different Departments (Computer Science, Mathematics, Physics, Social Education, and Economy) of the University of Turin, coordinated by the Research Laboratory for STEM Accessibility "S. Polin", as our proposed voice-based tool for individuals with visual impairments (Oliverio et al., 2024). This chatbot allows users to explore the Euler-Venn representation of mathematical functions through dialogue-driven interactions, enabling them to ask questions and receive responses from the software. Here is the provisional link to the tested prototype: https://delorean.di.unito.it/novagraph.

This paper shares insights from testing the chatbot with a blind participant with no prior experience using the tool. The results prompted several important reflections, particularly regarding mental representations, which were found to differ significantly from those of sighted users.

NoVAGraphS project

NoVAGraphS (Non-Visual Access to Graphical Structures – http://www.integr-abile.unito.it/en/progetto-novagraphs/) is an ongoing project devoted to the development of educational dialogue systems for users with visual impairments that specifically address the treatment of graphical structures. Scientific books frequently contain structured graphical information, such as tables and diagrams, which play a crucial role in conveying complex data and concepts. This project aims to establish standards through best practices and software solutions to enhance the interpretation and comprehension of structured graphical information, branching out into three main goals: (1) the study and the development of algorithms to convert graphical structures in dialogue scripts, which are prototypical dialogue fragments designed to communicate the information conveyed by the graphical structures; (2) the integration of these scripts in educational dialogue systems for users with visual impairments; (3) a comprehensive evaluation of these dialogue systems along different perspectives and disciplines. To fulfil the goals, the developer team designed a new dialogue system called NoVABOT. It is a web-based and accessible text-based dialogue system to which users with visual impairments can have access by using their own speech technologies, such as a screen reader and a keyboard. It is designed to understand and answer questions in Italian, in relation to three different topics: Euler-Venn function diagrams, electrical circuits, and finite state automata (https://delorean.di.unito.it/novagraph/tre_domini). These three topics were chosen as paradigmatic of the three scientific domains involved in the project: mathematics, physics and computer science. In particular, these topics are usually addressed precisely through diagrams: diagrams are both objects of study and tools or vehicles of knowledge. For relevance and conciseness, this contribution focuses on functions as a representative example. In this digital environment, NoVABOT engages in dialogues about functions on finite sets, represented through Euler-Venn diagrams. At this stage, several examples are available, with future plans to allow the software to "read" user-provided diagrams in input, e.g. through a photo. The user interface (see Figure 1) is designed with accessibility for users with visual impairments in mind. It is optimised for screen readers, featuring an input bar and a submit button directly beneath it. Responses to user queries appear below the button, and then a table displays the conversation history. This helps users to track and associate questions with answers. Webpage accessible navigation is simplified using HTML tags. The central feature of the interface will be an accessible and navigable SVG image, although this functionality is still under development. Currently, the image in focus is displayed on-screen but is not accessible (Figure 1).

NoVABOT is designed with a focus on inclusivity, featuring an interface that integrates both graphical and textual elements. This design choice aims to accommodate a diverse range of users with varying needs. For instance, the tool holds potential as a support for students with ADHD, benefiting from its visual aids and some colour-coded highlights of key elements of the function. Furthermore, the NoVABOT interface complies with the Web Content Accessibility Guidelines 2.1, ensuring comprehensive technical accessibility for users with visual impairments. The overall project design aims to enable them to access diagrams, fostering their development of diagrammatic thinking.

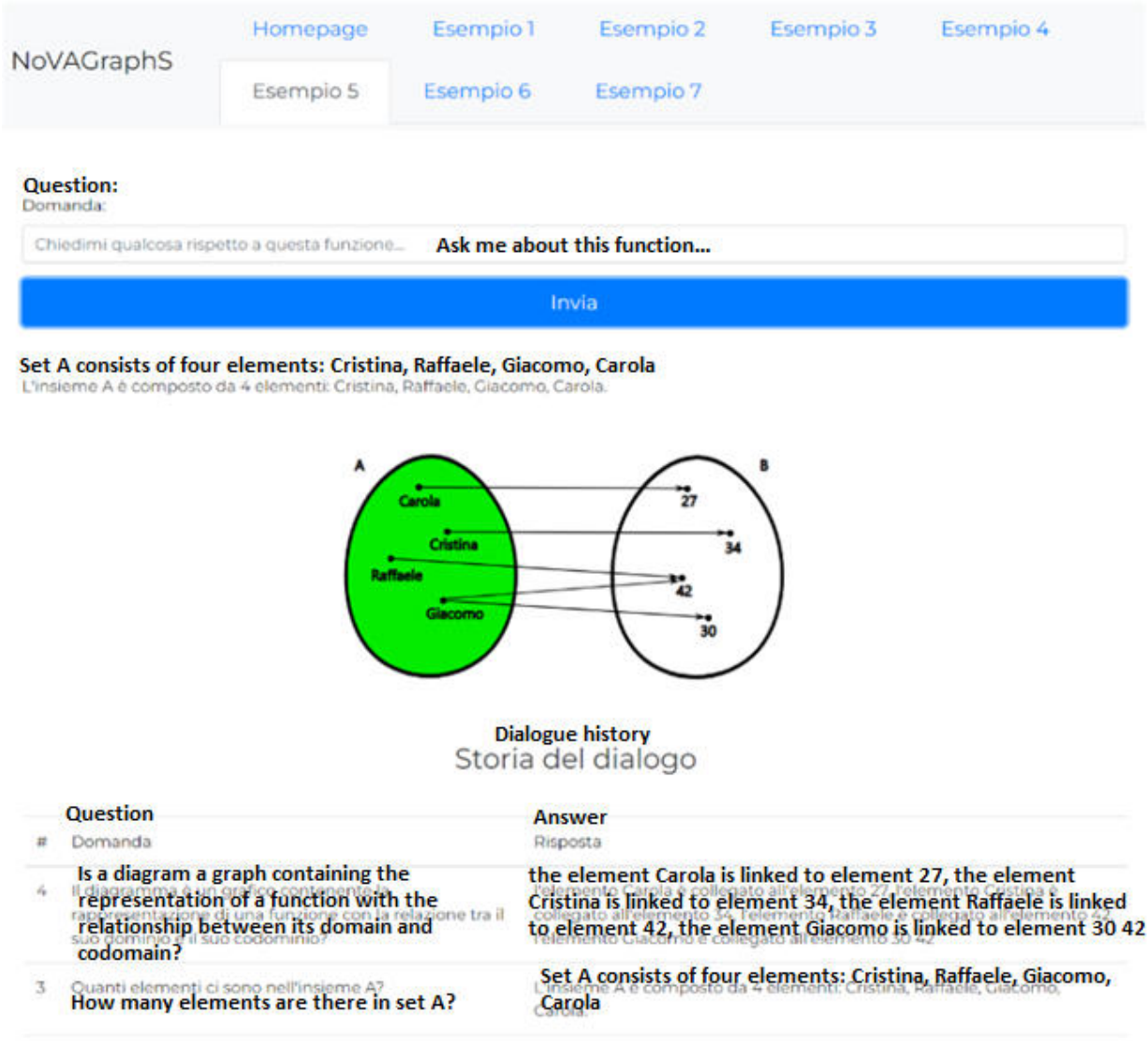

Figure 1. NoVAGraphS interface while exploring an example
[in bold authors' English translation of the texts on the screen, originally in Italian in grey]

Theoretical Framework

Charles Peirce suggests that much of mathematical thought involves observing or visualising results and patterns that emerge from manipulating various types of diagrams (Dörfler, 2005; Peirce, 1965). He defines a diagram as a sign "which is predominantly an icon of relations and is aided to be so by conventions. Indices are also more or less used" (Peirce, 1965, 4.418, 1903). Diagrams' key feature is their iconic nature, as they primarily serve to represent relationships (Bakker & Hoffmann, 2005). For Peirce, diagrammatic reasoning consists of three stages.

- The first stage is to construct diagram(s) using a representation system. This representation is required to show the relationships that are important in a given mathematical object or problem, and it can be done by different means, such as drawn sketches or software.

- The second stage is to experiment with the diagram(s). This experimentation must follow the rules of the specific representational system, so possible actions on a diagram are constrained. The important feature of this stage, as Bakker and Hoffman (2005) point out, is that thinking in general terms is not enough. The ability to observe is essential to act on the diagram.
- The third stage is to observe the results of experimenting and reflect on them.

Considering the concept of diagrammatic reasoning, our broader research aim is to investigate how it manifests in blind mathematics students, or more generally, in individuals with visual impairments when solving mathematical problems. In this paper, we specifically examine how NoVABOT can facilitate the exploration and interaction with diagrams. Consequently, our research question is: How does diagrammatic reasoning manifest in blind students or solvers through the use of NoVABOT?

Methods

We conducted task-based interviews with four blind users, letting them explore NoVABOT. We have used the three stages proposed by Peirce to analyse the process of diagrammatic reasoning of users. Regarding the first stage, "constructing a diagram", in this case, it was we who initially produced a diagram. That is, we prepared and uploaded to NoVABOT six different examples of functions between finite sets represented (graphically) via Euler-Venn diagrams. Given the embryonic nature of this study, we decided to provide the solvers with ready-made diagrams. However, the overall aim of the research project is to also examine, in future studies, the construction process by the users. In NoVABOT there is of course not only the graphical representation. The software implicitly knows the relationships between the different elements. It is therefore able to answer questions posed by users, e.g. about the domain and codomain elements, or the relationships between these elements.
In the first phase of the interviews, we let the users experiment and explore the diagrams in the different examples by asking them to identify characteristics they considered important of the mathematical objects explored. Subsequently, we asked them to reflect on the observed characteristics, to recognise which functions were injective, surjective, bijective, or which were not functions at all. This phase was followed by a meta-reflection moment in which respondents were asked to give feedback on the software and whether and how it can support users with visual impairments in exploring relationships between mathematical objects, where diagrammatic representations are particularly useful in highlighting such relationships, but are inaccessible to them.

In this paper, we present an excerpt from the task-based semi-structured interview, conducted through an online remote calling, with Antonio, a physics graduate who lost his sight six years ago due to a degenerative condition. He regularly uses his personal computer with the aid of speech synthesis. Since our aim is “to portray ‘what it is like’ to be in a particular situation, to catch the close up reality and ‘thick description’ [...] of participants’ lived experiences of [...] a situation” (Cohen et al., 2007, p. 182), to effectively capture the perspective of blind users, we chose an exploratory approach and utilised a case study design. This methodology parallels those used in previous research on expert blind solvers (e.g., Maffia et al., 2023; Miragliotta et al., 2023, Maffia et al., 2025). In particular the last part of the interview, comprising the interviewee’s meta-reflection on his own exploration, turns out to be particularly informative in the case of Antonio, as he is an expert solver of mathematics,

and not blind from birth therefore able to provide interesting comparisons between different forms of mental representation making use of images and not.

To construct diagram(s)

As mentioned, we prepared and uploaded six different examples of functions between finite sets to NoVABOT. Following the work of Martins et al. (2023), who studied 10th-grade students' functional thinking through their understanding of graphical representations of functions, we selected examples that illustrate functions with varying properties. For instance, Example 1 represents a function that is both injective and surjective, and thus bijective; Example 2 depicts a surjective but not injective function. Example 5 presents a relation that is not a function, as one element of the domain has two different images. These examples were carefully chosen to enable users to explore regularities in the relationships between elements and to distinguish the different properties of the explored functions.

Antonio's interview through the Diagrammatic Reasoning stages

To experiment with the diagram

At the beginning of the interview, Antonio is asked to explore NoVAGraphS interface. He observes several examples and chooses to examine Example 5. In this example (visible in Figure 1), as in the others, the diagram is presented alongside an overall description of the situation, which is read via a screen reader. The description states: "In the figure, there are two circles. Each circle contains some elements. Oriented arrows start from elements in the first circle (A) and point to elements in the second circle (B)". This description provides a preliminary overview of the diagram's notable features, as if it was a "first glance" at the diagram. Then, there is the area where the user can write and pose questions to the chatbot. Here is the initial interaction between Antonio and the chatbot.

Antonio: Is the diagram a graph containing a representation of a function with the relationship between its domain and its codomain?
Chatbot: Set B consists of 4 elements: 27,42,34, 30.
Antonio: In set A, how many elements are there?
Chatbot: In the first circle (in the domain) there are four elements: Cristina, Raffaele, Giacomo, Carola.
Antonio: How many arrows are there in the graph?
Chatbot: Forgive me, I didn't understand. Could you try rephrasing the question using keywords such as set, connected to, associated with, or naming some element of the sets?
Antonio: Describe how the elements of the two sets are connected, please.
Chatbot: The sets in our example are two. The first set, set A, called the domain in this case, consists of 4 elements: Cristina, Raffaele, Giacomo, Carola. The second set, set B, known as the codomain, on the other hand, is composed of 4 elements: 27, 42, 34, 30.
Antonio: Describe how the elements are connected.
Chatbot: Element Carola is connected to element 27, element Cristina is connected to element 34, element Raffaele is connected to element 42, element Giacomo is connected to element 30, 42.
Antonio: Giacomo to which element is related?
Chatbot: The element Giacomo of set A is connected to elements 30 and 42 of set B.

At this point, Antonio states that the situation is clear to him. One of the interviewers asks him whether he has therefore understood what is depicted in the diagram, or whether he needs to ask more questions. Antonio replies as follows.

Antonio: So, I have one doubt, though. As it's described, I imagine on one side the set A with the four names – Carola, Cristina, Raffaele, Giacomo, if I'm not mistaken –; on the other side I imagine the four numbers 27, 34, 30 and 42, with Carola connected to 27, Raffaele I think to 34, Cristina I don't remember…to 30 maybe, and Giacomo to 30 and 42. And I picture arrows starting from the names and going like in a direction to… from the set of names, from set A to set B. That is, I picture it in that way. So, the arrows with the arrow pointing from A to B.

Then, he is asked if he can say something more about the representation, considering the information he has gathered.

Antonio: But then, if that is the case, I would say that... I would have said initially that it could be a representation on one side of the names and on the other side of the age of these people. But having already like a two-element link, it's not the age. Which is not a biunivocal relationship, because anyway, it has more elements of the domain...To one element of the domain correspond more elements of the codomain, in the case of Giacomo. Yes, a little bit of this.

In the rest of the interview, Antonio explores additional examples of injective and non-injective, surjective and non-surjective functions. For each example, he is asked to express the properties he notices. For reasons of space, further sections of this interaction are not included here. Instead, we focus on the concluding part of the interview, where Antonio is asked to provide his feedback on the software's functionality and usefulness for a blind person.

To observe the results of experimenting and reflect on them

In response to the interviewer's question, Antonio shares his comments.

Antonio: In my opinion, the interaction with a conversational chatbot – so precisely that one, it has a memory of what has been said – is adequate for this kind of exploration of mathematically complex graphs, because it allows you to ask for information little by little, to mentally reconstructing... Generally, let's say, to mentally construct "the overall", what the whole diagram looks like. In my opinion, it's a good interaction system for the exploration of complex graphs. It's analytical, just like the exploration of the person who has a visual disability can be: analytical, rather than synthetic; later, you do the synthesis yourself, mentally. I'm doing that, though.

[...]

And as I said, for example, I knew before that there were two groups of two sets, connected with elements, and these elements connected with each other. In order to understand, to reconstruct how it was made, I asked at first "what are the elements in the set?", so that I could understand what was in one set, what was in another set, and mentally represent it to myself. Then, the second step, I asked about the connections, so how the elements were connected, in the way to add this information to what I already had in my head.

So I was analytical, in the sense that I proceeded piece by piece: first, the sets, then the elements within the sets, then the connections between the elements, and then mentally I made the synthesis of this information to try to construct by/for myself, precisely, this mental representation. That's it.

Discussion and conclusions

Antonio's reflection serves as a valuable starting point for analysing how he explores the graph and investigating a blind user's diagrammatic reasoning, in dealing with NoVABOT. He begins by examining the various components of the diagram. Knowing that there are two sets, he first seeks to understand their composition. Subsequently, he investigates the relationships between the elements

of these two sets. Observing that an element of set A corresponds to multiple elements of set B, he asks a specific question to confirm this observation. Only at this stage is he able to piece together the overall relationship, recognising that one element in the domain is associated with two elements in the codomain. Although his terminology is not entirely precise, Antonio concludes that the example does not represent a function. An interesting aspect of this deduction is the role played by the example's context. Though not explicitly stated, Antonio infers that the elements in the second set represent ages, and the relationship between the elements in the two sets is "Person is X years old". The fact that "Giacomo is 30 and 42 years old" leads Antonio to deduce that this cannot be a function, as one person cannot have two different ages. It is notable how this insight, which might not be immediately apparent visually, shifts the focus from the "content" of the elements to the presence of a link between domain and codomain elements.

In conclusion, interaction with NoVABOT facilitates targeted exploration. The user can analyse various aspects of the diagram according to their choice, combining the gathered information to understand the fundamental relationships. The software does not offer a generic description but requires the user to engage in analysis by asking specific questions. This process aligns with the intentional activity advocated by Bakker and Hoffmann (2005) in the exploration phase of diagrammatic reasoning. The limitations observed in this excerpt stem from the fact that the dialogue system does not fully capture or maintain the immediacy of the graphical content provided by the Euler-Venn representation of functions. Although NoVABOT does not manage to preserve the immediate visual impact of the diagram, it does sustain the diagrammatic reasoning and, most importantly, ensures user autonomy. Autonomy is a crucial aspect in the context of higher education. As reported for example in Bernareggi et al. (2016) and Piroi et al. (2023), blind students traditionally rely on human intermediaries to access and work with complex diagrams, such as those in focus within the NoVAGraphS project. NoVABOT represents indeed an innovative digital tool for fostering blind users' autonomous exploration of graphical mathematical representations. The benefits of this opportunity for autonomous exploration were highlighted by Antonio and the other participants interviewed in this study.

Further studies are needed to investigate the software usability and inclusive educational potentialities. At this stage of research and software development, some elements remain unstable: determining the extent of knowledge and interaction competence the software should possess to remain responsive without replacing the user's responsibility for knowledge construction; identifying the most appropriate vocabulary for the chatbot – whether it should adopt a neutral, descriptive register (circles, arrows, dots) or a content-related one (domain, image, counter-image); and understanding what can be lost or removed from such representations in order for them to be equally useful but accessible. Addressing these questions may require returning to foundational issues in mathematical epistemology, particularly the conceptual poles around which such representations are structured.

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Precedents and explorative participation in mathematical discourse

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The paper examines the case of Carlo, a low-achieving student, who participated in a remedial intervention in which he worked with one- and two-dimensional representations of functions in a digital environment. Using a commognitive lens, we analyze two episodes involving task situations that an expert would recognize as similar, but that Carlo approached differently. Our broad aim is to explore how previously learned mathematics can interfere with remedial interventions, such as the one in focus.

Keywords: Digital environments, remedial intervention, commognition, inclusion, low-achievement.

Introduction

In Europe (OECD, 2023), one in three 15-year-old students struggles with basic mathematical proficiency. For instance, although functions are a cornerstone of mathematics learning, students often find it challenging to analyze or describe their behavior through Cartesian graphs, which are not static images but dynamic representations of relationships between variations (Antonini et al., 2020). Several studies (e.g., Baccaglini-Frank, 2021) highlight the potential of digital environments in supporting mathematics learning, providing immediate feedback, and fostering transitions between different representations of mathematical concepts. Moreover, Nachlieli and Tabach (2012), highlight the significance of learners recognizing different mathematical representations as equivalent. These characteristics also seem optimal in the context of remedial interventions.

The study presented is part of the larger research project, DynaMat, which focuses on designing and implementing remedial interventions for learning algebra and functions within digital environments. It targets 10th-grade students in Italy with a persistent history of low achievement in mathematics (for details: https://www.carme.center/progetto-dynamat/). Here we explore a student recalling prior mathematical experiences during a remedial intervention in a digital environment. Such a phenomenon, which we have not found described widely in the literature, seems rather frequent on others of our case studies within the DynaMat project, making the case under focus prototypical.

Theoretical Framework

We have chosen Commognition (Sfard, 2008) as our theoretical framework because it enables us to describe students' participation in terms of their actual possibilities and forms of participation rather than focusing on their deficits. Commognition indeed provides tools for a fine-grained analysis of students' discourse, emphasizing their gradual and continuous (or interrupted) participation in mathematical discourse. This approach prevents assumptions on what learners "can't do", overcoming the duality between "what the student has in their mind" and "what the student actually says." The goal is to better understand difficulties, moving away from a deficit perspective and instead embracing a participationist approach. In this paper, we will see how the proposed activities enabled a student with a long history of difficulties to engage in explorative participation and with agency

(episode 1). However, this participation can be hindered when previously learned mathematics interferes with the activity that the student is undertaking (episode 2).

In this paper we will use a specific subset of commognitive analytic tools.

Lavie and colleagues (2019) have distinguished between *task situation* (TS) - when someone feels bound to act (in school, the situation typically refers to assignments that teachers give students); and *task* - the student's interpretation of a *given task situation* relying on *precedent*, a situation similar enough to the current one to justify applying the same approach or actions. The process of identifying precedents relies on a preselection within a *precedent search space*, however, only certain elements will be selected as precedents for a specific TS. This selection occurs through *precedent identifiers* - specific features of the current TS that lead a person to recognize a past TS as a precedent. The same TS can lead to different precedent identifiers, resulting in different interpreted tasks for different students. The pair of interpreted task and procedure is called *routine*, which is a pattern of action. Routines will also be our unit of analysis.

A task situation that is recognized as familiar to one student may instead be unfamiliar to another. In this context, we refer to a TS as *familiar* when the student seems to recognize the TS as similar to one they have previously faced . In this case the student has identified a precedent identifier within the TS, allowing them to retrieve a precedent from their mathematical precedent search space. On the other hand, if the TS is *unfamiliar* - not recognized as familiar - the student does not seem to immediately recall any precedent.

In Commognition, a mathematical object is always a discursive object, defined as a *signifier* with all its *realizations* (Sfard, 2008). For instance, a Cartesian graph or an input-output table can be recognized by an expert as different realizations of the same signifier function. Instead, for students entering mathematical discourse on function, these are initially only *visual mediators* of their discourse. This may also be the case for many low-achieving or failing students.

When participation involves merely executing memorized procedures focused on process, it is defined as *ritualistic*. In contrast, *explorative participation* involves constructing meaningful narratives, focused on product. Explorative and ritualistic participation represent the extremes of a continuum, and a student's discourse typically falls somewhere between these extremes. Processes of de-ritualization, such as *substantiability* (the shift of attention from process to product) and *flexibility* (more procedures can be used to solve the same task situation), can help infer if participation is more ritualistic or explorative (Lavie et al, 2019). Moreover, regarding *agency*, students who participate exploratively are more likely to propose new actions or outcomes (Baccaglini-Frank, 2021).

Carlo is a 10th-grade student who joined the DynaMat project. His case captured our interest because of his seemingly different forms of participation in mathematical discourse across TSs on functions during the intervention. Specifically, his participation seemed to differ significantly in two TSs that seemed to be asking quite similar tasks according to the design of the activities (and therefore in the eyes of experts). These TSs are designed to be not immediately recognizable as tasks previously encountered at school and designed to be accessible, even without a specific procedure, therefore

unfamiliar. This paper aims to describe the apparent difference in participation between the two TSs in the light of commognitive constructs outlined in the theoretical framework.

Methods

Carlo participated to the second experimental cycle of DynaMat project between October 2023 and December 2023. All students joined the project voluntarily during the afternoon in the CARME research center. They initially participated in an individual interview, followed by five two-hour activity sessions with digital environments, conducted by a researcher team. The sequence of activities proposed to Carlo involved dynamic representations of functions, known as dynagraphs (Antonini et al., 2020). It begins with a one-dimensional representation of function, as two variables moving along parallel lines (Figure 1a), to promote awareness of covariation. The activities build up to the Cartesian graph by first rotating the line of the dependent variable by 90° relative to the line of the independent variable, so that the variations of the two variables are now perpendicular to each other (Figure 1b). Finally, perpendicular lines are drawn from each variable's position, creating intersections that define each point on the Cartesian graph as the variables change (Figure 1c).

These are the task situations we are going to analyze here:

TS1: The student is given a dynagraph with Cartesian axes and five different Cartesian graphs printed on paper; the tutor asks: "While dragging tick mark A, can you imagine the trajectory of the point (A, B) on the Cartesian plane? Which of the following graphs represents the trajectory of point (A, B)?".

TS2: The student is given a dynagraph with parallel lines, and the tutor asks: "Explore the file and draw below the corresponding graph in the Cartesian plane."

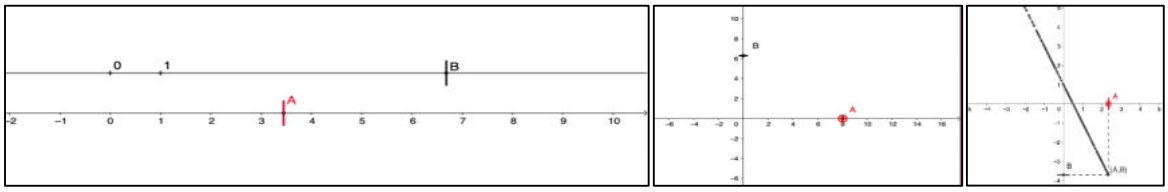

Figure 1: (a) dynagraph with parallel line, (b) dynagraph with Cartesian axes, (c) Cartesian Graph

Both TSs involve dynagraphs, which are usually not introduced in school, and Carlo, in the initial interview, states that he has only studied lines in the Cartesian plane. Moreover, none of the function realizations in these episodes (neither on paper nor in the dynagraph) are lines. For these reasons, we infer that the TSs presented unfamiliar tasks for him. In the TS design process it was essential for the researchers to share the same socio-cultural background as the students.

Both TSs aim to elicit discourse with transitions between multiple realizations of functions. The tutor tried to foster such discourse and she planned to introduce input-output tables. For example she would prompt Carlo by asking: "Could you use tables to write down some pairs of numbers that work?".

Carlo had access to a tablet displaying the dynagraph and to another one for writing or drawing. A tutor would pose the task situations, which could differ from the tasks interpreted by the student. We inferred Carlo's interpreted tasks by observing his procedure.

In the analyses below, both episodes (related to TS1 during session four and TS2 during session five) were transcribed with line numbering starting from 1. Between these two sessions, other activities were carried out.

Data analysis

Episode 1 – Carlo faced with TS1

The tutor reads TS1, which asks to envision the trajectory and identify the corresponding graph, between five different Cartesian graphs printed on paper (TS1). On the other hand, Carlo, after initially interacting with the dynagraph, immediately starts describing it. Thus, the task he interprets is to provide a verbal description of the movement of the tick marks.

3 Carlo: I see that, when I move into negative numbers with A, B rises very, very slowly [moves A to the left towards -∞], while when I lower it, it goes down a little faster [moves A towards zero, approaching from the left].
4 Tutor: Okay, so... when B goes down faster, I didn't understand.
5 Carlo: And then when A... when B goes down more, that is, when A keeps moving towards zero, B goes down faster. Then A, when it goes to 4, on the numbers, as the numbers increase [moves A around 5].

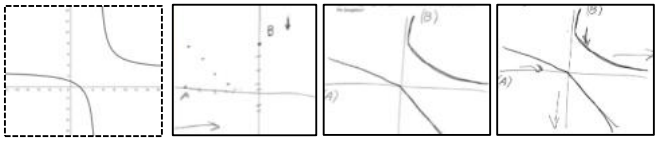

Figure 2: (a) selected Cartesian graph of TS1 ; (b) (c) (d) Carlo faced with TS1

Carlo spontaneously returns to the task situation (showing substantiability) and picks up the graphs chosen to answer the request [Figure 2a]. Now, is interpreted task seems to coincide with TS1.

19 Carlo: This [tracing the graph on the paper with a finger] is the movement of the dot, maybe of the intersection... like, okay, so if like, I don't know, A is here and goes this way, B is here and goes this way [draws the axes, inserts A and B, then the arrows in Figure 2b], this will gradually go like this, this, this, and like this, this, this. Then this goes down, like, I don't know, here, then here, then here, then here, and so on [inserts the ticks on the axes and the points].
20 Tutor: So these points you've drawn are the point (A, B)? The path it followed?
21 Carlo: And then when A reaches zero, B is below, so it forms something like this. This drawing, this is the line that A makes, no, wait. This is the line it makes.

In line 19 Carlo responds to the TS identifying the correct graph between the five different printed graphs, and he justifies his choice. First, he describes the movement of the point (A, B) by looking at the selected graph on paper, and then he starts plotting part of it on the Cartesian graph [Figure 2b]. This utterance shows both flexibility (as multiple procedures can be implemented to respond to the task situation) as well as substantiability (with the focus on the product, the selected graph). So, his participation is toward the explorative side of the spectrum. Then Carlo decides to return to the tablet and summarize his findings, showing agency. Carlo's interpreted task (line 29) seems to be to plot the movement of the point (A, B): he draws Figure 2c, confirming his initial choice (substantiability).

29 Carlo: [draws Figure 2c] It looks like the movement it makes. So this […] because when A increases… goes towards, that is, it grows because it goes towards zero, B decreases, as you can see. B goes this way and A goes this way [inserts the left arrows in Figure 2d]."

31 Carlo: Then when A, when A is in the positive range, B is also positive, and gradually as A moves away, going this way, B decreases [inserts the right arrows in Figure 2d]. Indeed, you can see that it goes lower and lower.

Initially, Carlo refers only to the movements of A and B, which he indicates with arrows in Figure 2b. Then, he makes a first attempt, even if discrete, to dynamically represent some of the positions reached by (A, B). Finally, thanks to the new dragging action, he manages to represent the trajectory as a static object that an expert would recognize as resembling the Cartesian graph of the function (it is not the graph of a function, but it closely approximates the trajectory of the graph shown in Figure 2a). All of Carlo's interpreted tasks are commensurable with those of the expert.

Episode 2 – Carlo faced with TS2

Initially, Carlo hesitated (line 5, "it's a bit difficult"), and the tutor, in accordance with the design, suggested introducing input-output tables as a further realization.

25 Tutor: Maybe it could help if you noted down some values where ticks pass through.

Line 25 changed the task interpreted by Carlo, who began solving the task in an unexpected way.

28 Carlo: Alright, I'll set it to 2.
29 Tutor: Okay, for example, when A is 2, B is on…?
30 Carlo: [He explores the dynagraph and marks the points (2;0) and (0;-2), then draws the ray from the origin passing through the point (2,-2) as shown in Figure 3a] How many points?
31 Tutor: Great, let's test it with other points, perhaps, to see how it behaves?
32 Carlo: At -8?

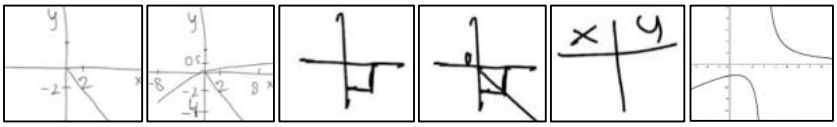

Figure 3: (a) (b) (c) (d) (e) Carlo faced with TS2; (f) expected Cartesian graph of TS2

Carlo marks -8 on the x-axis and -4 on the y-axis, then draws a ray from the origin to (-8, -4). He does the same for the point (8, 0.5), producing Figure 3b instead of the expected Cartesian graph (Figure 3f). The tutor tries to investigate his interpreted task:

37 Tutor: OK, so let's say what you drew what, what are these?
38 Carlo: Uh, various points, and I drew some lines
40 Carlo: All passing through zero
46 Carlo: [point (2;-2)] is an intersection
47 Tutor: Mmm, okay, so how do you find this intersection?
48 Carlo: By using these two numbers and drawing it like this [Figure 3c], you do something similar, and then this, being zero, makes a line like this [Figure 3d]
53 Tutor: Ok... Why did you draw the lines passing through the origin and that point?
54 Carlo: Because with a regular line… I didn't make a table like this x and y [Figure 3e], and I gave a value to x... I mean, I didn't have the equations; I just must look at the graph and take the values
55 Tutor: Okay, why do you make a table when you have the equations instead?
56 Carlo: Yes, I give a value to x, substitute it, that is, two values for x, two values for y, and it works out

By examining Carlo's implemented procedure, it appears that the interpreted task is related to plotting lines on a Cartesian graph, using an input-output table. Precedent identifiers that seem to elicit this

task are the tutor's request to "write down values" (line 25), the formulation of the tutor's question recalling the input-output procedure (line 29), and the tutor's suggestion to "test other points" (line 31). This inference is coherent with Carlo's mention of the input-output table in line 54 and Figure 3e. These precedent identifiers seem to trigger procedures that Carlo had previously memorized (at school), which he describes in detail, but that are not consistent with the TS proposed by the tutor. Since the very beginning, Carlo asks how many points he should consider (line 30) and whether starting with -8 is appropriate (line 32). In other words, he relies on the tutor to ask for validation of the procedure he is implementing. We interpret the tutor's question in line 53 as a request to explore the substantiability (using it to backtrack and address the task). Nonetheless, Carlo again goes through the procedure and compares it with another memorized one. The episode continues with Carlo plotting additional values on the graph and then exploring the behavior of the dynagraph near the vertical asymptote. At the end of the activity, the tutor explicitly asks him to clarify his use of the input-output table, and Carlo provides a detailed description:

99 Carlo: I put [the equation] in explicit form, so I write y equals and… then I change the value of x, perform the operation, and it gives me the result for y, and then with another value, it always gives me another result.

The tutor suggests giving an example for clarification, using the equation $y = x + 1$

105 Carlo: Like, for value 3, it becomes 4. Then another value is 4, and it becomes 5.

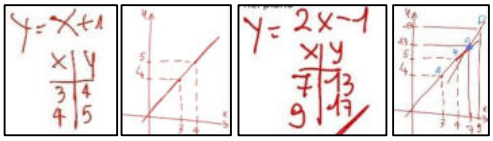

Figure 4: (a) (b) (c) (d) Carlo explains his procedure

Carlo writes Figure 4a and plots these points on a Cartesian plane [Figure 4b]. Then he asks the tutor for another equation to continue. The tutor suggests $\boldsymbol{y = 2x - 1}$.

113 Carlo: I put 7, so 7…13, and here 9,18, 17. So 7 and 9 and here [Figure 4c] 13, 17 [Figure 4d]. Uh, I do it like this, this would be like, so I need to connect these two points that are in the same table like this, and then find the common point they have, for example… this is the point they have in common.

Carlo's detailed description of the procedure allows us to infer that the task he interpreted might be about solving linear systems graphically. The procedure is described in lines 99-113 as a series of memorized steps, such as writing the line explicitly, substituting two x-values, finding the corresponding y-values etc. Carlo's focus is on the procedure itself, and hardly at all on its outcome - the point of intersection - which he mentions only once (line 113) when he repeats, upon the tutor's request, the procedure he had followed (lines 54 and 56). In line 55, the tutor's request is asking for substantiability; however Carlo's focus (line 56) remains on the procedure. So, the Carlo's participation in this TS is overall highly ritualistic.

Discussion

Carlo seems to perceive both tasks as unfamiliar with respect to what he was used to doing in school (where he certainly had never used a dynagraph). When facing an unfamiliar TS, he courageously

tries to respond, and in both TSs, Carlo recalls a precedent by that guides his discursive actions in the new TS he is currently facing. Let us take a closer look at what happens in each case.

Before addressing TS1, Carlo had already participated in activity sessions at CARME with dynagraphs involving parallel lines, where the TSs were mainly "describe the movement of the tick marks." He had also worked with dynagraphs with perpendicular lines, where the TSs were "imagine/describe/draw the trajectory of the point (A,B) in the Cartesian plane." Our hypothesis is that, when addressing TS1, despite it being unfamiliar, Carlo has sufficient prior precedents developed during the work at CARME, at his disposal. This allows him to recall a precedent consistent with TS1. His participation is therefore explorative, focused on the product, as intended in the design.

In TS2, however, Carlo's interpretation of the task does not align with the designed task (which is what the tutor had in mind). Hence, there appears to be a *commognitive gap* (a mismatch between the tasks interpreted by the participants in the same TS) between Carlo and the tutor. Carlo's procedure, which we only interpreted retrospectively, is similar to a procedure in textbooks for graphically solving systems of linear equations by identifying the intersection point of two lines on a graph. The tutor's intervention, which followed the design of the activities (line 25: “Maybe it could help if you noted down some values”) seems to offer Carlo a precedent identifier for the procedure he applies. Recalling such a precedent leads him to reinterpret (uncanonically) the task and carry out a routine ritualistically. We note that we can infer Carlo's interpreted task because we share his socio-cultural background (including typical school textbooks), and we are aware of the topics covered in Carlo’s class before the experimental activities.

Carlo’s different participation in the two TSs baffled us initially. However, the analyses carried out helped us reinterpret these episodes and make sense of them in a satisfying (at least for now) way. Summing up, in one case, Carlo recalls a precedent that is consistent with the TS and the designed task, probably thanks to his (brief) experience with the tutor and previous activities with dynagraphs. In the other case, he recalls a precedent from his school mathematics experience, which is not consistent with the TS and the designed task; such a precedent interferes with Carlo's success in our remedial intervention, locking him back into ritualistic participation.

Conclusion

Once we identified and described the phenomenon, we noticed that the interference of previous mathematical experiences that occurs as a student tries to act in an unfamiliar TS appears to be a significant factor for many of the students we worked with in our remedial interventions. In this sense, the present study is prototypical and it introduces a new perspective on previously observed phenomena (Funghi et al., 2023): students who, despite exhibiting both the capacity and willingness to create meaningful narratives through exploratory participation, nonetheless revert to a more ritualistic form of participation in certain task situations, in which previously learned mathematics seem to play a crucial role.

Although we cannot know for sure why Carlo's participation varied so significantly, we can advance some hypotheses. We inferred that the tutor's prompt, "find some values," serves as a precedent

identifier for the student and is, therefore, particularly significant from our perspective. This intervention occurs after a moment of difficulty for Carlo. We hypothesize that introducing new representations during challenging moments, along with a precedent identifier linked to a well-established ritual, may inhibit his authoring of endorsable narratives, preventing him from engaging in more exploratory participation (as observed in other instances, such as during TS1). This hypothesis encourages further research on the tutor's interventions during remedial interventions, with a specific focus on the timing of these interventions and the stability of the ritual routines.

In the DynaMat project we have only started to explore this direction, and hope to proceed in the future. For now we offer the notion of "interference phenomenon" as a potential dimension to consider when examining the effects of remedial interventions while taking into account the student's prior experiences.

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Fostering the unitizing concept in array representations by interpreting unconventional terms

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Active work with rectangular arrays plays a crucial role in developing the unitizing concept, which is the core idea of multiplicative thinking. Learners should learn to interpret composite units in the arrays flexibly. The research presented in this paper deals with the multiplicative thinking of second graders at an early stage of their learning of multiplication. In qualitative interviews they were asked to identify composite units in rectangular arrays. Verbal, gestural, and visual data were recorded. In-depth analyses of three children showed that they tended to look at the arrays in seemingly superficial ways. Productive irritations, such as identifying unconventional terms, often led them to explore composite units in a more flexible way. The findings underline the importance of discursive meaning-making processes for composite units in multiplicative representations.

Keywords: Multiplication, unitizing concept, eye movements, rectangular arrays, unconventional interpretation.

Mathematical representations and conceptual understanding

Mathematical representations play an important role in the mathematical learning process, especially of primary school children, as they visualise the underlying mathematical concepts and thus support meaning-making processes (Arcavi, 2003). If students are to understand and articulate how different representations are related, they must learn to connect these representations to each other (Duval, 2006; Prediger & Wessel, 2013). Since representations do not automatically reveal a concept, students must be supported in developing an interpretation of the underlying structure (Arcavi, 2003). As Arcavi (2003) writes, "learning to understand and be competent in the handling of multiple representations can be a long-winded, context dependent, nonlinear and even tortuous process for students" (p. 235). In-depth discussions about different mathematical interpretations of representations are particularly important in this process, so that an active exploration of the underlying mathematical structure can take place (Duval, 2006; Greeno & Hall, 1997).

Exploring the mathematical structure of multiplicative representations

Multiplicative thinking requires the ability to think in composite units (Steffe, 1992). In this so-called process of *unitizing* (Lamon, 1994), single units are formed into increasingly complex composite units. This requires a distinction between the multiplier, the multiplicand, and the product, with the multiplier indicating the number of units, the multiplicand indicating the size of the respective units, and the product reflecting a new composite unit (Götze & Baiker, 2023). For example, a rectangular array with three rows of four dots each yields twelve dots (Figure 1). The product must therefore be conceptualised as an entity in itself (Lamon, 1994). The independent coordination of the triad of 1) *number of units*, *2) size of units*, and *3) unit size of the product* can be described as multiplicative thinking. Because of the complexity of unit coordination, the multiplicative learning trajectory follows three broad stages (Boyce et al., 2024): In Stage 1, learners can only handle units of one.

Subsequently, in Stage 2, several of these units of one can be combined into units of the same size. Thus, children can construct composite units in the sense of unitizing (e.g., three fours), but this unitizing concept is not flexible yet. They have not internalised that there are other possible units. This distinguishes children at Stage 2 from children at Stage 3. In Stage 3, learners can work flexibly with composite units (Boyce et al., 2024). This means that students have a deep understanding that whole numbers can be interpreted as multiples of composite units so that they can, for example, reinterpret three fours as two sixes as well.

Rectangular arrays as representation for multiplicative thinking

Rectangular array representations (Figure 1) play a crucial role in developing multiplicative thinking, as they visualise the core idea of unitizing. They also allow flexible actions of regrouping. For example, a 3 x 4 array allows identification of other composite units such as two sixes, six twos, four threes, or two times three twos. Moreover, they provide insights into central properties of multiplication such as commutativity and distributivity and can therefore be used for flexible computation (van de Walle, 2020). Rectangular arrays are universal in the sense that they can be extended to other numerical ranges or number systems; thus they play a crucial role in multiple areas of mathematical content (Prediger et al., 2023). Because terms such as "5 x 3" do not show the unitizing concept in a meaningful way

> students should not only translate the multiplication 5 x 3 into an array model with five rows of 3 points each, but also [actively] explain how to see the unitizing structure in the rows ('five threes' or 'five sets of three') in order to verbalize the meaning of multiplication as unitizing (Erath et al., 2021, p. 249).

Directly addressing multiplicative structures in rectangular arrays using meaningful expressions such as "five threes" or "three fours" has proven to be supportive for all learners (Götze & Baiker, 2023). Although rectangular arrays represent the core idea of unitizing, and expressions such as "three fours" help to describe these representations, some children do not develop the capacity to think in composite units and are held back by a superficial view of arrays. These children have learned that a cursory glance at the array is sufficient, so they focus solely on the first column and the first row of an array to find the appropriate multiplicative term (Bolden et al., 2015). It is crucial that the idea of unitizing is actively developed while working with arrays to support children in reaching Stage 2 of the multiplicative learning trajectory. In order to reach Stage 3, it is necessary to be flexible in interpreting and identifying different possibilities for unitizing within the arrays.

But how should we design the individual learning pathways of primary school children, moving from Stage 1 to Stage 3? To gain further insight into this overarching research question, we draw on qualitative data from primary school children who were learning multiplication in the second grade. In this article, we focus on the following research question: *How can children's attention be drawn to (flexible) composite units when talking about multiplicative representations?*

Method

In our research project, we investigated the multiplicative learning pathways of 15 second graders. In diagnostic interviews at three points in time we examined the children's awareness of composite units

in rectangular arrays. The first interview took place before multiplication was introduced in mathematics classes, the second after 2 weeks of teaching and the third after a further 3 weeks. In the mathematics lessons that took place between the interviews, multiplication was taught with a focus on conceptual understanding, meaning that children were repeatedly asked to identify and describe the unitizing structure in rectangular arrays (e.g., "three fours"). Considering the current state of research (e.g., Götze & Baiker, 2023; Siemon, 2019), such activities help to develop children's multiplicative thinking. In the interviews, the interviewer presented different rectangular arrays on a flat screen and the children had to name one or more multiplication tasks that fitted the representation. For some arrays, different symbolic terms (e.g., "3 · 4" or "2 · 6") were presented for selection. The children then had to explain which term or terms matched the array. They were explicitly allowed to use gestures to support their explanations. In addition, eye-tracking was used to collect gaze data during the interviews. Eye-tracking provides further empirical insights into children's thought processes (Strohmaier et al., 2020). As primary school children are often challenged when attempting to explain the unitizing structure in rectangular arrays, eye-tracking data can provide further insights into children's conceptual understanding. This is because the recorded gaze movements may allow inferences to be made about ongoing cognitive processes. It has been shown that gaze data cannot always be interpreted unambiguously, so it is advisable to take other data into account when interpreting them (Strohmaier et al., 2020). Therefore, in addition to eye-tracking data, we also included verbal explanations and gestures to gain deeper insights into the children's multiplicative thinking. The focus of the analyses was on the extent to which the children flexibly addressed the unitizing concept visually, verbally and gesturally.

Insights in the data

In the following, we give empirical insights into the data collected from three children (Noyan, Edda, and Lia) in the third interview, after 5 weeks of teaching multiplication in mathematics classes. These three children were chosen because we considered them to be representative of the whole group. Their multiplicative interpretations could also be observed in other learners.

In the following scene, Noyan was shown a 3 x 4 array. In addition, six different symbolic terms were presented (Figure 1). Noyan had to explain which terms he thought fitted the array. The gaze data showed that before naming the term 3 x 4, Noyan looked at the first row and the first column of the rectangular array and then at the symbolic term of 3 x 4 below the representation (Figure 1).

After naming the term 3 x 4, the interviewer asked Noyan why this term fitted the array:

Noyan:	Three times four.
Interviewer:	Why does it fit?
Noyan:	Because there are three rows of four dots here (*points to the first column and the first row*), so you could also say there are three fours.

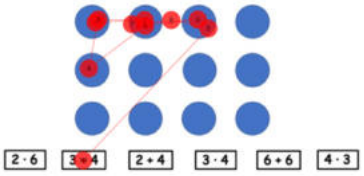

Figure 1: Noyan's gaze movement before naming the term 3 x 4 (the dot is typically used as a multiplication sign in Germany)

Noyan justified his choice by describing both the number of rows and the number of dots in the rows ("three rows of four dots") and emphasised this by saying "three fours". During this explanation, he pointed to the first column and the first row with his fingers. The eye-tracking data (Figure 1) showed that he focused visually only on the edges of the rectangular array. Since Noyan was gesturing to and visually focusing on the edges of the array, it was unclear at this point whether he was using the verbal expression "three fours" in an automated way because he had learned that he should use such expressions. However, the verbal data also demonstrated that Noyan may have had a deeper understanding of multiplicative structures. While his eye-movements and gestures could only reflect the process of identifying a task that fitted the array, Noyan's explanation of "three fours" indicated that he was aware of how the concept of unitizing was based in the array (Stage 2).

Aware of the problem that even triangulated data cannot always be interpreted unambiguously (e.g., Götze & Seidel, 2020), further questions were used to gain deeper insights into the children's multiplicative thinking. These were chosen to trigger "productive irritations" (Nührenbörger & Schwarzkopf, 2015, p. 319) that would help us gain further insights into the students' multiplicative thinking competencies at Stage 3. To this end, the interviewer asked Noyan if the term 2 x 6 also fitted the 3 x 4 array:

Interviewer:	Mhm. Would the term two times six also fit here?
Noyan:	(4 sec. pause) Yes.
Interviewer:	Why?
Noyan:	Because there are two sixes then.
Interviewer:	Where exactly?
Noyan:	So, if I do it like this (*points between the second and third column from top to bottom*), then there are six here and six there (*points first to the left and then to the right side of the array*).

Triggered by this question, gaze data showed that Noyan first looked at the two columns on the left side of the array and then at the first dot in the third column (Figure 2, left part):

Figure 2: Noyan's gaze movements while identifying (left) and explaining (right) why 2 x 6 fitted the 3 x 4 array

Noyan seemed to identify six dots in the left half of the array and the other six as a group in the right half, more or less simultaneously. After the interviewer asked why the term fitted, Noyan verbally

described “two sixes”. During this verbal explanation, he looked at the left half and the right half of the array with vertical gaze movements on the second and third columns, suggesting a vertical division of the picture into two sixes (Figure 2, right part). When asked where exactly these sixes could be seen, Noyan gesturally showed and verbally explained the vertical decomposition of the rectangular array. In summary, triggered by the impulse to look for an unconventional term in the array, Noyan's eye movements and thus the gaze data changed. He explicitly focused on multiplicative structures verbally, gesturally, and visually. He also showed that he was able to flexibly identify composite units in multiplicative arrays (Stage 3).

Other children in our data showed slightly different approaches. A typical example was Edda. Like Noyan, she perceived the 3 x 4 array by looking at the edges. She justified the fit between the array and the term 3 x 4 by saying “There are three fours.” During this verbal explanation, Edda glanced superficially at the edges of the rectangular array and did not focus visually on the composite units. In addition, she did not use any gestures. Again, it remained unclear whether she was aware of the units described (three fours) in the array or whether this was an automated response to the question of why the term fitted. However, the interviewer’s question as to whether 2 x 6 also matched the picture led Edda to take a different approach. The eye-tracking data showed that she focused on the first three dots in the first and second rows and then looked at the last column and the last row (Figure 3, left part) before verbally confirming that the term fitted the array. The gaze data suggested that she identified a six in the upper left part of the array and a six made up of the dots in the last column and row (Figure 3, right part).

Figure 3: Edda’s gaze movements when asked for an unconventional term in the array (left) and the compact interpretation of her data (right)

This visually generated interpretation was confirmed by her subsequent verbal explanation and gestures:

Interviewer: Why?
Edda: Because (...) if you (...) put these dots (*circles the first three dots of the first and second rows with her finger*) together, that's six and if you put these dots (*points to the last column and the last row*) together, that's also six.

Edda's verbal explanation, accompanied by her gestures and gaze movements during and before this explanation, showed that she reinterpreted the 3 x 4 array as representing two sixes. Triggered by the question of whether 2 x 6 fitted the picture, Edda structured the array individually and was able to reinterpret the three fours described at the beginning as two sixes in an individual way. As with Noyan in the previous example, the impulse to look for an unconventional term in the rectangular array led to a change in the gaze data and unfolded more flexible multiplicative thinking competencies (Stage 3). Edda explicitly focused on composite units verbally, gesturally and visually and was able to reinterpret the array in a multiplicative but in a slightly different way to Noyan.

While the two children described so far were able to move away from the seemingly superficial pattern of looking at the edges, this was not yet possible for Lia. She also focused on the edges of the rectangular array and named the term 3 x 4. Lia justified her choice in a very similar way to Noyan and Edda, while still visually focusing on the first row and column:

Lia: Eh, um, there are three rows and there are four dots in each row, so there are three fours.

From her verbal explanation it could be assumed that Lia, like Noyan and Edda, was thinking in composite units. Again, the question of whether the term 2 x 6 fitted the array was used for productive irritation:

Interviewer: Would the term two times six also fit here?
Lia: No.
Interviewer: Why not?
Lia: Because there are three rows and (…) yes.

The gaze data showed that Lia was still focusing exclusively on the edges of the array (Figure 4).

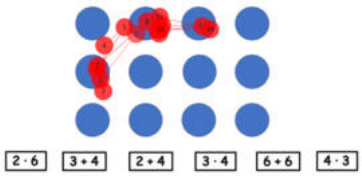

Figure 4: Lia's gaze movements when asked for an unconventional term

She verbally emphasised that there were "three rows," presumably meaning that 2 x 6 could not fit the array because there were not two rows. Consequently, unlike Noyan and Edda, Lia was not able to interpret the composite units in the array in a flexible way. Therefore, she had not yet reached Stage 3 of the multiplicative learning trajectory. The next learning step for children like Lia would be to disengage from such procedural approaches and focus on different possibilities of creating composite units represented in the array.

Conclusion

The data from the three children described in this paper and from other children in our study showed that many children take a superficial first look at multiplicative representations very early in their multiplicative learning trajectory. By asking for unconventional interpretations and thus by productively irritating them, these superficial glances can be transformed into deeper conceptual approaches that explicitly focus on different composite units in the rectangular arrays, which is indicative of Stage 3 of the multiplicative learning trajectory. Both verbal and visual data showed that requiring learners to make unconventional interpretations can be cognitively demanding. As a result, children often revise their initial interpretations and take an in-depth look at multiplicative representations. This creates learning opportunities not only to actively recognise and work with composite units in such representations but also to flexibly reinterpret them. In this way, a transition from Stage 2 to Stage 3 of the multiplicative learning trajectory can be achieved. Both Noyan and Edda changed their initial procedural approaches and were able to reinterpret the described "three fours" as "two sixes" by moving away from focusing solely on the edges of the array.

Moreover, Lia's example showed that even if learners can name a multiplicative term that fits an array, and even if they can justify their choice in a meaningful way, this does not automatically mean that they can identify those described composite units and handle them in a flexible way. The examples showed that array representations can run the risk of being superficial: Children may give a correct answer without full conceptual understanding. Lia's triangulated data showed that she was not yet able to identify composite units in the array as she was visually focusing only on the edges. She was somewhere between Stage 1 and Stage 2 of the multiplicative learning trajectory, as she knew that she needed composite units but could not identify them in the array representation. Teachers therefore need to be aware that to enable all learners to think multiplicatively in a flexible way, it is important to address composite units directly in concrete or graphical representations. Otherwise, there is a risk of developing and consolidating a superficial knowledge of composite units. This knowledge can prevent an in-depth understanding of composite units and prevent learners from reaching Stage 2 of the multiplicative learning trajectory. Consequently, the question "Which term fits the array?" combined with the request to show the units in the arrays with the help of gestures or markings seems to be essential.

However, this is not sufficient for achieving flexibility in unit coordination (Stage 3). Productive irritations and thus uncommon interpretations of multiplicative visualisations become necessary. Identifying different possibilities of grouping dots in a rectangular array and switching between these different interpretations seems to be very fruitful. Our data show that the unitizing concept in concrete or graphical representations must always be negotiated discursively, so that the different composite units become obvious. Otherwise, representations may become meaningless to children at an early stage of their learning and may inhibit further multiplicative learning. A limitation of our study is its small sample size. Future research should investigate whether the observed results can be applied to larger samples. In addition, because the interpretation of even triangulated data is not always clear, data should therefore be interpreted with caution and in the context of broader findings.

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Negative impact of making a drawing on problem solving: An eye-tracking study on non-linear geometry problems

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The ability to solve problems is an important part of the mathematical curriculum. Prior research has revealed that students often apply linear solutions to non-linear problems, a phenomenon known as linear overgeneralizations. Linear overgeneralizations were found to increase when students were asked to make a drawing. The present study aims to explore this negative effect by providing insight into the role of making a drawing in the solution process of non-linear problems. Eighteen secondary school students were asked either 1) to make a drawing or 2) to make a drawing and highlight the relevant area in the drawing before problem solving. Data were collected by recording the eye movements during task processing and using a stimulated recall interview. Analysis revealed difficulties with the scale of the self-generated drawings and a lack of conceptual content knowledge, which explained the observed negative effects of making a drawing on students' performance.

Keywords: Self-generated drawings, eye-tracking, non-linear geometry problems, linear overgeneralizations, problem solving.

Introduction

Promoting the ability to solve problems is an important goal of educational standards (Rott, 2015). Mathematical problem solving implies that a problem cannot be solved with directly available methods, i.e. barriers must be overcome while solving the task. Studies conducted by De Bock et al. (2007) revealed that students often have difficulties solving so-called non-linear geometry problems, where the area of similar figures has to be determined by a given scaling factor. Despite various teaching instructions examined in the past, many students try to solve such problems by linear overgeneralizations, i.e. by applying a linear model in an inappropriate context (De Bock et al., 2007; Krawitz & Schukajlow, 2020). Explanations discussed for this extensively researched phenomenon include the presence of a linear misconception that can make such tasks appear routine, but also the use of linear overgeneralizations as a surface strategy or in the absence of an alternative due to a lack of mathematical knowledge (De Bock et al., 2007; Greer, 2010; Modestou & Gagatsis, 2010). Self-generated drawings are considered a powerful heuristic for mathematical problem solving, with high drawing quality and strategic drawing knowledge being important indicators of a positive effect on performance (Krawitz & Schukajlow, 2020; Rellensmann et al., 2022). However, when solving non-linear geometry problems, De Bock et al. (2003) found that drawing had a surprisingly negative effect on task performance. Krawitz and Schukajlow (2020) confirmed this finding in a replication study, in which they also showed that improving drawing quality reduced the number of linear overgeneralizations, but did not improve performance. An in-depth analysis of the influence of drawing and drawing quality on students' solutions and the use of drawings revealed that calculating the area given in the drawing, which does not correspond to the measurements given in the task, was another source of errors. However, in most of the wrong solutions, no conclusions could be drawn about the role of the drawings in the solution process (Volbers et al., 2023). To clarify this research

gap, we designed a qualitative study using eye-tracking in combination with a stimulated recall interview to gain insight into the negative effects of making a drawing on performance and linear overgeneralizations. We pose the following research questions: 1. When and how do students use drawings in the solution process? 2. What are the reasons for linear overgeneralizations and what role do drawings have in the development of solutions that exhibit linear overgeneralization?

Theoretical background

Problem solving and non-linear geometry problems

Polya (1973) differentiates between (non-routine) problems and routine problems (p. 171). This article refers to the former as problems and the latter as routine tasks. A mathematical problem is a task that cannot be solved immediately using known solution methods and procedures, thus creating a barrier for the person who is supposed to solve the task. Mathematical problem solving describes the process of overcoming this barrier (Pólya, 1973). Important prerequisites for successful problem solving are, firstly, mathematical knowledge that has to be analyzed individually for each mathematical problem, secondly, the ability to select and use appropriate heuristics, e.g. by using self-generated drawings, and thirdly, metacognitive skills for planning, monitoring and reflecting on the individual solutions (Schoenfeld, 1985). In contrast, a task is a routine task, when a student can solve it by retrieving the solution directly from memory. The methods and operations for solving routine tasks are known and directly available. Whether a task is a problem or a routine task depends on the knowledge and skills of the task solver (Schoenfeld, 1985). The following model is based on the model of Pólya (1973) and describes the solution processes of mathematical tasks as follows (Figure 1):

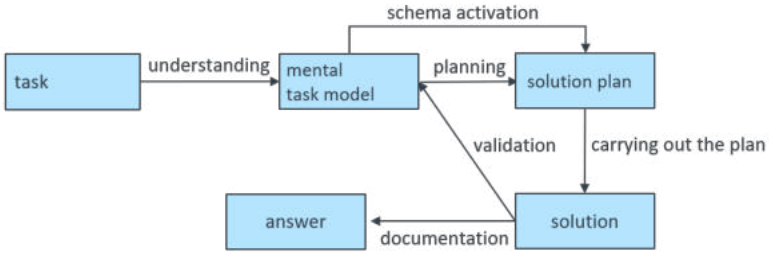

Figure 1: Problem-solving model based on Pólya (1973)

After reading the task, a problem solver creates a mental model. After that, one may activate a directly retrievable solution schema and stick to this schema for solving the problem, so that the planning phase can be skipped and a solution written down. This schema does not have to be correct and the task can be completed without a correct solution (Rott, 2015).

Non-linear geometry problems in this line of research usually involve calculating the area or volume of scaled surfaces or solids. In our study, we consider problems where the area of similar figures has to be determined by a given scaling factor. For example: “The side of square C is four times as large as the side of square D. If the area of square C is 1440 cm², what's the area of square D?” The solution to this problem can be found either by using the general scaling principle (when scaling by factor k, the area is enlarged/reduced by factor k^2, here D is 1440: $4^2 = 90$ [cm²]), or by identifying and applying appropriates formulae (e.g., $D = \left(\sqrt{1440:4}\right)^2 = 90$ [cm²]. These tasks are routine in the sense described above if one of these two solutions can be activated and executed directly as a schema. It has been

shown that students often have shortcomings in conceptual knowledge about areas and similar figures (Smith et al., 2011). Conceptual knowledge refers to an understanding of mathematical concepts, principles, and relationships and includes a deeper understanding of the ideas and structures on which operations and calculations are based (Rittle-Johnson et al., 2015). Conceptual knowledge about areas is based on the idea of measurement; it comprises an understanding of area formulas and their relationships as well as the ability to determine unusual shapes using appropriate paving strategies (Vollrath, 1999).

Linear overgeneralizations while solving non-linear geometry problems

Linear overgeneralizations are a widespread phenomenon that occurs in many content areas of mathematics, including solving the scaling problems described above, where the phenomenon has been shown to be resistant across different populations and countries. Several reasons are discussed for the occurrence and persistence of linear overgeneralizations: 1) There is a genuine linear misconception that is already present or is being developed while solving the problem. This misconception can be caused by the predominance of linear models for solving mathematical problems, which can give the impression that they are universally applicable, or by deficits in linear thinking itself (De Bock et al., 2007; Greer, 2010; Modestou & Gagatsis, 2010). 2) Linear overgeneralizations arise through the application of surface strategies, possibly facilitated by the processing of superficial task features. The root causes for this can be the intuitiveness of linear thinking on the one hand and the student's experiences in the context of school on the other hand: A decision about the operations required to solve a task is often successful when it is based on superficial task cues, such as certain keywords (e.g., “more” for addition or multiplication) or the given numbers (e.g. “all numbers have to be used for calculation”) (De Bock et al., 2007; Greer, 2010). 3) Linear solutions are given due to a lack of alternatives (Greer, 2010). If these reasons for linear overgeneralizations are related to the problem-solving model (Figure 1), a pre-existing misconception would imply the activation of an inappropriate schema (a schema for solving linear problems) and solving the task as a routine task. Linear overgeneralizations as a surface strategy or due to a lack of alternatives, as well as an acquired linear misconception, can be assigned to the planning phase. Concerning the above-mentioned skills required for problem solving, the use of linear overgeneralizations as a surface strategy can be seen in the context of difficulties related to metacognitive skills for planning, monitoring, and reflecting on the solutions. A linear misconception or linear overgeneralizations caused by a lack of alternatives may be due to deficiencies in conceptual knowledge about areas or the inability to activate appropriate heuristics.

Self-generated drawings as heuristics for problem solving

A drawing is an external visual representation, also called a visualization, that corresponds to the mathematical problem structure. If learners develop a drawing completely or in parts by themselves, these types of drawings are called self-generated drawings (Krawitz & Schukajlow, 2020; Van Meter & Garner, 2005). Like all visualizations, drawings have a product component, which refers to the visually represented result, and a process component, which refers to the process of creating the drawing (Arcavi, 2003). Learner-generated drawing describes the process and product of creating an illustration corresponding to objects and relationships described in a task (Rellensmann et al., 2022). Self-generated drawings are considered to be an important heuristic tool for problem solving as they can help in understanding the problem, in devising a plan, and in monitoring the solution process,

including validation of the result (De Bock et al. 2003; Pólya 1973; Stylianou, 2011). In the case of non-linear geometry problems, a scale drawing of the two areas could help to understand and analyze the situation. For example, learners may recognize which area is smaller or understand that the area does not grow linearly if the sides grow linearly. When planning and executing the plan, the scale drawing could support an appropriate paving strategy. Estimating the size ratio of the two sketches helps to validate the result (De Bock et al., 2007; De Bock et al., 2003). Studies investigating the effect of drawing on problem solving performance have produced mixed results: some studies have reported a positive effect of drawing, while others have found no effect or even a negative effect (Krawitz & Schukajlow, 2020). These inconsistent findings appear to be partly problem-specific and partly influenced by drawing quality and students' strategic knowledge about drawing (Krawitz & Schukajlow, 2020; Rellensmann et al., 2022). High drawing quality means that the drawing is correct and complete in terms of the important elements and their relationships (Krawitz & Schukajlow, 2020). Strategic knowledge about drawing includes knowing the characteristics that adequately represent the structure of the problem (Rellensmann et al., 2022).

Effects of making a drawing on linear overgeneralizations and performance

In line with the role of making a drawing as a heuristic for problem solving, De Bock et al. (2003) expected to improve performance in solving non-linear geometry problems through an intervention that made drawing an integral part of the solution process by asking students to generate a drawing before working on the non-linear geometry problem (see Figure 1, left). Surprisingly, the researchers found a strong negative effect of drawing on performance and an increase in linear overgeneralizations, despite drawings of good quality that students developed for solving problems. One possible explanation for the strong increase in linear overgeneralizations was that the students were focusing on linear activities by making a drawing (De Bock et al., 2003; Krawitz & Schukajlow, 2020). This negative effect of making a drawing was confirmed by Krawitz and Schukajlow (2020). Furthermore, it was shown that improving the quality of the drawing by highlighting the relevant area in the drawing (see Figure 2, right) did not improve performance, but reduced the number of linear overgeneralizations.

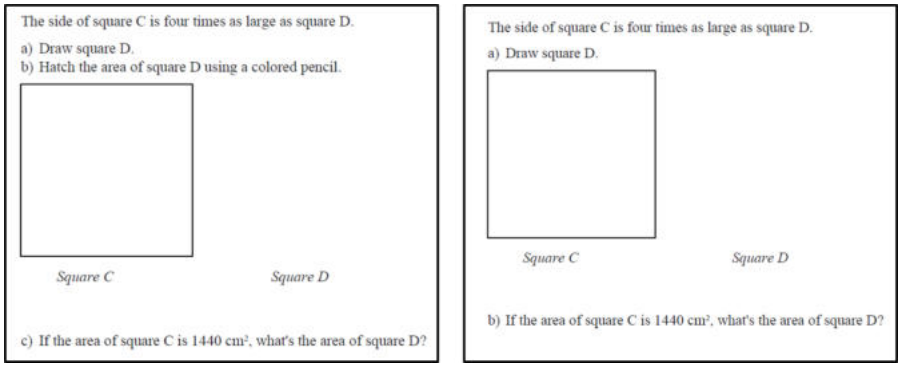

Figure 2: Sample items with drawing (left) and drawing with highlighting instructions (right)

A detailed analysis of the influence of drawing and drawing quality on the use of drawings while solving the non-linear geometry problem (such as paving or labeling with the information given in the calculation task) indicated the use of the self-generated drawing in only about 15% of cases (Volbers et al., 2023). In most of these cases, the students calculated the area of the figure in the

drawing, which does not correspond in scale with the measurements provided in the problem. This calculation of the area of the drawing (applied in about 10% of solutions) was found to be a second significant factor, after the linear overgeneralizations (applied in about 40% of solutions), that explained the negative effect of drawing on performance (Volbers et al., 2023). We did not find an explanation for this unusual error. For the majority of the solutions, and especially for the linear overgeneralizations, the role of the drawings in the solution process remained unclear.

Methods

The present study involved 18 students (10 female, mean age = 16.18 years) from upper secondary education, coming from two high-track schools and one comprehensive school. Students were randomly assigned to one of two groups. Before solving the problems, students in one group were asked to make a drawing, and students in the other group were asked to make a drawing and to highlight the quantity that they should calculate (see Figure 2). Both groups worked on a paper-and-pencil test. Students worked on the square problem described above (Figure 2), and then on a structurally similar problem about calculating the area of a circle (circle problem). We decided to include the circle problem because linear overgeneralizations occurred more frequently in circle problems than in square problems, and a paving strategy is difficult to apply to solve this problem (De Bock et al., 2003). Data were collected by recording student eye movements during task processing using Tobii Pro Glasses 3 and a stimulated recall interview (SRI) in which students were asked to explain their solutions using the gaze replay videos. To link eye movements and cognitive processes during problem solving, we used a qualitative content analysis following Thoemaneck et al. (2022). To minimize the ambiguity in interpretations of eye movements identified by the authors, the phases of problem solving (Figure 1) were taken into account. Eye movements were coded deductively-inductively based on category systems from prior research (Alemdag & Cagiltay, 2018; Thomaneck et al., 2022). Furthermore, the gaze data videos were triangulated with the statements from the SRI. The three categories of drawing use were “purely visual drawing use” (e.g., glances in the diagram area without corresponding action), “visual action drawing use” (e.g. measuring a side length), and “non-visual drawing use” (e.g. using the drawing lengths in calculations).

Findings and discussion

All students used one of the four solutions described in the theory section: They solved the problems by using formulas, applying the general principle, calculating the sketch area, or by linear overgeneralizations, which were the predominant solutions (17 of 36). Most of the drawings were correct. The analysis of students’ solution process to answer the first research question revealed the use of drawings by all students in at least one phase of solving the problem (see phases in Figure 1). Furthermore, we found evidence of the use of drawings for all phases of the problem-solving process. During the understanding phase, drawings were used purely visually to understand the task by recognizing which area is smaller, but not to recognize that the area does not grow linearly when scaled. During the planning phase, we observed the use of drawings, which were predominantly purely visual (Figure 3, left) and sometimes visually active, for each of the solutions described above. Students mostly explained these in the SRI by trying to remember calculation methods and formulas. This is in line with the tendency discussed above to solve area content problems by using formulas. Only one student developed the idea of paving as mentioned above when planning the solution to the

square problem (Figure 3, centre), but this student failed to transfer this strategy to the circle problem, which she explained as follows:

Elisa: Because the diagram didn't really help me much. I can't divide a circle in such a way that another circle results as with the square.

In addition, some other students expressed the idea of paving in the SRI but considered it suitable only for the square problem due to its shape. This result is in line with the shortcomings in the conceptual knowledge concerning areas and paving strategies described above. In the execution phase, this paving strategy was the only example of meaningful use of drawings. Eight students used the drawings to calculate the area of the smaller drawing for at least one problem. Five students recognized this error as part of validation by comparing the calculated result (2.25 cm^2) or the size of the drawn figure with the information given in the calculation task (1440 cm^2), but could not determine a correct result. Three students did not validate this incorrect result, indicating a lack of metacognitive skills to monitor the solution process. Some students who had determined the result using formulae or by linear overgeneralizations used the diagrams for validation by estimating the area ratios of the drawn figures (purely visual drawing use). For example, one student realized that the result of the linear overgeneralization must be wrong (Figure 3, right):

Niklas: I then realised that the result cannot be correct because the difference in size is relatively large.

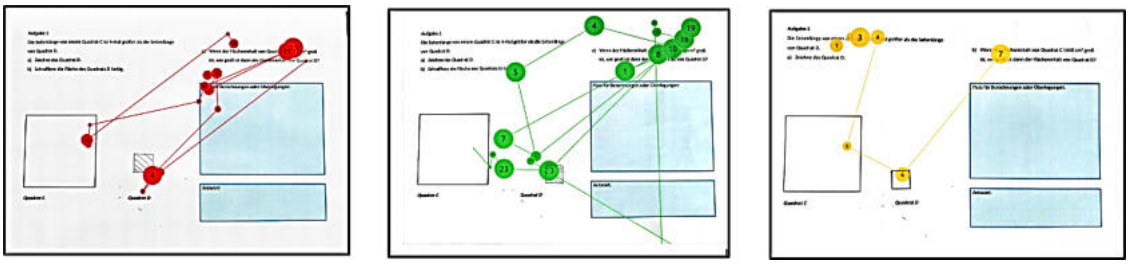

Figure 3: Examples for eye movements (purely visual drawing use)

With regard to the second research question (What are the reasons for linear overgeneralizations and what role do drawings have in the development of solutions that exhibit linear overgeneralization), we identified a plan development phase for all students who solved the square problem by linear overgeneralization; thus schema activation due to a genuine misconception and solution as a routine task was not identified. However, some students did not have a plan development phase and used linear overgeneralizations as schema by solving the circle task and justified this approach with the analogy between the two tasks. Concerning the square task we identified two types of 'linear solvers'. The first type applied linear overgeneralizations as a first and only solution strategy either as a surface strategy or due to a lack of alternative ideas for solving the problem. Three of these five students used the drawings unsuccessfully to remember calculation methods or formulas while planning their solutions. The second type applied linear overgeneralizations after an attempt to solve the problem using the drawing measures. After realizing the incorrectness of this solution, five students applied linear overgeneralizations due to a lack of alternative approaches. This seems to be another important cause of the increase in linear overgeneralizations after making a drawing, rather than an attentional bias towards linear variables caused by making a drawing, as suggested in previous studies (De Bock et al., 2003; Krawitz & Schukajlow, 2020). In only one case could the linear overgeneralizations be

reliably attributed to the expected transfer of linear reasoning from the drawing actions to the calculation of the area. Students of both types sometimes had doubts about the correctness of their solution and tried to validate it with the help of the diagrams. Three of them noticed that one figure was significantly larger than the other, but they did not realize their mistake. Two students recognized the incorrectness of the linear solution but were unable to correct it.

To summarize, for many students the mismatch between the scale of the drawings and the numerical values was a significant barrier to integrating information from the drawings and the text. A considerable amount of the incorrect solutions (calculation of the sketch area and some of the linear overgeneralizations) appeared to be due to this barrier. The main reason for this difficulty might be the lack of conceptual knowledge about areas and similar figures, which seems to be a major factor contributing to the negative effect of making a drawing. This lack of knowledge also led to a lack of positive effects of making a drawing. Although some students recognized their calculations as incorrect with the help of the drawings, they could not correct their mistakes. Furthermore, the paving strategy could not be transferred to the circle. Attempts to use formulas also limited the use of the drawings in planning, despite strategic knowledge of appropriate paving strategies. Other reasons are the lack of strategic knowledge about the use of a drawing during problem solving and a lack of metacognitive skills for problem solving. These results are in line with previous research and further extend prior studies by demonstrating that at least some students recognize their erroneous linear solutions but nonetheless adhere to them due to a lack of alternative approaches, with reference to specific content-related deficits. Our findings have both theoretical implications and practical consequences. The theoretical implications relate to the explanation of the negative effect of drawing in this particular task setting, which is related to a lack of conceptual knowledge rather than focusing on linear activities by making the drawing (De Bock et al., 2003; Krawitz & Schukajlow, 2020). A practical implication is a need to focus conceptual knowledge about areas in mathematics class, for example using visualizations to aid understanding.

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From research to practice: Visual representations as research-based tools and pedagogical mediators for teacher learning

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Visual representations play a pivotal role in effectively conveying research findings in reports, papers, and other publications. Research has also highlighted their importance in provoking productive conversations among teachers about their practice. This paper investigates how the Realization Tree Mediator (RTM), a specific visual representation, can serve both research and pedagogical purposes. Findings show that the RTM functions as a dual-purpose tool: 1) as a research instrument that explicates mathematical objects at the core of a task and evaluates learners' mathematical engagement, and 2) as a pedagogical tool that facilitates teacher learning towards ambitious teaching practices. This dual functionality is discussed in light of bridging the gap between research and practice in mathematics education.

Keywords: Teaching representations, research visualization, teacher learning, teaching practices.

Introduction

Visual representations play a crucial role in mathematics education research and teacher learning. In research, graphs and diagrams effectively communicate findings and encapsulate complex theories and data (e.g., Dai et al., 2022). In teacher education, representations like concept maps and video clips facilitate pedagogical discussions and help teachers learn about complex teaching practices (e.g., Herbst et al., 2011). While the value of such representations in both domains is recognized, more research is needed to understand the benefits of using the same representation across domains, bridging the gap between research and teacher practice. This paper explores how visual representations developed for research purposes can also serve as learning tools for teachers.

Visual representations for research and teacher learning

The power of visual representations extends beyond mere illustration and communication of findings (McCrudden et al., 2015). Researchers use them to summarize complex data, highlight patterns, and elucidate theoretical components (Few, 2009; Knox & Kontorovich, 2023). Theory guides research by providing foundations for formulating questions, selecting methodologies, and interpreting results (Bikner-Ahsbahs, 2019). While theory serves as a cohesive force that helps research inform educational practice and interventions, practitioners often find theoretical constructs difficult to access (Trowler & Cooper, 2002). Visual representations thus become particularly valuable in educational contexts, making research implications accessible to practitioners (Cai et al., 2019).

Teaching representations, such as video excerpts (Crespo, 2016) and animated teaching stories (Herbst et al., 2011), are critical for enhancing teacher learning (Grossman et al., 2009). These representations facilitate understanding of teaching practices and their classroom implementation. In teacher learning, they contextualize teacher conversations within instructional realities, simulate real-world scenarios, and enhance pedagogical skills by linking teaching aspects to future work. However, while these representations effectively discuss teaching-learning situations and specific practices, we

lack understanding of how they incorporate research findings and align with theoretical frameworks. When teaching representations are not explicitly connected to research, their potential impact on classroom practice may be limited. This disconnect could restrict the translation of theoretical insights into effective teacher development tools and might not fully capture the complexity of teaching situations as understood through contemporary research. Given the importance of visual representations in both domains and the need to contextualize research within teaching practice, this paper aims to explore the dual functionality of a visual representation – the Realization Tree Mediator (RTM), as both a research-based tool and a pedagogical mediator for teacher learning.

The Realization Tree Mediator (RTM)

The Realization Tree Mediator (RTM) (Weingarden & Heyd-Metzuyanim, 2023) is a research-based visual representation grounded in the commognitive theory (Sfard, 2008), which views mathematical learning as the development of proficiency in participating in mathematical discourse. The RTM (e.g., Figure 3) visually depicts classroom discussions by illustrating three key elements: the *mathematical object* at the core of the discussion, its various *realizations*, and the *saming narratives* produced during the discussion. *Mathematical objects* are discursive constructs that emerge through learners communicating their *realizations* – the different ways the object is manifested in the discourse. For example, the mathematical object "function" can be communicated through various realizations, such as symbolic expressions, verbal descriptions, or visual patterns. Developing proficiency in discourse involves talking about different realizations of the object as the same. This process is defined as *saming* and is operationalized in discourse by learners producing *saming narratives*, which involve the translation between realizations. The RTM visually organizes these elements, enabling the evaluation of the realizations mentioned during lessons and analysis of how they are connected through saming narratives. The research question of this study is thus: *How does the RTM, as a visual representation of mathematical activity, bridge research findings back to participating teachers?*

Method

Context: Professional development for teaching in heterogeneous classrooms

This study was conducted within a Research-Practice Partnership project – Heterogeneous Teaching of Mathematics, which aimed at developing pedagogies for teaching in heterogeneous classrooms. In 2023-2024, three Israeli middle schools participated, with 6 to 8 7th-grade mathematics teachers in each school joining Professional Learning Community (PLC) meetings every three weeks with two project researchers. These PLC meetings focused on addressing challenges in teaching heterogeneous classes, introducing effective practices, experiencing student-centered activities, and planning lessons. Between meetings, teachers implemented these activities and reflected on their experiences. This study examines one school and two PLC meetings centered on the Hanger Activity. In the first meeting, teachers experienced the activity as learners. Subsequently, one teacher, Tamara, tried it in her classroom. The second meeting focused on reflecting and discussing the activity's effectiveness.

The Hanger Activity

The Hanger Activity, developed by *Illustrative Mathematics* (illustrativemathematics.org), introduces the concept of equivalency using scales and equations. It begins with students observing

a picture of socks on two hangers illustrating equal and unequal weights, and answering open-ended questions about what they notice and wonder. Next, students analyze balanced and unbalanced scales, exploring the relationships between shapes representing different weights. This paper focuses on the third part of the activity, where students match hanger diagrams to equations (see Figure 1).

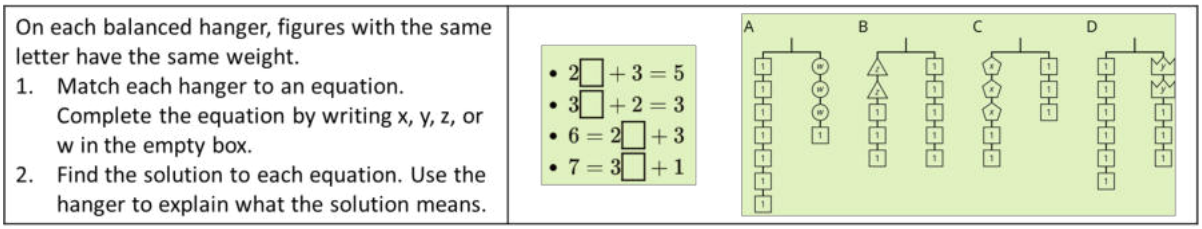

Figure 1: The third part of the Hanger Activity

Data collection and analysis

The two PLC meetings were recorded and transcribed. Field notes were taken during Tamara's implementation of the Hanger Activity in her classroom. Additionally, the data includes the full original lesson plan of the Hanger Activity as it appeared in *Illustrative Mathematics* (https://curriculum.illustrativemathematics.org/MS/teachers/2/6/7/index.html). The RTM was used as an analytic tool three times: 1) To explicate the activity's mathematical affordances by identifying the intended realizations and saming narratives; 2) To examine teachers' engagement as learners in the PLC meeting, explicating realizations and saming narratives they produced; and 3) To assess which realizations and saming narratives were produced during Tamara's lesson. The findings were then applied and redesigned as a pedagogical tool for teacher learning, resulting in two hypothetical RTMs depicting two Hanger Activity lessons (Figure 5), which were presented to teachers in the second PLC meeting. Teachers' conversations were analyzed based on their attention to different realizations and saming narratives appearing in the two contrasting RTMs.

Findings

As space is limited, I focus on Hanger A's analysis (Figure 1) and its matched equation $7 = 3w + 1$.

The activity's affordances: Saming between the balanced hangers and equations

The main mathematical idea of the Hanger Activity, as stated in the lesson plan, is to see how a balanced hanger is like an equation and how removing (taking off) weights is like solving the equation. It demonstrates two ways to show equality: a balanced hanger and an equation. The main concept embedded in such steps for finding the unknown amount relies on the idea of equivalency, through the principle of keeping the hangers balanced by "doing the same thing on both sides." The RTM, built for this study to analyze the task's mathematical affordances, shows how equivalency is reflected in different realizations of the unknown amount (Figure 2). At the top of the tree is the mathematical object at the core of the activity – the unknown amount. The branches show four types of realizations: hangers and equations in static situations (Branches 2 & 4), and in operational mode (Branches 1 & 3). The arcs between realizations show the potential saming narratives learners might produce while engaging with the activity, as depicted in the lesson plan.

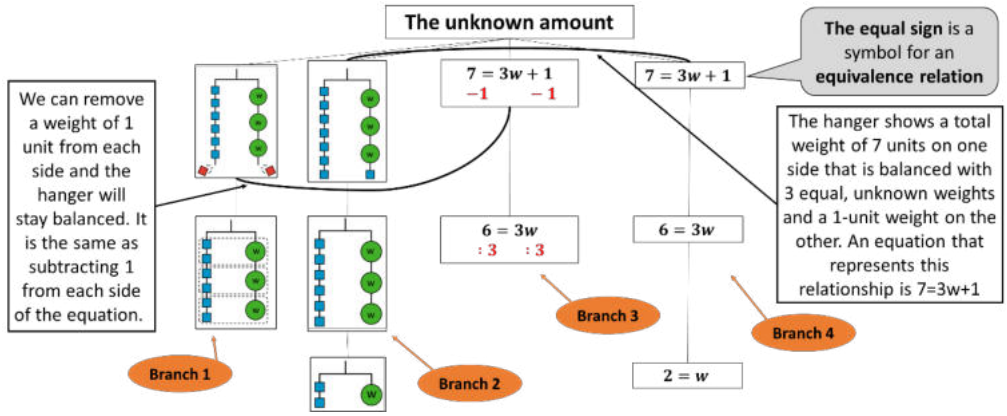

Figure 2: The RTM of the Hanger Activity: Emphasizing the mathematical idea of equivalency

Teachers experience the activity in the PLC: Unexpected evolvement of the covering method

Figure 3 displays the RTM depicting teachers' realizations and saming narratives produced as they engaged with the Hanger Activity during the PLC. Teacher-authored realizations are in black frames, with connecting arcs representing their saming narratives. Surrounding boxes provide examples of these narratives, such as Box 3-B-4 showing realizations 3 & 4 and the saming narrative (arc B) between them.

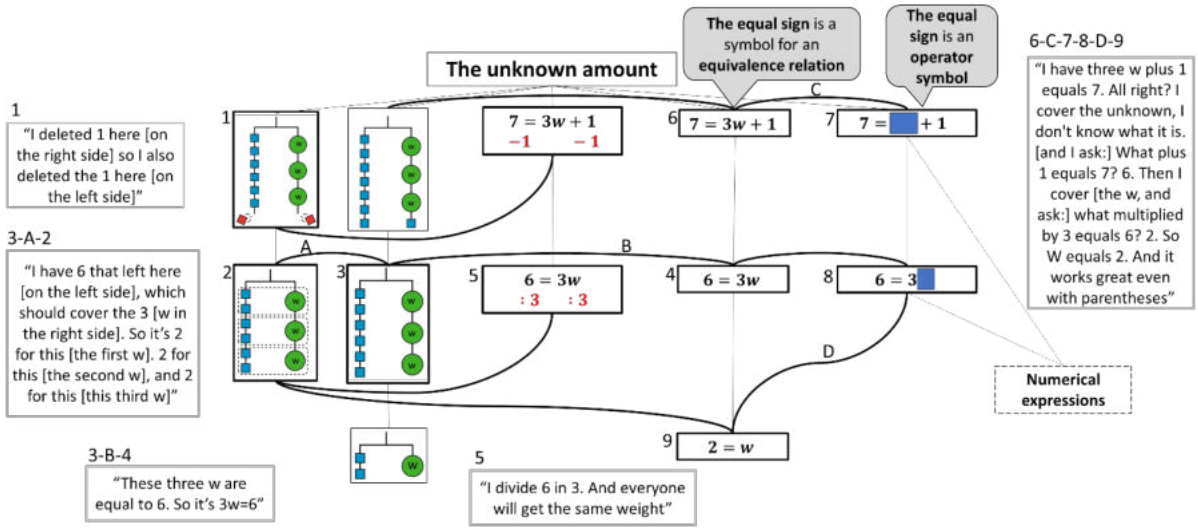

Figure 3: The RTM describing the realizations and the saming narratives teachers produced

Aligning with the lesson plan and the RTM in Figure 2, teachers found the unknown amount using both balanced hangers and symbolic notations. They operated on both hangers (e.g., Box 1) and equations (e.g., Box 5), producing saming narratives between symbolic expressions and hangers (Box 3-B-4), as well as between different hanger realizations (Box 3-A-2). The idea of equivalency remained consistent in teachers' discourse, whether working with equations or hangers. One teacher, for example, deleting a unit from each side of the hanger (Realization 1), explained: "What I do on one side, I do on the other side." In addition to equivalency narratives, the analysis revealed that teachers also produced narratives on numerical expressions, interpreting the equal sign as an operator

rather than a relational symbol (Figure 3, right branch). The numerical branch reflects the "covering method" – an additional method teachers introduced for solving equations. Box 6-C-7-8-D-9 (Figure 3) illustrates the covering method where teachers treat the equation as a sequence of operations to find a missing number, rather than as a relationship between equivalent expressions. While this method was valued by the teachers since it builds on students' familiar arithmetic operations ("they know this square from first grade"), it may emphasize a procedural view where the equal sign signals "find the answer" rather than expressing a relationship of equivalence between the expressions on both sides. This distinction matters because a relational understanding of the equal sign – seeing it as indicating that both sides have the same value – is crucial for advanced algebraic thinking. However, these distinctions in equation-solving methods and their implications for understanding the equal sign only became clear post-analysis. They were not discussed with the teachers during the PLC meeting.

Tamara implements the activity in her classroom: Insights from the classroom

After experiencing the Hanger Activity in the PLC, Tamara implemented it in her classroom. Figure 4 shows the RTM of her whole-class discussion. Similar to the PLC meeting, Tamara's class discussion involved three equation-solving methods. Guided by Tamara, her students first produced the covering method (Realizations 8-9) and algebraic manipulation (Realizations 6-7). As Tamara prepared to move on, two students introduced the balanced hangers approach (Realizations 3-4). Unlike the RTMs of the activity's potential (Figure 2) and the teachers' experience (Figure 3), the RTM of Tamara's lesson showed a few saming narratives. Students showed, for example, how the hanger (Box 2) can be written as a symbolic equation (Box 1), and how 6=3w (Box 7) leads to 2=w (Box 5) by dividing both 6 and 2w by 3. However, connections between balanced hangers (Boxes 3&4) and equations (Boxes 6&7) were absent. One of the explanations for such absence is in Tamara viewing the hangers as separate from "real mathematics," stating: "At some point, we have to say goodbye to the model of the hangers... In mathematical writing, I'm already building the equation, so I prefer the covering method."

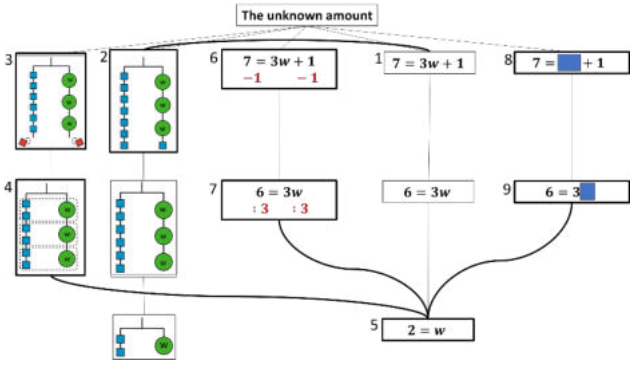

Figure 4: The RTM describing the whole-class discussion in Tamara's lesson

The RTMs for the activity's potential (Figure 2), teachers' experience (Figure 3), and Tamara's classroom (Figure 4) revealed two key findings. First, teachers valued the covering method and viewed it as a convenient approach to solving equations, overlooking the different functions of the equal sign in this method (as a numerical operator) versus identical operations on both sides (as an equivalency relation). Second, Tamara undervalued the hanger method, neither explicitly mentioning

nor connecting it to equations in her lesson. This suggests teachers may not recognize its potential for solving equations and communicating equivalency more accessibly (particularly in heterogeneous classrooms), potentially limiting students' meaningful engagement with core mathematical ideas.

Translating research outcomes into a visual representation for teacher learning purposes

The two findings—the equal sign's differing role in the covering method and using realistic representations in heterogeneous classrooms—are crucial for the project's participants as 7th-grade mathematics teachers who teach algebra in heterogeneous classrooms. Highlighting these mathematical and pedagogical insights can enhance their teaching practices, especially as they emerge from teachers' own experiences with the activity as learners and implementers. To effectively communicate these insights, two hypothetical RTMs of classroom discussion were created (Figure 5), conveying the analysis of teachers' experiences and Tamara's classroom implementation.

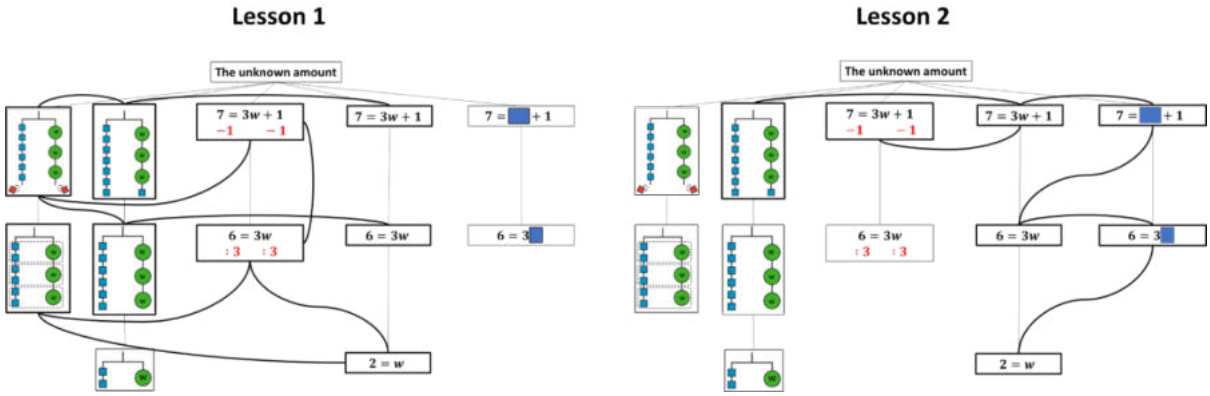

Figure 5: Two hypothetical RTMs used as a teaching representation for teachers' conversations

Lesson 1 exemplifies strong connections between balanced hangers and symbolic notations, emphasizing equivalency and accessibility for diverse students. Lesson 2 exemplifies the covering method alongside equations, minimizing hanger use. These hypothetical RTMs were presented at the next PLC meeting as a visual representation mapping imaginary students' mathematical engagement. Teachers discussed the differences between the two lessons and their mathematical opportunities. One teacher favored the covering method, stating that "it brings students back to a safe place... I love that the mathematics they learn is connected to what they've already learned." Another noted that the covering method does not emphasize "doing the same operations on both sides" as clearly as hangers and algebraic equations do. The PLC leader guided the conversation to focus on the different roles of the equal sign in each RTM. Teachers explored the learning opportunities and the added value of each method. A critical question emerged: "So, how should we teach it?" This prompted teachers to describe ideal lessons based on the realizations and saming narratives they deemed essential.

Discussion

This study explores the RTM as a dual-purpose visual representation for research analysis and teacher learning. Building on prior uses of realization trees for a priori task analysis of dynamic scale activity (Weingarden et al., 2024), the RTM was used in this study to examine the Hanger Activity's mathematical potential, teachers' experience with the activity as learners, and students' mathematical

opportunities when one teacher implemented the activity in her classrooms. The analysis revealed two main findings: First, in addition to operating on hangers and symbolic notations as described in the lesson plan, teachers unexpectedly mentioned the covering method when experiencing the activity. Second, Tamara primarily used numerical and algebraic equation-solving methods and made limited connections to the balanced hangers approach. The emergence of the covering method was surprising since it does not appear in the original lesson plan (Figure 2) and does not explicitly capture the activity's main mathematical idea of equivalency. While these findings are noteworthy, they primarily serve to illustrate the main goal of this paper: exploring how research outcomes, grounded in theoretical frameworks, can be effectively translated into teacher practices. This study shows how the RTM can visually translate theoretically-informed research into accessible, actionable, and practical knowledge for teachers. The utilization of research-oriented visual representations like the RTM demonstrates a dual potential contribution. First, it can help researchers better communicate complex research results and intricate theoretical constructs to readers – a crucial element in research dissemination (McCrudden et al., 2015). Second, and more pertinently to this study, it offers a means to address the research-practice gap – a concern raised by several researchers (Cai et al., 2019). These visual representations can be discussed in light of the evolving field of using theory in teacher education and its practical application (Lin et al., 2018). The theoretical foundations of the RTM played a significant role in revealing important insights about teaching practices, raising questions about how and to what extent theory should be made explicit in teacher learning programs. Visual tools like the RTM may potentially aid pre-service teachers in learning and applying complex educational theories, and ultimately improving their teaching practice. This study also contributes to the evolving landscape of teaching representations by exploring tools that are simultaneously visual, practice-oriented, and conceptually focused. These representations expand upon existing tools like video excerpts (Crespo, 2016) by integrating deeper, theoretically-grounded analysis within observed classroom practices. This approach creates a synergy between surface-level observations and deep conceptual analysis, providing a comprehensive view of teaching-learning situations and enhancing the current repertoire of teaching representations. While providing promising insights, this study is limited by its focus on a single activity and few teachers. Future research could explore the RTM and other research-based visual representations across various mathematical topics and with larger, diverse teacher groups. Comparative studies on the effectiveness of different visual representations in bridging research and practice could offer valuable insights for researchers and teacher educators.

Acknowledgment

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Estimating area: strategies and internal representations

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Previous research provides extensive insight into the development of students' strategies for external area measurement. However, less is known about strategies for area measurement on internal representations. Therefore, we investigated which strategies people report using when estimating area on internal representations of square shapes. We interviewed five participants estimating the area of squares flashing briefly on a screen to avoid external measurement. The results show that the participants reported using similar strategies for internal as for external area measurement. Further, they reported using both formal knowledge, such as multiplication learned in school, as well as more intuitive skills, not explicitly taught at school, such as visualizing and manipulating shapes internally.

Keywords: Geometry, representations, measurement, strategies, estimation.

Introduction

Area measurement plays a central role in school mathematics. It connects with other mathematical concepts, such as multiplication and composition of geometric figures, and has real-world applications, like determining the area of a wall to buy the right amount of paint (Clements et al., 2018; Outhred & Mitchelmore, 2000). Consequently, there is extensive research on the effects of instructional practices (e.g., Clements et al., 2018; Zacharos, 2006), students' understanding and misconceptions of area measurement (e.g., Zacharos, 2006), the developmental stages in students' understanding of area (e.g., Clements et al., 2018; Outhred & Mitchelmore, 2000), and the specific strategies that students use to measure and estimate area (e.g., Cullen & Barrett, 2020; Zacharos, 2006).

A notable strand of this research has focused specifically on describing strategies and representations for measuring area. For example, students may calculate area by drawing square units covering an entire space before counting the number of units (Cullen & Barrett, 2020; Zacharos, 2006). A similar strategy involves using physical objects, such as tiles or cardboard, to cover a rectangular space (Clements et al., 2018; Zacharos, 2006). Students may also use rulers to measure two sides of a rectangle and multiply the lengths of the sides to determine the area (Clements et al., 2018; Zacharos, 2006). Moreover, students sometimes rely on formulas, even when they are inappropriate for the given shape (Zacharos, 2006).

An essential aspect of learning mathematics is bringing one's internal representations of a mathematical object into coherence with its corresponding (and culturally agreed-upon) external representations (Goldin & Shteingold, 2001). While previous studies in area measurement provide crucial insights into how students' area measurement strategies develop—for instance, in response to instructional practices—they focus primarily on external representations, including visual diagrams on paper, physical objects like tiles or Lego bricks, or real-world spaces such as schoolyards or fields. Less is known about how students use internal representations of area and measurement. Estimating an area from an internal representation means visualizing an object in the mind's eye before

performing some mental operation on that internal representation, a process requiring general spatial visualization skills (Harris et al., 2023). Understanding how individuals operate on internal representations of area is essential, not only because it complements existing research on external representations, but also because it addresses the role of visualization in mathematics education more broadly (e.g., Harris et al., 2023). Research on people's internal representations of area might also address the ongoing debate on whether people have an innate magnitude sense (Leibovich et al., 2017), or whether our impressions of magnitudes, such as length and area, primarily result from internalizing external tools and strategies acquired in a socio-historical context (Núñez, 2017).

As a step towards understanding the role of internal representations in learning and doing mathematics within geometry and measurement, the aim of this study is *to characterize the strategies people report using when estimating the area of their internal representations of squares*.

Theoretical background

Mathematical objects are abstract and must, therefore, be represented in some form to be understood. According to Duval (2006), a mathematical representation is "something that stands for something else" (p. 104). It involves one or more semiotic forms, such as symbols, drawings, graphs, or language, that stand for—and must not be confused with—the mathematical object itself. In this view, representations are the core of mathematical understanding. Mastery of the different representations of concepts and the transitions between them is necessary both to communicate mathematical ideas and to perform mathematical procedures.

To better understand how people learn and do mathematics, Goldin and Shteingold (2001) distinguish between external and internal representations. External representations, such as physical graphs, diagrams, and algebraic expressions, exist beyond the human mind and communicate mathematical ideas; they can be observed and discussed in the classroom. In contrast, internal (mental) representations exist in the individual's mind, invisible to the external observer. They include mental images, strategies, and understandings that students develop in their minds when working with mathematics, and can vary from student to student, depending on their experiences and understanding (Duval, 2006). From this perspective, learning and doing mathematics involves interactions between external and internal representations: External representations can be internalized into the individuals' internal representations, and vice versa (Goldin & Shteingold, 2001), so bringing internal representations of mathematical objects into coherence with the corresponding external representations is an essential aspect of mathematical activity (Goldin & Shteingold, 2001).

The didactical literature mostly deals with representations constructed in cultural-historic contexts, such as graphs and symbolic expressions. However, people might also have innate representations of mathematical concepts. According to *magnitude sense theory*, our understanding of numbers and arithmetic depends on an evolutionarily ancient sensory system that automatically detects variations in magnitudes—and therefore provides internal representations of the natural numbers—without using verbal counting or any other culturally developed tools or strategies (e.g., Halberda et al., 2008). The human repertoire may include both innate and culturally acquired internal representations. However, in this study, we mainly focus on culturally acquired internal representations as described by Goldin & Shteingold (2001).

Methods

The study was designed to investigate how people estimate area. Five adults who had completed higher education (age 24-27, three women) were interviewed while performing estimation tasks on a computer following a think-aloud protocol (Reinhart et al., 2022).

The experiment was programmed with *JsPsych*, a JavaScript library for digital experiments. The tasks displayed a white background with a black square varying in area and rotation between tasks. A red 1-unit square of constant size was visible on the screen throughout the experiment. On each trial, a fixation cross first appeared for 500 ms at the center of the screen, indicating where the participants should place their gaze. Next, the square to be estimated appeared for 300 ms, also at the center, followed by a visual mask and a response box. The purpose of the mask and the short display duration of each square was to study the participants' internal representations by allowing enough time for perception but preventing precise calculations during the presentation. The experiments consisted of 15 trials with squares of varying rotation and varying area from 1 to 2401 units in random order.

During the experiments, the participants were interviewed individually for approximately 25 min. Following Kvale and Brinkman's (2009) framework for qualitative interviews, we used an interview guide to give the participants the same instructions and follow-up questions during the interviews. Before the experiment, the participants were given the following instructions: "We would like you to think aloud as you complete the tasks. This means that we want you to verbalize your thoughts as they occur, regardless of whether you believe your answer is correct or not. Our primary interest is not in the accuracy of your responses but in understanding your thought process as you work through the tasks. This includes any thoughts, even those that might seem insignificant or unrelated". The participants' task was to estimate the area of a black square by answering the question: "If the red square above has an area of 1, what is the area of the black square?". The response time was unlimited, but participants were asked to answer as quickly and accurately as possible.

The interviews were analyzed based on Braun and Clarke's (2006) guidelines for theoretical thematic analysis (TA). Before the analyses, a set of initial codes was adapted from Siegel et al. (1982), Dietrich et al. (2019), and Gandini et al. (2008). Examples of such codes were *eyeball*, *guess*, *counting strategies*, *calculation-based strategies,* and *benchmark comparison*. Subsequently, the interviews were transcribed and read in detail to identify codes relevant to the research question. These codes were extracted either inductively or deductively from the initial codes. In this process, we allowed both *semantic* (i.e., explicit) and *latent* (i.e., interpretative) codes.

After this initial analysis, some codes were combined to form an overarching theme, some were combined to form sub-themes, and others were discarded because of their lack of relevance to the research question. For example, all codes on "comparing" were united into a main theme. The resulting themes were systematically reviewed and assessed for internal consistency by comparing extracts from each code. Next, each theme was defined in an explicit coding guideline (Table 1). The analysis resulted in 5 main themes and 13 sub-themes (Table 2).

The initial steps of the analyses were conducted by the first author alone. To ensure intercoder reliability, twenty extracts were selected randomly by the first author and coded by the second and third authors based on the coding guidelines. The percentage of agreement was 88%.

Table 1: Examples of coding guidelines for two codes

Code	Coding guideline	Examples
Compare current square to previous squares	Use for a comparison with one or more of the previous squares and then adjust their estimate on the current square.	"I think this one is the same one I saw two times ago. The one of 225. It was awfully similar."
Iterating the unit square by counting	Use for counting the number of unit squares that can be placed inside the larger square.	"I am trying to count how many red squares I can fit inside the black one."

Results

The aim of this interview study was to identify how people estimate the area of internally represented squares. We found five main themes and 13 sub-themes of reported area estimation strategies (Table 2), which we describe below.

Table 2: Themes and sub-themes

Theme	Sub-themes	Statements (N)	Persons (N)
Compare to	Mental image, Mental measurement unit, Previous squares, Reference square, Other references on the screen	211	5
Iterating by	Eyeballing, Calculating, Counting	101	5
Sense	Certain, Uncertain	20	3
Spatial transformations	Spatial transformations	13	3
Evoke a visual memory	Successful, Unsuccessful	15	5

Compare

Comparing the square to a reference was the most frequently reported strategy for estimating area based on an internal representation (Table 2). We identified five main types of references: Participants compared the square to (a) previously estimated squares, (b) a measurement instrument, such as a ruler, (c) a mental image, such as an imaginary square, (d) the red unit square provided on the screen, or (e) other references on the screen, such as the text response box.

All the participants reported comparing the current square to previously estimated squares for at least one task (Table 2). They reported comparing it with the first black square they saw, the previous square, or any of the other squares they had seen previously. Interestingly, this suggests that the

participants can recall—at least approximately—what previous squares look like for an extended period of time.

> Participant 5: I want to say that this one also was bigger than the last black one. And since this one was bigger than the last one, then I said four, so it must be more than four, I say six. [comparison to the previous square].

Some participants described how they compared the square to a mental measurement instrument, for example, by mentally measuring the side of the red unit square or the side of the black square to be estimated in absolute units such as centimeters.

> Participant 1: Then I estimate the length of the box to approximately 10 cm.

Some participants reported comparing the squares to a mental image, meaning that they created a mental image used to assess the area of the square.

> Participant 3: Now I must create an image of how big a 25'er can be inside my head, that is, 5x5. And it can be really big. I answer 25 in area, 5 in side length.

The red reference square was also used as a basis for comparison.

> Participant 5: It was bigger than the red one. Quite a lot bigger than the red one, yes. I'm writing 10.

Some participants reported comparing the black square to other references on the screen, such as the response box, the fixation cross, or the entire screen size, to help them estimate the area.

> Participant 4: It takes up so much space. It took up a lot of space between where I write and the top. First, I think it looked like it took up half the space.

Iterating

All the participants reported the iterating strategy for estimating (Table 2). This involves iterating the red unit square to calculate the total area. Through the analysis, we identified three ways of iterating: counting, calculating, and eyeballing.

Some participants reported using the red unit square as a unit and counted how many times it could fit inside the black square.

> Participant 2: I start at the bottom, 1, 2, 3, 4 from the left, I also count 1, 2, 3, 4 upwards.

Others reported calculating strategies, like multiplying the width and the length of the square, often leading to square-number answers.

> Participant 4: 10 times 10, 100! There can be room for 10 of the red on each side, because it was a diamond figure. I can have room for 10 there, and 10 there, of the red one, and then there were 100.

Some reported making a rough, more direct estimate by eyeballing the square.

> Participant 4: Maybe there were 50 small ones inside the big one.

Sense

When the participants could not explain their estimation process, we categorized their strategy as sensing. In some cases, when the participants expressed a feeling or a guess about their answer, we described their estimate as uncertain.

> Participant 2: No, I have no idea. No, I don't know, I think I just have a feeling it was so much bigger and then, I don't know, it's just an assumption, a guess.

In other cases, participants said that they "just knew" or immediately saw the answer, with an expression of certainty.

> Participant 3: Now it's so clear to me, when you hear a word, you don't think about what it means, you just know it. Now I just see that it is a 2x2.

Spatial transformations

When explaining how they compared or iterated, some participants described that they made spatial transformations on the internal representations in their visual memory. That is, some participants described spatially manipulating the elements in their "visual scene".

> Participant 3: If I had a square on the PC, I would have pulled from one corner and pulled up so that a new square appears inside it, the same size that I felt that it was.

Evoke a visual memory

The participants typically reported retaining a memory of the square that they could visualize and refer to after it disappeared from the screen.

> Participant 5: So, I kind of have to remember what it looks like while I try to figure out how big it is. I try to store what the black one looks like inside my head.

Maintaining an internal mental representation of squares allows further cognitive processing, but the participants may lose the visual memory along the way, as illustrated in the following example:

> Participant 4: Now I've forgotten what it looks like.

Personal preferences

Several participants reported adjusting their estimates based on factors unrelated to the square, such as how aesthetically pleasing they found the numbers to be. For example, after making a temporary estimate, participants would round off or adjust the estimate to achieve a "nicer" answer.

> Participant 4: Let's say it was 49, and that's an ugly number, so I'll take 50.

This personal preference strategy is not included as a theme, as it was infrequent and does not explain the actual process of estimation, but rather how an estimate is modified (response bias). All answers were given unprompted as whole numbers.

Discussion

We have investigated how people estimate area from internal representations. The participants reported using several strategies for estimating the area, including retaining a visual image of the

square, comparing the square to previous squares, iterating a unit square to assess the area, directly sensing the area, and using spatial transformations.

The reported strategies include both formal knowledge acquired in school, such as multiplication and counting, and more intuitive skills that may not be explicitly taught, such as visualizing, manipulating, and comparing internal representations of shapes in visual memory. While all representations are important for the teaching and learning of mathematics (Goldin & Shteingold, 2001), further research is needed to determine how much each contributes to students' mathematical achievements. Answers to this question are important because they guide didactical practices, such as whether and how we emphasize internalizing external representations to support mathematical thinking and understanding. For instance, some education programs encourage building intuition through repeated practice on estimation tasks (e.g., Halberda et al., 2008), whereas culturally grounded practices emphasize children's reflection on structure, shape, or pattern to develop efficient strategies and strategy choices.

The results of this study also address the ongoing debate on whether humans have an innate magnitude sense. The magnitude sense is the ability to perceive, discriminate, and estimate continuous quantities such as size, density, and area (Leibovich et al., 2017), and just like our senses for light, smell, and taste, it is sometimes regarded as being an intuitive sense, inaccessible to introspection. Indeed, some participants in this study reported simply having a feeling about the area of a square, consistent with an intuitive magnitude sense. However, in most cases, the participants explained their internal strategies in detail, such as iterating a mental image of the reference square along a mental image of the black square. This result indicates that people primarily rely on cognitive procedures acquired in school to estimate internal representations of area, and that additional research into the mental processes that generate a rapid sense or feeling about a specific area are warranted.

Here, we investigated the internal estimation of squares. Squares are familiar from the school context and often associated with specific, learned formulas. This likely influenced the results, as participants often reported using school-learned procedures such as counting and multiplication. Thus, to validate the results reported in this paper, it would be valuable to investigate how people change their area estimation strategies depending on the shapes, e.g., in tasks involving arbitrary closed shapes. This replication can help identify whether participants use more sophisticated school measurement procedures, or whether participants are inclined to rely more on their basic mental visualization skills.

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TWG 25: Teaching and Learning of Calculus

An introduction to TWG25: The Teaching and Learning of Calculus

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This paper reports on the work of the Thematic Working Group 25 (TWG25) at CERME14, focused on the teaching and learning of Calculus. It explores the challenges related to conceptual understanding, formalization, institutional constraints, and interdisciplinary applications. TWG25 contributions covered a range of topics from real numbers to single and multivariable Calculus, integrating theoretical perspectives such as APOS theory, ATD, and Commognition. The report highlights innovative task designs, technological tools, and implications for instructional practices in diverse educational contexts.

Keywords: Calculus, Conceptual understanding, Task design, Institutional factors, Mathematics education.

1. Introduction and scope of the Working Group

The Thematic Working Group 25 (TWG25) on the Teaching and Learning of Calculus is a new group proposed in order to create a new community that investigates the teaching and learning of Calculus both at secondary and tertiary level, and at the transition between secondary school and university. Calculus plays a key role in mathematics education, both for its interesting features *per se* – difficulties in learning Calculus and infinitesimal processes have been investigated since the 70s' – and for its transversal role in society. Due to its powerful concepts, tools and theories developed to deal with graphs, variations, trends, and accumulation processes, Calculus is relevant for many other disciplines and professional fields (the so-called *non-mathematicians*) and is taught in most countries at secondary school, college and in the first year of university. The undoubtable value and flexibility of Calculus in modelling practices and in the conceptualization and formalization of several real-world problems does not turn into the recognition of its values by many teachers and students. Addressing the issue of teaching Calculus to most students in several different contexts is challenging. Indeed, its intrinsic complexity— including semiotic, conceptual, and argumentative aspects—combined with the didactical habit to formalize from the very beginning its notions, moving to merely procedural computational tasks or to theoretical issues, usually leads students to experience several difficulties (Trigueros et al, 2021; Biza et al., 2022).

The difference between the conceptual and semiotic aspects of key notions like continuous function, derivative, integral, limit, and the formal and theoretical evolution of their definitions and properties is often stressed using two different terms – Calculus and Analysis – to refer to different mathematical practices and the related didactical issues. Separating Calculus and Analysis sharply and definitively is not easy and can, in some sense, be misleading. A relevant and rather interesting attempt has been made by Bergé (2008), who identified some features of Analysis that are meaningful to distinguish from Calculus from the educational point of view, since they have an impact on curricula at secondary and college level and on syllabi of university courses in different countries. In Analysis, the notions and main concepts are the same as in Calculus but the attention is posed to generalization and formalization using limits and to theoretical issue; the presence of theorems and their proofs, the use of the epsilon/delta formalism to formulate definitions and in the proofs of theorems and the use of generalized notion of functions are examples of the switch from Calculus to Analysis courses. To clarify precisely what Calculus is, and what is not, is less clear and deserves careful attention, in particular from the educational point of view. Indeed, while Analysis is related to more traditional habits of mathematicians to organize and teach advanced knowledge in their field, Calculus teaching should pay more attention to the meaning of mathematical concepts at stake.

2. The main motivations to propose a new group on Calculus

There are at least two motivations to make an effort to create a new community on the teaching and learning of Calculus. First of all, it is necessary to provide new research, both empirical, practice-based and practice-oriented, and theoretical, to grasp the main variables affecting the conceptualization process, in particular the phase of representation and interpretation, that is necessary to ground the formal knowledge on strong and appropriated personal meanings. The disconnection between concept image and concept definition (Tall & Vinner, 1981), the need to connect the three worlds of mathematics (Tall, 2013; 2004), the rupture between manipulation of symbols and reasoning are solid findings and many undesired effects due to promoting the learning of Analysis without a meaning-making process and conceptual regulation processes behind have been documented by several researchers (e.g. Dreyfus, 2002; for a systematic review Bressoud et al., 2016).

The second motivation is that pre-Calculus reasoning and notions and other Calculus-related advanced mathematical practices (multivariable Calculus, modelling processes based on rate of change and accumulation) have specificities that usually are lost while moving to the formal world of Analysis and can not be simply disregarded as informal knowledge and processed to be refined or formalized as soon as possible pursuing mainly the formal rigor (Bressoud et al., 2016; Martínez-Planell & Trigueros, 2021).

Several contributions on Calculus had been proposed in other groups (in particular, TWG14 on University Mathematics Education), but some specificities were not addressed

in a systematic and pertinent way. The main motivation for proposing a thematic group on the teaching and learning of Calculus is precisely to clarify the specificities of the challenges related to its teaching and to explore ways to support meaning-making processes. This is seen as a strategy to address the deep institutional issue that underlies students' persistent failure in the learning of Calculus at secondary and tertiary levels, in particular in courses aimed at non-mathematicians.

The group inherits the work developed by researchers who worked on this topic for several years, including the Calculus and Analysis working groups attending the INDRUM conference (Trigueros et al., 2021; Hanke & Martínez-Planell, 2024) and the Calculus Conferences (CalcConf1 in 2019 in Norway; CalcConf2 in 2023 in Norway and CalcConf3 in 2025 in Italy) and new fresh perspectives by researchers of several countries with different theoretical perspectives and goals.

3. State of the art about the teaching and learning of Calculus

Recent literature stressed that these issues, that mainly regard the epistemic and cognitive level, are intertwined with institutional, linguistic and affective variables that make the challenge even harder to address. The main results are summarized in a chapter of the first handbook about the INDRUM community (International Network on Didactics of University Mathematics), resuming the current challenging issues university mathematics education is facing (Durand-Guerrier et al., 2021) – in particular by Trigueros et al. (2021) – and in the systematic review by Biza et al. (2021). At the institutional level, two important phenomena have been stressed by researchers that have a strong impact on the teaching process and on the analysis of learning outcomes.

Artigue (2016) emphasized that, among the various theoretical perspectives on investigating students' difficulties in mathematics during the transition from secondary school to university—as well as on innovations in university-level mathematics teaching—the institutional dimension has been less frequently addressed in the past. However, over the last two decades, there has been growing attention to the analysis of this aspect. In particular, the research community engaged in the research program developed by Chevallard since the late 1990s—adopting an anthropological perspective on the investigation of mathematical practices within institutional contexts (the Anthropological Theory of Didactic, or ATD; Chevallard et al., 2022)—has made significant contributions to the characterization of mathematical practices at the secondary and tertiary levels across various fields (mathematics, engineering, physics, economics, etc.). A significant contribution is highlighting the mismatches between praxeologies —that is, the structured relationships between tasks, techniques, and their justifications—carried out in different contexts that can be a major source of difficulties for students and teachers. Moreover, these analyses allow the development of new and more suitable teaching praxeologies that fit the educational and didactical purposes of different institutions.

Among many contributions in this direction, the work by Gueudet et al. (2016) is particularly important since the authors highlighted relevant features of such a mismatch at the transition towards university and the related implications for research. A different use of ATD is the analysis of praxeologies across different tertiary institutions, in particular comparing praxeologies and stressing the difference between the structural praxeologies that characterize the world of pure mathematics and the mathematical praxeologies carried out in institutional contexts where mathematical knowledge is intertwined with methods, goals and discourses that has been developed by other professional or scientific communities.

Biza et al. (2021) presented a relevant issue that relates to the intention itself of teaching Calculus to non-mathematicians in courses of the first year and on the assessment practices. Calculus is one of the main causes of dropout. The filtering and scaffolding perspective orient teachers and students towards mathematics teaching and learning in completely different ways. Unfortunately, Calculus is often intended, and used, as a filtering tool to test if students are good enough to attend the university courses. Regardless the features of mathematical practices related to a specific field, the students' interests and needs, Calculus is taught in a "standard" way, without any attempt the reconstruct mathematical praxeologies to better fit the needs of the students or the purposes, or to take care of those linguistic or affective needs of the students that educators need to consider in order to make Calculus meaningful to different categories of students. Taking into account these needs would turn Calculus concepts and praxeologies into scaffolding thinking tools that support students in their journey as university students. Moreover, the timely problem of inclusion and diversity in Calculus teaching and learning, that deserves particular attention in courses for *non-mathematicians* (Diaz-Eaton et al., 2019) and needs suitable lenses to be framed and investigated, is still underexplored in mathematics education.

Even if in recent times more attention has been posed to these issues and more research has been developed that can support innovation in the teaching and learning of Calculus at secondary level (Biza, Gonzales-Martin & Pinto, 2022) more research is needed in order to investigate the learning processes more deeply and design and implement more effective teaching-learning activities at secondary and tertiary level to accompany the students in a smooth and meaningful learning experience across different institutions.

4. Overview of TWG25: research *foci* and theoretical perspectives

In this contribution, we summarize how the papers presented in TWG25 contributed to addressing these challenges, combining different theoretical lenses and addressing several of the aspects highlighted in the literature, and the new issues that emerged during the discussion groups that opened up new avenues for research. The group brought together a broad range of research contributions focused on the learning and teaching of calculus, bridging foundational secondary-level concepts and advanced topics in tertiary education.

The authors addressed the development of students' understanding of key calculus ideas and examined instructional practices, theoretical frameworks, and methodologies that inform this field, building on research traditions spanning decades while responding to current curricular, technological, and interdisciplinary demands. Several contributors emphasized the importance of early conceptual readiness, focusing on variables, covariation, functional thinking, and graphical reasoning. These ideas form the cognitive substrate for understanding the rate of change and accumulation—central themes in calculus. Some papers deepened core topics such as the derivative as rate of change, the integral as accumulation, differentials, and the continuity and integrability of functions.

Among the diverse contributions to TWG25, we can identify three macrocategories: students' conceptualization of properties of real numbers relevant to Calculus; single variable Calculus; and multivariable Calculus.

Concerning students' conceptualization of real numbers, Borji and colleagues used the Schema part of APOS theory to analyze the convergence of infinite series of real numbers and the role of partial sums, identifying persistent challenges in students' understanding and highlighting tensions between formal definitions and intuitive conceptions of infinite series. Grisendi and Asenova investigated upper-secondary students' spontaneous conceptions of the density of $\mathbb{Q}$ in $\mathbb{R}$, an underexplored yet crucial topic for understanding continuity in Calculus. Using a classroom activity on continued fractions and thematic analysis of student responses, they reveal diverse, dynamic reasoning processes.

The notion of derivative received significant attention. Weigand and colleagues explored students' interpretations of this core concept, using the lens of Basic Mental Models analyzing how graphical and dynamic representations influence their evolving understanding. Nguyen and Greefrath contributed two studies on the same topic: one examined students' conceptualizations of derivatives using technology-enhanced environments, and another explored the affordances of dynamic representations in fostering deeper engagement with rate-of-change concepts. Lax and colleagues addressed students' thinking around continuity and accumulation, stressing the importance of connecting procedural fluency with a more grounded conceptual awareness of accumulation processes over time or space.

Gunia and Knipping analyzed the design and implementation of qualitative-graphical tasks in calculus classrooms. Their study emphasized the potential of such tasks to promote flexible mathematical reasoning and to shift learners away from narrow algorithmic approaches. Elias and colleagues explored the role of extra-mathematical contexts in supporting students' interpretations of derivatives as rates of change, demonstrating that real-world modeling can deepen students' grasp of abstract concepts by anchoring them in meaningful experiences.

Abed and colleagues presented a comparative case study of two teachers' instructional approaches to the derivative, linking their pedagogical choices to distinct mathematical worldviews and highlighting the impact of epistemological beliefs on teaching practices. Tasara and Monaghan scrutinized teachers' use of notation for gradients, revealing inconsistencies that could hinder students' conceptual clarity and pointing to a need for greater coherence between symbolic language and instructional intent. Finally, Izsak explored the teaching of inverse proportion and integration by substitution from a measurement perspective, proposing a reconceptualization that emphasizes the role of foundational reasoning structures in supporting students' calculus learning trajectories.

Four papers addressed issues related to Multivariable Calculus: Martínez-Planell and colleagues explored students' understanding of double and triple integrals in rectangular coordinates and proposed a conjectured model of the constructions students may make when learning this topic. Gabel and colleagues addressed the complexities of multivariable limits, reporting the challenges students face when evaluating limits of functions of two variables. Radmehr and Strømskag proposed a study and research paths (SRPs) model tailored to pre-service teachers, emphasizing multivariable functions and mathematical modeling. Lepellere examined the use of GeoGebra applets in the teaching of double integrals, illustrating how dynamic, visual tools can scaffold student reasoning and bridge the gap between symbolic manipulation and spatial intuition.

Some transversal issues emerged that are rather innovative in the field, in particular:

Extra-mathematical contexts: Several papers investigated how calculus is taught and learned in extra-mathematical contexts, including interdisciplinary courses for non-mathematicians, modeling applications, and STEM education. Elias and colleagues for example, highlighted how real-world contexts can support the interpretation of derivatives.

Conceptual challenges in Task Design: Gunia and Knipping analyzed tasks that integrate graphical reasoning, noting how qualitative features promote flexible thinking. Radmehr and Strømskag presented a SRP study for pre-service teachers centered on modeling and multivariable calculus. A recurring issue was calibrating the cognitive demand of tasks—striking a balance between challenge and accessibility.

The role of context and representation: contextualized tasks were shown to aid conceptual development. The use of real-world problems, especially those involving modeling, was seen to support the transition from intuition to formal understanding.

Technology, visualization, and emerging tools: visualization was a key cross-cutting theme. Lepellere demonstrated how GeoGebra can scaffold students' reasoning about double integrals, while Nguyen and Greefrath examined how dynamic representations enhance understanding of derivatives. The difficulty of visualizing 3D motion and accumulation was noted, particularly in student work.

Teacher Knowledge and instructional practice: Abed and colleagues revealed how teachers' mathematical worldviews influence their instructional strategies. Tasara and Monaghan uncovered inconsistencies in teachers' symbolic usage, highlighting a gap between expert knowledge and classroom discourse. Izsak's work on inverse proportion and substitution from a measurement standpoint added further depth to this theme. The findings point to an urgent need for more research-informed professional development.

The contributions showcased a wide range of theoretical perspectives, including APOS theory, Commognition, ATD, Concept Image and concept definition, and Semiotic Approaches. While this diversity enriched the dialogue, it also raised questions about coherence and integration. Participants debated whether future work should aim for theoretical unification or respect and encourage pluralism. Methodologies varied from thematic and content analysis to grounded theory and discourse analysis (Creswell & Poth, 2018). Participants emphasized the importance of aligning research methods with theoretical assumptions. Analysis of students 'scratch work' was highlighted as a valuable yet underutilized method for gaining insight into student thinking. Several recurring cross-cutting themes emerged, with particular attention to:

1. The tension between conceptual understanding and formal definitions
2. The role of visual and embodied reasoning in fostering insight
3. The influence of institutional and cultural contexts on curriculum
4. The need to bridge the research-practice divide
5. The foundational role of pre-calculus knowledge
6. The difficulty in assessing the cognitive demand of calculus tasks

Some participants addressed how calculus serves as a gateway to higher mathematical reasoning, including differential equations. This raised philosophical questions about how calculus differs from formal mathematical analysis, both in practice and epistemology.

During the discussion session, we explored several important research questions and considerations that warrant further investigation. Examples include: How should various institutional conditions (e.g., cultural factors, prior knowledge, and experiences of both STEM and non-STEM students) be taken into account in the teaching and learning of calculus and multivariable calculus? How can we enhance accessibility to calculus education while considering subject-specific norms? Do task design principles in calculus differ from those in other areas of mathematics? If so, what are the key design principles unique to calculus? For instance, what types of contextual tasks are most effective for teaching and learning calculus? Even within calculus, do these principles vary across topics—such as derivatives, limits, and integrals—or do they remain consistent?

5. Conclusion and future directions

TWG25 provided a vivid opportunity for discussing how students come to understand calculus, how teachers facilitate learning through lesson planning—including the design of mathematical tasks—, and how mathematics education researchers can better support both. Future research should deepen its engagement with classroom practice, explore the use of new technologies, refine the theoretical foundations of the field and address the issue of inclusion, in particular in the case of the teaching of Calculus for other disciplines and professional fields. Future research should also consider the future potential of AI and large language models (LLMs) in analyzing student responses and designing adaptive tasks. Cross-institutional collaboration and attention to diverse educational contexts will be key to addressing the evolving challenges in calculus education.

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Comparing two teachers' teaching approaches to the derivative concept with respect to mathematical worldviews

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Our study explores different teaching approaches to the concept of the derivative in high-school mathematics. In this paper, we employ the Goals-Actions model developed at a previous stage of the study to compare two teachers' teaching approaches to the derivative concept. Specifically, we focus on the mathematical-challenge actions they used before introducing the formal definition of the derivative, considering their respective mathematical worldviews. Our findings reveal notable differences between the two teaching approaches, which in turn offer students varied opportunities to engage with the derivative concept during the initial stages of its instruction.

Keywords: Teaching approach, the derivative concept, analysis, Goals-Actions Model, mathematical worldviews

Introduction

The derivative is a fundamental concept in high-school mathematics and many university courses (Rasmussen et al., 2014). Understanding such a concept is essential for mastering other mathematical concepts; however, many students have difficulty dealing with it, especially in connecting different representations and understanding their broader meaning (Prihandika & Perbowo, 2024). Often, students focus primarily on the technical rules of differentiation (ibid). While most studies on the topic have investigated students' *learning* of the derivative, very few have explored its *teaching* (Rasmussen et al., 2014). The current study addresses this discrepancy by examining various teaching approaches to the derivative concept. Teaching approaches significantly shape how students learn, and the manner in which the derivative is taught in class facilitate students' learning and understanding (Kendal & Stacey, 2000).

Recently, Abed and Ayalon (2021) proposed the 'Goals-Actions Model in Teaching the Derivative Concept', which encompasses teachers' approaches in two key aspects: goals and the actions taken to achieve them. The model identified three categories of actions: those related to mathematical challenge, those demonstrating sensitivity to students, and those focused on managing learning. This paper focuses on the actions related to mathematical challenge used by two teachers, Anna and Maya, before the formal definition of the derivative was introduced in class. Using Grigutsch and Törner's (1998) construct of 'mathematical worldview' to characterize each teacher's approach, we highlight differences in their teaching approaches and the distinct opportunities they potentially offer students to engage with the derivative concept during its initial stage.

Our research question is: *What are the differences between teachers' approaches to teaching the derivative, specifically in relation to the mathematical-challenge actions used during the first stage of the teaching sequence, prior to introducing the formal definition of the derivative?*

Theoretical Background

The Goals-Actions Model of Teaching the Concept of Derivative

An analysis of interviews conducted with ten high-school mathematics teachers in a preliminary study (Abed & Ayalon, 2021) resulted in the development of a model titled by us 'Goals-Actions Model in Teaching the Derivative Concept'. This model encompasses teachers' approaches to teaching the derivative concept in two key aspects: goals and the actions that help achieve them. It comprises three main goals categories, defined via directed analysis, by adapting the Mathematical Potential framework (Leikin, 2021): (1) cultivating students' mathematical abilities connected to learning the concept of the derivative; (2) affording students learning opportunities when teaching the concept of the derivative; (3) considering the emotional elements accompanying the process of learning the concept of the derivative. Additionally, the model includes three main categories for teaching actions, defined via directed analysis using the Teaching Triad framework (Jaworski, 1992; Potari & Jaworski, 2002): (1) actions connected to mathematical challenge in teaching the concept of the derivative; (2) actions related to sensitivity toward students when teaching the concept of derivative; (3) actions connected to the management of learning in teaching the concept of the derivative. For each of the categories there are sub-categories that were identified through inductive analysis.

Apart from the goals and actions identified in the interviews, we found that all the teachers reported using a similar sequence for teaching the derivative concept: (1) the introduction phase, prior to the presentation of the formal definition of the derivative, (2) the presentation of the formal definition of the derivative and arriving at the derivative rules, and (3) the application of the derivative in investigating functions and finding a tangent equation. In this paper, we present two differing approaches to teaching the derivative, identified by applying the model while focusing specifically on the introduction phase, before the formal definition is presented. We chose to focus on this phase because we found that it is less formalized and institutionalized, thus allowing teachers more freedom of action and more space to develop their professional individuality. Within this phase, we concentrate on the category of 'mathematical-challenge actions' in teaching the derivative as outlined in the model.

Mathematical Challenge Actions in Teaching the Derivative

The category of actions related to mathematical challenge in teaching the derivative concept aims to provide students with achievable challenges, encourage conceptual understanding, and foster connections between conceptual and procedural knowledge of the derivative (Abed & Ayalon, 2021). In the initial stage of teaching sequence, before introducing the formal definition of the derivative, the following actions were identified: Generating dilemmas; Repeated practicing of concepts known from the world of functions without referring to their role in drawing graphs of new functions; Using concepts related to functions learned before introducing the derivative to sketch graphs; and Using concepts from the field of physics. Below we provide examples for these categories of actions, categories that will be used to characterize the teaching approaches of the two teachers, Anna and Maya.

Mathematical worldviews

Grigutsch and Törner (1998) identified four aspects that are crucial in shaping the individual's understanding and perception of the nature of mathematics, thereby influencing their mathematical worldview:

The process aspect emphasizes the dynamic nature of mathematics, focusing on its emergence and development rather than viewing it as a static set of axioms and proofs. Mathematics is viewed as an ongoing process involving the exploration of problems and the development of ideas. This process has two important sub-aspects: discovery, which involves finding new knowledge or concepts, and understanding, which involves mastering the material and making connections. Approaching mathematics in this way requires reasoning, creativity, and a willingness to experiment. Teaching the concept of derivative in this way makes learning more exciting and meaningful, moving from passive memorization to active discovery and understanding.

The application aspect highlights the practical utility of mathematics, suggesting that it has direct relevance and applicability. It underscores mathematical knowledge as being beneficial to students in their future lives, aiding in solving everyday tasks and problems, or being useful in virtually every career. Additionally, this aspect points out that mathematics serves a fundamental and broad purpose in society. Teaching the concept of derivative with an emphasis on their practical applications fosters a deeper and more meaningful learning experience. It prepares students not only for academic success but also for real-world challenges, highlighting the significant role that mathematics plays in numerous aspects of life and society.

The schema aspect views mathematics as a comprehensive “toolbox” of schemas and algorithms, focusing on memorizing and applying definitions, rules, and procedures to solve problems. Teaching derivation from this perspective can provide students with a foundational approach to effective problem solving. However, to ensure comprehensive understanding, this approach should be complemented with other aspects that emphasize comprehension, application, and formal reasoning.

The formalism aspect underscores the importance of logical and objective reasoning in mathematics, characterized by its focus on rigor, accuracy, and precision. It helps students develop a thorough understanding of mathematical concepts, enhances their problem-solving and analytical skills, and prepares them for advanced studies and careers in a variety of scientific and technological fields.

In this study, we chose to sort the actions related to the ‘Mathematical Challenge Actions in Teaching Derivatives’ category in the Goals-Actions model according to the definitions of “mathematical worldview” because we found that each action belongs to one of the worldviews, as explained in the next section.

Methods

This study employs a multiple-case study approach to explore contrasting approaches to teaching the derivative concept. Each case, representing one teacher, is based on the prior work of Abed and Ayalon (2021), allowing for in-depth exploration of individual instructional approaches. We (ibid) analyzed semi-structured interviews conducted with 10 mathematics teachers who teach 10th grade in the advanced track. In this paper, we present two of these teachers: Maya and Anna. Both are

regarded as leading teachers who fill various roles in the Israeli mathematics education community. Anna and Maya are both mathematics teachers, Anna has 30 years of experience and a PhD in mathematics education, and Maya has 20 years of experience and a doctoral student in mathematics education. We focused on these two teachers as they were found to advocate different approaches to the teaching of the derivative concept at the first stage of the teaching sequence.

This study is limited by its reliance solely on teacher interviews without observational data, limiting insights into real-time classroom applications. Nevertheless, where available, interview details provide a richer understanding of each teacher's instructional approach. Specific examples discussed by the teachers have been included to contextualize each teaching approach

Study tools and data analysis

In the current study, we employed two tools: First, we employed the Goals-Actions model as a tool to analyze the interviews conducted with the interviewees and find out the actions in the first stage in the teaching sequence: Prior to the formal definition, which belongs to the "mathematical challenge" category. Second, we constructed an original tool (Table 1) by using the definitions of mathematical worldviews to associate each action with the appropriate worldview. The data analysis was performed with attention to the following two parameters:

1. We determined the time allocated to each stage according to the teacher's statements and according to the "Time allocation for each teaching sequence" section in the theoretical background.

2. We identified (by three steps) the mathematical worldview for the mathematical-challenge actions used in the first stage in the teaching sequence: Prior to introducing the formal definition. First, for each teacher, we identified the actions that belong to the category 'Mathematical Challenge Actions in Teaching the Derivative' in the Goals-Actions model. Second, we used definitions of mathematical worldviews to associate each action with the appropriate worldview according to Table 1. Finally, we identified the actions specifically linked to the first stage of the teaching sequence: prior to introducing the formal definition.

Time allocation for each teaching sequence

In the preliminary study (Abed & Ayalon, 2021) we found that all 10 participating teachers followed a similar sequence when teaching the concept of the derivative. This finding was reinforced by their interviews, where we observed variations in the duration of each stage. Specifically, the initial stage, "Prior to the introduction of the formal definition" varied in length among the teachers. They reported allocating a varying number of lessons to each stage, which we categorized as long, medium, or short duration. These categories reflect the relative importance each teacher places on these stages of teaching the derivative. Time allocation plays a crucial role in shaping mathematics lessons, as it influences both the depth of content coverage and the teaching methodologies used. Research suggests that increased instructional time positively impacts student achievement by providing greater exposure to essential academic skills and knowledge (O'Meara & Prendergast, 2018). The categorization of time allocation in this study reflects each teacher's instructional priorities, indicating 'Long' as exceeding four lessons, 'Medium' as 3–4 lessons, and 'Short' as 1–2 lessons. It represents the level of emphasis each teacher places on the derivative concept. This classification highlights the

differences in pedagogical focus, with 'Long' indicating extensive conceptual exploration, and 'Short' reflecting a more concise introduction. This categorization underscores how instructional time impacts teaching effectiveness and student engagement (O'Meara & Prendergast, 2018).

The connection between the 'Mathematical Challenge Actions in Teaching the Derivative' to the suitable worldview

In this section we will present (Table 1) the actions used by the two teachers, Maya and Anna, and we will explain the reason for associating each action with its corresponding worldview.

Table 1: Associating the 'Mathematical Challenge Actions' to the suitable worldview

Action	Example	Mathematical worldview	Explanation
Generating dilemmas	"Now what happens is that I write some polynomial on the board, usually fourth or fifth degree, and I tell them that I want to draw the graph of this function, and I ask them: what should I do now? … I evoke a need." (teacher Anna)	The process aspect	This action belongs to the process aspect because it describes a discovery, which involves revealing new knowledge or concepts.
Repeated practicing of concepts known from the world of functions without referring to their role in drawing graphs of new functions.	"According to the recommendation of the [Israeli] Ministry of Education, a basic introduction to analysis is taught as positivity and negativity, evenness and oddness … I do a kind of summative reminder of all the pre-analysis that they learned in middle school." (teacher Maya)	The schema aspect	This action belongs to the schema aspect because it views mathematics as a comprehensive "toolbox" of schemas and algorithms, focusing on memorizing and applying definitions, rules, and procedures to solve problems
Using concepts related to functions learned before introducing the derivative to sketch graphs	"So, first we start with a 'pre-analysis' segment; we learn from the quadratic function, absolute value, and polynomials. Basically, we get familiar with functions. We learn about functions in more depth, then we graph polynomials, working on graphs beyond the straight line and parabola. [In this way] we get to know other functions." (teacher Anna)	The process aspect	This action belongs to the process aspect because it highlights the dynamic nature of mathematics, involving continuous problem exploration and idea development. It fosters deeper understanding by promoting mastery of the material and facilitating connections between concepts.

Findings

In this section, we present findings regarding Anna's and Maya's unique approaches to teaching the derivative, specifically in relation to the mathematical-challenge actions used before the formal definition of the derivative is introduced.

Teacher Anna

Below we present a portion of Anna's description of her teaching before introducing the formal definition of the derivative. The index given in parentheses in the citation identifies the statement that will be referred to later in the citation analysis.

> First, I create in the students a sense of need for a new and effective tool like the derivative [i]. I don't just define the derivative as A, B, C, because that would sound completely foreign to them. I structure the classroom discourse so that concepts like the tangent or the slope of the tangent would seem useful from the outset. Therefore, I start by showing them the challenges of drawing a graph without the derivative. How do you draw a graph? We begin with a 'pre-analysis' segment, learning from quadratic functions, absolute value functions, and polynomials [ii]. We familiarize ourselves with various functions, delving deeper into their properties. We graph polynomials and explore graphs beyond just the straight line and parabola, getting to know other types of functions. This introductory phase takes more than two double lessons [iii]. By the end of the process, students understand the need for the tangent slope and begin searching for it. As we try to get as close as possible to the tangent slope, we naturally arrive at the formal definition of the derivative, through the concept of the limit of the difference quotient.

We can see that Anna uses two actions (i) *Generating dilemmas* and (ii) *Using concepts related to functions learned before introducing the derivative to sketch graphs*. From our point of view, as presented above, these actions are consistent with the *Process aspect* worldview according to the definition of this aspect (Table 1). The time allocation for this stage (iii) is considered "long".

Teacher Maya

Maya described her teaching before introducing the formal definition of the derivative:

> Following the Ministry of Education's recommendation, we start with a basic introduction to analysis, covering topics like positivity and negativity [of a function], evenness and oddness … [i]. This pre-analysis connects students to concepts they had learned previously, during the 9th grade. After 3-4 lessons [ii], we move on to the definition of the derivative. I also explain that when the function increases, the derivative is positive, and vice versa.

Table 2: **The differences between Anna's and Maya's teaching approaches, in the first stage**

	Anna	Maya
Actions	- Generating dilemmas - Using concepts related to functions learned prior to introducing the derivative to sketch graphs	Repeated practice of concepts known from the world of functions without referring to their role in drawing graphs of new functions
Mathematical worldview	The Process aspect	The Schema aspect
Time allocation	Long	Medium

We can see that Maya uses one action in this stage (i) *Repeated practicing of concepts known from the world of functions without referring to their role in drawing graphs of new functions*. From our

point of view, as presented above, this action is consistent with the *Schema aspect* worldview according to the definition of this aspect (Table 1), unlike Anna who uses two actions which belong to the *Process aspect*. Furthermore, the time allocation for this stage (ii) is considered "Medium". Table 2 summarizes the different teaching approaches of Anna and Maya.

Discussion

Recent studies on teaching the derivative concept highlight innovative strategies and challenges in student comprehension, providing a foundation for analyzing pedagogical approaches like those of Anna and Maya. These studies emphasize the importance of active learning, meaningful learning, and the use of technology in enhancing students' understanding of derivatives. The integration of these strategies can significantly impact students' conceptual understanding and application of derivatives in various contexts (Chavan & Kadam, 2022; Fazio, 2020). This study demonstrated differences between Anna's and Maya's approaches to teaching the derivative during the first phase, prior to introducing the definition of a derivative. In our interpretation, Maya's approach is related to the schematic mathematical worldview insofar as she focused on memorizing and applying definitions, rules, and procedures to introduce the derivative concept. In contrast, we regard Anna's approach as being related to the process mathematical worldview as she introduced the derivative concept in such a way that the students felt that there is missing information and felt the need for a useful tool to help them to be able to sketch any function graph. Following Grigutsch and Törner (1998), we argue that Anna's approach can imply more exciting and meaningful learning for students as it suggests moving from passive memorization to active discovery and understanding; furthermore, she provides her students with a foundational approach to effective problem solving. Maya, however, needs to ensure comprehensive understanding, so her approach should be complemented with other aspects that emphasize comprehension, application, and formal reasoning. Maya allocated a "Medium" time, i.e., 3-4 lessons, to teaching this phase, whereas Anna allocated a "Long" time, i.e., more than four lessons, to teaching this phase. Referring to the literature (O'Meara & Prendergast, 2018), we may conclude that Anna's approach is deeper, and the breadth of content coverage and the teaching methodologies employed surpass Maya's approach. This is because Anna spent more time introducing the derivative concept to her students, and it is generally observed that a longer allocated time may be associated with improved student achievement, as it provides more opportunities for exposure to key academic skills and knowledge development. It seems, therefore, that these two teaching approaches differ in terms of the opportunity provided to students to process and attain a sense of the importance of the concept of the derivative and build a need for using it, before presenting them with its formal definition.

The findings suggest that process-oriented teaching approaches, as demonstrated by Anna, may enhance conceptual understanding by fostering a sense of discovery. In contrast, schema-focused approaches, like Maya's, could benefit from additional application-based elements to enhance comprehensive understanding. These findings highlight potential areas for further research into balancing process and schema-based approaches in derivative teaching.

Our next step in the continuation of this research is to analyze interviews with additional teachers, focusing on the entire sequence for teaching the derivative concept, to explore further variations in

teaching approaches. Additionally, we are interested in examining whether expanding the study to different contexts and countries might reveal connections between teaching approaches and socio-cultural factors. Such an analysis could provide deeper insights into how local educational traditions, teacher beliefs, and classroom dynamics influence the teaching and learning of the derivative across diverse educational systems.

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Students' understanding of partial sums and convergence of infinite series

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The research literature shows that calculus students often struggle with infinite series due to the abstract nature of convergence and the complexities involved in defining and determining their sum. We used APOS theory to explore students' understanding of two important notions related to infinite series: sequence of partial sums and convergence. We designed a genetic decomposition (GD) grounded in the concept of Schema to conjecture cognitive structures and relations students might develop to understand infinite series. The GD was evaluated through semi-structured interviews with twelve students who had completed a calculus course. We used the data to exemplify the types of constructions observed based on the APOS notions of Schema, the Schema-triad, and the types of relations between Schema components. This study contributes to a better understanding of the notion of Schema in APOS theory, focusing on infinite series, and has implications for teaching practice.

Keywords: Calculus, infinite series, partial sums, schema, APOS theory.

Introduction

Incorporating infinite series into mathematics education builds a strong foundation for students, preparing them for more complex topics and real-world applications. Studying infinite series helps students grasp the concepts of limits and convergence, which are essential for calculus and analysis. Many advanced topics, including Fourier series, power series, and Taylor series, rely on the principles of infinite series. Working with infinite series enhances analytical thinking and problem-solving skills. It involves understanding the behaviour of sequences and series, performing algebraic manipulations, and applying various convergence tests. However, the research literature shows that students struggle to learn infinite series (e.g., Eckman & Roh, 2024; Martínez-Planell et al., 2012). Generally, infinite series questions can be categorised into two distinct scenarios. The first involves students working solely with the sequence of terms a_n, for example, using the root test $\lim_{n\to\infty} \sqrt[n]{a_n}$ to determine convergence, while the second requires students to utilize the sequence of partial sums S_n, for example, deriving a formula for the sum of a geometric series.

Martínez-Planell et al. (2012) showed that students may construct two fundamentally different notions of infinite series: one as an infinite, unending process of addition (SERLIST) and the other as a sequence of partial sums (SERFUNC). They argued that students have difficulty understanding series as a sequence of partial sums, leading to challenges in problem situations requiring this interpretation. In a case study, Eckman and Roh (2024) investigated understanding of the sequence of partial sums and the importance of this notion in learning infinite series. Their results corroborated Martínez-Planell et al.'s (2012) description of two distinct student interpretations of partial sums, ranging from considering partial sums as an additive process to viewing them as a function with a

corresponding rule. We build our study on these ideas using the notion of Schema of APOS theory to further investigate students' understanding of infinite series. This study aims to address the following research question: What kind of cognitive insights on infinite series conceptualisation can be gained from students' Schema development?

Theoretical framework

We use Action-Process-Object-Schema (APOS) theory as outlined by Arnon et al. (2014), with particular emphasis on the notion of Schema. Given that the APO portion of the theory is well-established, we will focus on explaining the Schema component. A Schema is a coherent collection of Actions, Processes, Objects, and other previously developed Schemas, connected by various types of relations, that individuals apply when encountering a problem situation. A Schema can be considered coherent when its different components are interrelated in a way that allows the individual to discern between those problems that are within its scope and those that are not.

To study Schemas, researchers may begin by defining the component structures of the Schema and describing the interrelations between these components that students need to construct to understand the didactic phenomenon being examined (Trigueros, 2019). Relations constructed between the components of a Schema can be classified as correspondence, transformation, or conservation relations. A *correspondence relation* occurs when the individual notices out of habit or repetition that, in some circumstances, one or more component structures are used in relation to another but cannot yet justify the relation, or when he/she superficially compares some Schema's components in terms of differences and similarities. A *transformation relation* is evidenced when an individual discovers that changes in one Schema component induce changes in other components and can identify groups of related structures. This occurs when one component mediates or is applied to explain or justify the relation between two or more other components. A relation between components is considered a *conservation relation* when it is consistently and normatively used across various problem situations, allowing one structure to be interchangeably or fluently used in place of another. The development of Schema can be examined through the Triad model (Piaget & García, 1983), which includes a three-level developmental mechanism: Intra-, Inter-, and Trans-. We will consider that a Schema is at the Intra-stage of development when its components are unrelated or mostly related by correspondence relations. A Schema is deemed to be at the Inter-stage of development when relations among most Schema components are constructed, and and these are mainly transformation relations. At the Trans-stage of Schema development, most components of the Schema are interrelated through conservation relations, and there is evidence of Schema coherence.

Methodology

Based on the research methodology of APOS theory, we designed a preliminary genetic decomposition and then created questions for semi-structured interviews to test it. Each researcher analysed students' constructions, followed by discussions to share and negotiate their analyses until a consensus was reached. Genetic Decomposition (GD) is a foundational notion within APOS theory. It serves as a conjectural model for how students may learn a particular mathematical concept and, in the APO portion of the theory, is described using APOS structures and mechanisms. When considering Schemas, a GD is represented as a list of component structures, a description of relations

between them, which students are expected to construct, and the stages of Schema development. We consider the following components for the Infinite Series Schema and add, in parentheses, a code to identify them and subsequently to determine the relations among those components.

- Sequence (Seq) as an Object. This includes a SEQLIST object (McDonald et al., 2000) that allows students to think of a numerical sequence as an infinite list of numbers and a SEQFUNC object that allows students to think of a sequence as a function with domain in the natural numbers.
- Limit of sequence (Lim) as a Process, including the definition of convergence and basic properties of limits.
- Finite sums and sigma notation (FS) as a Process that allows students to imagine forming finite sums and representing them with sigma notation without having to do so explicitly.
- Convergence tests (CT) sub-Schema consisting of Actions, Processes, and/or Objects corresponding to the different convergence tests.

The following are the proposed relations between Schema components that students are expected to construct. Note that each relation includes a code in parentheses to designate it. This code was used in our data analysis.

- Sequence and limit of a sequence (Seq-Lim): This relation is evidenced by thinking of a series as an infinite, unending process of adding terms. This is the construction called SERLIST in Martínez-Planell et al. (2012).
- Sequence and finite sums and sigma notation (Seq-FS): evidenced by relating the sequence of terms a_n and sequence of partial sums S_n, by obtaining one from the other in any representation.
- Finite sums and sigma notation, sequence, and limit of a sequence (FS-Seq-Lim): evidenced by students' recognition that the convergence or the sum of a series can be determined by taking the limit of the sequence of partial sums, that this limit (when it exists) may be denoted by $\sum_{n=1}^{\infty} a_n$, and that this same symbol may be used to denote the corresponding sequence of partial sums (a completely different object). This is the construction called SERFUNC in Martínez-Planell et al. (2012). This construction may also be evidenced by the recognition of using partial sums to find an overestimate or underestimate of the sum of a series (e.g., alternating series).
- Sequence and convergence tests (Seq-CT): evidenced by classifying different types of series (e.g., geometric, p-series, telescoping, alternating) based on the sequence of terms with the intention of applying a convergence test.
- Limit of a sequence, sequence, and convergence tests (Lim-Seq-CT): This relation can be evidenced by students' use of limit on the sequence of terms a_n to test for divergence and other tests such as comparison, limit comparison, ratio, and root tests. In establishing this relation, the student may think of an infinite series of numbers as an infinite sum, as described by the SERLIST construction described in Martínez-Planell et al. (2012).

The Schema's development can be described as:

Intra-Infinite Series stage. The student may demonstrate isolated components and has established only correspondence relations between some of them, if any.

Inter-Infinite Series stage. The student has started developing transformation relations between the Schema components. These relations are evidenced by explanations and using one component to justify the relation between other components. The use of these relations is mostly according to normative practice but might be inconsistent in different problem situations.

Trans-Infinite Series stage. Conservation relations interrelate all Schema components. These relations may be evidenced by their fluent use consistent with normative practice across different problem situations, as well as the student's ability to implement and clarify a suitable strategy based on the characteristics of the problem. The coherence of the Schema is manifested by flexibly using either SERLIST or SERFUNC conceptions whenever applicable or necessary.

Twelve students, spanning the range from above-average (3 students), average (6 students), to below-average (3 students), as chosen by the professor, were interviewed three weeks after completing a lecture-based calculus course (involving the topic of sequences and infinite series) at a top-tier university in Czechia. Each interview lasted approximately one hour and was video and audio recorded. The interview instrument consisted of eight questions, some of which were multipart. In this report, we will be referring to the following questions. Their description is followed by the codes of the relations in the GD they were intended to examine.

Q1.a. What does it mean when it is said that "an infinite series converges or diverges"? [Seq-Lim, FS-Seq-Lim]

Q3.a. Consider the infinite series $\sum_{n=1}^{\infty} a_n$. The graph of the sequence of the terms a_n is presented in Figure 1. Is the series $\sum_{n=1}^{\infty} a_n$ convergent or divergent? [Lim-Seq-CT, Seq-Lim, FS-Seq-Lim]

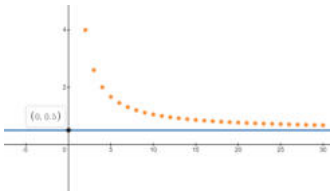

Figure 1: The graph of the sequence a_n

Q4. Given $\frac{1}{(1)(3)} + \frac{1}{(3)(5)} + \frac{1}{(5)(7)} + \cdots$. Let S be the sum of this series. Consider the following two arguments: a) The sum S of the series may be written as $S = \left(\frac{1}{1} - \frac{2}{3}\right) + \left(\frac{2}{3} - \frac{3}{5}\right) + \left(\frac{3}{5} - \frac{4}{7}\right) + \cdots = 1 + \left(-\frac{2}{3} + +\frac{2}{3}\right) + \left(-\frac{3}{5} + \frac{3}{5}\right) + \left(-\frac{4}{7} + \frac{4}{7}\right) + \cdots$ and one obtains that $S = 1$. b) $S_1 = 1 - \frac{2}{3}$, $S_2 = \left(1 - \frac{2}{3}\right) + \left(\frac{2}{3} - \frac{3}{5}\right) = 1 - \frac{3}{5}$, $S_3 = \left(1 - \frac{2}{3}\right) + \left(\frac{2}{3} - \frac{3}{5}\right) + \left(\frac{3}{5} - \frac{4}{7}\right) = 1 - \frac{4}{7}$, each of these partial sums may be written as $S_n = 1 - \frac{n+1}{2n+1}$, therefore, $\lim_{n\to\infty} S_n = S = \frac{1}{2}$. We have seen that for the series $\frac{1}{(1)(3)} + \frac{1}{(3)(5)} + \frac{1}{(5)(7)} + \cdots$, two results have been found: 1 and $\frac{1}{2}$. Is it possible that the series has two values? Clarify this situation. [Seq-Lim, FS-Seq-Lim]

Q5. If the nth partial sum of a series $\sum_{n=1}^{\infty} a_n$ is $S_n = 3 - n2^{-n}$, a) find a_n [Seq-FS] b) and $\sum_{n=1}^{\infty} a_n$. [FS-Seq-Lim]

Results

After analysing all the interviews, eight students were categorised in the Intra stage and four students in the Inter stage of Infinite Series Schema development. There were no students at the Trans stage. First, we examine Hana at the Intra stage, representing eight students, and then David at the Inter stage, representing four students. These two serve as paradigmatic examples of students whose Schemas were classified in the same category. In Q1.a, in which students explained their meaning of convergence or divergence of infinite series, Hana responded as follows:

> Hana: If the sum of all terms of the series equals a real number it converges, but if this sum equals ∞ it diverges. Usually the sum of an infinite number of terms is ∞ so they often diverge to infinity, but for some infinite series, like geometric series, we have a formula to find the sum as a real number. When we don't have a formula, it's very difficult to find the sum.

Hana explained the infinite series as the addition of infinitely many terms, showing evidence of the relation Seq-Lim. She understands that for some series, a direct formula can give their sum but does not mention the possibility of examining the sequence of partial sums in the absence of a formula. Her insistence on finding the sum of the series by adding all the terms or using a direct formula suggests that she has not constructed the relation FS-Seq-Lim. In Q3.a, where the graph of terms a_n was represented, Hana explained:

> Hana: Based on the graph, we can see the terms of the series become smaller and smaller, umm when we add them, the sum will have an upper limit. Therefore, the series converges to this upper limit.

We expected to see evidence of the relations Lim-Seq-CT, Seq-Lim, or FS-Seq-Lim in students' answers to Q3.a. Hana only demonstrated the relation Seq-Lim, but as a correspondence relation. She believed that the sum of an infinite number of decreasing terms, all of which are greater than 0.5, has an upper boundary. Now we consider Hana's response to Q4 (two values for the infinite series):

> Hana: I think the first one is correct because we are adding all the terms to find the sum. The second one isn't correct because instead of adding all the terms it used taking the limit of a sequence.

Hana once again (as in Q1.a and Q3.a) showed that she thinks of the value of the infinite series as the sum of its terms, providing evidence of the relation Seq-Lim. Even though a solution through partial sums was provided in the second solution, Hana's response further clarifies that she has not constructed the relation FS-Seq-Lim to see the sum of the infinite series as the limit of partial sums. In Q5 (finding a_n and the sum of the series through S_n), Hana responded as follows:

> Hana: I don't know how to find a_n, I only know that S_n is the sum of a_1 to a_n, the sum of this series is the sum of all of its terms, when we don't have a_n I don't know how to find the sum.

Hana was not able to find a_n from S_n, indicating she has not constructed the relation Seq-FS. Her understanding of infinite series is limited to the addition of infinitely many terms, as shown in all of her responses. Overall, considering Hana's responses to all the questions, which we did not have space to include in this paper, Hana showed evidence of having constructed some relations between Schema components, all of them as correspondence relations (e.g., Seq-Lim), while some components remained unrelated. We therefore concluded that Hana evidenced an Intra-Infinite Series

stage of Schema development. Overall, the students at the Intra stage encountered two significant cognitive challenges regarding the nature of series convergence. Some students believed that when terms in a series approach zero (i.e., $\lim_{n\to\infty} a_n = 0$), the accumulated sum would still increase indefinitely, meaning the series could not stabilize or converge. This misconception led them to wrongly conclude that a convergent series like $\sum_{n=1}^{\infty} \frac{1}{10^n}$ would diverge because the accumulated sum was perceived as perpetually increasing. Other students assumed that if a sequence a_n decreased and converged to a specific value, then the series must also converge. They believed that the decreasing nature of the terms would create an upper limit for the series, thus stabilizing the accumulated sum and causing convergence. They might mistakenly apply it to any series with decreasing terms, such as the harmonic series, which, despite smaller terms, actually diverges.

Now we consider David to represent the constructions of the students at the Inter-Infinite Series Schema development. In Q1.a (convergence and divergence of infinite series), David responded:

> David: If the limit of the sequence of partial sums of an infinite series is equal to a real number, I mean if it's a value, which is not ∞ or $-\infty$ then the series converges. If this limit is ∞ or $-\infty$ or doesn't exist the series diverges.

David clearly explained the convergence of a series by taking the limit of the sequence of partial sums. Furthermore, he was able to justify the conditions for divergence. This is consistent with having constructed the relation FS-Seq-Lim. In Q3.a, where the graph of the sequence a_n was given, David showed the construction of other relations:

> David: Here, we have terms a_n, umm they all are bigger than 0.5 but approach 0.5, so we have infinitely many numbers bigger than 0.5 so the sum of an infinite number of them can be as large as we want, approaching ∞, so the series diverges, or as a second solution we can use the test for divergence we can see $\lim_{n\to\infty} a_n$ is 0.5, umh it's not zero, therefore the series is divergent.

David first thought of a series as adding infinitely many terms of the sequence a_n, and using this construction (unlike Hana), he was able to explain why the series diverges. This gives evidence of the construction of the relation Seq-Lim. As his second solution, he realized the possibility of using the test for divergence by relating the components of limit of a sequence, the sequence, and convergence tests. This shows that in this question he related the components Lim-Seq-CT. Regarding Q4 (two sums for an infinite series), David explained as follows:

> David: It's not possible to compute the sum of infinitely many numbers of terms a_n just by adding, one by one umh the first solution isn't correct because we are ignoring something in infinity, umh to find the sum of the infinite series we need to find the sequence of partial sums and take the limit of S_n like the second solution, therefore the sum is $1/2$.

Once again, David demonstrated a SERFUNC construction of infinite series (Martínez-Planell et al., 2012) by justifying the computation of a series' sum through the limit of partial sums and highlighting the impossibility of summing an infinite number of terms directly. This question (along with Q1) suggests that David has constructed the relation for determining convergence or computing the sum of a series (FS-Seq-Lim) as a transformation relation. We continue the analysis of David's answers by examining his responses to Q5.a (finding a_n from S_n) and Q5.b (finding the sum of the infinite series using S_n).

David: [5.a] When we have a_n it's usually possible to find S_n by adding the first n terms, umh but I don't know how to find the general formula for a_n directly from S_n, umh maybe I can give some values to n and find a_1, a_2, a_3 and so on to find a pattern for a_n [David found $a_1 = 5/2$, $a_2 = 0$, $a_3 = 1/8$, $a_4 = 1/8$, $a_5 = 3/32$] but I can't find any pattern.

David: [5.b] We can find the sum of the series by taking the limit of its partial sums, the limit of $3 - n2^{-n}$ when n approaches infinity is 3 minus I think 0, because $n2^{-n}$ is n over 2^n and the geometric sequence 2^n grows much faster than the arithmetic sequence n, umh so the fraction $n/2^n$ approaches 0, therefore the sum is 3 minus 0 which is 3.

In part 5.b, David provided more evidence of the relation FS-Seq-Lim as a transformation relation. However, when asked to find a_n from $S_n = 3 - n2^{-n}$ (part 5.a), he demonstrated a correspondence relation between the components of sequence and finite sums and sigma notation (Seq-FS). David attempted to numerically determine the first few terms of the sequence a_n to identify a pattern for a general formula of a_n, but he was unsuccessful. For a student who has established a conservation relation between the previously mentioned structures (Seq-FS), computing a_n from S_n would be the same as finding a_2, a_3 and so on, by calculating the difference between the $(n-1)$th partial sum and the nth partial sum (i.e., $a_n = S_n - S_{n-1}$). However, David failed to establish this relation because in computing a_1, a_2, and a_3 he used the extended formulas $S_1 = a_1$, $S_2 = a_1 + a_2$, $S_3 = a_1 + a_2 + a_3$, rather than the difference between two consecutive terms of partial sums (e.g., $S_3 - S_2 = a_3$). In other questions not included in this paper, David showed evidence of the relations Seq-CT (recognizing the type of series based on the form a_n) and Lim-Seq-CT (applying suitable convergence tests by working with a_n) as transformation relations. In summary, considering David's responses to all questions, we can argue that the different Schema components were interrelated, some through correspondence relations and most through transformation relations. This led us to conclude that David demonstrated the construction of an Inter-Infinite Series stage Schema.

Discussion and conclusion

Considering that no student showed evidence of being in a Trans-Infinite Series Schema development and that eight out of twelve students were at the Intra-stage, it can be concluded that students have difficulties learning the notion of infinite series. Most participating students in our study interpreted the infinite series as an unending process of addition. This led them to fail to understand the sum of an infinite series as the limit of partial sums and, therefore, not construct the relation FS-Seq-Lim. Although some students, like David, used the interpretation of partial sums in their responses, none of the students were able to find a_n directly from S_n (in Q5), evidencing that they had not constructed the relation Seq-FS as a conservation relation. Some other difficulties frequently observed in this study include students' problems classifying the series based on the form of a_n (Seq-CT) and using a suitable convergence test (Lim-Seq-CT). The results suggest that to improve teaching practice, activities should be developed to help students construct the relations Seq-FS and FS-Seq-Lim and that more work is needed on the relations Seq-CT and Lim-Seq-CT.

Although we did not have any students at the Trans-stage of the Infinite Series Schema, a potential student at this stage, like an expert, would be able to flexibly determine when to work with the sequence of terms a_n and when to work with the sequence of partial sums S_n. In the former case, the expert would recognize the need to apply relations like Seq-Lim, Seq-CT, or Lim-Seq-CT, as in Q2,

where taking the limit of a_n helps assess the convergence of the series. In the latter case, the expert would understand the need to use relations such as Seq-FS or FS-Seq-Lim, as in Q4, where considering partial sums and taking their limit is essential to finding the series' sum. David mostly demonstrated this flexibility, except for finding a_n from S_n, and was able to transition between these two approaches based on the type of question.

The fact that Schemas were used in the analysis is in itself a contribution of this study, as it helps clarify the different theoretical notions involved. Our study extends the research carried out by Martínez-Planell et al. (2012), where the authors used the APO part of the theory. In this study, we used the Schema part of APOS, which enabled us to involve the components and their relations to better analyse students' understanding of infinite series in different problem situations. While Martínez-Planell et al. (2012) did not study the convergence tests, we examined some of them in our research. In this paper, we had space to present only one of the questions we designed about convergence tests, which was a geometrical interpretation of the test for divergence (Q3.a). Our study is also in line with Eckman and Roh (2024) in underscoring the importance of partial sums in learning infinite series. However, we further extended their study by considering other components such as sequences, limits, and finite sums and sigma notation. In future research, we aim to design activities based on the GD and implement them in class to help students understand the constructions and the relations between components of the Infinite Series Schema. As a limitation, although our instrument included some questions to study convergence tests, there are numerous convergence tests, which make it difficult for students to determine which test to use for different infinite series. In future research, we should include more questions related to convergence tests to further explore students' understanding of this important aspect of infinite series.

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Characteristics of extra-mathematical contexts for introducing derivative as rate of change

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This paper reports on a study which aims to shed light on the ways, in which experts rate different extra-mathematical contexts (EMC) as appropriate for introducing the concept of derivative as rate of change (RoC), and what characteristics of EMC affect their choices. The result is a beginning of a taxonomy of EMC, based on a survey of opinions and attitudes about EMC in general, and specifically EMC for introducing the concept of RoC.

Keywords: Derivative, rate of change, extra-mathematical contexts.

Introduction

A central issue for mathematics educators is how to make fundamental ideas of calculus meaningful for students. With respect to the derivative, the concept of rate of change (RoC) is important for developing a meaningful understanding but is usually difficult for students. Using extra-mathematical contexts (EMC) may assist in grounding this abstract concept. This paper reports on a study which concerns the suitability of different EMC based situations for introducing the concept of derivative as RoC, and the characteristics which make these situations less or more suitable for this purpose.

Background

Extra mathematical contexts

In the approach known as Realistic Mathematics Education (RME), context problems are defined as problems where the problem situation is experientially real to the student (Gravemeijer & Doorman, 1999), meaning that students can imagine themselves acting within the context. Through this approach, students are encouraged to develop mathematical concepts by building on their own experiences and reasoning within real-world contexts. Freudenthal (1973) argued that real-world contexts should be the starting point for developing mathematical reasoning.

Rubel and McCloskey (2021) listed rationales to contextualize mathematics. These include affective benefits such as motivating learners and helping them see the practical value of mathematics, but also the potential to serve as conceptual anchors (Reinke, 2020) or foundations, on which students rely in making sense of abstract mathematical concepts.

Aspects to consider when designing EMC based problems, include authenticity (Cooper and Harries, 2002), previous knowledge and experience with the context (Palm, 2008), success in imagining how the quantities in a problem's context change together (Moore & Carlson, 2012) and time as a variable (Swan (1980) – "time independent relationships between variables are much more difficult to visualize and work with than the straight time dependent relationships of a single variable" (p. 217)).

Rate of change

Students often find the concept of the derivative difficult to grasp, a challenge compounded by the widespread misconception that calculus is merely a set of procedural tasks (Thompson et al., 2013). According to Carlson et al. (2015), students tend to develop a shallow understanding of derivatives, focusing on formulas and rules without grasping the underlying concepts. A key idea for understanding calculus, including derivatives, is the concept of RoC (Confrey & Smith, 1994). RoC can be viewed in several ways, including as an observable phenomenon and a formal concept that includes slope and the difference quotient (Stroup, 2002). To fully comprehend RoC, students must integrate the different perspectives (Confrey & Smith, 1995). However, research indicates that both students and teachers face challenges in linking RoC to slope and derivative.

Goal and methodology

The goal of this study was to identify characteristics that make an EMC-based situation more (or less) suitable for introducing RoC, and what are the reasons for this, with the long-term aim of creating a tool for characterizing EMC (for the specific case of introducing RoC). In a semi-structured interview, teachers and other experts from the mathematics education community were asked to rank different situations and say to what extent they think the situation is appropriate for introducing the concept of derivative as RoC, and why. When designing the interviews, we used pre-defined characteristics for EMC, from the literature and pilot interviews. These characteristics are: (1) Time as the independent variable, (2) Familiarity, (3) Tangibility – The situation concerns a concrete process that one can see and touch, (4) Mathematically "clean" situation - No other variables, apart from the independent variable, influence the dependent variable. Guided by this list, we drafted the following situations (three are detailed and explained, others are mentioned briefly):

1. Car – 'While I drove my car on the way to work this morning, the distance I travelled changed over time. The RoC is the speed of the car as seen on my speedometer' - A familiar time-dependent 'clean' situation, which is not tangible (distance is abstract).
2. Pool – 'Yesterday I filled up my kids' pool. The volume of water in the pool changed over time, according to the rate of water flow from the garden hose. When they finished using the pool, we emptied the pool. While emptying the pool the volume of water changed over time as well, with a negative RoC.' - A familiar, tangible time-dependent clean situation.
3. Temperature – 'Yesterday morning the temperature outside changed over time. In the early morning it was cold, but at noon it was already quite warm.' - A familiar time-dependent situation, which is not tangible, nor clean (influencing factors are humidity, clouds etc.).

Other situation were: Taxi – A situation in which the cost of the ride changes with respect to the distance travelled; Photosynthesis – A situation regarding the concentration of dissolved oxygen in ponds which changes over time; Ice cream – A situation in which the amount of ice cream that melts changes with respect to the temperature outside; Buoyancy – A situation about an experiment in class using eggs to investigate the relationship between the concentration of saltwater and buoyancy.

These situations cover different combinations of the characteristics. This gives the interviewee an opportunity to experience the differences between the situations and relate to their personal

preferences for the EMC. In the interview, each situation was presented, and the interviewee was asked whether the situation is suitable for an opening lesson on derivative as RoC, and why. The characteristics were not mentioned by the interviewer at any point. The interviews were audio-recorded and transcribed.

The analysis of the transcripts was conducted in two stages. At first, we collected quotes that relate to a characteristic of the situation presented—either one of the pre-defined characteristics or a characteristic that had not yet been mentioned. In the next stage, we summarized the interviewee's apparent opinion about that characteristic. The summary provided insights into how the interviewee perceived each characteristic and how important it was in assessing the effectiveness of each situation. Through this process, we identified which characteristics are significant for the interviewees, thus validating the defined characteristics and revealing others that emerge from the interviews.

Population – 10 high school mathematics teachers and 5 mathematics education experts were interviewed. All teachers have taught advanced track high-school mathematics. Our choice was to interview both teachers and experts, since teachers might be those directly involved in the classroom, implementing teaching strategies, while experts might be those with deeper or broader knowledge, potentially having more experience in curriculum development. Teachers may provide practical, classroom-based insights into how students engage with different situations, while experts may contribute more theoretical or research-based perspectives.

Findings and their interpretation

For every characteristic, a choice of relevant quotes from different interviews will be presented, followed by a summary of the characteristic. Following each name is a letter in brackets stating whether that person is an expert from the mathematics education community (E) or teacher (T).

Time as the independent variable

Jimmy (E): The variable we choose, the derivative variable, is very, very important. It must have certain characteristics, I don't know what they are yet, but it's likely that time has the desired characteristics. [...] It should be something that somehow changes in a perceivably independent way. Independent of what, I don't know. Now, time, it's always advancing, so it's fine.

Donna (T): I think it's easier to understand these things when... when the variable is time. [...] That I have something where the x-axis is time, it seems simpler to me.

Sheila (T): Distance is something like... I don't know how much people feel it. I feel more in relation to time, for example.

While all 15 interviewees spoke of time-dependent variables as easier to make sense of, Jimmy's point of view reflects the core of the issue. The choice of the derivative variable is crucial, and it needs to have specific characteristics that facilitate comprehension. Jimmy suggests that time likely meets the necessary criteria. It is a variable that changes in a perceivably independent way, meaning it progresses naturally and continuously without relying on other variables. Sheila mentioned the physical aspect – we as humans can feel time going by.

Is the situation familiar?

Keren (T): [Good situations] not only are they not complex, they're also relevant to everyday life. For example, the taxi is less relevant in my opinion, because students don't really ride taxis much, or bother to look at the meter.

Yuri (E): I would introduce motion, maybe of bicycles, something that speaks to them, I don't know, electric scooter.

Gil (T): I wouldn't go for a car. I would explain, 'You move!' [...] They understand that if they're late for school, they increase their pace. […]. That's their world."

Alon (E): It's the effort, the effort to understand the phenomenon, it won't be too big, because ultimately, you want them to understand the concept [...] But there's some connection that these intuitions come from... the bodily experience of the person.

Interviewees recommended to use situations from students' everyday lives, ("something that speaks to them", from "their world"). The advantage of such situations, as Alon's words can be interpreted, is that the student has the knowledge required to understand the situation, and the effort is focused on understanding the new mathematical concept. Alon adds that the student has "intuitions" about such situations. Jimmy offered a nuanced view regarding the role of familiarity - the primary factor is not whether a situation is genuinely familiar, but rather how realistic the situation appears to them. While familiarity typically ensures a situation is realistic, a situation may be unfamiliar, yet still possess a high degree of realism if it can be easily imagined. Thus, according to Jimmy, realism, rather than familiarity alone, plays a critical role in the student's cognitive engagement with a task.

Is the situation tangible?

Grace (E): Why do I think about the pool much more than about temperature and it's because I see the water. Temperature I don't see, […], I feel, it fills up fast, it fills up slowly.

Jimmy (E): You feel the speed you're traveling at, and the speedometer reflects it as a number, which is excellent.

Donna (T): It's like here you say an experiment, you can really bring it to the classroom.

Jimmy (E): I would say that the height of the water in the pool, maybe I would choose that first, and not the amount of water in the pool, because the height of the water in the pool is something I see, I feel it with my senses, …

The quotes reveal a recurring theme regarding the relationship between visual and sensory perception and the abstract concept of RoC. Grace emphasized this distinction by comparing her ability to understand changes in water level versus temperature. She noted that she "sees" the water in the pool and can physically sense how fast it is filling up, whereas temperature lacks such a sensory aspect. The visual and tangible presence of the water, contrasted with the abstract nature of temperature, highlights how sensory engagement enhances understanding of dynamic processes. Jimmy emphasized how the "speed you're traveling at" can be physically felt, and this sensation is reinforced by the numerical feedback from the speedometer, connecting between physical sensation and numerical representation of RoC. Making a similar point, Jimmy reflects on the importance of sensory feedback when he distinguishes between perceiving the "height of the water" in the pool, which he can see and feel, versus the volume, which remains abstract. This distinction is significant because, while both height and volume measure water, only the height can be directly observed and felt, making it a more immediate representation of the process. Some interviewees offered representing invisible or abstract phenomena, such as temperature, with technical aids. Another idea in bridging the gap between abstract and concrete is physically experimenting, as Donna mentioned

Are there many external influencing factors?

Jimmy (E): Maybe I would, instead of my driving to work, I might construct a model, a situation that quiets down the noise, perhaps thinking about a racing track or a straight road or something. [...] I might try to minimize the things that hinder to the minimum.

Or (T): It depends quite a bit on other things, [...]it opens the opportunity to jokes and cleverness, and not necessarily something to conclude from the information.

Keren (T): I feel like these examples are less suitable, because they have a lot of considerations, which make these things much less absolute.

These quotes represent a common concern about simplifying the situations. Jimmy emphasized the importance of constructing models that "quiet down the noise" by removing unnecessary distractions, an focus attention on key variables. Or highlighted the risks of complex situations, which can lead to unproductive distractions ("jokes and cleverness") rather than meaningful conclusions. Keren reinforced these ideas by noting that certain examples, burdened with "a lot of considerations," become less effective for teaching, as they introduce complexities. Together, these reflections suggest that minimizing external factors is important. Specific situations raised these considerations (‘Taxi’ and ‘Ice cream’ the most). This also connects to the issue of whether time is the independent variable, since when some other factor is the independent variable, it is usually influenced by time, making the situation a complex function.

Emerging characteristics

In addition to the pre-defined characteristics, others emerged during the analysis of the interviews, among them the following three.

Is the situation interesting?

Donna (T): There's no... excitement in that [the Car situation]. What do I care.

Gil (T): When you try to present a non-mathematical position, at least it should be an interesting non-mathematical position. Okay, so the temperature changed, so what?

These quotes illustrate the importance of making examples engaging for students. Donna's comment, "There's no... excitement in that", underscores the need for examples that capture students' interest rather than mundane situations. Gil stated the bottom line – if EMC is to be used, it should be interesting. The topic and the phrasing of the situation should engage students’ interest.

Is the situation linear?

Or (T): But the rate of change here is constant, there is no change in the rate.

Alan (E): We need to look closely if this isn't a linear situation.

A characteristic which came up in several interviews was whether the situation is linear in real life, in the expert’s point of view, and thus less suitable for derivatives where we stress instantaneous RoC.

Is the situation authentic?

Grace (E): That it truly reflects reality and doesn't ridicule the connection between mathematics and reality.

Jimmy (E): The question needs to be authentic. That means, the questions we ask in our activities, they are questions that in some sense arise naturally from the context.

Gabriel (T): Let's say you have something sensitive in your house, that has difficulty to stand very large temperature changes, maybe because it shrinks, I don't know what. And

> you want to build some device that, let's say, alerts or beeps if there's too much of a sudden change in temperature.

Authenticity is a known characteristic from the literature, and several experts highlighted it as an important issue. It addresses the likelihood of asking the same question, not just within the context of a math class, but in studying a problem that is relevant to an actual real-world situation. Authenticity emphasizes the importance of ensuring that problems reflect real-world practical relevance, avoiding superficial or arbitrary examples. Gabriel provided a concrete example, involving a practical issue with temperature changes, which illustrates how even a temperature context that might seem unappealing can be used to formulate meaningful mathematical questions. Together, these quotes underscore that authentic problems are crucial for demonstrating the applicability of RoC in real life.

Discussion

The findings validate that the four key characteristics play a crucial role in the design of EMC based situations for introducing derivative as RoC in teachers' and experts' opinions. Additionally, several other characteristics emerged during the analysis, highlighting the complexity and subjective nature of contextual learning. In this discussion, we summarize the findings about the characteristics and suggest how EMC problems having these characteristics can effectively support students in constructing RoC as an abstract mathematical concept. The validated characteristics are:

1. Time as the Independent Variable - Time as the independent variable emerged as a critical characteristic that aligns with students' natural perception of events unfolding over time. This continuous and smooth progression of time allows students to see functions as evolving processes, rather than fixed objects, helping them better understand how variables co-vary. This dynamic view makes interpreting RoC more intuitive, as it connects directly to real-world experiences of change over time. When the denominator in the difference quotient represents a time interval, students may have less difficulty grasping the concept of two changes occurring over that interval.
2. Familiarity - Familiarity, was a key characteristic in interviewee's considerations. Previous knowledge of the situation frees cognitive resources to focus on the mathematical concepts rather than on the context itself. Jimmy argued that while familiarity helps, realism is important — students don't necessarily need firsthand experience but must be able to easily conceptualize the situation. When the situation is familiar, students are more likely to construct a mental image of it, as Thompson et al. (2022) describe. Whether the situation is something they have seen or can imagine vividly, this ability to mentally visualize the situation enhances students' understanding of how different variables interact. This supports their ability to connect abstract concepts to real-life interactions between variables.
3. Tangibility - The tangibility of a situation, referring to its physical and observable nature, further supports the learning process. When students can see and touch a process, they are better able to grasp the relationships between variables. Tangible situations, connected to what was just referred to as realistic, bridge the gap between abstract mathematical concepts and real-world experiences, making the learning process more concrete. We learned from the experts that it is beneficial to make the situation more tangible if needed – filming a video, using an app etc. Some experts emphasized bodily sensations, as feeling the speed.

4. Mathematically "Clean" Situations - A situation where no external factors influence the dependent variable, simplifies the learning process by removing unnecessary complexity. This characteristic ensures that students can focus solely on the relationship between the independent and dependent variables, free from the distractions of external influences. Such clarity helps to avoid idle discussions, or cleverness.

In addition to these validated characteristics, several others emerged during the study:

5. Interest - Engaging students' curiosity and motivation.
6. Presentation: How the lesson around the situation is built.
7. Nonlinearity: The potential for incorporating nonlinear relationships into the context.
8. Authenticity: The situation's relevance in the real world, beyond the classroom.

Among these characteristics, nonlinearity stands out as particularly important for introducing the concept of RoC, which involves instantaneous change. On the other hand, ‘Interest’ emphasizes emotional benefits, which is less crucial for the specific goal of understanding derivative as RoC. The way the lesson around the situation is built is a topic for further research.

The analysis reveals the subjectivity in how familiar a situation appears to different people. The 'Car' situation was intended to be familiar, assuming most people in the Western world have seen cars and experienced car rides. However, several interviewees argued that it wasn't familiar enough for students, suggesting alternatives like walking, biking, or riding electric scooters, which they felt were more relevant (“I would introduce motion, maybe of bicycles, something that speaks to them,”). It is important to recognize that some characteristics, like familiarity, are subjective and exist on a spectrum unlike time, which is either the independent variable or it is not. What is familiar to one student may not be to another, emphasizing the need for flexibility when designing context problems.

Furthermore, certain characteristics tend to overlap. For instance, less familiar situations are often more complex; contexts that are not time-dependent may never be ‘clean’; and situations that are familiar are generally more engaging and interesting to them. Possible dependencies among the characteristics, such as time-dependence making a situation more tangible, require further research.

This study contributes to the development of characteristics that guide the design of situations aimed at introducing the derivative as a rate of change. By considering both established and emerging characteristics, educators can design learning experiences that connect students' real-world experiences with the new abstract mathematical concept. For a first lesson on the derivative as a rate of change, these characteristics help ensure that the problems support students in understanding the concept. While later lessons should expose students to other contexts that may not include all of these characteristics, for an initial lesson, they are crucial in aiding students' comprehension.

A key contribution of this study is bringing order and structure to a topic that has not been systematically addressed before. Although some individual characteristics have been discussed in various studies, no prior work has proposed a consolidated, organized list of characteristics specifically aimed at a particular pedagogical goal. The framework presented provides a foundation for future research on characteristics for other pedagogical goals.

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Student strategies for deciding the limit of a multivariable function

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This is a grounded research study focusing on strategies students use to establish the limit of a multivariable function at a point, and student misconceptions. Open coding was used to analyze 688 exam answers in a multivariable calculus course. We provide a fine-grained analysis of 9 categories of student strategies, with a discussion of student understanding or misconceptions in each one. Most students approached the point along different curves or split the function as a sum of terms. Those that chose curves were twice as likely to succeed. Prevailing misconceptions were that checking only several paths approaching the point guarantees a limit, and an automated misuse of the sandwich theorem. We also detected misconceptions regarding the relations between the existence of a limit, continuity, partial derivatives and differentiability in multivariable functions.

Keywords: Multivariable calculus, function limit, student strategies, student misconceptions.

Introduction

Studying calculus teaching and learning is an ongoing line of study in mathematics education. Extensive research, dated as far back as the 1980's, explores student understanding and misconceptions of fundamental subjects such as limits, functions, continuity, derivatives and integration (Thompson & Harel, 2021). The growing body of research on tertiary mathematics teaching and learning has further extended the inquiry about calculus learning in different levels and disciplines, and research groups that focus on that subject are being established (e.g., the new TWG25 in CERME14). However, much of this research concerns single variable functions. Students' understanding of multivariable functions has been explored far less (Martínez-Planell & Trigueros-Gaisman, 2012; Mamona-Downs & Megalou, 2013). This study aims to provide new information concerning this issue, by studying strategies that students employ when trying to decide whether a function $f: \mathbb{R}^2 \to \mathbb{R}$ has a limit at a given point, namely whether the limit $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ exists.

Literature Review

There is a wide agreement that the concept of limit is one of the most fundamental concepts of calculus, whose comprehension is crucial to students' mastery of calculus. Yet, there is ample evidence that this concept, and its formal definition, presents major difficulties for most students (Cottrill et al., 1996; Thompson & Harel, 2021). Some difficulties arise because students attach colloquial meanings to the word 'limit', and those meanings obstruct both student conceptual understanding of limit and its definition. For example, students perceive 'approaching' a value as getting progressively closer to an unreachable value and interpret the term 'limit' as a boundary that cannot be passed (Denbel, 2014). Bezuidenhout (2001) found that students have conflicting conceptions of limit and continuity and claimed that these conflicts might have a negative influence on developing understanding of other calculus concepts. In addition, many first-year calculus students' "knowledge and understanding rest largely on isolated facts, routine calculation, memorizing algorithm, procedures and that their conceptual understanding of limits, continuity and

infinity is deficient" (Denbel, 2014, p. 38). Bezuidenhout (2001) adds that "the stereotyped exercises that are a feature of several calculus texts often encourage an instrumental approach…" (p. 498).

Alcock and Simpson (2004) explored approaches of first-year students who include visual imagery in their reasoning to solving problems about convergence of sequences and series. They conclude that "…visual images provide students who focus on them with a strong sense that they understand the material and can correctly answer questions about it. This can mitigate against any perceived necessity to seriously examine the complex formal material" (p. 29). Conversely, students who are internally motivated do successfully integrate visual and algebraic representations to support formal work. Consequently, it is unclear whether the common pedagogy of mathematics lecturers to refer regularly to visual images in real analysis courses has a positive or a negative outcome for students.

It is natural to inquire whether the documented difficulties of students regarding limit of a single variable function at a point persist in multivariable functions, and if so in what ways. Or, are there new misconceptions which arise, and their sources. However, there is considerably less research about students' difficulties regarding multivariable functions in general (Martínez-Planell & Trigueros-Gaisman, 2012) and about limits of multivariable functions at a given point in particular (Mamona-Downs & Megalou, 2013). Research on multivariable functions focused on students' mathematical knowledge and understanding and on students' mental constructs and schemas, using, for example, the theoretical framework of APOS (Martínez-Planell & Trigueros-Gaisman, 2012; Tatira, 2022), possibly following the work of Cottrill et al., (1996).

The studies on students' understanding of limits of multivariable functions were mainly based on task-based interviews of a relatively small number of students. Mamona-Downs and Megalou (2013) interviewed 4 students and asked them to examine whether the limit $\lim_{(x,y)\to(0,0)} \frac{7xy}{5y^2 - x^2}$ exists. Although they were not guided how to approach the problem, all 4 students chose a 'directional approach' (either approaching along lines passing through the origin or converting into polar coordinates). Tatira (2022) analyzed written assignments of 194 students seeking to characterize undergraduate students' conceptualization of multivariable limits. In the written assignments students were asked whether the limits of similar rational functions exist. Most of the students answered correctly, approaching the point (0,0) along lines and computing different limits. Tatira then interviewed 14 students who gave a partial or wrong answer and showed that students concluded the existence of a limit after considering only a few paths, such as lines passing through the origin (e.g., $y = 0$, $x = 0$, $y = mx$). Indeed, Tatira states that this is a common difficulty, as many students struggle to understand the nature of the infinite number of paths or curves along which a point in the plane can be approached, what can be deduced (nonexistence of limit if along different paths the function approaches different limits) and what cannot be deduced (existence of limit if along a single or along few different paths the function approaches the same limit).

Rationale and Research Questions

The authors teach a large calculus course at an engineering college. At this college, calculus is taught over two semesters (I/ II), where calculus I focuses on single variable functions and calculus II on multivariable calculus. The final exam for calculus II included a question about the limit and

continuity of a multivariable function at the origin. While grading the exam, the authors encountered a rich collection of strategies used by the students, that were not documented in previous research. This motivated an analysis of student strategies and misconceptions as they appeared in the written answers in the exam. Here, the term 'misconception' is used to indicate a mistake done consciously, and is the result of a misunderstanding, where a student believes a false concept or strategy to be true. The term 'strategy' is used to indicate the general way a student approached the problem. The authors analyzed students' written exam answers, to answer the following research questions:

1) What were the most common strategies students used to evaluate limits of multivariable functions?
2) Which of the strategies that students used were most likely to be successful?
3) What misconceptions did students have about the existence of limits of multivariable functions?

The study embraces the paradigm of grounded research theory, which was designed to generate theory from existing data, for example by using qualitative coding techniques. Czocher and Melhuish (2024) identify two kinds of what they name *Coding studies*: *Inductive studies* that "generate initial codes for analytic units, group like codes together into categories…" and *Deductive studies* that "apply an already-existing or literature-derived framework to analytic units or otherwise categorize analytic units according to clearly defined concepts" (p. 148). Thus, the current study is an Inductive study since it focuses on an initial categorizing of a new phenomenon.

Method

The study was conducted in a multivariable calculus course, taught in an engineering college. Students are in 7 different engineering majors, but all the students take the same calculus sequence with the same entry requirements. The course is large (~800 students), split between different lecturers and teaching assistants. A total of 688 students took the exam. In the final exam, students freely chose 5 of 6 questions to solve and be graded. One of the exam questions was as follows (translated): Does the function $f(x,y)=\begin{cases}\dfrac{3y^9+y^4x^2+x^5}{y^8+x^4} & (x,y)\neq(0,0)\\ 0 & 0\end{cases}$ have a limit at (0,0)? Is $f(x,y)$ continuous at (0,0)?

The definition of the limit at a point taught in the course is that "a function has a limit at a point if and only if the function yields the same limit when approaching the point along all possible curves". In the lessons, the instructors typically use a rational function similar to the exam question, and examine its limit at (0,0), to demonstrate that if along two (or more) curves approaching the point the function has different limits, then the function does not have a limit at the point. In the exam question, approaching (0,0) along straight lines produces the same limit, however a approaching along a parabolic curve produces a different limit, thus the function does not have a limit at (0,0). This is a challenging function, but students see similar examples in lectures and homework problems.

Our data was students' written answers in the exam. The analysis focused on strategies that students used to decide whether the limit exists, and students' misconceptions. Continuity was considered only if it was used as an argument for existence of a limit. Although the authors are experienced lecturers, they analyzed students' answers disregarding their own initial assumptions about common student

misconceptions. Instead, they scrutinized the written answers to find what strategies students used (and successfully), and what types of misconceptions were common. An open coding methodology was implemented: (i) 20 random student answers were examined by two authors and initial codes were set; (ii) the rest of the exams were split between the authors; any new code was discussed and agreed upon by all authors; (iii) each author coded students' answers and marked answers that did not fit neatly into one of the codes; (iv) two of the authors re-examined the marked ambivalent answers and decided how to classify them. At the end of this process there were 25 fine-grained codes, belonging to 5 main categories and 9 subcategories (we present only subcategories that appeared on more than 2% of the exams). Table 1 presents categories and subcategories of student strategies.

Table 1: Codes for Strategies – Main categories and subcategories

Category C - Use of different curves or paths to approach the origin	
C_✓ *Correct*	Approaching (0,0) along different paths (lines and parabolas) gives different limits so the limit does not exist.
C_part	Approaching (0,0) along some paths or lines (*x,y* axes or $y = kx$) gives a limit L=0 Or: Approaching (0,0) along $x=y^2$ gives a limit L=0.5
C_P	Improper use of polar coordinates
C_cts	Approaching (0,0) along a path ($x = y^2$ or another) that gives a limit L; since $L \neq 0 = f(0,0)$ the function is not continuous and hence the limit does not exist
Category D - Use of partial derivatives ($f_x(0,0) = 1, f_y(0,0) = 3$)	
D_exist	The partial derivatives exist so the limit of *f(x,y)* exists (not clear what the limit is)
D_neq	The partial derivatives exist but are not equal, so the limit of *f(x,y)* does not exist
Category T - Splitting the function into a sum of two/three terms and determining the limit of each term	
T_✓ *Correct*	Term 1 and Term 3 have a limit zero (using the sandwich theorem correctly) and Term 2 has no limit (argument using paths)
T_part	Term 1 and Term 3 have a limit zero (using the sandwich theorem correctly), Term 2 converges to 0.5, so the limit is 0.5 or does not exist
T_sand	All three terms have a limit zero (using the sandwich theorem correctly for Term 1 and Term 3, and incorrectly for Term 2), so the limit is zero

Strategies were classified in three main categories: (C) use of different curves or paths approaching the origin, (D) use of partial derivatives, (T) splitting the function into a sum of terms (e.g.,

$\frac{3y^9+y^4x^2+x^5}{y^8+x^4}=\underbrace{\frac{3y^9}{y^8+x^4}}_{Term\ 1}+\underbrace{\frac{3y^4x^2}{y^8+x^4}}_{Term\ 2}+\underbrace{\frac{x^5}{y^8+x^4}}_{Term\ 3}$) and examining limit of each term separately. There were two other categories: question not chosen (X); scratch work not leading anywhere (0).

As a final comment we wish to stress that the analysis did not evaluate the quality of students' argumentation or writing but focused on the chosen strategy. For example, students who split the function into three terms, demonstrate that two terms have a limit zero and the middle term does not have a limit thus conclude correctly that the function does not have a limit (T_✓), need to support this conclusion by adding an argument about the sum of functions with/without a limit. The lack of this argument was expressed in their exam grade but was not reflected in the analysis of strategies.

Findings

Out of 688 students who took the exam, 592 students (86%) chose to attempt the limit question. This suggests that students felt confident that they would be able to answer the question, at least partially. Table 2 presents the number and percentage of the students in each main category: paths of curves, partial derivatives, and sum of terms. Percentage in Tables 2 (and Tables 3-4) is computed relative to 592 students who tried to solve the question (i.e. N/592*100). For each category we also present the number of students for whom the use of this strategy produced a correct answer.

Table 2: Breakdown of main strategies

Category	C (Correct)		D (Correct)	T (Correct)		0	Total
N	278	(152)	49 (0)	178	(44)	87	592
%	47		8.3	30.0		14.7	100

Table 2 shows that of those who did answer, only about 15% made insufficient progress to classify a strategy (Category 0), so this leaves roughly 85% of the exams (505 students) to be assessed for strategies. Less than 10% chose to look at partial derivatives. 54.7% of the students who used the curves strategy answered the problem correctly. For students who tried a sum of terms (Category T), only 24.7% answered correctly. So, students who tried the curve strategy had twice the success rate over students who used the sum of terms strategy. This answers Research questions 1-2.

Table 3: Success and failure rates

Category	Correct	Partially correct	Wrong	0	Total
N	196	260	49	87	592
%	33.1	43.9	8.3	14.7	100

Table 3 presents the percentage of correct, partially correct and wrong answers. Partially correct are students in Category T who split the function to two/three terms and reached a correct conclusion about the limit of the first and third terms, and students in Category C that calculated correctly the limit of the function along different curves or paths but determined the limit incorrectly. We see that

approximately 52% of the students managed to make progress towards a solution (not counting the correct answers or the students who submitted scratch work), split over the different misconceptions.

Table 4: Breakdown of main Strategies by subcategories

Category	C_✓	C_part	T_sand	T_✓	D_exist	T_part	C_cts	D_neq	C_P
N	152	90	90	44	22	20	15	14	13
% (of 592)	25.7	15.2	15.2	7.4	3.7	3.4	2.5	2.4	2.2

Table 4 presents the number and percentages of each subcategory (above 2%, descending order). Each subcategory represents a slightly different strategy as explained in Table 1. Each of the strategies that led to a wrong answer represents a misconception, answering Research question 3. Of the students who tried paths of curves (Category C), the most common misconception was C_part, indicating students were not aware that they needed to check more paths. Understanding that checking only some paths does not guarantee a limit is one of most difficult ideas for students when learning multivariable limits. This misconception is echoed in T_part, which together with C_part gives 110 exams under this misconception, for 21.8% of the 505 who made progress. Of the remainder, the 2.5% of the students who used strategy C_cts, have the misconception that if the function is not continuous, then the limit does not exist. This misconception should have been treated in first semester calculus. Of the students who tried to split into a sum of terms, T_sand was the most common for 15% of the students trying to use the sandwich theorem incorrectly on Term 2.

Of the students who used the partial derivatives strategy (Category D), D_exist was the most common strategy, indicating that these students did not understand the limit definition of partial derivatives. This is particularly disturbing since the partial derivatives have different values, so it is not clear what these students believed the limit to be and why they were not conflicted. Students who used strategy D_neq had a similar misconception regarding what the limits they computed represent but relied on a wrong version of the argument that different limits indicate that the limit at a point does not exist.

Discussion and Conclusions

This study is an inductive coding study (Czocher & Melhuish, 2024). Open coding was used to analyze 592 written exam answers of 592, to determine whether a given multivariable rational function has a limit at a given point. We found most of the students tried either a strategy of analyzing the function along curves (Category C) or presenting the function as a sum of terms (Category T). Around 8% of students based their answer on the existence of partial derivatives (Category D).

Tatira's (2022) study also analyzed a large number of exam answers, rather than interviews, on the existence of limits of multivariable functions. Tatira found that over 74% students succeeded in proving that the given functions did not have a limit at the origin. In our study only 33.1% of the students got a correct answer. This might be because Tatira's rational functions required considering only straight lines through the origin to deduce that the limit does not exist. In our exam straight lines produced the same limit and proving nonexistence of the limit required parabolas. Overall, we agree with Tatira (2022) that "many students study multivariable limits evaluation without really

understanding indeterminate rational functions" (p. 160). Tatira categorized students' exam answers as correct/partially correct/wrong, whereas our open coding produced finer categorization, classifying different strategies within the categories of partially correct or correct. Still, since our study relates to a single rational function, our findings merit further investigation with other types of functions.

The most common misconception among students who analyzed along curves (Category C), is that checking only one path, or several paths, guarantees a limit (C_part, Table 4). This misconception was documented in previous studies based on a small number of task-based interviews (Mamona-Downs & Megalou, 2013; Tatira, 2022). Our analysis of a large body of data supports the prevalence of this misconception. Interestingly, for students who split the function into terms (Category T) the prevailing misconception was different: that all three terms have a limit 0 based on the sandwich theorem (T_sand, Table 4). We interpret this as procedural thinking about using the sandwich theorem, without understanding how to prove that a rational function is bounded. Finally, the success rate of students who split the functions was 24.7%, whereas for students who tried the curves strategy it was 54.7%. We suspect that students that provided a correct answer after splitting the function into terms have a deeper understanding of the limit concept, since they: (i) correctly detected the "problematic" Term 2; (ii) they did not automatically used the sandwich theorem for Term 2 like they did for Term 1 and Term 3, and (iii) used a complete path strategy to deduce that Term 2 does not have a limit at (0,0). Their analysis involves inquiry and proper use of mathematical theorems.

Another misconception that we found among 2.5% of answers, is that students claimed that if $f(x,y)$ is not continuous, then the limit does not exist. This misconception was also documented by Bezuidenhout (2001) in single variable functions. This misconception, together with the misuse of the existence of the partial derivatives (8.3%) reflects a confusion about the intricate relations between the existence of a limit, continuity, existence of partial derivatives, continuity of partial derivatives and differentiability. The use of derivatives to determine a limit of a function was not documented in other studies. We conjecture that since we analyzed written answers in the final exam, the students were already acquainted with all of these concepts, knew that differentiability is a sufficient condition for continuity and perhaps solved questions that combine the question on limits and continuity with the question on differentiability in preparation for the exam. Perhaps this, together with their previous knowledge about single variable functions, for which differentiability implies continuity, provided the basis of this wrong strategy. This is a matter for further research.

A major difference between single variable and multivariable functions is that students (and most instructors) can only visualize or sketch simple multivariable functions. For single variable functions there is a strong intuitive advantage to understanding the limit concept using the graph of the function. However, with a multivariable function most are "blind" in the sense that the intuition is gone, and students must rely on purely algebraic techniques without the ability to visualize what they are trying to compute. They have no conception of the infinite number of paths that approach a point, and how these level curves look on the graph of the surface. As noted by Tatira (2022), citing Fisher (2010), for multivariable functions even the use of computer graphics does not alleviate this problem. Alcock and Simpson (2004) stated that students get used to base their understanding on visual images and this somehow impedes their ability to solve problems relying solely on formal arguments. Overcoming this obstacle is a worthy goal for future research.

Overall, our study serves as a primary exploration and our findings can be used to design a pedagogical intervention addressing the misconceptions that were found or to plan learning activities that encourage student inquiry. Another future research direction is probing deeper into students' understanding of the concepts of limit and continuity using an appropriate theoretical framework, for example APOS that was used by Cottrill et al. (1996) and by Mamona-Downs and Megalou (2013), taking into account the new strategies that were documented. It seems that for students, the concept of limit in multivariable functions is not a simple extension of this concept for single variable functions but entails other challenges that are well worth exploring.

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Dynamic conceptualisation of density of $\mathbb{Q}$ in $\mathbb{R}$: Upper-secondary school students' spontaneous conceptions and arguments

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While research in Mathematics Education has mainly focused on density of $\mathbb{Q}$, little is known about density of $\mathbb{Q}$ in $\mathbb{R}$, despite its importance in grasping fundamental Calculus concepts such as continuity. This study investigates upper-secondary school students' spontaneous conceptions and arguments regarding the density of $\mathbb{Q}$ in $\mathbb{R}$. A classroom activity and group interviews focused on continued fractions, which provide a unified representation of rational and irrational numbers, were carried out. The data consist of the students' written and spoken statements, and the data analysis strategy follows a thematic analysis approach with some specific modifications. The research findings indicate that students' spontaneous conceptions are dynamic processes that range over different categories supported by different arguments and warrants.

Keywords: Density of $\mathbb{Q}$ in $\mathbb{R}$, spontaneous conceptions, arguments, continued fractions, warrant.

Rationale

The concept of density of $\mathbb{Q}$ in $\mathbb{R}$ is crucial because it ensures that irrational numbers can be conceived of as limits of sequences of rational numbers, guaranteeing that each point of the real line corresponds to a number. Students encounter the concept of density already when dealing with rational numbers, but while the set of rational numbers is dense, the set of real numbers is both dense and complete. If the density of $\mathbb{Q}$ becomes an intuitive model for the density of $\mathbb{Q}$ in $\mathbb{R}$, students may struggle to grasp the difference between $\mathbb{Q}$ and $\mathbb{R}$ and the role of completeness for $\mathbb{R}$ as the ideal domain for central Calculus phenomena such as continuity and completeness.

As research shows (Branchetti & Durand-Guerrier, 2018; McMullen & Van Hoof, 2020; Putra, 2019), exploring the transitions between numerical sets is crucial to avoid obstacles in future mathematical learning and the density-concept is crucial in this sense. Nevertheless, according to the literature, while the concept of density of $\mathbb{Q}$ is widely investigated (e.g., Hannula et al., 2012; Kim & Kwon, 2019), the density in the transition from $\mathbb{Q}$ to $\mathbb{R}$ – the one of density of $\mathbb{Q}$ in $\mathbb{R}$ – has received little attention from research in Mathematics Education (Marmur et al., 2020). Marmur et al. (2020) investigate how first-year mathematics undergraduates grasp the concept of density of $\mathbb{Q}$ in $\mathbb{R}$. However, students face this concept, at least intuitively, already in upper-secondary school. In this context, spontaneous conceptions, intended as intuitive beliefs formed from experiences in various contexts, including everyday life (Fujii, 2014), serve as a foundation for future institutionalised concepts. This paper, based on the first author's master's thesis (Grisendi, 2024), examines the spontaneous conceptions of the density of $\mathbb{Q}$ in $\mathbb{R}$ of two classes of Italian upper-secondary school students (aged 17/18). The students' spontaneous conceptions, the kind of arguments and justifications they use to support their reasoning in working on this concept, are categorised and integrated, focusing on the following research questions: (1) What are students' spontaneous

conceptions regarding the concept of density of $\mathbb{Q}$ in $\mathbb{R}$, and what arguments do they use to support them? (2) How do these arguments and spontaneous conceptions support students' reasoning processes as they evolve during a classroom activity focused on the concept of density of $\mathbb{Q}$ in $\mathbb{R}$?

Theoretical framework

The theoretical framework is composed of two elements: Balacheff's (1988) classification of different approaches to proof and Toulmin's (1958) concept of warrant. These two components are first introduced, and then their interrelations in the present investigation are discussed.

Balacheff (1988) distinguishes between *pragmatic proof* (based on actions) and *intellectual proof* (which employs verbalisations of the properties of objects and their relationships). Additionally, he identifies a taxonomy for the types of proof conceived as steps between *pragmatic proof* and *intellectual proof*, characterised by increasing sophistication: *naïve empiricism* (examples proposed are considered self-evident); *crucial experiment* (a paradigmatic example is used, suggesting that if it holds in one case, it holds in all cases); *generic example* and *thought experiment* (an example is used as a "support," but the reasoning is general).

According to Toulmin (1958), an argument has three constituent parts: claim, datum, and warrant. The claim refers to the main assertion one is attempting to prove; the datum includes the reasons, evidence, facts, and/or information that support the claim; the warrant represents the logical connection linking the datum to the claim supporting the provided argument. Our analysis focuses specifically on the warrant; instead, the claim and the datum are provided in the task and do not vary.

Balacheff's (1988) classification of different approaches to proof accounts for what students consider a reliable validation and the soundness of their inference and Toulmin's (1958) idea of warrant provides critical insights into the type of justification used to support the validation. In this sense, the two components mentioned above collectively support the analysis of student conceptions and arguments.

Research method

Our research is guided by an *ad hoc* constructed strategy based on different investigation themes. These themes include students' general arguments, with a specific focus on students' mathematical arguments, and finally, students' conceptions. To stimulate the emergence of spontaneous conceptions, an activity focused on continued fractions, which facilitates the transition from the countable ($\mathbb{Q}$) to the continuous ($\mathbb{R}$), was designed. Two classes for a total of 42 upper-secondary school students were involved. The activity, divided into two parts, was conducted under the lead of the first author in the presence of the classroom teachers but without their intervention. In the first part (two hours), students respond anonymously, working in small groups. This part involves a worksheet, which begins with a brief contextualised introduction to continued fractions, where a hypothetical student proposes a continued fraction as an approximation of an irrational solution of a second-grade equation; then, students are asked to argue for or against the statement: "[...] given any real number, can I always find a rational number that is as close to it as I want?" After reading the initial part of the worksheet, students complete two parts, called Assignment 1 and Assignment 2. In Assignment 1, students write down a points list of arguments for or against the statement read before,

while in Assignment 2, starting from the generated list, students are asked to produce a discursive response by writing an argumentative text. These tasks were chosen to track students' reasoning processes and identify the kinds of arguments they use to support their conceptions. Finally, students are interviewed by the researcher a few days later via semistructured group interviews, inviting them to deepen the arguments presented in their group. In this sense, the collected data were written or spoken sentences produced by the students while trying to argue their position concerning the task-claim.

The data analysis strategy used in this research partially overlaps with the thematic analysis described in Xu and Zammit (2020). However, it differs from the latter in several aspects, as it was constructed ad hoc for the data decoding process. Below, we explain our approach, showing analogies and differences to the thematic analysis approach. As revealed before, an *ad hoc* strategy was constructed because of the data set's complexity. We designed a structured data analysis process, i.e., a flowchart, that guides the coding of the data through structured questions. Thanks to this tool, some of the thematic analysis steps are unnecessary for our analysis.

The first step of our data analysis corresponds with the first step of the thematic analysis. We reviewed the students' sheets and prepared targeted questions for each group to explore their reasoning further. These questions are used to guide the initial part of the semistructured interviews. Indeed, only one question was repeated identically for every group: "How can you define, in your own words, a dense set?"

Also, our second step is analogous to the second step of thematic analysis (Xu and Zammit, 2020). We constructed a category structure, the theory-driven code, developed through both deductive (drawing from the relevant literature) and inductive (drawing from the data) approaches (Mayring, 2015), and we elaborated the above-mentioned flowchart as the data-driven code to simplify the process. The flowchart guided the classification process of the statements through structured questions. Following the flowchart's questions multiple times with the same students' statement, we explored its possibly different interpretations. This flexibility enabled us to observe a single statement considering different aspects of students' conceptions and arguments.

Below, we describe the category structure. Initially, we focused on classifying the types of arguments students proposed. Deductively, we referred to categories identified in the section Theoretical framework; inductively, we identified the remaining warrant-categories, supposing that each has a dual category. In this sense, a distinction was made between *pragmatic warrant* (related to utility and efficiency) and *theoretical warrant* (related to properties and definitions). Subsequently, for each of Inglis' (2007) warrant-categories – *inductive warrant* (the use of a certain number of examples), *intuitive warrant* (the use of observations or examples that reflect some underlying cognitive structure), and *deductive warrant* (the use of deduction from axioms, algebraic manipulations, and counterexamples), a distinction in *operational warrant* (related to a procedure) and *structural warrant* (related to properties and relationships between objects) was made. Finally, we categorised the spontaneous conceptions of density of $\mathbb{Q}$ in $\mathbb{R}$ that emerged during the activity. These conceptions were categorised inductively into two types: *analytic conception* (based on a fundamental idea of sequences, intuitively linked to the construction of the set $\mathbb{R}$ through Cauchy sequences), *synthetic*

conception (based on geometric-spatial concepts and related to the number line, intuitively linked to the construction of the set $\mathbb{R}$ through Dedekind cuts).

The following three steps of thematic analysis refer to using a thematic map developed in the third step, revised and refined in the fourth step, and organised "into a coherent and consistent account" (Xu & Zammit, 2020, p. 7) in the fifth step. Because our analysis develops by referring to the flowchart, the process is structured and makes the analysis shorter. Indeed, thanks to the data analysis thematic code, immediately after classifying a statement in the category structure and tracking how students move towards a new statement, we can reconstruct and represent the group's reasoning through specific kinds of thematic maps, called schema and graphs. In the following section, we show two different kinds of maps used to represent the groups' reasoning.

The last step of thematic analysis coincides with this paper's Discussion and conclusion section.

Results

Below, we provide a summary analysis of just one of the twelve groups examined in the thesis, suitable to exemplify the methodology of data analysis developed and used to this end. We focus on Group 10, whose responses are particularly interesting regarding the reasoning process the students proposed. Before Group 10's detailed analysis, we present two examples of the thematic maps used in our research. These maps are designed to represent "the meanings in the data set as a whole" (Xu & Zammit, 2020, p. 6) and "to fit each theme into a broader overall story about the data" (Xu & Zammit, 2020, p. 7), effectively facilitating the visualisation of data analysis for each group.

Figure 1 shows the schema that presents the analysed categories for Group 10's spontaneous conceptions and arguments. Each category is represented by a box with a distinct shape and colour. The coloured rectangles provide information on the context of data collection (group number, Assignment 1, Assignment 2, Interview).

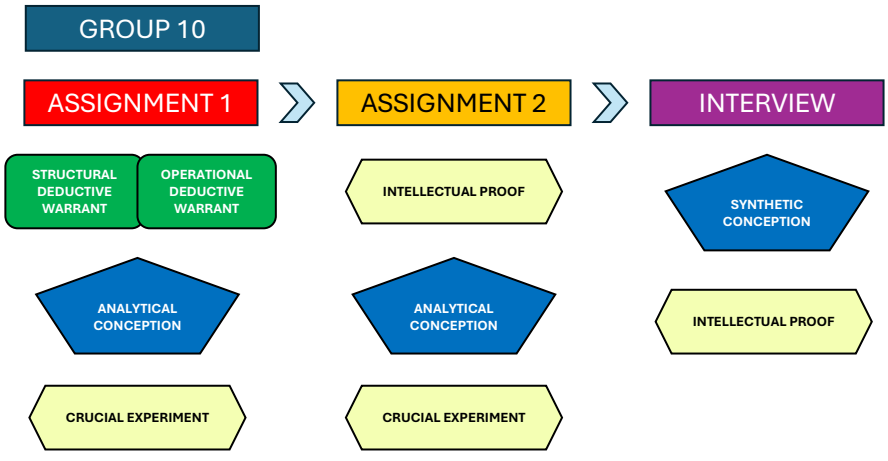

Figure 1: Schema of Group 10 students' arguments and spontaneous conceptions

Moreover, Group 10 students' reasoning can be effectively represented by a graph (Figure 2).

In the graph in Figure 2, coloured circles indicate the vertices, while dark blue arrows represent the edges. The logic behind the connections between the two statements is as follows: (1) If the edge

between two statements, *i* and *i*+1, is a head-tail arrow, it indicates a temporal sequence (i.e., the arrow's tail represents the statement at time *i*, while the head represents the statement at time *i*+1); (2) If the edge is a head-head arrow, the two statements are semantically similar, as they reflect analogous reasoning expressed at different moments, possibly with variations in wording. The students' statements are displayed in light blue rectangles; the coloured circles indicate when the statement was made during the activity, and the coloured triangles indicate the category in which the statement was classified.

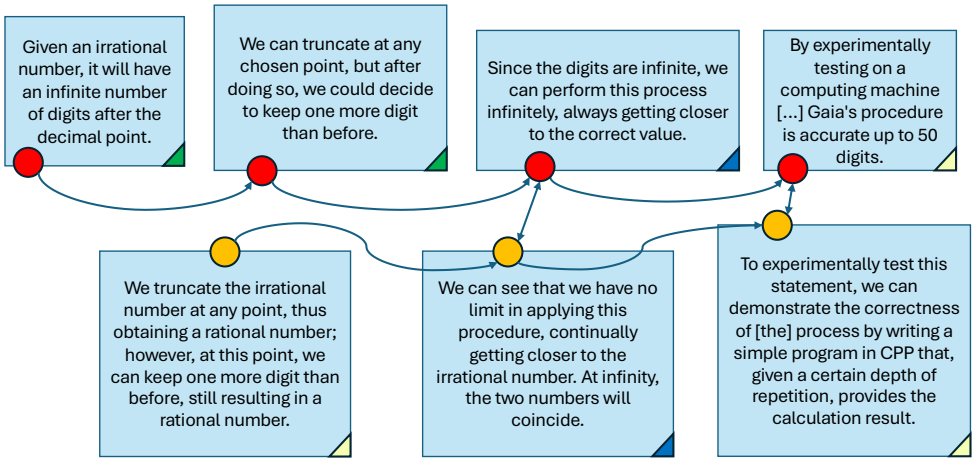

Figure 2: Graph of Group 10 students' arguments and spontaneous conceptions

With the support of the two previously presented schemas, we analyse Group 10's data collected. The students of this group initially employed a *deductive warrant* (Assignment 1). It first provided them with a theoretical reference (*structural deductive warrant*). Indeed, the statement, *Given an irrational number, it will have an infinite number of digits after the decimal point* (Figure 2), represents a fundamental property of irrational numbers. Additionally, an operational process (*operational deductive warrant*) was the foundation of their reasoning. Indeed, this is evident in the statement: *We can truncate* [the irrational number mentioned before] *at any chosen point, but after doing so, we could decide to keep one more digit than before* (Figure 2), which outlines a specific procedure.

Subsequently, in Assignment 2, Group 10 demonstrated a natural progression from a *theoretical warrant* (of which the *deductive warrant* is a specific case) to an instrument for justifying an *intellectual proof*. The statement classified as an *intellectual proof* is the following: *We truncate the irrational number at any point, thus obtaining a rational number; however, at this point, we can keep one more digit than before, still resulting in a rational number* (Figure 2). We can observe that, while in the *theoretical warrant* it is possible to distinguish between the reference to the theoretical concept and the proposal of an operational procedure, in the *intellectual proof*, the two components are no longer distinctly separable, producing a unified and coherent reasoning process.

It is evident that Group 10 is aware of the problem of generalisation. To validate their reasoning, the students conclude their analysis by providing a paradigmatic example, specifically a CPP program. Since the program is designed to generalise the process, Group 10's statements regarding this program are considered in the *crucial experiment* category.

The procedure proposed by Group 10 is further supported by the *analytical conception* that the students have regarding the concept of the density of $\mathbb{Q}$ in $\mathbb{R}$. In Assignment 1, the statement – *Since the digits are infinite, we can perform this process infinitely, always getting closer to the correct value* (Figure 2) – suggests a conception based on the image of a sequence of rational numbers obtained by the "truncation" process, converging towards the "*correct value*". In Assignment 2, the statement – *We can see that we have no limit in applying this procedure, continually getting closer to the irrational number. At infinity, the two numbers will coincide* (Figure 2) – intuitively expresses the idea of a sequence of rational numbers as a single number that "transforms" over time.

During the Interview, the *analytical conception* evolves into a *synthetic* one. Student A's statement – *There is a value at every point.* […] *You have a value for every point; you can always divide a point in half* (Figure 2) – explicitly refers to spatial and geometric concepts, as indicated by the terms "value *at every point,*" "value *for every point*," and "*divide a point* in half," which imply a correspondence between $\mathbb{R}$ and the number line. Furthermore, student B's statement – *I know that the cardinality of natural numbers and rational numbers is the same; whereas, for real numbers, it is different.* […] *It is larger. Between natural and rational numbers, we can find a "correlation," so we can say they have the same cardinality. However, real numbers are dense, meaning there is no space between one number and another* (Figure 2) – explicitly refers to a property of $\mathbb{R}$ set, namely the cardinality of the continuum. This becomes clear when student B states that the cardinality of $\mathbb{R}$ set is "*different*" from that of $\mathbb{N}$ and $\mathbb{Q}$ because it is "*larger*." This statement is classified as *intellectual proof* since the property of $\mathbb{Q}$ is used to argue that $\mathbb{Q}$ is a dense set. Moreover, it supports the emergence of the *synthetic conception*, as indicated by the terms "there is no *space* between one number and another".

Discussion and conclusion

In the Rationale section, we pointed out that only one of the referenced studies specifically addresses the concept of the density of $\mathbb{Q}$ in $\mathbb{R}$: the research conducted by Marmur et al. (2020). Marmur et al.'s (2020) findings are twofold. Firstly, the findings demonstrate "unconventional student understandings" (Marmur et al., 2020, p. 7) of the concept of density of $\mathbb{Q}$ in $\mathbb{R}$, e.g., some students use the concept of infinity only as a process, as authors write: "attend to the process of creating infinitely many rational numbers [...], rather than to the existence of these numbers" (authors' italics) (Marmur et al., 2020, p. 5). Secondly, rather than attending to general reasoning, students concentrate "on specific examples, contexts, processes, and personal meanings, consequently reducing the intended abstraction level of the task" (Marmur et al., 2020, p. 7). These authors' study focuses on university students, while our research participants are upper-secondary students. As the participants' linguistic and cognitive tools in the two investigations differ, the research findings differ, as detailed below.

On the one hand, according to our analysis, the students' understanding cannot be classified as "unconventional," as highlighted by Marmur et al. (2020). In our research, each group presents a different pathway and the term "unconventional" loses its sense. In particular, the groups that produced a less comprehensive conceptualisation tended to rely on categories of argument and proof that did not require intricate sentence structures or advanced reasoning (such as *crucial experiment*). Moreover, their spontaneous conceptions reflected only the *analytical conception*, likely because it

aligns more closely with the thesis presented in the worksheet, guiding their reasoning. Addressing the first research question, it becomes evident that the *analytical conception* is perceived as more intuitive and, therefore, primary compared to the *synthetic conception*. However, the coexistence of both conceptions ultimately facilitates significant advancements in the conceptual understanding of density of $\mathbb{Q}$ in $\mathbb{R}$.

On the other hand, our results shed light on when and why students need to refer to different kinds of warrants, e.g., a specific practical example or, instead, some intuitive properties of a mathematical object. So, we recognised that using examples is only sometimes about reducing abstraction, as suggested by Hazzan (1999); but, in some cases, students feel obligated to present an example, even if their reasoning is general because they do not have suitable linguistic tools to express their ideas in another way. This leads us to the second research question. We can state that the students' reasoning varies across different groups, each beginning from distinct starting points.

Moreover, groups capable of navigating the various typologies of arguments and proofs demonstrate a more in-depth reasoning process. It is important to note that some elements of the dual categories of warrants (e.g., the *intuitive theoretical warrant*) never appeared in the data. This is likely because these categories require cognitive and linguistic tools that are not yet fully accessible to upper-secondary students. During the written phases of the activity (Assignment 1 and Assignment 2), almost all categories of arguments were explored at least once in the students' responses. However, during the Interview, no categories related to warrant typologies emerged. This discrepancy may be attributed to the fact that the cognitive and linguistic tools available to students are less developed in oral communication than in written tasks. Additionally, during the written phases, students had the opportunity to collaborate in groups, allowing for the emergence of more complex and well-articulated arguments.

Ultimately, the data analysis reveals that students' spontaneous conceptions are dynamic and evolve during the activity. The categories employed in this analysis help identify connections between different conceptions and the arguments that connect them. A potential future research development is extending the investigation, using the developed methodology, to concepts contiguous to the one being examined, such as continuity.

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Challenging students' argumentations towards a flexible use of the concept of change in a qualitative-graphical approach to calculus

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Students' reasoning about functions and rate of change is challenged, when they face dynamic contexts, in which they have to predict how a quantity unfolds over time. We illustrate this by analysing high-school students' graphical and verbal solutions for dynamic problems, where population growth is dependent on population sizes (not on time). This touches several fundamental notions of calculus, including topics such as rate of change and ideas of differential equations. Our findings indicate that students need to develop flexible ways, how to use their concepts of function and rate of change in this context. This requires classroom mathematical practices, in particular argumentations that encourage transitions between the real-life context and graphical and verbal representations of the given situation. Negotiations of such argumentations seem promising for students' flexible use and deeper development of a concept of change.

Keywords: Calculus, functions, differential equations, concept formation, argumentation.

Introduction

Mathematically and in real-life applications, dynamical systems play an important role. For describing dynamical systems mathematically, autonomous ordinary differential equations are an important means. These equations establish a functional relationship between a quantity and its instantaneous rate of change. This perspective, relating a function and its derivative, is often still untypical in school contexts. Yet, it allows to address several fundamental concepts and notions of calculus, as well as possible application contexts. The concepts of derivative as a rate of change, integral as an accumulation and different ideas of differential equations can be related in this context.

Understanding these concepts requires more than procedures and computational routines, but meaningful contexts, in which a qualitative-graphical approach is emphasized. This approach, emphasizing the discovery and exploration of fundamental concepts of calculus in a qualitative way, has been characterized as 'qualitative calculus' in mathematics education (e.g., Stroup, 2002).

However, arguing qualitatively and in a 'differential equation perspective' also entails conceptual challenges that need to be addressed so that students can work successfully with this perspective. Students have to overcome several 'epistemological obstacles' in this context, as Sierpinska (1992) points out. For example, assuming that all changes of a variable are changes in time, is a typical functional understanding of students, which is mathematically an obstacle they need to overcome. Further, a qualitative-graphical approach requires argumentations based on graphical as well as verbal representations of a given situation, which includes constant transitions between these, when arguing in a real-life context. We illustrate how challenging this can be for students in this contribution.

Our central research question is: How do students' argumentations support their 'meaning-making' of fundamental concepts in calculus within a dynamic system context?

State of the art: A qualitative-graphical approach to calculus

In mathematics education, students' difficulties with functions and functional thinking are a broad area of research. The articles in Dubinsky & Harel (1992) provide an overview of typical misconceptions and problems of students related to functions. Tall (1993) gives such an overview for calculus. Especially, when it comes to real-life applications and the use of central concepts of calculus in different contexts, a flexible concept of change and other calculus related concepts are needed.

A qualitative-graphical approach has been suggested for the teaching and learning of calculus, with the intent to foster central concepts and conceptual reasoning instead of calculations and procedures (e.g., Hußmann, 2010). In this approach, which has been labelled as 'qualitative calculus' (e.g., Stroup, 2002), concepts and contexts are mainly represented graphically and verbally and according to their quality, rather than symbolically and quantitatively. The emphasis is not on numbers and computational routines, but on qualitative aspects of the given contexts in order to stimulate conceptual and visual reasoning and argumentations. Starting from real-life contexts, the development of context-related ideas is meant to encourage an understanding of basic concepts of calculus (e.g., Hußmann, 2010). This way, a concept-oriented perspective on calculus is intended.

In this tradition of a qualitative approach to calculus, Rasmussen and colleagues have suggested to introduce the idea of differential equations, to foster the flexibility of students' concepts of functions, derivatives and (rates of) change (e.g., Rasmussen & Keene, 2019). Engaging students in argumentations can help them to clarify the underlying concepts. Argumentation can also become a useful heuristic device here for the discovery of new mathematical concepts (e.g., de Villiers, 1990).

Theoretical framework: Argumentation as a means of discovery

The importance of argumentation in this context is not only that it may be a goal of learning, but that it becomes a means for learning mathematics and learning mathematical concepts (e.g., Knipping & Reid, 2019). Argumentation is seen here "as a means for discovery" (de Villiers, 1990, p. 21), which entails recognizing and exploring connections between new and existing knowledge. The assumption that social and communicative processes not only underlay learning processes, but constitute them, has widely been accepted in mathematics education. This led since the 1990s to a growing scientific interest in argumentation theory in mathematics education (e.g., Krummheuer, 1992). Following this line of research, reconstructing argumentation processes has become a methodological framework for the analysis of learning processes, with established methods. This approach allows to get deeper insights into the processes, by which students establish relationships between already existing knowledge to construct new concepts. Combining argumentation analyses with an epistemological perspective, has been suggested by several scholars (e.g., Tabach et al., 2014). We pursue such an integrative approach in our research to describe students' challenges, when they construct and negotiate meaning of the concept of change in a qualitative-graphical approach to calculus.

Content-related interpretations of mathematical objects and their negotiation occur in collective argumentations and are related to *framings*. This concept has widely been used by Krummheuer (1992). Krummheuer, referring to Bruner, conceptualizes 'framing' as a standardized and routinised definition of a situation that emerges from an individual experience in similar situations

(Krummheuer, 1992, p. 24). Since students' individual framings often do not align with each other, differences in framings not only become apparent in collective argumentations, but also constitute starting points for students' negotiations and collective argumentations, which may lead to substantial conceptual development. In this sense, the coordination and modification of framings is, what characterises mathematical argumentations and learning from this perspective (Bredow, 2023).

From an epistemological perspective, research on students' qualitative reasoning about graphs and functions goes back to the 1980s. Leinhardt et al. (1990) analysed interpretation and construction tasks associated with functions and their representations. A qualitative interpretation of a graph here means looking at the graph and gaining meaning related to the two variables displayed in the graph and related to their pattern of covariation. In this sense, interpretation tasks refer to an interactive meaning-making-process. The meaning gained by interpretation and argumentation can either reside within the symbolic space of the graph, or it can shift to a different *space* (e.g., the situation space or the algebraic space). Leinhardt et al. (1990) refer to these shifts as a type of *translation* between different spaces. Construction tasks refer to the act of generating a new representation from data or another representation. In this sense, also construction tasks build on an interactive interpretation and collective argumentation. Combining these theoretical perspectives is our aim in order to describe students' challenges, when deepening their understanding of function and rate of change.

Methods

The design experiment

In the presented research project, a learning environment addressing different perspectives on quantities, change and their interplay was designed. It was then implemented with 15- to 17-year-old students in different secondary schools in Bremen, Germany. The students were already familiar with the concept of derivative and the derivation rules. Change, in this context, had often been considered as a function of time. During five 90-minute lessons, the students worked in class and in groups on various tasks, and the lessons and group work were filmed. Since the analysis is focused on students' understanding and argumentations in classrooms, the students' work is reconstructed by argumentation analyses with an epistemological focus (e.g., Tabach et al., 2014). The corresponding mathematical learning processes are analysed from an interpretative-interactionist perspective. In this paper, the focus is on the students' interpretations of the given situation and their negotiations of the concepts of function, derivative and change that become visible in their argumentations.

Design of the learning environment

The learning environment used in this study is divided into three sections, each focusing on a different perspective on change. In the first perspective, the focus is on the development of a quantity over time ('perspective of quantity'). In the second perspective, statements about the change of a given quantity are the starting point for the students' investigation ('perspective of change'). The third perspective, which is the focus of this contribution, then focuses on the interdependence of a quantity and its instantaneous rate of change (as a function of quantity) ('differential equation perspective') (see also Gunia & Knipping, 2023). The task discussed in this paper is shown in Figure 1. It is used in the last of the five 90-minute lessons and addresses the 'differential equation perspective':

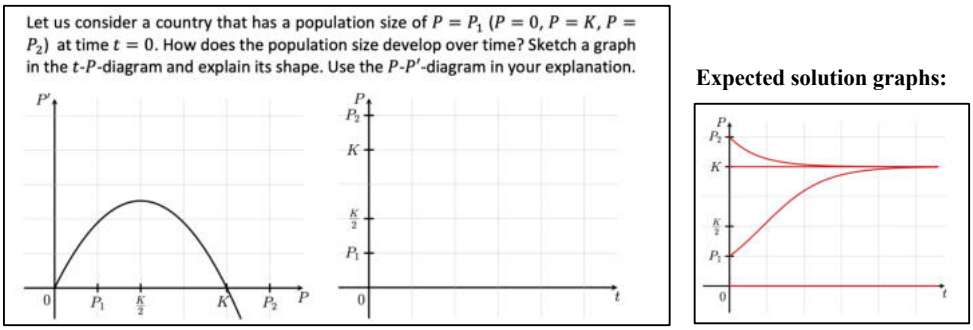

Figure 1: Task - "The growth of a population"

The graph in the P-P'-diagram describes a functional relationship between a population size P and its instantaneous rate of change P'. Independent of time, each population size P is attributed a certain instantaneous rate of change P'. For $P = 0$ and $P = K$, the corresponding rate of change is 0, which means that a population does not change for these values of P. This corresponds with a constant graph in the t-P-diagram (see Figure 1). For P-values $0 < P < K$, the instantaneous rates of change are positive, meaning that a population increases, if it reaches a population size in this interval. For $P(0) = P_1$, the resulting graph in the t-P-diagram represents a logistical growth. For P-values $P > K$, the instantaneous rates of change are negative, represented by a decreasing graph for $P = P_2$.

Results

When students in our research tried to solve this task, they faced various challenges and came up with diverse arguments to provide meaning to the given situation and overcome their struggles in this. The following example illustrates a typical approach, how students attempted to draw graphs into the t-P-diagram. Starting with $P(0) = 0$, the two students Lucy and Chloe discuss the following:

11 Lucy: In the beginning, the growth is zero. And then it is very steep up to K/2 and then it becomes flatter again.

12 Chloe: Yes, it first becomes steeper and steeper up to K/2, then it is at K/2 and then it becomes flatter again and then it has reached K.

13 Lucy: It first rises up to K/2 and then it does the same thing again, only reverse. Somehow like this. I now have an "S" somehow. [...] First, the slope increases, because P' also increases in the other diagram. And then the slope decreases again.

For drawing a graph into the t-P-diagram, Lucy starts by interpreting the P-P'-diagram for the initial value of $P = 0$. She translates her observations of the P-P'-diagram into the t-P-diagram by describing a solution graph (11). In doing so, she offers a temporal view, which is expressed in terms such as "in the beginning" and "and then". She also indicates a graphical view, when she uses terms such as "steep" and "flatter". Chloe adopts this temporal-graphical view in her comment (12). Both students do not talk about the context here, but start drawing the solution graph (see Figure 2).

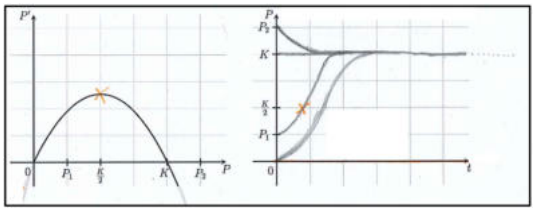

Figure 2: The students' graphical solutions for $P(0) = 0$, $P(0) = P_1$, $P(0) = K$ and $P(0) = P_2$

In turn 13, Lucy characterizes the emerging graph as an "S"-shaped curve and refers to the P-P'-graph for a justification. In the P-P'-diagram, she refers to the first increasing and then falling graph. She explains, what this means for the solution graph in the t-P-diagram, the "slope increases, because P' also increases" and then "the slope decreases" (13). The students start with an interpretation of the P-P'-diagram for the initial population size $P = 0$, which seems mathematically correct ("In the beginning, the growth is zero"). But then, they seem to read a time dependency into the first diagram ("And then … and then …"), instead of using the information about the current state of the system and its changing behaviour to make a prediction about the system's future behaviour (which would accordingly not change). The idea of representing change as a function of time becomes dominant in their interpretation of the P-P'-diagram. For the two students, a shift to the right on the horizontal axis seems to correspond here with a progression of time. When the students talk in the following about, what happens after the population size has reached the value $P = K$, this idea comes up again, but turns out to be problematic:

47 Lucy: And now the question is: If it goes above K, then the slope becomes negative. Then it should actually fall, but that's not possible.
48 Chloe: But, why isn't that possible?
49 Lucy: Because P is becoming more and more. P becomes more, but P' becomes less. That's not really possible. So, if it is negative, P', then P cannot become more. [...] Realistically, there can be no population size above K, as K is a growth limit. For values above K, the rate of change in the P-P' diagram becomes negative, but the population size still increases.

Lucy considers what happens, when P is bigger than K. She translates this information, given in the P-P'-diagram for P-values $P > K$, now into the t-P-diagram. Again, she adopts a temporal-graphical view by describing P-values next to $P = K$ as values following the state $P = K$ in time. According to her rationale, the slope is negative "above K" and the corresponding graph in the t-P-diagram should fall (47). She instantly describes this as impossible as "P becomes more, but P' becomes less" (49). Lucy's argument is based now on the assumption that the population size P is increasing over time. So, a contextual argument comes in here now, but the argument of time and population growth remains. Change is presented here as a function of time, which leads to a temporal interpretation of the horizontal axis in the P-P'-diagram. Accordingly, a progression in time corresponds with a shift to the right on the horizontal axis, which also means a growth in the population size. It seems that Lucy interprets the horizontal axis of the P-P'-diagram as both, a time axis and a population size axis. When Lucy interprets her temporal-graphical argument in the context of the given situation, she now talks about the meaning for the population and its growth (49). The temporal-graphical and the contextual view both occur as conflictive views at this point of their discussion, but remain in parallel.

When talking about the solution that starts in $P(0) = K$, Lucy and Chloe notice a lack of time dependency in the P-P'-diagram and start to solve the conflictive views:

71 Lucy: It should actually be constant there. First. But is it then falling or does it stay there? […] If you look at that point now, there is no time in it. But/ If there is no more growth at K, then P cannot increase anymore.
72 Chloe: Yes, so it is constant. Because it doesn't get bigger, which means the slope can't change either.
73 Lucy: Exactly. The graph is constant, because the instantaneous rate of change is zero.

Lucy states that the solution graph starting in $P(0) = K$ should be constant, at least in the beginning (71). She wonders, whether the graph then decreases or not. In her following (pointwise) interpretation of the P-P'-diagram, she notices the lack of time dependency on the horizontal axis and then starts with a contextual interpretation of the given situation: When there is no growth for $P = K$, the population size does not change and that means that the graph is constant. This observation illustrates a flexible transition between the real-life context and the graphical representation of the given situation. The two students succeed now in translating the information given in the P-P'-diagram into the real-life context and then to re-translate this contextual information into a graphical information in the t-P-diagram.

When they finally go on to the solution that starts in $P(0) = P_2$, the above-mentioned conflictive views related to the population size, which is bigger than K, comes up again:

106 Lucy: But, actually that doesn't work. It doesn't work in any world that P increases, but the rate of change is negative. The rate of change means that there are fewer people. But, you look further to the right on the axis, which means that there are more people. That doesn't work at all.

Lucy again interprets the function in the P-P'-diagram as a function of time. She also describes the function in the P-P'-diagram as a function of quantity, when the progression in time corresponds with a growth in the population size (106). Chloe on the other hand, describes the functional relationship represented in the P-P'-diagram as a given quantity and its dependent instantaneous rate of change, concentrating on a point-by-point interpretation of the graph:

107 Chloe: This is only about one point, which you might find here. If you have P2, right? That's a very huge population size. Then it is very far in the negative range. That means the slope, eh, the instantaneous change, is very strong. In the negative range, but it is still very strong. That means the decrease is very strong. But, since you can't go forward as here, when there is a decrease, because the population is then no longer increasing, but decreasing, you have to read the graph backwards. That means we are now going somehow backwards on the graph. That means, you go from steep to flat. And then it ends at K again.

For a P-value of $P = P_2$, Chloe identifies a very high negative rate of change. She directly transfers this information about the rate of change into a graphical information (when talking about the slope) and into a contextual information ("the decrease is very strong"). Next, she re-interprets the contextual information concerning the population growth into a graphical information concerning the P-P'-diagram. A decrease in the population size here means that one has to read the graph "backwards". Despite their different interpretations at the beginning, Chloe and Lucy finally develop a complete argument for the solution that starts in $P(0) = P_2$ together:

327 Chloe: For P=P2, the slope in the P-P'-diagram is very strongly in the negative range. The instantaneous rate of change is very strongly in the negative range.

328 Lucy: Thus, the population size decreases so that the x-axis has to be practically read backwards.

329 Chloe: Since P becomes smaller. [...] Then it is closer to the x-axis again, the graph in the P-P'-diagram, which means that the instantaneous rate of change is not so much in the negative range anymore and is getting closer and closer to zero, which is why the graph in the t-P-diagram becomes flatter.

Lucy adopts the aforementioned interpretation of the P-P'-diagram of Chloe. Together, they flexibly shift between graphical and contextual arguments. Their negotiation process is guided by the

question: “What does a negative rate of change mean for the development of a population size?” Related to the real-life context, the two students translate between the graphical representation of the given situation and its contextual meaning. This contextual interpretation is then again translated graphically into the t-P-graph, where the actual population growth over time is represented.

Discussion

The presented student approaches illustrate challenges students face, when confronted with a dynamic situation, in which change is depended on quantity (and not on time). Students have to overcome epistemological obstacles and conceptual difficulties, when looking at population growth, dependent on population sizes and not on time. This requires a shift in the students’ *framing* of the concept of change. Our data indicate, how challenging it is for students to interpret change as a function of quantity, not time. Students need to conceptualize this functional relationship to determine the change over time in a dynamic system. As students are not familiar with this perspective, this is challenging for them. Also, this conceptualization of change breaks conventions on how to read a graph: Following a “time dependency logic” of change, the students first read the graph from “left to right”. Reading the graph “backwards”, detached from a temporal point of view, is completely new to them.

In the course of the here presented group work, a significant shift in the students’ argumentations occurs. The students recognize the lack of time dependency in the P-P’-diagram, when they construct the solution that starts in $P(0) = K$. This constitutes a significant breaking point in the students’ interpretation and constructing process as described by Leinhard et al. (1990). From this point on, the students switch from a temporal-graphical interpretation of the given situation to a contextual-graphical perspective. In this sense, they *translate* between different *spaces*. This motivates the students to pursue a new direction in their argumentation, which takes into account the real-life context of the task. This shift supports a constant transition between the real-life context and the graphical and verbal representation of the situation, which is essential for their successful solution of the task. These transitions between the real-life context and a graphical and verbal representation of the given situation are challenging for the students, but allow them to adopt a new perspective.

Conclusion

The here presented results illustrate, how students can mobilize their knowledge about functions and rate of change, when working on the given task. This indicates, how tasks in dynamical contexts can support functional thinking and the discovery of new functional relationships. The students’ argumentations deepen and consolidate their understanding of functional dependency and change. Our findings also indicate that students are challenged, when they face dynamic contexts and that they need to develop flexible ways, how to use their concepts of function and rate of change in this situation. We provide evidence that argumentations help the students to develop more flexible ways to use these concepts, when they transit between different representations of the given situation.

The ways in which students deal with the challenges they face are diverse, as our data indicate. Looking closer into the arguments different students come up with and how these help them to overcome the described challenges, is our research goal. Characterizing distinct breaking points in the students’ negotiation processes and their argumentations will be a next milestone in this research.

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Opportunities for pre-structuralist reasoning at the crossroads from calculus – a case study on the equivalence between the continuity of a function and the path-connectedness of its graph

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Mathematical reasoning based on definitions and theorems and the attention towards generalisation are important for the transition from calculus to analysis or topology. We analyse proofs of a task about the equivalence of the continuity of a real-valued function on an interval and the path-connectedness of its graph. In doing so, we examine elements of "pre-structuralist praxeologies" at the crossroads from calculus to more advanced courses. The proofs take pre-structuralist aspects into account in different ways. This holds for the mobilisation of levels of pre-structuralist praxeologies as a means of generalisation. Thus, we contribute to calculus education by focusing on the transition from a more intuitive to a more formal way of reasoning based on underlying structures about functions and graphs.

Keywords: (Pre-)structuralist praxeologies, calculus, real analysis, continuity, path-connectedness.

Introduction

Many students have heard or emphasise themselves that the graph of a continuous real-valued function $I \to \mathbb{R}$ ($I \subseteq \mathbb{R}$ interval) – now called a *real function* – can be drawn "without lifting the pen from the paper" (Hanke, 2018). This intuitive characterisation is used to strengthen pupils' intuition for continuity in school, too. Research has also shown that one of the dominant strategies students at university follow to identify continuity is based on the appearance of the graph of the function (Hanke & Schäfer, 2017). One way to make this intuitive statement rigorous is the equivalence that a real function is continuous if and only if its graph is a path-connected subset of $\mathbb{R}^2$, which means that any two points on the graph can be joined by a continuous path in the graph. Here, the intuitive idea is made precise with the notion of path-connectedness of the graph, which is a more specific notion than connectedness. In particular, this formalisation underlines the importance of mathematical exactification of intuitive ideas into precise properties. This is relevant at the transition from calculus, where the more intuitive aspects are located, to more advanced courses. The context of our investigation are study programmes for mathematics majors or pre-service teachers that include the transition from calculus to more advanced courses. Furthermore, the equivalence is relevant for pre-service teachers as it provides background knowledge on how intuitive ideas as promoted in calculus at both university and school can be made precise.

Proving the equivalence is however rather challenging, at least when compared to the tasks posed in (school) calculus or in introductory calculus or real analysis courses. In particular, from a bird's eye view, continuity and path-connectedness are topological notions. One concern of this paper is thus to

highlight different approaches for reasoning about this statement that might be backed with structures from analysis or even topology. Specifically, we consider the following task:

> t: *For an interval* $I \subseteq \mathbb{R}$*, a function* $f: I \to \mathbb{R}$ *is continuous if and only if its graph* $\Gamma_f = \{(x, f(x)) : x \in I\}$ *is path-connected.*

This equivalence appears in sources, which are either explicitly devoted to real functions (e.g., van Rooij & Schikhof, 1982), which embed the notions of continuity and path-connectedness into metric space topology (e.g., Ross, 2013), or in sources on topology (e.g., Gathmann, 2023). This article's aim is to analyse proofs for t, rooted in real analysis as a formalised variant of calculus, with respect to their potential to foster students' awareness for possibilities to use definitions and propositions for verification of properties of objects (here: continuity and path-connectedness) and for possible generalisations to more abstract structures. In our analyses of proofs for t, we highlight to what degree the considerations could be interpreted as contextualised general principles, that is, structuralised ideas that can be more or less expressed verbatim in the context of metric or even topological spaces.

The paper is embedded into our previous work on this task (Hanke, 2018) and structuralist reasoning in real analysis (Hausberger & Hochmuth, 2023). In our novel methodological approach to the potential of tasks for generalisation, we draw on the concept of *(pre-)structuralist praxeologies* (Hausberger, 2018) in the *Anthropological Theory of the Didactic* (Chevallard & Bosch, 2020; Hausberger & Hochmuth, 2023; Laukert et al., 2023). More precisely, we extract so-called *techniques* and *technologies*, that is, ways to solve tasks and their mathematical underpinning, which constitute the architecture of proofs. These relate to reasoning based on the use of definitions and theorems in terms of two levels.

Theoretical framework

The *Anthropological Theory of the Didactic (ATD)* (Chevallard & Bosch, 2020) considers the relationships to an object of knowledge by praxeologies where the object intervenes at the level of at least one of the following components: "The anthropological theory of didactics considers that, *in the end*, all human activity consists of *accomplishing a task* t of a certain *type* T by means of a certain *technique* τ, *justified* by a *technology* θ which at the same time makes it possible *to think about* it, or even *to produce* it, and which is in turn *justifiable* by a *theory* Θ" (Chevallard, 2001, p. 3, own transl.).

Hausberger (2018) introduced the notion of *structuralist praxeologies* in the context of abstract algebra: They are praxeologies $P_s = [T^g, \tau, \theta, \Theta_{\text{structure}}]$, where the type of task T^g is a generalisation of T from a not-yet-structuralist praxeology $P = [T, ?, ?, \theta]$ (at least not when compared with the structure for P_s). Here, the adjective "structuralist" refers to the use of structures in the sense of Bourbaki (1950) rather than concrete instances (e.g., mobilising results about general metric spaces rather than a specific metric space) (Hausberger, 2018). In real analysis courses, generic objects such as real functions, sequences, or series may appear, but properties of functions and their domains ($\mathbb{R}$ or $\mathbb{R}^n$) that play a role may not be fully elucidated at that moment in terms of, for example, topological concepts. In this context, praxeologies were therefore called *pre-structuralist* (Laukert et al., 2023). That is, these praxeologies may be generalised with respect to a certain, not necessarily unique, structure (e.g., sets appearing in a pre-structuralist praxeology may be generalised to metric or

topological spaces). The structure may not have been made explicit in the institutional context of the pre-structuralist praxeology though and lurk in the background but may be made explicit in building up a structuralist praxeology. Regarding pre-structuralist praxeologies, we distinguish between *level 1*, where structuralist aspects mainly appear through definitions or as vocabulary, from *level 2* praxeologies, where a technique invokes general results on structures (Hausberger, 2018; Laukert et al., 2023). Accordingly, our research question is: *Which techniques and technologies can be enacted in proofs of* t *to foster the development of (pre-)structuralist praxeologies?*

Method

We performed a content analysis of German, English, and French textbooks and lecture notes on calculus, analysis, and topology for the equivalence in the task t and similar characterisations of continuous functions, as well as research literature and mathematics online forums. Based on these sources (e.g., JOJO, 2020; van Rooij & Schikhof, 1982; Ross, 2013), the first author broke down the argumentation in two proofs for each direction of t into techniques and technologies similar to a coding procedure (see Tables 1 and 2 below). Regarding reliability, the analysis of the proofs in terms of these techniques and technologies were discussed with all authors until a consensus was reached.

We emphasise that we do not present full praxeologies here in a *specific* institutional context. In fact, every institutional context gives rise to its own praxeologies. Rather, we focus on techniques and technologies and trace options for generalisation (e.g., as specified in different theories such as real analysis or topology). This aligns with our objective of exploring solutions that may be enacted in mathematics curricula with respect to the use of structures. This also means that the techniques and technologies we present contain *didactic variables*, that is, unspecified elements that are indeed specified within a full praxeology. As a result, the analysis below is presented in terms of chains of propositions like a skeleton and the linkages between these propositions are explained in terms of the techniques and technologies.

Results: Praxeological analyses of different proofs for the task

We denote the implication "f continuous $\Rightarrow$ Γ_f path-connected" in t by $t^{\Rightarrow}$ and the other by $t^{\Leftarrow}$. We recall that a subset A of a topological space X is called *path-connected* if for every $x, y \in A$ there is a path (i.e., a continuous function) $\gamma\colon [a, b] \to A$ with $\gamma(a) = x$ and $\gamma(b) = y$. For contextualisation, we also recall that A is called *connected* if it is not the union of two non-empty disjoint sets that are open in A (with respect to the relative topology), and that every path-connected set in every topological space is connected, but not the other way around.

The task t is a special case of two types of tasks, which may be embedded either in real analysis or topology, T_1: *Prove that a set* Y *is path-connected* and T_2: *Prove that a function is continuous*. Specifically, $t^{\Rightarrow}$ is an instance of T_1 for $Y = \Gamma_f$ and $t^{\Leftarrow}$ is an instance of T_2 for $f\colon I \to \mathbb{R}$. From a structuralist perspective, t also belongs to a type of tasks that is even currently investigated in research in analysis and topology, T: *Characterise properties of a function in terms of properties of its graph* (see Hausberger et al., these proceedings).

We will now present two proofs for $t^{\Rightarrow}$ and $t^{\Leftarrow}$. Tables 1 and 2 show techniques and technology that back these proofs and in which they occur. The analysis after each proof shows how these techniques

and technologies are enacted and highlight pre-structuralist levels and further aspects. We emphasise that different institutional contexts and theories lead to more exactified versions of the techniques and technologies we list here. For instance, θ_4 could mean "definition of continuity for a map $U \to \mathbb{R}$ for subsets $U \subseteq \mathbb{R}$" in a single-variable real analysis context or "definition of continuity for a map between two metric spaces" in a metric space context. In this sense, the techniques and technologies contain didactic variables to be concretised within each institutional context and theory.

Table 1: Techniques and their use in the proofs

τ_1: *Verify the definition of path-connectedness for a set*	$P_1^{\Rightarrow}$
τ_2: *Show that a set Y is the image of a path-connected set Z under a continuous function F*	$P_2^{\Rightarrow}$
τ_3: *Assume the discontinuity of a function and find a contradiction*	$P_1^{\Leftarrow}$, $P_2^{\Leftarrow}$

Table 2: Technologies and their use in the proofs

θ_1: *definition of path-connectedness*	$P_1^{\Rightarrow}$, $P_1^{\Leftarrow}$, $P_2^{\Leftarrow}$
θ_2: *product functions are continuous if and only if their component functions are continuous*	$P_1^{\Rightarrow}$, $P_2^{\Rightarrow}$, $P_1^{\Leftarrow}$, $P_2^{\Leftarrow}$
θ_3: *continuous maps preserve path-connectedness* (i.e., if Z is path-connected and F is a continuous map on Z, then $F(Z)$ is path-connected)	$P_2^{\Rightarrow}$
θ_4: *definition of continuity*	$P_1^{\Leftarrow}$, $P_2^{\Leftarrow}$
θ_5: *Bolzano-Weierstraß theorem* (bounded sequences in $\mathbb{R}^N$ contain convergent subsequences)	$P_1^{\Leftarrow}$, $P_2^{\Leftarrow}$
θ_6: *the intervals $[a,b] \subseteq \mathbb{R}$ are bounded and closed*	$P_1^{\Leftarrow}$, $P_2^{\Leftarrow}$
θ_7: *intermediate value property (IVP) for real-valued functions f defined on intervals $I \subseteq \mathbb{R}$* (i.e., for all $w_1, w_2 \in \mathrm{im}(f)$ and $c \in \mathbb{R}$ between w_1 and w_2 there is a $\xi \in I$ with $f(\xi) = c$)	$P_2^{\Leftarrow}$
θ_8: *intermediate value theorem for real functions* (i.e., every continuous function has the IVP)	$P_1^{\Leftarrow}$, $P_2^{\Leftarrow}$

Analysis of $t^{\Rightarrow}$: "f continuous ⇒ Γ_f path-connected"

Proof 1 ($P_1^{\Rightarrow}$) (Ross, 2013, p. 182):

1. Let $\big(a, f(a)\big), \big(b, f(b)\big) \in \Gamma_f$, and without loss of generality assume that $a \leq b$. The function $\gamma\colon [a,b] \to \mathbb{R}^2$, $x \mapsto \big(x, f(x)\big)$, is a path from $\big(a, f(a)\big)$ to $\big(b, f(b)\big)$ (i.e., γ is continuous and satisfies $\gamma(a) = \big(a, f(a)\big)$, $\gamma(b) = \big(b, f(b)\big)$ and $\mathrm{im}(\gamma) \subseteq \Gamma_f$).
2. Hence, Γ_f is path-connected.

The task $t^{\Rightarrow}$ is solved with the technique τ_1 to verify the definition of path-connectedness with the technology θ_1 for Γ_f. That is, this argument is on level 1. The continuity of γ is supported with θ_2 because $\gamma = \mathrm{id}_{|[a,b]} \times f_{|[a,b]}$ is continuous since $\mathrm{id}_{|[a,b]}$ and $f_{|[a,b]}$ are. The definition θ_4, although

part of theory blocks backing this technology, is not used explicitly. No structuralist proposition about path-connectedness is involved and the used technique corresponds to a property of product maps that happen to be needed as the codomain of the image of the path is in $\mathbb{R}^2$.

Proof 2 ($P_2^{\Rightarrow}$):

1. Γ_f is the image of the continuous function $\mathrm{id}_I \times f: I \to \mathbb{R}^2$, $x \mapsto \big(x, f(x)\big)$.
2. I is path-connected.
3. Hence, $\Gamma_f = \mathrm{im}(\mathrm{id}_I \times f)$ is path-connected.

Here, the technique τ_2 to identify Γ_f as the continuous image of a path-connected set, namely $\mathrm{im}(\mathrm{id}_I \times f)$, points to level 2 pre-structuralist reasoning. The application of this technique is backed with the technology θ_3 that asserts that path-connectedness of Γ_f is preserved under continuous maps for $Z = I$ under $F = \mathrm{id}_I \times f$. As in $P_1^{\Rightarrow}$, the continuity of the map $\mathrm{id}_I \times f$ is backed with θ_2.

We emphasise the different architectures of these two proofs: While the first proof amounts to a verification of the definition, the second is based on a pre-structuralist theorem (with respect to the metric space structure). It turns out, however, that the technology used to verify either the definition τ_2 and recognising Γ_f as a continuous image is in both cases θ_2. We note that the second more structuralist proof is, broadly speaking, not "more efficient", but that the work done is fundamentally different: verifying a defining property vs. verification the prerequisites of a theorem. We further note that the technologies θ_2 and θ_3 are also in principle valid in the context of general metric or topological spaces (e.g., for any path-connected topological space Z and continuous function F on Z). Hence, the specific properties of the spaces I and $\mathbb{R}^2$ are not important in $P_2^{\Rightarrow}$. As such, $P_2^{\Rightarrow}$ could be seen as an example for a proof, which is backed with real analysis and which mimics a more general metric space theory. In a similar spirit, the proof $P_1^{\Rightarrow}$ may be transformed into a proof of θ_3 for the context of metric or topological spaces. Hence, the argumentation in this proof of $t^{\Rightarrow}$ could, at least in principle, serve as a case to illustrate how a real analysis-based technology (i.e., if θ_3 is backed with real analysis) may be transformed into a more general structural one.

Analysis of $t^{\Leftarrow}$: "Γ_f path-connected ⇒ f continuous"

Proof 1 ($P_1^{\Leftarrow}$) (JOJO, 2020):

1. Assume that f is not continuous at some $x_0 \in I$. Then, there is an $\varepsilon > 0$ and a sequence $(x_n)_{n\in\mathbb{N}}$ in an interval $[a, b] \subseteq I$ such that $\lim_{n\to\infty} x_n = x_0$ and $|f(x_n) - f(x_0)| > \varepsilon$ for all $n \in \mathbb{N}$.
2. There is a path $\gamma = (u, v)$: $[0, 1] \to \Gamma_f$ with $v = f \circ u$, $\gamma(0) = \big(a, f(a)\big)$, $\gamma(1) = \big(b, f(b)\big)$.
3. There is a sequence $(t_n)_{n\in\mathbb{N}}$ in $[0, 1]$ with $u(t_n) = x_n$ for all $n \in \mathbb{N}$.
4. There is a convergent subsequence $\big(t_{n_k}\big)_{k\in\mathbb{N}}$ of $(t_n)_{n\in\mathbb{N}}$ with limit $t_0 \in [0, 1]$ such that $\lim_{k\to\infty} \gamma\big(t_{n_k}\big) = \lim_{k\to\infty} \Big(u\big(t_{n_k}\big), v\big(t_{n_k}\big)\Big) = \big(u(t_0), v(t_0)\big) = \Big(u(t_0), f\big(u(t_0)\big)\Big) = \big(x_0, f(x_0)\big)$.
5. In particular, $\lim_{k\to\infty} f\big(x_{n_k}\big) = f(x_0)$; contradiction to 1 (hence, f is continuous).

Here, the technique τ_3 is used and the discontinuity of f is assumed in a proof by contradiction. Mobilising the definition of continuity θ_4, here using sequences, and taking the logical negation, the numbers x_0 and ε, the interval $[a, b]$, and the sequence $(x_n)_{n\in\mathbb{N}}$ are initialised. Mobilising further the

technology θ_1 for $\Gamma_f \subseteq \mathbb{R}^2$, the points $\big(a, f(a)\big), \big(b, f(b)\big) \in \Gamma_f$, the path $\gamma = (u, v)$, and a sequence of preimages $(t_n)_{n\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ for the first coordinate function of γ is constructed using θ_8. Throughout, as in $P_1^{\Rightarrow}$ and $P_2^{\Rightarrow}$, θ_2 connects the continuity of γ to the continuity of the coordinate functions u and v. A convergent subsequence of these preimages is $\big(t_{n_k}\big)_{k\in\mathbb{N}}$ obtained by θ_5 and θ_6, whose limit t_0 is in $[0, 1]$ because $[0, 1]$ is closed by θ_6. The final computations use the convergence of $\big(t_{n_k}\big)_{k\in\mathbb{N}}$ and the sequential definition of continuity θ_4 for γ and f, which yield the desired contradiction to step 1. We note that the use of τ_3 and θ_4 is based on the sequence characterisation of continuity, hence tightly connected to one of the usual ways continuity is dealt with in analysis. These characterisations may be generalised to metric spaces, but not to general topological spaces, where a sequence-based characterisation is not available.

A variant of $P_1^{\Leftarrow}$ is due to van Rooij and Schikhof (1982, p. 53): Choose $(x_n)_{n\in\mathbb{N}}$ convergent to $x_0 \in I$ and contained in some interval $[a, b] \subseteq I$, then $\big(f(x_n)\big)_{n\in\mathbb{N}}$ is bounded (because $\Gamma_{f|[a,b]}$ is compact, which could be shown with the technology that surjective continuous images of compact to Hausdorff spaces are closed, i.e., a level 2 structuralist argument). It follows that $\alpha \coloneqq \limsup_{n\to\infty} f(x_n) = \lim_{k\to\infty} f\big(x_{n_k}\big)$ for some subsequence; hence $\lim_{k\to\infty} \Big(x_{n_k}, f\big(x_{n_k}\big)\Big) = (x_0, \alpha) \in \Gamma_{f|[a,b]}$ and $\alpha = f(x_0)$ (similarly for $\beta = \liminf_{n\to\infty} f(x_n)$). It can thus be concluded that $\lim_{n\to\infty} f(x_n)$ exists and is equal to $f(x_0)$.

Proof 2 ($P_2^{\Leftarrow}$) (Ross, 2013, pp. 182–183):

1. Assume that f is not continuous at some $x_0 \in I$. Then, there is an $\varepsilon > 0$ and a sequence $(x_n)_{n\in\mathbb{N}}$ in an interval $[a, b] \subseteq I$ such that $\lim_{n\to\infty} x_n = x_0$ and $|f(x_n) - f(x_0)| > \varepsilon$ for all $n \in \mathbb{N}$.
2. There is a path $\gamma\colon [0, 1] \to \Gamma_f$ with $v = f \circ u$, $\gamma(0) = \big(a, f(a)\big)$ and $\gamma(1) = \big(b, f(b)\big)$.
3. For any $z \in [a, b]$ there is a $t \in [0, 1]$ such that $u(t) = z$; thus $\big(z, f(z)\big) = \Big(u(t), f\big(u(t)\big)\Big) = \gamma(t)$ and $\Gamma_{f|[a,b]} \subseteq \mathrm{im}(\gamma)$.
4. f satisfies the IVP on $[a, b]$.
5. Without loss of generality, and potentially by replacing $(x_n)_{n\in\mathbb{N}}$ with a subsequence, assume that $f(x_n) > f(x_0) + \varepsilon$ for all $n \in \mathbb{N}$.
6. By 4., there is a sequence $(y_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ such that y_n is between x_0 and x_n and $f(y_n) = f(x_0) + \varepsilon$ for all $n \in \mathbb{N}$.
7. The sequence $\Big(\big(y_n, f(y_n)\big)\Big)_{n\in\mathbb{N}}$ satisfies $\lim_{n\in\mathbb{N}} \big(y_n, f(y_n)\big) = (x_0, f(x_0) + \varepsilon)$.
8. There is a sequence $(t_n)_{n\in\mathbb{N}}$ such that $\gamma(t_n) = \big(y_n, f(y_n)\big)$ for all $n \in \mathbb{N}$.
9. There is a subsequence $\big(t_{n_k}\big)_{k\in\mathbb{N}}$ of $(t_n)_{n\in\mathbb{N}}$ with limit $t_0 \in [0, 1]$ such that $\gamma(t_0) = \lim_{k\to\infty} \gamma\big(t_{n_k}\big) = \lim_{k\to\infty} \Big(y_{n_k}, f\big(y_{n_k}\big)\Big) = (x_0, f(x_0) + \varepsilon) \notin \Gamma_f$; contradicting 3 (hence, f is cont.).

As in $P_1^{\Leftarrow}$, τ_3, θ_1, and θ_4 are used to initialise the numbers x_0, ε, the interval $[a, b]$, the sequence $(x_n)_{n\in\mathbb{N}}$, and the path $\gamma = (u, v)$ between $\big(a, f(a)\big), \big(b, f(b)\big) \in \Gamma_f$. Throughout, θ_2 connects the continuity of γ to the continuity of the coordinate functions u and v. θ_7 (and θ_8) can be applied to u to show that $\Gamma_{f|[a,b]} \subseteq \mathrm{im}(f)$ (step 3; Ross, 2013, p. 182). Then, applying θ_7 to $v = f \circ u$ allows to conclude the IVP θ_8 for f. The construction of $(x_n)_{n\in\mathbb{N}}$ is based on step 1 (i.e., on θ_4) and $(y_n)_{n\in\mathbb{N}}$

on step 4. The construction of the preimages $(t_n)_{n\in\mathbb{N}}$ of $(y_n)_{n\in\mathbb{N}}$ and the convergent subsequence $(t_{n_k})_{k\in\mathbb{N}}$ is similar to $P_1^{\Leftarrow}$ and backed with θ_5 and θ_6. As in $P_1^{\Leftarrow}$, the computation of the limit $\lim_{k\to\infty} f(y_{n_k}) = f(x_0) + \varepsilon$ yields the desired contradiction to step 3 because $\gamma([0,1]) \subseteq \Gamma_f$.

Both proofs make use of level 1 pre-structuralist praxeologies, since the technique τ_3 is a check of the definition of continuity. Here, no level 2 (pre-)structuralist arguments of the form that a certain property of a graph of a function implies continuity is used; hence, the argumentation stays at level 1. Moreover, the arguments are strongly locally backed with real analysis. Also note that Ross' (2013) proof of $t^{\Leftarrow}$ is in a sense a more complicated version of JOJO's (2020) proof by adding the arguments about the sequence $(y_n)_{n\in\mathbb{N}}$. Since the (dis-)continuity of maps between metric spaces can be characterised in terms of sequences, the proofs here also work from the more abstract perspective of metric space topology. Indeed, and a bit surprising, metric space topology is developed in the chapter where Ross (2013) discusses the proposition from t, but he does not mention structural results nor metric space terminology in its proof.

Conclusion

The task t points to a school-related basic idea about continuous functions. The equivalence may be understood as an exactification using calculus and analysis concepts. Our analysis revealed that the task allows for different proofs, taking into account different techniques and technologies. Our novel methodological approach showed that the proofs point to structuralist aspects in different ways. This holds for the mobilisation of the two levels of pre-structuralist praxeologies, but also with regards to embedding the task in courses that transition to metric space or general topology. This is valuable for calculus education since a proof comparison, potentially made explicit to students, reveals how the techniques and technologies may be enacted in multiple ways and contexts. Our approach serves as a first step to analyse corresponding praxeologies. We would also like to refer the reader to Hausberger et al. (these proceedings) for an analysis of the task t through the lens of *external didactic transposition* (i.e., the process of transforming scholarly knowledge into knowledge to be taught).

A consequence for teaching is that students may be aided in making the transition from more local to more general arguments and in attending to structures by comparing techniques and technologies in proofs presented to or demanded from them. We addressed this issue with a proposition that students are likely familiar with from calculus, whose intuitive statement had to be made rigorous carefully by choosing a suitable topological property. We hypothesise that the analysed task may therefore contribute to students' development of an appropriate image of modern mathematics, particularly at the crossroads from calculus to analysis and further to (metric space) topology. Whether it is more beneficial to give the equivalence as a task to the students or to deal with it and its proofs in a lecture depends of course on respective institutional conditions. The aim of subsequent research is to examine student solutions to the task t in relation to their institutional context and how they proceed to focus on structures or use (pre-)structuralist praxeologies.

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Measurement, Inverse Proportions, and Integration by Substitution

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Interpreting definite integrals and their notation as summations of small amounts of a target quantity (Adding Up Pieces) is especially productive for applied problems. This report develops a new justification for integration by substitution based on Adding Up Pieces that makes three contributions. First, whereas past research on integration has rarely foregrounded explicit, quantitative meanings for multiplication, the analysis presented here is based on connections between multiplication and coordinated measurement with two units. Second, the analysis uses coordinated measurement to explain how, across diverse situations, small amounts of a target quantity can be understood as products represented by rectangular areas. This supports reasoning about Riemann sums as approximations for definite integrals. Finally, the analysis provides a new justification for integration by substitution that relies on rectangular areas and inversely proportional relationships.

Keywords: Calculus, Multiplication, Measurement, Integration.

Introduction

A growing body of research has reported on students' interpretation and construction of integrals, especially when solving problems in physics and other domains outside of pure mathematics (e.g., Jones & Ely, 2023; Nguyen & Rebello, 2011; Oehrtman & Simmons, 2023). Integration by substitution, or u-substitution, plays a central role in many applications yet, as Jones and Fonbuena (2024) pointed out, mathematics and science education research has largely overlooked this central topic.

In traditional calculus instruction, integration by substitution is often introduced as the "reversed chain rule," which in the case definite integrals can be expressed with the following equation: $\int_a^b f'\big(g(x)\big)\, g'(x)\, dx = \int_{g(a)}^{g(b)} f'(u)\, du$. (A more general statement holds when f is simply continuous.) Typically, students are presented with an integral whose form fits the left-hand side of the equation. Their task is to identify the functions f', g, and g' that afford transformation to the right-hand side of the equation, which then has an integrand that one (hopefully) knows how to integrate. A common justification for this transformation includes replacing $g(x)$ with u and $g'(x)\, dx$ with du, a justification that can be understood simply in terms of manipulating symbols.

This theoretical report presents a mathematical analysis that foregrounds reasoning with quantities to develop a new justification for integration by substitution. The analysis makes three contributions. First, it builds upon recent advances connecting multiplication and measurement of quantities to explain how rectangular areas can be used to model products across a wide range of situations that arise in physics and other domains. Second, the focus on rectangular areas challenges some recently articulated reservations about interpreting definite integrals as "areas under curves." Third, the new justification foregrounds rectangular areas and inversely proportional relationships and, thus, focuses on the Riemann sums used to approximate definite integrals. Finally, the analysis suggests a new

instructional pathway into integration by substitution that is based on reasoning with quantities and that may be accessible to students.

The rest of this report is organized into four sections. The first section provides an analysis of multiplication with quantities that lays the foundation for interpreting definite integrals as "areas under curves." The second section explains how a focus on quantitative meanings for multiplication contributes to research on integration. The third section builds directly on the first and presents the justification for integration by substitution based on rectangular areas and inversely proportional relationships. The fourth section provides examples of written work in which students construct explanations like the one developed across the first three sections of the report.

Multiplication with Quantities and Coordinated Measurement

Thompson (1994, p. 184) articulated the following definition of a quantity when introducing his theory of quantitative reasoning: A quantity consists of "an object, a quality of the object, an appropriate unit or dimension, and a process by which to assign a numerical value to the quality". This definition implies that numbers are outcomes of measurement: They express how many units, or how much of a unit, exhaust the extent of the "quality" in question. We often use the term "multiplication" to refer to this fusion of number and measurement—for instance, we might say that a taller building is "three times" the height of a shorter building when what we mean is that three of short height exhaust the tall height. In such cases, measurement, numbers, and multiplicative language are intertwined. This meaning for multiplication, based on measurement with a *single* unit, is insufficient for understanding how Riemann sums approximate definite integrals.

In contrast to using multiplicative language with a single measurement unit, Izsák and Beckmann (e.g., Beckmann & Izsák, 2015; Izsák & Beckmann, 2019) have explicated a meaning for multiplication based on measurement with *two* units they term base units and groups units. Figure 1 shows an equation where N, M, and P are outcomes of measurement, as discussed above. The product amount is measured simultaneously in base units and in group units: P base units and M group units are measures of the same product amount. N expresses the measure of 1 group unit in terms of base units. Izsák and Beckmann have referred to this perspective as the coordinated measurement meaning for multiplication, and they have argued that it works with positive real numbers, ties together diverse situations using a single quantitative structure, and challenges the field to rethink several established results, including results about classification of problem situations in which one typically multiplies or divides (e.g., Vergnaud, 1983).

N	•	M	=	P
How many base units make one group unit?		How many group units make one product amount?		How many base units make one product amount?

Figure 1: A coordinated measurement meaning for multiplication (Izsák & Beckmann, 2019, p. 91)

In preparation for interpreting integrals as areas under curves, I highlight Vergnaud's (1983) seminal approach to classification of problem situations. He analyzed mathematical structures for multiplying with quantities and distinguished two primary subtypes. *Isomorphism-of-measure-spaces* (I-o-M)

situations are ones where the multiplicative structure "consists of a simple direct proportion between two measure spaces, M_1 and M_2" (1983, p. 129). Here the direct proportion is often viewed as a rate of change. *Product-of-measure-spaces* situations (P-o-M) are "the Cartesian composition of two measure spaces, M_1 and M_2, into a third, M_3" (1983, p. 134).

Figure 2 shows one example of an I-o-M situation and one example of a P-o-M situation. Examination of the units attached to each number helps differentiate the two classes. In the I-o-M situation, one factor is expressed as a rate (miles per hour in Figure 2a) and the units that form that rate are also the units for the second factor (hours) and for the product (miles). In the P-o-M situation, none of the terms is expressed in rate units, and the units for each factor (Newtons and meters in Figure 2b) are independent of one another. Different patterns of units on factors and products raises as a question whether a single quantitative meaning for multiplication can be used across different situations or whether more than one quantitative meaning is required.

3 miles/hour • 7 hours = 21 miles (a)

3 Newtons • 7 meters = 21 Newton-meters (b)

Figure 2: Illustrating Vergnaud's (1983) classification. (a) An I-o-M situation. (b) A P-o-M situation.

Izsák and Beckmann (e.g., Beckmann & Izsák, 2015; Izsák & Beckmann, 2019) have argued that coordinated measurement provides a single meaning for multiplication that spans both I-o-M situations and P-o-M situations. Figure 3 shows how the coordinated measurement meaning for multiplication summarized in Figure 1, combined with rectangular areas, can fit both I-o-M and P-o-M situations. The same rectangular area represents a product amount measured both in base units (each small square represents 1 base unit) and in group units (each column represents 1 group unit). Figure 3a uses the same speed • time = distance example from Figure 2a, and Figure 3b uses the same force • distance = work example from Figure 2b. In Figure 3a, if each small square represents 1 mile (1 base unit) and each column corresponds to 1 hour (1 group unit), then the measure of one column in base units is 3, the change measured in hours is 7, and the change measured in miles is 21. In Figure 3b, if each small square represents 1 Newton-meter (1 base unit) and each column corresponds to 1 meter (1 group unit), then the measure of one column in base units is 3, the change measured in meters is 7, and the change measured in Newton-meters is 21.

The examples in Figure 3 indicate how attention to two measurement units, made explicit in the $N \bullet M = P$ equation, allows one to see parallels (i.e., there is an isomorphism) between rectangular areas and the two applied situations, one an I-o-M situation and one a P-o-M situation. This, in turn, justifies why in physics and other domains interpreting definite integrals in terms of areas can be applied both to I-o-M situations and to P-o-M situations. In the next section, I explain how explicit attention to coordinated measurement extends past research on integration.

Reasoning with Definite Integrals

Research on integration goes back to at least the 1980s but has grown considerably in the last decade (e.g., Jones & Ely, 2023). One line of research has identified alternative interpretations of integrals and their notation. Thompson and colleagues (e.g., Thompson et al., 2013) have argued that

interpretating integrals as accumulations from rates of change can facilitate reasoning about the Fundamental Theorem of Calculus. In other work, Jones (2013) found that college students can interpret the $\int_a^b f(x)\,dx$ notation as indicating addition of small pieces (Adding Up Pieces), denotating boundaries of a region in the plane, and signifying a function whose derivative is the integrand. Within this line, researchers have, at times, questioned whether the area-under-a-curve interpretation for definite integrals is sufficient for modeling physical situations (e.g., Jones & Ely, 2023; Nguyen & Rebello, 2011; Thompson et al., 2013). Moreover, there have been reports of students struggling to understand how areas can represent other types of quantities—for instance, how an area can represent a distance (Thompson et al., 2013, p. 125). As the examples in Figure 3 suggest, however, the coordinated measurement meaning for multiplication supports reasoning about rectangular areas as general products across a wide range of situations. Such a meaning could help students understand how rectangular areas—and Riemann sums—can be used to approximate definite integrals across both I-o-M and P-o-M situations. (As a side note, it is more precise to say that definite integrals can be interpreted as signed areas bounded by curves and the horizontal axis.)

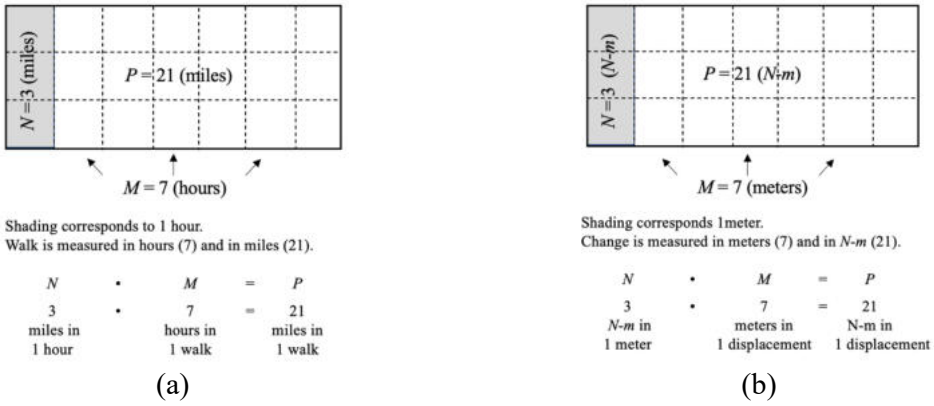


Figure 3: Coordinated measurement and area. (a) An I-o-M situation. (b) A P-o-M situation.

A second line of research on integration has decomposed reasoning about definite integrals into layers that include ones focused on local relationships (e.g., Oehrtman, & Simmons, 2023; Seeley, 2014). As one example, Oehrtman and Simmons proposed *Emergent Quantitative Models* as a tool for tracing how students' attention shifts across three scales—named the Basic, Local, and Global models—when constructing definite integrals. A Basic model relates quantities with constant values, a Local model restricts a basic model to small regions where varying quantities are approximately constant, and a Global model is a sum based on Local models. The Local model is where students would reason about products of the form $f(x) \bullet \Delta x$, but researchers have not articulated an accompanying quantitative meaning for multiplication.

Research on definite integrals summarized above has consistently highlighted the importance of reasoning about small pieces of a target quantity but, to the best of my knowledge, Jones and Fonbuena (2024) were the first to analyze such reasoning in the context of integration by substitution. They began their conceptual analysis by decomposing Adding-up-Pieces reasoning into three aspects—*partitioning* a quantity into small pieces (precursors to differentials), *determining the target quantity* in each of those small pieces (similar to the Local model mentioned above), and *summing*

the small pieces of the target quantity to get the total quantity (similar to the Global model mentioned above). Their analysis traced the consequences for each of these three aspects when one changes from one input variable to another. They described these consequences as "conversions" of the partition, integrand quantity, and interval of summation.

Similar to Oehrtman and Simmons (2023), Jones and Fonbuena (2024) did not articulate a quantitative meaning for multiplication. In the next section, I explain how multiplication understood in terms of coordinated measurement, rectangular areas, and inversely proportional relationships can be used to justify integration by substitution.

Coordinated Measurement and Integration by Substitution

Similar to the analysis of integration by substitution given by Jones and Fonbuena (2024), the analysis presented here is based on a change in the independent variable through function composition. Consider the Moving Truck situation presented below. Speed is piece-wise constant, and the above discussion of Figure 3a justifies using rectangular areas to represent distances. Think of the graph in Figure 4a as representing a Riemann sum that uses rectangular areas to show distance traveled for each segment of the trip.

Three friends are driving a moving truck from Detroit to Boston.

- *Steve takes his turn first and drives 4 hours at a steady speed of 60 miles each hour. At the end of his turn, the friends stop and rest for 1 hour.*
- *Chara takes her turn second and drives 4 hours at a steady speed of 72 miles each hour. At the end of her turn, the friends stop and rest for 1 hour.*
- *Zawadi takes her turn last and drives 4 hours at a steady speed of 54 miles each hour.*

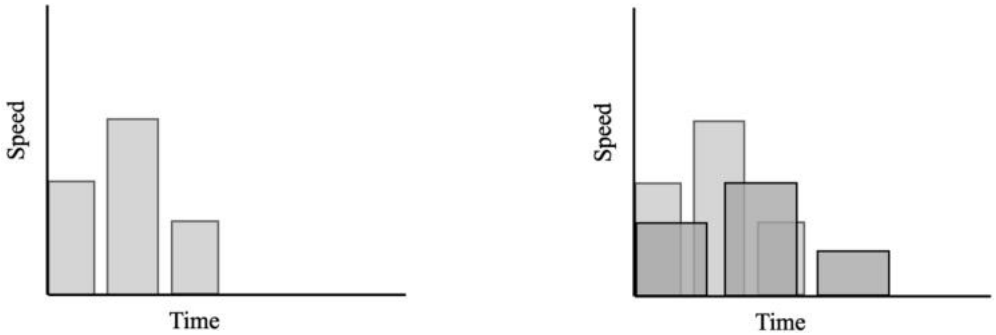


Figure 4: Moving Truck situation. (a) Time measured in hours. (b) Time measured in playlists

Now suppose that the three friends invent a new unit based on the time it takes to go once through a playlist that they listen to as they drive. Suppose further that 1 playlist is 2/3 of an hour. The miles driven in 1 playlist are fewer than the miles driven in 1 hour by a factor of 2/3. Thus, speeds measured in miles per playlist are 2/3 of the corresponding speeds measured in miles per hour. Simultaneously, the number of playlists it takes to measure any given duration of time is 3/2 the number of hours it takes to measure the same duration of time. Figure 4b superimposes onto Figure 4a a second graph of speeds and times for each segment of the trip based on playlists. Think of the graph in Figure 4b as representing a new Riemann sum that expresses distances when time is measured in playlists.

Notice that we have an inversely proportional relationship where shrinking in the vertical direction is offset by stretching in the horizontal direction. Thus, the total shaded areas corresponding to the two Riemann sums, and thus the distance for each segment of the trip measured in miles, is the same whether time is measured in hours or in playlists. This reasoning based a Basic model applies equally well to reasoning based on a Local model where the journey is partitioned into shorter and shorter segments.

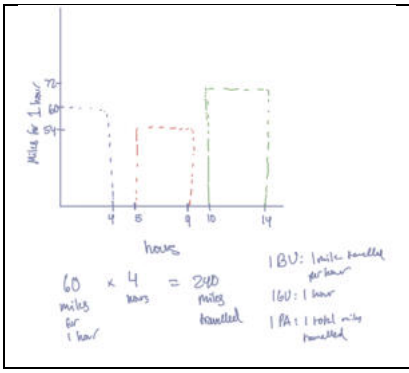
The two equal areas in Figures 4a and 4b relate to the commonly used integration-by-substitution equation as follows. If u stands for the number of hours elapsed since the beginning of the trip, then u ranges from 0 to 14 hours. In this case, $f'(u)$ is speed in miles per hour, and we can express the total distance traveled in miles with the integral $\int_0^{14} f'(u)\,du$ (Figure 4a). Continuing, if x stands for the number of playlists elapsed since the beginning of the trip, then x ranges from 0 to 21 playlists and $g(x) = \frac{2}{3}x$ expresses the number of hours for the same duration of time. Finally, $f'\left(\frac{2}{3}x\right)\frac{2}{3}$ is speed in miles per playlist, and we can express the total distance, again in miles, with the integral $\int_0^{21} f'\left(\frac{2}{3}x\right)\frac{2}{3}\,dx$ (Figure 4b). Thus, the integration-by-substitution equation

$$\int_a^b f'\big(g(x)\big)\,g'(x)\,dx \;=\; \int_{g(a)}^{g(b)} f'(u)\,du \text{ becomes } \int_0^{21} f'\left(\frac{2}{3}x\right)\frac{2}{3}\,dx \;=\; \int_0^{14} f'(u)\,du.$$

In the above example, there is a single linear relationship between the original input variable, hours, and the new input variable, playlists. If students used a different unit of time for each segment of their journey, the same argument would work, each rectangle associated with a given duration of time measured in hours (Figure 4a) would have the same area as the corresponding rectangle with the same duration of time measured in the new unit. Thus, the Riemann sum depicted in Figure 4a would still have the same area as the transformed Riemann sum depicted in Figure 4b. The statement of integration by substitution is then plausible because the two definite integrals are approximated by the same sequence of values.

Example of Student Work

For the past few years, I have been developing a course that introduces university students to coordinated measurement and that builds paths towards integration by substitution and other core calculus topics. The students, in general, are not STEM majors, and the close examination of multiplication proves challenging but accessible. Figure 5 shows sample work for the Moving Truck produced by one student on a final exam in Fall 2023. "Nancy" and "Mary" are *N* and *M* in Figure 1.



"In this graph, one base unit would be the column of one hour to one mile traveled in that hour, while the group unit would be that column extended to the 60 miles for one hour which is the amount that Steve drives in this problem. Nancy is represented by the height of the box which is 60 miles for one hour, and Mary is represented by the width of four hours, and the product amount is the area of the rectangle that those dimensions create."

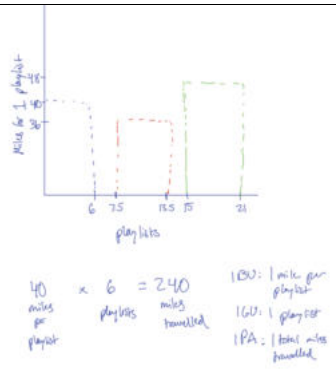
	"For the problem to be converted into playlists, I found that to find the same number of miles which is 240, I applied the 2/3 difference to be 2/3 of 60 miles per hour to be 40, which then increases the value of the group unit to be increased by a factor of 3/2 which is the inverse to then find the same Patty value."

Figure 5: An example of student work from a final exam, Fall 2023.

Discussion and Conclusion

To the best of my knowledge, the justification for integration by substitution presented above is both original and at odds with more traditional justifications. In particular, the justification presented here emphasizes why you can write an equal sign in between two numbers. In essence, the justification started with a Reimann sum for $\int_{g(a)}^{g(b)} f'(u)du$ and explained why the corresponding Riemann sum for $\int_a^b f'\big(g(x)\big)\, g'(x)dx$ gave the same number. This new justification requires reexamining several common ways of reasoning about integration. I have already discussed how the coordinated measurement meaning of multiplication supports using rectangular areas as generic models for products that span both I-o-M and P-o-M situations. In addition, the new justification consistently interprets integrands as rates. In particular, rather than associating $g'(x)$ first with dx, as is in more traditional explanations, the new justification first associates $g'(x)$ with $f'(g(x))$ and treats the entire $f'(g(x))\, g'(x)$ expression as a rate of change. This, in turn, supports consistent interpretations of integrands across related topics, including the Fundamental Theorem of Calculus. Finally, although the new analysis is not a proof, it highlights how inverse proportions, a topic from middle grades mathematics, makes the statement of integration by substitution plausible. It is not a rigorous explanation *that* the statement is true but, rather, offers a plausible explanation for *why* it is true. Furthermore, it is an explanation that students could plausibly generate.

Finally, the analysis of integration by substitution provided here extends recent research that examines ways in which students' reasoning about topics introduced in middle grades (~ ages 12-14) directly support and constrain ways they reason about calculus topics (e.g., Beckmann & Izsák, 2024; Byerley, 2019; Thompson & Harel, 2021). This emerging body of research suggests that a substantial revision to ways that topics related to multiplication are developed in earlier grades could help students be better prepared for studying calculus grounded in quantities.

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Accumulation: continuity and contexts

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We report on a study investigating how context affects students' understanding of accumulation and their ability to select a continuous graph for an accumulation function, given a discontinuous rate of change graph. Through quantitative and qualitative analyses, we found that extra-mathematical contexts significantly influence students' accumulative thinking.

Keywords: Integral, Accumulation, extra-mathematical context.

Theoretical Background

Integration as accumulation is at the core of understanding many ideas and applications. The accumulation function has the potential to serve as a model for various situations in continuous real-life processes, and vice versa, continuous real-life processes may demonstrate accumulation.

According to Thompson and Silverman (2008), in order to hold a well-structured understanding of the accumulation function, students need to coordinate three variables simultaneously: the value of x that changes, the value of the integrand f that changes accordingly, and the value of the quantity accumulated up to x, which derives from these changes of x and f and which changes according to both. Hence, one of the difficulties in understanding accumulation function arises from understanding accumulation as a function which depends on another function. Students typically encounter such a complex covariation for the first time when approaching accumulation.

Constructing the integral concept based on the idea of accumulation has been shown to be beneficial (Carlson et al., 2003; Kouropatov, 2016). Kouropatov (2016) built and researched a teaching unit for the integral concept based on approximation and accumulation; he concluded that this approach clarified the connection between integration and differentiation for the students and allowed them to realize the mathematical meaning of the integral.

Although the concept of accumulation is familiar to students from everyday life, they have difficulty grasping the 'bits' accumulating (Thompson & Silverman, 2008). Thompson (2013) argues that a significant reason for this is a lack of reference to meaning in teaching and research. Considering this, it has been suggested to teach integration with an approach that places the concept of accumulation at the center (Kouropatov & Dreyfus, 2013).

The effect of extra-mathematical context (EMC) on problem-solving has been widely studied in various subjects, for example, computer programming (Leinonen et al., 2021), algebra (Bottge, 1999) and proportional thinking (Lawton, 1993). Research in mathematics and science education reveals that it is difficult for students to apply their mathematical knowledge in different contexts (Delice & Roper, 2006) as the context requires an additional layer of the meaning to variables in the mathematical expression in order to be applied in the science context (Dray & Manogue, 2005), and particularly in integration problems (Alves et al., 2019; Jones, 2015).

Accumulation specifically has been found to be a concept which may be taught through EMC. Carlson et al. (2003) developed curricular materials aiming to promote students' understandings and reasoning abilities regarding the fundamental theorem of calculus, including tasks given in an EMC. The results showed that most students developed a strong understanding of the aspects regarding accumulation. Falach (2023) has defined the notion of accumulative thinking and investigated its development in a pool filling context.

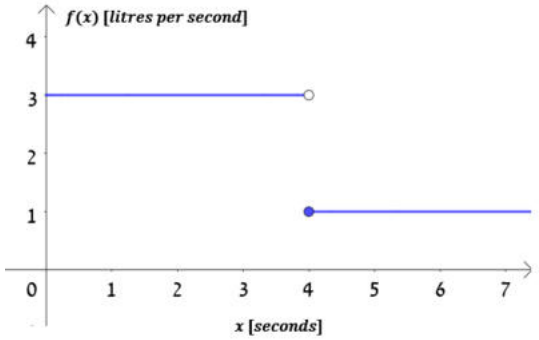
In the study presented here, we examine students' facility of expressing ideas related to accumulation in different intra- and extra mathematical contexts. We ask: In what contexts do students think of accumulation and express their accumulative thinking and what role does the context have in the process?

Tasks

This study is part of a larger project, in which we examine students' personal meanings for basic concepts of calculus. As part of the project, we developed various tasks, including one in which students are presented with a split-domain rate of change function, constant and positive in each subdomain, and are then asked about its accumulation function (see Figure 1 and below). The task was designed so that there is no need for technical work, but rather for understanding that if you know how fast a quantity is changing at every moment, you can know how much of that quantity has accumulated at every moment (Thompson et al., 2019). The quantity accumulates continuously, and because of the positive rate of change, the accumulation function is monotonously increasing.

For examining the role of context, we designed four parallel versions of the task, based on different intra- and extra mathematical contexts: (1) Formal; (2) Area; (3) Motion; and (4) Pool. We present (4) in detail (Figure 1).

The accompanying sketch shows the graph of the function $f(x)$, which represents the flow of water into a pool. The flow is measured in units of liters per second.
The pool was empty at the beginning.



Alona said that using these data it is possible to sketch the graph of the function $g(x) = \int_0^x f(t)dt,\ x \geq 0$, which represents the amount of water in the pool at time x.
The students discussed the meaning of her statement to them

Figure 1: The pool context version of the situation in which the task is set

Table 1 demonstrates the differences between Alona's statement in the four contexts.

Table 1: Alona's statement in the four contexts

Context	$f(x)$	g(x)	Alona said that using these data it is possible to sketch the graph of
Formal	Function	Integral of $f(x)$	the function $g(x) = \int_0^x f(t)dt$, $x \geq 0$.
Area	Function	The area that accumulated between the graph of $f(x)$ and the x-axis starting from $x = 0$	the function $g(x) = \int_0^x f(t)dt$, $x \geq 0$ which represents the area that accumulated between the graph of $f(x)$ and the x-axis starting from $x = 0$ up to $x = 7$
Motion	The speed of a tortoise (meters per second)	The distance the tortoise passed at time x	the function $g(x) = \int_0^x f(t)dt$, $x \geq 0$ which represents the distance the tortoise passed as a function of the time x.
Pool	The rate of flow of water into a pool (liters per second)	The amount of water accumulated in the pool at time x	the function $g(x) = \int_0^x f(t)dt$, $x \geq 0$ which represents the amount of water in the pool at time x.

In a first task students were asked to state how close different ways of thinking about accumulation are to their own. In this report, we focus on the second task which asked students to choose the graph they think represent the function $g(x) = \int_0^x f(t)dt$, $x \geq 0$, from among those in Figure 2. The graphs represent different ways to think of integration. Graph 2 suggests accumulative thinking because it is the only continuous graph.

Which of the following graphs, in your opinion, represents the function $g(x)$?

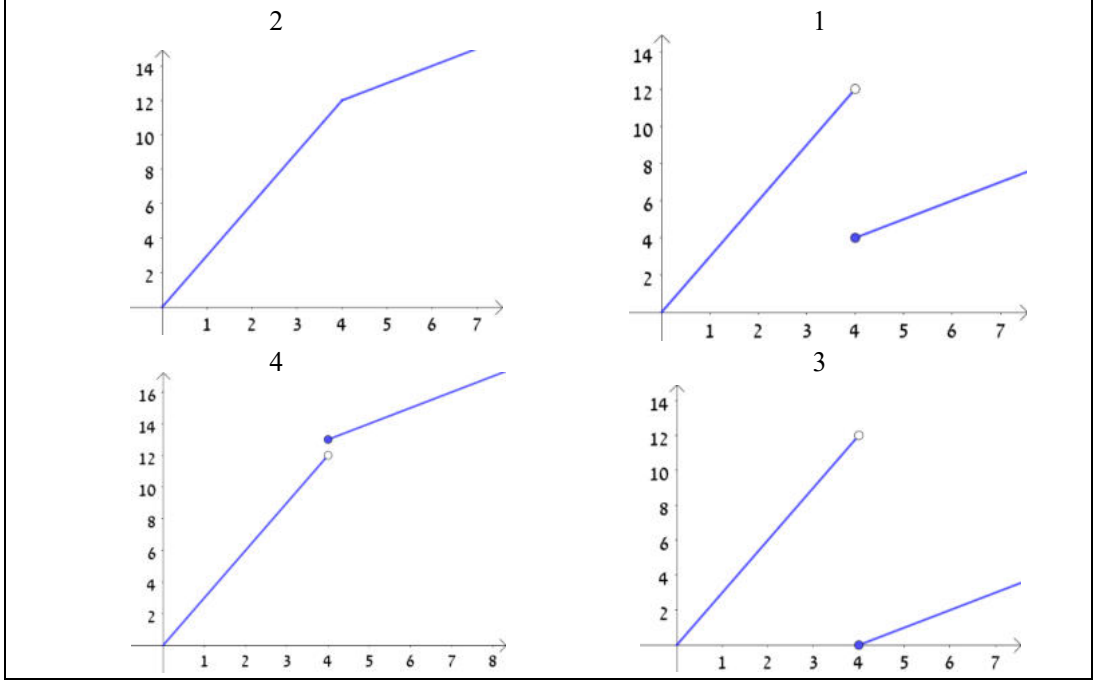


Figure 2: Proposed graph options for the function g(x)

Procedure and population

A total of 226 students from advanced track mathematics in grades 11 or 12, who had already studied integrals in school, completed a questionnaire, with approximately equal numbers in each context (see Table 2). In addition, 24 students were interviewed right after answering the questionnaire. The interviews were audio-recorded and transcribed. In the questionnaire, students were asked to mark their choice, and in the interviews, they were asked to explain their choices. In the analysis of the interviews, students' explanations were categorized by their statements, as detailed below. We report both quantitative and qualitative results from this task.

Findings

We first present some quantitative results for the 226 students who completed the questionnaire, focusing on their selection of the graph for *g(x)*. Table 2 presents the percentages of students who chose each of the graphs across different contexts.

Table 2: Graph choices for g(x) across different contexts

Context / Graph Choice	Pool	Motion	Area	Formal	Total
1	12.7%	18.9%	16.4%	23.6%	17.7%
2	55.6%	45.3%	40%	18.2%	40.2%
3	6.3%	18.9%	20%	25.5%	17.3%
4	11.1%	1.8%	9.1%	14.5%	9.3%
No Answer	14.3%	15.1%	14.5%	18.2%	15.5%
N	63	53	55	55	226

We see that about 40% of all students chose Graph 2, which represents accumulative thinking, with notable differences across contexts. In the Pool context, approximately 56% of students chose the Graph 2, while in the Formal context, only about 18% did so. A difference we found particularly interesting was between the Area context (about 40%) and the Formal context (only about 18%). We found this surprising because Area is an intra-mathematical context, and so we hypothesized that the results for the Area context would be closer to that of the Formal, and not as close to that of the EMCs.

When considering EMCs, the Pool context shows a higher percentage of selections for Graph 2 than the Motion context. We turn to the qualitative results to explore how students thought about these contexts. Due to space limitations, we focus on quotes for the Pool and Formal contexts to emphasize the distinct ways students approached accumulation in each context, as reflected in their explanations.

Formal context

Of the six students interviewed, only one chose Graph 2 and explained: "Plug in each value, make a point, and connect them. Then, something continuous will come out without a hole or anything."

While expressing thoughts of continuity, the student talks about a technical algorithm of computing and plotting.

The five others chose different mistaken graphs. Their explanations included:

S1: (chose Graph 4) The function, like, f, it's not... it's not continuous, so it feels to me like... like g isn't continuous either. I'm not sure, but it seems to make more sense to me.

S2: (chose Graph 3) Because here f is not regular, g should also be irregular.

S3: (chose Graph 1) Because I think that until x equals 4, the function obviously increases, and from x = 4 it still keeps increasing, but the value at the point 4 is lower than the point, like the hole that's here. [Interviewer: "Right, but this relates to the value of f(x), which is lower?"] S3: "Yes".

S4: (couldn't choose a graph) First of all, in Graph 2. I'll go by elimination. In Graph 2, they didn't address the breakpoint at x = 4, so I kind of ruled it out.

The most common reasoning for these students was categorized as "If f has this characteristic then so should g." S1 thought of f not being continuous and thus g must also have this characteristic, S2 thought of an irregularity that must carry from f to g. S3 said that if there is a decrease in f there must be a decrease in g. S4 had the same line of logic and said that a break point for f should be addressed in g, but didn't explain further. For these students, the discontinuity in f was an obstacle they couldn't overcome. They did not demonstrate accumulative thinking but instead relied on what seems like intuitive reasoning.

Pool context

Of the six interviewees, five ultimately chose Graph 2, either immediately or after considering and ruling out the other options.

S7: Because it's the accumulation of how much water is in the pool, it's like a line that continues. And like, Graph 2 shows something continuous, like it's rising, like it keeps rising because we're not emptying it. [...] What I did connect with in 2 is that, um... like, there are two different slopes. Like, from 0 to 4 there's a certain slope, and from 4 onwards, there's a different slope. [...] Like, the faster we fill the pool, the faster the water accumulates.

S8: The function g describes the amount of water per second, […] therefore, the flow will be the slope, but the amount is not something that decreases over time. It won't decrease at the point $x = 4$, only the slope will decrease. That's why I chose Graph 2, which is the only one where there's no change in the amount of water at the point $x = 4$.

S7 and S8 chose Graph 2 and explicitly connected their choice to the concept of accumulation or the amount of water in the pool ("it's the accumulation of how much water is in the pool", "there's no change in the amount of water at the point x = 4"). This demonstrates their ability to overcome the discontinuity in f by applying accumulative thinking rooted in the Pool context. Notably, they not only selected Graph 2 based on continuity but also reasoned about the slope of the accumulation function in relation to the value of the rate of change function ("The faster we fill the pool, the faster the water accumulates"). Their explanations combine real-world intuition ("We're not emptying it,' 'There's no change in the amount of water") with mathematical concepts such as slope, rate, function, and line.

S9: I know it starts with 3x. That means until 4, it simply increases at a rate of 3 per second. So, it stops at 12, and from there the slope changes, increasing by 1 per second until it reaches 15. That's it. […] There's no jump when the flow rate per second changes.

S10: At first there's an increase of 3x, and then it just drops to an increase of, like, 1x. […] In 4, they just add water without explanation, in 3 they ignore it, and the same with 1. 2 is the only one that makes sense.

S9 and S10 also reasoned based on the continuity of the amount of water in the pool. They chose Graph 2 due to its continuity or ruled out the others because of discontinuities ("There's no jump", "they just add water without explanation"). Their reasoning combined mathematical analysis, including computations and calculations, with real-world sense.

S11 similarly reasoned that: "there wasn't a jump in the amount of water. The amount of water continued from the same point. […] Only the rate of water filling up was changed." So, all 5 students used the Pool context in their explanations, leaning on their own experience and real-life common sense.

S12 was the only interviewee in the Pool context who chose a graph different than Graph 2, stating: "The domain of the function and the domain of the primitive function should be the same domain." It seems that S12 tends to look for similarities between the graphs of functions f and g, such as ensuring they share the same domain. This choice can be attributed to an attempt at formal mathematical thinking, rather than thinking in context. She focuses on domains, possibly recalling that the domain of the derivative is contained within the domain of the function being differentiated. However, she extends this idea to 'the same domain,' not realizing that the domain of the accumulation function may be larger than that of the derivative. Thinking within the context could help avoid such misconceptions.

Discussion

In this study we asked: In what contexts do students think of accumulation and express that accumulative thinking? What role does the context have in the process? According to the quantitative results, the context had a considerable influence on students' ability to think accumulatively and to choose a continuous graph for g. The qualitative results further confirm and explain this effect. EMCs typically facilitated accumulative thinking: The continuity of the accumulation process was more evident in the Pool context, than the Motion context, and more in the Motion context than in the Area context. In the Formal version of the task, a large majority of students did not choose the accumulative graph. The interview data, where we observed students transferring properties from the f graph to the g graph with limited interpretation, reinforces the claim made by Thompson and Silverman (2008) that one of the challenges in understanding the accumulation function arises from the need to comprehend it as a function dependent on another function.

Students who did not interpret the integral as accumulation may have relied on intuitive rules (Tsamir et al., 2002). Some of these rules seemed to influence the explanations of students who chose a graph other than Graph 2. For example, we observed students (S1, S2. S12) arguing that if a function has a discontinuity, its integral (or primitive function) must also have a discontinuity. This reflects the 'No Holes' rule, which states that students often believe a function and its derivative must be defined at

the same points, meaning if there is a discontinuity in the function, there must also be a discontinuity in its derivative at the same location. Additionally, the 'More A = More B' rule, which claims that students intuitively believe if a quantity is increasing, its derivative must also increase, appeared to play a role for students who argued that if a function increases or decreases, then its integral graph should behave similarly.

On the other hand, students working within the Pool context appear to rarely rely on these intuitive rules. Instead, they often use their everyday life experience, reasoning that if you do not empty the pool and continue filling it, the graph should be continuous and rising. This suggests that real-life contexts, especially tangible ones like the Pool, may help prevent mistakes based on intuitive rules, as students draw on real-world experience rather than relying solely on intuition.

Some of the students reasoning in the Pool context combined everyday common sense with mathematical concepts such as slopes and lines, suggesting that their reasoning may create connections to their mathematical knowledge, renewing and refining their understanding of accumulation.

The results of this study emphasize the importance of context in supporting students' understanding of complex mathematical concepts, like accumulation. It could suggest that using real-world, tangible contexts helps students make meaningful connections between their everyday experiences and formal mathematics, reducing reliance on intuitive but incorrect rules.

Acknowledgment

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Learning trajectory to teach double integrals with GeoGebra Applets

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In this contribution we propose the design and use of GeoGebra Applets and homework exercises to offer, for second year Civil and Environmental engineer students, an interactive teaching method in a second course of Calculus. We give here two examples of GeoGebra Applets, on normal domain and change of variable to polar coordinates respectively, referred to specific class exercises used during a double integral' lesson. The results of two homework exercises will be compared with the same one proposed in 22/23 where no specific GeoGebra applets were proposed but the underlying idea was already present, and in 23/24 when they were introduced. The theory of multivariational reasoning to design the applets and the exercises, and to analyse the errors of the students were used.

Keywords: Multivariational reasoning theory, Homework, Teaching and learning mathematics for engineers, GeoGebra, double integrals, normal domain, polar change of coordinates.

Introduction and theoretical background

There have been several attempts to improve students' learning and the teaching of mathematics in basic and multivariable calculus through moving away from remedial classes towards teaching to increase understanding. GeoGebra is a software that allows to create different interactive applications that can be used as teaching tools in math classes. The graphics in books, as well as the pictures drawn on the blackboard, are static and require the students to possess an adequately trained imagination. Nowadays, the existence of free programs with versatile capabilities and interactive representations helps to improve the presentation of contents taught, allowing dynamic visualizations. Hohenwarter and Fuchs (2004) listed the following four uses of GeoGebra: as a tool for demonstration and visualisation; as a construction tool; as a tool for investigation and inductive discovery; and lastly, as a tool for designing and preparing teaching and learning material.

Although research on the teaching and learning of single variable calculus is a field that has attracted much research in mathematics education, the number of studies examining the teaching and learning of multivariable calculus is considerably smaller. Double integration is a fundamental concept in many engineering applications and widely used in courses across the engineering disciplines. Students should know and correctly apply integration techniques, recognise, and describe the domain of integration, and evaluate using different available strategies. Limited research has been reported that focuses on multivariable integration (Gemechu et al., 2021; Martínez-Planell & Trigueros, 2020; Alessio et al., 2022). Worsley et al. (2008) commented on students' difficulties in double integrals, which included changing the order of integration. Gemechu et al. (2021) highlighted students' lack of understanding which manifested in their difficulty in algebraically representing regions of integration. Some find difficulty when dealing with integration first with respect to y and there is no understanding that the limits of integration of the inner integral are at most functions of x and similarly for the integration with respect to y first. This process involves the description of the boundaries of subsets in $\mathbb{R}^2$ and manipulation of its equations, possibly using different coordinates (Cartesian, polar), or simply put, a broader use of 2D geometry. After working almost exclusively

with a Cartesian coordinate system to represent the plane, we would be surprised if students did not come to think of the plane and the Cartesian coordinate system as essentially equivalent. However, the plane and the Cartesian coordinate system are not equivalent, and it is important that students construct the polar coordinate system in ways related to, but not constrained by, their ways of thinking about the Cartesian coordinate system. Moore et al. (2014) also describe three types of complexity that students, future high school teachers, encountered during the construction of the polar coordinate system: understanding the measurement of the angle in radians, overcoming the input conventions output for functions in the polar coordinate system and the differentiation of the pole of the polar coordinate system from the origin of the Cartesian coordinate system.

Homework is one of students' opportunities to learn mathematics. Findings from research indicate undergraduate calculus students spend more time doing homework than they do in class (Ellis et al., 2015). Mathematicians view homework as a critical component of learning (Dorko, 2021). Moreover, students report that homework and practice exercises are important for their learning, more than other forms of instruction such as lecture. However, undergraduate mathematics education researchers have only recently begun to study student learning from homework.

Covariational reasoning is "the cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other" (Carlson et al., 2002, p. 354). Moore et al. (2013) described how students' covariational reasoning supported them in conceiving graphs in the Cartesian coordinate system and polar coordinate system as representing invariant relationships. They have described students reasoning emergently to conceive a graph in a Cartesian coordinate system representing the same covariational relationship as a graph in the polar coordinate system despite perceptual differences in the curves. When more than two quantities are covarying, some authors speak of multivariational reasoning. Starting from the evidence that scientific phenomena often include more than two quantities and that the theoretical framework of covariational reasoning mainly focuses on two simultaneously varying quantities, Jones (2022) formally elaborated on the construct of multivariation to address the mental actions behind those "cases where more than two explicit variables relate to and change with one another" (p. 1).

The purpose of this article (continuing a work started in Lepellere, 2024), is twofold: to investigate difficulties encountered by students in solving two double integral homework exercises; to design two GeoGebra Applets one on normal domain and the other on the use of polar coordinate system to help the students to overcome these difficulties. These lend themselves to the four uses of GeoGebra mentioned by Hohenwarter and Fuchs (2004). The homework exercises and the Applets were designed considering the multivariational reasoning theory. The performance of students on two homework exercises in the academic year 22/23 will be compared with the same one given in 23/24 where the Applets were introduced and in the present one where their use were reproposed and most emphasized. The research questions are the following: 1) How did the integration of the Applets influenced the performance of the exercises proposed? 2) What were the mistakes made by the students in the homework in the three academic years? Were there any differences?

Methodology

Mathematical Analysis 2 is a second course of calculus (9 credits corresponding to 72 hours of lessons) held in the first semester of second year to Civil and Environmental Engineering students. It is preceded in the first year by Mathematical Analysis 1 (12 credits), Linear Algebra (6 credits) and Physics 1, and it is simultaneous with Rational Mechanics. The students come mainly from two different high schools: technical, especially with a focus on "construction, environment and territory" and scientific. The course was divided in three macro-sectors: differential calculus in several variables, free and constrained optimization study and implicit functions; multivariable and vector integrals; first and second order differential equations, systems of linear differential equations with simple stability studies. The students have the possibility to divide the exam into three partial tests, one for each macro-sector, but to participate is mandatory to solve several homework assignments.

During double integral lessons, in academic years 23/24 and 24/25 two GeoGebra applets were proposed, during the lesson and for the individual use at home as a test, to give students a tool to support the understanding of the concepts of normal domain and change in polar coordinate system. They were linked to two class examples, Example1 and Example2, one which required, after dividing the set into two parts, the use of normal domains, and the other required the use of the change in polar coordinates to obtain a non-rectangular domain. In 22/23 the same exercises had been proposed trying to undermine, with straight lines parallel to the axis or passing through the origin, what would later become the use of sliders in the applets. The use of applets has only made everything more dynamic. Two homework exercises, EX1 and EX2 included in a homework assignment containing 11 exercises on curvilinear and double integrals, and applications of the Gauss-Green formula) were designed to solve double integrals, the first in three different ways, thinking to normal domain with respect to x, normal domain with respect to y, change in polar coordinates, the second were expected the use of change in polar coordinates. To solve these exercises, the students were given a week, after which they had to send them us by email. The results of both exercises, for 37 students of this academic year were examined and the errors compared with the same ones proposed last year when the students were 23 and for 37 students in the academic year 22/23. For the applets and the exercises multivariational reasoning theory was considered.

The design of GeoGebra Applets

Let's look in detail the two applets (see Figure 1) linked to Example1 and Example2:

Example1. Drow the set $A = \left\{(x, y) \in \mathbb{R}^2 \mid 0 \leq \frac{x}{2} \leq y \leq 2x,\ y \leq -x + 3\right\}$ and determine its area.

The purpose of Applet1 for Example1 (https://www.geogebra.org/m/zpx5ypsv), see Figure 1a, was to show the dependence of the ordinate of the points D, E and F on the abscissa represented by the slider k and so what were the functions to take into consideration to see the set as a normal domain with respect to x and the necessity to divide the set into two parts. The "intersection" command and the slider were used to further highlight the change of the functions to take in consideration before and after the point C is crossed. The use of the slider k gives the possibility to create a multivariational track on the $x-$axis that moves simultaneously three points D and E that are in the domain of integration and F that is outside it before the intersection point C between the two straight lines is

crossed, after that the point D remains as the lower limit but the point E is replaced by the point F. Simultaneously, you could see how to express the two functions with respect to x to create the normal domains. Applet1 has the task of guiding students to understand which values follow x and which follow y based on the functions we take as the extremes of the double integral, and consequently the area (or the integral in general) we were interested to evaluate.

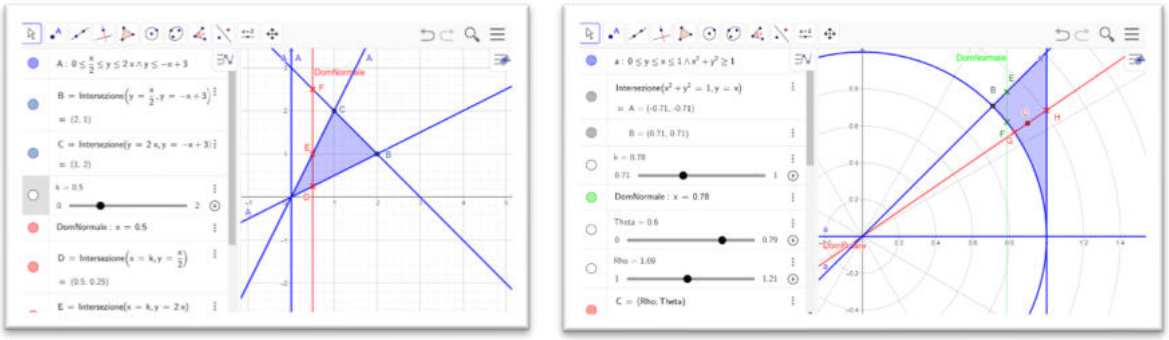

Figure 1: 1a on the left Applet1 on Normal domain, 1b on the right on Polar coordinates

Example2. Drow $A = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq y \leq x \leq 1, x^2 + y^2 \geq 1\}$ and determine $\iint_A \frac{x}{x^2+y^2} dxdy$.

The purpose of Applet2 for Example2 (https://www.geogebra.org/m/uhztazbu), see Figure 1b, was to see how to represent the domain using polar change of coordinates and to compare this to the use of normal domain. Also, in this case, we wanted to reproduce, using the sliders, a multivariational experience in the classroom. The vision of the set as a normal domain is resumed as in Example1, with the slider k and the intersection points F and E and in this case the multivariations of y as a function of x too. For the use of polar change of coordinates, we have taken the advantage that GeoGebra offers to change the reference system in polar form. In this case two sliders were used, one for rho, that is the distance of any point C from the origin, and the other for $theta$, the angle that the segment OC forms with the x −axis: we have shown that if only rho varies, the point C moves along the line containing OC; if only $theta$ varies, C moves along the circumference of radius rho. If both vary, while for the intersection point G rho is always constant and equal to 1, for the point H rho depends on $theta$. We wanted students to reflect that, when polar coordinates are used, the radius depends generally on the angle, so A is transformed in a normal domain with respect to the angle and not simply into a rectangle.

The Homework exercises

To test whether the students understood the use of normal domains and the relation to the change in polar coordinates for double integrals calculation, two homework exercises, EX1 and EX2, were designed and proposed. The work done at home had sent us by email. These were within a homework assignment containing 11 exercises including double integrals, curvilinear integrals, differential forms and applications of the Gauss-Green formula. The purpose of these exercises was to guide the students in their study in preparation for the second partial test. They were part of a learning trajectory designed for a gradual acquisition of the integration techniques based on the multivariational reasoning that lies behind them. Moreover, to reflect on the mistakes made or to discover alternative

strategies to solve the exercises, the best solutions of the students were provided to everyone as correction.

EX1. Drow $A = \{(x,y) \in \mathbb{R}^2 |\, x^2 + y^2 \leq 4,\ x \geq 1\}$ and calculate in 3 different ways $\iint_A\ x\, dxdy$.

The purpose of this exercise was to push students to think about the set as a normal domain with respect to x, i.e. $A = \left\{(x,y) \in \mathbb{R}^2 |\, 1 \leq x \leq 2;\ -\sqrt{4-x^2} \leq y \leq \sqrt{4-x^2}\right\}$, a normal domain with respect to y, i.e. $A = \left\{(x,y) \in \mathbb{R}^2 |\, -\sqrt{3} \leq y \leq \sqrt{3};\ 1 \leq x \leq \sqrt{4-y^2}\right\}$, and a change in polar coordinates i.e. $A = \left\{(\rho,\vartheta) \in \mathbb{R}^2 |\, -\frac{\pi}{3} \leq \vartheta \leq \frac{\pi}{3};\ \frac{1}{\cos(\vartheta)} \leq \rho \leq 2\ \right\}$. The three requests were proposed specifically in sequence: the first more intuitive apart from having to recognize that by explicating y as a function of x two values were obtained but, having already proposed this example when the implicit functions were introduced, no great difficulties were expected, furthermore the variability values of x were easily recognizable graphically too; the second instead required to find first the intersections between the circumference and the straight line $x = 1$ to have the limitations of y and then see the limitations of x between 1 and what is obtained by explicating x from the equation of the circumference $x^2 + y^2 = 4$; the last required the use of the change of polar coordinates and therefore it was necessary both to calculate the angle obtained by the two intersection points, calculated previously, and to transform the line $x = 1$ and the circumference into polar coordinates, so the set A becomes a normal domain with respect to the angle. The three requests could be thought of as achieving three levels of multivariational reasoning.

EX2. Drow $A = \left\{(x,y) \in \mathbb{R}^2 |\, x^2 + y^2 \leq 2y,\ x^2 + y^2 \leq 2\sqrt{3}\, x\right\}$, calculate $\iint_A\ x\sqrt{x^2+y^2}\, dxdy$.

In this case the presence of $\sqrt{x^2+y^2}$ in the integrating function, and the characteristic of A as intersection of two circles, had to suggest the use of the change in polar coordinates. The latter required to find the intersection points, $O = (0,0)$ and $P = \left(\frac{\sqrt{3}}{2}, \frac{3}{2}\right)$, between the two circumferences and the respective angle $\frac{\pi}{3}$ that the segment OP formed with the $x-$axis. In this case however it was necessary to realize that the dependence of the radius on the angle undergoes a change after $\frac{\pi}{3}$ was crossed dividing the set into two normal domains with respect to the angle, i.e. $A_1 = \left\{(\rho,\vartheta) \in \mathbb{R}^2 |\, 0 \leq \vartheta \leq \frac{\pi}{3};\ 0 \leq \rho \leq 2\sin(\vartheta)\right\}$ and $A_2 = \left\{(\rho,\vartheta) \in \mathbb{R}^2 |\, \frac{\pi}{3} \leq \vartheta \leq \frac{\pi}{2};\ 0 \leq \rho \leq 2\sqrt{3}\cos(\vartheta)\right\}$. Also, in this case, different levels of multivariational reasoning were required to obtain the result. We observe that using normal domains for EX2 it was not easy to solve the single integral.

The results

Let's analyse the results of EX1 and EX2 of the intermediate test in the academic years 22/23, 23/24 and 24/25. Let's start with EX1, the Table 1. shows, in the second column the percentage of those who tried to carry out the exercise considering the set as a normal domain with respect to x or with respect to y, with change in polar coordinates in the third and finally in the last column with correct use of all three methods. The percentage of those who made mistakes compared to the number of students who solved it is shown in brackets.

EX1	Normal to x or y	Polar coordinates	3 methods

22/23	89% (21%)	34% (38%)	3%
23/24	87% (15%)	48% (36%)	9%
24/25	92% (6%)	59% (23%)	32%

Table 1: Results of EX1

Let's immediately highlight that the change in polar coordinates was less immediate for students, but the choice has grown over the academic years. Between the choice of seeing the domain normal with respect to x or with respect to y, the first was more immediate. In 22/23 three students solved correctly with respect to x but not with respect to y, which did not happen in the two subsequent academic years this means that a higher level of multivariation reasoning was reached. Moreover, it was noted that this academic year, compared to the last, the second method had a greater choice. The use of the polar coordinate had a substantial increase, as did the number of those who performed the exercise correctly in all three methods, moreover the number of errors has decreased for both methods.

Let's look at the types of errors. For the use of normal domains with respect to x: in 22/23, apart from one student who considered a quarter of a circle as the domain, not taking into account the request that $x \geq 1$, the other two were not able to correctly explain y as a function of x; in 23/24, apart from one student who exchanged the order of integration, obtaining a function as result (the fact that the exercise was asked to be solved in different ways confused her, moreover, in the case of normal domain with respect to y, the order of the integrals was correct she only made a mistake in the limitations for y considered between -2 and 2), the other transformed the domain into a rectangle but he didn't try with normal domain with respect to y and with change in polar coordinates; in 24/25, both students who made mistakes transformed the set into a rectangle, but they had the same problem with the change of variables in polar coordinates. Regarding the use of normal domain with respect to y, in all three years the errors were in finding the limitations for y: in 22/23 between -1 and 1 and between 0 and $\frac{\sqrt{3}}{2}$ that are further from the correct solution; in 23/24 between -2 and 2 and between $-\sqrt{2}$ and $\sqrt{2}$ and in 24/25 between -2 and 2, perhaps due to a distraction since the limitations of the normal domain with respect to x of the two students who made mistakes were correctly found. It can therefore be observed that those who carried out the exercise using normal domains with respect to y achieved, for what was observed previously, greater awareness of multivariational reasoning. Finally, as regards the use of the change in polar coordinates, the common error in all three academic years was to transform the set into a rectangle but while in 22/23 this error was also committed by those who had correctly carried out the exercise in the other two methods in 23/24 and 24/25 it was committed only by those who had correctly solved the exercise only as a normal domain with respect to x this means that their level of multivariational reasoning was too low to be able to recognise the variation of rho from $theta$.

Regarding the results of EX2 in the three academic years 22/23, 23/24 and 24/25, there were respectively 59%, 70% and 81% the percentage of those who tried to do the exercise and therefore increased over the years. They set it correctly respectively for 23%, 22% and 33% but, in 22/23, 18% used normal domains and did not reach the result, in 23/24 there was only one student who tried to use normal domains but, realizing that the calculation of the single integral became too difficult, so

tried to use the change in polar coordinates. This year only three students tried to use normal domains with respect to x dividing the set into three parts and only one student, using the change in polar coordinates, transformed the set into a rectangle, all the others found it difficult to understand the variations of the angle but correctly find the dependence of rho on $theta$. There was therefore a progressive improvement of student performance over the years, but we must say that these exercises were given immediately after the explanation of the topic of double integrals and the Gauss-Green formula, and the students had only one week to hand in a long assignment containing 11 exercises.

Discussion and conclusions

Understanding double integrals is fundamental also because triple and surface integrals are based on them. The transformation of a set into a normal domain requires a student's multivariational capacity since they must be able to associate to a quantity that varies, two other quantities and not only one as they were used to do in Mathematical Analysis 1 and in secondary school, moreover this independent variable is not necessarily x. The difficulty in changing to polar coordinates within the resolution of a double integral is generally twofold: in the passage between the two reference systems and in recognizing that the result is in general a normal domain. This last part is the most delicate, for our experience there are no problems for our students to transform a part of a circumference centred in the origin into a rectangle. Furthermore, students are not generally used to dealing with inequalities when more than one variable comes into play, even more if the variables in question were not adequately explored in the past as the polar one. Two GeoGebra applets were proposed to guide students to understand these delicate passages. With their inclusion the results, over the years, were undoubtedly better. Their use helped students to reflect on what functions to consider for normal domains and what happens when the angle and radius vary simultaneously in the polar change of coordinates, increasing their multivariational reasoning. However, some difficulties remain. About homework exercises, compared to EX1, EX2 proved to be more suitable for testing the ability to correctly use the change in polar coordinates in the calculation of double integrals. Moreover, when was given the possibility of avoiding the use of change in polar coordinates, in the academic year 22/23 more than in 23/24 and 24/25 students did so. This study on one hand confirms the difficulties in calculating double integrals already known in literature, giving a different interpretative key, on the other hand the use of GeoGebra applets and homework go in the direction of improvement the delivery of teaching at university level of multivariable calculus courses.

The low number of students of this research does not allow us to generalise the results obtained but they are a starting point for future improvements of teaching tools and of in-depth research. In the future we will try to propose a questionnaire before the homework to guide the reflection of what the applets want to represent; an explanation in class is not always enough. We would also like to increase the use of applets and homework exercises and to study the impact they have on the partial test.

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Student understanding of double and triple integrals in rectangular coordinates

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We apply Action-Process-Object-Schema (APOS) theory to study student understanding of integrals of continuous functions over general regions using rectangular coordinates. The notion of schema is used to model students' constructions. We propose a simple model (genetic decomposition) consisting of three schema component structures and three relations between these components. The model was tested with semi-structured interviews with eleven students who had just finished a multivariable calculus course. The findings of our study can be used to improve the model and design activities to help students better understand double and triple integrals in rectangular coordinates. Findings show that it can be challenging for students to go from the case of finite Riemann sums over a rectangle to the infinite context of double and triple integrals over general regions. We documented different specific constructions where students can benefit from instructional support.

Keywords: Calculus, integration, multivariable functions, APOS.

Introduction and literature review

In the past twenty years, much work has been done on student understanding of multivariable functions, as discussed in the survey by Martínez-Planell and Trigueros (2021). They show that most of this research has dealt with the formal and geometric properties of two-variable functions and with the differential calculus of these functions. However, research on students' understanding of integral calculus lags. This study aims to contribute to addressing this gap.

Among the relatively scarce literature on multivariable integral calculus, one finds studies by Bašić and Milin Šipuš (2022), Borji et al. (2024), Henriquez (2006), Jones and Dorko (2015), Khemane et al. (2023), McGee and Martínez-Planell (2014), and Martínez-Planell and Trigueros (2020). The literature on integral calculus of one-variable functions is plentiful. Of particular interest is the study of Sealey (2014), who proposed a framework to consider students' understanding of the topic. Although conceived thinking about one-variable functions, Sealey's framework can be extended to the context of double and triple integrals, as shown by Martínez-Planell and Trigueros (2020) and Borji et al. (2024) in two research cycles. The present study builds on these two previous research cycles where students' understanding of double integrals of a continuous function over a rectangle was studied. In the first cycle (Martínez-Planell & Trigueros, 2020), a model (genetic decomposition) of how students may construct their understanding of Riemann sums was proposed and tested with student interviews. As a result of the study, the model was revised, and activities were designed to help students do the proposed construction, using the layers of Sealey (2014) and the semiotic chain of McGee and Martínez-Planell (2014). In the second research cycle (Borji et al., 2024), the authors showed that most students who worked on the research-based activities resulting from the first cycle

could successfully perform the proposed constructions of finite Riemann sums over a rectangle. The present study extends those previous research cycles by going from finite Riemann sums in rectangles to considering students' understanding of double and triple integrals in general regions.

Theoretical framework

We use Action-Process-Object-Schema (APOS) theory (Arnon et al., 2014). Since the APO portion of the theory is well-known, we will only describe the Schema portion of the theory, which is what we will use. In APOS, a Schema is a coherent collection (component structures) of Actions, Processes, Objects, and other previously constructed Schemas dealing with a specific mathematical notion. The Schema is coherent in the sense that its different components are interrelated in such a way that the individual can decide when a problem situation falls within the scope of the schema.

This definition suggests that to model Schemas, one may list their component structures and describe the relations between the components that students may construct to understand the mathematical notion. Schemas are dynamic structures constantly under construction as students' mathematical horizons expand. To discuss their evolution, the Intra-, Inter-, Trans- triad of Piaget and García (1983) is interpreted as stages of Schema development in APOS. In order to classify a Schema development into one of those three stages, the notion of types of relations between Schema components has shown to be helpful. This is innovative since this notion was only recently introduced by Trigueros (2019). A *correspondence* relation is established out of habit or repetition, but the student cannot justify it. It can result from repeatedly seeing two components used in tandem or from superficial comparisons of similarities between components. A *transformation* relation groups different components, with perhaps one of them being used to justify or explain the relation between the others. The relation is used normatively in a given problem situation, although not necessarily so across different problem situations. A *conservation* relation is used normatively and fluently across different problem situations, appearing whenever advantageous. Properties in one structure are conserved or translated to properties in other structures so that the structures seem to be interchangeably used.

A Schema is said to be at the Intra- stage of development when its components are isolated or mainly related by correspondence relations. At the Inter- stage of development, groupings of different components and transformation relations start to appear. It is at the Trans- stage of development when conservation relations interrelate most components. A Schema can appear to be in transition from one to another stage of development. A genetic decomposition (GD) for a schema is a model of a Schema given as a list of component structures, a description of relations between components, and a description of how the Schema may evolve. A GD is neither unique nor meant to describe the best way a student may learn a particular notion. It is only an initial hypothesis that must be tested with student interviews and rejected or revised based on the findings. This study contributes to the development of APOS theory since not many studies centre on the Schema portion of the theory.

Genetic decomposition for double and triple integrals

We propose a GD consisting of three components:

- Three-dimensional space (3D) as a Schema with points, fundamental planes, cylinders, and equations as Objects

- Two-dimensional space (2D) as a Schema with points, lines, curves, and the intersection of curves as Processes or Objects
- Riemann sums (RS) as a Schema including Processes for the geometric or contextual meaning of the factors of a typical term of a Riemann sum, partitions, Riemann sums and general regions of integration, integrals as limits of Riemann sums, and differential elements as "small" or "infinitesimal" rectangles or rectangular prisms as Processes or Objects

We consider the following three relations:

- 2D-3D: This relation is used to distinguish "floor" and "ceiling" (for double integrals in rectangular coordinates, the "floor" is the region of integration, and the "ceiling" is the integrand in algebraic or graphical representation; in the case of triple integrals in rectangular coordinates, the region of integration is that of the two outside integrals, the "floor" corresponds to the lower limit of the inner integral and the "ceiling" to the upper limit of the inner integral.) This relation only entails having the mental model of these surfaces and their role; it does not include the specific constructions needed to set up or give a detailed interpretation of the limits of integration in iterated integrals.
- RS-2D: Given a 2D region of integration (double integrals) in rectangular coordinates, this relation is used to do the conversions necessary to establish the limits of integration of the double integral or the limits of integration of the two outer integrals in a triple integral, and in treatments to verify that the limits of integration make algebraic sense. When establishing limits of integration, it is used to relate rectangles (may be thought of as infinitesimal rectangles) for Riemann sums to a given 2D region of integration and determine their orientation (horizontal or vertical), or equivalently, used to determine the order of integration.
- RS-3D: Given a 3D region of integration for a triple integral in rectangular coordinates or given an iterated triple integral, this relation is used to do the conversions necessary to establish the inner limits of integration or to establish or interpret the integrand in a triple integral; it is also used to relate rectangular prisms (may be infinitesimal prisms that may be imagined as line segments) to a given 3D region of integration and determine their orientation (parallel to x, y, or z axes) or determine the variable corresponding to the inner integral. For convenience and for the purpose of data analysis, we kept separate track of the representation or interpretation of the volume element $dxdydz$ (even as a memorized fact) or those corresponding to other orders of integration.

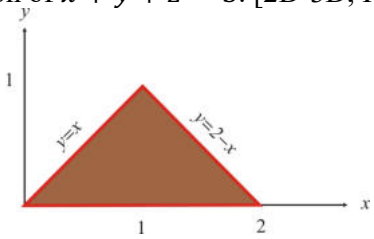
Our research questions are: What instructional implications can we find if we use the proposed GD to model students' understanding of integration? How can the GD be refined?

Method

Eleven of 21 engineering students taking an introductory multivariable calculus course at a mid-tier university in Iran were chosen by their professor (one of the researchers) and agreed to participate in the study. They were chosen based on their test average and course participation, so three were above average, five were average, and three were below average. This was done in order to obtain a wide range of different observations. The students participated in approximately one-hour-long semi-structured interviews one week after the semester ended.

During the semester, students worked on research-based activities designed to address the observations in Martínez-Planell and Trigueros (2020). The activities are described in Borji et al. (2024) and were successful in helping students construct finite Riemann sums over rectangles. Also, most students showed intuition of the limit of Riemann sums of a continuous non-negative function over a rectangle as the volume under the graph and over the rectangle. The present study considers the context of setting up iterated integrals on a general region. The course consisted of two 90-minute sessions per week. When dealing with a rectangular domain *R*, the course textbook (Stewart, 2010) motivates the double integral by first considering a continuous non-negative function *f* defined on a rectangle and the problem of finding volume under the graph and over the rectangle. After introducing a partition and suitable notation, it *defines* such a volume as $V = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f\left(x_{ij}^{*}, y_{ij}^{*}\right)\Delta A$, and then generalizes to functions that are not necessarily positive by defining $\iint_R f(x,y)dA$ as the above limit when it exists. After a brief instructor's presentation of these ideas, students were assigned tasks consistent with the textbook's presentation. The tasks were designed based on the study by Martínez-Planell and Trigueros (2020). In the tasks, students interpret a product of the form $f(a,b)\Delta x\Delta y$ as the volume of a rectangular prism in space. Additionally, when forming a partition of the rectangle *R* and the corresponding Riemann sum, they interpret this sum geometrically as the sum of the volumes of the corresponding rectangular prisms. Then, they express the double integral of f over the rectangle as the limit of these Riemann sums. The tasks did not explore setting up iterated integrals over general regions. An interview instrument with 17 questions was designed. For the sake of this short report, we only show two. Following each question, we include the relations it was meant to test in brackets.

1. Establish (but do not compute) an iterated double integral to find the volume of the solid that occupies all space over the region of the *xy* plane given by the figure below and under the graph of $x + y + z = 3$. [2D-3D, RS-2D or RS-3D]



2. What does the term *dxdydz* that appears in triple integrals in rectangular coordinates mean to you? Can you interpret it geometrically? [RS-3D]

The data was first analysed by introducing codes grounded on mathematical observations as they arose from the data. Then, on a second pass, the relations in the GD and the types of relations were used as codes. The initial codes supplement the ones based on the GD. A third pass on the data was needed to distinguish between transformation and conservation relations and classify students according to the Schema development stages of the Intra-, Inter-, and Trans-triad. Two researchers independently analysed the data, which all researchers then discussed until a consensus was reached.

Results

Tables 1 and 2 show that no student constructed a Trans- stage Schema and show relations in the GD that students need support to construct, particularly 2D-3D and RS-2D.

Table 1: Number of students at the different stages of Schema development

	Intra	Inter	Trans
Number of students/11	6/11	5/11	0/11

Table 2: Number of students constructing transformation or conservation relation

	RS-2D	RS-3D	2D-3D
Number of students/11	5/11	7/11	6/11

We now consider students' results in more detail. No student constructed an integration Schema at the Trans- stage of development mainly because students did not interpret the symbol *dxdydz* geometrically as the volume of an infinitesimal rectangular prism or give an equivalent description in terms of limits. The code ~geoRC ("no geometric rectangular coordinates") was introduced to denote this. We found that it was used once for each of the 11 students. The code "ord" was also introduced ("seems to believe *dxdydz* is used just to give the order of integration"). It was used 15 times involving 10 of the 11 students. One can think of these two codes as giving more information about the relation RS-3D. The appearance of these codes was so prevalent that we decided to separate the interpretation of the volume element from the rest of the relation RS-3D. The following is an example of an Inter- student (Navid) who showed all the constructions in the GD except those coded as "~geoRC" and "ord." When asked to interpret geometrically *dxdydz*:

> Navid: When we have a triple integral with the term *dxdydz*, then we need to find the boundaries for *x*, then the boundaries for *y*, and finally the boundaries for *z*. From a geometrical point of view, it means to me that we are describing a space in 3D using the limitations for *x*, *y*, and *z*.

Navid's answer does not inform us about the student's personal meaning of a triple integral, as in Jones and Dorko (2014), but it does inform us of their meaning for the rectangular volume element. Further, Navid's answer has practical implications as it suggests the need to explicitly discuss volume elements in class. We also show Navid's response to Question 1:

> Navid: I should try for the direction which needs only one integral. First trying for x and then for y, umm $dxdy$ is better than $dydx$. For x I need to go from left to right, so x will be from y to $2-y$. For y I need to go from down to up umm let me find the intersection point of the two lines, umm the point is $(1,1)$. So y is from the *x*-axis to the intersection point, umm I mean y is from 0 to 1, so the volume under the plane $x+y+z=3$ and above the given region can be computed by this double integral [see Figure 1]

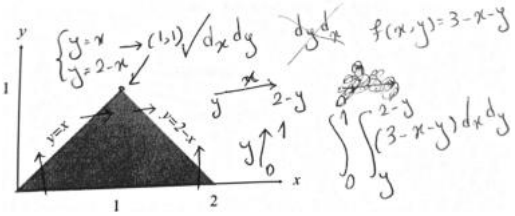


Figure 1: Navid's work on Question 1

Navid's response suggests that he had established the relation 2D-3D to distinguish "floor" and "ceiling" ("under the plane… and above the region"). Note that the explanation of the student and the arrows drawn in Figure 1 suggest that he might imagine vertical and horizontal "infinitesimal" rectangles to aid in determining the order of integration and the limits of integration. This supports the establishment of the relation RS-2D in this question. In the other interview questions, Navid consistently provided evidence of the relation RS-2D and of establishing the relation RS-3D.

Other codes we introduced to further detail RS-2D, for example, ~genx. This code stands for "no generic *x*." It means that the student seems not to imagine or use a "generic" rectangle with base *dx* when relating RS-2D (a rectangle is generic if, as you vary *x*, the corresponding rectangles fill up the region of integration and the base of the rectangles always touches the same curve below while the top of the rectangles always touches the same curve above). It was used more than once with six of the 11 students. This points to a specific difficulty in setting up limits of integration in double integrals (including the two outer integrals in a triple integral). The analogous code ~geny ("no generic *y*") was also introduced with similar results. The following example involves these and other codes and illustrates how the coding was done. In Question 1:

Samira: I divide this region into two small regions like this by drawing the line $x = 1$.
Interviewer: Why do you need to divide the region?
Samira: Because when we want to write the boundaries for *dy* from down to up we have two lines $y = x$ and $y = 2 - x$ for the upper boundary of *y*, umm but when I divide the region into two small regions then for each small region I have only one boundary line from the above.
Interviewer: Okay, write the double integral.
Samira: Since I have two regions I need to write one integral for each of them, umm for the left region I have *y* from 0 to *x* and also the boundaries for *x* are from 0 to 1. For the right region I have *y* from 0 to $2 - x$ and also *x* from 1 to 2 [right part of Figure 2].

Initially, it seemed that Samira (classified as Intra-) had established the relation RS-2D as the limits of integration were correct, and her explanation was consistent with an imagery of partitioning the region of integration and filling it up with vertical slices of base *dx* and height that goes from the lower curve $y = 0$ to the corresponding upper curve $y = x$ or $y = 2 - x$. We also note that Samira did not show having established the relation 2D-3D because the integrand was missing (see Figure 2). However, when pressed to use only one integral, Samira seemed to fall into a memorized routine that did not lead to the solution:

Interviewer: Is it possible to establish this iterated integral with only one integral?
Samira: No, because for *y* we have two different equations from above.
Interviewer: Can you establish the iterated integral *dxdy*?
Samira: If I first consider the limits for *x* I move from left to right, umm *x* is from *y* to $2 - y$, umm and for *y* we have 0 as the lower boundary and for the upper boundary we have two limitations, umm these two lines [$y = x$ and $y = 2 - x$] so we need again two double integrals.

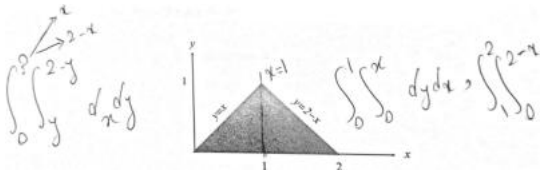


Figure 2: Samira's work on Question 1

This led us to classify the relation RS-2D in this problem as a correspondence relation and to use the code ~geny since the student seemed not to be imagining a horizontal (infinitesimal) rectangle with base *dy* stretching from the curve $y = x$ to the curve $y = 2 - x$, and filling the region as y changed from 0 to 1. We also used the code ~treat (meaning "no treatment") since a simple check of the symbolic form given as the answer (see left hand side of Figure 2) would suffice to note that the outside limits of integration should be numbers. Like in the case of Samira, the code ~treat was used 18 times with eight students, and more than once with five of those eight students.

Discussion and conclusion

Our study shows that even students who can be expected to have constructed finite Riemann sums with research-based activities can struggle when passing to the limit to consider integrals of continuous functions over general regions. One way of seeing this is by considering the understanding of volume elements. It was unexpected to us that students seemed not to have constructed a geometric mental image of volume elements of integration in triple integrals, as suggested by the results dealing with the "~geoRC" code, even after working on tasks where they drew such elements occurring in finite Riemann sums of triple integrals. This has instructional consequences since it suggests that volume elements must be explicitly discussed during instruction and in student activities.

Another contribution is the observation that many students do not undertake the simple treatment of determining if the limits of integration in double and triple integrals they have set up make algebraic sense. This observation, supported by the code "~treat," has instructional implications.

The frequent occurrence of the codes "~genx" and "~geny" and the number of students not constructing RS-2D as a transformation relation refer to the challenge of setting up double integrals. It suggests that when dealing with double integrals, or the outer two integrals of a triple integral, many students tend not to generate the imagery necessary to think of the region of integration as divided into "infinitesimal" rectangles whose base corresponds to the outer variable and that stretch from a single curve touching the base of the rectangle to another single curve touching the "top" of the rectangle. This type of imagery, aided with drawings of thin rectangles, line segments, or arrows in the 2D region of integration, can help students bridge the distance from the finite Riemann sums to the infinite integral over a general region. This supports the idea that instruction might consider going beyond the formal treatment of integration to help students construct the necessary imagery to examine and understand regions of integration prior to attempting to set up limits of integration.

The results dealing with the 2D-3D relation suggest that it might help students to develop the metaphors of "floor" and "ceiling" to think of the region of integration and the integrand in a double integral or the surfaces used to determine the inner limits of integration in a triple integral. The establishment of this relation can help students better organize the information used to determine the limits of integration. This is also supported by observations of the challenge in constructing the relation RS-3D.

In future research, we will refine the GD and use it to design and implement activities. We need to include more detail in the description of RS-2D and RS-3D so that the GD can better describe what students can do. New student data should reflect improved student understanding.

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Understanding the concept of derivative through dynamic visualisations by GeoGebra

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Many studies have investigated the challenges students face in understanding the derivative at the upper secondary level. The use of visualisations can prove beneficial in this regard. However, the studies on the effectiveness of dynamic compared to static visualisations in mathematics lessons yield contradictory results. This article thus examines the influence of dynamic versus static visualisations on the understanding of the derivative concept. Therefore, a sample size of 272 students was used to compare these two approaches in a pre-post control group design with a six-hour teaching sequence as an intervention. The results indicated that both groups exhibited an enhancement in their understanding of the derivative. However, there was a statistically significant difference in favour of the students who employed dynamic visualisations over those who used static visualisations on an understanding-oriented test about the derivative.

Keywords: derivative, understanding, dynamic visualisations, GeoGebra, static visualisations

Introduction

Calculus, and in particular the concept of derivative, has been part of upper secondary mathematics in numerous countries for over a century. In their literature review, Törner et al. (2014) found that the teaching of calculus continues to focus primarily on the development of procedural knowledge. In a study with high school and college students, Orton (1983) identified that, in principle, the formation of a derivative function works well; however difficulties tend to lie in the interpretation of the derivative as local rate of change or in the graphical understanding of the derivative (as tangent slope). In order to address the graphical comprehension difficulties, a teaching sequence for an understanding-oriented introduction of the derivative was developed in the study presented here. The sequence focuses on the use of visualisations, particularly function graphs, as a means of facilitating comprehension. Some studies have reported on the potential of dynamic visualisations to contribute to the understanding of the concept of derivative (e.g. Sari et al., 2018). Nevertheless, it remains unclear whether these can enhance understanding to a greater extent than static visualisations which can be employed with greater ease in class. This article aims to address this issue.

Theoretical Background

Students' understanding of the concept of derivative

A framework model that is still widely used today to describe an understanding of the concept of derivative was developed by Zandieh (2000) through an analysis of mathematical textbooks and discourses on the concept of derivative by mathematics educators, mathematicians and students. In order to capture the many facets of the concept, Zandieh (2000) distinguishes between the *graphical*, *verbal*, *physical* and *symbolic* representations/contexts, as well as the three layers (*ratio*, *limit* and *function*). The latter can be derived via the formal definition of the derivative at a point according to the limit of the difference quotient: "The derivative of f, f', is a *function* whose values at any point

is defined as the *limit* of a *ratio*" (Zandieh, 2000, p. 106). Table 1 presents the framework model used in this study, which includes the contexts and layers in their conception according to Zandieh (2000) and is extended by a *numeric* context based on Roundy et al. (2015). The latter allows for the formation of the derivative at a single point, circumventing the infinitesimal limit. Furthermore, the numeric context facilitates comprehension of the symbolic context.

Table 1: Framework for analyzing students' understanding of the derivative (modified after Roundy et al., 2015; Zandieh, 2000)

contexts / layers	**Graphical**	**Verbal**	**Symbolic**	**Numeric**	**Physical**
Ratio	secant slope	average rate of change	$\frac{f(x+h)-f(x)}{h}$	e.g. $\frac{f(2)-f(1)}{2-1}$	average speed
Limit	tangent slope	local rate of change	$f'(x_0) = \lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$	e.g. $\frac{f(1{,}001)-f(1)}{1{,}001-1}$	instantaneous speed
Function	tangent slope at any point	local rate of change at any point	$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$	limit at any point by a numeric approach	instantaneous speed at every moment

It is crucial that students are able to interpret the layers *ratio*, *limit* and *function* in different representations and utilise them in factual contexts in order to have a deep conceptual understanding of the derivative. There are a number of reasons why visualisations can be beneficial.

Learning (the derivative) with visualisations

Orton (1983) has already suggested that "graphical work is also of great importance in developing concepts of rate of change" (p. 243), since the use of the derivative in real situations happens mainly through graphs, figures and tables rather than through algebraic functional equations. A visualisation-based introduction to the derivative has the potential to facilitate comprehension of ideas such as those of decrease, increase and specific points, including extrema and inflection points, as these important properties can be readily discerned from the graph. Traditionally, visualisations employed in mathematics education have been static, such as illustrations that remain unaltered over time. However, with the increasing integration of technology in mathematics teaching and learning, dynamic visualisations are now becoming more prevalent. These are characterised by the fact that they can change when a user interacts with them. Dynamic visualisations can be generated via GeoGebra, affording students the opportunity to use the drag mode in calculus lessons, for instance, to move points or tangents along a graph or to play animations while controlling the playback speed. Furthermore, graphical, verbal and symbolic forms of representation can be linked dynamically, as well as two graphical representations – the function f and its derivative f' (Ainsworth, 1999).

A number of meta-studies have been conducted to compare the effectiveness of dynamic and static visualisations. The findings of Berney and Bétrancourt (2016) indicate an overall positive effect in favour of dynamic visualisations. However, when focusing on the eleven mathematics-specific studies, no statistically significant difference between these two kinds of visualisations could be

found. In accordance with the Cognitive Load Theory, two opposing explanations may be posited: Dynamic visualisations have the potential to increase cognitive load, for example, due to the rapid movements depicted in the animations and their transient nature. Conversely, the linking of different representations in a dynamic manner can facilitate cognitive processes (Sweller, 2020).

Tall (2009) has already emphasised the potential of computer software to make calculus concepts visible, e.g. by moving tangents along a graph f and simultaneously plotting the tangent slopes (graph f'). In a study conducted by Sari et al. (2018), a dynamic approach to introducing the derivative was investigated. The advantages of the dynamic visualisations generated via GeoGebra were observed in the construction of the derivative graph and in the integration of algebraic and geometric thinking. Fonseca and Henriques (2023) also reached similar conclusions with students who were provided with a discovery approach to the derivative via GeoGebra. Through point dragging explorations, the majority of students were able to describe the relationship between the tangent slope and the local rate of change and could apply it in several tasks.

Research question

It remains an unanswered question in the research literature whether dynamic visualisations facilitate a deeper understanding of the derivative concept. This includes, with reference to the framework in Table 1, a conceptual comprehension of the layers *ratio*, *limit* and *function* of the derivative in different contexts, the relationships between these contexts at each layer as well as the ability to solve corresponding tasks. A pilot study involving 110 students revealed no significant differences between students in the dynamic and static intervention groups with regard to their understanding of the concept of derivative. However, a discernible trend emerged in favour of the dynamic group.

Furthermore, the framework illustrated in Table 1 comprises multiple cells, which can be viewed from both a dynamic and a static perspective. In the graphical representation, the derivative at the layer *limit* is interpreted as a tangent slope. A dynamic view of the course of a graph can be obtained by observing it from a tangent moving along the graph, which represents the slope. However, the tangent slope can also be viewed in a static manner, without any movement, by zooming into the graph at a specific point. In this way, the tangent and the graph are approximately the same in a small interval. In a verbal context on the same layer, the interpretation of the derivative as a local rate of change is dynamic via the limit process of average rates of change. Alternatively, the local rate of change can be understood statically via the formal definition as the limit of the difference quotient.

Therefore, the following research question is examined in this study: To what extent does the understanding of the derivative differ in students who worked with either (1) GeoGebra-generated dynamic visualisations or (2) static visualisations after a lesson sequence on the derivative?

Methods

Sample and procedure

The study was conducted with students attending grade 11 in Germany during the school year 2023/24, between October 2023 and February 2024. To ensure a fair comparison between the dynamic and static approaches to the concept of derivative and to control for the potential influence of the teacher's personality, two groups were formed in each course, exhibiting a similar performance

level, following the administration of a pretest. One group employed dynamic visualisations throughout the six-hour teaching sequence (referred to as dynamic group), while the other group utilised static visualisations (referred to as static group). The two groups were taught in two different rooms. As illustrated in Figure 1, the teaching sequence was structured as follows: The three topics (i) interpreting the derivative as local rate of change, (ii) interpreting the derivative as tangent slope and the connection to the local rate of change, and (iii) derivative function and graphical differentiation were initially presented in a teacher-led lesson, conducted in accordance with predetermined lesson plans. Subsequently, students were required to engage in independent practice under the guidance of a research assistant. In particular, during these practice sessions, learners were tasked with completing exercises that reinforced the concepts covered in the previous lesson. This approach enabled the same teacher to hold a lesson to each group with a one-lesson delay.

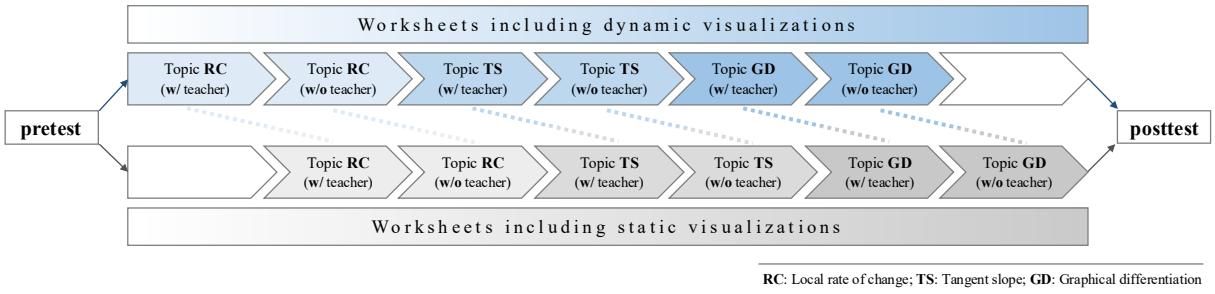

Figure 1: Design of the study

The data presented herewith pertain exclusively to students who were present for a minimum of half of the intervention period and completed both the pre- and posttest. This resulted in a total sample size of 272 students, of whom 140 were assigned to the dynamic group (80 female) and 132 to the static group (72 female).

Material

A folder containing all the requisite worksheets for the six-lesson sequence was provided to each student. The dynamic group employed visualisations created via GeoGebra, which offered a variety of interaction options (e.g. drag mode for moving points or tangents), accessible through QR codes included in the materials. In contrast, the static group completed tasks that were largely similar to those of the dynamic group, but with different instructions. They worked with a folder including static visualisations made from screenshots of the dynamic versions (see Figure 2). All tasks could be solved with the help of either dynamic or static visualisations, thereby circumventing the necessity for manual calculation of terms such as difference quotients.

In designing the lesson sequence for both groups, particular attention was paid to comparability. Castro-Alonso et al. (2016) identified biases in studies comparing dynamic and static visualisations. In order to minimise these biases as much as possible, we ensured that both types of visualisation were comparable in terms of attractiveness, variety, medium, size and realism. However, we were unable to address the issue of interactivity, given that this is an inherent feature of the dynamic visualisations used in this study.

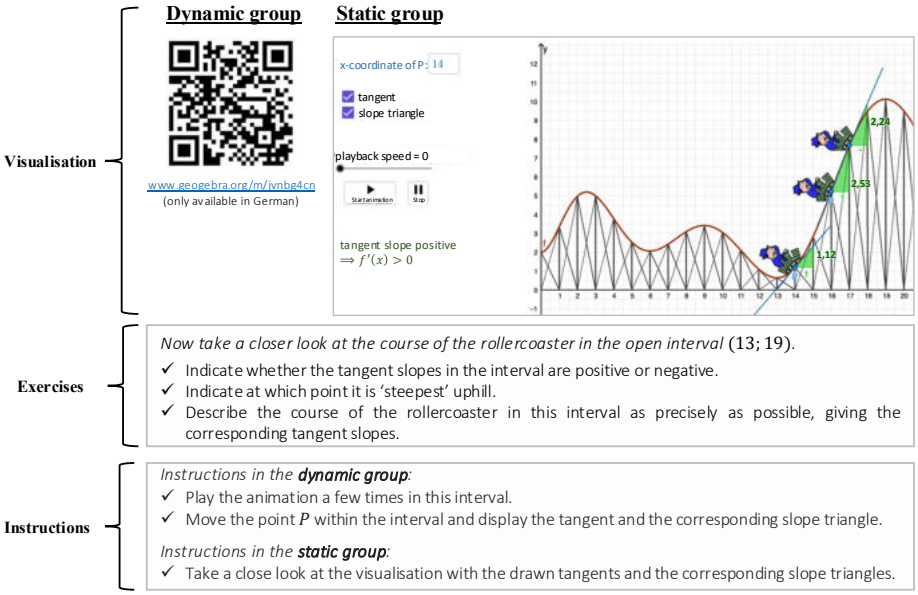

Figure 2: Example of a dynamic and a static visualisation, corresponding exercises and instructions

Understanding-oriented test on the derivative

To measure students' understanding of the concept of derivative, we used a total of 41 items. We tried to include as many layers and contexts as possible based on the modified model created by Roundy et al. (2015) and Zandieh (2000). Figure 3 shows that example 1 can be assigned to the tangent slope, while example 2 covers the layer *limit* in the verbal (and numeric) representation. In all the items used, fewer procedural skills such as determining the derivative of a function according to the derivative rules were tested; instead, the tasks focused on conceptual skills that require an understanding of the concept of derivative according to the framework in Table 1.

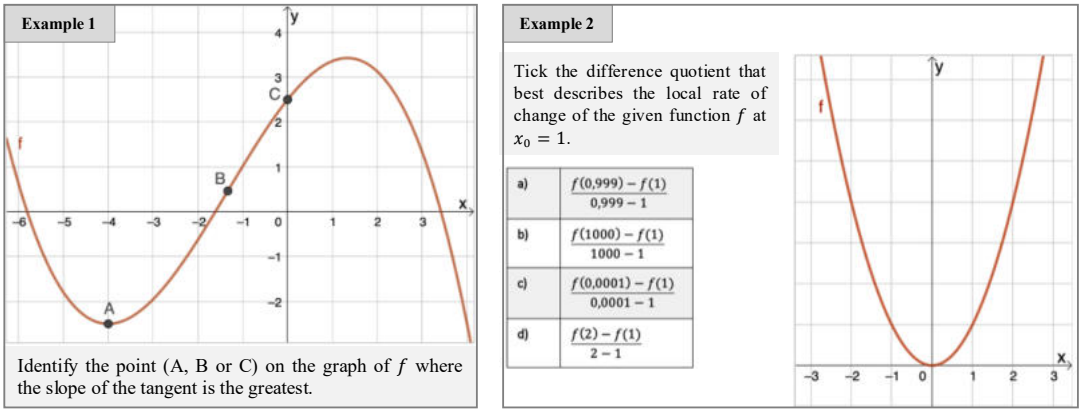

Figure 3: Sample items in the understanding-oriented pre- and posttest

At each measuring point, test booklets were developed comprising 27 items, each linked with eight anchor items. In the pretest (MP 1), a greater number of items was included that pertained to the layer *ratio*, with the objective of preventing demotivation during the testing process. In contrast, the posttest (MP 2) saw a reduction in the proportion of items related to the layer *ratio* and a shift towards

a greater focus on the layer *function*. The online questionnaire platform SoSciSurvey was employed for data processing. Subsequently, the data collected at both measuring points were scaled using a one-parameter IRT Rasch model. Overall, it demonstrated satisfactory fit values: The weighted mean squares of the individual items fall within the interval $[0.82; 1.18]$ and the item-total correlations range from 0.24 to 0.60. Furthermore, the test exhibits a WLE reliability of 0.82, which is sufficient for group comparisons. The statistical analysis was carried out using the program JASP.

Results

As illustrated in Figure 4, the descriptive statistics indicate that, on average, both groups have similar person parameters at the initial measuring point (MP 1). The dynamic group exhibited a mean of -0.44 logit with a standard deviation of 0.93 logit, while the static group demonstrated a mean of -0.45 logit ($\sigma = 0.89$). The non-significant t-test indicates that no statistically significant difference between the students in both groups at MP 1 could be found ($t(270) = 0.11, p = 0.91$). Therefore, the two groups were comparable and subsequently underwent an intervention with either dynamic or static visualisations. It can be determined that the students in the dynamic group exhibited an average increase of 1.02 logit in person parameters at the second measuring time (MP 2), reaching a value of 0.58 logit ($\sigma = 1.25$). In contrast, the static group demonstrated a smaller average increase of 0.76 logit, resulting in a value of 0.31 logit ($\sigma = 1.14$). It was thus observed that there was an overall increase in understanding the derivative in both groups.

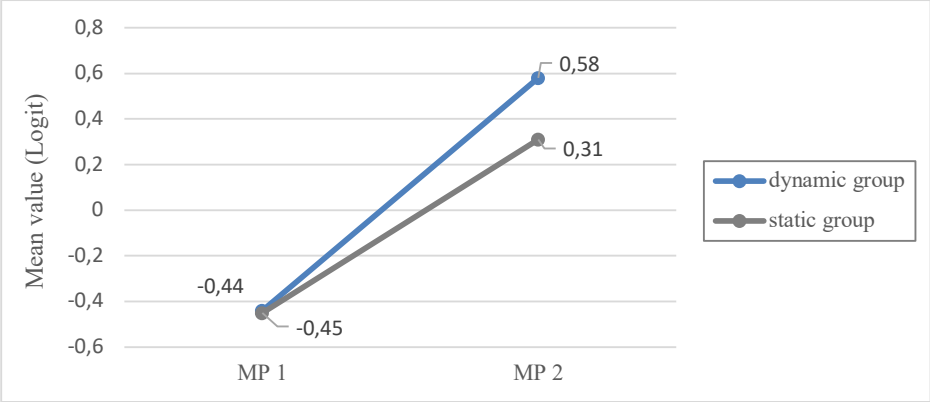

Figure 4: Graphical representation of the mean values (logit) of the two groups

To determine the influence on the independent variable posttest performance by the dependent variable group membership (dynamic or static group) while simultaneously controlling for pretest performance, we chose a multiple linear regression as a static analysis method. To do this, we used the equation $posttest = b_0 + b_1 \cdot pretest + b_2 \cdot group$, where the dynamic group was coded as 0 and the static group as 1.

Table 2: Coefficients of the regression model predicting students' posttest performance (at MP 2)

	b	SE	ß	p	R^2 (adj.)
Pretest (MP 1)	0.852	0.06	0.64	< 0.001	0.427 (0.423)
Group	-0.265	0.11	-0.11	0.02	

The coefficients of the multiple linear regression are presented in Table 2. When controlling for prior knowledge (MP 1), group membership is a significant predictor with a small effect size for the results at MP 2. This indicates that working with dynamic visualisations has at least a small advantage over working with static visualisations. Overall, the regression model explains approximately 43% of the variance in the results of the posttest.

Discussion and outlook

By more than doubling the number of participants compared to the pilot study and revising the materials, the original tendency in favour of the dynamic group was confirmed with a significant result in this study. The potential of dynamic visualisations according to Sari et al. (2018) and Fonseca and Henriques (2023) could also be supported with the results of this study. In regard to Cognitive Load Theory (Sweller, 2020), it can be assumed that the use of dynamic visualisations serves to reduce the cognitive load experienced by learners as a consequence of the interactivity and dynamic linking of content, which facilitates the learning and understanding of the derivative.

Nevertheless, the observed effect was found to be relatively minor. As previously discussed in the theory section, some facets of the framework model for understanding the derivative, as proposed by Roundy et al. (2015) and Zandieh (2000), can be viewed from both a static and a dynamic perspective. This includes the interpretation of the derivative as either a tangent slope or a local rate of change. The results of the quantitative study presented here do not provide information on the specific points during the teaching sequence at which dynamic visualisations have a clear advantage over static ones, nor do they indicate to what extent learning processes differ between students of both groups. It is conceivable that a combination of both forms of visualisation may lead to a deeper understanding of the derivative, as the layers can be handled flexibly in different representations from a dynamic and static perspective. Depending on the task, a dynamic or static view of the derivative may be advantageous. Moreover, further analysis is required to elucidate reasons behind the greater increase in understanding the derivative achieved by the dynamic group as a result of the intervention. The preliminary results indicate that the frequent use of the input field and the drag mode (e.g. moving points and tangents) by students in the dynamic group has a favourable impact on the observed development in understanding the derivative as measured in the pre- and posttest.

In conclusion, the findings of the study are exclusively limited to the influence of dynamic and static visualisations on the domain of derivatives. It is as yet unclear whether these findings can be generalised to other content areas, such as geometry. It may be hypothesised that the integral could also benefit from dynamic visualisations in calculus lessons due to the proximity of the content to the derivative. This is due to the fact that the concept of the integral, either through the accumulation of product sums or through the area, can be visualised particularly effectively. It should be noted, however, that the graphical approach to the derivative is not the only possibility. For example, Roundy et al. (2014) emphasise the physical context in which the derivative concept is applied in measurement experiments, while Tall (2009) describes an approach based on human embodiment, whereby gestures and hand movements can be used to determine the dynamically changing slopes of function graphs.

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Meanings of integrals and their compartmentalization: a case study

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The many meanings of integral calculus can result in students experiencing compartmentalization of knowledge. In this case study, we examine the interplay between meanings of integral held by a graduate student in mathematics education, after taking a calculus course focusing on the notions of rate of change and accumulation. To do this, we used a task-based interview designed to elicit a conflict between antiderivative and accumulation. We suggest a novel approach to analysing knowledge compartmentalization, exposing the connections that exist and the connections that are lacking between these two meanings of integral held by the student.

Keywords: Integral, Accumulation, Antiderivative, Compartmentalization, Meaning.

Introduction

In calculus, we often use 'integral' as an umbrella term subsuming definite integral (with numerical bounds), indefinite integral and accumulation function. While these are all integrals, they are mathematically connected yet distinct. It is therefore important to investigate how students relate to these integrals, what connections and distinctions they form between them, and whether they can identify when each is relevant.

In the Israeli high-school curriculum, up until recently, integrals were conceptualized as either antiderivative or as area-under-a-curve. In recent years, a new curriculum has been introduced, that also includes conceptualizing integral as accumulation (Dreyfus et al., 2021). This raises the question – how do students relate accumulation to their previously held conceptualizations? In this study, we focused on antiderivative and accumulation, and investigated the connections made between them. To investigate these connections, we used Thompson's framework for meaning (Thompson et al., 2024), in combination with compartmentalization of knowledge (Mandl et al., 1993; Shiakalli & Gagatsis, 2006; Vinner & Dreyfus, 1989).

Theoretical Background

Calculus is often regarded as a barrier in higher education, with many students experiencing difficulties. In particular, the notion of integral presents students with many challenges (e.g., Thompson & Harel, 2021). One challenge is that there are several different conceptualizations of integral, each appropriate in different circumstances. For example, Jones (2015) found that though conceptualizations of integral as area or antiderivative are less productive for contextualized integrals than Riemann sum-based conceptualizations, US college students draw on them more frequently. Thompson et al. (2019) developed a course that conceptualizes integral as accumulation to help students develop a dynamic conception of functions, quantities and change, that centres around the fundamental theorem of calculus.

Though there exists empirical research on compartmentalization (e.g., Hausberger et al., 2021; Shiakalli & Gagatsis, 2006), a single accepted definition of the term does not exist. Mandl et al. (1993) refer to three types of compartmentalization: when a learner's previously held beliefs are not in accordance with the formal scientific knowledge that the learner acquired; when different closely interconnected concepts are stored separately, and not viewed as parts of the more complex network that makes up the problem structure; and when learners ignore knowledge from everyday experiences and deem it irrelevant to the scientific situation they face.

On the other hand, Shiakalli & Gagatsis (2006) consider learners' mathematical knowledge regarding an object as compartmentalized if the learners identify two different representations of the object as two different mathematical objects. This can result in learners struggling to transition between different representations.

Lastly, Vinner and Dreyfus (1989) define compartmentalization as a phenomenon in which a learner has two different, potentially conflicting schemes in their cognitive structure, and certain situations stimulate one scheme, while other situations stimulate the other. These schemes can even be in contradiction with one another and can result in inconsistent or incoherent learner behaviour. This can result in learners possessing relevant knowledge but not perceiving it as relevant, or in learners using two different strategies to solve the same problem getting contradicting outcomes, but without cognitive conflict arising. In other words, using schemes that are cognitively disconnected results in outcomes that are cognitively disconnected, and so, even if the outcomes are mathematically contradictory, they will not be cognitively contradictory, and will not result in cognitive conflict.

Thompson et al. (2024) builds on the theories of Piaget to define meaning-in-the-moment as the space of implications existing at the moment of understanding. In other words, a person's personal meaning is what comes to their mind in a situation, as well as what could be implied, i.e., what could easily come to mind next. The implicative nature of meaning offers a lens through which to look at personal connections between different notions. Thus, it is a suitable framework for investigating compartmentalization as well. We therefore suggest defining compartmentalization as a cognitive state where actions, operations, images, schemes or meanings that might be expected to imply one another do not. This definition is compatible with previously observed behaviours described above: if different representations do not imply one another, the learner will struggle in transitioning between them; if relevant knowledge is not easily implied by the situation, it will not be deemed relevant; if different schemes are both implied by the situation but do not imply one another, a learner using both might get contradicting outcomes, without experiencing conflict. Using a framework that accounts for all behaviours could afford a more nuanced examination of the phenomenon, and potentially expose new such behaviours.

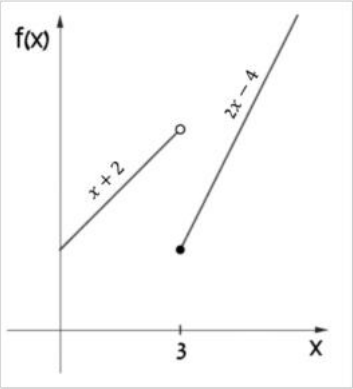
Methods

In this research we conducted an intrinsic case study (Creswell & Poth, 2018), by conducting a task-based interview with Alma, a graduate student in mathematics education, taking an analysis course centred around ideas of rate of change and accumulation. These notions were only recently introduced into Israeli high school curriculum, to be taught alongside the already established notions of antiderivative and area (Dreyfus et al., 2021). Hence, they were new to the students taking the course.

Alma was one of three students interviewed on this task; the students were chosen because they had high achievements in the course, and were thus more likely to have developed connections between their previous conceptions of integral, and accumulation. Due to space limitations, in this paper we focus only on Alma, who presented an interesting example of compartmentalization. The interview was recorded and transcribed. We also recorded all of Alma's writing and sketching.

In the task, Alma was given a non-continuous piecewise linear function. She was requested to graph the derivative of the given function, and then graph the accumulation function of that derivative [Figure 1]. Since the accumulation function $g(x)$ differs from the given function $f(x)$, this task is designed to induce conflict between conceptualizations of integral as antiderivative and as

The given graph describes the function $f(x)$ whose domain is $x \geq 0$.



a. Sketch the graph of the function $h(x) = f'(x)$.

b. Sketch the graph of the function $g(x) = \int_0^x f'(t)dt$.

Figure 1: The task

accumulation. The given function has a derivative for $[0,3)$ and $(3,\infty)$, and the derivative has no antiderivative in the original domain $[0,\infty)$, since an antiderivative function cannot be well-defined at $x = 3$. However, the accumulation function in the domain $[0,\infty)$ exists and is different from $f(x)$. In the interviews we focused on two differences between $f(x)$ and $g(x)$. First, $f(0) \neq g(0)$. Second, $f(x)$ has a discontinuity at $x = 3$ wheras $g(x)$ is continuous throughout.

We analysed the data in accordance with Thompson et al.'s definition of personal (in-the-moment) meaning (2024), looking for the implications brought to Alma's mind, how 'easily' they are brought to her mind, and what potential 'triggers' brought them to her mind. Due to the implicative nature of meaning, to identify Alma's personal meanings we paid attention to her distinct language, i.e., terms and phrases first brought up and used by her and thus indicative of her own implications. Ergo, we interpreted all mathematical terms and notations she used, particularly those used initially, intuitively, immediately; those that are unique. Since meaning is related to scheme, we examined Alma's claims and justifications, i.e., how she used the implied terms and notations to make sense of the

mathematics. We used this to expose the connections between the mathematical notions that comprise the meaning and possibly elucidate the structure of the scheme.

Thompson et al. (2024) refer to the space of implications as what is easily brought to mind. We interpret 'easily' as a scale – some implications are more easily brought to mind than others. We use several criteria to determine this. First, if an implication is recurring, if it is repeated at several points throughout the interview, then it is presumably more easily implied than an implication that occurs sparsely. Second, what is brought up immediately is likely more easily implied than what is brought up afterwards. Moreover, notions that require interviewer involvement to arise are less easily implied by the task than those that arise without intervention. Additionally, notions that are only brought up after the interviewee "gets stuck", deems their current scheme unhelpful, or encounters a cognitive conflict are even less easily implied than the previous ones.

As stated, we also paid attention to the possible 'triggers' that brought each notion to mind. These can be mathematical triggers, i.e., a mathematical notation or notion brings to mind another notion, a scheme, a claim etc. 'Triggers' can also be external, e.g., a social norm, a recent previous experience. To determine the possible 'triggers' we asked ourselves several questions: Which mathematical stimuli appeared just before the implication arose? What was the goal of the reasoning in which the implication appeared? In reference to which question\given\statement did the implication appear?

Findings

Alma has high achievements in the course, but low confidence. She began the interview exclaiming that she does not like analysis, and upon reading the task added "oh, you just killed me". Though she displayed a lack of confidence, Alma sketched an appropriate graph for the derivative [Figure 2], however she did not clearly address $x = 3$. Her sketch of $f'(x)$ was accompanied by consideration of linearly increasing functions, and ratios. The term 'Rate of Change' was not mentioned.

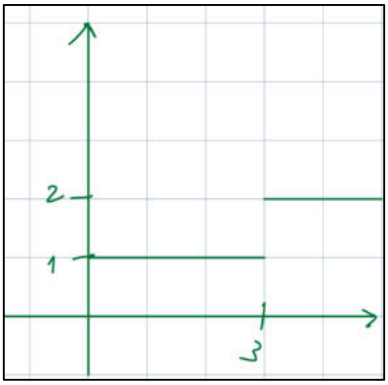


Figure 2: Alma's sketch of f'(x)

As Alma read the second part of the question, where she was asked to graph the accumulation function, her initial response was "I don't know how to do this. Sorry". However, she immediately followed with the statement "I mean, I want to say that it's this" while pointing at the graph of $f(x)$. This is indicative of Alma's implication for integral as antiderivative, though she struggles to elaborate, and upon prompting by the interviewer only adds: "simply because this is the connection between f' and h, the integral of … this is the connection between h and the integral of h."

From Alma's use of qualifiers in her statement – "I want to say that", it appears that this is her own most salient implication for integral. Further probing by the interviewer revealed what was holding Alma back:

> Alma: Look, if you asked me this a month and a half ago [before the analysis course started], I would just tell you it's that.
> Interviewer: Okay.
> Alma: It's just that because of [the professor] (sighs and laughs) we can't use the word integral. Just kidding, it's not forbidden, but we just like have to explain it in different terms.

Thus, we learn that Alma's immediate (presumably stable) implication for integral is "the opposite of derivative", and using the word integral is expressing this notion. Alma had also been exposed to conceptions of integral as accumulation in the course, however she struggled to connect these conceptions with the meaning she already possessed. This is further evident by her statement: "… what I want is to go back to this function ($f'(x)$ and to integrate it. Like, not to integrate it, to …". In this statement she references two different actions, one of which is integrating a function, and the other she does not name, but states it is not integration. The statement was made immediately after referencing the course, and immediately before reasoning that she expected the integral to start from (0,0). Since she justified this expectation by claiming that "at zero it doesn't have an accumulation yet", it appears that the second action referenced was accumulation.

Ergo, Alma seems to have difficulty in conceptualizing integral as accumulation. It appears that while accumulation is brought to her mind when reasoning about integrals, it is 'less easily' brought to mind than antiderivative, to the point where several times in the interview she even distinguished verbally between 'integral' and 'accumulation'. While accumulation was brought to her mind during the task, it appears that the trigger was in large part the context of the interview as part of the course.

As Alma attempted to reason using accumulation, she developed a cognitive conflict between her accumulation reasoning and her antiderivative reasoning. This is due to $f(0) = 2$ while $g(0) = 0$. Alma reasoned that at $x = 0$ the accumulation function should equal zero, since "… at $x = 0$ it doesn't have an accumulation yet". However, her antiderivative reasoning suggested to her that $g(x)$ and $f(x)$ should be one and the same, as previously observed.

> Alma: Why do I want to start from zero? (9 second silence) Look, the graph here is given from $x = 0$, right? So that means that if I want to check the accumulation of this function, then at zero it doesn't have an accumulation yet. It's the starting point of my accumulation, that's why I want … that's why there is this gap that I don't understand. … conceptually I don't know how to settle it with… with the fact that when I start to sum here, because like, I can draw the integral of this function without this constant (circles 2), without the 2 in the given function ($f(x)$).

Alma considered that the difference might be due to an integration constant – "what does it mean? Is it some kind of plus c? What? Don't remember, oh, it's this plus two". While she suggested an integration constant as a possible resolution to the conflict, her use of the word integral as distinct from accumulation suggests that she struggled connecting her conceptualization for integral as antiderivative with her conceptualization for integral as accumulation, even though both conceptualizations were implied.

As the interview continued, Alma proceeded to sketch a graph of $g(x)$ using accumulation considerations, starting at zero, and continuous at $x = 3$ (Figure 3). After sketching her graph, Alma gestured at the graph of $f(x)$ and exclaimed: "and then I don't know what this is". Thus, Alma still had not resolved the conflict. Moreover, Alma states "… it is almost forced upon me. I can only suppose that there's some piece of information that I don't, I don't understand all the way, and it's not here (gestures at the graph she drew)". It appears that for Alma, while both $f(x)$ and $g(x)$ are perceived as integrals of $h(x)$, $g(x)$ lacks some information that exists in $f(x)$. Ergo, this statement supports the hypothesis that for Alma there is a hierarchy between antiderivative and accumulation that favors the former. Furthermore, Alma explicitly stated: "right now there's like two types of attempts to think about things mixed in my head".

As she explained her conflict, Alma skirted around a possible resolution, however, she kept returning to the possibility of her sketches being mistaken. Interestingly, Alma first mentioned rate of change only at this point in the interview, suggesting it is less easily brought to her mind.

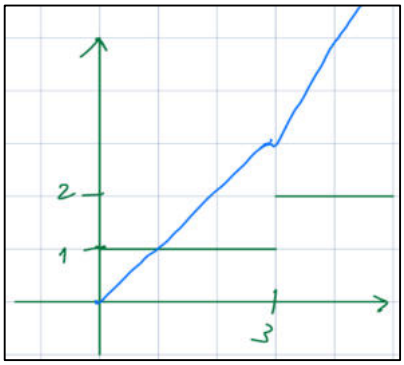


Figure 3: Alma's sketch of $g(x)$ (blue)

Alma: … And I know these constants have some role, and yes, the negative four, it moved us down, but I can't say how right now, what I drew wrong, or in the derivative itself, or maybe the derivative is drawn okay, but we know that the derivative, it actually only expresses rate of change, and there are a lot of different functions that can have the same rate of change, but something about them is different, and that thing is elusive to me right now.

To eliminate the doubt Alma displayed in the graph she drew of $g(x)$, the interviewer informed Alma that it is in fact correct. This led Alma to state:

Alma: … rate of change alone, presumably, is not enough to describe an integral. Or this is an integral and this is … I don't know. Umm … I need to think about it more.

Thus, it appears that Alma was contemplating two different solutions to her conflict. The first, that a given rate of change is not enough to 'describe an integral'. Here Alma was likely using integral in the sense of 'primitive function', closely connected with antiderivative. The second, that only one of the functions is in fact the integral of $h(x)$. Finally, she settled on the explanation that "there are two integrals here for the same rate of change". Additionally, Alma could not explain to the interviewer the difference between $\int_0^x f'(t)dt$ and $\int f'(x)dx$, claiming that to her they are the same.

What was immediately and most easily brought to Alma's mind by integrals is ideas of antiderivative and primitive function. This is in fact so salient to her, that she even struggled to explain it in terms

other than 'integral' or 'integrating'. Thus, at times she used 'integral' as synonymous with $f(x)$. Consequently, this meaning is both dominant and laconic. However, due to the course, Alma also developed another meaning – accumulation. While her accumulative reasoning might be intended to satisfy the interviewer to some extent, accumulation is brought to her mind independently, and is explicitly used in her reasoning during the task.

Though both meanings are brought to mind by '$g(x) = \int_0^x f'(t)dt$', they do not seem to be brought to mind by one another. Furthermore, Alma didn't seem to know when each meaning is relevant. Thus, Alma's reasoning with antiderivative is independent from her reasoning with accumulation, resulting in compartmentalization.

Discussion and Conclusion

In examining Alma's meanings, we found that she can employ reasoning related to both antiderivative and accumulation, however she cannot determine which is relevant in the situation. The immediacy of her use of antiderivative reasoning, and the apparent use of accumulation reasoning to satisfy the interviewer suggest a hierarchy of implications that favours antiderivative reasoning, which is the less relevant one in the given situation. This suggests compartmentalization between the two notions.

Another aspect suggesting compartmentalization is the apparent incompatibility between Alma's conceptions of antiderivative and accumulation. Though both notions were implied by the task, and both implied by 'integral', the connection Alma makes between both notions is mainly in name, and not in substance. Consequently, Alma does not recognize the requested function specifically as an accumulation function, but as an integral in a generic sense. This could also explain her struggle with referring to accumulation as integration, since she formed no meaningful connection between the meaning she held for integration prior to the course and accumulation.

We claim that though Alma experiences cognitive conflict due to her contradicting results, this behaviour still suggests compartmentalization, as the emergence of conflict is due to her tenuous and superficial connection between accumulation and antiderivative. Moreover, while she does recognize the relevant knowledge as relevant, we claim her inability to inhibit irrelevant knowledge is also a consequence of this compartmentalization. Thus, our suggested framework affords a more nuanced examination of compartmentalization, that enables the identification of more complex behaviours.

Like the students in Jones' survey (2015), Alma invokes less relevant knowledge. Our analysis of her meaning suggests the possibility that the low frequency of Riemann sum-based conceptualizations could be a result of weak connections between them and antiderivative and area conceptualizations. These results also suggest that in attending to meaning when teaching the different manifestations of integral, it is important to attend not only to the meanings we want students to develop, but also to the meanings they currently possess, and to the connections and implications we want to exist between them, otherwise students' knowledge could be compartmentalized. The approach suggested by Thompson et al. (2019), where a single notion of integral as an accumulation function is presented, could be a step in this direction; however, this is an issue for further research.

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Introducing preservice mathematics teachers to multivariable functions through modelling: A study and research paths approach

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This study explores the introduction of multivariable functions to preservice mathematics teachers (PSTs) through a modelling approach grounded in the Anthropological Theory of the Didactic. Implemented within a master's course, the study employs Study and Research Paths (SRPs) to guide PSTs in constructing models that serve as tools for determining the best football player in a league. Despite limited backgrounds in calculus, PSTs developed practical knowledge of multivariable functions, using weighted sums of variables to model the system studied. Through iterative refinement and empirical reasoning, PSTs enhanced the accuracy of their models. The findings suggest that SRPs can effectively facilitate the learning of multivariable functions, even in the absence of formal mathematical techniques. Questions remain about how further integration of SRPs and mathematical modelling can enhance PSTs' abilities to develop more sophisticated mathematical models.

Keywords: Mathematical modelling, multivariable function, pre-service teacher education, study and research paths, the anthropological theory of the didactic.

Introduction

Functions play an important role in mathematics. They are initially introduced in school algebra and are prevalent in calculus, real and complex analysis, abstract algebra, and many other branches of mathematics (Dorko, 2023). While there is a vast amount of research focusing on the teaching and learning of single-variable functions, comparatively less research has been dedicated to the teaching and learning of multiple-variable functions despite their fundamental role in modelling real-world systems and their widespread applications across many STEM (Science, Technology, Engineering, and Mathematics) and social science subjects (Martínez-Planell & Trigueros, 2021). In this paper, we report on the implementation of a teaching and learning approach grounded in the Anthropological Theory of the Didactic (ATD), known as *study and research paths* (Bosch, 2018; Chevallard, 2019). This method was taught and used in a master's course in a teacher education programme to introduce preservice mathematics teachers (PSTs) to multivariable functions and to the role variables and parameters play in modelling real-world systems. Our approach is relatively different from what has been proposed for introducing multivariable functions to university students and what can be found in traditional calculus textbooks, where vectors and the geometry of space are first introduced to students (e.g., Stewart et al., 2021). The strategy proposed here does not require comprehensive knowledge of single-variable calculus or visual representations of functions. As such, it can help PSTs and other university students with limited background in calculus to develop at least some partial knowledge of multivariable functions and their utility in solving real-world problems. The research question addressed in this paper is: *What knowledge about multivariable functions and their utility do preservice teachers without a rigorous background in calculus develop through study and research paths focused on modelling of a real-world system?*

Teaching and learning of multivariable functions

Previous research highlights that students' understanding of single-variable functions influences their understanding of multivariable functions, such as through their understanding of covariational reasoning from single-variable functions (e.g., Weber & Thompson, 2014). When generalising knowledge about functions from single-variable to multivariable cases, several considerations must be addressed, including the transformation in the nature of input, output, dependent and independent variables, as well as the domain and range (Dorko, 2023). For instance, while an x-value from a set A is an input for $y = f(x)$, an ordered pair (x, y) from a set $A \times B$ is an input for $z = f(x, y)$. Studies on student learning of multivariable functions highlight challenges such as difficulties in identifying domains and ranges, graphing, and reasoning with free variables, which hinder their ability to apply these functions in real-world settings (Borji et al., 2022; Dorko, 2023). Several teaching strategies have been suggested to enhance knowledge of multivariable functions. These include investigating the *global* behaviour of a two-variable function, fixing one variable at a time to analyse two-variable functions, using a function machine to generalise the concept from single to multivariable cases, and explicitly clarifying the roles of variables in different contexts (Martínez-Planell & Trigueros, 2021; Kabael, 2011). Furthermore, to address language-related issues in learning multivariable functions, the consistent use of embodied language is suggested: "move to the front-back instead of x increases-decreases, right-left instead of y increases-decreases, [and] up-down, instead of z increases-decreases" (Martínez-Planell & Trigueros, 2021, p. 698).

Theoretical tools from the ATD: Modelling and Study and Research Paths

In the ATD, two basic notions are "system" and "model" (Strømskag & Chevallard, 2022). A system $\mathcal{S}$ is any entity that operates according to its own laws. Given a system $\mathcal{S}$, a system $\mathcal{S}'$ is considered a model of $\mathcal{S}$ if studying $\mathcal{S}'$ allows us to answer specific questions about $\mathcal{S}$. Chevallard (1989) presents the original framework for modelling within the ATD, breaking the process down into three key stages that remain foundational today: *system definition*, which involves identifying the system and specifying the relevant variables and parameters in relation to the question being addressed; *model construction*, which entails establishing relationships between the defined variables; and *model exploration*, which involves manipulating and analysing the model to gain new knowledge about the original system. The third stage is driven by the modelling process's objectives, which might encompass solving equations, optimising functions, or exploring the model under different conditions. This framework has been further elaborated by Barquero (2023). Moreover, as discussed by Strømskag and Chevallard (2022), the relationship between systems and their algebraic models is crucial for translating complex real-world scenarios into mathematical representations. A key aspect of this translation lies in the explicit use of parameters, for example through multivariable functions, which allows for the flexible modelling of various aspects of a system. Tackling a generating question Q within the ATD means inquiring into Q, where the inquiry can develop along different study and research paths (SRPs) (Bosch, 2018). A notable characteristic of an SRP is the creation of what is known as a *didactic milieu*, which consists of the tools collected primarily by the students for studying Q. This milieu comprises existing answers, derived questions, works of different types (e.g., theories, reports, and historical studies), and datasets. In an SRP, there is a dialectic between *study* (consultation of existing works) and *research* (creation of elements for an answer to Q that are not

available in the existing works). The path of an inquiry may be illustrated by a directed graph along with a table containing the elements of the milieu used to generate an answer to Q (Strømskag, 2024).

Methods

This paper reports from a case study conducted at a Norwegian public university, involving SRPs carried out by PSTs enrolled in a master's course as part of a teacher education programme for Grades 5–10. Prior to this inquiry, which was their first encounter with an SRP, they were introduced to some ATD notions (e.g., didactic paradigm, model and system, SRP, and praxeology), as discussed by Strømskag and Chevallard (2022) and Strømskag (2023). The PSTs had no formal teaching of multivariable functions or vector algebra. They had only studied some basic single-variable calculus in their first year of study as part of a mathematics education course designed for this group of PSTs, including two 3-hour sessions on derivatives and two on integrals, following teaching on algebraic thinking, equations, programming, and functions. In their other mathematics education courses in the second year, they were taught statistics, probability, and geometry. It is worth noting that in the Norwegian teacher education system, PSTs in teacher education programmes for Grades 5–10 do not take any pure mathematics courses; instead, mathematical content knowledge is integrated into their mathematics education courses. Aligned with the ecological dimension of modelling proposed by Barquero et al. (2013), SRPs were used in this study as a teaching method to overcome curricular constraints where multivariable functions were introduced as tools to model real-world systems. The case consists of six groups of three PSTs, with each group undertaking its own SRP that was required to qualify for their final examination. The research data comprise these elements: three reports submitted by each group while working on the modelling assignment between February to April 2024, a questionnaire completed by the PSTs after completing the SRPs, and the lecturer's reflective notes from the course. Five of the six groups gave consent to use their data in our study. This paper presents an analysis of the reports from these five groups, with excerpts from Group 1 (hereafter G1) serving as representative of the others. The topic of the generating question was chosen in consultation with the PSTs to give them more agency in their study process. Consequently, the following generating question, here denoted Q_0, and guiding prompts were presented by the lecturer:

Q_0: How would you select the best football player in a football league [of your own choice]?

- What is the system studied? [Define the system and identify its variables and parameters.]
- Create a model of this system that enables you to answer Q_0.
- What knowledge can be inferred from your model and what limitations does this model have?

(The information in square brackets is a condensed version of what was provided in the assignment.)

Results

Defining the system: Discussing types of data and informational sources

In the context of deciding who the best football player in a league is and intending to build a model to make this decision, the system would be the set of all relevant factors and variables that define and influence the performance and evaluation of football players within that league. Although this definition was not explicitly given in the PSTs' first report, they focused on finding what possible data they could use from various sources to address Q_0, such as Eurosport.no (Myhre & Fredheim,

2022), FotMob.com (n.d.), and NIFS.no (n.d.). They discussed several questions, for example "Should different positions have different criteria?", which led to identifying four positions to include in the model—goalkeeper (hereafter keeper), defender, midfielder, and striker—along with additional factors to be considered. For example, G1 explained their criteria for selecting the best keeper as follows:

> We start by looking at the number of saves and goals conceded, and the ratio between these numbers. This is because the number of goals conceded can be influenced by the quality of the defence in front of the keeper … Furthermore, we consider the ratio of penalties saved to penalties conceded … Physical attributes such as height, agility, strength, and reaction time are also important … Additionally, we use the Goals Prevented metric, which indicates the number of goals conceded compared to the number of goals the keeper was expected to concede.

Constructing a model: Mathematising and using statistics to determine parameter values

In a lecture after the first reports were submitted, the lack of mathematising was discussed, and the PSTs were encouraged to study how multivariable functions could be used to find an answer to Q_0. Two examples were presented by the lecturer: $f(x_1, x_2) = ax_1 + bx_2$ and $y_{Gk} = 8x + 4y + 0.3z$. The PSTs were also reminded about the different roles that parameters and variables play in modelling systems, as previously discussed in the course, drawing on literature (e.g., Strømskag & Chevallard, 2022). All groups then used multivariable functions in their models, reflecting on this feedback. In what follows, we examine G1's construction of a model to determine the top football player, with a focus on the mathematical elements in this process. Section titles in italics are used for clarity.

Multivariable functions: The proposed model uses linear functions to evaluate players based on various parameters. Each player's performance is assessed using a formula that combines different weighted variables relevant to their position (i.e., keeper, defender, midfielder, striker).

$$y_k = ax_1 + bx_2 - cx_3 + dx_4 + ex_5 + fx_6 + gx_7$$
$$y_d = ax_1 + bx_2 - cx_3 + dx_4 + ex_5 + fx_6 + gx_7$$
$$y_m = ax_1 + bx_2 - cx_3 + dx_4 + ex_5 + fx_6 + gx_7$$
$$y_s = ax_1 + bx_2 - cx_3 + dx_4 + ex_5 + fx_6 + gx_7$$

G1 explained that y_k, y_d, y_m, and y_s are the performance scores for keepers, defenders, midfielders, and strikers, respectively. They further clarified that the variables x_i represent different numerical aspects of a player's performance: x_1 = goals scored; x_2 = team goals; x_3 = goals conceded; x_4 = assists; x_5 = matches over 45 min played; x_6 = player average rating; and x_7 = team average points.

Variables and parameters: The variables in these functions are drawn from key performance indicators (KPIs) that are relative to each football position. The parameters a, b, c, d, e, f and g are placeholders that G1 later assigned values to, reflecting the relative importance of each KPI for different positions, aiming to model each player's contribution to their team's success. However, this notation may have been problematic, as both variables and parameters should have been position specific. For example, $y_k = a_k x_{k1} + b_k x_{k2} + c_k x_{k3} + d_k x_{k4} + e_k x_{k5} + f_k x_{k6} + g_k x_{k7}$, or, more simply, $y_k = w_{k1} x_{k1} + \ldots + w_{k7} x_{k7}$ for keepers, with similar subscripts for the other positions. Here, w_{ki} denotes the weight of the variable x_{ki}.

Use of statistics: The model heavily relies on statistical data derived from various performance metrics (e.g., FotMob.com and Nifs.no), such as goals scored, assists, and player ratings. These data are used to populate the variables in the functions, which then produce a numerical score that represents the player's overall performance.

Exploring the model: Iterative testing and parameter refinement

Model validation: After testing the model on actual players, using "Best of the Season" (TV2, n.d.), a ranking of the best players in 2023, G1 explained their process of adjusting the parameters (or weights) of the multivariable functions to best match the observed data:

> In our model, it is easier for attacking players to achieve a high score compared to defenders. We also noticed that it was particularly difficult for the keeper to score many points … Currently, player ratings have a relatively small impact, while the number of goals scored and conceded by the team had a large effect … We have not found any variables that similarly count positively for defenders and keepers, so we have decided to further adjust the parameter *c*—the number of goals conceded by the team. We also felt that the player rating had too little impact and will adjust it slightly upwards. To ensure that the number of goals scored and conceded by the team does not have as much influence, we will slightly decrease these parameters.

Two successive iterations of adjustments like this were made to the model's parameters, with each round enhancing the accuracy of the scores to better reflect a player's performance, considering the different roles and expectations associated with their positions. G1's process, as described in the above quote, is an example of "model fitting," where the goal is to find the set of parameters that minimises the difference between the predicted values generated by the model and the actual data. The parameters of G1's model were not mathematically justified (e.g., through regression analysis); instead, the model fitting was conducted through trial and error, similar to what the other groups did.

Optimisation: The final step in the model development involved tweaking the parameters to minimise biases towards certain positions and ensure that the model could fairly evaluate players across different roles on the pitch. Here is G1's resulting model:

$$y_k = 10x_1 + 0.2x_2 - 0.3x_3 + 8x_4 + 0.5x_5 + 10x_6 + 10x_7$$
$$y_d = 3x_1 + 0.3x_2 - 0.5x_3 + 3x_4 + 0.5x_5 + 10x_6 + 10x_7$$
$$y_m = 1.5x_1 + 0.4x_2 - 0.8x_3 + x_4 + 0.5x_5 + 10x_6 + 10x_7$$
$$y_s = x_1 + 0.5x_2 - 1.1x_3 + x_4 + 0.5x_5 + 10x_6 + 10x_7$$

Modelling within the SRP framework: The milieu and path of the inquiry

Given space constraints, we briefly comment on the modelling assignment as an SRP, using G1 as an example. As an overview, Figure 1 depicts the path of the modelling process, while Table 1 lists the milieu elements incorporated throughout. At the start of the modelling process, there was no existing model available to the PSTs, so the concept of "existing answers" was applied not to Q_0 but to the derived questions, Q_k. The attributes of the variables in the constructed model were determined by the derived questions that addressed players' performances. Existing answers to these questions included player statistics and ratings, which served as works for creating a dataset to test the model and refine its parameters. Notably, G1 interpreted "works" (see Table 1) as the activities they

performed, rather than as products of human creation, as defined in the ATD. On a broader level, the challenge lay in modelling a complex scenario where inherent laws are elusive and difficult to define, unlike in physical or mathematical systems. This adds an additional layer of difficulty to the modelling process, highlighting the need for further research in this area.

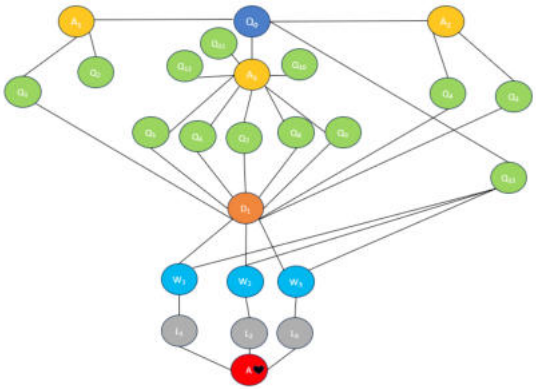

Figure 1: Directed graph displaying the didactic milieu and path of G1's SRP

Table 1: Overview of the elements of the didactic milieu created during G1's SRP

Generating question Q_0: How can we select the best football player in the Norwegian Eliteserien?
Existing answers $A_i^{\Diamond}$
$A_1^{\Diamond}$ – Eurosport.no (Myhre & Fredheim, 2022)
$A_2^{\Diamond}$ – Nifs.no (n.d.)
$A_3^{\Diamond}$ – FotMob.com (n.d.)
Derived questions Q_j (Derived questions not aligning with the model's variables are excluded below)
Q_3: What is the average rating of different players for the matches played? Q_4: How many matches over 45 min have different players played? Q_5: How many goals have different players scored? Q_6: How many goals has the team of different players scored? Q_7: How many goals has the team of different players conceded? Q_8: How many assists does the player have? Q_9: How many points per match (average points) has the player's team earned?
Dataset D_1: Statistics from $A_i^{\Diamond}$ on four selected players (chosen from TV2, n.d.)
Works W_k
W_1: Using D_1 to test the model W_2: Using D_1 to test a revised model W_3: Using D_1 on two selected keepers to revise the model
Partial answers l_m
l_1: Answers and evaluation of testing the model with the dataset *l_2: Answers and evaluation of testing the revised model with the dataset* *l_3: Answers and evaluation of testing another revised model with the dataset*

Discussion and conclusions

All five groups based their models on what we denote here as $y_R = \sum_{i=1}^{n} a_{Ri} x_{Ri}$, using between five and eight variables, and applied this to four different player positions, R (i.e., keeper, defender, midfielder, striker). This model provides a low-threshold introduction to multivariable functions. Regarding the notation y_R used for the dependent variable, it is particularly suitable for studying systems (as in this SRP) where there is no need to address the function as a mapping, considering

properties like continuity or operations such as differentiation and integration. In contrast, the notation $f_R(x_{R1},\cdots, x_{Rn})$ would be more effective in cases where such properties and operations are relevant. It should be recognised that the mathematical expression $a_{R1}x_{R1} + \cdots + a_{Rn}x_{Rn}$ (a weighted sum), can also be understood within the domain of linear algebra. In this context, it is interpreted as a linear combination of the variables, where the performance score for each player is represented as the scalar product of a vector of coefficients (weights) and a vector of player statistics. However, this conceptualisation was not introduced in the course since vector algebra had not been part of the secondary school curriculum for these PSTs. The PSTs developed some basic knowledge of multivariable functions through empirical, heuristic, and iterative reasoning, which reflects a practical approach to model fitting. By using trial-and-error methods, guided by intuition and domain-specific football knowledge, they iteratively adjusted the model's parameters to achieve fairness and accuracy across player positions. Although their approach lacked formal mathematical techniques like regression analysis, it effectively engaged the PSTs in hands-on problem-solving. This enabled them to explore the impact of various variables and parameters, deepening their knowledge of multivariable functions and their utility in modelling real-world systems. These findings align with Dorko (2023), who observed that students can develop meaningful knowledge of multivariable functions through practical engagement and intuitive reasoning, even without formal calculus training. This reasoning is crucial at this stage, laying the groundwork for more formal methods. In conclusion, the study suggests that PSTs without a strong calculus background can develop practical knowledge of multivariable functions through modelling, enabling them to create models where these functions serve as accessible tools for studying real-world systems through weighted sums of variables. However, while this approach has proven effective, open questions remain about how integrating SRPs with mathematical modelling can further enhance PSTs' capacities to develop and apply more complex models based on multivariable functions, particularly in scenarios that demand deeper mathematical insights and the handling of more intricate datasets.

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Inconsistencies in teachers' symbolism for the gradient and the gradient function - m and $\frac{dy}{dx}$

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This paper aims to contribute to ongoing research that seeks to understand and improve the teaching of elementary calculus. It draws from a qualitative study investigating how mathematics teachers in English schools introduce elementary differential calculus. The study examines the commognitive constructs in the teachers' pedagogical calculus discourse. Interviews with the teachers and observation of their introductory lessons on the derivative were used to collect qualitative data, and a commognitive thematic discourse analysis was used to deconstruct the teachers' pedagogical calculus discourse on the derivative. This paper reports on some inconsistency and ambiguity with calculus symbolism found in the teachers' pedagogical calculus discourse, particularly in the transition between the gradient (for straight-line graphs) and the gradient function (for curves).

Discourse, commognitive, derivative, calculus, symbolism.

Introduction

This paper presents some findings on calculus symbolism drawn from a doctoral study on the teaching of elementary calculus. It examines the calculus symbolism in the teachers' pedagogical calculus discourse, particularly focusing on the transition from (and between) the gradient (slope) to the gradient function (derivative as a function).

The doctoral study explores how teachers of mathematics introduce differential calculus to pre-university level students (16 to 18 years) in England. As a mathematics teacher educator, the researcher observed that many student teachers' understanding of elementary calculus concepts was often disjointed and limited to algorithmic rules of differentiating functions, which they had difficulty explaining. A preliminary review of the literature revealed that Berry & Nyman (2003), who had also worked with undergraduate and postgraduate students training to become mathematics teachers over several years in the UK and the USA, described their students' understanding of calculus as "a set of loosely connected actions based on a set of algebraic rules that can be applied in restricted, often artificial, algebraic situations" (p. 482). In a study, with 29 teachers, investigating teachers' beliefs towards calculus, Eichler & Erens (2014) report that all the teachers shared a view on calculus as a set of rules for students to be memorised and used in solving routine problems. The researcher's observations about student teachers' difficulties with calculus were, certainly, not isolated.

Students' difficulties with calculus are well-established in the literature. Tall (1992a) gave an extensive summary of the challenges that students encounter in learning calculus. A decade earlier, Orton's (1983) study with 16 to 22-year-olds found that the students lacked adequate intuition or understanding of the derivative concept. More recently, a survey conducted by Park (2013) with 12 elementary calculus university students found that they did not understand the distinction between the *derivative at a point* as a constant and the *derivative* as a function. In an examination of the calculus discourse of experts as reflected in the three most popular calculus textbooks in the US, Park

(2016) also found that nearly identical symbols were used in representations for the derivative at a point and the derivative of a function.

This paper focuses on the following research question: In teaching differential calculus, what visual mediators do teachers use? It discusses the visual mediators, particularly the calculus symbolism in the teachers' pedagogical calculus discourse. It advances knowledge on the researcher's previous PME paper (Tasara, 2018) that examines the word use and narratives (Sfard, 2008) in the teachers' pedagogical calculus discourse, which found some inconsistency in the teachers' pedagogical calculus discourse in their use of calculus words. This CERME paper will show evidence of some inconsistency in the teachers' pedagogical calculus discourse in their use of calculus symbolism.

The study and this paper are informed by the commognitive framework (Sfard, 2008) that views mathematics as a form of discourse, that is, a special type of communication with specific ways of saying and doing (Sfard, 2008). Thus, the study offers an alternative research perspective (to most past studies) on researching teachers and their teaching of elementary differential calculus (mathematics) that does not focus on teacher knowledge but on the teachers' mathematical discourses, herein referred to as the teachers' pedagogical calculus discourse; the amalgam of the teachers' mathematical and didactical discourse on calculus.

According to the commognitive perspective, visual mediators and word use are the tools with which the participants of a discourse "identify the object of their talk and coordinate their communication" (Sfard, 2008, p. 145). Visual mediators could be physical visible objects or symbolic artefacts such as algebraic notation and expressions, tables, graphs, diagrams or drawings that are used to mediate instruction (Ryve et al., 2013; Sfard, 2008). Sfard (2008) further asserts that visual mediators are an integral part of the act of communication in literate mathematics discourse. Indeed, visual mediators are an integral part of the teachers' pedagogical calculus discourse. Sfard (2015) describes symbolic artefacts as referring to numerals, algebraic expressions and graphs, and that they serve as "representations of impalpable mathematics objects" (p. 132).

Methods

Researching the teachers' calculus discourse requires examining their communicative activity, i.e. the teachers' forms of saying and doing in teaching calculus (Sfard, 2008). Therefore, a qualitative approach was adopted, in which interviews with seven different teachers of mathematics and observations of their introductory lessons on the derivative were used to generate qualitative data on the teachers' pedagogical calculus discourse. Although the methodology of my research was informed by the commognitive framework, analysing the huge amounts of qualitative data needed a systematic approach, i.e. thematic analysis (Braun & Clarke, 2006). By focusing on the commognitive constructs of word use, visual mediators, narratives and routines (Sfard, 2008) in the teachers' calculus discourse and following the process of thematic analysis, the study was able to analyse 'how and when' teachers used literate calculus words and calculus symbolism. Braun & Clarke (2006, p. 12) assert that thematic analysis (at the latent level) involves identifying and examining the "underlying ideas, assumptions, and conceptualisations [of the commognitive framework] – and ideologies - that are theorised as shaping or informing the semantic content of the data". Data analysis

followed the process of thematic analysis, in which the coding, defining and naming of overarching themes was informed by the epistemological assumptions and the four constructs of the commognitive framework. Hence, the process of thematic analysis was effectively combined with the theory of commognition to form the *commognitive thematic discourse analysis*, as a creative new application of the commognitive theory and thematic analysis.

Findings

Five out of the seven lessons observed displayed evidence of some inconsistencies and ambiguity with calculus symbolism (and word use) for or between the gradient for a straight line and the gradient function. For the purposes of this paper, exemplar excerpts from the lesson observation (LO) data transcripts, T4LO and T2LO, of two participant teachers, T4 and T2, respectively, are presented and discussed as examples that illustrate evidence of the inconsistencies in the transition from the derivative as a constant to the derivative as a function in introducing the concept of the derivative.

Observations from T4LO

In the lesson T4LO, the teacher used the function $y = x^2$ and a sketch diagram of the graph of $y = x^2$, see Figure 7.1 and Figure 7.2 below. [T4LO]:

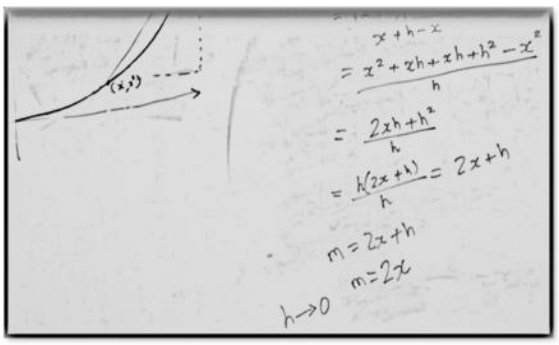

Figure 7.1 The gradient of secant through (x, x^2) and $[(x+h), (x+h)^2]$

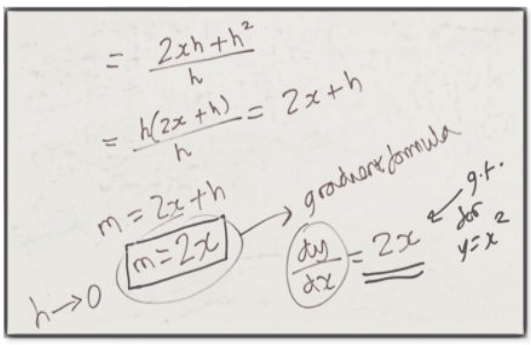

Figure 7.2 From m (gradient) to $\frac{dy}{dx}$ (gradient function)

Figures 7.1 and 7.2 capture the stage when the lesson had moved away from numerical coordinates to general points $(x, x^2)$and $[(x+h), (x+h)^2]$. Before this stage, the lesson had progressed from computing the gradient of a chord, by using two given points, (x_1, y_1) and (x_2, y_2) with specific numerical coordinates on the graph. The teacher used the formula $m = \frac{(y_2-y_1)}{(x_2-x_1)}$ for computing the gradients of secant lines (chords). The figures show the algebraic computation for the gradient of a secant line passing through the general points (x, x^2) and $[(x+h), (x+h)^2]$, which are substituted into the formula $m = \frac{(y_2-y_1)}{(x_2-x_1)}$, i.e. the algebraic manipulations and simplification of the quotient $m = \frac{((x+h)^2-x^2)}{((x+h)-x)}$ culminating into $m = 2x$.

Below is an excerpt from the lesson in which the teacher is explaining Figures 7.1 and 7.2. [T4LO]:

192 Teacher Okay. Time to think again guys. The gradient we found is that the gradient is equal to $2x + h$. Same, same question as last time. What happens as *h* gets

		closer to 0? What happens is, in other words, what happens as this point gets close, close to the original, the original point as *h* gets close to 0? What is the gradient of that line? [*Teacher pointing to the secant on the sketch diagram for the graph of* $y = 2x$]
193	Student	$2x$
194	Teacher	Yeah. $m = 2x$. Okay, because as *h* approaches 0, *h* becomes 0 and we're just left with $m = 2x$. Now that looks a bit weird because we've now got a gradient that isn't just a number, okay? What's gone on there? Why have we, usually in GCSE we get $m = 1$ or $m = 3$? Why have we got a bit of algebra for our gradient?
202	Teacher	That's right because this is a general formula for the gradient. This could tell us about the gradient at any point we want. It's not just *m* = 10. The gradient's not just 10, is it? Because Gxxx said right at the start of the lesson the gradient changes. This is a *general formula for the gradient at any point,* okay? Now we've got a name for that. We call it the gradient, we call this, we call this the *gradient formula*, okay. And we've got a special…we've got some *special notation* for it. Instead of saying $m = 2x$ we write $\frac{dy}{dx} = 2x$ and this is called the *gradient formula*, okay. So $\frac{dy}{dx} = 2x$, that's the gradient, that's the gradient formula for which curve?
203	Student	$y = x^2$
204	Teacher	$y = x^2$. This is always gonna be the gradient formula for the line $y = x^2$, okay, and that's it. That's differentiation, finding the gradient formula.

The symbolic mediator $h \to 0$ in Figure 7.2 depicts that as *h* approaches zero, the gradient $m = 2x$. Note the two signifiers in the teacher's utterance [192], "as *h* gets closer to 0" and "as this point gets close, close to the original", both describe $h \to 0$ (as *h* approaches *0*) without no mention the limit or explanation of the limit definition of the derivative. Based on Figure 7.2 the teacher introduces the derivative of the function $y = x^2$ as "$m = 2x$ is the gradient formula" and "$\frac{dy}{dx} = 2x$ is the gradient function". Here, it can be construed that these two utterances imply that $m = \frac{dy}{dx}$. Also, note that $\frac{dy}{dx}$ is introduced as "some special notation" [202] that is used instead of the m, the latter which is symbolism for slope (gradient). The teacher's utterance here, suggests that $\frac{dy}{dx}$ is the same as m, only that it is a special or different notation for the same object.

What is not done here, could be more important, a substantiation of the difference between $\frac{dy}{dx}$ and m. Consider these two utterances: (i) "We call it the *gradient*, we call this, we call this the *gradient formula*, okay" [202], and (ii) "So $\frac{dy}{dx} = 2x$, that's the *gradient*, that's the *gradient formula* for which curve?" [202]. Note that in both utterances the teacher gives $\frac{dy}{dx}$ two names, i.e. he calls $\frac{dy}{dx}$ *gradien*t and *gradient formula*. See how this is depicted in Figure 7.2. The teacher writes $\frac{dy}{dx} = 2x$ and $m = 2x$, and both are labelled as *gradient function*. The $\frac{dy}{dx}$ calculus symbolism was, thus introduced and subsequently adopted by the class. There were no questions from the students nor further explanation from the teacher about this new symbolism, $\frac{dy}{dx}$.

Observations from T2LO

In contrast to T4LO (and all the other lessons), in which the introduction to differentiation was built on the function y = x^2, T2LO used the function y = $x^2 - x - 6$, instead. Figure 7.3 is a snapshot from the lesson showing the algebraic computation for the gradient of a secant line through the general points $[(x), (x^2 - x - 6)]$ and $[(x + h), (x + h)^2 - (x + h) - 6]$. This was the procedure used to estimate the slope of the tangent to the graph of the function $y = x^2 - x - 6$ at the point $[(x), (x^2 - x - 6)]$. Like in T4LO, the teacher in T2LO used the gradient quotient $\frac{(y_2 - y_1)}{(x_2 - x_1)}$ to compute the gradient estimation for the tangent at $[(x), (x^2 - x - 6)]$. [T2LO]:

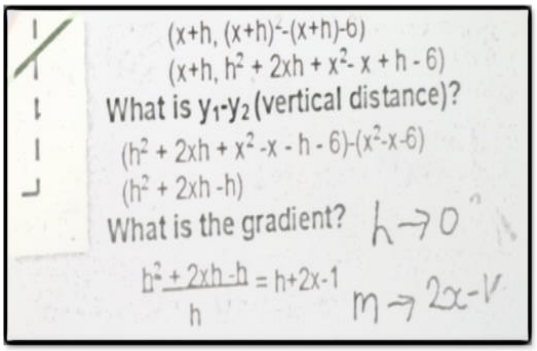

Figure 7.3 The gradient of a secant line

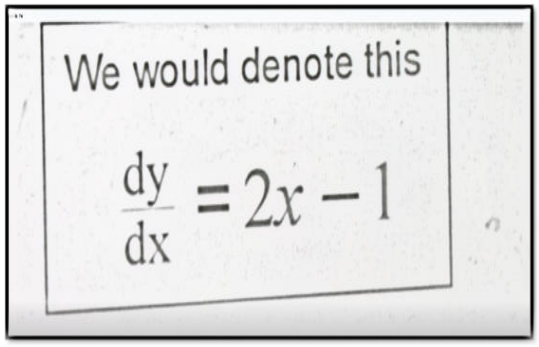

Figure 7.4 The gradient function of $y = x^2 - x - 6$

Below is an excerpt from T2LO about Figures 7.3 and 7.4. [T2LO]:

373	Teacher	We know as h gets smaller and smaller it goes to 0. So, then our gradient function, let's call it m for now, for the sake of *it will go to* $2x - 1$ [*my italics*]. … We found out the gradient function for this [*Teacher pointing to the function*]. … I thought it was nice to finish off with something like this [*Teacher displays Figure 7.4*] because I know you've seen this before. We think about the distances there [*Teacher pointing to the graph of the function*], being like $\frac{dy}{dx}$. Then we denote it like that [*Teacher pointing to Figure 7.4*].
374	Student	So, was the gradient $2x - 1$?
375	Teacher	The gradient function was $2x - 1$, yeah, for any point x.

Here, like in T4LO we see the introduction of new keywords, 'gradient function' and new symbolism, $\frac{dy}{dx}$. However, when a student describes $2x - 1$ as the *gradient* [374], the teacher immediately corrects the student with the endorsed narrative '*gradient function … for any point* x' [375], thus, reinforcing that $\frac{dy}{dx} = 2x - 1$ is the *gradient function* (not gradient, as in T4LO) for $y = x^2 - x - 6$. Like in T4LO, the $\frac{dy}{dx}$ symbolism was not adequately substantiated, apart from what is displayed in Figure 7.4, showing that $\frac{dy}{dx} = 2x - 1$. Neither were there any questions from the students about the meaning of $\frac{dy}{dx}$ symbolism.

Similar to T4LO, the symbolic mediators $h \to 0$ and m are used in T2LO, and m is used to denote the gradient function. Note that unlike in T4LO (where the teacher writes $h \to 0$, $m = 2x$), in

T2LO the teacher does not use the $=$ sign but writes $m \rightarrow 2x - 1$, instead (See Figure 7.3). T2 explains "as h gets smaller and smaller it goes to 0" and the mediator m (gradient function) is described as "… *it will go to* $2x - 1$" [373], which is consistent with the symbolism displayed in Figure 7.3.

Discussion

In teaching differential calculus, algebraic symbolic artefacts are an important aspect of visual mediation, and so are graphical mediators. In line with Sfard's (2008) argument for the multiplicity of visual realisations, and consistent with Tall (1992a) who argues for versatile transitions between representations in the teaching of calculus, the teachers used combinations of, and constant shifts between numerical, algebraic and graphical mediators in substantiating and constructing the definition of the derivative.

The evidence above shows that the teachers used the symbol m to signify gradient functions. The pupils at this stage (post-16) would be familiar with the symbol m as a signifier for gradient (slope). Yes, the parameter m is typically used to denote the slope (gradient) of a straight line (Styers et al., 2020) and it comes from the formula $y = mx + c$ for any linear functions (Stump, 1999). By describing $\frac{dy}{dx}$ simply as "some special notation" [T4LO; 202] that is used instead of the m, it implies that $\frac{dy}{dx}$ and m are mediators signifying the same object. Such use of m to signify the gradient function was common in other lessons too, e.g. in T2LO, m is used to mediate the gradient function, $2x - 1$. Teachers in this study used the parameter m, which is usually used to signify the slope of a straight line, to signify the gradient function. These findings are consistent with Park's (2016) analysis of the calculus discourse of experts on the derivative as reflected in the three most popular calculus textbooks in the US.

> The realisations of both the derivative at a point and the derivative of a function were mediated with nearly identical symbols suggesting a possible difficulty with understanding the difference between them. (Park, 2016, p. 417)

The findings also show evidence of ambiguity and inconsistency with calculus symbolism. The mediator $\frac{dy}{dx}$ was explained simply as "some special notation" in T4LO and "We denote it like that" in T2LO. There is a lack of adequate substantiation of the mediator $\frac{dy}{dx}$, e.g. what it means, where it comes from or whether it is a fraction. Furthermore, the evidence shows inconsistencies between the teachers' substantiation of the mediator $h \rightarrow 0$. As $h \rightarrow 0, m = 2x$ in T4LO, whereas in T2LO we see this: as $h \rightarrow 0, m \rightarrow 2x - 1$. It is important to point out that these two visual symbolic mediators ($=$ and $\rightarrow$) signify different discursive objects and that they convey different meanings. Thus, there is ambiguity here, around the concept of limit for the derivative: Does it go to … or does it equal to …? In literate calculus discourse, as $h \rightarrow 0, m(for\ \frac{dy}{d\ x}) \rightarrow 2x$, the limit. The symbolic artefacts that teachers use matter because they fulfil the role of visual mediators and serve as "representations of impalpable mathematical objects" (Sfard, 2015, p. 132). They (symbolic artefacts) are an integral part of the thinking and communication process in mathematical discourse (Sfard, 2008).

Learning and doing mathematics means becoming capable of participating in the literate discourse (Sfard, 2008). The research found evidence of some didactical ambiguity with some calculus symbolism in the teachers' pedagogical calculus discourse on the derivative. It is vitally important that the calculus symbolism teachers use in introducing the derivative is substantiated explicitly to facilitate students' induction into, and participation in, literate calculus discourse. Besides, past studies have shown that moving from and between the derivative at a given point on a curve, to the derivative as a function is not simple for students (e.g. Park, 2013). This research found evidence of teachers using both m and $\frac{dy}{dx}$ to signify the gradient function, thus, using the same signifier for the derivative at a point and for the derivative as a function. Park (2013) found students also describing 'the derivative as a tangent line', which suggests that the students "considered the 'derivative' as a point-specific object but also a (linear) function defined on an interval" (p. 624). Therefore, how differential calculus symbolism is introduced matters if students are to become capable of participating in literate calculus discourse.

Conclusion

The commognitive framework provided an analytical perspective for examining and explaining how teachers construct and substantiate the notion of a derivative and a derivative function in introducing differential calculus. On the face of it, these findings suggest that ambiguity and inconsistency with calculus symbolism in the teachers' calculus pedagogical discourse could be a factor behind some of the students' difficulties with calculus. The teachers in this study may or may not have difficulty understanding the difference between m and $\frac{dy}{dx}$. However, the evidence suggests that their students could potentially have difficulties understanding the difference between these two symbolic artefacts and their applications. For constructing and substantiating the definition of the derivative, teachers should pay close attention to calculus symbolism and explicitly explain the difference in application between the mediators $\frac{dy}{dx}$ and m. Symbolic mediation brings "generative power" (Sfard, 2008, p. 159) to the discourse and offers a "powerful manipulative ability" (Tall, 1992b, p. 9). However, calculus symbolism can be a source of some difficulties with calculus, for both teachers and students, alike. Nevertheless, teachers should seek to substantiate explicitly the symbolic mediators in their calculus discourse and do so consistently with literate mathematics discourse.

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Understanding and Basic Mental Models of the concept of derivative

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An empirical study with 501 university students of mathematics, mainly at the beginning of their studies, investigated the relation between the understanding—concentrated in basic mental models (BMMs)—of the derivative and the capability of solving derivative problems. The results show that the BMM of tangent slope is of central importance, it is much less for the local rate of change, while the local linearization plays no role. The study emphasizes the meaning of BMMs for successfully solving problems related to the concept of derivative.

Keywords: Basic Mental Model, concept understanding, concept image, derivative, tangent slope, rate of change, local linearity.

The understanding of the concept of the derivative is a central goal of the teaching of calculus. In the following, we refer to Basic Mental Models (BMMs) as a core part of the "concept image" and the framework of Zandieh (2000) concerning this understanding of this concept.

Concept definition – concept image

The relation *concept image – concept definition* has been studied in many different contexts since the early 1980s (Tall a. Vinner, 1981). "Concept definition (is) a form of words used to specify that concept" and "Concept image describes the total cognitive structure that is associated with the concept […]. It is built up over the years through experiences of all kinds, changing as the individual meets new stimuli and matures" (Tall & Vinner, 1981, p. 152).

This relation has been studied für many different concepts, for example, numbers, functions or limits. A problem concerning this framework is that the *concept definition* is very narrow, mainly concentrated on the definitions(s) of the concept, while the *concept image* is a too extensive concept. This calls for a possibility to have a concentrated core of a concept concerning the *concept image*.

Basic Mental Models of the concept of derivative

The concept of Basic Mental Model (BMM, in German: Grundvorstellung) has been well established in German-speaking didactics of mathematics for many years (vom Hofe & Blum, 2016). A BMM of a mathematical concept is a content-related interpretation that gives meaning to this concept, providing relations to meaningful contexts (see Greefrath et al., 2016). BMMs of mathematical concepts can be considered within the theoretical framework of *Concept Image–Concept Definition* (Tall & Vinner, 1981). They are parts or subsets of the *Concept Image* of a mathematical concept. While *Concept Image* refers to all individual mental images identified with the concept, BMMs are the *core* or *central components* of these images. *Normative BMMs* are derived from didactic analysis of mathematical concepts, *individual BMMs* are specific BMMs in a student's mind.

Based on didactical analyses, Greefrath et al. (2023) identified four BMMs of the first derivative of a function at a point (Table 2) and validated them in a comprehensive empirical study.

Table 2: Basic Mental Models of the first derivative of a function at a point (Greefrath et al., 2016)

BMM of the local rate of change (RC)	A varying quantity is considered. The derivative is the local rate of change of this quantity. If the quantity changes with time, the derivative has the meaning of an instantaneous velocity or speed.
BMM of the tangent slope (TS)	The graph of a function in a coordinate system is considered. The derivative is the slope of the tangent to the graph at a point. The development of this BMM is closely related to the development of the concept of a tangent to a graph.
BMM of the local linearity (LL)	If you "zoom in" more and more on the graph of a function at a point of interest, the graph locally appears more and more as a straight line. The derivative is the slope of this straight line. Software for dynamic mathematics with the possibility of zooming in on the screen is particularly useful for developing this BMM.
BMM of the amplification factor (AF)	A quantity y is dependent on another quantity x. The derivative indicates how strongly small changes in the quantity x affect the dependent quantity y. ($\Delta y \approx m \cdot \Delta x$)

Understanding the concept of derivative

BMMs are an important element for understanding and applying a concept. However, understanding a concept is much more than knowing one or some BMMs of the concept. There are many ways to describe the concept of *understanding*. Skemp (1976) distinguishes between instrumental and relational understanding. Another distinction is that in *conceptual knowledge*—about the properties of a concept, relationships between different properties and relationships to other concepts—and *procedural knowledge* – knowing how to solve equations, how to calculate limits or derivatives of functions (Hurrel, 2021). Zandieh (2000) proposed a comprehensive theoretical framework for analysing students' understanding of the concept of derivative. She distinguishes *multiple representations of the derivative* on the one hand, and on the other she considers the *derivative as a ratio, a limit and a function* (Table 1). The framework is developed for school mathematics, however, it might also be used for the university level (Feudel & Biehler, 2021). There are also some follow-up developments concerning this framework (for example, Radmehr & Turgut, 2024)

Table 1: Zandieh's framework for understanding the derivative

	Graphical	**Verbal**	**Symbolic**	**Physical**
Ratio	Slope of the secant line	Average rate of change	Difference quotient	Average velocity
Limit	Slope of the tangent line or slope of the curve under magnification	Instantaneous rate of change	Limit of the difference quotient	Instantaneous velocity
Function	Graph of the derivative function	Instantaneous rate of change as a function	Derivative as a function	Velocity as a function of time

Each column of this framework can be seen as an element of the concept image or/and of a BMM of the concept of derivative. We are interested in the understanding of the concept of derivative in a point of a function and in the relation between the function and its derivative function. This means, we refer to the limit and the function row of Table 1. The BMMs of the derivative are a part of the concept image and specify it in the sense of a core image or core perception.

Research questions

Our hypothesis is that BMMs are helpful–and maybe essential–for solving problems. In the following, firstly, we are interested how students solve mathematical derivative problems. Secondly, we want to quantify the relation between students' capability to solving problems and their BMMs. We are interested in the following research questions:

1. To what extend are students able to solve derivative problems?
2. What is the relation between students' capability to solve derivative problems and their BMMs?

Methods

A test with two groups of problems was developed. In the capability–test, mathematics students had to answer questions concerning the concept of derivative. In the following BMM–test, the students were classified with regard to their individual BMMs. The test is designed for university students of mathematics at the beginning of their teacher training courses or undergraduate courses in scientific disciplines like computer science, physics, chemistry or biology. A total of 501 students took part in the study. They were recruited in mathematics preparatory courses of teacher training programs (all school types) at four different German universities.

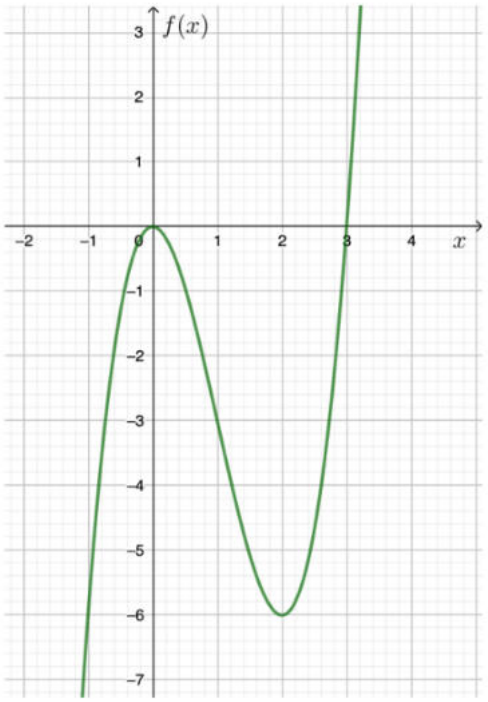
Problem Group 1: Capability–Test

Seven tasks were developed that relate to the first derivative of a real function at a specific point or in an interval of the domain of definition or to the entire derivative function. According to the classification scheme by Zandieh (2000) (cf. Table 1), the tasks refer to different representations – *Graphical*: relationship between graph and first derivative; *Verbal*: verbal descriptions of properties of the derivative; *Symbolic*: interpretation of a statement on the symbolic level; *Physical*: interpretation of an environmental situation.

Two types of tasks were used. For type 1 (tasks 1 to 4), three suggested solutions were given for each task, for which the student had to decide whether the statement given in each case was correct or incorrect.

Problem 1.1

The adjacent figure shows the graph of a real polynomial function f of third degree. Mark whether the following statements regarding the derivative function f' are true or false.



	true	false
All the values of the function f' are negative in the interval]0; 2[.		
The function f' is strictly monotonically increasing in the interval]0; 2[.		
The function f' has a local minimum at the position $x = 0$.		

Problem 1.2

The function values $f(t)$ give the outside temperature measured at time t. Mark which interpretation of the expression

$$\lim_{t \to t_o} \frac{f(t) - f(t_o)}{t - t_o}$$

is correct or incorrect.

	true	false
The expression indicates how the temperature changes between t_o and t.		
The expression indicates how the temperature changes at the time t_o.		
The expression indicates how high the temperature is at the time t_o.		

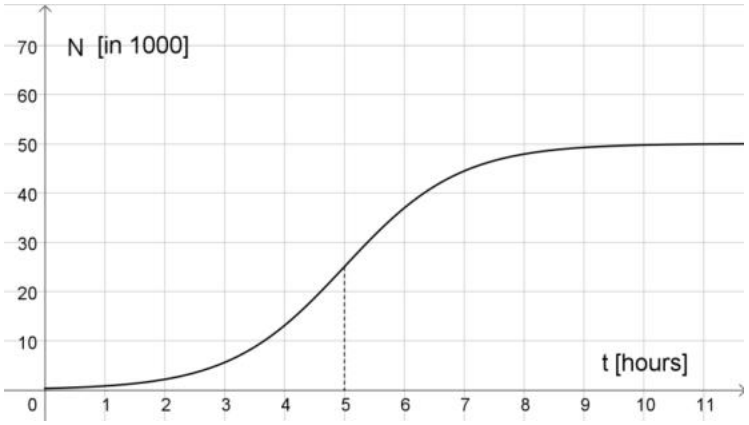
To reduce the guessing probability, a problem is only considered as correct if all subtasks are correct, otherwise, it is wrong (guessing probability 12.5 %).

Problem Group 2: BMM-Test

In Greefrath et al. (2021, 2023), a test was developed and tested to assess BMMs for university students of mathematics. The basic idea of this test is to present the students problems with argumentations that suggest certain BMMs. For each item, students are asked to mark on a five-point Likert scale from (– –) to (+ +) to indicate the extent to which the answer corresponds to their own thinking. The complete test consists of 13 tasks, eight items were selected for the test used here. Moreover, we skipped the BMM AF because it is not yet emphasized in high school and is useful only for quite limited kinds of problems. Students should be able to work on tasks with different BMMs and looking at problems from different perspectives. Therefore, test contexts were chosen that are conceivable with all BMMs. An example of the BMM–test is the following item:

Problem 2.1

The figure below shows the number of bacteria N in a nutrient solution as a function of the time t (in hours).



The special meaning of the time $t = 5$ can be explained in different ways.

	– –	–	o	+	+ +
If you zoom in on the graph, it appears like a straight line at every point. This has the greatest gradient for $t = 5$.					
The slope of the tangent to the graph is greatest at $t = 5$.					
The number of bacteria changes most rapidly at the time $t = 5$.					

Evaluation method

Homogeneity was checked with Cronbach's alpha. Correlations of sum scores were measured with Pearson's correlation coefficient. Linear regression was used in the analysis of dependencies. All calculations were carried out with the statistical program R.

Results

Capability of the Derivative

The mean values and standard deviations of the capability-test are shown in Table 3. Cronbach's α for problem 2 to 4 indicate good homogeneity, whereas these for problem 1 are more heterogeneous. For problems 5 to 7 exactly one answer is correct, determining Cronbach's alpha is not possible.

Tab. 3: Means and standard deviations of the capability–test

	Mean value	**Standard deviation**	**Cronbach α**
Task 1	0.24	0.43	0.26
Task 2	0.61	0.49	0.62
Task 3	0.28	0.45	0.60
Task 4	0.61	0.49	0.69
Task 5	0.62	0.49	–

Task 6	0.66	0.48	–
Task 7	0.84	0.37	–
Average	0.55	0.24	0.62

Overall, the capability to solve problems is not unidimensional. The value for Cronbach's alpha is 0.62. Especially the problems with a given graph (2, 4, 5, 7) are solved better on average than the others.

Basic Mental Models

Table 4 shows the mean values and standard deviations as well as Cronbach's alpha of the sum scores for TS, RC and LL. We confirmed the results in Greefrath et al. (2023), that a three-dimensional model for the BMMs fits quite well.

Table 4: Mean and standard deviations for the TS, RC and LL

	TS	RC	LL
Mean value	0.81	0.62	0.57
Standard deviation	0.20	0.21	0.22
Cronbach's alpha	0.86	0.79	0.82

Relation between Problem-solving capability and BMMs

All correlations – except the correlation of capability and LL – are (even highly) significantly different from 0. This shows that TS and RC are associated with capability of solving derivative problems, whereas it is weak for LL (see Table 5).

Table 5: Correlation matrix

	Problem-solving capability	TS	RC	LL
Problem-solving capability	1.00	0.32	0.17	0.05
TS	0.32**	1.00	0.29**	0.30**
RC	0.17**	0.29**	1.00	0.31**
LL	0.05	0.30**	0.31**	1.00

Discussion

The Capability-test covers many aspects of the two higher levels Limit and Function of Zandieh's (2000) framework, as well as all graphical, verbal, symbolic and physical representations. Furthermore, the test shows different levels of difficulty – mean values of 0.24 to 0.84 – with the items in the first column of the framework having higher solution rates than the items without graphical representation. In the test, more procedural aspects of knowledge such as understanding the derivation rules were not taken into account.

In the students' BMMs, the tangent slope is most prominent, while rate of change and local linearization occur much less frequently. Particularly in view of the stronger emphasis on modelling skills in school analysis (in curricula and textbooks, less in actual teaching), one might have expected a stronger emphasis on the local rate of change. The basic idea of local linearity plays a rather subordinate role. Presumably, typical learning processes in mathematics lessons play a central role here, in which the tangent concept is supported above all by the predominance of intra-mathematical problems. The repeatedly demanded view of the derivative as a local rate of change is probably not yet sufficiently developed due to too few modelling examples.

The ranking of the basic concepts tangent slope, local rate of change and local linearization does not change if only students with high or low problem-solving capabilities are considered. In the sub-group of students with lower problem solving abilities, the preference for the BMM tangent slope is much lower than in the high achieving group. but the other BMMs are even lower. Hence, even in this subgroup the BMM tangent slope is the best predictor of at least limited success. However, it can be assumed that BMMs are useful for increasing the capability to solve derivative problems and consequently the understanding of the concept of derivative (according to vom Hofe and Blum, 2016).

Furthermore, the results of a linear regression show that the tangent slope can predict the ability to solve derivative problems with a relatively high weight (again, especially for the items where a graph is given), this is the case for the rate of change to a lesser but also significant extent, while the concept of local linearization does not play any role.

Limitations

In terms of subject selection, this test clearly selects students by limiting them to mathematics students. However, this at least allows the assumption that the capability test would be worse with a non-subject-specific test group, for example, students from social studies.

Furthermore, it must be taken into account that the capability-test only consists of 7 questions and the BMM-test is based on self-reporting by the students. Developing a test which determines the "actual BMMs", is a research problem for the future.

Conclusion

The result of this study shows that the BMM of tangent slope is of highest priority for the solving of derivative problems. This justifies the special importance of the graphical representation, which is also prominently emphasized in Zandieh's framework. The rather low correlations between BMMs and the problem-solving capabilities shows that these are different categories, but they are not independent, as highly significant correlations exist. It still needs to be clarified what significance the different BMMs have for certain content areas and problems. This can provide indications how and when BMMs should be developed in school mathematics. Bednorz et al. (2023) have proposed a learning path for this purpose. Based on the results of this study, it can be assumed that the BMMs RC and LL are more difficult to teach and require well thought-out special strategies. However, the local rate of change is important in the school context, especially in relation to application tasks, and local linearization is important on university mathematics level. This means that the development of

all BMMs is important for the broad and global understanding of a concept, as they show different perspectives and offer different ways of using and working with the concept.

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TWG 26: Mathematics in the Context of STEM Education

Introduction to the papers and posters of TWG26: 'Mathematics in the context of STEM education'

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TWG26 met for the fifth time in Bolzano, at CERME14 to continue the work started at CERME11. Since our inception in 2019, our research interests have shifted from characterizing the nature of mathematical modeling, teacher preparation, and rigorous research methodologies in STEM to creating frameworks for interdisciplinary teaching, teacher and student STEM competencies, and STEM assessment. At CERME14, 16 papers and six posters were submitted. Among them 11 papers and four posters were presented, involving authors from nine different nationalities (Denmark, Germany, Italy, Greece, Portugal, Spain, Austria, Turkey, and USA). The papers and posters were grouped under four themes, (i) *The Nature of STEM* (two papers), (ii) *Student Learning in STEM* (four papers), (iii) *Working with Teachers in STEM* (three papers, four posters), and (iv) *AI in STEM* (two papers). Most of the presented research was about the role of mathematics in inter- and transdisciplinary STEM teaching and learning environments. Only one paper included Arts (STEAM), yet most of the posters did. The ideas and issues in the papers and posters will be presented under these four themes.

The Nature of STEM

The first paper by Spreitzer et al. investigated the presence and nature of mathematical activities within integrated STEM lessons as part of a nationwide school experiment in Austrian lower secondary education. Semi-structured interviews were conducted with 11 experienced teachers to examine what mathematics is integrated into the activities, how students respond to them, and which activities are considered unsuitable for STEM lessons. Findings indicate that activities such as measurement, construction, and modelling are frequently integrated, while practices like reasoning and justification are less common and perceived as challenging to implement. Mathematics was predominantly viewed as a means to an end rather than a learning objective in itself. Nonetheless, some teachers actively sought to broaden the role of mathematics showing the potential for mathematics in STEM.

The second study by Larsen and Svendsen focused on interdisciplinary STEM assessment, as part of the LabSTEM project in Danish secondary education. Drawing on the Anthropological Theory of the Didactic (ATD), the researchers analyzed student presentations to determine how mathematical tasks, techniques, and theoretical justifications were incorporated. The findings revealed that while students frequently applied mathematical techniques such as regression analysis and modelling using GeoGebra, theoretical components were often omitted or insufficiently explained. Integration of mathematics with scientific disciplines varied across groups, with some demonstrating coherent connections and others struggling to relate mathematical models to biological or chemical processes. The study highlights the need for clearer guidance in STEM assessment design to ensure that both practical and theoretical mathematical knowledge is addressed.

Student Learning in STEM

There were three papers presented that involved research associated with student learning. The papers had very little in common regarding the STEM topic being explored but all studies investigated student reasoning. Altindas et al. conducted a study that explored how university *life science* majors reason about graphs representing a STEM context. Researchers conducted semi-structured interviews with 16 students to learn how they reason about the graph of a logarithmic function depicting a titration curve. Students were asked to describe how the PH changes as a base solution is added at a constant rate. Researchers analyzed how students "chunked" the graph into particular sections and what those chunks revealed about the nature of the PH change. They found that students' reasoning with the chunked graph varied from surface level mathematics and science descriptions (e.g., describing the shape of the graph as increasing) to deep, intrinsic use (e.g., explaining how the PH changes at the inflection point of the logarithmic function). They concluded by suggesting that where students anchor their experiences (mathematics or science) may influence how they anchor their reasoning, on the mathematics or scientific characteristics of the titration graph.

Erbasan and Boston conducted a single case study of a 5th-grade student's computational thinking skills while participating in a STEM activity called the *Moonbase*, developed by LEGO®. Computational thinking has been identified as essential reasoning of all STEM disciplines. While no shared definition exists, they defined the main components of computational thinking as 1) decomposition, 2) pattern recognition, 3) algorithmic thinking, 4) abstraction, 5) generalization, 6) debugging, and 7) problem solving. The authors examined the components of computational thinking that emerged as the student built a robot to use hypothetically on the base of the moon and program it to move about the surface collecting "modules" to make it easier for astronauts to complete their tasks. The researchers found that the student engaged in debugging, decomposition, and pattern recognition, showing that this particular STEM activity has the potential to support some of the components of computational thinking.

Finally, Klaas et al. examined whether high school students' reason mathematically as they try to solve complex real-world problems constructed by employees at a local business, as well as to identify effective means of supporting their problem solving. Students selected from a palette of problems posed by workers at the business and are assigned a university student mentor and an employee to provide support. One of the most critical stages of the two-year challenge is the problem-solving stage in which the mentors provide students with an integrated problem-solving model (e.g., problem solving from chemistry and computer science). The researchers conjectured that an integrated problem-solving model, as opposed to mathematical problem-solving alone, may support students to create less superficial solutions to the authentic STEM challenges. In this paper, they compared problem solving in mathematics and computer science with chemistry to determine overlapping components that may be useful for students in the future. They found aspects of Polya's problem solving approach within each of the other STEM disciplines such as "devising a plan" but they are implemented slightly differently (e.g., in mathematics devising a plan, in computer science devising an algorithm). Nevertheless, the authors found enough overlap that they will continue to explore comparisons with other STEM disciplines so that they can implement this more general STEM problem solving approach with students.

Working with Teachers in STEM

This section included four papers: two studies focused on professional development through collaboration among STE(A)M teachers, one study on supporting teachers in primary education, and one study on teacher competences in STEM education. Additionally, there were four posters that fell into this category and were presented during our mini-poster session on the third day of the TWG.

The study by Adesso et al. investigated how STEAM teachers, working in teams, approached the solution of a real-world problem in different ways. As part of an Italian summer school training, 45 STEAM teachers engaged in activities such as building an electric network on a breadboard, leading to the discovery of the Fibonacci sequence. The researchers used the concept of disciplinary borders to explore interdisciplinarity and Habermas's construct of rationality to understand communication across disciplines. Data were collected through video recordings of group discussions and final interviews. The results revealed how teachers navigated disciplinary boundaries through epistemic, teleological, and communicative rationalities, with dialogue playing a key role in mediating and transforming understanding. The study highlights how interdisciplinary collaboration fosters professional development by enabling teachers to integrate diverse perspectives and co-construct knowledge at the borders of their disciplines.

The study by Bogiatzi et al. was part of an international programme and focused on professional development sessions involving Greek teachers (six mathematics and two science teachers). The aim was to explore the meanings teachers attribute to inquiry in STEM and how these evolve through collaborative dialogue. Using grounded theory, the researchers analyzed transcripts from teacher meetings to identify emerging meanings of inquiry, supported by theoretical constructs of dialogical moves and argumentation. The findings revealed diverse epistemological and didactical interpretations of inquiry, including tensions between mathematical and scientific inquiry, challenges in integrating digital tools, and concerns about student engagement. While some teachers prioritized disciplinary goals, others attempted integration, often facing practical constraints. The study underscores the complexity of implementing integrated inquiry-based STEM teaching and the value of structured dialogue in supporting teacher learning.

The study by Bennet and Rutchi explored how primary school teachers interpret and value a framework for integrated STEM education, focusing on four elaborated methods of integrating mathematics into science instruction. Conducted in a single school district in western United States, the mixed methods case study involved 24 primary teachers who completed a survey assessing their understanding, comfort, and perceived importance of each integration method. While most teachers recognized the value of integrated STEM and expressed confidence in applying three of the four elaborations, the construct "Accentuate Interdisciplinary Constructs" (AIC) revealed a notable discrepancy. Despite acknowledging its relevance for student learning, teachers rated AIC as least important for classroom integration, suggesting a gap between theoretical appreciation and practical implementation. The findings highlight how limited content knowledge, rigid lesson structures, and beliefs about student readiness may hinder deeper interdisciplinary teaching. This study underscores the need for targeted professional development that addresses both pedagogical and epistemological challenges in STEM integration, particularly in primary education contexts.

The fourth study by Stoffels investigated how mathematics teachers engaged in the MINTco@NRW project perceive and prioritize key competency areas required for mentoring students in authentic, interdisciplinary STEM learning environments. Using a multiple case study approach, six teachers from North Rhine-Westphalia participated in semi-structured interviews, reflecting on a competency model developed to support their mentoring roles. Teachers were asked to rank eight competency areas and discuss their relevance both within the project and in regular classroom teaching. Findings reveal considerable variation in how teachers prioritized competencies such as communication, project-based learning, and digital competence, with no clear consensus across cases. While most teachers found the model adequate, the ranking process highlighted tensions between theoretical importance and practical application. Teachers also noted that several competencies, particularly authentic problem-solving and digital literacy, are transferable to everyday teaching, though implementation is often constrained by institutional factors. The study underscores the complexity of teacher competencies in integrated STEM contexts and suggests that professional development should be tailored to individual needs and teaching environments.

All four posters were in this category as well. Berciano et al. shared a qualitative case study that explored the presence of mathematical process standards in three STEAM projects implemented in Basque secondary schools. The researchers analyzed teacher interviews and project activities to identify how mathematical processes—such as reasoning, representation, and communication—were integrated. While teachers did not explicitly prioritize mathematical processes during project design, several emerged organically through student activities. The findings suggest that mathematics often plays an instrumental role in STEAM contexts and highlights the need for more intentional integration of mathematical competencies in interdisciplinary project planning.

The poster by Costa et al. outlined the development of a STEAM lesson plan focused on water conservation and Sustainable Development Goals, created within the Erasmus+ ProSTEAM project. Using design research methodology, the plan was iteratively developed and tested across five cycles involving university experts and primary school teachers from multiple European countries. The plan included hands-on mathematical tasks to quantify water usage and promote sustainable habits. Teachers who implemented the plan in professional development programmes in Portugal and Lithuania reported high relevance to curricula and strong student engagement, supporting the value of integrated STEAM approaches in primary education.

The poster by Ortiz-Laso et al presented a study which explored how pre-service mathematics teachers identify school-relevant mathematical content in a real-world carpentry video sourced from Instagram. Fifteen Spanish pre-service teachers analyzed a video demonstrating how to divide a wooden plank into three equal parts using a geometric technique. While most participants correctly described the carpenter's method, only half recognized Thales' theorem as the underlying mathematical principle. The study highlights a gap between disciplinary knowledge and the ability to connect it with practical applications. The authors argued for the inclusion of real-world tasks, such as social media content, in teacher education to strengthen contextual understanding and support the implementation of integrated STEM education.

The last poster by Silvestri presented findings from the Erasmus+ STEAM-Connect project, which involved secondary mathematics teachers from five European countries in designing interdisciplinary STEAM activities. The study focused on the role of horizontal (teacher–teacher) and vertical (teacher–researcher) connections in shaping the design process. Through interviews and qualitative analysis, the authors found that teachers who engaged in both types of collaboration demonstrated greater evolution in their understanding and implementation of interdisciplinarity. Representations of interdisciplinarity ranged from metaphorical (e.g., a cooking pot) to structural (e.g., concept maps). The findings suggest that sustained collaboration fosters meta-disciplinary awareness and supports the development of meaningful STEAM practices.

AI in STEM

There were two papers on this theme presented. Both research projects involved exploring issues around the use of artificial intelligence (AI) and mathematics. Fock et al. argued that mathematics instruction should emphasize STEM contexts with a focus on education for sustainable development (ESD). STEM challenges require problems that promote high cognitive engagement and utilize meaningful contexts, yet most problems found in the literature are procedural and therefore must be altered. ChatGPT seems to be a potential alternative source for creating STEM and ESD tasks, so the authors designed a study to understand how mathematics pre-service teachers modified STEM problems generated by ChatGPT. They found that when PSTs intentionally modified a ChatGPT-generated STEM task, the cognitive demand indeed was higher. However, when PSTs did not intentionally modify the task, the cognitive demand only slightly raised in quality, showing that PSTs may still need to learn how to assess tasks' cognitive demand.

The second paper presented by Stenkilde et al. explored how research bias, particularly gender bias, can be incorporated into mathematics instruction for lower secondary grades. In particular, they attempt to address a pervasive STEM challenge: how to integrate STEM ideas (AI in this case) into Danish mathematics classrooms in such a way that teachers adhere to the mathematics standards they are obligated to teach. To explore such possibilities, they analyzed the content and interviewed the author of one particular research article that explains how to model algorithmic bias, in this case, modeling gender bias in ChatGPT's algorithm. Using the KOM-framework, they investigated the modeling competencies involved in determining algorithmic bias and found that these modeling competencies could indeed be replicated in lower secondary classes, albeit in less sophisticated forms (e.g., since students are not familiar with Python, they would need to calculate gender bias manually). Consequently, Stenkilde et al. concluded that it is possible to integrate AI and computational thinking through modeling competencies in lower secondary schools without taking time away from teaching mathematics standards.

Conclusions

We end this introduction by reflecting on the conversations generated by the paper and poster presentations, sharing the new insights and questions that arose as well as next steps of the Thematic Working Group. First, we articulated the ways in which research in the STEM TWG is distinct from other research at CERME. While science, engineering, and technology serve as contexts for mathematics problems, STEM education research focuses mainly on the *integration* across these four

disciplines. Special attention is given to the role of mathematics in this integration, whether it is merely instrumentally used in the service of the other disciplines or if it can also be developed as the primary content. A focus on the integration or interdisciplinary nature of STEM necessitates collaboration and coordination of practices across disciplines. Second, teaching and learning STEM ideas is new territory for researchers, teachers and students alike. There is very little STEM research to guide teachers who are already charged with designing, implementing and assessing STEM lessons and competencies and no consensus on the best way to design or define STEM learning. STEM researchers have some work to do; creating frameworks for designing STEM lessons, outlining STEM competencies for students and teachers, creating analytic tools to analyse students' STEM learning, determining sound methods for assessing those STEM competencies and providing research-based professional development for teachers is essential.

STEAM Teachers in TEAM shape their professional development at the border of disciplines

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This paper presents the idea of the border as a semiotic cross-cultural space to characterize interdisciplinarity in mathematics education exploring the intersections between mathematics and other disciplines, fostering meaningful communication between them. Using artifacts such as springs and networks we have investigated how STEAM teachers, working in team, approach in different manners the solution of a real problem discovering the Fibonacci sequence. We show how is it possible to interpret interdisciplinarity as a comparison between rationalities. Our arguments are framed by De Luca Picione and Valsiner's idea of the border as a metaphor and Habermas's construct of rationality. Results reveal the way the connections between the disciplines emerge and develop at the border.

Keywords: Interdisciplinarity, STEAM Education, border, Fibonacci, semiotic.

Introduction and Rationale

This paper presents the idea of the border as a semiotic cross-cultural space (Capone & Faggiano, 2024) to improve the STEAM teacher's professional development exploring the intersections between Science, Technology, Engineering, and Mathematics, mainly focusing on Mathematics and Physics, fostering meaningful communication between them. The starting point of this study is the premise that interdisciplinary teaching enables educators to integrate the knowledge and perspectives of various disciplines, thereby addressing the challenges of a complex society more effectively (Morin, 2001). However, it is common to argue that mathematics relies on the hypothetical-deductive method, while physics uses the experimental-inductive method, as if the two approaches to knowledge were mutually exclusive. In reality, this sharp distinction has been questioned by Jahnke (1978) and Winter (1983), who argued that the division between 'intuitive' and 'deductive' reasoning is flawed. They asserted that mathematical proof should not be viewed as a departure from observation and measurement but rather as a tool that guides the intelligent exploration of phenomena. Hanna and Jahnke (2002) examined the relationship between physics and mathematical proof by using concepts from physics to model a mathematical problem. Moreover, in recent decades, the role of technology has become essential for improving the teaching and learning of Mathematics and Physics (Artigue, 2010). After all, it is increasingly common to hear that an approach to teaching mathematics through engineering techniques enhances students' skills and stimulates interest in the subject (Pepin et al., 2021). Finally, art can often serve as an engaging tool for effective lessons. To support the previous assertions, an experiment was conducted with approximately 45 Italian STEAM teachers: an interdisciplinary training including Mathematics, Physics, Technology, Engineering, Art, was organized (Capone, 2022). Teachers were involved in the discovering of a Mathematical object,

starting from/including STEAM (mainly Physics labs activities) experiments. The involved teachers have different scientific backgrounds (they are physicists, Mathematicians, Engineers), and each one has its own teaching experience, in different contexts. In what sense is the proposed activity a STEAM activity? This interdisciplinary activity involves Science (S) for the concepts, computation and measurements of resistance, springs, etc.; Technology (T) for the use of the breadboard and the simulation software; Engineering (E) for the hands-on activities of constructing the springs system and the electrical networks using the breadboard; Art (A) is involved for all the connections with golden ratio in art; Mathematics (M) is, of course, used for studying the relationships between physical quantities and the discovery of the golden ratio.

The idea of interdisciplinarity and the metaphor of the border

Referring to Klein's (1990) definition of interdisciplinarity, this text explores disciplinary borders as dynamic, semiotic tools that shape human psychological development. Viewed through a psychological lens, borders facilitate the construction of meaning via classification, categorization, and interpretation, influencing human cognition and behavior. In this semiotic context, borders create a symbolic field where meaning is organized and transformed through cultural processes. De Luca Picione and Valsiner (2022) introduced the Generalized Semiotic Functions of Borders model, which operates along eight dynamic tension continuums. These continuums, with opposing forces, co-define and interact, reflecting the complexity and evolving nature of borders in shaping meaning and development: 1) Distinction: within a homogeneous and undifferentiated field, the border makes a possible difference, thus producing singularities (De Luca Picione & Valsiner, 2017). 2) Differentiation: the border helps to build the sense of identity starting from the difference with otherness. 3) Separation: the border acts ambivalently because it serves to connect and to create a bond. In a homogeneous and undifferentiated field, it is only possible to link something through a preliminary operation of separation. 4) Containment: the border allows for the creation of a frame within which it is possible to give sense and value to experience. This function makes it clear that borders are semiotic operators that generate meaning. 5) Protection: the border always acts as a function of protection, realizing a filter function from environmental/social perturbations. 6) Mediation: the border acts as a structure of mediation against the immediacy of action and reactive forms. 7) Transformation: the border acts as a function of connection, exchange, information, filtering and shock absorption. Borders are liminal zones of transformations, which make it possible to temporarily suspend too rigid distinctions and classifications, allowing fusion, merging, mixing and chaos. 8) Regulation: the operation of regulation makes possible to orient a system and its actions towards the future. In mathematics education, interdisciplinary collaboration relies on borders as semiotic spaces that both separate and connect disciplines. Borders highlight differences between disciplinary cultures while enabling their interaction. To fully understand interdisciplinarity, borders must be enriched with pragmatic and semiotic aspects, framing mathematics education as a mediation between diverse rationalities.

Habermas's construct of rationality

How could we understand and interpret the way the communication between disciplines can take place and how context contributes to construct meaning? In this attempt, we refer to pragmatic

linguistics and, in particular, to Habermas' Communicative rationality. According to Habermas (1981), the adjective rational can be attributed to a person who carries out a discursive activity intending to account for his choices according to criteria of validity. Rationality consists of three interrelated components: epistemic rationality (an opinion is rational when it is possible to justify it in a given context.), teleological rationality (one acts rationally when one acts according to a purpose by implementing intentionally chosen means), and communicative rationality (you aim to make the interlocutor share your communication content by choosing effective means of communication). The purpose of showing the same activity according to different rationalities is to create an understanding between the disciplines. Comparing different points of view leads to affirming all points of view as acts of truth through communicative interaction. Communication is a social action oriented towards understanding, sharing meanings concerning the world between the speakers involved, and sharing a lifeworld. In this paper, the Generalized Semiotic Functions of Borders model is used as a theoretical lens to characterize the border between disciplines. Habermas's construct of rationality is used to understand how interdisciplinarity is achieved through the communicative act that allows us to accept the other's point of view, which becomes part of our knowledge. We posed the following research question: how can the construct of rationality (adapted from linguistic pragmatics) give us a key to understanding educational activities crossing the border between STEAM disciplines?

Participants and Methods

To answer our research question, we have designed and implemented an interdisciplinary teaching activity (Adesso et al., 2024) based on the following principles: allow teachers' work in a fixed disciplinary contest; foster the crossing of disciplinary borders; promote debate between teachers, in circle time and after in plenary, that can be meaningful to connect the different activities and build up to recognize the other's point of view as a possible alternative to one's own point of view and not in opposition to it; recognize that there are different ways to the solution of a problem. The experiment was conducted during a summer training school for teachers, in which 45 participated in an interdisciplinary workshop, titled: 'From the Physical Phenomenon to the Mathematical Model or Vice Versa? Springs, Electrical Networks, and Fibonacci.' The workshop was led by one mathematics education researcher and two physics education researchers. The whole activity lasted four hours in a one-day meeting. The teachers were divided into three groups of seven (Groups 1,2,3) and three groups of eight (Groups 4,5,6), each including mathematicians, physicians, engineers, biologists, etc. The various steps of the lab activities were interspersed with group discussion moments. The research methodology is qualitative conducted with observational-interpretive techniques. The instruments used were the recording of the dialogues of the teachers engaged in the various working groups and a final brainstorming through an interview in which each group reported on their experience. The interview was videotaped. The video and pictures have been selected and coded, so that each group of multimedia files in each step of the activities is associated with a specific semiotic function of borders. Moreover, the dialogues have been organized in order to evidence the three rationalities. Specifically, in the final report each group summarized the role of each teacher in the group itself and how rationality evolved from epistemic to communicative ones. During the observation, the researchers tried to glean what elements emerged during the group activity according to the categorization of boundary processes (Valsiner) and rationales (Habermas). At the end of the

activities, there was a sharing session between the groups: a representative from each group communicated how the work evolved. We provided the teachers with a problem situation, along with worksheets, and each group could organize the activity according to the epistemology they felt closest to.

Activities

The activities have been divided into several steps. Teachers were involved in the equivalent resistance calculation of an infinite two-dimensional electrical network. In this step, they built a system of resistances using a breadboard, electrical resistances of the same value, jumps, and an ohmmeter.

Figure 1: Teachers build an infinitive network of resistance using a breadboard

The teachers built a network using a breadboard in such a way that two resistors in series form an elementary cell, to which resistors are alternately added in parallel and then in series. As they added the resistors, they measured and calculated the equivalent resistance and record the results in a table.

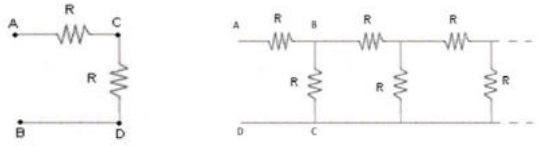

Figure 2: Single cell of an infinitive network of resistance

This experimental activity corresponds to the theoretical framework described below. As shown in figure 2, starting from a single cell of equivalent resistance 2R, adding to this a second elementary cell, we obtain an equivalent resistance equal to 5/3R. Adding cells to the starting one, we obtain the series of equivalent resistances between A and B, for each new added cell: $\frac{13}{8}R; \frac{34}{21}R;\ \frac{89}{55}R;$ ... If we now consider the succession consisting of the numerator and denominator elements of these fractions, the numerator occupies the even place and the denominator the odd place: 1,2,3,5,8,13,21,34,55,89,... which we recognize as a Fibonacci succession whose generic term is $a_{2n+1} = a_{2n} + a_{2n-1}$

having denoted n as the number of cells. The equivalent resistance of a network with n cells will be $R_{eq} = \frac{a_{2n}}{a_{2n-1}}R$. For n $\rightarrow \infty$ this ratio tends to the golden number 1.618..., therefore, the equivalent resistance of an infinite one-dimensional network of resistors is: $R_{eq} = 1{,}618\ldots . R$. During the second step, teachers cross from the practice by using electric resistances to that by using the elastic springs, i.e. from the electricity to the mechanics. We still work within the Physics, but at the border

between two of its areas.

Figure 3: Some cells of an infinitive network of springs

Parallel resistor connection corresponds to a series of springs and vice versa, so it is possible to compute the equivalent elasticity constant of a cell of springs (see Fig.3), thus obtaining the following terms: $2K, \frac{5}{3}K, \frac{13}{8}K$ ...

The division between the generic term a_{2n} and the previous term a_{2n-1} is: $K_{eq} = \frac{a_{2n}}{a_{2n-1}} K$. For n $\rightarrow$ ∞ this ratio tends to the golden number 1.618… The teachers built a system of springs with the geometry in Fig.3, using springs with the same spring constant, a dynamometer, and some mechanical rods. Also in this system, the teachers experimentally re-discovered the Fibonacci sequence, thus approximating the gold number, considering a system of springs analog to the system of resistance. At the conclusion of the activity, the connections between the golden number and some artworks from ancient Greece to the Renaissance were highlighted, so as to show students in the classroom an interdisciplinary activity in which, starting with experiments in the Physics lab (S) and using technology (T) to build appropriate artifacts (E), they could discover the presence of Mathematics in art (A) as well.

Discussion and results

The exploration of interdisciplinary collaboration among STEAM teachers in this study sheds light on how the construct of rationality (adapted from linguistic pragmatics) can give us a key to understanding educational activities crossing the border among disciplines. The three rationalities identified by Habermas: epistemic, teleological, and communicative emerged during the activities. Epistemic Rationality was evident when teachers engaged in discussions centered around the nature of knowledge within their respective disciplines. For instance, in the various groups, we noticed how some teachers approached the problem by first solving it numerically. Others approached it by neatly arranging the resistors on the breadboard in series and parallel, measuring the equivalent resistance step by step. In doing so, they worked by applying the specific epistemology and methodological approach of a particular discipline.

Here we show protocols about the video-recorded summary dialogue, held by one teacher per group (named by the number of the group to which they belong).

Teacher 1: In our group, some have worked a lot on the practical part of building the network. Some have calculated the results through physical laws, and someone has started working directly on it from a mathematical point of view.

From the perspective of border processes, three functions emerged: distinction, differentiation, and separation. Moreover, some remained within their disciplinary borders, a sort of comfort zone. In this, we recognized the bordering process called protection. Distinction refers to the initial recognition

and articulation of the unique identity and borders of each discipline. This recognition is critical because it ensures that each discipline retains its core identity and does not become diluted in the collaborative process. For instance, in the experiment described, the mathematics teachers emphasize the abstract, formula-based approach (e.g., Ohm's Law), while the Physics teachers rely on physical experimentation and measurement. A clear distinction between these approaches allows participants to bring valuable, discipline-specific insights to the table. Separation helps ensure that interdisciplinary collaboration does not collapse into a single homogeneous process but respects the differences that make each field valuable. In the context of the STEAM experiment, engineers measure directly, mathematicians calculate, and physicists experiment. By maintaining this separation in the early stages, teachers can approach the problem from their disciplinary strengths, ultimately contributing to the collective understanding. Containment refers to the ability of each discipline to hold and sustain its own knowledge structures, while also interacting with other disciplines. Even as teachers in this experiment engage in cross-disciplinary discussions, each group retains the capacity to function within its own disciplinary framework. The physicists continue experimenting with resistances, while the mathematicians proceed with calculations, both contained within their disciplinary paradigms, yet open to integrating findings from each other. Protection is the safeguarding of disciplinary methodologies and epistemologies during interdisciplinary engagement. It ensures that as different disciplines interact, none is compromised or diminished by the other. In the experiment, protection manifests when, for example, the mathematical rigor of Fibonacci calculations is preserved even as the concepts are applied in physical contexts like resistances and springs. The dialogue protects the methodological rigor of mathematics while enabling its application in physics. During the problem-solving activities that involved the construction of circuits and mechanical systems, teleological rationality became prominent. Teachers discussed the goals of the tasks, the intended outcomes, and the practical steps needed to achieve them. Here, the dialogue shifted to a more action-oriented, goal-directed form of reasoning. Throughout the process, communicative rationality played a crucial role in mediating between the epistemic and teleological aspects. It was through dialogue that teachers were able to negotiate their different rationalities, seeking mutual understanding and consensus. For example, when a disagreement arose about the application of the Fibonacci sequence in a physical context, the teachers engaged in a communicative process to reconcile their differing perspectives, thus allowing the border between Mathematics and Physics to be crossed and redefined (mediated by the use of technologies)

> Teacher 2: We have tried, by comparing ourselves, to put together different approaches so from the mathematical point of view and from the physical point of view and also go to look for that regularity and how to go about finding results in the various approaches also discovering some interesting things in the regularity clearly given by the Fibonacci succession. We have found some difficulties in the measurements depending on the instruments with compared to the precision of the mathematics, but finally we agreed.

The border, as a dynamic space, encompasses several dimensions that are activated through dialogue: differentiation, mediation, and transformation. In the early stages of the dialogues, the borders between disciplines were clearly marked by distinct epistemic rationalities. Each teacher's contributions reflected the unique knowledge structures and methodologies of their discipline, thereby highlighting the differentiation inherent in interdisciplinary collaboration.

Teacher 3: In our group, since there is no physicist, the approach was to start from the abstract, from the formulas that are known to us from Ohm's laws because the formula gave us confidence in proceeding with the calculations; then some began to build, others started from the abstract to get to the concrete. The experiment confirmed the accuracy that we had done the math right.

As the dialogue unfolded, the border became a space of mediation, where the teachers' rationalities intersected. This mediation was not merely about finding common ground but involved a deeper process of negotiating the meanings and applications of concepts across disciplinary lines. For instance, the teachers collectively explored how the Fibonacci sequence could be interpreted both mathematically and physically, using communicative rationality to bridge the epistemic and teleological divides.

Teacher 4: I'm really excited about how the golden ratio came out of our experiments. It wasn't something we had planned, but it was a natural find.

Teacher 5: Agree. This type of interdisciplinary approach is valuable because it allows you to explore and discover in a natural way, without preconceptions. It's a powerful way to teach not only Math, but also the scientific method.

The final dimension, the transformation, occurred as a result of the successful mediation of rationalities. The border was no longer a barrier but a site of transformation where new, integrated forms of knowledge emerged. The teachers' understanding of the Fibonacci sequence, for instance, was transformed as they collectively recontextualized it within the framework of physical systems. This transformation reflects the evolving nature of the border, as it is continuously reshaped by the dialogues that take place within it.

Teacher 6: And it also demonstrates the importance of dialogue between us as teachers. Without this collaboration, we might never have arrived at this discovery.

Regulation refers to the way disciplines negotiate rules, norms, and practices when collaborating across borders. This dimension ensures that interdisciplinary work is structured and guided by mutually agreed-upon frameworks or standards. Regulation helps in managing the interactions between disciplines to avoid confusion or conflict and ensures that the collaborative process remains coherent and productive. During the STEAM activity, regulation occurs in the structured steps of the experiment, such as dividing the teachers into groups and guiding them through the stages of building circuits and discussing their findings. These structured interactions regulate the collaboration, so that all participants follow a shared process while contributing their disciplinary perspectives.

Conclusions

The aim of the workshop was to create a homogeneous and undifferentiated environment aimed at discovering the golden number. Immediately the boundaries between disciplines emerged through the processes of distinction, differentiation and separation. Singularities emerged that made it possible to distinguish the disciplines which appeared differentiated according to the different approach that the teachers adopted. Then it came out how the disciplines, despite the singularities, communicated with each other. The results that emerged from this experimentation can be looked in a twofold aspect: from the point of view of STEAM teaching and STEAM teaching research. From a teaching point of view, the activity can be a stimulus to experiment in class and see if the same border functions emerge. From the point of view of research, we provided a key to interpretation through two theoretical

frameworks in order to characterize what happens at the boundary between disciplines, and to interpret the actions of the teachers during the experimentation. In conclusion, this study provides a framework for understanding interdisciplinarity in STEAM education through the metaphor of the border as a semiotic cross-cultural space. The collaborative processes observed among the participating teachers highlight the potential for professional development that transcends disciplinary borders, fostering a deeper, more integrated understanding of complex phenomena. This research contributes to a more nuanced understanding of how teachers can work together to build their professional development at the borders of their disciplines.

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GRAPHICAL reasoning: Students' ways of chunking a logistic function graph representing a titration

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We report students' graphical reasoning approaches to a logistic function graph depicting a titration curve. We conducted semi-structured interviews with 16 students, who were tasked to "explain the relationship between the pH of the solution and the amount of base added" modeled by a logistic function. Retrospective analysis showed several chunking approaches: (1) chunking based on graphical trends (e.g., "linear" (8 out of 16); (2) chunking based on the increasing and decreasing rate of change (4 out of 16); and (3) chunking based on the pH scale (4 out of 16). We discuss students' interpretations of surface-level or intrinsic properties of mathematical concepts or chemical ideas. We conclude with implications to help students move from viewing graphs as disconnected chunks to seeing them as continuous models of scientific phenomena.

Keywords: Graphical interpretation, titration experiment, chunking, reasoning.

Introduction

Representations of mathematical concepts, including symbols, graphs, and tables, are used as tools for communicating and making sense of mathematical ideas (Altindis, 2021;Dreyfus, 2002). Scientific disciplines employ similar representations to model and communicate about the phenomena under consideration. In this work, we focus on graphical representation. Students approach graph interpretation by drawing upon prior knowledge and experience. Their approach depends on familiarity with the context and mathematical properties of the graph, prior experiences related to the graph's context or its shape(s), and their ability to conceptualize and focus on the variables presented on the graph axes. We explore "students' graphical reasoning", which we define as their approach(es) to obtain meaning from a graph; more specifically, the cognitive activities involved in extracting information about represented data, finding trends and relationships in that data, and extrapolating or making inferences based on the data (Curcio, 1987; Glazer, 2011).

Visual chunking

Visual chunking refers to a process of breaking down visual information in representations into meaningful segments or "chunks" to facilitate understanding and interpretation. Visual chunking is predicated on structural regularities, where the patterns or groups are not arbitrary but rather derived from recurrent or significant structural features (Stieff et al., 2020; Talanquer, 2022). This facilitates the compression of substantial quantities of information into manageable chunks, allowing individuals to overcome the restricted capacity of working memory, thereby enhancing

both comprehension and cognitive retention (Thalmann et al., 2019; NRC, 1999). Experts encode meaningful groups of features or patterns in representations as "chunks" which activate relevant concepts or cognitive activities that facilitate interpretation and use of a representation (Carpenter & Shah 1998; Stieff et al., 2020). Frequent exposure to visual representations as part of gaining expertise in a particular domain supports the automatic and tacit recognition and interpretation of visual features (Cartrette & Bodner, 2010; Connor et al., 2021; Domin et al., 2008; Stieff et al., 2020). Novice and inexperienced students, on the other hand, may employ spatial chunking, a strategy that generates chunks by considering the spatial closeness of individual units to one another (Stieff et al., 2020). This approach may or may not correspond to expert chunking, as it may not be informed by appropriate domain knowledge. For novices, first interpretations of representations are often based on the most salient and easily processed features. They may retrieve knowledge based solely on surface similarities or recent experiences or elevate the importance of the most salient features during reasoning (Shah & Oppenheimer, 2008).

Graphical reasoning

Reasoning through a graphical task involves a series of cognitive activities, starting with extraction of information and progressing through construction of a sequence of meanings to generate claims and conclusions (Curcio, 1987; Glazer, 2011; Lithner, 2008; Thompson et al., 2024). Visually chunking the graphical information into smaller, more manageable chunks based on prior knowledge and experiences is a common first step for novices and experts alike (Rodriguez & Jones, 2024; Talanquer, 2022). A viewer may then make a claim about the chunks, either in isolation or as connected chunks, by utilizing whatever scientific or mathematics knowledge they activated. If the supporting arguments are situated into intrinsic properties of math, based on scientific principles, or both, then the argument is likely to be valid, and the reasoning could be classified as productive; if the argument is based on surface-level math, science, or both, then such reasoning is likely unproductive (Lithner, 2008).

Study purpose

We situate this study within the framework of Steffe and Thompson's (2000) definition of "students' mathematics." We employ the term "students' graphical reasoning" to describe the various ways in which students understand, interact with, and reason through graphical representations and the outcomes of these processes. Understanding how students visually chunk graphical representations and then use those chunks to draw conclusions about the represented phenomenon is a critical step in understanding "students' graphical reasoning". Only by recognizing the visual patterns and intuitive, mathematical or scientific ideas that guide students' graphical reasoning approaches can instructors develop strategies to support the development of sophisticated graphical reasoning. This study therefore asks the following: What chunking strategies are employed by STEM students when interpreting a logistic function that illustrates a chemistry titration experiment curve, and how is their interpretation related to intrinsic properties of math or chemical phenomena?

Method

This study is part of a larger study conducted at a research-intensive university in the northeastern United States. Approval was obtained from the university's Institutional Review Board (IRB), and informed consent was secured from all participants. Participants (N=16) were recruited from an introductory biology course for life science majors, an introductory chemistry course for life science and chemistry majors, and an introductory chemistry course for engineering majors. A stratified sampling approach was used: participants were contacted based on their performance (high, middle, low) on a course assignment involving proportional, covariational, and graphical reasoning (not described here). Interview data were collected via hour-long semi-structured interviews at the beginning of the academic year.

Interview Protocol

A semi-structured interview protocol was developed with input from chemistry, mathematics, and biology experts. The data presented relates to one task with a graph depicting a strong acid-strong base titration (Figure 1). Students watched a video of a titration, showing the collection of pH data as a strong base was added to an acid-containing beaker (MrGrodskiChemistry, 2020). Students were asked to describe the relationship between the pH of the solution (y-axis) and the amount of base added (x-axis). The interview was audio and video recorded. This figure was previously published in a related work (Altindis et al., 2024).

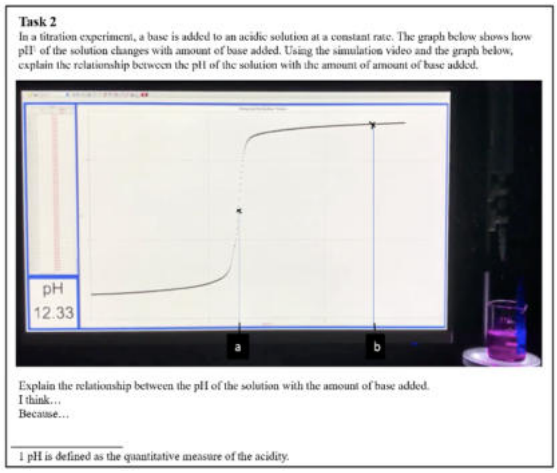

Figure 1: Titration curve task

Method of analysis

Interviews were analyzed for chunking approaches by looking for (1) how students chunked the graph, (2) their connections to underlying mathematical properties (e.g., slope, inflection points, asymptotes), and (3) their ability to construct scientifically grounded interpretations of the titration process, using a constant comparative analytical method (Corbin & Strauss, 2014). In some cases,

the chunking approach was obvious: students chunked and interpreted a specific section of the graph. In other cases, students' approach was buried in their speech and written artifacts.

Results

We identified three chunking strategies employed by students: (1) chunking based on graphical trends, (2) chunking based on the increasing and decreasing rate of change, and (3) chunking based on the pH scale and its logarithmic nature.

Chunking based on graphical trends

In the 'chunking based on graphical trends' approach, students chunked the graph into distinct segments based on observable trends in the graph's behavior. They used trend-focused language, such as describing sections of the graph as showing a "fast increase" when the slope is steep, or a "slow increase" or "flattening" when the line began to level off. Half (n=8 of 16) of the students chunked the graph into three chunks based on graphical trends (similar to Rodriguez & Jones, 2024), including Emma in the following excerpt.

Emma: ...when the reaction started, it was pretty level and then slowly when things are added, it slowly increases and then at one point it hit a tipping point, spiked up, and then when it hits a certain level, it can't go any higher and then it starts to level out gradually.

Emma describes how the line shows that the reaction starts off steadily, then gradually rises, hits a tipping point where it spikes, and finally levels out. She attempts to justify her observations by relating them to a chemical process, stating "a certain amount of solution was added to really engage in the reaction to cause a sudden spike" which suggested "a huge change happened". While her interpretation is seemingly grounded in the experimental context, it remains at a surface level without delving into chemical entities or mathematical principles.

Students who took this approach to understand the graph often identified the most salient patterns in the line, including the initial gradual increase, a spike, and a final leveling-off. Relying on the most salient features is a common strategy for novices (Rodriguez & Jones, 2024; Talanquer, 2022).

Chunking based on changing rate of change

Four out of 16 students chunked the graph based on changing rate of change. Carson examined the graph through a mathematical lens, specifically focusing on the rate of change and its connection to the steepness of individual segments.

Carson: ...the pH kind of slowly increased more and more until it got to a certain point. And then it kind of took off and increased a lot... with just a little bit of base and then it topped off and slowed down again. But it kept increasing the whole time that there's base being added. ...it rose with every drop whether it was increasing by a lot or a small amount of pH at a time. ...the amount of change of pH was a lot higher in the section between the tipping point [curve immediately before 'a'] and

when it slowed down compared to how much the pH change between the beginning and the tipping point. The middle section had a much higher rate of change than the beginning.

Carson's statement suggests that the pH displayed distinct patterns in different chunked sections of the graph. He highlights that the rate of pH change was notably more significant between the "tipping point" and where it begins to slow down (i.e. curve after 'a') compared to the pH change between the graph's initial point and that transition. However, Carson's emphasis on mathematics meant he overlooked chemistry-related information conveyed by the graph.

In contrast, Charlie also indicated chunks which differed in the direction and magnitude of the rate of change, but her explanation of the relationship between focused on the chemical process.

Charlie: ...the pH increases slowly until all the acid... or all the H+ ions are neutralized or turned into water and then it increases rapidly until it is very alkaline so the pH is very high…. because from the beginning the pH slowly increases and there are many H+ ions present because of the low pH. ...and then when most of the H+ ions are bound into oxygen or uh, into the water, the pH increases rapidly because the OH- concentration is increasing.

Charlie's initial description of the pH increasing slowly until neutralization, followed by a rapid increase to a high pH, indicates an understanding of key trends. She references the molecular entities represented ("H+ ions", "OH-") and their relative concentrations over the course of the titration, demonstrating an ability to connect the line behavior to the chemical process.

Chunking based on pH scale

Most students approached the pH scale as a series of numerical values without explicitly acknowledging the logarithmic nature of the scale. However, some students, like Kevin, attempted to relate their chunks to logarithmic change.

Kevin: ...as base is added... the solution rises in pH slowly. And then around 'a', abruptly increases possibly because the pH, which is 7.... And then maybe... once it gets past that point it goes a lot quicker because... the pH scale is... I don't know how to describe it. At 7 it's like a base of one, like a ten to the first. And then at 8, it's 10 times that. Then at 9, it's 10 times that again. So it keeps increasing exponentially from 7 to 0 and then from 7 to- I think 14. So I think that's why it has this really slow increase and then an abrupt increase here and then another slow increase.

Kevin seems to understand that the pH scale is nonlinear, and he tries to articulate that this is why the pH initially rises slowly, then more quickly around pH 7, and then slowly again. These words suggest that he understand the concavity of the inflection point. Kevin explains the sharp increase by suggesting that the graph's behavior reflects the exponential nature of the pH scale.

Discussion

This study explored how students reasoned through a logistic function graph depicting a titration curve, demonstrating how students' chunking approaches and connections to mathematical properties and scientific principles relate to their interpretation of a graphically-modeled phenomenon. We observed several chunking approaches, which varied depending on students' prior knowledge of mathematics and/or science. The most common approach involved chunking the graph based on trends (e.g., "linear", "exponential" (8 out of 16)). Students also chunked based on increasing and decreasing rate of change (4 of 16) and on the pH scale (4 of 16).

Students' use of mathematical properties during graphical interpretation varied from *surface-level* to *intrinsic* use. At surface-level, students relied on superficial and possibly inaccurate reasoning, such as misidentifying the logistic function as exponential or failing to recognize critical features like the changing rate of change at the inflection point. All students who chunked the graph based on trends chunked it into three chunks, anchoring their reasoning to graphical shape and employing trend-focused language (similar to Rodriguez & Jones, 2024). Relying on the explicit or salient features may have tacitly reduced cognitive load for these students, but without additional domain knowledge in mathematics (or science), their chunking approach may have limited what relationships they could find (Talanquer, 2022). Similarly, it is not granted that students will notice the (pH) log scale, so they may just focus on numerical values of pH, not recognizing the tenfold change occurring. In contrast, students who used *intrinsic* mathematical properties focused on critical properties of logistic functions, such as slope, inflection points, and asymptotes. They utilized these properties to anchor their reasoning and accurately justify interpretations but may not have connected them to the titration process (e.g. Carson's case).

Students' ability to construct scientifically grounded interpretations of the titration process also varied from *surface-level* to *deep*er use of scientific concepts. A *surface-level* explanation was disconnected from or only loosely tied to the titration phenomenon, such as chunking the graph based on the pH scale. Kevin, for example, conceived that each unit on the pH scale represents a tenfold change, but a tenfold change in what (H^+ ions) was not described. Some students tied the graph *deep*ly to the titration process, seeing behaviors and critical features of the line as representing changing concentrations of molecular entities and key chemical transition points (e.g. inflection point is the chemical equivalence point). For example, Charlie recognized the rapid change in pH represented when all initially present acidic H^+ ions were bound into water as part of the neutralization reaction. Most students' arguments were limited to surface-level use of concepts, demonstrating little integration of the experimental context and graph trends.

Conclusion

The students interviewed in this study came from introductory biology and chemistry, bringing different content knowledge, particularly in mathematics. While chunking makes it easier for students to analyze trends, relationships, or patterns within graphs, students' varied approaches demonstrate that instructors need to consider in what ways students' prior experiences may

influence how they anchor their reasoning: on the intrinsic mathematical properties of the graph, on scientific principles or features of the phenomenon represented on the graph, or on surface-level features. How students chunk may limit what kinds of inferences they can make or conclusions they can draw. A student chunking based on trends, for example, may miss that the rate of change is increasing or decreasing around an inflection point, which could have important mathematical or scientific implications depending on the context. Similarly, chunking based on mathematical properties may enable a student to accurately address tasks within a mathematical framework, but they may struggle to connect their reasoning to the experimental context.

Understanding these differing approaches can guide design of interventions that promote a holistic understanding of graphs, benefitting mathematics and science education. Students failing to move beyond surface-level graphical features or surface-level scientific concepts underscore the need for tasks that go beyond simply identifying shapes or regions of a graph. For students who rely on mathematical reasoning, tasks should include prompts that explicitly connect mathematical properties to chemical concepts. In short, instructors should encourage discussions around how meaningful features of graphs emerge, not only from the mathematical structure but also from the scientific context in which the data is situated.

Acknowledgments

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Primary teachers' understandings of integrated STEM elaborations: Discrepancies between importance and practice

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The intentional integration of science, technology, engineering and mathematics (STEM) is well understood to support student learning, especially as it relates to developing deeper conceptual learning. However, teachers' understanding of integration, and the ways in which integration can be implemented within the classroom, is not as strong. Building off a previous study (Bennett & Ruchti, 2022), for which an elaboration of Bybee's (2013) nine perspectives of STEM was proposed, this article describes primary teachers' understanding of the four specific and intentional methods in which integration in mathematics and science can be effectively implemented. This mixed methods case study, which included 24 primary teachers, only 10 of whom indicated they had some expertise in mathematics or science content, was conducted in the Intermountain-west of the United States. For the purpose of this paper, findings will focus on one elaboration as it presented a conflict between teachers' beliefs and their instructional values.

Keywords: STEM Education, Integrated Learning, Primary Grades.

Introduction

Supporting teachers in the integration of science, technology, engineering, and mathematics (STEM) in the classroom continues to be an area of much needed attention for several reasons. There continues to be no clear understanding or definition as to what STEM integration means or even what it looks like in practice (Bybee, 2013; Larsen et al., 2022). Given this lack of clarity from the research community, it is perhaps no surprise teachers struggle to effectively implement authentic, integrated STEM learning experiences (Aguilera et al., 2021). This means that in addition to being clear on operationally defining STEM, there is a need to support teachers with understanding the varied instructional approaches to STEM teaching and learning. However, all of this is predicated on teachers valuing varied instructional approaches and understanding how these approaches fit within their individual context. So, while different frameworks for integrating STEM may be valued by the larger research community, these frameworks cannot be studied to understand their effectiveness if teachers do not value them in the same way.

Review of Literature

The research field continues to grapple with a definition for STEM and thus the conceptual frameworks also vary (Roehrig et al. 2021). Although there are multiple models of integration, a common element often centres on the important role mathematics plays in supporting learning in science, either to enhance scientific content knowledge or scientific process understanding (National Research Council, 2011). The significant overlap and alignment of the Science and Engineering Practices, one dimension of the Next Generation Science Standards (NGSS Lead States, 2013), and the Standards of Mathematical Practice (National Governors Association, Center for Best Practices, & Council of Chief State School Officers, 2010) provides an area primed for integration (Bennett &

Ruchti, 2014). In addition, understanding science content often relies on an understanding of mathematics content and yet there are very few opportunities for teachers to participate in integrated STEM-related professional learning, and existing curricula are not currently designed to support teachers' effectively integrating STEM content within their day-to-day routines and instructional spaces (English, 2016). The nature of the integrations, and the scaffolding used, must be explored in more detail in order to support K-12 teachers' implementation of STEM education (Kelly & Knowles, 2016). Likewise, the integrations should be based on an understanding of exceptional pedagogical approaches from each discipline and key learning theories. This includes using integration to develop topics or content in more depth and apply content to real-world situations (NCSM: Leadership in Mathematics Education & National Council of Teachers of Mathematics, 2018). In essence, the integration of STEM content should allow for learning that extends each content area and creates a different kind of understanding.

However, the fluid and flexible nature of integrating disciplines can be challenging for classroom teachers without models of when and how this can be accomplished in the classroom. However, when teachers are able to integrate, and when they utilise a real-world contextualized application, the experience can create deeper connections and more authentic learning for students (Sandall et al., 2018). According to Moore and colleagues (2014), the role of the teacher in integrated STEM learning is to help students learn to see the abstract nuances in the content, including how to decontextualize content, before applying the learning in a variety of ways. Unfortunately, barriers exist for teachers that limit their ability to create these outcomes.

For example, teachers often lack the knowledge and/or equipment to bring integrated STEM to the classroom as well as finding the balance of developing problem-solving skills and teaching science content challenging (Dare et al., 2018). Furthermore, schedules and school structures, along with curriculum alignment, are often inflexible and therefore not conducive to quality integration (Margot & Kettler, 2019). To further complicate matters, mixed messages from experts in the field, colleagues, and school leaders about conceptions of STEM education can cause teachers to reject implementation (Aguilera et al., 2021). Dare and colleagues (2018) noted that the nature of integration, or how teachers perceived their role in making connections between multiple disciplines, was a struggle when attempting to implement integrated STEM in a science classroom. Specifically, teachers struggled to balance addressing each of the STEM disciplines, and how to move between disciplines. This means that missed opportunities for varied forms of integration can occur. In addition, teachers reported difficulty integrating mathematics into their science instruction and felt that their students were unable to do mathematics in their science classrooms; key science skills like computational thinking are often not included in their instruction (Rinke et al., 2016). For many teachers, they are often more comfortable teaching concepts separately or *adding on* to what they already teach as opposed to explicit integration (Crotty et al., 2017). Likewise, teachers often assume students will see the mathematical connections within the intended integrated lesson, rather than being explicit about the mathematics (Larsen et al., 2022); thereby making for more authentic STEM learning.

Theoretical Framework

Though conceptions of integrated STEM abound in the literature and policy documents, (Kelly & Knowles, 2016; Roehrig et al., 2021) most conceptions ultimately do not address the nature of the integrations specific to a teacher's context. That is, current frameworks often focus on broader structural aspects (Vasquez et al., 2013), curricular approaches (Roehrig et al., 2021) or traits that exemplify STEM characteristics such as being inquiry or problem based (Fan et al., 2021), whereas the framework understudy for this project focuses on the instructional techniques and pedagogical practices teachers would employ in the classroom and during the lesson to integrate STEM concepts. This study is based on Bybee's (2013) perspectives for STEM integration but incorporates an elaboration of these perspectives to better support teachers with STEM integration within their context. Specifically, these elaborations focus on four specific methods of integrating mathematics into science including 1) *Simultaneous Infusion*, 2) *Accentuate Interdisciplinary Constructs,* 3) *Temporary Shift in Discipline*, and 4) *Lateral Concept Connections* (Bennett & Ruchti, 2022). *Simultaneous Infusion* is when mathematical and scientific habits of mind and practices, such as the importance of using data and communicating ideas and understandings in STEM, are considered simultaneously during a science lesson. Likewise, *Accentuate Interdisciplinary Constructs* attends to the ways in which specific content understandings or skills from one discipline can be deliberately highlighted to support more purposeful learning. This differs from *Temporary Shift in Discipline*, which is a shift from one discipline (e.g., science) to conceptually explore a skill in another discipline (e.g., mathematics), and then a shift back to the original discipline (science) to bring new understandings to the primary science focus. Lastly, with *Lateral Concept Connections*, the focus of the learning moves from a primary science concept to a new science concept. For a more thorough description of the framework, see Bennett and Ruchti (2022). The purpose of this shift is to give students new content to reinforce a mathematical concept. When considered in light of Margot & Kettler's (2019) position that teachers are catalysts in advancing STEM talent, this study uses a constructivist epistemology to understand participants' perspectives on integrating STEM learning.

Methods

The purpose of this mixed methods case study was to understand how primary teachers understand and value an emerging Integrated STEM Elaborations framework (Bennett & Ruchti, 2022), which is built on Bybee's (2013) broader STEM integration framework. This study surveyed 24 primary teachers to understand the extent to which the four elaborations were: 1) clear to understand, 2) perceived as comfortable to implementation, 3) important for students in learning STEM, 4) instructionally important when integrating STEM content, and 5) valued in relation to the other elaborations. That is, the researchers wanted to better understand how this framework for STEM integration is understood by primary teachers to better support primary teachers with the integration of STEM content and practices. As such, the research question for this project was: How do primary teachers understand and value an emerging STEM integration framework within their context?

Participants

Participants in this study included 24 primary teachers from across one school district located in the Intermountain-west of the United States. Primary teachers were purposefully identified because they

typically teach all content areas to the same group of students, have more flexibility in their day to explore more project-based learning experiences. Of the 24 teachers in this group, ten were teachers in early childhood settings, which includes kindergarten through second grade, and the other 14 were teaching in third through fifth grade classrooms; none listed mathematics or science as a secondary area of expertise. With respect to years teaching, 12 were novice to early career teachers, with nine years of experience or less, and the other 12 had more than 10 years of experience, with seven indicating they had more than 15 years of experience. No demographic information on race, ethnicity, or gender identity were collected.

Data Collection & Analysis

Data collected included an anonymous cross-sectional survey (Creswell & Guetterman, 2019) sent to all primary teachers within the district. Respondents were given approximately one month to respond from when the initial invitation was sent out. The survey included closed-ended Likert-scale questions focusing on their understanding of the Integrated STEM Elaboration framework and open-ended questions seeking clarity on other aspects of STEM integration that should be considered. The Likert-scale items were coded numerically to allow for a descriptive analysis (Creswell & Guetterman, 2019), with less favourable beliefs coded as a 1 and stronger, more favourable beliefs coded as 5. The data were independently analysed by the research team to identify outlying or incongruous responses and to look for common themes within these incongruous responses. This included analysing responses according to specific sub-groups: 1) Number of years teaching, 2) content area emphasis, and 3) in which current grade level they taught. The research team then compared notes and re-categorized themes to appropriately capture the emergent patterns within the data set. Again, emphasis was placed on identifying outliers in the data set in order to develop additional questions for further areas of study; knowing where there are discrepancies, potential misunderstandings, or other outliers allows for more focused follow-up research.

Findings

Although there were common themes between the STEM Integration Constructs, one in particular provided data that seemed incongruous with anticipated results. That is, while the responses for *Temporary Shift in Discipline*, *Simultaneous Infusion*, and *Lateral Concept Connections* were all generally uniform and consistent between sub-groups, the data for *Accentuate Interdisciplinary Constructs* (*AIC*) was somewhat unexpected. Recall, *AIC* attends to the ways in which specific content understandings or skills from one discipline can be deliberately highlighted to support more purposeful learning. This is different from *Temporary Shift in Discipline* because the learning of one content area does not stop entirely; the teacher does not plan for a deliberate change in the content to support the overall learning goals. For example, the science content will be better understood because of conceptual understandings in the mathematics that are used, discussed, and highlighted during the science exploration.

Table 1: Percent of responses for greatest importance for sub-groups

	Simultaneous Infusion	Accentuate Interdisciplinary Constructs	Lateral Concept Connections	Temporary Shift in Discipline
Percentage for greatest importance (n=24)	42%	4%	25%	13%
Responses by sub-groups				
0-9 Years Teaching (n = 13)	29%	4%	8%	4%
10+ Years Teaching (n = 11)	8%	0%	17%	8%
Kindergarten through Second Grade (n = 11)	17%	0%	8%	4%
Third through Fifth Grade (n = 13)	21%	4%	17%	8%
STEM Content Emphasis (n = 10)	13%	4%	21%	13%
Non-STEM Content Emphasis (n = 14)	25%	0%	4%	0%

With respect to teachers' beliefs on which construct was the greatest importance for their context, *Simultaneous Infusion* received the highest percentage of responses across nearly all sub-groups (see Table 1) and *AIC* had the lowest percentage of responses for greatest importance within the teachers' context and there was only one response for *AIC* but this person belongs within three of the sub-groups listed.

Table 2: *Accentuate Interdisciplinary Construct* responses for all participants.

Accentuate Interdisciplinary Constructs	Percent Agree
After reading the description, I understand what it means.	71%
I am comfortable using this approach to integrate STEM learning experiences.	63%
I believe it is important for students to learn how to do this.	83%
I believe it is important is this for effectively integrating STEM learning in the classroom.	83%
Greatest Importance for Integration	4%

Participants were also asked to reflect on the importance of *AIC* for both teaching and learning STEM. That is, participants were asked to determine how important *AIC* was for students as they are learning STEM from an integrated approach and how well they understood the construct and their comfort in using this approach. When considering the data from all participants (see Table 2), and not by sub-groups, teachers indicated they generally understood what the *AIC* what the construct meant, generally believed that *AIC* is important for students to learn but did not strongly believe it was an important approach for integrating content within their context. This aligns with their beliefs about *AIC* as not being of great importance.

Essentially, all the teachers in the sample indicated that *AIC* is relatively important for both teaching and learning STEM in the classroom but only one early career, upper primary, STEM content emphasis teacher believed it was of the greatest importance.

Discussion of Results & Implications

The purpose of this study was to understand how primary teachers understand and value an emerging integrated STEM framework (Bennett & Ruchti, 2022). Participants indicated they understood three of the four elaborations well, that the elaborations were important for both teachers and students, and that the elaborations mattered when integrating STEM content. However, *AIC*, while understood, was not viewed as the greatest importance within their context. *AIC* attends to the ways in which specific content understandings or skills from one discipline can be deliberately highlighted to support more purposeful learning and this level of integration requires substantial content knowledge across disciplines (Schuchardt & Schunn, 2016) as well as a complex pedagogical understanding of how and when to purposefully highlight the cross disciplinary connections (Dare et al., 2018).

Since primary teachers create and deliver learning for multiple content areas, several concerns arise with respect to integration. First, and for many primary teachers, their mathematical content knowledge is underdeveloped (Campbell et al., 2014), which can impede their ability to appropriately accentuate the connections between interdisciplinary constructs. This may be why *AIC* was not viewed as being of great importance; it takes substantially deep content knowledge to implement it well. Likewise, lesson plans are often designed in a linear manner. They begin with an introductory set, an exploration or instructional episode, followed by some closing or summary of the learning. While such approaches are commonplace and often necessary, the complexity of integrated STEM learning with *AIC* is not linear. That is, *AIC* requires teachers to move fluidly between concepts as they arise. This kind of dynamic instruction is reminiscent of Smith and Stein's (2018) approach to facilitating mathematical discourse, wherein deliberate and purposeful planning allows teachers to skillfully improvise when important ideas and connections emerge. Furthermore, many teachers believe low-level skills and procedures should be taught before more complex, problem-based learning can be explored (NCTM, 2014). So even if an instructionally ripe moment occurs for which *AIC* might be ideal, a limiting belief that students "are not ready" for the learning prevents even the consideration for using *AIC*. So even if teachers have strong content knowledge and are open to more fluid lesson planning and delivery, their beliefs may limit *AIC* type integrations.

Understanding the importance of each construct, and teachers' beliefs on which are greater importance, may be influenced on how they define STEM (Larsen et al., 2022). For those who believe that integrated STEM learning is more aligned with *Simultaneous Infusion*, wherein the learning goals and outcomes of both content areas are well aligned, then other constructs may not seem as important within a classroom despite the authenticity and organic nature of the learning.

Limitations

A primary limitation in this study revolves around the fact that it included only one school district with only 24 participating teachers from this school district. It is possible that these 24 teachers are a non-representative sample, and some results may seem speculative. As such, this reduces the richness

of the data and thus the interpretations that can be made, and especially in different contexts, despite promising findings for future research.

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STEM teachers' meanings of inquiry in STEM teaching, in the context of a Professional Development Programme

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This study explores the meanings of inquiry STEM teaching that a group of 6 mathematics teachers and 2 science teachers working with teacher educators develop while participating in a professional development (PD) program aiming to support the design and enactment of an Inquiry STEM approach. We used grounded theory approaches to identify participants' meanings of inquiry STEM teaching and then the theoretical constructs of dialogical moves to provide further evidence of the meanings that the participants developed and of the process of this development. Our results indicate different emerging meanings of inquiry STEM teaching related to epistemological and didactical dimensions. The employment of the dialogical moves made by the participants enrich our understanding of how the above meanings may be constructed in a PD context.

Keywords: Inquiry STEM teaching, mathematics teachers' meanings, dialogical moves

Introduction

The meaning of inquiry in STEM (Science, Technology, Engineering and Mathematics) teaching is not well defined as there are differences between inquiry in science, mathematics, technology or engineering (Triantafillou et al., 2021). For example, inquiry in science teaching is related to experimentation, description of methods and processes that scientists use, set of specific cognitive abilities that students might develop, and means of developing the abilities of inquiry and understanding scientific concepts and principles (Bybee, 2006). Inquiry in mathematics teaching emphasises the process of questioning, hypothesising, identifying patterns, generalising, conjecturing, justifying and deducing conclusions and it develops in a classroom community that promotes mathematical communication among the students and the teacher (Artigue & Blomhøj, 2013; Goos, 2004). Thus, an integrated teaching approach to STEM requires connections among these different dimensions of inquiry that have been identified in the separate subjects' teaching (Goos et al., 2023). This is a demanding process for the teachers and the teacher educators since both need to cross boundaries among these practices (Bakogianni et al., 2021). Investigating teachers' meanings about inquiry STEM teaching and the process of developing them in a PD setting is essential for supporting teachers to enact STEM teaching approaches in their classroom. However, research in this area is rather limited. We try to address this gap by investigating the meanings of inquiry STEM teaching and learning that a group of teachers working with teacher educators develop while participating in an international workshop series of PD aiming to support the design and enactment of tasks and lessons that adopt an Inquiry STEM approach. We address the following research questions: RQ1: What are STEM teachers' meanings of inquiry in STEM teaching? RQ2: How do the meanings develop in a dialogical PD context?

Theoretical background and literature review

Inquiry-Based Teaching and Learning Approach is considered as an approach for improving teaching and learning in STEM disciplines. In their study Bell et al. (2010) identified nine main science inquiry processes in collaborative learning environments that could be used in STEM disciplines, namely: orienting and asking questions; generating hypotheses; planning; investigating; analysing and interpreting; exploring and creating models; evaluating and concluding; communicating; and predicting. The issue of collaborative inquiry learning is supported by researchers in mathematics education as well (Goos, 2004). Meanings of inquiry STEM teaching are related to the use of authentic resources and tools; making links with out-of-school activities; the development of problem-solving abilities and inquiry habits of mind; the autonomy and responsibility given to students, from the formulation of the questions to the production and validation of their answers; the use of tasks related to socio-scientific issues; and the use of technological tools (Artigue & Blomhøj, 2013). In a PD setting, as the teachers engage in discussions regarding inquiry STEM teaching, they participate in a social, verbal, and rational activity in which they may support, refute, or question arguments, share information, inquiry, clarify or negotiate meanings. The analysis of the process and the product of teachers' argumentation could assist us in understanding their construction of a shared meaning. We employ Martins and Macagno (2022)'s framework which considers arguments as elements of extended dialogues—and as dialogical units. Thus, analysing the turns of the dialogue and especially their contribution to the joint construction of common knowledge and meaning, we could draw some inferences regarding the effect of the dialogue in the shared meaning of a group of STEM teachers. In our study, we focus on teachers' dialogical and metadialogical moves, their relevance and their contribution to the construction of knowledge. The dialogical moves that we considered in this paper are: *dialogical inquiry move, dialogical persuasive move and meta-dialogical clarification and consensus moves*. The dialogical inquiry move aims to investigate the validity of hypotheses from evidence or request their investigation. The dialogical persuasive move aims to convince the other of the validity of an idea or request that this be done or resolve a conflict of ideas in order to analyse each idea or request that the conflict be resolved. The meta-dialogical clarification move, which aims to clarify the meaning of the purpose of other moves, or seek clarification of this meaning; Meta-Dialogical Move clarifies the meaning of an idea, which aims to clarify the meaning of an idea present in other moves, or seek clarification of that meaning; and Meta-Dialogical consensus Move, which aims to express the sharing of the same idea or ensure that this happens (Martins & Macagno, 2022). Regarding the relevance criterion, we adopt the three dimensions of Macagno (2022): dialogical dimension (there is relevance when speaker's intention is compatible with the current dialogue or dialogical setting, semantic dimension (there is relevance when the moves address the same topic and a common-ground dimension (there is relevance when the moves are understood by the interlocutor, and the relationship between a move and the goal that it is intended to achieve need to be grasped by the latter.

Methodology

This study was motivated during an International STEM workshop series for the PD on STEM of teachers coming from 13 European countries (https://icse.eu/icse-academy/). The focus of this workshop series was bringing participants up to date in research and teaching in the STEM subjects in connection with European priorities such as fostering sustainability, digital skills and intercultural learning. The teachers attending the modules were involved in plenary lectures; working groups; whole group discussion as well as in short homework assignments. The workshop series were held online, once a week for two hours in two PD cycles of about a semester each. The 1st workshop series workshops were attended by eight Greek teachers (six mathematics, one physics and one biology teacher), six mathematics teacher educators and one science teacher educator. Additionally, to the workshop series, five national reflective meetings were organised, once a month lasting about two hours, where STEM teachers presented their assignments, discussed their experiences and received feedback from all the participants. In this paper, we analyse the data of these national (Greek) five meetings of the 1st PD cycle. The data are transcribed discussions from those meetings. Using grounded theory techniques, we first analysed the transcripts line by line to identify teachers' meanings of inquiry STEM teaching. These meanings were coded, and the categories are reported below in Table 1. In the next phase of our analysis, we used the theoretical constructs of dialogical moves (Martins & Macagno, 2022) and arguments presented in the previous section to provide further evidence of the meanings that the participants developed and of the process of this development.

Findings

Emerging meanings of STEM inquiry teaching

In table 1, we summarise the emerging meanings of inquiry STEM teaching. Although the meanings are presented as separate categories, their epistemological and didactical dimensions are interwoven. In the table, we present the categories of the meanings, a detailed description and an illustrative example from the data. The epistemological characteristic of the meaning consists of the difference in the focus of inquiry (mathematical or/ and scientific) teachers give to the tasks, even the ready-made ones. There were significant difficulties to integrate mathematics along with the scientific aspect of the problems STEM inquiry needs. Some teachers first thought the openness of the problem could miss their mathematical learning goal: *"I don't know what mathematics is involved here, in which grade should I give it? »*, or in some other cases the mathematics were absent or sidelined because of the rich science context. We see that sometimes they agree about the context but the management of the task in the classroom and the digital tools provided discourage the teacher from implementing inquiry-based tasks: *"I don't see the point of giving them this arduino which is difficult to manage and time consuming, aren't there any other applications in order to measure such visual stimulus?".* Another big category of meanings are characterised as didactical because they address management issues that concern the teachers, such as the active engagement of the students, their work in teams and also the communication in the classroom.

Table 1: Emerging meanings of STEM teaching

Categories	Emerging Meaning	Description	Example
Epistemol ogical: Focus on inquiry	Mathematical inquiry is dominant while scientific inquiry is neglected.	The socio-scientific context is in the background while the mathematical inquiry comes in the foreground.	*"The emphasis in this task is on mathematics. The scientific phenomenon is rather neglected here although it has too many factors when you come to study it using Biology".* (biology teacher)
	Scientific inquiry is central while mathematical activity is trivial.	Emphasis is given on the exploration of the scientific aspect while the engagement with mathematics is poor.	*"The problem with the water consumption in a Greek island seems to me more inquiry based because it has nothing that betrays pure mathematics …"* (math teacher)
	Scientific and mathematical inquiry are integrated	Inquiry in science and in mathematics are parallel	*"I want the students to collect real data, do an experiment, I want them to start asking scientific questions. The mathematics involved is clearly statistics"* (math teacher)
Epistemol ogical: Focus on Technolog ical dimension / tools	Digital tools as means for supporting students' engagement and/or as a way to develop computational thinking	The technological dimension is restricted to the use of a digital tool for mathematical inquiry or links to mathematics through programming and computational thinking.	*"I could use an escape room just once a year in order to get the students interested"* (mathematics teacher) / *"I would prefer students to be involved in technology through programming not if only the teacher makes something more interactive through the technology …"* (math teacher)
Didactical: Focus on students' engageme nt	Students' active engagement in scientific/ technological and mathematical inquiry	The students themselves engage in scientific and/or mathematical inquiry by posing questions, using tools, forming hypotheses, justifying them and coming to conclusions.	*"We want the students to see a video about simple dance figures and after that they try to connect their moves to the notion of an angle and its possible values exploring many possibilities…"*(math teacher)

	Promoting Classroom communication	The students share their ideas in groups or in the whole classroom.	*"…the children should be in groups, discuss and negotiate so they can come to common proposals between the groups...and communicate it then in the classroom"* (physics teacher)
Focus on lesson design	The starting point for the lesson design follows different paths.	The starting point in the design of the lesson is the socio scientific phenomenon while the mathematical inquiry does not necessarily fit to a specific curriculum content or its the other way around	*"Here we see that you started with the maths and the notion of the corner and maybe then you saw the link to the video and put that rich content, in the contrast the other team started with the rich content and then they said we can have that and the other…"* (teacher educator) / *"I've never designed anything in the STEM context before nor something outside of the strictly mathematical framework… but for me the mathematical learning goal is very important to be seen in advance…"* (math teacher)

Although the teachers agreed that the students should engage in investigating the scientific/technological situation and set the parameters of the problem, they see difficulties in implementation. There were moments when they tried to combine science and mathematics, indicating tensions in classroom management, time limit, lack of experience handling the scientific situation along with unexpected answers from the students. Tensions also emerged when they could not see the integrated dimension of the inquiry needed to have successful inquiry in both mathematics and science. The mathematical learning goal is very strong along with the connection to the curriculum particularly in Greece, so sometimes the starting point is mathematics and not the social or the environmental phenomenon: *"I prefer the second problem because I think that in the context of mathematics it is important for students to read graphs and draw conclusions from them…"* (mathematics teacher).

The development of meaning through the dialogue

The two episodes presented below are related to the two tasks that represent aspects of the socio-scientific issue of global warming, the *Beach showers* and the *Global air temperature anomalies*. The two tasks differ in respect to their inquiry characteristics. The Beach showers task requires students to use data coming from official statistics in order to estimate the amount of fresh water consumed at the peak of the summer season in the open-air showers in specific Greek islands. In the Global air temperature anomalies task the data have already been elaborated and there is a graph that models the global temperature changes from 1840 to 2020. We try to illustrate the development of shared

meanings through the use of the dialogical moves. R, V, M are mathematics teachers while S and A are mathematics teacher educators.

S: We present to the children who are actually struggling to learn mathematics, extremely complex problems of the world and we expect them to be interested... I can see how in this way we educate them but I don't see how they would be interested in it and I wonder what the interest has to do with the familiarity. … where the mathematics is not visible, and the interest is not visible. They are far-off problems... can a 15-16 years old student handle them? The colleagues who work on them maybe could explain to us, and my second question is about inquiry as we see it here, it's like saying look for it, but is that it? It's more than that …

V: The two problems have an essential difference, the first problem with the showers puts the students on a quest, it doesn't tell them how to calculate, to start a discussion how long everyone lets the water run, the second problem is much more for the students to comment on a problem in which a specific strategy has been set, in the first they have to make their own and in the second they have to comment on the one that was followed if it was suitable to highlight the problem. They are not made to be solved at home because the students must discuss with each other, for example I gave the problem why the cola can is the size it is, as a homework, it brought me some excellent answers with maximisation and minimization….

In the above excerpt, S makes initially a metadialogical move, where he seeks to clarify the content and scope of an inquiry task for students. Thus, he shifts the focus of the dialogue from a mathematical meaning (related to the mathematics specifics of the particular tasks) to a re-evaluation of the general purpose of inquiry tasks for teaching, which relates to the didactical meaning. Then, he couples his move with a discovery move as he requests for an explanation (*…can a 15-16 years old student handle them? The colleagues who work on them maybe could explain to us…*) of the seemingly lack of interest on the part of students and the role of inquiry. His argument is source dependent (argumentation scheme: from position to know) based on experience that seems part of a common commitment store. In this sense he contributes to the common ground of the dialogue as his moves and the relationship between them and the goal can be understood by the interlocutors who share similar experiences. The shift to a different interpretation regarding the meaning of inquiry tasks is reflected in the next interlocutor (V) who, staying on the same line of thought, examines the inquiry characteristics of the two tasks. He is making a (meta)dialogical move to clarify the meaning of the objective of S's move which is supported by an argument (argument from illustration) that employs an illustration of his claim (*for example I gave the problem…).* Since he expands on S's utterance both in semantic and dialogical terms, V's response is characterised as dialogically relevant and contributes to the shared knowledge through an argument based on personal experience. The above interaction and the shift of meanings to a more detailed and teaching-oriented employment of the term inquiry, brings out the potential of language as a means for the common construction of meaning. Nevertheless, the different epistemological meaning that was brought into the foreground in the above shift, wasn't followed up by subsequent moves as they weren't semantically relevant to the themes raised by S. That is the learning potential of S's move was not fully exploited apart from V, since the teachers in the following parts of the dialogue turned again to more practical, didactically driven aspects of STEM teaching (making another shift of meanings).

M: In the task you presented, you had already searched for a lot of data that it would be better if the students had searched themselves to find, … I think you mentioned that we are getting data 30 years back, why do we go back 30 years? How … could be sure that the old data are comparable to the current since our present-day instruments' precision is higher than the old instruments had. Why shouldn't someone argue that if we had the same instruments these data could show that there is no global warming, …

A: where do you see that there is no global warming?

M: I say I don't see this question articulated or answered, if we don't learn to face these questions we lose a big element, the development of critical thinking.

Initially teacher's conversations up to M's intervention was mainly about more practical issues as time management and the curriculum topics concerned so they were too specific to promote learning in STEM. In the part of the dialogue prior to the above excerpt, there was an absence of pedagogically driven questions about 'why' preferring a certain task or approach over some other. In the above excerpt, M questions the scientific base of a task through a dialogical inquiry move as he calls for the investigation of the validity of the hypothesis that the measurements can be reliably compared. In dialogical terms, we note a shift on the semantic dimension of the dialogue as he focuses on the scientific validity of the data employed in the task. M introduces a new, scientific-driven meaning, that changes the common ground of the dialogue, as he disputes the scientific component of the task. Then (answering A's question) he makes a dialogical persuasive move which adds another parameter on the semantic dimension of dialogic relevance, which is student related. By focusing on the inquiry as a student's possible mode of thinking, he contributes to the goal of the dialogue by meeting the communicative purpose of the notion of inquiry in teaching. His two moves exhibit an integrated view of scientific and mathematical inquiry (epistemological meaning) and help bring a more general view of didactical meaning relating to the active engagement of students in developing inquiry reasoning. So, this move contributes to the construction of new didactical meaning regarding the inquiry component of a STEM task which is followed up by the next utterances of the dialogue, where interlocutors debate on that particular aspect of STEM tasks.

Discussion and concluding remarks

Our findings reveal multiple emerging interpretations of inquiry-based STEM teaching, shaped by both epistemological and pedagogical dimensions. These align with Goos et al. (2023), who highlight the importance of maintaining the disciplinary integrity of mathematics while embracing an integrated STEM approach. The presence of various dialogical and metadialogical shifts, along with diverse forms of argumentation and explanation, reflects the co-existence and evolution of these interpretations as teachers worked toward shared understanding. Collaborative reflection and the exchange of teaching practices—especially among teachers from different disciplines—supported this meaning-making process, as suggested by Hadar & Brody (2016). Disciplinary backgrounds sometimes led to differing perspectives, particularly in how knowledge is introduced, echoing findings from previous research on epistemological differences. Our analysis further shows that using a framework focused on dialogical context and moves can enhance our understanding of how meaning is collaboratively constructed.

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Exploring computational thinking in STEM education: Moonbase activity

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This paper describes a single-case study exploring the evidence of 5th-grade student's computational thinking skills while participating in a STEM activity. This STEM activity called the Moonbase was developed by LEGO® Education and requires using the WeDo 2.0 Core Set to design a robot that can move on the moon's surface. The participant student programmed the robot to collect modules while using both STEM-related knowledge and computational thinking. Data was collected in a three-hour implementation through semi-structured interview and observation. Qualitatively analyzed data reveal that computational thinking skills emerge when using the Moonbase STEM activity. Specifically, debugging, decomposition, and pattern recognition components are the pieces of evidence of computational thinking.

Keywords: STEM education, computational thinking, moonbase activity, LEGO education, WeDo 2.0 core set

Computational thinking and STEM education

In the 21st century, where information and technology are rapidly produced, changed, and consumed, the expectations of societies from individuals have also changed with the concern of keeping in step with the times. This concern has made it necessary for individuals to have different skills to respond to changing world conditions. These skills are identified as 21st-century skills, including problem-solving, critical thinking, analytical thinking, creativity, collaboration, communication, productivity, flexibility, being open to new ideas, and using technology effectively (Partnership for 21st Century Learning [P21], 2019). In addition to these skills, being an empowered learner, digital citizen, knowledge constructor, innovative designer, creative communicator, global collaborator, and computational thinker are seen as characteristics and standards for students to prepare them for the evolving world by The International Society for Technology in Education (ISTE) (2017). In a similar vein, computational thinking is one of the ten key skills demanded by employers as job skills in the 2020s (Institute for the Future [IFTF], 2011). In this context, numerous institutions and organizations have emphasized the concept of computational thinking as one of the high-level thinking skills needed in the 21st century in recent years (Çetin & Toluk-Uçar, 2020). By promoting computational thinking, which is an essential way of problem-solving, many 21st-century skills, such as critical thinking, creativity, and communication, can be covered in schools (Ding & Cai, 2023).

The concept of computational thinking, on which field experts could not reach a consensus in terms of definition, scope, and assessment (Grover & Pea, 2013; Kalelioğlu et al., 2016; Weintrop et al., 2016), seems to be open to interpretation (Barr & Stephenson, 2011) since it can be related to many different disciplines such as mathematics, science, computer science, and engineering (Li et al., 2020).

There is a reciprocal relationship between STEM education and computational thinking. STEM education provides a context and setting where students can use their computational thinking skills to solve real-life problems. Computational thinking supports the development of problem-solving skills needed in STEM education. By integrating these two concepts, teachers can prepare their students for their future life by equipping them with the 21st century skills, and offer real-life related learning activities that foster both mathematical and computational thinking (Weintrop et al., 2016)

Considering the components and competencies of computational thinking as well as its characteristics, they were identified by many researchers (Aho, 2012; Barr & Stephenson, 2011; Brennan & Resnick, 2012; Grover & Pea, 2013). Although there is no consensus on its components like its definition since the researchers formulated their understanding of computational thinking by using the existing definitions and related literature (Ch'ng et al., 2019), the most common components covered by computational thinking are decomposition, pattern recognition, algorithmic thinking, abstraction, making generalizations, debugging, and problem-solving (Grover & Pea, 2013; Kalelioğlu et al., 2016; Rose et al., 2017). Based on the studies mentioned earlier, the explanations of each common component are given in the following Table 1.

Table 1: Components of computational thinking

Component	Explanation
Decomposition	Breaking data, problems, or processes into smaller, more manageable, and more understandable parts
Pattern recognition	Identifying patterns, trends, and similarities in data, problems, or processes
Algorithmic thinking	Constructing a step-by-step procedure, including an ordered set of instructions, for solving similar problems
Abstraction	Making data, problems, or processes more understandable by identifying main ideas and reducing unnecessary information
Making generalizations	Creating rules, models, or principles of observed patterns to adapt solutions to other data, problems, or processes
Debugging	Determining the issues and problems in thinking, operations, or activities to fix them
Problem-solving	Identifying the problem and generating alternatives to solve the problem as the final step of logical thinking

In this context, the aim of the study is to reveal the computational thinking-related evidence that can be included in a STEM activity and thus understand the relationship between these two concepts. For this aim, the research question of the study is "What kind of computational thinking-related evidence emerges during a middle school student's engagement in a STEM activity?"

Method

This research was designed as a single case study, one of the qualitative research approaches. In a single case study, the focus is on an in-depth examination of an issue, concern, or individual in a bounded system (Creswell & Poth, 2018). Only one student participated in this research. So, the case of the current study is one 5th-grade male student studying in a public school in Ankara, Türkiye. The participant was selected through convenience sampling, as the student was readily accessible and willing to participate in the study. This approach allowed for in-depth exploration of a single case and efficient data collection within the time constraints of the research. Since the data was collected during the 2024 summer break, the student has finished 5th grade and is about to start 6th grade.

In data collection, Moonbase Activity developed by LEGO® Education was implemented to explore how computational thinking emerged in a STEM education context. Since this activity emphasizes a problem-solving approach where students build and program a robotic model, it stands out from other STEM-related resources due to its unique combination of computational thinking and hands-on learning integration. This dual focus makes it an ideal choice for investigating the relationship between STEM education and computational thinking. The participant student has experience in building models with the LEGO® sets, but he stated that this is the first time he has used the LEGO® Education set. The student tried to design a solution for a robot that set up a base on the moon by using the WeDo 2.0 Core Set. In this STEM activity, the aim is to build a robot that can move on the moon's surface, program it by using the programming blocks (PB), and collect small modules on the moon to make astronauts' tasks easier in space.

In a face-to-face environment on a volunteer basis, by closely observing the student's interaction with the activity for three hours, data were collected. Before implementing the Moonbase activity as the second session, the researcher met with the participant for three hours to introduce the materials in the set and the WeDo 2.0 programming tool. In this first session, the Cooling Fan activity developed by LEGO® Education was implemented to ensure that the participant became familiar with the set. Also, an activity called Rhythmic Counting Machine, created by the researcher, was implemented in the first session to understand and observe the participant's computational thinking skills.

Data were collected by discussing the programming process of the robot with the student, asking questions related to his PB, thinking processes, and solutions, and observing his reactions to robot movements. The questions posed by the first researcher to the participant were related to his first thoughts about PB to move the robot, his thoughts about the robot's movement after trying the PB, the whole implementation process, the relationship between the implementation and the mathematics concepts, and learning, and his reflective interpretations about the activity.

While collecting data, these two sessions were audio and video recorded. The student was encouraged to think aloud to discover what he did and why he did it during the whole process. After collecting data, the recordings were transcribed and analyzed by using the content analysis method and open coding. In this process, the data were systematically reviewed to identify recurring categories related to computational thinking skills exhibited during the activity. Codes were generated inductively based on the observed moves on the programming tool and the responses of the participant. Following this, content analysis was employed to group these codes into broader categories, focusing on evidence of debugging, decomposition, and pattern recognition. This approach allowed for a detailed examination of the participant's engagement and facilitated the interpretation of computational thinking processes in the context of the activity. The authors coded the transcribed data separately and then compared their coding by reviewing in case of disagreements. The discussions continued until a full consensus was reached.

Findings

The findings demonstrated that debugging, decomposition, and pattern recognition components emerged as pieces of evidence of computational thinking while implementing STEM activity with a middle school student. Some of the examples related to evidence of debugging and decomposition

are given in this paper. However, evidence of pattern recognition, along with more examples of each component, could not be included due to page limitation.

Evidence of debugging

After building the robot model and creating the PB by following the building and programming instructions, the student was asked how he expected the robot to move according to the PB by showing the robot's movement with his hand before starting the program.

Researcher:	Now let's talk before sending the program. What do the programming blocks say? I could read it as a sentence.
Student:	Start the robot.
Researcher:	Okay. Start the motor.
Student:	Start the motor. Go this way, go this way at a speed of 4 for 2 seconds. Still, at a speed of 4, go this way for 2 seconds. (He showed the robot's movement with his hand by turning the robot first to the left and then to the right.)
Researcher:	Okay, now, it says go this way, for example, do you understand it to go this way? (The researcher asked to turn the robot directly to the left where it was standing.) How do you expect the robot to go with this program?
Student:	The robot will come here. Turn this way (left). It will come for 2 seconds at speed 4. Then it will turn this way (right). It will come this way for 2 seconds.
Researcher:	Okay, go ahead. You can send the program.

When the PB were sent to the robot, the robot first went straight and then turned left. Since the robot did not move as expected, the student sent the same PB and tried to understand what happened by observing the robot's movement. While he was trying, it was suggested to pay attention to which part provided the movement and to observe the movement of the wheels.

Researcher:	I wonder if there is something in the programming blocks or in the model? There seems to be an error.
Student:	Maybe there is in the programming blocks.
Researcher:	Are you saying that? But the WeDo app told us the programming blocks. Now there may be errors in three things. First is the model. We did not make the model anyway; the app did. Then, there may be errors in the programming blocks, but the app gave the programming blocks. We also had an expectation. It would go like this, right? (By showing his expectation of the robot's movement.)
Student:	I think our expectation is wrong.
Researcher:	Is our expectation wrong? Where do you think we might have made a mistake in our expectations?

He observed the movement of the robot and the PB at the same time. Following several observations, he realized that the rotating mechanism was a part attached to the motor rather than the wheels, as he expected. After determining the issues in the discrepancy between the expectation and the robots' movement, he fixed his thinking and started to write PB by considering the movement of the wheels connected to the motor. Then, moving the robot by making a U-turn task was given to challenge him, and another piece of evidence related to the debugging component was revealed.

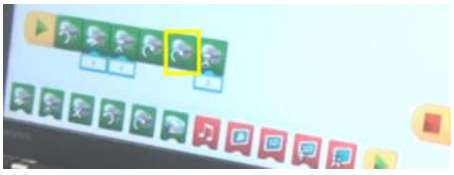

Figure 1: First changes in the programming blocks for making the U-turn

Researcher: If it turned directly towards me and not towards you, that is, made a U-turn, how would you do that?

Student: To do this, you need that programming block. (He added the highlighted programming block called "motor that way block" that demonstrated in Figure 1. Then he sent programming blocks to the robot and the robot moved in the same way, namely going straight and then turning left.)

Student: It did not work. We need to change this second (He changed the number input block under the motor on for block. As shown in Figure 1, the last programming block included 2 was changed to 5.) Now it will come back to you. There is no need for this anymore. (He deleted the added later and highlighted the turning block.)

Researcher: Okay.

Student: I started it. It will make a U-turn, get back to you. (When the program started, the robot made a U-turn by turning too much.) … It turned out a lot. It is necessary to write 4 rather than 5. (He changed the number input block to 4.)

Researcher: How did you decide it should be 4?

Student: Because it had to turn here. (He showed with his hand that the robot made a proper U-turn) But it turned a quarter more (He showed with his hand that the robot made a proper U-turn, and continued to turn for a while.) So, I need to make 4 to complete it.

The U-turn task enabled the student to determine his mistakes by trial and error and to fix the operation by using appropriate PB. This is evidence of the debugging component of the student's computational thinking skill in the activity, which requires reaching the moonbase modules.

Evidence of decomposition

Since the Moonbase Activity is a guided activity in the WeDo 2.0 programming tool, the app also gives the order of PB for how the robot will move after building the model. These ready PB made the robot move straight and then turn left as demonstrated in Figure 1. This means that the robot can collect the first module. After working on these ready PB, a new programming page was opened. The first module and the robot are aligned, and the second module is placed in such a way that it can be retrieved by turning left after picking up the first module. A new task was given, which was to collect the first module and then the second module. In this new task, the student broke the process into three parts: (i) going straight to collect the first module, (ii) turning left to reach the second module, and (iii) going straight to collect the second module. This decomposition process to make data more manageable is shown in Figure 2 by using screenshots.

Figure 2: The process of collecting two modules

He stated that the robot goes straight and turns left in the current program. So, he needed to "add something to the end of the first program." Then he added a motor power block with number input 4 and a motor on block with number input 2 at the end of the current program. However, this addition did not provide the necessary movement to get the second module, and the robot made a U-turn. He made changes in the PB and tried all programs to fix this issue.

In order for the student to realize the problem and solve it as evidence of the debugging component, the researcher asked him the following question: “I want to wash my hands. What steps should I follow?” While giving an ordered set of instructions for hand washing, he realized the importance of constructing a step-by-step procedure for reaching the second module. Then he revised the program by adding the programming block that included the change of direction of movement of the part connected to the motor to collect the second module. When he ran the program, he got the second module and stated that thinking about each step one by one made it easier to do the whole task by breaking it down into pieces.

Similar to the previous task, the next task was given by adding a third module aligning the second one as shown in Figure 3. The student preferred to work with a more understandable PB by dividing the task into smaller parts. He constructed a step-by-step procedure by considering the robot’s path to collect all modules in order. So, this new task provided pieces of evidence for the components of computational thinking used in the STEM activity as decomposition and algorithmic thinking.

Figure 3: The placement of the three modules and the robot that will collect them

Discussion

Many research studies have pointed out the strong relationship between computational thinking and STEM education (Hsu et al., 2018; Li et al., 2020; Swaid, 2015). Computational thinking-related and STEM education-related concepts can be implemented at all grade levels to enhance clearer understanding across STEM disciplines and develop related skills (Romainor et al., 2018). According to the content analysis study conducted by Kılıç (2022), STEM education can be used as an approach to develop students’ computational thinking skills. For example, Chiang et al. (2022) implemented an online STEM camp for primary school students, and they stated that STEM integration with an emphasis on medicinal knowledge and using educational robotics has the potential to improve students’ computational thinking skills. Similarly, Psycharis and Kotzampasaki (2019) introduced the concept of computational STEM pedagogy and investigated whether the STEM inquiry game learning scenario could improve primary school students’ computational thinking skills or not. As a result, they found that STEM inquiry scenarios increased students’ computational thinking skills.

While using a single case study limits the generalizability of the findings, it provides a valuable starting point for in-depth exploration of the relationship between STEM activities and computational thinking. So, in this research, the focus is on rich, contextual insights rather than broad applicability. In this direction, the results of this study shed valuable light on how computational thinking skills can be developed in the context of a STEM activity, emphasizing the emergence of debugging, decomposition, and pattern recognition. The presence of these three components of computational thinking, which are essential parts of problem solving, indicates that the participant student can engage in the STEM education-related task at a more profound cognitive level. The findings

demonstrate how the student applied computational thinking skills during the activity, particularly through these three components. These reveal the student's ability to approach the problem-solving process systematically, a key feature of computational thinking. This case study highlights the potential of STEM activities to foster such skills, providing valuable insights into the interplay between STEM education and computational thinking. In this context, the current study supported the strong relationship between these two concepts by providing evidence of computational thinking while implementing STEM activity with a middle school student. Knowing this relationship between these two concepts can guide curriculum studies. So, tasks and problems that address these two concepts can be presented in mathematics curricula and textbooks. More emphasis can be placed on STEM activities to develop students' computational thinking skills. Long-term studies with more participants from different backgrounds are needed to understand this relationship in more detail.

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Opportunities and limitations of ChatGPT in creating STEM tasks: A case study with mathematics pre-service teachers

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Global challenges are increasing the need to integrate STEM and Education for Sustainable Development (ESD) in mathematics education. Therefore, teachers should be empowered to incorporate these into their lessons, e.g., through suitable tasks, which poses a challenge for teachers. This case study uses an ad-hoc questionnaire and evaluative qualitative content analysis to investigate pre-service teachers' (PSTs; n = 20) use of ChatGPT as a tool to create STEM/ESD tasks for mathematics classrooms. The analysis demonstrates that while the modifications of the tasks implemented by PSTs generally resulted in tasks with higher cognitive demand and more meaningful context, not all their intended modifications were effective, and more modifications might be useful. The results highlight PSTs' active role in creating STEM tasks using ChatGPT, as well as the need to improve their abilities in task analysis and modification.

Keywords: Education for sustainable development, STEM, ChatGPT, task design, teacher education.

Introduction

In the face of contemporary global crises, such as climate change, water and sanitation, or clean energy, it is evident that there is a need for complex solutions that draw on expertise from various fields, including STEM. As a result, both STEM Education and Education for Sustainable Development (ESD) have become central to the educational agendas of schools. There is a significant overlap between these two initiatives in terms of their educational discourse, which includes both content such as climate change and teaching methods such as integrative learning approaches (Siller et al., 2024; Smith & Watson, 2019). Teachers are required to provide students with opportunities to understand how different subjects can contribute to addressing global and local economic, social, and ecological challenges and the importance of STEM disciplines (UNESCO, 2017). Thus, it is necessary to provide students with appropriate learning environments, including suitable tasks to ensure that students acquire relevant subject-specific knowledge and skills.

For STEM and ESD in mathematics education, it is important to go beyond solely applying existing knowledge and using simple procedural applications and incorporate mathematics in a meaningful way. To achieve this, tasks should encompass among others access to data, authentic real-world contexts, complexity, using technology, and interdisciplinarity (Mochizuki & Christodoulou, 2017; Just & Siller, 2022). It is crucial for mathematics teachers to be able to select, modify and create tasks (Chapman, 2013). However, particularly in (complex) reality-related contexts, solving and creating tasks and providing scaffolding can be a challenge for (pre-service) teachers (Greubel et al., 2024; Paredes et al., 2020; Siller et al., 2023). Moreover, reality-based mathematics tasks, especially in STEM contexts, found in the literature often require merely procedural skills (Greubel et al., 2024; Just & Siller, 2022). To overcome these challenges, using ChatGPT to create tasks may appear as an attractive solution. Indeed, 48 % of the teachers surveyed in Germany's largest federal state in Germany in January 2024 were using AI systems for work, also to create tasks (PhV NRW, 2024). It

is therefore of interest whether mathematics (pre-service) teachers are able to effectively and successfully use ChatGPT to create STEM tasks in an ESD context for mathematics classrooms.

Theoretical background

The opportunities and challenges of ChatGPT for teaching and the associated need for professional development have been acknowledged by several authors in general (e.g. Kasneci et al., 2023; SWK, 2024) and subject specifically (e.g. Faggiano et al., 2024). Despite teachers' frequent use of ChatGPT, research on pre-service teachers (PSTs) shows a lack of PSTs' capabilities to appropriately apply ChatGPT. For example, Noster et al. (2024) show that pre-service teachers (PSTs) may be influenced by wrong solutions when solving mathematics tasks using ChatGPT or that they may be unable to detect errors in an output of ChatGPT. Further Küchemann et al. (2023) show that PSTs tend to directly copy results from ChatGPT without modification, in this case created tasks.

Consequently, results of PSTs could differ depending on the quality of the output of ChatGPT. However, due to rapid changes in the performance of ChatGPT, and therefore varying outputs, research should focus on ChatGPT as a collaborator in the creation of STEM tasks, rather than solely concentrating on the technology itself. To investigate the interaction of PSTs with ChatGPT in creating tasks, it seems worth looking at task characteristics generated by ChatGPT, especially for STEM tasks.

Cognitive demand and context in STEM tasks in mathematics classrooms

Due to their relevance for reality-based/STEM tasks (Paredes et al., 2020; Just & Siller, 2022), this study uses two characteristics of mathematics tasks as a focus: cognitive demand and context. Cognitive demand can be classified into three levels using the TIMMS Framework (Mullis & Martin, 2017): *Knowing* refers to recalling factual knowledge, carrying out procedural calculations, or retrieving information from tables or graphs. *Applying* includes the creation of diagrams to model a problem situation or determining appropriate strategies to solve problems. *Reasoning* refers to using logical and systematic thinking to deal with new questions, drawing conclusions, generalising, and justifying solutions. Particularly, when implementing STEM tasks in mathematics classrooms, it is important that tasks not only focus on *knowing*.

For a meaningful context, Leuders et al. (2011) provide three characteristics: *Relevance* refers to whether the context is related to the experiences, interests, thoughts, and actions of the learners. *Authenticity* refers to the opportunity to work on authentic questions to gain insights into the context itself, not just the mathematical content. This description can be further subdivided into two aspects: authentic questions and the opportunity to learn about the context. *Richness* indicates that the context is complex and open-ended, encouraging deep, meaningful questions and exploration.

ChatGPT for (STEM) task creation in mathematics classrooms

Regarding ChatGPT, first results by Saptoka & Bondurant (2024) show that ChatGPT seems to be able to connect tasks to a variety of real-world contexts but most of the tasks were primarily procedural, not asking for conceptual understanding, which is problematic for holistic STEM tasks (Just & Siller, 2022). This was observed even after providing a description of cognitively demanding tasks with prompts. Moreover, van Long et al. (2024) show that ChatGPT may ask questions that are

trivial or hardly math-related. Further, Küchemann et al. (2023) show that assessment tasks in physics created using ChatGPT offer a poorer embedding into a suitable context than tasks created without the help of ChatGPT. Nevertheless, their study suggests that this shortcoming could potentially be remedied by improved prompting. These results underline the necessity for PSTs to evaluate the suitability of STEM/ESD tasks for mathematics classrooms that are created using ChatGPT. Additionally, PSTs should be able to modify these tasks where necessary to achieve changes regarding cognitive demand and context.

Research questions

The study is therefore centred around PSTs' modifications of STEM/ESD tasks for mathematics classrooms provided by ChatGPT. We pose the following research questions:

RQ1: Does the quality regarding cognitive demand and context within STEM/ESD tasks change through the modifications?

RQ2: To what extent do PSTs consider cognitive demand and context during task modification?

RQ3: To what extent do the changes identified in RQ1 relate to the PSTs' modifications?

Research design

To answer these questions, we had PSTs create a STEM task using ChatGPT-4o and complete an ad hoc questionnaire. The PSTs were supposed to create tasks tackling an ESD topic, namely climate action as one of the Sustainable Development Goals (SDGs) (UNESCO, 2017).

The research was conducted in the summer term of 2024 in three university lectures (*basic probability theory and statistics*, *teaching probability theory and statistics*, *applied probability theory and statistics*) with 26 PSTs who participated voluntarily and consented to data processing. However, six PSTs had to be excluded from further analysis as they did not create a task or did not hand in their initial task. Thus, the final sample for data analysis comprised 20 PSTs (m: 11, f: 9) for secondary school (16) and elementary school (4), on average in their fourth semester.

After an introduction to ESD in mathematics education, the PSTs received the following assignment, including the respective parts of the curriculum and a comprehensive description of SDG 13: *Use ChatGPT to create a task with a solution (bullet points) for a math lesson on the topic of evaluating data for year 7 (secondary school) in the context of SDG 13 (climate action).*

Then an ad hoc questionnaire with four open and 45 closed items designed for data collection to gather the PSTs' reasons to modify their tasks and some additional information such as their prior experiences with ChatGPT was handed to the PSTs. We used a four-point Likert scale to capture to what degree the PSTs modified their tasks based on different task characteristics. The validity of the questionnaire was assessed through expert evaluations conducted by subject area experts; however, a pilot test was not implemented.

Analysis

To answer the first research question, we analysed the initial and final task of the PSTs according to the level of cognitive demand and the context using an evaluative, deductive content analysis

(Kuckartz & Rädiker, 2023). The created tasks contained several sub-tasks, which were coded individually, as they often addressed different levels of cognitive demand. For each sub-task, we coded the highest level of cognitive demand contained in that sub-task and then categorised the overall tasks according to the levels present in the sub-tasks (e.g. *only knowing*, *knowing & applying*).

Table 1: Anchor examples for the category cognitive demand (translated by the authors)

Knowing	Calculate the arithmetic mean of the number of days on which the students go to school by bike.
Applying	How can you interpret the range in the context?
Reasoning	Is there a relation between [temperatures and rainfall]? What could you conclude from that?

Questions that were not mathematics-related were not coded in this analysis. To analyse the context, we used the four categories (*relevance*, *authentic questions*, *authentic context learning*, *richness*) described in the theoretical background. Within the categories, three different levels were distinguished: *does not apply*, *partly applies*, and *applies*. For each category, we evaluated the tasks' potential (P) and whether this potential was exploited by the questions (E). The tasks were coded consensually by the first and second author. Afterwards, we compared the initial and final tasks to answer (RQ1). For (RQ2), we analysed the closed items on task modification in the questionnaire. (RQ3) is answered by comparing the changes in cognitive demand and context to the modifications indicated in the questionnaire. Due to the small sample size, our study shows initial descriptive findings but is an important starting point for our further research.

Results

Evaluation of the tasks

To begin with, all students made at least one modification. Overall, the modifications of the tasks led to an increase in cognitive demand and an improvement regarding the context illustrated in Figure 1.

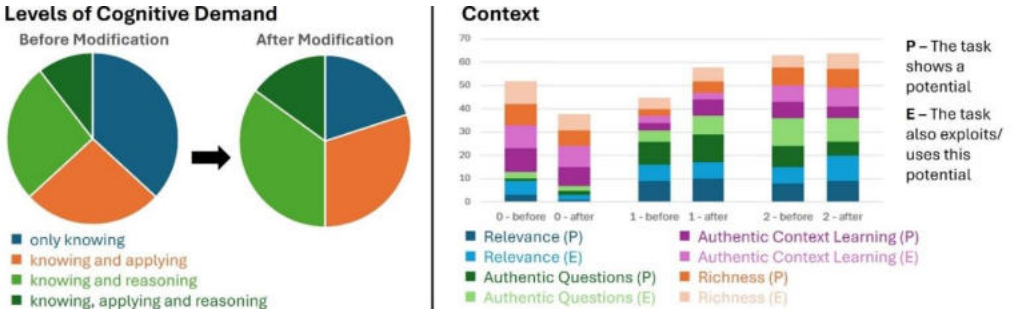

Figure 1: Changes in cognitive demand and contexts through PSTs' modifications

For the level of cognitive demand, the percentage of tasks only containing *knowing* was almost halved (seven to four tasks). Moreover, *reasoning* is contained in every second task of the final tasks, while only about one third of the initial tasks contains reasoning activities. If considering the inclusion of activities of higher cognitive demand as an improvement (e.g. *knowing & applying* is an increase from *knowing*), then eight tasks improved by the modification, while only two of the tasks declined.

Comparing the different context categories before and after modification, there is a decrease in the number of tasks that do not meet the criteria for a meaningful context (coded as 0) and an increase in those that partly meet the criteria (coded as 1). This is illustrated in detail in Figure 1, which shows the number of tasks coded as 0, 1, or 2 respectively for each context category.

Modifications of the tasks

Among the five task characteristics indicated most often as a reason for modification, three referred to subcategories of context, namely *relevance* (7 PSTs), *learning about the context* (8 PSTs), and *richness* (7 PSTs), and one referred to cognitive demand (7 PSTs). Two PSTs intended to both increase and decrease cognitive demand during their modification process. Despite being the most frequently modified characteristics, each of them was considered by less than half of the PSTs, while more than half of the initial tasks showed a need for improvement. For example, seven initial tasks not containing reasoning activities were not intentionally modified to increase in cognitive demand.

Relation between task characteristics and modification

The number of PSTs who modified the task with ChatGPT or by hand to increase or decrease cognitive demand and those who did not modify a task based on cognitive demand, separated by the type of change from the initial to the final task (increase, no change, decline), is shown in the left table in Figure 2. The first row shows that three out of the five tasks that were deliberately modified to only increase in cognitive demand actually show the respective change. It should be noted that the remaining two PSTs indicated to be dissatisfied with their final tasks. All other modifications do not coincide with a change at all, which seems comprehensible at least for those PSTs with both modifications. Interestingly, half of the tasks that were not explicitly modified to change cognitive demand still showed an improvement in this respect, while two showed a decline.

Level of Cognitive Demand (CD)

reasons to modify \ change	increase	no change	decline	total
increase CD	2 + 1*	1	1	4 + 1*
decrease CD	0	1	0	1
in- & decrease CD	0	2	0	2
no intended modification regarding CD	5	5	2	12

Context

Modification \ Change	improvement	no change	decline	total
Modification	8 + 1	13	4	25 +1*
no intended modification regarding context	10	32	12	54

*) ***+1*** indicates a modification by hand not using ChatGPT.

Figure 2: Comparison of modifications indicated in the questionnaire, cognitive demand, and context

For the context, each task was considered four times – once for each subcategory of context (leading to 4 x 20 aspects), summarised in the right table in Figure 2. Overall, the rate of improved tasks is higher among those that were actively modified (35% vs. 19%), while the rate of declined tasks is a bit lower (15% vs. 22%). These results indicate a slight tendency for greater improvement if a task was modified. Still, only 35% of the aspects the PSTs attempted to enhance did actually improve. However, *no change* is not necessarily negative if the initial task was already coded as 2. This was the case for four of the 13 aspects considered.

Discussion

The overall improvement between the initial and the final task could point towards the importance of PSTs' active contribution when creating STEM tasks using ChatGPT (SWK, 2024). While ChatGPT

can quickly provide various tasks, this case study indicates the importance of the ability of the (pre-service) teacher to modify and improve tasks regarding the level of cognitive demand and context (Sapkota & Bondurant, 2024).

However, even if the PSTs did not (consciously) modify for cognitive demand, there were still improvements in this respect. This could also point towards an opportunity of ChatGPT to provide more cognitively demanding tasks, even if no explicit modification concerning this characteristic was made. One might argue this makes PSTs' ability to assess cognitive demand of tasks less necessary when having access to ChatGPT. However, a closer look reveals that two out of the five increases only increased to an *applying* level, not to a *reasoning* level. Moreover, it should be noted that a task may also decrease in cognitive demand without modifications, as seen in two cases here. At the same time, there is a slight tendency that a purposeful modification to increase cognitive demand according to the questionnaire is related to the actual change. Therefore, it may still be important for PSTs to be familiar with assessing cognitive demand of tasks, particularly in a STEM context (Just & Siller, 2022; Sapkota & Bondurant, 2024).

Moreover, the results suggest that the PSTs were more attentive to context and cognitive demand than to other task characteristics, which seems reasonable given their importance in STEM and ESD tasks. Still, our data suggest PSTs' ability to consider these features should be strengthened even more. Further, our results could suggest that PSTs' modifications regarding context were at least partly successful. This would again underline the importance of (pre-service) teachers' competency in the analysis and modification of ChatGPT-generated STEM tasks. However, it must be noted that in a considerable number of cases, modifications were not associated with the respective changes. This might be due to the PSTs' difficulties to adequately communicate their intended modifications to ChatGPT, that is, to effectively use prompting strategies (SWK, 2024).

Limitations

The results of this case study are limited by several constraints. First, due to the small sample size and setting as well as the limitation to only one topic of ESD (climate action) and one mathematical content area, the results do not allow for generalisations. Second, the indication of the task modifications is limited to the PSTs' self-evaluation, not considering their actual interaction with ChatGPT. Analysing the chat logs might provide closer insights here. Third, our results are limited to one version of ChatGPT and may change as the performance of the system improves.

Conclusion

This study aims to analyse the opportunities and challenges of utilising ChatGPT in STEM Education by examining PSTs' current capabilities in using ChatGPT to develop STEM tasks and their motivations for modifying tasks.

The analysis revealed an overall improvement between the initial and final task, even though the PSTs did not always intentionally modify. However, in cases of unintentional changes, the level of cognitive demand was mostly limited to an *applying* level and even decreased in certain instances. Conversely, deliberate modifications to enhance cognitive demand as indicated by the questionnaire appeared to be rather successful.

From the perspective of teacher education, it is important to enable PSTs to critically utilise ChatGPT as a collaborator in creating STEM tasks. However, the fact that some PSTs did not intend to modify tasks generated by ChatGPT that had low cognitive demand or inappropriate context suggests a deficiency in their pedagogical content knowledge or a failure to critically evaluate responses provided by ChatGPT. This is highly problematic, especially considering the potential risk of ChatGPT generating errors or tasks limited to lower cognitive demand or inappropriate context.

Through our study, we have taken an initial step towards examining the capabilities of PSTs in creating STEM tasks using ChatGPT, leading to preliminary hypotheses. This study suggests that further research should focus on PSTs' proficiency in using ChatGPT, and how to empower them to view ChatGPT as a collaborator in the development of STEM tasks. This may involve teaching effective prompting strategies and enhancing their knowledge in evaluating STEM tasks to fully harness the potential of this innovative technology and enable meaningful STEM tasks in mathematics classrooms.

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Developing a Unified Framework for Authentic Problem Solving in STEM Education

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In the Authentic Optimizing research project, students from a German high school are confronted with real-world Problems from a company. The aim of the students is to solve the problems, for which neither they nor the researchers know the solution (or a way to solve the problem). The students are supported by coaches during his process, guided by mathematical problem-solving frameworks, such as Pólya (1945) or Schoenfeld (1985). Based on the first results of the project, we suspect that the students have major problems with the problem-solving (process), as well as with working with their mathematical resources. To improve the support of the coaches, it may be beneficial to integrate problem-solving frameworks from various disciplines. The goal is to describe these frameworks and combine, merge, or expand these to develop an interdisciplinary model. In this paper, we started with a view of problem-solving in mathematics, chemistry and computer science education.

Keywords: Problem-solving, Real-world problems, Pólya, Applied mathematics.

Introduction – Problem-Solving in Mathematics

As Halmos (1980, p. 519) aptly states, “mathematics consists of problems and solutions.” Problem-solving is undeniably a central component of mathematics, but what about other STEM disciplines and economics, which are also closely tied to mathematics? In our project, Authentic Optimizing, we aim to facilitate math-containing real-world problem-solving processes for students and provide support throughout their journey. Consequently, we are particularly interested in exploring problem-solving within mathematics, STEM, economics, and other fields, and understanding how these disciplines are interconnected, especially from a mathematical perspective.

Even within mathematics education, definitions of a "problem-solving" vary. One, which is important for us here, is given by Rott et al. (2021, p. 737). They describe a problem-solving

> in the sense of working on nonroutine tasks for which the solver knows no previously learned scheme or algorithm designed to solve them

based among others on Schoenfeld (1985), where it is described as working on non-routine tasks for which the solver has no pre-learned schema or algorithm. Rott et al. (2021, p. 737), referencing Halmos (1980) and Liljedahl et al. (2016), emphasize that problem-solving is not only essential in mathematics overall but is especially critical in the context of teaching and learning. In the field of mathematics education, the well-known work of Schoenfeld (1985) distinguishes four key components that contribute to problem-solving performance: resources, heuristics, control, and belief systems. Each of these categories plays a vital role in the problem-solving process. Resources refer to mathematical knowledge, including algorithms, routine procedures, facts, domain-specific information, and the connections between these elements. Heuristics involve strategies and techniques for tackling problems. Control relates to the global decisions made regarding strategies and resource allocation, and effective control requires monitoring and assessment (Schoenfeld, 1985,

p. 33). Lastly, belief systems encompass the individual's convictions, assumptions, and perspectives about mathematics and the problem-solving process. These four components are central to our approach in supporting students throughout the project Authentic Optimizing, which will be further discussed in the next section.

Probably the most famous problem-solving framework in mathematics is Pólya's four-step model (Pólya, 1945), which Schoenfeld (1985, p. 24) describes as a guide to effective problem-solving techniques. Pólya outlines four stages: understanding the problem, devising a plan, executing the plan, and reviewing the results. While this model provides a normative framework for problem-solving, it does not fully capture the iterative nature of actual problem-solving processes, where backtracking and revisiting previous steps are often necessary. To better capture the specifics of individual problem-solving processes, Rott et al. (2021) developed a descriptive model building on Pólya's framework. For this purpose, the added a fifth phase – the exploration. In this phase, students engage with the problem in an exploratory manner before formally planning or implementing a solution. A key feature of this model is its flexibility, allowing solvers to move back and forth between phases, better reflecting the reality of problem-solving processes.

While the presented frameworks were developed to support and investigate intra-mathematical problem-solving processes, the question arises as to how far these can be applied to authentic real-world problem-solving in STEM fields and beyond. To explore this, the next section introduces the Authentic Optimizing project as a critical case. Following this, problem-solving in other STEM disciplines will be discussed through two examples: chemistry education and computer science education. A final section will relate the presented frameworks to each other and outline initial insights regarding interdisciplinary and authentic STEM problem-solving.

Problem-Solving in Authentic Optimizing

Authentic Optimizing is a research project initiated by the mathematics education group at the University of Siegen. The central idea of this project is to present real-world problems from a company to school students, who then work on solving these problems with the support of coaches (students from the University of Siegen). Our business partner for this project is REWE digital, a subsidiary of the REWE Group, an international German retail corporation. REWE digital, located in Cologne, serves as the digital innovation think tank for the entire REWE Group.

Authentic Optimizing operates as a cyclical project, with overlapping cycles lasting one year each and consisting of three phases, punctuated by two important milestone events. Start of the project was in summer 2023. The first phase is the preparation phase, during which we promote the project in schools (Gymnasium; 7th to 11th grade) and select students to participate as problem solvers. This phase also includes the problem-posing process, where we collaborate with our business partner to define the problems and develop a student-oriented formulation. The preparation phase concludes with the kick-off event, in which the problems are introduced to the students, who then choose which problem they wish to work on. Each problem is assigned a stakeholder from the business partner who is responsible for the problem.

In the next step, the students form solver-teams (groups of four to five students) to tackle the problems in the problem-solving phase. These teams meet weekly in solver-hubs for 90-minute sessions. A coach (pre-service mathematics teacher) supports each team. During this phase, we apply problem-solving models, such as Pólya's (1945) model, to help students with strategies and control in their work (see Schoenfeld, 1985, p. 15). It is important to emphasize that our role is to support the students, not to solve the problems for them, encouraging them to implement their own ideas.

The problem-solving phase culminates in the Forum of Innovation, where students present their solutions to our business partner. By the forum, both parties have become experts in the topic. The final phase is the evaluation phase, where we interview students, coaches, and team members to refine the project and advance our research.

Posed Problems in Authentic Optimizing

To illustrate the types of problems the students address, we present two examples from the total of eight problems worked on in the first two cycles. The first example is the 'Last Mile', where students are tasked with solving the following problem: The last mile of delivery is often the most expensive. How can this be organized cost-effectively? This is an open-ended problem, allowing for a variety of solution approaches. Potential solutions might include drones, delivery boxes, drivers, or pickup stations. Students must also consider various constraints, such as chilled goods or EU return policies. In this case, students explored ideas like self-driving bots, drones, or an Uber-like delivery solution. The second example is the 'AI Project', where students explore how artificial intelligence (AI) could be utilized by a supermarket group. This problem is dynamic and changed during the solving process, as our business partner adjusted their compliance with ChatGPT in the problem-solving phase, requiring the students to adapt their approach accordingly. Here, students worked on the challenge of how AI could function as a co-worker. One group used ChatGPT to develop a custom AI tool to assist with organization and inventory management in a supermarket.

Challenges for Our Research in the Project Authentic Optimizing

While the students presented creative ideas, none of the solutions were profound. When analysing the solutions from the first cycle, both our university team and our business partner were disappointed with the overall quality and depth of the solutions. It became clear that the solutions were superficial. At the beginning of the problem-solving phase, we introduced students to problem-solving strategies, such as Pólya's model and the use of milestones. However, in our evaluation, students reported limited use of these strategies. For instance, one student stated:

Adam: No – everyone picked a topic and then researched it, and we entered the information into the PowerPoint.

In addition to these interviews, students were asked to keep a research diary and create a problem-solving plan at the start of the process. The analysis of both the plans and diaries supported the conclusion that established mathematical problem-solving frameworks were insufficient to describe the students' processes. In particular, students struggled with problem concretization and developing a plan (a kind of problem posing process), leading to superficial solutions. Given the real-world nature of the problems in this project, interdisciplinary approaches, particularly across STEM, may offer a solution to these issues.

Another challenge we observed was that, despite knowing we were mathematics educators, the students rarely utilized their mathematical knowledge to solve the problems. For instance, they did not apply mathematical methods to balance sheet calculations or calculate the radius of drone effectiveness, even when prompted. When asked in interviews to categorize the project within a school subject, very few students, if any, chose mathematics. This suggests that students may not perceive their mathematical knowledge as useful in solving real-world problems. Schoenfeld (1985) refers to this phenomenon as" they did not perceive their mathematical knowledge as being useful to them, and consequently did not call upon it" (p. 13).

Resulting Research Question

The "Authentic Optimizing" project serves as a critical case study for investigating interdisciplinary applications of mathematics and authentic, long-term problem-solving. While students engage in this project, our primary research aim is to examine how, and to what extent, they employ their mathematical competencies to address complex real-world problems that are not solely constructed for educational purposes, as well as to identify effective means of supporting them in this endeavour. In light of this, we pose the following research question: How can we better demonstrate, support or teach students in solving authentic real-world problems? The application of mathematics and problem-solving skills in real world situations is an important topic for mathematics teaching (Winter 1996, KMK 2012), which we want to investigate with this question as part of the project. One approach could be the integration of approaches from different disciplines, especially in the context of STEM. This addresses two of the four key components of Schoenfeld (1985): heuristics and control in the problem-solving process. We want to use the results of this research to improve the support for students in planning and implementing a more successful and better structured problem-solving process. Combining the different problem-solving frameworks from different disciplines will help us in planning and preparing the solving process for each interdisciplinary problem in the project. This part of the research is liked to a specific question: What are the differences and similarities of problem-solving frameworks from different disciplines and how can we combine them? In this paper, we will start with the two disciplines Chemistry and Computer Science, but we will continue with more STEM or economics disciplines in future research.

Problem-Solving in Chemistry

The German "Bildungsstandards" (educational standards) in chemistry mention problem-solving only marginally, which contrasts with the "Bildungsstandards" in mathematics, where problem-solving plays a major role and is defined as a competency (KMK, 2012, 2020). Despite this, if you look at educational research, problem-solving is vital for chemistry and its education. Bodner (2015, p. 181) emphasizes that "Problem solving is what chemists do." He supports this assertion with two key reasons. First, problem-solving helps to deepen one's understanding of the subject matter. Bodner argues that engaging with problems fosters a deeper comprehension of the topic at hand. The second reason relates to the nature of a problem itself. For many inexperienced students, tasks in their chemistry courses present challenges for which they neither know a process nor possess a plan to solve them (nor the solution itself). This aligns with the description: "and you don't know how to find a way to cross that gap" (Bodner, 2015, p. 194). Bodner (2015, p. 194) further defines problems using

a quote by Hayes (1989), explaining that a problem represents the gap between where you are and where you want to be, with no clear plan or method for bridging that gap. Streller et al. (2019, p. 52) offer a slightly different definition of a problem, characterizing it within the framework of problem-oriented teaching in chemistry. They describe a problem as the intellectual contradiction between what is known and what is observed, particularly from the students' perspective. Puhlmann and Tiemann (2008, p. 421) do not define problems directly but describe them through the lens of problem-solving, referencing the interdisciplinary work on problem-solving by Klieme (2001, p. 185). They define problem-solving as "goal-oriented thinking and action in situations for which no routine procedures are available." This complements Bodner's (2015, p. 194) definition of a problem, where he states that problem-solving is "what you do when you don't know what to do."

Although problem-solving is recognized in chemistry, there is a surprising lack of literature on the subject specificity in the field. Bodner (2015) developed a problem-solving model for chemistry, drawing on the models of Pólya (1945) and Dewey (1910). Bodner's model aims to be teachable, transferable across various chemistry problems, and useful in guiding the problem-solving process (Bodner, 2015, pp. 193–194). His model includes sixteen steps, with some repetitions. Steps 1, 2, 8, and 12, which instruct to "read the problem (again) ", align with Pólya's stage of "understanding the problem". Steps 3 and 4, "draw a picture that represents the structure of the system being modelled" and "write down what you hope is the relevant information", along with steps 5, 6, and 9, "try something (else)", correspond to "devising a plan". Steps 7 and 10, "see where this gets you", and step 11, "test intermediate results to see whether you are making any progress toward an answer", are consistent with "carrying out the plan". Finally, steps 14 ("write down 'an' answer"), 15 ("test the answer to see if it makes sense"), and 16 ("start over if you have to, celebrate if you don't") relate to "looking back". In comparison to Pólya, Bodner (2015) tries to expand and refine Pólya's (1945) original framework, adapting it for chemistry and creating a partially cyclic model that revisits key steps throughout the process. His research highlighted that "understanding the problem" is the most crucial step in chemistry problem-solving. He also identified strategies (notably "trial and error" as a powerful problem-solving strategy in chemistry), resources, and control as essential elements for the problem solver, though he did not directly reference Schoenfeld (1985). An interesting point for researchers in mathematics education is that Bodner (2015) initially developed his model for quantitative chemistry problems and only later adapted it to non-mathematical chemistry problems. Other researchers like Puhlmann and Tiemann (2008) used other problem-solving models like the OECD problem-solving model, which is also partially based on Pólya's work (OECD, 2004, p. 170).

Problem-Solving in Computer Science

Problem-solving is also a characteristic activity in computer science, generally with the goal of developing an algorithm and implementing it in a program. As Müller and Weichert (2013) describe:

> The goal of problem solving is to find a system, based on a given problem, which represents the solution. This approach can be broken down into three steps. The starting point is a given problem for which a solution is sought using a computer. Following this, through abstraction, an algorithm is ideally created from the problem. In the next step, the algorithm is transferred into a program,

which is then executed on a computer. The program represents the solution to the problem. (Müller & Weichert, 2013, p. 16, Translated by the authors)

The computer science problem-solving process, like the process in mathematics, can be described as a series of typical steps, which Müller and Weichert (2013) summarize in a model. First, the problem must be formulated or identified. The second step is to analyse the problem, which includes determining whether there is one, none, or multiple solutions. This step also involves abstraction, where variables are introduced for specific information. The problem may also be revised or broken down into sub-problems. The third step involves designing an algorithm for the sub-problems and the overall problem. In the fourth step, the correctness of the algorithm is checked (Does the algorithm terminate? Does it always produce the correct result?), and the fifth step involves analysing the implementation effort. Only in the sixth step does the actual programming take place, where the algorithm is written in a selected programming language. If the result is unsatisfactory, the algorithm and the program are improved through further problem-solving cycles. In computer science education, the entire problem-solving process does not always need to be completed. For instance, students may only develop an algorithm without implementing it in a program, or an algorithm may be provided for them to implement using a specific programming environment. In addition to the model by Müller and Weichert (2013) and similar process models for programming, Pólya's problem-solving model is also frequently used in computer science. However, Oehler and Ludwig (2024), based on a qualitative literature review, note that this is often done without adapting the model to the specificities of computer science, and that there has been no proper validation of Pólya's model in the context of computer science.

Comparison and Outlook

A comparison of the different frameworks from mathematics, chemistry, and computer science reveals that Pólya's problem-solving model is prominently referenced in each discipline. Pólya's model appears to be not only effective in mathematics but also applicable in chemistry and computer science. However, there are distinct differences in how these models are applied within each field. For instance, in the computer science model by Müller and Weichert (2013), "devising a plan" aligns with "devising an algorithm," but this could also be interpreted as the first step in "carrying out the plan". In chemistry, Bodner (2015, p. 195) emphasizes that the central aspect of the problem-solving process lies in "understanding the problem", which might be different in mathematics, where the focus is often on "devising a plan". Because of the connection to the Pólya framework, which is the basic or initial model for many of the problem-solving models in mathematics education (Rott et al. 2021), we can compare them with other models in mathematics education, too. For example, Schoenfeld (1985) added an additional exploration phase, which could be linked to "try something (else)". In the case of Rott et. al. (2021) whose model deals with both phases: planning and implementation and exploration, one has to categorise whether it belongs to one or both of these parts. Comparing the interdisciplinary frameworks with some of the mathematics education is also a goal for further research. For interdisciplinary problem-solving projects, such as Authentic Optimizing presented in this paper, it may be beneficial to integrate models from various disciplines especially from disciplines to which parts of the problem might be connected. We hope this will help us to improve the students' work in the problem-solving process, enhance their solutions, and give us

insights into the applied long-term problem-solving process. Some combinations of models of two disciplines already exist, e.g. for mathematics and computer science (Dilling et al., 2022b) or for mathematics and physics (Nguyen et al., 2020). Furthermore, expanding existing models like Schoenfeld's (1985) could be a promising direction. For example, Dilling et al. (2022a,2022b) applied Schoenfeld's model in a real-world problem-solving context and observed that students drew not only on mathematical knowledge but also on resources from physics, engineering, and the arts.

In the next steps of the theoretical development of Authentic Optimizing, we aim to systematically analyse problem-solving frameworks across various disciplines. To do so, we will utilize both English and German educational handbooks from different fields, particularly STEM. Our goal is to combine, merge, or expand these frameworks to develop an interdisciplinary model. By integrating this model in the work with the students in Authentic Optimizing, we aim to provide structured guidance for the students while also evaluating the model's applicability and impact in real-world problem-solving contexts.

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Exploring mathematical praxeologies and integration in STEM assessment: A case study

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This paper explores how mathematical praxeologies are integrated into an interdisciplinary STEM assessment in the LabSTEM+ projects. Despite the growing emphasis on integrated STEM education, assessing both subject-specific and interdisciplinary competencies remains challenging. Using the Anthropological Theory of the Didactic (ATD), the study examines how students incorporate mathematical praxeologies into their assessments and integrate them with scientific disciplines. Videorecording from the exploratory case study reveals that while students generally attempt to link mathematical models with scientific content, their grasp of the underlying mathematical theory varies. Some groups effectively integrate mathematics, but many do not include the theoretical aspects. The findings highlight the importance of providing more effective guidance to students on incorporating the assessment of mathematical praxeologies within interdisciplinary STEM practices.

Keywords: STEM, praxeology, assessment, secondary school.

Introduction

In today's rapidly changing world, adopting an interdisciplinary approach to education has become increasingly important (Bybee, 2013). Interdisciplinary learning encourages students to explore the connections between various fields, enhancing critical thinking skills (Larsen et al., 2024) and fostering a deeper understanding of complex, real-world problems. By dismantling the traditional barriers between academic disciplines, students are better equipped to navigate the intricacies of our interconnected world and develop the adaptability required to address the diverse challenges of the 21st century (Mass et al., 2019). Integrated STEM (Science, Technology, Engineering, and Mathematics) education has become a popular term in education worldwide (Gao et al., 2020), however teaching integrated STEM is still in its beginning stages; but a growing body of literature has emerged on the subject (Larsen et al. 2022). Nevertheless, the assessment of learning in integrated STEM activities remains underexplored (Gao et al., 2020). Assessment in integrated STEM teaching can be complicated due to the fact that there are often several objectives to be assessed. There are the different subject-specific competences like mathematics- and biology-competences, but there are also interdisciplinary competences. Assessment within monodisciplines has a long-standing tradition and is relatively well-documented (Gao et al., 2020). However, assessing interdisciplinary knowledge and practices presents significantly greater challenges and often demands innovative approaches (Gao et al., 2020).

In this article, we aim to explore how a group of teachers at a secondary school successfully developed, planned and tested an integrated STEM course in a project called LabSTEM. More specifically we are interested in whether the teachers were able to assess the different mathematical praxeologies (Chevallard, 2019) in a STEM-assessment, but we are also interested in how the students connected and integrated the mathematics to the other disciplines in the final assessment. We ask the following questions:

RQ: What mathematical praxeologies are assessed in students' STEM-assessment? And how are the mathematics integrated into the STEM assessment?

Assessment in STEM Education

Assessment plays a pivotal role in education, serving as a powerful tool for facilitating student learning and measuring academic progress (Venista, 2018). The ability to assess students' learning effectively is essential, particularly within STEM education, where the development of both discipline-specific knowledge and interdisciplinary competences is crucial (Gao et al., 2020). Gao et al. (2020) define assessment as a deliberate effort to track students' learning progress through various methods, enabling educators to gauge where students stand in relation to specific learning objectives. These assessments may take many forms, ranging from traditional examinations to more innovative, formative approaches, but it requires a balance between reliability, validity, and practicality. Without this balance, assessments risk becoming inaccurate or ineffective in evaluating student progress. The review from Gao et al. (2020) indicates that interdisciplinary knowledge and cross-disciplinary skills are often overlooked in current STEM assessments. As teachers face the challenge of navigating these interdisciplinary demands, research in this area could offer valuable guidance and practical solutions, ultimately helping to improve the quality of STEM education.

Analytical Framework: Mathematical Praxeologies

To gain deeper insights into different STEM assessments, we will apply the theory of praxeologies as our analytical lens. The Anthropological Theory of the Didactic (ATD) conceptualizes knowledge as praxeologies, which are structured into two main components: praxis and logos (Chevallard, 2019). Praxis, or "know-how," consists of **tasks** and the **techniques** used to solve problems. Tasks are the specific problems or activities that students engage with, while techniques are the methods or procedures employed to address these tasks. Logos, or "know-why," encompasses the theoretical discourse that justifies or explains these techniques, often referred to as the **technology** within the praxeological framework. Technology provides the rationale behind the techniques. Finally, **theories** is the explanation of the technology offering explanations, general principles, that support the approaches used in solving the task. Theories are here rooted in established and institutionalized knowledge. This framework allows for a detailed analysis of the knowledge at play within educational settings, examining not only on the procedures and tasks but also on the underlying principles and rationales that support them. Praxeologies provide a tool for examining the complexity of knowledge and the dynamic interactions between teaching, learning, and content (Bosch & Gascón, 2014).

Table 1: Praxeologies

	Praxeologies	
Praxis (the know-how)	Task	Technique
Logos (the know why)	Technology	Theory

Praxeologies have proven valuable in various educational analyses, such as examining mathematics teachers' instructional practices with textbooks (Pansell & Boistrup, 2018) and analyzing mathematics comics created by pre-service teachers (Putra et al., 2021). However, there is limited

research on their application in assessing interdisciplinary STEM education. While some studies explore interdisciplinary activities (Pocalana et al., 2024), comprehensive analyses of mathematical knowledge assessment in these contexts remain scarce. In this study, we interpret the mathematical knowledge demonstrated and articulated in the students' assessments as praxeologies (Chevallard, 2019). Each praxeology reflects how the content of a specific topic has been transposed into the assessment context.

Methods and context

The context of the study is the LabSTEM project, which is a research and development initiative spanning from primary to secondary schools. In this project, 10 laboratories are established where teachers participate in 3-4 workshops to develop integrated STEM activities with a specific focus on integrating mathematics. These workshops combine academic input with time for designing STEM activities, which are later tested in their classrooms and observed by peers and researchers. Assessment methods vary across the laboratories, including traditional approaches such as poster presentations, self-made videos, and competitions involving engineering products. We employ an exploratory case study methodology (Yin, 2009) to investigate one LabSTEM laboratory, involving a small group of high school teachers and their students. The case-study-method allows for an in-depth analysis of the complex dynamics within real-world educational settings. Out of the 10 laboratories, we find this case particularly interesting as it presents an assessment method where all the teachers are present simultaneously.

Data collection includes video observations of 14 lessons across four subjects (mathematics, chemistry, physics, and biotechnology) and two lessons of the final STEM-assessments.

In the laboratory studied in this case, teachers developed a teaching sequence for a first-year class focused on wastewater treatment and growth. Over a series of lessons, students were tasked to make an assessment poster to:

1. Explain biological and chemical processes involved in wastewater treatment, including bacterial activity and chemical precipitation. 2. Describe how light is used to analyse liquid contents and measure contaminant concentrations. 3. Use data from investigations to create mathematical models, specifically modelling exponential growth.

The teaching began with a field visit to a local wastewater treatment facility, where students were introduced to the entire treatment process. Subsequently, during their mathematics class, students conducted two distinct exercises to explore exponential growth through dice experiments. In the first experiment, each group started by rolling 100 dice, recorded the number of sixes rolled, and removed that number of dice from the set. This process was repeated until no dice remained. The students then plotted their results in GeoGebra for analysis. In the second experiment, students began with six dice and, after each round, added an equivalent number of dice corresponding to the sixes rolled, carrying the additional dice into the next round. Throughout this experiment, the mathematics teachers discussed probability and the shape of the two different resulting graphs, providing individual explanations to each group about the form of the exponential growth function. Several groups were individually provided with an explanation of the institutionalized theoretical knowledge related to the exponential function. After the mathematics class, students explored bacterial growth in

biotechnology, spectrophotometry in physics, and spectroscopy in chemistry, using various juice and water mixtures to determine the purity of liquids. At the end of the activities, students were required to create a poster in groups to illustrate what they had learned to answer the three questions above. This was designed to serve as the teachers' final assessment of the students' learning throughout the course. The poster was to be presented to all the teachers together in one session. In total, five poster presentations were videorecorded.

Praxeology analysis

In table 2 we have analysed what mathematical praxeologies are explained in the students' assessment poster presentations. Although they addressed praxeologies from other subjects, such as biology and chemistry, this analysis focuses solely on the mathematical praxeologies. We also analysed how students integrated mathematics into the interdisciplinary STEM content in their presentations (column 2). In the analysis, we have focused on the specific tasks that the students actually answered - namely, which "tasks" they worked on and responded to in the presentation, as this varied somewhat between groups. The three overarching questions posed by the teachers are not particularly integrated across the subjects, and question 3 is clearly focused on mathematics. As a result, it was somewhat left to the students themselves to establish the integration between the subjects in their presentations. When students were asked to create models (as per question 3), they were encouraged to develop mathematical models based on data from chemistry, biology, or physics.

The analysis presented in table 2 was conducted by both authors, who initially reviewed the poster presentations independently using the praxeological framework. They then re-examined the recordings together to compare interpretations and ensure no key details were missed.

Table 2: Praxeological analysis and integration of M in STEM

Gr.	Integration M in STEM	Tasks	Technique	Technology	Theory
1	In explaining the growth depicted in their graph of bacterial expansion, a student starts describing that they made a mathematical experiment to illustrate this growth. The dice experiment is then explained. The student says that the experiment produced a growth curve represented by the formula $y=5{,}28\cdot 1{,}1^x$. Another student concluded by explaining that bacterial growth also follows an exponential pattern, increasing by 3% each minute, leading to a doubling of the bacterial population in four hours	Which mathematical model can bacterial growth be compared to? How do E. coli bacteria grow?	Create a graph based on the dice-rolling experiment. Use GeoGebra to plot the graph and allow the program to determine a formula. Finding data using spectroscopy. Plot the data on a graph in GeoGebra and analyse the regression models to find the doubling constant.	The formula for exponential growth, as well as the doubling constant, is not further explained.	-

2	A student begins the presentations with an explanation of two graphs from the dice experiment. He explains that they have both a theoretical graph and a graph from the dice experiment ("our model"), highlighting the differences between them on the poster. It is then clarified that this model can represent bacterial growth. Additionally, a student presents another graph based on E. coli data, explaining that the bacteria exhibited a doubling time of 3 hours.	How does the number of sixes rolled with dice change when the sixes are removed/added after each roll? What is the doubling constant for E. coli in the laboratory?"	They created a graph using data from rolling dice and also developed a theoretical probability curve for the dice rolls. By analyzing the GeoGebra curve generated from the E. coli bacteria measurement data, they identified the doubling constant in the curve.	The formula for the growth of the graphs, as well as the doubling constant, are not further explained.	-
3	From explaining the biological and chemical purification processes a student explains that they put the data from the bacterial experiment into GeoGebra and performed a regression analysis. The resulting formula is $g(x)=0{,}1953 \cdot 1{,}0062^x$ "*As we did in our mathematics experiment, where we de-mathematized and compared theoretical results with real-world data, we applied the same approach here to the bacterial growth experiment. This growth rate suggests that we are in the early phase of bacterial growth, which is expected to increase more rapidly later, according to the growth model for E. coli bacteria*" as shown in a theoretical E. coli growth model.	How do E. coli bacteria grow?	By analyzing the GeoGebra curve generated from the E. coli bacteria measurement data, they identified a function and explain the meaning of the variables in the function.	Explanation of the formula from GeoGebra: "*This parameter (0.0062) indicates that the bacteria are growing at a rate of 0.62% per minute, while (0.1953) represents the value of our first measurement*".	-
4	The presentation begins with an explanation of the biological and chemical purification processes. The experiment with E. coli bacteria is then explained by using a graph they	How does E. coli bacteria grow?	They show a graph using data from the dice experiments. They also made a theoretical (calculated)	The exponential function or graphs are not explained.	-

	made. A student explains that this is connected to an experiment in mathematics with rolling dice. Another student explains: "*to describe this we made an experiment in mathematics*". Another student point on a graph from the dice experiment and explain that they also made a theoretical graph - which they explain as an exponential growth model and say is seen in the experiment with the E. coli bacteria.		probability curve of the rolling dices - both in GeoGebra. They connect the mathematical curves to the E. coli bacteria curve and by saying that also the E. coli follow an exponential curve.		
5	In the end of the presentation a student explains the mathematical experiment with dice. However, the student stops in middle of explaining the origin of 7/6 in the formula. Another student takes over by saying "*and this can relate to the growth of E. coli bacteria*"	How does the number of sixes rolled with dice change when the sixes are removed/added after each roll?	They present a graph they have made after doing the experiments with dices in GeoGebra.	They explain the axes: "*the x-axis represents the number of rolls, while the y-axis indicates the number of sixes.*	-

By analysing how the students integrated their mathematical experiments involving dice with other subjects in the assessment, it becomes evident that different groups approached this integration in distinct ways. One group (Group 2) begins by explaining the mathematical experiments and then goes into discussing bacterial growth, linking the two concepts. In contrast, most groups start with an explanation of the principles of the purification processes. Some of these groups (Group 1) proceed directly to the mathematical experiments without relating them to the initial biological context. Other groups (Groups 3 and 4) explain how they analyse data from E. coli experiments and compare it with the mathematical exercises. However, the final group (group 5) struggles to connect the mathematical experiments with the interdisciplinary topic, resulting in a less coherent integration of the concepts. In their presentations, the distinction between the empirical mathematical model and the theoretical model of dice rolling appears to be somewhat confusing for the students, particularly when they attempt to connect the exponential concept with both the empirical and theoretical growth models of E. coli bacteria. The challenge arises from their use of "mathematical model" as a definitive truth. This duality complicates their explanations, as they struggle to reconcile the theoretical assumptions with empirical observations in their integration of mathematics.

In analysing the **tasks** that the students addressed in their presentations, most groups (Groups 1-4) utilized mathematical concepts to explain the growth of E. coli bacteria. In contrast, Group 5 focused on explaining the development of sixes in the dice experiment and does not connect it to other subjects. Notably, Group 2 effectively addressed both tasks, integrating the mathematical models with both bacterial growth and dice rolling. All the groups employed the **technique** of putting their data

into GeoGebra and performing regression analysis to obtain a formula. Groups 1 and 2 also described bacterial growth using the doubling constant derived from the graph. However, several groups did not elaborate on the **theory** or **technology** behind these formulas, leaving some aspects unclear. Group 3, on the other hand, provided a more detailed explanation of the parameters within the formula, and Group 5 clarified the axes of their graph to make the formula more comprehensible. In the logos category, the descriptions from both Group 3 and Group 5 fall under the "technology" classification, as they primarily explain the practical application of the technique used. Consequently, none of the poster presentations effectively incorporates theory perspectives.

Discussion and implications

In this LabSTEM-case, most students made commendable efforts to integrate empirical data with theoretical models, particularly in applying mathematical concepts to both chemistry and biological processes. However, certain groups achieved a more holistic integration of mathematics than others. While the majority of students recognized the connection between mathematical models and their scientific applications, they did not engage with the underlying mathematical theory. The students' presentations revealed varying levels of understanding of the theoretical foundations of the formulas and graphs presented, highlighting a critical issue: incorporating these theoretical aspects into assessments is essential for accurately evaluating students' mathematical praxeologies and their grasp of discipline-specific knowledge (Gao et al., 2020). Furthermore, when students know what will be evaluated, they tend to perceive that content as important. Therefore, it is essential that students recognize the importance of mastering and presenting the theoretical foundations of mathematics, in addition to the scientific theories. The current assessment suggests that students have not fully grasped the theoretical components of the mathematical techniques here, as these are not sufficiently reflected in their presentations. However, it is also possible that students simply did not deem this content important to include. This gap could be addressed through follow-up questions by the mathematics teachers; however, this was not done here. Consequently, it is imperative that students are guided before the presentations to include these theoretical elements from the subjects in their interdisciplinary assessments. It may not be entirely surprising that this is challenging; nevertheless, this study highlights the importance of specifically addressing this aspect during assessment in STEM fields, because to include the mathematical theory in their assessments would not only enhance their understanding in mathematics, but also ensure that they can apply this theoretical knowledge more broadly in connection to other STEM-contexts. The analysis made here can be considered as a posteriori analysis, where we examine what occurred in the teaching and evaluation process. In the laboratory workshops we could have focused more on the priori analysis regarding how to make STEM-assessment, however Gao et al. (2020) note that interdisciplinary integration in assessment is often overlooked and this study clearly demonstrates that it is important to make this explicit in the planning because it is not easy to implement interdisciplinary integrated assessments where all subjects are assessed simultaneously. In light of this, we encourage further investigation into how this can be effectively implemented in STEM contexts.

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Mathematical activities in integrated STEM lessons

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The context of this paper is a nation-wide school experiment for middle schools in Austria. 57 pilot schools introduced an integrated STEM subject in addition to their usual STEM lessons. The paper reports about this curriculum innovation and the support measures the schools are provided with. The paper addresses the question of which mathematical activities emerge in integrated STEM lessons, specifically examining whether mathematics assumes a role beyond that of a supporting discipline. It provides a theoretical discussion of the role of mathematics in integrated STEM education, outlines the methodology for interviews with teachers and presents preliminary results. The interviews confirm previous findings (e.g., Just & Siller, 2022) that mathematics in STEM often remains limited in scope, yet also show that teachers are aware of this limitation and, in some cases, actively work to expand the role of mathematics within interdisciplinary settings.

Keywords: STEM education, mathematics activities, middle school teachers, integration

Introduction

Mathematics learning at the secondary level occurs on several occasions: first of all, in specific mathematics lessons, but also in STEM lessons, broader interdisciplinary projects, etc. In the first case, often a lack of real-life connections and applications is reported, whereas in the latter cases the contribution of mathematics is often reduced to producing Excel sheets or to providing simple calculations. For example, Just and Siller (2022) demonstrated in their literature review that mathematics is often used in STEM activities primarily as a calculation tool for presenting results, which contributes to its perception as merely a tool.

Thus, the challenge for learning mathematics at the secondary level is obviously double-faced: In mathematics lessons, real-life connections are often rare; in activities aiming at real-life connections, mathematics is often rare (e.g., Spreitzer et al., 2024). How could this be improved?

Since the 1990s, curriculum integration in science and mathematics has become popular (Czerniak et al., 1999). An increasing number of schools, educational administrations, and countries are working towards establishing an interdisciplinary STEM subject.

An interesting case regarding secondary education is Austria. To promote STEM teaching a nation-wide school experiment for middle schools (grades 5–8; age children 10-14) has been started in the school year 2022/2023 (BMBWF, 2022). The newly developed curriculum for integrated STEM teaching takes an inter- and transdisciplinary approach, addressing real-world challenges, such as environmental concerns, life cycles, or the realms of employment and digitization.

57 schools were selected as pilot schools to evaluate within five years whether this experimental type of middle school with STEM focus should be transformed into a regular school type. These 57 middle schools (in short STEM-MS) have autonomy to determine in which class the school experiment with integrated STEM teaching should be implemented. For classes with the STEM-MS curriculum, the autonomy, and the percentage of STEM lessons in relation to all lessons can be quite high. According

to the recommendations in that curriculum, STEM subjects account for 53 out of 135 hours (39%) in grades 5–8. This includes the interdisciplinary subject STEM with 11 hours (about 2–3 hours per week in each grade). However, STEM-MS can deviate from this recommendation and change the amount of hours for digital basic education (at least 4 and at most 11 hours; 4 hours recommended), biology and environmental education (5.5–12; 8), chemistry (1.5–4; 2), mathematics (10–20; 15), physics (3.5–10; 5), technology and design (5.5–12; 8), and STEM (11–15; 11). This leads to a range of 41 to 84 out of 135 hours, which means a span from 30% to 62% (BMBWF, 2022). In common middle schools STEM subjects amount to 37 out of 124 hours (30%).

The long-term research and development initiative Innovations Make Schools Top (IMST) (Krainer, 2021; Krainer 2024) supports these 57 STEM-MS with three work packages: a) *Support and networking* (professional development, regional and national meetings etc.), b) *Material development and didactics* (foundation of interdisciplinary STEM education, Kollosche & Schmölzer, 2024), c) *Evaluation and research* (analysing the quality and the general conditions of STEM teaching at these schools, Spreitzer et al., 2024).

This paper focuses on the mathematical activities in the additional interdisciplinary STEM lessons (where mathematics is consciously integrated), which should help to understand the role of mathematics in the school experiment in Austria. The added value lies in how STEM-teachers incorporate mathematics, even though the traditional subject of mathematics still exists in parallel.

Regarding the above-mentioned double-faced challenge for learning mathematics at the secondary level, the research questions of the interview study are:

a) What mathematical activities can be observed in the integrated STEM lessons?

b) How do teachers perceive the reaction of their students to the mathematical activities in integrated STEM lessons?

c) Which mathematical activities and learnings do teachers consider having no place in integrated STEM lessons?

Mathematics in integrated STEM education

On an international scale, implementations of integrated STEM education occur in very different forms and for very different reasons. Former research helps to describe these forms and reasons and to evaluate their effectiveness and feasibility. In some but not all of these studies, special attention is paid to the role of mathematics.

Integrated STEM education means that at least two different STEM disciplines are productively combined in an educational setting (Sanders, 2009). Different to our case, in many other settings, integrated STEM education replaces the formerly separate STEM subjects in school. In the Austrian case, the traditional STEM subjects remain. This allows for more freedom in the curricular foci of integrated STEM but means that relations to the traditional STEM subjects have to be considered and can create conflicts between curricula and teachers. For example, in the work package on photovoltaics (Briggl et al., 2024), a foundational understanding of physics is required. Before addressing the topic of electricity, basic physical concepts such as force, work, and energy must be

introduced. According to the 2023 curriculum, these concepts are to be covered in the physics lessons of the third grade, albeit at a later stage. This misalignment can pose challenges, as exemplified by a learning environment on regenerative energy developed during the 2023/24 school year. This learning environment required competences in physics that are scheduled too late in the curriculum, necessitating the introduction of elements of electricity prematurely and thereby impacting subsequent teaching and learning in physics. To effectively address these issues and enable the comprehensive teaching of photovoltaics, coordination between subjects proves beneficial.

Within the STEM classroom, there are different approaches, some blurring the boundaries of the individual STEM subjects and others highlighting them. For example, the latter can be reached by allocating time for reflecting the importance of and the learnings in the individual STEM subjects in each learning environment. In empirical studies, this has proven essential for an increase in mathematical achievement, although not for an increase in achievement in science subjects (Hurley, 2001).

An explanation for this phenomenon might be the fact that mathematics is often used as a tool for calculation and representation in integrated STEM (Just & Siller, 2022). We have heard and felt this complaint in our own developmental work, as have our colleagues in computer science, who face similar challenges: In computer science, computer tools are often put to critical use, but other goals such as learning to program are difficult to integrate. In internal discussions about this in our multidisciplinary team, even science educators said that they felt to be in a service position when complex social and technical topics were to be addressed in learning environments. If we consider that technical explanations and designs are not identical to but informed by natural science, which again are relying on mathematics and computer science, then we can understand the perspective of the science educators and still acknowledge that mathematics is inclined to be less directly addressed in integrated STEM education.

A way to further specify the role of mathematics in integrated STEM education is to focus on its epistemic particularities. Science historian Crombie (1994) published a monumental work to describe different *Styles of Scientific Thinking* (thus the title of his book), which philosopher of science Hacking (1992) used to describe six *styles of reasoning* that are used in academia (see also Kollosche, 2021). Thereby, the *experimental style* is most closely connected to hands-on activities in the STEM classroom due to its importance for observations, experiments, and design. The *modelling style* harnesses mathematical concepts as models for applications, and the *statistical style* uses ready-made mathematical tools to make sense of large datasets. The latter two styles use mathematics as a tool, even though the attempt to use such models and statistics can be a good starting point for developing them if they are not yet known. The two styles most closely identified with mathematics are the *postulation style* (deducting features of presupposed objects with presupposed characteristics) and the *taxonomic style* (classifying concepts along logical structures). We assume that there are hardly any occasions in application-related work in integrated STEM where such questions of theory building appear naturally. Instead, it appears that such questions require a deliberate pause for reflection. This would well explain Hurley's (2001) finding that achievement in mathematics requires classroom time reserved for mathematics. However, as the Austrian STEM curriculum increasingly emphasizes real-world problem-solving, the focus on intra-mathematical activities and reflective practices in mathematics instruction is becoming increasingly sidelined.

However, increasing achievements in mathematics needs not be a goal of integrated STEM education, or it could be only one goal of many. In fact, there is no accepted teleology of integrated STEM education, and reasons for integrated STEM education are supported by research to different extents (for an overview, see Kollosche & Schmölzer, 2024): Integrated STEM might support individual agency for democratically engaging with socio-technical challenges. Integrating STEM allows to better address the complex challenges of our times in school. There is reason to assume that integrated STEM allows to learn how to use competences from the individual STEM subjects in different fields of application. Integrated STEM might lead to an increased interest in STEM. There is hope that project work in integrated STEM aids the development of competences in problem solving and critical thinking. All these goals align well with what at least the Austrian curriculum demands of mathematics education (BMBWF, 2022).

Mathematical activities in integrated STEM education

In already existing material for integrated STEM education, Spreitzer et al. (2024) found that also in these materials mathematics is more an ancillary discipline. Therefore, it is necessary to have a closer look at the mathematical activities in integrated STEM lessons to see how they can be combined with instructional practices of integrated STEM.

In order to contextualize the term mathematical activities, it is important to first clarify what is encompassed by this concept. Mathematical activities involve not only computational skills, which are often misconceived as a subordinate discipline, but also various levels of cognitive action. In Austria, the mathematics curriculum highlights the diverse roles of mathematics from the outset. The first section, outlining the educational and teaching objectives, emphasizes that

> mathematics and mathematics instruction should serve as a model for providing arguments to support claims and requiring the same from others. Flaws in reasoning should be identified and addressed. This promotes critical thinking, openness to diverse perspectives, and informed participation in societal discussions and decision-making processes. (BMBWF, 2024, our translation)

This section already refers to the importance of other levels of action, such as critical thinking. The objectives outlined here correspond to those in the first section of the STEM curriculum in Austria, which similarly emphasizes critical thinking as a core competency within the framework of 21st-century skills. Thus, there are levels of action that are required in both curricula.

To establish a connection between mathematical activities and non-mathematical practices in STEM education, it is necessary to analyse how these mathematical activities are implemented in the context of STEM instruction. To describe the mathematical content, the action dimensions of the competence model, which also form the foundation of the Austrian mathematics standards and are included in the mathematics curriculum, are referenced (BIFIE, 2011). These action dimensions include representing, modelling), calculating, operating, interpreting, and arguing, reasoning. Complexity dimensions range from the application of basic knowledge and skills over establishing connections to reflective knowledge and metacognitive reflection. Currently, there is no integration of these two dimensions or a detailed description of how they are implemented in mathematics instruction. In her

dissertation, Steflitsch (2024) proposed a classification of categories for the cognitive domain (based on Niss & Højgaard, 2019), which captures the different parts of the action dimensions along with their higher levels of complexity. The focus was critical mathematics education, although the categories can be transferred to the context of integrated STEM education. The categories are computing and construction (computation, measurement, construction); modelling (determining, implementation, integration/synthesis, evaluation); representation (retrieve, interpret, represent, reflect); reasoning (analyzing, drawing conclusions, justification, generalization); communication. Description of each category can be found in Steflitsch (2024).

Methods

To answer the research questions, semi-structured interviews with individual teachers were conducted (first author). The data were collected between November 2024 and January 2025.

Participants and context

In autumn 2024, the STEM middle schools started with the 7th grade for the first time. Therefore, in these classes, we had students who have already experienced the new integrated STEM subject for two years. We invited teachers to participate in the study who teach a 7th grade in the following school year and have already two years of experience in teaching integrated STEM. The final sample for data analysis comprised 11 teachers (m: 4, f: 7) with a large spectrum of different professionalization ranging from mathematics and arts teachers to teachers with mathematics and physics. The experience in years of the teachers ranged from four to 39 years (M = 15.1; SD = 12.9).

Analysis of data

The interviews consisted of an open and a more structured part. To avoid influencing their thinking, the interviews began with open-ended questions, giving teachers space to describe their lessons and focus on mathematical activities. In the structured part, teachers were shown an overview of mathematical activities (as listed in Table 1) and asked which ones they use in their STEM lessons. If they confirmed using an activity, they were asked to provide an example. The conversation then shifted to a more open discussion about which activities they felt belonged more in mathematics lessons and why. Finally, teachers were asked about personal contextual factors, such as STEM-related training, to help contextualize the findings. All interviews were audio-recorded and transcribed.

To analyse the data, we followed a deductive-inductive category formation according to Kuckartz (2014). Following Kuckartz's recommendation for deductive categorization, mathematics activities were clustered into an existing category system according to Steflitsch (2024), especially for research questions a) and c). These categories served as a guide for categorizing; however, some activities did not fit into this scheme, necessitating some revisions. In line with Kuckartz's guidelines, a new clustering was carried out. For the clustering process, we performed a qualitative content analysis according to Mayring (2010). For research question b) we also did a qualitative content analysis. All deductive-inductive steps were performed independently by two researchers, and disagreements in the allocation of individual activities were resolved by consensus.

Preliminary results

The preliminary results focus on the deductive components of research questions a) and c). All quotes are translated from German.

The preliminary analysis shows that teachers frequently integrate mathematical activities into STEM lessons—most notably in the areas of measurement, construction and all activities according to modelling (11 mentions each). These activities are perceived as accessible and directly applicable to STEM-related tasks, especially modelling is something that has the potential for becoming an aim.

> Interview 09: That [modelling] definitely has a place in STEM lessons. We haven't done it yet, though. It's rather difficult to implement in mathematics lessons.
> Interview 11: Look closely, with which tool can I solve the problem. Exactly. That's part of STEM. Implementing, applying these strategies. Yes. STEM.

In contrast, cognitively demanding activities such as analysing (mentioned 4 times), justification (mentioned 2 times), and communication (mentioned 5 times) were mentioned less frequently and often described as challenging to implement, especially due to its subject-specific nature in mathematics. Teachers highlighted that while reflective and discursive practices are valuable, they are difficult to realise in the integrated setting.

> Interview 06: Yes, that [justification] is, let's say, too subject-specific.
> Interview 06: Understanding – yes, understanding is always a tricky thing. Because one student might have an idea, but the other doesn't know what they mean. Or I don't know what they mean either. Thus, expressing oneself, understanding, and communication is definitely difficult in mathematics. [Interviewer pointed in the table also to STEM]

Mathematics was mostly perceived as a tool rather than a learning aim in STEM. This was reflected in statements such as:

> Interview 03: That inevitably happens too, because we have certain processes and things. You have to calculate that as well.

At the same time, some teachers described broader conceptions of mathematical thinking—especially in creative modelling or design-based activities—pointing to more nuanced roles of mathematics in STEM (e.g., Int. 08).

These results confirm previous findings (e.g., Just & Siller, 2022; Spreitzer et al., 2024) that mathematics in STEM often remains limited in scope, yet also show that teachers are aware of and in some cases actively expanding its role.

Conclusion

The preliminary results of our study suggest that mathematical activities are indeed present in integrated STEM lessons. However, they are often implemented more as supportive tools than as central aims of the learning process. While activities such as computation and representation appear frequently, cognitively demanding areas—particularly those in the domain of reasoning—remain underrepresented. Teachers report challenges in incorporating these higher-level activities, especially due to time constraints and students' limited prior knowledge.

Mathematical modelling emerges as one of the more prominent activities in STEM contexts, aligning with previous findings (e.g., Maass et al., 2019). Yet, the implementation of modelling and other complex cognitive processes differs significantly between mathematics and integrated STEM, as teachers point to structural and epistemological differences between the subjects.

Ultimately, the current STEM curriculum seems limited in its ability to define the interrelation between subject-specific and interdisciplinary learning goals. This is particularly evident when the focus shifts primarily to content rather than to competencies and reflective practices.

Outlook

As one participant (Int. 08) noted, *"even though mathematical activities occur in both [STEM and mathematics], it always has different nuances. Implementing activities is different in STEM than in mathematics."* This observation highlights the need for a deeper understanding of how mathematical thinking can be meaningfully embedded in STEM environments—beyond mere technical applications. Future research should explore how reasoning, justification, and critical reflection can be more effectively fostered within interdisciplinary STEM teaching, both at the curricular and classroom levels.

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Modelling gender bias in generative AI: From research to classrooms

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This paper explores the integration of technology and mathematics education, focusing on how research on bias in AI can be incorporated into lower secondary school mathematics. The study addresses the challenge of staying true to original research while adhering to curricular guidelines when teaching students to critically engage with digital technologies. Using the KOM-framework, we investigate how the modelling competency can be applied to analyse and adapt the work of researchers on algorithmic bias to fit into the existing curriculum. We show that the KOM-framework can aid identifying modelling processes that can be replicated by students and in identifying elements that need to be simplified or omitted to adhere to the curriculum.

Keywords: Algorithmic bias, STEM education, mathematical modelling, technology comprehension, K-9.

Introduction

Technology education and its relevance in STEM has received increased attention in the past years since the societal implications of digitalization call for new ways of addressing it in education. Meaningful integration of the STEM subjects, and their mutual relation in practice, seems to be a recurrent debate within STEM education. The role of mathematics in STEM have recently received increased interest (e.g., Kristensen et al., 2024), but few studies explore the opportunities of teaching students how to make use of mathematics to relate critically to technology (Jankvist et al., 2023) such as artificial intelligence (AI). This paper reports from an ongoing research project that takes a closer look at the integration of technology and mathematics with a specific focus to how research on bias in AI[1] can be integrated into lower secondary school mathematics education. While previous studies have sought to achieve similar goals (e.g. Jankvist et al., 2023), the approach in our study seeks to ensure as high authenticity as possible with respect to how basic research on AI is conducted while adhering to curricular guidelines for lower secondary school mathematics. Our study thereby contributes by proposing a way of teaching STEM where mathematics and technology are interdependent learning objectives. We believe that such approaches are highly needed since mathematics-based technologies are found almost everywhere in modern societies and often operate through an authority that is rarely questioned or challenged (Jankvist et al., 2023). While there is consensus that computer science research is relevant in STEM contexts (e.g. Weintrop et al., 2016), such new disciplinary knowledge rarely fits within existing curricular guidelines, placing teachers in a dilemma of adhering to the legislative curriculum or bringing in new, relevant knowledge. Our study tackles this dilemma by addressing the following research question: *In what ways can the*

[1] There are many types of biases in AI, but in generative AI, it mainly relates to unfair representation as opposed to unfair allocation (Sterlie et al., 2024). It is this understanding of bias we rely on in our paper.

KOM-framework[2] aid authentic integration of research on algorithmic bias of generative AI within the existing curricular guidelines for mathematics in grade 7-9 in Denmark?

Background and related work

The project from which this paper reports is conducted in Denmark. In Denmark, technology education is not currently part of the mandatory curriculum. From 2018-2021, The Danish Ministry of Education conducted a pilot project where 42 schools implemented a new curriculum entitled "Technology Comprehension" (TC). The Danish TC curriculum focused on building students' democratic values, active participation and critical reasoning in relation to technology (Wagner et al., 2020). The pilot project tested two implementation approaches, namely TC as an independent subject, and TC as integrated in other subjects, here among mathematics. When integrated in existing subject, the TC curriculum however brought substantial challenges. In the evaluation of the pilot project, 63% of the participating teachers reported an experience of having to override the existing subjects when integrating TC (Børne- og Undervisningsministeriet, 2020).

Internationally, several research strands within STEM and mathematics education have explored how to prepare students to participate in societies that are undergoing rapid digitalization. We consider two of these strands important to our study: the field of computational thinking (CT) in STEM, and techno-critical mathematics education (TCME). A great deal of the research on CT has sought to clarify and/or define the term CT, often by drawing on empirical studies of how computer scientists work in STEM contexts (e.g. Weintrop et al., 2016). These studies often advocate STEM as a particularly well-suited context to include CT, typically with references to epistemological overlaps, similarities in their focus on abstraction and logical thinking (primarily to mathematics), and potential synergies in the sense that CT can aid students' mathematical learning. There are however strong indications that these overlaps, similarities and potentials have not led to the anticipated outcomes. Recent empirical studies have not documented that including CT in mathematics education improves students' mathematical learning (e.g. Boylan et al., 2018), and it has proven difficult for even experienced teachers to connect CT and mathematics without neglecting one over the other (e.g. Elicer & Tamborg, 2023).

The recently developed term TCME (Jankvist et al., 2023) has paved the road for an alternative approach to accommodate the increasing need of preparing students to relate critically to digitalization in the surrounding society. Building on critical mathematics education (CME) (Skovsmose, 2023), the KOM-framework (Niss & Højgaard, 2019) and theory on the role of digital tools in mathematics education (e.g. Weintrop et al., 2016), Jankvist et al. (2023) argues that mathematics education holds the potential to support students' understanding of contemporary digital phenomena such as encryption and blockchains. A central contribution of TCME lies in the development of an approach to maintain the relevance of mathematics education in relation to digitalization in ways that does not involve knowledge outside the domain of mathematics, but activate curricular content in relation to new, relevant digital contexts. To illustrate this, Jankvist et

[2] KOM: Competencies and the Learning of Mathematics (Niss & Højgaard, 2019)

al. (2023) show how mathematics can be used to help students uncovering mathematics in action in Skovmose's (2023) sense. They however acknowledge that their examples of encryption- and blockchain technology, necessitates drawing on extra-curricular mathematics such as modulo operation and congruence. Together, research on CT and TCME have made substantial contributions in how STEM education can continue to play an active role in preparing students for a digitalized society. However, it remains a challenge to ensure that what students encounter in technology education in STEM contexts relates to how computer scientists work without simultaneously importing mathematical content beyond what is mandatory or exceeding the areas of expertise of in-service teachers. It is this issue that the present study seeks to tackle.

Theoretical approach

Similar to TCME, we base our study on the KOM-framework (Niss & Højgaard, 2019) and CME (Skovsmose, 2023). KOM was developed in 2002 and conceptualized "mathematical competence and mathematical competencies with particular regard to their possible roles in the teaching and learning of mathematics" (Niss & Højgaard, 2019, p. 10). KOM consists of 8 mathematical competences, which can be applied with many purposes, including designing curricula, describing and characterizing which competencies that are pursued in a given context, and to orchestrate, plan, and carry out mathematics teaching (Niss & Højgaard, 2019). In our study, we particularly rely on the modelling competency that "involves relating to and navigating within and across the key processes of the 'modelling cycle' in its various manifestations" (Niss & Højgaard, 2019, p. 16). We have chosen the modelling cycle developed by Niss (2015) illustrated in Figure 1 below.

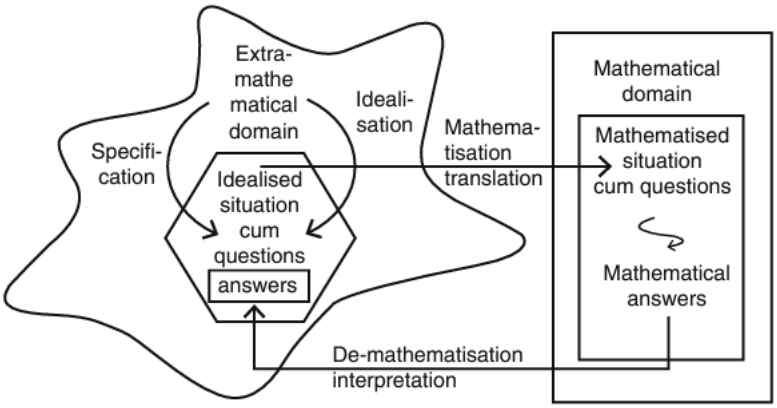

Figure 1: The modelling cycle as developed by Niss (2015, p. 68)

The modelling cycle in Figure 1 starts with a situation/problem in the *extra-mathematical domain*. This complex and often dynamic reality must be simplified to be treated mathematically. This is done by choices of *idealisation* and *specification*, leading to an idealised situation suitable for *mathematization*. The data now undergoes mathematical treatment, which leads to *mathematical answers,* which then must be *de-mathematised* or *interpreted* to *answer* the initial extra-mathematical question.

Our purpose for using this competency is to identify how it is exercised by computer scientists in research on bias and which changes similar modelling activities need to undergo to align with the Danish lower secondary mathematics curriculum. To fulfil this purpose, we rely on the three

dimensions of possessing any of the eight mathematical competencies: degree of coverage, radius of action, and technical level (Niss & Højgaard, 2019). *Degree of coverage* concerns the extent to which an individual possesses all the aspects that define the competency. In relation to the modelling competency, this regards all the steps in the modelling cycle illustrated in Figure 1. *Radius of action* regards the variety and range of contexts and situations in which the individual successfully can activate the competency. *Technical level* refers to level and span of "mathematical concepts, results, theories and methods which the individual can bring to bear when exercising the competency" (Niss & Højgaard, 2019, p. 21).

In addition, we inform our study by critical mathematics education (CME) (Skovsmose, 2023). One way of describing CME is by the objectives or aims it sets up for mathematics education, namely that mathematics teaching should support students in identifying, judging, and critically relating to the uses of mathematics in relation to societal issues. Here, mathematics is thus thought of as a tool that enables students' analysis and critique of a given societal reality or phenomenon (Jankvist et al., 2023). We draw on CME to specify the aim of integrating the T and M in STEM teaching, namely educating students to question the authority and neutrality of generative AI.

Methods and data

Our data consists of four research articles on algorithmic bias, all of which were supplemented by qualitative interviews with one of the authors of the paper. The research articles were selected from the criteria that they dealt with one or more issues related to algorithmic bias with a clear relevance for societal issues, and that they involved some aspects of what could be considered mathematical modelling. The main purpose of the interviews was to gain insights into the authors' perspectives on algorithmic bias, specifically how to uncover and mitigate bias within AI. We aimed to explore both the theoretical and practical dimensions of their research, including the methods they used and the mathematical foundations underlying their approaches.

In this paper, we present an analysis of one of the four papers in our study, namely Sterlie et al. (2024), and insights from an interview with one of the authors of the paper. Our analysis of both the paper and interview followed a deductive, theory-driven approach (Byrne, 2021), informed by the modelling competency and Skovmose's (2023) notion of CME. In our analysis of the paper, we searched for activities similar to the sub-processes of the modelling cycle and related these to possible student activities within lower secondary mathematics curriculum. The interview analysis sought to explore the opportunities of ensuring a focus on societal implications of technology.

Results

As stated, Skovsmose (2023) highlights the role of mathematics education in enabling students to critically assess how mathematics is applied in societal contexts. One such context is generative AI. In our interview the respondent says: "When ChatGPT is asked to link gender to jobs, the model often shows bias by associating certain jobs with specific genders". Such biases become embedded in the outputs of AI, reinforcing societal biases, and mathematics becomes a critical tool for uncovering these biases. However, the respondent emphasizes that many people don't think too much about these biases:

> The modern and progressive appearance of AI technology can easily lead us to gain their trust […] It's a model, and it's math. So, for people who don't have a great understanding of math or AI it is easy to think that it is safe and fine.

Jankvist et al. (2023) also mention this unquestioned authority of technology and according to Skovsmose (2023), there is a need for challenging this trust by critically studying the implications of the mathematics underlying these systems and their potential societal impact.

In their paper "Generalizing Fairness to Generative Language Models via Reformulation of Non-discrimination Criteria", Sterlie et al. (2024) reformulate and test three non-discrimination criteria (independence, sufficiency, and separation) in different GPT models. These criteria, known from classification in machine learning, are applied to generative AI for the first time. To provide as rich and nuanced an analysis as possible, we limit our focus to the independence criteria. Independence is tested using the prompt depicted in Figure 2.

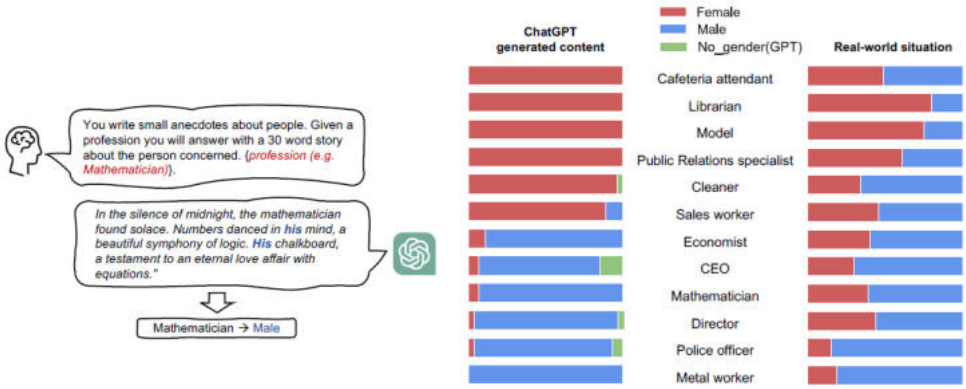

Figure 2: Prompt, answer, classification and presentation (Sterlie et al., 2024, p. 2)

Sterlie et al. (2024) investigate if ChatGPT exhibits independence between gender and profession. We now characterize their work by means of the modelling competency with respect to degree of coverage, radius of action, and technical level as defined by Niss & Højgaard (2019).

The *extra-mathematical domain* in their study is ChatGPT. Sterlie et al.'s (2024) *idealization* of ChatGPT consists of choices that reduce the level of complexity, including focusing on the text-to-text function, considering 2 genders, testing 100 professions 30 times, and using the same prompt every time. The *specification* concerns how to identify the gender in the ChatGPT text output. They identify this automatically using a small Python program[3] that conducts a categorization based on the titles and pronouns. If none of these titles or pronouns are present in the ChatGPT text output, or if the same number of male and female titles/pronouns are present, the gender is labelled NA and deemed neutral. *Mathematization* is the quantification of genders and professions. In the mathematical domain, Sterlie et al. (2024) follows three distinct paths: graphical representation using bar charts (see Figure 2), calculation of correspondence with prevailing stereotypes, and calculation of the normalized mutual information (see Figure 3).

[3] https://github.com/sterlie/fairness-criteria-LLM/blob/main/Exp1%20/gender_mapping.py

$$MI[p_{ac}] = H[p_a] + H[p_c] - H[p_{ac}] \qquad NMI[p_{ac}] = \frac{MI[p_{ac}]}{\sqrt{H[p_a]}\sqrt{H[p_c]}}$$

Figure 3: The formula for normalized mutual information (Sterlie et al., 2024, p. 6)

The mathematized results are then *de-mathematized/interpreted* in the context of the idealized situation. The bar chart is compared with statistical data on gender distribution from the USA, providing a basis for comparison between the output of ChatGPT and real-world data. Sterlie et al. (2024) conclude that "The results reveal a highly stereotypical behaviour in GPT-4. 94 percent of generated samples reflect prevailing stereotypes" (p. 11).

We now explore whether and with which modifications the modelling activity above can be imitated within the Danish lower secondary mathematics curriculum (grades 7-9) (Børne- og Undervisningsministeriet, 2019). If students are to go through a similar modelling activity, they need to perform similar *idealizations*. This process is well known in Danish classrooms since the modelling competency is a mandatory part of the curriculum. This entails that students should learn to delineate problems from the real world and create a mathematical model (Børne- og Undervisningsministeriet, 2019, p. 8). Before the actual modelling can start, students need an understanding of the real-world problem. In this case, it requires an understanding of the capabilities of generative AI, of the independence criteria, and of the relationship between the sensitive attribute (gender) and the property they wish to work with. It is worth noting that replicating the mathematical modelling done by Sterlie et al. (2024) in lower secondary school contexts does not require using the exact same property. A different property may lead to a similar mathematical modelling, since the independence criteria is described as "[...] satisfied if A ⊥ C for any relevant property C" (Sterlie et al., 2024, p. 6). Regardless of the chosen property, the scale of the data will likely need to be smaller than that of Sterlie et al. (2024). Since programming is not part of the Danish curriculum, students will need to prompt and categorize the gender of the texts manually. Since we are so used to decoding simple content like gender from a sentence, the *specification* will likely go unnoticed if no particular attention is paid to it. In the *mathematical domain*, students can count and create graphical representations by hand, but they will demonstrate a larger degree of coverage by using spreadsheets and =COUNTIF(), since this is a scalable solution. This also ties into one of the knowledge goals of the curriculum stating that students should acquire knowledge about statistical descriptors, charts, and digital tools to process large amounts of data (Børne- og Undervisningsministeriet, 2019, p. 8). In accordance with the curriculum, students must choose a representation they find adequate for assessing and communicating their findings. This *de-mathematization* will be different depending on whether real-world data exists or not for the given property. If no real-world data exists, they will need to rely on intuition when judging whether and how the output is biased. Using the same *idealization* and *mathematization* but different mathematical analysis, the students can also calculate how many percent of the stories created that reflect the prevailing stereotypes and compare directly to the results of Sterlie et al. (2024).

It is our hypotheses that students are likely to go through the steps of the modelling process in a similar way to Sterlie et al. (2024), but adaptations to fit lower secondary school mathematics are needed at the *technical level*. Sterlie et al. (2024) utilize programming to automate the data processing, but the same processing can be done in a more handheld way, suitable for lower secondary

students. The calculation of mutual information (Figure 3) relies on entropy calculations, which are beyond the technical level of the lower secondary curriculum but are taught in upper secondary. Therefore, this type of data processing should not be part of the students' work. Both Sterlie et al. (2024) and lower secondary students are expanding their *radius of action* albeit in different ways. Sterlie et al. (2024) are familiar with the non-discrimination criteria but are looking to expand how it can be used thus getting a tool to measure bias in generative AI. Students are likely to have worked with modelling of taxi-drives or mobile phone subscriptions before, but bias in generative AI is almost certainly a new context thus expanding their radius of action of the modelling competency.

Discussion and conclusion

Our approach tackles several previous difficulties encountered in research on CT, particularly balancing disciplinary knowledge from CT and mathematics (Børne- og Undervisningsministeriet, 2020; Elicer & Tamborg, 2023). The modelling competency, radius of action, degree of coverage, and technical level can provide a terminology that helps to identify modelling processes that can be replicated by students, while also highlighting technical elements that may need to be simplified or omitted to adhere to the curriculum. We consider it pivotal that students experience the power of mathematics and gain an understanding of the issues computer scientists study, but not that students perform exactly the same tasks. Our paper suggests that it is possible for students to work with similar problems and reach similar results. Even within the curricular guidelines, students can likely conduct similar modelling activities, although at a different technical level. Algorithmic bias can be seen as emerging from a composition between a domain specific context and mathematical modelling that is built into an algorithm. The research analysed in our paper requires a combination of understanding the capabilities of generative AI, the independence criteria, and intuition of gender distribution in various profession. A consequence of this is that our study is limited to addressing the T and M of the STEM acronym. However, algorithmic bias appears in many contexts, not least such that involves the natural sciences. As an example, automated image recognition of x-rays has proven to be more effective for medical diagnostics of men than women (Weng et al., 2023). Addressing such problems is likely to call for activation of the S, along with the T and the M. Seen in this light, STEM education and its interdisciplinary nature could hold the potential to constitute a highly constructive starting point for educating students to critically and constructively relate to algorithmic bias. It is our hope that the approach presented in this paper can aid future studies that explore how STEM can prepare and empower students to participate and contribute to an increasingly digitalized society.

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(STE)M teachers competence areas in MINTco@NRW: Evaluation of and reflections induced by a mathematics educational model

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The demands on teachers in supporting STEM teaching and learning are complex. In mathematics education, specific research questions arise in teacher education and training: What competencies should teachers possess? How can teachers be supported in their work? To what extent do these aspects go beyond discipline-specific teaching? This article addresses how teachers rank the importance of various competence areas in the MINTco@NRW project through a multiple instrumental case study. The results show a big variation in teachers' rankings, which cannot be easily linked to specific characteristics of the cases in the sample. The similarities and differences in the necessary competence areas in the project and regular classroom settings depend on the specific competence area in question.

Keywords: Reflection process, mentors, teachers' competencies, STEM education.

Teachers' competencies in mathematics-related STEM teaching and learning

The importance of STEM literacy and competencies among students is particularly emphasized in the context of 21st-century skills and similar initiatives (Basham et al., 2010; Falloon et al., 2020). The desire for integrated STEM education is also a focal point (Roberts et al., 2022), raising questions about how STEM should be conceived and organized–whether as individual disciplines or as an interconnected network (Blum & Niss, 1991; Falloon et al., 2020). This brings up the challenge of ensuring that disciplinary knowledge and skills, especially in mathematics, continue to be developed, as mathematics is often perceived as underrepresented in integrated STEM settings (Maass et al., 2019; Roberts et al., 2022). One key approach could be through "high-quality STEM learning experiences" (Roberts et al., 2022, p. 2), where students use different STEM disciplines collaboratively to solve real-world problems. These learning environments pose significant challenges for teachers, adding to the usual complexity of their work (Beswick & Fraser, 2019). Therefore, it is crucial to consider what competencies (math) teachers need, how they can be trained, and what support they require to meet these challenges (Maass et al., 2019). In integrated STEM settings, teachers often need to collaborate with various partners and address STEM disciplines beyond their expertise–and possibly their comfort zone (Priemer et al., 2020). Besides the necessary competencies and skills, teachers' beliefs also play a crucial role. Reflecting on their own role (Artzt et al., 2015) and competencies is essential for successful teaching in these "high-quality STEM learning experiences". These diverse challenges necessitate a closer look at the complex nature of teacher competencies.

There are numerous models of teacher competence. For example, Bloemeke et al. (2019) propose a general model describing teacher competence, explaining the continuum from teachers' dispositions through situation-specific abilities to performance. This model identifies general areas of competence, bridging general pedagogy and subject-specific aspects. In STEM education research, there are also specific models. Mondisa et al. (2021) describe the A STEM-ME framework, which

outlines interactions between different stakeholders in STEM ecosystems, such as mentors from various subjects. Beswick & Fraser (2019) developed the STEMCrAfT framework to support (novice) teachers in STEM teaching at different levels through reflective questions on specific teacher activities and learning environmental aspects, including preparation, planning, context, resource use, support considerations, and implementation evaluation. Other models focus often on specific competence areas, like digital competencies. Each model has a different scope of application and offers various perspectives on teacher competence. When teachers engage with a model, it may be useful to choose those that are relatively easy to grasp and allow reflection on specific areas of competence in their practice. A proposal for such a model was developed as part of MINTco@NRW.

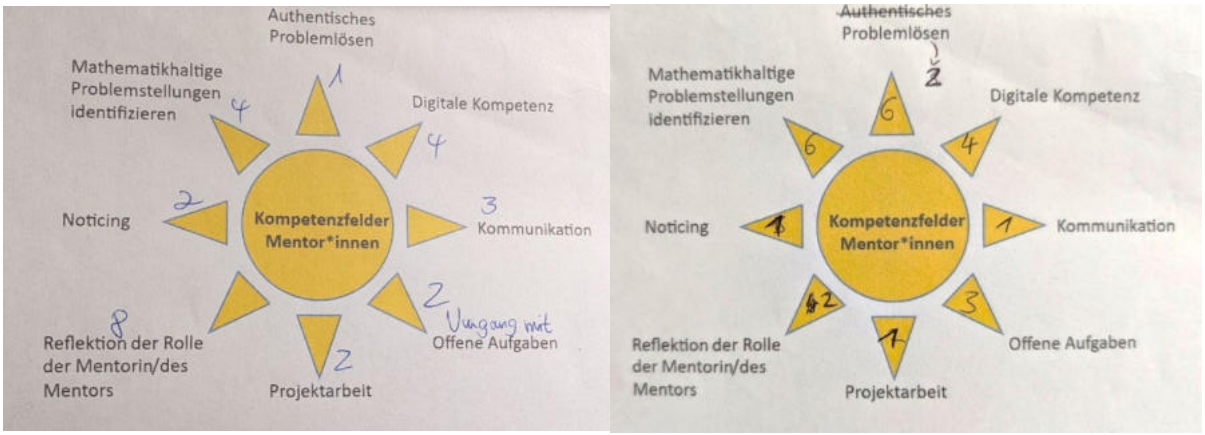

Figure 1: Ranking (1 is most important to lower numbers less important) of competence areas by T1 (left) and T2 (right)

The MINTco@NRW project creates a unique situation for teachers in North Rhine-Westphalia (NRW) by implementing long-term problem-solving of real mathematics-related problems in regular classes. It focuses on unsolved mathematics-related problems from companies, which are collaboratively solved by learners in solver teams of 4-6 German and US students. Thus, teachers are expected to act as cooperation partners for companies and mentors for their students, similar to Mondisa et al.'s (2021) conception of mentoring that supports the growth of learners. To support teachers in this multifaceted and challenging situation, a simplified model (Marx & Stoffels, 2023) was developed based on existing models and experiences from previous projects regarding the central activities of preservice teacher-student mentors (Stoffels, 2023). The aim of constructing the model by Marx & Stoffels (2023) was to emphasize central areas of competence for mentoring in the project, make them visible to teachers, and encourage reflection on these competence areas.

The identified competence areas in the model are (cf. Figure 1): (i) mathematics-related problem posing (Mathematikhaltige Problemstellungen identifizieren), (ii) solving authentic problems and mentoring authentic problem-solving processes (Authentisches Problemlösen), (iii) digital competence/digital literacy (Digitale Kompetenz), (iv) communication (Kommunikation), (v) teaching and learning using open-ended tasks (Offene Aufgaben), (vi) project-based learning (Projektarbeit), (vii) noticing, and (viii) reflection on the role as a mentor (Reflektion der Rolle der Mentorin/des Mentors).

The competence areas (ii)-(vi) are parts of models addressing teacher competencies for 21st-century learning (Beswick & Fraser, 2019). Subject-specific competence areas (i) and (ii) can be connected to the cognitive competence facets of teacher competence related to MPCK (Bloemeke et al., 2020). Noticing, understood in this model as attending to aspects in problem-solving situations and interpreting them based on professional knowledge and beliefs (Dreher et al., 2021), is connected to reflecting on the role of the teacher as a mentor (Artzt et al., 2015), focusing on student growth and supporting their solving process without leading to the teacher's solution of the problem. Both competence areas were added due to less effective mentoring in predecessor projects but are also implicitly addressed in already mentioned models (Bloemeke et al., 2020; Beswick & Fraser, 2019).

Based on these theoretical considerations and the aim of constructing the model, the general question arises whether, from the teachers' perspective, the model by Marx & Stoffels (2023) is adequate for the project. More precise research questions are:

- Do teachers, looking back at the project, think that the model by Marx & Stoffels (2023) sufficiently identifies the necessary competencies of a teacher in the project?
- Can the competencies be organized according to (perceived) importance? If so, what would this ranking look like?

From a broader perspective, it is also interesting to explore to what extent the model is perceived as project-specific or whether teachers consider it transferable to regular teaching. Therefore, another research question is:

- In what ways do teachers address similarities and differences between the competence areas necessary in the project and regular (mathematics) lessons?

Methods

The teachers' mentoring in the MINTco@NRW project is investigated through an instrumental embedded multiple case study (Yin, 2014). This case study aims to examine the overarching questions of what competencies teachers need to support such long-term problem-solving processes, how these competencies differ from those required in their regular teaching, and what support and further training needs they have according to the analysis. In this paper, I discuss only a part of the case study, focusing on the previously formed research questions and specifically on teachers from NRW. Teachers from the USA are also considered in the broader case study and will be discussed in further publications. After a brief description of the study's context, the data collection and analysis are presented.

Context of the case study

The general structure of MINTco@NRW is similar to its predecessors (Stoffels, 2023). The project started in January 2024 with a "Kick Off" where companies hand-over their real problems to students. After that the students had four months to solve the problems, which includes also a company visit. The "Forum of Innovation" in May 2024 was the culmination and end of the project, in which the Solver Teams present their solutions to the companies.

Table 1: Number of participating students and their age per school

	T1, T2	T3	T4	T5, T6	T7
Number of participating students	13	14	6	12	4
grade level	9	8-9	8-11	9	9
age	14-15	13-15	13-17	14-15	14-15

An important difference from the predecessor projects is that in MINTco@NRW, mathematics teachers served as the mentors, presenting them with the challenges already addressed in the first section. The hope is that mathematics teachers, due to their teaching experience but also professional development, can improve the support of the students compared to preservice teacher students' support in the predecessor. This article focuses on four schools from NRW and one other German school. In these schools, six teachers from NRW (T1-T6) and two teachers from outside NRW, who jointly participated in the interview and ranked the competencies together (T7), were part of the project. Throughout the project, all teachers supported their students in extracurricular STEM clubs, which met for 90 minutes per week. The students regularly meet with their counterparts in the USA for 20-60 minutes per week via videocall, depending on the schedule of the partner school in the USA. The number of participating students, grade levels, and age groups varied (cf. Table 1).

Data collection

The data discussed in this paper is part of the final semi-structured interview (Naz et al., 2022) with the German teachers. Each interview consisted of three parts and lasted between 1 to 1.5 hours. All interviews were conducted in person and videotaped, except for the interview with both teachers in case T7, which was conducted via Zoom, resulting in a joint ranking of the competence areas (cf. Table 2). In the first part, general aspects of the project implementation were discussed. The second part focused on the areas of competence shown in Figure 1. A diagram of the model was used to elicit responses from the teachers, allowing them to grasp all competence areas simultaneously without a hierarchy, and the ranking could be done on paper. This visual prompting leads to the collection of richer data, even when diagrams or figures with text seem to encourage respondents to focus on the text in the diagram (Page et al., 2022). Participants were asked to review the competence areas and, if they wish to add additional competence areas to the model. The next task was to rank the competence areas in order of importance for their work in the project, starting with 1 as the most important competence area. Afterwards, the teachers explained how they chose their ranking. At the end of the second block of the interview, the teachers were asked to what extent these competencies are also relevant in regular lessons. Figure 1 was available to the interviewees in German, so the ranking could usually be recorded in writing on the model.

Data analysis

For ranking the given competence areas, the ranks were transferred for each case into Table 2 to compare the teachers' ranking. The data of the ranking is ordered on an ordinal scale, as the importance is reflected in different or same ranks. To get an overview of the different rankings by the teachers and compare them, the last row of Table 2 shows the mode of the rankings of the different

competence areas. The mode is the value that appears most often for each competence area, providing insights into the variation in the ranking per competence area and indicating which competence area seems to be most or least important. All interviews were AI-supported transcribed using aTrain (Haberl et al., 2024) and analyzed using MaxQDA. The transcripts were revised, and the second part of the interview was then analyzed using an inductive summarizing qualitative content analysis (Mayring, 2015) to answer the first and third research questions.

Results

The first research question addressed at the beginning of the second interview part was whether teachers, looking back at the project, think that the model by Marx & Stoffels (2023) sufficiently identifies the necessary competencies of a teacher in the project. Most participants did not want to make any additions to the present model. This might be due to the number of competence areas in the model, which requires time to fully grasp. As part of project work, T7 discussed that it should be emphasized that deadlines must be met, and that attention must be paid to compliance with the companies.

Table 2: Number of participating students and their age per school

	authentic problem solving	**digital comp-etence**	**com-muni-cation**	**open tasks**	**project-based learning**	**reflection of the role as mentor**	**noticing**	**problem posing**
T1	1	4	3	2	2	8	2	4
T2	2	4	1	3	1	2	1	6
T3	5	5	1	3	3	4	2	5
T4	3	2	2	4	5	3	1	4
T5	3	7	1	5	2	4	8	6
T6	2	2	1	-	1	-	-	2
T7	5	5	4	5	1	3	2	3
mode T; Germany	2;3;5	2;4;5	1	3;5	1	3;4	2	4;6

The second research question was how teachers rank the different competence areas in the model. For most teachers it was not an easy decision to rank the competence areas. For example, one teacher states after she did her ranking: “Yes, it also just seems wrong, ... because afterwards something like posing the problem, ... gets the biggest ... number, ... feels wrong. ... But on the other hand, I think ... maybe that's ... rather, or I left it to the students. ... And more or less kept me out of it” (T4, pos. 83 [translated by G.S.]).

Nevertheless, almost all teachers ranked the competence areas (cf. Table 2). T6 did not rank all given competence areas. Overall, the ranking of the competence areas is very diverse. Except for the

competence areas digital competence, open tasks, reflection on the role of the mentor, and problem posing, each competence area is rated as most important at least once. Except for the competence area communication, every other competence area is also assigned the lowest importance at least once. To obtain a general overview of the importance of the individual competence area, the statistical mode of the rankings per competence area is given in the last row of Table 2. Accordingly, communication and project-based learning are rated as most important by most teachers, followed by noticing. All other competence areas are in the middle range, except for problem posing at the end of the spectrum. This could be explained by the fact that in this project cycle, the problems were identified with the companies prior to connecting them with the teachers. Reflection on the role as a mentor was also considered at the very end of the spectrum. The wide variability in the rankings may be due to specific requirements and experiences in the project. For example, the evaluation of the importance of the competence area "digital competence" seems to be dependent on how successful the work with the digital cooperation media was experienced. For an intensive analysis of the reasons, it seems necessary to also connect the single cases to their U.S. counterparts.

The last research question was how teachers address similarities and differences between the competence areas necessary in the project and regular mathematics lessons based on the given model. Some teachers considered all competence areas to be important for regular mathematics lessons. All competencies were also explicitly mentioned at least once by a teacher as being relevant for regular mathematics lessons. Authentic problem solving (Vos, 2018) and digital expertise were mentioned particularly often. Project work and open tasks were either mentioned as less connected to the classroom or as highly teacher-dependent in their use. Regarding open tasks, assessment of students, time restrictions, institutional framework conditions, or a lack of student independence were often referred to as challenges for their implementation. These aspects are well-known challenges for problem-based learning and open tasks in regular classroom settings (Beswick & Fraser, 2019). Particularly in view of the mathematics-related nature of the transfer of competencies to regular classroom settings, specific competencies from the model were often addressed similarly to pedagogical knowledge as well as subject-specific in the sense of mathematical pedagogical content knowledge (Shulman, 1986): "Communication, understanding, ... again students who misunderstand you ... and I don't just mean in terms of content, but ... at the level of what you do in terms of pedagogical work ... and so on. So that's definitely not ... only related to math, but ... everywhere. If I understand communication, math as a language, ... that is a different aspect" (T4, pos. 148 [translated by G.S.]).

Conclusions & Outlook

The teachers in NRW who participated in MINTco@NRW report that the model by Marx & Stoffels (2023) is adequate for describing key competence areas of teachers in the project. The teachers' ranking of the importance of the competence areas in the model varies widely, and the ranking process shows that it was not easy for most teachers to find a definitive ranking of the importance of the given competence areas. This difficulty might be due to the general complexity of teacher competencies (Bloemeke et al., 2020; Beswick & Fraser, 2019) or the new context of the project, similar to Nicol (2002), who describes challenges faced by pre-service teachers when first encountering the workplace. From the teachers' perspective, the competence areas of the model are

also relevant in regular classes. Individual differences between the participating teachers can be seen, ranging from broad agreement to major differences depending on the competence area in question. These differences can partly be traced to traditional arguments about why it is not possible to implement project- or problem-based learning in schools.

Of course, the presented study has some limitations. First, the cases are special, as all interviewed teachers were highly motivated to participate in the project and experienced intensive and long-term mentoring of students solving real problems from companies. Additionally, the perspectives of the U.S. counterparts of the teachers could not be discussed in more detail in this paper but might provide further insights into the reasons for the teachers' rankings, as exemplified by the competence area "digital competence". Nevertheless, the results suggest that future support for teachers in mastering their tasks while mentoring STEM learning needs to be individualized. In following research cycles of the MINTco@NRW project, the aim is to mentor in regular classes with regular curriculums, which might provide further insights into STEM learning environments similar to MINTco@NRW.

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Presence of mathematical processes in STEAM high school projects: a case study

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Keywords: STEAM, mathematical process standards, high school education, mathematical competence.

STEAM Education

In the current educational context, STEAM projects, which integrate science, technology, engineering, arts and mathematics, offer the possibility of working with complex concepts in a practical and meaningful way. Several research studies support the benefits of STEAM projects in secondary education with respect to different aspects of education; among them, Wahyuningsih et al. (2020) highlight that STEAM projects require the application of knowledge in real-world contexts and often involve problem solving that requires critical thinking and creativity. However, Ubuz et al. (2023) state that '*it is difficult to know how to integrate STE(A)M subject matters, how to design STE(A)M tasks and implement them with students'*. Thus, restricting our interest to mathematical competence, Diego-Mantecon et al. (2021) raise doubts about whether interdisciplinary projects are adequate to exploit school mathematics effectively; where, according to the National Council of Teachers of Mathematics (2000), the acquisition of mathematical competence should include work on the following mathematical process standards: reasoning, representation, connection, communication and problem solving.

Method

The aim of this study is to analyse the mathematical process standards that emerge in three STEAM projects implemented in three compulsory secondary schools in the Basque Country (see table 1). For this purpose, a qualitative methodology with an interpretative approach (Cohen, Lawrence, & Keith, 2007) has been selected to analyse: 1) the mathematical content and mathematical processes that teachers have considered in the design of the project and 2) the mathematical processes that emerge from the design of the activities created by the teachers in each project. To answer 1), semi-structured interviews were carried out with the teachers involved, and to answer 2), the type of activity defined and its relationship with the mathematical processes were analysed.

Table 1: Characteristics of the projects analysed

Project	STEAM context	Disciplines involved	Course	Age of students
P1	Witchcraft and cooking	History, physics, chemistry, language, mathematics	1st course	12-13 years
P2	Ecosystems in parks	Biology, language, mathematics	1st course	12-13 years
P3	Blood pressure in different contexts	Physics, chemistry, language, mathematics	3rd course	14-15 years

Results and conclusions

Table 2: Incorporation of the mathematical process standards

Project	1a) Mathematical content/sense	1b) Mathematical Processes in the design	2) Mathematical processes in the activities
P1	Proportionality. Numerical sense	Problem solving, connections	Problem solving, connections, reasoning, communication (written), representation
P2	Geometrical measures. Spatial sense.	-	Communication, representation
P3	Data analysis. Statistical sense	-	Communication, representation

From the analysis of the interviews, we can conclude that, in general, the teaching staff do not consider mathematical processes to be a fundamental part of the design of the projects and the incorporation of mathematics is more instrumental in nature. However, from the analysis of the activities proposed, we can see that some of the mathematical processes are involved (see table 2), which shows the need to further explore the importance and necessity of their inclusion in these projects.

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Designing a lesson plan to prevent the waste of water in the context of the ProSTEAM Erasmus + project

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Keywords: Mathematics education, STEAM education, Sustainable Development Goals, clean water and sanitation, hands-on.

There is an increasing call for the promotion of STEAM (Science, Technology, Engineering, Arts and Mathematics) education. This approach is recommended to improve students' performance and better prepare them for real life challenges and can also contribute to improve mathematics teaching (Costa et al., 2022; Stohlmann, 2018). However, despite an increasing number of projects in this field, there are several issues about its implementation, namely about an integrated approach of STEAM and the role of mathematics in this context, as well as teachers' knowledge and skills to be able to implement this approach in class (Costa et al., 2022; Maass et al., 2019).

This poster presents a lesson plan designed in the context of the Erasmus + project, ProSTEAM - PROmoting STEAM in primary school: partnership for disseminating good practices, with the duration of three years from February 2022 to January 2025. In the first Intellectual Output (IO) of the project, it was necessary to verify the curricula of the partner countries to find similar contents to develop the project. In the second IO, a pedagogical framework was designed to guide the development of the lesson plans, which were designed with a Design Research (DR) methodology (Plomp, 2013) by university experts in collaboration with primary school teachers from nine European partners. The plans are intended for teachers to autonomously be able to implement an integrated approach of STEM education in class. It is the case of the "Water and Sustainable Development Goals (SDG)" plan that was developed by the authors of this Poster. This plan is focused on SDG 6, drinking water and sanitation that stands out, drawing attention to the importance of water, the need for responsible consumption, as well as finding ways to store and reuse water for efficient management of this resource (Mateus et al., 2022). Table 1 presents the five cycles of design research that were conducted to develop the plan. In the end, all plans will be included in a teachers' guide to be disseminated in an online community of practice for everyone to have access to the available resources. It was only after being validated by all the partners of the ProSTEAM project that the plan was implemented in a teachers Professional Development Programme in Portugal (PDP) and in Lithuania with a total of 40 teachers. All the plans were implemented in at least two countries.

Table 1: Cycles of design research of the Water & SDG plan

DR cycles	Development of the Water & SDG plan in the ProSTEAM project	Duration
First cycle	Design of the plan by experts from university and school teachers	3 to 4 months

Second cycle	Implementation of the plan in the primary school teachers classes	3 months
Third cycle	Revision of the plan based on the feedback of all the parners	2 months
Fourth cycle	Implementation of the plan in a teacher PDP	4 months
Fifth cycle	Plan revision based on the feedback of the participants in the PDP	2 months

Besides several STEAM tasks proposed in the plan, tables were proposed to be filled by students for them to understand the water that is wasted for example when washing teeth and hands. Table 2 is an example filled by students in class supervised by the teacher who participated in the PDP.

Table 2: Brushing teeth with the water running or closing the tap

	Water running	Closing the tap	Water saved
1 student once a day	4 litres	1 litre	3 litres
Class with 20 students	80 litres	20 litres	60 litres
Class twice a day	160 litres	40 litres	120 litres

Based on the five cycles of DR and on a qualitative methodology and interpretative approach, with participant observation (Cohen et al., 2007; Plomp, 2013), it is concluded that the plan was approved by teachers who found it to be appropriate for school curricula. Also, they recognized the importance of this approach to motivate students to learn. Finally, they referred that they intend to keep developing this approach in the next school years, namely implementing other plans of the project.

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Identifying mathematical content in a carpentry Instagram video: A study with pre-service secondary education teachers

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Keywords: Integrated education, Instagram, social media, STEM education, world of work.

Introduction

Integrated science, technology, engineering and mathematics education (i-STEM) has been incorporated into the curricula of various countries. This approach seeks to provide contextualised or real tasks for students to connect knowledge and skills from the four disciplines (Ortiz-Laso et al., 2023). Central to this approach is the establishment of connections between the world of work and the disciplines (Maass & Engeln, 2019). In carpentry, various mathematical fields, including geometry, are employed (Saló i Nevado & Pehkonen, 2018), and carpenters use techniques to optimise the design of any product (Diego-Mantecón et al., 2021).

Tasks involving the i-STEM approach and connections to the world of work are scarce in current curricular materials (e.g., textbooks). This is why professional development programs have focused on providing these types of tasks (Maass & Engeln, 2019; Ortiz-Laso et al., 2023). Social media platforms, particularly Instagram, widely used by in-service and pre-service teachers (Carpenter et al., 2020), increasingly serve as resources for real-world tasks associated with i-STEM projects. For teachers to effectively implement these real-world tasks into their teaching and thus engage with i-STEM projects, not only mathematical knowledge is required but also contextual knowledge related to the techniques employed by professionals (Diego-Mantecón et al., 2021).

Objectives and method

This study examines how fifteen Spanish pre-service secondary education teachers analyse the technique employed by a carpenter in an Instagram video and how they identify the underlying school mathematical content associated with this real-world task. The participants, who took part voluntarily, were enrolled in a one-year master's degree to become secondary education mathematics teachers. The video, sourced from the *Carpintero de los Cerros*' (2022) Instagram account, showcases a professional demonstrating how to divide a wooden plank of decimal width into three equal parts. The Cerros' account was not initially intended for teacher training but rather to display effective carpentry techniques and tips.

Data were collected using a questionnaire after viewing the video with no sound as many times as necessary. The questionnaire comprised two open-ended questions: the first asked for a narration of the video, while the second focused on identifying the mathematical content underlying the carpenter's technique. The analysis involved an interpretative qualitative examination of the written responses. The first question was coded as correct when explaining that the video demonstrated how

to divide a segment into three parts. The second question was coded as correct when identifying Thales' theorem as the mathematical content underlying the carpenter's technique.

Results and conclusion

Concerning the first question, most pre-service teachers correctly identified that the carpenter was dividing a wooden plank into three equal parts. They noted that, rather than performing decimal division, he used an alternative strategy to avoid calculating $5\ cm\ \div\ 3$. Additionally, most described the procedure, explaining that he drew a segment ($\overline{AB}$) whose length is a multiple of 3 and depicted two points on it. Some interpreted the two points as the addition of the vector $\frac{1}{3}\overrightarrow{AB}$ and the vector $\frac{2}{3}\overrightarrow{AB}$ to point A, whereas the others understood them as the result of a division. All the pre-service teachers referred to the parallel lines drawn by the carpenter that contained the two points.

Regarding the second question, only about half of them infer Thales' theorem as the principle behind the carpenter technique. This study reveals that pre-service secondary teachers with strong disciplinary knowledge struggle to connect school mathematics with real-world applications. Teachers need to be familiar not only with mathematical content but also with their applications, as they are expected to teach these in context. We recommend using real-world tasks to help pre-service teachers understand the mathematics underlying professional techniques. This approach will support the classroom implementation of i-STEM projects, as many school curricula recommend. Given the challenges universities face in collaborating with workplaces, we suggest using social media to find videos like the one shown.

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Teachers' design of STEAM activities: different levels of connections

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Keywords: STEAM; teacher professional development; connections; interdisciplinarity.

Introduction: different types of connections in the STEAM-Connect project

This research is conducted within the framework of the Erasmus+ Project STEAM-Connect, involving scholars of five European universities in collaboration with mathematics secondary school teachers of the same countries (from 2 to 5 per country, participating on a voluntary basis). Teachers of different disciplines involved by math teachers during the project. The goal of this collaboration is to design and implement STEAM activities for secondary schools (Pocalana et al., 2024). The shared meaning given to the letter A in STEAM includes art, music, and humanities, according to literature (e.g., Preignat & Katz, 2019). One of the main aims of the project is the creation of different types of connections between the actors involved in the project. In this study we investigated two types: the *vertical connections* (between teachers and researchers, as subjects belonging to different institutions) and the *horizontal connections* (between teachers of different disciplines or schools).

Research aim and methods: teachers' interviews and representations

In this study, we aim to investigate how connections influence the design and implementation of interdisciplinary STEAM activities. Interdisciplinary teaching is a form of collaborative teaching in which, according to Lester and Evans (2009), all the teachers are responsible for planning the activities, deciding how the class material should be presented, providing the instruction, assessing student learning. According to the case study methodology, we analyse the work of six math teachers involved in the STEAM-Connect project of three different countries (Italy, Austria, Luxembourg) and from different school programs (no STEAM curriculum). We conducted online interviews with the teachers and analysed them through a qualitative content analysis (Cho & Lee, 2014) approach. We asked the teachers to represent and to explain their idea of interdisciplinarity and to verify the use of this teaching approach in the design of their STEAM activity. We grouped the interview data in categories (Semeraro, 2011) highlighting the development of connections in teachers' teamwork. We focused on the role of math teacher in collaboration with teachers of other disciplines. Finally, we investigated the sustainability of the use of this teaching approach in daily school practice.

Results and discussion: towards a multidisciplinary approach

From the data analysis we derived teachers' ideas on the collaboration of different disciplines and teachers in the design of the STEAM activity and the different types of connection between teachers and researchers. T1 (high school) told us about three design phases of the activity. The changes made in the evolution of the activity were due to the involvement of teachers from different disciplines (*horizontal connections*) and reflections with researchers (*vertical connections*) on didactic approaches and interdisciplinary activities. T1 provided various definitions and representations of

interdisciplinarity that reflect the evolution of the activity design. T2 (high school) worked alone from the beginning: no *horizontal connections* and no *vertical connections*. T2 reports no evolution in the design of the activity and represents the need of interdisciplinarity to "put everything we've torn apart back together" through a cylinder and its orthogonal projections says: "if we approach reality from different disciplines, we see just a part of it". T3 and T4 (technical institute) worked at the same activity in different stages: T3 designed the activity by himself, T4 started collaborating in subsequent moments (no *horizontal connections*). They represented interdisciplinarity in a concept map with the disciplines involved in the activity. T5 (middle school) worked with three different disciplines teachers on the design of the activity (*horizontal connections*). The definition of interdisciplinarity was presented by the researcher but T5 recognized this teaching approach in his own activity. T5 represents a big pot with many ingredients: "to put them together you can cook something delicious". T6 (middle school) reports changes in the design process thanks to the involvement of different teachers (*horizontal connections*) and the exchange with the researchers (*vertical connections*). T6 provides different representations of interdisciplinarity that changed during the design phases. We conclude that the development of connections, both *vertical* and *horizontal*, allowed for a process of evolution in the design of STEAM activities leading towards an increasing level of meta-disciplinary competence (William & Roth, 2019), that is the awareness of the role of each discipline in dialogue and activity with the others. This can be seen in T1 and T6, who described an evolution in the designs following discussions and reflections with researchers and other teachers and in T5 consciousness of a transdisciplinary approach. These results encourage us to foster continuous exchange through *horizontal* and *vertical connections* and to investigate how each connection affects didactic planning.

Acknowledgment

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TWG 27: The Professional Practices, Preparation and Support of Mathematics Teacher Educators

Introduction to the papers of TWG27: The Professional Practices, Preparation and Support of Mathematics Teacher Educators

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Current state of TWG27

TWG27, focused on mathematics teacher educators (henceforth: MTEs), has now been active throughout three consecutive CERME meetings. The group first met in CERME12, and has since seen a growth in its extent: from 17 submitted contributions and 21 participants in CERME12, to 26 submitted contributions and 29 participants in CERME14 (see additional details below). This growth reflects the increasing interest of the Mathematics Education community in MTE research, an interest manifested in various forms – for example, the Topic Study Group dedicated to MTEs in ICME-15 (held in 2024 in Sydney); the Special Issue on MTE expertise, published on March 2025 in *Journal of Mathematics Teacher Education*; dozens of papers focusing on MTEs (both educators and facilitators[1]), published in 2024-25 alone and revealing results from MTE research worldwide (e.g., Okumus & Vondrová, 2025; Thurm et al., 2024). Accordingly, within CERME, TWG27 provides an opportunity for researchers interested in this relatively new field to meet and discuss current and emerging issues related to MTEs' roles, knowledge, practices, challenges, professionalization processes, and more.

The work of TWG27 during CERME14

Of the 26 contributions submitted to TWG27, 20 were accepted (16 papers and 4 posters). Figure 1 shows the distribution of these contributions by the country of their first author. As can be seen, 10 countries were represented, with European contributions constituting almost half of the total number (45%), followed by North and South America (35%), Asia (10%), Australia and Africa (5% each). During the conference, the group met for 7 sessions. Papers and posters were allocated to sessions according to common themes. For most sessions a similar structure was applied: a short presentation and questions by the presenting author, a reaction and additional questions by a pre-assigned discussant, then a small-group discussion (either straight after a presentation or after two consecutive ones) and finally a whole-group discussion. For each session the groups were reshuffled, with a deliberate effort to achieve a mixture of countries represented in each group, for maximizing communication and deeper acquaintance among individuals. This model was noted by TWG27 participants as productive, enabling diversified voices to be heard and considered.

[1] In TWG27, we use the terms *facilitators* and *educators* to denote MTEs who support the learning of practicing and prospective teachers, respectively.

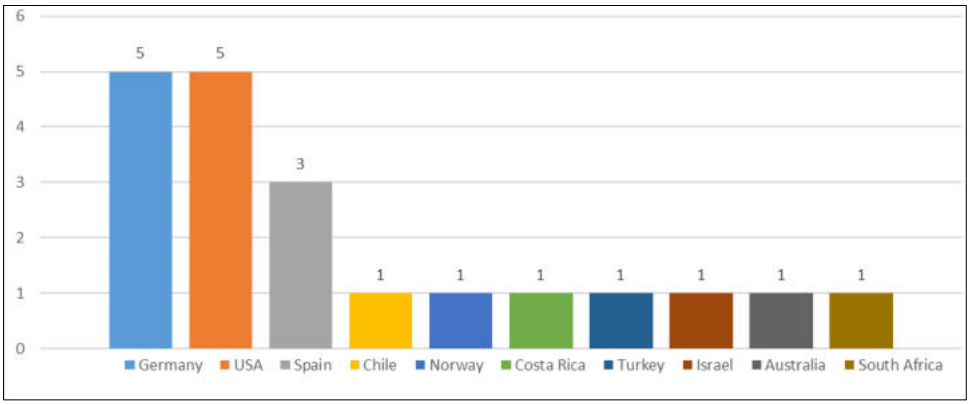

Figure 1: Distribution of TWG27 contributions by countries (refers to first authors, n=20)

Issues discussed in the contributions accepted to TWG27

The call for papers and posters raised the following issues:

1. Theoretical and conceptual frameworks that can be used as lenses in MTE research.
2. MTEs' knowledge, skills and practices, and changes thereof along MTEs' professionalization.
3. Beliefs and identities of MTEs, especially during the transition period of becoming MTEs, and along the professional path from a novice to an experienced MTE.
4. MTEs' role as bridging between theory and practice.
5. Role-modelling as a practice that MTEs may use, explicitly or implicitly; researching the extent of coherence between MTEs' practices and the teaching practices they are meant to support.
6. Integrating generic and mathematical content-specific aspects within the work of MTEs.
7. Preparing MTEs: formal and informal qualifications; different kinds of PD programs designed for MTEs; learning mechanisms of MTEs (e.g., reflection, rehearsals); tools and resources for supporting MTEs.
8. Scaling up programs for mathematics teachers: building institutional capacity through focusing on facilitators; institutional factors that support or hinder the effectiveness of PD facilitators.
9. The influence of current global issues and STEM-related topics on the role and practices of MTEs.

Some of these issues were already present in the previous two calls for TWG27 (but not necessarily addressed by the contributions submitted), whereas other issues or sub-issues emerged from the discussions in the previous meeting of the group in CERME13 (2023). For example, theoretical and conceptual frameworks for MTE research (Issue #1) were widely discussed in the 2023 meeting, yielding an additional emerging question: To what degree are frameworks useful for purposes other than research, e.g., when preparing and supporting MTEs? This question was addressed in the present meeting by Borko et al., who presented the Learning to Facilitate Framework (LTFF) as a conceptual tool that can guide both the analysis and the design of facilitator professional development (FPD) activities. Several other contributions from Spanish-speaking countries used the framework of Mathematics Teacher Educators' Pedagogical Content Knowledge (MTEPCK) to analyze the specific knowledge utilized by MTEs in the preservice level, thus demonstrating a combination of Issues #1 and #2 (Alfaro-Víquez; Giadas et al.; Pérez-Montilla et al.). Other authors addressing Issue #2 mainly delved into MTEs' practices: McKie et al. explored how MTEs structured their conversations with

teachers to support them in using educative curriculum materials. Elliott and Nieman examined how facilitators learned "on the job" to press for learning goals and to respond to teachers' contributions during instructional conversations. Jarry-Shore et al. looked into practices of a coach and a meta-coach in the process of learning to broach candid discussions with teachers about their teaching, and Munson et al. investigated how a coach and a teacher collaborated within side-by-side coaching to better respond to emergent problems of practice. Interestingly, all these four contributions came from the USA, implying that there may be culture-based trends in MTE research. Still within Issue #2, two contributions related to facilitators' noticing skills: Shure et al. explored German facilitators' noticing of a fictional PD situation, finding that nearly half of them still needed further support to develop knowledge-based, relevant interpretations. Rodríguez-Segarra et al. looked into similarities and differences in the noticing of Spanish MTEs with varying academic backgrounds. Issue #3 was addressed by three contributions: Montenegro et al. reported on an intervention in which MTEs in Chile were encouraged to investigate their own practice, with the results showing that participants perceived this experience as highly contributive to their professional path. Erens explored self-perceptions and beliefs of German facilitators regarding their role as adaptors and implementors of a professional development (PD) program for practicing teachers. Zastrow et al. analyzed responses of 333 facilitators in the QuaMath PD program in Germany, revealing that many of the facilitators' professional beliefs aligned with the program's core principles, a finding seen as essential for the success of this large-scale PD. MTEs' role as bridging between theory and practice (Issue #4) was addressed in Opheim's paper, focusing on how a PD course she designed and taught as MTE in Norway took into account elements of theory to support Teacher Specialists in becoming successful change agents in their schools. Issue #5 was addressed by several contributions: Lovemore reported on the use of modelling in her work as MTE with prospective teachers in South Africa, while Alfaro-Víquez explored the different forms of modelling employed by two MTEs in Costa Rica. Sarılıcan and Aslan-Tutak deliberated on modelling as one of key professional roles, unpacked in a case study of a Turkish MTE. Both Holzäpfel et al. and Baumanns et al. discussed issues of coherence, relating to the QuaMath PD program in Germany. Issue #6 was addressed by Marshman and Goos, describing a long-term project of collaboration between mathematicians and MTEs in Australia, aiming at integration of mathematical content-specific aspects and pedagogical aspects in university courses for prospective teachers. The preparation and support of facilitators (Issue #7) was addressed by three papers: Baumanns et al. presented 'teaching experiments' conducted by facilitators as a central design element of FPD, that was found to contribute to improving and refining PD materials for teachers. Borko at el. analyzed a facilitator preparation session using the LTFF mentioned above. Karsenty et al. introduced a tool named 'the FPD Structure Map' that enables comparing different FPD programs in terms of their structural and institutional characteristics. Issue #8 was not directly addressed by the contributions of TWG27, but was present in those dealing with large-scale PD programs, especially in the German context. Finally, unlike in the previous two conferences, MTE research linked to current global issues and STEM-related topics (Issue #9) was not addressed.

Emerging issues

In the course of discussions within the group's sessions, five issues that did not appear in the TWG27 call for contributions emerged as being relevant to this group. We briefly describe these below.

Emotional knowledge

This issue refers to developing MTEs' sensitivity to the specific needs and problems of the teachers they work with. It was noted that MTEs need this kind of *emotional knowledge* (about teachers' concerns, interests, beliefs, etc.); therefore, it should be part of MTEs' preparation and learning.

Ways of practicing specific MTE tasks before the actual leading of PD sessions with teachers

The group elicited various means and strategies to support MTEs' practicing of tasks, to be further researched for their effectiveness. Among these were providing opportunities for leading candid conversations; developing norms for analyzing teaching to understand potential tensions; using vignettes, videos, rehearsals, and other types of simulations to deeply unpack possible PD scenarios.

The triad of MTEs, prospective teachers and the schools

Understanding the relationships between the three vertices of this triad was elicited as highly relevant for MTE research at the preservice level. The group talked about brokering and addressing tensions; developing means to improve MTEs' work with the schools (the possibility of co-teaching was mentioned), and unpacking roles and practices that MTEs can adopt during the teaching practicum.

Eye on the mathematics in our MTE research

The group discussed the need to put more emphasis on mathematics as a subject matter when researching MTEs, and raised the following questions: What part should mathematics play in preparing and supporting MTEs? How might the facilitation of mathematics teachers' professional learning be distinct from the facilitation of teachers' professional learning in other subject areas?

Macro and micro analysis in MTE research

Different grainsizes of data analysis were discussed, with questions such as: How can we efficiently analyze a large volume of data? When we look at detailed qualitative data (e.g., referring to MTEs' decisions, practices. etc.), how can we balance the micro-analysis without missing the "macro" picture? What may we learn about this from other lines of research (e.g., student- or teacher-focused)?

Summarizing words

At CERME14, the spectrum of issues discussed in TWG27 has widened. We explored theoretical and practical aspects of MTE research, elaborated mathematical vs. general pedagogical aspects of MTEs' work, and delved into fine-grained issues such as noticing, role-modelling, and specific tasks for supporting MTE learning. We look forward to further development of this TWG in the future.

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Examining pedagogical content knowledge in teacher educators' modelling practices within didactics-specific courses

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This study explores the modelling practices employed by two mathematics teacher educators (MTEs) while teaching courses on the didactics of algebra and functions. Through qualitative content analysis of semi-structured interviews, the research investigates the different forms of explicit modelling used by MTEs and the MTE pedagogical content knowledge (MTEPCK) demonstrated. The findings highlight modelling as a key strategy for bridging theory and practice, showcasing effective teaching methods and addressing student misconceptions. The study underscores the importance of developing MTEs' competencies in designing and implementing modelling strategies and calls for further research into the relationship between MTEPCK and modelling as a means of advancing MTE professional development.

Keywords: Mathematics teacher educators, modelling, didactic-specific courses, pedagogical content knowledge.

Introduction

Mathematics Teacher Educators (MTEs) play a crucial role in preparing preservice mathematics teachers (PSTs) to become proficient educators. As Rojas et al. (2021) emphasize, MTEs are responsible for designing and implementing curricula that deepen PSTs' mathematical understanding while equipping them with the knowledge, skills, and strategies essential for effective teaching. This requires crafting learning experiences that simultaneously support mathematical learning and pedagogical development (Ball et al., 2008) and fostering situation-specific cognitive skills crucial for teaching (Blömeke & Kaiser, 2017). To achieve this, learning opportunities must seamlessly integrate theoretical and practical knowledge (Boyd, 2014), helping PSTs bridge the gap between formal mathematics and the mathematics they will teach in schools (Buchholtz & Kaiser, 2013).

In the Mathematics Education program at the University of Costa Rica, MTEs must teach five didactic-specific courses that focus on both mathematical knowledge for teaching and essential pedagogical tools. These courses aim to transform theoretical knowledge into practical, actionable skills through situational reconstruction (Valverde et al., 2020). Teaching these courses becomes particularly challenging for Mathematics Teacher Educators (MTEs), as they often lack specialized training for preparing PSTs (Lunenberg et al., 2007). This absence of formal preparation forces MTEs to navigate complex responsibilities on their own. To address this, some MTEs have adopted modelling as a teaching methodology, providing PSTs with concrete examples of effective teaching practices and demonstrating how theoretical concepts can guide practical decision-making.

This paper aims to explore the modelling strategies employed by two MTEs in teaching courses on the didactics of algebra and the didactics of functions, as well as the MTEs' PCK mobilized for these practices.

Theoretical background

Modelling as a teaching strategy

The role of MTEs goes beyond organizing and supporting PSTs' learning through direct instruction. In every class, the pedagogical and methodological choices they make when presenting content or explaining teaching strategies, they are modelling teaching practices themselves. This means that the "teaching process in which mathematics teacher educators engage appears to be as influential as the knowledge imparted" (Rojas et al., 2021, p. 368). Lunenberg et al. (2007) describe modelling as the practice of deliberately demonstrating specific teaching behaviours intended to foster the professional learning of PSTs.

There are four forms of modelling can be found in teacher educators' practice: a) Implicit, b) explicit, c) explicit and connected with PST's own practice and d) explicitly connecting exemplary behaviours with the theory (Lunenberg et al., 2007). *Implicit modelling* is often associated with congruent teaching, where MTEs employ "good practices" in their own teaching not only because they are effective for instructing PSTs, but also because they can be valuable for PSTs when teaching their future students (Boyd, 2014). However, this modelling practice can be ineffective if PSTs fail to recognize the demonstrated teaching strategies as a learning objective (Lunenberg et al., 2007). *Explicit modelling* requires MTEs to make explicit the choices they make for designing the lesson or during the implementation, and why (Boyd, 2014). This form of modelling can be performed by MTEs by thinking out loud. *Explicit and connected with PST's own practice modelling* is intended to help them recognize how the demonstrated teaching methods can be adapted and applied to various teaching situations, including those they will encounter in their future classrooms (Rojas et al., 2021). It is important to emphasize that the goal is not for PSTs to simply accumulate a repertoire of teaching practices, but rather to critically reflect on which practices will be most useful and how they can align with their own teaching objectives. Finally, *explicitly connecting practice with theory modelling* refers to MTEs clearly linking their teaching choices to relevant academic theory on pedagogical knowledge (Boyd, 2014).

Mathematics teacher educator pedagogical content knowledge

Since MTEs are responsible for "teaching mathematics teachers to teach" it makes sense to base the study of their knowledge for teaching PSTs on the same foundational concepts of PCK that apply to PSTs (Chapman, 2021). On that ground, and following Chick and Beswick (2018) framework, MTEs' knowledge can be described as "PCK for teaching the PCK for teaching mathematics" (p. 476). The Mathematics Teacher Educators' Pedagogical Content Knowledge (MTEPCK) framework organizes knowledge into three categories that are neither hierarchical nor strictly separate, but rather overlap and exist on a continuum: *clearly PCK, content knowledge (CK) in a pedagogical context, and pedagogical knowledge (PK) in a content context*. Additionally, the framework recognizes that MTEs approach teaching from two distinct perspectives: the PCK required for school mathematics teachers (SMTPCK) and the PCK necessary for teacher educators. For instance, when addressing misconceptions, MTEs must consider both the potential misunderstandings that PSTs may have regarding mathematical or pedagogical knowledge, as well as the common misconceptions school students might have about mathematical concepts (Chick & Beswick, 2018). This dual focus is

essential for MTEs to effectively teach their PSTs. Table 1 summarizes the framework, and the examples illustrating when the PCK types are evident should be interpreted within the context of the two perspectives mentioned. Consequently, the concepts or procedures may align with both SMTPCK and MTEPCK.

Table 1: Framework for mathematics teacher educators pedagogical content knowledge (MTEPCK)

Clearly PCK	CK in a pedagogical context	PK in a content context
Teaching strategies: Discusses or uses general or specific strategies or approaches for teaching a mathematical (or an SMTPCK) concept or skill. *Student thinking:* Discusses of addresses student or PST's ways of thinking about a concept, recognises typical levels of understanding. *Student thinking -misconceptions:* Discusses or addresses student or PST misconceptions about a concept. *Student affect (in relation to content):* Discusses or addresses students' *or* PSTs' affective responses to particular mathematics or SMTPCK topics. *Cognitive demand of task:* Identifies aspects of the (SMT *or* MTE) task that affect its complexity. *Representations of concepts:* Describes or demonstrates ways to model or illustrate a concept (can include materials or diagrams) *Explanations:* Explains a topic, concept or procedure. *Knowledge of examples:* Uses an example that highlights a concept or procedure. *Knowledge of resources:* Discusses/uses resources available to support teaching. *Curriculum knowledge:* Discusses how topics fit into the school curriculum *or* the teacher education curriculum *Purpose of content of knowledge:* Discusses reasons for content being included in the school or the teacher education curriculum or how it might be used.	*(Beliefs about) the nature of content*: Expresses an appreciation of the nature of mathematics that goes beyond the school curriculum and aligns with mathematicians' view of the discipline/ *and* of the SMTPCK that goes beyond the teacher education curriculum to include critique of the content and its conceptualisations. *Profound understanding of fundamental content:* Exhibits deep and thorough conceptual understanding of identified aspects of mathematics and SMTPCK. *Deconstructing content to key components:* Identifies critical mathematical or SMTPCK components within a concept that are fundamental for understanding and applying that concept. *Structure and connections:* Makes connections between mathematical or SMTPCK concepts and topics, including interdependence of concepts. *Procedural knowledge:* Displays skills for solving mathematical or SMTPCK problems (conceptual understanding need not be evident) *Methods of solution:* Demonstrates a method for solving a mathematical or SMTPCK problem.	*Assessment approaches:* Discusses or designs tasks, activities or interactions that assess learning outcomes/ Discusses or uses assessment tasks purposefully to evaluate PSTs' knowledge of SMTPCK. *Goals for learning:* Describes a goal for students' or PSTs' learning *Getting and maintaining student focus:* Discusses or uses strategies for engaging students or PSTs. *Classroom techniques:* Discusses or uses generic classroom practices. *Student affect (general):* Describes how student affect influences pedagogical approach/ Discusses typical PST affective reactions to learning about SMTPCK

Source: Adapted from Chick and Beswick (2018)

This study analyses the modelling forms used by two mathematics teacher educators when teaching didactic-specific courses, as well as the types of MTEPCK that they incorporate in their modelling to answer the following questions:

RQ1: What forms of modelling are employed by MTEs in didactic-specific courses?

RQ2: Which elements of the MTEPCK framework become evident through MTEs' use of modeling practices?

Context and Methods

This qualitative study uses content analysis (Kuckartz, 2019) to examine transcripts from semi-structured online interviews conducted in 2023 as part of a larger research project. With participants' consent, the interviews were video recorded, transcribed, and handled confidentially. The study involves two MTEs, referred to as Paola and Roberto. Paola, teaching *Didactics of Algebra to* second-year PSTs, while Roberto, teaching *Didactics of Functions*, taught third-year PSTs. The interviews, conducted at the end of the semester, reflect the MTEs' recollections of their teaching practices in those courses.The interviews aimed to explore the MTEs' decisions on content development, assessment, methodologies, and course assignments. Consequently, the interview protocol did not include specific questions about the use of modelling. However, modelling practices emerged naturally during the interviews. Paola recalled additional instances of modelling after the interviewer prompted her, while Roberto independently described all his modelling practices when asked to describe his course-related decisions, making further prompts unnecessary.

The analysis process started with a review of the interviews to identify all modeling episodes, based on the forms of modeling outlined earlier. Each episode was then categorized according to the modeling form(s) it exemplified. Additionally, the modelling episodes were examined through the lens of the MTEPCK framework to identify the types of MTEPCK that were demonstrated.

Results and discussion

MTEs employed modelling practices for various purposes. Consequently, the analysis is structured based on the form of modelling episodes identified in the MTEs' interview narratives and the corresponding elements of MTEPCK associated with each type.

The episode of *explicit modelling* was recalled by Paola, when she was trying to demonstrate "good practices" for reading an academic article, showing the PSTs the process, she goes through when reading a paper for teaching purposes. Paola recounted aloud how she had approached the article multiple times throughout the course, adapting her reading strategies based on different objectives

> Paola: This is the fourth time I read the paper (..). The first time I read it to see what it was about, what was in there related to the course contents. The second time, I read it more carefully, thinking, 'Okay, now it's my turn to explain it. What am I going to select from this?' (…) And then I read it again (...) to see what I select from it to explain to you.

Her aim was to encourage PSTs to engage in purposeful reading by considering the reason for reading, what they hoped to learn, and how the content could inform their future teaching. In this modelling example, is possible to recognize several elements of MTEPCK of the category of clearly PCK. For instance, Paola knew that reading academic articles is a task of high *cognitive demand* for the PSTs. She also considered the *student-thinking misconception* element, because she knows that PSTs usually read the course theory expecting the MTE to make the connections with the practice.

An example of *explicit modelling that connected theory to practice* was identified in Roberto's class during the activities he called "exploratory sessions." The course included five such sessions, each comprising a sequence of mathematical tasks designed to demonstrate how theoretical frameworks, such as didactic phenomenology or functional thinking, could inform the design and implementation of classroom activities. After PSTs completed these tailored learning experiences, Roberto explicitly highlighted how his decisions and the examples he selected illustrated the theoretical foundations guiding his approach.

> Roberto: Alright, in this exploratory session, we worked together on an example of a phenomenological math task. Now, let's move on to reading about phenomenology.

Roberto's goal was for PSTs to evaluate the tasks they had completed considering the newly introduced theory. He also recalled similar instances where he explained how he applied theoretical knowledge—such as the history of functions and the effective use of technological resources—to align task design with specific learning goals. This example of modelling highlights two MTEPCK elements within the category of clearly PCK: *knowledge of resources* and *knowledge of explanations*, both of which are tied to topics within SMTPCK

Finally, both interviews revealed instances of explicit modelling closely connected to PSTs' own teaching practice. Paola described two episodes that occurred during classes involving role-playing dynamics, while Roberto's examples were tied to the exploratory sessions. This form of modelling demonstrates MTEPCK elements spanning all three categories: clearly PCK, CK in a pedagogical context, and PC in a content context.

Paola's modelling examples included a class on systems of equations and another on lesson planning. In the first, she demonstrated problem-solving strategies by posing a task with multiple solutions and representations, taking on the role of a mathematics teacher while PSTs acted as students. She guided them with targeted questions and introduced a tabular representation to scaffold their understanding. Paola guided the PSTs' work by posing questions such as, "Find different prices for the pizzas and the salads," encouraging them to engage with the task. She then introduced a tabular representation to help the PSTs organize the information and scaffold their understanding toward finding a general formula. At this stage, Paola's knowledge in *teaching strategies* and *representation of concepts* both in the category of clearly PCK became evident, particularly from the perspective of secondary mathematics teacher PCK. Paola intentionally used incorrect variable names when drawing the table on the board to integrate her students' previous notational mistakes. This approach incorporated their knowledge about *student- thinking misconceptions* into the task. As the PSTs found different formulas to solve the problem, Paola asked them to create and interpret a graphic representation, discussing the validity of values and the context in which they applied. These actions demonstrated her knowledge of *representation of concepts*, *profound understanding of fundamental content*, and awareness of *structures and connections.* The last two MTEPCK elements belong to the CK in a pedagogical context category.

Paola emphasized her didactical and methodological decisions, prompting PSTs to reflect on mistakes, her questions' role in achieving learning goals, and the reasoning behind her choices, such as asking them to find five prices instead of one. She applied her knowledge of *cognitive demand of*

a task and *goals for learning*, recognizing that asking relevant questions is challenging but essential for class discussions. Additionally, Paola used questions as an *assessment approach* to evaluate PSTs' understanding of PCK content. *Goals for learning* and *assessment approaches* are elements of the PCK in a content context category. This shows that, at this stage, Paola has demonstrated knowledge across all three MTEPCK categories in an integrated manner, which are evident also in the following example.

In the lesson planning class, Paola modelled the process by creating a lesson plan on a secondary-level topic. Recognizing the high cognitive demand of this task and knowing that PSTs were designing a lesson for the first time, she chose the modelling approach to guide them. She meticulously crafted her lesson plan, completing all the required sections that PSTs would need to include in their own plans, mobilizing her knowledge about *deconstructing content to key components,* in this case the content is lesson plan. The MTE used her knowledge about describing *goals for learning, teaching strategies* and *curriculum knowledge* for crafting the lesson plan, she then conducted the class in the role of a secondary school teacher, with the PSTs acting as eighth-grade students. After the class, Paola reviewed the lesson plan with the PSTs, highlighting each phase and helping them connect these elements to their own recent experience. She then refined the plan for clarity and used it as a guide for PSTs to design their own lessons, offering them a concrete example of how to structure each phase and what to include. Additionally, she modelled how to guide individual student work and incorporate *assessment approaches* in real-time, providing feedback as she observed PSTs' solutions. At the end of the class, she discussed with the PSTs her movements during their individual work and the decisions she made, evidencing her knowledge about *classroom techniques*.

> Paola: Well, while you were doing the independent work, I evaluated you. But I wasn't just evaluating you for the sake of evaluating, (..) I'm identifying "hey look, this one has such an error, that one has such a difficulty" and then I'm going to comment on it to the whole group. Or this one has this solution and it's different from that one's and then when I'm going to move on to the discussion, I'm going to strategically ask these two to come to the board, because I need the class discussion to generate more resolution methods, and it wouldn't make sense to send two who did the same thing.

Roberto's modelling examples took place during the exploratory sessions, as explained earlier. The tasks focused on mathematical content PSTs might need to teach and addressed areas where PSTs had previously struggled, reflecting Roberto's awareness of *student-thinking misconceptions*. Unlike Paola's classes, in these sessions, PSTs worked as prospective teachers rather than secondary students. Roberto increased the difficulty by incorporating formal mathematical knowledge, such as determining whether the injectivity of a function is a definitional property or something that can be proven, and if so, how. This demonstrated his *profound understanding of fundamental content* and his consideration of the PSTs' knowledge. The mathematical tasks illustrated how to pose questions to PSTs, demonstrating Roberto's knowledge of *assessment approaches*. For example, he explained:

> Roberto: In this case, about the exponential function, you can see that there were open questions, closed questions, where students were asked to mark with an "x", but justify', and then with this, we showed the PSTs what type of tasks, what type of request can be proposed to a student.

During the session, Roberto's role as a teacher was to guide the PSTs' work and pose of relevant questions. He modelled how teachers should support student work, *goals of learning*, and manage *classroom aspects*, such as time control. PSTs were also tasked with formulating learning goals for which the proposed tasks were suitable. This demonstrated Roberto's understanding of the *cognitive demands* involved in setting learning objectives and designing mathematical tasks that effectively support their development. In addition, he evidenced his *procedural knowledge*, by asking PSTs to create the learning goal for the task, rather than designing the task to fit a pre-established learning goal, as is typically done. Roberto's modelling examples also demonstrate elements of MTEPCK across all three categories. He summarized the rationale behind his choices as follow:

> Roberto: Why did I think of that? as I was saying about this model of offering PSTs professional learning opportunities, in the sense that it would make evident both the theoretical mobilization of the topics that we are discussing in class, as well as a classroom proposal to work with their students; where the PSTs will reinforce knowledge that they were showing (...) to be weak.

Conclusion

As noted by Lunenberg et al. (2007), MTEs face a highly complex task in training PSTs. They must be proficient in elements of the CK and PCK typical of secondary mathematics teachers, and in the CK and PCK specific to teaching prospective teachers. Moreover, they must address the challenge of effectively bridging the gap between theory and practical application (Chick & Beswick, 2018). The modelling practice has proven to be an effective approach for addressing these challenges (Boyd, 2014). The interviewed MTEs employed *explicit modelling, explicit modelling related to PSTs' own practice* and *explicit modelling linking theory with the teaching competences and decision* to illustrate multiple theoretical and methodological aspects. As shown in the results, to design and execute modelling practices effectively, MTEs drew upon their PCK. Their PCK informed and illuminate the MTEs to recognize what it is important to model, and which is the best way of doing it. In Paola and Roberto's examples, multiple elements of MTEPCK were evident, highlighting the interaction between all three MTEPCK categories. However, the form that revealed the most MTEPCK elements was *explicit modelling connected to PSTs' own practice*.

Modelling by MTEs can be considered as a vital practice for advancing the professional development of PSTs, as it enables them to experience firsthand the strategies, knowledge, and decision-making processes essential for effective teaching (Lunenberg et al., 2007). Effective modelling requires considerable self-awareness, pedagogical expertise, and sensitivity to student needs (Loughran & Berry, 2005), underscoring the complexity and intentionality behind this approach. Moreover, modelling is not innate; it must be learned, practiced, and continually refined by MTEs.

Finally, considering that the theory of teacher educator modelling is not content-specific, it is crucial to integrate it with a framework like MTEPCK. Such a framework enables the intentional design of modelling episodes, ensuring awareness of their potential implications and maximizing their impact on PST learning. By developing their modelling strategies within this context, MTEs can better support PSTs in bridging theory and practice. Further research into the relationship between MTEPCK and modelling could have an impact on advancing MTE professional development.

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Teaching experiments as a central design element of facilitators' PD

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The professional development (PD) of facilitators (FPD) who lead teacher PD (TPD) workshops is essential for high-quality mathematics teaching. This article describes how teaching experiments conducted by the facilitators themselves can become a central design element of FPD. Following the strategies of lifting, nesting, and unpacking, we explored how this design element can contribute to improving and refining PD materials. To do this, we analysed the written reflections of 140 primary school facilitators who participated in FPD during the 10-year QuaMath project in Germany. These reflections showed the facilitators' experiences with the teaching experiments in their mathematics classes. The analyses showed that integrating teaching experiments at the FPD level has great potential for improving FPD and can also provide valuable insights for revising the PD materials.

Keywords: Facilitators, professional development, teaching experiments, primary school.

Implementing approaches for high-quality mathematics teaching has remained a major challenge for professional development (PD) programs (Ahl et al., 2023). A shared vision of what constitutes high-quality teaching and a coherent set of core principles have been shown to be essential (Cobb & Jackson, 2021; Prediger & Selter, 2024). This forms the basis of the 10-year German implementation project QuaMath (Developing *Qua*lity in *Math*ematics Classrooms and Teacher PD). A consortium of 28 mathematics education professors, 40 academic project team members, 60 state project leaders, and 400 facilitators has aimed to reach 10,000 schools in 10 years (Prediger & Selter, 2024).

The first year of QuaMath (Sep 2023–Sep 2024) focused on facilitator professional development (FPD) to qualify facilitators for the teacher PD (TPD) workshops. The facilitators participated as learners, taking on the dual roles of teacher and facilitator and moving between these roles as part of their qualification. The TPD workshops have included planning and conducting teaching experiments (Timperley et al., 2007) aimed at applying the content of the PD program directly in the math classroom. In the QuaMath project, these have been primarily small, easy-to-implement suggestions to create a strong incentive for their immediate use. For example, it can be different representations of a task, where one representation does not match the others. The children have to explain why one representation does not fit and how it would have to be adapted to make it fit. Teachers can reflect on how in-depth discussions about representations supports student conceptual understanding. As part of their FPD program in QuaMath, facilitators were asked to conduct and reflect on these teaching experiments themselves. This paper thus explores the use of teaching experiments as a central design element in facilitators' PD. These self-experiences in the teaching experiments have contributed to the qualification of the facilitators and to the improvement of the PD materials.

Theoretical background and FPD context

Multi-layered structure of PD programs

The PD program of QuaMath considers the learning processes of three different groups of participants—students, mathematics teachers, and PD facilitators—and works with them within the implementation strategies of the well-established three-tetrahedron model of PD (Prediger et al., 2019). This model was developed as part of the joint research activities at the German Centre for Mathematics Teacher Education (DZLM) and provides a framework for the multi-layered research and development structures within complex PD programs. According to Borko et al. (2021), the three tetrahedrons, their elements, and their interrelationships, both within individual tetrahedrons and between them, illustrate the complex nature of learning and teaching across classroom, TPD, and FPD levels. The first tetrahedron focuses on the classroom level and is based on the didactic triangle of students, teachers, and mathematical content. The classroom resources extend the triangle to a didactic tetrahedron (Prediger et al., 2019). These four elements constitute the content at the TPD level and are thus the starting point of the second tetrahedron. At this level, the teachers are the learners, and facilitators take the role of the teachers; the TPD content focuses on mathematics teaching and learning, and the TPD resources include the materials and activities in the TPD program. Finally, the third tetrahedron addresses the facilitator level, using the same idea of role change: Facilitators become learners, facilitator educators are their teachers, the content is on FPD level, and the resources are the materials and activities for qualifying the facilitators (Borko et al., 2021). For a detailed description of the model, including its elements and interrelationships within and between the tetrahedrons, we refer to Prediger et al. (2019).

Raising awareness of the interconnections across the three tetrahedrons is not only important for the design of FPD and TPD programs but also represents a central perspective for conducting research. In this respect, Prediger et al. (2019) highlighted the three central overarching strategies of *nesting*, *lifting*, and *unpacking* for embedding the three tetrahedrons both in the design and in the research of PD programs. If certain research questions and/or methods from the classroom level are used on the TPD level and transferred from this level to the FPD level, this is referred to as a *lifting* strategy (Prediger et al., 2019). The *nesting* strategy is closely linked to this but considers all relevant aspects of the lower tetrahedron. This means that the complete classroom tetrahedron is nested in the TPD tetrahedron, and the complete TPD tetrahedron is nested in the FPD content (Borko et al., 2021). The *unpacking* strategy focuses the reverse way: Aspects from the FPD content need to be unpacked to the lower levels, for example, to determine how facilitators' knowledge grows from the FPD and how they implement their knowledge in their TPD workshops (Prediger et al., 2019).

Conceptual focus as a principle for high-quality mathematics teaching

For the long-term success of the QuaMath program, it is essential to think and act within a coherent framework of high-quality mathematics teaching on all levels of the three-tetrahedron model. To establish this coherence from the outset, normative, epistemological, empirical, and pragmatic perspectives were considered to define five principles of high-quality mathematics teaching (for details, see Prediger et al., 2022; Prediger & Selter, 2024): cognitive demand (initiate active learning processes), conceptual focus (build a basis for concepts, strategies, and procedures), longitudinal

coherence (prepare for sustainable learning), student focus and adaptivity (work with students' perspectives), and enhanced communication (talk about mathematical ideas). These five QuaMath principles were developed by systematising central research findings and by empirical justification. These core principles of QuaMath for high-quality mathematics teaching were consistently integrated into all PD modules (Prediger & Selter, 2024).

Considering the crucial importance of conceptual understanding, this paper focuses on the principle of conceptual focus. High-quality mathematics teaching should support conceptual understanding, which is essential for linking mathematical concepts to procedural skills and fostering long-term learning in mathematics (Cobb & Jackson, 2021; Kilpatrick et al., 2001). Prediger et al. (2022) stated that conceptual understanding can evolve from "using and connecting everyday contexts and multiple representations", "starting with meaning construction before deriving procedures", and "connecting the procedures back to the underlying meanings" (p. 6). However, many teachers struggle to design or select teaching materials that facilitate conceptual understanding for all students (Cobb & Jackson, 2021). This is compounded by the fact that many teachers measure their teaching success not by children's conceptual understanding but by correctly solved procedural tasks (Watson & Geest, 2005). Furthermore, tasks that support conceptual understanding are often only selected for high-achieving children (Prediger et al., 2022). However, conceptual focus cannot be addressed through conceptual tasks alone. Instead, it relies on the teacher's ability to foster rich classroom discussions that promote conceptual understanding among students (Erath, 2017). In the QuaMath program, we aimed to address these challenges at the TPD level by helping teachers learn how to support conceptual understanding in everyday mathematics teaching. Therefore, we particularly emphasised the proven effectiveness of teaching experiments and their focused reflection as central design elements of successful TPD programs (Prediger & Selter, 2024; Timperley et al., 2007).

Teaching experiments and focused reflection

Teaching experiments provide the opportunity to observe how the implementation of specific TPD content in mathematical classes can influence students' mathematical learning. However, it is not only important to prepare and conduct the teaching experiments but also to reflect on the experiences made in the next TPD workshop and with other teachers (Timperley et al., 2007). Facilitators have to guide the preparation and reflection phases of the teaching experiments to 'explicitly connect the teachers' plans and experiences with the underlying ideas' (Prediger & Selter, 2024, p. 36) of the TPD content. Preparing facilitators for this challenge is crucial and should not be underestimated in the design of the FPD program. The following section describes how we prepared the facilitators in the QuaMath program for this task and in the context of conceptual focus as TPD content. In doing so, we refer to the three strategies of lifting, nesting, and unpacking.

Primary-school facilitators' qualification in the QuaMath program

The participants of the 10-year QuaMath program are about 150 facilitators for primary school TPD courses who have voluntarily enrolled for the qualification. One hundred and forty consented to have their data and documents from the FPD program analysed for research purposes. The facilitators are mostly primary school teachers with varying degrees of experience (experience as teachers in years:

M = 15.80, SD = 9.74) or work in a school-related field (e.g., teacher educators for pre-service teachers). Some had already worked as facilitators (experience in years: M = 3.77, SD = 4.94).

In the 1st year of the facilitators' qualification (Sep 2023–Sep 2024), the Basic Module of QuaMath focused on the five principles of high-quality mathematics teaching. The central aim of the FPD qualification was to prepare facilitators to run the TPD workshops in the following year. For facilitators' qualification, two workshops were held for each principle. The first workshop dealt with the respective TPD materials, while the second one, which took place a few weeks after the first, focused on the teaching experiments. Special FPD materials were developed for the qualification workshops. In these materials, the classroom level and the TPD level were consistently linked, and reflections on the change of perspective were taken into account. The aim was to connect the classroom level and the TPD level systematically in the FPD so that facilitators could effectively experience the PD content as teachers and facilitators as well (Prediger & Selter, 2024). In the following, we illustrate how we implemented the consistent linking of classroom and TPD levels with focused reflection, particularly on the teaching experiments, in the design of the FPD program. After the first qualification workshop, the facilitators were asked to carry out the classroom teaching experiment in their own mathematics classes. It should be noted that the core idea of a teaching experiment in QuaMath is a small, easy-to-implement suggestion designed to apply the targeted principle of high-quality mathematics teaching in everyday lessons across various primary school grades. The experiences gained in the teaching experiment were supposed to be reflected upon in written form and sent to us before the second workshop. So, we *lifted* the teaching experiments as part of the classroom level on the level of FPD.

Since the experiences at the classroom level were made the explicit subject of the FPD in the second workshop, and the complexity of the classroom and the TPD tetrahedron was thereby addressed in the FPD workshops, we also followed the *nesting* strategy. Thus, the FPD qualification in QuaMath considered that our facilitators, as teachers, should know how teaching with the materials of the teaching experiments directly addresses the principles of high-quality mathematics teaching and supports students' learning. On the one hand, this was self-experience and on the other, facilitators have to train how to involve the teachers in the teaching experiments with a focus on students' conceptual understanding. Our *nesting* strategy therefore addressed the entire three-tetrahedron model. The facilitators' written reflections on the teaching experiments *unpacked* the principle of conceptual focus on TPD and classroom levels. Finally, the written reflections reveal how facilitators as teachers had actually understood the principle of conceptual focus. For the PD designers, the reflections also provided an opportunity to improve the PD material in terms of the focused principle. This unpacking was explicitly part of the second workshop. In the following, we provide some empirical insights into the data on the teaching experiment on the principle of conceptual focus.

Teaching experiments as part of the FPD on the principle of conceptual focus

The PD materials on the principle of conceptual focus emphasise the importance of manipulatives and visualisations in learning basic mathematical skills in primary school. Particular attention is paid to the discursive negotiation of meaning among the children. This is the starting point of the teaching experiment. Teachers in the TPD should establish so-called *nonsense stories* as ritualised in-depth

communication about conceptual understanding. In 'nonsense stories', teachers formulate mathematical statements that sometimes make sense but are sometimes nonsensical, such as '27 minus 9 is one more than 27 minus 10. Am I right or am I wrong?' The subsequent discussion about the statements is of main importance. The children are asked to explain and argue why the statement is or is not nonsense, and visualisations or manipulatives are often necessary to underpin the explanations. In line with the *lifting* strategy, the primary school facilitators should implement their own nonsense stories in their mathematics classes. They were then asked to reflect in writing on their teaching experiments using guiding questions. For example, they were asked:

- What nonsense stories did you use in your mathematics classes to address the principle of conceptual focus? (exact wording)
- To what extent could conceptual understanding be observed in children's explanations?
- How did the use of manipulatives or visualisations support children's understanding?
- To what extent did the nonsense stories address the principle of conceptual focus in your teaching experiment and in general?

In line with a combination of *nesting* and *unpacking* strategies, we used the reflections of the facilitators for the design of the second workshop. Facilitators' experiences were used to discuss the principle of cognitive demand at the classroom and TPD levels. Unpacking was also a valuable research strategy for identifying facilitators' challenges in understanding this principle and drawing consequences for the TPD content. In accordance with Prediger et al. (2019), the *nesting* strategy considered the multi-level character on the design side, while the *unpacking* strategy took the multi-level character in the content-related research into account.

Procedure of analysis

We analysed the written reflections in two steps. First, we inductively categorised facilitators' nonsense stories used in their teaching experiment (see the first guiding question) based on the following criteria: (a) Do the facilitators' nonsense stories have the potential to address the principle of conceptual focus in mathematics classes (e.g., through the use and connection of everyday contexts and multiple representations)? (b) If the stories lacked this potential, how can they be categorised? Second, we qualitatively analysed the guided reflection (see the second to fourth guiding questions) to understand how the facilitators dealt with the nonsense stories.

Insights into facilitators' reflections of teaching experiments

Analysis Focus 1: Nonsense stories with and without potential to support conceptual focus

Group 1: Nonsense stories considering conceptual focus: Some facilitators used appropriate nonsense stories that offered the potential to discuss mathematical meaning and thus to address the principle of conceptual focus: 'If I add a counter to a place-value table, the number always increases by 1', '6 times 7 is one sixes more than 5 times 7', or '10 plus 8 is 2 more than 9 plus 7'. These nonsense stories show the potential for joint reflection of basic mathematical concepts (in this case, place value and understanding of mathematical operations).

Group 2.1: Captain problems: Some facilitators used so-called captain problems (see Verschaffel et al., 2000) as nonsense stories: 'Mrs. Meyer has 3 blue and 2 red pencils. How old is Mrs. Meyer?' It is well known that many children in primary school do not question whether such tasks can be answered at all but sum the values of the task, for example, to get a solution. Although these tasks have a noteworthy educational purpose, they rarely allow an in-depth discussion about mathematical concepts and thus show limitations in addressing the principle of conceptual focus.

Group 2.2: Technical terms: Instead of mathematical concepts, technical terms were in focus, such as: 'A triangle has four sides' or 'The number 354 has four 10s.' The answer to these statements is based on definitions of what a triangle and what '10s' are. Such definitional matters rarely allow an in-depth discussion about the underlying meaning of mathematical concepts.

Group 2.3: Understanding material or simple calculation tasks: Nonsense stories of this group dealt with the understanding of visualisations or formulated classic word-problem tasks. Examples include: 'On the hundreds table, the 28 is below the 38' or 'Peter is 112 cm tall. His sister Susi is 200 mm shorter. How tall is Susi?' Understanding the material (in this case, the hundreds table), on the one hand, is usually not accompanied by a conceptual focus as it does not allow in-depth discussions about mathematical concepts (in this case, about place value). Simple word problems, on the other hand, focus primarily on simple calculation skills and generally do not enable in-depth discussions about mathematical meanings.

Analysis Focus 2: Reflection on the use of nonsense stories to address conceptual focus

The reflections on the implementation of the teaching experiments revealed a broad spectrum. Some insights into typical cases are given below. In some cases, it was not clear from the reflections whether the nonsense stories were discussed beyond a yes/no question in terms of conceptual focus. Facilitator Marc used nonsense stories about the divisors of a number (e.g., 7 is a divisor of 47). Students were asked to apply divisibility rules to answer the nonsense story. The facilitator reflected that 'the tasks are only easy to understand if the mathematical rules are applied automatically'. Conceptual understanding regarding the divisibility rules was not addressed in the reflection. Nonsense stories seemed to *remain as yes/no questions*. In contrast, the reflection by Lisa emphasised the importance of the discussion after the nonsense stories for addressing conceptual focus: 'Although children gave quick feedback with "thumbs up or down", this sometimes did not allow any conclusions to be drawn as to whether they had understood the connections and were able to explain them.'

Another reflection on using the nonsense story about multiplicative decomposition strategies (as discussed in Group 1) revealed a more thoughtful approach to addressing conceptual focus. The facilitator Emma emphasised the importance of linking representations for conceptual understanding. She suggested representing 6 times 7 as a rectangular array, marking the groups of sixes, and asking: 'How many sixes are in 6 times 7?' The children then showed 5 times 7 in the same way and compared the representations to reveal that it is not useful to look for sixes in 5 times 7. This approach aims to connect 'math language' (6 x 7), visual representation ('rectangular arrays'), and verbal meaningful explanations ('6 sevens'). This approach leads to a situation where, as Emma reflects, 'one is never satisfied with a simple result or an answer, but always demands explanations.'

Implications for the FPD program

Implication 1: Clarification of *conceptual focus* for facilitators

The use of teaching experiments within the FPD showed that the idea of conceptual focus needed to be clarified at the TPD level. To this end, examples of captain problems and simple calculation tasks were discussed in the FPD workshops, with the question of why these stories do not address the principle of conceptual focus. We therefore used the knowledge from the analyses to address the facilitators' misconceptions about the principle of conceptual focus in the second workshop.

Implication 2: Implementation of *conceptual focus* for teachers

Teachers' activities highlighting the importance of manipulatives and visual representations to address the principle of conceptual focus have been added to the TPD material. These help to illustrate the importance of discursively negotiating mathematical meaning of visual representations. For reflecting on the teaching experiments in the next TP workshop, we implemented some slides illustrating how nonsense stories help to encourage discussions about mathematical meaning.

Discussion

In this paper, we have explored the use of teaching experiments in the context of conceptual focus as a design element in facilitators' PD. The teaching experiments and facilitators' written reflections provided valuable insights into revising the PD materials and facilitator qualification. The analysis of the facilitators' written reflections helped identifying facilitators' misconceptions about the principle of conceptual focus. We were therefore able to respond to these difficulties in the second workshop (see Implication 1). The results also helped improving the TPD material (see Implication 2).

Additionally, the iterative design of FPD along the entire three-tetrahedron model as a research and development strategy seems very promising. Lifting the teaching experiment from the classroom to the FPD level was the starting point. The second workshop reflected on these implications for the TPD workshops, aligning with the nesting strategy of the three-tetrahedron model. The teaching experiment also unpacked the principle of conceptual focus, from the FPD to the TPD and then to the classroom level. The lessons learned from conducting teaching experiments at the FPD level offer several broader considerations for mathematics FPD programs. First, teaching experiments conducted by facilitators provide a valuable opportunity to clarify and unpack the content of PD workshops, leading to necessary revisions of PD materials. Second, written reflections on the teaching experiments provide insights into how the content, is implemented at the classroom level (Timperley et al., 2007). These reflections guide the revision of PD materials and help to ensure that PD content is applied in practice as intended. As QuaMath is a 10-year project, these analyses will also inform the design of workshops for subsequent years aimed at the long-term improvement of PD materials.

Finally, for designing PD workshops, the three-tetrahedron model (Prediger et al., 2019) and the lifting, nesting, and unpacking strategies through all levels provide a valuable perspective. Our research shows how intertwined the three strategies appear in PD. Especially in the multi-level structure of the QuaMath PD, it seems to be useful to consider all levels of the three-tetrahedron model for facilitators' qualification in order to achieve the overarching goal of developing quality in mathematics classrooms and TPD.

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The Learning to Facilitate Framework: A model for preparing professional development facilitators

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This paper presents the Learning to Facilitate Framework (LTFF) which highlights four pedagogies of facilitator preparation programs: selecting a central task of facilitating professional development, representing the practices that comprise the task, approximating these practices, and unpacking the representation and approximation. In this paper, we analyse one facilitator preparation session using the framework, considering whether and how the four pedagogies are enacted during each FPD activity in the session. We then suggest possible uses of the framework to design and research facilitator professional development. The facilitator professional development session is from the Problem-Solving Cycle program; the central task is facilitating a video-based discussion.

Keywords: Professional development, facilitator learning, mathematics teacher educators.

Skillful teacher professional development (TPD) facilitators are critical to the success of any scaled-up TPD program (Borko et al., 2014; Even, 2008). The preparation of TPD facilitators is therefore essential to providing continued TPD opportunities for all teachers. Research has identified the knowledge and skills that facilitators of mathematics TPD need to conduct productive TPD activities. However, we know little about the characteristics of effective facilitator preparation programs and, as Lesseig et al. (2017) observed, we "lack research-based principles to guide the design of leader preparation" (p. 592). Although the situation has improved since Lesseig and colleagues made this observation, there is still much work to be done. Our project, "Preparing Facilitators to Lead Teacher Professional Development: An Initial Conceptual Framework and Exploratory Investigation," addresses this issue. Our team of scholars from three countries, who have conducted extensive research on TPD with mathematics teachers and initial investigations of facilitator preparation, worked collaboratively to develop the Learning to Facilitate Framework (LTFF) – a conceptual framework and set of design principles for facilitator preparation (Jarry-Shore et al., submitted). This paper presents an introduction to the framework and uses it to analyse the activities that comprise a complete facilitator professional development (FPD) session from one of our programs.

Background

Situative perspective on learning

The LTFF is grounded in a situative perspective on knowing and learning, which conceptualizes knowing as situated in particular physical and social contexts; social in nature; and distributed across the individual, other persons, and tools. Learning is conceptualized as changes in participation in

socially organized practices (Lave & Wenger, 1991), and the social context in which a person learns is a fundamental part of what is learned (Putnam & Borko, 2000). A situative perspective on learning of practices thus suggests the importance of focusing on authentic activities in which the practices occur. One approach to situating teacher learning in the authentic activities of teaching is to coach teachers in classroom settings (Gibbons & Cobb, 2017). Another is to use artifacts such as lesson plans and videotaped lessons to bring experiences from the classroom into the TPD setting (Putnam & Borko, 2000). Similarly, artifacts and other representations of *facilitation* practices can be used to situate learning experiences for TPD facilitators in the authentic activities of facilitating TPD.

Facilitation practices and the situational demands of facilitating TPD

In recent years, facilitator educators and TPD designers have identified the central tasks of facilitating TPD, or what we refer to as the situational demands of facilitation (Prediger et al., 2022). These include multifaceted tasks such as guiding teachers through the process of solving mathematics problems together (e.g., Borko et al., 2021; Elliott et al., 2009) and orchestrating discussions of classroom videos with groups of teachers (Borko et al., 2021; van Es et al., 2014), or more specific tasks such as handling difficult moments in TPD discussions (Schwarts et al., 2022). The practices that are enacted to meet these situational demands, and all that underlies them (Karsenty et al., 2023), are essential elements of FPD programs (Prediger et al., 2022). This multidimensionality characterizes the activities we considered when developing the LTFF.

The Learning to Facilitate Framework

Our approach to preparing facilitators to meet the situational demands of facilitation draws upon the literature on preparing teachers to meet the situational demands of teaching. Grossman et al. (2009) developed a framework for preparing teachers that consists of three teacher education pedagogies – representations, decompositions, and approximations. They argue that teachers can learn to enact practices for meeting the central situational demands of teaching when the demands are represented, decomposed, and approximated. This framework has been used in pre-service teacher education and TPD settings, and there is some evidence for its success in supporting teachers' professional learning (e.g., Ghousseini, 2017; Kazemi et al., 2016). Our project lifted Grossman et al.'s framework from the TPD level to the FPD level and adapted the pedagogies to create the Learning to Facilitate Framework (LTFF) (Jarry-Shore et al., submitted), based on an assumption that these adapted pedagogies may also support TPD facilitators' professional learning.

The LTFF includes three key adaptations: a) adding a fourth pedagogy of *selecting a situational demand* central to TPD facilitation; b) replacing decomposing with unpacking – whereas decomposing focuses on practitioners' observable practices, unpacking also surfaces the rationales underlying visible practices and the planning that precedes these practices; and c) allowing flexibility in the order and recurrence of pedagogies (e.g., unpacking may occur after representing and approximating). Thus, the LTFF consists of four pedagogies: selecting a TPD situational demand (i.e., a TPD task) and the practices to address this TPD task, representing practices for carrying out the task, approximating these practices, and unpacking the represented and approximated practices. A given FPD activity may not include all four pedagogies in the LTFF, and the pedagogies can reoccur in a FPD activity or occur simultaneously rather than in sequence.

Many decisions are made for each LTFF pedagogy in planning and enacting the FPD activities. Here we provide a few examples. The facilitator educator who is designing an activity must first select the focal situational demand and TPD task. What is selected may vary in grain size, ranging from the broad TPD task of leading a discussion of a classroom video to the sub-task of ensuring equitable participation during such a discussion. In representing the practices for meeting the selected demand, decisions must be made regarding the modality of the representation, for example, a simulation, video, or physical artifact. Facilitator educators also decide whether to unpack decisions made in planning for the represented TPD task, enacting the represented practices for this TPD task, or both. Prospective facilitators may have the opportunity to approximate the entire TPD task or a particular aspect of it, and the group may ask them to unpack the decisions they made when planning or enacting the approximation. Figure 1 depicts the LTFF pedagogies and key decisions for each one.

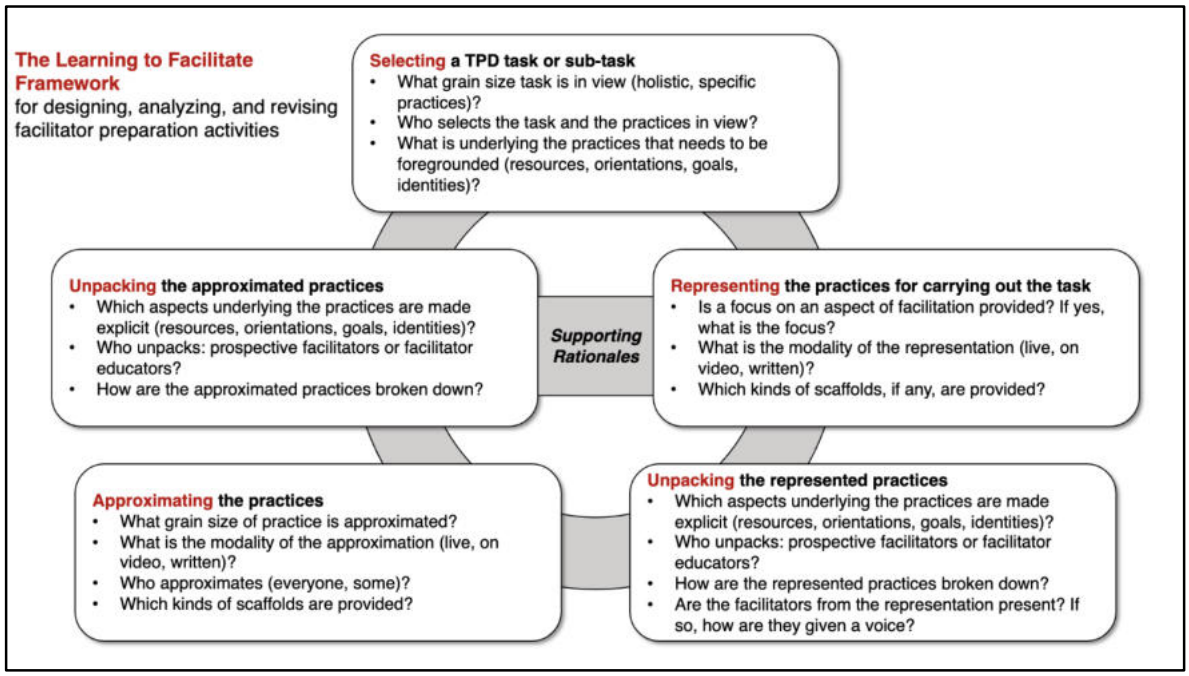

Figure 1: The Learning to Facilitate Framework (from Jarry-Shore et al., submitted)

Analysis of a FPD session: Preparing to facilitate a video-based discussion

In this section we use the LTFF to analyse one full-day FPD session from the Problem-Solving Cycle (PSC)[1] TPD program for middle-grades mathematics teachers (Borko et al., 2015). The PSC is a school-based program in which TPD workshops are led by facilitators at their schools. To support the sustained implementation of the PSC program at scale, Borko et al. (2015) developed a FPD program in which facilitators receive support in preparing to facilitate the TPD workshops at their schools. Each FPD session focuses on one central TPD task for the upcoming workshop. One or more

[1] In this description, we use the LTFF terminology of TPD, FPD, and facilitators, though in our discussions of the program in other articles and presentations, we refer to the TPD as the *Problem-Solving Cycle* program, the FPD as the *Teacher Leader Preparation* program, and facilitators as *Teacher Leaders*.

facilitator educators typically model facilitation of the central task, and the facilitators and other facilitator educators (not leading the current activity) serve as teacher participants. Next, this live representation is unpacked to surface the decisions and rationales underlying the facilitator educators' visible practices, as well as their decisions and rationales in planning the activity. The facilitators then prepare for and rehearse their own versions of the task; these rehearsals may also be unpacked.

The FPD session we analyse, which included five FPD activities, took place in the first year of a partnership between a university and an urban school district. The university and district partners adapted the PSC program to address the district's mission of providing every student with high-quality instruction and equitable learning opportunities in mathematics by building the district's capacity to conduct school-based TPD for mathematics teachers. FPD and TPD sessions were conducted for 4 years, beginning in Fall, 2016. The facilitators in the project were practicing teachers who also served in the role of facilitator. They had a range of experience in facilitating school-based TPD prior to participating in the project. Facilitators in the participating schools typically worked in pairs (Borko et al., 2021). Figure 2 depicts the LTFF for this FPD session. Our analysis explains how each FPD pedagogy was enacted during the session.

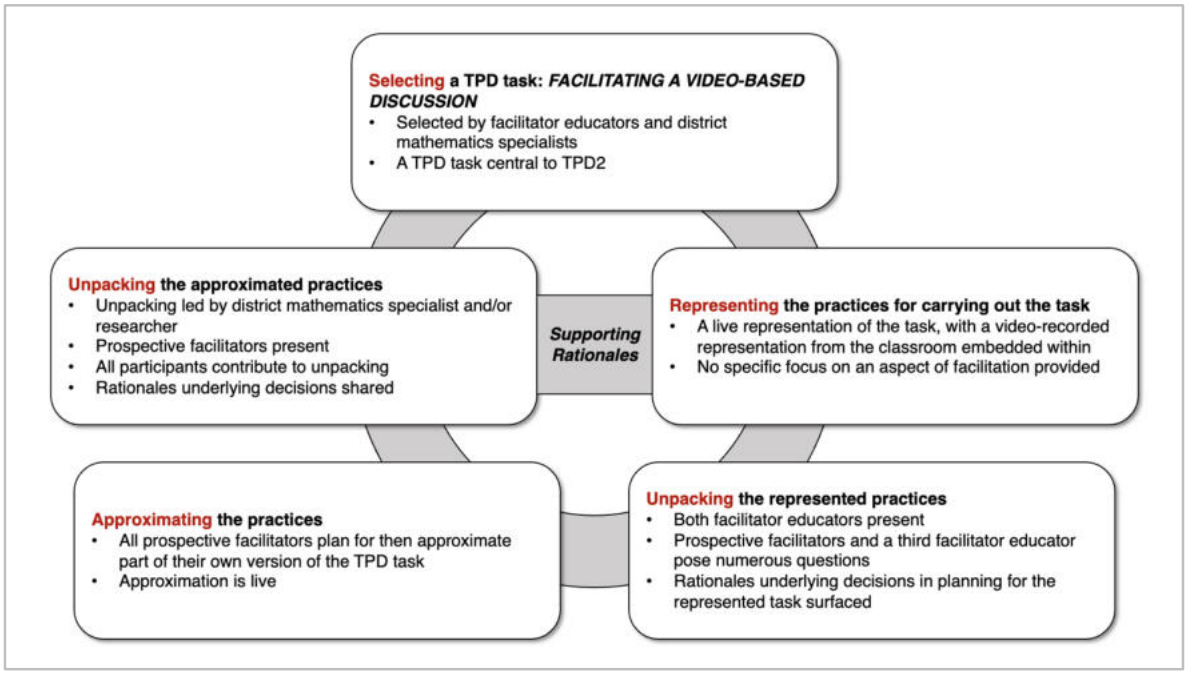

Figure 2. Analysis of a FPD Session from the PSC Using the LTFF

Selecting the focal TPD task

The TPD task selected for the FPD session was the facilitation of a video-based discussion. This task, selected during the planning of the session, was represented by two facilitator educators, one of whom is an author of this paper (FE2, Michael), during the second facilitator preparation session of the 2016-2017 school year. The unpacking of the representation was led by a third facilitator educator (FE3, Hilda), also an author. Participants were a group of 15 facilitators.

Representing the TPD task and unpacking the representation

The two facilitator educators represented the TPD task by modeling a video-based discussion. To begin this live representation, they reminded the facilitators to "wear their teacher hats" (i.e., assume the role of teacher). In that role, the facilitators first solved the mathematics problem in the video they would be viewing. Next, the facilitator educators reviewed a set of norms for discussing classroom video and showed the video without a specific viewing focus. They then shared a poster displaying the Common Core State Standards for Mathematical Practice (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010), a set of eight mathematical practices that mathematics teachers in the US are asked to teach their students. They focused the facilitators' attention on Mathematical Practice 3: Construct viable arguments and critique the reasoning of others. The video was then shown a second time with the following focus: What evidence do we have of students engaging with Mathematical Practice 3? After watching the video, participants shared their observations in small groups. FE1 then facilitated a whole-group discussion as FE2 recorded participants' contributions on a whiteboard at the front of the room.

Following this live representation, FE3 orchestrated an unpacking of the representation. She began by asking the facilitators to switch from wearing a "teacher hat" to a "facilitator hat" and to write down any questions they wanted to ask FE1 and FE2 about the activity or how they planned it. Next FE3 asked FE1 and FE2 questions about the planning and enactment of the video-based discussion, including: a) their goals for the video-based discussion; b) how and why they selected the video; and c) what back-up questions they had prepared for the discussion.

FE3 then invited the facilitators to ask questions of FE1 and FE2. They asked about facilitation practices, the planning for the video-based discussion, and the underlying rationales for both. These questions addressed, for example: a) how the two FEs decided who would facilitate the discussion and who would record contributions; b) why they chose to focus on Mathematical Practice 3; c) their goals for the discussion; d) how many times they viewed the video in preparing to facilitate the discussion; and e) how they kept the conversation on track. This unpacking made transparent the goals underlying the facilitator educators' visible practices, as well as their belief that preparing to facilitate a TPD task requires substantial prior planning and careful thought.

Approximating the TPD task and unpacking the approximation

After the representing and unpacking activities, the facilitators worked in school teams to plan their rehearsals of a video-based discussion, supported by one or more of the facilitator educators. Each team then rehearsed their video-based discussion with the other participants assuming the role of teachers. In one rehearsal, the two facilitators showed a video of a small group of students working on a mathematics problem. They showed it twice, the first time without a specific viewing lens, and the second focused on how the students supported each other's participation. After the second viewing one facilitator led a video-based discussion, asking questions about what the participants noticed in the video and how what they noticed supported student participation. In response, one participant observed that there was a piece of poster paper between the students, "which was a shared space to do the work." They suggested that it supported participation because the students "were making their mathematical work visible to each other," allowing them to "co-construct understanding." As participants shared ideas the facilitator wrote what they shared on the whiteboard.

The discussion to unpack the rehearsal followed a Praise-Question-Polish protocol, a peer feedback strategy designed to guide participants to provide specific, constructive feedback. The facilitator educators in the group guided the discussion, asking the other participants to share what went well in the activity, ask clarifying questions, and suggest future modifications. The participants asked clarifying questions about the planning and the video-based discussion, such as how the facilitators decided to use participation as the lens for viewing the video, and why they introduced the first viewing by asking the participants to write down what they noticed, whether or not it was about the math. Suggestions focused on both the preparation and the workshop itself, for example, that the facilitators bring copies of the math problem to the workshop as a reminder for the teachers, and that they introduce the second viewing by explaining the benefits of watching a video with a specific lens.

Discussion

Our team developed the Learning to Facilitate Framework (Jarry-Shore et al., submitted) by lifting an existing framework for supporting teacher learning from the TPD level (Grossman et al., 2009) to the FPD level and adapting its pedagogies to support novice facilitators in leading TPD. In this paper, we analyse an entire FPD session in the Problem-Solving Cycle program and make explicit both the facilitator educators' decisions in designing and leading FPD activities and the rationales underlying them. In this way, we exemplify the value of the LTFF as an analytic tool. Here we share examples of the insights we gained by using the LTFF to analyse pedagogies enacted in PSC's FPD program.

Using the LTFF to analyse unpacking of the representation revealed the importance of improvisation in a live representation. FE3 planned several questions in advance, which focused almost exclusively on FE1 and FE2's planning, in order to make their decisions and rationales visible to participants. However, equally important were the questions she asked that she had jotted down during the live representation, which made the facilitator educators' in-the-moment decisions and rationales visible. In reflecting on the representation and its unpacking, we also realized that participants' questions differed according to their prior experience facilitating TPD. Based on this insight, we plan to elicit contributions from participants with a range of prior facilitation experience in future FPD sessions, as that could impact the focus of the questions and relevance of the conversation for the facilitators. Finally, we saw that by explicitly asking facilitators to "wear a facilitator hat" in each unpacking, we were implicitly supporting their development of what was a new identity for many, that of facilitator.

Analyzing activities from our respective FPD programs using the LTFF allowed our team to see how the framework's four pedagogies – selecting, representing, approximating, and unpacking – appear in our work and support our diverse goals as facilitator educators. Reflecting on these analyses also resulted in several insights about both the LTFF and FPD programs in general. For example, we realized that although all four pedagogies are relevant and essential for supporting facilitators' learning, they may not all be present in a particular FPD activity. They may occur either in a single activity or a coordinated set of activities within a FPD session, or even across several sessions in a FPD program. The PSC's FPD session we analysed is illustrative. All four pedagogies were present over the course of planning and enacting the FPD session, however, no activity included all of them. During the initial planning for the session, the central TPD task of leading a video-based discussion was selected as the focus of the entire FPD session. Then, the morning portion of the session included

only representing and unpacking, while the afternoon included approximating and unpacking.

A decade ago, McDonald et al. (2013) called for scholars and practitioners in teacher education "to collaborate in the development of a common language for describing (a) how teachers learn to practice and (b) the pedagogies teacher educators enact to support teachers in learning to practice" (p. 381). The LTFF serves a similar purpose in the more nascent field of facilitator education by offering designers and researchers a common language for discussing the pedagogies of FPD. Prior to developing this framework, we recognized that our respective FPD programs had several similarities and differences, but we lacked the language necessary to capture and describe those program features. In our conversations, it felt at times as if we were talking past one another, struggling to convey our ideas in terms that our colleagues would understand. The LTFF provided the language we needed to describe and compare our FPD programs, and we have gained important insights in our broader research program by using it to analyse similarities and differences in the enactment of the four pedagogies in our work. We hope that other facilitator educators will find the LTFF useful in designing or researching their own FPD activities. And we look forward to it supporting communication between FPD designers and scholars in diverse contexts, to foster a shared understanding of the complexity of FPD and help the field to move forward.

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Studio Facilitators' Learning on the Job

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Abstract: Limited research has examined facilitators' learning on the job to support teacher learning in professional learning (PL). We examined how two facilitators' practices pressed for PL goals and were responsive to teachers' contributions in instructional conversations and what guided those practices. Our analysis identified shifts in facilitator and teacher interactions across PL cycles using these practices guided by four principles of teacher learning: goal-directed facilitation is essential, teachers' ideas are sensible assets, rich representations of teaching are essential, and teacher learning is a process developed through teaching and reflection. We contribute to an emerging body of research on facilitators learning on the job and discuss implications for the adaptations of our theoretical perspective and future research.

Keywords: Facilitators, Professional learning, Principles for teacher learning

Introduction

Previous research highlights that professional learning (PL) conversations centred on the instructional system – content, student reasoning, and teaching practice – to enhance future practice are generative for teacher learning (Horn & Garner, 2022). Facilitating such conversation demands that facilitators simultaneously press for professional learning (PL) goals linked to this system and act responsively to participants' contributions for there to be significant opportunities for teacher learning (Jackson et al., 2015; Pöhler-Friedrich et al., 2024). However, limited research has examined the practices of facilitators who have learned on the job and have not been involved in leader PL, which often guides leaders to take up goals linked to the instructional system in responsive ways (Even et al., 2020). This study examines mathematics facilitators' learning on the job to lead teachers in instructional conversations in Studio, a classroom-embedded PL model. Studio facilitators and teachers use lesson artefacts and live enactment to examine how the nexus of content and pedagogy afford and constrain student learning (Lesseig, 2016). We sought to understand *how two Studio facilitators pressed for PL goals and were responsive to teachers' contributions in instructional conversations and what guided their practices*.

Literature Review and Framing

Because of the limited studies of facilitators' practices learned on the job, we review expert facilitation and facilitators enrolled in leader PL research. In one of the most widely-cited facilitation studies, van Es et al. (2014) identified facilitators coordinating across practices[1] to reach their goals, which led to productive discussions. Coles (2019) similarly identified connected practices that supported his goal of teachers describing features of the instructional system that lead to productive instructional conversations. Jackson et al. (2015) identified facilitators' practices of pressing teachers

[1] Of note, the field does not use consistent terms for practices and underlying principles. For clarity, we have modified terms to align meanings and to create consistency across studies.

to articulate their reasoning and build on contributions while attending to the PL activity goals. This is similar to numerous facilitator studies using various leader PL designs (Borko et al., 2021; Elliott & Lesseig, 2023; Pöhler-Friedrich et al., 2024). Jackson et al. also identified a set of principles that guided facilitators' decision-making, providing a rationale for moves made in–the–moment linked to general principles for teacher learning. These authors found that leaders ascribed to modelling instructional practice rather than creating a series of experiences that built, pressed, and sustained focus and dialogue on core features of the instructional system. As a result, their analysis showed that leaders made progress but still needed support with the principle that *teacher learning is a progression.*

Limited extant research examines the facilitation practices learned on the job. Lewis' (2016) study of novice Lesson Study facilitators showed how facilitators' practices shifted across PL cycles to include more content-specific conversations involving the instructional system. The facilitators were challenged to balance redirecting, standing back, and pressing for goals. Gibbons and Nieman (2024) studied facilitation practices in teacher workgroup conversations. They identified that a facilitator's *press* to specify students' mathematical reasoning, like Jackson et al. (2015), was linked to broader principles. Studies of facilitators' practices across PL activities and designs document the importance of pressing on teachers' reasoning, focusing on the instructional system, and responsively building on contributions to support innovative future practice. These practices are generative for teacher learning. Fewer studies reveal the principles guiding facilitators' practices.

Building on Coles (2019) and Jackson et al. (2015), *principled facilitation practice* guides facilitators' decisions for supporting teacher learning. This framing comes from lifting a widely used construct of *principled teaching practice* to frame how researchers talk about what guides classroom instruction (Horn & Garner, 2022; Prediger et al., 2019). For example, a facilitator's decision to press teachers to specify teacher-student interactions may be driven by different purposes, such as to clarify mathematical ideas or to examine how a teacher's actions build on students' ideas and align with the lesson goal. Such purposes may be guided by different principles for teacher learning, such as *specifying and describing instruction, which allows for deeper subsequent teacher conversations* (Coles, 2019), or *teacher learning is a process of reorganising teachers' current practices to orient them to new practices* (Jackson et al., 2015). Studies of principled practice show that we gain insights into these principles underpinning decisions by analysing the facilitators' and teachers' contributions over multiple teacher conversations (Gibbons & Nieman, 2024; Jackson et al., 2015). We use the construct of *principled facilitation practice* to examine *what* guided facilitators' practices.

Methods

Mesa High School (9th–12th grade, students aged 15–18) Studio PL was the site of this one-year study. Faculty "detracked" classes, so students were not assigned to mathematics classes based on presumed ability. Two teacher-leaders, Georgia and Jasper, who had taught for nearly 15 years, facilitated five of the past 10 years of Studio. Studio is a predictable cycle of lesson planning, rehearsing with peers, enacting with students, debriefing, and generalising to future practice. Two *Studio teachers*, Oliver (online Studio 1, fall 2020) and Benson (in-person Studio 2, spring 2021), volunteered to teach. Seven other faculty (called *residents*) participated. In 2020, Oliver was a first-

year teacher; the faculty had a range of seven to 26 years of teaching experience. Six of the 11 current faculty members participated in Studio for 10 years, and only one teacher was new to Studio.

Mesa's Studio cycles centred on a data literacy lesson involving slowly revealing parts of a data representation. In the lesson, students first notice and wonder about a limited view of the data representation. As the representation is revealed, students reflect on how their thinking changed. Finally, seeing the whole representation, students make sense of the data's quantities and variables and reflect on their data literacy. Students work individually, in pairs, and as a whole class to discuss their ideas and reflect verbally and in writing.

Facilitators named two PL goals for the two Studio cycles: to support students' data literacy, which meant that students learned to describe and interpret data representations and reflect on changes in their reasoning. A second goal was to support teachers in lesson development focused on data literacy. By rehearsing, enacting, and debriefing multiple times across Studio cycles, facilitators aimed to support teachers using students' data literacy contributions to move lesson goals forward. Facilitators aimed for teachers to adapt the data literacy lesson, meaning they would select data representations aligned to their course's data reasoning goals and use the lesson in their classrooms. In Studio 1, facilitators and a Studio teacher rehearsed the lesson and debriefed the rehearsal to refine it. The Studio teacher taught the lesson online due to COVID-19. Residents observed and took notes to support debriefing the lesson afterward. In Studio 2, a Studio teacher rehearsed the lesson; everyone debriefed it to refine it, and the Studio teacher enacted it with students in a classroom, followed by the lesson debrief.

Data Collection and Analysis

Primary data sources include field notes, transcribed video records, and artefacts from two Studio cycles (9 h). We first reviewed field notes and applied structural codes to mark cycle activities (e.g., rehearsal). We segmented the data by topic changes that directed the conversation to a new observation, claim, or idea. Each segment was coded for *elements of the instructional system* (content, pedagogy, or student reasoning), *discursive practices* (e.g., eliciting reasoning, building on contributions, offering different perspectives; van Es et al., 2014), and *conversation forms* (e.g., replaying, rehearsing, anticipating students' responses; Horn & Garner, 2022). To reach inter-rater agreement, we coded a subset of the data and discussed coding disagreements to resolve them.

We then looked for patterns and shifts in facilitators' practices to uncover the *principles* underpinning their contributions. We wrote memos (Miles & Huberman, 1994), tracing patterns and shifts in their discursive practices and conversation forms across the two cycles. We pressed ourselves to articulate purposes for their contributions (e.g., norm-setting, pressing on goals; Coles, 2019; van Es et al., 2014). We then used an iterative process to identify principles associated with these purposes; we share these in the Results.

Results

We illustrate patterns of interaction and shifts in how Studio facilitators pressed for PL goals and were responsive to teachers' contributions in instructional conversations through excerpts across Studios 1 and 2. In doing so, we highlight four *principles* guiding facilitators' practices: (1) goal-

directed facilitation is essential for teachers' learning; (2) teachers' ideas are assets to connect to; (3) rich representation of teaching is essential for teachers' learning; and (4) teacher-learning is a process developed by teaching and reflecting.

One key pattern concerned a series of facilitator interjections to make explicit the PL goals of the data literacy lesson. Midway through the facilitators' rehearsal introducing the residents to the lesson, Jasper highlighted residents' mathematical contributions: "I'm noticing that people are thinking about a couple of different variables here: the size of the circle and the numbers inside the circles, and attempting to make some sort of connection or relationship between those." Jasper's description modelled how a teacher might highlight students' data literacy, which is a PL goal. Later, Jasper asked residents in the debrief to reflect on a PL goal of how to support students' data literacy: "What else do we need to say …in terms of …the goals, the language we use when we're talking to kids about how do we get them better [at] reading graphs and interpreting graphs?". Resident Tyson responded, "We're seeing graphs that are …not traditional,... multiple variables and multiple things going on." Georgia added, "Tyson said the multiple variables, that was what Jasper had talked about…. taking all these multiple variables and essentially putting them together to make a conjecture." Georgia made explicit the PL goal for the lesson, building from Tyson's comment and enacting Principle 1, goal-directed facilitation. Her building on residents' ideas is also linked to Principle 2, which states that teachers' contributions are assets. Georgia followed up, asking residents to reflect as a segue to the next Studio activity, "So, really, how do we support kids in coordinating variables?" Residents planned a data literacy lesson, followed by Studio teacher Oliver rehearsing before teaching his lesson.

In Oliver's rehearsal debrief, facilitators' practices were shaped by Principle 1; they attempted to make PL goals evident by asking about data literacy contributions and teaching moves. However, in Oliver's rehearsal debrief, residents focused on the social nature of eliciting students' discourse. In an attempt to shift from the social focus to PL goals, Jasper, guided by Principle 1, said, "I'm going to pause us," and asked, "What's Oliver saying to help [us] focus on data literacy?" Residents continued highlighting Oliver's social scaffolds. Jasper made one last interjection in the rehearsal debrief, asking Oliver and resident Henry, who planned with him, to elaborate on how their design leveraged data literacy ideas. Jasper's prompting oriented the group toward a PL goal (Principle 1), also creating a rich representation of mathematics (Principle 3) related to that PL goal.

In the lesson debrief, residents initially *replayed* how Oliver elicited and positioned students' contributions, again a social focus. For several contextual reasons, residents' contributions made sense. Oliver was a novice teacher, and the students had limited opportunities to interact due to online learning during COVID-19. Midway through the debrief, resident Brooke inquired about two students, one who revised their thinking based on the complete graph (a key idea in the lesson) and one who did not. Jasper leveraged Brooke's inquiry, explaining how the students were being data literate. This sparked resident Justine to interject on how she saw students revising their thinking based on a deeper look at the graph.

> In [the second reveal], Oliver, you were really encouraging them to revise their thinking, and you brought reference to the sentence frame. This was that piece of establishing purpose, and that part

about why we're even asking students to revise their thinking, and this being a place that we can help instill that.

Justine highlighted the lesson goal and an element supporting students' data literacy, the sentence frames for students' sense-making of the graph. After others reflected on Justine's comments, Jasper recentred her ideas, elaborating now on the teaching moves that supported data literacy.

> Like Justine is saying, be really explicit about telling them that the next time you interpret this, I want you to ask yourself, what do you notice? What do you wonder? How has your thinking changed? When you look at it a second time more closely and ask yourself, what's the data telling us?

Across Studio 1, facilitators' practices shifted in response to residents' contributions. For example, Georgia and Jasper initially made the PL goals explicit since the lesson and its content were new to residents. In response to residents' focus on students' social interactions in Oliver's rehearsal debrief, Jasper paused the conversation to re-orient residents to the PL goals. Because this interjection did not connect to residents' contributions, residents did not take it up. Later in the cycle, during the lesson debrief, Jasper built on residents' contributions to orient them toward the PL goals. This shift in Jasper's practice appeared grounded in Principle 4: Teacher learning is a process developed by teaching and reflecting, and it led to conversations grounded in PL goals and connecting aspects of the instructional system.

In Studio 2, facilitators continued to press for PL goals, and we saw more residents' contributions shift to reflect on goal-focused ideas. This opened up ways for facilitators to be responsive to them. Studio 2 focused on the same data literacy lesson (using a new data representation) and PL goals. Studio teacher Benson rehearsed and taught the lesson. Georgia and Jasper pressed the group during the rehearsal debrief to specify how the lesson supported students' data literacy, guided by Principle 1. When responses moved away from data literacy, facilitators continued to press, inviting the group to consider how the lesson could take up data literacy goals. As a result, the group engaged in conversation, making robust connections between the data literacy goals, pedagogy, and anticipated student reasoning related to the data literacy lesson.

While Studio 2 rehearsal debrief was a shift from Studio 1, there were still bumps or stumbles. Benson initially focused on "tweaking the task" based on the rehearsal. However, residents and the facilitators encouraged him to take up the data literacy purpose rather than focus on his stated goal of students' arguments. Jasper and Georgia invited him to consider data literacy goals through multiple interjections. Residents Brooke, Kay, Tyson, Justine, and Zandra supported these ideas by unpacking the lesson and discussing how students might use data reasoning as the lesson unfolded. Jasper reflected, "When I see this stuff, I naturally tend to, and I think we all tend to, leave the data literacy off. And it's partly because we're still developing [our reasoning on the topic]." Jasper used personal and plural pronouns to create a sense that this was collective work to understand data literacy. Georgia said to Benson as she encouraged him to consider students' arguments related to data literacy, "Why don't we try? …[T]hat is why we are here, to try it, because it doesn't have to be an exemplar." Facilitators keenly pressed Benson, a seasoned teacher who was a part of the statistics team and who was teaching data science the following year. Through the support of residents and the facilitators,

Benson designed a path informed by two goals: data literacy and students' argumentation. During the Studio lesson, the facilitators coached Benson to take up students' contributions without assessing them. Jasper interjected with Benson's invitation when a student reflection prompt was sidestepped, eliciting students' data literacy reflections.

Resident Justine started the lesson debrief and focused on the lesson's purpose, elaborating on how students were prompted to think about what they would do the next time they saw a graph. "And all of a sudden, they all started talking about [what] we would want them to do every time they look at graphs." Jasper used that as an opening to elicit residents' replaying specific student contributions related to data literacy. Residents then shared what they heard from students (e.g., considering the context of the data and looking at the representations' features).

Across Studio 2, Jasper and Georgia were responsive to residents' contributions by building on their ideas (Principle 2). In doing so, they oriented the group toward students' data literacy and the PL goal of teachers lifting and describing students' contributions to demonstrate data literacy (Principle 1). As a result, the group's conversation made connections between students' data literacy contributions, discussion prompts, and how they could build on students' contributions in future instruction. The facilitation shifts across Studio cycles evidenced how these practices were guided by t*eachers' learning is a process developed by teaching and reflection (Principle 4).*

Discussion and Conclusions

We provided an image of facilitators' learning on the job as they pressed for PL goals linked to the instructional system responsive to teachers' contributions guided by principles for teacher learning. Findings indicate that as facilitators' interjections shifted within a Studio cycle and across cycles, building on and connecting to residents' contributions, more residents' contributions linked to the core goals of data literacy, supporting generative conversations at the nexus of content, pedagogy, and student reasoning (Borko et al., 2021; Horn & Garner, 2022). Studio 1 facilitator interjections lifted mathematical and pedagogical ideas to highlight how they were related to PL goals. Across the cycle elements, we saw residents' attention to data literacy slowly grow. Studio 2, carried out with the same PL goals, allowed facilitators to elevate students' data literacy, which empowered residents to contribute to generative conversations focused on mathematics, pedagogy, and students' reasoning.

Like in other research studies, Georgia and Jasper faced tensions as they pressed on PL goals and were responsive to teachers' contributions to support generative teacher learning (Elliott & Lesseig, 2023; Karsenty et al., 2023; Lewis, 2016). They were responsive to the experiential realities of Studio teachers, positioning new teacher Oliver as competent to build trust and pressing Benson, a seasoned teacher, to experiment, collaborate with residents, and take up classroom coaching to uncover students' reasoning (Gibbons & Nieman, 2024; Jackson et al., 2015; Lesseig, 2016). The interjections guided by overlapping principles for teacher learning showed how facilitators may be goal-directed and frame teachers' contributions as assets as they press for specificity and rich descriptions. By tracing facilitators' practices across activities within and across cycles, we illustrated how they were also guided by Principle 4, teacher learning as a process supported by teaching and reflection.

The four principles guiding facilitators' practices are well-rooted in PL research. Principle 1 is rooted in the goals of the instructional system and reflects numerous studies of facilitation that highlight the

challenges of goal-directed facilitation (e.g., Elliott & Lesseig, 2023; Jackson et al., 2015; Karsenty et al., 2023; Pöhler-Friedrich et al., 2024). Principle 2, teachers' ideas are assets, links to studies of teachers' asset-based language on students' reasoning (Horn & Garner, 2022). Principle 3, rich representations, echoes Coles' (2019) and Horn's and Garner's (2023) ideas that specifying and describing instruction leads to deeper teacher conversations, and Principle 4 is similar to Jackson et al.'s (2015) teacher learning is a process which was facilitated by the Studios' coherent design where residents developed and reflected on multiple lessons with similar goals and structures within and across cycles. These linkages to previous research validate the importance and utility of the principles guiding the Studio facilitators' practices to press for goals responsive to teachers' contributions supporting generative teacher learning.

Our study contributes to the growing body of literature on facilitators' efforts to press for PL goals and be responsive to teachers' contributions by providing an image of facilitators' learning these practices on the job and identifying the principles guiding that practice. We show how these experienced facilitators of classroom-embedded PL have transcended some of the tensions of being goal-directed *and* responsive to teachers' instructional conversations, a persistent challenge documented in facilitation research (Elliott & Lesseig, 2023; Jackson et al., 2015; Schwarts et al., 2023; Pöhler-Friedrich et al., 2024).

This study's limitations include its narrow focus on instructional conversations and one PL context. Future research should examine whether and how the principles identified might support future facilitator PL design and the scaling up of research on facilitators' learning on the job. In addition, future research should explore facilitators' principled practice in other PL contexts.

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Self-perception of facilitators in PD

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Professional development (PD) courses are developed by researchers and PD designers who are often not the persons who conduct the PD course (Borko et al., 2014). The facilitating of teacher educators is commonly the means of scaling up PD courses to in-service teachers. The professionalization of in-service facilitators is an intervention to promote facilitators´ learning and their development of knowledge, as the facilitators' realizations vary according to their professional views and beliefs relating to PD programs and teaching practice. However, little empirical insights exist about facilitators´ self-perception as educators and their beliefs. This contribution presents some empirical results on the way facilitators perceive their role in the process of adapting and implementing of a PD program for in-service teachers. The goal of this study was to explore the self-perception of the facilitator, based on the underlying and connected beliefs that were reconstructed.

Keywords: Beliefs, professional development, teacher education, roles.

Professional development (PD) of facilitators

Design principles of content-related professional development (PD) courses for in-service mathematics teachers have become a vivid research area (Karsenty et al., 2023). The professional behaviour and beliefs of facilitators have been identified as an almost blind spot of PD research (Jackson et al., 2015). Some pioneering existing research studies focus on facilitators´ professional knowledge (Jaworski & Huang, 2014) and practices (Borko et al., 2014). Given the consensus on the decisive function of facilitators in the process of scaling up, the present study investigates the professional roles and beliefs of facilitators who educate in-service teachers with the aim of supporting their professional development.

As a first step, the concept of a facilitator will be shortly outlined. Based on a review of the literature, Lunenberg et al. (2014) broadly defined the heterogeneous group of facilitators to consist of teacher educators who are responsible for the continuing professional development of in-service teachers (in contrast to teacher educators for pre-service teachers). Like in-service teachers, facilitators face many challenges. They come from different backgrounds and work in different settings of teacher education. Their personal interpretation of their position as teacher educators is based on expectations from the environment (personal and institutional) which defines the concept of professional role (Dengerink et al., 2015). This definition of the professional role of facilitators should not be mingled with concepts of professional identity (ibid.).

Learning and especially lifelong learning is and has always been a fundamental issue in professional development (Jacobs et al., 2017). In research about models of teacher professional competence, knowledge as one dimension is complemented by the notion of beliefs (Philipp, 2007). As beliefs potentially guide decisions and aspects of practice (Fives & Buehl, 2012), it is therefore significant

to encompass beliefs in inquiries about decision-making and the professional behaviour of facilitators (Karsenty et al., 2023).

Synthesizing what is known from research about beliefs in education literature, this report adopts Philipp´s definition of beliefs as "psychologically held understandings, premises, or propositions about the world that are thought to be true" (Philipp, 2007, p. 259). Beliefs tend to be context dependent and differ from knowledge in that beliefs are thought to be true (while knowledge is known to be true or false). Moreover, beliefs serve as filters through which information is interpreted, influencing what one notices and how one interprets it (Philipp, 2007). Thus, beliefs about mathematics and its teaching play an important role in teacher learning and professional development.

This research on facilitators´ roles and beliefs is motivated by relevant adaptation processes similar to teachers on the classroom level. With their professional knowledge (on both: teacher PD and classroom level) facilitators might be seen as experts initiating appropriate learning processes to achieve specific targets in their PD sessions. Subjectively expressed beliefs as a part of facilitators' competence are of particular importance in trying to better understand what exactly determines their decisions in planning and performing PD sessions (selection of content and methods in PD). These decisions are to a large degree determined by what they think about the PD content and additionally by their individual perceptions (e.g. roles and beliefs) and goals (cf. Karsenty et al., 2023).

When facilitating the learning processes of schools and teachers, facilitators may have various roles which may also depend on the relevant expectations of the target group. Distinguishing various professional roles of facilitators, each requiring relevant skills, the differentiation of Fransson et al. (2009, p. 80f) of the following role descriptions is used as a starting point:

- Catalysts: who make people move, who bring about change, who inspire.
- Experts: knowledgeable and competent in acquiring or producing new knowledge.
- Problem solvers: who help learners identify and solve problems they are facing.
- Process helpers: who support learners in their learning processes.
- Developers: capable to develop learning environments, material, and tools.

The descriptions above do not completely cover the various roles of facilitators as a role can be an object of discussion. In practice several roles may often be combined. It seems obvious that facilitators indeed combine several roles and role combinations can possibly be a source of tensions, even conflict and modifications, because facilitators have to meet expectations from the setting of PD programs which may sometimes be hard to combine. Facilitators in this study were encouraged to communicate their own descriptions of professional roles as in-service educators. The deductive descriptions above thus serve as a mirror scope for facilitators' personal interpretation of their role which may then be derived inductively from the data in this study.

Research context and research questions

With the above considerations in mind, facilitators select and adapt PD materials to their own PD course within the given setting and context (specific teacher group, content of PD, cf. Dengerink et al., 2015) with respect to their personal choices, professional views and beliefs.

Within this research context, the aim was to examine how PD facilitators perceive their own double-role as educators and learners, and to reconstruct underlying beliefs with respect to their role. Among others in this project, the main research questions addressed in this paper are thus:

a) How do facilitators see their specific role in the PD courses regarding the target group?

b) What are facilitators´ beliefs with respect to the implementation of their own PD sessions?

To prepare and further qualify the facilitators for their PDs, a qualification course for these facilitators was conducted which consisted of continuous sessions over a period of one and a half years. Facilitators were provided with course material as well as possible design principles for PD sessions, video studies and relevant research literature. The participating facilitators in this project are all teachers who still teach at school and work as teacher educators simultaneously. Their task in the institutional context also contains the implementation of PD sessions concerning competence-based teacher education and the transmission of new principles of the curriculum (e.g. core practices).

Methodology of the study

For pursuing the research questions, a qualitative survey instrument in the form of a problem-centred interview (PCI), one of many techniques within the landscape of qualitative interviewing, was used as the PCI has an explicit interest in reasons why particular views, opinions or beliefs are prevalent (Witzel & Reiter, 2012). The individual interviews were conducted by the first author with N=10 facilitators, who voluntarily participated in this study. Various clusters of questions concerning relevant subjects (content and goals of particular PD session, self-perception) were used as well as narrative prompts concentrating on making facilitators aware of beliefs by means of reflection on intended action in their PD sessions. As an example, one activity in the PCI was a prompt to make participants comment on their priorities and views on design principles (e.g. participant-orientation) of their professional practice in PD sessions which guide their decisions from an individual perspective.

The interviews were transcribed verbatim, and the data was analysed following established conceptions of qualitative content analysis (Mayring, 2021). Especially an open content analysis seemed to be appropriate to maintain an openness approaching the research questions described above. The facilitators´ remarks on the selection and prioritization of contents for their PD courses, their strategies, underlying beliefs, and their roles as facilitators were reconstructed by a systematic comparison of cases.

Methodologically the PCI aims to facilitate a structured dialogue which allows to uncover views and perspectives of participants on particular issues. Additionally, the orientation on relevant subjects makes it possible to pose follow-up questions and precisely address the research focus (on particular roles and beliefs). The reconstruction of beliefs is thus furthered by the participants' capability and motivation of self-reflection.

Results

The results show that all facilitators considered appropriate learning activities and settings important in prioritizing various adaptions of the PD content resources provided. Facilitators mentioned several

issues (e.g. community and context), considering holistic challenges and multi-facet complexities of their PD sessions. Identifying specific facilitators´ views is always established in various parts of a single interview. We report only facilitators´ views that are coherent throughout the whole interview. In general, facilitators have to be able to adopt multiple roles, each requiring a high level of competence and skills (cf. Dengerink et al., 2015). Based on our data, some major roles and connected beliefs have been identified: three roles are highlighted exemplarily, appearing in most of the sample.

Facilitators consider their coaching role as very influential. They see themselves as process helpers who support teachers in their professional learning. In particular, facilitators try to meet the needs of the participants which they first have to analyse and identify themselves:

> Mr R: These are precisely the topics I mean. You come across certain things with participants in your PD sessions, then you can identify problems and possibly move them to reconsider their existing paradigms. And here I can accompany them in this process with research-based approaches. […] But being a problem solver, no, I wouldn´t presume to do that.

In terms of participant-orientation Mr. R. considers different orientations of the participants and intends to mediate between teachers´ interests and the need to communicate ideas based on PD resources, encourage reflection, and give productive feedback to participants in their learning community. Facilitators can reflect on their own actions and associated beliefs regarding the limit of one´s expertise and area of responsibility, as the example of Mr R. shows, who additionally mentions that he does not see his task description to be a problem solver as a facilitator. In meeting the needs of participants and with an explicit approach in trying to stimulate the learning process with an adaptation of the PD material, a clear belief can be reconstructed that Mr. R sees his role as a cooperating facilitator and coach. Offering research-based knowledge and professional support are key factors for this facilitator and shows that he determines his role as coach in an aspired symbiosis of various points - providing subject matter knowledge, experience in professional support and skilful communication.

The role and belief of a facilitator as a transmitter of input can clearly be found in the data. In their self-perception, some facilitators consider themselves to be competent in producing new knowledge and underline these beliefs with regard to the role as an expert, as the following examples illustrate:

> Mr S: Yes, I am definitely a provider of new ideas and try to give fresh impetus and impulses to my PD participants. […] Whether this results in any changes in teaching practice it remains to be seen.
>
> Mr B: …then [I see myself] as a source for transmitting input by raising relevant issues and questions, or stimulate discussions by outlining various teaching scenarios along with a critical reflection.

Both facilitators claim to be an initiator and catalyst who want to bring about change and who want to be a source of inspiration for teachers in their PD sessions. A connected belief to this role is that facilitators draw upon a sound knowledge base and their PD activities seem to be closely linked to the provided PD resources. However, the subjective evaluation of Mr. S. demonstrates the awareness of potential dilemmas between transmitting and PD efficacy. Sometimes facilitators try to model the implementation of both – content and teaching methods – in their PD sessions. Regarded as being particularly effective within the context of a PD session and as an adaptation of the PD material, Mr

B. enacted a discussion in one of his sessions about raising student awareness of thinking processes in teachers' classroom lessons:

> Mr B: We want our students to be able to use and choose between different problem-solving strategies, but first they need to learn to generate alternatives for any given mathematical problem or task. So, don´t we need to practice that for our own teaching methods in PD? […] When we design our actual [PD] courses, we must consider the goals and activities so that participants can promote thinking of students by critically considering various pedagogical approaches in actual teaching.

The modelling of learning approaches in PD sessions shows that Mr B. tries to be both – a transmitter of input and an initiator by transferring a specific issue from the discussion in PD sessions into the classroom that is one of his main concerns in professional learning. Thus, it can be reconstructed one of his beliefs is that modelling learning processes is one way of transferring research-based theory into practice. His role as a facilitator certainly includes being an initiator.

The adaptation of curriculum materials and content on the teacher PD level to the classroom level tries to combine research-based expectations with participating teachers' knowledge of practice. The goal of developing and fostering interactions between facilitator and participants becomes apparent here. The remark about critical reflection shows that Mr B. articulates his role as an initiator to create a community of learning and shows his belief that with an initiated focus on specific PD themes and content, the potential of his PD sessions can lead to consolidated joint insights and product on the classroom level. Of course, this is a case-specific individual perception of being an initiator which is rather questionable in terms of efficacy as can be seen by the remark of Mr S above. It should be noted that both facilitators seem to fulfil the role of a broker in the professional development and learning of in-service teachers. This role implies that a facilitator needs communication and negotiation skills to transfer and mediate curriculum material and content from a PD level to a classroom level using expertise on both levels. Stimulating inquiry and a joint attention on specific themes as a defining feature of the facilitator role as a broker confirm this interpretation (cf. Lunenberg et al., 2014, p.58f).

As a linking pin in the transfer of methods and content from research-based PD design to teacher adaptation in heterogeneous classroom the facilitators in this project contribute to the necessary process of theorizing and adaptation. However, they do not see themselves in the role as a developer who is capable to supplement materials and tools:

> Mrs K: I wouldn´t see myself as a developer. I would not have sufficient skill or knowledge to develop materials, but rather implement findings from research into practice.

With respect to the principle of participant-orientation facilitators stress that they want to fulfil teachers´ expectations in their PD sessions and some facilitators rather tend to focus on task examples as a form of transmitting practical knowledge. This does not exclude brief processes of theorizing, but participants in this sample rather hold the belief that their competence profile as a facilitator needs further qualification in this respect. The knowledge domains of facilitators in our sample are dominated by knowledge of practice with a focus on skills and teaching practice as well as an awareness of reflective beliefs and orientations in which facilitators go beyond a mere consideration of teaching skills and core practices towards a better connection between theory and practice.

Participating teachers' concerns and needs are more central than compliance issues about curriculum development or enactment in their adaptations of the PD curriculum materials.

> Mr C: ...well, if this [being a developer] was successful, this would be really good, but regarding myself this is not a direct objective and certainly this is not covered by my role in the system.

It could be argued that one principle for the design of education curricula is the common principle of linking theory with practice. In a role of being a "seconding developer" the facilitator needs to have a clear vision of the relevant knowledge and professional learning (being acquired on a facilitator PD level). Then facilitators adapt and model facilitator PD curriculum materials about teaching and learning approaches, which are advocated in the PD program, in their own PD sessions.

Yet this potential role of a "seconding developer" cannot be found at all in the data of our sample as the quotation from Mr. C. exemplarily shows – with an implied limitation. The potential role of being part of a development process or a developer seems to be restricted by institutional norms and role assignments in the educational system. Further hints of a desired role of being a developer cannot be found in Mr. C.'s interview data. The one given, however, is an indication that an active adaptation process between the different levels is taking place. It supports the argument given by Cochran-Smith (2006) who contents that any teacher educator or facilitator is a "public intellectual" who should play an active role in debates and processes about curriculum development and adaptation. Our data rather confirms the hypothesis that in this respect facilitators are more likely to be following the role they have been assigned by education institutions and authorities.

Discussion and conclusion

Based on the literature and our findings, it can be said that facilitators in this study consider themselves adopting a role of being professional learners. They achieve to attain an individual level of being a role model in professional learning in which facilitators hold a set of beliefs concerning their roles which they try to organize in a way that makes sense to the individual and is thus subjectively coherent (Leatham, 2006). The different roles, however, imply both, the challenge of how to cope with these and making decisions when a particular role is appropriate in the specific setting of a PD context.

The beliefs of facilitators which could be reconstructed in this sample illustrate a variety of role models and challenges which are perceived and reflected by our participants with different modes of experience in facilitation. The identification of facilitators' roles and beliefs is important in considering the adaptation processes of practices on a teacher PD level. These beliefs and their connection to enacted PD practices is however a tentative one.

Although the presented study is limited in scope it already provides some interesting answers to the research questions. The results show that all facilitators considered participant-orientation and stimulating cooperation as highly important design principles for their various adaptions of the PD content resources provided in their qualification course. These beliefs and associated role reflection are seen as an important and characteristic competence of facilitators in their capability to incorporate the needs and expectations of in-service teachers (target group) in several stages of the facilitation process. Facilitators usually have different backgrounds and have different paths how they found their

way into being in-service teacher educators. All participants of our sample are or have been active classroom teachers and are thus experienced in the profession. As teachers as clients in PD sessions prefer to be supported by someone who knows the profession from the inside, it is an expected result that facilitators subjectively hold beliefs that emphasize their preference for a cooperative setting and want to provide rich learning opportunities for their clients.

Using research-based textbooks and hands-on material was a major facilitator goal to support in-service teachers in coping with heterogeneity in the classroom. Participants see themselves as tutoring process helpers, experts, and transmitters of input, who provide impulses for innovations rather than being problem solvers or developers. This is consistent with their beliefs with respect to their own PD sessions and their roles therein.

What becomes apparent in the data of this study is that facilitators experience various tensions when they report about their beliefs and roles. Combinations of roles and potentially conflicting beliefs can be a source of tensions – be it in teaching at school or in professional development settings. Facilitators must meet various expectations and norms which are sometimes hard to combine. Dilemmas which have been mentioned by participants in this study are: the tension between theory and practice as well as the tension between approaches being advocated in the teacher PD curriculum material and their personal adaptation processes in relation to their beliefs and role reflections as an actor in this professional development program. In dealing with such tensions and dilemmas, facilitators require comprehensive theoretical knowledge as well as experience and reflective decisions. By finding the right balance between the factors causing the tensions, facilitators can make sense of their role of a teacher of teachers.

The limitations of this study are obvious: a small sample of facilitators can only illustrate facilitators' beliefs and roles exemplarily and should be taken as some illuminating cases for investigating significant factors of professional development practices in any other PD context. Overall, the results suggest that further connections between facilitators' competences, their role in adult education, their self-perception, beliefs, and efficacy should be identified in future research.

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Learning to broach potentially difficult discussions about teaching through meta-coaching

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Coaches can help teachers improve their teaching, yet little is known about how coaches broach potentially difficult discussions with teachers about aspects of their teaching to improve or how they learn to do this work. We examined a coach learning initiative, in which a mentor coach, or meta-coach, helped a coach broach difficult discussions with teachers about their teaching. We analyzed meetings between the meta-coach, coach, and one teacher before and after the teacher's lessons. Analysis revealed that the coach was unsure how to talk with the teacher about a lack of student engagement in her lessons. The meta-coach played a crucial role in helping the coach broach a potentially difficult discussion of this topic using "entry points" in the form of classroom data. This approach led the teacher to reflect on her teaching, absent the negative emotions that may arise when discussing improvements to one's teaching. We discuss implications for coach and teacher learning.

Keywords: coaching, meta-coaching, professional development, responsive coaching

If teachers are to make the substantial shifts in practice they are currently urged to make (Cobb & Jackson, 2021), they will require access to sustained professional learning opportunities. One approach to supporting teachers' professional learning that has become increasingly prominent of late is coaching, in which a mentor teacher, or *coach*, helps teachers develop their teaching practice in ways likely to enhance students' learning (Desimone & Pak, 2017; Gibbons & Cobb, 2017).

To support teachers' learning, coaches must engage teachers in potentially difficult discussions about aspects of their practice to improve. In observing a teacher's lesson, a coach might recognize an aspect of a teacher's practice that may be in need of growth, yet merely pointing this out to the teacher could seem evaluative and engender negative emotions like frustration or shame (McGugan et al., 2023). Coaches may thus avoid such discussions, maintaining the coach-teacher relationship (Anderson et al., 2014), but leaving the practice in need of improvement unchanged.

The field has begun to examine how coaches navigate such tensions in their work, yet still lacks knowledge of productive approaches coaches may implement in broaching difficult discussions with teachers. Also lacking is an understanding of how coaches learn to broach these sorts of difficult discussions. We examined an approach to supporting coaches' professional learning we call meta-coaching, in which an experienced coach, or *meta-coach*, helped a coach learn to broach difficult discussions with a teacher about her teaching. We addressed the following questions: 1) How did a coach broach difficult discussions with a teacher about aspects of her teaching to improve? 2) How did a meta-coach prepare a coach to broach difficult discussions with a teacher about her teaching?

Conceptual framework

Supporting teachers to develop their teaching practice through coaching

Coaching is an increasingly common approach to helping teachers develop their teaching practice in ways likely to enhance students' learning (Desimone & Pak, 2017; Russell et al., 2020). With regard to teaching mathematics, coaches can help teachers implement an ambitious and equitable instructional approach (Cobb & Jackson, 2021), in which they: a) launch a cognitively demanding task, b) ask students to solve the task in small groups, and c) select student strategies to be compared in a whole-class discussion. As this teaching approach is quite unfamiliar to many who teach mathematics (Ball et al., 2001), teachers may need help implementing it. Coaches can offer such help.

To support teachers in implementing this instructional approach, coaches may enact various routines (Gibbons & Cobb, 2017). For instance, they may facilitate a coaching cycle (Kochmanski & Cobb, 2023), in which they: a) co-plan a lesson with the teacher, b) observe the lesson and gather artifacts from it (e.g., video; student work), and c) examine these artifacts with the teacher in a lesson debrief.

Supporting coaches to develop their coaching practice through meta-coaching

Coaches must learn a great deal to support teachers in developing their teaching, yet with few exceptions (e.g., Russell et al., 2020; Saclarides & Kane, 2021), little work has studied approaches to supporting coaches' professional learning. In this study, we show how a new approach to supporting coaches' learning that we call *meta-coaching* supported a coach in developing her coaching practice.

The role of meta-coach is something we conceived in collaboration with our district partner. For us, a meta-coach is a mentor coach who supports individual coaches in developing their coaching practice. A meta-coach is also an experienced coach with substantial prior teaching experience. As meta-coaches are experienced coaches, they are likely to have prior experience broaching difficult discussions with teachers and can thus share lessons learned from these experiences with coaches.

To understand the work of meta-coaching, we draw on Prediger et al.'s (2019) Three-Tetrahedron Model, which describes learning at three levels. At each level, there are learners and someone supporting the learning. At the classroom level, the learners are students, whose learning is supported by a teacher. At the teacher professional development level, the learners are teachers, whose learning is supported by a facilitator. Finally, at the facilitator professional development level, the learners are prospective facilitators, whose learning is supported by a facilitator educator. A meta-coach is like a facilitator educator in that they prepare others to support teachers' learning.

Much as a facilitator educator can draw upon their experience as a facilitator, a meta-coach may draw upon their experience as a coach. For instance, they may lift the practice of facilitating coaching cycles with teachers to enact a variation of the coaching cycle with a coach, leading a planning session with the coach before a coaching cycle, then debriefing the coaching cycle with the coach afterwards.

Conversational stances in coaching cycles

Coaching cycles are one coaching routine a meta-coach might support coaches in implementing. As part of these efforts, a meta-coach may support a coach in adopting a directive or responsive conversational stance (Heineke, 2013; Ippolito, 2010). These stances position the coach and teacher

in different, and consequential, ways. With a responsive stance, a coach places a teacher's ideas center stage, thereby positioning the teacher as knowledgeable while decentering their own thinking, ideas, and opinions as a coach. Such a stance also allows teachers' interests to guide the coaching process (Dozier, 2006). With a directive stance, on the other hand, a coach offers advice to teachers about which teaching practices to enact and how, thereby positioning themselves as the primary knower, centering their own thinking, and setting the focus for the teacher's learning.

Both stances have affordances and limitations. While a responsive stance may enhance teachers' participation in coaching (Gillespie & Amador, 2024; Ippolito, 2010), it could result in discussion of an aspect of teaching practice other than the one on which a coach may wish to focus. A responsive stance may thus maintain a sense of comfort, but could leave an aspect of a teacher's practice in need of growth remaining unchanged (Dozier, 2006; Ippolito, 2010). With a directive stance, this is less of an issue, as such a stance allows a coach to explicitly name the teaching practice they would like to discuss. Such explicitness, however, may understandably elicit negative emotions (McGugan et al., 2023) and leave a teacher feeling judged, potentially withdrawing from the discussion or even choosing not to work further with a coach (Heineke, 2013). Given this, some have suggested moving between these two stances or adopting a combination of the two (Heineke, 2013; Ippolito, 2010).

Methods

Setting and participants

We examined a meta-coaching initiative that took place in the 2019-2020 and 2020-2021 school years in a large district in the western United States. In total, three coaches and six teachers took part in this study. We focus here on data for only one coach and the teachers she worked with in 2020-2021, as the problem of coaching practice integral to this study – broaching difficult discussions – was most salient to this coach and came increasingly into focus for her in 2020-2021. We present our analysis here of a coaching cycle with one of the coach's teachers in which this problem was especially salient.

The meta-coach was an experienced coach and former teacher. She was previously responsible for ensuring coherence across the district's professional learning initiatives and for supporting implementation of the district's new mathematics curriculum. She then began serving in the role of meta-coach, a role we conceived together and that was funded by a grant to our research team. The coach, on the other hand, was a former teacher with five years of coaching experience. The teacher taught 7th-grade mathematics and was in her 4th year in the classroom. All participants were familiar with one another through their prior participation in other district professional learning initiatives.

The meta-coach helped the coach facilitate coaching cycles with several teachers, including the focal teacher. In the cycles, the coach worked alongside teachers in planning a lesson, before then observing the lesson and debriefing it with the teacher; the meta-coach also attended these phases. In this work, they supported the teachers with student engagement, launching tasks, and fostering student discussion. The meta-coach and coach met together at the start and end of each coaching cycle, as well as between the observe and debrief phases of the coaching cycle in what we refer to as the *meta-coaching cycle.* The coaching cycle was thus embedded in the meta-coaching cycle (Figure 1).

Data collection

The meta-coach collected audio-recordings of each phase of the cycles except the lesson enactment. She had been provided an audio-recorder and a brief script to read at the start of each audio-recording stating the date and phase of the cycle being recorded. This generated audio-recordings of the following phases: plan for the coaching cycle, plan for the lesson, prepare for the lesson debrief, debrief the lesson, and debrief the coaching cycle. Phases 1, 4, and 6 involved the meta-coach and coach. Phases 2, 3, and 5, on the other hand, involved the meta-coach, coach, and teacher. The recordings of each phase were 20 minutes or longer. Transcripts were created for all audio-recordings.

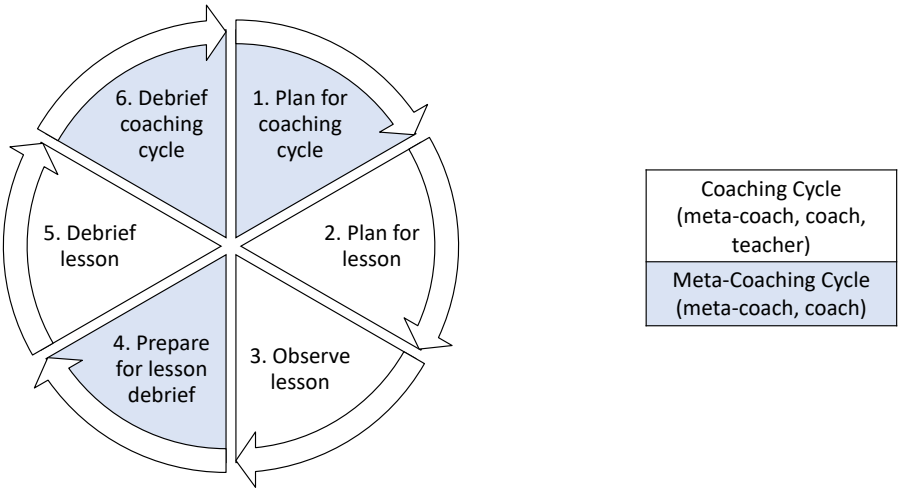

Figure 1: The meta-coaching cycle

Brief student surveys figured into the coach's work and were a secondary data source in this study. These surveys were short, took fewer than five minutes to answer, and consisted of questions asking students about their experiences of lesson launches and small-group or whole-class mathematics discussions (Nieman et al., 2020). The surveys were developed to offer teachers feedback that could be referenced during ongoing, job-embedded professional development like coaching. With their coaches, the teachers in this study identified a specific survey and survey items to focus on in each coaching cycle related to the goals they had set for their teaching.

Data analysis

The first and second authors began analysis by reading transcripts of the lesson debrief preparation phase for the coach's 16 coaching cycles in 2020-2021. Their goal was to identify cycles where the coach mentioned an aspect of teaching practice she wanted to discuss with a teacher in the debrief, but was unsure how to broach. This led them to identify one cycle where this problem and efforts to address it were especially salient. They next read transcripts for all phases of this cycle to see if the aspect of practice was discussed elsewhere, before writing and comparing analytic memos (Maxwell, 2013) in which they reflected on discussion of this aspect of practice in the cycle. This allowed them to identify: 1) the nature of the teaching practice the coach wanted to discuss, 2) how the meta-coach supported the coach in preparing to broach discussion of this aspect of practice, 3) how the coach broached discussion of this aspect of practice in the lesson debrief, and 4) how the coach and meta-coach felt things went. Throughout, the first and second authors attended to the conversational stance advocated by the meta-coach and adopted by the coach in broaching discussion with the focal teacher.

Results

To illustrate how the meta-coach prepared the coach to broach difficult discussions, we present excerpts from phases 4, 5, and 6 of the cycle we analyzed, in order, starting with the lesson debrief preparation phase. We chose these excerpts as they were especially illustrative of key elements of the meta-coach's approach to broaching difficult discussions and her efforts to help the coach enact it.

In preparing for the lesson debrief (i.e., phase 4), the coach expressed disappointment regarding a lack of student engagement she had perceived in the teacher's preceding lesson, stating the following:

> We kind of planned on putting a premise [i.e., a problem context] for kids to get excited about the math problem. I thought the teacher was going to make it more engaging about splitting pizza, like an actual story. I imagined visuals and pictures and here's these three groups of friends and, I mean, honestly, the question that – I know I can't ask this – but I really want to ask this question, which is, "what would make you excited in your own classroom, when were you excited in this classroom?" Because I was not excited in this classroom.

The coach seemed disempowered here to do much about the lack of engagement she had perceived. She also noted that, while she wanted to speak to this issue directly, she could not, implying that such a directive approach might come across as evaluative and result in an unproductive lesson debrief.

The meta-coach agreed that student engagement had been low and that this should be the focus of the ensuing lesson debrief. To help the coach broach discussion of the lack of student engagement, the meta-coach suggested the following to the coach:

> The teacher gave you an entry point to that. I know you're not going to say those words, but she gave you an entry point because she said, "hey, let's look at the [survey] question that asked, 'Were you excited about the problem?'" The teacher asked to look at that specific question. And the results were close to 50-50 … so, she gave you, she opened the door for you to have that discussion … So, I would open the conversation with her in the lesson debrief with something to surface that.

The meta-coach then provided additional suggestions for what to do in the lesson debrief depending on the teacher's response to being asked to examine students' responses to this survey question:

> And if we don't get, "[the results are like that] because the lesson was boring for me, too," if we don't get that statement from her, [but instead get], "oh yeah, I should have expected that," [we can ask], "why do you say that, what made you say that?" And I think, with asking those two questions in that sequence, I think that'll get her to say, "because it wasn't exciting" and then you can say, "well, what could we do differently to hook the kids?", because the kids weren't hooked.

In the lesson debrief (i.e., phase 5), the coach followed the meta-coach's advice, saying to the teacher:

> There was a question on the survey that you were wanting to look at … and I think it was the last question of the survey, ["Were you excited about the problem?"]. And I just wanted to hear more about your thoughts about what drew you to that particular question.

A conversation about student engagement ensued, in which the teacher acknowledged the importance of engagement, but also expressed concern that making every lesson exciting is unrealistic:

> So, I'm going to be really honest right here that, yes, I truly do believe that getting kids curious about math problems is a really good way to up engagement. But I also think that it is one of several hundred responsibilities that I have going on right now.

Later in the debrief, the teacher shared a similar sentiment, saying, "I feel that the problems in the lesson were as engaging as dividing with decimals ever gets." As the conversation continued, however, she described how, alongside a colleague, she had developed a different lesson for students related to Escape Rooms, also noting that students found that lesson very engaging. In response, the coach stated, "when you started talking about Escape Rooms, did you notice that your face physically lit up and you started to smile and you just got really excited?" The coach continued by stating:

> I think the kids will follow you in your passion. So, whatever you find that is exciting and new about a problem, even if it's the same old dividing decimals, like you're saying, I can make something in here be new and exciting. I think our kids get excited about whatever you're excited about, I think, is my theory.

As the preceding shows, the coach managed to broach a prolonged discussion about student engagement – the topic she wanted to discuss – using student survey data as an entry point. She also did so in a way that did not seem to elicit negative emotions on the part of the teacher.

In debriefing the coaching cycle (i.e., phase 6), however, the coach expressed doubt about whether she had managed to accomplish much in debriefing the lesson with the teacher. She reflected on her efforts to encourage the teacher to increase student engagement and display more enthusiasm in her lessons, saying, "I think that's when she called me a Pollyanna." The meta-coach then stated:

> I know you probably feel like you have not had an impact on her, but I feel like you have. Even from last year to this year. She agreed to do this. She could have easily said, "no, not this year." And then, when you said, "for our next cycle," she said, "okay, alright." There was no resistance to you being with her, for us being with her, no resistance whatsoever.

Discussion

We make two primary contributions to the growing literature on coaching. First, we describe a productive approach to initiating potentially difficult discussions with teachers about aspects of their teaching to improve. This approach was directive in that the aspect of practice in focus was something the coach had identified (Gillespie & Amador, 2024; Heineke, 2013; Ippolito, 2010). However, it was also responsive as the coach did not bring up this aspect of practice directly, but allowed discussion of it to emerge through the use of "entry points" in the form of classroom data. As this approach resulted in the teacher reflecting on an aspect of practice the coach believed could be improved, but without the experience of negative emotions like frustration or shame, we would regard it as particularly productive. Scholars have argued that the experience of negative emotions may be a necessary part of the process of developing one's teaching via coaching (McGugan et al., 2023), yet coaches may understandably be reluctant to engender such emotions with teachers. Why such emotions were absent in the current study, yet present in the work of others, seems related to differences in the responsiveness of each approach. In the work of McGugan et al. (2023), the coach broached discussion of students being overwhelmed with a challenging problem, something the

teacher had not previously identified as an aspect of her practice to work on. As such, when this topic came up for discussion with the coach, it seemed to catch the teacher off-guard. In the current study, by contrast, the teacher had previously identified student curiosity as a topic of interest, so was perhaps less taken aback when a discussion of student engagement arose in the debrief. It was somewhat fortuitous that the aspect of practice of interest to the teacher aligned with that which the coach believed needed improvement, and we wonder how the discussion would have played out had the teacher not already expressed interest in students' curiosity.

This study also demonstrates the crucial role of meta-coaching in helping a coach learn to broach difficult discussions with teachers. It thus adds to the field's growing understanding of approaches to supporting coaches' professional learning (Russell et al., 2020; Saclarides & Kane, 2021). The meta-coach's support seemed especially critical in preparing for the lesson debrief. During this phase, the coach expressed disappointment with the lack of student engagement in the preceding lesson, seeming disempowered to address this with the teacher. In response, the meta-coach provided encouragement to the coach and reminded her that the teacher had provided an "entry point" into a discussion of this topic by having previously expressed interest in the survey question about students' curiosity. The meta-coach was very adept at recognizing such conversational entry points. With regard to teaching, experienced teachers have been found to be particularly attuned to student contributions that can be leveraged in steering a lesson toward consideration of a lesson's key objective (Ghousseini, 2015). Likewise, the meta-coach seemed particularly attuned to *teacher* contributions that could be leveraged as entry points into a discussion of an aspect of practice set as a goal for the teacher's learning. Had the meta-coach not been present to offer encouragement and identify such entry points, we question whether the ensuing discussion of student engagement would have even taken place.

This raises questions about the infrastructure required for meta-coaching to become established and sustained within districts. Moreover, while we argue that the teacher in this study reflected on important shifts in her teaching practice, we lack the evidence to say that the teacher indeed made such changes. Future research could examine these and other unanswered questions as the potential of meta-coaching to support both coaches' and teachers' learning is examined further.

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Mapping structural characteristics of preparation and support programs for mathematics PD facilitators

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In recent years, facilitator professional development (FPD) programs have increasingly been institutionalized to prepare and support mathematics PD facilitators in many different ways, which have not yet been systematically compared. In this paper, we suggest a map of structural and institutional characteristics of FPD programs, aiming to make external and internal structural decisions for FPD programs explicit, to compare programs across contexts and to provide a tool to support decisions about new programs. We illustrate relevant characteristics in three programs from Israel, USA, and Germany and discuss interesting differences and commonalities the maps revealed.

Keywords: Facilitators' Professional Development (FPD) programs, FPD characteristics, support.

Introduction

The past decade has seen an advancement in researching professionalization trajectories of those who lead professional development (PD) programs for mathematics teachers, namely, mathematics PD facilitators (Karsenty et al., 2024). Ten years ago, Even (2014) noted that professionalization processes of facilitators are commonly not the result of organized activities but rather occur as a byproduct of the projects that facilitators work in. She pointed to the need to better understand how facilitators should learn and develop as professionals. Since then, several institutionalized efforts of preparing and supporting facilitators, in the form of facilitator professional development (FPD) programs, have been reported (e.g., Borko et al., 2021; Bruns et al., 2022; Valoyes-Chavez & Felmer, 2021). However, as stated by Karsenty et al. (2024), there is still much need for expanding our knowledge about FPD programs, how they are constructed, what purposes they serve and the affordances and limitations involved in decisions made while designing them.

One way to gain such knowledge is by comparing existing FPD programs, in order to distill possibly relevant characteristics that shape these programs. Whereas another paper submitted to this TWG presents the Learning to Facilitate Framework (LTFF), a tool that focuses on the *content* and *agenda* of FPD activities (Jarry-Shore et al., 2024), our aim here is to focus on *structural* and *institutional* characteristics of FPD programs. This paper is therefore a conceptual contribution, rather than an empirical one. We draw on experiences gained from three different FPD programs in three countries, posing the following research question: *Which structural and institutional characteristics shape FPD programs?*

Potentially relevant structural and institutional characteristics of FPDs

Existing publications on FPD programs allow us to identify various aspects to consider when attempting to map structural and institutional characteristics: First, we assume that the nature of the

teacher professional development (TPD) program in which facilitators work may affect how the FPD is structured. For example, research points to the difference in facilitating highly-adaptive TPD programs vs. highly-specified programs (Koellner & Jacobs, 2015). Second, facilitators might act differently if they had a first-hand experience with the TPD as learners, or if they were themselves involved in the development and/or adaptations of the TPD materials (e.g., Valoyes-Chavez & Felmer, 2021). Third, projecting from our knowledge about how work conditions shape professionalization processes at the teacher level (e.g., Brodie & Borko, 2016), it is reasonable to assume that professionalization processes at the facilitator level may also depend on the conditions in which facilitators work. Fourth, research on FPD programs (e.g., Borko et al., 2015), shows that they can emphasize teacher learning of mathematical content knowledge, teacher learning of pedagogical content knowledge or particular facilitation skills, all of which are needed yet can be prioritized differently. Fifth, it is significant if the FPD program targets novice or experienced facilitators, or both, as novice facilitators encounter unique challenges (e.g., Schwarts et al., 2021). Finally, a crucial question concerns the issue of sequencing: are facilitators introduced to the TPD program simultaneously to, or before, the FPD program? While we already intuitively assumed that all these aspects might impact decisions about structural and institutional characteristics, only the systematic comparison of the three FPD programs allowed us to articulate them concisely.

Approach and context for comparing structural and institutional FPD characteristics

Context of inquiry

The research question we pose and the need to develop a language for structural and institutional characteristics of FPDs emerged during an intense collaboration between three FPD design research teams from USA, Germany, and Israel. While conducting comparative empirical research on facilitators' learning in two FPD programs (Schwarts et al., 2021) and on the ways we design and enact FPD activities (Jarry-Shore et al., 2024), we became increasingly aware that some structural and institutional commonalities and differences between our FPDs substantially shape the professional learning experiences we provide for facilitators. Thus, we tried to articulate what exactly these characteristics were, by comparing the structures and institutional aspects of the FPDs we work in.

Comparative approach

The three FPD research teams met in two intensive workshops, led by the authors of this paper. In the workshops we discussed the structure of each FPD program and identified seven elements that we agreed are necessary for capturing the essence of FPD structures. For each element, we identified the different categories of choices made or external conditions provided, for the three programs, and possible options in each category. By comparing the program structures, we were successively able to refine the initial elements into finer-grained characteristics that seem important for distinguishing among FPDs. The outcome is a comparative tool, the *FPD Structure Map* which we introduce here, and a first tentative discussion of its potential contribution to our understanding of FPD structures.

Three FPD programs to be compared and contrasted with respect to their institutional settings

The Problem-Solving Cycle FPD program. The Problem-Solving Cycle (PSC) is a TPD program for middle-grades mathematics teachers, implemented in the USA. It typically consists of two cycles per

school year, each cycle including three workshops that revolve around a focal mathematics problem (Borko et al., 2015). The FPD program, designed to prepare prospective facilitators to lead PSC workshops, consists of three sessions that precede the three TPD workshops in a given cycle. FPD activities are led by mathematics facilitator educators who may be university researchers or district mathematics specialists. The sessions include modeling of facilitation tasks around the focal mathematical problem (e.g., leading a discussion of a classroom video) by the facilitator educators, collectively unpacking the modeled facilitation tasks, and rehearsing versions of leading the task by the prospective facilitators (Borko et al., 2021).

The VIDEO-LM FPD program. VIDEO-LM (Viewing, Investigating and Discussing Environments of Learning Mathematics), is a TPD project implemented in Israel. Its central aims are to enhance the reflective skills of secondary mathematics teachers and to enrich their mathematical knowledge for teaching. In the TPD, typically a 30-hour course, videotaped lessons serve as objects for teachers' peer-discussions, directed by a framework of analysis developed for this purpose (Karsenty & Arcavi, 2017). The FPD for preparing facilitators to lead VIDEO-LM TPDs was designed as a one-year course, consisting of seven 4-hour sessions, facilitated by two facilitator educators. The course included activities such as simulations, "Live Labs" of facilitation, and session planning.

Mastering Math FPD program. Mastering Math is a TPD program implemented in Germany, aimed at developing mathematics teachers' expertise to foster students' understanding of arithmetic concepts in Grade 5–7 (Prediger et al., 2023). The TPD, consisting of six 3-hour sessions followed by classroom experiments, focuses on teachers' practices in specifying, monitoring, and enhancing students' understanding. During the FPD program, preparing Mastering Math leaders, the participants also experienced the TPD as teachers and reflected on what teachers should learn. To establish facilitation practices and their underlying rationales, the FPD workshops focused on selected situational demands related to leading the TPD, with reference to noticing and leveraging teachers' typical practices and underlying knowledge and orientations about fostering students' understanding.

Outcome: The FPD Structure Map as a tool for comparing and planning programs

In Figure 1, we present the *FPD Structure Map* which depicts the structural and institutional characteristics that we have identified as crucial in shaping FPD programs. As noted, this mapping tool pertains to seven elements, each with one or more categories that include different options - either design decisions made or external conditions to be taken into account when FPD program structures are developed. In the following we detail these elements in the order they appear in the rows of Figure 1. The various categories and options are indicated in corresponding rows of the map. Importantly, we note that options are not mutually-exclusive. The list below identifies the elements and categories.

1. *Target group of facilitators*. This element includes four categories pertaining to the FPD participants: their experience, the nature of their employment, whether they are expected to work alone or in teams, and by whom they are recruited.
2. *Nature of the TPD program*. This element pertains to the TPD that facilitators are being prepared to lead, in terms of duration, degree of specification of the TPD materials, and the TPD orientation.
3. *Facilitators' initial involvement with TPD materials*. This element refers to facilitators' familiarity with the TPD content and materials when they enter the FPD program.

4. *Form of integration of facilitation experience in TPD*. This element maps five different options for sequenced vs. parallel phases of facilitators' participation in TPD, FPD, and their own TPDs.
5. *Focus of the FPD content*. This element refers to design choices about the nature of what facilitators learn in the FPD.
6. *Duration of preparation and support period*. This element specifies the nature of the FPD length, and the model of preparation and support for facilitators.
7. *Certification and attendance*. This element refers to choices regrading whether formal certification is granted upon completing the FPD, rules referring to facilitators' attendance in the FPD, and the work expected from participants between FPD sessions.

Target group of facilitators:	Novice facilitators	Experienced facilitators	Long-term contracts in well-defined institutions	Freelancers	Appointed specifically for the PD program	Working alone	Working in teams	Selected by PD team	Selected by external authority
Nature of the TPD program:	Short term program (one or just a few sessions)	Long term program along at least several months to a year	Long term program along several years	Highly specified PD materials	Basic PD ideas can be used in an open-ended manner	Instructionist	Adaptive to teachers' starting points	Content-oriented	Process-oriented
Facilitators' initial involvement with TPD materials:	TPD content new to prospective facilitators when entering FPD		TPD content well known by prospective facilitators when entering FPD				TPD materials are developed in collaboration with facilitators		
Form of integration of facilitation experience in TPD:	Successive TPD⇨FPD⇨own TPD: First facilitators participate in the TPD, then in FPD, then they start their own TPD		Successive FPD⇨own TPD: Facilitators participate in the FPD, then they start their own TPD		Simultaneous TPD | FPD⇨own TPD: Facilitators experience the TPD as part of the FPD, and after the FPD they start their own TPD		Simultaneous FPD | own TPD: Facilitators are prepared during already giving their own TPD		Simultaneous TPD | FPD | own TPD: Perhaps with mini-successions within the FPD
Focus of the FPD content:	Mathematical content		Pedagogical content knowledge and teaching practices		Processes that teachers need to engage in (e.g., reflection)		Facilitation knowledge and practices		
Duration of preparation and support period:	Short term preparation FPD (one or just a few sessions)		Long term preparation FPD – along at least several months to a year		Few or no opportunities for support once the preparation period is over		Support through continuous FPD	A combined model of preparation and support	Personalized support by experienced facilitators
Certification and attendance:	Formal certification	No formal certification	Attendance in FPD sessions is mandatory		Attendance is expected but not mandatory		Participants' work in between sessions is expected		No or very little work is expected in between sessions

Figure 1: The FPD Structure Map

Figure 2 depicts small versions of three filled-in FPD Structure Maps for the three FPD programs (note that these are not meant to be read; the purpose here is to understand the procedure of personalizing the FPD Structure Map for a specific FPD program).

VIDEO-LM FPD Structure Map | Mastering Math FPD Structure Map | Problem-Solving Cycle FPD Structure Map

Figure 2: Filled-in Structure Maps of three FPD programs

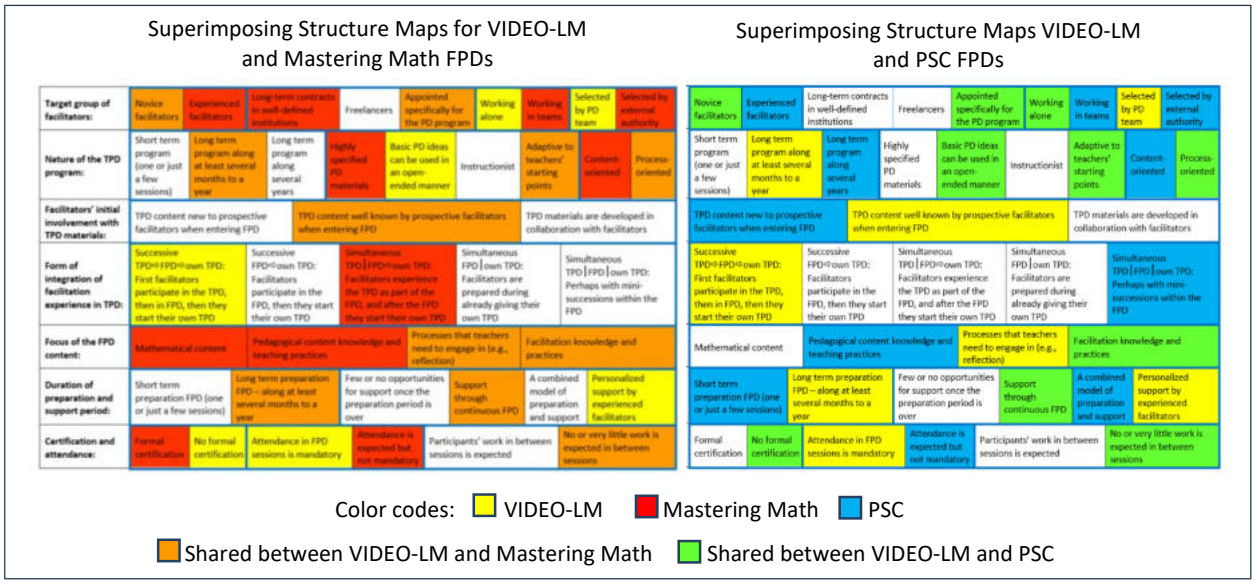

Figure 3: Bilateral comparisons by superimposing filled-in FPD Structure Maps

The two maps presented in Figures 3 were created by superimposing the Structure Maps of two FPDs. The superimposing reveals that the VIDEO-LM and the Mastering Math FPDs have several commonalities (orange colored cells), for example with respect to target groups and length of the TPD and FPD programs. However, there are differences in the nature of the TPD: while the Mastering Math TPD program is highly specified and offers TPD materials with detailed TPD activities (thoroughly designed to work towards TPD content goals in clearly planned processes), the VIDEO-LM TPD program is adaptive and process-oriented, emphasizing reflection on classroom videos with a high degree of flexibility regarding the contents for which the reflective lenses are used and the specific activities offered to teachers. As a consequence, although both programs focus on facilitation knowledge and practices as well as processes that teachers need to engage in (row 5 in the Structure Map), only the Mastering Math FPD focuses also on particular mathematics contents and specific pedagogical content knowledge and teaching practices to be leveraged with the teachers. These programmatic differences lead to different forms of integration of TPD and FPD experiences (row 4 in the Structure Map): whereas VIDEO-LM adopts the successive model of "TPD⇨FPD⇨own TPD", Mastering Math adopts the simultaneous model of "TPD|FPD⇨own TPD". This means that in VIDEO-LM, only teachers who have previously participated in the 1-year TPD program are invited to be novice facilitators for the FPD program. Thus, they go through the modules first as teachers, and only at a later year as prospective facilitators focusing on how to lead reflective VIDEO-LM sessions themselves. In contrast, in Mastering Math the novice facilitators have mini-successions in working through the TPD as teacher participants, then reflecting on these experiences with respect to what they are expected as facilitators of the same sessions, taking into account the backgrounds of teachers' professional learning. After a year of this simultaneous TPD | FPD experience, the facilitators start their first own TPD. These differences resonate with the respective TPD focus: for unpacking the details of the TPD content in Mastering Math, the conjoint TPD and FPD experience is more suitable. In VIDEO-LM, for establishing norms of reflection and high-quality processes in which teachers shall engage, one year of TPD participation and a second year of FPD participation

gives more opportunities for socializing into, and gaining ownership of, the intended reflection processes.

Sometimes, however, it is not the inherent characteristics of the TPD that determine the form of integration of TPD-FPD experiences, but rather external conditions, for example a district under pressure to start the program immediately. In the case of implementing the PSC program in one district in California, such external conditions necessitated that FPD designers choose the model of simultaneous TPD|FPD|own TPD ("on the job" model), preceded only by a 3-day summer institute and with mini-successions during the year. Thus, the induction into the TPD program, the FPD unpacking, and the facilitators' own first TPD experiences were all taking place in the first year.

The superimposing of the VIDEO-LM and PCS FPDs (see right-hand side of Figure 3) reveals a different set of commonalities (green colored cells) and differences between these two programs. The greatest similarity is in the nature of the TPD program. In both programs, the TPD ideas can be addressed in an open-ended manner and are adaptive to the teachers' experiences and needs, although whereas VIDEO-LM focuses only on process, the PSC centers both content and process. Neither program provides formal certification and neither expects much work between sessions. Interestingly, for several elements the two FPDs show *both* overlaps and differences, in the sense that VIDEO-LM choices are subsumed within those of PCS: for example, both programs recruit novice facilitators, however the PSC additionally targets experienced facilitators; in both programs facilitators work alone, yet in the PSC program the possibility to work in teams is also available. For the latter element, external conditions necessitate that FPD designers be flexible. Since the PSC program is designed to work with facilitators who conduct PSC workshops in their own schools, and since schools within the district vary in size, it follows that in smaller schools it is not possible (or necessary) to have more than one facilitator.

Looking at both superimposed Structure Maps in Figure 3, we can also see that each FPD program has several unique characteristics. The PSC program is unique in that it is designed to work with a single school district for several years, with the aim of building both TPD and FPD capacities within that district. External conditions resulting from this design characteristic led to several ways in which it differs from both the VIDEO-LM and the Mastering Math FPD programs. For example, it is the only program of the three in which the TPD content is possibly new to prospective facilitators when they begin the program. The program takes that possibility into account by providing a combined model of preparation and ongoing support, beginning, as mentioned, with a summer institute to introduce new facilitators to the TPD program. Throughout the simultaneous TPD|FPD|own TPD experience, the design of mini-successions allows the facilitators to engage with the TPD as learners prior to enacting it as facilitators in the workshops they lead.

The Mastering Math FPD is unique in several features, which we suggest are related to the systemic and large-scale implementation of the program (Prediger, 2022): it is the only FPD of the three that grants facilitators formal certification, and the only one that offers long-term contracts in well-defined institutions. The choice to have facilitators use highly-specified TPD materials may also be linked to the institutionalized nature of the project.

The VIDEO-LM is the only one of the three FPDs where prospective facilitators are selected by the design team, and the only one where experienced facilitators provide personalized support to novice facilitators. This may be due to scale considerations in more than one sense: Israel is a much smaller country than the US and Germany; the community of mathematics leaders is relatively small, and there is a sense that 'everyone knows (almost) everyone'. Selecting candidates to lead PD on the basis of prior acquaintance is a very common practice in the Israeli Mathematics Education community. In addition, the scale in which VIDEO-LM was implemented (about 70 TPD courses given, by about 30 facilitators), although quite large in Israeli terms, still allows for personalized support.

Discussion

This paper is not an empirical one. We do not claim that the comparison of three FPD programs in which we (as FPD designers and facilitator educators) engaged, is highly methodologically controlled, or that we would not come to other results if other programs were in view. We emphasize that the results of this comparison are tentative and preliminary, and we hope that the suggested Structure Map will further develop through the contributions of other FPD design experiences.

However, we are convinced that already in its first version, the FPD Structure Map can be of value for other FPD designers and FPD researchers, as it has been for us, for several different tasks:

- *Communication.* The FPD Structure Map (perhaps in further iterations in the future) can provide a descriptive language for structural and institutional characteristics, that can enable an efficient exchange of ideas among FPD designers and researchers when they report their research contexts, processes and program effects. This will help in comparing different FPD research findings.
- *Systematized catalogue of typical models.* We envision a catalogue of different FPD models made explicit according to the FPD Structure Maps, so that, as a field, we can learn from other contexts.
- *Reflection.* The Structure Map, together with a catalogue of realized examples, would help FPD designers not to take the familiar structural and institutional settings in their own context (program / country / district) for granted, but to consider (at least some of) them as design choices that may be taken differently in other contexts.
- *Deliberate design.* The specified characteristics and the catalogue of other examples can support FPD designers in taking design decisions more deliberately when planning a new FPD program.
- *Understanding complexities and dependencies.* When we start to investigate the emerging catalogue, we can aspire to deepen our understanding about how some external conditions, as well as the inherent nature of the TPD program, may have consequences for the design decisions regarding both the structural aspects of FPD we discussed here and others to be revealed in future research.

In particular for the last point, we will need many more comparisons, with higher methodological control, to investigate possible complexities and dependencies more explicitly and systematically.

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Ten years on: The learnings from a project nurturing collaboration between mathematicians and mathematics educators

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The IMSITE project provided significant funding to six Australian universities to nurture partnerships between mathematicians and mathematics educators who educate prospective teachers and to foster integration of mathematics content and pedagogy in university coursework. This longitudinal study explores how the IMSITE initiatives have survived, focusing on one of the participating universities. Valsiner's zone theory was used to analyse project publications and a recent interview with a mathematician still working at the university. Whilst ten years later the last of the IMSITE initiatives was still in place, it will soon be discontinued. Nevertheless, the mathematician who brokered interdisciplinary collaboration still works with mathematics educators, albeit subversively. Maintaining lasting institutional relationships remains challenging, even when being championed.

Keywords: Mathematics teacher educators, Valsiner's zone theory, disciplinary boundaries, identity.

In Australian universities, as in many other countries, there are strong disciplinary boundaries in initial teacher education. Prospective teachers are generally taught mathematics content in the university's mathematics department and how to teach mathematics in the education department: often, there is little or no communication between these departments. Prospective teachers describe the tensions caused by the conflicting beliefs and practices of mathematicians and mathematics educators who teach them in their initial teacher education program (Marshman, 2021).

In 2013 the Australian Government funded several projects as part of the *Enhancing the Training of Mathematics and Science Teachers (ETMST)* Program. The *ETMST* Program aimed to lead:

> a major improvement in the quality of mathematics and science teachers by supporting new preservice programs in which faculties, schools or departments of science, mathematics and education collaborate on course design and delivery, combining content and pedagogy so that mathematics and science are taught as dynamic, forward-looking, and collaborative human endeavours. (DET, 2016, as cited in Goos et al., 2017, p. vi)

The *Inspiring mathematics and science in teacher education* (*IMSITE*) project, conducted between 2013-2017, provided significant funding to nurture partnerships between mathematicians and mathematics educators who educate prospective mathematics teachers and to find new ways for mathematicians and teacher educators to integrate mathematics content and pedagogy and embed these practices in their institution. Mathematicians and mathematics educators at six Australian universities were involved. While the project was successful in supporting new forms of collaboration between mathematicians and mathematics educators (Goos & Bennison, 2018, 2019), its longer-term impact has not been monitored. Our current project exploring the identity trajectories (Wenger, 1998) of Australian mathematics teacher educators (MTEs) provided an opportunity to consider a longitudinal study of one mathematician ten years after she was interviewed as part of the *IMSITE*

project. We used zone theory (Valsiner, 1997) as a lens to explore this MTE's identity as she moved along her professional path to becoming an experienced MTE. The analysis also allowed us to identify changing institutional conditions that enabled or constrained the survival of *IMSITE* initiatives. Our research question is: What is the long-term impact of interdisciplinary collaboration between mathematicians and mathematics educators on one MTE's identity?

Theoretical perspectives

Collaboration between mathematicians and mathematics educators is increasingly recognised as valuable for improving initial teacher education (Marshman, 2021) and whilst collaboration can enhance the quality of mathematics teaching and learning, it is often complex and challenging (Marshall et al., 2023). The *IMSITE* project (Goos & Bennison, 2018) identified *personal qualities* including "included open-mindedness, trust, mutual respect, and shared beliefs and values" (p. 265), working towards *common goals,* and a *common or shared problem* as sustaining collaboration whilst the *physical separation* of buildings, "*workload formulas or financial models* that did not recognise or reward interdisciplinary collaboration" (p. 267) and entrenched *cultural differences* hindered collaboration. Successful collaborations required acknowledging the personal and professional experiences of all participants, centring identities and humanness (Marshall et al., 2023) as well as motivation, acknowledging each other's strengths, trusting that each person wants to improve student learning, and helping each other to reach their mutual goals (Sultan & Artzt, 2005). These interdisciplinary efforts can lead to more coherent experiences for prospective teachers, promote better student outcomes, offer personal and professional rewards for the collaborators involved (Marshall et al., 2023: Marshman, 2021) and lead to changes in beliefs about and practices of teaching and learning mathematics (Inch & McVarish, 2003). Collaboration can also lead to new learning and practices including: *confrontation* (forcing a reevaluation of current practice); *hybridisation* (combining practices from different contexts); *crystallisation* (embedding new procedures into practices); and *continuing to negotiate practices* that work against collaboration (Goos et al., 2017).

This study is underpinned by a sociocultural perspective, utilising Wenger's (1998) ideas about identity development and an adaptation of Valsiner's (1997) zone theory. Wenger (1998) stated that identity development entails a temporal learning trajectory connecting the past, present, and future. This trajectory can be traced by using zone theory to examine the changing interactions between an individual's learning environment and the people within it.

Extending Vygotsky's Zone of Proximal Development (ZPD), Valsiner (1997) integrated the goals, social environment, and actions of participants by introducing two additional zones – Zone of Free Movement (ZFM) and Zone of Promoted Action (ZPA). The ZFM denotes what the environment allows whilst the ZPA denotes activities or resources within the environment with respect to which a person's actions are promoted. For a MTE, the ZFM includes characteristics of teacher education students (e.g., students' beliefs about mathematics and how it is best taught and learnt and the depth of their mathematical knowledge), structural characteristics of teacher education programs (e.g., extent of connections between mathematics content courses, mathematics teaching courses and other general education courses), organisational structures (e.g., timetabling, workload, budgets), and university culture (e.g., lack of value placed on teaching) (Goos & Bennison, 2019). Similarly, the

MTE's ZPA might include reflections on practice (e.g., Chapman et al., 2020), research with teachers (e.g., Jaworski, 2001), professional development (e.g., the professional diploma for mathematics teacher educators offered by the Norwegian University of Science and Technology), and informal interactions with colleagues. For example, the informal collaboration between Bass and Ball (2014) focused on the interplay between mathematical and pedagogical aspects of classroom practice. Similarly, Jaworski (2001) described the interplay between mathematical, pedagogical, and educative reflection when teachers and MTEs work in a community of inquiry. The ZFM and ZPA work as a ZFM/ZPA complex that represents the affordances offered by people and resources within an environment, thus shaping an individual's developing identity as a MTE, whilst acknowledging individuals' agency to change their environment or their interactions with the people in it.

Research design and methods

This longitudinal case study is part of a larger study exploring the identities of Australian MTEs, both university-based mathematicians and mathematics educators, using a survey and semi-structured interviews. The interviews were designed to reveal the MTEs' identity trajectories using Valsiner's zone theory, and thus asked questions about their ZPD (Can you tell me about your background? Can you tell me about your teaching?), ZFM (Can you tell me about your professional context?), and ZPA (How have your knowledge and practices as a mathematics teacher educator developed over time?). One of those interviewed had participated in the *IMSITE* project ten years ago (Goos et al., 2017), and so this provided an opportunity for monitoring of the project's impact. Ethical clearance was provided by the university Human Ethics Research Committee (A242040), consent was given, and a pseudonym used to refer to the participant.

In this paper we report on one MTE, Carol, a mathematician who acted a broker between the mathematics and education discipline communities. Carol's interview from this current study, interview transcripts in the *IMSITE* project report (Goos et al., 2017), and a publication (Goos & Bennison, 2018) from the *IMSITE* project were analysed using Valsiner's zone theory to explore the temporal development of her identity from *past* to *present* to *possible future* as in Goos and Bennison, (2019). Data were analysed by identifying excerpts that clearly identified examples of ZPD, ZFM, and ZPA (listed above) from the interview, publication, and project report. The MTEs' responses were inserted in a table with columns for ZPD, ZFM, and ZPA and rows for past, present, and possible future. Inspecting the table horizontally, within time periods, enabled us to identify alignments and misalignments between the zones within that time period, whilst inspecting the table vertically enabled us to identify incidents that sparked changes from past to present or could potentially trigger a future change within the ZPD and ZFM/ZPA complex. From this table we could identify the enablers and constraints for collaboration between the discipline communities. The findings are summarised in Table 1 and discussed below. Interview excerpts are tagged with the year in which they occurred and categorised as relating to the MTE's ZPD, ZFM, and ZPA to illustrate the data/theory relationship.

Findings

Beginning with the ***past*** temporal dimension, Carol studied a mathematics degree and a Diploma of Education, followed later by a PhD in mathematics. She said that she identifies as a mathematician.

Reflecting on her background, Carol said she began collaborating with Tess, a mathematics educator, in 2011 when she was assigned to teach into the primary teacher education program. Thus, her *past ZPD* offered possibilities for her development as a teacher educator.

> Carol (2024); About 2011, I was assigned to teach preservice primary. Oh, that was interesting. And then I started working with Tess from education in that space, in that primary Ed space, and then Merrilyn's big *IMSITE* project came on. (ZPD)

Carol's *past ZFM* was largely structured by her perception of student characteristics. She recognised that many had mathematics anxiety and poor mathematical knowledge from school.

> Carol (2024); I had no idea that people knew so little mathematics coming out of year 12. So, I was somewhat flabbergasted by my experiences. The first time I taught, … I had my tutorial class, and I could not get them to answer questions for love nor money. I had all the classic avoidance symptoms of leaning over their work so I couldn't see it. I gave them hands on things, and they wouldn't try, necessarily. I'm drawing them out to get them to do anything, and this almost a resistance to learning. (ZFM)

Carol "did a bit of research" to understand the issue of mathematics anxiety and the "whole I can't do it. I don't want to show anybody that I can't do it." This was confronting for Carol – the "maths anxiety, maths phobia, and how our education system is failing people." The following year she made lectures compulsory, encouraged students to work with materials, explained concepts in multiple ways to build "relational understanding" and introduced mastery exams, indicating that Carol had the *personal qualities* needed to work in this interdisciplinary space.

Although Carol successfully shared teaching of the primary mathematics course with Tess, a mathematics education academic, she nevertheless observed *cultural differences* arising *within* the education discipline, where the non-mathematics educators did not understand or like mathematics.

> Carol (2024); The non-maths people in education, … they have a real interest in how students learn … When it comes to mathematics, they've got no idea. They were not good at mathematics. They don't like it either, and they feel for the students. (ZFM)

In the secondary teacher education space, the *IMSITE* project offered a ZPA providing the time for Carol and Tess to collaborate, during which they identified a *common or shared problem* in that prospective teachers did not have a good conceptual understanding of secondary school mathematics.

> Carol (2024); Tess and I identified in *IMSITE* was that they were not good with their high school mathematics, our preservice secondary maths teachers, that they're weak with their maths skills from high school. And partly it was because while they could do it, they had never investigated the whys or the wherefores or they've not made links. So, they just could do it their way. They couldn't do it any other way. (ZFM/ZPA)

IMSITE allowed Carol and Tess to codevelop and coteach a course integrating content and pedagogy - *Reflective Communication in Mathematics* to overcome *common problems* of poor student conceptual understanding of school mathematics and increase the number of students studying teaching. Originally designed for noneducation students and studied by education students as an elective, the course aimed for students to be able to analyse their own understanding of mathematical concepts, demonstrate pedagogical content knowledge by explaining mathematical concepts and explain mathematical ideas in creative ways (Goos & Bennison, 2018). The course became part of the Bachelor of Mathematics Education (Goos et al., 2017).

Carol (2024): I did succeed in getting one subject up and running through *IMSITE* that we've held on to. And that's a taste of teaching subject. So, I do talk about instrumental versus relational understanding. I do talk about reflective communication. (ZPA)

Carol (2014): We both realised that [these students] had not made the connections between their maths subjects, their pedagogy subjects, and the maths that they were going to be teaching at school. This was the first subject that they'd had where we were talking about both at the same time, taking it further than anything had been taken – like take the syllabus from high school, push it into where it goes to university where they come back and talk about how might you teach it so that you get those outcomes. (Goos & Bennison, 2018, p.264) (ZFM/ZPA)

Carol and Tess developed an understanding of the different cultures in mathematics and education:

Carol (2015): We broke down some of the skepticism that both sides can have. (Goos et al., 2017, p.15) (ZFM/ZPA)

When coteaching the mathematics subject they developed *hybrid* practices that recognise the "uniqueness of their established practices as a mathematician and mathematics educator" and their "complementary expertise" (Carol in Goos et al., 2017, p.15):

Carol (2015): We go to class and there are times when she says to me 'That's all yours because it's beyond what I understand' and that's fine. Likewise, she'll come in and talk about the greats of education and I'm just going blank, no idea. (Goos et al., 2017, p.15) (ZFM/ZPA)

Carol and Tess continued to work together at the boundary between their disciplines, shaping their shared ZFM/ZPA with weekly project meetings, attending and teaching into each other's tutorials in mathematics and mathematics education subjects, co-supervising honours students, and conducting professional development for practising teachers together (Goos et al., 2017).

Moving to the ***present*** temporal dimension, Carol's credibility in both disciplines continues to be recognised by mathematics educators at her university, potentially nurturing her development (ZPD).

Carol (2024): I've had lots of influence on the maths ed here. … I regularly get, oh, we need to catch up. … they go in to fight for these things that they truly believe in. And then they come out of the meeting, wondering … Should I keep fighting for it, … And they get from me, okay, these are the reasons why this is a good thing to fight for, okay, because of this and because of that. … I'm behind them, but it's tough. (ZPD)

In a recent review of teacher education programs the Education department decided that mathematicians were not suited to teaching mathematics to prospective teachers. At the same time, a Mathematics department review of programs decided to discontinue *Reflective Communication* course.

Carol (2024): Unfortunately, next year, with the new changes, I'm out, they've kicked me out. I think they've been trying to find a way to do it for 10 years. (ZFM/ZPA)

Carol (2024): That's where I come up against my culture, right? … They [mathematicians] just want to, it's almost like they want to open students' head and cram in as much information as they possibly can, and then sit on the head while they sew it, zip it back up again, right? ... There's so much maths to teach. (ZFM/ZPA)

The entrenched *cultural differences* between education and mathematics that had been partially mitigated by the *IMSITE* project have now become a much more significant constraint on Carol's development as a MTE. Yet she continues to seek out new ZPAs that affirm her MTE identity, such as the First Year in Mathematics network for mathematicians who are interested in teaching.

Table 1: Zone theory analysis of Carol's MTE identity trajectory

	ZPD	ZFM	ZPA
Past	- Mathematician with secondary mathematics teaching qualifications -Interested in teacher education	-Teacher education programs generally separate content and pedagogy. -One mathematics and two pedagogy courses cotaught -Prospective teachers poor conceptual understanding of secondary mathematics	-Some Mathematics Department support for co-teaching and mathematics education research -Reflections on MTE practice -*IMSITE* project provided resources for collaboration (ideas, funding, time, encouragement)
Present	-Credibility in mathematics and mathematics education communities -Most research in education	*TENSION: Disciplinary cultures prevail over interdisciplinary collaboration once IMSITE* resources end -Education academics do not understand mathematics -Mathematicians no longer value education	**Change** -Retains & increases respect of mathematics education colleagues -First Year in Mathematics
	TENSION: Institutional cultures discourage collaboration		
Possible Future	-Maintains MTE identity -Continue education research	-Cultural differences between mathematics and education	-Develop further mathematics teacher professional development
	TENSION: Carol and mathematics educators continue to collaborate		

Despite Carol's rather discouraging situation, we can catch glimpses of an optimistic ***future*** temporal dimension in her MTE identity development. She continues to pursue her goal of developing her teaching practice by reflecting on student feedback and what she has heard at conferences:

> Carol (2024): I do self-reflect lots and lots, … thinking about how I can do it better, … I regularly go to the First Year in Maths network. So, I need to go to those and get ideas from other people. Mind you, … most of the time I sit there and pull them apart and go, "Well, this is a problem with that, and this is a problem with that." But is there some gem that I could take that will help me improve. (ZPA)

Thus, Carol, is attempting to navigate through her increasingly constrained university ZFM by seeking external ZPA where she can interact with like-minded people from both the mathematics and the mathematics education communities.

Discussion and conclusion

Having qualifications in both mathematics and education, Carol's ZPD positioned her to value interdisciplinary collaboration. However, it was the realisation that many prospective primary school teachers had so little mathematical knowledge after completing twelve years of schooling (ZFM) that provided the *confrontation* between the school mathematics and university mathematics (Goos et al., 2017) that perhaps strengthened Carol's commitment to collaboration between the disciplines, similar to the co-learning of mathematics teachers and MTEs through collaborative critical inquiry (Inch & McVarish, 2003; Jaworski, 2010). Carol was perhaps driven to improve the schooling experience that was failing so many students (ZFM/ZPA).

IMSITE provided time (ZPA) for Carol and Tess to collaborate in transforming secondary mathematics teacher education so that there were education and mathematics courses that were cotaught and contained cross-disciplinary content, providing a *hybridised* practice (Goos et al., 2017). New cultural dialogues had *crystallised* (Goos et al., 2017), so that after Tess retired Carol continued to collaborate with the mathematics educators who seek her advice (ZFM/ZPA).

Whilst Carol's ZPD and strong *personal qualities* have allowed her to build sustained relationships with mathematics educators that are focused on the *shared problem* of improving student learning, this cross-disciplinary *trust* and *valuing* has not extended to all education academics where *cultural differences* mean she is seen as being on the margins. Although Carol had developed a successful primary mathematics education course, the education academics felt threatened by her presence. The Education department decided that the primary mathematics course needed to be taught by someone from the education discipline as they couldn't "rely on the mathematician to do a good job." Carol gave her course resources to Tess and continued to work with her so that the primary prospective teachers were well supported, and connections were made between mathematics and pedagogy, similar to the work of Bass and Ball (2014).

More recently, the *cultural differences* between mathematics and education arose in the review of the education pedagogy courses: the Education department no longer wanted to share course *workloads,* meaning there was no space for the mathematics discipline expert to contribute to teaching. Similarly, the mathematics program review means that, from next year, the *Reflective Communication in Mathematics* course will be discontinued. Carol's teaching is not valued by either community.

Ten years later there is still a strong sense of community between Carol and the mathematics educators who have continued the collaboration. This relationship is built on working towards the *common goal* (Goos & Bennison, 2018) of improving the learning of prospective mathematics teachers – both primary and secondary – and the "*personal qualities* of open-mindedness, trust, mutual respect, and shared beliefs and values" (Goos & Bennison, 2018, p. 265). However, within the wider mathematics and education communities, *workload formulas* and the entrenched *cultural differences* between the disciplines have not recognised or rewarded disciplinary collaboration (Goos & Bennison, 2018). Hence interdisciplinary collaborations need to be worked on continually because their success depends on more than just those who are collaborating.

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Structuring deep and specific conversations: The work of mathematics teacher educators in professional learning

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This paper explores how mathematics teacher educators routinely structured examination of educative curriculum materials that led to detailed, fine-grained analyses of teaching in order to support teachers' development. The study examined lesson analysis cycles, which involved lesson planning, enactment, and reflection. We found that the structure supported mathematics teacher educators to leverage curriculum materials to orchestrate discussions with teachers that focus on pedagogy, mathematical content, and student thinking. Implications for the practices of mathematics teacher educators are discussed.

Keywords: Mathematics teacher educators, professional development, MTE practices

Introduction

The goal of this paper is to describe a design for professional learning that enables mathematics teacher educators (MTEs) to lead conversations among teachers shown to support teacher learning and, in turn, more ambitious teaching (Lampert et al., 2013). Central to the design is an intense focus on the details of individual lessons from an educative curriculum (Ball & Cohen, 1999). In the United States curriculum refers to published student textbooks with teachers' manuals and accompanying materials, including unit overviews and lesson plans, designed to meet jurisdictional educational standards. Many different curricula are created and sold to school districts; often with neighboring districts teaching from different curricular materials. Curriculum materials are not all created the same. Educative curriculum materials, such as the one used in this study, support teachers in developing deeper understanding of mathematical content and pedagogy, which in turn supports students' deeper understanding of mathematics (Remillard, 2000).

In this paper, we describe an aspect of a project aimed at understanding the ways in which professional learning activities supported teachers' learning and changes in practices. We focus on the role of MTEs as they led groups of same-grade level teachers to collaboratively plan to teach a lesson from their curriculum. We provide examples that show how MTEs used a structured routine to analyse curricular lessons in a way that encouraged detailed, fine-grained analyses of teaching.

Our thesis is that deep and specific conversations about mathematical content provide rich learning opportunities for teachers (Russell et al., 2020) and that MTEs play a crucial role in the orchestration of these conversations. In the U.S., these conversations are difficult to support as teachers are necessarily trained to teach their subject apart from a particular curriculum, owing to the plethora of different materials used across the country. As Cohen observed, teachers in the U.S. typically "are taught how to teach no particular version of their subjects" (2010, p. 45). We aim to show how MTEs can leverage the analysis of a well-designed lesson to create opportunities for teachers to learn to teach a very specific version of mathematics.

Background and conceptual framework

MTEs providing coaching and expert support to teachers has become more common in the U.S. However, uncertainty exists about the optimal work of this role (Peters-Dasdemir et al., 2023). While research on the incorporation of MTEs in professional learning shows promise for improving teaching and student outcomes (Russell et al., 2020), this is not always the case (Witherspoon et al., 2021). Evidence indicates that effective facilitation is complicated work, with some approaches more effective than others (e.g., Karsenty et al., 2023; Şahin-Gür & Prediger, 2023).

One area of research has focused on conversations among MTEs and teachers that lead to more ambitious teaching (Ferreira et al., 2023). A significant finding is that the most productive conversations are deep and specific (e.g. Lefstein et al., 2020) and "focus on the interactions between teachers, students, and mathematics" rather than on "what the teacher will do" (Russell et al., 2020, p. 443). Imagine teachers slicing an individual lesson into thin pieces, identifying a critical piece, and spending significant time examining the key mathematical idea, the way students might think about that idea, and teachers' role in supporting the richest thinking possible. Examples of such conversations exist in the literature (e.g., Melville, in press; Stigler & Hiebert, 1999).

Building on this work, we examined the nature of conversations that MTEs facilitated that reached this standard of depth and specificity about mathematics content, student thinking, and pedagogy, with special interest in the conditions that supported these conversations. In our view, an important question for the field is: How do MTEs structure planning conversations that facilitate deep and specific conversations among teachers?

Context

We report preliminary findings from a project seeking to understand how teachers translate experiences during professional learning into teaching that ultimately improve student learning. The project, which emerged from a partnership between MTEs at a statewide mathematics coalition and researchers at a state university, employs a lesson analysis cycle guided by MTEs using educative curricula to engage Grades 4-7 teachers in a video-based professional development (PD) program.

The lesson analysis cycles include a highly structured process, developed by the MTEs, and centred on lessons from the *Bridges in Mathematics* curriculum (Math Learning Centre, 2019). Each cycle involved three phases: (1) planning (2) enacting and (3) reflecting. The planning involved MTEs and 3-5 same-grade level teachers collaboratively analysing and planning to teach a lesson (approx. 3.8 hours). The enactment involved the MTE videotaping a teacher teaching the lesson (approx. 45 mins). The reflection involved the MTE and each teacher watching video clips from the lesson, discussing student thinking, teacher moves, and reflecting on any changes to improve teaching of the lesson the following year (approx. 45 mins). The cycle occurred three times throughout the school year; the project was in its second year of implementation when this data was collected. Our analysis focuses on the first phase of the cycle, when the MTE facilitated planning with the teachers.

Methods

Participants

The project included 13 MTEs. For this study, we shadowed three MTEs working with 4th and 5th grade teachers. All three are considered expert MTEs with an average of 30 years teaching experience

and a range of facilitating experience from 4-20 years. As we explain below, although we analysed data for three MTEs, we pulled examples from one MTE to describe our findings—Nina, a former elementary teacher with four years of facilitating experience, and her group of three 5th grade teachers, Carla, Dana, and Tracy, with a range of teaching experience from 2-17 years.

Data collection

Researchers collected audio recordings, observational notes, and artefacts such as the planning structure, curriculum documents, the group's written record of the planning process, and the final annotated lesson plan. Recordings were collected using audio recorders.

Data analysis

All data were uploaded, transcribed, and marked using Vosaic software. We were interested in understanding how MTEs structured planning conversations. Therefore, we listened to audio recordings of the planning sessions and carried out direct content analysis (Hsieh & Shannon, 2005) by marking segments that captured interactions about content, pedagogy, and/or student thinking. We then came together and compared selections and agreed on segments that focused on at least two of these aspects (e.g., the MTE facilitated a conversation about anticipating how students thought about using money to understand decimals). We then created memos that identified specific components of the structure and moves that the MTE employed to orchestrate these interactions.

Findings

In each of the planning sessions, we repeatedly observed how MTEs used the structure they designed to orchestrate a detailed and comprehensive analysis of the lesson the teachers were preparing to teach. We first describe the MTEs' collaborative lesson planning process they co-developed (Table 1), then we provide two examples from the data to illustrate the considerable specificity with which the MTE supported teachers to analyse teaching. In our view, these conversations enabled teachers to learn to teach a specific version of mathematics.

Table 1: Planning structure used by MTEs

Duration	Components of planning structure
60 mins	Study the unit overview
20 mins	Read the lesson and engage in the mathematics in the lesson (individual)
20 mins	Identify and agree on the lesson learning goal(s)
20 mins	Unpack the key mathematical understandings and connections they want students to make
20 mins	Identify the critical moment, where the learning goal will come to life
90 mins	Anticipate student thinking and identify the pedagogical strategies that will support students to make connections and reach the learning goal

The planning process began with participants individually reading the unit overview and then collaboratively discussing how the lesson fit into a larger sequence. Following this, MTEs asked

teachers to engage with the mathematics of the lesson and identify the learning goal(s) and the critical moment of the lesson–the point of the lesson at which students would have an opportunity to grapple with the mathematical idea identified in the learning goal. After individually considering these ideas, the MTEs supported the group to reach consensus on the learning goal and the critical moment. MTEs then facilitated a longer discussion among the group, unpacking the mathematical understandings and key connections they wanted students to make. MTEs would prompt teachers to anticipate students' responses and formulate questions they could ask students to support the intended mathematical connections. Teachers identified points in the lesson where they wanted to create and foster discussion among their students, and they planned how to record students' thinking to orient other students toward the mathematical ideas.

To illuminate how MTEs utilised the planning structure to engage teachers in analysing teaching in deep and specific ways, we consider two episodes. In these episodes Nina and the three teachers were analysing a lesson designed to support students' understanding of decimal quantities. These episodes demonstrate how the MTE supported teachers to consider what they wanted students to understand in relation to decimal quantities and how they could prompt students to think about these ideas. Prior to the analyzed episodes, teachers had reached consensus on the lesson's learning goal.

The team planned for a Grade 5 mathematics lesson titled *Draw and Compare Decimals.* This lesson was part of a module that focuses on adding and subtracting decimals and supporting students to "deepen their understanding of place value in decimal numbers, equivalencies between decimals, fractions, and money, and adding and subtracting decimals" (Math Learning Centre, 2019, p. 573). As written, the first part of the lesson involves the teacher leading the class through a decimal subtraction number string (Figure 1), one problem at a time, while asking students to share their thinking and model strategies students share publicly using a number line.

3.7 - 0.7
3.7 - 0.9
4.2 - 0.2
4.2 - 0.3
7.46 - 0.46
7.46 - 0.52
1.82 - 0.83
28.32 - 0.48

Figure 1: Number string

The team agreed upon the following learning goals for the lesson: "(1) when using the removal strategy for subtraction, the subtrahend can be decomposed and removed in parts in order to reach a friendly number, and (2) when comparing decimals, the value of the tenths digit determines which decimal is greater than or less than".

Example 1: Facilitation of unpacking the key mathematical understandings and connections teachers want students to make

Nina, the MTE, asked teachers what they needed to know in order to teach the lesson well. In response, Carla shared a problem that she had dealt with in the past. She explained that sometimes students will say "I subtracted one" without referring to the unit the one represents. Anticipating the language their own students might use, the group then discussed different ways they might respond.

Carla: I don't know…where this should be added [in our lesson plan], but this thought popped in my mind…I want to make sure I'm intentional when we're getting to these prompts. When…we're moving to the whole number, that when [the students] say "and then I removed one more" to be intentional [with our response]: "What do you mean one more?" But it's not one whole one… I've just been thinking about that as we're approaching unit three… that's always a misconception that I need to

	do better at, making sure when they say "one," do they mean one or one hundredth? Do they mean one tenth? Do they mean one whole?
Nina:	So, would you call that, the language of decimals?
Carla:	Yes
Nina:	…That kind of reminds me of the last lesson we did before: 30. Thirty what? Thirty minutes? Or thirty-sixtieths? How are you thinking of that?
Carla:	And I think that the work we did on that last lesson – like being intentional with my language did improve their understanding. I've been thinking about that with decimals about how I want to be intentional because like, of course, when they say "remove", I removed one more, I know what they mean. But I think I need to encourage that kind of talk so that when we get to other removal problems, there isn't a confusion there. Because I was just thinking about that with the critical moment. That, you know, hopefully, what we'll see is that they remove the 82/100 or the 82 cents. And my kids will, in previous years or even in this group, I could see them say "and then, so I knew that that got me into one whole. So, then I removed one more." And then I would be like, "yeah." And I would just put that, but if I don't show that to them and the [students] that aren't quite there are going to be removing a tenth instead of a hundredth. And in that problem, they're actually removing a hundredth, or a penny. So, I just kind of put a note for myself that I want to listen for that misconception and correct it in that moment.
Nina:	And question.
Carla:	Mm-hmm and question to get to it

In this excerpt, Nina prompted teachers to identify specific aspects of mathematics in the lesson which were essential for them to understand. This led to several outcomes. To anticipate students' thinking and plan their responses, the teachers reflected on previous ways students had engaged with similar concepts. Carla spoke in detail describing what students might do when subtracting decimal quantities and how she planned to respond when students say they subtracted "one more". Nina connected the problem Carla identified with a lesson they had discussed earlier in the school year–a lesson in which they explored different representations when solving addition or subtraction problems with unlike denominators. This led Carla to recall that, in that lesson, it was helpful to press her students to describe the referent to which the quantities referred. Nina's questions and responses prompted the teachers to consider student thinking, mathematical content, and pedagogy. Teachers anticipated that students would name a value without being specific about what the quantity represented. The specificity with which teachers anticipated students' responses allowed them to prepare to ask questions of the students that would help students use the appropriate decimal naming.

Example 2: Facilitation of the identification of pedagogical strategies that will support students to reach the learning goal

This example took place 7 minutes later, after the teachers negotiated and identified the following mathematical connection: "I can break apart decimals into parts that make it easy to subtract". Nina led a discussion on each expression in the problem string (see Figure 1). The group anticipated how students might interact with each expression, any challenges students might have, and what the teachers wanted to pay attention to as they introduced each expression. This conversation lasted approximately 37 minutes. The excerpt below came after Nina drew the group's attention to the expression: 7.46 - 0.52, the step in the lesson they had identified as the critical moment. Nina asked the group to anticipate how students might reason through the expression. Carla offered what she thought students might struggle with (related to removing 6 hundredths from 7).

Nina: So, what are you expecting then on the 7 dollars and 46 cents minus 52 cents? [long pause]. We'll be able to get that jump of 46 cents, right? To get to the 7.

Carla: I think that the struggle… could possibly come in when they then have to remove the other 6 hundredths. And that their starting point is the whole number 7.

As the conversation continued, multiple teachers agreed that students would struggle with removing the remaining 6 hundredths from 7 dollars, after removing 46 cents from the initial amount. Nina invited the teachers to think collectively and share ideas about possible pedagogical moves that could support students with this struggle. The group members offered different ideas: comparing 7.46 - 0.52 to an expression with whole numbers (746 - 52); asking students "What if you subtracted another penny?" from $7.00; using a number line to represent the values between 6.90 and 7.00; and, simply asking students "What is in between $6.90 and $7.00?". After three minutes of sharing ideas, Nina wondered if they should consider changing the expression so students could engage with the mathematics in a way that still met the learning goal and, at the same time, would be accessible to more students.

Nina: If it [referring to the subtrahend] was just 50 cents, would I then know it's 4 more pennies pretty easily? That gives some access but keeps the goal the same…Like if this wasn't 52 cents, and it was 50 cents [7.46 - 0.50], I definitely feel like kids know there's 4 pennies. I still have to subtract 4 pennies. Some will just struggle with knowing it's 6 pennies to subtract.

Carla: Right.

Nina: I think that could… give some access to kids with still that same goal. I still have to get from 7 and I have to subtract 4 more pennies, but at least for some of them it's not struggling with figuring out its 6 pennies. I just know it's 4 pennies. If taking away one little struggle

Dana: Um-hum.

Nina: But I still have that idea of $7, take away 4 pennies. I think that money's...

Tracy: Yeah.

Nina: going to be important for this one .... And Tracy…with your kids… it could be 47 cents. Now I'm just taking away 1 penny. That's still having to know it's $6.99. That's, that's accessible.

The group continued to discuss how best to modify the problem. Because anticipating students' thinking is part of the planning structure, Nina intentionally drew teachers' attention to the critical moment of the lesson. She pushed the group to consider how best to create opportunities for students to grapple with the mathematical ideas of the learning goal. They discussed modifying the problem because they predicted that students might struggle removing 6 hundredths if they did not yet understand the big ideas of the lesson. Nina's orchestration supported teachers in developing a deeper understanding of mathematics through their fine-grained analysis of the problem string.

Discussion

MTEs in this project supported teachers to engage in deep and specific conversations. What supported their facilitation of these conversations was following a highly structured planning process using educative curriculum materials aligned with ambitious teaching. This planning process contained repetitive, recognizable interdependent actions (Feldman & Pentland, 2003) that allowed them to make "the complex work of teaching visible" (Grossman, 2020, p. 8).

The planning process involved MTEs guiding teachers to first understand the mathematical content of the lesson. This foundational step identified the key learning goal and critical moment of the lesson. The structure next supported MTEs to facilitate conversations about how students would engage with the mathematics and what actions the teachers would take to support students' learning. This approach ensured that teaching decisions and discussions remain grounded in the mathematics, rather than being driven solely by pedagogical techniques or classroom management strategies. This finding aligns with accumulating research that suggests facilitating repeated conversations among teachers characterised by a focus on content, pedagogy, and student thinking at a specific, fine-grained level is one path toward teacher learning that yields important changes in practices (Russell et al., 2020).

The planning process enabled the MTEs to support depth and specificity in the conversations, which intentionally increased with each step. Over the course of the planning process, the MTEs pressed for more specificity as they anticipated students' mathematical thinking and considered what pedagogical actions they would take to enrich that thinking. This depth and specificity was evident in the time the group spent discussing the language they would use to name decimals in episode one (5 mins) and discussing an expression in the number string (8.5 mins). This dissection of lesson elements supported teachers to develop nuanced strategies focused on student understanding of mathematical concepts.

Our analysis revealed that MTEs can facilitate opportunities for teacher learning through a highly structured planning process in which teaching becomes an object of study (Stigler & Hiebert, 1999). When MTEs and teachers examine shared problems of teaching, they create a shared language with which to communicate. These MTE-facilitated interactions can lead to learning to teach very specific versions of their subject (Cohen, 2010). For U.S. teachers, this defines a learning opportunity strikingly different from the norm.

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Mathematics teacher educators professional learning: Evidence from inquiry into their own practice

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Mathematics teacher educators play a crucial role in teacher education, and examining the opportunities that facilitate their professional learning is imperative. In this study, we report how inquiry into one's own practice can be a suitable strategy for this, characterizing the improvements that mathematics teacher educators identify when participating in an e–learning self–instructional course and workshops on inquiry into their own practice. We employed a quasi–experimental, mixed methods, pre–post design; data were collected through an online questionnaire and focus groups. Findings evidence that participants in this professional learning experience perceived they developed new research skills and expanded their knowledge and positioning about research, highlighting that inquiry into one's own practice is a valid genre of research that allows them to improve their prospective teachers' learning. Finally, the implications of these findings are discussed.

Keywords: Professional learning, mathematics teacher educator, inquiry, practice–based research.

Introduction

Teacher educators are seen as crucial agents of change in preparing teachers for increasingly complex and uncertain educational contexts. They ought to model effective teaching practices, engage in research, and navigate multiple roles as researchers, mentors, and career curriculum managers (Lunenberg et al., 2014). In addition, they are responsible for implementing educational policies, reforms, and quality teaching and learning experiences inside preservice teacher education programs (Cochran–Smith et al., 2020). Considering their crucial role in educating teachers, several studies have reported the relevance of supporting teacher educators' professional learning (Kelchtermans et al., 2018; Ping et al., 2018; Tack et al., 2018).

One way to improve teacher educators' professional learning is by developing their inquiry stance and development as scholars and researchers (Cochran–Smith et al., 2020; Murray et al., 2020). According to Willemse and Boei (2013), teacher educators' research is seen as a means to achieve professional development, improve their classroom practices, and contribute to the body of their knowledge. In terms of mathematics teacher educators doing research, there has been much interest in practice–based research (such as action research, narrative inquiry, and self–studies) to develop a deep understanding of their practice to enhance it (Chapman et al., 2020). When mathematics teacher educators inquire into their own practice, they can develop an inquiring mind, maintaining research-informed teaching and building awareness about the complex task of teaching future teachers (Cochran–Smith et al., 2020; Murray et al., 2020). In other words, practice–based research allows mathematics teacher educators to scrutinize their own practice and, therefore, to support their own growth and professional development.

The RedFID project is an original model of professional learning and development for Chilean mathematics teacher educators. It offers a comprehensive support system, featuring a technological platform with three distinct digital environments: Professional Learning, Communities, and Resources (https://www.redfid.cl/). These environments provide mathematics teacher educators with access to a self–instructional professional development course, synchronous workshops, various communities of practice, and a digital library of resources.

This proposal reports the professional development outcomes related to two activities of the Professional Learning environment: the e–learning self–instructional "Inquiry into teaching practice" course and six synchronous workshops to strengthen different teaching and research skills to inquire about their teaching practices. The following research question guides this study: What are the main gains of professional learning in mathematics teacher educators who participate in a formal program to improve their abilities to inquire into their own practice? The research on this topic becomes relevant because it will allow us to systematize the improvements mathematics teacher educators gain by participating in professional learning opportunities focused on strengthening their professional development as teacher educators.

Conceptual Framework

Professional learning for teacher educators

Professional learning for teacher educators encompasses a range of activities and processes designed to enhance their knowledge, beliefs, skills, and practices (Ping et al., 2018; Tack et al., 2018) and is essential for teacher educators to maintain and enhance their teaching practices, research skills, and overall professional growth. Several authors have identified various strategies for improving professional learning among teacher educators: participation in communities of practice that share common interests and goals (Ping et al., 2018), engagement in collaborative learning and reflective practice with other teacher educators (Tack et al., 2018), and conducting research on one's own teaching practices through practitioner research and self–study (Kelchtermans et al., 2018; Tack et al., 2018). Professional learning is more effective when it is participatory, situated in the workplace, and contextually relevant (Murray et al., 2020; Ping et al., 2018).

Furthermore, the professional learning needs of teacher educators are diverse, reflecting the multifaceted and complex nature of their roles. First, it is crucial to develop specific knowledge and skills related to their dual roles as teacher educators and researchers (Ping et al., 2018) and research skills associated with conducting research and integrating research into their teaching practices (Czerniawski et al., 2017). It is also essential to build a strong professional identity and confidence in their roles as teacher educators (Tack et al., 2018). Finally, staying updated with current research in mathematics education and understanding how to apply these findings to their teaching, ensures they are well–informed and knowledgeable in their field (Martignone et al., 2022).

Inquiry own practice as professional learning

Teacher educators' engagement in conducting research is a fundamental aspect of their professional development. Ping et al. (2018) in their study point out that many teacher educators highly value research knowledge as essential for their work. This ability to conduct research includes formulating

theoretical frameworks, developing research designs, selecting research methods, and publishing results. However, despite recognizing research's importance for professional development, teacher educators face several challenges. These challenges include a lack of time and insufficient support, knowledge, and skills necessary for undertaking research (Czerniawski et al., 2017; Lunenberg et al., 2014). On the other hand, Lunenberg et al. (2014) highlight another challenge: while institutions expect teacher educators to engage in research, many feel ambivalent about this role. Indeed, they often identify more with their role as educators of future teachers and tend to show greater confidence in this aspect of their work than in their research role (Lunenberg et al., 2014).

Self–study is highlighted as an essential type of practitioner research that enables teacher educators to reflect on and improve their teaching practices, helping them better understand the relationship between teaching and learning in substantive ways (Loughran, 2014; Tack et al., 2018). The outcomes of such inquiries often highlight the role of reframing taken–for–granted practices to become more informed and knowledgeable about teaching and teacher education. Similarly, Murray et al. (2020) discuss how researching one's own practice can lead to transformative practice changes and a renewed sense of agency. This expansive learning involves creating new professional knowledge through research–informed practices, leading to new insights and changes in their professional identities and practices. Through reflection on their own practice and engagement in self–study, teacher educators develop the capacity to examine their teaching practices critically, enhancing their teaching and research skills (Kelchtermans et al., 2018; Loughran, 2014).

Methods

To identify gains in mathematics teacher educators' professional learning, we adopted a quasi–experimental, mixed methods, pre–post design (Creswell, 2014). The sample comprised 28 mathematics teacher educators from 14 Chilean universities, 19 (67.9%) females and nine males (32.1%). The average age of the sample is 47 years, ranging from 33 to 75, and the number of years of professional experience in this group ranged between 1 and 50 years, with a mean of 11 years. Regarding professional and academic qualifications, 16 (57.1%) were mathematics secondary school teachers, 7 (25%) were primary school teachers, 5 (17.9%) had a bachelor's degree in mathematics, and 18 (64.3%) had a master's degree and 10 (35.7%) had a doctorate. Concerning their affiliation with a teacher education program, 13 (46.4%) taught math courses in primary programs, 12 (42.9%) taught math courses in secondary programs, and 3 (10.7%) taught math courses in both primary and secondary programs. Each participant was assigned an identification number (MTE1–MTE28), replacing their names. All participants signed a consent form that followed national ethical standards.

Participants completed an approximately 20–hour online self–instructional course. The course is divided into three modules. In the first module, titled "Why investigate your own practice?" participants explore the concept of teaching practice and the challenges that may arise within it through a case. Each participant must identify and describe a problem of practice based on their experiences educating future teachers. The second module, titled "Approaches to innovate or investigate one's own practice," continues the case discussion by presenting a dilemma: whether to follow an innovation or research approach to address an identified problem of practice. This module clarifies the definitions of researching and innovating in one's practice and analyzes how these

approaches can complement each other when tackling a problem. Each participant is also tasked with proposing an approach to address the problem of practice they identified in the first module. In the third module, "Generating an inquiry proposal," participants are introduced to key aspects of establishing a clear purpose and analyzing the feasibility of developing an inquiry project. By working on their identified problem of practice, participants create a proposal that includes the problem they wish to address, the purpose of their inquiry, the chosen approach, and an analysis of how feasible it is to implement their proposal. After completing the course, mathematics teacher educators participated in 6 synchronous workshops to strengthen different research skills. These workshops allowed participants to complement the knowledge and skills gained from the online course in a practical setting. Figure 1 summarizes the course and workshops in which mathematics teacher educators participated.

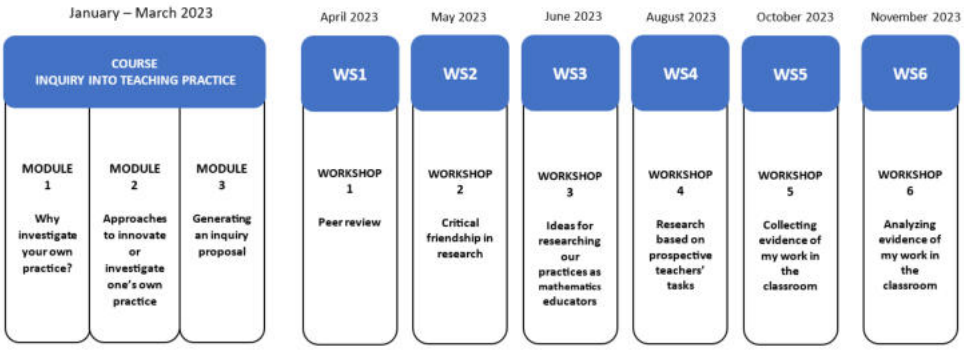

Figure 1: The course and workshops that mathematics teacher educators participated in

The research strategy used for collecting pre and post–test quantitative data was a questionnaire, whereas focus groups were used for collecting pre and post–test qualitative data. The questionnaire contained 30 items rated on a 5–point scale, from "I fully disagree" to "I fully agree". The items were adapted from the survey on teacher educators' professional learning developed by Ping et al. (2020) and the teacher educators' researcherly disposition scale developed by Tack and Vanderlinde (2016). The questionnaire has five scales: disposition to research (6 items), appraisal to research practice (5 items), appraisal to innovation practice (5 items), teaching for teacher education (8 items), and research for teacher education (6 items). The pre–test questionnaire data were collected at the beginning of the course in January 2023, and the post–test data a year later in January 2024. Descriptive statistics and paired t–tests were applied to analyze the data gathered.

The focus groups were organized to be conducted during two National Seminars of Mathematics Teacher Educators held by RedFID. Pre–test focus groups were conducted with participants who would take the course in December 2022 (n=31), and post–test focus groups were performed with participants who finished the course and actively participated in the different workshops in December 2023 (n=28). In pre–test and post–test focus groups, questions focused on how participants view professional learning and what they consider crucial topics and opportunities for strengthening mathematics teacher educators' professional development. A thematic analysis was done to identify recurring themes in the data (Clarke & Braun, 2017). While analyzing the data, two research team members cross–verified the coding of answers and jointly refined the different codes and categories

of analysis wherever necessary. That way, a high degree of consensus was achieved in the coding strategy, increasing the validity and trustworthiness of the data analysis (O'Connor & Joffe, 2020).

Findings

Quantitative results

The results of the questionnaire show relevant findings. First, as indicated in Table 1, the average scores in the post–test measurement increase in all the scales that measure aspects related to skills and interest in research. Second, only in the "Appraisal to research practice" scale the t–test was statistically significant ($t=2.372$; df 27; $p<0.05$). Third, we estimated the t–test of the pre–test and post–test scores of all the questionnaire items, finding statistically significant differences in only five items from three scales. From disposition to research scale item 5 "I am familiar with the results of recent research on teacher education" ($t=2.458$; df 27; $p=0.021$); from appraisal to research practice item 13 "I have experience in developing research on my teaching practice" ($t=3.400$; df 27; $p=0.002$), and item 14 "I have the methodological knowledge necessary to research my teaching practice" ($t=2.073$; df 27; $p=0.048$). And from research for teacher education scale item 11 "I have learned how to research my own practice" ($t=3.777$; df 27; $p=0.001$); and item 12 "I have learned the role of the mathematics teacher educator as a researcher" ($t=2.105$; df 27; $p=0.045$). These results evidence that participants in this professional learning experience perceived developing new research skills, especially related to researching their own practice.

Table 1: Results of the t–test on the different scales of the questionnaire

Scale	Pre–test score	Post–test score	t	df	*p* value
Disposition to research	3.77	3.88	0.837	27	0.410
Appraisal to research practice	**3.97**	**4.20**	**2.372**	**27**	**0.025**
Appraisal to innovation practice	4.13	4.23	0.699	27	0.491
Teaching for teacher education	3.60	3.42	-0.904	27	0.374
Research for teacher education	3.36	3. 58	1.027	27	0.314

n=28

Qualitative results

Mathematics teacher educators highlighted some key features of professional learning and some interesting changes in their approach to practice from the focus group analysis. Participants in the pre–test focus group mentioned two key issues regarding professional learning. On the one hand, mathematics teacher educators emphasise that professional learning allows the construction of different types of knowledge, such as disciplinary (mathematical), pedagogical, and socio emotional. However, it is noteworthy that the professional learning needs expressed by the mathematics teacher educators did not include those related to research and even less so to the research of their own practice: "Professional learning is not only disciplinary, but it has to do with competencies, skills, characteristics and everything that involves my job…" (MTE8, female, 22 years as a mathematics

teacher educator, pre–test FG3). Moreover, professional learning through research is associated with academic research and carried out by a privileged group of teacher educators who have the funds and protected time to conduct it: "With the privileged conditions of the assigned time, the teacher educator can devote himself to research. If these conditions do not exist in an institution, research should be forgotten" (MTE1, male, 40 years as a mathematics teacher educator, pre–test FG2).

On the other hand, since mathematics teacher educators have been involved in this professional learning instance, two main advances have been observed that we consider relevant to highlight. At the beginning (pre–test focus group), the mathematics teacher educators emphasised the need for constant reflection on their practice without necessarily specifying how this would be translated into concrete actions or a working methodology: "I think this is a very practical reflection, to look at oneself and questioning Wow, this year did not work so well for me, what happened? or What should I do?" (MTE21, female, 15 years as a mathematics teacher educator, pre–test FG4). In contrast, in the post–test focus group, participants emphasised not only the reflection of the practice but also that they had already acquired the necessary tools to engage in the process of inquiry into their own practice: "[RedFID] gave me skills, especially related to inquiry one's practice, knowing yourself, and the knowledge you have […] I really developed skills that I did not think I had" (MTE10, female, 4 years as a mathematics teacher educator, post–test FG3).

A second advance that is deemed significant pertains to the notion of research as mathematics teacher educators conceptualise it. In the beginning, in the pre–test focus group, several participants realized that research was something foreign to them, of an academic nature, or that it was not directly related to their teaching practice: "We are stuck with academic research... we do not do practitioner research that contributes more to teaching" (MTE11, male, 12 years as a mathematics teacher educator, pre–test FG3). Conversely, a year later, the participants considered research more broadly and recognised its value in fostering their academic careers: "I have broadened what I understood by research, what research was. It has changed my paradigms… Now, I can also investigate myself [...]. It has allowed me to understand that there are other ways of doing research." (MTE18, female, 15 years as a mathematics teacher educator, post–test FG2). In addition, researching one's own practice is seen as an option not only to improve teaching practices but also to impact student learning: "Another thing that I learned is that much evidence of our practice allows us to inquire about our own practice, and this is going to have implications in our way of teaching and, obviously, it has an impact on the outcomes of our prospective teachers" (MTE23, female, 4 years as a mathematics teacher educator, post–test FG4).

Conclusions

The present study aimed to investigate the professional learning of mathematics teacher educators who participated in the "Inquiry into teaching practice" course and six workshops as part of the RedFID's Professional Learning digital environment during a year. We adopted a quasi–experimental, mixed method, pre–post design, collecting quantitative and qualitative data. We found that participants scored significantly higher in their perceptions of gains related to research on teacher education and methodological knowledge and skills for researching their own practice. Furthermore, they expressed new insights about the research practice, emphasizing that researching their practice

as a teacher educator contributes to their own and prospective teachers' learning, evidencing participants expanded their knowledge about doing research and inquiry into one' practice.

These findings are in line with Willemse and Boei (2013), who state that research allows teacher educators to achieve professional development, improve classroom practices, and contribute to the knowledge base of the teacher profession. In addition, these results, despite being in line with the discussion about the research on one's practice as a means to improve the professional learning of teacher educators (Kelchtermans et al., 2018; Loughran, 2014; Murray et al., 2020; Ping et al., 2018), provide new insights of the challenges in doing this genre of research due to the necessity to acquire methodological knowledge and research skills to be able to do it. However, it is relevant to point out that, in general terms, the interest of investigating one's practice in participants has a greater focus on generic pedagogical issues rather than didactic or disciplinary issues associated with the teaching and learning of mathematics. It would be interesting to encourage research into one's own practice linked to disciplinary issues in teaching and learning mathematics to generate bodies of knowledge linked to the teaching of this discipline.

In sum, the present study provides new insights into professional learning in mathematics teacher educators through practice–based research and inquiry into one's practice. However, we must acknowledge the limitations and scope of the results obtained. For instance, the small number of participants in a specific context (Chile) limits the generalizability of the findings to a broader group of mathematics teacher educators. Moreover, this study relied solely on data gathered from questionnaires and focus groups. Incorporating additional qualitative data (such as in–depth interviews and reflective journals) could have offered a more comprehensive examination of the participants' learning experiences.

Despite these limitations, the findings of this study have the potential to contribute new knowledge and practical suggestions for the field of professional learning in teacher educators. We recommend continuing the study of the professional learning of mathematics teacher education through formal professional development programs situated in the workplace and focused on research and inquiry–based learning (Murray et al., 2020; Ping et al., 2018). It is essential to begin systematizing the results of initiatives and professional development programs for teacher educators, identifying good practices and relevant results that strengthen the professional learning of mathematics teacher educators.

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Responding to emergent problems of practice in side-by-side coaching

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Side-by-side coaching, one form of practice-embedded PD, is designed to bridge the stubborn gap between professional development (PD) and classroom implementation. In this study, we investigate the problems of practice that emerge while teachers are participating in side-by-side coaching focused on ambitious mathematics instruction and their participation shifts from appropriation of practice to guided participation with the coach. We examine how the teacher and coach collaborate to respond to these problems of practice. Our analysis identified three emergent problems of practice, four pathways that the teacher and coach used for moving out of appropriation, and a suite of tools the teacher and coach used for responding. These findings have implications for designing practice-embedded PD that can support teacher learning of ambitious mathematics teaching practices.

Keywords: Teacher learning, Coaching, Problems of practice, Guided participation.

Prior Literature

For more than two decades, studies of effective professional development (PD) design have sought to address the enduring challenge of how to support teachers to take what they have learned in PD and implement it in their classrooms (e.g., Borko, 2004). Efforts to address the gap between PD and practice are evident in the proliferation of practice-embedded teacher learning, which situates PD closer to or within practice (Gibbons et al., 2021). Facilitators of practice-embedded teacher learning carry a variety of culturally bound titles, such as coach, mathematics leader, and numeracy coordinator (e.g., Gibbons & Cobb, 2017; Sexton & Downton, 2024). In this paper, we refer to the activities of supporting practice-embedded teacher learning as *coaching*, regardless of professional title. To understand how coaching can support teacher learning of ambitious mathematics practices, the field needs to decompose practice-embedded professional learning designs (Sztajn et al., 2020). In this paper, we examine side-by-side coaching, one form of practice-embedded teacher learning. In the context of side-by-side coaching, we specifically aim to understand what meaningful problems of practice emerge while teachers are enacting ambitious mathematics instruction and how those problems are addressed when a coach is positioned to support teacher learning in their classroom.

Decomposing Practice-Embedded Teacher Learning

Practice-embedded teacher learning designs create opportunities for in-service teachers to learn in, through, and from practice (Lampert, 2010) as teachers are able to enact and study practice in their classrooms (e.g., Gibbons et al., 2021). In these designs, the teacher educators (TEs) might be traditional university-based TEs, school-based professional developers, coaches, or any more knowledgeable other who engages in coaching. Practice-embedded PD typically involves cycles of

co-planning, teaching, and reflection, often with groups of teacher learners. Following sociocultural theory, these designs can create opportunities for learners to co-participate in practices as they learn to enact them (e.g., Wenger, 1998). Recent work has attempted to characterise the facilitation of practice-embedded PD to identify approaches that support teachers' learning (Gibbons et al., 2021).

In our study, we examine a one-on-one practice-embedded teacher learning design: side-by-side coaching. In side-by-side coaching, a coach is embedded in a single teacher's classroom, working side-by-side with the teacher to enact, problematize, and refine a focal pedagogical practice. Focal practices require in-the-moment responsiveness to student thinking, the improvisational work inherent to ambitious mathematics teaching. The coach and teacher interact with one another and with students, moving fluidly between forms of participation. To more precisely understand these interactions, we drew on Rogoff's (1995) constructs of *guided participation*, in which learners engage in shared activity with guidance to communicate and coordinate efforts, and *appropriation*, in which learners take up, transform, and enact practice, making these practices part of their repertoires. We used these two constructs to characterise structures of co-participation used during side-by-side coaching, thereby offering a detailed, empirical depiction of this practice-embedded teacher learning design (Munson et al., 2024). In doing so, we found seven patterns in the ways that the coach and teacher moved between forms of guided participation and appropriation. In three of these patterns, teachers shifted from appropriation of the focal practice to guided participation. We hypothesised that preceding these shifts, one or more problems of practice emerged that the teacher and coach attempted to address through guided participation, and it is these instances that we explore in this study.

Problems of Practice as Teacher Learning Opportunities

The pedagogical dilemmas teachers face are often referred to as *problems of practice* (Lampert, 1985). Problems of practice are not resolved by routine decisions with clearly established pathways. Rather, addressing these dilemmas requires adaptive expertise (Hatano & Inagaki, 1984), or professional judgement guided by the teacher's interpretation of data and understanding of the context. As such, problems of practice have been viewed as rich opportunities for inquiry, professional discourse, and learning (Horn & Little, 2010). One paradox of problems of practice is that they can be both perpetual, ongoing dilemmas, and immediate, singular challenges. For instance, one problem of practice in ambitious mathematics teaching is how to respond to student thinking so as to deepen students' mathematical inquiry without taking over (Munson, 2019). Teachers may experience this challenge as a daily tightrope; however, at any point the teacher must be ready to respond to a single student contribution. Problems of practice are not only theoretical; teachers have the "responsibility to act" (Lampert, 1985, p. 180). Decisions must be made and implemented in the midst of ongoing dilemmas for which there may be no singular, conclusive solution.

We agree with prior teacher education researchers that professional discourse about problems of practice can be fruitful for teacher learning (Horn & Little, 2010; Lefstein et al., 2020). In practice-embedded teacher learning designs, like the side-by-side coaching under investigation here, problems of practice can emerge from the act of teaching and can open opportunities for shared inquiry and decision making, which must be immediately tested by the responsibility of enactment. We

investigate the nature of these emergent problems of practice in side-by-side coaching to understand how they might be immediately addressed by the teacher and coach.

This analysis is situated in a larger study of side-by-side coaching in the classrooms of three primary mathematics teachers in the US. This analysis explores three linked research questions: (1) How often is a problem of practice present in the shift from appropriation to guided participation with students? (2) What problems of practice do the coach and teacher address in the transition from appropriation to guided participation with students? and (3) How do the coach and teacher address problems of practice in shifting from appropriation to guided participation with students?

Methods

Our study draws on a total of 21 side-by-side coaching sessions with three primary mathematics teachers (6-8 sessions per teacher) in the US in which the teachers aimed to develop their practice of conferring, or interacting with students during collaborative problem-solving to support sensemaking (Munson, 2019). Each coaching session occurred during a single mathematics lesson; the side-by-side coaching took place during the collaborative problem-solving portion of the lesson when teachers conferred with students. The three teachers who elected to participate in side-by-side coaching spanned different grades and career stages. Hope was a first grade (students aged 6-7 years) early-career teacher; Carmen was a second grade (students aged 7-8 years) veteran teacher seeking to make meaningful changes in her mathematics teaching practices; and Jane, a fourth grade (students aged 9-10 years) teacher, was selected by the school as a mathematics teacher leader. While all three shared the goal of supporting students to learn collaboratively through inquiry, they each had different teaching practices and motivations. As a second year teacher, Hope sought support to reflect on the decisions made during teaching. Carmen had no prior experience in inquiry-based mathematics instruction but was a confident literacy teacher. She aimed to make mathematics teaching and learning as nourishing as literacy. Jane felt she lacked access to a more knowledgeable other who could push her practice forward. She saw coaching as a way to connect with a like-minded practitioner. The first author served as the coach. Each session was video recorded, with a stationary camera located high in one corner of the classroom. Both the teacher and coach wore microphones. These records were professionally transcribed for analysis in qualitative video analysis software.

Because the focal pedagogical practice was conferring, we parsed the data into conferring interactions, which were operationalized as teacher interactions with students during collaborative problem solving, focused on students' mathematical thinking. There were a total of 50 of these conferring interactions; the teacher and coach were co-present and interacting with one another in all of them. We then segmented each conferring interaction into guided participation and appropriation. We operationalized guided participation as instances where the teacher and coach were working together on the focal pedagogy, sharing instructional leadership, or where the coach led and the teacher listened in. We identified five forms of guided participation during conferring interactions. Two of these forms—joint teaching and modelling—occurred while the teacher and coach were actively interacting with students. The other three forms—coordination of activity, pausing instruction to make sense or decisions, and reflection—occurred when the teacher and coach stepped

away from students and engaged with only each other. We operationalized appropriation as moments when the teacher took instructional leadership and the coach listened in.

After segmenting all conferring interactions, we mapped the sequences of these activities and looked for patterns in how each conferring interaction was structured. We found that when appropriation occurred, (n = 22 of 50), it was located at the beginning of conferring interactions and often (n = 17 of 22) was followed by one or more forms of guided participation with students: joint teaching or modelling. In the current analysis, we focused on these 17 conferring interactions, whose pattern of participation included a shift from appropriation to joint teaching or modelling. For each of these 17 interactions, memos were constructed which included four features: a description of what occurred during the teacher's appropriation of conferring, a description of what occurred immediately after the shift to guided participation, who initiated the shift (teacher or coach) and, where applicable, the problem of practice being addressed in that shift. We developed categories for the problems of practice and the ways the teacher or coach initiated the shift to guided participation. Where appropriate, problems of practice and responses were double-coded. We identified themes in the ways the teacher and coach responded to each problem of practice, which are reported in the findings.

Findings

In 15 of the 17 instances (88%) in which side-by-side coaching included a shift from appropriation to guided participation with students (joint teaching or modelling), the shift in participation addressed an emergent problem of practice. In the two other cases, an interruption from a student not involved in the conferring interaction precipitated the shift. In all other instances, the teacher and coach addressed an emergent problem of practice at the transition to guided participation. Our analyses focus on these 15 instances to illuminate the nature of these problems of practice, finding three distinct and related problems: (1) how to elicit, interpret, and respond to student thinking, (2) how to support students to make sense of the mathematical task or the mathematics within it, and (3) how to support students to develop models or strategies for the mathematical task at hand. In the following sections, we detail each of these problems, then turn to how the teacher and coach addressed them through side-by-side coaching.

Problems of Practice Addressed in the Shift from Appropriation to Guided Participation

In six instances, the shift to guided participation from appropriation occurred when teachers grappled with *how to elicit, interpret, and respond to student thinking*. When teachers approached students to confer, they were in the midst of problem solving, and their thinking was often still in formation or murky. Teachers began these interactions by attempting to elicit student thinking as they worked to orient themselves to what students understood, what strategies they were attempting, and where, if at all, they were struggling and why. In these six instances, teachers were variously challenged to elicit enough clear information about student thinking, make sense of the information they elicited, or develop an interpretation of the source of students' mathematical struggles, all of which are precursors to responding to and supporting student thinking. For instance, Carmen began a conferring interaction by eliciting students' thinking about the task: "Rosie had 67 flowers. Andre gave her some more flowers. Now Rosie has 90 flowers. How many flowers did Andre give her?" The pair of

students she spoke with gave contradictory and overlapping explanations of what they were trying, making it unclear what they understood and how they were working. Carmen turned to the coach and said simply, "I don't know." In that moment, the challenge was both how to glean enough information and how to form a justifiable interpretation such that a response could be developed.

In seven instances, the emergent problem of practice teachers faced was *how to support students to make sense of the mathematical task or the mathematics within it*. In these cases, teachers formed interpretations of student thinking that suggested that students were struggling to understand the mathematical task itself, interpret the context of the task, or struggling with the mathematical concepts that undergirded the task. Teachers then faced the dilemma of how to support student sensemaking without resorting to directives or prescribing procedures. For instance, Jane began an interaction with students who had been asked to determine if a specific situation involving filling up a vending machine with bottles, which came in six-packs packed into boxes, involved multiplication or division. The interaction came to a point of silence, and the teacher was not sure how to support the students to interpret the meaning of the task. Ultimately, this problem of practice involved deciding what to say that might move students to construct sustainable mathematical understanding in the moment.

In five instances, teachers confronted *how to support students to develop models or strategies* for the mathematical task at hand. In these cases, teachers saw that students needed to create models of the mathematical situation or develop a strategy for moving through the problem. In some cases, students expressed being stuck, and the teacher's dilemma was how to discursively create space for students to develop a pathway with some support while not dictating what pathway they should take. In other cases, students had selected a strategy or model but needed support to use it in ways that supported mathematical sensemaking. For instance, Hope posed a problem to four students: "I made 10 cookies and Ms. J ate 4 cookies. How many cookies are left?" When all students instantly declared the answer to be 6, Hope pressed them to construct models to provide proof. One student began to model the task with blocks by constructing a group of 10 and a second group of 4, then joining them. When Hope pressed the students to say why it did or did not make sense to make 4 more cookies in this context, one student said, "You take some out," and another added, "Because, if you make more cookies, then we [don't] have the number we need to make." But the student who constructed the joining model was not convinced. Hope faced the challenge of how to support the group to develop models and strategies that aligned to the task, their answer, and one another's understanding.

How the Coach and Teacher Addressed Problems of Practice

We explored how the teacher and coach moved between appropriation and guided participation by examining who initiated the shift out of appropriation and how they did so. In 12 of the 15 instances (80%) the coach initiated the shift, indicating that one role the coach played when the teacher was appropriating practice was to look for emergent problems of practice that could be elevated for inquiry or joint exploration. The teacher and coach used four pathways when moving out of appropriation: pausing for teacher-coach talk before shifting into joint teaching ($n = 5$) or modelling ($n = 3$), shifting directly into joint teaching ($n = 6$), or shifting directly into modelling ($n = 1$). Here, we focus on the most prevalent pathways: pauses leading to other forms of guided participation, and joint teaching.

The teacher and coach used pauses in three ways. First, the coach might initiate a pause to suggest an instructional move or line of questioning, with the intent that the teacher could implement those moves (n = 4). For instance, in the earlier example of Hope's interaction in which she encountered the problem of practice of how to support students to develop models or strategies for the cookie task, the coach initiated a pause to suggest that a productive next move might be to "tell the story with manipulatives." The teacher agreed; together the coach and teacher prompted students to prove their answers by retelling the story, modelling each part with the manipulatives available as they went. Second, the coach or teacher might initiate a pause to offer an interpretation of student thinking that the pair could discuss or debate (n = 4; note: one of these instances was double coded with the previous code). For instance, in the previous example of Jane's interaction, when she confronted the challenge of how to support students to make sense of the task, the coach initiated a pause in which she offered an interpretation that the students in the interaction were focused on defending their numerical answer of 10, while Jane was trying to focus them on the units involved; the coach posited that this communicative struggle was complicating their interaction. The teacher then invited the coach to model how to move through this challenge by saying, "Go ahead." Third, the teacher might initiate the pause to elicit the coach's interpretation of student thinking (n = 1). In each pathway involving a pause, the interaction during the pause precipitated the teacher and coach re-entering the conferring interaction with students in joint teaching, in which they collectively implemented something discussed, or modelling, in which the coach enacted the ideas generated at the teacher's request. These pathways allowed the teacher and coach to develop a hypothesis about student thinking, create a plan for how to respond, or both, which they then tested through implementation.

Alternatively, in more than one-third of instances (n = 6), the coach entered the conferring interaction directly, shifting the participation to joint teaching. In each case, the coach enacted an instructional move that addressed the problem of practice, such as posing an eliciting question, making a connection between what students said and their written work, or suggesting that constructing a model might be useful. Teachers picked up on these cues and co-participated in the interaction, and both adults shared responsibility for moving student thinking forward. For instance, in the previous example of Carmen's interaction in which she confronted the dilemma of how to elicit, interpret, and respond to student thinking, she turned to the coach and stated, "I don't know." The coach then entered the interaction with students, pointing out that their written representation was similar to their work on the 100-bead frame. The coach elicited student thinking about these models and the teacher joined in, ultimately supporting students to articulate a compensation strategy. Joint teaching was a key way to seamlessly address the emergent problem of practice, pointing the teacher down a potentially fruitful instructional pathway that the teacher and coach could traverse together.

Discussion

During side-by-side coaching, one prevalent pattern of participation was for the teacher to initiate instruction with students by appropriating the focal practice and then for the interaction to shift into guided participation. In most instances, this shift occurred after a problem of practice emerged. This study identified three problems of practice precipitating these shifts: (1) how to elicit, interpret, and respond to student thinking, (2) how to support students to make sense of the mathematical task or

the mathematics within it, and (3) how to support students to develop models or strategies for the mathematical task at hand. Through pauses, joint teaching, and modelling, the coach and teacher addressed these problems of practice using a suite of tools: suggesting or eliciting an interpretation of student thinking, suggesting a pedagogical move, and implementing pedagogical moves to advance the interaction with students. Taken together, these emergent problems of practice and ensuing responses paint a detailed picture of the dilemmas teachers face and how coaches might work with teachers to address them in the moment, simultaneously supporting teacher and student learning, extending earlier work (Gibbons et al., 2021; Munson et al., 2024).

The findings suggest appropriation can create opportunities to surface problems of practice that can then be addressed through guided participation. Appropriation was not an endpoint but a key part of the learning process, in line with Rogoff's (1995) argument that guided participation and appropriation are intertwined and mutually supportive. Further, problems of practice typically precipitated the transitions between appropriation and guided participation. Appropriation opened opportunities for problems of practice to emerge while the teacher was supported by an attentive coach, enabling the teacher and coach to respond to those problems of practice collaboratively. This suggests that the role of the coach during appropriation is to observe for problems of practice that can become opportunities for teacher learning through guided participation. Knowing the potential problems of practice that could arise when teachers are learning to enact a practice, such as a conferring, could support coaches in attending to teachers' appropriation such that they can recognize these dilemmas when they emerge. The coach then must select the right moment and tool for addressing that problem of practice. Given the challenges of responding to student thinking during instruction, these findings highlight the potential for side-by-side coaching to play an important role in teacher learning of ambitious mathematics instructional practices alongside other potentially productive coaching activities (Gibbons & Cobb, 2017).

These findings are bound by important limitations. The problems of practice we identified were directly linked to the focal practice: conferring with students during collaborative problem solving. We would expect other problems of practice to emerge during side-by-side coaching of a different focal practice, such as facilitating whole class discussions. Further, our sample was limited to 17 interactions across three classrooms with a single coach; a larger data set, with additional coaches, might reveal a wider suite of tools for responding to these problems of practice. We view these limitations as key opportunities for future research on practice-embedded teacher learning designs.

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Empowering mathematics teacher specialist students as change agents: bridging theory and practice on professional development

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This paper reports on a mathematics teacher educator (MTE) conducting action research with the aim of developing a course for teacher specialists that empowers them as change agents in the schools they work in, thus bridging theory and practice. Change agents can support both top-down and bottom-up changes, and the latter is the focus of this paper. Factors that can enable or inhibit teacher led change efforts are presented and discussed with respect to the phases of action research. Through the research, important factors are identified, and how these are dealt with collaboratively between the MTE and the students, securing results that leads to changes. Most notably, collectively we found ways to successfully get support by the principal or school leader to ensure bottom-up changes in school.

Keywords: Mathematics teacher educator, professional development, action research, change agents.

Introduction

Professional development programs can take on many forms and I will in this paper present some key ideas of designing and redesigning a professional development course in the Teacher Specialist program in Norway. Teacher Specialist was a government initiative with the aim of having dedicated teachers leading professional development with their colleagues in school.

I was involved in all phases of the process of giving a Teacher Specialist course in mathematics at my University; from participating in the working group who formulated the application to the tender from the Norwegian Directorate for Education and Training, to deciding the courses included in the program, writing one of the course descriptions, and finally lecturing this specific course. This provided me a unique opportunity to design a course built upon knowledge from research, and I will present the choices, development and learning in this paper using action research as a methodology.

Work culture in Norway has a noticeable flat organizational structure, where workers often have a great deal of autonomy; this is also so for teachers. While the curriculum specifies what the pupils should learn in mathematics, it is up to the teachers to choose the most appropriate methods to achieve this. Consequently, a professional development program called Teacher Specialist was not well received in the beginning because of the name (Hoem et al., 2017). In Norway, there is a skepticism to anyone claiming to 'be better' than others and it can therefore be challenging for these teachers to find their place as a leader in mathematics teaching at their school. An evaluation of the pilot for the Teacher Specialist education points to a need for improvement in the schools' involvement in the Teacher Specialist student's assignments. In addition, the Teacher Specialist students themselves ask for tools to initiate change processes at their own workplace and more awareness and training on colleague guidance (Seland et al., 2017). Considering these challenges Hoem et al. (2017) argue that we need to find ways to make these teacher specialist students change agents at their own workplace.

This is the background for designing a course in the Teacher Specialist program, and this led me to the following research question: How can I design a professional development course for Teacher Specialists in mathematics that will make them successful change agents at their workplace?

Change agents are here defined as "individuals who can successfully transform aspects of how organisations operate" (Brown et al., 2021, p. 1).

Theoretical perspectives

A goal of the Teacher Specialist program in Norway is to educate teachers who can act as educational change agents in mathematics at their school. So, this professional development program is not solely meant for individual learning, but to support teachers to successfully transform aspects of how their school operate. Being a change agent can be challenging, and most of the research literature generally conceptualize change agents as "those individuals who are best able to support the introduction of new top-down (as opposed to bottom-up) change" (Brown et al., 2021, p. 11). A top-down change is characterized by school leaders or policy-makers initiating change, and a factor for successful change in this perspective is distributive leadership (Brown et al., 2021). On the other hand, bottom-up change describes changes initiated by teachers trying to influence a school organization. The research on bottom-up change agents is limited, and so is research on the effectiveness of all types of change agents (Brown et al., 2021). However, Brown and colleagues (2021) identified in their literature review four main factors that may enable or inhibit teacher leaders' change efforts. These are:

> 1) principal or school leadership support; 2) buy-in to the role by practitioner colleagues; 3) access to training and professional development; and 4) perceived autonomy and teacher leaders' own positioning in the role. (Brown et al., 2021, p. 10)

These main factors that may enable or inhibit are also recognizable in the Norwegian Teacher Specialist initiative. Lorentzen (2020) has investigated Teacher Specialists and the boundaries they perceive to encounter in the Norwegian system. She has identified three types of boundaries:

> The knowledge boundary: Concerns perceived variations in the specialists' and the teacher staffs' perspectives of the specialists' performances. The specialists feel anxious about whether other teachers perceive their activities, especially their plenary presentations, as useful.
>
> The structural boundary: Concerns the specialists' perceptions of falling outside the schools' organisational structures, resulting in a mixture of positions and roles and the impression that teachers are forgetting about their role.
>
> The relational boundary: Concerns perceived variations in specialists' and the teacher staffs' perspectives of prominence. The specialists feel lonely and are anxious about being too pushy or outreaching to the other teachers. (Lorentzen, 2020, p. 683)

So, the research literature is clear about theoretical knowledge from universities is not enough to make teachers effective change agents in their workplace. Therefore, a course in the Teacher Specialist program should preferably include work that will empower the teacher specialists also in structural and relational areas which research literature point to regarding successful change agents.

Methods

I have conducted action research to answer my research question. According to Kemmis (2009), "action research aims at changing three things: practitioners' *practices*, their *understandings* of their practices, and the *conditions* in which they practise" (2009, p. 463). As previously stated, my research question is: How can I design a professional development course for teacher specialists in mathematics that will make them successful change agents at their workplace? In order to investigate this research question, I need to engage in a critical and self-critical process together with the Teacher Specialist students. There are various ways of describing action research, and Kemmis (2009) distinguishes between technical, practical and critical action research (pp. 469-471). These variations of action research are distinguished by the relationship between the participant-researcher and the others involved in or affected by the research. They vary from technical action research where the participant-researcher makes all the choices and decisions, to critical action research which is viewed as a collective effort where all voices have the same decisive power (Kemmis, 2009). I position this research as a practical action research, following the descriptions of Kemmis (2009) as a project that is self-directed, but in this case the others involved also have a voice (p. 470).

The context of this research is the course I am teaching to the Teacher Specialist students, and the aim of the research is to develop and adjust this course to provide support for the students to be successful change agents at their workplace. The number of students taking the course has been 16, 8 and 16, and they are required to be certified teachers and be employed as a teacher when they are taking the course. The teacher specialist professional development programme I taught was given as an opportunity from 2019–2021, and the Norwegian Directorate for Education and Training funded this by paying close to 40 % of the teachers' salary, thus freeing time to study. The teachers applied for the professional development through their principal, who had to actively agree to let teachers at their school take part in the teacher specialist study programme. The course I designed and taught was called 'selected professional-relevant methods and perspectives on learning'.

The data analyzed are extensive, and include the researcher's considerations when designing the course, course requirements, students' reactions and feedbacks and course outcomes. These are analyzed with respect to factors that may enable or inhibit teacher leaders' change efforts, and the results are structured according to the phases of actions research, which can be seen in Figure 1.

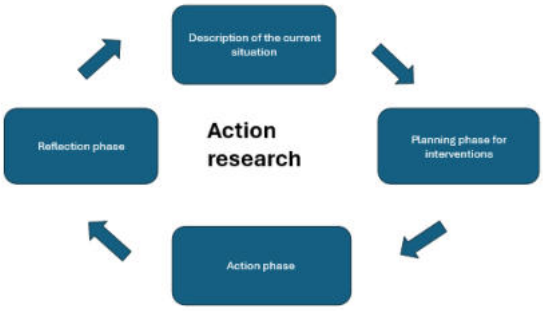

Figure 1: Phases of action research

First, I describe the current situation with respect to the research question I wish to address. This is followed by a planning phase of the interventions needed. The third and fourth phases are the action phase and the reflection phase. The findings and choices made in the phases of action research are discussed with respect to theory on change agents in the discussion part.

Results

Description of the current situation

Given that the model did not give the teachers extra money to study, but freed their time to develop their own teaching, I assumed that the students applying would mostly have an internal motivation to learn and improve their own teaching. Also, I assumed that while these teachers probably are highly motivated and with a high standard of work ethics, their workload is most likely not evenly distributed and there might be very intense periods teaching their classes. Therefore, the course needed some flexibility. One element I was unsure about was how they themselves thought of the term 'teacher specialist', their own role now and in the future, and also how their leader and colleagues looked upon this. We know from research that this can vary.

In 2020 a new curriculum was implemented in Norway, and there were some notable changes to mathematics in this curriculum. Previously, there were only learning goals in mathematics, but with the new curriculum, core elements were also introduced. These core elements are defined as the most important aspects of the subject that the pupils need to learn through school. In mathematics these core elements are: exploration and problem solving, modelling and applications, reasoning and argumentation, representation and communication, abstraction and generalisation, and mathematical fields of knowledge. While the curriculum was not implemented before 2020, the core elements where already known in 2019. The mathematical topics teachers were used to, like numbers and geometry, were now all part of the last core element, while the other core elements describe processes the pupils need to develop and master to be competent in mathematics. Also, these core elements should be an integrated part of teaching mathematics at all levels, from first grade until the pupils finish school.

While many teachers have taught mathematics and paid attention to these core elements before the new national curriculum was introduced, this is no longer optional. From 2020 all teachers are required to teach according to this curriculum. Teaching according to these core elements is challenging (Reinhardtsen et al., 2025), and I therefore assume many teachers would want support on how to develop such teaching. Consequently, the timing for teacher specialists to take on a leading role among colleagues is good.

Planning phase for intervention

Since I know how challenging it can be to be an effective change agent at your own school, I wanted to initiate some changes while the teacher specialist students had support both from me and their fellow students. Therefore, I designed a course where they not only learned about Lesson Study as a way of professional development, but they also had a compulsory task to lead colleagues in cycles of Lesson Study in mathematics at their workplace. While many courses in teacher education introduce Lesson Study as a professional development method and challenge their students to test it out

(Munthe et al., 2015), I have not heard of any other course challenging students to both learn about the method and lead their colleagues at the same time, before they have had any experience themselves with Lesson Study.

I knew that requiring the Teacher Specialist students to lead Lesson Study meant that I would have to be pragmatic on how it was done. At the university, the students learned about key aspects of Lesson Study and that the method is about systematizing teachers learning (Munthe et al., 2015). The first stage of Lesson Study is to do a GAP-analysis, that is to identify teachers need to improve their own teaching. This means it is a bottom-up method. The next stage is planning the intervention. This stage should take at least six hours to complete and I urged them to challenge their usual practice. The third stage is to conduct the intervention, and the teachers should draw about who teaches the lesson while all others have designated observation roles, observing the pupils not the teaching. The fourth stage is to discuss their observations up against their research question and how they can improve the lesson even further. The final stage, a crucial stage, is dissemination to colleagues at school.

The Teacher Specialist students are taking a course they get credits for, and the exam result is graded. I am therefore assuming they want to work for good grades, and thus need to design their assignment so they are willing to challenge their own practice and colleagues, without this resulting in a bad grade. Consequently, leading and reporting on how they conducted Lesson Study with their colleagues was a course requirement, but not relevant for their grade. In addition, I allowed them to deviate from any of the key aspects of Lesson Study, if they saw it as necessary to be able to do this with their colleagues. However, they would have to reflect upon possible benefits and losses by making such decisions.

While the written report was just a course requirement the Teacher Specialist Students received feedback on, the work they had put into it would still influence their exam grade. They had the first 5-10 minutes of their oral examination to present how they had led the Lesson Study process at their workplace, and to reflect upon the process and the choices they had made. This way, the students could use their theoretical insights and knowledge to reflect upon their own work as change agents in practice.

Action phase

Most of the Teacher Specialist students reacted positively when they learned about how they were expected to lead professional development at their own workplace, with a few exceptions. However, some of the students quickly made me aware that it might be challenging to get their school principal to set aside time for them and their colleagues to complete such a task. We therefore discussed how this best could be solved, and we agreed that I should formulate a letter to the principal. This letter was formulated in the 'Norwegian spirit', meaning that I did not demand the principal to give time for this. Instead I thanked them for allowing an employee and the school to commit to the Teacher Specialist education, expressed an understanding of tight schedules in schools, explained how Lesson Study could be beneficial to implementing the new curriculum, including references to The Norwegian Directorate for Education and Training emphasizing this method , and carefully reminded them that Teacher Specialist education would have course requirements that needed to be done in

schools. In Norway, it is mostly more efficient to let the principal feel they have a choice, but to argue so well and convincing that they see it as beneficial to support the Teacher Specialist student.

After I had written the letter, I sent it to the Teacher Specialist students so they could give feedback on it, and whether they believed it would be sufficient. For some of the students, I added text to address individual situations at their school. Whether the students used the letter was voluntarily, and not all students felt the need. However, all students now managed to get enough time and participating colleagues at their workplace to complete the required assignment on leading Lesson Study at their school.

Another notable thing in the action phase, was how much time we spent discussing the key aspects of lesson study. Because I allowed them to deviate from the method, they felt a need to understand why the requirements of Lesson Study were formulated the way they were, and this made the class discussions fruitful and closely connected to practice at the students' schools.

Reflection phase

It seems that allowing the Teacher Specialist students to deviate from key aspects of Lesson Study might have made them even more aware of why these requirements are there and the purpose. Several of them commented that they would not deviate from certain key aspects again, if they were to repeat the process. For instance, requiring that everyone in the group was present when working, spending more than six hours planning, and not deciding who would teach the lesson until after it was planned, were all key aspects it seemed like the students only fully saw the potential of after having omitted them.

The letter to the principal was something I had not thought of, but I think it was beneficial on more than one level. The most obvious was giving the students time to do the assignment, but it also provided the principal more insight into what Teacher Specialist student learns, and how this can be beneficial for the whole school, thus bridging the gap between the course and practice of the school.

Discussion

The research question guiding this action research was: How can I design a professional development course for Teacher Specialists in mathematics that will make them successful change agents at their workplace? I will in the following discuss my findings with respect to the theories on change agents.

There were four main factors identified in the literature by Brown et al. (2021) that may enable or inhibit teacher leaders' change efforts. The first of these was support by the principal or school leader. This factor also includes Lorentzen's (2020) structural boundary, where the Teacher Specialists can feel anxious about falling outside the school's structure. When I designed the Teacher Specialist course, I knew that the school principals had approved the study application by the Teacher Specialist student, and I assumed they had noticed and were aware that some of the course work would take place in this school. I assumed that the principal would be positive and support the teachers, while I was more unsure about the teacher colleagues. However, the Teacher Specialists made me aware that this was not necessarily the case, and we therefore wrote the letter to the principal. In hindsight, this was probably a good way of not only allowing the Teacher Specialists time for their assignment, but to ensure that school leaders were more aware of what they learned and how they could be a resource

for the school in the future. While we at the University can anticipate and make plans for how to facilitate our students school leadership support, I have learned that this is something that should be continuously discussed with the Teacher Specialist students. It can vary a lot, but it is also possible to find solutions, like the letter, with joint effort.

The second factor described by Brown et al. (2021), is the buy-in to the role by practitioner colleagues. I would say this factor includes Lorentzen's (2020) description of the relational boundary, but the relational boundary can also be part of the fourth factor discussed below. Finding a balance between giving teachers space and autonomy and giving them guidance and advice, is a difficulty for Teacher Specialist students, in all countries and especially in Norway with an egalitarian organisational structure. If the Teacher Specialist students are to be successful, they need to find this balance. I had chosen Lesson Study as a method for professional development with this in mind, because it is centred around the idea that it is the teachers that need to identify what they need to learn and further develop. Consequently, I am not asking the Teacher specialists to implement a top-down reform, but giving them a developmental frame they can use to support their colleagues in a bottom-up professional development at their school. This seemed to be a successful approach, given that several of the initiatives started with this Lesson Study continued in various ways. The Teacher Specialist students found colleagues to work with and got them interested, and this was reported both on the oral exams, but also further developed and researched through master projects.

The third factor Brown et al. (2021) draw attention to is access to training and professional development. This factor is also relatable to Lorentzen's (2020) knowledge boundary, although the Teacher Specialists are more worried about whether their colleagues view their knowledge as useful. Since the Teacher Specialist education is given as a professional development where the students simultaneously are working as teachers, this provides opportunities for access to training and professional development while they are acting as change agents. Experiences from this course also emphasise the importance of challenging the students to act as change agents while they are still doing the course, thus giving more opportunities to solve issues from practice with the support of fellow students and me as an educator.

Perceived autonomy and teacher leaders' own positioning in the role is the fourth factor Brown et al. (2021) point to. This factor is also relevant all of Lorentzen's (2020) boundaries. The system in Norway and the egalitarian organisational structure makes this factor even more important, because the Teacher Specialists need to take an active role in defining their position among colleagues. This was something we concurrently discussed throughout the course, and it was important to the Teacher Specialist students to both have these discussions, but also to have the autonomy to define this role back at their school.

Conclusion

When giving a professional development course where one of the aims is to empower the students to become effective change agents at their school, we need to address more than just the theoretical perspectives. We need to find ways of working that can provide support for the students with respect to all factors listed by Brown et al. (2021). While we as mathematics teacher educators can anticipate and make adjustments with respect to theories and research, we also need to include the students if

we want to be successful in finding ways where they can make sure they get support from their school leaders and colleagues. This is especially so if we want a self-preserving environment of bottom-up development in mathematics at schools. While this research demonstrates how all factors that might enable or inhibit teacher leaders' change efforts need to be addressed in such a course, most notably, collectively we found ways to successfully get support by the principal or school leader to ensure bottom-up changes in school.

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Exploring mathematics teacher educators' pedagogical content knowledge: a case study

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This study is situated within the research line of Mathematics Teacher Educators (MTEs). Specifically, it aims to characterize aspects of the pedagogical content knowledge (PCK) used by an educator in their practice. To achieve this goal, the work begins with a literature review that presents various perspectives on the PCK required by an MTE. It then details the research design, which is characterized by adopting a qualitative-interpretative approach through a single case study, with the context being a pre-service teacher program for secondary school mathematics teachers. The results highlight three components of the MTE's knowledge that supported his practices. We hope that this paper contributes to a better understanding of the knowledge required by MTEs and their practices.

Keywords: Mathematics teacher educators, professional knowledge, teacher education, case studies.

Introduction

In recent decades, the research community has sought to understand the processes of teaching mathematics from a perspective focused on students' learning outcomes, with the teacher being a key actor. However, as noted by Krainer et al. (2021), attention has shifted towards the teacher, giving rise to a new research field: teacher education, in which teacher educators play an essential role. This paper is situated within the research line on teacher educators' knowledge and aims to characterize the pedagogical content knowledge (PCK) employed by a secondary mathematics teacher educator while delivering a pre-service teacher training course.

In this regard, research finds that the community of mathematics teacher educators (MTEs) is heterogeneous in terms of profiles, backgrounds, and training contexts (both pre-service and in-service). This diversity implies that there are multiple perspectives on what should be taught, the goals to be achieved, the resources used, and the challenges faced by MTEs (Li & Castro-Superfine, 2018). Consequently, there are various perspectives (Castro-Superfine et al., 2020; Peters-Dasdemir et al., 2023) attempting to characterize the knowledge required by an MTE, although there is a consensus that both the content knowledge for teacher education (MTECK) and the pedagogical content knowledge for teacher education (MTEPCK) are considered two relevant dimensions.

Thus, the research question we propose is: what knowledge does an MTE mobilize when teaching pre-service teachers? More specifically, and for the sake of brevity, we will focus on a specific dimension of PCK referred to as knowledge of the features of teachers' professional development (Escudero-Ávila et al., 2021).

Literature review

Research on the knowledge required by MTEs remains in its nascent stages, but there is a consensus that the knowledge necessary for teachers to effectively practice their profession constitutes a

foundational component or an essential dimension of a prospective framework or model for teacher educator knowledge (Chapman, 2021).

The role of a teacher educator is to provide support or guide to teachers within the context of adult education, which requires specific knowledge and competencies. In the case of MTEs, given that mathematics teaching involves highly specialized domain-specific knowledge, it is crucial to define the knowledge required by a MTE (Peters-Dasdemir et al., 2023). In this regard, Pérez-Montilla & Cardeñoso (2023) analyzed several frameworks related to MTE's knowledge, concluding the content they teach differs from that taught by mathematics teachers, and their audience is not schoolchildren, but teachers in training or in service. From this perspective, MTEs require a specific PCK that helps them "represent the subject in such a way as to make it accessible to their students" (Escudero-Ávila et al., 2021, p. 31). Therefore, and through a reinterpretation of Shulman's (1987) work considering the professional duties of MTEs, MTEPCK could be organized around three dimensions: knowledge of teaching teachers, knowledge of teachers' professional learning, and knowledge of the syllabus, with this study focusing on the second dimension (Escudero-Ávila et al., 2021). In this study we will focus on MTE'S knowledge of teachers' professional learning.

In their work, Escudero-Ávila et al. (2021) refer to knowledge of teachers' professional learning as *features of the professional development of mathematics teachers* (which we abbreviate as KFPD). This knowledge provides MTEs with insights into the evolving nature of teachers' learning processes and how these processes can be characterized. The authors also address the challenges teachers face during the specialization of their knowledge, the prior knowledge and preconceived notions they bring, and how these influence their learning. Similarly, scholars such as Prediger et al. (2022) use the term *knowledge of teachers' typical difficulties* to describe a facilitator's awareness of how teachers' preconceptions or prior knowledge may impede the attainment of professional development goals. On the other hand, Leikin et al. (2018) introduce another element of knowledge related to how MTEs (in their study, professional mathematicians) demonstrated *sensitivity* to mathematics teachers by addressing psychological aspects integrated into their courses, encompassing both cognitive and affective components. Additionally, *knowledge of teachers' typical content-specific learning pathways*, as introduced by Prediger et al. (2022), would provide MTEs with critical information regarding key aspects to focus on. This would also enable them to infer the stage at which teachers are in relation to a specific content area within a professional development program.

In any event, all the previously described approaches underscore the necessity for MTEs to possess a distinct type of knowledge compared to that of classroom teachers. This distinction arises from the unique characteristics of both the content being taught and the specific audience being targeted.

Design research

In this section, we describe the participant in our study, the context, the types of data collected, and the methodology of the analysis. This study is part of a larger ongoing investigation aimed at characterizing the professional knowledge of a mathematics teacher educator within the context of pre-service secondary teacher education.

Participant

Our participant, hereafter referred to as José, is an active secondary school teacher with over 20 years of experience and has been serving as a teacher educator for the past 5 years. He possesses extensive experience in delivering professional development courses to in-service teachers and has recently embarked on research in mathematics education. José was selected based on criteria of availability and intentionality, due to his significant professional background in teacher training and his dual role as both a teacher and educator.

Context

Our research was conducted during the first six sessions of a pre-service secondary teacher training program delivered (with 20 PSTs enrolled) at a Spanish university, due to resource constraints. The focus was specifically on the course *Learning and Teaching in the Mathematics Specialization* within the master's degree in Secondary Teacher Education—an essential qualification in Spain for becoming a certified teacher-. Each session lasted 4 hours, and the content of each session covered during the research is detailed in Table 1.

Table 1: Session schedule for the curriculum and learning analysis module

Session 1	Purposes of mathematics education	Session 4	Analysis of the Spanish curriculum III: specific mathematical competencies
Session 2	Analysis of the Spanish curriculum I: general overview	Session 5	Mathematical competencies: perspectives from PISA and Niss & Højgaard (2019) (design of a task analysis instrument)
Session 3	Analysis of the Spanish Curriculum II: general ideas on competencies	Session 6	Mathematical competencies: perspectives from PISA and Niss & Højgaard (2019) (design of tasks to develop competencies)

Information gathering

The sessions were recorded in both audio and video, and the audio recordings also captured the collective discussions that took place in small groups during these sessions. Additionally, a summary of the main learning objectives was provided by José to the researchers before each session. The data were segmented by selecting episodes whose content focused on the development of the PSTs' professional identity, their professional knowledge for teaching mathematics or effective teaching practices. This selection was based on the framework proposed by Escudero-Ávila et al. (2021), which identifies these three dimensions as essential components of the knowledge that an MTE should possess in relation to the content to be taught in a mathematics teacher education program.

Method

This study is conducted within an interpretive paradigm and adopts a qualitative approach through a single-case study design (Bassey, 2003), and the method for data analysis is of the top-down/bottom-up type (Grbich, 2003). The phenomenon to be studied is to characterize the PCK that informs an educator about how pre-service teachers learn during a PD program which we have denoted as KFPD.

We began the top-down process building on Prediger et al. (2022) components: knowledge about teachers' typical difficulties and knowledge of teachers' typical learning pathways considered as elements of KFPD and described previously. The bottom-up approach focused on systematically organizing and integrating emerging data into these predefined categories, with the goal of connecting the available information to the relevant theoretical concepts. At the end of the two processes, the initial category system was refined including a new component: knowledge about emotional and affective aspects of pre-service teachers' professional development. The credibility of the findings was ensured through member validation (Bryman, 2001). In the first stage, one member of the research group coded the information contained in the recordings. Subsequently, this coding was presented to the other members to gather their perspectives. Discrepancies were discussed, and consensus was reached by repeating the analysis, which is the one presented in this work.

Results

José's interventions have been coded as MTE_X-S, where X indicates the episode and S the order of the intervention, while the pre-service teachers (PTs) are identified as PT_X-M-S, with M representing a unique identifier for each PT.

Knowledge about teachers' typical difficulties

Episode 1 was selected from session 5 because it illustrates the role that knowledge about teachers' typical difficulties plays in the management of collective discussions, specifically in making sense of the interventions of the PTs. Segment MTE_1-2 shows a moment in which our participant counterposes PT's preconceptions revealed in PT_1-3-2 regarding the use of counterexamples in teaching, questioning and challenging to help them.

1	PT_1-3-1:	If I give the students $\sqrt{25}$ and then $\sqrt{9}+\sqrt{16}$, the result is clearly different. End of the problem, I mean, they'll understand that they can't do that anymore [add the radicands].
2	MTE_1-1:	So, you're trying to tell me that with the strength of your argument you're going to achieve… (smiles) I'm not sure how to say it…
3	PT_1-3-2:	What I'm trying to say is that counterexamples help to [see] that it's not a general rule. Therefore, if they're working abstractly or algebraically rather than numerically, they can't rely on those preconceived ideas.
4	MTE_1-2:	Keep in mind that the concept of a counterexample that a mathematician has, they don't [have it]… right? Well, okay, so the example you gave me doesn't work, but who's to say it's only that one? And you, as a teacher, know it. It's not that simple because the notion of a counterexample, like other mathematical proof techniques, has a certain level of sophistication, right? They don't necessarily have that. What's clear is that you need to suggest the correct answer; you can give an example or a counterexample. But that doesn't guarantee that the problem will be solved.

Episode 2, extracted from a later moment in session 5, illustrates how knowledge about teachers' typical difficulties enables our participant to anticipate the specific challenges that PTs face, which must be addressed. In this case, José highlights the difficulty of distinguishing which mathematical competency a student is drawing upon, due to the overlap of the skills involved and their supportive role.

5	MTE_2_1:	[…] Communicating with mathematics and about mathematics. Here, we understand that, on one hand, the pupil is able to establish communication

with mathematics. They need to understand texts in mathematical language, which can be visual or algebraic, right? For example, here we have two representations [points to the ppt] of the Pythagorean theorem, and they are two different ways of understanding the same concept. Being able to grasp that duality is crucial, and obviously, you can connect it with previous competencies, which often makes it difficult to draw the line between them, right? One competency influences another…

Knowledge of teachers' typical learning pathways

The third episode, selected from session 6, illustrates how José is aware of the different ways in which PTs reason, act, and react when selecting technological resources for the tasks they design. This knowledge enables to our educator to identify relevant focal points that can trigger moments of PD.

6	MTE_3-1:	But when you tell me that you've focused more on the IT aspect…
7	PT_3-4-1:	We're going to have a problem-solving session, and the next day [...] all those problems they did by hand the day before [...] (interrupted)
8	MTE_3-2:	Let me see if I understand you. You're supposed to have been working on... (interrupted)
9	PT_3-4:	The topic, the content...
10	MTE_3-3:	And now you're going to use a technological tool—what for?
11	PT_3-4:	To solve the problems and verify that the solution is correct. More than just checking, so they also learn how to use the computer to solve those same problems with different software.
12	MTE_3-4:	Okay. What's interesting, in my opinion, would be to say, alright, that seems like a good approach, […] How could the inclusion of that tool not only serve to tell me if the solution is correct or not, but also how could that tool help me pose questions that I wouldn't ask if I didn't have that tool [...]?

In episode 4 (extracted from session 1), José selects excerpts from the reports written by groups of PTs, around which he organizes whole-group discussions. Knowledge of teachers' typical learning pathways helps our participant to identify key areas that PTs need to explore and focus on.

13	MTE_4-1:	[…] You wrote down: incorporation of tools to assist with calculations. So, I understand from what you've said that you don't want to abandon calculations. Calculations must be present [in teaching], but somehow, they need to be supported.
14	PT_4-5-1:	A middle ground. I think it's also important to know how to do calculations by hand because otherwise, you don't fully understand... You don't completely grasp what's necessary for the computer to do in the background. That's what I believe.
15	MTE_4-2:	So, if one isn't knowing what the machine is doing, are we missing something?
16	PT_4-5-2:	It's not quite that, but I think it's important... I mean, if you're in a technology or engineering course and you need to apply mathematics and having a computer is great because otherwise it would be impossible, right? But for someone who is going to dedicate themselves to mathematics not to know how the computer works until they start working in the field is, I think, unfortunate. […]

Knowledge about emotional and affective aspects of teachers' professional development

During episode 5, which took place in session 6, José showed interest in a mathematical task developed by a small group of PTs. The task aimed to enhance mathematical competencies (Niss & Højgaard, 2019). This excerpt illustrates how knowledge of the emotional and affective aspects of

teachers' professional development was useful when providing feedback to PTs. It highlighted areas for improvement while also recognizing their efforts and strengths.

17 MTE_5-1: Okay, but let's not forget that what we want is for the students to become competent in some way. I'm not sure what specific competencies you're aiming to develop.

18 PT_5-2-1: It's because the activity we're considering involves us proposing real-life problems, like "If Juan and Pedro together have 5 euros, but the difference between them is 2 euros, how much money does each have?" We want them to model it. This way, we already have a competency.

19 MTE_5-2: […] But I think it's a good approach, understand me. However, you need to come up with something that has a bit more substance. I find the approach interesting, and I also think it's great point that you're using ICTs, even to critique them.

Finally, episode 6, extracted from session 1, places José in a group discussion. He shared a report with the whole group that had been prepared by a small group of PTs. This excerpt is illustrative because it shows how José draws on his knowledge of the emotional and affective aspects of teachers' professional development. During the public reading, he focused on the strengths of the report, drawing on them to introduce clarifications. His approach was accompanied by positive comments, which helped reinforce the PTs' confidence and encourage further development.

20 MTE_6-1: "Using mathematics to develop logic." Well, I find this interesting. The idea of having an exchange of ideas, right? For someone to develop their ability to think and critique, right? Very good.

Discussion

The aim of this study was to investigate the KFPD (Knowledge of the Features of Professional Development) mobilized by an educator in practice. Three elements of KFPD were identified. Firstly, José's knowledge of PTs' typical difficulties proved useful in managing collective discussions (episode 1). Specifically, it helped him understand PTs' preconceptions and how these can be an obstacle in their PD which led our participant to challenge PT-3's ideas about counterexamples. Therefore, we hypothesize that this knowledge would inform MTEs about when to introduce facilitation moves to sustain a focus on specific aspects of teaching related to students' thinking and reasoning (Gibbons & Nieman, 2024) and support productive discussions. In addition, our findings (see episode 2) suggest that this knowledge would have guided José in determining the most appropriate moments to introduce clarifying comments on aspects that could potentially be problematic, that is, it would support the practice of anticipating potential difficulties for teachers.

In another context, our findings regarding José's knowledge of PTs' typical learning pathways showed that our educator was aware of PTs' typical use of ICT resources in teaching in episode 3. In this respect, we hypothesize that José knew how PTs use teaching resources when designing a mathematical task, tailoring his feedback to challenge their preconceptions. As a result, we share with Prediger et al. (2022) the view that this knowledge is key to reacting in participant- and goal-oriented ways since it helps facilitators and educators prompt opportunities for PD.

Finally, José's awareness of the emotional and affective aspects of his pre-service teachers significantly supported his management of group discussions. He created comfortable environments that foster the exchange of ideas and practices, ultimately establishing a learning community (Ferreira

et al., 2024). Therefore, this element of knowledge could help MTEs in their practice by making it clear when it is appropriate to praise positive aspects of PTs' contributions, demonstrating their sensitivity to PTs' emotions. In line with Prediger et al. (2022), our results point out that this sensitivity is crucial, as it helps MTEs know when to prioritize atmospheric goals to maintain PTs' engagement.

Conclusions

Our empirical findings contributed to a deeper understanding of the knowledge required by an educator in practice. The findings presented suggest that an MTE's expertise requires multiple knowledge resources to navigate between different content goals, to notice what PTs can already bring in, and to build upon the fruitful starting points to advance their learning pathways (Prediger et al. 2022). However, several limitations must be highlighted. The analysed episodes present only very small extracts, which is why we consider it necessary to include other sources of information, such as semi-structured interviews, that capture the participant's reflections, allowing us to triangulate the descriptions made. Nevertheless, we hope this study will serve as a valuable contribution to the research community in understanding the complexity of MTEs' professional knowledge and practices.

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Case Study on Mathematics Teacher Educator's Professional Roles During Teaching Practicum Course

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While the role of teacher education is considered as a critical element for the education of prospective teachers, there is less known about teacher educators. Teacher educators are individuals who support the learning of both teachers and prospective teachers. The importance of teaching courses in the education of prospective teachers such as teaching practicum is well known. According to the literature on teacher educators, a person's classification as a teacher educator depends on the tasks and responsibilities that come with the profession and six professional roles for teacher educators were determined accordingly. Therefore, the study aimed to understand how these professional roles function in the context of teaching practicum by examining the professional roles of mathematics teacher educators. As a result, it was supported by the findings that the professional roles of teacher educators can be explained in accordance with the existing literature. However, findings showed that there are other aspects which need to be investigated in the context of teaching practicum.

Keywords: Teacher education, mathematics teacher educator, professional roles, teaching practicum.

Introduction

There is a growing, but still limited, amount of information regarding teacher educators as the role of teacher education is becoming more recognized as an important tool for the education of prospective teachers (Swennen et al., 2010). The European Commission (2013) defines teacher educators as anyone who constantly supports teachers and student teachers' formal learning. In Türkiye, teacher educators who hold a PhD in mathematics education or mathematics offer teaching courses at teaching departments. According to the comprehensive literature review conducted by Lunenberg et al. (2014) on the professional roles of teacher educators, the categorization of individuals as teacher educators is contingent upon the tasks and responsibilities inherent in this profession. One of teacher educators' responsibilities at institutions is offering teaching practicum courses, and it is well known that the teaching practicum plays a pivotal role in the education of pre-service teachers. Consequently, the way a teacher educator facilitates this course holds significance in shaping the learning experiences of future teachers. Therefore, understanding how teacher educators offer this course, focusing on the tasks and hence the roles they adopt may shed light on this essential part of teacher education. This qualitative study aims to investigate the professional roles of mathematics teacher educators' (MTE) in the context of teaching practicum. The study mainly describes MTE's professional roles who offers teaching practicum course in the field of mathematics teacher education in the lenses of six professional roles explained by Lunenberg et al. (2014) as: *teacher of teachers*, *researcher, coach, curriculum developer, gatekeeper* and *broker*. The study seeks to address the gap in understanding the context-specific professional roles of mathematics teacher educators, particularly within practicum courses. By focusing on how these roles manifest in practice, the

research aims to provide deeper insights into the interplay between professional roles of MTE's and teaching practicum course. Moreover, study aims to empirically investigate these professional roles that are theoretically given.

Teacher Educator

The occupational group of teacher educators are broad. Therefore, researchers claim that the definition of a teacher educator is uncertain to some extent (Hamilton et al., 2016; Livingston, 2014; Swennen et al., 2009). According to the European Commission (2013), teacher educators are "all those who actively facilitate the (formal) learning of student teachers and teacher educators" (p.8). Murray (2016) argues that teacher educators stand out by supporting students and assisting teachers in improving their practice. She also views them as second-order practitioners, responsible for teaching first-order practitioners. Therefore different from teachers, the role of teacher educators necessitates the acquisition of new and distinct forms of professional knowledge and understanding, as well as expanded and varied pedagogical abilities (Korthagen, Loughran & Lunenberg, 2005; Loughran, 2006). Lunenberg et al. (2014) defined the term teacher educator broadly and considered the ones that are instructing and guiding pre-service and in-service teachers in schools and in institutions of teacher education as per the definition of European Commission (2013). The term "teacher educator" has been defined in various ways, however, for the case of Türkiye, the current research focuses on university-based mathematics teacher educators specializing in mathematics education within the Faculty of Education.

Professional Roles of Teacher Educators

Lunenberg et al. (2014) identified six distinct professional roles undertaken by teacher educators in their analysis of 137 articles across diverse countries. The objective of their review study was to establish a comprehensive understanding of characteristics of professional roles of teacher educators and behaviours related to those roles and the teacher educators' professional development in relation to these roles and the behaviours that go along with them. They identified the professional roles as *teacher of teachers, researcher, coach, curriculum developer, broker, and gatekeeper*.

Majority of the research has focused on the role *Teacher of Teachers.* Murray and Male (2005) have made a widely recognized differentiation between *first-order teaching* and *second-order teaching*. A particular teacher education pedagogy is needed for second order teaching, and according to Loughran and Berry (2005) key components of this pedagogy are *modelling* in other words, practicing what is preached and explicating. To promote the development of prospective teachers, this entails teacher educators being able to reflect on and express their own actions and feeling. There is a notable increase in recognition of the teacher educator's role as a *researcher.* The consensus that teacher educators should be involved in research is becoming more widespread globally, extending beyond traditional research universities (Murray et al., 2011). However, teacher educators currently lack agreement on whether they should take on the role of a researcher, and if so, there is uncertainty about the specific responsibilities associated with this role in addition to the lack of consensus on the focus of such research (Lunenberg et al., 2014). The third role identified in the literature review is the role of *Coach.* This term is frequently employed interchangeably with other terms such as facilitator,

mentor, cooperating teacher, and school-based teacher educator. However, Lunenberg et al. (2014) pointed out that there is no universally agreed-upon definition for coaching. From the studies related to school practice in educational institutions, they have defined the main purpose of a coach as the individual who fosters the educational growth of the pre-service teacher. There is relatively less emphasis on the fourth professional role, *Curriculum Developer*, in the existing literature. The preliminary literature search of Lunenberg et al. (2014) shows that there is a considerable body of research dedicated to the development of teacher education curricula. However, according to these studies, researchers discovered that a small number of them specifically examine the teacher educator as the subject of study in their role as curriculum developers. *Gatekeeper* is the fifth professional role that researchers determined which relates to the duty of teacher educators in facilitating the transition of student teachers into the teaching profession. Teacher educator plays a crucial role in guiding and facilitating the entry of students into the teaching profession by providing them with the necessary knowledge, skills, and insights. The sixth professional role is the *Broker.* Lunenberg et al. (2014) mentioned that previously, the collaboration between a school and a teacher education institution typically involved interaction among an institution-based teacher educator, a mentor, and a pre-service teacher. The primary objective of this collaboration was primarily focused on the mentorship and guidance of the pre-service teacher throughout the teaching practicum course.

Methods

Design and the participant

This research is conducted as a case study which is a qualitative method where the researcher investigates a bounded system (referred to as a case) in real-life and contemporary settings (Yin, 2009). This can involve exploring a single bounded system or multiple such systems over an extended period. A university-based mathematics teacher educator was selected as a participant to examine how the professional roles of mathematics teacher educators occurred during teaching practicum. Participant of this study is Yasmin (pseudonym name) who has a PhD in mathematics education, and she has been offering the teaching practicum course for 14 years. A year before this study, I had been observing her classes for exploratory purposes. It was her classes that prompted my decision to concentrate on the professional roles of teacher educators.

Data collection

Data was collected through researcher's notes from classroom observations and an individual interview that was conducted with Yasmin. Firstly, I observed her teaching practicum classes and took field notes for one semester. Furthermore, I have joined Yasmin while observing the students' teaching at practicum schools including the feedback sessions after their teaching. Third, I conducted an individual interview via Zoom at the end of the semester to gain data on Yasmin's views on professional roles. The questions were drawn on the methodological framework offered by Lunenberg et al. (2014).

Data Analysis

The classroom observations of 11 lessons were analysed using an open coding strategy. Each information unit is labelled with a code that represents the essence of the data pieces. The open codes are compared to indicators from the Professional Role framework. The individual interview about the professional roles was also analysed through the framework, and interview data was interpreted in conjunction with classroom observations to familiarize both data sets for triangulation. In the analysis, for instance, the open code 'Developing the teacher education program' aligns with the indicator 'Curriculum development in practice' under the curriculum developer role. However, open codes such as 'Some pre-service teachers still struggle to explain mathematical big ideas' do not align directly with any predefined indicator, although they fit within the gatekeeper role. In such cases, new indicators were identified in accordance with the specific context of the practicum**.**

Findings

The analysis of classroom observations shows that Yasmin extensively exemplifies the professional role of a *teacher of teachers* in addition to the role of *coach*, *curriculum developer, gatekeeper and broker* as a mathematics teacher educator. However, the role of *researcher* could not be observed during the classroom observations. On the other hand, the analysis of interview questions shows that all roles including the *researcher* role can be observed in MTE's practices in teaching practicum course and interview data. Below, the findings for each role will be discussed separately by combining data from observations and interview.

Teacher of teachers

The role of teacher of teachers consisted of four main indicators: *(1) second order teaching; (2) promotion of active learning; (3) modelling;* and *(4) dealing with tensions and dilemmas.* The ability to work with adult learners in higher education and the ability to articulate tacit knowledge and underlying theory are features critical to second order teaching. Most of the open codes were associated with second order teaching. According to the open codes from the classroom observations such as "Asks questions to candidates" and "Forces them to think about definitions", it can be said that Yasmin was working effectively with adult learners and contributing to their reflection, development of conceptual understanding, and professional learning.

The data show that Yasmin concretizes articulating knowledge and underlying theory by explaining theoretical concepts such as cognitive demand of a mathematical task to students. In addition, statements such as "Big idea settled in the students' minds" and "Big idea is explained in the lesson plans" show that the teacher candidates develop the ability to include these theoretical concepts in their lesson plans. As a result, Yasmin contributes to the candidates' understanding and application of theory by expressing implicit knowledge explicitly. The open code "Yasmin gave time to prospective teachers to think" indicates that her directing prospective teachers to think and analyse their own strengths and weaknesses. This approach supports active learning and helps prospective teachers take responsibility for their learning. During the modelling process of the Yasmin, it is clearly seen that the advice she gave to the teacher candidates coincides with her own teaching style. For example, the codes "Have a plan B for teaching" and "The content is not the only important thing

in the lesson plan" reveal that the educator tried to instil in the candidates the flexibility and attention she showed in her own teaching. This shows that what she does and what she teaches are in harmony. Interview data was aligned with how the teacher educator lives the role of "teacher of teachers" and practices various aspects of this role. She determines herself as more of a teacher educator in teaching practicum course. In the interview, her statement, "Teacher educator research is my field" reflects her expertise in the field of research. These codes show that the educator takes a comprehensive approach to developing teacher candidates based on her own experiences.

Researcher

The researcher role included three main indicators: (1) Views of the role of the researcher; (2) the practical elaboration of the role of researcher; and (3) the focus of the research. The role of researcher could not be observed in classroom observations. This may be due to the structure of the teaching practicum course, since the course heavily includes practical orientations of teaching rather than theoretical knowledge. Moreover, it may be due to observer's inability to comprehend the underlying theory of Yasmin's decision-making processes. So, this was one of the reasons for having an interview with Yasmin. The interview showed that Yasmin emphasizes her research perspective with the phrases "My teaching and research" and "Teacher education is my research area." In addition, where she says "I also know from literature" how she integrates her existing knowledge.

Coach

The role of coach was explained under two main indicators: *(1) the task of workplace facilitators* and *(2) The task of institution-based coaches.* The identified behaviours of *the task of workplace facilitators* have been described with the behaviours as: offering local knowledge, practical orientation: giving advice about curriculum and classroom practice and promoting reflection aimed at transfer. Based on the frequency of these behaviours, it can be concluded that Yasmin's notable strengths in the role of a coach lie in offering local knowledge and promoting reflection aimed at transfer. For example, the codes "Provides students with detailed information about their school" and "Considers school culture" indicate that Yasmin guides teacher candidates by considering the socio-cultural characteristics of the school. This knowledge helps candidates better adapt to the school environment. Yasmin mentioning "How should prospective teachers dress" and "Classroom management" indicate that she provides concrete advice to student teachers about professional behaviour and classroom practices. "Preservice teachers shared their observations" and "Differences in how teachers conduct their lessons" indicate that preservice teachers observed different teaching approaches and discussed how they could transfer them to their own teaching practice. This type of reflection helps preservice teachers integrate practical knowledge with theoretical knowledge which supports "Promoting reflection aimed at transfer". The features of the behaviours related to the task of institution-based coaches are the caring therapist/belief and attitude therapist, which was observed in classroom observations, and fading boundaries: overlapping tasks, which unfortunately could not be observed. This behaviour refers to the role of providing support to teacher candidates and encouraging their professional development. The interview data revealed that coaching role means providing customized and professional support to teacher candidates to Yasmin. As she mentioned,

"I felt very close to coaching" and "The definition in the internship course may be different" indicate that the coaching process includes both academic and practical aspects to her.

Curriculum developer

The role of curriculum developer is investigated under one main category: *variety of approaches and practices.* Behaviours of the related role were determined as: societal discussions, principles guiding teacher education and curriculum development in practice. It can be asserted that societal discussion stands out as a prominent aspect within the role of the curriculum developer. Yasmin supports the development of teacher candidates by making comparisons on national and international curricula. Observations emphasize that teacher candidates should exhibit a skill-based approach in their lessons, and that technology integration, classroom management and the use of mathematical language are important for the development of these skills. While Yasmin provides guidance on the basic principles of the curriculum, she also attaches importance to reflective learning processes. The interview data shows that the role of a mathematics teacher educator as a curriculum developer involves various levels of experience and knowledge. Codes such as "being able to discuss IB-AP-MEB curriculum differences" and "figuring out the Big Ideas of learning objectives" reveal that the Yasmin develops a deep understanding by comparing international and local curricula. In addition, the expressions "developing the teacher education program" and "knowing the substructures of other courses" supports the “curriculum development in practice” behaviour, which was previously not observed during the classroom observations.

Gatekeeper

The variety of approaches and practices of the role of gatekeeper are: *standards and profiles, portfolios, and assessment of school practices.* As the assessment of school practices was observed, behaviours linked to standards, profiles, and portfolios were not observed in the classroom observations. The "gatekeeper" role of the teacher educator becomes particularly evident in the process of providing feedback on lesson plans and guiding teacher candidates in the right direction. Yasmin organized feedback meetings with the candidates, evaluated the candidates' lesson plans and focused on the big idea. For example, in one meeting, the candidate's relaxed attitude and mastery of the subject during the lesson were praised, while the points that needed to be developed were conveyed in a sweet manner. In this process, feedback on the candidates' mathematical content knowledge and pedagogical content knowledge played an important role. The gatekeeper role emerges more clearly and explicitly in the oral interview. Yasmin stated that the responsibility for measuring the candidates' competencies was her responsibility and she took care to be fair in the evaluation process. Making decisions about which schools the candidates could do their internships and the impact of this process on the candidates' future professional lives were emphasized as an important part of this role.

Broker

The broker, identified as the final professional role of teacher educators, was classified into two main categories: *(1) goals of cooperation and (2) competencies of the broker*. The goals of cooperation were further delineated into two behaviours: the student as the central focus and becoming a

collective. Consequently, it can be asserted that becoming a collective stands out prominently in the role of the broker. Observations reveal that the "Student as the central focus" behaviour emphasizes the importance of teacher candidates establishing effective communication with their mentors and receiving feedback on their lesson plans during the internship process. The student's collaboration with the mentor and developing joint projects increases the quality of their educational experiences. In addition, co-teaching practices and mentor-student interaction play a critical role in strengthening an approach where the student is kept at the centre. On the other hand, the "Becoming a collective" behaviour reveals the importance of cooperation and a sense of belonging between the university, teacher educator and the internship school. The close contact of mentors and teacher educators with students, acting in cooperation and encouraging a sense of belonging have been determined as factors that positively affect the development of interns The interview expands the scope of the "broker" role. Yasmin has close relationships with the internship schools and maintains these relationships with her personal efforts and existing connections. It is seen that she coordinates the student placement process by considering the different requests of the internship schools and offers solutions suitable for the specific demands of the school. These efforts reveal that the "broker" role is critical for optimizing the students' internship experiences. In addition, Yasmin constantly manages and tries to develop her relationships with these schools by paying attention to the requests of the internship schools. In this context, the importance of the coaching and broker role in the internship processes comes to the fore as a part of the educator identity.

Conclusion and Discussion

The findings of this study indicate that teacher educator's professional roles can be explained according to the extensive literature review by Lunenberg et al. (2014). For instance, *teacher of teachers* role was strongly observed through both the classroom setting and the interview, as being able to teach teachers which was mentioned as second-order teaching by Murray and Male (2005). These findings suggest that the role of researcher, which appeared less prominent, warrants further exploration to determine its significance in the preparation of teacher educators, and this will be a focus of my future research. The role of researcher on the other hand, became more prominent in the interview compared to classroom observations, where the MTE mentioned teacher education as her research area. This may be due to the practicum course's emphasis on practical teaching over theoretical knowledge. Therefore, a follow-up interview was conducted to gain further insights. Moreover, she highlighted that adopting teacher education research also feeds her knowledge about MTE practices. As it was mentioned to be the most important teacher educator role in the literature (Wold et al., 2011), the role of coach was strongly observed during the classroom observations and the interview as the MTE was providing; knowledge and experience about the local context of the school and concrete advice to prospective teachers about professional behaviour and classroom practices. Yasmin's role as a curriculum developer involves different levels of experience and knowledge as she informs her students about the national and international curricula. So, it can be said that she is aware of this role as it was mentioned in the study of Bouckaert and Kools (2018). However, curriculum development in practice behaviour was not seen clearly in the classroom observations which may be since the teacher educator, although taking an active role in curriculum

development, does not share these experiences in the classroom or does not see herself as an authority on this issue and prefers to only make general suggestions about what should happen. The gatekeeper role emerges more clearly in the oral conversation. Yasmin states that it is her responsibility to measure the competencies of the candidates and that she acts fairly in this process. Determining internship schools and the impact of this decision on the candidates' professional lives is an important part of this role. Lastly, the role of broker was outstanding where she oversees the student placement procedure by considering the many requests made by the internship schools and providing solutions that are appropriate for their needs, which are behaviours that are aligned with the broker definition of He An (2009). Besides, it is obvious that there are other aspects in addition to these roles, particularly in the context of teaching practicum and the case of Türkiye. These aspects will be further analysed through thematic analyses regarding the practicum and cultural differences.

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Facilitators' noticing of teachers' responses to student thinking concerning a multiplicative thinking task

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Professional development (PD) courses are crucial for the advancement of mathematics education and require facilitators to have specific expertise in navigating complex instructional scenarios. Through a situated examination, this study explores facilitators' (N = 148) noticing of a fictional PD situation concerning multiplication, particularly their use of knowledge elements. Overall, when perceiving student thinking in a fictional PD situation, facilitators demonstrated limited skills with regard to noticing the fictional PD situation. In terms of the interpretation of and decision-making related to teachers' responses to students' solutions, about half of the facilitators superficially interpreted a student's solution and focused on aspects not relevant to the situation. These findings provide insights for aligning future PD courses to better serve the distinct needs of facilitators.

Keywords: Facilitators, noticing, professional development.

Introduction

Continuing professional development (PD) for in-service teachers has been increasingly emphasised in recent years, with PD courses seen as crucial for the advancement of mathematics education, by enriching teachers' skills, and improving mathematics instruction overall (Prediger et al., 2022). Facilitators play a key role in this process, as they have a significant impact on teacher learning (e.g., Borko et al., 2011). At the same time, PD facilitation has been described as "ambitious and challenging work" (Jacobs et al., 2017, p. 3). The expertise of facilitators to engage in this work, and in particular their ability to notice during PD courses, is critical for facilitating teachers' learning. In line with van Es and Sherin's (2002) framework, noticing comprises the recognition of the significance of a situation, the establishment of connections between interactions and broader teaching principles, and using contextual knowledge to reason about interactions. Facilitator expertise in PD settings, essential for navigating complex instructional scenarios, with a focus on the mathematics involved, has been explored by Zaslavsky and Leikin (1999). Recent research, however, has often not regarded content-specific considerations, focusing instead on generic aspects that are relevant for different subjects (Prediger et al., 2022). Our study addresses this gap by focusing on content-related PD and, specifically, by investigating the knowledge elements facilitators use in their noticing of a PD situation, including their pedagogical content knowledge on the PD level (PCK-PD).

Theoretical background

The concept of teacher noticing (van Es & Sherin, 2002), underlining the identification of crucial aspects of instruction, establishing connections, and applying contextual knowledge for reasoning, is from our point of view transferable to facilitators and PD settings. This transferability is particularly

evident in the alignment of this framework with the PID model (perception, interpretation, and decision-making from Kaiser et al., 2015), which highlights situation-specific skills such as the ability to perceive events, interpret activities, and make decisions. In a PD setting, analogously to a classroom situation, perception, interpretation, and decision-making are key aspects of a diagnostic competence and capacity to skilfully facilitate and teach (Hoth et al., 2016).

As a means of considering the expertise that facilitators bring to PD settings, including of these aspects, frameworks for examining teachers' expertise (Prediger, 2019) have been lifted to the facilitator level (e.g., Karsenty et al., 2023). Prediger et al. (2022) applied a content-related framework for mathematics teacher expertise (Prediger, 2019) to the facilitator level. This framework encompasses tasks of facilitating as typical and complex situational demands associated with the facilitation of specific mathematics PD content. Facilitators handle these tasks with practices that are recurring patterns of facilitators' utterances and actions that are based on underlying knowledge elements, pedagogical tools, orientations, and situative goals. Facilitators' pedagogical content knowledge for teachers' professional development (PCK-PD) is one facet of their knowledge elements and thus underlies their noticing. Such PCK-PD consists of the knowledge the facilitators have in relation to teachers' learning. Jacobs et al. (2017) describe PCK-PD as "the ability to engage teachers in purposeful activities and conversations about those mathematical concepts and relationships and to help teachers gain a better understanding of how students are likely to approach related tasks" (p. 3). Thereby, the PCK-PD that facilitators bring is important to consider in relation to facilitators' noticing to better understand their role in initiating teachers' learning. Furthermore, in understanding facilitators' use of PCK-PD, facilitator qualification programs can be better tailored to meet the needs of facilitators and better prepare them for such complex situational demands.

Against this background and, in particular, the underpinning aspects concerning noticing and facilitator expertise, we investigate the expertise and needs of the facilitators in a mathematics PD and qualification program, and specifically, facilitators' use of PCK-PD when they notice. We thereby pursue the following research questions: *RQ1a) How do facilitators perceive a fictional PD situation involving multiplicative thinking? RQ1b) How do facilitators interpret a fictional PD situation involving multiplicative thinking? RQ1c) How do facilitators make decisions concerning a fictional PD situation involving multiplicative thinking?*

Methods

Participants, data collection and analysis

In total, 148 facilitators participated in this study. The facilitators were primary mathematics teachers involved in the *QuaMath* (Developing *Qua*lity in *Math*ematics Classrooms and Teacher PD) facilitator PD (FPD) qualification and teacher PD (TPD) program (for program details, see Prediger & Selter, 2024). At the time of the survey, the facilitators were about to begin the FPD program that encompasses the first year of the ten-year program for the qualification and professional development of mathematics facilitators and teachers. After this first year, the facilitators' PD will continue. At the same time, they start to lead the TPD of the program themselves using the materials provided. In the last three years, sixty-eight of the facilitators had experience leading 10 or more mathematics PD courses, 36 of the facilitators had experience facilitating fewer than 10 PD courses and 44 facilitators

did not provide any PD. The facilitators' teaching experience ranged from one to 39 years (M=15.7) and their facilitation experience ranged from no experience to 23 years (M=3.9).

The facilitators were provided with a situated multi-part instrument (Figure 1) prior to the beginning of the FPD program. They were first asked to consider a classroom-level task addressing multiplicative thinking: "Draw a picture that fits the task. Why does your picture fit the task?" Then they were asked to describe a typical solution that they would expect from a) a student who had understood multiplication and b) a student who had not understood multiplication (Figure 1, Part 1), thereby showing their perception of student thinking that is essential for leading PD. This first part requires the use of necessary PCK-PD for the explanations of student (mis)understanding of the multiplication task with an explicit description of this (mis)understanding with language relating to, e.g., units, equal groups, and the differentiation between multiplier and multiplicand, and by referring to students' explanations of these concepts and their ability or challenges to connect these concepts in different representations (e.g., verbal, symbolic, pictorial) for the multiplicative thinking task.

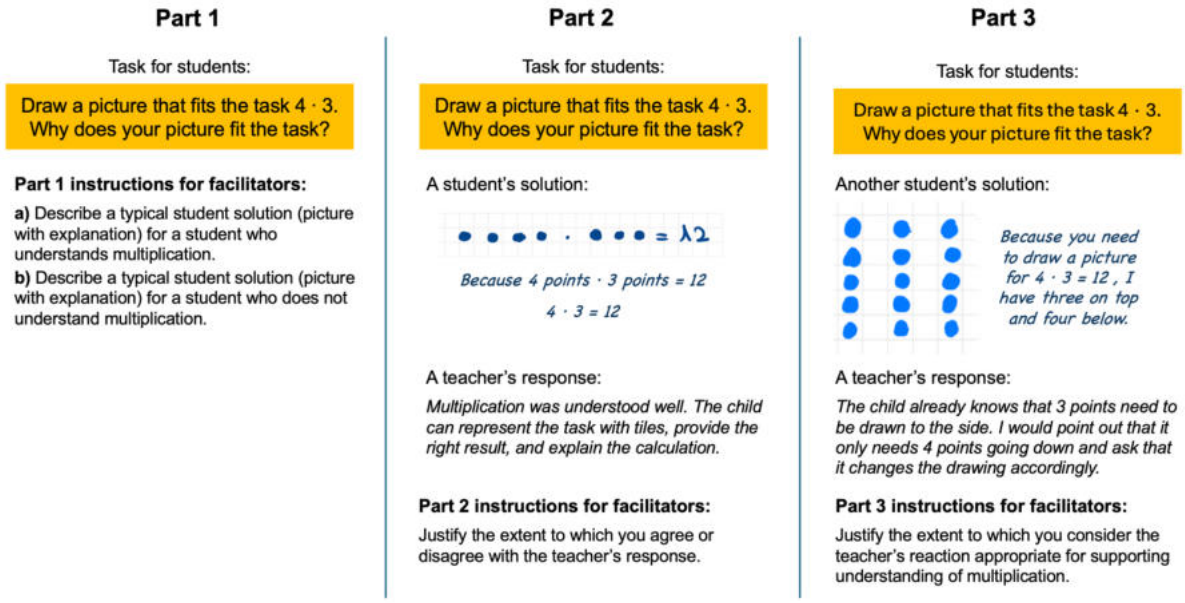

Figure 1: Situated instrument provided to facilitators for investigating their perception, interpretation, and decision-making

In the second part of the instrument (Figure 1, Part 2), the facilitators were provided with an incorrect student solution to the multiplicative thinking task in addition to a teacher's response to the student's thinking. The facilitators were asked to justify the extent to which they agree or disagree with the teacher's response to the student's solution. This part enabled the examination of the facilitators' diagnostic/interpretive knowledge elements concerning the teacher's response to the student's thinking and the facilitators' ability to interpret the student's challenges as relating to conceptual understanding, and specifically the idea of unitising, with connecting representations and with an overemphasis on procedural, automatised handling of multiplication.

For the third part of the instrument (Figure 1, Part 3), a further incorrect student solution to the task was provided along with a teacher's reaction. The facilitators were asked to justify the extent to which

they consider the teacher's reaction appropriate for supporting multiplicative thinking. This third part allowed for investigating the facilitators' decision-making regarding teachers' ideas about student learning, requiring recognition that the teacher's reaction is not constructive for learning, does not allow the student to understand why the representation is incorrect, that the teacher's response emphasises procedural knowledge and does not examine conceptual understanding.

The analysis to answer the research questions was based on the knowledge elements used for noticing from the facilitator expertise framework (Prediger et al., 2022) to reveal how the facilitators perceived, interpreted, and made decisions concerning the three parts of the situated instrument. A coding system, incorporating the main PCK-PD knowledge elements, was developed and a sum score of 22 points possible was obtained for the situated instrument, with between 0 and 12 points based on the extent to which the facilitators addressed the knowledge elements in their perception of student thinking (Part 1), 0 and 6 points for interpreting the teacher's response (Part 2), and 0 and 4 points for decision-making concerning the teacher's reaction to the student solution (Part 3). For example, a response for Part 1 with "a) understood ♦♦♦♦ · ♦♦♦♦ · ♦♦♦♦ b) ♦♦♦♦ · ♦♦♦" would score 1 point (only for Part 1b), as only a fitting solution in that part was provided. For Part 2, a response like "no understanding of 'thinking in groups', which is the basis of multiplication - the result is probably memorised - the justification does not really fit the presentation," would score 1 point for noting memorisation, 1 point for mentioning not connecting representations, and 2 points for PCK-PD detail concerning the lack of conceptual understanding. In Part 3, a response that finds the teacher's reaction unsuitable and highlights finding out why the student drew that representation and whether he or she has deeper understanding would score 3 points (1 point for each aspect). Twenty percent of the facilitators' completed instruments were coded by two members of the research team to ensure inter-rater-reliability ($\kappa = 0.88$).

Results

Considering how the facilitators noticed a PD situation addressing students' multiplicative thinking (RQ1), the mean scores for Part 1 (Perception: $M = 2.61$ ($SD = 1.27$)), Part 2 (Interpretation: $M = 1.60$ ($SD = 1.02$)) and Part 3 (Decision-making: $M = 1.48$ ($SD = 0.91$)) as well as the overall mean sum score of 5.27 ($SD = 2.26$) (out of 22 total) displayed the facilitators' need for support in perceiving and interpreting the situation as well as in making decisions while utilising the relevant knowledge elements that were embedded in the fictional PD situation.

Facilitators' perception of student thinking (RQ1a)

In terms of the facilitators' perception concerning student thinking in the fictional PD situation (Part 1), the facilitators often provided a general description of a pictorial representation of the task, did not include an explanation of the reasoning behind the representation and did not refer to a student's explanation or ability to connect different representations that fit the task. The following example that scored a total of 2 points for Part 1 demonstrates this frequently provided description of student thinking when multiplication is understood and not understood: "a) The child draws four times three flowers. b) The child draws four flowers once and three flowers next to it" (Fac._1). This type of description of a task that did not include an explanation of why the pictorial representation fits the task, was provided by 114 facilitators for Part 1a and 134 facilitators for Part 1b.

In contrast, a small number of the facilitators identified the significance that understanding multiplication encapsulates an explanation in addition to a drawing of the task and, aside from providing a description of a correct drawing of the task, referred to students' explanations of the concept (10 facilitators) or their abilities to use/connect different representations (4 facilitators). The following example demonstrates the use of the appropriate knowledge elements:

> Child who has developed multiplicative thinking: - understands 4 times 3 as four 3s - he/she could, for example, draw 4 groups of 3 dots or choose the arrangement as a rectangular field of dots with 4 rows of 3 dots each […] and is therefore flexible in the way they are presented (e.g., contextual relationships). The description would make it clear that the child gets the equal group idea with the respective power (four 3s) and can explain the tasks using different representations. b) Child who has not developed multiplicative thinking: - it draws 4 dots and next to it 3 further dots - in the explanation it would explain that with the 4 dots you find the 4 from the multiplication task and with the 3 dots you find the 3 from the multiplication task - it is therefore stuck on the purely symbolic level and only represents the single number with dots - but doesn't have a conceptualisation about the meaning of the multiplication task. (Fac._2)

For Part 1a, the above example scored 2 points for the solution, 1 point for the explanation, 1 point for connecting representations; and for Part 1 b, 1 point for the solution, 1 point for the explanation, 1 point for connecting representations. In sum, most facilitators focused on a general description of solutions to the task. A small number of facilitators provided accompanying explanations along with their descriptions of the solutions and emphasised the importance of connecting different representations while using concepts such as unitising or equal groups, as seen above.

Facilitators' interpretation of a teacher's response to student thinking (RQ1b)

Concerning the interpretation of the teacher's response to the student's thinking, the majority of the facilitators disagreed with the teacher's positive assessment of the student's solution showing four dots with a multiplication symbol and three dots as being equal to twelve. In terms of the level of detail of the facilitators' interpretations of the teacher's response, approximately half of the facilitators focused on the student's solution and interpreted superficially as reflecting a memorised solution/overemphasis on procedures. The following highlights this common interpretation, scoring 1 point: "I do not agree. The solution is not clear from the drawing. The task is not shown with tiles. It could simply have been memorised. The result is correct, but the calculation is not explained" (Fac._3). Similarly, the facilitators often went into limited detail concerning the interpretation of the student's challenges as being related to conceptual understanding (71 facilitators) or with connecting/going between different representations (53 facilitators), as exemplified by the following example that received 1 point for noting the lack of conceptual understanding: "I disagree because the drawing shows that the child has not understood multiplication. It should always draw 3 dots 4 times, only then can you read the result from the diagram" (Fac._4).

Furthermore, a large portion of the facilitators (n = 91) did not apply PCK-PD in their interpretation of the teacher's response. In contrast, a small group provided an interpretation using knowledge elements explaining the student's difficulties as being related to conceptual understanding, and

specifically difficulties with unitising (Lamon, 1994) or with multipliers/multiplicands, and with connecting representations, as displayed in the following response:

> I cannot agree with this statement. The child has not yet understood multiplication as thinking in units. There is a mixture of material and symbol here. (Multiplier 4 shown as four tiles, multiplicand 3 as 3 tiles and the drawing symbol placed in between but not as a multiplicative idea – in units.) The explanation could go in the right direction, if not misunderstood – but does not fit the picture drawn. (Fac. P_5)

In comparison to the facilitators' interpretation responses that superficially addressed or did not address PCK-PD aspects, the above response shows the facilitator's focus on the student's difficulty as being tied to not understanding the concept of unitising (2 points). The facilitator also explains the mixed use of symbolic notation and pictorial representation as being connected to the student's lack of understanding (1 point). Such interpretations were seen infrequently, in less than 10% of responses.

Facilitators' decision-making concerning a teacher's response to student thinking (RQ1c)

In making decisions concerning the teacher's response to student thinking in the third part of the situated instrument, the facilitators overwhelmingly agreed that the teacher's response was not conducive for promoting student learning. In total, only 25 of the facilitators provided a positive or neutral assessment of the teacher's response. In regard to providing reasons for their decision-making about the teacher's response, overall, almost half of the facilitators (48%) did not identify the embedded PCK-PD knowledge elements and instead focused on aspects that were not relevant to the situation, visible in the following example that scored 0 points: "This is not a representation of multiplication for me but only a drawing of dots!!!" (Fac._6). When the facilitators were able to reason using PCK-PD, the most frequently applied knowledge element emphasised letting the student be active in the learning process, (n = 44), as seen in a response that received 1 point for this knowledge element:

> To promote understanding, the child should become active and not be presented with the solution. More suitable would be: Count how many points you have. Is it 12? How many more are there? Think about how this came about? What do you have to do to get it right? What would you pay attention to next time? (Fac._7)

In addition to responses to the teacher's reaction underscoring the importance of letting the student be active in correcting the mistake, some facilitators emphasised that the student's incorrect pictorial representation was not explored by the teacher in the fictional situation (n = 29). Furthermore, some facilitators highlighted that the teacher did not investigate if the student possessed conceptual knowledge in addition to procedural knowledge as their rationale for finding the teacher's response unsuitable (n = 23). The following example scored 3 points for the application of these two knowledge elements and the correct assessment that the teacher's reaction was unsuitable:

> I think the teacher's continuation is unsuitable. Instead of just pointing out how the picture should be changed, it would be more effective to enter into a dialogue with the child. I would have the dot picture drawn again and have the child observe or explain to me what they are doing and why […] In any case, it can be said that the child's explanation is too imprecise and needs to be

reworked (together with the teacher). In my opinion, the rather weak explanation also indicates that the principle of the two-dimensional representation of multiplication was insufficiently understood. (Fac._8)

Overall, the analysis of the facilitators' decision-making highlighted the facilitators' recognition of the teacher's response not being conducive for supporting learning progress and especially emphasised their focus on enabling the student to take a more active role in the learning process and in constructing a viable conceptualisation of multiplication.

Discussion and conclusions

In summary, the facilitators were able to successfully provide general descriptions of correct pictorial representations of the multiplication task in Part 1, thereby demonstrating their perception of student thinking. A perception of the situation that used knowledge elements to connect understanding to being able to explain the task and connect/switch between different representations was not identifiable in the responses from most facilitators. While the facilitators overwhelmingly correctly interpreted the teacher's response in the second part of the situated instrument as not constructive, they tended to provide superficial details in their interpretation of the student's challenges. Finally, almost all facilitators were able to correctly assess the teacher's response in Part 3. More than half of the facilitators did not draw on PCK-PD knowledge elements in their decision-making with respect to the fictional PD situation. Instead, they focused on aspects that were not relevant to the situation or mainly focused on pushing the teacher to let the student self-correct the representation as the key reason that the teacher's response was not appropriate for supporting multiplicative thinking.

These findings highlight the needs of the participating facilitators in relation to the necessary PCK-PD knowledge elements for the teaching and learning of multiplication. Thereby, more targeted activities in the FPD program can address these needs to better prepare the facilitators for the complex endeavour of leading TPD. In utilising a situated approach to assess the facilitators' approximated noticing, this study was able to gain insights into facilitators' situation-specific skills. While only one fictional situation was presented, future research will consider the facilitators' noticing in further situations. This study did, however, offer insights into facilitators' use of knowledge elements in their noticing of a fictional situation at the beginning of the FPD course, thus contributing relevant information for the future design of FPD courses as based on the facilitators' abilities and needs. Meanwhile, facilitators took part in a one-year qualification course, and we investigated their noticing of teachers' responses to student thinking concerning the multiplicative thinking task again. So far, these results indicate a development towards highly relevant PCK-PD knowledge elements. In our ongoing research we will continue to unpack facilitators' learning.

Acknowledgement

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Teaching quality in mathematics lessons from the perspective of facilitators and how they perceive the teachers' perspective

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What does quality in mathematics teaching mean to you? This question was given to facilitators in the nationwide QuaMath (Developing Quality of Mathematics Classrooms and Teacher PD) program in Germany (N = 243). The facilitators were also asked what answers they would expect from those teachers who will participate in their PD courses. The data analysis was carried out using qualitative content analysis, inductive categories were formed and compared with the five principles for high-quality mathematics teaching (Prediger et al., 2022). Analyses indicate differences between facilitators own understanding of quality and their view on how teachers would answer in this question.

Keywords: Professional development, facilitators, teaching quality, professional learning, mathematics.

Theoretical background

Quality of teaching in mathematics

One aim of professional development (PD) programs in mathematics is to improve the quality of teaching. Teaching quality has been the focus of both general and subject-specific empirical educational research for several years (Praetorius et al., 2018). In recent years, a distinction between surface and deep structures of teaching has become increasingly popular for describing and analysing teaching (Decristan et al., 2020). Surface structures of teaching are therefore easily identifiable features that are diverse and interchangeable: methodological and organisational design features, social forms and easily observable teacher and student behaviour (Pauli, 2020). In contrast, deep structures of teaching focus more on the content-related engagement with the learning content and the interactions which is indirectly observable and play a crucial role in student learning (Decristan et al., 2020). These deep structures provide answers to the question of how the content and methods should be implemented (ibid.). In the context of mathematics the German Centre for Teacher Education in Mathematics (DZLM) developed 'five principles for high-quality mathematics teaching' (Prediger et al., 2022), that can be assigned to the deep structures of learning. These principles are based on a combination of normative, epistemological, empirical and practical perspectives and relate to the subject of mathematics. The five principles are as follows: Conceptual Focus, Cognitive Demand, Student Focus and Adaptivity, Longitudinal Coherence, and Enhanced Communication (ibid.).

Facilitators' views of the quality of teaching in mathematics

In order to improve the quality of teaching, it is necessary to approach various levels: the classroom level, the teacher PD level, and the qualification of facilitators who carry out the PD for the teachers: the facilitator PD level (Figure 1). Our study is part of QuaMath, a nationwide 10-year implementation program in Germany aimed at enhancing the quality of mathematics classrooms and teacher PD. The program first focuses on qualifying facilitators in alignment with the five principles for high-quality mathematics teaching (see above) (facilitator PD level). Subsequently, the facilitators work directly with teachers, delivering PD sessions grounded in the QuaMath program (teacher PD level). In our study we refer to the facilitator PD level. Effective PD for facilitators of mathematics teachers requires a clear understanding of the competencies they need to support and inspire teachers in improving instructional quality. To address this, we draw on the framework by Peters-Dasdemir et al. (2023) (Figure 2), which outlines key dimensions of facilitators' competencies. This framework provides a systematic approach to examining how facilitators are prepared to guide teachers in developing high-quality mathematics instruction and how their professionalization can be supported effectively.

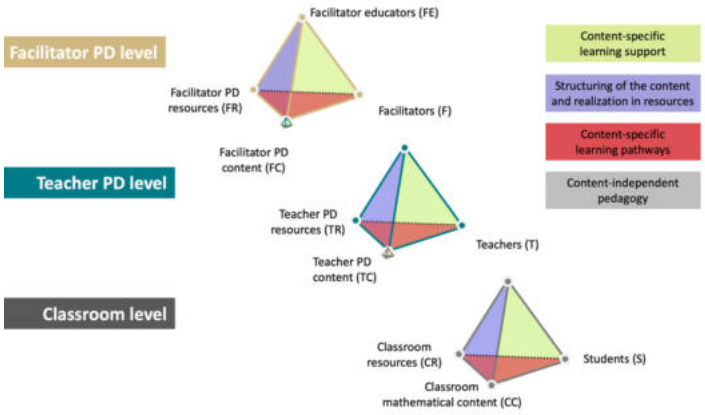

Figure 1: Three-Tetrahedron Model (Prediger et al., 2019, pp. 410)

Figure 2: Competency framework of facilitators in mathematics education (Peters-Dasdemir et al., 2023, pp. 329)

In the following, we focus on two dimensions of the framework. (1) Research shows that generic or content-related beliefs, along with pedagogical attitudes, implicitly or explicitly shape facilitators' practices (Prediger et al., 2021) (dimension: **professional beliefs**). Since QuaMath facilitators are responsible for conveying QuaMath content in their PD sessions, it is crucial that their understanding aligns closely with the core principles of QuaMath. This raises an important question: What do facilitators actually think about quality mathematics teaching? Specifically, is there already a shared understanding of quality mathematics teaching among all participants involved in QuaMath? While there is substantial research on teachers' beliefs and orientations in general, little attention has been paid to what teachers, teacher educators, or facilitators specifically think about effective teaching practices (Wisniewski & Siegel, 2020). This aspect can also be viewed through the lens of participant-orientation: understanding the knowledge, experiences, and ideas that facilitators bring to the

QuaMath program is crucial for us. Participant-orientation focuses on tailoring PD to the unique needs, beliefs, and prior knowledge of the participants. Adopting this participant-centred approach has been shown to significantly enhance the effectiveness and impact of PD programs (Barzel & Selter, 2015). (2) On the other hand, we are interested in how the facilitators see their future participants: what answers do they expect teachers to give to the question: what is good mathematics teaching? This question not only allows us to assess the facilitators' competence in the field of participant-orientation, but also provides insight into the dimension **role identity**: As facilitators often hold multiple professional roles, such as being both classroom teachers and PD facilitators, they must navigate between these roles, which can be particularly challenging for novice facilitators (Perry & Boylan, 2018). A key question for us is how facilitators transition into their (new) facilitator role. Comparing facilitators' own perceptions of good mathematics teaching with their understanding of teachers' expectations provides insights into the dynamics of this role shift. Their self-perceptions, in contrast to teachers' expectations, reflect both their understanding of the profession and the progression of their expertise development.

Research questions

The above considerations lead to the following research questions:

Research question 1: What do facilitators think of good mathematics teaching? To what extent do the characteristics mentioned by the facilitators correspond to the QuaMath principles (five principles for high-quality mathematics teaching) and what kind of understanding of teaching do their answers indicate?

Research question 2: How do the facilitators anticipate the teachers' answers in comparison? To what extent do these anticipated characteristics correspond to the five QuaMath principles and what kind of understanding of teaching do the anticipated answers suggest?

Methods

The data was collected during the kick-off workshop of the facilitator PD in the QuaMath program. The facilitators were asked the following open-ended questions, which they answered freely, without predetermined options:

1. What do you think constitutes quality mathematics teaching? Write down three characteristics of quality.
2. Consider potential teachers' responses to the question: What do you think constitutes quality mathematics teaching?

$N = 333$ facilitators were questioned, 155 from primary schools (46.5%) and 178 from secondary schools (53.5%). Of these, 300 have studied a university teaching program, 18 had not, and 15 did not specify. Teaching experience ranged from 1 to 40 years, with a median of 14 years. The median experience as a facilitator is two years and varies from 0 to a maximum of 34 years.

This study uses a mixed-methods approach, integrating both qualitative and quantitative data to offer a comprehensive analysis of the research question. Initially, the data were analyzed through qualitative content analysis (Kuckartz & Rädiker, 2023). To identify the key characteristics of

effective mathematics teaching as perceived by the facilitators, an inductive category system was developed. This system, consisting of 30 categories, was informed by a literature review on (mathematics) teaching quality.

To evaluate the understanding of teaching reflected in the facilitators' responses, the data were analyzed a second time according to two categories: (1) *aspects related to surface structures of teaching,* where the facilitators' descriptions of effective mathematics teaching align with an understanding focused on surface-level elements, such as teaching methods, and (2) *aspects related to deep structures of teaching,* where the facilitators' references to quality characteristics indicate a more profound understanding of learning and teaching, such as cognitive demand.

The answers of each facilitator were then summarized and rated on a scale between 1 – 4 (1 = all answers pertain to surface structures of teaching, 2 = the majority of answers relate to surface structures of teaching, 3 = most answers refer to deep structures of teaching, 4 = all answers reflect deep structures of teaching). Table 1 presents two sample responses and their corresponding rating.

Table 1: Examples of the rating

	Named characteristics	Does the characteristics refer to surface or deep structures of teaching?	Rating
Facilitator 1	Cognitive activation	refers to deep structures of teaching (cognitive demand)	4 (all answers refer to deep structures of teaching)
	Promotion of communication	refers to deep structures of teaching	
	Orientation towards understanding	refers to deep structures of teaching	
Facilitator 2	Silence	refers to surface structures of teaching	2 (the majority of answers refer to surface structures of teaching)
	Group work	refers to surface structures of teaching (method)	
	Different levels	refers to deep structures of teaching (adaptivity)	
	Learning videos	refers to surface structures of teaching (media & learning materials)	

25% of the data was independently coded by two researchers with Cohen's kappa values ranging from .70 to .77 for the different analyses, indicating substantial agreement (Landis & Koch, 1977). To examine differences between facilitators' own perceptions of good mathematics teaching with their understanding of teachers' expectations, a paired t-test was conducted.

Results

RQ1: What are the facilitators' ideas of effective mathematics teaching? Figure 3 presents the categories, ranked by their frequency of mention by the facilitators (N = 243). The top five categories—adaptivity, conceptual focus, learning atmosphere, structure/clarity, and cognitive demand—are highlighted, with adaptivity being by far the most frequently mentioned, appearing in 96 responses. Notably, the categories corresponding to the five QuaMath principles for high-quality mathematics teaching (highlighted) are among the top seven most frequently cited. A total of 39% of the responses align with these QuaMath principles.

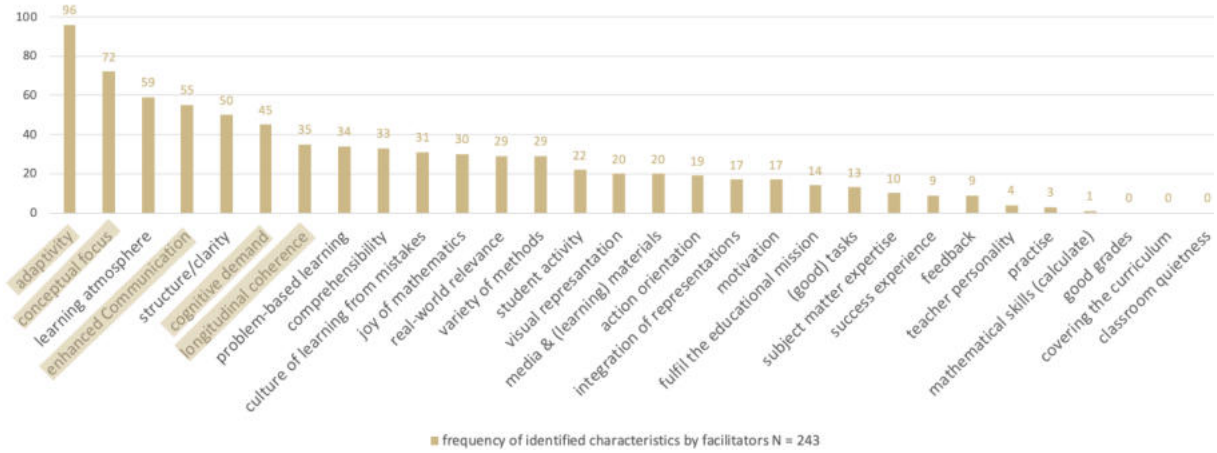

Figure 3: The facilitators' own understanding of mathematic teaching quality

When the facilitators' responses are rated on the scale between 1 (all answers refer to surface structures of teaching) and 4 (all answers correspond to deep structures of teaching) the average rating is $M = 3.37$ ($SD = 0.795$). This suggests that the facilitators' perceptions of effective mathematical teaching are more strongly aligned with an understanding that emphasizes deep structures of teaching.

RQ2: How do facilitators anticipate the teachers' responses in comparison to their own perspectives? Figure 4 illustrates the differences between the two perspectives: that of the facilitators' own views and their expectations of the teachers' responses. The data reveals an unequal distribution across the categories. From the facilitators' perspective ($N = 243$), the three most frequently mentioned categories are adaptivity, comprehension orientation, and learning atmosphere. In contrast, facilitators expect that the teachers ($N = 209$) would most often mention adaptivity, good grades, and media & materials. Notably, only 16% of the anticipated responses align with the QuaMath principles for high-quality mathematics teaching, compared to 39% for the facilitators' own beliefs.

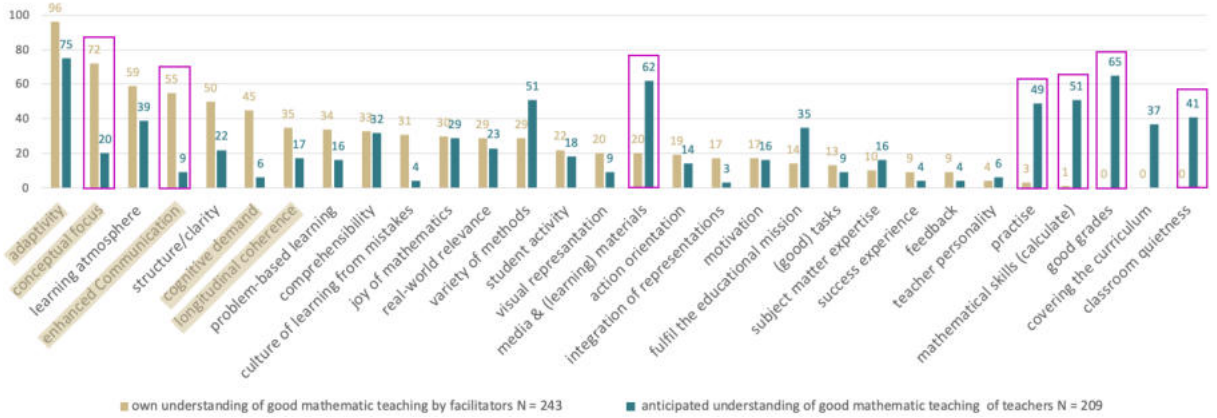

Figure 4: Comparison of facilitators' own ideas and facilitators' expectations of teachers' responses

The main disparities between the two perspectives are highlighted in the following categories: Conceptual focus, enhanced communication, media & material, practice, mathematical skills, good grades and classroom quietness.

The expected responses were rated based on their alignment with surface (1) or deep (4) structures of teaching. The average rating for the anticipated responses was $M = 2.03$ ($SD = 0.82$), indicating that the facilitators tend to attribute a more surface-level understanding of teaching to the teachers.

The comparison of the two perspectives reveals a distinct contrast (Figure 5): While facilitators' own ideas of good mathematic teaching are more aligned with deep structures, their anticipated responses from teachers tend to reflect surface-level structures. This distinction was also confirmed statistically ($t(171) = 14.78$, $p < .001$) with a strong effect size ($d = 1.27$).

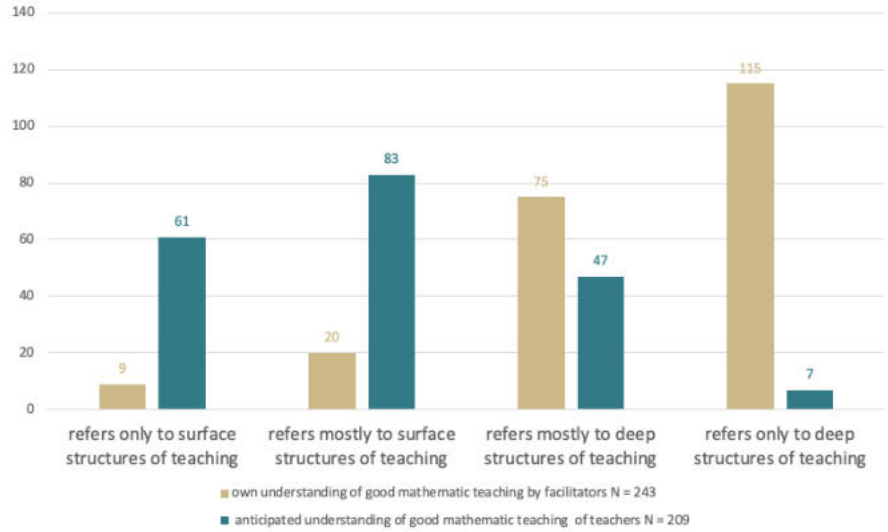

Figure 5: Comparison of perspectives regarding surface and deep structures of learning

Discussion

The findings reveal that a significant portion of the facilitators' responses align with the five QuaMath principles for high-quality mathematics teaching. This alignment indicates that facilitators possess professional beliefs and orientations that closely reflect the program's core principles, a critical factor given their role in promoting these ideas during PD sessions. Such a shared understanding among facilitators is essential for the success of a large-scale program like QuaMath, as it fosters coherence across their practices. Coherence, in turn, is a key driver of the effectiveness and impact of PD initiatives (Prediger & Selter, 2024). The finding that facilitators predominantly associated quality aspects of teaching with deep structures highlights a strong alignment between their professional beliefs and key insights from empirical teaching research, which emphasize the critical role of deep structures in improving student learning outcomes. This reflects the potential of QuaMath to bridge research and practice, ensuring that its facilitators are positioned to enact research-based instructional approaches in their PD sessions.

When examining facilitators' perceptions of their future participants, a distinct pattern emerges: facilitators tend to assume that teachers focus primarily on surface structures of teaching, such as lesson materials, while deeper structures, such as cognitive demand, are less frequently anticipated. This perception may stem from facilitators' prior PD experiences, where teachers often expressed needs tied to practical, surface-level aspects, and reflects facilitators positioning themselves as

advocates for an understanding of learning that refers to the deep structures of teaching. The divergence between facilitators' views of good teaching and their perceptions of teachers' priorities highlights an emerging professional identity. Notably, these findings, collected at the start of the QuaMath program, suggest that facilitators are already beginning to distinguish their new role from that of the teachers they aim to support.

The question of whether teachers primarily assess teaching quality based on surface characteristics cannot be answered from this survey. However, we anticipate gathering relevant data in the coming months as facilitators conduct their PD courses, focusing on the topic of high-quality mathematics teaching and the five principles. Additionally, it will be of interest to examine how facilitators' perspectives evolve over the course of the QuaMath program, with further data to be collected at its conclusion.

One limitation of our study is that facilitators' responses were provided as keywords, leaving some ambiguity about their precise meanings. Additionally, the five principles were introduced during the opening of the QuaMath program, and facilitators may have recalled them when completing the questionnaire. However, our analysis of their responses revealed that only a small portion (15%) directly referenced the exact wording of the QuaMath principles. These mentions primarily related to the concept of cognitive demand, a widely recognized aspect of teaching quality that extends beyond the specific principles of the QuaMath program.

The aim of QuaMath is to convey the underlying aspects of teaching quality of mathematics to teachers and to anchor them in teaching practice in the long term. The results show that this understanding has already been initiated in the facilitators and we can assume that they strive to strengthen and communicate these aspects of deep quality of mathematic teaching in their future PD courses.

Acknowledgment

We would like to thank all our colleagues who held the QuaMath workshop 'pedagogical content knowledge on PD' in Berlin, where the data for this study was collected.

The QuaMath program has been funded in Phase 1 (2023–2028) with 17 million € from the Standing Conference of State Educational Departments in Germany, and additional staff support from the federal states for facilitators and state coordinators has been provided (about 20 million €).

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A Study of Mathematics Teacher Educators' Knowledge from the Perspective of In-Service Teachers

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Keywords: Mathematics teacher, Mathematics teacher educator, Knowledge.

Introduction

In recent decades Mathematics Teacher Educators (MTEs) have been the subject of study due to their role in the preparation of future mathematics teachers (Beswick & Goos, 2018). International researchers in mathematics education have studied MTEs' knowledge with the aim of constructing a theoretical model. In this regard, Even (2008) noted that these studies are mostly carried out by MTEs themselves through self-study techniques. This, along with the idea highlighted by Beswick and Goos (2018) that there is no clear answer as to which professionals should investigate MTEs' knowledge, suggests that the right questions may not be being asked to explore MTEs' knowledge. Following the approach in Giadas et al. (2023), this research is top-down conducted, based on In-Service Mathematics Teachers (ISMTs) trained by MTEs. The aim of the research is to analyse the perceptions that ISMTs have regarding MTEs' knowledge. Specifically, this study seeks to answer the following research question: What are ISMTs' perceptions of MTEs' knowledge? By addressing this question, existing MTEs' knowledge models will be deepened. This research considers the term MTEs in a broad sense, including terms used in TWG27 such as educators or facilitators.

Theoretical background

MTEs' knowledge has been explored through various methodological techniques due to the variety in MTEs' profiles (Giadas et al., 2024). The literature seems to lack clarity on MTEs' training background and experiences. This fact hardens the task of reaching a definition of MTE and establishing a theoretical model that characterises their knowledge. Additionally, this situation is influenced by the fact that different profiles of MTEs exist, such as facilitators or mentors (Karsenty et al., 2020), each associated with different knowledge depending on the professionals they train and their research activities (Martignone et al., 2022). In this sense, some of the aspects used to characterise the MTEs' knowledge are the standards expected of MTEs' work, the presumed MTEs' knowledge, the practices carried out by MTEs, and the existing knowledge models for mathematics teachers (Karsenty, 2020). In this study, the Mathematics Teacher Educators Specialised Knowledge (MTESK) model proposed by Martignone et al. (2022) is used because it takes into account multiple MTEs profiles. This model considers the MTEs' knowledge divided into Mathematical Knowledge (MK), Pedagogical Content Knowledge (PCK) and beliefs related to both types of knowledge.

Methods

This research presents the results of an exploratory study conducted with eleven ISMTs from different Italian regions, all teaching in lower-secondary schools (grades 8 to 10). The used instrument was an anonymous open-ended questionnaire, distributed via Google Forms and designed by the authors to

understand and explore the participants' perceptions on the MTEs' knowledge. The questionnaire comprised three questions related to the general aspect about MTEs' characteristics and knowledge. The questions were: (Q1) What are the most important characteristics you think a 'good' MTE should have? (Q2) What are the most important knowledge you think a 'good' MTE should have? (Q3) Is it necessary for a 'good' MTE to be familiar with the main research results in mathematics education? If so, which ones? This instrument allowed us to collect data in the same way Martignone et al. (2022) did in their research and to explore whether ISMTs consider necessary for MTEs to possess knowledge of research in mathematics education. The participants' responses (units of analysis) were collected in a spreadsheet, and they were analysed using deductive analysis (Patton, 2014). In this regard, the division proposed by Martignone et al. (2022) was used. The different units of analysis were classified in two phases: firstly, into MK and PCK domains, and then in more detail into the different subdomains of the MTESK model.

Acknowledgment

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From the idea to the implementation: Ensuring coherence and vision transfer in the development of teacher professional development

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Keywords: Professional development program, coherence, teacher educators.

How do you design an effective, coherent professional development (PD) program? Empirical research in recent years has shown that certain quality characteristics significantly influence the effectiveness of PD (a.o. Borko et al., 2004; Darling-Hammond et al, 2017; Desimone, 2009). Based on this extensive research, the German Centre for Mathematics Teacher Education (DZLM) has developed six design principles for effective PD for mathematics teaching (Barzel & Biehler, 2020). The principles include competence orientation, participant-orientation, teaching-learning diversity, case reference, promotion of cooperation and promotion of reflection (ibid.). Since then the principals are used in the design, implementation and research of PD concepts and PD materials in different projects.

On of them is the 10-year national implementation program QuaMath (Developing Quality of Mathematics Classrooms and Teacher PD) in Germany, which aims to develop the quality of mathematics education and teacher PD. In QuaMath the six design principles for effective PD serve (among other aspects) as the basis for the design of the QuaMath PD modules, with an additional principle added to the six characteristics: Coherence. As early as 2001, Newman et al. found that training program were often fragmented, that different objectives prevailed and that participants found it difficult to connect the content. A coherent vision of good teaching is often lacking, training topics are not linked to a larger context and there are no references to each other. Accordingly, coherence has been identified in several studies as a quality feature for effective PD (a.o. Darling-Hammond et al., 2017; Desimone, 2009). Coherence often refers to the consistency between PD content and teachers' school contexts (with curricula, school goals and education system reforms). In addition, Borko et al. (2004), for example, argue that training programs that offer thematically related modules and a coherent sequence of activities increase the likelihood that teachers will integrate what they have learned into their teaching. Hill et al. (2013) and Cobb et al. (2013) also show in their studies that coherent PD programs based on a clear, overarching vision are more effective than programs consisting of loose, disjointed modules and materials.

Prediger & Selter (2024) already described the PD concept of QuaMath, its vision, goals and coherent core elements. As the project is quite large, involving mathematics education experts from 12 different universities across Germany and the development of 26 PD modules, the question arises, how the development of a coherent PD program can be implemented. In our poster we present elements that contribute to the development of coherence. We also examine the extent to which the visions (goals, core elements) are already anchored in the minds of the teacher educators involved - a prerequisite for the creation of coherence.

The research questions include:

- Which elements contribute to the coherence of the training program?
- What challenges and difficulties can be observed in the creation of coherence among the participants?
- What frameworks, guidelines and support can help to create coherence and at the same time take into account the ideas of the participants?

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Reflecting on action research cycles to enhance my teaching of jump strategy to pre-service teachers

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Keywords: Reflective practice, Mathematics teacher educator, Empty number line, Action Research.

Introduction

In this poster presentation, I share my experience as a mathematics teacher educator (MTE) reflecting on an intervention programme, and action research study, with pre-service teachers (PSTs). The intervention trialled the jump strategy to support mental computation. I modelled using and teaching the jump strategy with the empty number line (ENL) and facilitated PSTs in creating micro-teaching videos of implementing the strategy. Analysis from Cycle 1 of the intervention lead me to consider how I could enhance my practice in Cycle 2. Hence, the research question is, 'In what ways did I enhance my teaching of the jump strategy through reflection in action research cycles?'.

Literature review and theoretical framing

The mathematical knowledge and number sense of PSTs is of concern in South Africa and internationally (Bowie et al., 2019). Hence, there is a need for MTEs to reflect on ways in which they can enhance their teaching of mathematics didactics to support PSTs. Garcīa et al. (2006) use reflection-on-action to simultaneously engage in teaching and research for enhancing learning as both MTEs and researchers. I too was guided by Schön's (1983), reflection-in-action and reflection-on-action. After Cycle 1 of implementation, I reflected *on* the success of the intervention, which lead me to consider ways in which I might enhance my own teaching of the strategy. In Cycle 2 of the intervention, the findings from Cycle 1 guided my reflection-*in*-action.

Methods

In this action research study, I trialled the jump strategy as an intervention during a mathematics didactics course with 120 primary school PSTs. Cycle 1 took place over 6 lecture sessions in Term 4 of 2023. From this cycle, I analysed PSTs' micro-teaching videos with an adherence checklist (Smith et al., 2019), based on four identified key elements (ENL, jumps, key phrases, gestures). PSTs were mostly able to implement the strategy with some level of fidelity. However, I noticed ways in which I could adapt my own teaching of the strategy. Therefore, in Cycle 2, in Term 4 of 2024, also with first year PSTs, I kept a reflective journal to guide me in exploring and enhancing my practice (see Figure 1). I thematically analysed the reflective journal and one video-recorded demonstration using codes based on the adherence checklist of the four key elements as well as coding themes that emerged from the data.

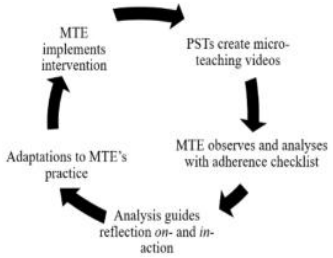

Figure 1: Action research

Findings

Three themes emerged in relation to how I enhanced my teaching in Cycle 2: making key elements explicit; creating opportunities to practise through micro-teaching; and guiding reflection.

Based on analysis from Cycle 1, as the MTE, I became aware of aspects in each of the four key elements that I could make more explicit when teaching the jump strategy on the ENL. During Cycle 2, I kept these in mind during my teaching and reflection (on- and in- action). After lecture 4, I noticed that I still needed to make the key elements explicit for the PSTs. I reflected, "I did a recap with the four key elements. I did not name them or label them as such before, maybe I should use a checklist for the PSTs to check their own examples" (RJ24-10-03). This initiated a plan to use the analytical tool (adherence checklist), from Cycle 1, to support PSTs in Cycle 2, by labelling each key element and guiding the PSTs to reflect on their own examples.

I also noticed the need for PSTs to have more opportunities to practise teaching the strategy, as evidenced in my reflection referring to PSTs demonstrating to the class: "I think that this is a useful strategy to get the PSTs to practise in class and to reflect on each other's examples early on in the intervention" (RJ24-08-20). Reflecting on this, I realised that I should not be the only one demonstrating the strategy during class. I used micro-teaching more frequently for PSTs to get more opportunities to practise teaching the strategy and I used this to initiate discussions around the pedagogical value of the key elements of jump strategy. I noted in my reflective journal that PSTs tried to reflect on and compare their strategies, and that "…it was good that students were questioning the strategy while working in pairs" (RJ24-09-17). Intentionally planning for reflective opportunities was important. I encouraged further reflective feedback amongst PSTs by guiding them to use the adherence checklist to evaluate and reflect on each other's micro-teaching examples.

Conclusion

The findings from Cycle 1 indicated areas where I could improve my teaching of the jump strategy. My reflections on and in Cycle 2 showed the need for MTEs to focus on making key elements of a strategy explicit in their teaching, to create opportunities for PSTs to practise through micro-teaching and to intentionally guide reflection based on PSTs' micro-teaching. This example of using action research to enhance the MTE's practice could guide other MTE-researchers in their research, reflection, and practice of implementing similar interventions.

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Mathematics teacher educators' noticing in a problem posing task

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Keywords: Noticing, mathematics teacher educator, problem posing, fractions.

Background

Noticing is an essential teaching skill which can be developed by an individual depending largely on their knowledge and experience. Van Es and Sherin (2021) revised their original framework for learning to notice, identifying three stages: *attending*, which consists in identifying noteworthy features of classroom interactions and disregarding selected features of classroom interactions; *interpreting,* i.e. using one's knowledge and experience to make sense of what is observed and adopting a stance of inquiry; and finally the stage of *shaping*, where the aim is to construct interactions and contexts which provide access to additional information.

In the context of the implementation of a problem posing task using fractions with pre-service primary teachers (PPTs), a comparative study was carried out into the noticing of two teacher educators. The aim of this study is to identify potential similarities and differences in the noticing of two of the MTEs from varying academic backgrounds through the knowledge they bring into play in a specific episode.

Noticing in a specific episode: methodological aspects

The focus of this paper is on two MTE's noticing of one PPT solution for a training task, in which the PPTs had to devise a problem with fractions, structured into two additive stages involving an addition and a subtraction. In addition to the problem itself, the PPTs were required to provide a solution with an accompanying illustration, and to explain what the fractions meant and predict the possible difficulties the pupils might face. The implementation was recorded and the focal point is an episode where a PPT was asked for reading and explaining his problem writing on the blackboard in order to invite other PPTs to discuss and offer solutions. (Figure 1).

PPT: There are ten balls. After a match, half of them have been lost. Out of ten. And a fifth of those remaining have been punctured. To rectify this, half of the remaining balls have been bought. How many balls are left?

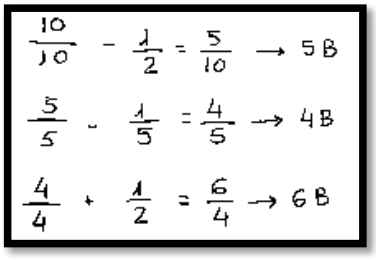

Figure 1: PPT's solution to his fractions problem

The clip containing the PPT solution was then separately played to two MTEs, of differing backgrounds, who had designed and implemented the task in another moment. After the two had watched the video, they were given a semi-structured interview designed to draw out their considered reflections on the three stages of noticing. A transcript of the video recording was then provided for

further comment. Following this analysis, an interview was conducted in order to triangulate interpretations and a consensus was achieved.

Regarding informants' backgrounds, the first (MTE 1), with 13 years' experience as primary educator and researcher, had studied a first degree (5 years) in mathematics, followed by a master's degree in secondary level mathematics teaching and a doctorate in mathematics education. MTE 2, with 11 years as primary educator and researcher, had originally studied (general) primary education (4 years), and after a master's degree and doctorate in mathematics education.

Results

Results are organised according to the stages of noticing. Although the MTEs' commentaries in the first stage (*attending)* are along similar lines, drawing on their mathematical knowledge (MK) of fractions, the aspects they focus on differ and amount of time spent considering them. Thus, whilst MTE 1 shows a deeper substantive knowledge of the topic (types of arithmetic problems, the meaning of fractions, definitions of fractions, phenomena associated with fractions as operator, and so on), MTE 2 demonstrates a pedagogically oriented perspective firmly rooted in the primary classroom.

In their interpretation of the video recording, both MTEs link their knowledge of how PPTs tend to understand mathematical content to their experience as researchers and trainers. They account for the episode in terms of the generally held part-whole conceptualisation of fractions that intuitively attracts students and tends to prevail in school mathematics. Nevertheless, while MTE 1 relates the use of part-whole to a difficulty in identifying operator, MTE 2 identifies as the main obstacle what is meant by unit at any given moment. Moreover, both take into account pedagogical aspects. While MTE 1 foregrounds knowing how to formulate a problem for pedagogical purposes, while MTE 2 regards the formulation as a means of evaluating the PPT's content knowledge:

In *shaping* stage, both MTEs orient their responses towards constructing the PPT's mathematical content knowledge. Although both would manage their responses in broadly similar ways, MTE 1 describes a sequence of interventions to encourage reflection on the aspects identified and interpreted in the previous stages of noticing. For his part, MTE 2 does not explicitly state their intention to encourage the PPTs to consider potential misunderstandings around the different meanings fractions can take, but rather focuses his response on a solution in keeping with the context of the set problem.

There are similarities between the informants, largely with respect to the interpreting stage, although differences were found in the attending and shaping stages, as well as in terms of the deployment of examples and the use of mathematical vocabulary. Hence, this study foregrounds potential differences in the profiles of teacher educators. More in-depth studies might uncover different possibilities connected with particular profiles that promote more effective work among teams of teachers. In addition, this study could be approached from other theoretical perspectives.

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TWG 28: Collaborative Settings in Mathematics Teacher Education

An introduction to TWG 28: Collaborative settings in mathematics teacher education

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The second convening of TWG 28 at CERME 14 was stimulated by seventeen papers and one poster documenting research from eleven countries. Discussions were framed around four interrelated themes: the role and preparation of facilitators, collaborative early-career and pre-service teacher education, discourse and socio-cultural dimensions of collaborative research, and didactical theory and knowledge development. Contributors highlighted the complex competencies required of facilitators, spanning from orchestrating reflective questioning to nurturing collegial trust. A range of theoretical framings of different contributions including consideration of paradidactic systems and the ATD underscored the global relevance of collaborative inquiry, and pointed to how deep analyses of textbooks and student work can surface tacit pedagogical knowledge. Learning studies that focused on variation theory and lesson studies in the Japanese tradition were prevalent in the work reported.

Keywords: Teacher education, collaborative settings, professional development, lesson study

Introduction

This was the second meeting of TWG 28, which was established at CERME 13. The group builds on the work of an informal network of European mathematics education researchers that has developed around studies of collaborative settings in which teachers and researchers, together with educators, more widely explore mathematics teaching and learning. The work of the TWG at CERME 14 involved in the majority those who attended CERME 13 whilst also welcoming colleagues for whom this was their first involvement. In this way the group built on their previous work whilst also expanding in new directions. The group was focused around seventeen papers and one poster presentation. The contributions came from eleven European countries with a good distribution between North and South, and East and West confirming the continuing interest, particularly in lesson and learning studies in mathematics education, internationally.

The contributions to our discussions collectively portrayed research in collaborative settings in teacher education as a multifaceted field of inquiry for advancing mathematics teacher education, revealing both the promise and the complexities inherent in collaborative professional learning. The programme for the conference was organised around four main areas of study: that of facilitating collaborative work including the role and work of those who do this; collaborative teacher education of those in early stages of their career, including the pre-service phase; talk and discourse in

collaborative settings, including consideration of socio-cultural issues; and didactical theory and didactical knowledge.

In the remainder of our overview, we provide a brief insight into the flavour of the contributions to the group and our discussions which in addition to brief presentation of each paper encouraged questioning around context, findings and conclusions that might be drawn as well as thematic exploration providing opportunities for synthesis and considering directions for future research.

Facilitating collaboration

Issues facing those who facilitate collaborative research was a major part of the discussion of the group in the previous meeting of CERME, and this focus was continued in 2025. The competencies demanded of study facilitators, especially in lesson and learning studies were identified and considered as major areas of concern. Discussions were assisted by the overviews provided by Quaresma and colleagues, and Kanbolat and Arslan.

Beyond organisational acumen and logistical coordination, facilitators must also cultivate a truly collaborative climate, especially through developing considerable expertise in using reflective questioning strategies, backed up by robust mathematical, pedagogical, and didactical knowledge. Our reflections underscored that effective facilitation of collaborative research focused on close-to-practice activity, that is often in classrooms and entails more than managing schedules or materials. Our conversations outlined that effective facilitation requires fostering open dialogue, guiding inquiry, and sustaining collegial trust, while also navigating shifting personal and professional pressures and institutional constraints. It was recognised that without administrative backing and management support, together with finding sufficient time to work adaptations that are sensitive to socio-cultural contexts, facilitators may struggle to maintain momentum or to engage all teachers meaningfully in the process.

There was agreement that, for research in collaborative settings researchers must attend to the cultivation of skilled facilitators capable of guiding the intricate cycle of lesson planning, enactment, and reflection (see, for example, the paper of Gomes and colleagues). While existing work has identified the core competencies of facilitation, ranging from orchestrating collaborative talk to structuring post-lesson debrief, the processes by which novices acquire and refine these capabilities remain opaque. It was considered that directions for future research could include longitudinal studies that might examine how targeted training programs shape facilitators' use of probing questions during reflection meetings and how these questions, in turn, scaffold teachers' professional growth as indicated by Bahn and colleagues. By mapping intended lines of inquiry against practitioners' actual reflective trajectories, scholars could illuminate the mechanisms that undergird robust Lesson Study implementations and thereby design more effective preparatory experiences. In relation to this the study of Békési and colleagues points to how collaboration is something that needs to be learned by participants in ways that are productive for those participating.

Collaborative talk

Related to facilitation is the critical role of discourse in collaborative lesson and learning study settings. This was addressed by contributions from Clivaz and colleagues, and Faria and colleagues. Analyses of post-lesson reflection meetings and teacher talk reveal that the quality of facilitator questioning can bridge the gap between teaching practice and broader professional development, deepening participants' practical and theoretical understanding. Discourse analysis distinguishes cumulative, disputational, and exploratory talk, designating exploratory talk, marked by critical yet constructive engagement, as particularly potent for collective knowledge construction. As pre-service and in-service educators negotiate the evolving micro-didactical contract within classrooms (that is, the implicit agreements governing teaching and learning interactions) they shift from traditional, instructor-centered paradigms toward participatory, teacher-led inquiry. This transition demands time and mutual adjustment but paves the way for richer, more collaborative professional learning. Roorda and de Vries, in their poster presentation, explored classroom discussion in the context of Japanese Problem Solving lessons. The contribution of da Ponte and colleagues illustrates the issues inherent in developing from pre-service teacher to become more experienced in leading whole-class discussions.

Included in our discussions in this theme was the research of collaborative talk focused on knowledge for teaching by Leakey. Linked to analysis of talk of this nature, issues of deep-seated socio-cultural practices at both societal and educational professional levels and interactions with translation of different source languages were raised in the contribution of Wake and Lander. This discussed issues around how we, as researchers, are interpreting each other's work.

Early stages of, including pre-service, teacher education

Research in initial teacher education (for example, see the contributions of Presutti, and Andriano and Robuti) shows that the micro-didactical contracts that become established in classrooms, influenced by general patterns of teaching that emerge in our different countries, and in response to teacher beliefs, classroom relationships and their interactions with students, are important in collaborative settings. These are particularly important to our understanding of how early career teachers transition from traditional university training to a more teacher-led, collective activity with a focus on ongoing professional learning. This evolution requires time for pre-service teachers, educators, and more well-established teachers to adjust to the characteristics of collaborative professional learning such as those focused on lesson and learning studies (see the work here of Békési and colleagues).

Some variants of Lesson Study (including bi-institutional models that connect university and school settings, as well as Learning Study approaches that emphasize problem analysis and iterative refinement of teaching material) demonstrate how the group are using adaptable lesson studies across contexts and content areas, as well as in supporting the development of early career teachers. Detailed studies in this area, for example, see the research on algebra instruction by Cosan reveals that collaboration across institutions can catalyse a shift from mere procedural proficiency to deeper technological and theoretical reasoning, prompting students themselves to articulate and justify their

solution strategies. Yet these structured collaborations can also provoke discomfort among teachers when sequences become overly regimented, suggesting a delicate balance must be struck between fidelity to collective planning and the autonomy needed for teachers to tailor implementation (for example, the contribution of Antoine).

Didactical theory and knowledge

Another strand of our discussions focussed on how collaborative activity can function as 'paradidactic' systems, wherein pre-service and early career as well as more experienced teachers and teacher-educators can form inquiry communities whose collaborative planning and reflection in ways that can sharpen their Mathematical Knowledge for Teaching. De Haan and colleagues, Østergaard, and Mårtensson and colleagues shared their work using learning studies and variation theory to explore these issues. By analysing student work, leveraging planning templates, and interrogating textbook content through the process known in Japan as kyozai kenkyu, participants make tacit pedagogical knowledge visible and can extend their knowledge for teaching including their horizon content knowledge. Comparative studies of textbooks from diverse cultural contexts further illustrate how critical engagement with instructional materials can inform new modes of support. For example, our engagement with consideration of such activity underscores the global relevance of the methodologies of our work we employ in collaborative research of this nature and the theoretical framings we use in exploring and extending our thinking in relation to teacher knowledge. The work of Garcia and colleagues, Østergaard, Bahn and Cosan,in particular drew on the Anthropological Theory of Didactics to frame their thinking.

Conclusion

In conclusion, the second meeting of TWG 28 reaffirmed the vibrancy and complexity of collaborative inquiry in mathematics teacher education and provided promising avenues for future enquiry. Participants illuminated the multifaceted roles of facilitators, the nuanced dynamics of teacher discourse, and the evolving didactical contracts that shape pre-service and in-service learning alike. Our discussions underscored the imperative of cultivating facilitation expertise through longitudinal, context-sensitive training; of deepening our theoretical grasp of paradidactic systems and variation theory; and of attending rigorously to socio-cultural dimensions of our work across different contexts. Future comparative and longitudinal research that considers both university-based teacher preparation and school-embedded practice appears potentially very attractive. By sustaining our dialogue across borders and career stages, TWG 28 can continue to refine collaborative frameworks that can help advance mathematical knowledge for teaching and how this might be best facilitated.

Disciplinary and meta-disciplinary noticing of prospective teachers in a lesson study experiment

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Abstract *This study involves secondary school prospective teachers in an Italian master's degree in mathematics, when they are involved in a course of mathematics education, while doing an experiment of lesson study. A group of four prospective teachers are observed, while engaged in the planning phase of a lesson study, in which they have chosen to work on the introduction to functions in grade 11. The study aims to analyse prospective teachers' discussions, focusing on their noticing as a theoretical lens to qualitatively describe teachers' reflections. The results suggest that the transition between disciplinary and meta-disciplinary ways of noticing during the discussion improves teachers' awareness and enables them to make decisions about the lesson plan.*

Keywords: Lesson study, prospective teacher education, noticing, disciplinary, meta-disciplinary.

Introduction and literature review

Several studies suggest that Prospective Teachers Education (PTE) programs should include activities that engage teachers in researching their practice (Potari & Ponte, 2017). According to Mason (2021), the concept of *noticing* in mathematics teaching serves as a valuable tool for encouraging teacher reflection and enhancing professional expertise, as the way teachers observe and reflect on their practice can significantly impact the quality of their instruction (Jacobs et al., 2010; Sherin et al., 2011). This is enhanced when prospective teachers have the opportunity to work in professional communities. Several studies point out Lesson Study (LS) as a professional development process where prospective teachers develop their knowledge through collaboration in a reflective setting (Ponte, 2017; Chen & Zhang, 2019; Gomes et al., 2022; Minisola et al., 2024). Ponte (2017) pointed out that given the nature of national policy on PTE, it could be problematic to implement LS in PTE, and the process should be adapted.

In the last decades there has been an increasing attention on teacher noticing (Konig et al., 2022). Some studies testify a growing interest in applying the concept of noticing to LS (Lee & Choy, 2017; Choy et al., 2017; Ylmaz & Yetkin Ozdemir 2023; Andriano et al., 2023). Among them, only a few studies apply noticing in LS in PTE (Bakker et al., 2022; Ylmaz & Yetkin Ozdemir 2023), and secondary school mathematics teachers noticing in LS in PTE is underexplored.

The aim of this study is to explore noticing skills in prospective secondary school teachers during lesson study. We focus our attention on the planning phase to understand which elements of noticing are fundamental in planning the lesson.

Theoretical framework

The theoretical background is based on the practice of *noticing* (Mason, 2021). According to Mason, being a teacher requires sensitivity to learners, and this is possible because of awareness-in-discipline. As Mason sustains, awareness is educable and can be improved by the *discipline of noticing*, i.e. "a

systematic method for conducting research into one's own practice" (Mason 2021, p. 231). There are different conceptualizations of noticing: some authors define noticing as that to which teachers attend (Mason, 2021), for others noticing involve two processes, attending, and making sense. Moreover, making sense for some researchers is solely interpreting (Sherin et al., 2011), whereas for others it involves interpreting and deciding how to respond (Jacobs et al., 2010). Among the growing number of research studies on teachers noticing, many of these studies explore noticing skills when reviewing teaching videos (Rosaen et al., 2008; Sherin & van Es, 2009) and only a few in the context of lesson study (Choy et al., 2017; Ylmaz & Yetkin Ozdemir, 2023). Choy (2017) extended the notion of teacher noticing to examine what and how teachers notice during the planning of lessons. However, much of the discussion is about specific attributes, such as student thinking, student production, teacher-student interaction. When teachers plan a lesson in the context of LS, they must take into consideration multiple aspects: the mathematical content, the curriculum and other institutional constraints, such as textbooks, resources, classroom setting, the choice of pedagogical approaches, the task, student thinking. In this research we use the notion of teacher *noticing* to examine what and how teachers notice in the planning phase of a LS related to mathematical content.

When noticing regards mathematics as a discipline, there is a spectrum in which one could distinguish, among others, two different levels: disciplinary and meta-disciplinary (Williams & Roth, 2019). According to Williams and Roth, the disciplinary regards the knowledge of mathematics seen from the inside, with all the rules, definitions, practices and the corresponding terminology and linguistic forms. The meta-disciplinary regards mathematics seen from the outside and involves awareness of the nature of the discipline. It is the level at which the epistemic quality of the discipline becomes clearer, and a conscious theoretical control becomes possible. It is the meta-knowledge level that emerges from the reflection on the knowledge inside the discipline.

To frame our analysis of teacher noticing about mathematics, we consider two dimensions related to how teachers use noticing. The first is related to what teachers notice at the disciplinary level, the second to what teachers notice at the meta-disciplinary level. Moreover, we are interested to investigate how the used noticing helps them in making the corresponding choices in the lesson plan. Therefore, our research questions are:

RQ1: How do prospective teachers use noticing when planning a lesson during LS?

RQ2: How does this noticing enable them to make decisions about the lesson plan?

Context

This research is a case study developed in Italy in the academic year 2023-24. The prospective teachers were attending the first year of the 2-year master's degree in mathematics in a public University. All of them already have a degree in Mathematics and none of them have teaching experience. LS was introduced in three 2-hour lessons, as part of a course in Mathematics Education (the second author is the professor of the course). 20 teachers were involved in the LS, split in five groups of four. All the participants were contacted in advance and indicated their willingness to participate in the study. The first lesson was devoted to a general introduction to LS.

The LS had the following structure:

1. two planning meetings (one in person and one on-line)
2. one research lesson
3. one debriefing meeting

After the general introduction to LS every group chose one among four given topics related to a research problem (Table 1).

Table 1: Topics proposed to the groups

Research problem	Topic	Observation focus
Distinguish functions among other mathematical relations	Introduction to function	Strategies and different semiotic register used by students
Combinatorial thinking	Introduction to combinatorics	Students' combinatorial thinking
Calculate the area of a region bounded by curves	Applications of the integral	Students' geometrical thinking and use of signs
Give meaning to the equation of the plane in analytical geometry	Plane equation in 3D geometry	How students move from geometric concept to the equation

Therefore, the groups have not been engaged in the first step of a LS cycle, i.e. selecting a topic and defining the goal. The reasons are two: for time constraints and because the participants had no teaching experience and therefore the research problem would not have come from a concrete teaching problem. The lesson planning phase had two sessions, one in person and one online. Then, each group taught the lesson to the class of prospective teachers: one component of the group performed the lesson whereas the other components observed the lesson, and the other prospective teachers acted as if they were the students. Then, each group had one post-lesson discussion in person, in which they shared what they noticed and how they could improve the lesson, and a second session online in which they wrote a second version of the lesson plan. Finally, there was a whole class discussion to share some final considerations. Since there was one teacher educator and one facilitator in the room at the time, the groups worked autonomously, and the facilitator went around the groups to supervise the work.

In this study we analyse the video recordings of only one of the four groups. The topic chosen by the group was: introduction to functions. The participants were four prospective teachers, henceforth referred to as Teacher A, B, C, D.

Introducing LS in prospective teacher education needs adaptation of LS, and these adaptations raise some issues (Ponte, 2017). In our adaptation there are some advantages, because participants are engaged in a truly collaborative relationship, they are at the same level and there is not an expert so there is a balanced relationship among participants. Moreover, it has been performed with 20

participants and two teachers' educators, so it proved to be scaling. On the other hand, a limitation is that the research lesson was not performed with secondary students, due to the institutional constraints of prospective teacher education at university level in Italy.

Method

The aim of this research is to analyse prospective teachers' noticing during the planning phase of a lesson study. Therefore, our data is the videorecording of all LS meetings and lessons along with teachers' observation notes and lesson plans collected and saved during the meetings. The research follows a qualitative approach. To carry out the analysis the authors viewed many times the recorded video, first individually and then together, looking for episodes of noticing. With an episode of noticing we mean an episode in the discussion where we could find three distinct parts: attending, reflecting, and acting (Andriano et al., 2023). We looked for moments of attending, then we keep track of this attending: do they start a critical reflection within the group? And does this reflection enable the group to act, or make decisions? When we find attending, reflecting, and acting, we call it an episode of noticing. Then we transcribed these episodes. Having identified an episode of noticing, we analyse what teachers notice in the episode. If the noticing regards mathematics knowledge, namely a disciplinary aspect, we analyse also if teachers' noticing is at a disciplinary level, i.e. regarding mathematics from the inside (rules, definitions, etc…), or at a meta-disciplinary level, i.e. regarding mathematics from the outside (meta-knowledge). In the analysis of video recordings of the first group's planning meetings we identified 11 episodes of noticing. Among them, 9 regards mathematics knowledge. In this study we carry out the analysis of one moment of teachers' noticing regarding mathematics knowledge.

Data analysis

In this study we analyse one episode of noticing taken from the first meeting. The lesson is about the distinction between functions and relations that are not functions. Teachers are discussing and drawing graphs of functions (e.g., parabola) and graphs of curves that are not functions (e.g., circles). We will present one episode of noticing regarding mathematics knowledge in relation to one specific epistemological node, the concept of function, which is discussed by the prospective teachers.

Attending

Teacher D: The thing that would interest me is to understand what the focus is. That is, what is the element that makes us say that a function suits us in one way and doesn't suit us in the other.

Teacher D started attending a new thing. The graph she drew (Figure 1) shows that she knows how to recognise a function from the graph, so she knows the concept at the disciplinary level. The point is at the meta-disciplinary level as she will explain in the following.

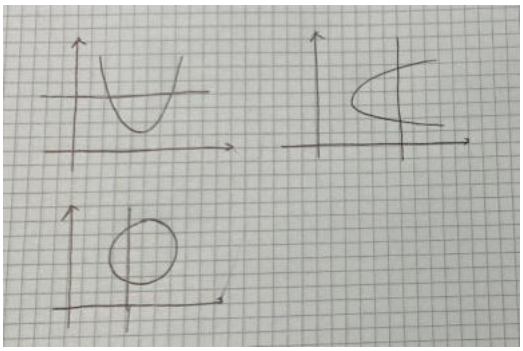

Figure 1: Teacher D drawings

Reflecting

From now on starts the critical reflection within the group:

Teacher D: Mathematically why is it a problem that the vertical axis encounters twice, and it is not a problem that the horizontal one does?
Teacher A: Because that's how it is from the definition of function
Teacher B: Yes, right.
Teacher A: You understand this from the function definition.
Teacher B: That the domain is on this here axis and the codomain is on the y axis.
Teacher A: And we associate with each element of the domain one and only one element of the codomain. [together with B from "to each element..."]
Teacher D: No, I agree that this is the definition, but I'm wondering why, and I've never asked myself that, so it's good that we're doing it, I'm just wondering why the definition is put like this, because we're really interested in those facts in this way.

In the last sentence teacher D explains clearly that she is attending to other things, she says “why the definition is put like this”. She wants to understand why the definition of function is important in mathematics, why it is given in that way. Her noticing is at a meta level: it is meta-disciplinary. This is further reinforced by the next sentence: “and I never asked myself, so it’s good that we are doing it”. She had a shift of attention, and she seems aware of the fact that she is thinking about function from a new point of view. On the other hand, the other two teachers’ reflections are at a disciplinary level. They know the definition and, repeating the same words together, they know it by heart. They know how to recognise a function, but their sentences are within the disciplinary level. They know the concept from the inside of the discipline, with all the rules, the definitions, the techniques. There is an asymmetry between teacher D and the other two: they are reflecting at different levels. Then the critical reflection of the group of teachers goes on by continuing to move between the two levels, but this asymmetry makes the other teachers gradually get the point of teacher D, until all teachers of the group move at the meta-disciplinary level:

Teacher A: I think she wants to find a motivational aspect.
Teacher D: Why will this thing function well and this one not? I don't know if I can explain myself, so as not to just learn mechanically.
Teacher B: I don't know why they invented this concept.

Teacher C: Maybe that's the problem. If we have thought of the function to describe one thing as a function of another, the problem lies in the moment in which... that is, if I have to study one thing that depends on another.

Teacher B: Ah, wait, I have an idea. If you think that with this function, we are representing the movement, the various positions, then you really cannot be in two positions at the same time, that is, if you indicate time here and space here then you cannot necessarily be in two positions at the same time.

Teacher D: Exactly. So, we have an independent variable that moves and then the rest that depends on this independent variable and can only have one value.

During the discussion teacher D explains her point of view and in the first sentence teacher A gets the point and passes at a meta-disciplinary level. In the third sentence also teacher B moves to the meta-disciplinary level "I don't know why they invented this concept". And then all the teachers start reasoning at a meta-disciplinary level. Reasoning at a meta-disciplinary level opens new perspectives to teachers. This enables for example teacher B to find a different point of view on function that gives an answer to teacher D's first question. The group of teachers, through the discussion at a meta-disciplinary level improves their awareness on the nature of the discipline.

Acting

The very rich dialectics between disciplinary and meta-disciplinary levels of noticing brings a new awareness to teachers, which enables them to be in action, namely, to consequently decide the kind of task and how to design it.

Teacher B: We could start from the space-time graph, precisely, time below and space above
Teacher A: Yes, I like that.
Teacher C: Yes.
Teacher D: We could start with a physical example of a space-time graph.
Teacher B: Let's start with a physical example, a person who moves.
Teacher D: And so, let's say that the ordinate tells us about the distance from the starting point.

The group discussion on a meta-disciplinary level improves teachers' awareness and opens new perspectives on the discipline. This enables teachers to act, to make the decision of designing a task starting from a space-time diagram.

Discussion

The aim of this study is to analyse the reflections of prospective secondary school teachers during the planning phase of lesson study. To carry out the analysis we used noticing as a theoretical lens. We have identified episodes of noticing characterised by three steps: attending, reflecting, acting. In this study we analysed one episode of noticing regarding mathematical knowledge. We analysed the critical reflection with respect to two categories: disciplinary and meta-disciplinary (Williams & Roth, 2019). The first research question is: how do prospective teachers use noticing when planning a lesson during LS? We found that teachers' noticing regards mathematical knowledge at a disciplinary and meta-disciplinary level. The episode of noticing starts with one teacher attending to mathematical knowledge at a meta-disciplinary level. The teacher had a shift of attention from disciplinary to meta-disciplinary. At the beginning of the critical reflection the other teachers'

sentences are at a disciplinary level. At this moment there is an asymmetry in the group between reasoning at two different levels. During the discussion there is a continuous shift between the disciplinary and the meta-disciplinary level until all teachers' reasoning is meta-disciplinary.

Regarding the second research question: How does this noticing enable them to make decisions about the lesson plan? We found that the collective reasoning at the meta-disciplinary level changes teachers' point of view and opens new perspectives. It is a way to see the discipline from another point of view. During the discussion teachers become more aware of the nature of the concept of function. This new awareness enables them to act, and they decide to design the lesson starting from a space/time diagram. Moreover, the choice to start from a space/time diagram shows that teachers have a broader perspective that includes another discipline, physics. Therefore, this study suggests that the dialectic between disciplinary and meta-disciplinary could improve teachers' awareness on mathematics and could offer new perspectives towards other disciplines. This could be particularly significant in the design of interdisciplinary lessons as in the case of STEM or STEAM activities.

In this paper only one episode has been analysed. We need to expand the analysis to other episodes where the disciplinary/meta-disciplinary dialectic emerges. Moreover, further research is needed to analyse which are the characteristics of the applied design and the facilitator's moves that could emphasise this dialectic between disciplinary and meta-disciplinary.

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“I’ve really seen myself as a robot”: professional development and perception of teachers involved in a collaborative setting after a second implementation loop

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This PhD research investigates the professional development of schoolteachers involved in a collaborative research with science and mathematics education researchers. For two years, this group has coconstructed a complete sequence of lessons on the attribute mass for pupils from 4 to 7 years old, based on a children’s book. This paper presents the first results of this study. It appears that all the teachers have increased their abilities to identify the difficulties of their pupils and clarify their didactical objectives and issues. Nevertheless, as the sequence became more structured, they felt a sort of a discomfort in their teaching practices and found their pupils less motivated by the lesson than the first years. This observation leads us to discuss the balance between the collective development of a structured, detailed common scenario and the latitude left to the teachers in their implementation.

Attribute and measurement, collaborative research, teachers’ professional development, elementary school, concept of mass.

Context

At the interface between mathematics and physics, “attributes and measurement” is an essential domain in sciences. In the French curricula for elementary school, “attributes and measurement” is one of the four main themes of mathematics teaching. These two concepts also appear in the kindergarten’s curricula in the area entitled “Exploring shapes, magnitudes and organized sequences”. A progression is recommended to enhance the attribute’s conceptualization: the discovery of a specific attribute begins with direct perception, followed by activities involving direct and indirect comparisons. Its measurement is then introduced, initially with arbitrary units and then with the usual units. In the line of Piaget studies, many research point out the relevance of this progression for the cognitive development of children (e.g. Sarama et al., 2011).

However, many schoolteachers face difficulties to follow this progression (Munier & Passelaigue, 2012). When they follow it, they often do so only for the length and very little for other attributes (mass, volume, duration, angle, etc.), which may be explained by the difficulties they have in distinguishing attributes from measurement (Passelaigue & Munier, 2015). One question seems essential: how to enhance the schoolteacher’s knowledge and practices concerning the teaching of attributes and measurement and foster their professional development on this specific domain? Various studies show that collaborative settings can enhance the improvement of didactical knowledge of teachers (e.g. Chanudet & Favier, 2023) and, more generally, the professional development of teachers (e.g. Bednarz, 2013). Moreover, our previous research (Antoine et al., 2023) shows that it is possible to foster pupils’ learning thanks to children's books in which attribute

is central to the story. In particular, they favour working on attribute apart from its measurement and also encourage pupils' language activities. Consequently, our hypothesis is that the use of children's books in a collaborative research setting involving schoolteachers and sciences education researchers could be a lever for teachers' professional development on the attribute and measurement teaching process.

Theoretical framework: a collaborative research setting on mass teaching with a children's book

In the line of *collaborative research* which works "with" the teachers rather than "on" them (Bednarz, 2013), our collaborative setting is based on the four principles identified by Allard & al. (2022): collective construction of a common problematic; support on teaching contexts and practices; iterative loop of conception and implementation of the resources; sharing responsibilities and tasks among the different participants. As Abboud, Robert and Rogalski (2022) suggest, we rely on the usual practices of the teachers involved in order to remain in their *Zone of Proximal Development of Practices* (ZPDP). We focus on the *didactical* professional development that we define as enhancing teacher's knowledge and practice on a specific teaching subject: mass teaching and more generally attributes and measurement teaching. In the line of many research which study knowledge needed to teach, we consider both subject matter knowledge (SMK) and pedagogical content knowledge (PCK). We rely on Magnusson's model to categorize the PCK into five different components: orientation to teaching science, knowledge of science curricula, knowledge of students's understanding of science, knowledge of instructional strategies and knowledge of assessment of scientific literacy (Magnusson et al., 1999).

The group is composed of five schoolteachers (Célia, Coralie, Déborah, Victoire and Yaëlle, having pupils from 4 to 7 years old, and facing various socio-cultural context, as three of them teach in "educational priority zones") and three science and mathematics education researchers (including the author) and meets once a month. At the beginning of the setting, researchers brought several children's books. The schoolteachers decided to work on the attribute mass and chose the book *Just a little bit* (text by Ann Tompert, illustrations by Lynn Munsinger). *Just a little bit* tells the story of an elephant and a mouse who want to play on a seesaw despite their weigh difference. Various animals come and sit on the mouse's side to help them until finally the board tips over when a beetle arrives. So, in this book, the concept of mass is essential to understand the story. Various aspects of this concept can be worked: mass additivity, distinction mass-quantity, balance and unbalance states of the seesaw, language formulations to express a comparison ("A is heavier than B"), schematisation of a seesaw thanks to the book's illustrations, etc.

During the first year of work, the group designed and implemented lessons in their five classes. These experiments were recorded, and the group then analysed what happened from the point of view of the pupils' mathematical learning. Thus, real-time feedback enabled adjustments to be made to the sequence. The teachers' objectives have become refined (Antoine, 2023) as they improved their skills to identify pupils' difficulties and the didactical issues, which led the group to work on a more detailed lesson plan for the second iteration loop, on the teachers' demand.

Problematic and method

Based on these observations, our research question are the following: what does the restructuring of this sequence between the first and the second year reveal on the professional development of the five schoolteachers? What impact has it had on their practices? And how did the teachers experience the second iteration loop?

All the group's meetings were recorded and transcribed. In the verbatim, we analyse each moment of the discussions by identifying the mathematical topic questioned and what we call a *look* on this topic. These looks are inspired and adapted from the PCK model in order to categorize the different ways of questioning a specific mathematical topic. We characterized each *look* with one of the Magnusson's categories, and added the "SMK *look*" and a "*look* on pupils' procedures". Sentences were coded (using the software Transana) according to two dimensions: the specific mathematical topic discussed (the list of all the topics was determined from our didactical analysis), and the *look* on this topic. Four examples of this code are given in the figure 1.

Topic / "Look"	Binary formulations of comparison	Schematisation: coding from observation
Pupils' difficulties	"I don't think he could use that formulation, so he said 'because the lion is heavy and the giraffe is light'. But you see, it's a way of saying which is heavier."	"The drawing contradicts the physical reality. So it's not just a question of vocabulary, it's that they actually draw it up there."
Teaching strategies	"We worked on the formulations 'heavier than', 'lighter than', 'less heavy than', 'as much as', and what I wanted to do was get each pupil to make a sentence choosing two animals."	"I'll tell them 'we're going to take two animals'. We anticipate, discuss, pose, check and then I'll ask 'face the scale, draw what you see'. So first I make sure that the drawing is consistent with what's there."

Figure 1: Examples of quotations coded with both a 'look' and a topic

We now present the evolution of the lesson plan between the first and second implementations in order to identify traces of the teachers' professional development in the restructuring process. Then, we will question the origins of a "discomfort" expressed by the teachers during the second iteration loop, which leads us to reflect on the teachers' professional development process and the effects of the collaboration on their practices.

Evolution of lesson plan between the first and the second year

During the first year, the work of the group was exploratory: the different lessons based on the children's book were designed and implemented progressively in the class. As there were three different levels of class and in order to remain in the teachers' ZPDP, the group elaborated a plan of the lesson, called "episodes". Whereas the instructions and the precise scenario of the episodes were not detailed, the group discussed and decided their contents and their articulation to facilitate each teacher's appropriation of those episodes. The plan was structured in four phases: the discovery of the book and its vocabulary, the reproduction of the story thanks to a small seesaw built by the group, the introduction of a Roberval scale as another instrument to reproduce the story and compare masses and, finally, the introduction of mass measurement procedures for the older pupils.

At the beginning of the collaborative work, the teachers felt unable to distinguish an attribute from its measurement, and did not identify precise didactical objectives on this topic. Progressively, in the teachers' speeches, the objectives of the global sequence have become more and more specific

and focused on the particularities of the mass teaching, thanks to their analysis of the children's book (figure 2). This analysis, combined with classroom experiments, enabled the teachers to better understand and spot pupils' difficulties, establish remedial strategies and "unfold" all the didactic issues involved in teaching mass. In particular, the teachers expressed the need to "elementarise" objectives which could serve as prerequisites for certain lessons in the sequence, to facilitate the progression of pupils' learning. This demand led the group to rework on the plan of the lessons during the second year.

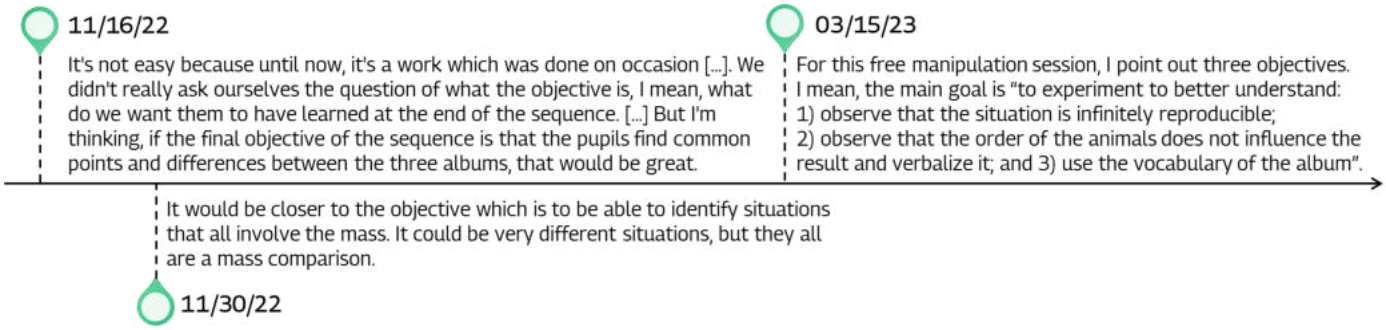

Figure 2: Evolution of Yaëlle's speech about the objectives during year one (Antoine, 2023)

The different phases of the sequence and their order did not change in the second year. Yet, three major themes have been identified to structure the learning objectives: conceptualization of mass, language issues and schematisation issues. For each of these themes, an overall progression was worked out, in conjunction with the other two, and specific activities were reorganized and placed to fulfil those new objectives. Specific attention was paid to the articulation of the objectives, and in particular to the identification of certain 'must-haves' that became prerequisites for other objectives. For example, in the language issues theme, the group identified several steps in the learning process of the pupils: understanding of the book's vocabulary; reuse of this vocabulary to describe the seesaw's state of unbalance; using of "heavy" or "light" instead of "big" or "little", etc. The group also noted that the French expression "*la planche bascule*" [the board rocks] could be confusing for the pupils. Indeed, it can convoy the idea of movement but can also be used in "*la planche bascule* du côté du plus lourd" [the board tilts on the side of the heaviest object], which refers to a static state of the seesaw. Moreover, the French word "*bascule*" can also refer to the seesaw itself! To avoid this confusion, they decided to use this expression only to designate a movement, a change in the state of the seesaw, and to use another expression to designate a static tilted state of the seesaw.

Quickly in the restructuring of the plan, a key moment was identified: the moment when pupils must leave the story to reinvest their learning outside any fictional context, right before the introduction of the Roberval balance. Therefore, on several meetings, the group focused on the elaboration of a common intermediate assessment. Three tasks were designed to verify the good mastering of certain skills for all pupils, based on the articulation of the three themes of the plan.

The details brought to the common plan around the three major themes and the elaboration of an intermediate assessment point out an improvement of the teacher's PCK in at least three of the five components identified by Magnusson. By refining the sequence objectives, they show a better understanding of the concepts of attribute, measurement and mass (SMK). They can also anticipate

the difficulties their pupils could encounter (PCK – student's understanding of science) and they develop several strategies to deal with them (PCK – strategies) and to assess their pupils' learning (PCK – assessment), mainly by identifying a progression of the learning and prerequisites to some specific activities and by using more precise vocabulary and instructions.

The second experiments loop: analyse of a "discomfort" felt by the teachers

Expression of a "discomfort" feeling. The teachers experimented the lessons for the second time in their classes with a more detailed and complete lesson plan. But, at the end of the sixth meeting of the second year, in February 2024, they expressed a feeling that the experiments were not going as well as the previous year. From their perspective, their pupils felt less free to experiment with the material and less motivated by the different lessons, whereas it was the same book and similar activities than the first year. We transcribe a part of the group's discussion below:

Célia	I have the impression that my way of doing things is less interesting than last year. I have the feeling that I over-frame them [her pupils]. I have the feeling that last year I was able to create research situations, and less this year. I don't know why.
Victoire	What I'm doing is a lot more polished on my side. And while last year we were all about enthusiasm and self-discovery, this year I feel it's more step-by-step, and we're losing a bit of the magic, but ... […]
Yaëlle	I wonder if we haven't locked certain things up too tightly. In particular, this precision of language that we've imposed on ourselves this year, hasn't that made us lose a bit of...
Victoire	Spontaneity?
Yaëlle	Of spontaneity and enthusiasm. This year, I've really seen myself as a ***robot***, being very careful… One do not say "rock"… […] We pay attention and, inevitably, we're less…
Victoire	Emotional. So maybe that's what the pupils are missing.
Yaëlle	Maybe they're thinking… they act like us. […] "We're careful, we can't do anything stupid, we have to say the right things". […] It's a bit like those classroom situations where, when you haven't planned exactly what you're going to do, sometimes you do some brilliant stuff. And when you come up with something, and you're really into it, you know, you're an expert in it and you've read all the literature on it, and in fact you just flop.
Coralie	I think that there's a balance. We need to keep a common thread because it was also a request from us to have a guide. But how do we build this guide to keep the spontaneity of the children, the research activity of the children, and also to be able to bounce off what's daily done in our classes, because we have children who are all different. And I think that's perhaps why we need a third year now, a third experiment to find just the right balance.

They particularly pointed out one moment when the pupils did not act with the same commitment: when the seesaw is introduced to reproduce the story. On their demand, during the next meeting, the group watched videos of this phase from the first and second year to compare the instructions given. This viewing enabled them to identify a lack of problematisation when introducing the seesaw during the second year, and led them to adjust their instructions and their focus for the following lessons. This may appear paradoxical: the second iteration loop was more structured overall but this specific lesson was not, as it had happened "naturally" during the first experiment.

The teachers' perception. At the end of the year, in individual interviews, when asked about this particular moment when they felt discomfort, they agreed (at different levels) that, after all, the

second version of the sequence enhanced their pupil's understanding, but some of them still kept the feeling that their pupils were less motivated.

Déborah — This year, I've got to grips with the sequence. I've reinvested it. […] And so I got to the bottom of things, in fact. […] I think that almost all my students understood. They went where we wanted them to go. Because they were more focused. Because I was clearer about my objectives. […] It's true that there were times when we said to ourselves, they're less curious, they say fewer things. But in fact, afterwards, I say to myself that maybe they say fewer things, but maybe they also say things of better quality.

Yaëlle — Was it really because I was […] more in control in relation to the fact that the sequence had to be like that, that we'd all set ourselves objectives, that we'd said to ourselves the next time we see each other, we've got to be there, there and there. So, I might have been the one who was less opened. And yes, the children are less involved and less enthusiastic. […] Still, I have the impression that, in terms of vocabulary, things have been much more successful this year.

Coralie — Last year, I felt I'd been a bit... Yes, I felt I'd been a bit messy. This time it was clearer. And therefore certainly clearer for my students too.

Our hypotheses. The discussion above shows that the teachers, and Yaëlle in particular, question the restructuring of the sequence, considering that it may have restricted their practices and reduced their latitude in responding to the pupils' suggestions, successes or difficulties. The use of the noun "robot" to refer to herself is strong: it is as if Yaëlle felt trapped within the plan and had lost a part of her didactical and pedagogical freedom. It appeared very paradoxical to all of them: they felt more confident and "expert" about the mass teaching process but also felt like they failed to implement it in their classes. The lesson of the introduction of the seesaw is a great example of this paradox. We can formulate four hypotheses to explain this phenomena:

"*First-time magic effect*": as this was no longer the first time, they were able to compare their classes and their pupils' achievements, even though they were different. They also might have "idealised" the first experiment as it was new and exploratory and as they had no specific expectations.

"A*cademical research effect*": my own PhD study might have been a hindrance to teachers. During informal discussions, some of them expressed their worry that deviating from the planned sequence would influence my research results in a negative way. This matter was then clarified with them.

"*Resource design effect*": at the beginning of the second year, the group began to discuss the possibility of disseminating the sequence to other teachers. Consequently, the teachers might have thought that they needed to test and verify the relevance of the sequence. The research focus changed: the first year was very exploratory while the second year was intended to confirm and adjust some of the group's hypotheses about the pupils' learning process with the children's book.

"*High standards effect*": on the base of the first-year experiments, the teachers got a broader view of all the didactical issues involved in the mass teaching and learning processes. They might even have felt overwhelmed by such an extent, that they did not perceive before, as some of those issues are so "naturalised" for adults that they have become very difficult to spot. As we mentioned it, they wanted to work on the learning progression with more precise, almost "elementary", objectives in

order to clarify and articulate those issues. They also wanted to make sure they would go deep enough into the mass topic. For them, the plan was supposed to be reassuring and helping to propose the different lessons in the "right" order. Their knowledge contributed to improve but also disturb their practices, and they therefore wanted to follow a more precise plan for fear of "failing".

It seems that the collaborative setting and the experiments interfered with the teacher's preexisting knowledge and practices about this specific teaching process. To our point of view, the discomfort they felt illustrates that the didactical professional development is a complex process, in which improvement of the didactical knowledge and changes on the practices are closely related. We can even assume that those moments of discomfort or disruption may be necessary to a professional development as they can push the teachers to question their practices, and are inherent, in fact, to the development of new knowledge. This reflexive attitude also reveals a professional development.

Discussion

The analysis of the evolution of the sequence and of teacher's interactions and speeches shows that the teachers involved in this collaborative research have improved their knowledge on the teaching process of attributes and measurement. They have a better understanding of the concepts of attribute and measurement, have developed teaching and assessment strategies relating to mass and can more easily spot and remedy their pupils' difficulties. Nevertheless, this improvement in their knowledge led to a feeling of discomfort in their practices, which was generally shared by the teachers. To our opinion, this disruption was necessary, as it enhanced them to adopt a reflexive posture on their practices and then, to adjust their instructions and their activities to facilitate pupils' learning, and shows a real professional development. For the third year, the group wants to explore another attribute with another book, but it is likely that the teachers will also implement the sequence on the mass in their classes. At this point, they plan to follow the structured sequence they designed in the second experiment, while allowing themselves to get out of the sequence, if this seems more relevant to them, according to their pupils' activities and their classes' situations.

Our results raise major open questions for this TWG. What level of detail is needed, and what specific types of details are necessary in the construction of teaching sequences to ensure global didactical coherence? How to balance the structuration of these sequences in order to disseminate them within the educational community and the need for teachers of flexibility in their classrooms? What implications might this expressed feeling of "discomfort" with overly structured resources have on teachers' professional development programs? What visions of the teacher implicitly underpin the different collaborative settings which aim to construct such sequences?

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Potential roles of facilitators' questioning during post-lesson reflection

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We wish to explore the role of the facilitator's questioning during post-lesson reflection meetings and how this can further or delimit the growth of the participating teachers' professional knowledge. We analyse the dialogue from a reflection meeting, which took place after a grade 8 geometry lesson, held by a group of teachers working with lesson study for years. We draw on the Anthropological Theory of the Didactic to map potential questioning and knowledge emerging from lesson plan and lesson observation to compare with the question and answers actually raised during the reflection meeting. From this we see crucial differences, and the need for further research regarding the role of facilitators' questioning being somewhere between the didactic system, in which teaching takes place, and the paradidactic system, in which professional development such as lesson study takes place.

Keywords: Lesson Study, practice knowledge, facilitator role, study and research paths.

Introduction

Our contribution is about the roles of facilitators' questioning during reflection meetings in lesson study (LS) and deals with different foci in TWG28, especially organizational contexts in collaborative settings in mathematics teacher education (CSMTE), and the roles of different participants in CSMTE. We are interested in how the questioning as part of the facilitator's practice can support the development of practice knowledge among the teachers participating in LS as an example of CSMTE. However, we hypothesise that this requires explicit knowledge about the role as facilitator of others' learning in the specific context. In CERME 13, several studies in TWG28 provided insight into the facilitators' roles in LS conducted outside Japan. Masselin & Artigue (2023) presented a French case of long-term professional development for facilitators and future facilitators of LS regarding the process of their work with so-called *situation seeds* which are formulations of open problems adapted to the context of each teaching situation of the teachers in LS. Gomes, et al. (2023) reported from Portugal the importance of the facilitator's *balanced leading* actions of LS sessions, within the preparations of resources, task design and planning the research lesson. Finding a good balance between their guiding roles as project leaders and orientating teachers towards the thinking of others and exploring their thinking are crucial for the professional development (PD) of teachers' collaborative work in LS. Asami-Johansson et al. (2023) modelled teachers' collaborative learning in PD, specifically on *board work* in a case of a Facebook group in Japan. At the Facebook forum, teachers who post and lead the discussions on the forum play a role of facilitator of the study group. The more passive participants *study* how the experienced group members deal with different aspects on the posted board work and carry out their didactic discussions in a constructive manner. We wish to understand the functioning of the facilitators'

questioning better, and propose more explicit uses of notions from the Anthropological Theory of the Didactic (ATD, Chevallard, 2019) to a priori map this for a specific lesson. We consider the questioning having the function of situation seeds and as supporting the facilitators balanced leading of actions in LS, helping teachers to study and learn from their own and others' work related to the specific lesson.

When we use the term "facilitator" in the context of CSMTE, it may include slightly different levels and functions of their roles, depending on the PD system for in-service teachers, and the specific situations their studies or projects are realised in. In Japan, the role of facilitators in their PD systems spans multiple layers from national, municipal and local school levels, as well as degrees of collaboration (Miyakawa & Winsløw, 2019). Indeed, the roles and functions of facilitators are complex, and it can be challenging to foresee specific phenomena to observe in the processes of PD, and which questioning from the facilitator will develop practice knowledge among the participating teachers. Within ATD, researchers have tried to model this complex structure of organizational contexts and roles of different participants within LS as PD (e.g. Winsløw et al, 2018; Miyakawa & Winsløw, 2019; Otaki et al, 2020). In this contribution, we propose to link this with the concept of *study and research paths* (SRP) for the analysis of the questioning as situation seeds and supporting the balanced leading of the process mirroring the relation between didactic and paradidactic systems. Below we elaborate on these notions.

Theoretical background of the Anthropological Theory of the Didactic

Studying the nature and functioning of LS, Winsløw (2012) proposes the notions of paradidactic system and infrastructure to analyse and compare teacher collaboration regarding realisations of specific didactic systems. A didactic system (colloquially known as lesson) consists of a group of learners (potentially guided by a teacher or facilitator), addressing a problem or question to be studied e.g. finding the area of a geometric shape not earlier addressed by students. During the process learners and teachers may draw on relevant resources. This holds true for students learning mathematics, or teachers developing their practice knowledge (Winsløw, 2012; Miyakawa & Winsløw, 2019). The paradidactic system consists of the pre-didactic system, where teachers study for and plan the study lesson, the didactic observation system, where teachers observe the study lesson, and the post-didactic system where the realisation of the didactic system is evaluated and potentially revised (Winsløw, 2012). The work conditions, available resources, and traditions, habits and policies linked to the teachers' work naturally affects the study and learning processes of the teachers, when participating in the PD. Thus, Winsløw "defines the paradidactic infrastructure […] as everything which conditions and constraints the [paradidactic system] in its different phases […]" (Winsløw, 2012, p. 293). Though the nature of how Japanese teachers learn from and realise LS may be rather different from how LS is interpreted outside of Japan (Miyakawa & Winsløw, 2019), Winsløw and colleagues (2018) argue that detailed analyses of knowledge developed through the interaction between didactic and paradidactic systems are key to understand the potentials of exploring and exposing LS outside Japan. They propose to understand the functioning and dynamic of didactic and paradidactic systems by analysing the type of tasks or questions which are pursued and what answers are developed. Developing (practice) knowledge through question-answer dialectics is considered learning within ATD, and therefore in this paper we will focus on

question-answer dynamics when proposing the need for a priori mapping of facilitators questioning. In ATD, this mapping is linked to Study and Research Paths (SRP). Considering LS a SRP, the study processes means to pursue a teaching question (with the aim of developing students' learning) by drawing on existing knowledge, study new knowledge from resources (e.g. textbooks, web-pages, teacher journals, lesson plans, etc.). The knowledge is combined in the research process, when developing the teaching activity and presenting it in the lesson plan. This is the result of the work within the predidactic system. It represents a hypothetical answer materialised in the lesson plan, which is then realised (didactic system), observed (didactic observation system) and reflected upon (postdidactic system). All the systems will give rise to questions of both mathematical and didactic nature, some addressing students learning of the specific mathematical content. Other questions will emerge from the paradidactic system and aim at teachers' development of their practice knowledge. Thus, learning is developed when learners are engaged in the dialectics between question and answers, between study and research, meaning the dialectic of decomposition and reconstruction of knowledge (Winsløw, Matheron & Mercier, 2013; Jessen, 2017).

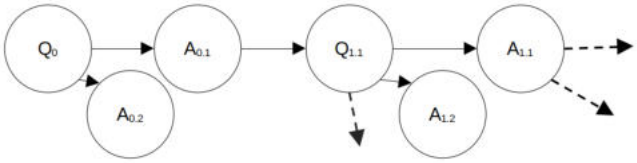

Figure 1: Illustration of a question-answer map (QA-map) to support teachers in SRP

An SRP is initiated, often by the teacher/facilitator, posing a generating question, $Q_{0,}$ in LS a teaching question. This is an open question which the learner understands, but it requires the learner to engage in study and research processes if the learner is to develop an answer, A, to the question. By means of its answers, the generating question will lead to derived questions, e.g. $Q_{1,1}$, addressing parts of the initial question. Therefore, an answer to a derived question, e.g. $A_{1.1}$ will be a partial answer to the initial question, Q_0. When working with SRP we use QA-maps (see Figure 1) to show both a priori the learning potential of posing Q_0 to certain learners, and a posteriori to map the knowledge addressed and developed during the SRP. We hypothesise that SRP can be a strong tool for analysing SRP, in particular for capturing the role of facilitators' orchestration of the learning of participating teachers. In Japan, expert teachers or researchers take the role as facilitators in different settings of LS (Miyakawa & Winsløw, 2019) and they have been part of this practice for years in what we consider apprenticeship. When initiating LS as PD outside of Japan, we lack this apprenticeship, and to further the *balanced leading* of *study* (and research) activities and productive uses of *situation seeds,* we consider it relevant to map facilitators' questioning

This leads to the research question of this paper: What is the impact of facilitators' questioning during the post lesson reflection meeting when analysed as a study and research path compared to a similar a priori analysis?

Context of the study

Our data stems from a research lesson at an ordinary public school in greater Copenhagen, where the mathematics teachers have worked with lesson study in various forms for more than nine years

(see Bahn et al., 2022). For the first four years, an external facilitator, $F_{ext,}$ supervised the activities. During the next three years, two supervising teachers were trained by the F_{ext}, and since 2022, they have been acting as facilitators. This paper, is based on one of these *internal facilitators*', F_{int}, work. Since 2020, it has been mandatory for all teachers from their school to participate in LS.

The LS-activities at this school started as empirical studies for the first author's Ph.D.-studies (Bahn, 2018), but were expanded and developed into a systematic programme for PD. Despite the LS efforts running for 9 years, we still consider the PD immature compared to Japanese LS e.g. with respect to facilitation. Concurrently three types of practices are being developed: teachers' practices of teaching (linked to the didactic system), teachers' practices of developing teaching (intertwining the didactic and the paradidactic system) and facilitators' practices of supporting teachers' development of teaching (linked to the paradidactic system). Recognising that new practices are hard to develop from scratch or from vague descriptions, it has been part of the LS-activities to invite Japanese expert teachers and teacher educators to demonstrate their principles and practices of teaching and facilitation once or twice a year. Researchers of didactics have been involved in the activities and final comments to the LS several times.

As there is no education to become a LS-facilitator, the Japanese system of apprenticeship has been adopted including learning from the Japanese visits. Thus, the training of F_{int} has been based on the accumulated knowledge and experiences from the 9 years running. Prior to facilitating LS, the F_{int} has worked as a mathematics teacher for more than 30 years, and more than 10 of those as a trained supervisor for other teachers. Furthermore, F_{int} has participated in international LS activities (Japan and UK) and numerous professional discussions with facilitators.

The data comprise the lesson plan (generated in the predidactic system), and F_{ext}'s notes from and video recordings of the study lesson (didactic observation system) and the post-lesson reflection meeting (postdidactic system). F_{ext} was not involved before the study lesson and the post-lesson reflection. In both he was solely present as an observer until the very end, where he give some final comments. The lesson plan was send to F_{ext} a few days in advance. Based on these data we can map examples of F_{ext}'s (tentative) a priori questioning and F_{int}'s a posteriori questioning during the realised facilitation and compare those referring to knowledge linked to the didactic and paradidactic systems.

During the post lesson reflection, in the transition from sharing observations to reflecting on specific elements, F_{int} proposed: "[…] From our reflections I have marked these 'Can we do something to get the conversations going?' […] and [...] 'Could we do something to make the goal clearer to the students?' or?". We consider these first derived questions in our a posteriori analyses, stemming from the overarching "Q_0: How to decide on what to discuss further?".

The lesson – intentions, realisations and reflections

In the lesson plan, the learning goals are stated as "an understanding of information needed to draw identical right-angled triangles" and "an emerging understanding of cosine without mentioning the word in this lesson. The larger the angle, the shorter the adjacent side". During the lesson, students were working with three tasks (t_1-t_3) on identical right angled triangles and one task (t_4) on the

connection between the angle and the adjacent side: t_1 Draw a triangle with a 10 cm hypotenuse, t_2 Draw a triangle with a 10 cm hypotenuse and a 5 cm side, t_3 Draw a triangle with a 10 cm hypotenuse and an angle of 60°, and t_4 Draw a triangle with a 10 cm hypotenuse and an angle of 45° in a diagram with two right angled triangles drawn, one with an angle of 30° and one with an angle of 60° all sharing the same origin (0,0). The teachers anticipated that based on the three first tasks, the students would realise the connection between the size of the angle (at 0,0) and the length of the (adjacent) side in t_4. This was intended to form basis for later learning about cosine.

Method

Following Jessen (2017) we draw on existing methodology for using SRP when we compare an a posteriori analysis of the questioning from the post-lesson reflection and an a priori analysis based on the lesson plan and observations. We see a potential in facilitators gradually developing the mapping of questions and answers derived from Q_0: How to teach students cosine in lower secondary mathematics? The lesson plan represents a preliminary answer to Q_0, which the group of teachers implement and test in a classroom. When receiving the lesson plan, key elements to be observed can be noted, but also it gives rise to specific questions linked to choices made.

Based on reading and analysing the lesson plan, F_{ext} starts mapping possible questions and answers. During observation of the study lesson, the QA-map is revised as written notes due to observations. Adjustments are made from analysing teachers' observations and comments. Final adjustments are made 'on the fly' based on teachers' dialogues. In this case, we have furthermore been able to consider other adjustments, as we have been analysing the data set a posteriori of the whole lesson study. Through the discourse analysis (cf. Jessen, 2017) we identify questions raised explicitly or implicitly and potential answers developed by the group. For this paper we present one branch of the QA-map to show how the analysis can shed light on the importance of the paradidactic system and knowledge surrounding a PD like lesson study.

Results

Below we present our analyses of expected and realised QA dialectics during the reflection meeting. We paraphrase the transcribed dialogue and Figure 1 show the mapping of the reflection.

A priori analysis – expected facilitator questioning

From reading the lesson plan F_{ext} expected some challenges for the teacher and the students stemming from an unclear connection between the two learning goals as well as between the tasks and the learning goals. For instance, while right angled triangles are fundamental to the function of cosine, the two learning goals are unconnected.

Some of the expected challenges indeed did arise in the lesson. Based on notes from observing the lesson and the sharing of observations, F_{ext} would have wanted the teachers to consider the connection between the (final) learning goal and how the teaching supports or hinders the students in achieving that goal. The didactical idea (Winsløw et al., 2018) of the lesson seemed to be (based on the assumption) that if the students see and hear a series of illustrative cases (in $t_{1\text{-}3}$) they will deduce the relevant knowledge about the relation between angle and length (in t_4), in the triangles. This clearly did not happen, according to shared observations. When starting the reflections, F_{ext}

would have wanted to help teachers realise the weak or missing connection between the tasks and the learnings goals, and that these two learning goals are unrelated. In fact, each of them could be better achieved individually.

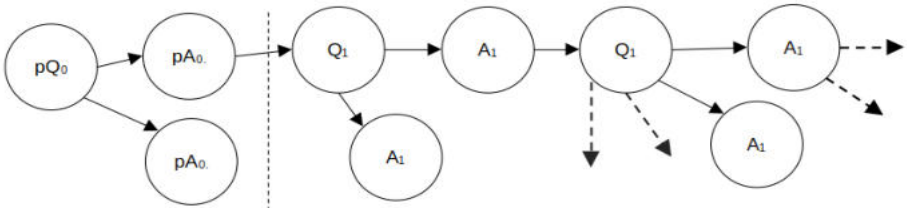

Figure *2*: Illustration of the QA-map of F_{ext}'s a priori analysis. The dotted line indicates a shift from paradidactic QAs to didactic QAs

After sharing observation data in the post-lesson reflection, the fundamental question of *how to decide on what to discuss* for the remaining.

g time arises. One approach to answering this fundamental and generative question, pQ_0, , is for F_{ext} to identify if there is an "overlap between teachers' observations and F_{ext}'s learning goal for the teachers" ($pA_{0.1}$). Since this was the case, F_{ext} could ask questions which would lead teachers to realise the missing connection. For instance, such questions as "how many students did actually realise the connection between angle and length by themselves?" as Q_1 and the mathematical question of "what *is* the connection?" as $Q_{1.1}$. Such questions should guide the teachers to formulate how such connections could (or could not) be established (first mathematically, then didactically). In this sense, F_{ext}'s questioning for others' learning bears a clear intention and critical concurrent analyses of the potential and realised situations, anchored in F_{ext}'s paradidactic knowledge.

A posteriori analysis – realised questioning

After the sharing of observations (in reality blended with autonomous reflections and evaluations), F_{int} sought to narrow down the reflections. In order to "decide on what to discuss" (pQ_0), F_{int} posed some suggestions which he had emphasised by himself from the pool of shared observations. Accordingly, F_{int} proposed "Can we do something to get students' conversations going?", as Q_1, "Could we do something to make the goals clearer to the students?" as Q_2, and (indirectly) "something else?" as Q_3. As has been validated in a following conversation between F_{int} and F_{ext}, F_{int} did not have a specific intention for these proposals other than they seemed representative of what had been shared and somewhat considered relevant. The following response by a participant, which seems to answer the question about getting students' conversation going (Q_1), was that "during the relatively short time, too much content was being worked with, which then again does not encourage students' reflections" (A_1). Underpinning the lack of paradidactic intention from F_{int}, he – as a teacher – responded in the affirmative, that the balance between the content at stake and time allocated is difficult to strike ($A_{1.2}$). The question of how to get students conversations going was not touched upon again, and not related to the mathematical content knowledge of the lesson.

Discussion and concluding remarks

So, what is the impact of facilitators' questioning during the post lesson reflection meeting when analysed as a study and research path compared to a similar a priori analysis? From the very brief

analysis presented above, we see at the reflection meeting, questions and answers shared among the participants, were linked to didactic knowledge and more practical aspects of the lesson as time management. The initial question regarding decision on what to discuss, was chosen based on F_{int}'s paradidactic knowledge and the functioning of the infrastructures build around the specific lesson study activities. But what comes after, is all based on specific observations from the lesson, and participants draw on their prior didactic knowledge and experience as teachers. The following conversation between F_{ext} and F_{int} even indicate unclear definition of the role as facilitator.

From the a priori analysis we see the potential of developing knowledge regarding the nature and function of intended learning outcomes, and how this relates more deeply to mathematical knowledge concerning geometry. Therefore, our preliminary conclusion is that the lack of explicit uses of paradidactic knowledge (how to develop the LS participants' didactic knowledge), together with the facilitator's mathematical knowledge to form questions, we see missed learning potentials for the participating teachers. Of course, this is only a pilot of the full analysis of the questioning during a post-lesson reflection meeting and more data needs to be analysed, as well as similar studies conducted. However, we see a potential of elaborating research on the role of facilitator's questioning during the reflection meeting in LS as an example of CSMTE. By considering the nature of facilitators' questioning as didactic, paradidactic or mathematical in nature, we may be able to explain why some questions function well as *situation seeds*, or *balanced leading* of LS.

Observing lesson study in Japan, much knowledge regarding facilitators' role and questioning appear as tacit knowledge developed through different forms of apprenticeship (cf. Asami-Johansson, et al., 2023). This is possible because LS is integral (though in various forms) to teachers' practice, and have been developed over decades, with vast amounts of documentation form former LS to get inspired form. When LS is initiated outside Japan, we lack the infrastructures, we lack the documentation, we lack the resources to be studied and the existence of a rich milieu where questions and answers can develop. Therefore, we might need to be much more explicit about the role of the facilitator and the knowledge needed to be taken into consideration, when learning the role of facilitator. In Denmark, teachers have a mixed background mainly consisting of (generic) didactic knowledge, little mathematical knowledge, and no paradidactic knowledge (Jessen & Rasmussen, 2020). We consider this a constraint for implementing LS and educate Danish teachers to become facilitators creating productive reasoning. More research into ways of mapping potential reasoning and reflection and compare with actual reflections made, we may be able to build stronger implementations of LS in Denmark and other countries.

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A teacher's journey in lesson studies regarding whole-class discussion

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Whole-class discussions are a significant challenge for teachers. Lesson study is recognised as a process of professional development that may contribute to gradual changes in teacher practice and which values whole-class discussions. This study aims to know how a teacher's understanding regarding whole-class discussion develops when participating in three lesson studies. This case study is interpretative, qualitative and uses discourse analysis. The participant is an in-service teacher of grades 5-6 with 20 years of experience. Data collection included audio recording of an initial interview, as well as of the lesson planning and reflection over the lesson studies. The results show that: i) after leading the first discussion, the teacher already considered important activities collaboratively done while planning; and ii) over the lesson studies, her discourse anchored on previous lesson study experiences by noticing changes in her practice and on students and suggesting new actions.

Keywords: Mathematics teaching, whole-class discussion, planning, reflection, lesson study.

Introduction

Research contributions to understanding whole-class discussions in mathematics teaching and learning are significant. According to Mosvold (2024), research has paid attention to aspects that support those discussions, such as frameworks, simulations prior to the discussion, and to leading discussions, focusing on teachers and students' actions. The author indicates that it is necessary to "lay the foundation for developing a professional practice that acknowledges the skills and knowing that are involved in leading mathematical discussions" (p. 487), because nowadays traditional patterns of communication are still widely observed in mathematical teaching and learning. Therefore, it is necessary to support teachers in preparing and leading whole-class discussions, namely through professional development such as lesson study. Lesson study has been identified as a process that gradually but significantly fosters teachers' practice and students' learning (Fujii, 2018). In this process whole-class discussions are considered the heart of the lesson (Takahashi, 2021). However, even though the development of teacher knowledge when participating lesson study is the most reported aspect in research, it "is unclear whether there is a long-term impact on teaching and learning" (Ding et al., 2023, p. 95). Thus, we seek to know how a teacher's understanding regarding whole-class discussion develops when participating in three lesson studies, by answering the following question: what changes occur in a teacher's discourse about whole-class discussions when preparing or reflecting about this lesson moment?

Whole-class discussion in mathematics

Currently, the curriculum in Portugal states the importance of promoting a dialogical learning environment, particularly in mathematics (Canavarro et al., 2021). In the curriculum, whole-class discussion is suggested as a way of working, a possible moment of assessment, a strategic teaching action and an opportunity to develop students' mathematical capacity. In fact, whole-class discussion

may promote these dimensions, but it requires teacher engagement to plan and lead a discussion and even to reframe their understanding of whole-class discussion, if necessary.

Students develop their mathematical communication capacity if this also emerges in the dialogue with the teacher. Therefore, recognizing the importance of the language that students use to describe and reason is not enough and teachers must create opportunities for students to participate in the mathematical discourse (Planas et al., 2023). Also from a social constructive perspective, Kooloos (2022) considers that discourse-based teaching, which means "teaching through whole-class discourse that builds on students' various ways of thinking" (p. 15), helps to develop students' mathematical thinking. In his study, Kooloos (2022) also considers the distinction between a whole-class discourse and a whole-class discussion: the first concerns the dialogue about students' different ideas, while the second occurs when students actively engage with their colleagues' mathematical ideas. So, regardless of the structure (Takahashi, 2021), a productive whole-class discussion is necessarily based in a discursive practice that engages students. For that to happen, interactions students-students and teacher-students must occur. Thus, we characterize a whole-class discussion in mathematics as a dynamic and collaborative discussion, facilitated by the teacher and based on the students' solutions. It is a process that seeks to support the exploration and negotiation of mathematical concepts and procedures, enhancing the development of students' learning.

Research supports the idea that it is important to promote discussions to support students' mathematical thinking (Kooloos, 2022; Mosvold, 2024), and establishes this dialogue as a tool for teaching-learning (Planas et al., 2023). However, traditional communication patterns, like recitation and lecture, are still dominant (Mosvold, 2024). Contributing to the prevalence of a traditional pattern of communication may be the challenges experienced by teachers about whole-class discussion, such as promoting students' active participation, managing time, unpredictability and diversity, and maintaining an appropriate cognitive challenge (Faria et al., 2023). These challenges, in turn, are anchored in practices that are in tension with whole-class discussions: restrict the time for interpretation, solution and discussion; questions the students mostly to clarify answers and correct errors; and registers on the board with no planning and no monitoring (Faria et al., 2023).

Lesson study and whole-class discussion

Lesson study is a teacher professional development process that is practice-based, teacher-oriented, student-focused and collaborative (Ding et al., 2024). Participant teachers collaborate in a reflective and dialogical environment with a common goal of improving their students' learning, that can be composed with five phases: goal setting, lesson planning, research lesson enacting and observation, post-lesson discussion, and reflection (Fujii, 2018). Considering teaching as a social practice, we note that teachers' discourse during all the phases of lesson study is more than speech, it is a way of (re)building meanings, professional identities and social relationships. This understanding is inspired by Fairclough (1992) definition of discourse. Therefore, although collaboration can also be challenging (Ding et al., 2024), the exchange of ideas, questions or noticing aspects is significant for teachers, since in various studies it is mentioned by the participants as a relevant part of the process.

The planned lesson in a lesson study should have a structure and flow that allow access to students' mathematical thinking (Fujii, 2018; Takahashi, 2021). For this to happen, it is important to foresee

how to promote students' interventions, namely by planning whole-class discussions as part of teaching practice. During the planning phase, participant teachers work collaboratively with various mathematical tasks to carefully choose one with a clear learning focus and that will promote important mathematical ideas to emerge in the discussion (Fujii, 2018). In this professional development process, even the board is a dimension that needs to be planned "in order to best compare and contrast students' solutions" and "to organise both thought processes and results" (Fujii, 2018, p. 7). In addition, according to Takahashi (2021), it is also important that these lesson plans have flexibility and openness, so that questions and answers arise from students' interventions, going beyond show and tell. Planning the whole-class discussion in a lesson study process goes beyond the written lesson plan. In fact, while the students are working autonomously on the task, teachers must monitor their mathematical activity so that they "can use the information they gather during this time to plan who they want to call on during the whole-class discussion, and in what sequence" (Takahashi, 2021, p. 26).

Methods

This study follows an interpretative and qualitative approach, with participant observation design. We analyse the case of Marta, who has been teaching mathematics at grades 5 and 6 (10-12 years old), for around 20 years. She was interviewed before participating in the first lesson study, carried out with a group of 9 teachers (2021/22), and participated in two more lesson studies (2022/23), the second one with three teachers and the third with two. The first author was the facilitator and lead the interview. All the authors participated in data analysis and manuscript' writing.

The data were collected i) in Marta's initial individual interview (2021-2022); and ii) in the planning and reflection phases of her first, second and third participation on lesson study (2021-2023). The interview had the purpose of getting to know several aspects of Marta' practice, with a particular focus on whole-class discussions. Data are analysed through Fairclough's (1992) critical discourse analysis, which considers discourse to be understood as a mode of action, even if only intended. The concept of intertextuality states that all discourses are anchored in previous ones. Thus, the change of each individual discourse can provide evidence of an intended change in teaching practice. Studying this intertextuality means identifying discursive changes arising from a new combination of different discourses or noticing the absence of new elements in the discourse (Fairclough, 1992). Therefore, after the transcription of the sessions, we selected excerpts associated with the preparation of the whole-class discussion and the reflection on discussions. After, we selected Marta's interventions in which it was possible to identify changes in her discourse regarding whole-class discussions.

Results

Initial interview

When asked about students' communication during discussions, in her interview, Marta replied:

> Marta: Communication, how so? To solve an activity, sometimes they even speak in a large group because I launch an activity and they have the opportunity to explain, right?... I think they talk. I try to work on it, but sometimes it's not easy. They struggle to communicate.

There is a strong association of student communication with talking or explaining strategies. Even though Marta considers that students speak, she identifies that this is a skill that presents challenges for students. About this struggle, she considers:

> Marta: I think it's more a question of language and mathematical concepts… Then I say: "But you have to know how you thought. Explain, how did you think?".... And then, sometimes, I also make corrections and ask again for them to explain.

Thus, Marta links with the students' struggle to communicate to their lack of conceptual understanding. In her intervention, it is not clear how she supports students to develop this skill, particularly during the discussion, since she gives them the responsibility of knowing how they thought and verbalizing this thought. Regarding students communicating with each other in situations of disagreement, Marta stated:

> Marta: Sometimes it's even better than having to explain it ourselves. We can't always do this… We ask those students who finished first or who already have a more formed idea to help others. They accept this help.

In these excerpts from the initial interview, Marta considers that students struggle to communicate their mathematical ideas, particularly during discussions. In this regard, she corrects students or asks them to repeat the explanation and, although she considers positive that students communicate with each other in situations of disagreement, this is understood has a moment of accepting help.

First lesson study

The teachers began a dialogue about how to act when questions arise about the task. Regarding the work on Geometry and Measurement, the topic selected for the research lesson, Marta says:

> Marta: My tendency is when I see them measuring, if is not necessary, for example, I say 'Well, that's not for measuring!', and then I don't let them continue the task alone.

By not letting her students work on the task by themselves, Marta may be lacking to promote their mathematical activity and authorship, important aspects to enrich a whole-class discussion. On this first lesson study, when reflecting on the lesson led, Marta focused a lot on the whole-class discussion, specifically on how planning and leading it felt like an innovation to her practice:

> Marta: When I facilitated the lesson, the whole-class discussion was what most deserved my commitment and attention as it were, perhaps, the most innovative factor in my teaching practice. Leading students in whole-class discussions in order to highlight and connect their contribution to the aim of the lesson is to leave my comfort zone because, in this way, there will always be aspects that I cannot anticipate, but that I will have to know how to attend in order to guide a productive whole-class discussion.

When reflecting, Marta gave special attention also to the tasks used:

> Marta: Tasks should have a more exploratory character to be more interesting and challenging, as this will encourage greater student participation… In the sessions we analysed several student responses to a given task. This analysis allowed us to see those different ways of communicating. The student must be able to explain their reasoning and be valued for it.

Instead of speaking about not letting students do the task alone if questions arise, Marta reflects on the need to have exploratory tasks and how this is connected to students participating and communicating in a discussion.

Second lesson study

In response to a question from the facilitator, Marta began to analyse a diagnostic task, but she turned into a broader reflection, by focusing on students' difficulty in explaining and recording their mathematical ideas so that they could later be mobilized in the whole-class discussion:

> Marta: They're already beginning to ask each other questions and I think this is good. But I'm also a bit more alert, because of the last year's lesson study and research lesson... We're more aware of certain things. Now, I move around, look at different strategies and ask them to go to the board. When they go to the board, I ask them to explain to their classmates.

Marta went on to specify some aspects associated with the students' activity during the discussion. She identified changes in the interaction between the students during the discussion, which she seems to associate with changes in her practice, influenced by her first participation in lesson study:

> Marta: A difficulty they have is putting ideas down on paper and organising them in writing, because they can convey ideas orally or sometimes some of them just stay in their heads... We have to encourage that what they talk about in their small group is actually what is discussed afterwards and that it is explained as well as possible.

Marta pointed out that, in relation to the whole-class discussion, the students tend to verbalize other aspects or only the initial part of what they explored during the autonomous small group work. This struggle to communicate during the whole-class discussion was already mentioned by her during the interview. However, during this planning phase Marta was linking the teacher' actions during autonomous work and the students' communication during the discussion.

Third lesson study

Marta schematized how the board could be organised during the whole-class discussion (Figure 1). In the previous lesson studies, although the board organisation was considered, Marta and the other teachers participating in the lesson study never made a drawing or a scheme.

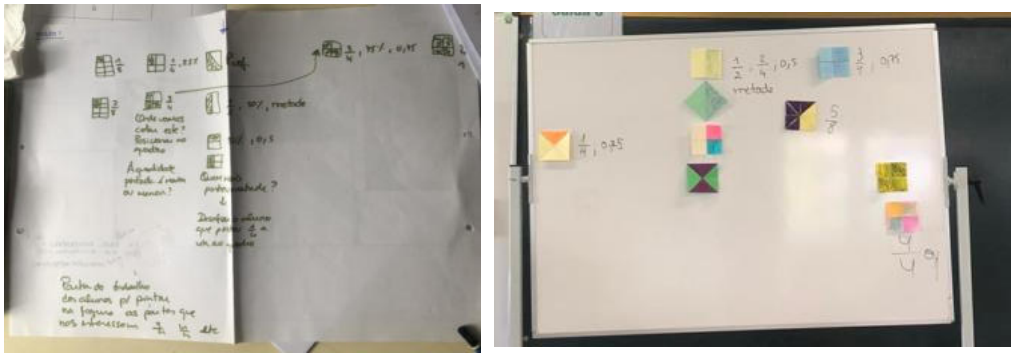

Figure 1: Board scheme and board in the actual lesson

When reflecting on the lesson, Marta focused on the board organisation as a main aspect of the whole-class discussion preparation and as support for students learning.

Marta: That's right, that was one of the things I had also noted down. And it was quite curious, because Filipa always focused a lot of the sessions on organising the board and here I really had the feeling that having thought about and organised the board, what it would look like when we stuck things on the board, was very important because the fact that we had already done it and thought about it, helped. It made a difference because maybe I was going to stick things up randomly and the organisation of ideas for the students wouldn't be so good. So, the fact that we had already rehearsed and thought about it was good, because I stuck mine in the middle and then asked the students where they should stick it. The way I called students out was also even more intentional.

Besides the research lesson that was observed by all participants, Marta had the chance to lead the lesson to another class, before our post-lesson discussion two days after, where she shared:

Marta: Well, I was just remembering that too. I was already remembering that from 1/4, when I said it was 0.25... Because if it's the whole square, and then we divide, and half of 100 is 50, and half of that is 25... I was also thinking that it could have been explored a bit and that didn't happen. And it already came up today, so there are already differences from a [discussion] leading to the next.

From one lesson to another, she adjusted her interventions during the whole-class discussion in order to prompt the connection between different representations. Besides reflecting on the particular whole-class discussion that was observed, she extended her reflection to the possibility of improving her leading of this moment in other lessons.

Discussion

Our aim was to know how a teacher's understanding regarding whole-class discussion develops when participating in three lesson studies, by identifying what changes occur in her discourse about whole-class discussions when preparing and reflecting about this lesson moment. Her discourse does indeed show changes over time, supporting the idea that discourses are dynamic and are not neutral, carrying ideologies that can be shaped through social relations (Fairclough, 1992), as it happens in highly collaborative professional development contexts like lesson study.

In the initial interview, Marta associated communication during the whole-class discussion with students speaking or explaining. She considered that students struggle with communication in discussions because they have conceptual difficulties. Alongside this difficulty, she stated in the interview that she corrects the students when this happens, reinforces the request for an explanation and gives the student the responsibility of knowing how they thought and knowing how to explain it during the discussion. Regarding mathematical disagreements in a whole-class discussion, Marta said that those students who have finished or who have clearer mathematical ideas can help the others, considering that their colleagues are mostly willing to accept help. Thus, this does not necessarily mean that a negotiation of mathematical ideas is being prompted. Therefore, we may consider that Marta's initial understanding of whole-class discussion was more aligned with a whole-class discourse that with a whole-class discussion (Kooloos, 2022).

Throughout the planning and reflection that took place in the *first lesson study*, Marta's discourse continued to associate the teacher's interventions with helping the students, i.e., if there is a misunderstand of the task, they solve it together with the teacher. This practice may run counter to promoting the students' mathematical authorship and impoverishes the discussion, since it restricts

students' solution and discussion time (Faria et al., 2023). In addition, during the reflection after the first research lesson and the whole-class discussion that followed, Marta's discourse began to consider important elements of lesson study such as anticipating strategies, selecting challenging tasks, managing unforeseen events and leading productive whole-class discussions, encouraging student participation and valuing different ways of communicating (Fujii, 2018; Takahashi, 2021).

In the *second lesson study*, Marta stated, while preparing the whole-class discussion, that the students were starting to pose questions, which is something that when considered and managed by the teacher can enrich the discussion (Kooloos, 2022; Planas et al. 2023; Takahashi, 2021). Students' questioning is an aspect brought into the planning by Marta, an action that she did not associate to the students until this moment. As for her practice, she considered herself more alert to whole-class discussion, sharing that now she circulates during autonomous work, identifies different strategies, selects them and asks the students to explain (Takahashi, 2021). Also, in this second lesson study in which Marta participated, she understood more broadly the students' difficulty in communicating, going beyond difficulties in conceptual understanding. By then, when planning the whole-class discussion, she provided suggestions for her action and the actions of her colleagues, such as encouraging that the ideas discussed in the small group were brought to the discussion by the students and explained as well as possible. This new discourse illustrates an intention to develop her practice, which is shaped by lesson study as Marta clearly illustrates (Fairclough, 1992).

When taking part in the *third lesson study* Marta planned the board for the whole-class discussion with more detail that ever before (Figure 1). When reflecting on this aspect and its impact on the discussion, Marta related the teacher's aim regarding the board organisation and the sequence of students' interventions to the quality of the students' mathematical understanding (Takahashi, 2021). Marta also added to her reflection how teachers' interventions during the whole-class discussion may support or not the achievement of the lessons' aim, namely the establishment of connections, which is something that she considers to be able to improve from a class to another.

Marta's discourse also illustrates aspects that seems to have contributed to these changes, such as a) exploring tasks and anticipating students' solutions and challenges; b) the presence of a facilitator who supports the exploration of aspects like the board organisation; c) participation in more than one lesson study enriching her reflections. Overall, throughout the sessions of the lesson studies, it is possible to identify that Marta's discourse about whole-class discussion changes. By the moment she begins to reflect about the whole-class discussion that she led on the first lesson study, she begins to anchor her discourse to particular moments and activities that are important in lesson study. Over the next two lesson studies, her discourse when preparing and reflecting about the whole-class discussion is anchored also in her previous experience within the context of lesson study.

Conclusion

The results illustrate two main aspects: i) after leading the first discussion, Marta's discourse begun already to consider activities collaboratively done during the lesson study planning phase, like selecting tasks; and ii) while planning and reflecting over time, Marta's discourse was anchored on her previous lesson study experiences. This aspect becomes apparent when Marta notices change in students such as posing questions and changes also on her practice, like monitoring students'

mathematical activity. Continuous participation in lesson study has allowed Marta to gradually plan and reflect on different aspects related to whole-class discussions: the tasks, the students' solutions and encouraging their participation and, finally, the organisation of the board and it connection to the quality of the students' learning, an aspect so highly highlighted by the literature on lesson study. Therefore, even though the contribution of lesson study may be traceable during the teacher first participation, sustained involvement in this process is important. Due to limited space, we focused on how Marta's understanding of whole-class discussions developed over time. The triangulation of her understanding, planning and leading, considering lesson study features such as collaboration, will be object of further work to better understand the contribution of lesson study to whole-class discussions.

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Reasons to teach and practise collaboration at university and school level

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Research has shown that collaborative settings and activities positively impact motivation and learning in both teacher education and schools. Collaborative skills are also considered essential 21st-century skills. However, there is limited research on the necessity of teaching these skills and effective methods for doing so. It is often assumed that pre-service and in-service teachers, as well as students, are inherently capable of collaboration. We hypothesise that collaboration requires practice, as it is not an innate skill. Further, how pre-service teachers perceive collaboration, and its use seems to impact their readiness to collaborate and to implement collaborative tasks. We will also discuss activities we designed to enhance collaboration.

Keywords: collaboration, perceived usefulness, perceived ease of use, pre-service teachers

Introduction

Education and its focus are evolving, and in the last few decades competencies and 21st-century skills have gained prominence. Schools now focus not only on knowledge transfer but also on preparing students for their adult life. Collaboration and cooperation are crucial in both educational and work settings. As Robutti and colleagues (2016) emphasise in their study, these terms are similar but differ in meaning. Both collaboration and cooperation involve working together, but they differ in their approach, much like multidisciplinary and transdisciplinary settings. Cooperation aligns with multidisciplinary settings where all participants contribute towards the same goal. In contrast, collaboration aligns with transdisciplinary settings, requiring negotiation among participants. This implies that collaboration is more complex, demanding negotiation, communication, networking and finding consensus. The growing societal demand for collaboration may explain why collaborative settings had an extra working group, TWG28, at the 13th CERME conference and the growing research interest in collaborative settings (Robutti et al., 2016). As noted in the introduction to TWG28 in the proceedings of CERME13 (Wake et al., 2023), the group was split into two sub-groups to facilitate discussion of the numerous papers and posters, many of which focused on Lesson Study (LS). LS, a method of teachers' professional development originating from Japan going back to 1872, has become a key area of educational research (Robutti et al., 2016). This suggests that the main interest is teachers' professional development, and the studies build on pre-service and in-service teachers' ability and willingness to collaborate. We aim to study if this willingness and ability, the prerequisites of collaboration, are there or need to be developed and enhanced and how this can be achieved. While young children in many countries must attend kindergarten to socialise before

entering school, during school time, teachers' methods decide if a student is exposed to collaborative activities; however, nobody questions the necessity of learning collaboration. Consequently, not all students are exposed to collaborative activities. This raises the question of whether pre-service teachers can collaborate. Research indicates that childhood experiences influence teachers' teaching methods (Castro, 2010) and that the underlying beliefs about teaching and learning are stable (Liljedahl et al., 2012). To examine this, we will conduct research involving students, pre-service and in-service teachers. First, we will review the literature to define collaboration, then establish a theoretical framework, describe data collection and analysis, and finally, discuss our findings.

Literature review

While research exists on preparing teachers to collaborate (Santagata & Guarino, 2012), it often focuses on discussion skills and lesson study. However, collaboration has numerous forms requiring more active participation. In educational settings, teachers can collaborate by co-teaching a class or collaborating across disciplines. Regardless of the form, collaboration involves negotiation, agreeing on goals, working together and sharing responsibilities (Barfield, 2016). Collaborative tasks, such as project-based and STEAM (Science, Technology, Engineering, Art and Mathematics) activities enhance learning (Houghton et al., 2024). Project-based learning, defined as students engaged in a science-related topic over a longer time working in groups (Krajcik & Blumenfeld, 2005), inherently requires collaboration. These transdisciplinary activities necessitate networking among teachers, schools and even beyond. This challenges the traditional classroom setting and requires teachers to practise boundary crossing (Akkerman & Bakker, 2011). In this context, boundary crossing means that teachers step outside their areas of expertise and collaborate with colleagues from unfamiliar disciplines to find solutions through negotiation. Akkerman and Bakker (2011) carried out a literature review and studied what boundary crossing in different research areas means. They identified four basic mechanisms: (1) Identification: noticing differences and similarities; (2) Coordination: developing strategies; (3) Reflection: examining perspectives; (4) Transformation: changing one's perspectives. In all cases, boundary crossing also involves a potential learning opportunity (Akkerman & Bakker, 2011). Although collaborative teaching practices and boundary-crossing have been researched, there seems to be a research gap in understanding how collaboration skills are acquired. A suitable theoretical framework is needed to explore this, one particularly focusing on the social aspect of knowledge creation and teachers' beliefs, as these influence their teaching methods (Schoenfeld, 2011). We understand beliefs as Goldin and colleagues, i.e., finding something true or false depending on knowledge, social or mental state and affective status (Goldin et al., op. 2009).

Theoretical framework

Many studies on collaborative settings rely on Activity Theory, which stems from Vygotsky's work in the 1920s. This theory claims that learning occurs through interaction with tools, culture and experts who provide scaffolding (Hashim & Jones, 2014). Noticing that learning requires sociocultural and historical contexts, this theory has been recently referred to as Cultural Historical Activity Theory (CHAT) emphasising the connection between thoughts and actions (Nussbaumer, 2012). CHAT views a whole activity as a unit comprising a subject (involved person), an object (the

task) and tools (mediating device) (Hashim & Jones, 2014). In educational settings, CHAT often frames studies on Lesson Study (LS) (Wake et al., 2023; Robutti et al., 2016). However, this already requires collaborative skills, and we aim to investigate how these skills are acquired. As boundary crossing requires negotiation, we employ the expanded version of CHAT, Engeström's Theory of Expansive Learning (TEL) (Engeström, 2014). TEL calls for a multi-dimensional approach and a community of learners who act collectively to construct new knowledge crossing boundaries. This aligns with Social Constructivism, which views knowledge creation as a social process (Amineh & Asl, 2015). To understand teachers' beliefs, we adapt Davis's Technological Acceptance Model (TAM) (Davis, 1985) into the Innovation Acceptance Model (IAM) (Békési et al., in progress). Davis studied user acceptance of technology focusing on perceived usefulness and ease of use to predict future implementation. The same variables apply to teachers and students in an educational context. Innovation, in this sense, encompasses anything new, including collaborative settings. Once teachers are convinced of the usefulness and ease of use of an activity or a teaching method, they are more likely to implement it. TEL and Social Constructivism guide our study on collaboration, while TAM and IAM inform our understanding of teachers' beliefs. This theoretical network guides our study to address the following questions:

RQ1: How do pre-service teachers' school experiences impact their attitudes towards collaboration?

RQ2: How can teacher training enhance pre-service teachers' beliefs about collaborative settings?

RQ3: How can we support students, pre-service, and in-service teachers in collaborative activities?

Methods and data collection

We studied three different groups: students, pre-service teachers, and in-service teachers, using different approaches and methods. Over two school years (2022-2024), we conducted collaborative STEAM projects with three lower secondary school classes. The two or four-month-long projects involved more subject teachers and required collaboration among students, teachers, and between students and teachers, and occasionally even beyond. These project-based collaborative STEAM activities explored topics such as settling down on MARS, renewable energy, the power of water and designing a house. These activities were carried out by a teacher team teaching English, German, Biology, Geography, Physics and Mathematics, the Mathematics teacher, also a researcher in the team, initiated the projects. The teacher team evolved after the success of the first project in 2022. Four teachers were involved in the first cycle and six more joined the team in the second cycle with varying levels of participation. TEL's four mechanisms guided the teacher team's work: identification, coordination, reflection, and transformation. We identified shared goals and coordinated our approaches in regular meetings. We reflected on each other's views and adapted ours to find an agreement. Collaboration between students and teachers followed a cyclical pattern (Lewis et al., 2019): (1) study: establishing a team and selecting a topic; (2) plan: identifying teaching/learning goals; (3) teaching/carrying out the activity; (4) reflection: evaluating the activity.

As for pre-service teachers, the researcher, who is also a teacher trainer, designed and conducted a course focusing on innovative teaching methods involving collaborative teaching methods, technology implementation and STEAM activities. This course aimed to familiarise pre-service

teachers with these methods and influence their beliefs about their usefulness and ease of use. The course provided examples, practice opportunities and had a collaborative learning environment during the whole course. Pre-service teachers worked in groups, which were regularly rearranged, and the groups also shared their views on Padlet, a collaborative platform. Pre-service teachers' work also followed the same cyclical pattern but in a slightly different way. As the course participants didn't know each other, the researcher created the teams, defined the topics and took an active part in the reflection to make pre-service teachers aware of possible changes in their beliefs.

To answer the first two research questions, data were collected from students, teachers, and pre-service teachers through Likert-scale questionnaires and open-ended questions. All stakeholders, even external contributors, completed questionnaires during and after completing each project or the course. Pre-service teachers were also questioned after their first implementations. To answer RQ1 and RQ3, we conducted semi-structured interviews with a smaller group of students, including one who was particularly involved in the projects, as well as two teachers and four pre-service teachers.

Data analysis and findings

The questionnaires included general questions about participants' likes and dislikes, as well as their expected or experienced learning effects across all groups. Some questions were more specific depending on the group. In this section, we present only the data relevant to our research questions. Figure 1 and Figure 2 display some general questions and the responses from all stakeholders. Note that the pre-service teachers answered these questions after the course but before implementing the activities.

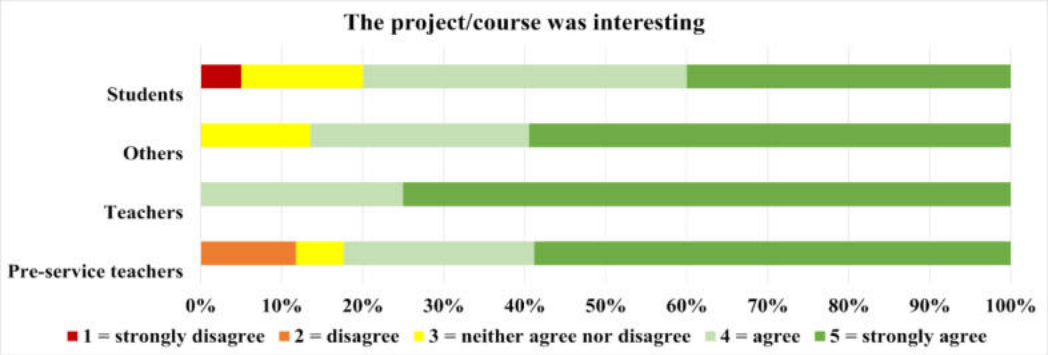

Figure 1: Stakeholders' interest in the project or the course

Comparing the results, the stakeholders seem to differ in their beliefs. Students found the project interesting but were unsure about the learning impact. Others, mainly parents attending the presentation day, found the project both interesting and beneficial for learning. While all teachers found the project interesting, they were unsure about the causes for the learning impact if it was due to the project or another variable.

Pre-service teachers found the course focusing on these projects just as interesting as students found the projects themselves; however, pre-service teachers were not fully convinced about the projects' positive learning impact. Figure 3 depicts pre-service teachers' responses collected after their first implementation of the activities in their teaching.

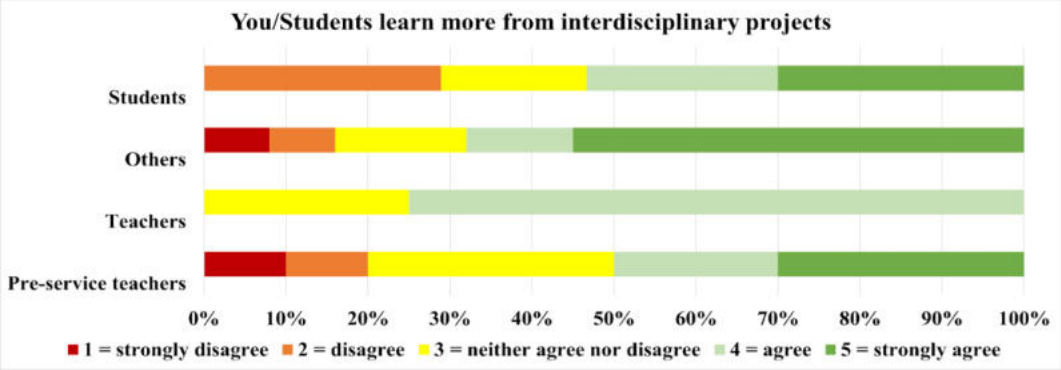

Figure 2: Stakeholders' beliefs about usefulness

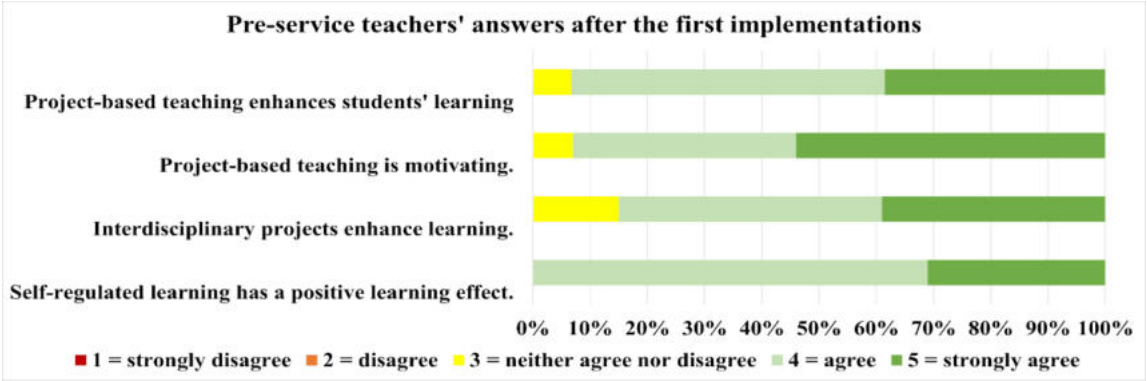

Figure 3: pre-service teachers' beliefs after the first implementations

A comparison between Figure 2 and Figure 3 reveals a clear shift in pre-service teachers' beliefs about interdisciplinary projects. Negative beliefs disappeared, and approximately 90% of teachers had positive or very positive beliefs after implementing the activities. To investigate this shift further, teachers and pre-service teachers also received questions regarding their school experience and whether they participated in interdisciplinary projects as students or at the university. Only two teachers in the team are under 35, and only one of them participated in a project still at elementary school as she mentioned in her interview, otherwise the teacher team had no prior experience with project-based collaborative teaching. Pre-service teachers' answers showed a similar trend. During the 4 cycles starting in 2021, we questioned 54 pre-service teachers, and only two in the latest cycle in 2024 said that they participated in an interdisciplinary school project.

To further explore the influence of prior experience with collaborative settings, we conducted interviews. The two youngest members of the teacher team seemed most likely to have experienced collaborative settings during their studies, so we interviewed them. They have similar backgrounds: both are in their early thirties; both are head teachers of a class and teach more subjects. We will refer to them with the fictitious names Anne and Kathy. One significant difference is that Anne participated in a project as a primary school student, but Kathy didn't as we discovered during the interview. Then, both had a traditional secondary school and university education. As for their participation in

the project, Anne joined the teacher team in the first project already, participated in teacher team discussions and contributed actively to the project. We interviewed her to understand her motivation.

Researcher: Why did you agree to join the project? What was your motivation?
Anne: Why did I agree to it? Well, first of all, because I believe the students benefit tremendously from it. It's something completely new for them, and it helps foster cohesion and cooperation within the class. I also think it's beneficial for the teaching team to have more points of contact and to encourage more communication among us. I believe that parents also view it very positively, as it creates a good impression, showing what we are doing at the school, and the parents can see that as well.

In contrast, Kathy started a separate project in her class and followed a different structure. She worked independently and joined forces with the teacher team just for the sake of a presentation. Although Anne and Kathy had different approaches, they both considered project-based interdisciplinary teaching as beneficial. Kathy emphasised in her interview that the greatest gain is the social development experienced in her class during the project.

Researcher: What do you think the students learned during the project?
Kathy: Well, just working independently, working in a team. And simply planning so that it has to be finished by Friday and presented, and within the production just coordinating who does what and how we do it. So, it has a lot of— I mean, I find it has a lot of social advantages. They learn to work independently. It's really about the social aspect and not about any specific expertise.

Kathy's actions seem contradictory. While she finds the social aspects of her project the most important for her students, she is reluctant to engage in collaboration with her colleagues.

Researcher: Could you imagine collaborating with other colleagues and working together?
Kathy: It always depends on who is involved. Well, I must admit that I am a bit of a perfectionist, and I always like to have a certain structure. That's a small problem where I need to let go a bit, I have to say. When several people are involved, we have to do it completely differently. You can't do it the way we did it now.

Kathy gave a detailed description of her school experiences referring to a lack of options, sitting all the time and experiencing just frontal teaching, which illuminates her attitude. Kathy's case proves that school experience impacts future teaching practices. Even though Kathy recognises the potential benefits of collaboration for her students, she cannot cross the boundary between individual work and collaborative engagement within a community.

Regarding RQ1, the answers to the questionnaires (Figure 1, 2 and 3) along with the interview data suggest that early experiences impact both pre-service and in-service teachers' beliefs. However, recognising the value and usefulness of collaboration can motivate teachers to implement collaborative activities, and this addresses RQ2. Experiencing collaboration's positive effects both in learning and teaching facilitates a shift in beliefs.

We also questioned the students about their likes and dislikes regarding the projects. This was presented in detail at the MEDA4 conference in Bari (Békési et al., 2024). Here we highlight just our findings: students valued the social aspects: they praised working in groups in the first place and the freedom to select a topic and to create in the second.

The data reveals the acceptance of the use of collaborative activities by all stakeholders, but it varies if one considers the positive learning impact, the social aspects or the increased motivation. Students who participated in multiple projects learned to collaborate through direct experience (learning by doing) and appreciated its benefits. Pre-service teachers and teachers belong to a generation that has no such an experience from their school life and crossing the boundary to start collaboration seems an obstacle to overcome. Interviews and questionnaire responses suggest that early school experiences significantly impact teachers' beliefs and practices. However, pre-service teachers became more open to collaboration after experiencing its benefits during the course resulting in a belief shift demonstrated by the questionnaire administered after the first implementations. This finding addressing RQ3 implies that collaboration needs to be practised and through practice, you can learn to collaborate.

Discussion and conclusion

Our findings indicate that a lack of experience with collaborative settings during their own schooling significantly impacts teachers' practices as they have doubts about the use of innovation and the ease of use as described by the IAM model. This aligns with existing research emphasising the crucial role of teachers' beliefs, which are strongly shaped by their own school experiences, and determine their instructional choices (Castro, 2010; Liljedahl et al., 2012). However, we found that providing practice opportunities can foster a shift in beliefs regarding the use of collaboration. This finding is consistent with Santagata and Guarino (2012), who argue that pre-service teachers need preparation for collaborative settings, and with Engeström's (2014) TEL theory suggesting that scaffolding is necessary for teachers to cross boundaries. Additionally, our research supports the idea that collaboration is valuable not only among students but also between pre-service and in-service teachers, as demonstrated in previous studies (Houghton et al., 2024; Akkerman & Bakker, 2011). Overall, our findings confirm that both experience and structured support are essential for fostering a collaborative mindset among educators.

Since practice shapes beliefs, it is essential to design projects and courses that offer practice opportunities for collaborative engagement. We intend to study students' artefacts and their interviews to triangulate our data. Future research should also explore the long-term effects of collaborative practices on both students' learning and teachers' instructional practices.

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Types of talk and mathematical knowledge for teaching problem-solving: Dialogic analysis of a lesson study

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This study investigates mathematical knowledge for teaching problem-solving in lesson study among primary school teachers. Eight meetings were analysed, revealing mainly three types of talk, cumulative, disputational, and exploratory, evolving across the lesson study cycle. Knowledge levels differed among the types of talk, offering insights into teacher knowledge development.

Keywords: Type of talk, lesson study, mathematical knowledge for teaching problem-solving.

Introduction

This research aims to analyse the type of mathematics problem-solving knowledge for teaching used when primary school teachers work collaboratively in a lesson study process and examine how dialogic interactions contribute to knowledge construction.

Two papers and one conference paper have already been published about the first part of the study. The first paper outlines the theoretical and methodological aspects (Clivaz, Daina, et al., 2023), while the second presents the results of the analysis of the first part of the LS cycle (covering five out of eight meetings, Clivaz, Batteau, et al., 2023). Following the CERME13 conference paper (Clivaz, Presutti, et al., 2023), this contribution links the micro-level utterances' dialogic features to types of speech at the meso-level of longer episodes during LS meetings. These results have not been published yet.

Theoretical framework

This article lacks the necessary length to comprehensively expound upon the associated research's theoretical framework. Consequently, we direct the reader to our previously published works elucidating the concepts of Lesson Study (Clivaz, Batteau, et al., 2023, p. 2); knowledge levels (Clivaz, Batteau, et al., 2023, pp. 2–4); as well as Lesson Study Dialogic Analysis (LSDA, Clivaz, Daina, et al., 2023, pp. 24–28; 37). We present briefly here the Mathematical Knowledge for Teaching Problem-Solving, the Lesson Study Dialogic Analysis, and its link to the types of talk.

Mathematical Knowledge for Teaching Problem-Solving (MKTPS)

Research on problem-solving (PS) has mainly focused on students' perspective. However, some studies have explored teachers' understanding of problem-solving and their mathematical knowledge for teaching (Ball et al., 2008). Based on a literature review of studies from 1922 to 2013, Chapman (2015) identified six categories of problem-solving knowledge for teaching, including problem-solving content knowledge and pedagogical problem-solving knowledge. These categories are influenced by teachers' problem-solving proficiency, affective factors, and beliefs. Adopting Chapman's findings, we propose a graphical representation of this categorisation (Figure 1) to bridge

the categories of mathematics problem-solving knowledge for teaching and mathematical knowledge for teaching.

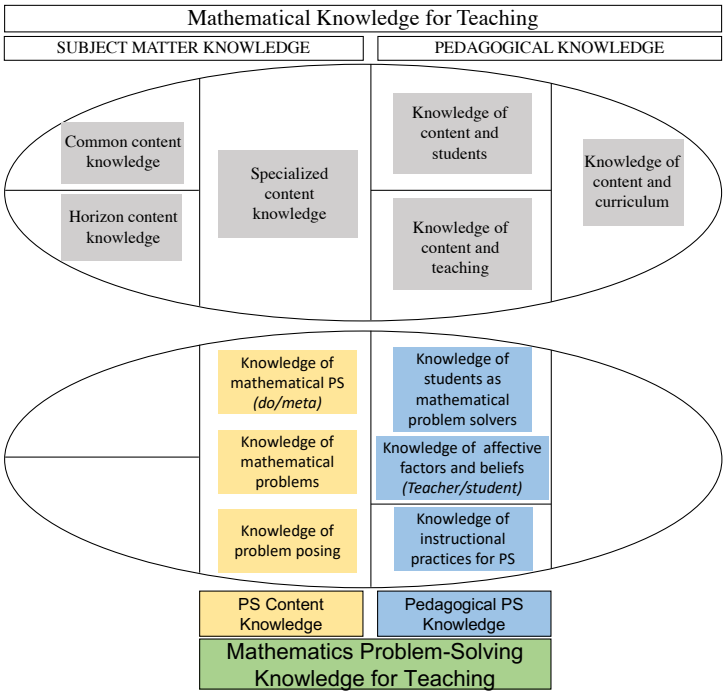

Figure 1: Mathematical Knowledge for Teaching Problem-Solving (MKTPS). Mathematical knowledge for teaching (the upper grey part of the figure) is taken from Ball et al. (2008); mathematics problem-solving knowledge for teaching (coloured categories) is from Chapman (2015). For the description of the categories, refer to (Clivaz, Daina, et al., 2023).

We then needed to determine knowledge levels (Table 1) to identify the participants' knowledge evolution throughout the meetings or differences related to the roles (teacher, facilitator). These levels do not constitute a hierarchy: contextualised knowledge or questioning is as valuable as generalised knowledge. The knowledge levels are labelled from 1 to 5.

Table 1: Knowledge levels (Clivaz, Daina, et al., 2023)

Knowledge levels	
1	Inaccurate knowledge, lack of knowledge, self-assumed ignorance, and/or debatable personal representation
2	Unexplained knowledge. Observation, testimony
3	Incomplete knowledge, knowledge with a low degree of certainty. Explicit questioning
4	Contextualised explicit knowledge. Speaker knows or appears to know the rationale
5	Generalised, decontextualised knowledge (decontextualisation process, possibly not fully completed). Generic example

Types of talk

Our analysis of the utterances and types of talk is based on the work of the Cambridge Educational Dialogue Research (CEDiR) team who developed a Scheme for Educational Dialogue Analysis particularly the SEDA group (SEDA, Hennessy et al., 2016). We adapted their micro-level utterance

analysis to our context (Clivaz, Daina, et al., 2023). Following the SEDA method, we linked this micro-level analysis at the utterance level to the meso-level, defined by a series of speaking turns in which the topic remains constant. At this level, we adopted the three types of speech characterised by Mercer and colleagues (Littleton & Mercer, 2013, pp. 15-16; Wegerif, 2020, pp. 28–29):

> Cumulative talk: Characterised by speakers building positively but uncritically on what the others have said. Everyone expresses their knowledge in the conversation, but there is no critical position or evaluation. Cumulative discourse is characterised by repetitions, confirmations and elaborations.
>
> Disputational /qualificatory talk: Characterised by disagreements between discussants and a competitive atmosphere in which each person makes their own decisions. There is little sharing of resources or constructive criticism, and interactions are often about taking short positions without justification.
>
> Exploratory talk: Characterised by everyone engaging critically but constructively in the conversation. Everyone brings information they consider relevant; this is questioned and argued, and group members seek to reach agreement before moving on to another stage. It is possible for an outside observer to follow the line of reasoning.

These types of talk have been determined by our LSDA indicators (Clivaz, Daina, et al., 2023, p. 37). In order to move from the classroom to the LS context, the members of the CEDiR group made some adjustments to these types of talk (Dudley, 2013). We also added the qualifier *qualificatory* to disputational and considered two more types of talk: Structuring conversation and managing understanding. These two types were less prevalent in our results, and we don't discuss them in this paper.

These types of discourse can be seen as different ways of thinking together. The exploratory talk is considered the most 'educationally effective' (Littleton & Mercer, 2013, p. 16).

Data, method and research question

Context, data collection and method

A LS group of ten teachers and facilitators from the Lausanne region was formed for this study. The facilitators, one math educator (the first author of this paper) and one teacher, aimed to focus on problem-solving in the classroom, without necessarily focusing on a particular mathematical topic. After discussing difficulties in teaching problem-solving, the group concentrated on helping students understand the problem representation. During the next meeting, they selected a problem from a state test that students struggled with in the past. The group then discussed representation, student approaches, and mathematics involved before planning a research lesson. After teaching and observing the lesson, the group discussed their observations of student performance and the importance of manipulatives, whole-class discussions, mathematics, and support for struggling students. Based on this discussion, they planned a second lesson in a second class, observed and discussed it. The last meeting allowed a discussion about the two lessons, the finalisation of the lesson plan, the assessment of the cycle and the desire for the next cycle.

The eight meetings were video recorded, transcribed and coded with the help of qualitative data analysis software. The coding of the data and the method are described in detail in Clivaz, Daina, et al. (2023). The three levels of analysis are nested, and provide a systematic structure for the analysis of the data. Systematic coding is carried out at micro-level. Then, the aim is to characterise the communicative events in such a way as to highlight the "interaction function" in the sense of Dudley (2013). Therefore, each communicative event can be characterised as a type of talk (arrow 2 in Figure 2). This type of discourse allows analysing the LS phases (arrow 3 in Figure 2).

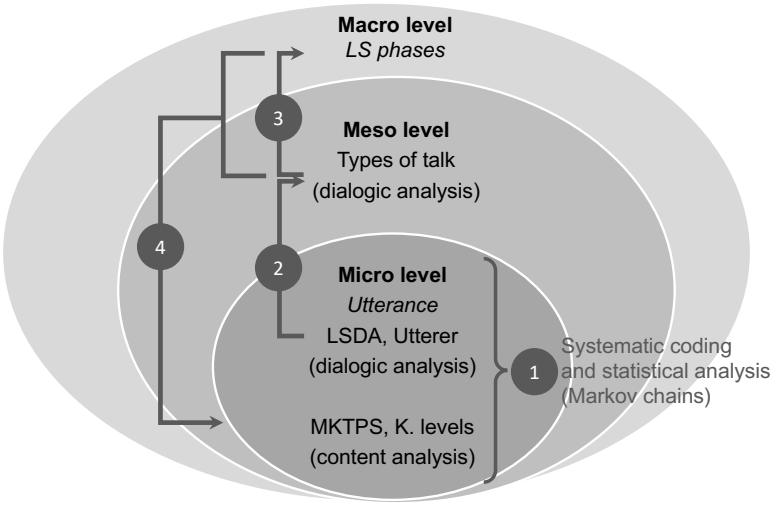

Figure 2: Nesting of analysis levels

Research question

Our main research question is: In what way is MKTPS built collectively during the LS process? For this article, we will focus on the sub-question: "What are the types of talk? Are they evolving? Is the MKTPS building related to the type of talk (arrow 4 in Figure 2)?"

Results

Types of talk during the LS meetings

The various types of talk can be associated with specific phases within the LS cycle, the types of MKTPS, and the knowledge levels. Our initial hypothesis posited that the specific MKPTS category would not correlate significantly with the types of talk, and our analysis validated this assertion. We also anticipated that the types of talk would undergo evolutionary changes during the course of meetings, contingent upon the specific nature and focus of each meeting. Furthermore, we expected that certain levels of knowledge would manifest more prominently within specific types of talk. The ensuing sections present the findings about these two dimensions.

Figure 3 illustrates the predominant discourse modalities observed during the meetings: exploratory talk (53%), cumulative talk (23%), and disputational talk (10%). A closer examination of individual meetings reveals distinctive characteristics. For example, Meeting 1 (M1), a general discussion about problem-solving teaching, is distinguished by the prevalence of cumulative talk (58%). In this meeting, participants expressed their ideas and questions independently, with limited cross-

referencing or interdependence in their utterances. This contrasts significantly with M8, a discussion about the two lessons, lesson plan, cycle assessment, and considerations for the next cycle. In M8, participants exhibited a marked reduction in cumulative talk and a notable surge in exploratory talk (70%) and disputational/qualificatory talk (24%, both highest incidence within the LS cycle). During this meeting, participants actively engaged in building upon each other's ideas and occasionally expressed disagreements with other participants. This escalation in exploratory discourse is also conspicuous when comparing meetings with similar themes. M2 and M3, (study of the curriculum material) witnessed an increase in exploratory discourse during the latter meeting, transitioning from 37% to 62%. Similarly, the planning of the second research lesson (M6) was characterised by a predominant presence of exploratory talk (76%) compared to the planning of the first research lesson (M4), with a lower incidence at 50%. The contrast is even more pronounced when comparing the post-lesson discussion following the first research lesson (M5) with the one following the second research lesson (M6). In the first discussion, each teacher successively reported their observations, which were predominantly cumulative (48%). In contrast, during the second discussion, almost every observation was related to those of colleagues, comparing or contrasting the observations in an exploratory way (65%), or sometimes disagreeing in a disputational/qualificatory way (14%).

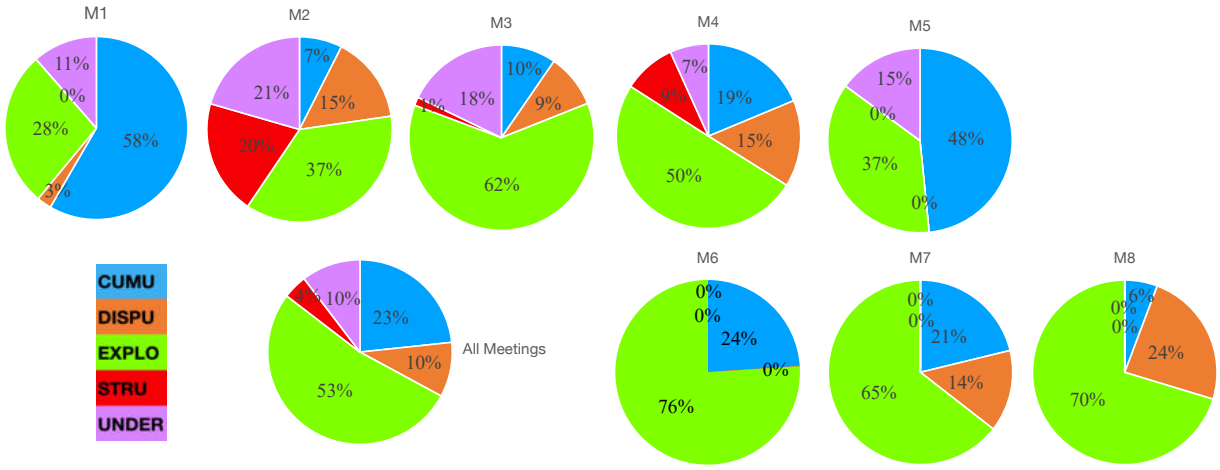

Figure 3: Type of talk for each meeting and for all the meetings

Levels of knowledge present in the types of talk

Upon examining the knowledge level in the three primary types of talk, a distinction emerges between cumulative talk and exploratory talk (see Figure 4). Specifically, there is a higher prevalence of level 4 (Contextualised explicit knowledge. Speaker knows or appears to know the rationale) in exploratory talk (29% compared to 19%). In contrast, level 2 (Unexplained knowledge. Observation, testimony) is relatively less prominent. This difference can be attributed to the definitions of type of talk and knowledge levels. But it can also be rationalised by the observation that during phases of a meeting characterised by exploratory discourse (at the meso level), knowledge tends to be more comprehensively justified (at the micro level). The observation that disputation talk is close to exploratory talk in terms of levels of knowledge is more unexpected. However, a closer examination of the transcripts reveals that, despite taking “short positions without justification” in response to

previous utterances, participants in disputation talk often demonstrate that they know the rationale of this knowledge, with a reduced occurrence of unexplained knowledge compared to cumulative talk.

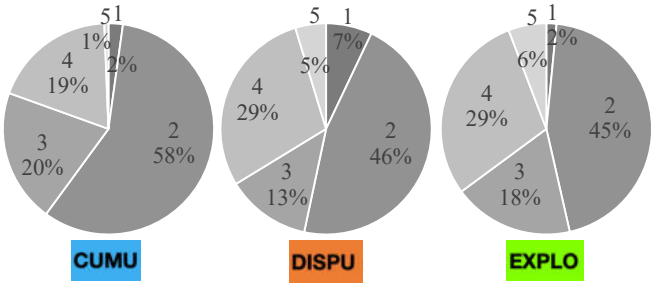

Figure 4: Knowledge levels for the three main types of talk

We refined our analyses by studying the transitions between levels of knowledge and this by type of talk (Figure 5).

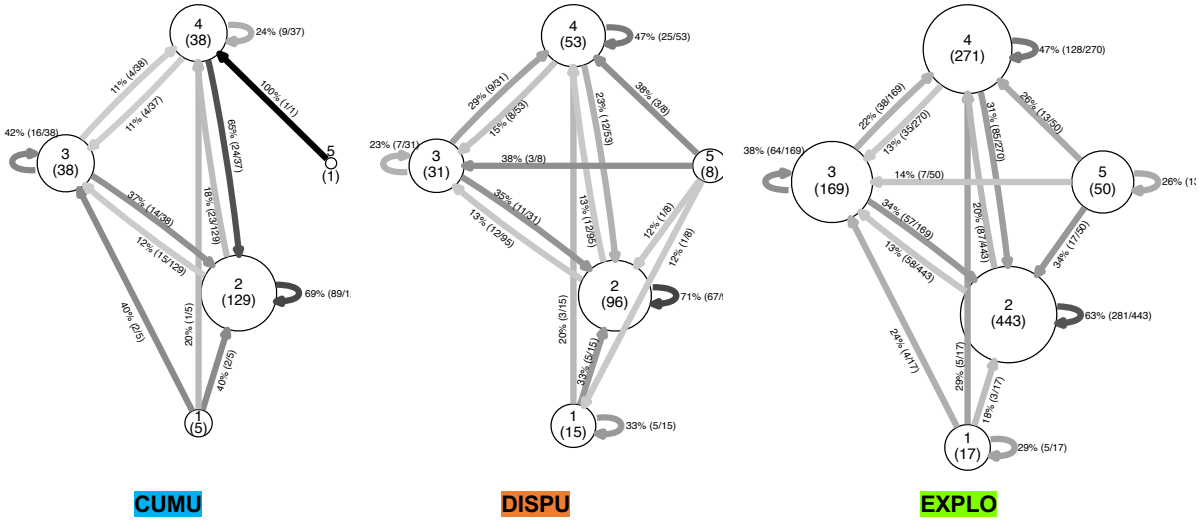

Figure 5. Markov chains for the levels of knowledge by type of talk

These Markov chains show that the exploratory and cumulative talk differ on the transitions from level 3 to level 4, going from 22% for the exploratory talk to 11% for the cumulative. This result can be interpreted as follows: when one speaker expresses incomplete knowledge or a question, the next speaker will express explicit contextualised knowledge. Similarly, the transition from level 4 to level 2 (unexplained knowledge, of the order of observation or testimony) is much more significant in the cumulative type of talk (65%) than in the exploratory type (31%).

These results from analyses of transitions between levels of knowledge illustrate how knowledge is constructed in exploratory talk in contrast to cumulative talk. These results confirm our research hypothesis based on the work of Littleton and Mercer (2013, p. 16) according to which the exploratory talk was judged to be the most "educationally effective". These results also allow us to analyse the dynamics in the construction of knowledge, through transitions between levels of knowledge.

In addition, there is no level 5 generalised, decontextualised knowledge in the cumulative talk, unlike the exploratory and disputational one. The transitions from levels 2, 3 and 4 to level 5 knowledge are all below 10% for the three types of talk. There is therefore no significant transition to level 5 of decontextualised knowledge, which we might have expected in the exploratory talk. On the other hand, there were transitions from level 5 to level 2 for the exploratory (34%) and disputational (12%) talk. Similarly, there were transitions from level 5 to level 3 for the exploratory (14%) and disputational (38%) talk, and to level 4 for the exploratory (26%) and disputational (38%) talk. As the number of occurrences is low, we can emphasise that in the exploratory and disputational talk, the decontextualised knowledge mobilised in one round is reexamined or recontextualised to classroom situations in the next round. These results illustrate the dynamics between decontextualised knowledge and knowledge of the order of questioning or classroom observation.

Level 2 knowledge is most present in the cumulative talk, in contrast to the other two. In a cumulative talk, participants express their knowledge without taking a critical or evaluative stance, and this knowledge is mainly level 2. In this type of talk, the participants talk about what they observed during the research lessons or refer to their classroom practices without considering the previous interaction.

The types of cumulative and exploratory talk are similar in the transition from level 2 to level 4, at 18% and 20%, respectively. Following a classroom observation, the turn to speak carries contextualised knowledge. This can be explained by the search for consensus in these two types of talk (Littleton & Mercer, 2013).

The transitions from levels 2 to 3 (12 to 13%) and 3 to 2 (34 to 37%) are similar for the three types of talk. This means that the type of talk do not differ with regard to the transition of knowledge linked to a class observation that is followed or preceded by a question or incomplete knowledge. Similarly, the transitions from level 4 to level 3 (between 11% and 15%) are similar for the three types of talk. These results may be explained by the very nature of the level 3 knowledge code, which involves either questioning, which may be conducive to knowledge building, or incomplete knowledge, which may be less conducive to knowledge building if it is not followed by level 4 or 5 knowledge.

Conclusions and perspectives

The development of teachers' mathematical knowledge about the teaching of problem-solving in a collaborative setting is a complex process. The findings of this case study are specific to the group under examination, and individual factors have impacted the results. Nevertheless, the mixed-method approach provides a comprehensive and interactive perspective on the collective construction of MKTPS. The unique characteristics of LS and MKTPS stressed in our results reinforce the idea that LS and PS teaching are interdependent and demonstrate how teachers' professional knowledge is shaped through dialogic engagement during LS. This supports the notion that LS and PS teaching are "two wheels of a cart" (Fujii, 2018) and highlights the significance of a dialogic process in developing teachers' professional knowledge.

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Bi-institutional lesson study on algebra: what is it and what can lower secondary school teachers learn?

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Abstract

This study explores the use of bi-institutional lesson study to address transition issues between two institutions, and the case of algebra in Danish lower and upper secondary schools. This new variant of lesson study is presented based on our theoretical framework, the anthropological theory of the didactic. It is investigated what teachers can learn by engaging in this approach and what challenges may arise. Preliminary results indicate that this approach may enhance teachers' awareness of students' relationship with a specific content area and further the participating institutions' alignment of expectations.

Keywords: Lesson study, bi-institutional lesson study, algebra, institutional transition problem.

Introduction

Transition problems from Danish lower secondary to upper secondary school are experienced as more acute in mathematics than in subjects like English or Danish, where more students experience more continuity between the disciplines as taught in these two schools (Ebbensgaard et al., 2014). Transition problems can be due to organizational, social or developmental reasons (Cantley et al., 2021). The cited difference motivated us (Cosan, 2024) to focus on more specific, subject-related or academic differences. Cosan (2024) presents a model for school algebra as consisting of the following three aspects: 1) setting up an algebraic model, 2) substituting in an algebraic model and 3) rewriting or operating on an algebraic model. According to Cosan (2024), the transition problem in algebra between Danish lower secondary and upper secondary school is predominantly related to the theory and practice of rewriting or operating on an algebraic model.

Whenever transition problems exist between two consecutively attended school institutions, this may encourage teachers from the institutions to collaborate, aiming to understand and ease the transition by establishing a shared knowledge of the requirements and expectations at each level.

In this context, considering the potential of lesson study is relevant. Lesson study began in Japan and is a collaborative professional development approach that centers on students and is designed to develop and support teachers' practices. Research shows that the implementation of lesson study impacts teaching practices, teacher learning, student learning, and the development of professional learning communities (Ding et al., 2024). Lesson study is a widely recognized teacher professional development format that is utilized in both lower and upper secondary schools even beyond Japan; however, in many countries including Japan, it is somehow different and less common in upper secondary school (Nishimura et al., 2018).

This paper introduces a new form of lesson study - *bi-institutional lesson study* - involving two connected (consecutive or neighboring) institutions, which was tested in Danish lower secondary and

upper secondary school, to aid students' transition, focusing on algebra (which, for reasons not developed in this paper, appears to be particularly central to the overall transition). The hypothesis is that teacher professional knowledge can be developed through collaborative lesson study with the aim to facilitate a specific transition occurring between the institutions.

Background: Lesson Study

Lesson study, or *Jugyo Kenkyu*, is a form of collaborative professional development widely used in Japan and other East Asian countries to improve teaching and learning. It involves teachers working collaboratively to plan, observe, and reflect on a lesson. The lesson study process consists of following a cycle known as the lesson study cycle in the following manner: A team of teachers identifies a learning goal and collaboratively plans a detailed research lesson. One member of the team of teachers teaches while the others observe and collect data. They then analyse, reflect on, revise, and report the lesson (Lewis, 2002).

As pointed out in Miyakawa & Winsløw (2019), lesson study can facilitate both the development and use of didactic knowledge and help it to become shared across a larger community of teachers. In a Western context, teachers are used to work in relative isolation, having individual responsibility for their teaching. Implementing lesson study in their practice allows them to work more collaborative and as pointed out by Lewis et al. (2009), the use of lesson study "makes various types of knowledge more visible, such as colleagues' ideas about pedagogy and students' mathematical thinking, thereby enabling teachers to encounter new or different ideas, and to refine their knowledge" (p. 286). Thus, lesson study as a professional development allows the teachers to develop and strengthen shared didactic theory.

Ding et al. (2024) review a large body of studies that show how lesson study contributes to teachers' professional knowledge, and how this has resulted in improvements in their teaching methods and also their common knowledge of mathematics. They cite research papers demonstrating how lesson study has led to changes in teaching methods, including better focus on students' work and interaction during the lessons. At the same time, this review also highlights that studies have shown how lesson study has improved students' mastery of mathematical content, such as both theoretical and practical aspects of fractions. The authors also stress that lesson study is often limited to teachers within a single school and that it is more frequently utilized by teachers in elementary and middle schools, and to a lesser extent in secondary schools.

Theoretical framework

In this paper, the work is situated within the theoretical framework: the Anthropological Theory of the Didactic (hereafter ATD) introduced by Yves Chevallard (2019). ATD suggests that the notion of praxeologies can be used to model human knowledge and activity. A praxeology consists of two parts, praxis and logos. The praxis refers to practical knowledge while the logos refers to explicit knowledge about the praxis. Concretely, a praxis consists of types of tasks T and techniques τ utilized to solve these tasks whereas the logos consists of a technology θ, which refers to the discourse used to explain, justify, and describe the techniques, and a theory Θ which refers to the more formal justification of the technology (Miyakawa & Winsløw, 2019). While a mathematical praxeology is

related to the praxis and logos about mathematical tasks and techniques, a didactical praxeology refers to the praxis and logos about the teaching of a mathematical praxeology. According to ATD, teaching mathematics involves developing and sharing a mathematical praxeology, and teaching itself constitutes didactic praxeologies. The mathematics teachers' knowledge is fundamentally rooted in and developed through engaging in didactical praxeologies. However, these didactical praxeologies are always based on mathematical praxeologies, so these two kinds of praxeology are inseparable (Miyakawa & Winsløw, 2019). A key concept in ATD is *didactic infrastructure* which encompasses the conditions and constraints influencing and shaping teachers' classroom activities (Miyakawa & Winsløw, 2019). Praxis which is related to teaching but occurs outside the classroom for instance the preparation of a lesson or participation in professional development, is called *paradidactic praxis* and the *paradidactic infrastructure* then relates to the conditions and constraints under which the paradidactic praxis takes place. So, how teachers participate in paradidactic practices is influenced by and shaped by this paradidactic infrastructure (Miyakawa & Winsløw, 2019).

Lesson study is focused on didactic praxis, the research lesson. Planning the lesson, observing it, and reflecting on it are three key tasks within lesson study and these tasks are according to ATD not didactic but paradidactic tasks (Winsløw et al., 2018). Lesson study itself therefore is a possible element of paradidactic infrastructure according to ATD (Miyakawa & Winsløw, 2019).

Bi-institutional lesson study

In a *bi-institutional lesson study*, teachers from two consecutively attended institutions, referred to as I_1 and I_2, collaborate to plan, observe, and review a research lesson in I_1, to facilitate the transition between these two institutions and to develop the participating teachers' common didactic knowledge. Bi-institutional lesson study can be used to address specific problems that arise when students transition between two educational institutions such as lower secondary and upper secondary school. This approach enables teachers from I_1 and I_2 to collaboratively reflect and improve on their teaching practices, focusing on developing a shared knowledge of students' thinking and learning in challenging mathematical praxeologies during these transitions. Like a regular lesson study, carried out within one institution, a bi-institutional lesson study follows a lesson study cycle and provides the opportunity to develop didactic and mathematical knowledge about teachers' own practice. However, the bi-institutional lesson study also affords the opportunity to build didactic and mathematical knowledge about practices in the neighboring institution.

Research question

This paper will address the following question:

Can lower secondary school teachers' didactical and mathematical praxeologies be developed by engaging in a bi-institutional lesson study focused on algebra? If so, which praxeologies and what challenges can appear?

With this question, the focus is specifically on investigating whether Danish lower secondary and upper secondary school teachers develop new didactical and mathematical knowledge related to their own and each other's practices after participating in bi-institutional lesson studies.

Context and Methods

In the 2023/2024 school year, six lower secondary school teachers and two upper secondary school teachers participated in bi-institutional lesson studies. To ensure that all teachers had the opportunity to participate in the planning of the lesson, the teachers were divided into two smaller groups, each consisting of three lower secondary school teachers and one upper secondary school teacher. These teachers work at a lower secondary school and an upper secondary school in the same municipality, where most of the students who choose upper secondary school actually end up in the one participating in this study. The upper secondary school teachers had previously been involved in the Erasmus+ TIME (Teachers' Inquiry in Mathematics Education) project, which aimed to explore how upper secondary school teachers can collaboratively develop their practice through lesson studies (Jessen, Bos, Doorman & Winsløw, 2023). Therefore, the two participating upper secondary school teachers in this project already had some experience with lesson study.

Throughout the school year, each group planned and conducted three lesson studies, resulting in a total of six lesson studies. Each lesson study was planned and carried out over a period of 4-5 weeks and followed the lesson study cycle. The two groups also participated in observation and reflection related to each other's research lessons. Each lesson study included a first try and a research lesson. The first try could be taught by all the lower secondary school teachers while the research lesson was always taught by the lower secondary school teacher who normally taught the class.

In the first bi-institutional lesson study, I, as the author, facilitated the planning meetings, helping teachers set goals, choosing a task, defining learning objectives, filling out the lesson plan, and identifying observation points for the research lessons. As the teachers gained experience, my role shifted to becoming more of an observer. For the research lesson, another facilitator and a knowledgeable other from the university and university college were always present.

In the following section, the method for data analysis will briefly be presented. After the implementation and observation of the lesson studies, the teachers were interviewed. Two lower secondary school teachers did not participate in the interviews due to illness and a change of workplace. Notice that this paper is a part of a larger project, where we are interested in the data from all six semi-structured interviews with two upper secondary and four lower secondary school teachers. However, the data for this paper consists of an interview with one of the lower secondary teachers, who was also the one teaching the research lesson, due to space constraints. A more comprehensive analysis will be presented elsewhere, based on interviews with all the teachers to provide broader answers to the research question.

All the teachers involved in the project were experienced teachers with many years of teaching practice. The interview questions were designed to gain knowledge about teachers' experiences from participating in the bi-institutional lesson studies. The questions were divided into two parts: first, the teachers' individual experiences were explored, focusing on both the benefits they gained and the challenges they encountered. Then, their experience with the collaboration was examined, both within their institutions and across institutions.

After transcribing the interview data, the content was reviewed multiple times to gain an overview of the data material. This process, which is still ongoing, helped to identify key statements and observations from the teachers that could contribute to answering the research question.

A thematic approach, drawing on ATD, was applied to analyze the teacher's experiences and the development of their didactical and mathematical praxeologies through participation in bi-institutional lesson studies. Central statements from the transcriptions were identified to highlight both the benefits and challenges faced by the teacher, as well as how these challenges were connected to the development of both mathematical and didactical praxeologies. These statements were then organized into two main themes: 1. *Increased professional awareness and facilitation of student involvement*, and 2. *alignment of expectations between institutions*. The framework ATD was applied here to interpret these themes in terms of how the teacher's didactical and mathematical praxeologies evolved through the bi-institutional collaboration. Specific quotes from the analysis were used to support the identified themes and to illustrate the teacher's experiences and learning in relation to both the practical and theoretical aspects of their teaching.

Analysis and results

Generic lesson study outcomes: professional awareness

The teacher points out that her participation in the project has impacted her teaching by making her more aware of the importance of actively incorporating students' knowledge into the lessons:

"It is more important for the students to get involved rather than for me to be in the spotlight during the lessons—both in terms of them actively solving tasks and also being verbally active."

This illustrates how participation in the bi-institutional lesson studies has helped her recognize the value of shifting focus from teacher-led instruction, which is more or less her usual practice, to a more facilitative role where students explore and solve problems together.

In this context, she expresses that "there is often something that is right and wrong [referring to the result of a math problem], but there are many ways to approach solving things, such as methods or, what you might call, procedures". So, the teachers' view of mathematical praxeologies developed. The teacher refers to the fact that there can be different techniques for solving the same task and similarly, during validation (in the sense of Brousseau, 1997, p. 15), she argues that one must consider different technologies and more theoretical aspects, including how to justify the various technologies. Based on the bi-institutional lesson study, it can be observed that the teacher experiences a shift from a more technical level to a more technological/theoretical level, and a demand for students to explain their techniques and methods. In open lessons, the teacher then has to address a greater variety of solutions, and reasoning of a more theoretical nature. This is a mathematical praxeology, but it requires a didactic praxeology to generate and manage this mathematical praxeology, which is what is discussed during the planning meetings. Especially during lesson planning, the paradidactic infrastructure facilitates the interaction and development of the teachers' mathematical and didactic praxeologies, from two different institutions.

So, she acknowledges that the identification and validation of different approaches students use to find a solution for a mathematical task are just as important as the final answer. This indicates a development in the teacher's professional awareness, extending a focus on mathematically correct results to include students' methods and thinking. She now recognizes the value of recognizing and exploring the many different ways by which students can solve a task in mathematics.

A central focus of the bi-institutional lesson studies has been how the teacher can facilitate a teaching approach that increasingly involves students. In the interview, the teacher expresses that the bi-institutional lesson study has allowed her to reflect on how she should balance control and freedom in the classroom:

"It is a challenge to let go of the control and see where it leads... how can you keep it controlled while still allowing the students to actually contribute knowledge and participate in the lesson."

This reflection highlights one of the major challenges in teaching, according to the teacher, namely finding the right balance between guiding students and giving them the space to develop their own understanding of algebra. Indeed, this could be a general outcome of a first lesson study experience even within one school, but was here accentuated by the participation of teachers from another school who had both previous experience with lesson study, and helped focusing on topics (like algebraic techniques) which are of particular importance in the future of students.

Specifically bi-institutional outcomes: alignment of expectations between institutions

Collaboration with upper secondary school teachers has also made the teacher aware of differing expectations between the two institutions. The teacher frequently mentions this:

"It has been rewarding to gain insight into my expectations and their expectations, which were not exactly the same. My expectations for what I send my students out of school with, and their expectations for what they receive in terms of what the students' are capable of. It was like, 'Wow.'"

A concrete example of this difference in expectations concerns students' knowledge related to linear equations: "Their expectations of what the students can do... at the upper secondary school, they expected the students to know more... they had higher expectations than I do." This example shows how the two institutions can have different perceptions of students' academic levels, which may create a need for discussion.

Another important realization for the lower secondary school teacher was how comprehensive algebra is: "Even something like linear equations, what they expect the students to be able to do and what I expect my students to manage when they leave school. There was a difference." This difference became particularly evident in the way linear equations were approached, reflecting distinct mathematical praxeologies. This gap in understanding can only be fully grasped by considering the broader institutional conditions that affect each institution. While the lower secondary school teacher focused on the technique of solving equations by guessing, upper secondary school teachers emphasized the theoretical dimension of setting up and solving equations, expecting students to handle more complicated tasks, at least according to the lower secondary school teacher.

During the planning meetings, the participation of upper secondary school teachers highlighted a critical gap between practices across the two institutional levels. The conversations between the teachers helped lower secondary teachers prepare their students better for future challenges. These conversations emphasized the different meanings, value, and importance assigned to algebra in the two institutions, highlighting its extensive scope and its varying importance in each institutional context. The reflection prompted a broader conversation on how to bridge these praxeological differences and align teaching practices more effectively.

The mere extent of the subject algebra is emphasized by the teacher: "It may have dawned on me how extensive it actually is compared to how little time is actually given to it in lower secondary school." Thus, through the bi-institutional lesson studies, the teacher has become aware that school algebra is more complex and extensive than she previously assumed. This contributed to a deeper understanding of what is required to prepare students for the transition from lower to upper secondary school.

The collaboration with upper secondary school teachers in lesson studies has helped lower secondary school teachers become more aware of specific algebraic skills that students need later. If the lower secondary school teacher had not previously seen algebra as such an extensive subject area, it could indicate a general lack of communication between lower secondary and upper secondary school institutions, regarding what is required and expected of students in algebra. This could strengthen the argument for why bi-institutional lesson study is effective, as it forms a "practice near" infrastructure for teachers to share their views and knowledge of the academic content across institutions.

Based on the interview, it can be seen that the bi-institutional lesson studies have provided the teacher with new and valuable knowledge. She has gained new academic awareness, recognized the importance of aligning expectations between institutions, reflected on how to best involve students in teaching, and acknowledged the extent of algebra. At the same time, she has faced challenges, particularly in balancing control with student involvement in the classroom.

Conclusion

This study examined the use of a bi-institutional lesson study to address the transition problem between Danish lower and upper secondary schools, focusing on algebra. Motivated by the difficulties in algebra during this transition, the study works within the Anthropological Theory of the Didactic (ATD) to understand the interaction between mathematical and didactical practices across institutions. The analysis of a lower secondary school teacher's experiences highlighted several key findings. The teacher gained knowledge about the nature and extent of algebra, and the corresponding expectations of teachers in upper secondary education. This collaborative process also facilitated a better understanding of how different teaching practices impact student learning. The teacher faced challenges in balancing classroom management with encouraging student contributions. However, overall, the bi-institutional lesson study approach appears to have promising potential for improving the alignment of teaching practices and otherwise addressing transition problems.

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Preparing and leading a whole-class discussion in a lesson study in initial teacher education

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Based on the analysis of a lesson study carried out during an initial teacher education program, we seek to understand how this teacher education process can promote opportunities for a prospective teacher to develop her practice about preparing and leading whole-class discussions in the classroom. We follow a qualitative approach, and the data was collected through observation and recording of the lesson study sessions. The analysis of data is made based on a comprehensive model for preparing and leading whole-class discussions. The results show that the lesson study seems to have contributed to the development of the prospective teacher's practice of how to prepare and lead whole-class discussions. She managed to promote an interesting whole-class dynamic but she found it difficult to get most pupils to participate in the whole-class discussion.

Keywords: Whole-class discussion, lesson study, mathematics teaching, initial teacher education.

Introduction

Whole-class discussions assume great relevance in pupils' learning, enabling the exploration of mathematical ideas, anchored in the pupils' solutions and of the tasks proposed by the teacher. The role of the teacher is central, encouraging pupils to present and justify their ideas, requiring that the teacher prepares well the enactment of a whole-class discussion (Fujii, 2018; Stein et al., 2008). However, leading these discussions has been identified by research as a challenge for teachers and, especially, for prospective teachers, since it requires a broad knowledge in content, didactics and pupils and the way they learn (Fujii, 2018; Stein et al., 2008). In addition, leading whole-class discussions also requires "solid expectations for what is likely to happen, [and] novices are regularly surprised by what pupils say and do, and therefore often do not know how to respond to pupils in the midst of a discussion" (Stein et al., 2008, p. 321). According to Fujii (2018), one way to deal with the challenges of leading whole-class discussions is to anticipate possible strategies for dealing with pupils' difficulties, preparing how to act in the different moments of the lesson and the teacher interventions to do.

Lesson study is recognized as a formative process with potential to promote the development of knowledge of prospective teachers, namely on leading whole-class discussions (Ni Shuilleabhain & Bjuland, 2019). Focused on pupil learning, lesson study provides opportunities for prospective teachers to prepare lessons in detail, anticipating pupil work and teacher actions. This teacher education process allows not only prospective teachers to prepare the lesson, but also to lead it and reflect on the pupils' work and learning (Bjuland & Mosvold, 2015). In this study, we seek to answer the following research question: What practices a prospective teacher enacts in the preparation and leading of a whole-class discussion in a research lesson?

Preparing and leading the whole-class discussion through lesson study

The quality of a whole-class discussion is strongly influenced by the teacher's preparation for the lesson (Fujii, 2018; Takahashi, 2008). This preparation begins when the teacher identifies the

mathematical content to be addressed and defines the learning goal to be developed with the pupils in the lesson (Boerst et al., 2011). From there, the teacher needs to select tasks for which the pupils do not have an immediate solution method, which creates opportunities to build or deepen their understanding of mathematical concepts, procedures, representations and ideas. After that, the teacher needs to anticipate various solutions and difficulties that pupils may present. To do this, the teacher should consider the diversity of strategies and how they may relate and establish connections between them and with the learning goal to develop in that lesson. Also, it is important to identify possible difficulties the pupils may have in interpreting the task, analyse articles with suggestions on pupils' usual responses and difficulties with similar tasks and analyse the curriculum documents (Stein et al., 2008). This study also allows the teacher to realise what pupils' strategies and answers are most likely to emerge and what errors or misconceptions they may have (Boerst et al., 2011). It is also important for the teacher to prepare the questioning and interventions when leading the whole-class discussion. Such questions should contain a certain degree of challenge, promoting disagreements and allowing the teacher to understand the pupils' difficulties.

Then, the teacher should think about how to begin the discussion of the strategies used by the pupils, already selected and sequenced in the preparation phase (Boerst et al., 2011). The teacher leads the whole-class discussion with special attention to i) the pupils' work; ii) mathematical ideas; and iii) classroom interactions. By analysing the pupils' work, the teacher is able to obtain, understand and make connections between the different ideas, solutions and strategies used by the pupils. When the pupils explain how they solved the task, the teacher may ask questions to support them in expressing their reasoning, particularly to highlight important mathematical ideas related to the task and the lesson. And with regard to interactions, the teacher's interventions may also encourage pupils to participate in the discussion, encouraging them to listen, analyse and respond to their peers' reasoning (Boerst et al., 2011). Since a whole-class discussion, in order to be productive, results from pupils sharing their thoughts, the teacher may reformulate the ideas they present in an incomplete or erroneous way. Sharing incorrect or incomplete ideas can lead to disagreements between pupils, which is a great opportunity to argue, criticise and explore errors (Stein et al., 2008). The whole-class discussion ends with a final synthesis that may be enriched with the collaboration of the pupils. This moment aims to highlight and summarise the main mathematical ideas explored throughout the lesson and point connections between the established goal and other ideas previously known by the pupils (Duarte et al., 2024).

The model of Stein et al. (2008) says very little about how to lead a whole-class discussion. There are other models about this classroom moment (e.g., Boerst et al., 2011; Kooloos et al., 2023), but they tend to emphasize particular aspects. This indicates the need of a comprehensive model to account for all the main aspects of this lesson activity. To this end, we developed a model that serves as a basis for data analysis that we present in the next section.

Research methodology

This research follows the qualitative approach and the interpretive paradigm (Bogdan & Biklen, 2007) and considers as its data source the sessions of a lesson study carried out at a higher education institution in Portugal in the initial teacher education of primary teachers. The lesson study was carried out in the 2023/2024 academic year and involved two prospective teachers who were attending the first semester of the first year of their master's degree program in teaching (Júlia and Alda, pseudonyms). There was a total of eleven sessions (S*n*), held weekly and lasting approximately two hours each, organised as

follows: goal setting – S1, S2; lesson planning – S3, S4, S5, S6; research lesson – S7; post-lesson discussion – S8; reflection – S9; and preparing to disseminate the results – S10, S11. The lesson study was an integral part of the internship programme, which was being carried out in a grade 2 class with pupils that were between the age of six and seven. The prospective teachers were supervised and guided by a teacher from the higher education institution, who took on the role of researcher, preparing and conducting the lesson study sessions. After analysing the medium-term planning provided by the school where the prospective teachers were doing their internship, the participants decided that the research lesson would focus on the theme of numbers and the topic of multiplication. The lesson's learning goal would be to solve problems with rectangular arrangements that fostered an understanding of the relationship between multiplication and successive addition of equal numbers. All the work carried out for the research lesson considered the exploratory approach, namely in the selection of the task, which is shown in Figure 1. There was much reflection about which task to choose (as indicated by Fujii, 2018) and in anticipating students' responses (as indicated by Stein et al., 2008).

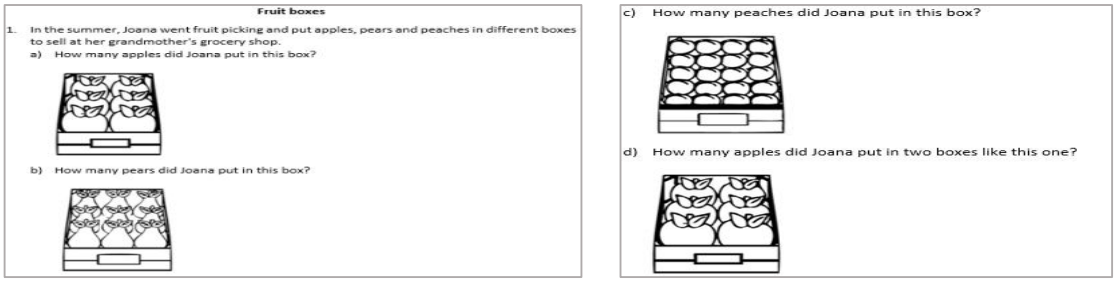

Figure 1 - Task proposed to the pupils in the research lesson (adapted from Rocha & Menino, 2009)

To promote the final synthesis, the prospective teachers decided to explore with the pupils a table, which systematises some of the strategies that may be used to solve the task and includes a space to fill in with the pupils' suggestions.

We analyse how Júlia prepared, conducted and reflected on the research lesson. Although she conducted exploratory lessons before in pursuit of her degree, valuing their contribution to pupil learning, her participation throughout the lesson study showed a pronounced development of self-reflection on her own practice. The data was collected through observation of the planning sessions and the carrying out of the research lesson. The lesson study sessions were recorded and relevant excerpts for this study were transcribed. The data was analysed using a deductive method, considering the model presented in Table 1, which organises the teacher practices to be considered when preparing and leading a whole-class discussion, final synthesis. In this model, the preparation of the discussion includes aspects that have long been recognised in the teacher education literature (e.g., Fujii, 2018; Stein et al., 2008). With regard to leading the whole-class discussion, the model proposes the identification of three practices that need to be conducted simultaneously: initiating discussion and intervention according to the situation and goal, analysing pupils' work with a focus on mathematical ideas, and promoting pupil participation. As for the final synthesis, we have added the action of summarising and highlighting the main mathematical ideas or procedures explored to the establishment of connections previously indicated by Stein et al. (2008). In our analysis, we identify significant episodes from the preparation and leading of the research lesson conducted by Júlia, through which we sought to understand how the prospective teacher led the whole-class discussion during her

participation in the lesson study. In addition, we analyse how the prospective teacher led the whole-class discussion and seek to understand how the lesson study contributed to the development of her knowledge about leading whole-class discussions. (Pupils' names are pseudonyms).

Table 1 - Practices to consider when preparing and leading the whole-class discussion and final synthesis (adapted from Duarte et al., 2024)

To prepare the whole-class discussion		To lead the whole-class discussion	To conclude the whole-class discussion with a final synthesis
Before class	**During class**		
Initial preparation Choosing the task Solving the task Preparing to carry out the task in class	Presenting the task Monitoring the pupils' work Selecting pupils' answers Sequencing	Initiate the whole-class discussion and intervene according to the situation and the goal Analyse pupils' work with a focus on mathematical ideas Promote pupil participation	Conclude, with the collaboration of the pupils

Results

Preparing the whole-class discussion

After selecting and adapting the task to propose to the pupils in the research lesson, the participants anticipated possible strategies that the pupils could use. As such, they solved the task in several ways, regardless of the degree of correctness or sophistication of each solution strategy:

Júlia: I, sincerely, think that on b) they [pupils] may use…
Alda: Counting one by one.
Júlia: [Counting] by rows and by columns: 3+3+3 and 3+3+3. Here is the same. To say 3 columns of 3 or 3 rows of 3 or 3 horizontally, and 3 vertically… And whereas in the previous question the factors changed, replacing their position, but the result was the same, here [question b)], no. The factors are equal, does not matter to replace position. In question c), in addition to these, there are two products: 4×5 or 5×4 and the position of the factors matters again. (S3)

Faced with the diversity of possible strategies for solving the task, the participants discussed how they could select and sequence the pupils' answers for the whole-class discussion, namely considering the wrong or incomplete strategies, which "should be presented first, if we want disagreements to arise" (Júlia, S4) and, therefore, lead the pupils to explain and justify their solutions. In addition to beginning the discussion with wrong or incomplete strategies, the group also chose to sequence the pupils' answers according to an increasing degree of complexity and sophistication:

Alda: Counting one by one is simpler, I think it makes sense to begin by that one.
Júlia: And it must be the more used, at least in this [first] question. The multiplication should be the most difficult for the pupils and it is also where we want to get, so it makes sense to be the last one. And if we want that they connect addition and multiplication, I think that is the order. From additions to multiplications.
Supervisor: And in b)?
Júlia: I would do the same. However, I would not select counting one by one, because that was talked before… The aim of the lesson is not counting one by one… Therefore, I would not select it. (S4)

Júlia also suggested giving priority to strategies that made it possible to explore the aim defined for the lesson, choosing not to value strategies such as counting one by one from the first question of the task. Depending on the different solution strategies that the participants anticipated, they also prepared questions and teacher interventions to use during the whole-class discussion to promote

pupil learning (Duarte et al., 2024; Stein et al., 2008). Júlia considered important to intervene not too much, allowing pupils to take a central role in explaining their reasoning, nor to validate or induce the answers, and often challenge other pupils to participate in the discussion:

Supervisor:	And these interventions [support and challenge] are directed towards a group, a pupil, the whole class?
Júlia:	To the groups, I think. The class must be provoked to always participate. Moreover, they must listen with attention what a group says to be able to criticize. To say: "Are there other ways to count, group x? That group counted in another way. Explain how you thought" or, for question b), "the group A added 3 with 3 and group B did 2 times 3. However, the result is 6 in the two solutions… Is this right? Why? Group C, explain why it yields the same result". (S4)

Leading the whole-class discussion

In the research lesson that she led, Júlia begun the discussion with an invitation to the group that she selected to be the first to present a solving strategy: "the group of Carolina, how did you do it? Explain to your colleagues" (S7). This invitation action led the pupils to share their solutions and explain them, as it occurred with the discussion of question a):

Júlia:	Ricardo did 2+2+2. Is there another way to say this?
Ricardo:	2×2.
Júlia:	2×2? So, let us pick here in this operation that Ricardo told us.
Hugo:	2×2 is 4!
Júlia:	2×2 is 4. Very well. However, you know how we call the operation that has this signal [point towards ×]?
Ricardo:	Is "times"!
Júlia:	Is "times". And how do we call the operation?
Fabiana:	Is "x".
Júlia:	But it has a name… the "x" is called "times", not "x". But what is the name of the operation? This here is an addition [points towards +], and this one uses "times", how we call it? You do not know? Say, Hugo?
Hugo:	Mult…?!
Júlia:	Multiplication! When we use "times", we are doing a multiplication. Let us go back to your responses. Hugo, it is your turn, say it.
Hugo:	It can be 2×3.
Júlia:	Very well. So, it is 2×3 and not 2×2. What does it yield?
Hugo:	6.
Júlia:	6. But look at one thing. Do we have two times three or three times two?
Pupils:	Three times two!
Júlia:	Then, the correct operation is 3×2. (S7)

In this episode, Júlia also analysed the pupils' work with a focus on mathematical ideas, interpreting Ricardo's reasoning and actions of initiating and intervening when she invited him to think of another answer that would allow making connections between the addition of equal numbers and multiplication. At Júlia's invitation, and although he gave a wrong answer, Ricardo made the connection with multiplication. Hugo said that the product of 2×2 is 4 and mentioned the name of the operation at sake. Apparently, these two pupils already knew multiplication, possibly because they had private mathematics lessons. As this was the first moment of the lesson in which the operation appeared, Júlia chose not to immediately highlight the error, but took advantage of Ricardo's intervention to explore multiplication with the pupils. Then, she resumed to the answer 2×2, again with intervention actions, now challenging the pupils to validate this response, with Hugo correcting the operation enunciated by Ricardo. In this episode of the discussion, Júlia also interrupted the pupils' interventions, again through interventions in order to support/guide them, encouraging pupils

to clarify their explanation, but also to promote the continuation of their involvement in the discussion, seeking to promote their participation (Duarte et al., 2024; Kooloos et al., 2023). The prospective teacher also used intervening actions, highlighting important mathematical ideas, introducing information and validating pupils' answers, and analysing pupils' work when she promoted connections between the pupils' answers 2×3 and 3×2.

Another episode, referring to the discussion of question c), highlights how Júlia promoted the participation of the pupils, when faced with a situation of disagreement in which Francisco, when explaining his reasoning to his colleagues, had a question about the position of the factors:

Francisco:	First, we counted, 1, 2, 3, 4, 5 [a column. After 1, 2, 3, 4 [a row]. And we understood that it could be 5×4, I mean 4×5.
Júlia:	And why it could not be 5×4?
Igor:	It would yield I don't know.
Júlia:	Francisco said that you thought that way [by columns] 4 times, and in this way [by rows] 5 times. And it could not be also 5×4?
Igor:	It would not be. It would yield other number. I need to do the computation.
Júlia:	Children, what do you think?
Maria:	Would be another thing.
Júlia:	They thought 5, 5, 5, 5, four times [columns]. And would not yield 5×4?
Rui:	It would be 20. So, that would be another thing.
Júlia:	It would be? How that would be?
Rui:	I think it is in both ways.
Júlia:	Why?
Rui:	I don't know.
Júlia:	Let's all discuss. Ricardo, say it?
Ricardo:	Because there are there 5 rows with 4 up front [rows] and 4 rows with 5 downwards [columns].
Júlia:	Do you want to come here [to the board] explain?
Ricardo:	Here [points the board] there is 4+4+4+4+4 [points to the columns], equal 20. [Points to the rows] 5+5+5+5 equals 20.
Júlia:	So, we can do 5×4 if we think as Francisco and we can do 4×5 if we think as Ricardo. How is it? Explain once more.
Ricardo:	4+4+4+4+4 [points, at the board, to the columns downwards] and 5+5+5+5 [points at the board, left to right]. (S7)

Faced with the difficulty of Francisco, Júlia, analysing the pupil's work, sought to promote connections between the mathematical aspects of his strategy with the aim of the lesson and (Duarte et al., 2024), to this end, made an intervention, challenging and insisting with the pupils to explain why "it could not be 5×4". When she realized that the pupils were not so involved and participative in the discussion, she made an intervention, inviting them to share their opinion about what their colleagues said (Boerst et al., 2011). Júlia also made actions related to the analysis of the pupils' work, apparent in her mathematical activity during the discussion and, in the connections, she promoted between the pupils' strategies and the aim of the lesson. Such actions, often in the form of challenges, presented pupils with mathematical questions and situations in which they were called upon to justify their explanations (Stein et al., 2008). The prospective teacher also showed that she analysed the pupils' work, trying to understand their reasoning, choosing to respond to them in a more explicit way, informing/suggesting by highlighting information. The situation of disagreement, motivated by the difficulty of some pupils in understanding that the position of the factors does not change the product, was solved through the explanation of a pupil, Ricardo. The prospective teacher, although she made several interventions, allowed the pupils to justify their reasoning and reach a consensus.

Concluding with a final synthesis

The whole-class discussion ended with a final synthesis, promoted through the exploration of the previously prepared board organization, which also led to the discussion of some of the pupils' ideas:

Júlia:	This first part is correct? 2+2+2 is equal to 3×2?
Some pupils:	No!
Júlia:	No? Why not? What is wrong here?
João:	It is the same.
Júlia:	Is it the same? You all agree?
Pupils:	Yes.
Júlia:	Why so? I want to know why you agree that 2+2+2 is equal to 3×2. Why that is correct?
Tomás:	Because it is 3 times 2.
Júlia:	We have 3×2. We are adding 3 times 2. So, now 3+3 is equal to what?
Tomás:	To 2×3.
Júlia:	Mariana?
Mariana:	6.
Júlia:	Yes, is 6. However, I do not want to know the result. I want to know 3+3 is equal to what in a multiplication?
Sofia:	6.
Júlia:	Sofia, I just said to Mariana that I don't want to know the result. Carolina, how is 3+3?
Carolina:	It is 2×3.
Júlia:	All agree?
Pupils:	Yes.
Júlia:	Because we are adding three twice. So, now, 3+3+3. How is it in multiplication?
Francisco:	3×3. (S7)

In this segment of the final synthesis, Júlia encouraged the involvement of the pupils, who explained their reasoning and expressed themselves in relation to the explanation of their colleagues, validating the answers presented. In this way, the final synthesis was promoted through the collaboration of the pupils in filling out the table where they summarized and highlighted the main ideas explored in the lesson (Duarte et al., 2024) and evidenced connections between successive addition and multiplication, namely between the pupils' answers 2+2+2 and 3×2.

Conclusion

The preparation of a whole-class discussion is the object of attention in many studies, and there is a broad consensus around the usefulness of the model of Stein et al. (2008). However, it seems necessary to better clarify everything that has to do with the first phase of the preparation of the lesson, namely the choice and solving of the task, which we seek to do in this study. Regarding leading the lesson, we suggest that three practices intertwine, initiating the discussion and intervening according to the situation and the aim, analysing the pupils' work with a focus on mathematical ideas, and promoting pupil participation. Several of these aspects are mentioned in previous studies, with emphasis on a dimension or another, and often under other terms (e.g., Fujii, 2018; Kooloos et al., 2023; Stein et al., 2018). The model we present also highlights the importance of concluding the whole-class discussion through a final synthesis, led with the collaboration of the pupils, where we highlight two important aspects, the summary of the essential ideas of the lesson and the connections of these ideas with other mathematical ideas already known by pupils.

In this study it was possible to verify that a prospective teacher of the first years, already with some previous experience in leading whole-class discussions within the framework of an exploratory approach,

managed to achieve an interesting dynamic of discussion, inviting pupils to participate and intervene according to the situation and the aim and analysing the pupils' work with a focus on mathematical ideas. However, she did not always make the most appropriate interventions and, above all, she manifested difficulty in promoting a broad participation of most pupils. The contribution of the lesson study to the development of her knowledge about leading whole-class discussions seems to have been mainly in the organization of the preparatory work to do for the enactment of the discussion. Thus, in future lesson studies to carry out with other prospective teachers, we will strive to pay more attention to ways to promote a broader pupil participation.

Acknowledgment

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Applying variation theory: a challenging case in pre-service teacher education

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Although research has shown that conducting a learning study – a lesson study using variation theory – can enhance mathematical knowledge for teaching, the specific role of variation theory in this process remains unclear. The aim of this study was to get more insight in the relation between the use of variation theory and the development of participants' mathematical knowledge for teaching. A team of three pre-service teachers carried out a learning study, as part of a last-year module within mathematics teacher education. Pre-tests, student work and lesson plans were analyzed on how variation theory was applied and how this related to the participants' mathematical knowledge for teaching. Findings suggest that applying key elements of variation theory contributes to aspects of this knowledge, particularly during the first phase of the learning study. However, implementing variation theory in lesson design remains challenging, indicating a need to strengthen the support provided within the learning study module in pre-service teacher education.

Keywords: Learning study, preservice teacher education, mathematical knowledge for teaching, variation theory.

Introduction

Several studies have shown that conducting a learning study, a lesson study in which variation theory is used as a learning theory, can lead to a deepening of knowledge for teaching (e.g., Cheng, 2014; Lai & Lo-Fu, 2013; Mårtensson, 2023). Variation theory fundamentally addresses how concepts, or phenomena, should be understood, by focusing on its critical aspects and by structuring teaching to ensure that these critical aspects can be discerned by the learner. However, the relation between the use of variation theory and fostering mathematical knowledge for teaching remains unclear. Royea and Nicol (2019) found that it helped pre-service teachers (PSTs) in mathematics education in understanding mathematics, and in teaching mathematics in different ways, but they also found that PSTs perceived applying variation theory as difficult. Based on these findings, we designed a learning study module for pre-service teachers in their last year of pre-service mathematics education for lower secondary. We examined experiences of one team of three PSTs who participated in this module. The aim of this study is to shed more light on the way variation theory is used by the PSTs in each phase of the learning study, and on the relation with the development of their mathematical knowledge for teaching. Our research question is: In what way is applying variation theory in a learning study related to fostering pre-service teachers' mathematical knowledge for teaching?

Theoretical framework

Mathematical knowledge for teaching

The model of Mathematical Knowledge for Teaching (Ball et al., 2008) describes what kind of knowledge and skills teachers need when teaching mathematics. The knowledge and skills are divided into several domains. The central domains Specialized Content Knowledge (SCK), Knowledge of Content and Students (KCS) and Knowledge of Content and Teaching (KCT) are the domains most connected to the actual teaching in classroom: SCK is the deep knowledge of a topic within mathematics (relational understanding, according to Skemp (1976)), in order to be able to explain the topic; KCS is the knowledge of how students think about a topic, how they understand the topic, and KCT is the knowledge of the organization of the lesson, the way tasks are designed, in order for students to understand the topic. In 2017, Ball elaborates further on what teachers need to be able to do when teaching mathematics and identifies this as the *mathematical work of teaching* (Ball, 2017). A central aspect of the work of teaching is "being aware of and oriented to learning about and coordinating with others' perspectives" (Ball, 2017, p. 17). All students are different, everyone is shaped by their background, which causes different ways of thinking, of reacting on questions, on combining information. This perspective on teaching and learning is in line with the perspective on teaching and learning provided by variation theory.

Variation theory and its central concepts in a learning study

Variation theory presumes that individuals understand phenomena from their own perspective and that learning takes place when the way of understanding a phenomenon, or a so-called Object of Learning, has changed (Lo, 2012). For this learning to occur, some aspects of the Object of Learning must vary while other aspects remain constant (Marton, 2015). If a learner understands an Object of Learning in terms of its critical aspects, a learner has gained relational understanding of the Object of Learning (Marton, 2015). To make this more concrete, the following three elements of variation theory are critical in a learning study (Lai & Lo-Fu, 2013; Lo, 2012; Marton, 2015):

First, learning is always directed at something: the Object of Learning (OL). Unlike a learning objective, an OL not only tells what to learn, but also how to get there. This is therefore the key distinction between an OL and a learning objective: the OL includes the aspects that need to be considered, to create the possibility for learners to identify what is important (or: critical) and what is not, depending on their existing knowledge.

Second: investigate your students' (mis)conceptions, as these play a key role in determining the final OL and the critical aspects to address. For instance, if a student believes that in equation $y = ax + b$ the slope is equal to a, and in $y = d + cx$ the slope is equal to d, this suggests a misconception: the student likely assumes that the first term of the formula always represents the slope. However, in linear equations, the order of terms is not critical for identifying the slope; what *is* critical, is the coefficient of the independent variable.

Third: use patterns of variation in teaching, based on critical aspects derived from the misconceptions of students. For instance, related to the example above, by varying the order of terms and leaving all

other aspects of the equation invariant, students can discern themselves what is critical and what is not, in connecting the slope to the numbers in the equation.

Methods

The PSTs that participated in the learning study course were in their fourth and last year of pre-service mathematics teacher education for lower secondary at NHLStenden Hogeschool, University of Applied Sciences, in the Netherlands. Two teacher educators (TEs) taught the course, one of them in a dual role as TE and first author of this paper. Twelve PSTs participated in the course (four groups of each three PSTs). All meetings were class meetings, until the implementation of the research lesson - from that moment on, the TE met with each group seperately (each TE supervised two groups). The course lasted from September 2022 – May 2023. The course meetings took place every two or three weeks, during one morning, and lasted two to three hours. During the rest of the week, the PSTs were in their school-based practicum, in which they conducted the research lessons as part of the learning study course.

In this study, we gathered data from one group of PSTs (ages 21-24).They all taught in Grade 7 (first grade of Pre-vocational Secondary Education). The PSTs had no experience in applying variation theory, nor with doing research. The implementation of the course is presented in Figure 1.

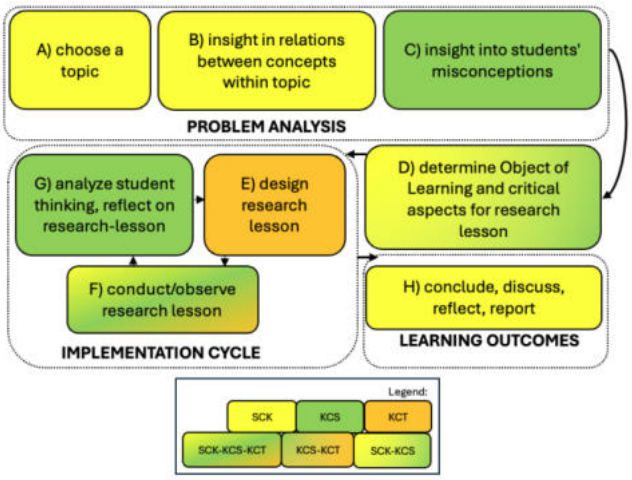


Figure 1: Implementation of the learning study course, September 2022 – May 2023

Figure 1 outlines the different phases of the learning study course and illustrates how the associated activities are connected to the knowledge domains identified by Ball et al. (2008), represented by different colours. The phase of the Problem Analysis takes seven meetings, from September to January. During this phase, PSTs choose a topic and design a concept map, with critical aspects, connections, structures, using the notions of relational and instrumental understanding (to get from A to B, in Figure 1). Then they design a pre-test. For each pre-test task, the guiding question is: "What do you know about students' understanding of the Object of Learning when completing the task?" Pre-tests are iteratively revised, based on feedback from peers and TEs, to address this question. Once approved by the TEs, PSTs administer the final pre-test during their teaching practice and subsequently analyze the results (from B to C, in Figure 1).

The instructional strategies employed were mainly presenting examples and non-examples (of concept-maps, of pre-test questions, of research lessons). This way the PSTs could identify the criteria of a good concept map, pre-test question, or research lesson. For instance, the NON-EXAMPLE of a pre-test question in Figure 2 does not give information on possible misconceptions: a right answer for a) does not give enough information on the learners' conceptions of negative numbers, and a right answer for b) does not give information on why a student thinks -5 is smaller than 6. However, the EXAMPLE does give a lot of information on misconceptions, which also incorporates decimal numbers and well chosen whole numbers.

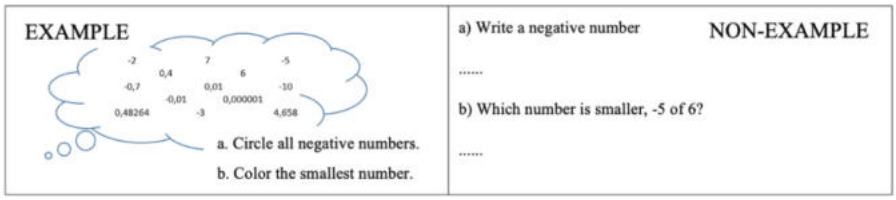


Figure 2: Example and non-example of a pre-test question

During the implementation phase, which takes six meetings from February - May, PSTs design a research lesson, focusing on OL and critical aspects, with use of targeted variation. They reflect on teaching and observing with the TE after each cycle of implementation. The TE provides resources to enhance the research lesson if necessary; each PST conducts the research lesson.

We gathered PSTs' written material during the learning study process: (revised) pre-tests (with justifications), analysis of results of implemented pre-tests, choice of OL with critical aspects, (revised) lesson plans (with justifications), their final report. In our analysis, we focused on the development of the pre-test and the research lesson. From the perspective of variation theory, we examined the extent to which the OL and critical aspects were addressed. Following Kullberg et al. (2024), we examined which (critical) aspects of the OL were made possible for students to discern (and which were not), and we connected this to the respective domains of mathematical knowledge for teaching. For instance, if PSTs designed a pre-test task to identify a specific misconception, but in the analysis, they found a different misconception they had not predicted - yet they understood its origin - their KCS improved.

Results

We present a few examples from the pre-test and the research lesson.

Choice and development of Object of Learning and pre-test by the PSTs

The pupils just got acquainted with representations of linear relationships, during the first weeks of Grade 7, without a focus on 'translation skills' from one representation (situation, formula, graph or table) to another. Therefore, the PSTs chose *linear relationships* as their temporary OL, and focused on translation skills, by tasks like the one from the first version of the pre-test in figure 3.

1. To the formula $m = 3n + 5$, Klaas makes the table below:

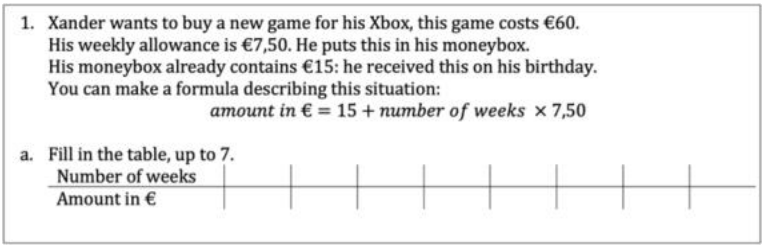
m	0	1	2	3	4	5
n	0	8	13	18	23	28

a) Mention two things that are not right about Klaas' table.

Figure 3: Task from pre-test version 1

PSTs' justification of this task was that they were curious whether students would see that m and n were interchanged, in the table; that 5 and 3 were interchanged, and that $(0,0)$ was incorrect. They did not justify why they wanted students to mention two incorrect things.

Peers and supervisor provided feedback, such as that too many answers were possible (for instance, $(0,0)$ and $(2,13)$ are not right), but that all these answers do not give enough information on whether the students would make the kind of mistakes the PSTs were looking for. Based on feedback, the team changed their pre-test-task; see figure 4.


1. Xander wants to buy a new game for his Xbox, this game costs €60.
His weekly allowance is €7,50. He puts this in his moneybox.
His moneybox already contains €15: he received this on his birthday.
You can make a formula describing this situation:

$$amount\ in\ € = 15 + number\ of\ weeks \times 7{,}50$$

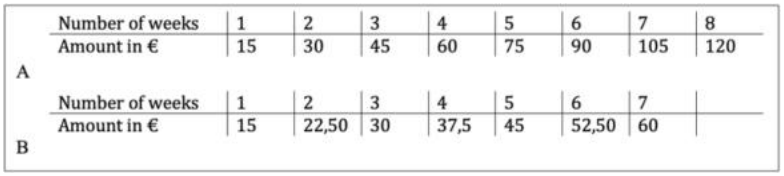
a. Fill in the table, up to 7.

Number of weeks							
Amount in €							

Figure 4: Task from final pre-test

PSTs justified these changes by explaining they wanted to see what kind of mistakes students make themselves, instead of having students guess what kind of mistakes are made by someone else, as in Figure 3.

The PSTs implemented the pre-test in their classes. Although some students gave a correct answer, there were also mistakes. In Figure 5, two types of mistakes are presented:


A

Number of weeks	1	2	3	4	5	6	7	8
Amount in €	15	30	45	60	75	90	105	120

B

Number of weeks	1	2	3	4	5	6	7	
Amount in €	15	22,50	30	37,5	45	52,50	60	

Figure 5: Two different student-answers to the task of Figure 3

In their analysis, the PSTs noted that they had not anticipated students starting the table with week 1 rather than week 0. In retrospect, however, they considered this choice understandable - after all, why begin with zero weeks? They also observed that these responses began with an initial amount of €15. While this value is difficult to justify when applying the formula correctly, it appears more reasonable when considering the context, particularly in the case of Answer B. The PSTs came to realize that presenting both the context and the formula simultaneously made it more difficult to interpret students' reasoning. In the case of Answer A, they were unable to determine how the students arrived at their solution; they only observed that €15 was added for each week, and that €7.50 played no role in the table. They concluded: "Students do not yet have a sufficient understanding of the concepts of slope and y-intercept."

Their final OL was *the translation from formula to table*, with critical aspects that students should be able to recognize the dependent and the independent variable in both formula and table and recognize the slope and the y-intercept in both formula and table.

Analysis of the development of Object of Learning and pre-test

Looking at Figure 3, we see a task with the assignment "find the mistakes" in a table to a given formula. The task is focusing on "translation from formula to table", and the critical aspects addressed are to recognize the dependent and the independent variable in both formula and table, and to recognize that the number of the slope in the formula is used as the increase in the table. However, both aspects are simultaneously varied in the same task: the top-row in the table is labelled with the independent variable, and the bottom-row is labelled with the dependent variable; the increase of the numbers of the bottom-row of the table is related to the the y-intercept instead of the slope. Since both aspects are varied in the same task, it is hard to determine how students think, from analyzing their answers; furthermore, yet another mistake ($(0,0)$ is incorrect) is included, without an explanation how this reflects a critical aspect.

In Figure 4, the final pre-test task, some aspects are changed:

- a context and a formula are presented, instead of just a formula;
- the labels of the rows are kept invariant, instead of interchanged;
- students have to fill in the values in the table themselves, instead of deciding about the way someone else thought: no focus on a critical aspect here, but still a possibility to show how you think.

Based on the answers as presented in Figure 5, the PSTs could detect whether students choose the right number for the increase of money per week - and they detected that students start a table with 1, and not with 0 - which made them think about whether writing 0 in a table is critical, or not. This reflection can be identified as SCK. The analysis of their students' thinking shows an increase of KCS, and from the arguments for re-designing the pre-test task, we identify KCT.

Research lesson task developed by the PSTs

In the opening of the research lesson, the PST put a task on the board, as presented in Figure 6. The class was divided in teams of four students each, and they had to answer the question: *Which table belongs to the formula of the taxi costs?*

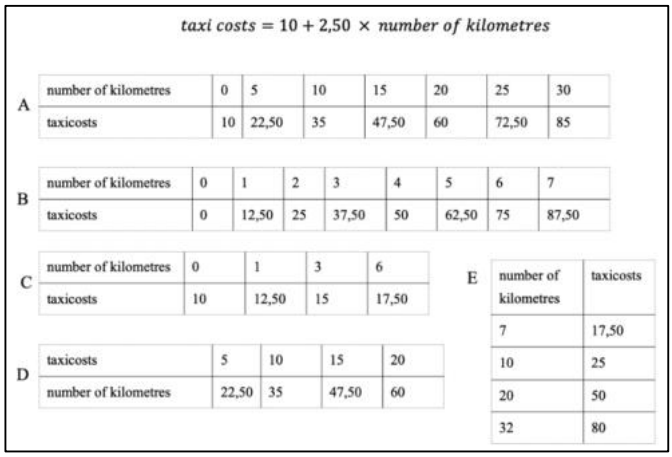


Figure 6: Opening task from research lesson

Based on the observations in the classroom, the PSTs noticed that students did not examine the remaining tables once they had identified the correct one. As a result, they failed to recognize the errors in the other tables. When prompted, students thought table E was incorrect, solely because it was vertically oriented. To address this issue, the assignment *"Find the misconception in the other tables"* was added in the subsequent lessons.

Analysis of the research lesson task

As mentioned before: the OL was "the translation from formula to table", with critical aspects: 1) students should be able to recognize the dependent and the independent variable in both formula and table, and 2) students should be able to recognize the slope and the y-intercept in both formula and table.

In this task, several aspects of the OL are varied simultaneously: the labelling of the rows (table D), the attention to the step-size (tables A,C and E), the value of slope and of y-intercept (tables B and E). The orientation of the table was also varied. The critical aspects were addressed in different ways - at the same time numerous other aspects were dealt with, both critical and non-critical.

By incorporating various possible misconceptions into the task, and by making incremental adjustments to the task during successive implementations of the research lesson - with the aim of encouraging students to reflect on these misconceptions - PSTs demonstrated both KCS and KCT.

Discussion and conclusion

We presented results of the written materials of one team of PSTs. From this, we see some development in mathematical knowledge for teaching in the problem analysis phase of the learning study course, and during the implementation phase. This is in line with the findings of Lai and Lo-Fu (2013), who found that the phases of the learning study in which student thinking is central, are successful in fostering mathematical knowledge for teaching. However, the design of task in the research lesson is not tightly aligned with OL and critical aspects chosen. Is this due to a lack of experience in applying variation theory ("It takes time to come to understand the aspects of variation theory and how those aspects are related" (Royea & Nicol, 2019, p. 14)), or to the inherent complexity of variation theory itself, or to the structure of the learning study course? This study does not provide enough information to answer the questions stated above, since it has significant limitations: the analysis is based solely on written materials from just one group. Incorporating qualitative data, such as interviews with the PSTs, could have offered a richer understanding of their learning experiences. Analyzing similar data from other groups in the course could also have provided a deeper understanding of the learning outcomes. Although this study is therefore limited in scope, we aim to answer the research question based on the data available to us: In what way is applying variation theory in a learning study related to fostering pre-service teachers' mathematical knowledge for teaching? We can conclude that applying the first two elements of variation theory (determining the OL, and finding misconceptions to decide on critical aspects of the OL) is related to fostering mathematical knowledge for teaching in analyzing student answers (KCS), reflecting on student answers (SCK) and reflecting on the design of the task (KCT). The third element of variation theory (using targeted variation in the research lesson, focused on the OL and its critical aspects) is partly

related to the KCS that was fostered as a result of analyzing student answers. To provide explanations for this, we require additional data and a more in-depth study.

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Reconsidering textbooks as representations of current didactic paradigms in lesson study

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Textbooks are important resources for mathematics education in many educational institutions. They also play a crucial role in LS in the kyozai kenkyu phase. We introduce the notion of the current didactic paradigm to analyze how textbooks represent the prevailing modes of study in institutions. We particularize this analysis in the case of the introduction of multiplication in Primary Education in Japan and Spain. We discuss how textbooks propose different approaches based on distinct epistemological models, pursuing different didactic aims, and based on diverse didactic means. We conclude by hypothesizing how LS transformative potential might be enhanced by the incorporation of textbooks that represent alternative paradigms in the kyozai kenkyu phase.

Keywords: lesson study, kyozai kenkyu, textbooks, didactic paradigms.

Introduction

Twenty-five years after the publication of Stigler & Hiebert's (1999) book, there is a vast body of research that shows how teacher educators and researchers have promoted LS in many countries around the world, together with several attempts to theorise it. This resulted in a complex scenario in which different LS models coexist. To clarify the situation, Seleznyov (2018) summarised what are the seven critical components of LS when it is more closely connected to the Japanese tradition, with *kyozai kenkyu* as one of them. *Kyozai kenkyu* is understood as the "planning" part of LS (Kim et al., 2021). However, it is much more than just the planning of the lesson. In *kyozai kenkyu, the* teachers "investigate all aspects of the content and instructional materials in the context of how students think about and understand the concepts they are going to learn" (Takahashi et al., 2005, p. 101). The process "involves examining teaching materials and tasks from mathematical and educational points of view as well as from the students' point of view" (Fujii, 2015, p. 278).

The importance of *kyozai kenkyu*, as a deep-rooted collaborative practice of the teaching profession (Shinno & Mizoguchi, 2021), is not always recognised beyond Japan (Choy & Lee, 2020; Fujii, 2015). Existing research does not always document it accurately (Seleznyov, 2018).

Although the teaching materials considered in *kyozai kenkyu* are not restricted to textbooks, they play a prominent role as primary resources for teaching in Japan (Shinno & Mizoguchi, 2021; Watanabe et al., 2008). These textbooks, which are carefully designed, represent an interpretation of the current curriculum: presenting tasks and problems for studying content using different instructional tools, suggesting paces across different contents, emphasising how the current topic relates to what came before and will come later, and giving a rationale for the selection of a particular material or tool (Watanabe et al., 2008).

Textbooks may be an advantageous gateway to understanding and comparing different cultures. In turn, they may contribute to a deeper understanding of *kyozai kenkyu* and to better support teachers

outside Japan, considering their cultural context. In addition, an enhanced understanding of textbooks might inform new ways of supporting teachers during *kyozai kenkyu* by incorporating and comparing Japanese textbooks with those in their local context.

In view of the above, the first aim of this paper is to introduce some theoretical tools from the Anthropological Theory of the Didactics (ATD), arguing how these could be helpful to uncover "cultural specificities". Then, to narrow the scope, we focused on a particular topic (multiplication of whole numbers in Primary School) and on Spanish and Japanese textbooks. By analyzing how this topic is introduced in each textbook, we aim to uncover some essential features of the existing culture in each country. We will end by briefly discussing how *kyozai kenkyu* mediated by these textbooks might be different in each country. Thus, we aim to contribute to discussions on knowledge creation in collaborative settings and how research informs the analysis of such settings, in coherence with domains 3 and 9 of the TWG28.

Theoretical framework and methods

Let us consider a school institution I and an ill-defined field $\mathcal{F}$ of mathematics to be taught in I. The notion of the *current didactic paradigm* in I related to $\mathcal{F}$ ($CDP_I(\mathcal{F})$) was proposed by Gascón and Nicolás (2022) as a model of how mathematics education around $\mathcal{F}$ is interpreted in I. A didactic paradigm is taken as a functional unit composed of four components: (1) a current epistemological model of $\mathcal{F}$, (2) the didactic ends pursued by I about the learning of $\mathcal{F}$, (3) the didactic means considered in I to achieve these ends, and (4) the didactic phenomena that the epistemological model unravels and to which the paradigm, globally considered, offers an alternative answer.

To perform the didactic analysis of the $CDP_I(\mathcal{F})$, it is assumed, as a fundamental hypothesis, that the researcher must adopt an explicit "point of view". This is embodied in the notion of *the reference didactic paradigm* (RDP), which expands the former notion of the reference epistemological model (Gascón, 2014). It is a fundamental instrument for researchers' emancipation from institutional dynamics that can be taken for granted, hindering the identification of didactic phenomena. Following the above-mentioned structure, a $RDP_I(\mathcal{F})$ should always be interpreted as a scientific hypothesis (Gascón & Nicolás, 2022). It is a tool for researchers to uncover undesired didactic phenomena linked to the current didactic paradigm in I, conjecturing that a study process ruled by the $RDP_I(\mathcal{F})$ might reduce or even avoid them.

Methodologically, to analyse the $CDP_I(\mathcal{F})$, the first step is the construction of a hypothetical $RDP_I(\mathcal{F})$, adopting thus an explicit position about a possible way of studying $\mathcal{F}$ in I. Then, the researchers take into consideration different sources that might give them access to relevant features of the $CDP_I(\mathcal{F})$, interpreted always from the perspective of the $RDP_I(\mathcal{F})$. Textbooks, as mediators between the intended and the implemented curriculum, offer a privileged entry to the analysis of the $CDP_I(\mathcal{F})$.

To narrow the scope of this paper, we focus on how multiplication is presented to pupils in Spanish and Japanese textbooks. First, we will outline the key features of a possible reference didactic paradigm to be used. Then, we will consider excerpts from two textbooks and conduct a qualitative analysis based on the basic components of a didactic paradigm, seeking to identify distinctive features of the current didactic paradigm in each country. Finally, we discuss how this might critically affect *kyozai kenkyu* in each country and LS as a whole. By doing this, we attempt to address the question: how cultural assumptions embodied in textbooks affect *kyozai kenkyu* in LS?

Components of a reference didactic paradigm of multiplication in Primary Education

In this section, we outline some fundamental features of a possible RDP that includes the multiplication of natural numbers in the Primary Education (PE) institution. Methodologically, we considered different sources for constructing a RDP (Gascón & Nicolás, 2022), which we also describe. Provisionally, we interpret multiplication within a broader field that includes what is commonly known as "arithmetic" in PE. We use the symbol $\mathcal{Q}$ to denote this blurry conglomerate of praxeological components, which we assume is not clearly determined in advance. From the perspective we adopt here, arithmetic is not limited to the study of numbers, operations, and their properties, but it should be integrated within the study of discrete and continuous magnitudes and measurements (Chambris, 2022; Chevallard & Bosch, 2001). Besides, the RDP needs to be compatible with a global disciplinary paradigm about the study of mathematics assumed by the researchers. In conjunction with the position of the ATD, we globally assume the mathematical modelling paradigm (Gascón, 2014), which postulates that any mathematical activity can be interpreted as the construction of mathematical models of intra- or extra-mathematical systems.

Thus, the reference didactic model $RDP_{PE}(\mathcal{Q})$ proposes that the genesis of school arithmetic is embedded in the universe of magnitudes and measures. School arithmetic is initially based on the study of extra mathematical systems of physical objects with measurable qualities (like quantity, length, area, volume, mass, etc.). This does not mean that school arithmetic is necessarily restricted to the study of these systems. The recursive nature of the system-model relationship postulated by the ATD would lead to a progressive distancing from them, to consider intermediate systems of more formal objects (like systems made of symbols that evoke the physical objects) and even the study of pure arithmetic systems (such as the system of prime numbers, or of the divisors of a number). The main didactic end promoted by the $RDP_{EP}(\mathcal{Q})$ is precisely to engage students in authentic modelling processes. This way, the genesis of arithmetic objects in schools will be linked to questions that arise from the study of these systems, like how to compare objects in terms of a certain measurable quantity, how to communicate to others about measurable qualities in objects, how to make this communication effective, and how to express measurable qualities of objects that are created from a certain combination of others.

In terms of didactic phenomena, our $RDP_{EP}(\mathcal{Q})$ emerges as a reaction to didactic phenomena in current educational systems already described in research, like the progressive degeneration of mathematical practices around magnitudes and measures, which isolates arithmetic from magnitudes, forgetting the role of magnitudes in the construction of arithmetical objects and weakening their rationale (Chambris, 2022). According to Chevallard & Bosch (2001), in school arithmetic, the study of magnitudes and measurement processes tend to fade away. The mathematical work prioritises the study of arithmetical objects on their own, at the risk of compromising their meaning.

The introduction of multiplication in Spanish and Japanese textbooks

In Spain, we will consider a grade 2 (7-8 years old) textbook[1] (other textbooks and grades have also been revised). As shown in Figure 1, multiplication is introduced in the context of the measurement

[1] Fraile, J., & Pineda, A. (2023). *Matemáticas 2 Primaria* [Mathematics 2 Primary]. *Proyecto abanico*. Vicens Vives.

(quantification) of collections of objects arranged in equal size subgroups. The underlying problem is how to determine and express the measure of the whole. Multiplication is presented as a model of the sum of equal addends, as shown in Figure 1. Thus, it is linked to the additive meaning of the composition of measures (Vergnaud, 1982).

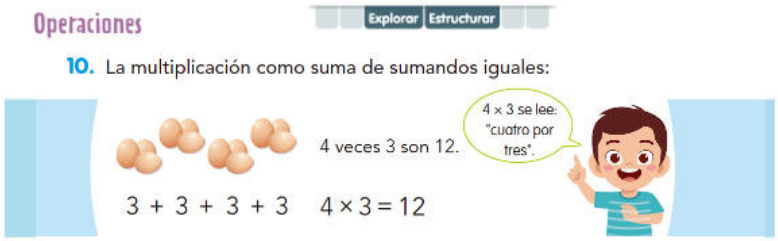

Figure 1. Introduction to multiplication in a Spanish grade 2 textbook (p. 214)

This meaning is reinforced later (Figure 2) through different exercises in which collections similar to the one in Figure 1 are displayed, chains of sums are presented, and pupils are asked to express this situation using the "new" operation. Multiplication appears as a new operation useful for representing (model) already known situations. In fact, it is nothing else but a new expression of an old meaning.

Figure 2. Introduction to multiplication in a Spanish grade 2 textbook (p. 214)

In Japan, the introduction of multiplication is widely described in the literature (Hino & Kato, 2019; Isoda & Olfos, 2021). At first sight, the Japanese grade 2 book we are considering[2] resembles the Spanish one. The introduction of multiplication is also based on the measurement of collections structured in subcollections of equal size (Figure 3). However, now, pupils are asked to explore these systems without any reference to addition. In fact, the focus is on the structure of the system, giving rise to verbal models that express such structure (Figure 3, right-hand side). In the absence of new techniques, pupils are expected to obtain the total measure of the given collections by counting.

Multiplication is presented as a new operation that models new situations: the total measure of a discrete system formed by groups of equal size (Figure 4, left-hand side and middle). Discursive models, called "math sentences", are introduced, and the students' activities are focused on the writing of sentences following this model (Figure 4).

[2] Fujii, T., & Majima, H. (2020). *New Mathematics 2B for Elementary School*. Tokyo Shoseki.

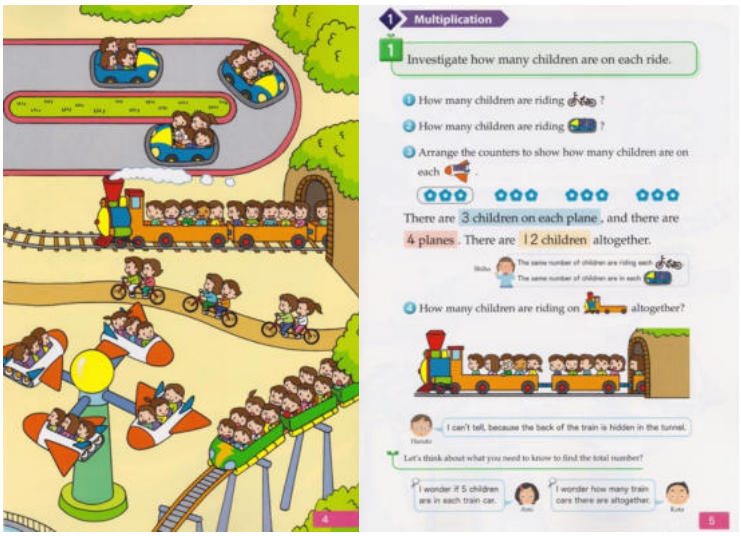

Figure 3. Introduction to multiplication in a Japanese grade 2 textbook (p. 4-5)

From a mathematical perspective, pupils are faced with the measurement of collections considering different kinds of units (for instance, pupils and planes), considering the relation that exist between them (3 children on each plane). As Hino & Kato (2019, p. 129) explain: "the meaning of multiplication of whole numbers, decimals, and fractions is represented as B × P = A, where B is the base quantity, P is the relative value, and A is the corresponding quantity that has the specified relative size with respect to B".

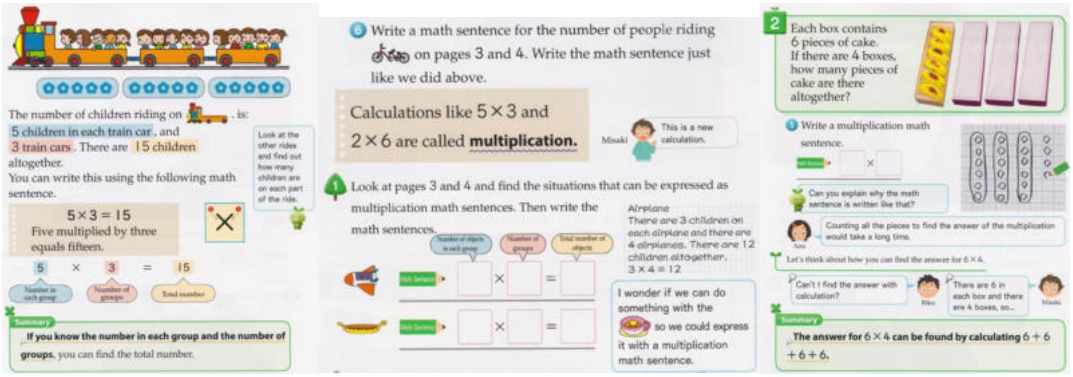

Figure 4. Multiplication in a Japanese grade 2 textbook (Fujii & Majima, 2020, p. 7, 8 and 10)

Only when this meaning has been established do the pupils think of strategies other than counting to determine the total measure of the collections, considering that counting would take a long time. Thus, addition appears as a more effective strategy to obtain the measure of the given collections but never as the meaning of multiplication (Figure 4, right-hand side).

Analysis and discussion

The Spanish and Japanese textbooks introduce multiplication in the context of systems of discrete collections of objects, arranged in subcollections of equal size, and the problem of how to find their measure. From this, each textbook follows a very different path, which we interpret here considering the components of two essentially different current didactic paradigms: $CDP_{SP}(\mathcal{A})$ and $CDP_{JP}(\mathcal{A})$.

From the perspective of the underlying epistemological model, in the $CDP_{SP}(\mathfrak{Q})$ multiplication is reduced to addition in the sense of the composition of measures (Vergnaud, 1982). Presented as a new operation, it is nothing but a new expression for "old" situations. In contrast, in the $CDP_{JP}(\mathfrak{Q})$ multiplication is introduced in the sense of isomorphism of measures (Vergnaud, 1983), as a new meaning that cannot be reduced to previous meanings. In both cases, discrete magnitudes (collections) are presented. However, in the Spanish case, these mainly play a decorative role. Multiplication, as repeated addition, could be introduced without any reference to them and, in fact, it is more than likely that the collections play a marginal role in classroom work, deepening the separation between numbers and magnitudes (Chambris, 2022). In the Japanese case, using collections is crucial. Numbers do not represent only measures, but also the function operator (e.g., in Figure 4, "5" means, at the same time, "5 children" and "5 children in each train car"). This new meaning needs to be introduced and understood. It is modelled discursively by the math sentence "number of objects in each group".

The previous relates to the didactic ends of each paradigm. In the $CDP_{SP}(\mathfrak{Q})$, we interpret that the end is to keep children very close to their previous knowledge of addition, avoiding an epistemological break. Thus, the focus is on the translation between new expressions of the form $a \times b$ and old expressions such as $b + \cdots + b$. Thus, it is not surprising that a short and direct introduction, followed by examples, is sufficient. On the contrary, the $CDP_{JP}(\mathfrak{Q})$ pursues students' entry into a new world, which implies thinking differently about quantities and, in particular, considering relations between them. Although the situations presented in the textbook resemble old ones, the mathematical activities around them are new. This explains, at least to some extent, why textbook authors need 6 pages to familiarize pupils with multiplication, as well as the emphasis placed on the "math sentences" as intermediate models. In terms of further aims, in the Spanish case, multiplication as repeated addition is a direct and cost-effective way of introducing it. However, in the middle term, this will provoke epistemological issues when continuous magnitudes and proportionality are introduced. In contrast, the Japanese approach is more demanding but better suited to further developments, in line with the "extension and integration" principle (Isoda & Olfos, 2021).

A full analysis of the didactic means within each didactic paradigm is not possible here. However, we see how, in the $CDP_{SP}(\mathfrak{Q})$, the direct introduction of multiplication through an example plus similar exercises seems to be sufficient. We interpret this in the context of a disciplinary current didactic paradigm strongly based on "show and tell" methodologies. In Japan, we identify features of the *structured problem-solving* paradigm (Stigler & Hiebert, 1999). The book proposes different situations that resemble open-closed ones. It is expected that they will use their previous knowledge (like counting, addition or whatever) and engage in collective discussion and evaluation of their strategies before a final strategy is declared (Figure 4, right-hand side). Nonetheless, what the book proposes needs to be reconsidered from the perspective of "using the textbook to teach" instead of "teaching the textbook" (Shinno & Mizoguchi, 2021). The latter is more common in countries like Spain, whereas the former directly connects with LS practices in Japan.

Conclusions

Our analysis, although brief, points out essential differences in the $CDP_{SP}(\mathfrak{Q})$ and $CDP_{JP}(\mathfrak{Q})$ from the perspective of the $RDP_{PE}(\mathfrak{Q})$. Will this impact LS processes?

In the case of Japan, Isoda & Olfos (2021) described two LS experiences regarding multiplication carried out under the conditions of the $CDP_{JP}(\mathcal{Q})$. In both cases, the research lessons were created around the idea of working directly with collections (sweets in a box and dots, respectively), finding equal-sized groups, and expressing the total amount of objects in terms of those groups. The students were asked to think about different ways of grouping objects, how to express these groupings, and how to obtain the total amount. Different approaches were used to achieve understanding through discussion. In short, we consider that the essential features of the $CDP_{JP}(\mathcal{Q})$ followed by the textbook have been maintained through the *kyozai kenkyu* process, which has led to the design of these lessons.

What could happen if a group of Spanish teachers engage in *kyozai kenkyu* from the perspective of a Spanish textbook? We do not know for sure, but it is more than likely that the fundamental features of the $CDP_{SP}(\mathcal{Q})$ are barely questioned, thus limiting the transformative power of LS. That is, LS will reinforce or, at most, refine the relation between teachers and $CDP_{SP}(\mathcal{Q})$. This phenomenon may not be unique to the Spanish case. In the Japanese case, we wonder if LS also leads to an improved relationship between teachers and the $CDP_{JP}(\mathcal{Q})$, but it does not really challenge it.

On the other hand, what could have happened if in *kyozai kenkyu* the teachers engaged in a deep analysis of materials (textbooks, in particular) that followed didactic paradigms that offered alternative perspectives? We hypothesise that it would enhance the transformative power of LS, making it possible to question how mathematics education (around the topic under consideration) is organised within their institution. We plan in exploring this line of research in future work.

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Stepping up from an experienced teacher to a lesson study facilitator

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In this paper, we aim to understand how an experienced teacher, Vanda, who had already participated in two LS, has an initial experience as a lesson study facilitator. She was challenged to facilitate LS by a more experienced facilitator, who was alongside with her all the LS sessions. The two facilitators worked collaboratively to prepare the LS. Data collection included participant observation, with audio recordings of the preparation and LS sessions, and audio recordings of semi-structured interviews before and after the LS and included document collection. The results show how the dimensions of resources, orientations, goals and identity (ROGI framework) interplay in framing Vanda's concerns and practices as a novice facilitator.

Keywords: Novice facilitator, lesson study, professional development.

Introduction

Lesson study (LS) is a collaborative professional development (PD) process aimed at enhancing student learning through planning, teaching, observing, and reflecting on a research lesson. In Japan, a 'knowledgeable other' is invited as an external expert to provide extensive feedback especially at the end of research lessons (Hourigan & Leavy, 2021). In contrast, in Western LS there is often a facilitator who organizes and leads the process (Hourigan & Leavy, 2021), balancing expertise, teaching, and guidance, while allowing teachers to express their questions and collaboratively build knowledge (Mynott & Michel, 2022). The fact that "teachers drive the agenda and co-design the learning opportunities in real time" (Lewis, 2016, p. 538) means that many aspects of the LS sessions cannot be script in advance by the facilitator. So, preparing and facilitating the LS sessions is complex (Lewis, 2016; Mynott & Michel, 2022). It requires facilitators to understand the many facets of school life and teaching, to have knowledge about LS, content and pedagogy for collaborative and sustained teacher learning (Akiba et al., 2019; Hourigan & Leavy, 2021; Lewis, 2016). Thus, it is very important to know how such facilitators can be prepared.

Framework

Facilitators of PD for mathematics teachers play a pivotal role in strengthening teachers' professional expertise and in bringing educational innovations into schools (Karsenty et al., 2023). They are responsible for managing PD activities, setting norms, promoting teacher collaboration, and guiding discussions. To be effective, facilitators require adequate preparation and ongoing support, as their effectiveness improves with experience (Karsenty et al., 2023; Lewis, 2016). Considering that it is important to focus on enhancing the facilitator' decision-making abilities, which develop over time, Karsenty et al. (2023) developed the ROGI framework with four components: resources (R) involves intellectual, material, contextual, and social tools available to facilitators, highlighting the

significance of content knowledge and pedagogical content knowledge for effective facilitation; orientations (O) refers to the facilitators' beliefs and values about teaching and learning; goals (G) includes both explicit and implicit purposes for PD, such as content, process, and environment goals; and identities (I) defined as how facilitators perceive their roles within a given context.

In LS, as a collaborative PD process where teachers work together to plan, teach, observe, and reflect on lessons, the facilitator's role is even more complex. LS facilitators are expected to navigate the dynamic and often unpredictable nature of LS sessions, support teachers in real-time co-design of learning experiences, and foster a collaborative environment where teachers feel comfortable sharing ideas and reflecting (Akiba et al., 2019; Lewis, 2016). Mynott and Michel (2022) highlight the often "invisible" nature of LS facilitation, pointing the importance of the facilitator being able to subtly guide the process without dominating it. This requires facilitators to have not only content and pedagogical knowledge but also the ability to manage group dynamics, support reflective practice, and adapt to the emergent needs of the group and to the often-unpredictable nature of LS sessions (Lewis, 2016; Mynott & Michel, 2022). Additionally, novice facilitators must model effective teaching practices while simultaneously learning from the process (Hourigan & Leavy, 2021).

Akiba et al. (2019) and Clivaz et al. (2023) emphasize the importance of facilitators to engage in continuous learning and reflection to improve their practice. Mynott and Michel (2022) argue that the "invisible" work of facilitation requires ongoing learning and support, particularly for those new to the role. This aligns with the idea that facilitators often develop their expertise through a combination of formal training or collaboration with more experienced facilitators, and on-the-job experience, where they can experiment with different facilitation strategies and learn from their experiences (Clivaz et al., 2023; Hourigan & Leavy, 2021; Lewis, 2016). Therefore, understanding how facilitators develop the skills and knowledge to effectively lead LS and focusing on the evolving practice and learning of novice facilitators is essential. So, our research question is: How does an experienced teacher navigate her initial experience as a lesson study facilitator?

Methods

This research is qualitative and interpretive, with a case study design of a novice LS facilitator, Vanda (pseudonym) who had already participated in two LS. She was challenged to facilitate LS by a more experienced facilitator (Paula), who was alongside with Vanda all the LS sessions. Preparing the LS sessions involved collaborative work of the two facilitators, both teachers with several years of mathematics teaching experience (grades 7-12), who had already planned lessons together. Vanda had previously participated in two LS facilitated by Paula. In the LS facilitated by Vanda, after the first session, preparation always started with Vanda reflecting on the previous session. Eight mathematics teachers volunteered to participate in the LS. All had more than twenty years of teaching experience, except for one first-time teacher, and all but two teachers had previously participated in same LS as Vanda. The teachers decided to plan lessons on the topic "functions", to grade 10 students.

Data collection included participant observation (in preparing for and during LS sessions), with audio recordings of the preparation sessions (PS1, PS2, …), and of the semi-structured interviews before (II) and after the LS (FI). We focus on Vanda's concerns because they shaped her actions and reflections during LS preparation and interviews, particularly in clarifying the LS process, leading

whole-class discussions, and her transition from an experienced teacher to a novice facilitator.

Table 1: Categories and subcategories considered in data analysis

Category	Subcategory	Indicators
Resources	Knowledge about and for the LS process	Refers to previous LS; uses Paula as a support to consolidate/validate a particular issue/idea
	Knowledge about the teaching approach	Refers to the different kinds of tasks; the structure of the lesson; the whole-class discussion; the dynamics of the lesson
	Material resources	Proposes training tasks; shares tasks for the classroom
	Social resources	Acknowledges the support and sharing among participants (collaboration) as fundamental to their learning
Orientations	Facilitator orientation for teacher learning	Promotes situations that she considers as important for teachers' learning; identifies/remembers episodes of practice that contribute to teachers' PD; makes decisions about the content from previous teachers' interventions
	Teacher orientation for student learning	Proposes orientations/resources/dynamics/examples that contribute to students' learning; values a particular teaching approach
Goals	Content goals	Identifies important aspects for teacher learning
	LS process goals	Makes decisions to support teachers learning about the LS process
Identity	The way she sees herself	Recognises/positions herself as a teacher or/and as a facilitator
	How others see him/her	She is recognised as a teacher or/and as a facilitator

Based on Vanda's concerns, and as we focused on her role as LS facilitator, to analyse her processes of decision-making we adjusted the components of ROGI framework (Karsenty et al., 2023) using a deductive and inductive coding process (Table 1): to resources we added knowledge mobilized regarding and for LS as well as knowledge mobilized about the exploratory teaching approach; in orientations we considered the facilitator's guidance for teacher learning, which encompasses ideas/conceptions about how teachers learn (the interplay with practice, discussions/reflections with the construction of ideas, "teachers learn by engaging in practice"), along with the teacher's orientations for student learning; in goals, we considered the goals for the LS process, namely the facilitator's decisions for teachers to learn about the LS process; and in identities we considered the data referring to how the facilitator perceives herself, both as a teacher and a LS facilitator.

Results

Clarifying the LS process

In the initial interview, Vanda spoke about her experience as a LS participant and expressed her concern in clarifying what work would be done at each stage of the LS so that all participants got a clear idea of the process (goals).

> Vanda: I think we must present the concept of LS, don't we? And for everyone to reflect on what they think... Either those who have already been [in LS] or those who are new to it (II).

To address this concern, Paula asked her about what resources they could use, when preparing session 1, immediately after the interview. Vanda suggested using a form that the teachers could fill in with questions about LS, based on their previous experience as LS participants. Paula questioned whether, before this form, it would be important for the teachers to read a policy brief which she shared with Vanda before the LS. After some discussion, Vanda suggested using the form as a resource for participants to record their ideas about LS after the session. To "present the concept of LS" (PS_1), she suggested asking the participants to read the document before the session and to adapt presentation (ppt) they used to share their work on another LS (material resources).

> Vanda: To share this [policy brief] for them to read before the session... In the LS session, presenting the [LS] diagram... We can give some examples, or don't you think so? Maybe not, because we are going give them answers, aren't we?... [We could] ask people who have experienced LS… What they would do differently or what they would do the same? … In our planning, we write "steps on the diagram… To be completed collectively", that's it. This way we can understand if everyone, at least, identifies the LS stages correctly... (PS_1)

Vanda facilitated the session as planned. Although not all participants did the suggested reading, they all shared their ideas, mainly those who had participated in LS before, or asked questions, particularly the first-time teachers. Reflecting on how she facilitated session 1, Vanda acknowledged the collaboration among the participants as key to their learning (social resources), and emphasised their active role in the LS sessions, drawing an analogy with the work she does with her students (teacher orientation to student learning).

> Vanda: [As] there were people who had already been [to LS] who shared their ideas, those who had not, may have got a much better idea than the one shared by a facilitator... As with our students... They are much better at sharing with each other than we are, sometimes… It was in line with what I'd hoped would happen. (PS2)

Thus, drawing on her knowledge of LS and participants, and on the discussions with Paula (resources), Vanda selected different resources (material and social resources) and made several decisions about how to facilitate the session, aiming to clarifying the LS process for all participants. Emerging from her concern to clarify the goals for each session, Vanda suggested using a PowerPoint presentation with the LS stages to be completed in the session, based on participants' ideas, rather than presenting a pre-filled slide. In doing so, Vanda created situations she believed would be more valuable for participants' learning (facilitator's orientation for teacher learning).

To lead productive whole-class discussions

Referring to her teaching practice (the way she sees herself as a teacher), Vanda emphasised her

increased attention to students' solving strategies and representations, as well as to whole-class discussions, because of participating in LS:

> Vanda: The way I begun to see the kids' answers and allow them to share... My teaching practice changed a lot... I used to focus on a particular answer, and it would go on the board. Last year, I found myself wanting them to do it, to show it in a different way, and I began to see things that I had never really thought about... I think that the more we work in this format, so to speak, the more we can innovate... It is only by doing it that the teacher realizes whether it works or not. (II)

Reflecting on her previous LS experience, Vanda noted that the task they designed was too extensive and did not lead to much discussion, prompting participants to suggested that in future LS more attention should be paid to leading whole-class discussions. In the initial interview, Vanda emphasised that in the planning should be important consider time for the students' autonomous work while leaving time for them to "explain their reasoning, and I think this is also important... We want them to expose themselves and... Discuss the results in the lesson" (II).

Considering this, when preparing session 1, Paula shared with Vanda a video in which a teacher led a whole-class discussion, starting from a task that the students solved using different strategies. Vanda stressed the importance of recording in the session plan and discussing with the teachers "how do we... record and choose questions [solving strategies] for the discussion... 'How to select strategies for the whole-class discussion?'" (PS_1). In session 1, the teachers solved the proposed task, but they struggled to align the task with the curriculum, which limited the time for whole-class discussion. Reflecting in this session, Vanda stressed the importance of active student participation in lessons and leading the whole-class discussion: "One of the things that I think has to be a concern, really, is that in the lesson we are going to observe, the students must play an active role, and we have to see them debating ideas" (PS_2). From this concern, her teaching practice, and the way she sees herself as a teacher, Vanda, "very keen on 'seeing is believing, experiencing is knowing'", as she stated in the initial interview, suggested having the teachers play the role of students:

> Vanda: Getting them to work on the task and run the simulation... 'Kids [colleagues], we have a task for you. We present the task, ... 'You have *x* time to complete it.' We collect the answers, show them if needed for them to explain their ideas, and then we explain that this is what we are aiming for [in the research lesson]. (PS2)

In session 2, Vanda monitored the teacher's work on a task she proposed, and encouraged them to share their ideas, but she ended up explaining teacher's answers. Reflecting on this session, she identified the preparation to leading the whole-class discussion as a less successful aspect. She pointed to leading this moment of the lesson among peers and managing time as challenges:

> Vanda: What went wrong? Time management, on my part, as the 'leader'... We had so much planned... And we didn't do half of it... Perhaps there should have been more time for the colleagues to clearly explain how and why they did what they did... It's very difficult to simulate a teacher explaining to a teacher... When I look at my colleague's solving strategy, I'm following it, right? (PS3).

From this reflection, when planning session 3, Paula and Vanda discussed how to plan the whole-class discussion, based on the task they planned to propose. Initially, Vanda suggested a short discussion, similar to session 2, making more detailed planning for the research lesson. However, as leading whole-class discussions was a challenge pointed by Vanda and these participants in previous LS, Paula questioned whether preparing such discussion could be part of the planning process "Is it very different to do it with this task or another ...?" (PS3). She then suggested adding students' answers for teachers to analyse, select and sequence as a starting point for planning the whole-class

discussion. Vanda agreed that it was important for teachers to plan the whole-class discussion more thoroughly, analysing possible students' answers. She also emphasized the importance of teachers anticipating students' strategies and difficulties and using them to prepare the whole-class discussion. When facilitating session 3, Vanda monitored teachers' work, photographed their answers and displayed those that she considered interesting, giving teachers the opportunity to explain them, to ask questions about their colleagues' answers and to reflect on their teaching practice. There was also time for her to show on the board the students' answers, previously selected on PS3, prompting further discussions about selecting and sequencing students' answers and leading whole-class discussions.

Thus, Vanda identified leading whole-class discussion as a key aspect for teachers' learning (content goals), based on teachers' feedback from a prior LS (facilitator's orientation for teacher learning), and valuing it in her lessons (identity). She suggested leading whole-class discussions based on the teacher' work in tasks she proposed, as she values teacher learning through practice (orientations). Although she faced challenges in leading the discussion with the participants without dominating the discourse, Vanda drew on her pedagogical knowledge and discussions with Paula (resources), to lead discussions and create a safe environment for participants to share ideas, ask questions, and reflect on their teaching practice.

Becoming a LS facilitator being an experienced teacher

Referring to her view of herself as a facilitator (identity), in the initial interview, Vanda stated that during the teacher education process she led, the aim was to "teach [the teachers] how to use a specific software". She added that facilitating LS would be quite different, as it requires "being able to tailor what you are going to work on in each session to meet the participants' needs" (goals), emphasizing this as both a challenge for the facilitator and a valuable aspect of LS: "I don't think I have this... getting people to think, to go along with what you need, while also considering their needs... I don't have this role of facilitator." (II)

In her view, it would be important to collaboratively plan the sessions with Paula and to develop her knowledge about LS by reading articles (resources):

> Vanda: The one thing I know I'm really going to have to do is to be constantly... Working with you. Because I can't get there on my own. Will I have to read a lot? Yes... I'm going to have to read a lot about this. Maybe the reading that I didn't do initially [as a participant], I will have to do carefully. Because I think it's important, right? To share knowledge, one must be knowledgeable. (II)

Afterwards, Vanda read an article about LS carried out in Portugal. She also had other articles from a LS in which she participated. However, referring to herself as a facilitator (identity) in the first LS sessions, Vanda kept saying "I know how to teach kids, I don't know how to teach colleagues" (PS_3).

In the final interview, Vanda stated that she felt more like a colleague than a facilitator, pointing to planning and making the global organisation of the LS as challenging: "I feel like a colleague. Facilitator ... OK, I know how to hold a conversation, in quotes, with colleagues... Doing this work myself [e.g., planning the session] that you do so well, I don't feel I can do it."

Vanda emphasised the collaborative work with Paula as essential for preparing sessions (resources):

> Vanda: Our discussions before the sessions were essential, weren't they? I think that if they hadn't existed, it would have been very different... And it was thinking and finding

> strategies for getting round issues, problems, in other words, challenges, that the benefit lies... For me it was very rich... It wasn't just one person thinking about the session, was it? We were sharing ideas. (FI)

Although Vanda felt she could not prepare a LS session independently, she became comfortable drawing session plans in later sessions, emphasizing the collaborative work with Paula and having her available during the sessions (resources). She added that, even though she still had a lot to learn as a facilitator, working collaboratively on session preparation enhanced her learning as a LS facilitator:

> Vanda: [It was] very easy to do [facilitator's role] because you're always there… It went far beyond my expectations... I think I have a lot to learn, but I learnt a lot... I didn't even feel like a facilitator at first, did I? Clearly, the preparation we did in the last sessions had much more input from me than the earlier ones. (FI)

Vanda pointed out that LS facilitators "have to deal with too many unpredictable things" (final interview) and need to have in-depth knowledge about LS. She emphasized the importance of facilitators having prior experience as LS participants "to then be able to be on the other side" (final interview), and of learning from practice:

> Vanda: Training [facilitators] would maybe involve doing it [facilitating LS], wouldn't it? … Or having training, or someone working with the facilitator... I like 'doing it'... You have the training, and you get a working group at school, and you, at school, play this role... Planning, doing, planning, doing. I think it would be important, yes, for anyone who wants to become a LS facilitator. (FI)

In Vanda's perspective, facilitators learn through practice, either with teacher educators or other LS facilitators. She highlighted the importance of the "plan, do, plan, do" cycle, similar to the work carried out in the LS mentioned in this paper.

Thus, Vanda reflected on the preparations of the LS sessions and her identity as facilitator, noting the challenge of tailoring the sessions to participants' needs and emphasising the importance of collaborative planning and discussing with Paula (knowledge about and for the LS process). Initially she considered herself unprepared for planning sessions and for her role as facilitator, and by the final interview, she felt more prepared to handle the complexities of facilitating LS (identity), recognizing that each session's preparation offered an opportunity for her growth and learning as LS facilitator (knowledge about the LS process). She emphasized that LS facilitators need to have extensive knowledge of LS, prior experience as LS participants, and learn through practice.

Conclusion

The analysis of Vanda's initial experience as a LS facilitator illustrates the intricate balance of identity, orientation, goals, and resources in her journey from a novice to a more confident facilitator. Navigating the complexities of facilitating LS, she continuously sought to refine her practice, drawing on her experiences, and relying on both material and social resources, such as collaborative planning with Paula, that she pointed as essential in shaping her facilitation practice, and relevant literature. Although initially Vanda expressed uncertainty about facilitating the learning of colleagues, in contrast to her confidence in teaching students, her orientations, particularly her belief in teacher learning through practice and collaborative work, informed her decision to simulate classroom scenarios and encourage collaborative learning among teachers (as in Clivaz et al. (2023) and Akiba et al. (2019)). Key goals for Vanda were clarifying the LS process and enhance participants' knowledge, particularly in leading whole-class discussions. As she gained experience as a facilitator, she provided more time for teachers sharing their ideas and reflecting on their teaching practice. This reflects the complexities of assuming

the 'teacher of teachers' role (Hourigan & Leavy, 2021) and highlights the importance of aligning session goals with participants needs. Despite her initial uncertainties about her role as a facilitator and her ability to lead colleagues, through continuous reflection, collaborative planning with Paula, and practical experience, Vanda's identity as a facilitator evolved, shaped by her initial concerns and experiences and later by her growing confidence in leading LS sessions. By the final interview, she felt more confident in her ability to tailor LS sessions to participants' needs and acknowledged the importance of "doing" as central to her learning and growth as LS facilitator, as in Lewis (2016) and Hourigan and Leavy (2021). Vanda underscored the importance of extensive knowledge of LS, prior LS experience as a participant and the importance of continuous practice, reinforcing the "plan, do, plan, do" cycle as essential for growth as LS facilitator.

Vanda's reflection further enhanced her belief in learning through practice, the value of reflective practice and collaboration in professional learning. Her experiences highlight the dynamic nature of the LS process, where facilitators not only guide others but also engage in continuous learning themselves, adapting to the needs of participants and the demands of each session. By the end of the LS, Vanda acknowledged that while she still had much to learn, she had stepped up from being a hesitant facilitator to a more confident one, feeling more prepared to handle the complexities of facilitating LS.

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The Roles of Knowledgeable Others in a Mathematics Lesson Study

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This study investigates the roles of knowledgeable others (KO) in a lesson study (LS) model. The research involved an academician, a teacher, and three pre-service mathematics teachers over a six-weeks period. Data were collected through video recordings of meetings, field notes, semi-structured interviews, focus group interviews, and reflective reports. The obtained data were analyzed with the help of the qualitative data analysis program NVivo and taking into account the LS sharing contents stated by Kanbolat (2015). As a result, it was determined that KOs predominantly assume roles as informants, critics, and, less frequently, learners. The study also highlights the differing contributions of the academician and the teacher as KOs, examined through the lens of their sharing contents.

Keywords: Knowledgeable others, lesson study, sharing contents, teacher education.

Introduction

LS, originating in Japan, has rapidly gained prominence as a model for teacher education and professional development worldwide (Fernandez & Yoshida, 2004). A distinguishing feature of LS, is its structural process, which involves planning, implementing the plan, observing, and evaluating the implementation. This collaborative approach fosters professional growth, as teachers engage with a diverse group of participants, including colleagues, administrators, and other stakeholders in education (Kanbolat & Arslan, 2022).

Another appealing aspect of LS attractive is its flexibility in participant profiles. LS group members may include teachers, pre-service teachers, students, administrators, academics, parents, educational policy makers, in short, everyone involved in the learning-teaching process (Kanbolat & Arslan, 2022). This diversity not only enhances the impact and functionality of LS but also allows for the integration of varied perspectives and expertise.

Among the participants, knowledgeable others (KO) play a pivotal role by contributing their expertise in subject matter, pedagogy, or LS methodology. Their involvement enriches the exchange of ideas and experiences, enabling deeper insights into teaching practices. While the roles of KO are well-defined in some contexts where LS is widely practiced, in others, these roles remain ambiguous (Clivaz & Takahashi, 2018).

This study addresses this gap by exploring the roles of KO in LS processes and aims to reply to the following question : What are the roles of KO in the mathematics-focused LS?

On knowledgeable others

KO play a crucial role in the LS model, enhancing the quality of discussions (Wood and Sithamparam, 2014) and contributing to its effective execution (Bjuland & Mosvold, 2015; Kanbolat, 2015; Takahashi, 2013). In Japan, where LS is widely practiced, KO are integral to LS groups, offereing

expertise in subject content, teaching and LS process (Hourigan & Leavy, 2021). Depending on the context, KO may serve as experts, teacher trainers or members of the LS group (Jacques, 2023), leveraging their strong content and pedagogical knowledge, and LS experience (Fernandez, 2002; Seino & Foster, 2021; Takahashi & McDougal, 2016).

KO contribute significantly to discussions by addressing curriculum-related issues, students' mathematical thinking, and other aspects of teaching knowledge (Amador & Weiland, 2015). They effectively bridge theory and practice, a core principle of LS. In cases, KO also provide participants with access to theoretical insights or recent research developments. (Fernandez, 2002), helping teachers shape and deepen their learning outcomes during the LS process (Clivaz & Georgy, 2020; Hourigan & Leavy, 2021). Moreover, KOs encourage group members to engage in meaningful dialogue, stimulate critical discussions, and question established practices within the LS environment (Jacques, 2023).

The LS Research Group at Teachers College, Columbia University, identified three key reasons for KOs to provide feedback during research lessons: (1) offering a fresh perspective on the group's LS process; (2) contributing subject-specific insights, new ideas, or reforms discussions, and (3) sharing experiences from other lesson study groups. (Fernandez, Yoshida, Chokshi & Cannon, 2001). These components also help define KO roles. In addition, Clivaz and Clerc-Georgy (2020) defined four roles for KO: meeting organizer, teacher trainer, researcher and group member.

Understanding KO roles is essential both for those undertaking this role and for refining their definitions in the literature. Unlike previous studies, this provides a detailed analysis of KO roles through categories such as classroom management, students recognition, feedback, mathematical knowledge, unexpected situations, problem-solving instruction, material usage, contextual knowledge, learning outcomes, assessment process, and group/individual teaching (Kanbolat, 2015). Furthermore, the distinct contributions, as a KO, of the teacher and the academician offer valuable insights by examining their roles both throughout the LS process and within the context of sharing contents. This dual perspective highlights how KO can benefit from the LS environment and from each other's expertise.

Method

The study involved an academician and a mathematics teacher serving as KO, along with three pre-service teachers who were assigned with pseudonyms: Esma, Erkan and Hakan. The academician, a graduate from the primary school mathematics teaching program, had one year of teaching experience, completed a master's and doctoral degree, and has been teaching mathematics education at a public university for five years. His research focuses on training mathematics teachers, and he teaches courses such as General Mathematics and Linear Algebra.

The participating teacher had three years of teaching experience and was working on a master's thesis in mathematics education. The teacher also served as KO during the pilot study and was the only participant with prior lesson study experience. The researcher facilitated the LS process by preparing the environment but did not intervene as a KO.

The LS process was conducted over 6 weeks as part of a teaching practice course. Three cycles of LS were carried out in the study, with each cycle comprising planning, implementation, reflection, re-planning, re-implementation, re-reflection and plan finalization (Fernandez et al., 2001; Lewis, 2000). Each cycle focused on instructional plan, resulting in six classroom practices, each lasting two hours, for a total of 12 hours. Each pre-service teacher conducted two classroom practices.

Data collection involved video recordings of LS meetings, researcher field notes, semi-structured interviews, focus group discussions, and reflective reports. The data were analyzed through content analysis, focusing on the research question, what roles do KOs play in these sharing environments? Using NVivo qualitative analysis software, the researcher coded meeting recordings, informed by initial role codes derived from field notes. Three primary roles of KOs emerged from the analysis:

Informant: Providing information, explanations, or answering questions related to the content.

Critic: Offering constructive criticism of statements or practices related to the content;

Learner: Asking questions to deepen understanding or exchange ideas about the content.

The researchers revisited the video segments to the coding of Kos roles, incorporating the LS sharing content categories identified by Kanbolat (2015). These categories included classroom management, students recognition, feedback, mathematical knowledge, unexpected situations, problem-solving instruction, material usage, context information, learning outcomes, assessment process, and group/individual teaching.

The analysis revealed the specific roles KOs assumed within these sharing environments and the frequency of their contributions. Figures were createdto illustrate the roles of KOs and the dynamics of their participation in LS environments.

Findings

When analyzing Figure 1, which was created by considering the sharing contents and the roles of KO and using of the meeting records, it is evident that the KOs predominantly play the role of *informant* and rarely taht of *learner*. Specifically, KOs mainly assumed the role of *informant* in conversations about *student recognition*, the role of *critic* in discussions on *giving feedback*, and the role of *learner* in dialogues concerning *student recognition* and *material usage*.

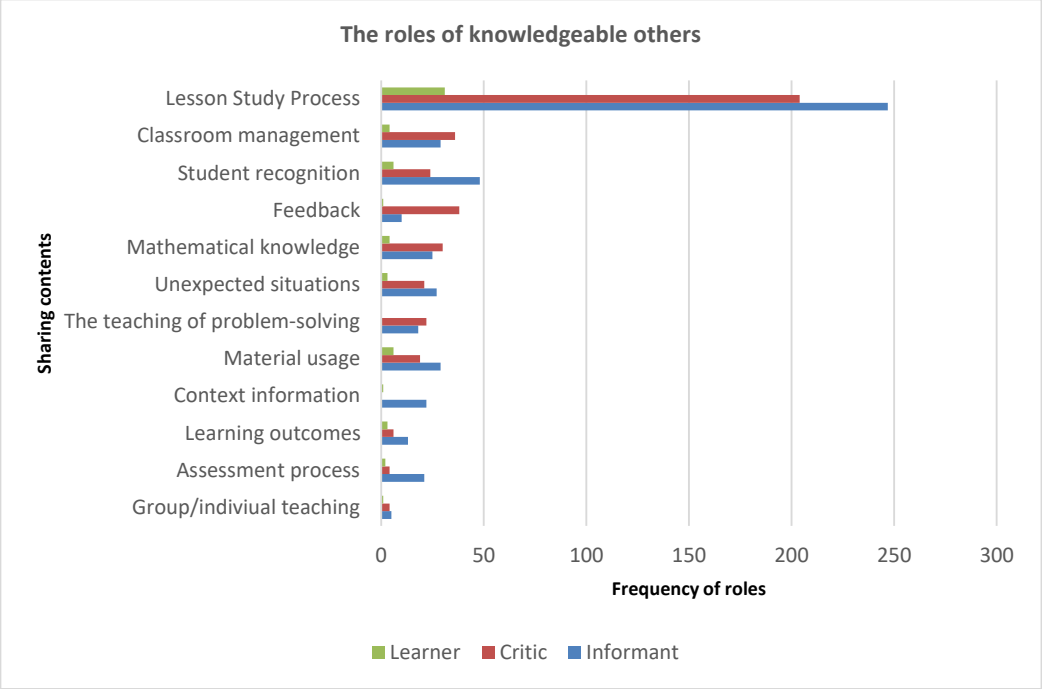

Figure 1: The roles of knowledgeable others

Figure 2, which separates the roles of the academician and the teacher as KOs, indicates that the teacher exhibited the roles of *informant*, *critic* and *learner* more frequently than the academician throughoutthe LS process. In addition, when the content of the sharing is examined, it is noteworthy that both of KOs displayed the role of *critic* equally in contexts related to *mathematical knowledge*, *assessment process* and *learning outcomes*. However, neither exhibited the role of *critic* in relation to *context information* and nor the role of *learner* in the context of *teaching of problem solving*.

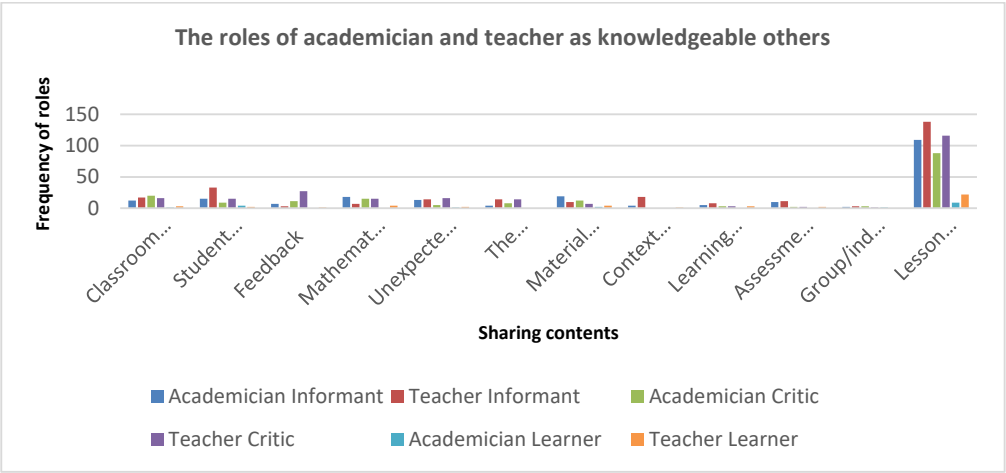

Figure 2: The roles of academician and teacher as knowledgeable others

Analyzing Figure 2 in terms of the academician's role in the LS process reveals that the academician is more informative than the teacher in discussions about giving *feedback*, *mathematical knowledge*

and *material usage*. The teacher's statements during the focus group interview support the finding that the academician acts as an *informant* in the context of *mathematical knowledge*:

> Different opinions, especially the views of our academician, were very valuable for me. In our usual practice, we often focus on achieving specific outcomes—if this is the case, then that must be true, and so on. However, I was exposed to a range of new viewpoints. For example, concepts like length, distance concepts... were not only approached from an educational standpoint, but also explored as pure definitions. Is a square? To me, a square is defined as a shape with four equal sides. Yet, I learned that a square can also be considered a type of trapezoid... or a rectangle. If I were working on this alone, I would not have ventured into such a discussion.

In terms of the critic role, Figure 2 shows that the academician assumes this role more frequently than the teacher in discussions about *classroom management, material usage, and group/individual teaching techniques.* The reflection report from Erkan, one of the pre-service teachers, provides an example of the academician's *critical* stance *in classroom management*:

> Our academician also suggested that it would have sufficed to include 5–6 tables instead of printing the entire table. Additionally, he mentioned, I wish you had solved all 3 questions in time measurement because you moved to the field with 10 minutes remaining... (Excerpt from Erkan's 2nd cycle reflection report)

Additionally, Figure 2 indicates that the academician exhibited the role of *learner* by asking more questions than the teacher in contexts related to student recognition and group/individual teaching techniques in the LS process. An example of the academician's *learner* role is evident in the following question directed to the teacher:

> Teacher, you know better, but it is not the sixth grade, right? Algebraic expressions?...

Figure 2 demonstrates that the teacher took on the role of *informant* more than the academician in the discussions about *classroom management, student recognition, unexpected situations, teaching problem solving, context information*, *learning outcomes*, *assessment* and *group/individual teaching.* The academician's statements about the teacher's role as an *informant* in *student recognition* are particularly noteworthy:

> In that environment, the classroom teacher anticipated potential questions from students, misconceptions or mistakes that could arise or evolve in the future... While discussing the foundational aspects of the subject and the elements that would form the basis, we relied entirely on the teacher's guidance. We must acknowledge this, as it became clear that we could not adequately predict students' prior knowledge and potential issues when approaching the topic.

Furthermore, Figure 2 shows that the teacher assumes the *critic* role more often than the academician in terms of *student recognition*, *feedback, unexpected situations* and *teaching problem solving*. The teacher's field note about the classroom practice supports this *critic* role of the teacher regarding the *teaching of problem solving:*

> We introduced problem-solving tasks without solving any problems beforehand. Students were not fully motivated to engage in setting up problems. (Excerpt from the teacher's field note of 2I2C).

Lastly, Figure 2 illustrates that the teacher exhibited the *learner* role by asking more questions than the academician in contexts related to *classroom management, feedback, mathematical knowledge, unexpected situations, material usage, context information, learning outcomes* and *assessment process*. The teacher's query to the academician — *If there is a difference between a rectangle and a square, and they are called by different names, how can they be the same? I still find this contradictory* — exemplifies the teacher's role as a learner within the context of mathematical knowledge.

Discussion and conclusion

The research identified that the KO in the LS process assumed the roles of *informant, critic* and *learner*. These findings highlight that KOs contribute not only as experts but also as group members who actively learn and benefit from the LS environment. Notably, the academician and the teacher primarily served as *informants* with their roles as *learners* being less frequent.

The findings further reveal that the KOs most often assumed the role of *informant* in discussions related to *student recognition*, the role of *critic* in during *feedback-related* conversations, and the role of *learner* in discussions concerning *student recognition* and *material usage*. The relative experience of pre-service teachers in *student recognition* and providing feedback, which develops as a result of one-to-one communication and interaction with the student, may have caused the KO to assume the roles of *informant* and *critic*.

The academician consistently displayed the role of informant across all sharing environments, particularly in discussions about *feedback, mathematical knowledge and material usage*. This dominance may stem from the academician's university-level teaching responsibilities and expertise in pure mathematics and related courses. The literature similarly emphasizes that KOs in LS groups often provide insights into *mathematical knowledge* and innovative ideas (Fernandez et all., 2001; Lewis & Hurd, 2011; Watanable, 2005).

In addition, the academician demonstrated the *critic* role across most contexts, particularly in discussions about *classroom management*, *material usage* and *group/individual teaching techniques*. However, the academician displayed the role of *learner* by asking more frequently than the teacher in discussions about *student recognition* and *group/individual teaching technique*, as evidenced by their inquiries aimed at deepening understanding.

Conversely, the teacher exhibited the role of *informant* in every sharing environment and more frequently than the academician in discussions about *classroom management*, *student recognition, unexpected situations, teaching problem solving, context information, learning outcomes, assessment process and group/individual teaching technique.* It is noteworthy that the sharing contents in which the teacher is more active informant have the characteristic of being nourished and developed by in-class experiences. This is because the teacher interacts directly with the classes in which the application is performed and with the relevant age group. For example, the teacher has more input to offer in explaining and limiting the achievements in conversations because he is an individual practitioner (Cheach, 2010).

The teacher can also contribute more about the class atmosphere, the students' pre-learning experiences and readiness levels, and some special students, because the classes in which the discussed practices will be applied are, in fact, the teacher's classes. As a matter of fact, in the LS

environment, the teacher is expected to guide other group members on how students learn the relevant content by making use of their experiences (Watanable, 2005). In the process, in addition to the role of *informant*, the teacher assumed the role of *critic* in all sharing contents except context information. Especially in the discussions about *student recognition, giving feedback, unexpected situations and teaching of problem solving*, in which he exhibited the *informant* role more than the academician, he also assumed the *critic* role more than the academician. In addition to all these, the teacher exhibited the *learner* role more than the academician with the questions she asked in the environments related to *classroom management, feedback, unexpected situations, context information, learning outcomes and assessment process.* In particular, the presence of an academic conducting courses on pure mathematics and material use at the university may have paved the way for the teacher to exhibit the *learner* role more than the academic in environments related to *mathematical knowledge* and *material usage*.

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Collaborative planning in lesson study: The role of mathematical knowledge for teaching

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This qualitative analysis of recorded teacher meetings during a lesson study cycle on fractions identifies the mathematical knowledge for teaching used and developed by a group of primary school teachers. The analysis aims to understand how mathematical knowledge for teaching and collaborative planning influence one another. The results show that throughout the cycle teacher participants draw upon and build all elements of mathematical knowledge for teaching. Input factors such as analysis of student assessment responses, the use of planning templates and textbooks helps to make tacit knowledge visible and supports the development of new knowledge through discussion. Despite this, difficulties transcending limitations to horizon content knowledge and specialised content knowledge were evident. Considering what influenced these difficulties has implications for designing and adapting lesson study locally to maximise learning opportunities for teachers.

Keywords: Lesson study, professional learning, teacher collaboration, teachers' mathematical knowledge, fractions.

Introduction and background

Lesson study (LS) has been described as a "collaboration-based teacher professional development approach that originated in Japan" (Murata, 2011, p. 1). As the breadth and depth of LS research expands, increased attention has been placed on its connection with the use and development of mathematical knowledge for teaching (MKT). However, the locus of attention is often on *what* MKT is drawn upon and incorporated during a LS cycle (Ní Shúilleabháin & Clivaz, 2017). Research into *how* this knowledge development occurs during LS (Clivaz et al., 2023) is still in its infancy. Processes found to support teachers' knowledge development include studying curriculum and research and observing students during research lessons (Lewis & Perry, 2014). However, it is unclear how these processes contributed to the learning. In contrast, Ní Shúilleabháin's (2015) findings show how Knowledge of Content and Students (KCS) and Knowledge of Content and Teaching (KCT) was developed by focusing on the role of viewing teaching and learning from student perspectives. The research presented here contributes to the emerging literature on *how* MKT influences and is influenced by teachers collaborative planning in LS including a discussion about the role of Horizon Content Knowledge (HCK) and Specialised Content Knowledge (SCK). This is framed in contexts where local conditions do not afford the availability of an experienced facilitator or knowledgeable other prior to the post-lesson discussion.

This is of particular interest and relevance in Scotland – the location of this research – as Scotland's Curriculum for Excellence (CfE) tasks teachers with building their own curriculum from broad experiences and outcomes (Es and Os). These Es and Os are grouped into curriculum levels that each span multiple years of schooling rather than outlining a year-by-year progression. By the end of primary school, most children will have experienced the first three levels: Early, First and Second Level. While the intention is that these less prescriptive, broad experiences provide professional space

for teachers to meet the varied needs of all children, in some situations, this has led to teachers relying on a smorgasbord of materials of varying quality (Leakey & Mynott, 2025). This may have implications for curriculum coherence and consequently learning outcomes for children.

This research was conducted with a group of four teachers from different schools who were utilising the Collaborative Lesson Research (CLR) model (Takahashi & McDougal, 2016). The group collaboratively planned a fractions unit and LS research lesson for a Primary 4 class (students aged 8–9). The qualitative analysis of this case study utilises information from all five recorded teacher meetings with a focus predominately on meetings three and four which involved collaboratively planning the fractions unit and the detailed lesson plan for the research lesson. This study is framed around the research question (RQ), how do MKT and collaborative planning influence one another in a LS research cycle on fractions? Following an analysis of the results, which focus' on four of the six MKT domains in more detail, there is a discussion specifically about the role of SCK and HCK.

Theoretical framing

When trying to implement LS, teachers' existing mathematical subject matter knowledge (SMK) and pedagogical content knowledge (PCK) is an important consideration, as is the SMK and PCK that they develop. Fernandez (2005) questions whether teachers were able to take full advantage of opportunities to learn through LS due to a lack of content knowledge. Ball and colleagues (2008) MKT framework outlines six domains that can be used to categorise knowledge for teaching which may help to understand the interconnectedness between MKT and collaborative planning in LS:

Subject Matter Knowledge (SMK)

- Common Content Knowledge (CCK) – "the mathematical knowledge and skill used in settings other than teaching" (Ball et al., 2008, p. 399)
- Specialized Content Knowledge (SCK) – "the mathematical knowledge and skill unique to teaching" (Ball et al., 2008, p. 400)
- Horizon Content Knowledge (HCK) – "an awareness of how mathematical topics are related over the span of mathematics included in the curriculum" (Ball et al., 2008)

Pedagogical Content Knowledge (PCK)

- Knowledge of Content and Students (KCS) – "combines knowing about students and knowing about mathematics" (Ball et al., 2008, p. 401)
- Knowledge of Content and Teaching (KCT) – "knowledge that combines knowing about teaching and knowing about mathematics" (Ball et al., 2008, p. 400)
- Knowledge of Content and Curriculum (KCC) – a "grasp of the materials and programs that serve as 'tools of the trade' for teachers" (Shulman, 1987, p. 8)

In relation to fractions, five subconstructs (Kieren, 1976), have been further developed including to create a theoretical model connecting basic operations of fractions to these subconstructs (Behr et al., 1983). Watanabe et al., (2017) used this framework to analyse how textbooks from Japan, Korea and Taiwan introduce and develop fraction concepts (Table 1). Some variation exists in the ordering of the five subconstructs; however, many fraction topics, such as fraction as number (Grade 3) and equivalent fractions (Grade 4) were first introduced at the same grade level in all three textbooks.

Table 1: An interpretation of $\frac{3}{4}$ according to the five fraction subconstructs (Lamon, 2007) mapped against their first appearance within three Asian textbooks (Watanabe et al., 2017)

Fraction subconstruct	Interpretation and meaning of $\frac{3}{4}$	Grade		
		Japan	Korea	Taiwan
Part-whole	3 parts out of 4 equal parts of the unit	2	3	3
Measure	3 ($\frac{1}{4}$-units); a distance of 3 ($\frac{1}{4}$-units) from 0 on the number line or 3 ($\frac{1}{4}$-units) of a given area	3	3	3
Operator	$\frac{3}{4}$ of something; is a rule that tells how to operate on a unit	6	3	5
Quotient	3 divided by 4; is the amount each person receives when 4 people share a 3-unit of something.	5	4	4
Ratio	3 of A are compared to 4 of B; 3:4 is a relationship in which 3 As are compared, in a multiplicative rather than additive sense to 4 Bs.	6	6	6

If the goal of fraction instruction is to help students "recognize nuances in meaning; to associate each meaning with appropriate situations and operations; and, in general, to develop insight, comfort, and flexibility in dealing with the rational numbers" (Lamon, 2007, p. 636), then teachers must also have a sound knowledge of these meanings (SCK); how they connect to different fraction topics and how these different meanings and topics are related over the mathematics curriculum (HCK). This brings us back to the RQ under consideration here but with a specific focus on the role of SCK and HCK – how do MKT and collaborative planning influence one another in a LS research cycle on fractions?

Context, data and analysis

This qualitative case study of a LS planning team took place over five weeks in May/June 2023. Four primary school generalists working in different schools in the north of Scotland volunteered their own time to be part of the LS cycle. Two participants had never been involved in LS and two had limited experience. Due to local constraints, an experienced facilitator was not present in any of the meetings and neither was the researcher. One team member took up the notional role of facilitator using a loose facilitator guide (created by the author of this paper) intended to support novice groups like this to understand and maintain faithfulness to the general principles of LS. One team member volunteered to teach the fraction unit (three lessons) to their class. The team chose the second lesson as their research lesson as they decided it would give them most information for their LS RQ. The team also had access to a research proposal template and an English translation of a Japanese textbook which they could use to support their planning if they wished. Due to the geographical distance between them, they conducted their meetings online using Microsoft Teams. The cycle consisted of four planning meetings (approximately 90-minutes each); a LS research lesson which was recorded using Microsoft Teams and watched asynchronously by the planning team; and a post-lesson discussion (90-minutes) which included an invited knowledgeable other who provided final comments during

the final 20 minutes of the meeting. One teacher was unable to attend meetings three and four but returned to watch the recorded research lesson and contribute to the post-lesson discussion.

Prior to analysis, the automated meeting transcriptions were adjusted for inaccuracies. As this paper's RQ pertains to collaborative planning, sections of the transcribed data relating to planning the wider fractions unit (three lessons) and the detailed research lesson (lesson two in the unit) were analysed identifying instances where planning decisions were proposed, discussed, revised and finalised. The identified data was coded using the MKT framework (Ball et al., 2008) to establish the knowledge utilised and developed during the collaborative planning. To understand how MKT was developed, the data was further analysed to understand the relationships between planning decisions and MKT.

Results

Prior to the LS cycle commencing, team members had expressed an interest in focussing on fractions; an area of teaching they found challenging. In the first meeting, the group solved a series of fraction problems themselves and discussed unknown pupils' responses (CCK, SCK, KCS). They also discussed perceived strengths and difficulties in relation to teaching fractions (SCK, KCS, HCK). Jamie, acting as note taker, periodically summarised their discussions. After summarising the group discussion about the challenges associated with a lack of school wide understanding of progression over the year (KCC, HCK), Jamie draws on his own experiences working across a group of schools.

> Jamie: …we're [the group of schools] noticing that fractions outcomes are lower than other areas, but we're not changing how we're doing or when we're doing it or the frequency of what we're doing. So I think that sort of taking it as a one year approach every year, rather than looking at as a seven year sort of longer term, is also a challenge with fractions.

This reflection suggests an awareness of the importance of understanding how fraction topics are related over the curriculum. However, at that time, details about what those relationships might be (HCK) did not surface. The shared challenges were reflected in Sam's experiences with her class. In Primary 1 and 2 her students had no experiences of fractions and only some came to Primary 4 with formal experiences of fractions from Primary 3. This was attributed to Covid and teacher differences. Sam shared an overview of the fraction topics taught over two occasions this year (KCC). Based on this, and with reference to curriculum and planning documents, Sam suggested a focus on equivalent fractions. The limited elaboration makes it difficult to interpretate the nature of SCK and HCK.

> Sam: Yeah. So my class they are familiar with recognizing fractions; are familiar with ordering fractions from the smallest to the largest; they're familiar with fractions of a number, some of them are, fractions of a quantity. I'm trying to think what else we've done so far, but we spent a lot of time on that. I try to use the language of fractions as much as I can when we're doing times tables and stuff like that, as well.

During the first two meetings, utilising all six MKT domains, the team had solved, analysed then selected eight assessment questions for Sam's Primary 4 class so they could evaluate the students existing knowledge to support their planning. Prior to the third meeting, Jamie summarised the correct and incorrect responses and analysed the data; including, looking for patterns and outlining the types of errors and misconceptions that were evident (SCK, KCS). In discussion with Jamie, Sam had reviewed these results alongside the CfE Es and Os and the Japanese textbook they were utilising (KCC) to suggest a revised focus (HCK). At the start of meeting three, Jamie and Alex were

discussing this (Sam had not yet arrived). Meeting three then consisted of more detailed discussions in relation to the unit plan using this information.

Jamie: I think we've maybe, I sort of I wondered this at the start, but I think we've overshot and I think the equivalent fractions will be too much based on the results.

Alex: Your notes on the assessment were great, and I do agree. I think, you know, like the question, they're not even recognising equal parts for some of them. So that's huge, isn't it? That's a big thing that needs covered, I suppose.

Jamie: [Sam] was talking about she thinks the better sort of plan of attack would be to look at this sort of basic unit fractions up to tenths with the specific vocabulary about whole, part and equal parts. Looking at the sort of activities that were on 2B [referring to Japanese textbook] about the folding and that sort of like practical applications of the fractions and then looking at how you can then develop that into comparison, so using the tape or the like paper and looking at the vocabulary of greater than, less than and equal to with regards to fraction sizes. And she was hopeful that that might then lead into some level of introduction to equivalence, where they might say that two halves and, sorry two quarters, and a half are the same and then using all of that above to then start placing them on the number line. So having like a one strip which is classified as the whole and then cutting it down and having half and building two of the, so you've got that sort of build up towards the whole or from the whole reduction down. So that's what she thinks would be the best series of tasks.

These exchanges revealed strengths in relation to their use of new KCS and to some extent with KCC HCK and SCK as they were thinking about the order that fraction topics should be taught, how they relate to one another and possible representations. However, potential limitations in SCK and HCK were also revealed. In the exchange above, four different fraction topics are highlighted: fractions as equal shares, comparison, equivalence and fractions as number. In the Japanese textbook, these topics are introduced over three different grade levels; however, the proposal here is to cover them within a single unit. It is not clear how much the group understand about the interconnections between these topics (HCK) or how they relate to different fraction interpretations and meanings (SCK).

A pivotal point in meeting three, which supported the group to discuss tasks in more detail and streamline the many tasks they were trying to utilise, was when Alex shared her understanding of the pedagogical approach that she felt the textbook was trying to adopt. It also seemed to act as a stimulus for the utilisation of KCS as Jamie started to role play an anticipated pupil response and considered the affordances that the task might offer (KCT).

Alex: I think the way the textbook is sort of implying it is that you give them the paper and just say fold this square paper to make half the size of the square and then you look at what everyone's done and see how different people have done it.

Jamie: That then opens that opportunity for the discussion, though. I think this is still a half even though it's different to your half because it's, the whole's partitioned into two equal parts, blah blah blah. So I would say that would be fine.

Although it waxed and waned, this more in-depth discussion increased into the final planning meeting. Discussions that drew upon and built KCS and KCT included deciding on the exact wording of tasks; anticipating responses; and planning potential follow-up questions based on what these responses revealed about student understanding. The planning template structure appeared to be a stimulus for these discussions. It included column headings relating to 'Learning task and activities'; 'Teacher support'; and 'Assessment' and rows subdivided into different phases of the lesson. This

structure is part of a pedagogical approach to teaching known as teaching through problem solving. When discussing the 'summing up' part of the lesson, for example, Sam justified the use of a model they had been discussing including its affordances and how these link to the lesson success criteria.

An ongoing issue, that related to both SCK and HCK, was the number and type of tasks they tried to include within a lesson. These tasks were often addressing different fraction topics and drew upon different fraction subconstructs. One example was the task involving placing fractions on a number line. Firstly, the lesson goals were framed around the topic 'fractions as equal shares' (part-whole subconstruct); however, the number line relates to the topic 'fractions as numbers' (measurement subconstruct). The team had discussed differences between part-whole and linear conceptualisations and Jamie had demonstrated knowledge of some of the fraction subconstructs from a previous professional learning experience; however, it appeared challenging to apply this to their planning.

Jamie: …it [the professional learning course] talks about the different schemes of fractions and how you learn them from, like, you know, partitioning and iteration up to like whole, part-whole schemes and measurement schemes and so on. And I'd never really thought of fractions as a number, and it's bad in my sense, like I've always thought of it as a quantity of an amount which has always been that context of shaded in boxes, etcetera, etcetera…

Secondly, the complexity of fractions as numbers appeared to be underappreciated, as demonstrated by Alex's remark in meeting four.

Alex: …we obviously had the Frayer model kind of as a class plenary together. Do we just pop number line in there as well?

And finally, they had different representations of a number line. Again, this confusion appeared to be related to SCK and the different interpretations (subconstructs) of fractions.

Sam: So when you've got the number line by pieces of paper it becomes your fraction wall, but then if you've got it by number, you've got your number line that you can have. So you can show both.

Jamie: Yeah, because if you had a zero at the front and like this is one strip and this is one strip being partitioned into four and we've got one of those four.

It is not clear that the others made sense of Sam's second description of the number line until the research lesson where a tape measure (from 0 cm to 100 cm) is used as the number line. Students placed fractions (e.g. one half) on the 'number line' therefore positioning one half at 50 cm. However, this no longer relates to the number one half where one is considered as the whole (although it would have done if the tape was labelled from 0 m to 1 m). Instead, the one half represents a relationship between two numbers where the original size is 100 cm.

In the post-lesson discussion, the group acknowledge that the number line was a bit of an afterthought and agree this should have been a lesson in its own right. Alex also makes a point about the potential confusion of the centimetres but also reflects on how this offered other learning opportunities.

Alex: Obviously kind of the idea of us using the linear context was kind of to get them into that knowledge and idea and understanding of, you know, a half is the kind of biggest unit fraction and then you go down to the third, the quarter. So, I was wondering, did the numbers on the number line make them focus more on the kind of centimetre aspect of it and take away from the unit fractions getting smaller or do you think it. I think in some ways it did help because they could say well twenty-

> five is smaller than fifty, so a quarter is smaller than a half. But then did we, you know, did we account for that connection, really, when we were planning the lesson. Because I don't think we did, we didn't talk about how, as much about, you know, using the measuring tape and how that would, how that would affect it.

However, the group do not appear to appreciate that the use of the tape measure does not address what they had originally wanted to draw out, which was this idea of fractions as numbers. The knowledgeable other does draw this out in their final comments in the post-lesson discussion.

Discussion and conclusion

In summary, there was evidence that teachers drew upon and developed all elements of MKT while collaboratively planning during the LS cycle as they learnt from and with each other. The planning team members' individual experiences allowed them to bring different perspectives and knowledge to the LS cycle. Specific input factors also appeared to have an important role in developing this knowledge. For example, the implementation and analysis of a fractions assessment and the use of a Japanese textbook allowed the team to consider new KCS and KCT that helped them to plan a more appropriate starting point for their fraction unit. Additionally, input factors such as a research proposal template and the Japanese textbook appeared to stimulate opportunities to make tacit knowledge visible and develop new knowledge through discussion including the analysis and anticipation of pupil responses. This also appeared to have a particular impact on KCS and KCT.

In the interests of understanding how we can maximise learning opportunities for teachers through LS, two main limitations the group appeared to have difficulty transcending will be explored. Firstly, a lack of deeper understanding of the different fraction interpretations and how these correspond to different fraction topics. Secondly, challenges in understanding how to sequence topics in the longer term and design instruction that supports learning at a deeper level. The broadness of the CfE Es and Os and lack of associated guidance for mathematics teaching may be contributing to these limitations without a universally accessible way for teachers to address this. The structure of the Japanese textbook appeared to offer some support in the development of these two areas. However, a lack of familiarity with this textbook and lack of access to associated resources like the Japanese teaching manuals may have limited its potential to support the group within the time frame of this LS cycle. A question also arises about the nature of the relationship between SCK and HCK. Do limitations to SCK, make it challenging for teachers to develop and use HCK? Understanding how fraction interpretations and topics relate to one another may be challenging if fraction understanding in general is limited in the first place. Could more robust curriculum documentation, alongside LS as a vehicle for understanding and using this documentation in practise further develop teachers' SCK and HCK?

Moving forward, a longitudinal study tracking a planning team over time may provide clarity over how SCK and HCK develop when groups have a longer period of time over which to develop this knowledge using shared materials to support their research and planning.

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Learning to teach early arithmetic skills through learning study: The case of four prospective teachers

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Abstract

This paper presents findings from an intervention where prospective teachers participated in a learning study during a course in teacher education. During the iterative process the prospective teachers collaboratively designed one lesson using variation theory as a theoretical tool, and enacted, analysed, revised the lesson, and enacted the revised lesson. The aim of this paper is to identify what one team of prospective teachers learned from the intervention, shown as the changes made from the first to the second lesson. Variation theory is used as a theoretical framework to analyse their teaching and thereby what the prospective teachers learned. The findings show that the prospective teachers from the first to the second lesson; a) identified more critical aspects of the object of learning and specified the critical aspects, b) focused more on pupils' difficulties, c) developed more demanding tasks. The findings have implications for teacher education.

Keywords: Learning study, prospective teachers, teacher education, early arithmetic skills.

Introduction

Prospective teachers' (PT) teaching skills is primarily developed during periods of school-based practicum. It is argued that teacher education needs to develop pedagogies for the enactment of teaching and combine them with pedagogies of investigation (Grossman et al., 2009). Studies suggest that working in a collaborative process with peers in investigations of teaching and learning can support PTs' development of teaching skills (Cheng, 2014; Hiebert et al., 2007). There are good grounds for expecting that variation theory-driven learning studies (Pang & Marton, 2003) can support PTs in such a way (e.g., Royea & Nicol, 2019). Studies have reported on PTs' own experiences of taking part of learning study in teacher education, showing a common opinion that the learning study process supports a deeper understanding both of how to prepare and improve lessons and into pupils' ways of understanding (Cheng, 2014; Royea & Nicol, 2019; Turu, 2017). Although we know a lot about PTs' own experiences of learning studies, we know less about how learning study affects their teaching skills. More extensive studies investigating PTs' teaching of subject matter are lacking. The aim of this paper is to identify what one team of PTs learned from taking part of a learning study. The research question addressed in this paper concerns in what ways the PTs developed their teaching about part-whole relations of numbers when participating in a learning study incorporated in a mathematics education course?

Learning study in teacher education

Learning study is a teacher-driven, collaborative and iterative inquiry of practice that is used by in-service teachers to develop teaching that enhance pupil learning (e.g., Pang & Lo, 2012). The learning

study cycles of planning and analysing lessons and pupil learning are framed within a theory of learning, commonly variation theory (Marton, 2015). Variation theory can be seen as an explanatory and pedagogical tool that guides teachers in understanding and analysing the relationship between teaching and learning a particular object of learning (the content and capability to which learning is directed during a lesson). As learning study has come to play a valuable role in supporting professional development for in-service teachers, the approach has been incorporated in teacher education programmes at several universities worldwide (e.g., Cheng, 2014; Mårtensson & Ekdahl, 2021; Tan et al., 2020) to help PTs developing their instructional design skills and teaching competence. It has been argued that participation in learning studies during education also could facilitate continuous professional development in future careers (Pang & Runesson Kempe, 2019). There have been a number of studies reporting on experiences of implementations of learning study in initial teacher education. Most of these studies are based on self-reported reflections and/or interviews focusing on PTs' own experiences and learning from participating in learning studies. Most of the PTs in Royea and Nicol's (2019) study expressed that variation theory has engineering power, since it is directly applicable to teaching and stands out in comparison with other learning theories they have previously learned. In that and other studies (Cheng, 2014; Davies & Dunnill, 2008; Wood, 2013), PTs report that learning study and variation theory have the potential to support them in focusing on pupils' learning and their different ways of understanding, as well as in preparing and improving lessons.

However, research shows that it is challenging for prospective teachers to learn fundamental principles of variation theory (Ko, 2012; Royea & Nicol, 2019), for instance it seems easier (or more common) for PTs to focus on activities and methods when planning lessons than on how to use variation theory to structure the content (Turu, 2017). Some fundamental principles of variation theory are critical aspects and variation/invariance as the theory stipulate that learning is being able to discern new aspects of what is being learned (critical aspects), and that such discernment presupposes an experienced variation against a background of invariance. In the context of teaching, it is therefore, necessary to identify and teach the critical aspects of the object of learning (Kullberg et al., 2023; Marton, 2015). Even though, it may be demanding for prospective teachers to learn and practice these principles it is apparent that principles from variation theory may be learned even in courses that last only a few weeks (Brante et al., 2015; Ko, 2012; Royea & Nicol 2019). However, how prospective teachers' make use of variation theory and develop their teaching skills is not much known.

Methods

The analysis reported on is a part of the ongoing research project *Prospective teachers' ability to teach early arithmetic skills* that aims to determine how PTs' (grades K-3) teaching of early arithmetic develops through participation in learning studies (100 PTs at two universities in Sweden, divided in 25 groups). In this paper we draw on data from one of the groups, the rationale behind the selection is two folded: First, we wanted to generate hypotheses for the wider study by focusing on concrete in-depth knowledge of the issue. Second, we expected that this group most likely could maximize the information and highlight prototypical characteristics. This particular group of four PTs was assigned to work with the object of learning "Discovering the inverse relationship between addition and

subtraction, that is the complement principle ($a + b = c$ then $c - a = b$)". As part of the course design, the PTs took part of two intervention cycles closely following the steps of planning, teaching, analysing, and revising lessons in a learning study set out by Pang and Marton (2003). During this period, five learning study tutorial sessions were conducted (each of them lasted for 3,5 hours), oriented towards a systematic inquiry into pupils' learning difficulties, the nature of the object of learning and critical aspects and, teaching the content from a variation theory perspective. Moreover, resources such as course literature, two research articles on pupils' learning regarding the object of learning, two lectures on variation theory and one lecture on task construction were given. Based on insights from analysing a pretest that was administrated to two Grade 3 classes, the PTs planned and enacted one lesson (L1) and a post-test in their first class. After analysing the post-test and L1, they made revisions in the lesson plan and enacted the revised lesson (L2) in their second class. Both lessons lasted for 45 minutes. Data was gathered during the course, for this paper we have analysed two video filmed lessons (L1 and L2) and the groups' written documentation of their planning and analysis of teaching and pupils' learning. In our analysis we focused on 1) what the PTs considered to be critical aspects, the rationale behind this and how critical aspects were expressed by the PTs 2) what was made possible to learn in each sequence of the lessons from a variation theory perspective (Marton, 2015), that is how the PTs handled the critical aspects in terms of variation and invariance, 3) differences and similarities between L1 and L2.

Results

Lesson 1

Sequence 1: Marking and naming parts and wholes

The lesson started with four equations (figure 1a) to illustrate the part-whole relation 22|14|8. First, the position of the whole (the sum) varied, and the numbers were kept invariant in the two first addition sentences, hence it was possible for pupils to discern that the whole can be placed in the right-hand side as well as in left-hand side. The PT emphasized this by letting the pupils express the difference between the two addition sentences and concluded that "The equal sign means that it is the same on both sides, so it doesn't matter which side to place the whole".

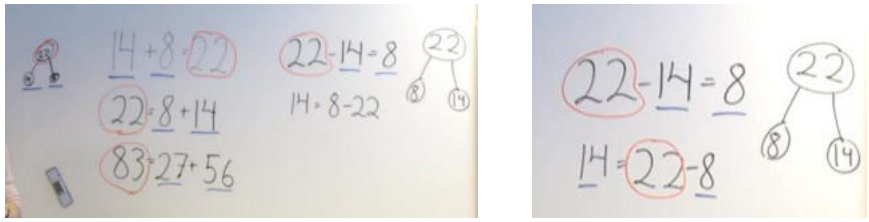


Figure 1a and b: The part-whole relation written on the board by the PT

However, as *discern parts and the whole* was identified as a critical aspect from the pretest, several questions focusing on naming the terms and sum as parts and wholes were posed, e.g. "What are these called, which words do you use [pointing at 14 and 8] and "What do we call this [pointing at 22] as shown in figure 1. In parallel the PT circled the whole in red and underlined the parts in blue. To generalize the difference between parts and wholes, the same procedure was performed on $83 = 27 + 56$. Hence, the numbers varied while the position of the whole were kept invariant.

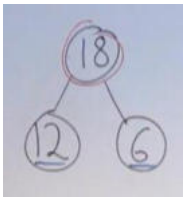
Next, the operation sign was varied, and the numbers remained invariant when introducing $22 - 14 = 8$. To make it possible for pupils to *discern the reading order* (subtraction is not commutative) which was considered a critical aspect, the incorrect number sentence $14 = 8 - 22$ was noted on the board. The PT asked the pupils to explain in different ways why the subtraction was incorrect. After concluding that the value is not the same when changing the position of the terms, the expression was erased and replaced with $14 = 22 - 8$ (Figure 1b). However, as the whole and the parts once again were named and marked in different ways and as the incorrect number sentence was erased and replaced, it can be questioned whether it was possible to discern the critical aspect.

Sequence 2: Marking parts and wholes

Pupils worked in pairs with items on a worksheet. In a similar way as in sequence 1, the pupils were asked to underline the parts and circle the wholes, and to write the relation between numbers in triads.

Sequence 3: Filling in the missing number from the triad and showing a number relation

In this sequence, *discern parts and wholes* was handled once again. The triad (figure 2a) was on the board, and in a similar way as before, the whole was circled, and the parts were underlined. The PT wrote four items on the board (figure 2b) and asked the pupils to use the triad to solve each item without counting. Furthermore, she invited the pupils to explain whether the missing number is being a part or the whole. Each item was erased and replaced with another item, so that just one item at the time was on the board. Even though there were a variation between the parts and the whole (the part 12 was missing in the two first items and the whole 18 was missing in the third), the pupils were not offered possibilities to pay simultaneous attention to the relations due the fact that there was just one item on the board at a time and that they just could pick the missing number from the triad.



$$18 = \_ + 6$$
$$18 - \_ = 6$$
$$6 = \_ - 12$$
$$12 = 6 - 18$$

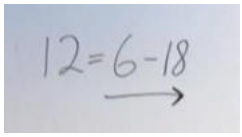
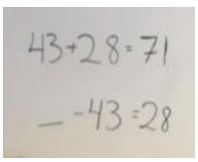


Figure 2a, b and c: The items on the board written by the PT

When introducing the fourth item *the reading order* was once again handled. The PT challenged the pupils and red the last subtraction from the right to the left (18 minus 6 equals 12) and asked if and why it was right or not. Several pupils expressed that "you must read the subtraction from left to right" and the PT added an arrow (figure 2c) to mark the pupil's statement. Next, to make it possible for pupils to *discern the relationship between addition and subtraction* a new part-whole relation was used. Based on the addition $43 + 28 = 71$, the pupils were asked to find the missing number in the subtraction $\_ - 43 = 28$.



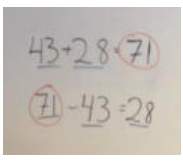


Figure 3a and b: PT asked the pupils to find the missing number in the subtraction

The PT summarized by pointing out that "you get the whole by adding the parts in addition. And here you have the same whole [pointing to 71 in the subtraction in figure 3b and circling the whole in both items], and by taking away one part you have the other part left." Thereafter, the pupils practiced their knowledge (filling in missing numbers from triads) in a worksheet for the rest of the lesson.

Lesson 2

Sequence 1: Focusing more directly on the object of learning and making more challenging tasks

Based on the analysis of L1, the PTs concluded that they should focus more directly on the object of learning and challenge the pupils more. As a consequence of this, the critical aspect *discern parts and the whole* was brought out in a different way. The same relationship 22|14|8 as in L1 was used, but the PT directly varied the operation sign to focus on subtraction ($14 + 8 = 22$ and $22 - 14 = 8$). To draw the pupils' attention to whole and parts, these were marked in the same way as in L1, however, the pupils were not asked repeatedly to name the numbers as parts and wholes. In a similar way to L1, another part-whole relation was discussed, this time 78|6|72 (figure 4).

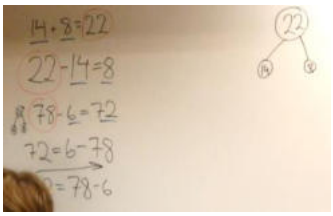


Figure 4: A new critical aspect was handled in L2, even though the same part-whole relation was used

But, as the PTs after the analysis of L1 had identified a new critical aspect *that subtraction must be made from the whole*, and furthermore, that *the reading order* had to be more emphasized, the subtractions were handled differently in L2. After $78 - 6 = 72$ was brought up and parts and whole were marked, the incorrect subtraction $72 = 6 - 78$ was contrasted to $72 = 78 - 6$ (the placement of the minuend and subtrahend varied). The incorrect subtraction was not immediately erased as in L1, therefore, it was possible for pupils to experience a simultaneous variation of the critical aspects when the two subtractions were juxtaposed. After the pupils had concluded that $72 = 6 - 78$ is incorrect as "you must subtract from the whole", *the reading order* was handled. The PT asked, "but what if I read the number sentence from the right to the left" pointing at $72 = 6 - 78$ saying "seventy-eight minus six equals seventy-two". In the discussion, an arrow from left to right was drawn under the incorrect subtraction to emphasize the right way.

Sequence 2: Adding a new task to link to the next sequence

As in L1, pupils worked in pairs on a worksheet. However, a new task was added as the PTs had identified a new critical aspect *that subtraction must be made from the whole*. In this new task pupils were asked to create as many subtractions as possible by using the triad 18|12|6.

Sequence 3: Creating part-whole relationships and handling the unexpected

The subtractions from the worksheet were noted on the board (figure 5a). This time, the triad was not used to ask the pupils to pick the missing number from the triad but to handle the critical aspect *discern that subtractions must be made from the whole*. Therefore, the placement of the whole varied

from the right- to the left-hand side. The number 18 was circled in each subtraction and the PT asked the pupils to explain which patterns they noticed. The pupils noticed that "The subtraction sign must always come after the whole" and "You must subtract from the whole". Next, in a similar way as in L1, the PT made a contrasted by varying the minuend (last item in figure 5b). A difference, though, was that both items were on the board at the same time so that it was made possible for pupils to discern similarities and differences between subtracting from 18 or 6. The item highlighting the relationship between addition and subtraction (43|28|71) was handled in the same way as in L1, but in L2 it was not just showed. As the PTs had noted from the analysis of L1 that pupils also must *discern how part whole relationships can be used* (i.e discern the different meaning of the whole—from the sum for addition to the minuend for subtraction) the pupils were once again asked to create subtractions, now by using the parts 21 and 13 (figure 5c). After pair work, the PT asked the class "How did you start?" The class suggested to add the two parts to get the whole. The PT confirmed by saying "Right, you must start with an addition to get the whole, and then it is possible to create a subtraction" and she wrote 21+13=34 on the board. The PT continued "Given this, which subtractions did you made?" The example $34 - 21 = 13$ was given, hence, the meaning of the whole 34 was varied—from a sum to a minuend.

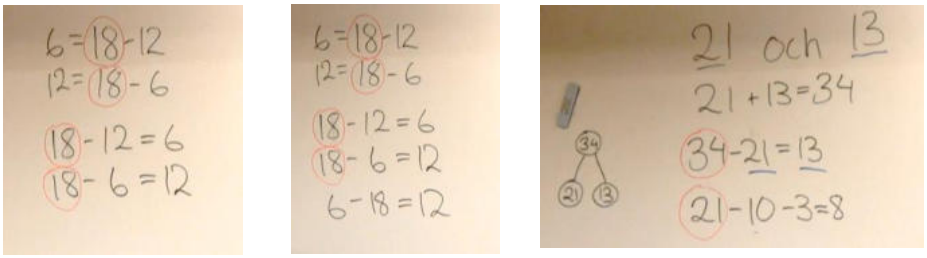


Figure 5: The pupils created subtractions by using the triad 18|12|6 and a subtraction by just using the parts 21 and 13

One of the pupils objected and gave another suggestion. Even though this solution was incorrect and not something the PTs had counted on the PT listen carefully and wrote the incorrect suggestion on the board $21 - 10 - 3 = 8$. Furthermore, she asked the class first to compare the whole 21 with the whole 34 in the item above and also the number 21 in both items. Hence, it was made possible for pupils to discern the contrast between 21 as a part and a whole. After summarizing that the whole must be found before knowing which number to subtract from, the pupils practiced their knowledge in a worksheet.

Discussion

Our analysis of the teaching showed that the PT developed their teaching about part-whole relations in a notable way during the learning study process. The findings show that the prospective teachers from the first to the second lesson identified and handled more critical aspects of the object of learning, and also, specified the critical aspects. The PT also focused more on pupils' difficulties and developed more demanding tasks. The PT group developed the tasks used in the second lesson to be more in line with the object of learning. For example, when they asked the pupils to write a number sentence in subtraction when only two parts were known. Hence, the PT seemed to become more aware of the object of learning and what the pupils need to experience, that is the critical aspects. In

this sense they became more sensitive to the pupils' learning when planning and enacting L2. Previous findings on PTs' learning from learning studies have shown that PTs learn to use and reflect about the variation theory when they plan their lessons (Royea & Nicol, 2019). We add to this research by showing the changes made in teaching during the learning study process. In this study we only analysed one group of prospective teachers' teaching of the same content in two lessons. Studies with more groups of PT are needed to see if similar changes are found in other groups as well. What prospective teachers learn during teacher education is of utmost importance since this most likely will affect their teaching when working as teachers in schools and in turn pupils' learning. We suggest, as other (e.g., Royea & Nicol, 2019; Cheng, 2014) that conducting learning study in teacher education programme likely is one way to develop more powerful teaching skills.

The findings about critical aspects and specification of them can be used by prospective and in-service teachers to improve instruction about the object of learning. However, the critical aspects should be seen as something that can be further develop regarding different classes. Hence, it is possible that there are more critical aspects and that there could be different critical aspects for different groups depending on what the group have already or not discerned (Kullberg et al., 2024).

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Collaborative Lesson Study in initial teacher education: Joint focus on students' learning

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In this paper, we present a new inquiry-based approach in initial teacher education: Collaborative Lesson Study. The study is a case study conducted in Denmark, with data consisting of video observations and field notes from open lessons, reflective discussions, and related coursework activities. Collaborative Lesson Study belongs to the paradigm of questioning the world, where pre-service teachers and teacher educators collaboratively explore how to teach mathematics and how students learn mathematics. This approach appears to offer valuable insights into bridging the gap between traditional coursework, theory, and teaching practice. Collaborative Lesson Study also exposes some unaddressed shortcomings, and new approaches are needed that involve cooperating teachers, aligning theory more closely with practical teaching experience.

Keywords: Collaborative Lesson Study, inquiry stance, the Anthropological Theory of the Didactic, study and research paths, praxeologies.

Introduction

Understanding the nature of mathematical knowledge for teaching is key to grasping the complexity of the mathematics teaching profession. However, mathematical knowledge for teaching does not develop in insolation; it is shaped by broader professional contexts, including activities beyond the classroom (Clivaz, 2018; Miyakawa & Winsløw, 2019), and requires an *inquiry stance*, where collaboration becomes a norm within a teaching community (Jaworski, 2006). In this regard, frameworks such as lesson study provides a structure for collective inquiry, enabling teachers to refine their practice through joint planning, observation, and feedback –thereby fostering both professional growth and contributing to elevation of the teaching profession (Sachs, 2016).

The benefits of using lesson study for both initial teacher education and professional development are well-documented, but several conditions and constraints remain. At the micro level, success depends on teachers' mathematical knowledge for teaching, their inquiry-based approach, and the quality of collaboration within professional learning communities (Chokshi & Fernandez, 2004). Teachers' commitment to growth and support from knowledgeable colleagues are also essential for promoting student-focused instructional practices (Takahashi, 2014). At the macro level, cultural values, professional development systems, and leadership play a critical role. Barriers such as cultural norms and bureaucratic challenges can hinder implementation, requiring a shift toward a more collaborative teaching culture (Kusanagi, 2014). Within initial teacher education, lesson study has been adapted in innovative ways to navigate these challenges. Some studies involve pre-service teachers teaching research lessons to peers in university settings, rather than in traditional classrooms (Fernandez, 2010). Others integrate lesson study into school internships, allowing pre-service teachers to engage in lesson planning, although they may not always have the opportunity to teach the lessons themselves (McMahon & Hines, 2008). In other cases, pre-service teachers engage in full

lesson study cycles, teaching research lessons in actual school environments (Skott & Østergaard, 2015). Despite these innovations, integrating lesson study into initial teacher education remains complex.

This paper presents the design and analysis of the initial teacher education project, *Collaborative Lesson Study*. The project addresses on both micro and macro levels, emphasizing a collaborative and dynamic network for teacher engagement. Our research is guided by a key question posed by Hiebert and colleagues (2007): "To learn from teaching, what would such a program look like? What knowledge, skills, and dispositions would teachers need to learn from teaching – not in an informal, haphazard way but in an intentional, systematic way" (p. 48). Using the Anthropological Theory of Didactics as a framework, we address two research questions: RQ1: How can Collaborative Lesson Study be conceptualized as a paradidactic system, and to which paradigm does it belong? RQ2: What mathematical and didactical praxeologies emerge in the context of Collaborative Lesson Study?

Collaborative lesson study

In Denmark, initial teacher education is structured according to the grade levels teachers intends to teach. To qualify for teaching primary and lower secondary school (grades 1–9), candidates complete a four-year professional bachelor's degree at a university college, combining subject-specific content, general didactics, pedagogical theories, and internships. Studying educational theories in isolation, without connecting them to classroom practices, is inherently problematic. For theories to be meaningful and effective, they must be grounded in authentic learning experiences that bridge the gap between theoretical learning and practical experience in real-world teaching contexts. University College Copenhagen has introduced a Collaborative Lesson Study design into its three-year mathematics education program. This includes a series of open lessons where approximately 200 pre-service mathematics teachers and 15 teacher educators observe and analyse structured problem-solving lessons. The lessons are taught by teacher educators, experienced in-service teachers, and pre-service teachers (see Figure 1).

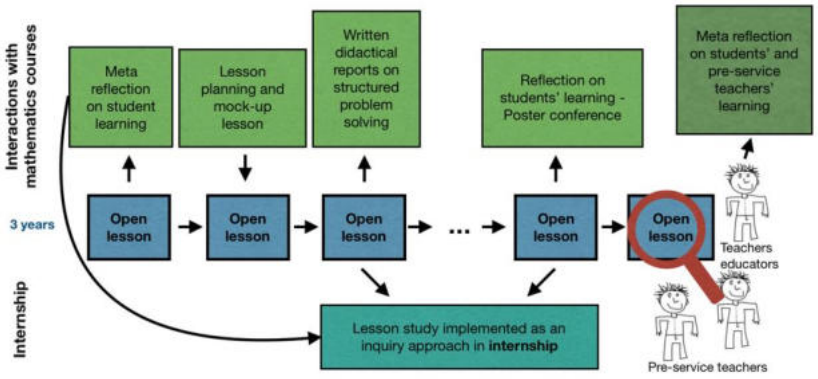

Figure 1: Collaborative lesson study more than a sequence of open lessons

A central focus of this collaboration is students' learning of mathematics. Drawing on Hiebert and colleagues (2007), we propose that focusing on students' learning and understanding the factors

behind their success or challenges “provides a targeted but comprehensive and systematic path to becoming an effective teacher over time” (p. 48). The open lesson serves as a pivotal element of this collaboration, offering a shared foundation for joint reflections and further elaboration. Observations and insights from the open lesson are integrated into coursework, providing opportunities for meta-reflection and written analyses of both didactical and mathematical issues. Additionally, the open lesson enables pre-service teachers and teacher educators to engage in complete lesson study cycles, encompassing studying, planning, teaching, and reflecting—some of which are taught during the open lessons or during internship.

The Anthropological Theory of the Didactics

Our study is grounded in the Anthropological Theory of the Didactic (ATD; Chevallard, 2019), which models knowledge as *praxeologies*, categorized into mathematical, didactical, and paradidactic praxeologies. In praxeologies, *praxis* refers to practical knowledge or "know-how," encompassing tasks and techniques for solving the tasks. *Logos,* on the other hand, represents explicit knowledge about praxis, including technology—knowledge of the technique—and theory, which provides abstract discourse to further explain or justify the technology. The praxeologies of mathematics teachers, including pre-service teachers and teacher educators, form a complex structure that integrate these three types. Both mathematical and didactical praxeologies are shaped by the conditions and constraints of the institutional systems in which pre-service teachers study and teacher educators work. Paradidactic praxeologies refer to the practices and explicit knowledge developed outside the classroom (Miyakawa & Winsløw, 2019).

The current educational paradigm, also in initial teacher education, is often referred to as the *paradigm of visiting monuments* (Chevallard, 2015). The paradigm of visiting monuments conceptualises education as the transmission of established knowledge—referred to as "monuments"—with the curriculum organised as a collection of prescribed works for pre-service (mathematic)s teachers to study. Within this paradigm, learning resembles a guided tour orchestrated by the teacher educator, where pre-service teachers engage with carefully selected pieces of knowledge, such as algebra concepts or approaches to classroom discussions. While paradigm of visiting monuments may allow pre-service teachers to develop praxeologies, it often lacks critical reflection on the relevance of the answers obtained. As a result, pre-service teachers may perceive the knowledge encountered as disconnected from their future roles as in-service teachers. The alternative paradigm, the *paradigm of questioning the world* (Chevallard, 2015), focuses on addressing meaningful questions – here about mathematics teaching and student learning. These questions are taken seriously and not just used to revisit pre-established knowledge. Paradigm of questioning the world aims to engage pre-service teachers and teacher educators in real inquiry, where they learn to pose and solve authentic teaching problems and critically reflect on the knowledge produced (Bosch, 2018).

The concept of *study and research paths* serves as a design tool for teaching within paradigm of questioning the world (Bosch, 2018; Chevallard, 2015). Within this framework pre-service teachers engage in milieus to explore a generating question, Q_0, examine established answers, $A^{\diamondsuit}$, and interact with relevant works, W. The goal is to collaboratively discuss and validate a final answer, $A^{\heartsuit}$.

The paradidactic system in initial teacher education can be modelled as a structure involving pre-service teachers X and mathematics teacher educators Y around a didactic goal ♠, denoted $S_{TE}(X, Y, ♠)$. When examining ♠ more closely, we find that it represent the didactic system of teaching mathematics, S(x, y, ♥), where x is the class of students, y is the teacher or pre-service teacher who teach the lesson and ♥ is the didactic objects of the lesson (Otaki & Asami-Johansson, 2019). This relationship can be summarized as $[S_{TE}(X, Y, S(x, y, ♥)) ➦ M] ➥ A^{♥}$, indication that the didactic system interacts with the *milieu,* M. In ATD, the milieu includes existing answers, $A^{◊}$, from media or prior knowledge, and encourage critical assessment by pre-service teachers and teacher educators, who can provide objective feedback about possible answers. *Media*, such as books, lesson plans, journal papers, convey information and answers, while milieus are designed to facilitate exploration and investigations of Q.

Methods

This study employs a descriptive case study design (Yin, 2013) to provide an in-depth analysis of local Danish practices in initial teacher education. The data collection process incorporated a range of sources, including lesson plans, audio and video recordings of seven open lessons and subsequent joint reflections, field notes, meta-reflections, and 12 didactic analysis reports authored by pre-service teachers. Selected video recordings, particularly those capturing lesson reflections, were transcribed and analysed using NVivo software. The study also incorporated official curricula and contextual information about initial teacher education in Denmark. By integrating these diverse data sources, the case was systematically constructed, organized, and refined into a comprehensive account centred on Collaborative Lesson Study.

In this study, we utilised theoretical tools from the ATD to analyse the complex network of systems and praxeologies involved in Collaborative Lesson Study. Our approach consisted of two main steps. First, we applied the concept of study and research paths to model the paradidactic system Based on the analysis, we concluded which paradigm it belongs to. Second, we conducted a detailed examination of specific mathematical and didactical praxeologies, focusing on the tasks, techniques, and underlying logos. In the praxeological analysis presented in this paper, we did not examine all open lessons, associated reflections, or the broader coursework. Instead, we focused on a single open lesson and a written didactic analysis produced by a group of pre-service teachers. The analyses allowed us to capture both the broader structural dynamics and the finer details of how Collaborative Lesson Study fosters professional development.

Analysis of collaborative lesson study

In Collaborative Lesson Study pre-service teachers, X_1, and teacher educators, X_2, jointly engaged in the overarching inquiry, Q_0: How do we teach mathematics? Q_0 was approached with seriousness, engaging both pre-service teachers and teacher educators in a joint inquiry into students' learning. The didactical contract between X_1 and X_2 differed from traditional coursework, which typically involves a predetermined curriculum and answers already known by the teacher educator. Instead X_1 and X_2 adopted a shared an inquiry stance, concentrated on the study of students' mathematical learning. In Collaborative Lesson Study X_1 and X_2 posed a series of derived questions (Q_1.. Q_n) to guide their inquiry. These included broad questions, such as Q_1: How do students learn mathematics?

And Q_2: How can structured problem-solving serve as a milieu for students' learning? Additionally, they formulated more specific student-focused questions, such as Q_3: How do students identify the relationship between steps and perimeter of a staircase figure? And Q_4: How can *bansho*, organisation of the blackboard, support students' learning and their understanding of early algebra?

Collaborative Lesson Study was non-linear and took various pathways. The works, W, consulted were elected based on their relevance to Q_0 and the subsequent derived questions. These questions were explored across different milieus, M, such as in the open lessons, M_1, and subsequent lesson reflections, M_2, while other derived questions were posed, studied and researched as a part of the coursework content, M_3. X_1 and X_2 continued their (re)search for already existing answers, $A^{\Diamond}$, using various media (e.g., textbooks, research papers, teacher journals, classroom videos), such as $A^{\Diamond 1}$: The framework of structured problem-solving, $A^{\Diamond 2}$: Bansho, $A^{\Diamond 3}$: Five Practices for teaching. They studied relevant mathematical works, W, and validated the answers, $A^{\Diamond}$, which were then implemented in resources such as lesson plans and subsequently taught as mock lessons in initial teacher education or during internships. In Collaborative Lesson Study X_1 and X_2 worked toward a final answer, $A^{\blacktriangledown}$, a question that can remain central and be continuously explored throughout their teaching careers.

$$S_{TE}(X_1, X_2, S(x, y, \blacktriangledown)) \rightarrow M_1, M_2, M_3] \rightarrow A^{\blacktriangledown}$$

Building on the preceding analysis, we conclude that Collaborative Lesson Study is situated within the paradigm of questioning the world.

In the following sections, we will zoom in and analyse various aspects of the paradidactic system.

In the first zoom, we examine the mathematical praxeology developed in an open lesson. The system, $\boldsymbol{S}(x, y, \blacktriangledown)$, refers to an open lesson, where x represents16 fifth-grade students, y is a second-year pre-service teacher, and ♥ refers to early algebra. The task, T_1, of the research lesson was: What is the relationship between the number of steps and the perimeter of a staircase figure? The students' techniques were primarily based on inquiry into the development of the figure, drawing staircase figures and counting perimeter (see staircase figures in figure 2). They then constructed a table, identified that increasing the steps by one add four to the perimeter, and further concluded that if the number of steps is known, multiplying it by four gives the perimeter (see table on figure 2).

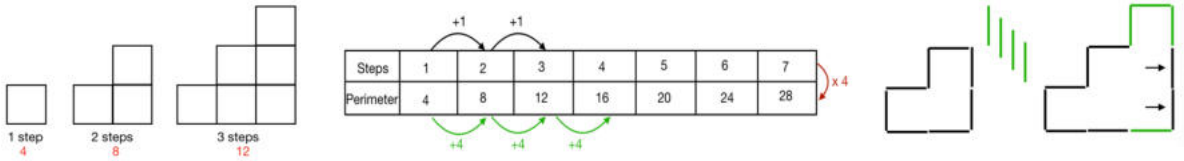

Figure 2: Grade five students' mathematical praxeologies: techniques, technology and theory

A derived question, T_2, from T_1 is Why '+ 4'? The techniques used include exploring the relation in the table and constructing new staircases by adding four more sticks (green) to transition from one staircase to the next. During the whole-class discussion students and pre-service teacher worked to develop geometric reasoning and attempted to generalize the +4 pattern. They did this by adding four sticks, one stick for the case and three sticks for the top, a technique that applies generically when

moved from an n-step staircase to an n+1 staircase. However, some students proposed techniques that could not be generalized. Students' learning was discussed during the post-lesson reflection: $S_{TE}(X_1, X_2, S(x, y, \heartsuit)) \rightarrow M_2] \rightarrow A^{\heartsuit}$.

In the second zoom, we examine a didactical praxeology developed in $S_{TE}(X_1, X_2, S(x, y, \heartsuit)) \rightarrow M_3] \rightarrow A^{\heartsuit}$. The System, S_{TE}, operated between the open lesson and coursework, where X_1, under the supervision of X_2, posed derived questions (re)study and (re)research students' learning, in the system $S(x, y, \heartsuit)$, guided by specific didactical perspectives (from media: research papers (Tan et al., 2023)). One group of X_1 explored the tasks, $T_{1'}$: How can *bansho*, organization of the blackboard, support students' learning and their understanding of early algebra? The techniques primarily relied on observations of a research lesson, such as the use of the blackboard (Figure 3). In their analysis, the pre-service teachers incorporated various theoretical concepts, including the four steps of organising the blackboard, the use of multiple representations (such as drawings, algebra, and color coding), and subsequent reflections on students' learning. The techniques and theories of bansho were explored in a written didactical report communicated $A^{\heartsuit}$ through the concept of multiple representation. In the report the pre-service teachers concluded that "by positioning students as active contributors, bansho enhances their engagement and fosters a dynamic, student-centered learning environment." (Report, p. 10).

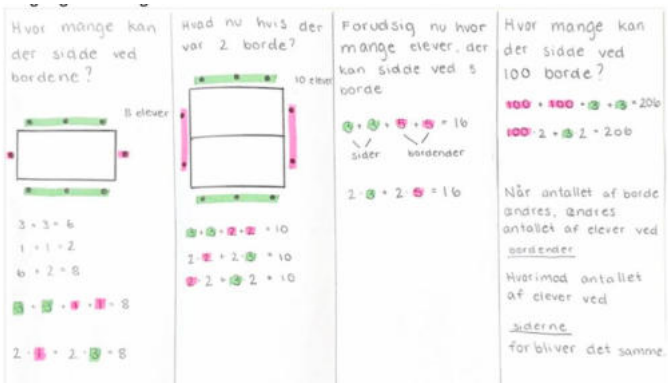

Figure 3: Bansho from research lesson 'how many students are seated at the tables?'

Discussion

Collaborative lesson study brings important insights but also reveals significant shortcomings that remain unaddressed in this study. While pre-service teachers and teacher educators jointly explore how to teach mathematics and promote student learning through inquiry, the realities encountered during internships often stand in sharp contrast. In school practice, pre-service teachers are typically exposed to more traditional forms of teaching, where inquiry-based mathematics is largely absent. The emphasis during internship tends to be on general teaching strategies, such as classroom management, rather than on mathematical subject-specific instruction or students' learning. This disconnect is strongly felt by pre-service teachers, who experience a mismatch between campus-based learning and internship practices. Bridging this gap calls for new forms of collaborative lesson study that integrate coursework, school-based practice, and the internship experience. Involving cooperating teachers in the inquiry process—alongside pre-service teachers and teacher educators—

may help create a more cohesive and aligned learning environment where theory and practice mutually inform one another.

Conclusion

This study examines Collaborative Lesson Study through the lens of the ATD, addressing two central research questions. First, we conceptualise Collaborative Lesson Study as a paradidactic system situated within the paradigm of questioning the world (RQ1). This framework fosters inquiry communities where pre-service teachers and teacher educators collaborate to develop and refine mathematics teaching practices. By positioning Collaborative Lesson Study within this paradigm, we underscore its potential to integrate inquiry-based approaches into initial teacher education. Second, our analysis reveals the emergence of specific mathematical and didactical praxeologies within the context of Collaborative Lesson Study (RQ2). At the micro level, these praxeologies are shaped and refined through collaborative observations, discussions, and reflections between pre-service teachers and teacher educators. Concrete examples illustrate how tasks, techniques, and justifications are negotiated. At the macro level, Collaborative Lesson Study demonstrates its broader impact by promoting a more transparent and inquiry-driven teaching culture within initial teacher education, bridging the gap between theoretical knowledge and practical application. However, addressing the disconnect between campus-based learning and internship realities is crucial, highlighting the need for further development of the Collaborative Lesson Study intervention. Involving cooperating teachers in future Collaborative Lesson Study interations could create a more cohesive environment.

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Using the didactical contract to untangle the complex dynamics of a lesson study group in initial teacher education

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This paper addresses the issue of understanding the dynamics between the participants of a collaborative setting like lesson study (LS) in an initial teacher education program. The case study presented is a LS course held for prospective mathematics teachers at secondary level at Lausanne University of teacher education, in Switzerland. The changes in the didactical contract and in the postures of both the educator and the preservice teachers are analysed using qualitative methods. Results show a gradual change in the group dynamics through the process, and the progressive reinforcement of clauses of the didactical contract that are characteristics to LS.

Keywords: Lesson study, initial teacher education, didactical contract, facilitator, role.

Introduction

At the beginning of the 2010s, Krainer and Llinares (2010) literature review identified the social dimension and the reflective practice as key features for mathematics teacher education, as well as new challenges for the future. Since then, the interest towards collaborative settings in this field has significantly increased (Borko & Potari, 2020), especially regarding initial teacher education (ITE). Due to its features, *lesson study* (LS) is one of such settings. This form of professional development, founded upon the Japanese practice of *jugyou kenkyuu*, gained popularity outside of the Asian continent at the end of the 1990s. Although in Japan LS is used for both in-service and preservice training, outside this country, preservice implementation has taken place only at a later stage. The reason of this late development is that LS is subject to several constraints and challenges when practiced in ITE, like time limitations or the difficulty of access to real classes (Ni Shuilleabhain & Bjuland, 2019). In this context, the task of teacher educators is particularly challenging, as the imbalance of power and experience between the participants may hinder the establishment of truly collaborative relationships (Ponte, 2017). Moreover, they are at the same time facilitators, experts and educators to the LS group, which implies a different approach to knowledge depending on the role they chose to take.

For these reasons, in order to explore LS potential for ITE, it is important to better understand how its implementation affects the interactions and the dynamics between the members of the program. The study presented here is part of a broader doctoral research (Presutti, 2024) addressing two main issues regarding LS in ITE. On the one side, it seeks to examine the conditions and adaptations that affect LS key features, in the context of an ITE program held at Lausanne University of Teacher Education, Switzerland. On the other side, it aims to explore the possible contributions of LS and the learning outcomes for its participants. This paper addresses the first aspect, providing a follow-up to a precedent study presented at CERME13 (Presutti, 2023), with a specific focus on the evolution of a LS group dynamics through the analysis of its didactical contract.

Theoretical background

The main object of study of this research is, as previously stated, LS. This process is based on the collaboration of a group of teachers trying to solve a teaching or learning problem. To achieve this, they engage in an action-research process: they study the problem and analyse the curriculum, then plan a lesson and collectively observe its outcomes. Subsequently, they discuss their observations and either disseminate the results to the professional community or re-engage in the first phase of the process, giving it a cyclical structure.

LS is usually coordinated by a *facilitator*, who helps the flowing of the process by involving all participants and promoting the dialogue (Lewis & Hurd, 2011). During some phases, and particularly the post-lesson discussion, the group's work can be supported by an external expert, or *knowledgeable other*. These two figures have hence different attitudes towards the knowledge at stance, with the former potentially retaining some of it, while the latter is invited to state it explicitly. It is although important to note that in countries where LS is not yet well-established, these two roles may overlap, since the facilitator should possess knowledge about the LS process, as well as pedagogical and disciplinary expertise (Hourigan & Leavy, 2021). To account for this complexity, some authors have modelled the facilitator as a figure with multifaceted roles (Clivaz & Clerc-Georgy, 2020). The picture is even more complex when LS takes place in ITE, as the role of educator come into play. According to Clivaz and Clerc-Georgy (2020), whose research was on LS with in-service teachers, all the roles can coexist and evolve throughout the LS, depending on the relationships within the participants, but also the subject of the LS and the facilitator's expertise in it. Using the framework of Brousseau's Theory of Situations (1997), it is possible to interpret this evolution as changes in the *didactical contract*.

The didactical contract is in fact a set of norms, mostly implicit, that regulate the mutual expectations of the teacher and the students concerning the mathematical knowledge at stake (Hersant & Perrin-Glorian, 2005). It encompasses two dimensions: one epistemological, related to the mathematical knowledge, and one social, concerning the reciprocal expectations between the stakeholders in the production of knowledge. Hersant and Perrin-Glorian (2005) developed Brousseau's concept of didactical contract by considering four aspects that characterize it at different scales. The first one, acting on a *macro-level*, is the teaching object. The second and third aspects are the didactic status of the knowledge and the existence of an adidactical potential of the milieu. These aspects characterize the contract at the *meso-level*, i.e. the realization of an activity (for example the resolution of a problem). The fourth aspect is the distribution of responsibility in the construction of knowledge, acting at a *micro-level* (at the scale of an episode: the set of interactions around a unit of knowledge). According to this framework, the complex interactions between the participants of LS in ITE influence both the epistemological and the social dimensions, particularly in the distribution of responsibility, i.e. at the micro-level.

With respect to this theoretical framework, the research question may be stated as follows: *how does the (micro)-didactical contract of a LS group held in a ITE program evolve through the process?*

Data and method

The case study presented in this paper concerns a university course based on LS that took place at Lausanne University of Teacher Education during the autumn semester of 2021. The LS group was composed by five preservice middle-school mathematics teachers. It was facilitated by a university educator with experience in LS, who was also in charge of their evaluation, while two experienced schoolteachers participated to some of the sessions. The LS consisted of two cycles, with one mock-up research lesson and one lesson held in the class of one of the schoolteachers. The total was of 12 sessions of 1.5 hours each, plus one 45-minutes lesson. As a researcher, I designed the LS and each session with the teacher educator, and I observed their implementation without intervening. The LS subject was decided in advance during the design of the course, to better anticipate the teaching and reading material. The chosen topic was the teaching of integers at grade 7, a pivotal moment for the extension of the number system and the passage to algebra (Rezat, 2019). During LS, the preservice teachers decided to focus on integer multiplication and its mathematical justification, one of the first examples the pupils encounter where the justification relies only on intra-mathematical arguments, namely Henkel's permanence principle[1]. For the teacher educator, the goal was for the preservice teachers to acquire mathematical knowledge about the permanence principle, and to be able to justify their decisions regarding its didactical transposition to the class.

Data collected for this part of the PhD research consist in video recordings of all the meetings. Other data consist in texts written by each participant at the end of the course, the lesson plan, and all the documents and the artefacts produced during the process. All the video recordings were summarized in a synopsis, and arranged according to three levels (macro, meso, micro), following Hersant and Perrin-Glorian (2005) method (see Presutti, 2023, 2024 for further details). Macro-level consisted of LS phases. Meso-level consisted of training tasks proposed by the educator to the preservice teachers. Micro-level consisted of episodes. To grasp the interactions and the evolution of the didactical contract, each episode involving the permanence principle or its didactical transposition was qualitatively coded according to the knowledge at stake, the elements of the milieu, the kind of micro-contract[2], the preservice teachers postures (namely pupil, student-teacher, teacher, and researcher-practitioner, following Guille-Biel Winder et al., 2018), and the educator's posture.

Coding of the teacher educator's posture required a specific procedure, since very little research is done about his role in the ITE case. The first step consisted in theoretically defining and separating the *roles* from the *postures*. The educator's *roles* stem from the institutions to which he or she is subjected. *Postures*, on the other hand, depend on his or her positioning within the roles and influence how the educator acts, thinks or speaks in a given situation, as well as his or her attitude towards knowledge. All postures hence coexist, but for each episode it is possible to identify a predominant

[1] The idea that algebraic operations should be consistent in all number systems, particularly in the development of their extensions.

[2] Hersant and Perrin-Glorian highlight seven kinds of micro-didactical contract, depending on whether the teacher retains full responsibility (four), shares it with the whole class (two) or with each student in the class (two)-see Table 1.

one based on the educator's actions and speech. The second step was the creation of a list of roles and postures. This was made according to a literature review, from which all the possible roles and postures were listed (most of the articles didn't differentiate between the two terms). The first list, L_0, comprehended 20 terms. Step three was the refining of L_0 by associating similar terms, e.g. *koshi*, external expert and knowledgeable other. During step four, the terms indicating institutional roles were extracted: educator, facilitator, and teacher. The resulting list was then tested on the data set and refined by taking away the remaining ambiguous terms. The retained roles and postures are shown in Figure 1.

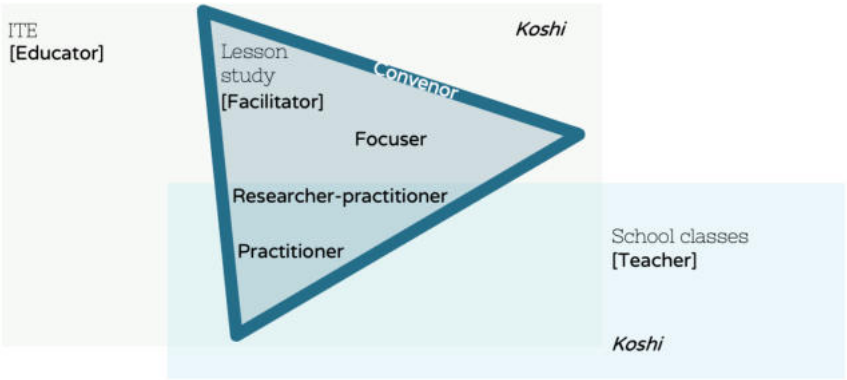

Figure 1: Institutional roles [in brackets] and postures of an educator doing LS in ITE

The analysis at the micro-level highlighted some particular moves of the educator on the didactical contract and on the milieu (presented in Presutti, 2023), as well as some key moments in LS process with regard to the knowledge at stake. Those key moments were analysed at the meso-level to better understand the dynamics of the interactions between the participants during the mathematical and didactical activity. Some results of these analysis are presented in the following section.

Results

A first result concerns the evolution of the micro-contract of distribution of responsibility with respect to the construction of the knowledge at stake.

Table 1: Frequence of the micro-contracts of distribution of responsibility

Micro-contract category	Type of micro-contract	Number of episodes identified (total=161)
The educator retains full responsibility	Information	4
	Assumed ostension	34
	Disguised ostension	4
Responsibility is shared between the educator and the group of preservice teachers	Adherence	28
	Collective production	83
Responsibility is shared between the educator and each preservice teacher	Individual production	8
	Tutoring	0

In total, 83 out of 161 episodes were coded with the micro-contract of *collective production*, more than each of the other kinds of contract. Most of these episodes (63) are distributed in the second part

of the LS course, between session 6 and session 12. Besides, in sessions 5, 10 and 11, many episodes show a didactical contract of explicit ostension of knowledge: these episodes are marked by the presence of the schoolteachers, who undertake the posture of *koshi*, or experts, in most episodes.

To get a closer look of the evolution of the social and epistemological dimension of the didactical contract, three training tasks have been chosen from the beginning, the middle and the end of the LS course. Task 4 took place at the beginning of the LS process, during session 2 (00:40:39-1:05:35). This session was the beginning of the study phase and aimed at understanding the mathematical reasoning behind the sign rule through the analysis of some textbook tasks. After a discussion about textbooks, Thomas, one of the preservice teachers, directedly asked the educator why $(-1)\cdot(-1)=+1$. The educator first proposed to the group to look for some common answers (Figure 2, E01), to which the preservice teachers answered by suggesting a pattern explanation (E02). The repeated moments of silence and hesitation during this episode led the educator to propose a task found in a lesson plan of a previous LS group (E03). The preservice teachers' expectation of an answer from the educator led him to take an expert posture (*koshi*) and to set up a kind of dialogue lesson in which he finally stated the concept of the permanence principle. In E03, preservice teachers assumed a pupil posture. This kind of move of the educator on the didactical contract, named M1, has been presented in a previous study (Presutti, 2023).

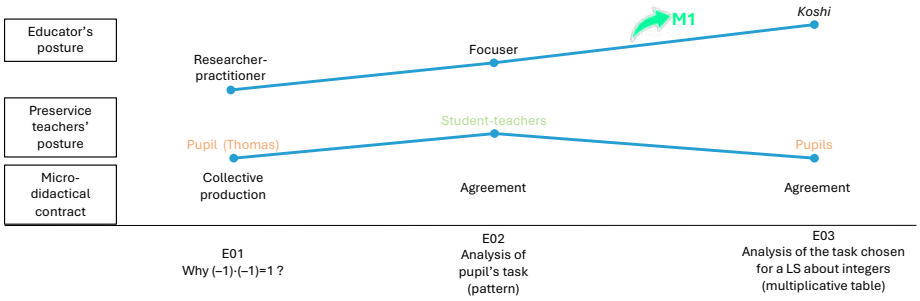

Figure 2: Educator and preservice teachers' posture changes in session 2, task 4

Task 3 (0:35:56-1:17:12) took place in session 6, devoted to the planning of the first research lesson. During this task, the LS group discussed the first part of the lesson, in which an illustration of the integer multiplication is shown through the use of arrows made of cardboard representing vectors on the number line in four different cases. After the discussion upon the first case, $2\cdot3$, the group analysed the two multiplications $(-2)\cdot3$, $2\cdot(-3)$ and their respective representations with the arrows. During the discussion, in episode 2 the educator took a researcher-practitioner posture by saying that he *really* didn't know the best way to present it to the pupils, and the group could only make assumptions and validate them during the research lesson. With this move (M2) the educator took a participant perspective and strengthen the didactical milieu, as it was the only way to validate the mathematical and didactical knowledge at stake. In episode 3, the prospective teachers took themselves a researcher-practitioner posture and produced hypothesis about the vectorial representations of the multiplication, which led in episode 4 to the emergence of knowledge about the commutative property of the multiplication, the "symmetry" of the representation and the minus

sign as an opposite operator. After M2, the micro-didactical contract turned to one of collective production of knowledge for the rest of the training task (Figure 3).

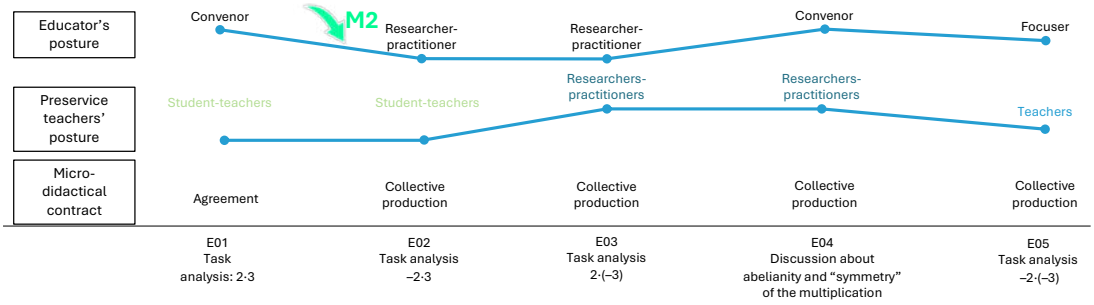

Figure 3: Educator and preservice teachers' posture changes in session 6, task 3

Task 3 (0:48:26-1:22:51) of session 11 took place during the analysis of the second research lesson. Session 10 was devoted to a first debriefing, while session 11 was devoted to a deeper analysis of the observations. One of the schoolteachers, Yves, joined the meeting. During this task, the preservice teachers assumed a teacher posture and put into perspective their observations about the timing, the place for "mathematics theory" during the lesson (episode 1, Figure 4), or pupils' activity (episode 2). During episode 3, Yves was invited to express his thoughts on these topics. He explained that when he taught integer multiplication, he made different choices, questioning the group's will to tackle the mathematical justification of the sign rule. In this case, the educator felt it appropriate to regain his *koshi* posture by taking explicitly his "teacher educator hat" (Presutti, 2023) and mediate between the positions. This move (M3), together with Yves intervention, marked the micro-didactical contract as one of explicit ostension of the knowledge at stake (in this case, didactical knowledge). During episodes 3 and 4, the preservice teachers maintained their teacher postures by drawing parallels with their previous teaching experiences and analysing classes similarities and differences. In episode 5, the educator finally asked the participants how they would redo the research lesson, taking a convenor posture. Eventually, each participant answered with a different version of the lesson. All but one preservice teacher maintained their teacher posture, while one of them took a research-practitioner one by directedly linking her lesson to the LS issue. It is important to observe that in this phase of the LS, the teacher posture was predominant over the researcher-practitioner one. Episode 5 was also the only episode in which the schoolteacher took a practitioner posture.

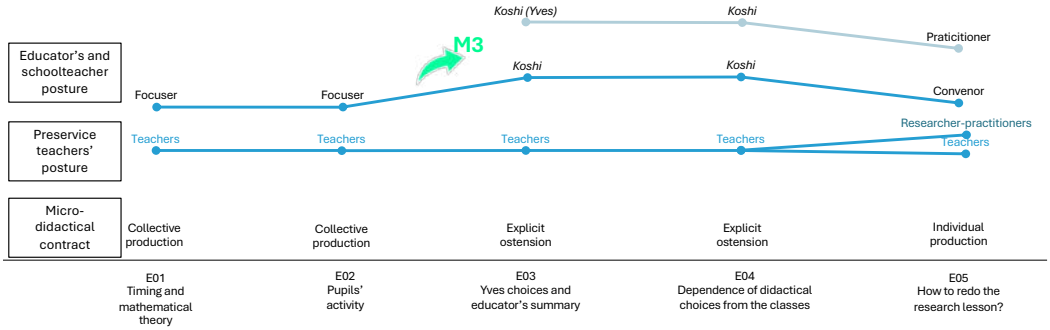

Figure 4: Educator and preservice teachers' posture changes in session 11, task 3

Discussion

The focus of this case study was the understanding of the dynamics between the participants of a LS set in ITE through the theoretical lens of the didactical contract. For this purpose, the analysis of the changes in the micro-didactical contract was enriched by the study of the postures of both the teacher educator, the preservice teachers and, when present, the experienced schoolteachers. The observation of the evolution of the didactical contract shows that there is a significant shift in the second part of the LS from the point of view of the sharing of responsibility in the construction of knowledge, which is more and more a collective effort instead that an initiative of the teacher educator. This aspect is also illustrated by the comparison between the training tasks in sessions 2 and 6. In fact, in session 2, the educator's move (M1) led to the assumption of an expert posture, ending up in an explicit clarification of the attended knowledge. On the contrary, in meeting 6, the educator move (M2) consisted of the assumption of a researcher-practitioner posture, resulting in a change in the epistemological and social dimensions of the didactical contract and in a stronger interaction of the preservice teachers with the milieu. Besides, the distribution of these two moves during the LS course (Figure 5) show that M1 is more present in the first half of the LS process, while M2 in more present in the last half.

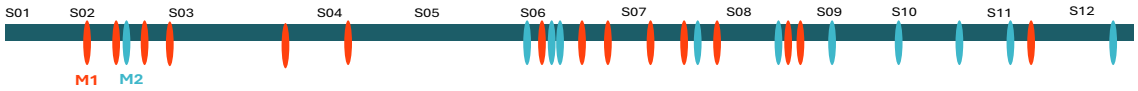

Figure 5: Distribution of M1 and M2 during LS

Session 2 and 6 also show a "mirror effect" between the educator and the preservice teachers' postures, resulting in a distancing of the postures (M1) or an approaching (M2). This can also be read as a more teacher-student dynamic (M1), or a researcher-researcher one (M2). Interestingly, this mirror effect is no longer present at the end of the LS, as shown in session 11. In this case, the assumption of an expert posture by the educator (and the schoolteacher) doesn't end up with preservice teachers taking a pupil or student-teacher posture. They persist instead in their teacher posture. This may suggest a "maturing" of the teacher posture over time, which also assumes a more reflective standpoint, but doesn't become a researcher-practitioner one.

In conclusion, this study shows a significant evolution of the group dynamics during LS, marked by the transition from a more traditional university training to a LS one (i.e. a collective, teacher-led activity). These analyses also highlight the fact that, in ITE, adapting to LS might require significant time for the participants to adjust to the "LS clauses" of the didactical contract. Regarding this aspect, it might be insightful to test other forms of LS in ITE, such as practicum-based. Finally, it is important to note that in this study the posture of the prospective teachers and of the educator are not symmetrical, as they come from different frameworks. Further work is needed on investigate these postures and their consequences on the actual effectiveness of LS in ITE.

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Necessary skills and knowledge and challenges of the mathematics lesson study facilitator. A literature review

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In lesson studies carried out in Western countries, a key role is played by the facilitator, who organizes and leads the entire process. Despite the contributions regarding this issue provided in the literature, the knowledge about the preparation that facilitators need and the challenges that they face in their activity is still fragmented. Thus, the aim of this communication is to present a systematic literature review on the role of the facilitator in lesson studies in mathematics, identifying and synthesizing the necessary skills and knowledge and the challenges faced by facilitators and suggesting directions for future research.

Keywords: Lesson study facilitator, knowledge, skills, challenges.

Introduction

Lesson study (LS) is a collaborative and reflective professional development process focused on improving student learning. In a LS, a group of teachers works in cycles of planning, teaching and observing a research lesson, and reflecting on this lesson. In the lesson studies carried out in Japan, the so-called "knowledgeable other" plays an important role, as an external expert who, in most cases, intervenes mainly in post-lesson reflection in order to enlarge and deepen the participants' understandings about students learning. In lesson studies carried out in Western countries, another figure tends to assume a key role, the facilitator, who is responsible for organizing and leading the entire process (Hourigan & Leavy, 2021; Huang et al., 2021; Lewis, 2016).

Recent studies indicate that the role of this facilitator is complex and underline its relevance in structuring LS and leading its sessions (Hourigan & Leavy, 2021; Lewis, 2016; Mynott & Michel, 2022). This facilitator "must be strong enough to guide, yet invisible enough to give participants agency to determine their learning" (Mynott & Michel, 2022, p. 57). To perform these functions, the facilitator needs to understand the many facets of school life and teaching, have knowledge about LS, and a broad and deep mathematical and didactic knowledge to be able to guide teachers in a process of collaborative learning and continuous improvement (Hourigan & Leavy, 2021; Lewis, 2016).

Despite the various contributions on the facilitators' role in LS provided in the literature, the knowledge about the preparation necessary and the challenges they face in their activity is still fragmented (Moeller et al., 2022; Mynott & Michel, 2022). A systematic review of the literature therefore can help to provide a clearer and more articulated view of the various facets of the facilitator's role in a LS. Thus, the aim of this review is to synthesize the existing literature on the role of the facilitator in LS in mathematics, seeking to identify what this literature says about the

necessary skills and knowledge and challenges faced by facilitators in LS and to suggest directions for future research.

Methods

On November 14, 2023, we conducted a systematic search of the Scopus, Web of Science, and ERIC databases, using a specific search strategy to identify relevant articles on the topic of study. The search strategy was configured to search in the title, abstract and keywords of the articles, with the following formulation: ("lesson study" OR "learning study") AND ("facilitator" OR "facilitation" OR "teacher educator" OR "teacher of teacher"). To ensure the quality and relevance of the results, we apply filters to restrict the results to peer-reviewed scientific articles, with no limits on year of publication, written in English. We included articles that met the following criteria: (i) empirical studies; (ii) studies focused on LS with in-service teachers, from primary to secondary education, or in the preparation of facilitators. Articles related to LS in the initial teacher education, pre-school or higher education educators were excluded.

The process of identification and selection of articles was conducted by two independent researchers. Initially, the researchers analysed the titles, abstracts, and keywords to select articles for the eligibility study. The selected articles underwent a comprehensive analysis to determine their inclusion in the final corpus of analysis. In case of disagreement between the researchers, a third researcher was consulted to solve the issue. Cohen's Kappa coefficient was calculated to measure inter-rater agreement, resulting in strong agreement for the inclusion of articles in the eligibility study ($k = 0.895$, $p < 0.001$) and inclusion in the final analysis corpus ($k = 0.885$, $p < 0.001$). The steps of this identification and selection process are given in Figure 1. This process led to the identification of thirteen articles which summary is provided in Table 1.

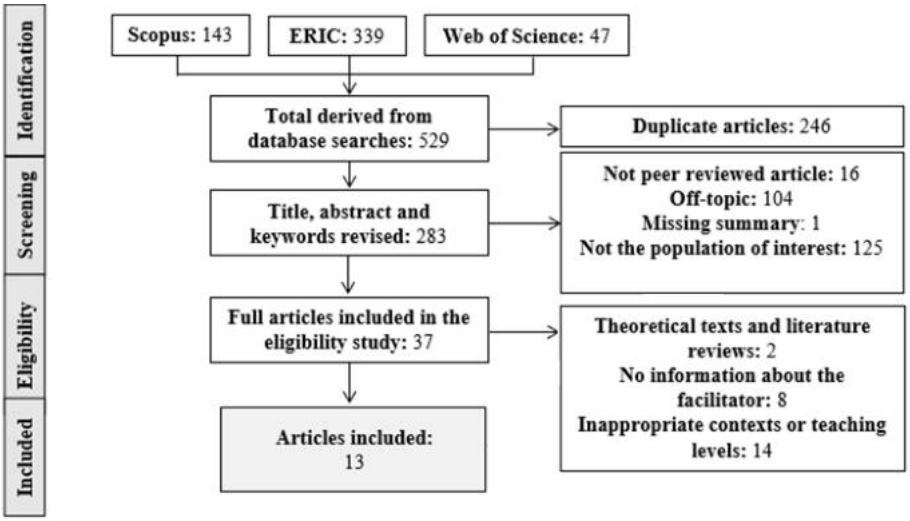

Figure 1: Identification and selection process of the articles for the literature review

In the first stage of the analysis, SciSpace software was used to identify relevant information regarding the two dimensions of interest: the "necessary skills and knowledge" and the "challenges faced by the facilitator". Based on this, three researchers carried out a detailed analysis of the articles,

identifying main categories in the previously identified dimensions.

Table 1: Brief description of the articles

Authors	Aim of articles	Main results
Akiba et al., 2019	To examine which design features of mathematics LS support effective teacher collaboration and professional development	Facilitators significantly influenced teachers' learning by focusing on student thinking and providing relevant resources. Effective facilitation of in-depth discussions enhanced critical thinking and fostered collaboration. Facilitators needed strong knowledge of content and student reasoning to lead discussions for meaningful teacher learning.
Clivaz et al., 2023	To examine how dialogic interactions contribute to knowledge construction	In lesson planning, both facilitators and teachers used their knowledge of instructional practices and of students. The facilitators, particularly the math teacher educator, mainly adopted a guidance role, although new ideas came mainly from teachers. As a teacher, the other facilitator played a key role in the dialogue for arousing reasoning development.
Hourigan & Leavy, 2021	To explore the practice of two mathematics teacher educators acting as novice facilitators of LS	Having noted participants' resistance in the first session, the facilitators sought resources to fit teachers' needs and context, while balancing supporting and promoting their agency. Managing the participants' involvement and considering their busy schedules, is challenging for facilitators and can limit teachers' learning.
Huang et al., 2021	To analyze how digital platforms facilitate teacher learning and collaboration by an online cross-cultural LS	The LS cycle and expert facilitators were crucial for teachers to develop their content knowledge (concept of probability). Although several technical challenges emerged, teachers and facilitators collaborated to improve the lessons, linking research to practice. Teachers' learning also included technology, pedagogy, and instructional design skills.
Jiang et al., 2022	To understand how and what teacher and teacher educators learn collaboratively in LS	The facilitator realised the importance of addressing teachers' concerns, moving from being the one "to learn from" to the expert who works with teachers to understand how they learn and the decisions they make in practice. This form of collaboration may impact teachers' work and professional learning, as well as the facilitators' knowledge.
Joubert et al., 2020	To explore the use of LS in a blended format to help isolated teachers to integrate technology into their teaching	Facilitators were essential in providing guidance and prompting teachers' reflective practices, along with peer-facilitators that emerged during LS. Smaller groups, reflection questions and lesson planning forms enhanced collaborative learning. Nonetheless, the allocation of teacher time was vital for successful collaboration and teacher learning.
Lewis, 2016	To analyse how novice facilitators learn to lead LS	Facilitators enhanced their knowledge by drawing on their prior LS experience and addressing challenges such as teacher resistance and time

		management, while balancing guidance with teachers' autonomy. They emphasized collaboration and combining pedagogical, content, and school-based knowledge to address participants' needs.
Lim et al., 2016	To investigate how LS can enhance teaching practices and improve student learning	Facilitators played a crucial role promoting the development of teachers' pedagogical practices and beliefs, namely by promoting collaborative work at lesson planning, facilitating productive discussions, providing feedback and encouraging reflective practices. The authors also highlight the importance of considering the local cultural needs.
Moeller et al., 2022	To understand how facilitators influence the implementation and effectiveness of a professional learning program (PLP)	Local facilitators played a critical role in adapting PLP, like LS, ensuring fidelity to its core principles while making contextual adjustments. Collaboration between teachers, school, district and external facilitators was crucial for scaling up PLP effectively. Facilitators benefited from program experience, but their needs varied based on roles and expertise.
Mynott & Michel, 2022	To analyse how two facilitators from different continents support and lead LS cycles	Facilitators needed to prepare teams for effective collaboration, avoid dominating discussions, and guide participants, yet be invisible enough to give teachers agency to determine their learning. Sustaining LS cycles required participant-focused facilitation, establishing strong relationships, norms, and considering standards of LS facilitation.
Preciado-Babb & Liljedahl, 2012	To understand common and differential elements of groups of teachers, involved in collaborative design processes	The authors highlight the value of collaboration in empowering teachers and improving their ability to creatively address classroom challenges. Facilitators played a crucial role in guiding teachers into a collaborative process by structuring the sessions, organizing discussions focusing on student learning, and maintaining focus on goals.
Richit, 2020	To analyse LS in Portugal, identifying their challenges, adaptations, and specificities.	Facilitation of LS by an external team and adjusting the number of sessions according to the LS cycle goals are important adaptations to the context. The specificities include the exploratory approach as teaching approach, the development of micro-investigations on teaching, and collaboration, highlighting the importance of understanding the context.
Schipper et al., 2017	To investigate whether LS promotes teachers' professional development in adaptive teaching	The LS facilitator significantly influenced teacher professional growth by managing the process, stimulating teachers to apply theoretical knowledge, and fostering a focus on student learning and collaboration. Time constraints remained a challenge for full teacher participation.

Results

In these articles we find several categories, with some articles fitting in more than one of them. For the issue *necessary skills and knowledge*, the categories emerging concerned: (1) the creation of a

collaborative work environment and promotion of communication (with a subcategory of guiding teachers) and (2) the organization of professional learning situations, appearing in 10 papers, (3) the general planning and organization, appearing in 8 papers, (4) mathematical, pedagogical and didactic knowledge and knowledge about the LS process in 7 papers, and (5) reflection skills in 5 papers. As for the *challenges faced by facilitators*, the categories identified were: (1) managing the logistics of the process in 11 papers, (2) promoting the teachers involvement in the LS, in 9 papers, (3) promoting communication and creating a collaborative environment (with a subcategory of conducting discussions without dominating them) appearing in 5 papers, (4) making the necessary adaptations, (5) dealing with work pressure and other professional issues, both in 4 papers, and (6) getting the support of the administration in 3 papers. In the following, we provide a brief description of what we found in these categories and indicate their frequency in the articles.

Necessary skills and knowledge

All thirteen articles on the LS facilitator have relevant ideas about necessary skills and knowledge required for effective facilitation. First, 10 articles give especial attention to the capacity of the facilitator to create collaborative environment in the group in which there is a stimulating communication among the participants. One example is Lewis (2016), who highlights that facilitators should be able to co-design learning opportunities with teachers so they can tailor the learning experiences to the specific needs and interests of the group, thereby enhancing the relevance and significance of the learning process. Within this category, some articles refer that facilitators need the capacity of playing a pivotal role in guiding teachers (7 articles). For instance, Schipper et al. (2017) state that "the LS facilitator seems to play an important role in guiding teachers ... and stimulating the teachers to use theory and new insights in terms of PCK" (p. 299). They highlight the importance of facilitators "introducing the team members to LS, stimulating the LS team to focus on content and to exchange ideas" (p. 297).

Second, another group of articles concern the role of the facilitator in organizing the content of sessions, providing the opportunities for professional learning and the necessary materials to use in each session (10 articles). For example, Jiang et al. (2022) suggests the facilitator should create a learning space for mutual learning, pointing that "using teaching language with care to reach students", "anticipating students' misconceptions with forward-thinking", "addressing students' misconceptions", and "understanding students' perspectives/voices" (p. 66) prompt it. This is also referenced by other researchers (e.g., Clivaz et al., 2023; Schipper et al., 2017) that emphasize the importance of promoting "in-depth discussion of student thinking and actively engaging teachers in discussions" (Akiba et al., 2019, p. 7). Additionally, Lim et al. (2016) and Richit (2020) point to the importance of discussions in lesson planning and post-lesson reflection, which led to changes in teachers' beliefs and practices, and allowed them to acquire new professional learning.

Third, a set of articles refer to the facilitator's skill of planning and making the global organization of the LS, including the structure of different phases (8 articles). One example is the paper by Akiba et al (2019) which identifies specific design features that enhance teacher participation and learning in LS, namely facilitator's focus on student thinking and active teacher participation in the inquiry process. Richit (2020) emphasizes the significance of understanding the needs and cultural features

of the setting where LS is carried out, while Joubert et al (2020) focus on organizing LS in virtual settings and conclude that, while online LS is feasible, it requires smaller teacher groups for success. By organizing the different phases of the process, selecting appropriate resources, and adapting LS to the specific context of each school, facilitators contribute to the professional development of teachers and the improvement of teaching quality.

Fourth, several articles refer that facilitators need mathematical, pedagogical and didactical knowledge as well as a comprehensive understanding of the LS process (7 articles). For example, Akiba et al. (2019) and Clivaz et al. (2023) identify content and pedagogical content knowledge as fundamental, encompassing the selection of tasks with high cognitive demand and the ability to select materials based on research, as well as knowing students and teaching practices. Akiba et al. (2019) stress the need for expertise in LS. This knowledge is vital for selecting challenging tasks, guiding student-centred discussions, and ensuring in-depth dialogues, thus enhancing teachers' learning.

Fifth, there is also reference to the reflective capacity that facilitators need to have, to improve their facilitation practice (5 articles). This is seen as a capacity that allows the facilitators to deepen their knowledge about their role and improve their practice. For example, Lewis (2016) highlights the iterative nature of facilitator development, as they gain confidence and adaptability over successive cycles, being "more comfortable presenting less, [and] expressed more confidence in their ability to respond to teachers in the moment" (p. 535).

Challenges faced by facilitators

Concerning the challenges faced by facilitators, eleven of the thirteen articles point issues to consider. First, the most common category related to the challenges faced was managing logistical aspects (11 articles), primarily involving the management of available resources. For example, Hourigan and Leavy (2021) highlight the "inflexible timeframe" (p. 663) as a key constraint, while Jiang et al. (2022) pointed out the difficulty of coordinating lesson observation schedules.

Second, another category is to achieve the involvement of teachers in the LS (9 articles). This includes recruiting teachers to participate and also to get an effective participation of teachers in the sessions. For example, Hourigan and Leavy (2021) acknowledge that they "under-estimated the complexities of engaging teachers in LS" (p. 658), noting resistance from teachers. Moreover, they perceived a lack of interest in individual professional development among teachers. Lewis (2016) highlights the challenges associated with managing teacher resistance, particularly when facilitators possess inadequate training to address this issue.

Third, another challenge identified in five articles is to promote the communication withing the group and create a collaborative environment (5 articles). Four of these articles, as Jiang et al. (2022) and Mynott and Michel (2022), address a specific aspect of communication: to lead discussions without dominating them, allowing the participants to actively engage and make independent decisions. Lewis (2016) and Lim et al. (2016) highlight how a dominant facilitator can hinder teacher participation.

Fourth, another category concerns the need to make adaptations taking into account the cultural context and the curriculum (4 articles). This is pointed, for example, by Richit (2020). Sometimes, modifications are necessary to accommodate local needs, as demonstrated by Lim et al. (2016), who

encountered challenges related to finding availability to re-teaching the lesson.

Fifth, another challenge for facilitators is to manage teachers' work pressure (4 articles). In Hourigan and Leavy (2021) research, teachers were unwilling to dedicate their free time to professional development which, additionally to schedule issues, led to a significant number of teachers missing one or more sessions and limited the teachers' involvement. Participants in the study of Schipper et al. (2017) also reported "being under pressure to realize the objectives within the given time frames" (p. 298).

Sixth, there is yet another aspect mentioned in three articles: the need to get support from the school leadership and other administrative instances. For example, Moeller et al. (2016) emphasize the need for funding and time allocation as well as the importance of school administration to provide time for collaboration. Schipper et al. (2017) and Richit (2020) similarly underscore the importance of administrative support, arguing that without it, teachers lack the time and resources necessary for full engagement in professional development.

Conclusion

This systematic literature review provides the identification of important features regarding the necessary skills and knowledge of facilitators and the key challenges that they face. Taken overall, the articles reviewed indicate that facilitators need mathematical, pedagogical and didactic knowledge as well as knowledge about the LS process, need knowledge about the general planning and organization of LS, need to know how to organize professional learning opportunities as well as how to create collaborative work environments and how to promote communication in the group, guiding the teachers along the way, and, finally, as in any other complex activity, need reflection skills. Taken together, these are all relevant aspects regarding the skills and knowledge of facilitators, and the issue that remains is the depth and specific features of each of them.

As for the challenges faced by facilitators, the categories identified were making adaptations in the LS structure, content and process, promoting communication and creating a collaborative environment, including the challenge of leading discussions without dominating them, and getting teachers involved. Further challenges were managing the logistics of the process, dealing with work pressure and with professional issues and getting the support of the administration. Another issue, that we do not address here, is to know what research says about how these challenges may be faced in order to deal with them. That may be the subject of further study.

This review shows that there is already important knowledge about the facilitator's necessary knowledge and skills and about challenges faced by them. This knowledge, which is scattered through many different journals, provides a rather impressive global picture. However, other issues remain to be studied, such as to know what the literature says about how facilitators may be prepared to undertake this complex activity.

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Socio-cultural practices and behaviours in collaborative inquiry

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Centrally here we report a conversation between academics from the United Kingdom and Japan. It arises after a study visit of a group of colleagues from the UK to mathematics classrooms in Elementary Schools in Tokyo. This visit brought together the authors of the paper for the first time. Both are experienced in Japanese Lesson Study (JLS) but work in the different socio-cultural settings of the U.K. and Japan. The other visitors were meeting JLS for the first time and this prompted much discussion of the different socio-cultural environments of maths classrooms, schools and JLS as well as society more widely. The conversation draws on the experiences of the visit and JLS more widely to consider what is lost in translation in the interpretation of JLS to Lesson Study as practised in the U.K. and other socio-cultural contexts more widely. Drawing on a specific but limited number of issues we discuss the problems of translating JLS as a socio-cultural practice into different countries.

Keywords: Lesson Study, socio-cultural practices, cultural awareness, translation

Introduction

This article is concerned with issues that the authors consider should be more prominently visible and more widely discussed in research that is informed by the practice of teachers in Japan that has become known internationally as Lesson Study (LS). Throughout this article we refer to LS as practised in Japan, as Japanese Lesson Study (JLS) to differentiate from LS as manifested more widely in other jurisdictions. We use JLS as a catch-all term, however, we wish to recognise and emphasise that although there are aspects of JLS that vary considerably across space and time. For example, JLS that takes place within and across schools nationwide, can at times be more exploratory whereas at other times is more of an opportunity to be summative and a showcase for what has been learned. Here we minimise these differences to be able to consider the socio-cultural context(s) of JLS and LS more widely.

It is our intention to consider how the translation of terms and words associated with JLS, and the adoption or adaptation of this practice as LS, can obscure to some extent deeply embedded cultural norms and behaviours.

As a mathematics education community internationally, we increasingly look to other countries and jurisdictions for ways in which we might better educate young learners. In this paper, we caution that we need to look deeper than taking a surface view of practice and focussing primarily on the visible, overt and explicit. Here we argue that understanding socio-cultural practices such as JLS and other manifestations of LS requires more than a surface view and is a crucial element to certain areas of LS. Consequently, the translation of lesson study across the world whilst not necessarily misinformed may be under-informed and may be lacking sensitivity in translation if you are to acquire a deeper understanding of the wider picture of LS.

The main body of the article comprises an ongoing email conversation between us as authors. This discussion began in June of 2024 while on a lesson study trip to Japan then continued by email again until around mid-September the same year. The empirical material for this study consisted entirely of the conversations therein. Through this exchange we start to explore some of the issues that are emerging as we discuss the socio-cultural world of lesson study that we had experienced together during a study visit that focussed on schools, maths lessons, and lesson study in Tokyo, Japan.

At this point in our ongoing discussions, we draw some emerging conclusions and signal a warning for future lesson study researchers and practitioners. Such studies, conducted in different international settings we believe need to be more aware of, and sensitive to issues of socio-cultural practices and as researchers we should consider how to handle these in our work that draws on JLS and Japanese cultural understanding, the cultural origins of LS.

Beyond the translation of words

This article emanates from a collaborative exploration of lesson study that the authors undertook in Tokyo, visiting a number of schools with UK based colleagues that were encountering JLS as school-based collaborative research for the first time. Both authors had previous extensive experience of lesson studies in Japan with the first author having little knowledge of the Japanese language and the second author, although being a first language English speaker having lived and worked in the Japanese education system for almost thirty years. The second author carried out real-time interpretation from Japanese to English in classrooms and in discussions with teachers and academics and in discussions outside of the classroom, also interpreted from English to Japanese. All the JLS activity took place in mathematics lessons in elementary school classrooms and in addition some mathematics lessons that were not part of lesson study cycles were also observed.

The authors have not found discussion of translation issues prominent in the LS literature. However, of course, there is much to read regarding issues of translation elsewhere. Here, our main concern is that of the socio-cultural context of translation, not merely of language and we have found academic work of relevance in the socio-cultural activity academic community.

In discussion of culture in translation Teilanyo (2007) takes a literary work, an African saga and, as a starting point, considers its translation from its source text in the Izon language to a target text in English language. There is general agreement that the translation by Clarke (1977) is well crafted and captures the meaning and nuances of the saga in many ways. There is, however, criticism that the translation does not capture the relevance of the socio-cultural context. In discussion, Teilanyo points to possible different approaches to translation as being lexical translation, literal translation, free translation and figurative translation, suggesting that the latter two are target language focused and discusses how free translation has problems because words and expressions in the source language when there are culture-bound notions/constructs are involved, especially those imbued with philosophical meaning.

As researchers in LS this issue is immediately apparent as words such as *koushi*, *hatsumon*, *neriage*, *hansei*, plus many more that do not necessarily have cultural equivalents in other languages. In our discussion of JLS we often include these terms as untranslatable words which are often seen to remain

in their original Japanese form in English language literature. This may appear confusing to many as we believe their current English equivalents do not suffice as a sufficient translation of a deep-rooted Japanese cultural concept. As Teilanyo in his writing (ibid. p.7) further discusses, the use of such words from the source language that remain untranslated, in the target language helps achieve a "foreign flavour" that is recognised by those as the audience of the lingual and cultural translation. Before turning to our conversation that unpicks some of these ideas in the context of JLS translation to LS we note that some readers may meet the ideas that we explore after a double translation process. Translation first from Japanese as the source language to English as the target language, and then translation from English as the source language to a further target language of the researcher.

Conversation

GW: Following our observations of teaching and learning in Japanese classrooms, for which you kindly acted as interpreter, I have been thinking quite carefully of the socio-cultural practices involved in the lessons and realise that I need some further 'translation' of these so that I can better understand what we observed and experienced. For our understanding we need more than mere translation of the words spoken. At times the socio-cultural practices involved are openly overt, such as in the post-lesson discussion with certain formalities being clear, for example the turn-taking of participants, with the observed teacher always starting the formal discussion with some reflections on the lesson, but at other times I feel that I am potentially misunderstanding much of what is going on.

Maybe we can start our conversation by considering the reflective nature of the post-lesson discussion. I've been reading some of the literature around early years education and revisiting Catherine Lewis's seminal text, Educating Hearts and Minds (Lewis, 1995). In this volume she writes of children's pre-school experiences and how much freedom children are provided with to play and through such play learn to become responsible class members. This is facilitated with quite structured and teacher orchestrated class reflections that explore what has been done during the day, and significantly, events and issues that might have occurred that caused tensions are discussed, such as when individual children might be selfish when sharing with classmates. My further explorations uncover that this process has a Japanese term, *hansei*, which translates as something like critical self-reflection. Rholen and Le Tendre (1998) write of this suggesting that *hansei* involves asking questions about what individuals have done that might be considered as kind/helpful/naughty and the implications for others. These reflective practices seem to be formalised into group/class practices in pre-school and beyond and, to date, our observations of lesson studies of maths classes, haven't unearthed this aspect of school life at all. My interest in *hansei* is not entirely in relation to school practices per se, but the socio-cultural development of the identity of young people and implications for their future interactions as individuals in collectives more widely. For example, how do early practices such as *hansei* manifest in future lesson study behaviours? In relation to this I have also come across the practice *kaizen* which relates to "continuous improvement" and seems, according to my brief reading, to be more focussed on the activities of the collective. Maybe you could comment?

BL: Interpreting for you and colleagues during your study visit was a very rewarding experience. Throughout that week of interpreting during all the class observations and in the pre- and post- discussions, I made several important discoveries. The interpreting task during the class

lessons themselves was relatively easy, but the post-lesson discussions after each session were much harder due to the many cultural concepts that were raised and hidden meanings that discussants perhaps subconsciously used. During each post-lesson discussion there was a ritual of customs that were quite evident to me that had an impact on the order of speech and hierarchy within the room. The level of authority in the room indicated who spoke first for example and how much or little each teacher could add to the discussion.

Through my involvement in LS over the years I have found that there are in fact many cases of words and customs being misinterpreted. Words that you often see in the LS literature such as *koushi*, (teacher, or knowledgeable other), *hansei* (reflection) and *kaizen* (improvement) are deep-rooted cultural concepts that are often interpreted with just one word or expression in the literature leading to confusion or misguidance. Yes, I would agree with Catherine Lewis's thoughts on this, the idea of "*hansei*" is a cultural concept that doesn't exist in English in the same way as it does in Japanese, and it is something that is taught to children from a young age in Japan. The concept is quite complex and in LS refers to reflecting on decisions made in the classroom, either good or bad, by the teacher or the student and what the impact of those decisions are. *Hansei* has a very deep meaning and translating it with just one word in another does not do it justice. *Hansei-Kai* for example, is usually the period after a lesson study where teachers, researchers get together to discuss the lesson, a collaborative exercise that helps highlight why certain decisions were made. The meaning of the hansei-kai is not only to reflect but also to learn collaboratively and collectively and to discuss in depth concepts that arose in the research lesson. This is usually done in a positive manner with lots of praise by the observers and denial of such praise by the teachers who respond by saying that they could be better. It is not an evaluative in any way and is a predominantly positive exercise.

Another example of this is, as you mention, the word *kaizen*, or to improve. In English the word 'improve' has negative connotations, i.e. that something could be better, that there was a flaw somewhere and needs to be fixed. However, in Japanese that word does not have a negative connotation. I would say that kaizen refers to making something even better with the idea that nothing is ever perfect, a somewhat Buddhist belief. No lesson no matter how good it was conducted could always be improved, for example. It is culturally taboo to gloat or boast in Japan. When praised in Japan for example, it is common to decline or politely brush off any praise.

GW: It seems that my reading in my attempts to better understand both general behaviours and those that are particular to LS keeps returning to the development of young children as it seems that many behaviours are inculcated from the very earliest years. You mention hierarchy for example. This is something that can be overtly observed to some extent in Japanese and other cultures. An obvious example is that of there being more overt respect of age in Eastern than in Western societies. I read (Hendry, 1987) that children are expected from an early age to acknowledge that older children are more experienced, and that as younger children they can learn from them. Further, I believe that this distinction and expected respect is reinforced by language with terms being used to identify 'older brother/sister' and to show respect to grandparents, for example. I suppose the terms *sensei*, *san*, and *sama*, in professional life also overtly position those to whom we are talking.

I think it's worth emphasising at this point that it's not just how these terms and how they might be understood in a rather abstract translation, but rather how they result in both collective and individual behaviours, and these are introduced in the very earliest collectives into which we are socialised at very young ages. Tobin et al's volume (2009), *Preschool in Three Cultures Revisited*, provides some insight into how, from the very earliest age, children are expected to develop a sense of responsibility and control for the good of the collective

BL: Yes, Japanese society is a very hierarchical one. You're right, there is a different suffix at the end of people's names depending on the relationship between the two parties, with *kun*, *chan*, *san*, *sama* and *sensei* given to the various levels of respect when addressing others. This is implicitly taught from a young age and these elements of hierarchy are quite noticeable in all walks of life, but particularly in education circles with dialogue between student to student, teacher to student and of course depending on the level of experience and age of the teachers between teacher to teacher.

Japan is also well known as a country that ranks relatively highly in *Hofstede's* cultural dimensions theory as a collectivist society (Hofstede, 2011). This refers to the importance placed on goals and the well-being of the group/collective as opposed to the individual. Yes, it is true that these relationships are also evident in schools between students. In many schools, particularly primary schools where students in the upper years will look after those in the lower years passing on simple tasks. There is usually a gap of several years between those playing the senior role to those in the junior role. This trait continues in high school, higher education and in the corporate community with the *kouhai/senpai* custom where new recruits to a company, or younger students (*kouhai*) are expected to look up to their elders (*senpai*) with more experience, knowledge and respect. In this context the *kouhai* (younger or newer members to the group) use more respectful language with the *senpai* (superiors, although a poor translation). We witnessed this first hand several times during the post-lesson discussions in Tokyo regarding the order of contributions made. All this is highly visible in lesson study circles and is something that is perhaps not fully understood outside of Japan.

Stigler and Hiebert's idea of the cultural code, or the DNA of teaching seems relevant here: "*The teaching methods that are handed down from generation to generation through a cultural code that is present in most classrooms*". This code they say, "*is the main reason why changing teachers does not automatically produce changes in teaching*" (Stigler and Hiebert, 2009). These socio-cultural customs are very relevant to the whole body of JLS. Many of the words that are used to describe them are insufficient and need more explaining. Every language is immersed in cultural expressions that may have deep meaning in the language they are spoken but cannot perhaps be well translated into a second language due to cultural differences. Here are a few more words that may not be fully understood in countries outside of Japan: *tokkatsu* and *neriage* which you may find in lots of LS related literature of late.

I think that as LS gains popularity globally it has morphed into slightly different methodologies in each specific country. In each country they use the words we have discussed above but adapt the procedure to the cultural norms of that particular place. Good examples of that and relevant to what has been discussed here is the example of *Koushi* and *Hansei*. *Koushi* refers to a 'knowledgeable other', or a professor from a university perhaps that holds an element of importance and pride,

something in Japan society values and respects. 'Knowledgeable other' though holds a slightly pompous connotation implying that the Koushi is more knowledgeable than everyone else which could be interpreted as taboo in English. Hansei, or the *hansei-kai* (post-lesson discussion) on the other hand are being conducted in ways that are socio-culturally acceptable to the country where they are held in. In both cases, there is a slight difference to the hierarchical socio-context of Japan where they originated.

GW: Further to our discussions around socio-cultural practices more generally I have been thinking about where the maths fits in. Let me take the conversation in a mathematical direction. Attached is a screenshot of a Japanese maths textbook that has been translated into English (see Figure 1). You will notice it talks about the mathematics of "times as much". I think that there is a Japanese word for this phrase? I am assuming that "times as much" is a translation of this word and the translation is an attempt to capture a whole different cultural way of thinking about what I might call "the constant of direct proportion" in maths situations that can be expressed as a mathematical function by the relationship $y=kx$.

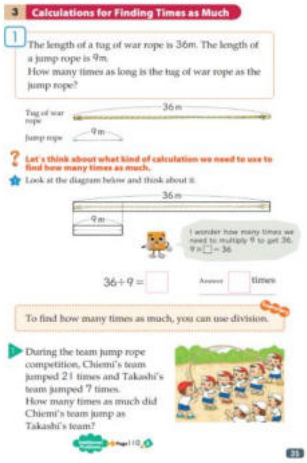

Figure 1: Page from Japanese Grade 3 mathematics textbook.

BL: I see. Yes, I can see now where the confusion with this term arises. I think the term "times as much" here is referring to the Japanese word 倍, *bai*, which refers to double, or a multiplication of a number, however the word *bai* does not have the exact same connotation as "multiply" in English. The word bai usually succeeds a number, as in: 2-bai, 3-bai, 4-bai etc etc. this would mean 2 'times as much, 3 'times as much', multiplied by two, three etc. When explaining the original text in Japanese into another language I can see where the confusion lies now. People often use this word as a noun as well which there is no equivalent of in English. For example: when talking about the price of things, one could reply simply with the word *2-bai*, or *3-bai*, meaning 2-3 times as much as it was. This could be another case of 'lost in translation'.

GW: It is noticeable that Japanese maths textbooks take a consistent approach to the development of these mathematical ideas (as directed by the Japanese mathematics curriculum guide (Isoda, 2010), and as some educators such as Tad Watanabe point out multiplication "sentences" are always written and introduced from the perspective that the number in a group is multiplied by the number of groups,

hence the “times as much”. It is my contention that this may be very different to approaches that are less consistent, one might even say, “exact”. Of course, in later stages this is extended to ideas of division. (Full details can be found in the text by Isoda and Olfos (2021).

Discussion

Mathematics is, in the main, considered to be ‘culture free’. International studies such as those of the O.E.C.D.’s PISA programme, although giving much background detail of the framework of how mathematics is conceptualised as a study domain, do little to dispel the notion that mathematical thinking and learning can be abstracted from the cultural differences of societies across the world. Our discussions reported here provide some insight into how making sense of teaching and learning, and indeed JLS as a collaborative research activity, might only be understood by having deeper knowledge and sensitivity to mathematics learning as a social, cultural and historically situated activity. Researchers such as Wenger (1989) have developed whole theoretical frameworks that consider learning from such a perspective, considering learning as involving apprenticeship into community practices but these approaches do not particularly signal that different national settings have their own peculiarities. This seems, however, to be a particularly important issue for those of us interested in LS given its provenance in Japan.

Egodawatte (2012) gives a helpful overview of how socio-cultural practices influence mathematical cognition. Our discussions as reported here provide some insight into some of the ways in which this might be relevant and important in our understanding of JLS. It seems that behaviours of teachers, both in and out of the classroom, are deeply seated in culturally and historically developed practices. Translation of words spoken is just one aspect of what is happening when teachers and students interact. Perhaps the most significant occurrence of this in relation to JLS is the practice of *hansei* which is not something that is significantly different from practices that are introduced to Japanese children from the very earliest of years. The post-lesson discussion, then, is something that one might expect to occur naturally. This is distinctively different to when it is introduced in LS in England, for example. In our use of LS, we have to work as a collective to establish ground rules for such reflective discussions so that we all adopt new individual and collective behaviours. For example, in England, class observations are often judgemental in nature, and associated with quality control of school leaders’, and the inspectorate’s expectations in relation to teaching and learning. For this reason, many teachers may oppose the idea of being observed which they may equate to being evaluated and critiqued where this is not the case at all in JLS.

We have also touched on how mathematical ideas might be mediated by linguistic tools that signify cultural models of mathematical thinking such as in the case of *bai* (times as much). In such cases we might consider the very mathematics we observe in Japanese classrooms as being situated in the cultural setting in which it is being learned and worked with and on.

Conclusion

The ideas we present here following our experiences of observing and studying lesson study in Japan provide what is inevitably a brief glimpse into how as a collaborative research collective working with lesson study we need a greater understanding of the socio-cultural aspects of JLS. This raises

issues of how we might adopt and adapt JLS in different countries and cultures, and our work in this regard might also be sensitive to the different ways in which we constitute and signify mathematical concepts in our different cultures. Alongside lesson study, there seems to be a growing tendency, certainly in England, to borrow from other countries' teaching artifacts, such as different representations, that can provide insight into mathematical structure. Again such borrowing seems to obscure how learning mathematics is in reality a deeply embedded socio-cultural practice. We advocate that future research is needed to better understand JLS and inform our own work in LS.

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The classroom discussion in a TTP lesson: The challenges of a lesson study team

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Introduction

Teaching mathematics Through Problem solving (TTP) is an approach to teach mathematics through problem solving activities. A TTP lesson consists of four phases: (1) posing a mathematical problem, (2) student problem solving, (3) a guided classroom discussion to compare and discuss students' solutions and (4) summarizing (Takahashi, 2021). The third phase, the classroom discussion, is mentioned as an important and central phase of a TTP lesson (Takahashi, 2021), but is also challenging. In this phase, teachers should avoid the often used Initiation-Response-Evaluation (IRE) patterns (Schoenfeld, 2002), but should guide interaction between students in a more pedagogical way to come to a shared understanding of possible solutions of the initial mathematical problem. Teachers appear to be able to develop TTP-lessons by collaboratively working in a Lesson study (LS)–team (e.g., Groves et al., 2016; Roorda et al., 2024, Takahashi, 2021). But how do teachers in a TTP-LS team improve the guided classroom discussion?

Methods

We implemented a TTP-LS intervention in which three teams of teachers of two secondary schools collaboratively designed TTP lessons in two consecutive LS-cycles (see Roorda et al., 2024). For this poster, we present data from one team of five teachers, during their second cycle, because this team explicitly discussed how to improve classroom discussion. The team was introduced to TTP-LS in three start-up meetings. The LS-meetings were facilitated by a mathematics educator (first author). The team prepared all four phases of the TTP lesson. The teachers choose to start the lesson with the problem in Figure 1. Teachers agreed that in classroom discussion the teacher would mainly discuss the different 'formulas' that could be used. The teacher would also discuss problem solving skills, such as making a sketch of the situation.

You see the table of a quadratic function:

a. Calculate the y-value for $x = 7$.

b. Calculate also the y-value for $x = 100$.

x	…..	1	2	3	4	……	7
y		0	-3	-4	-3		?

Figure 1: The problem *Table of a parabola*

The research lesson was conducted twice in respectively one grade 12 and one grade 11 class. Minutes and logbooks were made of the preparatory meetings. The research lessons were audio-recorded,

transcribed and analysed for patterns of initiation by de teacher (I), response by a student x (R_x) and evaluation by the teacher (E).

Results

In research lesson 1, it turned out that the teacher made the students work too long. In the limited time left (only 6 minutes) for the classroom discussion, the teacher shortly discussed different formulas, using three IRE-patterns for three solutions and one more interactive IR_aIR_bE-pattern. There was one minute left for discussion about problem solving strategies.

During the reflection meeting and an additional preparatory meeting for the second research lesson, teachers agreed that more time should be reserved for classroom discussion. Additional questions were formulated to help focussing on the central mathematical goals of the lesson.

In research lesson 2 the classroom discussion lasted 13 minutes. The teacher started with more interactive patterns (p.e. $IR_aIR_aIR_bIR_bE$), in which students were asked to reflect on ideas of other students. After a while the lesson became less interactive and the teacher used more IRE patterns. The additional questions proved helpful in organizing the focus on the essential mathematical aspects. Discussing and explicating problem-solving strategies, such as 'sketching' and 'checking answers' took 8 minutes. Our poster will present the team-development with respect to classroom discussion.

Conclusion and discussion

With respect to classroom discussion in a TTP-lesson, we conclude that the discussions in the LS team resulted in improvements, such as, more interaction, asking specific questions to highlight central mathematical goals, but that the teacher still frequently used IRE-patterns in the classroom. We presume that part of the problem lies in the social norms in the classroom. In addition to a different attitude from the teacher, this also requires a different attitude from the students. This challenging change requires explicit attention from the LS team. We therefore recommend to specifically train student centred classroom discussions, for example in mock-lessons (see Takahashi, 2021). Teachers could also train their skills by using lesson plans from already tested and described TTP lessons.

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TWG 29: Embodied and Material Studies of Mathematical Behaviour

Searching for formats: How to share embodied and material studies. Introduction to TWG29 at CERME14

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The development of the TWG focus

This year, we held the TWG Embodied and material studies of mathematical behaviour for the second time. We welcomed all researchers interested in mathematical embodiment and materiality. The range of topics is diverse, covering a very broad phenomena in mathematics teaching, learning, and doing. Just as before, we invited researchers investigating multimodality and sensory-motor processes, including gestures, eye movements, drawing, touch, language use, proprioception, physiological processes, etc. We specifically hoped to gather studies not only on sensory-motor processes performed by a student or a teacher, but also running across the bodies. The second focus lay in making and using material artefacts, technologies, as well as engaging in dance and other aesthetic forms. Affect and emotion as socio-material phenomena, pointing at atmospheric conditions of learning, were of great importance in our call for papers: already from the previous year, we felt how studying cognitive aspects of learning as isolated from affective energy is limiting for understanding mathematics learning. Even more than previous year, we hoped to gather researchers who appropriate embodied and/or material lens to study the diversity of racialized, gendered, dis/abled bodies in formal and informal learning settings, thus opening the room for discussions of political as indispensable from our attention to bodily processes that highlight idiosyncrasy and freedom over control. Discussions at the previous CERME brought in two new focuses. Firstly, we noticed how the investigation of teaching practices and teacher education lags behind the studies of students and learning environments, so we specifically invited the researchers to focus on teacher-oriented inquiries to support in transforming mathematical classrooms toward embodied teaching (Macrine & Fugate, 2021). Finally, we strived to discuss alternative research methodologies and means of dissemination that better address the embodied nature of mathematical activity, and/or avoid representationalism and strive for difference making (Barad, 2007), such as, diffractive, performative, historical, design-based, activist methodologies, etc.

Overall, the coherence of the thematic group lies in the perspective of consideration rather than on specific topics, as we are committed to a wide range of theoretical perspectives that highlight the role of the body and material environment in fostering mathematical behaviour. We intended to work on clarifying core theoretical and methodological issues per se, as well as their educational implications. Bringing the diversity of phenomena and theoretical lenses, yet staying committed to appreciating embodied and material processes, we aim to re-imagine future education as passionate and accessible.

Exploring alternative conference formats

At the previous CERME, a format of presenting research on embodied and material aspects of mathematics education was a recurrent theme. We discussed that the textual format of articles, including transcripts of activities, limits the possibility of transferring our findings and struggles. We

also noticed how a standard format of conference presentation is helpful by allowing the dynamics of videos to explicate embodied processes. Nevertheless, while listening to a presentation, we still felt alien from the real creative process that the authors of each contribution and their participants were living through—as it might be characteristic to a positivist rather than participatory paradigm (Heron & Reason, 1997). CERME has a long-lasting experience in avoiding long boring presentations, as participants are always invited to read the papers ahead and discuss them at the conference, sometimes with the help of pre-assigned opponents or thematic panels.

Yet, this year, we decided to radically experiment with the conference format, inviting our participants to invent new forms for sharing their work. In line with the spirit of embodied and material studies, we do not believe in transmitting mathematical knowledge from one head to another; instead we commit to an affective, creative process of engaging in mathematics (re)invention. Similarly, we thought, research findings and struggles shall be *lived through* rather than told about (Sinclair, 2024). So, we tried to provoke the authors to create collaborative environments consonant with each paper, where participants could *feel and explore* rather than hear about other researchers' results. Such a collaborative environment might include a brief reminder of a paper's focus, followed by an embodied/material exploration in the form of activity tryouts and/or moderated discussions.

As exploration requires sufficient time, we split the group into parallel sessions, offering the authors of each paper 30-minute timeslots. To secure some coherence along parallel sessions, we divided all the papers into five thematic lines of 4 papers: two pairs of thematically coherent papers were presented in parallel, so that everyone would be in touch with each thematic line. Each thematic line was supported by a set of leading questions that we derived from reading the corresponding papers. After parallel sessions, we would gather all together to have a short exchange on what had been happening. Furthermore, we planned two times an hour for the small group work, in which participants who attended different sessions would bring their insights together. Again, striving to support knowledge exchange as a material and embodied process, we invited participants to express their insight in *artistic form*, supplied with colorful paper and crayons.

As a thought-provoking kick-off, we gathered participants for a pre-conference online meeting where we invited participants to join this format exploration. We discussed a few ideas of possible ways to deliver the papers through exploration, showcasing a possibility of a moderated discussion or exploration of learning materials from the paper. Yet, we expressed our uncertainty and exploratory stance, inviting the participant to co-create their own conference experience. The commitment and enthusiasm of our authors were unprecedented. Many met before the conference to discuss how the panel stays coherent; others worked hard on adapting and transforming their materials so that a short exploration would make sense and on setting up the environment during the conference itself.

Thematic overview of the contributions and presentation formats

Overall, 30 authors of 20 papers and 5 posters joined TWG, coming from the UK, Italy, Brazil, Greece, Hungary, the Netherlands, USA, Canada, Israel, Austria, Switzerland, the Netherlands/China, Chile, Germany, and Australia. We started from a traditional ice-breaker in the form of a speed dating where participants meet each other while randomly walking in a room and

exchanging their various preferences and experiences, from food to research focus. After this, the parallel sessions dedicated to thematic panels began.

The first panel was on *Teaching and learning environments*. One of the parallel sessions was mostly arranged around teaching, and the other around learning. Alessandra Boscolo invited participants to discuss the limitations and major concerns for the effective implementation of Active Bodily Experience Mathematics learning activities in schools. The interviews in Boscolo's paper highlighted the need to transform school culture and classroom practices in a situation where institutional support is lacking and teacher awareness of existing methods is limited. During the CERME discussions, the issue of implementing ready-made embodied activities versus co-creating those activities directly with the teachers was raised. Claudia Corriveau and Benjamin Nicolas-Noir invited the participants into a mini design challenge, similar to the one that pre-service teachers in their research engaged in: using provided making materials, such as various papers, cardboard, wires, paperclips, we needed to create a manipulative that would allow exploration of integers. The unexpected complexity of this seemingly trivial task struck us, revealing how designing reveals to us, as experts, students' potential challenges. In a parallel room, Andrea Ghersi and colleagues engaged the audience with a selection of tasks using GGB+, which interprets gestures and shapes indicated by their arms, in relation to quadratic functions. They also introduced the theoretical framework of conceptual blending. The presentation was active and thought-provoking. Further, Jennifer Way shared examples of activities with groups of children collectively enacting mathematical representations. The children switched between egocentric and allocentric perspectives, formed number sequences by aligning themselves after number counting, and drew a polygonal shape using a loop of string.

The second panel was dedicated to *Mathematical concepts in embodied and material mathematics*, challenging traditional understanding of concepts as idealistic and abstract. Michael Rumbelow introduced Friedrich Froebel's mathematical objects and invited participants to observe how block construction is ultimately a process of breaking symmetry. Karel Boonstra and Marjolien Kool engage participants in symmetry exploration through see-through mirrors, inviting participants to reflect on teaching and learning opportunities that the environment-task provides, and how the shape of human bodies supports the exploration of symmetry with a see-through mirror. In the *joint discussion*, participants exchanged the excitement of playing and seeing mathematics in a new way in the suggested artifacts, noticing how embodied and materiality support mathematical exploration in various forms: from movement through interconnections and rotations to observing the materiality of polyhedra and crystals. Hang Wei introduced the group to the activities that required them to engage with bi-manual coordination in a digital environment where dynamic nomograms could be explored through multitouch dragging. The participants had the occasion to focus on the emerging feeling when synchronous movement was achieved and on the color-changing feedback, which would be probably difficult to gain in a standard presentation. Such experience also allowed the participants to gain a deeper understanding of the diagrammatic rendering of the data collected during the study, which illustrate how a student achieved fluency in the process of coordinating hands (and with the technology in use). Then, Christina Gkreka and Chronis Kynigos challenged us to reproduce specific patterns shown on the screen where relative positions of objects were to be preserved. The experience created space for thinking about angles as turns, and more broadly in relation to spatial orientation,

which was at the core of their contribution. The *discussion* around the papers presented in this panel also gravitated around the materiality of space, the diverse demands of the technological tools at play, and the different scales at which the body might be involved: from minute hand coordination to whole-body experiences.

The next panel, *Philosophy of embodiment, materiality, and maths*, was meant to create space for a deeper dive into what mathematics is and how it operates in a politically charged society, drawing from non-dualist embodied and material perspectives. Elizabeth de Freitas and Nathalie Sinclair presented a paper that addressed a concern raised at our previous 2023 CERME TWG meeting: the need to reckon with a more political perspective when discussing embodiment. They showed how embodiment and materiality are crucial concepts for making sense of the politics of mathematics education. The authors emphasized how mathematical concepts from geometry can be studied for how they embody political and cultural norms about space and interaction. They started by showing a series of slides (a two-column proof of the Isosceles Triangle theorem, an image of indigenous Quipu calculations, an image of a gnomon) and asked the other group members to discuss, "How are these political, as well as mathematical?" Anna Chronaki then presented a paper about the affective intensity of mathematical concepts enacted by young people in Greek schools, where convergent and divergent movements were studied as ways of engendering mathematical forms of belonging. She showed images from her research, and asked the audience to reflect on their assumptions about basic shapes. *The two papers* shared an interest in the aesthetics of mathematical practices, and the ways that mathematics is always lived through materiality. Together, they brought attention to political and ethical questions about inclusion and diversity in mathematics Education. They both emphasized monist theories of mind/matter, drawing on ideas found in Deleuze, Whitehead, and Spinoza, as well as feminist thinkers such as Donna Haraway, Rosi Braidotti, and Claire Colebrook. In a parallel session, Jorge Soto-Andrade asked participants to work on experimental situations with diverse materials and instructions, such as a collection of sticks and an algorithm to be bodily enacted. The results were extremely diverse, illustrating ideas under development by the Chilean School of Enactivism in Mathematics Education Research on creativity of mathematical perception as metaphorizing. Ricardo Nemirovsky and colleagues asked the participants to create trefoils with a piece of rope. After introducing the Gauss notations for knots, they discussed the correspondence between a trefoil and a number sequence of three odd numbers. *Both presentations* revealed how mathematics is present in engagement with material and how mathematical aspects open up in action and perception, thus fundamentally criticizing the view on mathematics as abstract and suggesting instead that processes of abstraction/concretion are constantly present in perception and action that reveal structures and patterns.

The third thematic panel was *Mathematics and movement*. Here we gathered specifically the papers that focused on how mathematics emerges from diverse types of movement: full body exploration of space, dance, large gestures grasped by technology, finger movements in exploration of tangible figures, etc. Myrto Karavakou highlighted the movement involved in students' creation of dances using trigonometric functions in a constructionist learning environment MaLT2 designed by Chronis Kynigos. Myrto highlighted the aesthetic responses guiding the students' "choreographies", which were expressed in the coding language of the learning environment, showing how the dancing was

used to express students' sensorimotor ideas about periodic functions. The authors draw on inclusive materialism to highlight the novel conceptual meanings of covariation that emerged in the students' creations. Then, Nadav Marco presented his research on "Blindfolded Mathematics," where learners engage in mathematical tasks using tangible artifacts while temporarily deprived of sight. One task involved determining whether three sticks glued to a card were parallel. Nadav highlighted the ways in which the body learns to move in new ways, becoming more efficient in solving the task. Participants could experience shifts in their own movements as they worked on different cards, and observed others doing so as well. In the parallel session, Réka Szász presented her work with co-authors on exploring plane transformations through large arm movements captured by GeoGebra Dance AR technology. Participants were invited to try out different modules and contribute to the design improvements during the small group discussions. The issue of congruency between mathematical action and participant movement emerged as critically relevant in such movement-oriented designs. Lena Radünz engaged participants in a large game on understanding place values through finding numbers that form a 100 square on the floor. Participants were surprised by unexpected struggles but also insights into how numbers below a hundred relate to each other. Embodied participation in the game differed drastically not only from reading about it, but also from simply watching the others playing the game. *All presentations* highlighted the importance of movement in mathematical activity, but focused on different aspects of movement. For Myrto, movement was a way of communicating ideas, while for Nadav, movement played a more epistemic role. Presentations by Lena and Reka specifically raised the issue of passivity versus agency, and how even repeating some action through a gesture is an active process. Discussion touched upon the issues of mirror neurons that activate not as a copy of another person's movement, but as an active interpretation of their movement aims (Rizzolatti et al., 2014).

The final panel was dedicated to *Methods for the study of embodiment and materiality*. As we hoped in the call, methodological considerations covered a variety of aspects. In their contribution, Francesca Ferrara and Giulia Ferrari approached the idea of the materiality of data and proposed a methodological reflection on data presentation, focusing on the creation and use of diagrams in our field, and their potential to transform data. During the panel, the experience proposed to the participants engaged them in choosing a classroom situation or a diagram presented in one of the papers discussed in the TWG. The groups of participants were then (respectively) tasked with the creation of a diagram or with a close analysis of the chosen diagram. The following discussion fueled the ongoing issue of the TWG around the ways of reporting and sharing our research. Christoph Oberbucher created a paper version of the digital task he reported on in his article, which concerned the concept of continuity. We were all engaged in placing paper strips on different graphs of functions, in order to delimit the vertical and horizontal regions (δ or ϵ- region respectively) that are at play in evaluating the continuity of a function represented in the cartesian plane. This activity (and the questions that accompanied it) served as an introduction to talk about the implementation of "impurities" in the design of tasks (referred to as doping) in the specific case of continuity within the framework of radical embodiment. Anna Baccaglini-Frank and Nathalie Sinclair focused on a new methodological approach for data analysis, namely re-enactments, introduced in Sinclair (2024). They invited participants to re-enact a small video clip with dynamic digital environment exploration

so they could experience how certain body movements themselves—movements of the hand on the screen, posture changes, tone of voice, etc.—differed significantly from reading about them in a transcript. This difference points to the need for more "embodied validity" (to use Patti Lather's term) in research that seeks to understand the nature of embodied meanings in mathematics thinking and learning. An activity by Anna Shvarts and Karel Boonstra was intended to showcase a similar point but on a broader scale. Participants were invited to role-play researchers, teachers, and students in embodied studies, working within different paradigms: strict measurement and controlled environments of positivist science, interviews in constructivist science, and, finally, collaborative co-creation of learning environments in transformative/participatory science. Comparing their experiences, participants could exchange how positivist and constructivist paradigms are limited when studying flexible, multimodal, discovery-based embodied processes. *Overall*, re-enactment of the data as analytical methodology, the role of diagramming in data presentation, the specificity of design research, and broad issues of embodied and material studies surfaced. These discussions presented a paradigmatic shift in social science which highlights how embodied and material studies of mathematical behaviour repositions not only study participants, but also researchers who are similarly engaged in material and embodied activities, and yet often forced to translate these into mostly disembodied and reified scientific discourse.

Finally, we had an embedded poster session, as such a format allowed participants to get into more detailed conversations around the projects that are still at a very early phase. Particularly, the insights from non-Western traditions, such as, Al-Khwarizmi's work on mathematical visualization of algebra and challenges of creating mathematical education consonant with Amerindian cosmology where relational dynamic ontology dominates (Thiago Pinto and João Viola; Vinay Kathotia and colleagues). Those issues formed a unique contribution to the scope of the group and pointed to important future directions for research in this area. Also, the core questions on embedding teachers' perspectives (Benjamin Nicolas-Noir and Karel Boonstra), methods to study affect (Jacoliene van Wijk and colleagues), and the theme of mathematics and art (Claudia Flores and colleagues) inspired many discussions during the poster sessions and afterwards.

Form and content: Looking back and forward

To integrate the richness of gained experiences, participants worked in small groups on artistic projects where they had an opportunity to express their insights not only in verbal but also in affective and diagrammatic formats. The outcomes varied from word clouds and networks to colorful metaphoric presentations and videos, which are difficult to capture in text (see for example Figure 1). The following themes were recurrent.

Firstly, we noticed very many aspects and perspectives on conceptualization of the *body* as taking part in mathematics learning and teaching: body as a whole and as a special part of assemblages in doing mathematics, extension of the body through tools and discourses, body as a tool itself, body as senses and as a source of emotions as feelings, as well as body of mathematics and mathematical knowledge presented as artifacts and physical objects such as crystals, wooden blocks and sticks; moreover, body was thought as relational as it participated in framing interactions and perception of physical objects. Similarly, the *movement* was present in very many forms: dance, walking actions in

finding mathematically specific positions, arm gestures in interaction with technology and with each other, hand manipulations and dragging, touch explorations, and so on. The variability of those perspectives shaped our experience of diversity in possible senses, materials, learning paths, technologies—while traditional research might strive to choose the best options, our group illuminated the variability and richness of possibilities that research opens, letting the community engage and evaluate them rather than presenting a final judgment.

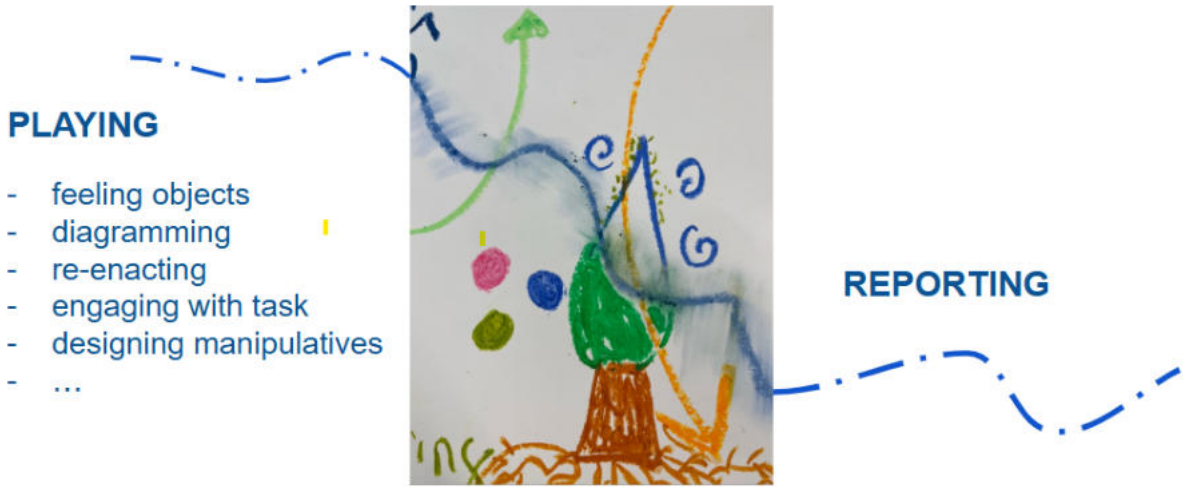

Figure 1 The drastic difference between playing with mathematics, educational design and research versus reporting about them

We sensed a *methodological* shift from static to dynamic that lit research results and mathematics education in practice as vibrant, unpredictable, fluent, affective, untransferable. Our methods called for alternative temporalities in a search for slowing down strategies, dwelling in, and thus disrupting typical ways of making sense. New methodological directions pointed towards connecting with arts-based practices, re-enacting, re-making, performing, symbolizing relations, co-creating, participating, transforming, rather than representing, assessing, judging, or taking for granted. These new directions raise numerous challenges, such as: How to capture and report a genuine and a holistic view? How to address and justify subjectivity? How to deal with policy and privacy issues? How to relate to established research norms and which new quality criteria might emerge?

Critically, such a radical reconsideration of what research might look like calls for thinking about possible infrastructures that would bridge the gaps between policy-makers, researchers, teachers, and students. We need forms that would engage those often isolated communities in a respectful and creative embodied flux where relations are of mutual learning rather than hierarchy. It also calls for a radical reconsideration of the nature of mathematics itself - the need to problematize, as always, assumptions about where and how mathematical activity occurs.

Such a transition is fundamentally *political*, challenging the established political system and revealing the political in small educational choices. For example, a choice of a task format or educational design: contrasting open-ended and closed problems, designs that privilege specific learning paths versus those that open diverse learning landscapes, technologies that respond to specific sensory-motor sensitivities and ignore others. Such pedagogical research needs to also engage with sociological, philosophical and historical investigations into the nature of mathematical concepts,

exposing the political framing of mathematical embodiment at the level of curriculum and policy, underscoring concerns about *which bodies produce which mathematics*.

The experimental *format* brought great enthusiasm and inspiration—we felt positive affect running through the bodies of the conference participants. The sessions became affectionate and memorable. Participants were free to raise their fundamental concerns about freedom, joy, and restrictions of normative mathematical culture. During the conference, we could not find a participant who was critical of our format experiment, apart from ourselves within the co-leaders team. We also enjoyed the freedom and playfulness that the formal experiment brought in. Yet we also felt we were sometimes lacking depth in our discussions and careful work on discursive components of theoretical work that embodied and material studies assume. The invitation to feel and explore fitted perfectly with some contributions, yet was challenging for others. How can the diversity and affectivity of bodily exploration and enactment complement careful theoretical work and reflection on past research in this and related fields, to allow for community synergies, mutual recognition, and some advancement in our collective understanding? This question, in application to both mathematics education and mathematics education research, will inform future iterations of our TWG. At future CERMEs, we will continue to experiment with formats, and search for collaborations with teachers, policymakers, and scholars with kindred interests. We hope to further enrich our meetings by attracting researchers who study learning processes in diverse cultures and contexts, attending to gendered and racialized bodies, as well as alternative sensory and motor profiles

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Re-enacting a digital environment tutoring episode: New insights

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This paper focuses on an episode in which two 10th-grade low-achieving students are learning about unknowns and equations with a tutor in a dynamic digital environment. We initially conducted a commognitive analysis, which provided insight into the students' emerging mathematical discourses. To address issues of embodied validity, we then used the method of re-enactment, which enabled us to understand techno-affective aspects of the interactions in the episode.

Keywords: Low-achievement, digital technology, inclusive education, embodiment, re-enactment.

Introduction

Within the ongoing research project DynaMat (https://www.carme.center/progetto-dynamat/), we are conducting case studies of Italian high school students (10th grade) with a history of low achievement in mathematics. Our study concerns the use of a dynamic digital tool designed to address students' difficulties in algebra (Weingarden et al., 2024). The tool is a digital version of the balance model, commonly used to introduce linear equations and to deal with the equal sign and with unknowns (Otten et al., 2019). The episode discussed in this paper caught our attention. It features two classmates, Alba and Elsa, who spent considerable time during an early part of the didactical sequence, on finding the weight of the "red balls" even though they were not asked to do that, and who interacted with each other and with the tutor in interesting ways. The activity is developed within Desmos and is embedded in a sequence dealing with a dynamic scale that has known (squares labelled with numbers) and unknowns (red circles) weights (Figure 1). The user can move them, together with the ones in the blue box on the left, and the scale adjusts accordingly.

The sequence of activities, which is effective even for students with severe arithmetical difficulties (Robotti et al., 2023), promotes strategies of adding or removing object(s) from each pan of a scale while keeping the scale balanced. It then aims to shift students' attention to objects with an unknown weight and to ways of finding its value, which can be akin to the procedure of solving linear equations (Bonadiman, 2023). Such a shift was not yet promoted when the following activity was assigned: *Here you see a balanced scale. If possible, remove 3 objects from the scale so that it is still balanced afterwards. If it is not possible, explain why* (Figure 1). We expected students to find a combination of objects to remove, such as the 4 from the left pan and the two 2s from the right pan, to keep the scale balanced. Moreover, the tutor followed a script which included prompts for both justifying the strategies and finding the value of the red balls.

This study investigates Elsa and Alba's participation in the activity. We focus on: (A1.1) Why all this interest in the red balls? (A1.2) What did they want to achieve by moving them? (A1.3) Were they successful (from their point of view)? (A1.4) Were they successful (from the tutor's point of view)?

In addressing these questions, we worked first with the discursive tools of commognition, which yielded some insights, but did not seem to adequately capture other aspects of the students' actions and interactions. We therefore decided to try re-enactment, a methodological approach that aims towards the embodied validity of Lather (1993), as recently undertaken in mathematics education (Sinclair, 2024). The re-enactments enabled us to experience what Elsa and Alba had both done and possibly felt, which provided us with additional insights into these questions.

Figure 1: The activity described in the paper

Theoretical framework

In accordance with the DynaMat project, the initial analysis was commognitive (Sfard, 2008). According to Lavie et al. (2019), a *task situation* is a situation in which an individual feels compelled to act. Task situations in a classroom typically involve the teacher's assignments and questions. The *task* is how the student interprets the task situation. Students may perform different *routines* (each comprises a task plus an implemented procedure) in the same task situation because their tasks are different; these can be considered *rituals* (applying memorized procedures to please someone else without concern of the products) or *explorations* (producing meaningful narratives with a focus on the products). Commognition also provides lenses for discussing identities, which are defined as "a collection of stories" that are reified, endorsed, and significant (Sfard & Prusak, 2005).

With the re-enactments, we focussed not only on discourses, but also on affects and senses, which are seen as being important in theories of embodiment, and which often refuse cognitive/affective and mind/body binaries that are common in most mathematics education theories (see de Freitas & Sinclair, 2014). In digital environments (and even more so in physical environments, but these were not a part of our study), the senses are often more significantly at play than in typical paper-and-pencil environments, which invites students and teachers into novel sensory experiences, including not just the visual, but the haptic, the kinesthetics, and much more (Güneş et al., 2024). A re-enactment involves simulating an event as closely as possible, with the aim to aestheticise experience (Fuller & Weizman, 2021), which is to say, heightening the sensory dimensions that are frequently ignored when the focus remains on words and written transcripts. The way in which people are affected by what happens on the screen, or what their hands do as they drag objects can be very hard to see in a video, which is why the re-enactments were undertaken.

Methods

Alba and Elsa, the protagonists of this study, consider themselves to be low achievers, as does their teacher (the two students come from the same class). They took part together in five two-hour sessions with digital environments. For Alba, Italian is a second language. Data collection took place in an out of school learning centre (CARME) and includes audio-video recordings of the sessions and screen recordings of the iPads (students' written works and digital environments they interact with). The

participation of both during the interview was highly ritualistic and focused on recalling procedures. Both students developed stable failure identities over time. Elsa tended to describe herself using catchphrases such as dyscalculic, dyslexic, or not being cut out for math. Alba explicitly referred to her feelings while doing math, expressing that she felt overwhelmed, tired, and got headaches.

Data were initially processed using MAXQDA, a computer-assisted mixed methods research software. We conducted a discursive analysis led by the aims of identifying task situations proposed by the tutor (T) and of inferring tasks addressed by Alba (A) and Elsa (E) (Figure 2). We labelled the transcript considering the prompt of the tutor as task situations, while the students' tasks were identified looking at the procedure they implemented (what they said and what they did, on the scale or in the notes). The aim of this analysis was to gain insight into what triggered Alba's focus on finding the weight of the red balls (had it been a shift promoted by the tutor?), as well as to establish each student's discourse about scales and its analogies with discourse about principles of equivalence of equations (which is not the focus of this paper). In response to interesting phenomena arising from this original analysis, that we describe below, we also conducted two re-enactments (see Sinclair, 2024). The first re-enactment was done to zoom into the students' (and Alba in particular) interaction with the dynamic virtual scale; the second one to further explore the interactions between the participants (Elsa, Alba and the tutor) and the technology.

The first re-enactment was done by the first (AA) and second authors (AN), concentrating on Alba and Elsa's actions in the digital environment and words accompanying such actions. AA and AN sat next to each other on a couch, each holding a laptop computer: on AA's they played a video made of a collage of footage of Elsa and Alba's, synchronised with the screen capture of the iPad with the digital environment of the activity. In the analysis section, we refer to Elsa/AA and Alba/AN to highlight the re-enactment-created data. The re-enactment took about 2 hours, during which time AA and AN took notes on what they newly noticed and felt. The second re-enactment was different: it was done by first author AA playing Elsa, the third author AE playing Alba, while the fifth author AC played the tutor and the fourth author ACH the director. It took place in a similar setting as Elsa and Alba's original one. AA and AE sat next to each other on two chairs and had a table in front of them with two iPads, one having the same activity assigned to Elsa and Alba and the other having a blank page. AC had some space to walk behind them. On the sides, there was a laptop showing the same collage video as in re-enactment 1, and another laptop with the transcript of the episode. The footage was made with a third iPad handled by ACH. The actors watched some seconds of the original video and reproduced as closely as possible actions, words, gestures and interactions with the iPads. The re-enactment took about 2 hours, after which a discussion about the actors' impressions took place. The footage was then edited similarly to the original video (by making a synchronised collage of the footage and the two screen recordings of the iPads) (Figure 3). Even though the original episode was in Italian, as were both re-enactments, we have translated them to English for this paper. We will use a composite labelling to refer to the players in the re-enactment: Elsa/AA Alba/AE, the tutor/AC.

Findings

In Figure 2 we report on the MAXQDA tool *Codeline* in which the columns display the ranges of paragraphs of the transcript while in the rows there are the speaker-sensitive labels. We differentiate

between labels starting with "TS" and "T". The former denotes instances in which the tutor directly formulated task situations. On the other hand, the latter refers to parts of the discourse where we can observe a procedure carried out by Elsa (E) or Alba (A), which allows us to infer the task being interpreted at that moment. For example, the label *TS-initial (T)* refers to the instances in which the initial task situation is recalled by the tutor, as in line 20. The label *T-initial (E)* refers to the instances in which we infer that Elsa task is the initial one, as in line 19.

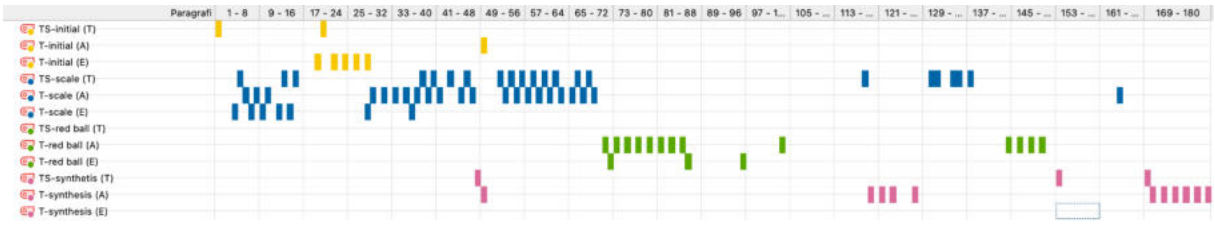

Figure 2: The *Codeline* of the episode used for the commognitive analysis

19 Elsa: Thus we have to remove three objects…
20 Tutor: It says to remove three objects. I leave you to work alone

Looking at the episode, we realise that the students are eager to investigate the relationship between the scale and the three red balls. We labelled these tasks as *T-scale*. Observing the fourth line of the Codeline, we notice how, following Elsa's prompt (line 4 "Why are there two little balls? I mean three little balls"), the tutor formulates TSs about the movements of the scale related to the three red balls and about justifying the dynamic feedback (we labelled these with *TS-scale (T)*). For instance, in line 38 the tutor asks what happens if all the red balls are removed from the pan. The aim of her intervention was to focus on the feedback of the scale and in particular to find a justification for it (line 40) based on their previous experience and on the scale's features. This leads to an exchange between the tutor and Alba: the tutor aims at promoting meaningful narratives about the feedback of the scale, Alba at answering the question, but seeming focused on something else.

37 Alba: There's something... okay, let's remove all the little balls [removes all the little balls from the pans]. So, if we remove all the little balls…
38 Tutor: What happens if you remove all the little balls?
39 Alba: So, there's one that's lighter, and there's, yes, the lighter one and the heavier one, so…
40 Tutor: Does this make sense to us?

Indeed, in line 71 Alba starts to deal with the red ball left on the right pan. The label T-red ball, which appears from that moment, signals the students' shift of attention towards the task of finding the value of a red ball. As highlighted in the Codeline, while Elsa does not take part in the process, Alba addresses this task of hers (line 71-85). At this point no task situation about finding the exact weight of a red ball is formulated by the tutor (line 7 of the Codeline is empty). Alba claims that "the sum of this one [the left pan] and the sum of that one [the right pan] make the same product" as she adds the weights on the left pan, finding 18. Thus, by subtracting gradually from 18 the weights on the right, she reaches 7 and she says, "So with this logic it would be seven" (line 81).

71 Alba: Wait, wait. I think this little ball has something... these have the s... the sum? Uh, the sum of this one and the sum of that one do the same product, gosh, my Italian. So I'd like to try here, no. If we do 4 plus 5 plus 1 plus 8, and here there's 3 plus little ball plus…

72 Elsa: But how much is the value of the little ball?
73 Alba: We'll find out now. Because, uh, if we add this 4 times 5, 9, 10, 18.

This analysis provides evidence of Alba's involvement in the activity and her continuous effort to discover and explore, but most of all to understand what the red ball "has" (line 71). Additionally, thanks to the Codeline, we notice that Elsa tries to participate in the discourse, too (e.g., in line 19 she goes back to the initial task and tries to accomplish it), but she gives up. From line 28, she does not act on the scale anymore and just watches Alba and tries to follow (line 72). Moreover, Elsa never responds to the tutor's request of synthesis (*TS-synthesis (T)*), such as "After you removed the same weight (red balls) from both pans, you did other things. Tell me more" (line 169). Alba instead offers a synthesis of her action and adds other considerations (line 49, lines 119-127, lines 170-180).

New findings after the re-enactments

The first re-enactment offered two main insights. The first relates to A1.1. By re-enacting all of the movements of objects that Alba and Elsa made on the screen, many of them not captured in the transcript because they were not associated with utterances, we realised that the red balls were moved extremely frequently (27 times in less than a minute, way more than other pairs of students). This movement is of course visible in the video, but we only attended to it because of the effort required to drag the ball on the screen. The varying ways in which the red balls were moved make it seem that Alba/AN, especially, did not have a particular strategy, but was trying to gain information about the red balls by seeing what changes they would provoke. In other words, for Alba/AN, finding out about the unknown had to happen by moving the unknown itself; instead, in solving a (linear) equation, the value of the unknown becomes known by acting on the numerical parts of the expressions, that corresponds to moving other known weights on the scale.

The red ball was also frequently moved back onto the scale, where it had already been. By re-enacting this movement, which at first seemed unnecessary, it became clear that Alba/AN was expecting something different to occur. As with a real scale, she was expecting that adding a weight to an unbalanced scale would make it even more unbalanced, which is not the case in the design of the digital tool (line 29). This is not an expectation that is ever uttered, but one that became felt in the re-enactment. The red ball becomes interesting not (only) because it is the unknown that the tutor is asking about, but because not much can be learned about it by manipulating it and because moving it does not always produce an effect on the screen (A1.1). There is a sense in which it was hoped, perhaps not even consciously, that moving it would divulge information about it, or causal relations it had with other objects on the screen.

Re-enacting the video, which involved not just simulating the actions on the screen, but also the tone of voice and the facial expressions, also dramatised Alba/AN's enthusiasm and sense of satisfaction. This satisfaction is not only felt in relation to solving a task, but also when she smiles, heightens the pitch of her voice and sits back to admire what has been done, like when she succeeds in putting a single red ball on the right. An exclusive focus on solving the task or "learning the math" neglects to adequately capture the affective nature of her experience, and the positive emotions that are so crucial in creating strong (positive or negative) memories about mathematics (Nemirovsky, 2011). The satisfaction does not stop at solving the task. Alba/AN also finds different ways of solving problems

and different ways of proving that her solution is correct. For example, when she works out a solution by hand, shifting to pencil-and-paper on her own accord, to work out the difference between 18 and 11, which gives her the weight of the ball (lines 148-150).

In relation to Elsa/AA, who seems to participate so little throughout the episode, the re-enactment offers insight as well. She may not say very much, but she is involved in manipulating the objects on the screen as well. And in fact, these actions sometimes come in response to something said by Alba/AN and therefore constitute a non-verbal interaction. Indeed, the re-enactment revealed that it was Elsa/AA's action that enabled the pair to find a way of removing only weights with given values of which two sum up to be the other.

One of the motivations for doing the second re-enactment was the realisation that the role of the tutor had been ignored in the first re-enactment, which made it difficult to understand the contributions of both Elsa/AA and Alba/AE. In the second re-enactment, it became more evident to the actors that both Alba/AE and the tutor/AC hindered Elsa's learning opportunities. This was felt in two significant ways. First, the re-enactment highlighted the fact that Alba/AE frequently completed Elsa/AA sentences. Sometimes Alba/AE would "speak for" Elsa/AA. For example, in line 164, "You took one from here and one from here, so you took the same weight. That is what you wanted to say?": here Alba/AE draws the tutor/AC's attention to herself, while Elsa/AA confirms with a "Yes" (line 165) what her mate says and leaves the pencil on the table. Alba/AE also invades Elsa/AA's space at a certain point, saying, "I need to think, sorry" (line 33, Figure 3). Alba/AE physically dominates the space, taking away opportunities for Elsa/AA to participate, or forcing her to participate passively.

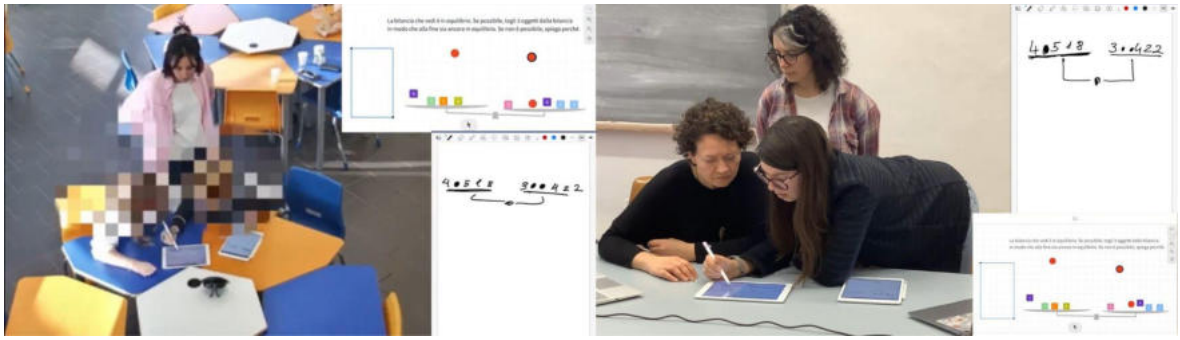

Figure 3: Space invasion of Alba/AE into Elsa's/AA's (comparing re-enactment videos)

Second, Alba/AE positions herself as the teacher and Elsa/AA as the learner, as when she explains to Elsa how she found the weight of the balls in response to her request ("Wait, but how did you find it?", line 86). In this, there's a sense that Alba/AE and the tutor/AC participate and position themselves more as equals. The tutor/AC tends to direct her gaze towards Alba/AE, even if Elsa/AA frequently looked towards the tutor/AC. Elsa/AA, in the beginning, is interested and engages to some extent (e.g., line 19, line 72). But then she ends up participating only through a few words and interacting with the digital environment. Elsa/AA is never asked to write, and she never answers tutor's/AC request of synthesis explicitly directed to her (line 153). She is willing to "give up her words" and leave sentences to be completed by Alba/AE. At a certain point, when Alba/AE explained to her how she had found the weight of the red ball the first time, Elsa/AA puts her head down, like in the initial interview with Anna, which felt like continued antipathy or disengagement.

Discussion and conclusions

Through the combination of the commognitive analysis, which highlighted significant aspects of the students' discourse, and the analysis of the re-enactment, through which we noticed different actions, interactions and feelings, we were able to address our four questions. Overall, we discovered that it was Elsa who first mentioned the red balls (line 4) and shifted the discourse in that direction. Only much later (line 71) does Alba's task shift to finding the weight of a red ball, which does not justify all the movement of the red balls at the beginning. The re-enactment helped us notice how Alba was affected by the red balls (A1.1). The discourse analysis allowed us to infer several tasks related to the red balls, taken on especially by Alba (A1.2), who was successful in solving them (A1.3). Elsa seems to be focused on the initial task situation, but it is unclear to what extent she managed to follow Alba in finding the weight of the red balls. In re-enacting her actions, Anna realizes that Elsa let Alba speak for her and did just a few actions on the scale (A1.3). The tutor seems to acknowledge Alba's success, directing her gaze and attention mainly towards Alba (A1.4).

These findings provoke reflection on the kind of inclusive learning setting we had designed. First, Alba's "speaking for" Elsa may perpetuate discursive routines from the classroom. While these two students had been intentionally paired so that Alba could help Elsa, the result was a "taking away" of opportunities for learning. Elsa ends up contributing "only" to simple answers. Second, in terms of the teaching: with all the best intentions, the tutor treated the students differently. This propagated discursive routines in which the weaker student gets fewer chances to participate. All this in a context in which we were all "trying our best" (through design principles, interactive digital environments, etc.) concerning the inclusion and participation of struggling students. We maintain that the design of the environments can enrich learning for students like Alba and Elsa, but that we need to further study how to the students' actual needs and ways of participating in mathematics.

On a methodological note, we acknowledge the practical and ethical tensions that arises from trying to experience other people's lives (see Sinclair, 2024). There are significant differences in terms of position and knowledge between the actors and the students. That said, the re-enactments enabled us both (1) to surface things that the discursive analysis left invisible and (2) to gain conviction about the significance of emotions and other non-verbal events that cannot be expressed in words. These felt experiences bring to light significant affective aspects of mathematical activity, as de Freitas et al. (2019) evoked in their study of sympathetic bonds, such as Alba's compulsion of moving the red balls and the nuanced shifts in positioning people's roles in interaction.

Acknowledgment

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Moving towards abstraction: An embodied design for reflection symmetry

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This paper reports about a design study on creating a learning environment for fostering the process of abstraction in geometry. Through manipulating chess-towers and a see-through mirror, primary school children showed conceptualization of equidistance and orthogonality as properties of reflection symmetry. The fostering of embodied abstraction benefits from introducing relevant artifacts, promoting the verbalization by the students of the apparent structures in the action space.

Keywords: Geometry, abstraction, embodied cognition, field of promoted actions.

Introduction

Inspired by the research field of embodied cognition we seek to design a learning process, based on realistic geometrical experiences, which invites students to engage in a process of abstraction. Generally, embodied cognition posits that we learn through interaction with our body and the environment (Wilson & Golonka, 2013). Focusing on the role of the body in interaction with the environment can bring the inherent geometry into experience. A learning environment that invites a student towards discovering, through promoting certain directions of moving, is referred to as a 'field of promoted action' (Abrahamson & Garcia, 2016). The action-based embodied design genre (Abrahamson, 2014), offers guidelines for arranging such a field of promoted action. Where usually action-based embodied design studies make use of digitally arranged environments (touchscreens, augmented reality, motor sensors), we in this study, seek to use a non-digital environment. Although recent studies on applying action-based embodied design ideas in various domains, show that conceptual understanding indeed can originate from students' actions elicited in a field of promoted action; see for example, studies on sine-graph (Shvarts et al., 2021) and gradient (Bos et al., 2022), the notion of abstraction is not addressed explicitly. In this study we aim to arrange a non-digital action-based embodied design where the fostering of abstraction is central. To this end we set up a design study targeted at developing such a learning environment while gaining knowledge on the process of abstraction through an embodied approach. The learning environment targets the geometrical subject of reflection symmetry.

Theoretical background

Reflection symmetry is an important element of primary school geometry (Ferrari & Saldano, 2023; Sinclair et al., 2016). Where reflection symmetry is often presented in modern curricula as a static concept (Ng & Sinclair, 2015), we seek to use a dynamical approach. Imagine standing in front of the mirror with arms stretched forward. What do you see? There seems to be an image of you on the other side of the mirror. Each part of the 'image' you is positioned exactly opposite to, and at equal distance from the mirror as, yourself. We seek to develop a field of promoted actions in which the aspects, equidistance of object and image, and orthogonality of object and image to the mirror line,

are discovered. In a field of promoted action (Abrahamson & Sánchez-García, 2016), the student is presented a specific motor control problem in such a way that self-exploration is encouraged. Such a motor control problem is one of the three criteria for an action-based embodied design as described by Alberto and colleagues (2022). To both encourage and guide students' self-exploration, constraints are applied to the environment and the task. In this way students are guided to reinvent ways that "were never dictated, demonstrated, explained, or cued," (p. 214). The second criterion for the embodied design is to provide some form of continuous feedback on students' movements. In this way the student can easily observe whether the objective of moving is met. The third criterion is to not use measuring artifacts in the initial phase of the design. Of particular importance for embodied design is the notion of attentional anchors (Abrahamson & Sánchez-García, 2016). An attentional anchor is a perceptual structure that emerges to the student when acting, and helps students' steer their movement. The aim of the current study is to clarify how such type of designs can be enhanced to trigger a process of abstraction. We follow Dreyfus (2014) in considering abstraction as a process of understanding the structure of a mathematical concept. Bringing in the embodied perspectives to the process of abstraction led Boonstra and colleagues (2023) to describe a definition of embodied abstraction as "a process of reflecting on, describing, explaining, and structuring of sensory-motor actions in the experienced world through developing and using mathematical artifacts" (p. 8). Just like in all instances of action-based embodied design, to foster the development of a mathematical discourse on student's actions and perceptions, a careful stimulation by a tutor is needed (Abrahamson, 2014; Alberto et al., 2022). By using artifacts such as mathematical words, or objects (sticks, ruler) a re-description of the (transformed) action space is brought about that elicits the verbalization of perceptual structures inherent, leading to a conceptualization and abstraction of the concepts at hand. In fostering abstraction, the tutor, when noticing such structure emerging to the student, can build on this notion through reflecting and re-description. Since abstraction is an idiosyncratic process, it does not necessarily require the student to use a particular (formal) wording or description, instead the appropriateness of the wordings used, is discussed and agreed upon within the tutor-student dyad. The central question in this study is, how can a non-digital action-based embodied design foster students' process of embodied abstraction regarding reflection symmetry?

Method

Our research methodology is that of design research. We develop and test an action-based embodied design for reflection symmetry. We hypothesize that this embodied design invites the student, with aid of a tutor, to engage in a process of embodied abstraction. The design process, consisting of three cycles, is driven by a hypothetical learning trajectory (Simon, 1995). A Hypothetical Learning Trajectory (HLT) consists of three components (1) a learning goal, (2) a set of learning tasks, and (3) a hypothesized learning process. How these components are used in this study is described next. The central idea in this embodied design is that the mirrored image of a white chess tower can be made manifest by using a see-through mirror and a second tower. When this second tower is placed in a reflective symmetry position of the first, the image of the white tower coincides with the second, see Figure 1a. This 'coinciding' forms the basis for the continuous feedback of the 'system'. This set-up makes it possible for a student, when physically manipulating the towers in order to let these coincide,

to discover two aspects of reflective symmetry: equidistance and orthogonality, which are the learning goals (1) of the HLT.

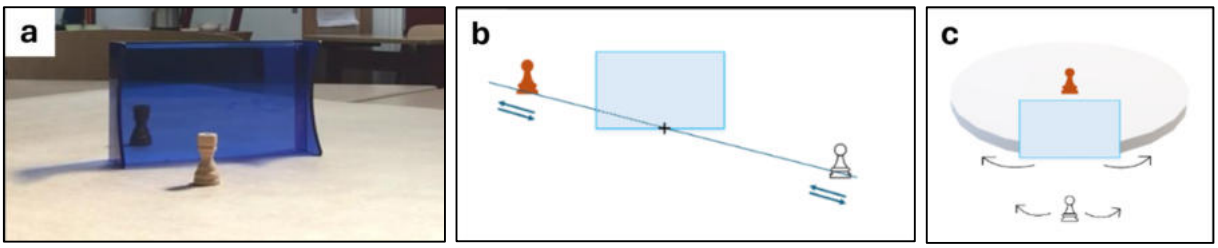

Figure 1: a-b-c the setup of the towers and disk: (a) Task 1: free dance; (b) Tasks 2 and 3: equidistance; (c) Tasks 4 and 5: orthogonality

The learning tasks (2) of the HLT are the following. *Task 1*. 'A free dance'. When looking in the mirror, the student moves both towers (as in Figure 1a, but towers not yet coinciding), thus freely exploring the setup. The goal is to discover the coinciding of both towers. *Task 2*. Goal: discover equidistance. Student moves the towers continuously on the line and keeps them coinciding, see figure 1b. *Task 3*. Goal: apply the experience from task 2. A stick is placed orthogonal to the line to resemble the mirror (not in the exact position as the mirror to avoid copying). The brown tower is placed on the line. The student places the white tower so that the towers 'coincide'. *Task 4*. Goal: discover orthogonality. Student moves the mirror and the white tower (standing on a base) and keeps towers coinciding. The brown tower is placed in the center of the disk, the mirror against the disk, see figure 1c. *Task 5*. Goal: apply experience from task 4. The mirror is removed. The stick is placed alongside the disk, see figure 4b. The student places the white tower, so the towers 'coincide'. *Task 6*. Goal: apply experiences from tasks 2 and 4 outside of the action field. On paper, a dot, resembling the brown tower, and a line, resembling the mirror, are drawn. The student writes an instruction, as to where to put the white tower. *Task 7*. Goal: apply experiences from tasks 2 and 4 outside of the action field. On paper two dots are drawn resembling the towers. The student writes an instruction on where to place the mirror. The hypothesized learning process (3) of our HLT is that, in engaging in the series of tasks and with the aid of the tutor, a process of embodied abstraction (Boonstra et al., 2023) is brought about: the student's discoveries (perceived structures) in the action space in task 2 and 4 are applied in task 3, 5, 6 and 7, where re-arrangements and (semi)-abstractions of the initial field of promoted actions are presented. The student, making use of artifacts, then verbalizes these structures, leading to the conceptualization of equidistance and orthogonality. For each task, the student's expected actions and multimodal behavior were aligned with the other two elements of the HLT and the tutor's expected interventions, leading to an exemplified and elaborated hypothesis of the embodied design. The testing of the embodied design was done in three cycles and eleven tryouts (3-3-5), each with a single student from grade 4 or 5, the first author acted as tutor. These tryouts were videorecorded. After each cycle, design sessions with the first, second and third author were held, during which observations from the videos were shared and discussed in relation to the elaborated arrangement mentioned in the above. From here adjustments were made to the design.

Results

In this section we share a series of vignettes from the tryouts. We describe key observations on how students engage in the process of abstraction related equidistance and orthogonality.

Equidistance: Abe confirms his measuring with his perception

In task 1, Abe has found multiple places where the towers coincide. When asked how he found these, Abe says, "by setting them opposite to each other." It is not clear whether Abe means orthogonal to the mirror, nor if he refers to the same distance. The tutor asks, "how do you do that? I see you doing different things with each hand." Abe: "No, you need to do something like," Abe grabs the towers, moves both in opposite directions alongside the mirror, (see figure 2a) and then to and from the mirror. "One needs to go to the front and the other to the back." Through trial-and-error Abe has found a way of letting the towers coincide, yet this pragmatic achievement is not yet supported by any abstraction. In task 2, Abe moves both towers one after the other further away on the line (not

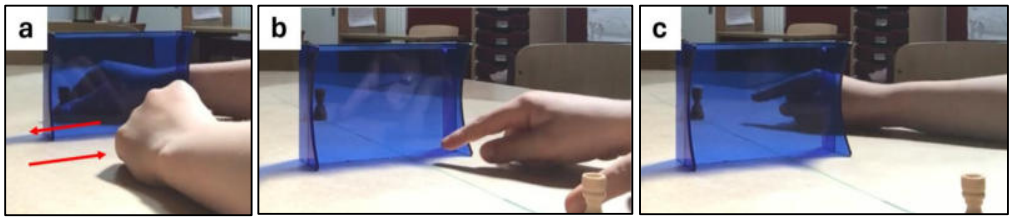

Figure 2: a-b-c Abe moves and gestures

simultaneously). Tutor: "Where do you pay attention to?" Abe: "that they are in each other." Tutor: "how do you do that?" Abe: "by placing them equally far from the mirror." Tutor: "what do you mean with equally far?" Abe goes on to measure with his hands. He first grabs the distance between the towers with both index fingers, see figure 2b. And then points to the position of the brown tower, see figure 2c. He relates distance to position, which is in consonance with his actions; first 'making a distance' by moving the white tower forward and then adjusting the position of the brown tower. Tutor: "Do you know this is so by looking in the mirror, or do you just know?" Abe: "by looking in the mirror it is clearer." Abe uses what he perceived in the mirror to motivate 'equally far.' A sign that the process of abstraction is grounded in experience.

Orthogonality: Abe's difficulty in being precise on 'opposite' and orthogonal

Where in task 1 Abe already talked about 'opposite to each other', he struggles in task 7, raising uncertainty about his conceptualization of orthogonality. Abe places the stick correctly between the two towers. When asked how he can see that the stick is placed correctly, his response is: "they are opposite to each other and on the same distance." He gestures with his arm, see figure 3a. Note that he refers to the towers and not the stick; he says 'they'. Where Abe is able to correctly 'place' the mirror, he has difficulty in writing the instruction. At first, he writes, 'in the middle of the towers.'

Noticing that Abe does not refer to distance nor oppositeness, the tutor places the stick skewed and points out that now the stick also goes through the middle of the towers. Abe says: “Ehm, how do you say that?” Where Abe was able to mention the towers being opposite, he is unable to find a suitable term for the stick related to the towers, or he thinks ‘opposite’ is not accurate. “Where do you pay attention to apart from distance?” Abe replies, “that he goes straight to the middle.” Tutor: “Through the middle of what?”, Abe, hesitatingly: “through the middle of the distance between the towers?” Here Abe shows being aware of a cross-like figure that would be formed by a mirror and a line connecting a tower with its reflection, which serves to him as an attentional anchor. After writing the instruction, the tutor asks to draw what he means with distance between the towers. Abe, while drawing, says, “ehm, if this is the distance, then you should also draw the distance there, and where they meet the stick goes,” see figure 3b. Note that Abe uses distance in the same way as in task 1, first ‘making’ a distance on one side of the mirror and then applying this on the other side. When the tutor asks how the stick should go, Abe cannot find a verbal expression that suits his drawing, yet the drawing itself perfectly represents the geometrical structure of the action field. The above shows that the term ‘opposite’ does not suit the notion of orthogonality. As we will show in the following stories on orthogonality, the use of artifacts is key. In this case the elicitation of verbalizing a cross would help Abe, since he showed to use such attentional anchor.

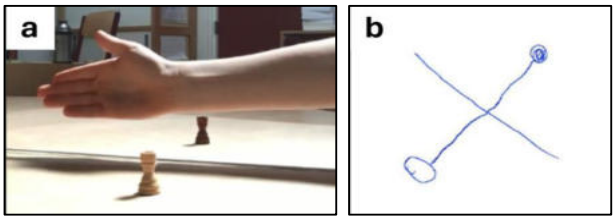

Figure 3: a-b Abe’s gesture and drawing in task 7

Orthogonality: Matt’s motivation gives an indication for a perceptual structure

In task 4, Matt moves the mirror and the white tower carefully clockwise, see figure 4a. The tutor, wanting to trigger Matt’s awareness of the orthogonal position of the line between the towers and the mirror, asks, “do you notice something?” Matt: “No.” The tutor decides to invite the student to change perspective, and asks, “what do you notice about how the towers and mirror are placed, when looking from above?” Matt looks from above and says, “hm, almost directly opposite to each other.” Tutor: “Can you check that in the mirror.” Matt looks again from sitting position, moves the white tower a little and says, “so, now they are opposite to each other.” Tutor: “Can you move again, the mirror and tower, to see if this remains so?” After moving the mirror and the tower for a while, the tutor asks, “what do you notice?” Matt says, “they are still directly opposite to each other.” It seems that the concept of orthogonality, in the form of ‘directly opposite to each other’, is noticed by Matt. The structure is noticed when Matt looks from above, and coupled with his actions and perception, when moving. In task 5 Matt uses this structure again. Here the mirror is replaced with a stick, see figure 4b. When the tutor asks if the stick is placed correctly, Matt says “no, it should go directly opposite,” and gestures between the two towers, see figure 4b. Tutor: “Opposite to what?” Matt: “to the tower.”

After Matt has moved the stick in the correct position, the tutor asks, “how do you know this is opposite to each other?”

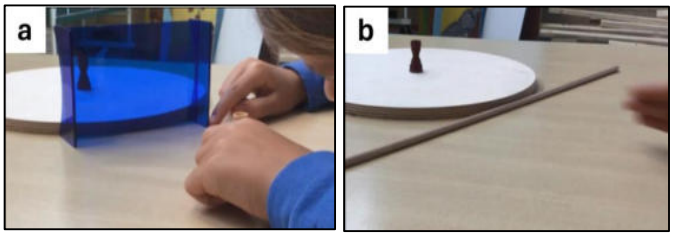

Figure 4: a-b Matt moves and gestures

Matt: “because this is also opposite.” (camera did not catch what Matt was referring to). Tutor: “how do you see that?” Matt: “well, it was like just before.” Here Matt, who showed no difficulty in placing the stick and tower correctly, motivates ‘directly opposite’ with his earlier experience with the disk. Matt uses his perceptual structure in a more abstract situation. The term ‘directly opposite’ however is not a clear indication for orthogonality, as we also saw in Abe’s story. In making the transition from student’s perceptual structure to the concept of orthogonality, the embodied design would benefit from using suitable artifacts, for example by referring to a cross, as we see in the next vignette.

Orthogonality: Tania uses artifact and makes a cross

The accompanying questions for task 4 developped in the design to open-ended questions like: ‘what do you notice?’, ‘where do you look at?’, ‘where do you pay attention to?’ The aim is to trigger the student’s perceptual structure of the orthogonal line between the towers and the mirror. The case of Tania shows that such questions do not lead to immediate result per se, but could very well pave the way for students’ verbalization of the structure in a re-arrangement of the action space. This is clearly visible in the tryout with Tania. In task 4 the above questions did not lead Tania to refer to the structure, she only referred to the towers coinciding. When the mirror is replaced with the stick (task

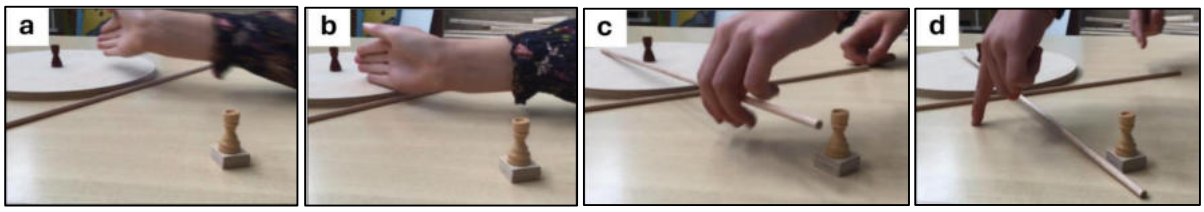

Figure 5: a-b-c-d Tania gestures the orthogonal line, makes a cross and points out the quadrants

5), it becomes apparent that Tania was aware of the perceptual structure of orthogonality earlier. In task 5, Tania checks whether the stick is placed correctly. She says: “if it goes here, and this,” she points at the touching point of the stick with the disk, “would be the center of the mirror, then it will look past it,” she gestures with her hand from the white tower to the place where she imagines the image would be, see figure 5a. She then turns the stick clockwise in the correct position and says, “if it would go here, then it would look in this way,” gesturing a line from the white over the touching point of the stick with the disk to the brown tower (figure 5b). Tania gestures a line orthogonal to the stick, physically representing the perceptual structure from task 4. Considering that verbalization of the structure is key in the process of abstraction, the tutor asks: “and why is it positioned correctly now? You can also look from above.” Note that here the tutor invites to change perspective. Tania

stands up, looks from above, “if the mirror would be straight,” she gestures with her left hand along the stick, “then you look at it like this,” and gestures a line from the white to the brown tower. Tania just gestured a cross! To let Tania verbalize this structure and relate this to an artifact, the tutor decides to ask, “do you recognize a shape, in what you just pointed to?” Tania gestures again over the line between the towers. The tutor asks, “together with the straight line of the mirror it forms a…?” When Tania does not respond the tutor hands a second stick to her, “maybe this could help?” Tania places the second stick on the line between the towers, see figure 5c, and immediately says, “a cross.” Tutor: “a cross. What is special about a cross?” Tania: “it has two straight lines.” The tutor places the second stick skewed and says, “but these are also two straight lines.” Tania gestures over the sticks and says, “now they are not equal, because,” she places the second stick again correctly, “if you place it here, then you have one-quarter here, there a quarter, there and there,” and touches each quadrant with her index finger, see figure 5d. Tutor: “ah, could we call this a straight cross?” Tania: “yes.”

Discussion

Our embodied design focuses on the fostering of student’s process of embodied abstraction related to reflection symmetry and led us to developing a sequence of tasks and the hypothesized learning process. Although the tasks successfully elicit actions and perceptions related to the concepts equidistance and orthogonality, the process of abstraction does not happen entirely by itself; as expected, careful fostering by the tutor is necessary for conceptualization (Abrahamson, 2014; Alberto et al., 2022). This encompasses re-arranging the action space (changing perspective or placement of objects), and bringing in artifacts (stick, words) to highlight perceived structures and elicit reflecting and verbalization. Incorporating these considerations into the embodied design improved students’ abstraction process. Although in tasks 1 and 2 students are able to position the towers correctly, this does not mean per se that equidistance is discovered, as Matt, who stayed in action-perspective, showed. More attention needs be paid to verbalizing the concept of equidistance. For this, it is helpful to direct the student’s attention to the relative position of the towers to the mirror, for example by letting them observe the action space from above. ‘How are the towers positioned?’, ‘what do you noticed how the towers are positioned to the mirror?’ could serve as accompanying questions. When this does not lead to relevant verbalization, measuring artifacts like a ruler, or a grid can be introduced. Regarding orthogonality, the design showed that most students grasp the idea of ‘opposite’ but struggle with the concept of orthogonality. Although students do not mention a perceptual structure when moving the mirror and the white tower, the performance in the subsequent task shows that such attentional anchors often were present, since in most cases a cross-like structure was indicated. Although verbalization often lacks, this does not mean that the actions performed did not bring about perceptual structuring of the action space in a new way. As the story of Tania showed, bringing in the concept of a cross by using artifacts such as a second stick, assists in the student progressing in the process of abstraction. In embodied abstraction structuring by the student is key for conceptualization. It is therefore crucial that the teacher elicits the perception and description of the inherent structures. Even when students’ actions lead to correct positioning, eliciting verbalization of the structure is crucial, as several stories in the above show. Overall, answering the research question, how can a non-digital action-based embodied design foster students' process of embodied abstraction regarding reflection symmetry, we highlight three new design heuristics for the HLT

relevant for the tutor. (1) Stimulate the changing of perspective on the restructured action space, (2) bringing in artifacts to depict the structure of action space and (3) elicit verbalization of the structures and conceptualization, even when students actions lead to successful accomplishments in the task. Incorporating these into the embodied design we believe improves on student's abstraction process.

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Factors influencing the implementation of active, bodily experience mathematics learning activities: the perspective of educators

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The study explores the implementation of active, bodily experience mathematics learning activities (ABM activities) within schools, focusing on factors that hinder or support teachers in integrating them. The exploratory research involves online interviews with selected mathematics education researchers from Australia and Italy. They identified, as determinant factors, individual characteristics of teachers, including their knowledge, beliefs, and openness to change, as well as contextual factors like school culture, availability of resources, class size, and organizational flexibility. Furthermore, limitations strictly inherent to the specific attributes of ABM activities have been identified. The results suggest the need for familiarization with embodied education and that promoting dedicated training courses, including reflective teaching practices, examples of activities aligned with curricular content learning, and direct observations, can facilitate the implementation.

Keywords: Embodied cognition, enactive learning, implementation research

Introduction

In recent decades, there has been a growing research interest in embodied education (Abrahamson et al., 2020). Even if a variety of perspectives and profound differences are recognized, the PME-46 working group (Palatnik et al., 2023) testifies to a nascent attempt to bring together research groups interested in the design of embodied mathematics learning activity to address common challenges with a collective effort. Hence, it seems the time is ripe to delve into how research in mathematics education can actively promote a wide integration of the embodied education perspective in schools. As expressively remarked by Felmer (2023), in its process of evolution, research in mathematics education should advocate for a change that seamlessly encompasses teaching-learning in schooling.

The direct involvement of teachers is essential to address the challenge of implementing in school contexts learning activities possibly consistent with the embodied education design (Abrahamson et al., 2020). Teachers play a pivotal role in spreading educational innovations in schools, acting both as promoters of their introduction and as primary users (Century & Cassata, 2016). Particularly, bridging the gap between research and practice requires investigating the factors that teachers perceive as barriers or facilitators for the implementation. To this end, a teacher survey was planned to mind the gap. However, the specificities of a teacher survey design (i.e., a questionnaire reaching a no-specific audience of teachers) do not allow to refer to a specific theoretical background (e.g. the enactive pedagogy, the inclusive materialism, the multimodal approaches, etc.). Therefore, this necessitated the involvement of researchers to identify an operational construct easily communicable with teachers, overcoming the theoretical differences within the research field. The resulting operative construct is the ABM activities (Active, Bodily experience Mathematics learning activities) (Boscolo, 2022), defined as activities that actively engage students in exploring mathematical concepts through manipulatives, tools (virtual or physical), or whole-body movements. This construct

encapsulates two primary components: students' active engagement in mathematical exploration and their perceptual-motor involvement as connected to mathematical meaning (Boscolo, 2023). Although this definition may appear trivial, it encompasses commonalities across various theoretical perspectives within the field of embodied cognition in mathematics education, with some limitations.

It is further relevant to characterize ABM activities at the research level. For this purpose, a selected group of researchers in mathematics education —whose interests are possibly related to the study topic and who had extended experience working with teachers—were interviewed. The objective of these interviews is twofold: first, to identify the inherent characteristics of ABM activities (including their importance, limitations, and core features) and of their implementation (e.g., teachers' necessary knowledge, awareness, and beliefs); second, to explore possible directions for investigating teachers' beliefs through a survey. It will be relevant to examine whether factors deemed critical by researchers are also considered significant by teachers, to inform the research efforts delve into implementation. Furthermore, researchers' perspectives may offer valuable insights into interpreting the survey results, especially in light of the specificities of the educational contexts under investigation. Indeed, cultural differences may play a role in shaping the conceptualization and implementation of ABM activities.

In light of this complexity, researchers from two distinct educational contexts, Italy and Australia, were involved. Their involvement may provide insights into cross-cultural and cultural-specific characteristics relevant to ABM activities conceptualization and implementation (Boscolo, 2023), and enhance the interpretation of survey results. The presented study focuses, in particular, on researchers' perspectives concerning factors that may influence ABM activities implementation. Specifically, it investigates both the limitations considered specific to ABM activities, along with factors that could hinder or support teachers in incorporating them into their practice. The following research questions guide the study: According to educators, what factors could inhibit or support teachers to incorporate ABM activities in their school practice? What are the main limitations of the integration and use of ABM activities in teaching practice?

From the meta-analysis offered by Century and Cassata (2016), the main typologies of factors considered for investigating the transfer of research findings in school environments are the following: 1. Characteristics of the individuals involved, which can concern students (e.g., school levels, motivations, knowledge) and teachers, including knowledge, experience, beliefs, values, attitude, motivation, and self-efficacy related to the change they face or psychological characteristics generally related to change (e.g., readiness, openness, and the perceived capability to handle change); 2. Contextual factors, encompassing organizational factors internal to the school (e.g., class size, available resources, spaces, lesson schedules), managerial factors (decision-making freedom for individual teachers), and factors determining the organizational culture (collective attitudes and behaviour within the school system); 3. Attributes of innovation, both features inherent to the proposal itself (e.g., complexity, objectives, empirical evidence, costs, design, specificity), or perceived by users (clarity, attractiveness, ease of use, familiarity, perceived relevance, perceived advantage over current practices). Some attributes are hybrid, like the adaptability of the proposal to circumstances; 4. Support strategies for implementation, which can vary from structural proposals to sporadic ones. Examples include operational plans integrating the proposal into practice, resource provision, training courses, mentoring periods, and sharing assessment processes; 5. Implementation

of the proposal over time, which involves assessing the implementation phases, adaptation times, the time before becoming potentially routine, and modifications over time.

The findings of this study highlight factors belonging to all these five dimensions.

Method

The exploratory study involves online semi-structured interviews with six selected mathematics education researchers, familiar with the Australian context, belonging to MERGA (Mathematics Education Research Group of Australasia) and nine Italian experts, seven academics and two teacher-researchers. Seven of them are members of the AIRDM (Italian Association of Researchers in Mathematics Education). They have experience in implementing innovation at school and research interests akin to the object under study. The selection process consisted of contacting researchers with these characteristics: the interviewees in the sample are the ones who voluntarily joint the project.

As part of a more extensive interview (see Boscolo, 2023), to understand what factors, according to researchers, might influence teachers to incorporate in their practice ABM activities, we asked them to answer the following questions: 1. *In your opinion, what are factors that could either inhibit or support a teacher to implement active, bodily experience mathematics learning activities in school?* 2. *What are the main limitations of the ABM activities integration and use in teaching practice?* Researchers' answers were transcribed and further analysed according to an inductive thematic content analysis methodology (Patton, 2002). The narrative materials were organized into units of meaning (i.e., core themes derived from the analysis of researchers' answers), encoded using MAXQDA software. The resulting framework was represented in maps (realized with XMap, MAXQDA Analytics Pro 2022) illustrating an overview of the totality and variety of researchers' opinions. Each node in the map corresponds to a code that signifies a unit of meaning. The numbers attached to each link indicate the cumulative frequency with which the core theme (represented by the code on the node) recurs in the answers, accounting for multiple mentions by interviewees. Further details on the data analysis and reliability process can be found in Boscolo (2023).

In the following section, two resulting maps and a narration with exemplary contributions that relate to these themes (attributed to the researchers with a pseudonym, e.g., AR1 for the first Australian one, and accompanied by the paragraph number of reference) showcase the main themes highlighted.

Results

Influencing factors

The concept map (Figure 1) summarizes the possible factors negatively influencing a teacher to integrate ABM activities into school practice, identified by the researchers interviewed. Hindering factors can be external to teachers (FOE, in darker yellow, at the top of the map in Fig. 1). They could be determined by the specific context, as organizational factors internal to the school, like desk disposition, "I learnt mathematics in classrooms where the desks were arranged in twos […] And I sat next to one person and we didn't move from that desk ever" (AR3, p. 72), or "how is the school timetable structured" (AR4, p. 42). However, researchers mainly referred to an unsupportive school culture (FOE6): "[…] the teachers feeling is they can't implement those kinds of activities, and that comes back to the culture of the school and the culture of their colleagues" (AR1, p. 62).

> Every school has its own version of what's expected from the teachers, what's expected from the students, and when you go and you want to do something different, it's like you've challenging the contract. (AR3, p. 46)

This contract involves often implicit norms that obstacle the implementation of such activities (FOE1), i.e., no written rules inconsistent with the principles of these activities: "[…] the teachers and the school beliefs about what a math lesson should look like, and what a math class should look like" (AR3, p. 106), "[…] based on students' beliefs which come from many years of experience of having been taught mathematics" (AR4, p. 28). Often, hostile cultures are related to insufficient institutional support (FOE2). As highlighted by multiple researchers, e.g., "[…] there are so many pressures on teachers' time to actually learn how to use and find out about the activities" (AR1, p. 34). This pressure encompasses wondering about time to cover the curriculum, and reach standard in the national assessment program, which disincentives these initiatives (FOE3): "[…] there's a lot of pressure on schools to improve performance on Naplan [National Assessment Program Literacy and Numeracy]. So, […] there can be an emphasis on the preparation for Naplan, rather than spending time developing students' conceptual understanding. (AR1, p. 34). Finally, lack of resources and classroom size (FOE4) is identified as one of the main hostile conditions: "there's limited resources or basically the organization of bigger classes to have each kid manipulating" (AR6, p. 27).

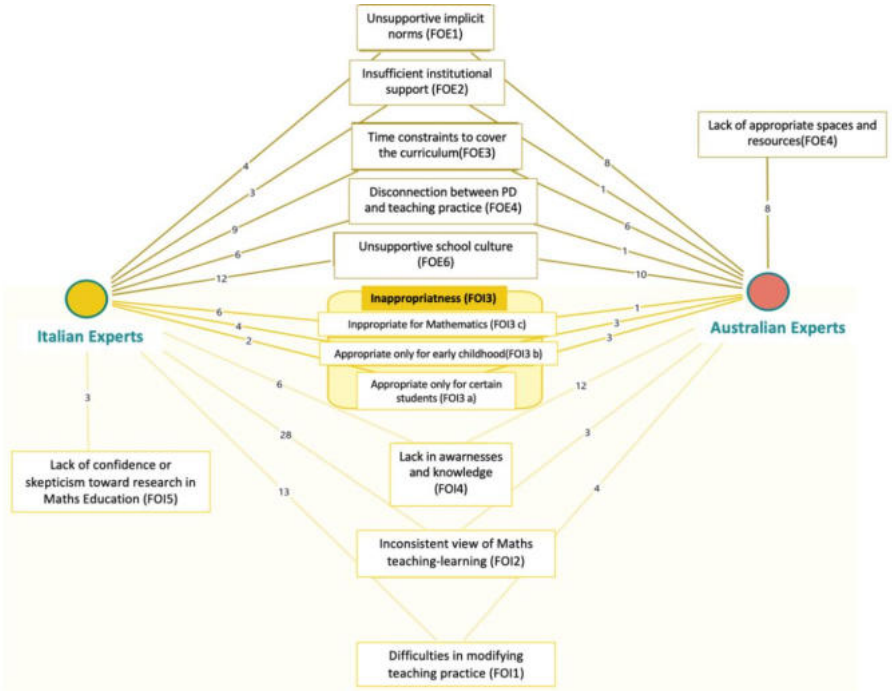

Figure 1. Overview of researchers' opinions on factors that could inhibit to implement ABM activities.

Otherwise, hindering factors can come from the individual characteristics of teachers like beliefs, awareness, and knowledge (FOI, in light yellow, in the lower part of the map in Figure 1). They may regard difficulties in modifying their own teaching practice (FOI1), in general, or specifically referring to the change requested for ABM implementation. Moreover, teachers may have a view of mathematics teaching-learning that is inconsistent with the one promoted through ABM activities (FOI2): "If you got a teacher who has strong beliefs that the way to learn mathematics is explicit

teaching and coughing examples, you probably not going to get them on board" (AR3, p. 80). Some researchers mentioned other specific beliefs: "You have to believe that all students can do mathematics. Not in the same way, not the same speed, not the same level, but everyone can do mathematics" (AR4, p. 34). Further, teachers may not have appropriate knowledge (FOI4), and this influences their teaching practice, "If you are not even a qualified mathematics teachers, then you are not going to or want to do anything out of the ordinary" (AR4, p. 40), particularly toward ABM activities implementation: "[…] teachers are open to embodiment, but they maybe lack the knowledge mathematically of how they might link that to the math lessons" (AR3, p. 112). Finally, they can perceive some limitations in the attributes of innovation itself (FOI3), such as they are only appropriate for lower grades students or they cannot fit with the mathematical contents teachers have to cover: "[…] like secondary teachers, they need to see that it's appropriate to use those materials and not only in primary school" (AR5, p. 30). Further, they can believe ABM activities are only appropriate for certain students, e.g., either lower or high-achieving students.

Among factors that possibly foster teachers to implement ABM activities, there are conditions and initiatives that may contribute to creating a supportive environment and providing guidance and resources (FFE). One is the promotion of PD (Professional Development) training courses (FFE5) which are not sporadic, in which examples and materials are provided but, otherwise, a reflection on the own teaching practices is promoted: "To truly impact the way teachers teach, the first step is to make them aware of how they teach" (IR8, p. 19). In particular, Italian researchers even indicated the importance of enhancing the experience of exchange between schools and universities, rewarding the teachers involved. At the organizational level, researchers mentioned the institutional support (FFE2) and it involves different roles in the school: "I think the role of the principal, head of department, the leadership in the school is very important. Because that establishes the culture about what is possible and what is not possible" (AR4, p. 40), "It's important to have some technical support and not all schools might have a person who does that job" (AR4, p. 50). Moreover, flexibility in the school's internal organization and collaboration with other colleagues is considered a high-impacting factor (FFE4). Further, researchers mentioned initiatives and experiences that can change teachers' individual characteristics (FFC), such as their beliefs towards ABM activities. For instance, showing the consistency with the curriculum and educational policies of such activities (FFC2) "In Australia, the biggest thing that enables teachers to try something is if they can connect it clearly to the curriculum" (AR3, p. 84), or:

> […] Helping teachers understand that what I'm proposing can actually replace some of the things you are doing, without losing anything. So, by doing things this way instead of some other things you will still be able to achieve - or the students will still be able to achieve - the learning outcomes in the curriculum (AR4, p. 34).

Moreover, especially Italian researchers emphasized the relevance of enhancing teachers' awareness of the complexity of the teaching-learning process and bringing research evidence to teachers. Even Australian researchers suggested providing evidence of the efficacy of ABM activities implementation toward a conceptual understanding of mathematics (FFC4): "[…] They actually have a valid reason for using them […] There is value in the material being able to be used for a purpose, to actually further student understanding that you couldn't easily achieve in the past" (AR5, p. 30).

Also, promoting initiatives to directly experience and observe the positive effect can be precious (FFC5): "Before they try an activity, they would like to know from another teacher 'how it works'. So that, it would be good if I can see the activity being implemented by another teacher" (AR4, p. 50).

The only two factors indicated as ambivalent are the availability in schools of appropriate spaces and resources (FA1), listed also among hostile factors when interviewees refer to as an obstacle, and, only by Italian researchers, the stability of teachers in relation to the opportunity to plan long-term (FA2).

Limitations

As Century and Cassata (2016) indicated, the implementation is mainly influenced by the specific *attributes of the innovation* itself. Among these, there are possible limitations.

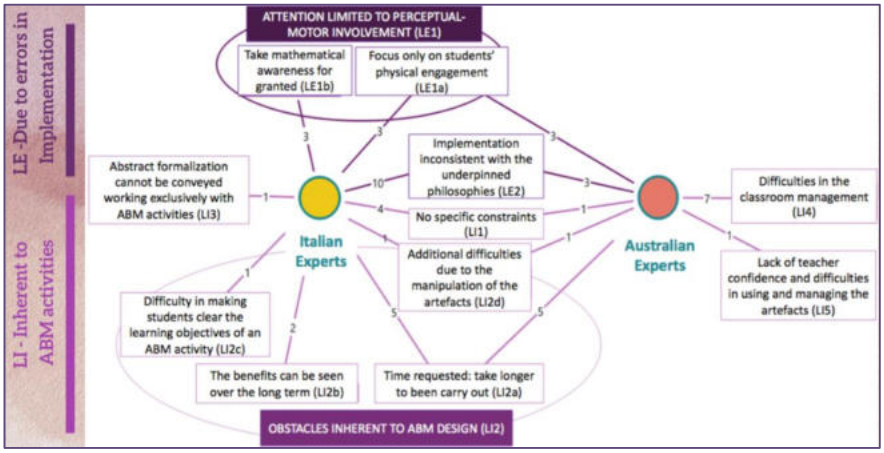

Figure 2. Overview of researchers' opinions on limitations for ABM activities implementation.

Both Italian and Australian researchers stated possible limitations due to an incorrect realization of ABM activities in the classroom (LE, in darker purple, on the upper part of the concept map in Figure 2). Those can be related to an inconsistent implementation, due to a distortion, on the part of the teacher, of the philosophy underlying these activities. That limitation is a lack of fidelity with respect to certain characteristics identified as essential components (Century & Cassata, 2016). It can result from a lack of awareness of the philosophy underpinning ABM activities design (such as an exploratory, inquiry-based learning approach, constructivist or socio-constructivist paradigm, ...) (LE2), and insufficient attention to link core mathematics meanings involved in the activity and the students' perceptual involvement. Indeed, referring to primary school, in particular, considering this proposal limited to the perceptual-motor aspects of student engagement, neglecting the bonding and awareness-building of mathematical knowledge (LE1), is considered a second major limitation. Other limitations are due to inherent characteristics of ABM activities (LI, in lighter purple, on the lower part of the map in Figure 2). Among the obstacles structurally characterizing the activity (LI2), the class time that needs to be invested in implementation (LI1 a) constitutes the most critical point according to the experts, while the difficulty in classroom management (LI4) comes up as a feature that seems to be of some concern, but on which the Australian experts focus particularly. Others can be students' difficulty in focusing on the learning goal (LI2c) and in manipulating artefacts (LI5). Especially the Italian researchers highlighted obstacles that may concern a lack of theoretical

systematization of mathematical knowledge (LI3). Finally, the advantages of spending this time may be seen over the long term (LI2b) and the schooling often requests an evaluation over a short period.

Conclusions and further directions

In the exploratory study, we have identified a hypothesis on the main factors that possibly influence Italian and Australian teachers for the implementation of ABM activities in schools, analysing the opinions of selected researchers in two different educational contexts.

Among the individual characteristics of teachers, knowledge and awareness of both mathematics and the principles of embodied education, as well as how to connect these two aspects, were highlighted. Additionally, the reflection on the own teaching practices and the openness to change were emphasized. Beliefs concerning the effectiveness of ABM activities, especially regarding curricular content learning, involving all students, are widely mentioned.

Researchers identified contextual factors, like class size, the availability of resources, and the flexibility of the school organization as extremely relevant. Other significant factors are supportive school culture, the presence of reference figures (both technical and managerial) to support teachers' initiatives, and the opportunity for dedicated Professional Development (PD) time, particularly focused on collaborative planning.

Concerning innovation attributes, besides the challenges related to having materials and tools available, along with technical support personnel, the need to familiarize teachers with the perspective of embodied education was stressed. Further than being appropriate for all students, the proposal should demonstrate its attractiveness by showcasing perceived advantages over current practices, making it appear integrated rather than an additional activity.

In terms of support strategies, researchers pointed out the necessity of offering specific PD training. This training should not be sporadic and should include reflection on one's teaching practice, accompanied by concrete examples of activities integrated into the curriculum. Teachers should have firsthand experience with these activities: e.g., directly observing their implementation in the classroom and engaging in reflection sessions with more experienced teachers.

Regarding the implementation over time, the period of familiarization with this teaching approach is considered a challenging initial investment. Some of these factors may be generally involved in the implementation of educational innovations, while others might be more specifically related to the ABM activities, e.g. limitations concerning attributes of the innovation. However, when considering the implementation, both specific and non-specific influencing factors need to be taken into account.

Some needs highlighted have already been considered by Palatnik and colleagues propose (2023), e.g., the creation of a resource database. Others may give new insight to researchers in the field of embodied education, who are delving into the diffusion in schools of the activities designed from this perspective. Through researchers' answers, many features of ABM activities conceptualization emerged: e.g., they are relevant for inclusion and appropriate for all school grades, they require much time to be implemented and planned, the benefits can mainly be seen in long-term period and for conceptual understanding. Further, there are some assumptions concerning their implementation: for instance, it is supposed that their implementation implied a disruptive change in how teaching-

learning maths is generally conceptualized in schools. Moreover, the issue of the lack of fidelity, i.e., a potential misalignment in ABM activities implementation, prompts us to investigate how such activities should be designed to ensure consistence with actual practice.

Considering groups of experts from different Countries can provide insights into contextual and cross-cultural factors, both shedding light on features of educational environments immersed in different cultures and aiding in decision-making for interventions aimed at fostering change in schools. Considering the two groups involved, greater attention to the exchange between universities and schools, as a key component for fostering implementation, emerges in the Italian context, while Australians highlighted to link these activities with the curriculum demand. Looking at attributes of innovation, while Australian researchers emphasized the issue of classroom management, the Italians focused on teachers' awareness of how to link mathematical knowledge to ABM activities. However, since the study is exploratory, the insights provided may only represent initial hypotheses. Furthermore, the overall framework mixes together the interviewees' different theoretical perspectives influencing their answers; therefore, it is strictly dependent on the researchers involved.

A further direction may be to compare the findings with the ones that may emerge from other groups of experts and in different countries, which may complete and differentiate the overall framework. Relating these findings with the results of a teacher survey, built upon the hypothesized influencing factors, is the natural next step.

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Concepts and Bodies: Creating together in affective bodying

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The present paper discusses the importance of making space and time for affective bodying in mathematics education so that to open conditions for concepts and bodies to engage together and for sad affects to be affirmed as core for the learning experience. Affective bodying allows to explore our capacity to act together in-common and for our commons by moving together with our bodies, languages and minds toward exploring ways to sense and express thinking and feeling. It can be a response to oppressive learning experiences with mathematical concepts in early year teacher education. With affective bodying, the research aims to move beyond the deadening bipolar discourses of children failing and succeeding in mathematics -a dichotomy that perpetuates alienation from the materiality of knowledge. Instead, the paper argues for the need to affirm creative experimentation in affective bodying with mathematical concepts, bodies and difference.

Keywords: affect, affective-bodying, failure-and-success, mathematical concepts, teacher education

How (not) to create a concept?

> How can you teach 'area' to so young children!? (…) This class has many Roma children and children with special needs... You cannot do normal math with them.
>
> **Schoolteacher addressing three female student-teachers, January 2010*

> We cannot continue... We are afraid that we will fail.
>
> **Student-teachers addressing their educator after meeting the schoolteacher, January 2010*

Figure 1. Young children experiment with the concept of area

The concept of 'area' is created out of cultural, communal, earthly and collective knowledge experiences around problems focusing on how a certain surface (of two, three or n-dimensions) can be studied, compared and calculated. Historically, the idea of area sparkled within the social needs of

addressing the just distribution or redistribution of land when, for example, flooding events changed or destroyed cartographic boundaries at the banks of the Nile, Tigris and Euphrates rivers. Although, the concept of 'area' started by focusing on questions concerning the earth's de/re/formation, it has proceeded toward mapping and studying ways of recording and measuring the qualitative and quantitative characteristics of geometric and topological surfaces. Thus, the art and craft of geometry and topology is grounded in embodied contact, understanding and exploring issues that directly concern our bodying with soil, water, air, cultivation, sharing, but also the commonly shared as practical or metaphysic inquiry with the concrete and abstract nature of concepts such as unit, length, perimeter, radius, area, volume, etc. Through critical problematizations, such embodied experiences create conditions for the invention of appropriate artifacts, tools, concepts and techniques, facilitating calculations in direct relation to questions of measurement, sometimes by means of estimation and other times with the need for pursuing a relative accuracy, for detecting potential limits, for proving, and for arguing the value of utopian quests or the truth of answers. For a detailed discussion of measurement as relational, intensive and analogical see also de Freitas and Sinclair (2020).

However, despite the importance of a physical contract (Serres, 1990/1998) where geometric concepts are included and studied through bodily relations, today's curricula and textbooks accelerate the rigorous foundation of geometry by examining concepts such as 'area' through abstract geometric and algebraic formulas tied with the principles, values and aspirations of a royal science as they emphasize axiomatic definitions, accuracy of descriptions and validity of proofs (Deleuze and Guattari, 1994). At the same time, certain pedagogic and didactic interventions for young children have appropriated, for more than five decades now, the developmental logico-mathematical thinking offered by Jean Piaget (1896-1980) where the possibility of concept creation is related with specific stages of developing logical and mathematical behavior and is tied with children's mental, moral and health development. It is argued that the transition from one stage to another requires maturity (or child-readiness) which is recognized through children's capacity to coordinate bodily kinesthetic movement, language-use and moral behavior in appropriately organized environments.

Despite critique (Donaldson, 1978; Walkerdine, 1990), the influence of such research perspectives on educational policy and reform has been decisive toward determining a rhetorical relationship of 'readiness' between children and mathematics in the field of early mathematics education in linearly structured stages where children develop their relation to number and space by developing maturity from sensory motor to language related activity. Specifically, the relation to geometry thinking is seen, by and large, as developing around the van Hiele learning stages of: supervisory perception, analysis of forms and relationships, formal reasoning and inductive rigor (Pegg, 2014). When such cognitive stages serve mainly the intended curriculum, mathematics learning becomes recognized through discourses of readiness in relation to age, ability, and maturity. Moreover, it cannot accept the virtual/actual potentialities of chaotic passages where orality (reasoning as verbal use of mathematical register) strives to function through body movement. Such agonistic strivings expressed by children as they attempt to transverse learning stages need to be recognized as opening up emancipatory lines of flight (Deleuze & Guattari, 1994).

The formal teaching and learning of mathematical concepts run the risk to lose their connection with the limits and affordances presented by the natural and social environment and, together, to lose their

ability to be read culturally, socially and politically as Michel Serres (1990) urges. At the same time, we, as educators, researchers, and learners, lose the oortunity to get in touch with conceptual creation processes that arise effortlessly through the unexpected corporeal re/connection with different fields of knowledge releasing new life-giving forces. In the field of education, including mathematics education, the biopolitical colonization of mind and body dominates through a psycho-cognitive culture of developmental stages that prescribes what might be appropriate mathematics teaching. And formal curricula tend to either silence difference amongst children, student-teachers and teachers. This is where alienation resides as disconnection from concept creation and as fear of difference perpetuating the fear of failure. It is, thus, important to explore a return to minor mathematics as also discussed by de Freitas and Sinclair (2014).

Fear of Failure: Dis/Connection from concept creation

The fear of failure expressed by student-teachers, teachers, parents and, ultimately, all participants in the education of young children becomes visible in the two quoted passages at the paper entry. The nursery schoolteacher is critical of the student-teachers's desire to experiment with a material-discursive apparatus for the teaching and learning of area. In this, the concept of area is introduced *first* through the known story of the three little pigs where the floors in rooms of their pig-house need restoration and they must calculate the number of tiles needed and *second* though material models of room floors and tiles. The apparatus design goes beyond the didactic norm as it requires the capacity to transverse motor and mental functions such as the filling up of given model-shapes with smaller tiles of equal shape, counting the tiles and comparing the results. When the student-teachers presented the schoolteacher these ideas and before, even, trying with children, she said 'you cannot do normal math with them' (see entry and Figure 1). The student-teachers felt unable to defend the desire to act experimentally with children. As they encounter the schoolteacher, they confront the authority of curricula repeated in the normative discourse of what children could and should do. Thus, fear takes over their bodies, prompting them to resign. What is, then, fear and how does it move so comfortably among us? Gabriel Tarde (1843-1904) describes the body's submission to fear as follows:

> The momentary paralysis of spirit, language and limbs, this deep agitation that descends to the core of our being, this deprivation of self that we call bullying (...) It is a nascent social condition that occurs every time we pass from one society to another. (quoted in Massumi, 2005, p. 31).

In a similar way, the schoolteacher mediates to the student-teachers her embodied fear of a possible failure involved in the timid passage from a cultural situation dominated by discourses of learning bounded to strictly defined developmental stages to another situation that strives to open toward the indeterminacy of learning and its relation to thinking as feeling. It is there, where the need to create new concepts becomes grounded. In these passages, a dense articulation of knowledge/power is experienced which does not hesitate to humiliate itself for safeguarding the established order. It is perceived physically as "*the unfolding reality of an emotion of fear"* (Massumi 2005, p. 38), making resistance difficult. Here, we must ask: Is there a way out of such an oppression creating alienation and disconnection from concept creation? How could one re/connect with concept creation?

Affective Bodying: Re/Connection with concept creation

The mind-body distinction can be traced back to the seventeenth-century philosopher Descartes (1596-1650) who argues that the mind can exist without the body (I think, therefore I am). Cartesian dualism requires the unruly emotional body to submit to the rational mind which becomes a vehicle for disciplined thinking, reliable knowledge and truth (Burkitt, 1999). Today, post/Marxist feminist approaches focus on the body as the primary tool of capitalist economy to increase production and productivity in the workplace where the female body (of teachers and student teachers) experiences the precariousness of unpaid reproductive labor, vulnerability, and violent behavior in care sectors. Furthermore, the Foucauldian genealogies for sexuality, the prison system, and the clinic recognize the biopolitical inscription of legislative, cultural, and pedagogical norms on the body itself (Martin et.al., 1988). The body is increasingly becoming the product of bio/psycho/socio/technological processes mediating current norms in medicine, psychiatry, biology, technology, administration, and education capturing the self-regulated, self-controlled, and self-disciplined subject. As these norms are mainly supported by mathematics, artificial intelligence, and digital media, it is crucial to ask the question: How can we envisage the mind-body assemblage and the body itself beyond learning norms that tend to reproduce dichotomies, hierarchies, oppression, and fear?

Baruch Spinoza (1632-1677), at about the same time as Descartes, denounced the mind-body distinction, proposing a return to viewing the body as an assembly of elementary units intertwined in trajectories of motion that mobilize both thought and emotion. There is no thought and emotion without the body and its movement in the world, argues Spinoza in Ethics, giving precedence to affect through beneficial reconnections of bodies and as Deleuze explains:

> A body, of any kind, is defined by Spinoza in two simultaneous ways. First, a body, however small, consists of an infinite number of particles, it is the relations of movement and rest, velocity and slowness between particles that define a body, the individuality of the body. Second, a body affects other bodies or is influenced by other bodies, it is this ability to influence and be influenced that also defines a body in its individuality (Deleuze, 2005, p. 58).

A crucial issue here concerns how the body's difficulty in performing the norm and/or reacting to oppressive normality can be reconsidered as a fluid continuum of body-concept-movement-world-thought-affects? Brian Massumi (2005) distinguishes between affect and emotion arguing that whilst emotion signifies affect in a social structure, affect is a pre-conscious autonomous intensity of the body supporting to appreciate the non-rational aspects of sociopolitical phenomena despite critics who call for bridging the affect/emotion divide. Here, the artistic and philosophical work of Erin Manning and Brian Massumi (2014) responds to Spinoza's exhortation *"We don't know what a body can do!"* and inquire affect through the creative process of an open experimentation that emphasizes the movement of bodies along with concepts. By asking "*What else can a body do?"* they create moving practices such as improvised choreographies that are directed toward processes of exploration and co-synthesis of the empty spaces inhabited between bodies and concepts. These processes highlight how the body remembers, expands its boundaries, realizes its limits, and allows to become different. They involve a kinesthetic slowing down of body movement along with concept movement aiming for the aesthetic expression and the political mobilization of thought itself.

The above theorizing has motivated me, in my teacher-education work, to organize a choreographic workshop that diffracts the student-teachers' school experience where their material/discursive apparatus design on area was enacted. Even though the young children were engaged playfully, and the activity was evaluated as success, the student-teachers stayed with the fear of failure. Thus, the choreographic workshop aimed for memory work with the material/discursive components of their experience. Together with student teachers we worked choreographically (for more details see Chronaki, 2019) with our collective memories of feeling and thinking in the early years classroom. These memories were unfolded as affective bodying with moments of teaching the concept of area with the children. The student-teachers worked in pairs and in the whole group with choreography, video and geometry techniques allowing for a caring return to school-time and go deeper into experiences with the mathematical concept itself.

For example, while working with the mathematical concept of area they moved from the apparatus of floors and tiles toward their own bodies attending the very act of touching, sensing, feeling. By using the palm of their hand as a kinesthetic tool or a measuring they explored a specific surface. The palm as a tool (or unit) goes beyond the normative practice of area calculation working as minor mathematics. Their contact with the concept of area returns to a physical experience and its choreographic performance sets the concept of area in movement and e-motion as it mobilizes the bodies through a return to sensing specific relations and correlations felt during school-time. This, potentially, expands towards a joint creative synthesis and redistribution of the sensible by means of the concept of 'area' as affective bodying moving them beyond the 'sad affect' of the fear of failure. With Massumi (2005), this is a move toward experiencing e-motion (i.e. setting the emotion of fear of failure in motion) and concentrating more on the movement itself and how it motivates the in-between bodies assemblage to be set in motion. The emphasis on working with concept creation as an affective process where bodies-concepts-movements come together is what has been discussed as affective bodying in Chronaki (2019).

Figure 2. The choreographing workshop after the area classroom activity with children

As already discussed in Chronaki (2019) when the bodies come together in these choreographies, they act beyond a tool for communicative clarity and representational accuracy of the concept of area.

Instead, they consist of an assemblage that disrupts the force of developmental models that tend to determine what (who or when) are the bodies able to perform and/or fail to cognitively perform the concept. They, instead, focus on how else the bodies could act and enact together by encountering the mathematical concepts and, especially, their anarchic and chaotic motion trajectories moving beyond normative movements of thought expression. This condition stands in contrast to an axiomatic logic of mathematics that subjects the mind with discourses of precise definitions, conjectures and hypothesis linked with truthful theorems and modelling application. In contrast, it opens toward the anarchic potential of diverse bodies sensing together the concept otherwise through the modest act of moving and producing a novel politico-aesthetic redistribution of what is taken as sensible.

Concept Inquiry in Affective Bodying: *Beyond Curricula Discourses of Failure-Success*

Through this choreographic workshop experience, the student teachers become re/connected with the concept of area and its diverse expressions through a novel material/discursive relation with their bodies producing a political and aesthetic inquiry of the concept of area and its elements. Their thinking-feeling of area is being expressed in the following excerpts (Student teacher 1 and 2) but also in the videography and the series of photographs in Figure 2.

> *Movement, movement of the body greatly affects conception, understanding of the concept (...). Through the sense of touch we come into direct contact with the objects we are* called *to imagine (...). Following the outline of the tiles we realize their existence and realize the space they occupy.* (Student-teacher 1, diary notes, February 2010).

> *A key element in measuring a surface was the sense of touch. Through touch we could perceive the boundaries of any place. From floor to ceiling. From one side of the room to the other. And from our feet to our heads. In the end, we were able to define the outline (...) using small pieces of paper that provided opportunities through the sense of seeing and comparing different shapes* (Student-teacher 2, diary notes, February 2010).

The above excerpts note how the choreographic workshop of affective bodying focuses directly on performing the political and aesthetic aspects of mathematical concept inquiry through memory work in school-time and improvisation in choreography movement. The affective bodying assembles the process of studying the teaching and learning of mathematics by, strategically, en/countering the failure-success dipolar discourse. As such, it aims to counter the fear of failure. Both failure and success as teleological positions for learning become un-important. As the political and aesthetic dimension of an affective bodying with concepts and difference opens, the mathematical inquiry expands through the continuum of body-concept-movement-world-thought-affects. As already explained before: This process involves a kinesthetic slowing down of feeling and thinking as a synchronous movement of bodies and concepts aiming together for aesthetic expression and political mobilization of mathematical thought that counters the oppression embodied through institutional bodies of formal and informal educational structures.

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Designing a tangible manipulative as a socio-material process integrating empathic and mathematical imagination

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This paper analyses the design process of a prospective teacher in a course focused on creating tangible manipulatives for mathematical learning. Building on considerations of empathy in design and on material frameworks of mathematical activity, we aim at shifting from an individual to a socio-material perspective on the design process. Using the prospective teacher's design journal, we analyse how empathy and imagination partake to the design phases. When defining the design problem, lived experiences of teaching ground the manipulative (empathy); when ideating solutions, possible learners' actions are entertained to evaluate manipulatives (empathic imagination); when prototyping a solution, new body-tool-symbol interactions are entertained (mathematical imagination). From the analysis, we finally highlight design drawings as sites of socio-material mathematics.

Keywords: Design, manipulatives, empathy, imagination, number concepts.

Problematising the design of tangible manipulatives

Our paper explores the working group's ongoing reflection on the material and embodied nature of mathematics, focusing on material creation in mathematics teacher education. We explore how designing a tangible manipulative can be approached as a socio-material process integrating empathy and imagination. Let us first situate the study with respect to its context and to the literature on designing tangible manipulatives.

The design process took place at Laval University (Canada), within a secondary school mathematics education program. This four-year program culminates with a course where students analyse their journey and outline short- and medium-term plans for ongoing professional development through various cultural activities (museum visits, book fairs, cinema, etc.). As part of this course, they also conduct investigative work on a question related to the teaching and learning of mathematics. In 2023, we explored with five students the possibility of engaging in a design activity: creating a tangible manipulative for learning mathematics, based on an analysis of a specific mathematical content.

As part of an appraisal and perspectives course, designing a tangible manipulative aimed to present teaching as creating resources, rather than implementing existing ones (Akuom & Greenstein, 2022). This aim can be situated in the field of research on the design of tangible manipulatives. In a review of this emerging field (17 articles, 7 of which from the same team), Stigberg (2022) found that most research takes place in the context of teacher education, focusing on the knowledge prospective teachers leverage in design settings (p. 7). The theoretical grounds of such research are oriented towards learning, especially constructionism with roots in constructivism (Papert & Harel, 1991) and Learning by Design (Koehler & Mishra, 2005) (Stigberg, 2022, p. 8). Frequently, designing a tangible

manipulative is studied as solving a teaching and learning problem, by "embedding" curricular and pedagogical knowledge into an artefact (Akuom & Greenstein, 2022).

In this paper, we aim to broaden the frame of designing a tangible manipulative, shifting the focus from an individual endeavour of "transforming" ideas into artefacts (Akuom & Greenstein, 2022) to a social and material process. The previously mentioned Learning by Design approach (Koehler & Mishra, 2005) lies in the research field connecting *design thinking* to teaching (Henriksen et al., 2020). Inherent to this approach is the consideration of "teachers as designers", as professionals faced to creatively solve open-ended problems, aimed at addressing others' needs (ibid., p. 213). Thus, a core feature of "teaching as designing" is the consideration of learners' needs across the design process. We find this social dimension lacking in research on the design of tangible manipulatives, focusing on "describing learning from an individual perspective" (Stigberg, 2022, p. 7). Moreover, we are concerned by the polarization of the design process around an immaterial "idea", transformed into a material "artefact". First, because the consideration of others (learners) does not seem to settle into these two poles, but to pervade the whole design process. Second, because this polarization risks locating mathematics in an immaterial human mind, as "ideas" pre-existing manipulatives and outliving them. Thus, we strive to study the design of a tangible manipulative as a social and material process, investigating how these dimensions might be entangled.

More specifically, we find resonance with Henriksen et al. (2020) design thinking model for teacher education, including *empathy* as a first phase: "empathize, define, ideate, prototype, and test". The process of "understanding" learners' needs is deemed crucial and challenging (p. 215) and invites researchers to study the influence of "empathy strategies" (as observations, interviews) on the design process of prospective teachers (Chen et al., 2023). Building on these considerations, we aim to understand how prospective teachers empathize with learners in the design of tangible manipulatives. In other words, how do they "think and deal with problems from the perspective of others, experience their emotions and thoughts, and understand their positions and feelings" (Chen et al., 2023, p. 471, as they refer to "human-driven innovation")? Let us ground this aim on existing theoretical ideas.

Theoretical background

As a first step, we define the "design process" according to Henriksen et al. (2020) five phases. The first phase is "empathising", as relating to the people involved with the created product. This might be accomplished by several strategies, as observing people's behaviour, interviewing them about their perspectives and needs, or thinking about an experience from their perspective. The second phase is "defining", as synthesising the diverse perspectives into a defined problem to be addressed with the creation. Third is the "ideating" phase, as envisioning and evaluating potential solutions to the defined problem. From there, the fourth phase, "prototyping", describes the choice of a solution and its creation. The last phase is "testing", as trying out the prototype to inform future modifications.

Having detailed the process of designing, we now deepen the question of empathy. Framing empathy as an embodied and material phenomenon gains interest in studies of social cognition. On the one hand, based on neurocognitive data, social behaviour can be framed as an intercorporeal process, involving a "second-person perspective" on others through "embodied simulation" (Gallese, 2014). Experiencing otherness, from this perspective, can happen by activating neural circuits to simulate

others' physical states. Empathising would not only involve explaining others' behaviour (taking a "third-person perspective" on them, Gallese, 2014) but also simulating their motor goals through neural activation. These mechanisms of simulation deploy when observing others' actions and may also be triggered when "imagining" oneself or others performing actions (Alibali & Nathan, 2018).

On the other hand, investigating empathy through the neuronal activity of the designing pre-service teacher would risk centring back the design process on the individual[1]. Our aim is to understand how the perspective of learners is involved in the process rather than in the individual brain. Nemirovsky and Ferrara (2009) put forth that grasping others' behaviour also relies on being aware of the culture, norms, immediate past and goals it relates to (their "lived context/culture and surroundings", p. 163). Thus, *empathy* might intervene in the design process of a tangible manipulative through contextual elements grounding its creation, attempting to grasp the learners' perspective. These elements might stem from observations, direct interactions with students, or indirect accounts of their behaviour from other actors in education (as in the empathising stage of Henriksen et al., 2020).

This consideration of the social and the material through empathy is still close to the analyses of Akuom and Greenstein (2022): the design appears as a creation process grounded in learners' needs. However, the design process not only involves defining a problem, but creating a solution to it. When designing manipulatives, the material process of design is not only social when "understanding" the learners' needs, but also when "dealing with" them (Chen et al., 2023, p. 471). There is an interaction of empathy with the material process of designing manipulatives, beyond grounding a handleable object into a social problem. The defined problem triggers and determines a material creation, as much as the existing and the possible material creations trigger and determine the solution. The social dimension of the creation process is entangled with its materiality. This is cyclically experienced by designers through a "reflective conversation" with the materials (Schön, 1992, p. 5).

We find *imagination* (Nemirovsky & Ferrara, 2009) to account for this creative interplay between the social and material dimensions of designing manipulatives. "Imagining", as the authors propose, is "entertaining possibilities for action; entertaining (in the sense of 'holding' or 'keeping') a state of readiness for the enactment of possible actions" (p. 159). In this perspective, manipulatives do not embody mathematical ideas, but "expand the possibilities for action students can entertain" (p. 173). When manipulatives are perceived in terms of possibilities for learners' actions, their evaluation and creation may involve imagination. Thus, the "ideating" phase of design may involve *empathic imagination*, as entertaining possibilities for learners' actions for evaluating potential solutions. In turn, the "prototyping" phase may involve *mathematical imagination*, as entertaining possibilities for actions that constitute, from the perspective of the pre-service teacher, "new ways of moving with the tool as well as new mathematical relations" (Ferrari & Ferrara, 2023, p. 7).

Given these theoretical groundings, our research question reads: How are empathy, empathic imagination and mathematical imagination involved in the design of a tangible manipulative?

[1] A similar risk is highlighted by de Freitas and Sinclair (2014), in the context of accounting for the role of the body in learning (pp. 19-20).

Analysis of a prospective teacher's design of a manipulative

The analysis tracks instances of empathy, empathic and mathematical imaginations in the design process of Camille, a prospective teacher engaged in designing a tangible manipulative. Her journal constitutes our analysis material. As part of the course, the prospective teachers were required to keep a journal recording their design experience. The purpose was to gather information on the design process to understand the project's evolution, the questions and challenges faced, how they were addressed and any collaboration that occurred, as it was highly recommended. The layout of the journal differed from one student to another. Nonetheless, it encompassed both textual (reflections, summaries, syntheses, work notes) and pictorial elements (drawings, diagrams, screenshots and photos). In Camille's case, we analyse a 35-page document including "Annexes 1 and 2" and a 23-slide PowerPoint that documents her work with a colleague, named "Annexe 3".

The analysis is guided by the theoretical framework. First, in the "defining" phase of the design process, we identify instances of *empathy* when Camille relies on contextual elements to account for the learners' perspective. Second, in the "ideating" phase, we conceive as *empathic imagination* when the prospective teacher entertains actions students can explore with different tangible manipulatives. Third, Ferrara and Ferrari (2023) highlight the materiality of *mathematical imagination*, as new ways of coordinating the bodies and the tools. Thus, to analyse mathematical imagination, we analyse how, in the "prototyping" phase, body-tool interactions relate in new ways to mathematical relations.

Defining phase: empathy as grounding the manipulative in contextual learning needs. The design process starts with the analysis of a mathematical content deemed challenging by Camille. She chooses integers, according to reasons underlined in her journal:

> I decided to explore this concept further since **operations with negative numbers are a challenge for students, according to my observations**, even in secondary school. I am particularly thinking about the famous "two negatives make one positive" or the distributive property of subtraction in algebra. Many rules concerning negative numbers are applied as if they were recipes or algorithms, rather than being rooted in understanding. **My experiences during internships and tutoring have revealed that students often confuse the different meanings of the "-"** (the sign of the number, the operator, the opposite). Therefore, **I hope to design a manipulative that will help clarify the various meanings of "-"** (Journal, p. 4).

Relying on observations from teaching internship and tutoring experiences, Camille defines the problem she perceives for the learners. She gradually precises it, from operations with negative numbers being "a challenge for learners", to operative rules conceived as "recipes or algorithms", finally centring on the "confusion" between "the different meanings" of the "-" sign. We see here accounts of the learners' needs, resembling those produced by an empathy strategy advocated by Henriksen et al. (2020) and Chen et al. (2023): observing the learners in their own environment (here school and tutoring sessions). We note that this observation is active, as Camille was involved in these teaching situations, so the problem is also defined through her interactions. These experiences construct what she perceived as the learners' needs, situated in a specific culture (mathematics teaching in primary and secondary school of Québec) and in related norms (operative rules applied as "recipes"). Thus, the material process of creating the manipulative is grounded in empathy, in the sense that the design problem is defined in the "lived context/culture and surroundings" (Nemirovsky

& Ferrara, 2009) of learners. Contextual social challenges determine the "hope" to help learners understand the unary (sign of a number), binary (subtraction operator) and symmetric (sign of being opposite) aspects of the "-" sign.

Ideating phase: empathic imagination as comparing and evaluating manipulatives. Camille then compares different manipulatives related to integers and evaluate them as potential solutions to the defined problem. These manipulatives correspond to existing "models" of integers found in research articles (e.g. Vig et al., 2014): the "thermometer", the "two-colour" or "double-sided tokens", the "number line" and the "hot-air balloon" (Journal, p. 5). For each model, Camille entertains possibilities of actions with it. These possibilities are entertained to address the problem and especially to anticipate learners' difficulties. Regarding the two-colour token model, she writes:

> Students understand that a red token cancels out a blue token (and therefore 1 + (-1) equals zero) after **overcoming the difficulty that zero can be represented by multiple tokens**, which is not the case with a collection of objects (represented by a natural number). The tokens thus allow to work on the concept of opposites. However, the problem arises when solving operations such as 3 – 5. Here, it is necessary to add pairs of tokens (1 red, 1 blue) until it is possible to remove 5 blue tokens. However, this adaptation does not necessarily correspond to the mathematical process involved in this operation. Here, **the adaptation would be seen by the students as an algorithm and would not necessarily promote understanding** (Journal, p. 6, we underline).

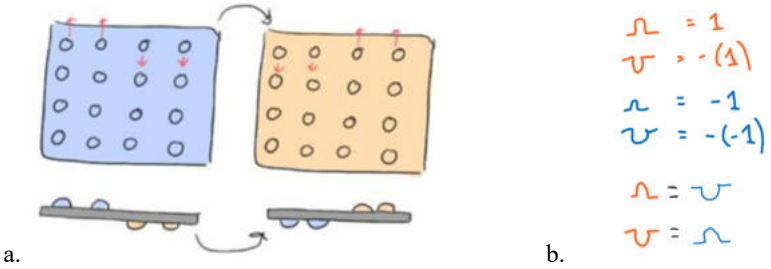
Several possibilities of actions are entertained: "cancelling out" a red token with a blue one, "representing" zero and "solving operations" with tokens. Each time, these possibilities are put in the perspective of learners, anticipating the potential difficulties they involve. First, the "cancelling out" and "representing" actions may conflict with a previous definition of numbers as cardinals of sets. Zero becomes not only associated with the absence of objects, but with pairs of objects that "cancel out". Second, provided this difficulty is overcome, "solving" a subtraction with the tokens may conflict with the "mathematical process involved". It might be that the strategy of adding "cancelling pairs" for performing 3 – 5 is deemed unlikely to align with future written computations (as it would correspond to 3 – 5 = 3 – 5 + (2 – 2)).

We see here an entanglement of the social and the material dimensions of the design process, which we termed empathic imagination. In this ideation phase, the social intention of addressing learning challenges takes the form of entertaining possible actions with manipulatives. Conversely, entertaining possible actions exist as an evaluation of manipulatives with respect to the learning challenge. This seems to be an instance of imagining as "juxtaposing" partial viewpoints of the same situation, that Nemirovsky and Ferrara (2009) analyse in the case of mathematical imagination. Indeed, in the above paragraph, the perspective of the learner, entertaining actions, is juxtaposed with the prospective teacher's perspective, anticipating difficulties for learning.

Prototyping phase: mathematical imagination as entertaining new body-tool interactions. Entertaining possible actions from the perspective of learners allows Camille to choose a solution for the design, as she brings possibilities of actions that avoid the pitfalls of existing ones. Camille names the manipulative "Numeric Pop-it", drawing inspiration from the fidget toy. In her Journal and in

Annexe 3—created when reflecting with Simone, a fellow prospective teacher—she outlines what they consider as the foundation of their solution (see also Figure 1a):

> Pushing down a "pop" is equivalent to subtracting a value. For example, if it's orange, it corresponds to subtracting a positive integer (pressing two orange "pops" represents – 2). On the other side, the same number is represented: two blue "popped" out, which implies +(-2). Its main advantage over other models comes from the fact that it's possible to visually understand the concept of opposites when flipping the Pop-it, for example, 6 = -(-6). It therefore provides a visual aid to the symmetrical meaning of the minus sign, making it easier to understand without going through subtraction's binary meaning (Annexe 3, slide 4).



a. b.

Figure 1: a. Drawing of the Numeric Pop-it (Annexe 3); b. Correspondences between actions and mathematical symbols (Journal, p. 11)

As one of their "desires for the manipulative" they add: "the movements/interactions directly engage with the concept at play" (Annexe 3, slide 5). It might be that, compared to the previous manipulatives, both the "binary" and "symmetric" meanings of the "-" sign involve the body-tool interaction of opposite displacements. Indeed, from any side of the manipulative, pulling and pushing a button constitute the "subtraction's binary meaning", by respectively adding and subtracting units. Moreover, as the two sides of the manipulative are connected, pushing a button from one side is equivalent to pulling a button from the other, such that adding a unit from one side (say 1) is equivalent to subtracting a unit from the other (here (-1)). Thus, the actions of pushing and pulling buttons and of "flipping" the plate also constitute the "symmetric" meaning of the "-" sign.

These body-tool interactions emerge as new ways of constituting relations between integers, involving mathematical imagination. Further, the new interactions rely on a "juxtaposition" (Nemirovsky & Ferrara, 2009) of different spatial and temporal perspectives. As shown in Figure 1a, the juxtaposition of front and top views of the manipulative constitutes the opposite relationship between adding and subtracting a unit (pulling and pushing). Similarly, juxtaposing views of the blue and orange sides constitutes the equivalence between actions performed on both sides (pulling a blue button and pushing an orange one). Importantly, empathic imagination is also at play, as new body-tool interactions are entertained as solutions to the defined learning problem.

We can develop how body-tool interactions here constitute new mathematical relations. The initial problem was strongly related to the symbol "-". The concept of Symbolically Structured Environment (Coles & Sinclair, 2019) usefully describes how mathematical relations would emerge in the use of this manipulative. It frames how mathematical relations can emerge in a mutual structuring of body-tool dynamics and mathematical symbols. Figure 1b, produced by the prospective teachers, shows how body-tool interactions directly relate to the binary (first four correspondences), and symmetric

(last two correspondences) aspects of the "-" symbol. Further, all the tasks they envisioned involve generating written symbols from the start (relating opposite numbers or adding and subtracting integers, Journal, Annexes 1 and 2). Thus, mathematical relations not only emerge as body-tool interactions, but as written symbolic relationships. As body-tool interactions and relationships among written integers structure each other, we see here how mathematical symbols can also partake in emerging "assemblages" of bodies, tools and mathematical concepts (Ferrara & Ferrari, 2023).

Discussion

Across the analysis, we centred on the interactions between the social and the material dimensions of designing a tangible manipulative. One might say that this interaction is resumed by involving the learners' perspective in the definition of the design problem, which we termed empathy. However, we argued that several phases of the design process involve and entangle social and material aspects. Ideation entailed empathic imagination, as entertaining possibilities of actions with manipulatives from the leaners' perspective. Prototyping involved mathematical and empathic imagination, as entertaining new body-tool-symbol interactions depending on the learners' needs.

Socio-material frameworks of mathematical activity greatly informed this research. Mathematical imagination allowed to analyse the design drawing of Figure 1a as a site of emergence for new body-tool interactions. This might be a rich way of pursuing the investigation of designing a tangible manipulative as a socio-material process. Indeed, mathematical diagrams are central to understanding mathematical activity as socio-material, as framed by de Freitas and Sinclair (2012). Diagrams are seen as material entities animated by "potential", prone to generate gestures and mathematical relationships. From this perspective, we understand how the materiality of Figure 1a allows new mathematical relations to emerge: inviting a third dimension into the planar sheet allows to "capture" (ibid.) possibilities of actions, using vertical arrows for pushing and pulling buttons and curved ones for flipping the plate. The same potentiality allows juxtaposing several spatial perspectives, involving front and top views of the same object. As these juxtapositions also address the learners, design drawings may be seen as sites of "mathematical agency as a socio-material encounter" (ibid.).

Importantly, this decentres the "agency of creating" (in teaching as designing) from the individual teacher leveraging resources, to a "distributed agency" (ibid.) across perspectives, individuals and materials. Thus, it enriches how designing can be situated at the "interplay" between "designer and materials, and between designer and audience" (Akuom & Greenstein, 2022, p. 111). Conversely, design technologies may enrich ways of thinking about the material potential of drawings. Consider, for example, how the "extruding" function in a 3D-design software can create a solid by "stretching" out a planar figure into a third dimension (creating a prism from a polygon, a cylinder from a circle…) and "carving out" a third dimension out of the plane (de Freitas & Sinclair, 2012).

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The politics of mathematical embodiment

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What constitutes the political? How does the political designate something different from the social or cultural? How does the designation 'political' come to be seen within mathematics education research? In a historical moment of polarized 'identity politics', digital imperialisms, and 'post-truth' conditions, how might we need to rethink the space and manner of the political? In this paper, we push 'the political' past the ideological and the psychic, and argue for a political agenda that more adequately concerns the body in all its unruliness. We discuss new materialist philosophies of mathematics that emphasize the 'political ecology' of mathematical participation. We ask two entangled questions — Whose bodies become individuated and marked (gendered, racialized, disabled) through mathematical agency? And how are mathematical concepts themselves political?

Keywords: Embodiment, Materiality, Political, Aesthetic, Ecological.

Introduction

In the Topic Study Group (TSG) on visualisation and embodiment at the 2024 ICME conference, after three days of group meetings, a new researcher asked, "does anyone ever talk about the political in this group?" An extensive discussion ensued, during which different participants talked about how they could imagine the political being relevant to questions of embodiment in mathematics education research. It became evident, however, as the discussion evolved, that the ideas we have been working on for over a decade (see de Freitas & Sinclair, 2013), which have been centred on the notion of inclusive materialism (IM), were not seen as political, despite the critical, feminist origins of this work. In search of allies, we have decided to highlight, in direct and succinct terms, both the initial political intentions of IM and its continued articulation with scholarship more easily recognized as political—in Disability Studies, Anthropocene Feminisms, Posthumanisms and Anti-Colonialisms.

What constitutes the political? How does the term designate something different from the social or cultural? How does the designation 'political' come to be seen within mathematics education research? In a historical moment of polarized 'identity politics', digital imperialisms, and post-truth conditions, how might we need to rethink the space and manner of the political? The field of mathematics education research often falls back onto simplistic ideas about what constitutes the political, with a bias towards theories of the political that target (a) sociological perspectives on policy and ideology and (b) psychic perspectives on subjectivity and identity. Such work has often drawn from Marx, Althusser, Freire, Foucault, Butler, and Lacan. This is all necessary and excellent work, but also limited. All too often contributors designating their work as political position themselves as critics or pessimists, without taking a creative action or risking a proposal. The work of researchers who design alternative mathematic curricular practices or proposals are then seen as apolitical (or naively complicit) because they are working within the confines of the system. This means that the field typically bifurcates along two axes, each encoded in terms of varying degrees of political

engagement. The first axis is *criticality*, where research that simply seeks to understand or describe an as-yet unexamined context of learning is seen as apolitical (neutral or complicit, depending on your view), while research that explicitly critiques that which it describes or studies is seen as political. This axis seems to sanction a separation between description and critique, which needs to be problematized. The second axis is *scale*, where research that seeks to study the micro-ethnographic or physiological human body is seen as apolitical, while research that studies groups or people is seen as attending to the political. In our 2019 edited ZDM volume, *Body Studies in Mathematics Education: Diverse Scales of Mattering*, we tried to open up a discussion about how the political operates across these different scales.

In our opinion, this sorting of the field according to these two axes is incredibly dysfunctional. Not only does it overlook the radical contributions to political theory from feminist thinkers ("the personal is the political") but it also overlooks the widespread shift in 21st century theory towards different scales of political action and agency, as well as the fragmentation of typical humanist and Marxist notions of political voice and agency. The bias in the field has, despite its good intentions, affirmed a problematic image of the political, even as it claimed to offer insight into political dimensions. Beccuti (2024) discusses the importance of ideology critique, while also troubling the simplistic assumption that ideology functions deterministically, and that critical mathematics education will always be a kind of false consciousness. Various post-colonial and anti-racist revisions in the humanities (e.g., *Becoming Human*, by Zakiyyah Iman Jackson, 2020 or *Posthuman feminisms*, by Rosi Braidotti, 2022) have shown how such a sorting of the political often reflects a *ressentiment* that keeps one trapped in subjection to the very thing one needs to overthrow. We believe the critical theory that formed the foundation for much of the political work in mathematics education research (MER) in the last decades, and which has often fuelled the work on identity and subjectivity, needs to embrace new theories of molecular and more-than-human geo-politics, in order to deal with new modes of governmentality in an algorithmic-control society, under a new climatic regime. In our case, "the ontological turn" brought to our attention thinkers in science and technology studies (STS) like Karen Barad, Myra Hird, Bruno Latour, Donna Haraway, and Isabelle Stengers, who offered new ways of studying the politics of more-than-human agencies, as well as more recent scholars working on the "creolization" and post-colonial revisions of STS, scholars like Juan José Saldana and Viveiros de Castro.

Most importantly, we believe this shift in the theoretical underpinning of the political better equips us to examine mathematical embodiment as a political practice. Rather than simply study the way that experiences in mathematics education shape identity and subjectivity, leaving the mathematical ideas outside and untouched by critique, and leaving the body as the mere passive carrier of social imprinting, we show how mathematical concepts are themselves historical and political and materially bound up in bodying processes. This has been our way of demonstrating how the political runs deep, past the cultural framing of subjects and institutions, and into the very mixtures of concept and matter – what Latour calls our "political ecology". For instance, when we analyze how the number line is a historical apparatus for merging discrete and continuous notions of magnitude, we are exposing the way that particular humans try to manage the material mixture of these onto-epistemic concepts, which are themselves bound up with other competing agencies. When we

examine concepts like proportionality (de Freitas & Sinclair, 2022), the infinitesimal (de Freitas, 2020), conditional probability (de Freitas & Sinclair, 2018), ordinality (de Freitas & Sinclair, 2016), and spheres (de Freitas, 2023), we show how these mathematical concepts reflect and feed-into cultural and political space. Our efforts follow other thinkers with kindred methods, like Michel Serres (2017) and Châtelet (2000), who shows how Western mathematical and scientific concepts are bound up with historical and political and cultural values. Similar lines of inquiry have propelled the IM-infused and more racially oriented political work of Dominguez et al. (2023) and Peralta (2024).

In the sections that follow, we highlight some of the main political implications of Inclusive Materialism. We discuss some of the ways that IM recasts the body as ecological – as we articulated over a decade ago, we understand the "human body" as "an assemblage of human and more-than-human mathematical concepts" (de Freitas & Sinclair, 2013, p. 453) – which has enabled us to contribute to the recent socio-ecological movement (see de Freitas et al., 2022; Coles et al., 2024).

Not constructionism

Given the early work in what became known as embodied cognition, particularly that of Lakoff and Núñez (2000), but also the enactivism of Maturana and Varela (1980), and the growing importance of the work within phenomenological-informed studies, none of which explicitly attended to what MER conventionally names as 'the political', perhaps it is not surprising that a book entitled *Mathematics and the Body* (2014) would be seen, wrongly, as extending the constructivist/constructionist paradigm. In this paper, we aim to articulate the specific philosophical assumptions and political ambitions of IM, both as it was framed in 2014, and as we (and our colleagues) have elaborated since then.

IM was not motivated by the question of how students learn mathematics or how they can be taught in better ways. It grew from our own historically informed and critically oriented philosophical investigation of what constitutes situated practices associated with mathematical knowledge writ large, whether it be by experts, learners, maverick engineers, ancient Mesopotamians, etc. This objective entailed examining examples from both in and outside of contemporary schools, a pluralist perspective that pushed past epistemic relativism to more ontological questions about material practices (an ethico-onto-epistemological perspective). IM is a philosophy of mathematics, first, with an accompanying philosophical method for investigation. While we attend to pedagogical issues and agendas (studying problem-posing habits in classroom discourse, tracking visual and dynamic manifestations of concepts, destabilizing the essentialism associated with psychometric research, etc.), IM does this work as part of a larger philosophical and historical project, exploring how mathematical concepts are material, and have/will/can never be fully extracted from the material contexts that give rise to them. This is not constructivism or constructionism because the ontology is literally at odds with the anthropocentric tenets of all such approaches, derived as they are from the Kantian Enlightenment image of reason. Constructionisms and constructivisms tend to centre the human as the fundamental synthesizer of sensory information by positing mental faculties and internal schema that derive from psychological premises about individuated cognition. The specific theorists we draw from, especially Gilles Deleuze and Karen Barad, offer a radically different ontology, which better recognizes the agency and virtual potentiality of more-than-human ecologies.

Moreover, our work has always been deeply historical. If there is a European theoretical lineage we inherit from, it is that of Deleuze (Leibniz, Spinoza, Nietzsche, Bergson, Foucault) which informs our *archeological* approach to the history and onto-genesis of mathematical concepts and allows us to expose the multiplicity of mathematical practices, and the resurgence of "minor mathematics" as a constant destabilizing force in mathematical communities over time (Deleuze & Guattari, 1987). This focus on *minor mathematics*, rather than state-sanctioned mathematics, has helped us surface all kinds of neglected and erased mathematical practices, and to both destabilize the conventions of school mathematics, and offer images of alternative approaches.

In IM, mathematical concepts embody or express political forms because mathematics is as much of the earth and our situated material practices as any other practice. A sphere is never just a sphere. It is part of a cross-cultural and historically repeated fascination with particular spatial territory (de Freitas, 2023). Spheres are *both* abstract forms and political forms. In other words, our project is a political analysis that seeks the force of the political *within* the mathematical, rather than only in the institutions where it occasionally lives. This takes back from those who would guard mathematics as their own pure or ideal politically neutral activity, to show how mathematics is itself bound up in the messy material conditions of its diverse formulations.

Aesthetics and dis/ability

The new materialism of Karen Barad (2007) came as part of a chorus of critical responses to second wave feminism, which had emphasized gender as a performance, to use Butler's (1993) term. Second wave feminism was intended to free women from being defined *a priori* by their biological bodies, and enable women to access many new forms of participation in society – the exigencies of a fixed Nature gave way to the possibilities of a more flexible Culture. This was crucial work, in shifting civil rights towards more inclusion and opportunity. However, IM looked instead to third wave feminism which problematized the nature-culture divide that had shaped the second wave, in part because the second wave positioned the body as a passive container (Bennett, 2010; Haraway, 1991), ceding all power to human discourse and socio-cultural norms, and in part because it had failed to address the specific intersections of race and class, and the heteronormative images of female sexuality. Both the emphasis on discourse and the lack of attention to intersectionality led to important theoretical developments, as a way to correct those two aspects of second wave feminism. Third wave feminism sought to agentialise the intersectional body by decentring the human (as in Haraway's cyborgianism) and fashioning a non-dualist NatureCulture mix. With Da Silva, Wynter, Haraway, Stengers and other feminists, Barad refused to simplistically position the sciences as the enemy of Woman and instead worked towards creating new forms of science (in her case, physics) that ceded fantasies of Humanist objectivity and universality to feminist commitments to emergent and agential, but no less realist, material-discourse mixtures.

Scholars in critical disability studies in mathematics education understood our agenda and took up our ideas as political interventions, particularly those working with neurodiversity, and therefore interested in our alternative image of mind-body relations (Tan et al., 2019). In 2015, we published research interrogating the faulty assumptions of neurological research on dyscalculia, carefully examining the experimental apparatus and its built-in bias towards a limited conceptualisation of

number itself. In other words, we showed how one could critique such research for its assumptions about the nature of mathematical concepts, to better undermine their simplistic assessments of human number sense and disability. We saw our effort as strategic, getting under the hood of their research, looking at the details and the specific image of number sense that was assumed. The target was their flawed image of mathematics – which was a target that mattered to them. Their built-in bias about number was significantly over-determining what the researchers were able to see in their data, continuing decades-long Piagetian-informed narratives of developmental cognition. To critique the medical research at this level – the level of their mathematical concepts – meant they would take notice. Identifying these assumptions allowed us to then disrupt them, and proceed to lay bare the many alternative ways that number thrives in diverse bodies and *neurodiverse learners*. From this work, it is possible to see how no developmental trajectory (or pathway) is neutral because it privileges certain conceptual constellations over others, typically those aligned with the practices of white, anglo-euro, male, heterosexual human bodies. Thus our research aligns with the efforts of critical mathematics education research, but goes about this work differently.

By drawing on the political aesthetics of Rancière (2004), we traced some of the sensory-material aspects of classroom environments in terms of a 'distribution of the sensible'. Rancière insists on the way in which the aesthetic functions politically to produce communities of sense-making. In other words, there is a significant connection between what can be sensed – in mathematics, we might take a simple example of what can be seen – and what is taken to make sense within a certain set of practices. For the Bourbaki, for instance, mathematics was to be a discursive, symbol-driven practice in which diagrams were not permitted. This is an aesthetic move (restricting what can be known through the senses) that has political consequences (determining what counts as knowledge). The history of mathematics is chock full of such political-aesthetic conjunctions and these play out in mathematics education as well. A recent example is the shift from banning the use of fingers in the classroom to promoting them as ways of making sense of number. We have explored other material-aesthetic practices in the mathematics curriculum relating to proof practices, where the formal manner of engaging with the diagram in classical geometric deduction is also a kind of aesthetic practice (de Freitas & Sinclair, 2014).

Conclusion

Because mathematical concepts are linked to specific material-sign-making practices, we think it's important to complement the recent calls for (re)humanizing school mathematics (see Gutiérrez, 2017; Tan et al., 2019) with specific proposals for problematizing the centering and continued emphasis on the exceptionality of the human as a deeply gendered and racialized category (Jackson, 2022). This entails revealing the ways in which mathematics has never been modern (Latour, 1993) – i.e. it has never been only the achievement of the human. In the face of mass extinctions and climate crises, IM places the human in the midst of other life agencies. Drawing on both "sympoiesis" (see Haraway, 2016), and imagination, we describe more-than-human mathematical worlds (that of wolves, spiders, ants) which allow us to further reflect on the precise ways in which human mathematics is (and is not) a reflection of human agency (de Freitas, 2024; Barwell, 2022). Indeed, we have argued that it is only by addressing the onto-ethico-epistemological nature of "majority" mathematics that we can dethrone it from its current hierarchical, discipline-defining status, allied

with a particular 'human' (with all the resulting violence entailed), not to cast it away, but to understand it historically and philosophically, and thereby become empowered to engage with it.

We take as given that the universalising impulse of movements such as "math for all" are fictitious at best and viciously reinforcing of existing exclusions at worse (Martin, 2015). With IM, we are interested in the force of difference and disorder within Western mathematics: what differences were not actualised in the differentiating of mathematics from religion or from magic? What differences were sidelined in the convergence of calculus towards a particular set-theoretic formulation? But also, which differences emerged precisely and powerfully at these historical junctures when mathematics was forced to become something different and unrecognizable? We seek different *minor mathematics* that bubbles up everywhere, to show how mathematics is not the monolithic static or dead thing it is taken to be. Our work complements or supplements other work in ethnomathematics and socio-cultural studies of non-Western mathematics, but our recent efforts are distinctly focused on exposing *the minor in the major*: we find a disruptive minor mathematics in the work of European mathematicians who pushed back at the conventions of the field and invented new controversial mathematical concepts; we find it in the decolonizing efforts of Creole scholars in 17th century New Spain who offer an entirely new "mixed mathematics" more responsive to the particular environment of the Americas; we find it in the speculative mathematics of Lewis Carrol and Charles Hinton (de Freitas, 2019). In this way, we treat mathematics as a kind of non-innocent artful and evolving material practice whose objectivity or claim to universality is itself situated and grounded in a vast ecology of people, atoms, guns, water and planets.

Our effort is fundamentally ecological because it recognizes the way that *temporal/spatial scales* of relevance expand out from the personal to the climate crisis, a crisis that is experienced differently depending on the specificities of racialized and gendered bodies. Who is amalgamated into this common stratum of "human being" at the moment of the Anthropocene? Who is counted in this newly globalised planet-wrecking species? How does this planetary concern wash away other socioeconomic differences that have mattered (Da Silva, 2022; Latour, 2017; Yusoff, 2018)? Without denying the effects of the massive redistribution of the earth's biomass, or carbon emissions or nuclear waste, Colebrook (2017) points out that the very idea of the human (and thus also the idea of the posthuman) is caught up in entangled differences (racialized, gendered, sexualized, commodified) that have come to produce both the conception of a stable, predictable Nature (with orderly seasons, orbits, migrations) and the concept of the Anthropocene which disrupts that stability. "We" are called on to reckon with the human in ways we never did before, precisely because of this contested term "Anthropocene". The claim that the personal is geological is an assertion about scales of mattering that are often ignored in the separation of disciplines and the phenomenological focus on the intact individuated human body. As a gesture of both collapse and opening, the geo-personal reminds us of how bodies straddle different modes and scales of geo-political relevance.

A research focus on the body contributes to political analyses of mathematics education for all kinds of reasons. In our case, this focus is sourced in a distinctive philosophy of mathematics. We ask: How might this philosophy change mathematics education, and cultivate different and more inclusive images of Nature, images that affirm distinctive singularities, exceptions, and also multi-scale bodying and relationality? In following the personal to the political, geological, and ecological, we

have doubled down on the historical entanglements that expose the fundamental earthboundedness of mathematics, with the aim of crafting a foundation for a future mathematics education.

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Diagrams and mathematics education research: A methodological reflection

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In this paper we address a methodological reflection on data presentation and visualization in our field, especially concerned with ways to display bodily and material engagements as relevant to mathematical activity. We narrate a story of methods in mathematics education research, which focuses on the need for alternative ways of showing and presenting data if we want to adequately address the body and the interaction body-tool in mathematical activity. We focus on diagrams as a research method to explore learning events where the bodily, the tactile, the kinaesthetic, the affective, the cognitive, etc. are all entangled in shaping the activity. We discuss examples of the use of image and diagrams in research practice and develop lines of flight around a desire to use them creatively, how they transform data and the issue of the entanglement between researcher and data.

Keywords: Body, diagrams, material, mathematics education, research methods.

The embodied turn and its demand on research methods

In this paper, we want to address a methodological issue that, in our view, lies at the heart of the Thematic Working Group "Embodied and material studies of mathematical behaviour" (TWG29), that is, the presentation and visualization of data in mathematics education research. Our aim is to exemplify this issue by drawing particular attention to ways in which researchers in the field have dealt with the broad question of how to visually display bodily engagement and materiality in research papers as relevant to a discussion of mathematics teaching and learning. Moreover, we want to discuss lines of thought concerning the fact that the way data is presented and visualized influences the analysis, while capturing the entanglement of the researcher with data.

Theories of embodiment (Lakoff & Núñez, 2000) have brought forth the role of the body in mathematics thinking and have sprouted a great interest in how sensory motor experiences shape and ground the ways in which we create mathematical meanings. Since the 2000s, many researchers have drawn on the idea that mathematics learning could be better theorized as a multimodal activity, in which not only verbal utterances, but the whole complex entwinement of words, gestures, gazes, movements, voice modulations, and so on, play a crucial role (e.g., Radford et al., 2009). This theoretical perspective has posed for researchers the methodological question of enriching the field notes or transcripts of the learning situation under study with a rich description of gestures, movements and actions in it. These were necessary ingredients for discussing "what was going on" besides the sole verbal register, and for capturing the complexity of the learning event. In most cases, gestures and movements were not interpreted as a mere subsidiary of what was being told, but as the texture of the cognitive processes at play in the classroom: the need to include such descriptions in the papers was therefore not accessory, but a direct consequence of a change in the theoretical paradigm. However, there are differences in the interpretation of the role that tactile and kinaesthetic bodily experiences and their interaction with artifacts and material culture play in how we make sense of the world. In this paper, we strive to make these differences emerge, when possible, but we stress

our interest in a somewhat transversal line of investigation, which is not independent from a specific theory or conception of cognition, but also not bound to it. In the following, we articulate our reflections around strands that we consider significant for grounding a discourse on the methodological aspects of data presentation in mathematics education research. We do not intend to provide a literature review on this topic, but rather to discuss innovative elements that have emerged from some studies that could stimulate rich meta-level discussions about our work as researchers.

The materiality of data and editorial format

Video is probably the most widely used kind of research data in educational practice nowadays, partly because it is now easier (than in the past) to gather videos from the (mathematics) classroom. Even though it is often perceived as the best resource for adequately grasping the learning event, it is still not so widely problematized or theorized (Hall, 2000; Derry et al., 2010). The attention towards the body and bodily engagement surely increased the interest towards this type of media, as a video could be replayed several times, and many aspects (like gestures) could be observed, scrutinized and analysed more closely. Still, the video captures only a particular perspective, with all its limitations, on the complexity of mathematical experiences and does not provide alone all the answers to our research problems. Furthermore, to "appear" on paper, students' and teachers' interactions are then documented through detailed transcripts, paying attention to expressions, gazes, and movements, which reconstruct a description of what was captured in the video. As an example, the methodology of the semiotic bundle (Arzarello, 2006) accounted for the multimodal nature of mathematical cognition, beyond the focus on linguistic elements. The analysis involves a "timeline", a specific table where the actions of different actors at play are thoroughly described over time by unpacking the different modalities they engage with (Arzarello et al., 2011).

Another interesting element for our discourse here is the role of the technical aspects of video data. Think of the use of multiple recordings from different perspectives (which can be synchronized later by audio and displayed close to each other); or the use of video recordings using cameras placed on participants (like GoPro cameras). All these material constraints or possibilities, created by the very means in which data can be captured, influence the ways in which we, as researchers, are able to tackle research questions (Nemirovsky et al., 2018).

A very well-known special issue of Educational Studies in Mathematics published in 2004 and entitled "PME Special Issue: Bodily Activity and Imagination in Mathematics Learning", which arose from a previously conducted PME Research forum, proposed an interactive editorial format called "video papers". The papers published in the special issue were not only made up of words and static pictures, but also of animated and interactive graphical representations, and motion pictures with sounds ("video-clips") of the tools and devices involved and of students, teachers and researchers working together (Nemirovsky et al., 2004). While this was a first-time (and, at present, unique) attempt in history, not so much has changed in publication formats since then, even though some journals are now experimenting with more visual templates like the "graphical abstract" and more interactive resources (e.g., digital applet) are commonly shared by researchers via web links as supplementary materials of the paper.

On the other hand, different attempts have been made in writing by different researchers to better capture and interpret the materiality of the mathematical activity, and to profit from the materiality of data. For example, Staats (2018) has proposed poetic structure as an additional discursive modality, alongside prosody, gesture and writing, relevant to multimodal analyses of mathematical activities that value a broader theory of embodied knowledge, for which mathematical concepts are also amplified by layers of syntax. Mikulan and Sinclair (2019) have introduced educational stratigraphy as a method (and meta-method) to address various scales of mattering in mathematical activity. They propose to see an event, an utterance or a gesture or drawing, as issued in a time and space detached from those that came before and overlapping with past events so that each is a novelty. Rather than thinking of events in a continuous, chronological time, events may therefore be seen as superimposed and granted their own force of folding and unfolding or mode of existence, actualized or potential (that might have been or still to come, that did not happen or might be imagined). In this way, we can read the movements, the tool, the concept (or the biological and the social, the affective and the cognitive, the abstract and the concrete, etc.) as different modes of mattering alongside each other in mathematical activity and better account for the multiplicity or plurality of ontological possibilities that shape the activity. More recently, Rumbelow and Coles (2024) draw methodologically on theories of embodiment to study children's interactions with physical materials (Cuisenaire rods) and an AI app to work with number as length (or measure). In this case, interactions are taken to exist in complex dynamic (body-artefacts-task) systems that never reach states of equilibrium (following Shvarts et al., 2021), so they can only be locally stable, for a while. These researchers introduce diagrammatic notation of the body-rods-app triad as a research method to explore the presence and flow of forces such as affect in the activity. They use different colours to capture where the flow of activity (the interaction) is smooth or disrupted (thus, generating tension or frustration).

All these few examples offer occasions to experiment with data and pose new points of view for their interpretation. Looking at them, it is already clear that the visual dimension is not the only one that has been explored and exploited from the research methodology side. However, it is the one we are most interested in discussing in this paper due to the force of image in our knowing of the world nowadays. Briefly speaking, we see diagram as a potential, powerful way of capturing data and producing readings that are not necessarily representational, sequential or unidirectional. In the following, we will refer to some diagrammatic uses of data while exemplifying issues around the relevance of visualization and experimentation in the research methodology.

Diagrams that transform data

Diagrams are ubiquitous in mathematics as well as in mathematics education: in the latter, they are often used to describe relationships among actors, institutions, cognitive processes or pieces of knowledge. In a semiotic mediation perspective, for example, a diagram with lines illustrates the links between tasks, the mathematical knowledge, the material used, the individual and collective productions (e.g., Bartolini Bussi & Baccaglini-Frank, 2015). Here, we are interested in different kinds of diagrams, namely diagrams used by researchers in their practice to present or analyse data. Since we would have had the possibility of including only a few diagrams in the paper, we decided not to include any. We particularly focus on their role in generating new meanings and actively partaking in the transformation of data, describing only certain aspects of the mentioned diagrams.

The first two examples we discuss are related in that they share the need to study movement as a broad concept, involving human and non-human bodies, including mathematical concepts, and the complex entanglement of affects circulating among different agencies (de Freitas & Sinclair, 2014). De Freitas and colleagues (2019) used diagrams with arrows to describe the movements of students who were coordinating to create a circle by moving two remote controls, which were part of a particular technology, WiiGraph. The diagrams helped the researchers to analyse the microscales at which coordination was achieved or lost and to highlight the different scales of mattering along which the concept of circle emerged in the girls' sympathetic negotiation. Ferrari (2019) crafted a movement notation derived from Labanotation (a system for recording dance moves) to capture the motion experiments that students performed in the classroom with the same technology. The creation of a notation for the specific case at hand was driven by both a theoretical and a methodological question: on the one hand, treating movement as an overarching concept (which includes, but is not limited to, bodily movement) required new ways of "talking about" it; on the other hand, in most research studies presenting motion sensor activities, the motion experiments themselves are extensively described in words, and the qualitative aspects of movement involved in the activity are often lost.

These challenges required the research to open towards diagrammatic attempts. Interestingly, in trying to capture the qualitative dynamics of movement, both attempts necessarily deal with the interplay between discrete and continuum, which is not fully resolved inside the diagram, but is evoked in its virtual dimension. The researchers' work in creating the diagram entails a different stance towards data than simply coding it, as well as decision-making on the choices regarding the diagram's articulation. At the same time, the diagram requires the reader to approach the situation under study through a new medium, which actively demands interpretation.

Beyond the inevitable examples' limitations and inherent pitfalls in representing data, we consider them relevant to our discourse as the diagrams offer a novel perspective on the event, striving to bridge the gap with the complexity of movement and expand it, from a post-human perspective, beyond the consideration of the human body alone as the centre of the activity.

Another example we wish to discuss is the diagrammatic rendering of eye movement, which is made possible by tracking technology (Strohmaier et al., 2020). We do not want here to enter the details of how data are captured and then interpreted: rather, we want to draw attention to two significant aspects. First, it is worthy of consideration that this technology allows for the creation of a new representation that the researcher can use to make sense of the aspects of a task/activity that have the greatest influence on the subject's orientation. For example, Krichevets and colleagues (2014) perform an analysis of eye movements as a method for investigating the transformation of perception at different levels of competence: by looking at gaze paths and the number of visual fixations (rendered through diagrams) of experts and novices when solving a mathematical task, the authors observe that the two groups of participants *see* very different environments. Secondly, we can say that eye-tracking technology uses sensors that allow perception to be studied beyond our perception, possibly expanding the researcher's perception of an event. This aspect poses theoretical challenges to the nature of perception, our knowledge of it and the role of the researcher in interpreting and questioning the data captured by the technology.

The entanglement of data and researcher

Recent post-human perspectives, also drawing on feminist studies (e.g., Haraway, 1988), have been questioning the objectivity of the researchers and positivistic stances on science. This implies resisting claims of authority and the understanding that the proposed account is only one among many possible ones. Related to this, a key point of data collection, within a critical postmodern paradigm, is that the researcher is no longer at the centre of the process, but s/he provides space for new and different data, without imposing her/his point of view (Stinson & Bullock, 2015). Dominguez and colleagues (2023), for example, draw on both feminist materialisms and Indigenous Knowledges to account for the possible as emerging in philosophical conversations with students around the concept of space. Instead of following traditional research patterns, for which video data is first translated into a transcript, then coded and subsumed into main themes and finally summarized into key findings, the data is treated differently, expanded rather than reduced. The paper contains comics, which act not just as "a representational device, or a mediational tool at the service of language." (Dominguez et al., p. 1157). Comics allows the authors to break the linearity of language and embrace data through the "agentic materiality of drawing" (*ibid*., p. 1157), engaging the researcher in a creative process. This is in line with considering "data as alive, agentic, and full of possibilities." (*ibid*., p. 1156). While it is now common to see line drawing reproducing on paper a sketched version of human bodies' interactions (e.g., Radford, 2021), the method employed by Dominguez and colleagues serves as a reconfiguring of data, offering new opportunities for interpreting them through a reworking of language and drawing.

We see a similar attempt to different experimentation with data in Sinclair (2024), where the author discusses narrative and re-enactment as research methods that allow for recovering the flow and richness of the experience and increase situated and embodied validity. She considers a short video clip of a four-year-old child interacting with the multitouch app *TouchCounts* to work on number sense. The attempt here consists first in providing a narrative account of the event to give a sense of the whole event and to situate it, then in inviting the reader to re-enact the event through selected images, focused on a particular aspect of the experience: movement (of arms, hands, fingers). The image component is still fundamental in the qualitative experience of research and for an intensive encounter with data. One novelty in Sinclair's (2024) approach is that her reading of data strives to engage the reader in the same research process or, according to the author, to enable the reader "to experience the experience" (p. 32). Being invited to re-enact images taken from videos, readers in fact explicitly *mimic* the learning event rather than simply *read about* it.

In this way, re-enactment becomes a way of positioning (see Herbel-Eisenmann et al., 2016) the reader beyond the researcher and valuing what may make sense or become sensible to the reader. It becomes a way of positioning the reader as partaking in the research process, engaged 'in conversation with' the researcher, therefore enabling a relationship between researcher and reader. This implicates "the potential for the reader to create their own interpretations" (Sinclair, 2024, p. 34) or narratives around the event. We can ask: How does it change the reader's experience? But it is also a way of positioning mathematics education research so that alternative, challenging storylines, for example about the role of the body, or the tools or the methods, in mathematical activity might be possible, might come or be imagined. The issue is strictly related to the theme of the working group

since it speaks directly to the material dimension of the researcher's act of knowing and to the virtuality of research as a process, or of research data and research methods.

Final remarks

In this paper, we have tried to narrate a story of methods in mathematics education research, which focuses on the need for alternative ways of showing and presenting data, for the sake of adequately addressing the body, as well as the interaction body-tool in mathematical activity. We have drawn attention to diagrams as a research method to explore classroom situations and learning events where the bodily, the tactile, the kinaesthetic, the affective, the cognitive, etc. are entangled in shaping the activity. We have seen examples of the use of diagrams in this direction and have discussed them to develop lines of flight around a desire to use them creatively, the way that they transform data and the issue of the entanglement between researcher and data. Following Stinson and Bullock (2015), the examples we presented seem to fall within paradigms of inquiry more related to both *emancipation* and *deconstruction*, without exhausting or completely fulfilling them. We consider these two aspects as essential to our research practice if we want to treat (as we do) research as an open-ended process rather than a product-oriented activity. What we really see as valuable in the examples is precisely that they give rise to new questions, issues, and directions more than to definite answers, generating knowledge differently and stimulating new knowledge. The approaches are then relevant to research practice in the field because they open room for dealing with the agency and the potentialities of data, allowing us to reimagine our encounters with data and to engage in creative research processes that move "toward the probable and away from the possible" (Dominguez et al., 2023, p. 1169) and toward a broad consideration of what matters in research practice studying the role of the body, movement and materiality in mathematical activity.

Finally, if we think of methods as "technologies of research", as Sinclair (2024) suggests, then we can question how future new technologies will impact research. For example: What will be the implications in terms of/for research design(s)? Or: How will AI change our data and will implicate the need for alternative methods / futures / imaginations / storylines in educational research? These are just some possible questions that may be raised in relation to our methodological reflection, but we feel that they are relevant to a discourse of educational research in post-human times, when we are also engaged with demanding contemporary issues involving science education research: "climate change, multiculturalism, the flourishing of new interdisciplinary domains (like cognitive neuroscience, artificial intelligence, digital humanities to name a few), as well as issues stemming from living in the digital and post-truth era" (Levrini et al., 2021, p. viii). Levrini and colleagues point out that the preferred way to address these challenges is deepening our understanding and that science education can play a key role in this effort. As they stress, and we agree with them, the deep and novel character of these challenges "necessitates collectively searching for new narratives, languages, and forms of beauty that capture our attention and trigger new ways of thinking", while demanding "exploring new theoretical perspectives and crafting novel methods as appropriate" (*ibid*., p. x).

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Use of conceptual blending to investigate students approaching functions in an emerging embodied technology

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When emerging technologies meet embodied activities, new mathematical learning opportunities arise together with new questions about underlying conceptualization processes. In this paper, we present the results of a study involving secondary school students developed upon the use of an emerging embodied technology, GeoGebra plus. To analyse the students' cognitive processes, we used conceptual blending (Fauconnier & Turner) and we built a model to describe the complexity of integrating embodiment, a new technology and mathematics. In this paper we show that GeoGebra plus can support the object-oriented and dynamic view of functions. Moreover, we suggest that such technologies are the engine to move from the paradigm of 'humans-with-media', introduced by Borba and Villarreal, towards a 'humans-in-media' one.

Keywords: Conceptual blending, emerging embodied technology, GeoGebra plus, function, secondary education.

Introduction

Over the last two decades, mathematics education research has provided evidence for the benefit of using digital technologies in mathematics learning (Clark-Wilson et al., 2020; Drijvers & Sinclair, 2024; Lagrange & Kynigos, 2014). In these digital environments, several embodied approaches have been adopted to foster functional thinking, for instance, bimanual coordination connected to a monitor to engage with proportions (Abrahamson & Sánchez-García, 2016), bimanual coordination through a multi-touch digital interface to promote correspondence and covariation approach (Wei et al., 2023). More recently, even the so-called emerging technologies (Veletsianos, 2010) (e.g., augmented reality, virtual reality, and artificial intelligence) have started to be used in mathematics learning, and some of their affordances exploited to incorporate the use of the body in the digital environments and support the learning process.

In this paper, we will examine a pilot study (consisting of two experiments) designed to support an object-oriented and dynamic view on functions making use of the emerging embodied technology GeoGebra plus (GGB+ from now on). GGB+ is an extension of GeoGebra (GGB) developed by David Hornsby (JKU, Linz, Austria). By adding a pose-estimation model (Posenet) via camera to GeoGebra, GGB+ enables embodied interaction with digital mathematical elements. GGB+ identifies specific points of the user's body (e.g., eyes, shoulders, elbows) and builds an avatar of the user in the GGB graphics view that moves in accordance with the user's movements. In this way, the avatar coexists with representations of mathematical objects (e.g. points, lines, or shapes) interacting with them both statically and dynamically. As a result, when the user moves in front of the screen, their own avatar mirrors the movements, allowing them to interact with mathematical objects and to

receive different visual types of immediate feedback from GGB+ (e.g., the simultaneous movement of the avatar with the user's body, a green Ok! appearing on the screen, or numerical values associated with some coefficients changing with the body's movements). The immediate feedback loop guides the user to align with specific poses or movements and, from now on, we will refer to the process of intentionally adjusting the body posture or pose in response to real-time feedback displayed on a screen with the word *positioning*. This experience goes beyond typical multisensory interactions with technology in mathematics education (Arzarello & Robutti, 2010; Nemirovsky & Borba, 2003), because here the user, through the avatar, co-exists with the representation of mathematical objects.

The pilot study here presented was carried out in the context of the European Widera project TransEET – Transforming Education with Emerging Technologies – and we use the *conceptual blending* model (Fauconnier & Turner, 2002) to describe the students' cognitive processes that emerge in the learning experiences with GGB+. Such a model was chosen for its suitability to describe the complexity of integrating embodiment, a new emerging embodied technology and mathematics.

Theoretical framework

The theory of conceptual blending, developed by Fauconnier and Turner (2002), explains how we unconsciously and instantly integrate inputs from diverse knowledge domains to create new knowledge. This process is represented through a network of mental spaces: *input spaces* (knowledge domains), the *cross-mapping space* (connecting corresponding elements), and the *blended space* (the outcome of merging and elaborating inputs). The same inputs can produce different blends. Fauconnier and Turner also introduced *double-scope blends*, where inputs with distinct organizing frames (e.g., a computer desktop blending a desk and software) combine to form unique structures. These blends support specific actions or meanings, such as "moving a file to the trash bin."

Earliest studies on multimodality, related to the field of cognitive science on embodied cognition (Wilson, 2002), supported mathematics education research in deepening and outlining the relevance of gestures and body movements in teaching and learning processes (e.g., Arzarello et al., 2015; Ferrari, 2020; Shvarts et al., 2021). Similarly, the construct of *grounded blends* was born (Yoon et al., 2011) within the framework of conceptual blending, with the aim of investigating how gestures can be endowed with and used to communicate mathematical meanings. In this type of conceptual blend network, one of the input spaces is the person's *real space*, including gestures and non-verbal communication that actively contributes to the elaboration of the final blend. Several studies adopting the conceptual blending model have also carefully reflected on the role of technology in conceptualization activities: for instance, Gregorcic and Haglund (2021) analyse a learning activity on celestial motion in which several digital tools are involved, offering various external stimuli, and each one of them (e.g., computer simulation or touchscreen device) is considered as an input space.

In light of the previous considerations, the research question guiding our study is: *What are the blending processes of secondary school students while exploring linear and quadratic functions in an emerging embodied technology such as GGB+?*

Method

Participants and research context

In this contribution, we will adopt the blending network introduced and described before to analyse two short pieces of data from the pilot study in the TransEET project.

The first experiment was conducted in Italy with a pair of 10th-grade female students from a linguistic secondary school. The experiment took place in the context of a summer maths stage and lasted 75 minutes. The activities were administered by the first and third authors who were present during the whole duration of the experiment, with the role of participant observers. The teacher informed the authors that the students had not studied the Cartesian plane or functions, but they may have encountered them in their previous grades.

The second experiment was conducted in the Netherlands and involved six 10th-grade students (four girls and two boys) of a secondary school devoting special attention to dance education. The activity took place during an extracurricular class, lasted 90 minutes, and was structured as a field experiment, with the second and fourth authors present with the role of participant observers. The students had already studied functions (e.g., linear, polynomial, exponential functions), their representations and their general properties. In the data selected for this study, we will present the activity of the two girls.

Task design

In both experiments, students received tasks to be solved in GGB+. After an initial exploration, aimed at letting students familiarise themselves with the new technology GGB+, three structured tasks followed, each to be solved in a specifically designed applet: (1) *GeoGebra Mirror*, in which students should fit the graph of given linear or quadratic functions on the screen with their body; (2) *Get the coefficient*, in which students' positioning affects linear/quadratic function coefficients, whose values vary in real time (Figure 1, this to enable students connecting how the changing of the coefficients affect the graphical representation; and (3) *Level Up*, in which students should reproduce the graphs of given linear or quadratic functions whose formulas are shown on the screen and so connect two different types of function representation.

In the first experiment, given the little mathematical background of the two students, the task focused only on the case of linear functions, while in the second experiment, the task involved both linear and quadratic functions.

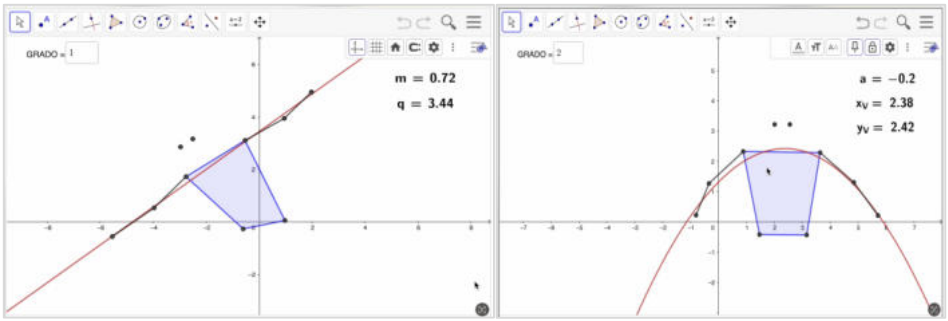


Figure 1: Interface of a GeoGebra+ applet called *Get the coefficient*

In this contribution, we focus on the second task (see Figure 1). In this task, groups of two students were involved in solving the task in GGB+ and filling in a worksheet which asked for a reflection on what they observed. One student at a time moved in front of the screen to position her body in order to: statically represent a function (linear or quadratic) and dynamically vary her positioning. Then both the students observed how the variation of the position of the one in front of the screen determines a change in the values of the function's parameters (m and q if $y=mx+q$, or a and vertex coordinates if $y=ax^2+bx+c$), and which of these variations were involved in this change. After this, students filled in their worksheet asking to reflect first on the static representation of the student position in front of the screen and corresponding to a linear or quadratic function, then to the dynamic representation of the variation of such positions corresponding to function transformations (e.g., translation). In this task, by means of the avatar in GGB+, students can link the meaning of the parameters' values to the graphs of the functions, building a mathematical meaning for them.

Our adaptation of the conceptual blending model

In our study, the emerging embodied technology GGB+ is considered a micro-learning environment by itself. The construct that seems most appropriate to frame the richness of GGB+ is that of the *double-scope blend*. Indeed, it can be seen as the result of the blend of input spaces with different organising frames, in our case the mathematical Cartesian system offered in GeoGebra and the virtual avatar controlled by the human body.

The cognitive processes can be represented through a conceptual blending network, adapted from the one by Fauconnier and Turner (2002) in which continuous lines indicate the matching and the cross-space mapping between the inputs and dashed lines indicate the connections between inputs and the blended space (see Figures 2 and 3); we coloured some words and images to highlight the corresponding elements between different spaces. The network consists of three input spaces: (i) the *maths space* referring to the mathematical knowledge that students possess and on which they draw, (ii) the *real space* with the bodily expressions of conceptualization, and (iii) the GGB+ environment intended as both an input and a blended space, that is a double-scope blend (the input spaces contributing to GGB+ as a double-scope blend in Figures 2 and 3 are grey). While we recognize (i) and (ii) the role of input spaces, (iii) is the result of a further blend involving (iii-a) maths in GGB and (iii-b) the user's avatar positioning as input spaces. Therefore (iii) should not be seen as a standalone technology, but as an integration of students' thoughts, words and actions that makes sense only within the GGB+ environment (e.g., a student saying "If I lift the function…"). It also works as an interface providing multiple forms of feedback, triggered by both students' body movements and the GGB tools.

Data collection and data analysis

In both experiments, we video-recorded the students' activity with a camera and the computer screen recording. The videos of students' activity and the screen recording for each experiment were synchronously coupled, to be seen side by side. For the preliminary analysis presented in this contribution, we selected two critical incidents (Tripp, 2011), namely representative episodes of the students' activity, relevant to our research question. Such incidents were selected because they

revealed the blending of the three main conceptual components at play: maths, body, and technology. These two episodes were transcribed and then qualitatively analysed using the blending network.

Results

In this section, we contextualise each of the two critical incidents (the first focused on linear functions and from the Italian context, the second focused on quadratic functions from the Dutch context), and provide a qualitative analysis both in a descriptive form and using a blending network representation, adapted from Fauconnier and Turner (2002).

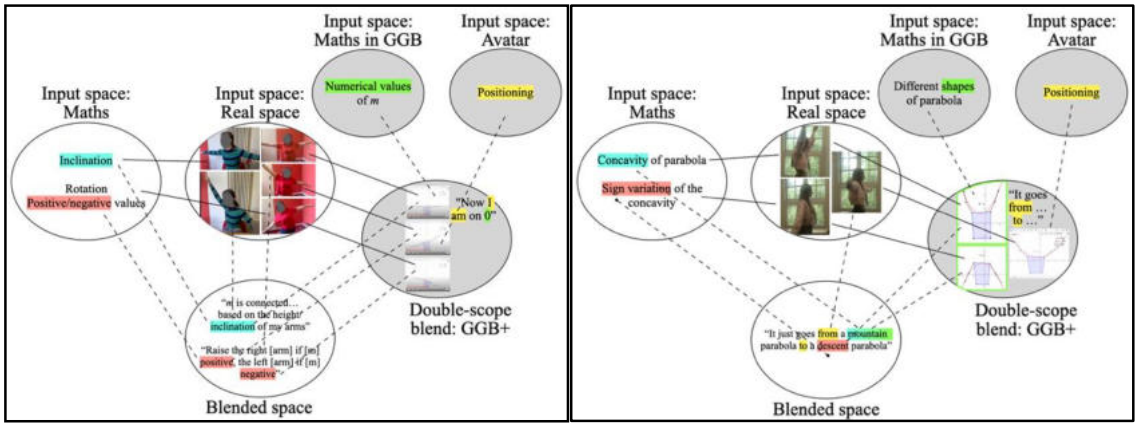
Critical incident 1 on linear functions

During the task, the researcher asked the two students (S1 and S2) how the number *m,* displayed in GGB+ while working with linear functions, was associated with their body position. S2, who was sitting at the table, started moving her open arms in different positions, describing then her variation of position as a "circular movement". S1, who instead was in front of the screen, positioned her arms horizontally ($m = 0$) and then started raising her shoulders while claiming that it was connected to her arms height, and then to her arms inclination. The researcher then asked "How is this number [*m*] connected to your positioning?". And S2 claimed: "Now I am on 0", a sentence making sense only *in* the GGB+ environment where her avatar's positioning can be connected to a numerical value (0). The horizontal positioning of the arms was simultaneously assumed also by S1. And again, under the researcher's prompts, S2 inclined her arms differently to have on the screen a representation of *m,* first positive and then negative. S1 intervened by adding that she raised "the right [arm], if [*m*] positive, the left, if [*m*] negative". Thereby the blending network representation (see Figure 2) shows that S1 and S2 linked: (i) their mathematical knowledge about positive/negative values and inclination (*maths space*); (ii) the arm movements done by S2 in front of the computer (*real space*); and (iii) the numerical values of *m* provided by GGB+ and controlling the linear function created through her arms (*GGB+ space*). The conceptual blending of these elements generates new knowledge in which the students collectively elaborate on the mathematical meaning of *m* by connecting the variations of body position to its corresponding numerical values.

Critical incident 2 on quadratic functions

In the task students were asked to think about how the shape of the quadratic function had changed when the concavity changed from positive to negative. In this fragment, after having read the task, S3 asked S4, "What changed in the parabola with [...that] movement?". While speaking, S3 was moving her arms up and down in front of the computer, in order to test and explore how the quadratic function's graph was changing on the screen. S4 looked at S3, while she was moving, and then at the avatar on the screen, which was embodying a quadratic function fitting to the avatar's arms points. Hence, S4 replied "It just goes from a mountain parabola to a descent parabola". S4 elaborated on already known mathematical concepts, i.e. a mathematical property of the quadratic function (mountain and descendent, to refer to concave-down and concave-up quadratic function) conceiving it as a mathematical object. The network representation of the blending process (see Figure 3) shows that S4 linked: (i) her mathematical knowledge of the concavity of a parabola and its sign variation (*maths space*); (ii) the arm movements done by S3 in front of the computer (*real space*); and (iii) the

feedback provided by the avatar controlling a quadratic function through its arms (*GGB+ space*). The conceptual blending of these elements generates new knowledge: S4 elaborated on a mathematical transformation by linking bodily movements to mathematical expressions.



Figures 2 and 3: Network representation for critical incident 1 and 2 on linear and quadratic functions

Concluding discussion

In this study we introduced an emerging embodied technology, GGB+, developed within the TransEET project, that can support meaningful learning of abstract mathematical concepts by providing students with means to explore and to grasp these concepts in embodied forms that make sense to them. We provided two critical incidents of the use of this technology, and we discussed its potential for learning about functions adopting the analytical lens of conceptual blending.

First, GGB+ allows the students to build their own embodied representation of the mathematical objects, which were in this case linear and quadratic functions, and explore their properties. The technology, together with the proposed tasks, engaged students in making conjectures, testing and adapting them starting from their embodied experience. The novel aspect of GGB+ is that students position themselves as the mathematical object, thereby simulating specific functions with their bodies. For instance, in critical incident 1, students discovered the slope *m* by moving their arms and causing a change in the avatar's position, while in critical incident 2 they explored a change of concavity in a quadratic function by means of variation of their own position. The reasoning accompanying these variations is directed to noticing properties rather than calculations helping students to build and develop an object-oriented and dynamic view of the function concept. Wei et al. (2023) pointed out that these approaches to functional thinking are scarcely promoted in learning practices and so our study practically contributes to research on this matter.

Secondly, in this experiment, we elaborated on the model of conceptual blending to include learning situations in which technology allows for the embodiment of mathematical objects. Thanks to affordances of GBB+, which allow the user to "live" within a world of mathematical representations, this model aligns with the perspectives like those of Sinclair et al. (2023), and as earlier proposed by Borba and Villarreal (2005) with the *humans-with-media* paradigm, fostering a more integrated view

of human and technological interaction. For that reason, we frame this new kind of technology as emerging embodied technology, in a paradigm that we want to call *humans-in-media*, to strengthen the one by Borba and Villarreal. This model provided the analytical method to study the critical incidents and to reveal the special role of GGB+ in them. Framing GGB+ as a double-scope blend within our conceptual blending network (see Figures 2 and 3) provides a means to describe the mathematical concepts as understood by the students while keeping track of the elements that contribute to it (the conceptualization process). This was valuable in our experiment because it highlighted a new form of technology which students are working *in* rather than just working *with*. The network revealed how students were interacting and reasoning *in* a technology and the mathematical meanings built through it. Future research is needed to analyse longer-term use and learning with GGB+ and compare it to other humans-in-media technologies. Further deepening of our study should also consider how the cultural background of the involved students determines their processes of building mathematical concepts, the way they approach the tasks and engage within the technology.

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Navigating Angles: A Spatial Approach to Angle Measurement

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The notion of angle as turn is fundamental for understanding angle measurement. Given the close relationship of mathematical and spatial thinking, in this paper, we adopt a spatial approach, aiming to study what kind of meanings about angles 12-year-old students develop when using multiple linked spatial representations during navigation in physical and digitally simulated space. In our design students initially navigated physical space and tracked their paths using a Global Positioning System application. They then recreated them using a textual programming language in a digital environment which simulated the physical space they physically navigated. In the results we analyse an indicative critical episode that shows how students exploited their sensimotor experience in order to develop their understanding about angles as turns and apply angle measurement with the protractor.

Keywords: angle, angle measurement, protractor, spatial navigation, GPS, textual programming

Introduction

Angle is considered one the most abstract and difficult concepts not only for students to conceptualize (Mitchelmore & White, 2000) but also for mathematicians to define (Keiser, 2004). It has been defined as a geometric figure (a pair of rays with a common endpoint), as a dynamic figure (a turn or rotation), and as a measure (Smith et al., 2014). In typical school education, angle is basically approached as a static geometric figure, a shape formed by two-line segments which meet at a common point and the surface between these segments. This emphasis on the visible parts of the angle often results in students' confusing the size of an angle with the size of the surface that is included by its two-line segments (Latsi & Kynigos, 2022) or thinking that the size of an angle depends on the length of its sides (Psycharis & Kynigos, 2008). On the other hand, approaching angle as a turn refers to the process of changing direction in the plane or the space. In this case students identify easily the vertex of an angle but they find it difficult to identify the angular relationship (Keiser, 2004) which is essential for understanding angle as a measure, represented by a number (Psycharis & Kynigos, 2008; Smith et al., 2014;).

Recently researchers have turned their attention on the role of the sensimotor experience in understanding mathematical concepts (Hall et.al., 2014). While most of the research in the field has focused on the role of gestures (Latsi & Kynigos, 2022), few studies have investigated how the whole-body experience can influence the way students develop mathematical meanings. At the same time a growing number of studies highlights the spatial aspects of mathematical learning and the need to bring forward the connection of school mathematics with students' spatial thinking (Mulligan, 2015). Even though the close relationship of spatial thinking with the development of mathematical concepts is well established by researchers (Lowrie et al., 2020), empirical data on how spatial thinking aids geometry understanding is insufficient (Bruce et al., 2016). This study employs an embodied design to explore the role of spatial thinking in mathematical understanding focusing on the meanings

students develop about angles when they navigate the physical and digitally simulated space using multiple spatial representations that embed all characteristics of angle definitions. The study aimed to provide insights in two research questions (RQ):

RQ1: What is the role of physical experience during navigation in the development of angle meanings?

RQ2: What kind of meanings about the angle do students develop in the context of spatial navigation in physical and digitally simulated space?

The angle as a turn

Angle as a turn refers to the process of changing direction in the plane or the space and it is considered essential for understanding angles as measures (Psycharis & Kynigos, 2008). It is considered the most instinctive aspect of angle (Latsi & Kynigos, 2022) bringing to the surface students' meanings that derive from their embodied experiences. One example where angles are represented as turns is LOGO programming where students program the movements of a digital entity, traditionally a Turtle, in a digital scene. In this context the digital entity functions as an enactive metaphor (Gallagher & Lindgren, 2015), allowing students to relate angles with bodily actions (Papert, 1980) and develop a relationship between "turn-as-body-motion" and "turn-as-number." (Clements et.al, 1996; Latsi & Kynigos, 2022).

However, researchers highlight the difficulties students face in applying their knowledge of the static representation of angles to activities involving the formation of angles as a result of turning (Clements et.al., 1996). Turning in LOGO-like environments represents the relationship between two different orientations of the digital entity, so the sides of the respective angle are not clearly distinguishable. Consequently, students find it difficult to identify the angular relationship to be measured in order to represent the angle as measure. In our research we aim to address this difficulty by contextualizing the digital entity movements into spatial orientation context where students can refer to the change of direction by using the spatial relationships of distinctive landmarks.

Methods

Even though, spatial thinking has been associated with the development of geometric concepts (Lowrie et al., 2020), there is lack of qualitative research describing the processes and ways in which spatial reasoning is related to mathematics (Bruce et al., 2016). In our study we followed a design-based research methodology (Bakker & van Eerde, 2014) aiming to analyse the ways students employ spatial reasoning in order to understand angles as turns and thus angle measurement. In our design we propose spatial orientation as a context for the development of meanings about angles as turns, bringing forward the strong connection between mathematical understanding and spatial thinking. Following we provide the outline of the activities implemented.

The activities

In our study 22 12-year-old students, divided in groups of four, firstly navigated physically in the surrounding neighbourhood of their school. They used a GPS application to track their paths, which they later saved as .kml files. In the next phase they uploaded their paths in Google Earth. In Google

Earth map they could see the map of their school's neighbourhood, the landmarks of the neighbourhood which were indicated with pin points of different colours and the paths they walked during the first phase (Figure 1).

Figure 1: Students could link the spatial representations through the landmarks which were represented with the same colour in both digital environments, Google Earth (on the left) and MaLT2 (on the right)

Students were then asked to recreate the paths they walked physically in a MaLT2 (http://etl.ppp.uoa.gr/malt2/) (Kynigos & Grizioti, 2018) a textual programming environment where the user programs the movements of a digital entity, using simple LOGO commands like forward, left and right. (Figure 2a, 2c) on a scene that simulated the physical space they navigated initially through corresponding-coloured landmarks (Figure 1). In order to insert in the LOGO commands the values that represented the according distances in Google Earth, students could measure the length of straight paths using the Google Earth measurement tool (Figure 2a, 2c). They could measure the angles that represented their turns using a digital protractor which was added as an extension in the browser students used (Figure 2b).

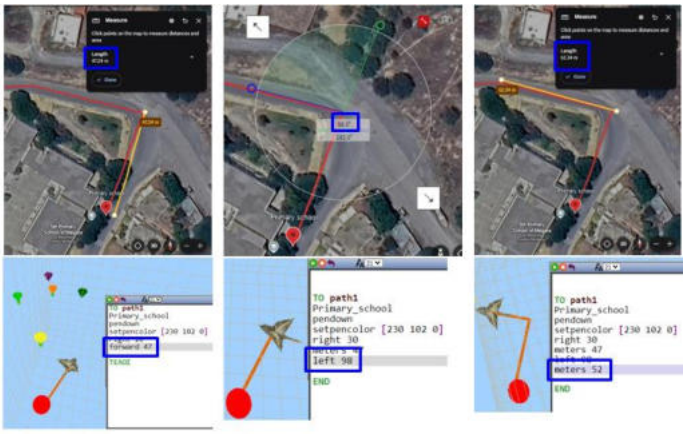

Figure 2: Students could measure the distances with the Google Earth measurement tool (a, c) and the angles using a digital protractor (b)

In this context the concept of angle is addressed as a body turn during physical navigation but also as a shape on the digital environments formed by the lines that represent paths walked, encompassing two out of three angle definitions. The LOGO microworld brings into play the third definition of the

angle, angle as a measure (Smith et al., 2014), since students have to express the turns in the LOGO editor window using arithmetic values (Figure 2a and 2c). At the same time, we contextualized students body movements in a digital space that simulated the physical space they navigated during the first phase, allowing them to use the spatial relationships of distinctive landmarks to conclude angular relationships.

Data collection and analysis

The data collected included audio recordings of children's conversations during the activities, screen recordings from the devices they used, and video observations from the activity. For the analysis we used the critical episodes approach (Angelides, 2001), identifying in our data episodes where children faced a problem and came up with a solution regarding the recreation of their turns as represented in the Google Earth environment, as geometric figures drawn by the movements of the digital entity on MaLT2 scene.

Results

As expected, students faced difficulties in grasping angles as turns and using the protractor effectively. Intuitively students measured the angle formed by the line segments representing their paths, rather than the angle of the turn itself. An indicative episode is the following. Students try to program the entity to draw the path represented by the red line in Google Earth, beginning from their school (red pin), and ending to the Lyceum (yellow pin) (Figure 3). They have led the entity in the node depicted by the blue arrow in Figure 3 and now they need to head south.

Figure 3: On the left the figure displays the path students followed beginning from the Primary school (red pin) and heading to the Lyceum (yellow pin) as represented in Google Earth environment. On the right, the path students have programmed the entity to draw is shown. Students have programmed the entity's movements path until the node indicated by the blue arrow

Episode

Student3:	We have to turn, place there the protractor.
Student4:	Ok but how?
Student3:	The one is going to be this line (Figure 4, segment 1); we are coming from here (means they are coming from northeast).
Student4:	Then the other one is the other line (Figure 4, segment 2), we go towards there (towards the south).
Student3:	Yes, so the blue shows in front of us (Figure 4, segment 2).

The protractor counts 134 degrees and students program the entity to turn right 134 degrees (Figure 4c).

Figure 4: Students initially measure the angle formed by the two-line segments that meet at the node on the map

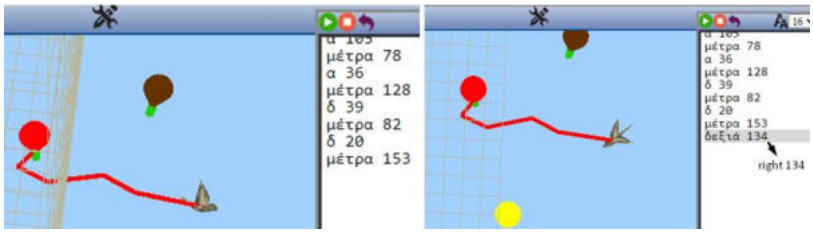

Figure 5: Students program the entity to turn right 134 degrees

Student4: No, it's too much, here we turned a little, look we are already like this (Figure 6a)…And we just want to do this (Figure 6b)…

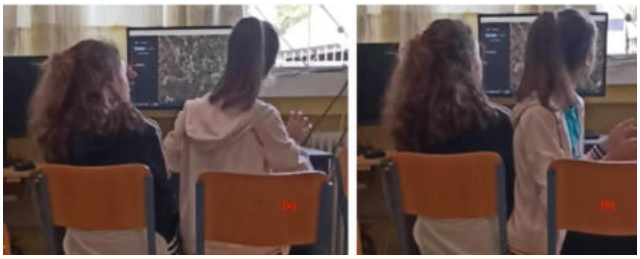

Figure 6: The student represents the turn they have to program the entity to take turning her corso

Student4 realizes that the turn they have measured does not correspond to the turn they want to make and explains her opinion representing with her body the change in direction they made when they were physically at the node aiming to show that it was 'a little' while 134 degrees is 'too much'. She also uses her palm to indicate the initial and final direction making clearer the angular relationship. Taking into account other episodes in our data, while not directly mentioned in this episode students realized that the turn the entity took was not correct taking also into account the spatial relationship between the entity's direction and the Lyceum landmark (yellow pin) (Figure 5). Aiming to correct the value of the angle, students used their previous physical experience as a starting point. They finally managed to relate the angle as a turn which they had previously physically experienced with the value that had to be used to represent angle as a measure. S3 observes her peer and analyses her movements.

Student3: Wait so we look there (shows southwest) and then we turned like you said, so it is like this, we look there (places the axes of the protractor, Figure 7, segment 1) and then we turned like this (places the other axis of the protractor (Figure 7, segment 2)…try 60 degrees.

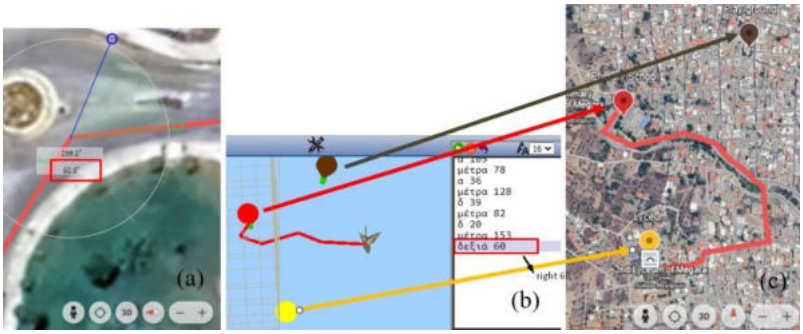

Figure 7: Students insert the correct value in MaLT2 editor. They conclude that the value is correct since they can compare the representations using the known landmarks (b). The spatial relationship between the landmarks and the direction of the entity proves indicates to them that they are heading south as they were supposed to

Student4: Yes, this was correct, now we are heading south like we are supposed to (Figure 7b).

Observing her peer's movements, S3 realizes that they have to start measuring the angle from the axis that corresponds to their initial direction. She places the centre of the protractor to their initial position on the map, the vertex of the angle represented by the red path. She finally places the blue axis of the protractor towards the direction 'they look'. (Figure 7a) and the green axis towards the direction the 'turned', their final direction. The student combined thus elements from different definitions of the angle, which are embedded in the multiple representations they used. They used the previous experience of the angle as turn, an enactive representation in order to represent angle as a measure.

Concluding Remarks

Given the strong relationship between spatial and mathematical thinking, in this paper we aimed to study students understanding about angles in the context of spatial navigation activities in physical and digitally simulated space. While previous research has identified challenges in understanding angles as dynamic figures (Clements et al., 1996), our findings demonstrate the effectiveness of spatial navigation contexts, where students link iconic representations of angles with embodied experiences of turning in physical space, aligning with research on the role of spatial reasoning in mathematics (Lowrie et al., 2020), providing though a detailed description of the processes through which students establish spatio-mathematical connections.

Previous research has also shown that engaging the human body in the learning process plays a crucial role in students' understanding of angles (Smith et al., 2014). However, most of these studies focus on the use of gestures (Robutti, et al., 2022) or study angle concept development through synchronous embodied interaction with technology (Smith et al., 2014). Our study highlights how the whole-body movement in physical space provided students with an experiential basis to understand the angular

relationship between two different directions and conceptualize angles as changes in direction. This also allowed them to use the protractor effectively, one of the most challenging tools for the students to use (Van de Walle et al., 2014). In line with previous research (Bartolini Bussi & Baccaglini-Frank, 2014; Baccaglini-Frank & Mariotti, 2022) the combination of different kinds of representation including the embodied metaphor of the digital entity with which students could synchronize their bodies supported students' understanding. Framing this activity into a spatial representation that simulated the physical environment encouraged students to use spatial orientation in applying the concept of angle as a turn in the LOGO environment addressing difficulties that previous research with LOGO-like activities has demonstrated (Clements et al., 1996). Concluding, studying geometry concept development in spatial contexts can provide further insights on how spatial reasoning can be exploited to facilitate students mathematical understanding.

Acknowledgment

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Experimenting mathematical transformations with GeoGebra Dance: An embodied learning approach

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This study explores the integration of embodied learning in mathematics education through the development and implementation of "GeoGebra Dance," a novel educational tool combining geometric transformations with Augmented Reality (AR) induced physical movement. Grounded in the theoretical framework of embodied cognition and body-artefact functional systems, the research investigates how the designed embodied interactions enhance students' understanding of mathematical concepts, particularly spatial abilities. A pilot intervention was conducted with sixth-grade students, employing thematic analysis of qualitative data. Findings indicate that AR induced embodied interactions improve student engagement, the Interactive Geometry environment enhances conceptual understanding, while further iterations of embodied design are needed in order to specify the effect of the embodied AR element on conceptual understanding.

Keywords: Embodied learning, educational technology, spatial ability, transformations (mathematics).

Introduction

Spatial abilities play a critical role in the cognitive and motor development of learners, enabling them to effectively engage with both every day and academically focused spatial tasks (Kwon et al., 2022). Enhanced spatial skills are strongly correlated with academic success, particularly in the Science, Technology, Engineering, Arts, and Mathematics (STEAM) disciplines (Cheng & Mix, 2014). Recent advancements in educational methodologies emphasise the significance of embodied learning, where physical movement and sensory experiences are integrated into the learning process to enhance cognitive development and knowledge retention (Abrahamson et al., 2020). Incorporating gestures and movements into mathematics instruction has been shown to deepen students' understanding and improve problem-solving skills by making abstract concepts more tangible and accessible. However, challenges remain in designing effective embodied learning experiences that are accessible, scalable, and adaptable to various educational contexts. This study seeks to address these challenges by developing and evaluating a prototype that leverages GeoGebra and Augmented Reality induced body movement into the prototype "GeoGebra Dance" to teach mathematical transformations, contributing to both theoretical understanding and practical application in the field. The primary research question guiding this study is: “How can embodied interactions with GeoGebra be designed to support and improve students' understanding of mathematical concepts?”

Theoretical framework

We use the framework of embodied cognition, which posits that thinking, learning, and understanding mathematical concepts are deeply interconnected with bodily experiences, movements, and sensory engagement (Abrahamson et al., 2020). Duval (2006) claims that understanding of a mathematical concept happens if learners are able to *treat* the concept (transform representations within the same register), or *change* (transform representations to a different register). Shvarts et al. (2021) theorize embodied cognition and design through the genesis of body-artifact functional systems, and Shvarts (2023) looks at embodiment on three levels: (1) Mathematical concepts are "coordinated systems of cultural artifacts as they participate in cultural actions" (p. 45). (2) Embodied learning in mathematics with a technological tool is a genesis of a body-artifact functional system, where the students' body-mind is in a constant perception-action loop with the artifact (technological tool), and through the artifact with the environment. In a perception-action loop the student constantly readjusts their actions based on their perceptions (Figure 1). (3) Embodied design fosters the genesis of body-artifact functional systems.

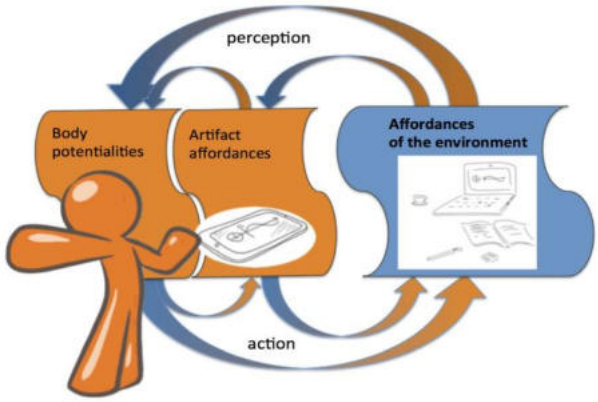

Figure 1: Body-artifact functional system in interaction with the environment (Shvarts et al., 2021

Methodology

The current study was our first attempt at the design, implementation, analysis, redesign cycle of embodied design outlined above (Shvarts et al., 2021).

Design

We designed the GeoGebra Dance artefact by integrating Augmented Reality (AR) embodied interaction features into GeoGebra. Previous research has explored similar integrations of augmented and virtual reality in mathematics education, demonstrating their potential to significantly improve student engagement and understanding. For example, a narrative review by Mouali, Lamarti Sefian, and Bakkali (2024) discusses how these technologies, when combined with tools like GeoGebra, can provide immersive and interactive learning experiences that enhance the comprehension of complex mathematical concepts.

The prototype development was guided by the following criteria. (1) Authorability: Designers of lesson activities such as teachers should be able to create new materials without requiring high

technological expertise. (2) Low technological requirements: The prototype should only require technology that is typically available in a classroom that engages in digital activities. (3) Privacy: The prototype should collect and store as little data as necessary to answer the research questions. All the data should be processed on the device and no student data should be sent to external services. Since many excellent authorable technology solutions already exist for mathematics education, extending existing tools with embodied interaction possibilities was preferred over creating new software. The GeoGebra Calculator Suite[1] was chosen as the basis for the prototype for its wide range of possible interactions and its open licensing model.

To enable embodied interactions, a web app was developed which maps the human body key points (limbs such as eyes, wrists, shoulders) obtained from the webcam with a pose estimation model[2] to the GeoGebra coordinate grid. The software only relies on the computer webcam, no further equipment is needed. The mapped points (limbs) from the person facing the camera can be used to create geometrical constructions like any other point created in GeoGebra. Additionally, a menu to configure simple interactions was developed to lower the barrier for creating new activities. The interactions are defined by 4 parameters: a limb or key point, two-dimensional object, slider, and a speed value. Whenever the limb (e.g., right shoulder) is aligning with the area of the object (e.g., circle, Figure 2) the speed value is added to the slider value. If the slider value is then used in the definition of an object, that object will undergo a transformation based on that value. E.g., the circle centre, a point with x and y definition, if the x value is defined by the slider value, moving the slider will create a horizontal translation of the circle. In short, if the user moves their right shoulder in a way that the corresponding point aligns with the area of the circle, the circle will translate right. We will refer to these interactions further in this paper as “limb commands”.

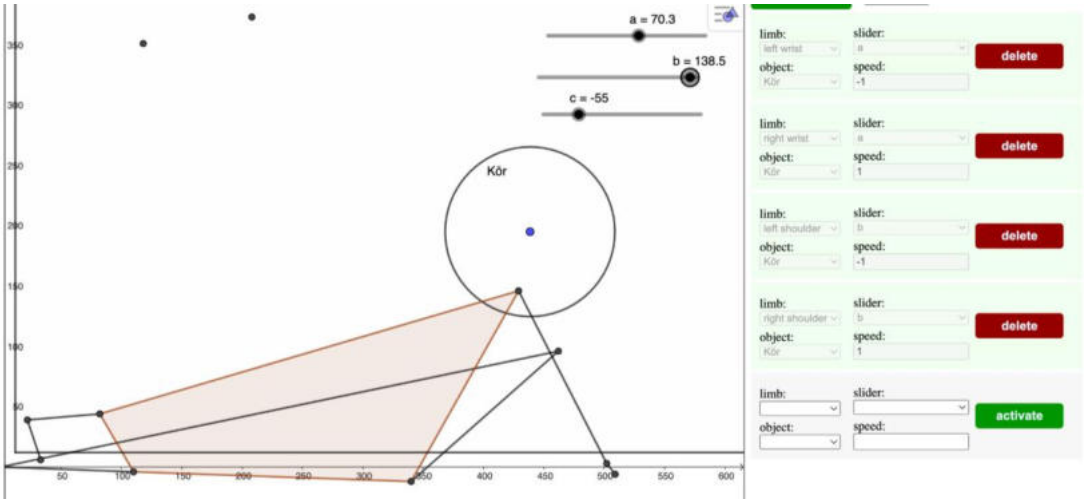

Figure 2: Screenshot from the activity: the right shoulder is presently inside the circle, hence slider ‘b’ is increasing, which causes the circle to translate right.

[1] GeoGebra Calculator Suite: https://www.geogebra.org, 2 September 2024

[2] MoveNet pose detection: https://github.com/tensorflow/tfjs-models/tree/master/pose-detection, 2 September 2024

Implementation

We implemented an activity with GeoGebra Dance in a grade 6 classroom in Hungary, whose mathematics teacher is one of the authors. The students have varying mathematical attitudes and skills, including various special needs (dyscalculia, learning disability, ADHD, autism). For the topic of the lesson we chose the properties of two specific geometrical transformations: translation and dilation. The implemented lesson of 60 minutes length had the following activities (1) students experimented in pairs with a premade geometrical shape of their choice (e.g. circle, pentagon, parallelogram), which was programmed with limb commands for translation and dilation, as described in the *Technology design* section. (2) Students worked in pairs to add further limb commands for translation. (3) Students in a think-pair-share structure worked on a printed worksheet where they needed to decide if translation and dilation have certain properties (e.g., "distance of two points stays the same", "angles stay the same"). (4) There was a whole class wrap-up discussion of the worksheet.

We chose the topic because on the one hand transformations involve complex spatial manipulation, on the other hand it was a curricular requirement for the group. When designing the lesson we sought to meet the embodied design principles of Palatnik et al. (2023): DP1: Involve students' bodies in the learning process DP2: Offer immediate sensorimotor interactions with artifacts DP3: Attend to the semiotic sensitivity of the design DP4: Include a variety of semiotic registers and artifacts that potentiate mathematical perception and discourse DP5: Foster multimodal engagement and "languaging". Engaging with the software in the first part of the lesson involved bodily experience (DP1) and immediate sensorimotor interaction (DP2). Carrying out a more abstract, discussion based worksheet task relying on embodied experiences, and immersing students in further representations in later lessons (collaborative whole body movement, manipulating with tracing paper, using ruler and compass) are in line with DP3, DP4, and DP5.

Data Collection and Analysis

A qualitative study using thematic analysis (Braun & Clarke, 2012) was carried out related to the classroom implementation. The following data was collected:

- Observer notes: an external researcher observed the lesson and noted descriptive and reflective observations about the lesson.
- Post-lesson interview with observer: the observer was interviewed after the lesson about his reflections about the lesson and his ideas about possible ways to develop and use the software.
- Teacher reflections: the teacher wrote reflective notes after the lesson about her experiences of students' attitudes, learning, and the role of the software.
- Student questionnaire: students were asked in an anonymous google form about their likes, dislikes and difficulties concerning the lesson, their learning, and their suggestions.
- Student group interview: students were interviewed as a group about their experiences, learning and suggestions.

During the thematic analysis (Braun & Clarke, 2012) recordings were transcribed and translated into English. One of the researchers coded the data with inductive codes related to the research questions. Then two consecutive researchers recorded the data and revised the codes. We organised codes into themes and used the themes to represent the narrative in this paper.

Results and discussion

We identified the following themes:

- Movement & technology
- Motivation / engagement
- Learning
- Suggestions / improvement

The first three themes describe learning through the body-artifact functional system, while the last theme points towards the redesign phase of the embodied design (Shvarts et al., 2021). We restructured the narrative in a way that we distributed suggestions among related themes. Therefore, for the first three themes we present a summary of the results, an analysis in the body-artifact functional system framework, and related ideas from the Suggestions / Improvement theme. We refer to the students with alphabets in this narrative.

Movement & technology

The theme describes how technology catalysed movement in the classroom. According to the observer "students started experimenting right away, they discovered the software by themselves". First, they experimented with how the stick figure on the screen followed their movements without any instructions from the teacher. When later the teacher instructed them to move the geometrical shapes with their limbs, some of them experimented with the task, but some of them kept engaging only with the stick figure: "H is dancing, I, J are experimenting translation, enlargement", "B found out that they can move it with their face" (observer). In the framework of the body-artifact functional system (Shvarts et al., 2021), we can see perception-action loops on two levels. On level 1, when students interact only with the stick figure, their action is moving their bodies, and the perception is seeing the stick figure moving according to the way they move their body parts (the ones visible to the webcam). During the perception-action loop they try to activate the stick figure as much as they can by showing more body parts to the camera (standing on a chair) and moving intensively (dancing). On level 2, they use limb commands to transform the geometrical shape. Here the action is putting an appropriate limb in the area of the shape to perceive that the dot representing the limb is within the shape, and the resulting transformation of the shape. Level 1 is necessary to move onto level 2, but mathematical learning is only happening on level 2. Therefore, we deduced that the teacher needs to create circumstances in which the student moves to level 2 using the software. Hence, suggestions that may aid students in moving from level 1 to level 2 includes making it easier to use limb commands and make the software assign tasks itself.

Motivation / engagement

The theme describes the levels and sources of student engagement and motivation during the lesson. In the questionnaire and the group interview students expressed a highly positive attitude about the software, they said they liked it because "it was unconventional", "very funny and interesting", and most of them said they liked it because it was tracking their movement. In the observation protocol the observer noted many instances of individual students being engaged with the software, and in the post-lesson interview he said:

Observer: It definitely increased their motivation to do it, I think that was true for everybody. Let's say that everybody without exception tried this interface and they were interested in it. I think it was such an exciting thing for them.

The teacher also noted that even students who are less active usually were involved, including a student with dyscalculia. “They kept being engaged during the mathy-worksheet part.” There were also instances when some students were disengaged. The sources of engagement through the framework of the body-artifact functional system (Shvarts et al., 2021) seem to be the perception-action loops (both level 1 and level 2 described in the previous theme) themselves. It is novel and surprising for students to see their own body movements and their effect as transformations on the screen. The word “experiment” is linked to engagement multiple times, which expresses directly the nature of perception-action loops: students experiment with the consequences of their actions perceived on the screen. Many of the students' suggestions for improvement related to how to make the software even more motivating to use: e.g., track the whole body and show a real body shape with contours rather than a stick figure.

Learning

The theme describes mathematical learning taking place in the lesson, and in what ways and to what extent GeoGebra Dance contributed to it. Some instances of mathematical understanding came from observing the screen directly (*treating*, in the framework of Duval, 2006). For example, when the teacher asked why dilation is angle preserving, the reply was:

Student: The form doesn’t change, only the size [that is why dilation is angle preserving].

In other instances further mental abstract manipulation was needed (*conversion* in the framework of Duval, 2006). For example, the pair manipulating a circle had difficulty checking whether transformations are angle preserving, which then they were able to do by considering sectors of the circle:

Teacher: If you have a circle, how do you check it? [whether translation is angle preserving]
Student: Slice it like a cake.

When the students were asked about what they learned in mathematics in the questionnaire and in the interview, answers included “I got a better overview of geometry”, “I was able to visualise the shapes better”, “not much in maths” and “thinking”.

Both the observer and the teacher pointed out that students did not use the software during the worksheet task, but they referred back to their experiences with the software during the whole-class wrap-up discussion.

Observer: [based on] what they said, they had the experience with the concepts, they weren't just thinking about them speculatively, but they could see them, they could imagine the transformations better through trying them.

Hence they inferred that GeoGebra supported mathematical learning. They both said that it is difficult to judge how much the added AR element (transforming geometrical objects with movements rather than with the mouse) added to the mathematical learning directly. According to the observer it mostly added to learning through motivation, while the teacher supposed it had more direct influence.

Let us interpret these observations through the framework of the body-artifact functional system (Shvarts et al., 2021). Both the students and the observer mentioned that "visualising" and "trying" transformations aided learning, and these two words correspond directly to "perception" and "action". It seemed evident to all research participants that engagement catalysed by embodied instrumentation had a positive effect on learning. Students perceived learning various general skills, but they did not perceive learning about the curricular topic of transformations. This aligns with the decentralised and multilevel nature of embodied instrumentation: students do learn complex skills besides the curricular topic (e.g., "thinking" and "overview"), but learning about the specific curricular concept is less explicitly visible. Both the observer and the teacher stated that they could not directly perceive the effect of the added embodied element of GeoGebra Dance (gross motor control through AR) versus the original GeoGebra (fine motor control through the mouse), and their indirect inference is probably influenced by their beliefs.

Suggestions / improvement

In several suggestions embodiment through AR has a strong connection to mathematical concepts, hence the direct effect on mathematical learning is more apparent, for example:

- making limb commands more intuitive by relating them to the transformation itself (software), and creating a choreography related to the composition of transformations (activity)
- tracking the movements on the screen (software), and students creating a mirror image of a given shape (activity)
- coding movements, e.g. projection (software), pairing movements to codes or decoding movements (activity)

The AR nature of the software makes it a challenge to implement some of these changes, as it is difficult for the software to recognise the intention of the user, for example, to assess when a certain movement starts and finishes.

Conclusion

The aim of this study was to find ways GeoGebra can support embodied learning in mathematics, and as a first step of the embodied design process we added an AR element to GeoGebra and used it in an activity with geometrical transformations in a grade 6 mathematics classroom. We found that the use of Geogebra enhanced mathematical understanding and the AR had a motivational effect, but it was not clear to what extent embodiment through AR supported mathematical understanding. In the redesign phase we will aim at making embodiment more explicit by integrating suggestions received during the study, e.g. making limb commands more intuitive and inline with the mathematical concepts, and involving various other forms of embodiment and further mathematical topics.

A limitation of the study was the limited classroom data collected. In the next implementation phase we are hoping to gather data that reflects classroom actions closer, e.g., video-recording, eye-tracking and screen recordings, providing a more comprehensive analysis of the instructional approach.

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Dancing with trigonometric functions: aesthetically guided covariational reasoning while creating periodic dynamic models

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We draw upon a materialist perspective to report on 11th grade students' meaning making on periodic functions guided by their aesthetic responses while interacting with digital tools to create dancing animations. The students acted as 'choreographers' and coded digital models to move according to periodic functions (simple trigonometric functions or trigonometric polynomials) of their choice, based on their graph, in order to capture their sensory-motor ideas of what periodic movement better matches a song. They became involved in a continuous shaping of their covariational reasoning expressed multimodally and framed as the iterative re-formation of their evaluative lens regarding mathematical attributes. After addressing the synthesis of theoretical ideas, we discuss initial insights on the interplay between the material activity afforded by the digital tools and students' shaping of aesthetically-guided covariational reasoning on periodic functions.

Keywords: Covariational reasoning, periodic functions, aesthetics, music, educational technology.

Introduction

Technology has revolutionised the types of interactions with abstract mathematical concepts, as they uniquely provide them with material, dynamic, perceivable forms, bridging them with students' embodied experience (de Freitas & Sinclair, 2013). New materialistic approaches have introduced new ways of viewing mathematical meaning-making connected to material acts enabled by the digital tools (Latsi & Kynigos, 2022; Sinclair et al., 2013). Learning is situated within a complex system of collective interactions among human (students, teachers) and material (concepts, artefacts and tools) actors with distributed agency. Thus, technology functions as active participant in the meaning-making process. This aligns with Pickering's (1995) metaphor of a "dance of agency" that ascribes the elements of dynamicity, symmetry and interconnectedness among different actors in the creation of scientific knowledge. Recently, concerns have been raised over the way tool-dependence carrying the designer's intentions could lead to students' passive participation (White, 2019). We address this concern for the case of the concept of function, which has gained increased research attention. The majority of studies use digital simulations of physical phenomena and explore students' reasoning as they interpret or construct graphs within problem-solving tasks (Moore, 2014; Thompson & Carlson, 2017). Material acts are pre-determined by the researchers-designers, while students take the role of the observers and passive actors with limited space for creative activity. Following the dancing metaphor and adopting a constructionist approach, we claim that students should not just take the role of the actors (dancers), but also that of the choreographers, i.e., to be creative actors (Sinclair et al., 2013). In this study, we took this metaphor even further by exploring a case where students became actual choreographers of dynamic figural artefacts that move according to periodic functions of their choice, in order to match to a specific song. Drawing on the key role of aesthetics in enabling this shift from passive to embodied, affective and creative engagement, we placed the material act of

aesthetic evaluation in the centre of the designed environment (de Freitas & Sinclair, 2013; Karavakou et al., 2023). Periodic functions were 'animated' (in a dual sense) and put under iterative evaluation, in the context of free exploration and artistic creation. Students' creative activity was guided by their own sensory-motor ideas of *what periodic movement better matches a song*, shaped by the digital tools. Reasoning on periodic covariation, defined as 'periodic covariational reasoning' (PCR), is analysed as part of the formation of students' aesthetic criteria (i.e., subjective evaluative lens regarding mathematical attributes) rather than respecting objective norms. To shed light on the relationship between meaning-making and the material activity, we discuss the questions: What types of PCR do students develop as they form and express aesthetic criteria on aspects of periodic functions? How the material acts afforded by the technologies influenced the shaping of their PCR?

Aesthetically guided covariational reasoning within a materialist perspective

Covariational reasoning (CR) has been defined as the ability to coordinate the parallel changes of two quantities that covary (Thompson & Carlson, 2017). Researchers have introduced models with developmental levels for the description of students' evolution of CR. Such levels involve students' ability to adequately describe aspects of covariation, such as direction and (discrete, average or instantaneous) rate of change of the two quantities. However, these models have been developed within discursive theoretical approaches, that prioritise mental processes over the material interactions. Drawing upon a neo-materialist perspective, CR emerges from students' embodied interactions within an assemblage of materials (graphs, diagrams, manipulatives, artefacts) and contexts (e.g., physical phenomena) with distributed agency. Rather than framing CR as the alignment of pre-defined variables, new materialism highlights its multiplicity and fluidity within a process of becoming (de Freitas & Sinclair, 2013). CR is framed as students' creative acts, which, according to Sinclair et al. (2013) involve 'acts that introduce the new in an unpredictable way that transgresses current habits of behavior and exceed existent meanings', and are 'sensible to the distributed and collective enterprise of the classroom' (p. 252).

New materialism integrates aesthetics into students' reasoning by emphasizing the embodied, sensory and affective dimensions of their creative acts. In this perspective, aesthetics is not confined to subjective judgments of beauty or elegance, neither attached to fixed, inner properties of mathematical results, but is interwoven with the dynamic interplay of bodies, concepts (functions) and (digital) tools (de Freitas & Sinclair, 2013). Students form aesthetic criteria, i.e. evaluative lens for something to be liked or not, on a mathematical concept (e.g., of a function), which is materialised and under assessment. Aesthetic criteria and mathematical meanings are under co-formation in an appropriate learning environment. The latter can be 'aesthetically rich' as it fosters appreciation for the beauty and coherence of the mathematical concepts. Such an environment emphasizes embodied and material entanglements that evoke wonder, curiosity and a sense of harmony or disruption. It can be supported by technologies with constructionist, dynamic, exploratory features, and through open artistic expression and creation (Karavakou et al., 2023; Sinclair, 2004). In this context, aesthetic criteria serve as heuristics, guiding students' CR. They explore what is *beautiful*, *harmonic* and *aesthetically pleasing* within the subjective context of their artistic creation, rather than what is the *correct* solution in a pre-fixed context (e.g., a physical situation with predetermined conditions).

Methodology

Periodic functions were selected for bridging the abstract concept of covariation with the dynamic arts of dance and music through their periodic nature. Students explored and used periodic functions of two types (a. simple trigonometric functions of sine and cosine, and b. trigonometric polynomials) in order to create animated figural models of periodically varying geometric magnitudes in a digital resource, called MaLT2. MaLT2 (http://etl.ppp.uoa.gr/malt2/) is a freely available online Logo programming tool that embeds 3D Turtle Geometry, but also allows for dynamic manipulation of the graphical output through sliders changing variable-procedure values, temporalizing the functional behaviour (Kynigos & Grizioti, 2018). Students could use a periodic function as input in a logo-movement command (e.g., *forward 10*sin(2*:t)*, or *right 12sin(:t)+50*cos(2*:t)+30*cos(4*:t)*) and, by steadily-moving the slider of the variable *t* to the right (through the right-arrow key), they could create a simple oscillating movement of specific amplitude and frequency-period or a more complex periodic movement (in the case of polynomials). MaLT2 provided means for open expression and exploration of functional properties, which were materialised, animated and under aesthetic evaluation. For selecting the appropriate function formula to input within the code, they were also given a supplementary tool for plotting the function graphs by dynamically changing coefficients values through sliders; the GeoGebra Graphing Tool (https://www.geogebra.org/calculator).

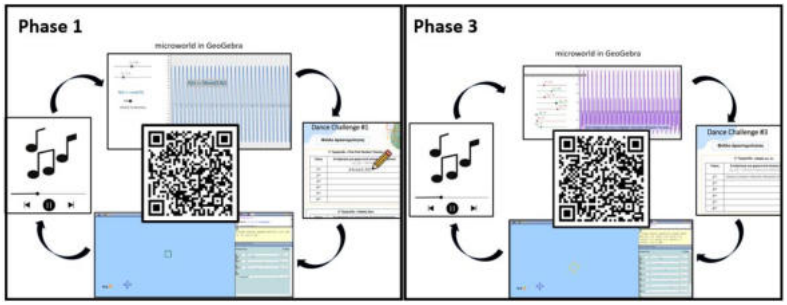

Figure 1: Resources designed for Phase 1 and Phase 3 (can be also viewed by scanning the QR codes)

The designed tasks were divided into four phases, all framed as "dance challenges". Phase 1 and 3 involved four and three songs respectively, with six rounds (for each song) of: 1) exploring the parameters through sliders to adapt the graphical characteristics of the function, to capture their dancing-movement idea that matches the music; 2) noting down the formula on a worksheet; and 3) creating the animation in MaLT2, using the formula generated in 1 (as seen in the videos of Figure 1). Phase 2 and 4 composed the "main challenge" that involved free creation of two in parallel dynamic figures through free programming in MaLT2, to match one song: starting with simple trigonometric functions (Phase 2) and continuing it with trigonometric polynomials (Phase 4).

This study is part of a design-based project that consists of two cycles of design, implementation and thematic analysis to generate empirically grounded theories and frameworks for action. In this paper we analyse a part of the main cycle, in which eight 11th grade students (S1-S8) divided in groups of two (G1-G4) participated. Empirical data included voice and screen recordings, and students' and attending researchers' written notes. The results were framed as different patterns of aesthetically-guided PCR, whose flow is reflected in the diagram of Figure 2. Students followed similar circular patterns of engagement, where their PCR, expressed through aesthetic criteria, was continuously

under re-formation, affected by their interactions with the digital tools. While listening to the song, they expressed, either verbally or bodily, an initial sensory-motor idea of *what motion better matches the song*. Next, they addressed aesthetic criteria on the periodic functions under use, regarding graph's attributes or parameters' values on four aspects: *period*, *range*, *direction* and *rate of change*. Aesthetic criteria were both of subjective and relational nature, as they were evaluative filters on specific materialised aspects of periodic functions, under the condition to create a movement that matches the song. They then selected a formula, based on its graph, that satisfied their criteria for inputting into the code of their dynamic model in MaLT2 to create an animation that captured their motor idea. Observing and aesthetically evaluating the animated feedback while listening to the song, as well as recognizing the song's attributes, would then feed the reformation of either the initial idea or their aesthetic criteria. The mathematical properties, as materialised by the tools, influenced this creative process in these two levels, serving as agents in the process of PCR formation.

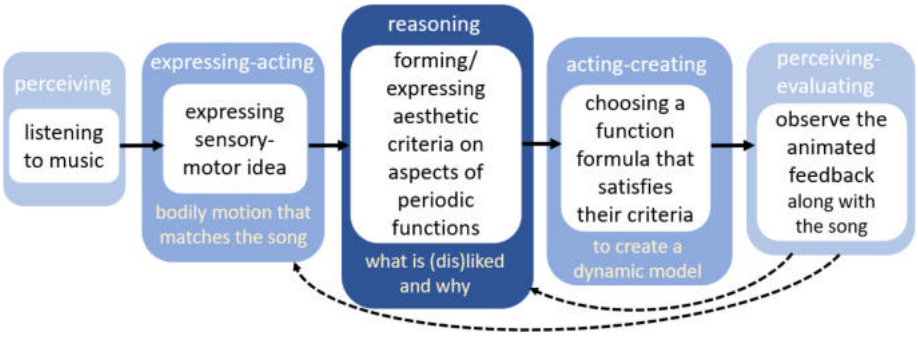

Figure 2: Flow of students' engagement in aesthetically-guided PCR

Results

In the first rounds of phase 1, three groups expressed sensory-motor ideas of dancing movements that followed the song's main melody of not steady rhythm. Students evaluated positively any plotted trigonometric function, regardless its amplitude or frequency value, without addressing any criteria. They developed PCR of *varying periodicity*, as they predicted that the trigonometric covariation would be of non-steady temporal periodicity, which matched the varying melody's rhythm. After one or two rounds of observing and disliking the animation, they altered their sensory-motor idea of dancing movement, by adjusting it to the restrictions provided by the trigonometric functions and developed PCR of *steady periodicity*. The following episodes capture the way that G3 developed this pattern of PCR. In the 2nd round of the first song, S5 claimed that $f(t)=22\sin(t)$ would perfectly fit the melody. S5 moved the mouse along the graph according to the varying rhythm of the melody (video in Figure 3a). By testing the animation in MaLT2 (video in Figure 3b), S5 revised her idea:

S5: No, no. I was wrong. It seems even slower.
S6: We were thinking wrongly about it. We need to catch the steady tempo from the background. This tountoun-tountoun (sings the song's tempo).

S6 turned to GeoGebra and changed the coefficients values, choosing the function $f(t)=20\sin(3t)$. Then she moved the mouse across the graph at a constant rate, as shown in the video in Figure 3c.

S5: You're not moving it right. We want to catch the main melody.
S6: We have to catch the steady rhythm. There is no point in catching the main melody. The motion will always be steady.

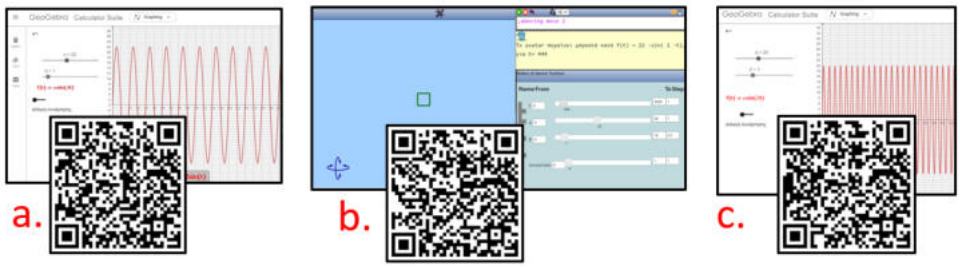

Figure 3. Videos of G3's actions: a) initial sensory-motor action captured in GeoGebra (link); b) animating the function *22sin(t)* in MaLT2 (link); c) altered sensory-motor action in GeoGebra (link).

S6's motion shows that she thought of the covariation of $f(t)=20\sin(3t)$ steadily, developing PCR of *steady periodicity*. After evaluating the animation in MaLT2, she reconsidered the initial sensory-motor idea of matching movement, based on the materialised property of steady temporal periodicity.

Following the emergence of this type of PCR, all groups started addressing specific aesthetic criteria regarding the graph's form and parameters' values, in order to capture the preferable dancing motion of steady period. Thus, in phase 1, the majority of PCR developed was focused on the concept of the period of trigonometric functions, following a similar pattern for every group. All four groups initially developed reasoning on *amplitude-dependent period*, i.e. on the increase or decrease of the amplitude value or graph height in order to respectively increase or decrease the covariation period. The following example showcasing the above case of PCR comes from G1:

S1: It's really slow. It's not even close to the beat.
S2: How will we fix it?
S1: It needs to be quicker. Make it do smaller moves.
S2: Ok, then… Should I reduce the alpha? It will take less time to reach the edges. (Changes the amplitude value from 8 to 6, plotting the graph of $f(t)=6\sin(2.5t)$.) Ok, nice. It looks much quicker. It will go like this.. (Sings the melody and moves hand).

In the above episode, G1 predicted that the graph with lower peaks would produce a quicker periodic motion, resulting in a smaller period. They bodily expressed PCR on the *proportional dependency of the period from the amplitude*. Then S1 tested its dynamic representation in MaLT2 while listening to the song (movement a of the video of Figure 4a-b). The unexpected result redirected their thinking:

S2: No, no, no! It seems even slower to me. (Opens the GeoGebra tab.)
S1: This doesn't make any sense. Here it has less points than before. So it was supposed to go quicker. Try a higher alpha.
S2: What should I do with the beta?
S1: Leave it as it is for now. We need to understand what alpha does. Try 10.
S2: (Changes the amplitude value from 6 to 10, plotting the $f(t)=10\sin(2.5t)$). Now?
S1: It certainly does not look quicker. It will need much more time to reach the edges.

S1 noted down the formula of the function and tested its dynamic representation in MaLT2, while listening to the song (movement b of the video of Figure 4a-b). After observing it, they commented:

S2: Now I feel it got a bit faster. Let's raise it even more!
S1: No, it seems just as slow. Let's not risk it. I think we should focus on the beta.

In this episode, S2 developed reasoning on the *inverse-proportional dependency of the period from the amplitude*, possibly influenced by perceiving the animation's higher rate of change. On the contrary, S1 developed reasoning on the *independency between period and amplitude* and suggested

to 'focus on the beta' in order to change the period of the movement. Both students changed their reasoning based on the way they perceived the animated feedback and reformed their aesthetic criteria regarding the parameters' values. During the subsequent episodes, the majority of PCR types was on the *inverse-proportional dependency of the period from the frequency*, based on which they managed to capture the appropriate period of the trigonometric covariation. However, *amplitude-dependent period* types of PCR kept emerging in phase 1 in reduced frequency. In phase 2, they were fully replaced by *amplitude-independent period* ones. Students' aesthetic criteria exceeded the aspect of the period and also integrated aspects of range and rate of change. G1 expressed the idea to create two synchronized squares that moved with the same period by inputting two trigonometric functions into two different logo-commands. After four trials, they chose the same function (*f(t)=60sin(9t)*) in both commands, which satisfied their criteria regarding the period. After observing the animated outcome, S1 formed another idea on one dancing figure, whose motion was an oscillating turn:

S1: It would be better if it made a full turn. (Changes the command *right 60*sin(9*:t)* to *right 180*sin(9*:t)* in the code.) 180 to the right, 180 to the left (moves hand).
S2: Now you need to change the nine to keep the same pace. Make it six. No, three.
S1: Beta should be equal to 9 for both figures. It is non-negotiable. 9 catches the rhythm.

S1 animated the two models, while playing the song in MaLT2 (video in Figure 4c). The audiovisual outcome led S2 to develop *amplitude-independent period* type of PCR and extend their dancing-turning movement idea by raising criteria to the amplitude in order to increase the intensity:

S2: Ok, I got it. They have the same period… no matter how big the front number is. But it gets more abrupt. Let's increase it more. Put 300. No, 600. 600 will be awesome! Can you imagine? (Laughs.) It will match the song's vibe completely.

In phase 3 and 4, all groups expressed complex aesthetic criteria, enabled by the diverse possibilities of trigonometric polynomials. They raised their aesthetic criteria to generalised graph's attributes, such as horizontal or vertical distance between consecutive extrema, monotony, curvature and slope, in order to capture diverse song's perceived attributes through the dynamic models, such as the tonic sequence, intensity and energy-vibe. The following episode of G4 from phase 4 includes part of their engagement to create a dancing animation of two figures. In the beginning S8 expressed his sensory-motor idea for the second figure, after they had already chosen a well-rated function for both figures' motion (Figure 4d). His idea changed while observing the animated outcome of synchronized figures:

S8: I've got a better idea. Now that I envisioned the choreography. They start synchronized, I like that. But then the blue star follows with a slower move. (Replays the song along with the animation and moves his hand next to the screen.) Look at my hand. See what I mean? I imagine it slower. Smoother. It's more natural if they lose their synchronicity for some parts.

S8's idea of 'more natural' asynchronous combination of movement was shaped through his bodily interaction with the animated figure. S8 took a photo of the first graph with his phone and used it as basis to create the one that would satisfy the aesthetic criteria. He ended up creating the graph of *60sin(t)+4cos(t)+10sin(2t)+4cos(2t)+11cos(3t)+10sin(4t)+3cos(4t)*(Figure 4e).

S7: This is really high! Are you sure? Are you sure it will be slower? It seems abrupt.
S8: Yes, I have this feeling. Here it leans more. And it has this part here (shows the turning point using the mouse) that makes it even smoother, like.. (moves hands).

While viewing the new animation (video in Figure 4d-e), they reflected on the graph interpretation:

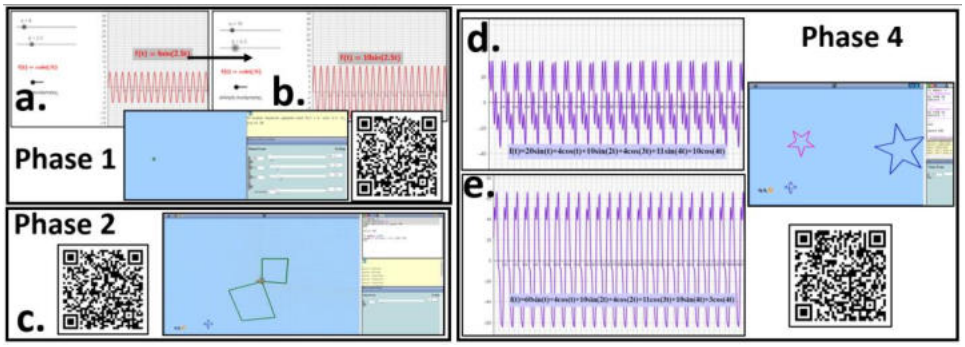

Figure 4: Dancing animations made in a-b) Phase 1; c) Phase 2; and d-e) Phase 4

S8: See the blue star? The more the line leans horizontally, the smoother it moves.
S7: It's like.. Yes! I can feel the music! The combination is wow! (…) Maybe we should smoothen this part also. (Shows the higher part of the graph-period). So that it would go like … (hand movement) calmer than the other the whole time.

S7 initially rated the graph poorly while expressing PCR on *amplitude-dependent speed*, as he predicted that bigger height between extreme points would lead to 'abrupt' motion. Then S8 defended his choice based on the bigger slope, forming *slope-dependent speed* type of PCR. After observing the generated animation, S7 extended his idea on the motion of the second figure, expressed through hand motion in coordination to the animated artefact. He addressed new aesthetic criteria for graph attributes (to smoothen its upper part) that further guided the group's material and creative activity.

Discussion

This study offers initial insights regarding the interplay between students' PCR and the material activity within the designed aesthetically rich environment. PCR was traced as creative acts arising through the material act of aesthetic evaluation. It involved meaning making on the materialised aspects of periodic covariation under evaluation. Students engaged in iterative re-formation of both their sensory-motor choreographic idea that guided their actions, and the aesthetic criteria regarding mathematical properties (Figure 2). These dynamic changes were grounded on the way they perceived (not always in the same way as shown in the case of G1) the animated feedback in relation to the music. Their ideas were adaptive to the material agents (type of functions and tools) of each phase. In phase 1, all groups focused on matching the period of covariation to the song's tempo. This focus did not emerge automatically, but arose after discriminating the possibilities of sine and cosine functions, as shown in the first episode of G3. Their criteria were consequently focused on the role of the parameters; starting with reasoning on the *proportional dependency of the period from the amplitude*, moving to reasoning on their *inverse-proportional dependency*, and finally to *amplitude-independent period* and *inverse-proportional dependency of period from frequency* PCR. In phase 2, open exploration of different types of synchronicities of two parallel dynamic models was pivotal for establishing the last two PCR types around the aspect of *period*, but also for enriching the PCR aspects towards the *range* and *rate of change* (intensity), as shown in the second episode of G1. In phases 3-4, all groups expressed more complex sensory-motor ideas for dancing movements, which captured the song's attributes in a richer way, e.g. by following its tonic sequence, or by creating a combination of asynchronized models of different intensity, like G4 did in the last episode. In order to capture these ideas, students expressed diverse aesthetic criteria on all four PCR aspects regarding generalised

graphical properties (e.g., its slope). This enrichment of ideas and criteria was enabled by the polynomials' widened possibilities. Future steps involve the integration of motion-capture technology that would provide means for richer variety of bodily ways for expressing aesthetically guided PCR.

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Learning is moving in better ways in and across task contexts

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In this paper, I aim to contribute to the mathematics education literature by suggesting an operational definition of embodied learning. I outline three general facets of learning that a desired definition should reflect: regulation, improvement, and transfer. Based on Abrahamson and Sánchez-García (2016), I propose the operational definition in the title and demonstrate its applicability in a case study from my project, "Blindfolded Mathematics," where learners engage in mathematical tasks using tangible artifacts while temporarily deprived of sight. I conclude by calling researchers wishing to capture in-the-moment emergence of learning to distinguish between different "ways of moving" and specify which ways are better within their contexts, along with justifying why.

Keywords: Embodied cognition, embodied learning, blindfolded mathematics, learning definition.

Embodiment in mathematics education—searching for simplicity in a world of complex adaptive systems

Embodiment in mathematics education has emerged as a key theoretical framework, emphasizing the integration of sensorimotor experiences and physical engagement with mathematical concepts. This theoretical framework, based on theories such as ecological psychology and complex dynamic systems, posits that cognition is not a disembodied process but is deeply rooted in the learner's interaction with the environment (Chemero, 2013; Gibson, 1986/2015; Varela et al., 1991). The embodied learning paradigm brings to light complex phenomena, such as self-organization, distributed cognition, and non-linear learning processes (Bernstein, 1967), emphasizing the interactions between bodies and the material world (de Freitas & Sinclair, 2013; Shvarts et al., 2021). Although the embodiment literature provides rich and profound descriptions of learning, these descriptions seldom cohere with the conventional understanding of classroom teaching practices.

Recent embodiment research raises concerns that many teachers, particularly those without a substantial theoretical background, may find it challenging to incorporate these theories into their teaching (Cobb & Jackson, 2011). This highlights the need for simplification and accessibility of such theories for broader pedagogical use (Macrine & Fugate, 2021). This paper addresses this need by proposing a concise *operational definition* of embodied learning. An operational definition explains a concept by specifying the exact procedures or tests used to identify and measure it. The measured properties must be publicly observable so others can independently verify or assess them (Harel & Koichu, 2010). Therefore, an operational definition of embodied learning may assist researchers in capturing and describing learning processes across various contexts.

One of the most straightforward and concise descriptions of learning was suggested by Abrahamson and Sánchez-García (2016, p. 203): "Learning is moving in new ways." These authors used it as a thematic slogan based on their investigation into the complex relationship between physical action, perception, and conceptual understanding in mathematics education. Drawing on theories from

ecological dynamics and embodied cognition, they argue that learning, particularly in fields like mathematics, is not simply the acquisition of abstract symbols and rules but is rooted in sensorimotor interactions with the environment. Through engaging in designed "fields of promoted action" holding environmental and task constraints (e.g., Mathematical Imagery Trainer), learners develop new motor-action routines and attentional strategies to solve problems. Their gradually emergent physical interactions—such as adjusting hand movements to achieve proportionality—allowed students to internalize mathematical concepts through action. In essence, learning involves students finding new ways to engage their bodies and senses in problem-solving, which leads to a deeper, more intuitive grasp of mathematical ideas represented in normative semiotic displays. Thus, "learning is moving in new ways" suggests that cognitive development is deeply intertwined with physical movement and interaction, bridging action and thought (Abrahamson & Sánchez-García. 2016). The succinctness of Abrahamson & Sánchez-García (2016) slogan and the elaborated descriptions of embodied learning they provide drove me to adopt their work as a basis for an operational definition of embodied learning for mathematics education.

Three requirements from an operational definition of learning

Typically, operational definitions are crafted to align with broader conceptual definitions (Harel & Koichu, 2010). Accordingly, I outline vital aspects the operational definition of learning should reflect.

(a) Regulation: Learning is directed towards organized behavior, instilling order and stability where disorder or nothingness previously prevailed. The operationalization of this trait is that to determine whether learning has occurred, we search for repeating patterns of behavior—for dynamic equilibrium in body-environment interactions (Harel & Koichu, 2010). This is perhaps the reason Abrahamson and Sánchez-García (2016) did not maintain learning is "performing new movements" but instead talked about "ways"—schemes of movement—emphasizing the regulatory aspect of distinguishable repeating movement patterns.

(b) Improvement: Learning is generally incremental. In an educational context, we do not want to consider any change in one's behavior as learning. Learning is about a specific kind of change: a change for the better. Learning is about improving at something. So, if we want to talk about learning meaningfully, we ought to be clear about what this "something" is: A task, problem-solving (see Harel & Koichu, 2010), discourse practices, physical exercises, or mastery of some mathematical content. In other words, learning can only be determined concerning goal-oriented activities with better and worse actions, evaluated by their efficacy in goal achievements. For instance, Abrahamson & Sánchez-García (2016) work improvement is illustrated by students adopting techno-scientific frames of reference and action (i.e., attentional anchors) and developing mathematical explanations in concert with their movement strategy.

Although we would like to think of learning as incremental, it does not always mean it is strictly linearly incremental. On the contrary, it is widely acknowledged that learning processes are non-linear and involve regressions in ability/performance towards a solution of some situated problem (Palatnik & Marco, 2024; Tancredi et al., 2021). Operationally, I suggest that regressions be considered part of learning whenever they are linked to progressions in future iterations. For instance,

if I once solved differential equations efficiently but could no longer do so after years without practice, we would not call this decline "learning." However, if I make a failed attempt, learn from my mistakes, and improve, we will consider the entire episode as learning.

(c) Transfer: Learning involves generalizations and adjustments. If attaining some ability in a first situation seems to be applied in another similar situation, we would be propelled to maintain that learning occurred in the first situation. The notion of transfer (Barnett & Ceci, 2002) here means that a first situation/environment and the behavior developed in interaction with it were generalized to allow a similar environment to be interpreted as having similar affordances (Gibson, 1986/2015)—inviting the enactment of similar patterns of productive behavior in the first situation. Meanwhile, since every situation puts different constraints (Abrahamson & Sánchez-García, 2016), adjustments are almost always essential for achieving one's goals. For example, an infant first learns to walk on her house floor. The next day, she walks on the high grass in the park. Doing so, she generalized the walkability of the environment and slightly adjusted her walking pattern to the new environment (e.g., raising her knees higher).

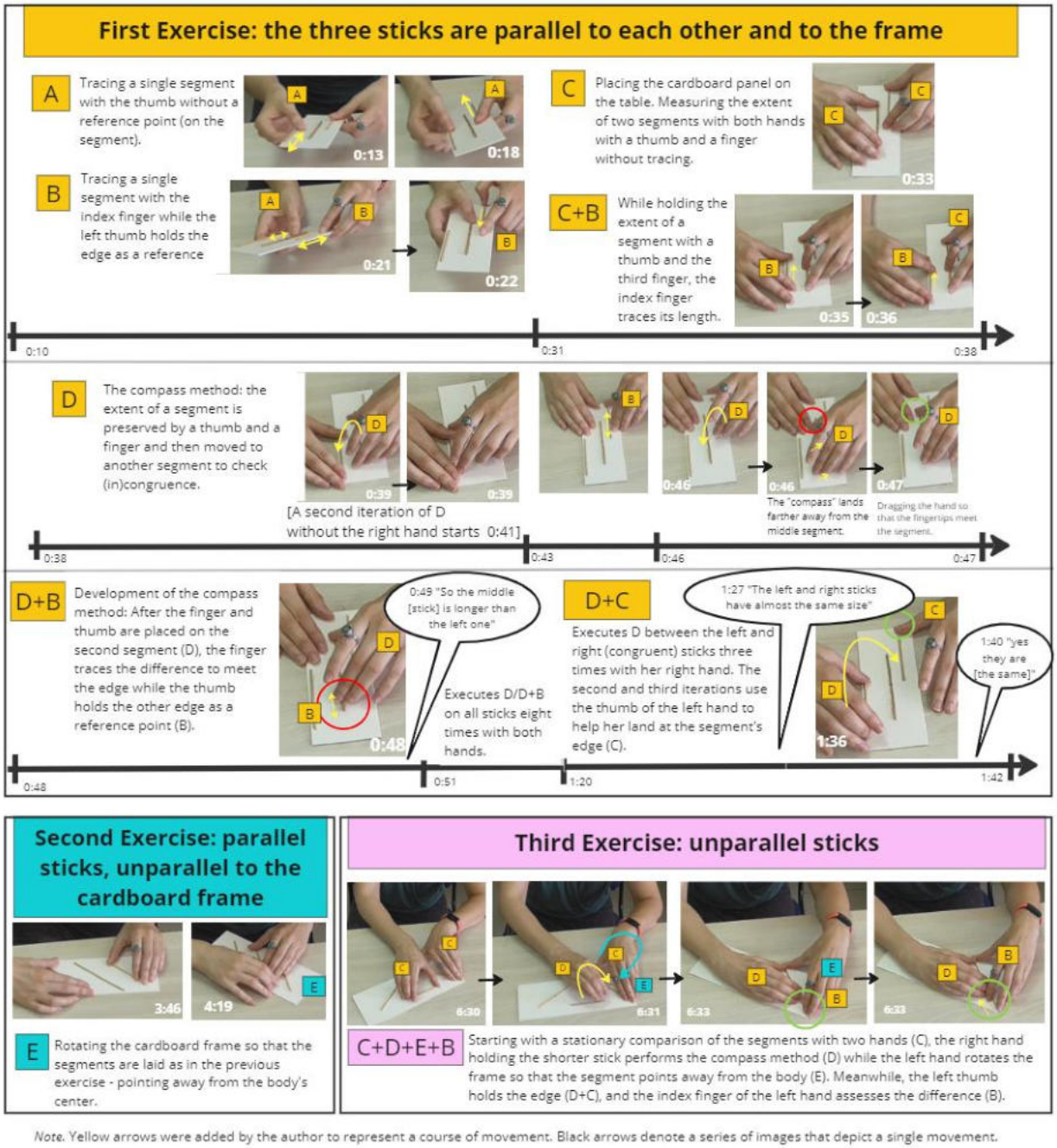
Therefore, building on Abrahamson and Sánchez-García (2016), I suggest the following operational definition of embodied learning: *"Learning is moving in better ways in and across task contexts."* This definition is embodied, conceptualizing learning through "movement," and incorporates the three operational requirements I have mentioned above: (a) *regulation*: identifying repetitions and the emergence of (relative) stability of behavior ("ways"); (b) *improvement*: identifying progression regarding specific goal-oriented activity ("better"); (c) *transfer*: capturing the generalization and adjustment aspect of learning ("in and across task contexts"). In "task contexts," I mean that transfer may be evident both when one performs a similar task in a different context or a slightly different task in a similar context. In the next section, I use this definition to analyze an episode and demonstrate its applicability.

Learning as moving in better ways—a case study

The following episode is taken from the "Blindfolded Mathematics" project, which aims to develop mathematical learning by circumventing temporal sight deprivation when engaging learners with mathematical tasks that include tangible artifacts. In a nutshell, I argue that learning occurs whenever a "better" moving scheme appears and is repeated. By "better," I mean more adapted and efficient in achieving the activity's goals. Additionally, I show that the appearance of merely "new" movement schemes that are less productive can, at best, be classified as "part of learning" if they lead to "better" movement schemes down the road to task completion.

The participant, Ella, a Ph.D. in astronomy, was a math lecturer in a math teacher education program. Her task was simple: With eyes covered, she sensed and determined the relative length of three sticks glued to a rectangular cardboard panel. The task included three phases hereafter called "exercises": (1) the three sticks are parallel to each other and to the cardboard panel sides (Figure 1, top), (2) the three sticks are parallel to each other but not to the rectangular sides (Figure 1, bottom left), (3) none of the three sticks is parallel to another (Figure 1, bottom right). After each exercise, Ella was asked to open her eyes and was presented with a ruler measurement of the three sticks to verify her answers

visually. The task served as a preparation for a sequence of more complex mathematical tasks involving geometry proofs that will be reported elsewhere.



Note. Yellow arrows were added by the author to represent a course of movement. Black arrows denote a series of images that depict a single movement.

Figure 1: Milestones in the task: "What is the relative length of three sticks?"[1]

[1] For an enlarged view of Figure 1 and to leave comments, readers are encouraged to visit the Miro board at https://miro.com/app/board/uXjVKhJCmEE=/?share_link_id=127905518767

Overall, it took Ella 6:35 minutes to complete this preparatory task successfully. Based on the literature, I expected the third exercise to be more difficult as it involves assessing the extent of the unparallel segment (Armstrong & Marks, 1999). However, it took Ella 20 seconds less to solve it (2:12 min.) than the first exercise (2:32 min.). This alone suggests that she has learned something during the performance of the first and second exercises that enabled her to complete the third, more arduous task faster. Considering her extensive mathematical experience and knowledge and the basic mathematics involved in this task, she is unlikely to learn any new mathematical content (e.g., segment length, pairwise comparisons of magnitudes, transitivity of inequalities). In what follows, I use my proposed definition: "learning is moving in better ways in and across task contexts" to argue that Ella learned during the episode and specify what she learned.

When Ella first received the panel, she held it in the air and traced the sticks with her thumb while placing all the other fingers underneath it (Figure 1, A). This movement perhaps helped her to allocate the three sticks and have a general sense of their place and relative layout. However, at 0:21 (Figure 1), movement B first appeared—assessing the sticks' extent with the thumb used as a reference point from which the index finger traces the stick. The transition from A to B is gradual, and at 0:21, Ella performs both movements simultaneously with the right and left hands, correspondingly. However, A seems inferior to B in relation to the task goal (measuring extents), and thus, it is abandoned and never re-enacted in the first exercise. On the other hand, B is reiterated many times and combined with other ways of moving throughout the entire episode (e.g., 0:35, 0:43, and 6:33 in Figure 1). Using "learning is moving in new ways" as a definition would posit that Ella learned already when performing A for the first time as it was a new way of moving and it helped her somewhat to make progress—gaining some sensory experience of the sticks. However, my proposed definition emphasizes that learning has occurred by transforming from way-of-moving A (tracing without a reference point) to the more efficient way-of-moving B (tracing with a reference point).

At 0:31, the way-of-moving C first appears: Ella places the panel on the table and, with two hands, touches the edges of two segments with her thumbs and index fingers (without tracing). Embracing "learning is moving in new ways" would determine that learning has occurred. However, Ella does not seem to gain any increased confidence regarding the stick's length performing C; she immediately elaborates on this way of moving by incorporating B into it (C+B, 0:35, Figure 1)—holding the edges of a segment with a thumb and third finger whereas the index finger traces its length. The proposed definition would say that this new way of moving could, at best, be considered part of learning as B+C is an unproductive variant immediately abandoned and replaced by D—the way of moving that will be the game changer in this episode.

In 0:39, Ella first enacts D, which I call the "compass" method: the thumb and finger hold the edges of a segment, then, while preserving the stick's extent, are moved to hold another segment for checking (in)congruence. Ella will reiterate and refine D multiple times throughout the episode (fourteen times only in the first exercise, Figure 1). At 0:43, after two iterations of D, Ella performs B on a single stick with her left hand. At first glance, it seems like a regression as B alone seemed unproductive for length comparisons in previous enactments (0:22, 0:35). Considering learning as merely "moving in new ways" would say this reiteration does not entail learning, as at this point is no longer "new." However, considering what happens next suggests that this seemingly "regression"

to B was performed to make a significant step forward—so my proposed definition would include this iteration of B as part of learning.

At 0:48, Ella incorporates B into D: After the finger and thumb are placed on the second segment (D), the finger traces the segments' difference until it meets the edge while the thumb holds the other edge as a reference point (B). B+D was found to be a very efficient movement for comparing the relative lengths of two segments, and immediately after performing it (0:49), Ella correctly stated that "the middle [stick] is longer than the left one." Accordingly, she reuses D+B multiple times throughout the episode (e.g., 0:48, 2:15, 3:57, 6:33).

When performing D, Ella faces a challenge to land with the thumb precisely on the second stick's edge (see red circle at 0:46, Figure 1). Dragging the "compass hand" to the second stick's edge may reduce precision as the fingers' extent may be shaken (0:47 green circle, Figure 1). Another improvement Ella incorporated into D is D+C, which appears first at 1:36. There, Ella uses the thumb of the other hand to indicate the landing point for the thumb of the "compass hand."

Due to space restrictions, I spare some more minor improvements and nuances found in Ella's work on the second exercise, except for one interesting way of moving, E, that first appeared in 4:19. There, Ella rotated the cardboard frame so that the segments will be laid as in the first exercise—pointing away from her body's center. The image of 3:46 (Figure 1) represents the uncomfortable layout of the sticks she worked with until she performed E. The rotation itself is not new—Ella had performed many rotations of the panel before (compare 0:18 with 0:21 Figure 1). So here it seems she made this small adjustment to allow her to move in an "old way"—actions on sticks pointing away from her, repeatedly practiced and developed in the first exercise. This progress suggests that sometimes learning can be manifested in moving in "old ways" if they contribute to effective goal achievement.

In the last seconds prior to completing the third exercise, Ella performed a very elaborate way of moving, combining C, D, E, and B in a single maneuver with two hands and five active fingers (Figure 1, 6:30-6:33). She started with a stationary comparison of the segments with two hands (C); then the right hand, which held the shorter stick, performed the compass method (D) while the left hand rotated the frame so that the target stick pointed away from Ella's body (E). Meanwhile, the left thumb held the edge (D+C), and the index finger of the left hand traced the difference (D+B). Taking separately, none of the movements in this elaborated maneuver was new. However, together, C+D+E+B satisfies my proposed definition's aspect of transfer and explains how Ella managed to solve the more difficult third exercise faster than the first one.

In one of the subsequent blindfolded tasks, Ella tried to determine whether a quadrilateral made of EVA foam, which she knew had a right angle, was of a special type (e.g., a rectangle, a trapezoid, or a square). The quadrilateral was, in fact, a rectangle. If she applied D to compare its opposite sides, she would have probably discovered they were congruent (as in Exercise 1, 1:42) and inferred it was a right-angled parallelogram—thus a rectangle. However, she did not use D. Instead, she employed a new way of moving (to be reported elsewhere). This new way was imprecise, leading her to incorrectly conclude that the quadrilateral was a general one. Hence, by my proposed definition, her in-task (i.e., in and across the three exercises) learning did not transfer to the slightly altered task context, which limits the overall significance of her learning.

Discussion

In this paper, building on Abrahamson and Sánchez-García (2016), I proposed a concise operational definition of embodied learning: "Learning is moving in better ways in and across task contexts." I strived to incorporate three desirable traits of learning in this endeavor: regulation, improvement, and transfer. Based on a case study, I showed that this definition is applicable and adequate for capturing evidence of embodied learning compared to Abrahamson and Sánchez-García (2016) appealing thematic slogan ("learning is moving in new ways").

Recently, building on Abrahamson and Sánchez-García's (2016), Palatnik and his colleagues (2023, p.162) suggested "moving in a new way, perceiving in a new way, and naming in a new way" as a possible generalized description of embodied learning. I chose not to build the proposed definition based on this description since it is less concise than the original slogan, and "moving" can be considered as including moving for perception (e.g., body positioning and repositioning; Marco & Shvarts, 2024) and verbal utterances (e.g., using mathematical terms verbatim; Palatnik & Marco, 2024). Although I did not use Palatnik and colleagues' (2023) description, their attempt to break down "movement" into smaller categories and modalities is a valuable and essential methodological step. Moreover, to apply my proposed definition of learning for assessing embodied learning, researchers would have to predefine some local specifications of the elements compounding it.

First, studies need to present the goal of the activity/task and account for in-the-moment shifts in learners' intentionality, as they may vary constantly during the activity. Second, researchers should explain how "ways" of moving are identified and distinguished based on observed repetitions. Third, researchers should provide task-oriented justifications for claiming why and by what means some ways of moving are "better" than others (e.g., take less time and effort, yield better results). Fourth, when regressions are observed, researchers should try to link them to later developments and include them in the non-linear learning trajectories. Lastly, researchers should check if transfer has occurred to provide strong evidence for learning. That necessitates capturing evidence of previously developed moving patterns in a slightly alternated task context.

I believe my proposed definition of learning as "moving in better ways in and across task contexts" has significant practical implications for mathematics education, as it can guide the design of learning experiences that promote deeper understanding and engagement. However, it has some drawbacks that I would like to mention. First, putting much emphasis on "tasks" is less applicable to broader contexts in which tasks are not salient and explicit, as in mathematics education. Second, analyzing learning through this lens may inadvertently emphasize observable evidence at the expense of participants' lived experiences, which could provide additional insights into their internal cognitive and emotional processes. A third limitation is the definition's lack of a cultural dimension to capture collaborative learning settings. I hope future studies will find it helpful enough to be improved and adapted into their studies.

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Concretion, abstraction, and mathematical experiences

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As with most words, 'concrete' and 'abstract' are polysemic and used in myriad ways and situational variations. The intent of this paper is not to define them as if it were possible to place them in suitable self-enclosed boxes but to elaborate and suggest a particular philosophical perspective, whereby we think of coexisting processes of concretion and abstraction, rather than consider concrete and abstract as opposing concepts. This might offer a productive usage of these words with the potential to enrich the practices of mathematics and mathematics education, as well as draw attention to commonalities in how these processes are used in a variety of settings, and are part of the everyday activity of all, including mathematical activities at all ages and social contexts.

Keywords: Concretion, Abstraction, Experience, Making, Craft

Introduction

Several years ago, as part of the "Forces in Translation" project, we began to explore how engagement in craft projects can foster novel and rich ways of encountering mathematical ideas. This work is part of a nascent worldwide movement interlacing mathematics and craftwork, e.g., The Bridges Organization (https://www.bridgesmathart.org) or the Journal of Mathematics and the Arts. In our practices, craftwork is neither an "application" of mathematics, nor a preliminary concrete phase launching an ensuing phase dealing with mathematical symbols and its presumed abstractions. For us, mathematics and craft offer a potential for productive interactions both in terms of their correspondences, i.e., the overlapping areas of investigations and their profound differences, which offer opportunities for reflecting and for raising fundamental questions. In this sense, we are trying to move away from arguments that reduce crafts to mathematics or mathematics to crafts, and we are interested in the productive contrasts and provocations that can arise when mathematics and crafts are made with one another. In our efforts to position crafts and mathematics on equal ground, some of us felt a need to reframe, in a philosophical sense, what we mean with words such as 'concrete' and 'abstract'. How could we rethink these concepts so that craftwork and mathematics can both be concrete and abstract? Overall, the common intuitions that concretion and abstraction are dichotomic oppositions, that understanding abstractions is incredibly difficult or impossible for some people, or that the concrete is material and the abstract mental, seem to petrify the potential mobility and richness interlacing craftwork and mathematics. In this paper, we attempt to articulate an alternative perspective about the philosophy of concretion and abstraction.

The next section ("Preliminaries") articulates philosophical premises that we adopt to grasp our understanding of concretion and abstraction. The two subsequent sections elaborate respectively on our renewed understanding of concretion and abstraction. Finally, in a fourth section we reflect on potential similarities and differences with selected texts from the literature in mathematics education.

As we will conclude in the discussion, we hope that, together, these four sections will allow us to take a step forward in recognising that both abstraction and concretion can simultaneously manifest at all ages, genders, ethnicities, and disciplines and transcend a variety of personal and social categories.

Preliminaries

The terms "concrete" and "abstract" are inherently polysemic, shaped by diverse contexts and usages. This paper does not aim to rigidly define them, but proposes a philosophical perspective that, we hope, offers a productive interpretation, enriching both mathematics and its education. Our philosophy of concretion and abstraction grows around two basic premises that will permeate our forthcoming investigation. The first one is to envisage concretion and abstraction as experiential and processual. Inspired by James' thesis in "radical empiricism": "To be radical, an empiricism must neither admit into its constructions any element that is not directly experienced, nor exclude from them any element that is directly experienced" (James, 1904, p. 534), we conceive of abstraction and concretion as submerged in flows of experiences. This experiential focus allows us to eschew qualifying objects, ideas, actions, or symbols as themselves being concrete or abstract. The same object or idea can be experienced as concrete or abstract depending on the kind of processes living beings are engaged in as they encounter or enact them.

The second premise is that concretion and abstraction are not opposed as if they were the terms of a dichotomy. More of one does not indicate less of the other. They are different kinds of experiences that may co-exist in harmony or tension, but not as mutually exclusive processes. To be consistent with this premise, one needs to question common usages of these words, according to which they almost automatically elicit the expectation of their mutual denial. It is often suggested that children are "concrete" or that only adults develop abstract thinking, and, similarly, assumed that artisans think concretely and mathematicians abstractly. These claims are part of pervasive discourses according to which intelligence is distributed unequally along the lines of genetic inheritance, development stages, and educational orientations. Of course, to argue that this is not the case, it does not suffice to just claim it. We must advance and renew conceptions of abstraction and concretion along which we can reasonably assume that they simultaneously manifest with no discrimination for demographic, cultural, or educational factors. These are some of the aims of the three ensuing sections.

On concretion

According to a criterion advanced by Simondon (2017), a technical object undergoes concretion as it gets integrated, so that its parts participate simultaneously in multiple aspects of its functioning, achieving an overall coherence that makes it more reliable and self-regulated. From this point of view, biological organisms are exemplars of concrete entities. One can discern organs as parts of an organism, but organs are not *partes extra partes*, that is, they do not just work alongside each other, like adjoined Lego blocks, nor are they replaceable in a perfunctory way, as a carburettor can be replaced in an old car. The inner whole coherence of a biological organism, plunged in a milieu of matter, energy and information bathing its organs, allows for the joint organic functioning.

Since our interest is to situate concreteness amid pervasive and common experiential processes, we will consider the process of getting to know a city as an experience of concretion. At first, a city may

feel like a collection of unrelated and isolated places – a hotel, a park, a shop – dispersed across a vast, indistinct network of streets. As one navigates and connects these locations, a sense of orientation gradually develops: relationships between landmarks take shape, and a dynamic, meaningful whole begins to form and transform. This process of concretion never starts from scratch, nor does it end; it is always 'in the middle'. It never 'starts' per se, as it is always invoking prior experiences, habits, and frames of reference – such as familiarity with or differences from other cities – as well as cultural connections and disconnections that may shape one's integration with the milieu. Concretion also has no definitive end, not only because one's experience with the city continues to evolve, but because the city itself is not a fixed or static entity. Rhythms of life that animate the city are in constant flux, as streets change, shops open and close, and patterns of movement reshape its spaces. Like Simondon's concrete technical objects, the experience of the city moves toward coherence through the interrelation of its parts, which participate in multiple layers of its functioning and transformation, deepening its integration for the traveller.

In the case of mathematical experiences, instead of cities, we travel and explore what we might call "mathematical worlds". Let's think of mathematical worlds as fields of practice with certain families of symbols, diagrams and tools. A mathematical world may be populated by number sequences and limits, another by polynomials and their roots, others by algebraic operations with matrices, polyhedra, graphs, knots, and countless others. Travelling a mathematical world may entail going from certain diagrams and tools to others by means of geometric transformations, algebraic procedures, hypothetical assumptions, solving systems of equations, broadening a family of cases, encountering paradoxical objects, and many others. During such travels and particularly if occupying the "driver's seat", the explorer – whether a child playing with wooden polyhedra or an adult exploring their properties – experiences concretion, which implies a richer sense of orientation, creating significant landmarks, becoming more capable to imagine corners and the overall landscape, as well as appreciating more acutely the surrounding atmosphere.

Figure 1. Gauss notation workshop

During one of our online studio sessions, we explored a knot notation developed by Gauss. This notation can be understood as a "world" in the landscape of knot theory, where knots exist as sequences of small integers. Looking at Figure 1, a central idea of this notation is to choose a starting point on a knot (red point) and a direction of travel (red arrow) and to number the crossings encountered sequentially until back at the starting point. Within this world of small integers, we made several observations that contributed to the development of a sense of orientation. A central observation in our conversations was that, regardless of the starting point or direction chosen on our

trefoil knot, certain crossings consistently paired: 1 with 4, 2 with 5 and 3 with 6. Gauss's notation also highlights the nature of the crossings themselves by using a + or − sign to indicate whether the crossing goes "over" or "under." For example, in Figure 1 (left), the first crossing encountered goes "under" crossing number 4, resulting in −4 for the first term of the sequence. The second crossing goes "over" crossing number 5, resulting in +5, and so on with −6 (since the third crossing goes under crossing number 6), 1, −2, and 3. This sequence of numbers (−4, 5, −6, 1, −2, 3) characterises the trefoil knot in this mathematical world of numbered knots. Within this knot-number world, several questions became salient. For example, what kinds of knots could we obtain by altering the number sequence? Figure 1 (right) shows one such exploration during our studio session, where the second and third crossings (relative to the red starting point and indicated direction) are made consecutive. This results in a "twist" that can be undone, producing the "unknot"—a knot with zero crossings.

There would be much more to say, of course but we want to draw attention to how we can think of this investigation of Gauss notation as an example of concretion. At first, this world of numbered knots seemed fragmented—a set of disconnected numbers and arbitrary rules to follow to generate sequences. Yet, by engaging with the inhabitants of this world, patterns began to emerge, and a sense of orientation gradually develops. Observations such as the consistent pairing of crossings or the effects of altering sequences revealed possibilities within the notation. Importantly, this process did not start from scratch. Our prior experiences with small integers, sequences, and symbolic reasoning provided insights for engaging with this mathematical world. And, this process has no end as endless possibilities to alter, explore, and reinterpret the notation continue to unfold.

On abstraction

We begin this section on abstraction by bringing forth common and pervasive experiences of play that conjure up make-believe or as-if worlds, such as puppetry, theatre, or story-telling. These worlds provide a felt sense of abstraction, offering inspiring parallels to mathematical practices. Drama and puppet plays are acts of as-if world-making, endowed, we suggest, with those two processes: immersion and correspondences. A drama play strives to allow for immersion, captivation, or enthrallment which requires that, for the most part, everything that happens can make sense from the play itself, without posing the need for spectators to step back and consult other sources to figure out what might be going on, which would entail interrupting the immersion in the make-believe. The leeway that a play offers for engrossment is relative to the cultural background of the audience. At the same time, a drama play resonates with the world at large to the extent that it awakens diverse and often surprising correspondences with daily life, with known stories, characters, and legends, as well as shared personal and collective experiences. These correspondences make the characters, say, funny, sad, silly, naïve, inspiring, or surprising.

This idea of world-making through immersion and correspondences forms the basis of how we think about the experience of abstraction in mathematics. Abstraction can be understood as a process of as-if world-making, where one is immersed in the mathematics at play and draws on correspondances. In mathematics we draw a circle and treat it as-if it were a geometric circle, or write a short sequence of numbers and treat it as-if it were an infinite sequence. Such practices cannot be accounted for in

terms of signified-signifier pairs because inmersion entails as-if ways of gesturing and perceiving affordances fully permeated by affective currents traversing them.

During a workshop at Ruthin Craft Centre draws we explored braid theory in the context of braiding activities. During the session, we referred to Artin (1950), which historically introduced the mathematical concept of braids. The activities we ran allowed us to examine certain correspondences between braiding and braid theory. Such correspondances, as Ingold (2020) suggests, are not one-to-one mappings but involve partial and often tenuous co-responses along different worlds. Drawing on such correspondances we began to familiarise ourselves with the world of mathematical braids by exploring their properties with pieces of string of various colours, cork boards and pins. A braid in braid group spans between two parallel horizontal bars as shown on the left of 2, where 5 highlighted threads of different colours travel from the top to bottom of the cork board to form a braid of order 5. The braid in this case is made of 5 threads, but it could include any number of threads.

The threads can travel under or over other threads until each of them reaches a point on the other side. In the case shown in Figure 2 (left), the green and grey strings meet on their journey from their starting point to their end points. However, and this is a central idea in the definition of a mathematical braid, this intersection is "combable", i.e. it is possible to simply lift or move one of the strands without unpinning it so that they travel alongside each other without touching. In this respect, Figure 2 (right) shows braids that are not combable because they have a combination of non-trivial overs and unders.

A mathematical group is formed by a set of elements, and an operation that transforms a pair of elements into another one belonging to the same set (i.e., "closure"). The operation of the Braid Group is called "composition" and entails taking two braids and joining their threads one after another. Figure 2 (right) shows an example where a 5-threads braid (1) is joined to another 5-threads braid. The outcome of such a composition is also a member of the 5-thread braid family, signifying closure.

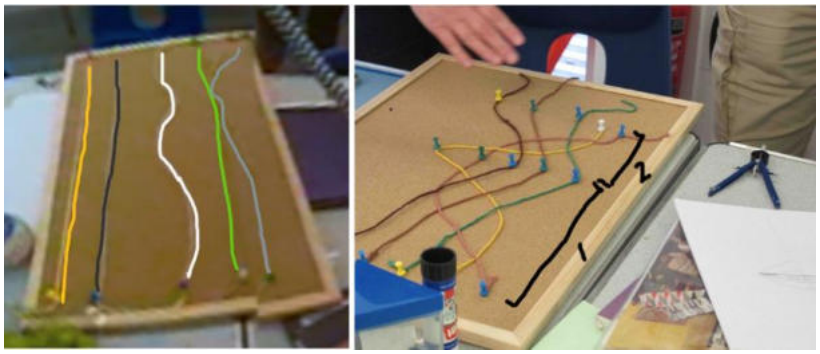

Figure 2. Braid Group workshop

In addition to closure, the operation of a group must satisfy some of the properties that are standard for addition and multiplication. One of them is that the set of elements of a group must include one that is called "identity" or "neutral" because operating with it does not change the original structure. In the case of addition, the neutral element is zero, since n + 0 = n. The neutral element of a Braid Group is a braid with all the threads parallel to each other, without any crossing as the one shown in

Figure 2 (left). Another required property is that, given a Braid Group of a certain order, for any braid the set must include another one that is its "inverse", such that composing both obtains the neutral braid — the one that can be combed. During the workshop one of the investigations was figuring out, for a given braid, another one that is its inverse. It turns out that a braid and its mirror reflection (in a horizontal mirror line for braids running vertically) are inverses of each other, so that what is obtained by composing them can be arranged as a set of parallel threads. Many other questions related to the Braid Group could be investigated, but we will turn to reflecting on how the example of the Braid Group may help us grasp the experience of mathematical abstraction.

In a similar way to discovering a puppet play, getting acquainted with the Braid Group involved for us a process of immersion in a realm populated by characters, histories, relationships such as inverses, mirror braids and properties, each with its own distinctive "personality". For example, in a braid group the threads making up the braids can only travel from one side to the other without going back in their journey to intersect with others or themselves. This behaviour, its roots and its consequences, are not simply a matter of memorisation, but the work of a continuous and never fully finished journey into the world of the Braid Group and, more broadly, of Group Theory. To experience the Braid Group as an abstraction is to feel immersed in it and to meet, question and imagine its inhabitants and their relationships. An abstract experience is immersive in the sense that it is full of potential to experiment with and welcome new inhabitants or to rethink those who already live there.

Throughout these processes Group theorists, students or mathematics enthusiasts are also likely to encounter, make and be inspired by multiple correspondences, like we did, not only with braiding crafts, but also with previously unforeseen practices, such as constraining the movement of points on a plane. As mentioned above, these correspondences do not mean one-to-one mappings, but rather areas where two practices can travel together, in the same direction, questioning and stimulating each other (Ingold, 2020). An example of this type of correspondence is the experience of the "combability" of a mathematical braid, which evokes, both in terms of language and the actions it calls for, the experience of a woven braid. And with this idea of correspondence, neither of the two practices, weaving or mathematics, prevails in the sense that the experience of abstraction does not reduce one to the other. Rather, they are stimulated by these differences by challenging each other. The thesis of mathematical abstraction as a type of as-if world-making, involving correspondences and immersion, pivots on the presumption that groups, as well as numbers, functions, symmetries, and so forth, are as much part of the world as baskets, planets, and stars. In the same ways that there are infinite possibilities for the weaving of baskets, and that different materials and tools facilitate some and resist others, there is no closure for the discovery/invention of, say, numbers or groups, and they also resist and suggest skilled courses of action.

Connections to conversations in Mathematics education

Our reflections on abstraction and concretion have certain points of resonance and dissonance with conversations in the world of mathematics education that have inspired us and provoked reflection (see Shvarts et al., 2022 and Coles & Sinclair, 2019 for some overviews). Reference to abstraction or concretion is often made to describe learning trajectories, some focusing on the movement from

concrete to abstract (Piaget and Bruner) and others on more relational and bi-directional experiences (Davydov, Freudenthal and others). In a direction that aligns with our objectives, Coles and Sinclair (2019) challenge these abstract-concrete learning trajectories, taking a more relational and enactive view, with "attending to things as a concrete mode of mathematical activity and attending to relations as an abstract one" (p. 468) which can be in play simultaneously. Their work is focused on "symbolically structured environments" which support this interplay, whereas we are considering different processes of concretion (being better oriented, grounded and connected) and abstraction (as-if world-making) which provide a framing beyond disciplinary confines or environments. Our approach finds affinities with Wilensky (1991), who sees concretion as the process of coming into 'a relationship with', though he sees abstract as being in opposition to this, where one has not yet made concretising connections.

Discussion

The notions of concretion and abstraction we have been articulating are accessible to all and yet can be very personal. Concretion has neither beginning nor end, nor a goal of easing understanding, but providing a richer sense of orientation, awareness and connectedness. Abstraction is experience of as-if world-making – the development of correspondences (similarities or incongruencies) and adding memories, which over time enables us to experience, shape and express our lives in unique ways.

These processes of concretion and abstraction do not exist in opposition to each other and can occur simultaneously. In our example of working with braids and the Braid Groups at the Ruthin Craft Centre, immersing ourselves in constructing braids to come to grips with the definitions and properties of mathematical braids, and seeing the correspondences with braids in craft work, suggests a process of abstraction (world-making) for the case of Braid Theory. Some participants saw in the requirement that braids always flow downwards, the downward passage of time while braiding. Others made a braid followed by its inverse (therefore, in totality, the neutral or combable braid) and were able to relate that to the fabric braiding technique of Sprang (en.wikipedia.org/wiki/Sprang).

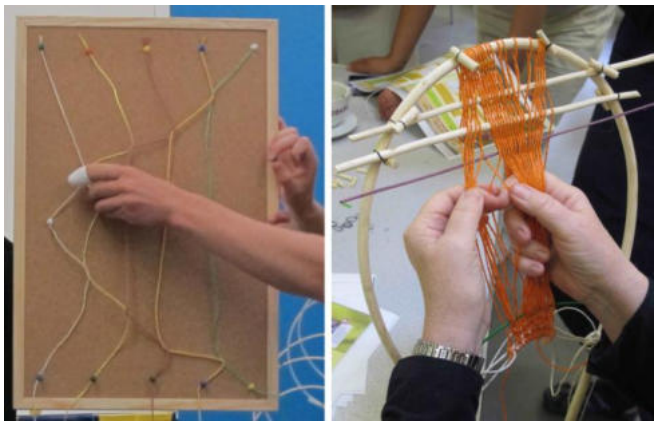

Figure 3. Sprang related work at the Braid Group workshop in Ruthin

In Figure 3, the sub-figure on the left shows the construction of a braid followed by its (mirrored) inverse, together constituting the neutral/combable braid. A pin, halfway down the board, prevents the (neutral) braid from collapsing into combable strands, foreshadowing construction of Sprang. The

sub-figure on the right shows the construction of a Sprang on a newly-crafted Sprang loom. These examples of correspondence, of seeing the flow of time in braids and of connecting braid inverses with the ancient technique of Sprang, show abstraction in play, creating a richer world of braids. Simultaneously, the work involved recognising braids that were similar to each other, developing increasing familiarity with some basic examples (such as a pair of strands crossing each other), and noticing that physically different braids (indeed a veritable infinity of them) are all equivalent to the same mathematical braid. These latter processes manifest the concretising nature of these experiences, as one gets oriented in the landscape of braids.

Thus, such a framing may allow us to recognise that both abstraction and concretion can simultaneously manifest *for all*, across disciplines, and provide an inclusive approach to mathematics. Our interest emerged from wanting to find a way of having mathematics and craft work alongside each other on equal ground without reducing one to the other. We hope this article provides an opportunity to re-engage with how the philosophies of abstraction and concretion can inform contemporary work in mathematics education around social justice and inclusion in mathematics.

Acknowledgment

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Doping an embodied design

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Students in both school and university often encounter challenging ready-made mathematical artifacts – especially in the context of the concept of continuity. Within the radical embodiment paradigm actions reify and crystallize in artifacts. Inspired by the theory of concept development of the German subject-matter tradition, the process of crystallization and the involved processes are analyzed, connections between the process of crystallization and concept development are illustrated and a methodological extension of an embodied design called 'doping' is presented. As a result, a duality of reified actions comes to the fore. An embodied design for continuity will be introduced, along with the implementation of 'doping'. The embodied design for continuity illustrates how 'doping' is able to broaden the scope of concepts and determine "what is a mathematical concept of one kind" and "what is a mathematical concept of another kind".

Keywords: Radical embodiment, crystallization, doping, embodiment

Introduction

New research discoveries question the separation of the mind and body in cognition, demonstrating that the body plays a crucial role in processes of sense-making. Embodiment, or embodied cognition, seeks to acknowledge these discoveries and proposes that our bodies are more than just physical entities in space. Within embodiment, current research explores how our physical experiences (embodied processes) connect with or reflect cultural artifacts and has been established as a process of reification (Shvarts et al., 2024). According to cultural-historical ideas, over time, cultural artifacts encapsulate or reflect the essence of processes that fostered crystallization of those artifacts. In this sense, artifacts are crystallized in history (Boels et al., 2023). This paper aims to analyze the process of crystallization as depicted in current literature in the context of the theory of concept development within the German subject-matter didactics tradition. As a result, the process of crystallization should be looked at with nuance. The differentiated view leads to methodological suggestions and also contributes to the theoretical understanding, as a dual role of actions comes to the fore. A concrete implementation with empirical data of the methodological proposal is presented in the context of a learning environment addressing the epsilon-delta criterion of continuity (Oberbucher, 2024).

Reification from a radical embodied perspective

The process of reification within the radical embodiment paradigm is well described by Shvarts et al. (2024). As a first step of the enactment, a body-artifact functional dynamic system arises, which emerges from a pool of potentials (affordances and body potentialities). This can be supported by highlighting relevant affordances (Shvarts & Abrahamson, 2023). As actions become more fluent and stable, new affordances or attentional anchors can arise (Abrahamson & Sánchez-García, 2016). These fluent and stable actions reify in artifacts – the artefacts thus are seen as crystallized actions. (Shvarts et al., 2024). The preservation of past actions in the form of artifacts can also be reversed to

reveal those actions once again, as demonstrated by Boels et al. (2023) with the example of the sine graph and histogram. This reversal of the crystallization process is referred to as melting.

Classical views on concept development

Mathematical concepts have been conceptualized in various ways: Coles emphasizes the idea that concepts are material/ abstract relations, while de Freitas and Sinclair highlight that concepts undergo a process of becoming shaped by activity (de Freitas et al., 2017). In the German tradition of subject-matter didactics, the concept development is highly esteemed. According to Ruwisch & Weigand (2023) a (mathematical) concept in this tradition refers to mathematical objects, their characteristics, the relationships between these objects and mathematical activities. The process of forming these concepts is regarded as dynamic. Several psychological theoretical frameworks serve as references when describing individual concept development. *Classical Theory* holds that categories are established by creating definitions of concepts, which then help to determine whether a particular mathematical object fits into a particular category or not. In contrast, *Prototype Theory* emphasizes the identification of the typical characteristics of prototypical representatives of a concept. The assessment of whether an object belongs to a category is done by comparing it with established prototypes. *Exemplar Theory*, on the other hand, proposes that a category comprises an array of individual instances, each distinguished by their distinctive features (Ruwisch & Weigand, 2023).

Within the dynamic process of concept development *concept content*, *concept scope* and *concept network* are distinguished (Ruwisch & Weigand, 2023). These components are interrelated and serve to facilitate concept development. The *concept content* refers to the defining attributes of a concept, whereas the *concept scope* is related to the totality of all objects that can be included in this concept and the *concept network* addresses the awareness of relations between mathematical concepts.

Concept content und crystallization

By reversing the crystallization process (see Boels et al., 2023), the embedded actions within artifacts are revealed, which are the fundamental actions that define the nature of an artefact. This is pretty much in line with the *concept content,* which includes the entirety of the concept's properties and their relationships to each other. The goal, according to Ruwisch & Weigand (2023), is to know typical representatives (Protoype Theory) and defining properties (Classical Theory) of that protypes in order to classify them in categories (Exemplar Theory). The theory of concept development then emphasizes expanding the *concept content* by going beyond the concept's defining properties and highlighting its connections to other concepts. This is particularly achieved through the formation/ extension of the *concept scope* and the *concept networks* (Ruwisch & Weigand, 2023).

In contrast to this, within the context of crystallized artifacts, only the defining/essential actions – *concept content* – are elucidated. The implicitly occurring actions that have significantly contributed to the (mathematical) artifacts are not addressed. Consider, the epsilon-delta criterion of continuity, which emerged from the arithmetization of mathematics, driven by the limitations of the original definition of continuity (in the form of the *natural continuity*, according to Núñez (1998)), which could not answer certain questions. The processes involved in finding a proper definition of continuity are "hidden social practice[s]" (Leontiev, 1978, as cited by Radford, 2003, p. 124). These practices

of exploring and setting boundaries of a concept are part of the genesis of the epsilon-delta criterion and are addressed in German subject-matter didactics – but are not yet seen as part of the process of crystallization. These actions could be emphasized as a part of the crystallization process, as these actions contributed to the origin of the concept as "hidden social practice[s]" (Radford, 2003). Therefore, the process of crystallization is not reduced to the "essential" actions, but instead expanded.

Expanding crystallization process

The scope of a concept from my perspective can only be expanded if one enacts the concept – conceptualized as "coordinated systems of cultural artifacts" (Shvarts, 2024) – within the context of other concepts, by interacting with them, reflecting on them, and relating them to one another. Differences in concepts emerge when examining the actions that differ in the process of crystallization. (At which level of crystallization are differences, and how do these differences affect the finished artifacts which constitute the concept.). The idea is hence to broaden the view on a mathematical concept (*concept content*) in order to investigate "What is a mathematical concept of one kind?" and "What is a mathematical concept of another kind" (*concept scope*) and how different concepts are related (*concept networks*).

To broaden the scope of a concept – the concept of continuity in this case – the proposed methodological extension of an embodied design is a structured process of a sequence of enactments (see Figure 2). First, stable and fluid actions should be developed around the target concept of continuity, leading to a crystallization in *ready-to-act* artefacts, which facilitate further actions (Shvarts et al., 2024). Once mastered, these *ready-to-act* artifacts are further challenged by introducing deviations in the task that were not previously implemented into an embodied design. The interaction problem (IP) is therefore extended through the deliberate introduction of discontinuities. This introduction of impurities is in analogy to physics referred to as *doping*. In the same way that *doping* in physics alters the properties of materials in order to discover new potentials, *doping* an embodied design generates new action-perception loops, thereby expanding the *concept scope*. The original task without deviations will be referred to as *undoped* task – the task with deviations will be referred to a *doped* task (see Figure 2). For this reason, a learning opportunity designed to broaden the scope of a concept – to answer the question "What is a concept of one kind and what is a concept of another kind?" – must provide the opportunity to interact with other but related concepts – to perceive boundaries and to address the *concept scope* within radical embodiment.

Methodology

Within a design-based research (DBR) framework, a technology-enhanced embodied design in the form of a browser-based application was developed, allowing individuals of various levels of expertise (school students, master's students, professors) to experience the epsilon-delta criterion of continuity as an IP. The similarities and differences in engagement of individuals as well as interactions with artifacts and the interplay of these elements will be investigated in order to explore how an *doped/ undoped* embodied design contributed to the crystallization of mathematical artifacts and the learning of the concept of continuity. Using interpretative analysis following

phenomenografic methodology (Marton & Booth, 1997), the following research question is of interest: How does *doping* an embodied design affect the enactment of individuals of various levels of expertise in the context of the concept of continuity/ discontinuity?

Learning opportunity as embodied design

A digital enhanced embodied design (Abrahamson, 2014) has been created in order to promote the learning of the concept of continuity (Oberbucher, 2024) A function $f: D \rightarrow \mathbb{R}$ with $D \subseteq \mathbb{R}$ is continuous at $x_0 \in D$, if and only if $\forall \epsilon > 0\ \exists \delta > 0\ \forall x \in D: |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon$

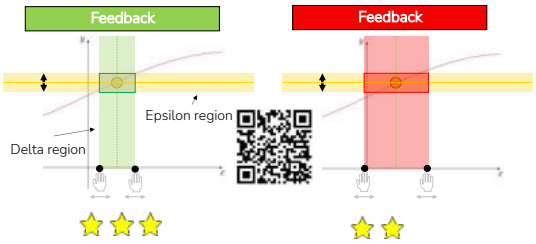


Fig 1.: Interaction Problem. Horizontal region: Epsilon region; Vertical region: Delta region; Left: solved interaction problem – green feedback; Right: unresolved interaction problem – red feedback; QR-Code: Link (Oberbucher, 2024)

This definition is basically saying that if one can narrow down the vertical region (delta region) to the extent that all of the function values within that region fall within the given horizontal region (epsilon region), then your function can be considered continuous at that point. The approach described is implemented as an interaction problem (IP) (Abrahamson, 2014) with highlighted relevant affordances – providing a specified horizontal region – and the student's task is to adapt the corresponding vertical region in order to fulfill the epsilon-delta criterion of continuity. Within the IP (see Figure 1: Scan the QR-code for iPad use) graphs of functions will be shown (see Figure 2). At specific locations (see Figure 2: squares) a horizontal epsilon region (see Figure 1: horizontal yellow region) is dynamically decreasing in size such that students are supposed to align the delta region (see Figure 1: vertical green/ red region) such that the epsilon-delta criterion is fulfilled. Meeting the epsilon-delta requirement results in positive green feedback displayed on the screen. If the green feedback is sustained for an adequate duration, it allows for the accumulation of stars. This serves as the system's feedback mechanism.

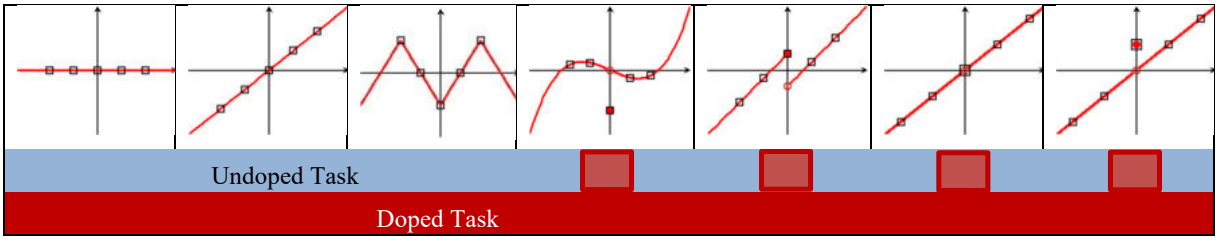


Fig 2.: Sequence of five graphs (continuous graphs: graph 1-3) used as interaction problems; blue: continuous; red: discontinuous; square: indicates locations to fulfill epsilon-delta criterion

In the first part of the learning opportunity (see Figure 2: Undoped Task), the objective is to achieve stable and fluent actions so that defining actions of continuity are crystallized as *ready-to-act* artifacts. In graph number four, for the first time a discontinuous location is shown in the IP – this expands the

concept content and offers a starting point for reflection processes. When attempting to meet the epsilon-delta criterion at this particular point and not succeeding, there are two potential scenarios from an individual's perspective: either the inability to meet the criterion is (a) due to a change in the environment, indicating a graph of a new kind, or (b) there is a need to adjust one's actions to meet the criterion, because not all the essential characteristics/ processes/ actions to define continuity have come to the fore. The first (a) refers to broadening the scope of the concept - the second (b) to deepening the content of the concept.

Research findings

In the following Episodes, research findings in the context of the role of *doping* the embodied design – introduction of discontinuities – will be exemplified and discussed. Episodes 1 and 2 are related to the enactment of a school student with no prior experience in the context of the concept of continuity, and Episodes 3 and 4 are related to the enactment of a professor in mathematics at a university. Interactions with the application will be highlighted in blue.

Episode 1 – undoped task

Episode 1 depicts the student's enactment after completing of the IP with graphs 1–3 (Figure 2).

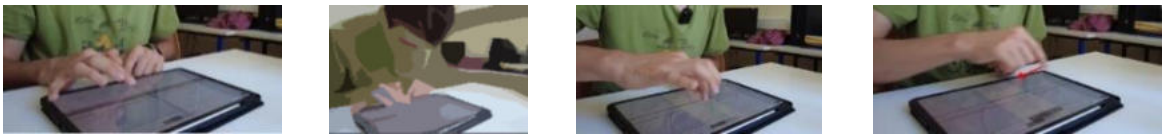
Tutor:	Okay. How does it stay green?
Student:	[…] So I have to stay within the region with the dotted red line *(Moves delta region centered over the point)*, which is now green, inside it. And the black/brown box *(points to the intersection of the epsilon and delta region)* inside there [note: in the epsilon region] must exactly contain the line. So there, the green line must be exactly within it *(points to the graph at the intersection of the epsilon and delta regions)*. And (...) if the period of time *(both index fingers are positioned at a horizontal distance from each other)*, so the region [note: delta region] becomes too big, you can see that it turns red *(points to the edges of the delta area)* and that doesn't work. And then you have to change it again (*reduces the width of the delta region on the screen)* so that it fits the size.

Episode 1 illustrates the enacted dynamics by the student in order to solve the interaction problem as perception-action loops. First, the center line of the delta region is placed over the point, and then the region is reduced in size (aligning the center line on the point) such that all function values within the delta region are green. At continuous points, this dynamics was typically enacted by the student.

Episode 2 – doped task

Episode 2 depicts the student's enactment of the IP at the discontinuous point (Figure 2: graph 4).

Student:	(The point jumps further to the location of discontinuity.) [Figure 2: graph 4] *(The delta range is positioned symmetrically around the point. The delta region is made very small.)* [Figure 3 (a)] Hm. *(Laughs.)* *(The delta region is positioned symmetrically around the point - the graph is partially red. The edges of the delta region are overlaid.) (Leans closer to the screen.)* [Figure 3 (b)] This is stupid now.
Tutor:	What do you mean by "This is stupid now"?
Student:	Well, it's just that now it's so small there. *(Points at the dashed line in the middle of the epsilon region.)* And especially, now the point would be down below *(Points with the index finger to the filled circle)* [Figure 3 (c)] at the dashed line *(Moves the index finger from left to right along the dashed line of the epsilon region)* [Figure 3 (d)], so at the gray dashed line. (Tutor: Mhm.) And the graph just goes along the top. *(Traces the graph with the index finger)*.



(a) (b) (c) (d)

Fig 3.: Enactment of Episode 2

At first, the initial stable and fluent actions (the enacted schema in Episode 1) are carried out. Subsequently, *"Hm*" suggests that there is an issue with employing the previously chosen actions to address the interaction problem. What follows is again an enactment of previous stable and fluent actions, which led to the solution to the interaction problem before (see Episode 1). This is a reflection in the original sense of the word, *epistrophé*, from the Greek meaning "to start something anew" (Bolle, 2018). Leaning closer to the screen indicates increased level of interest, because the student moves closer to initiate physical connection/ interaction. This part points to the process where the student assesses the current situation as being new and of different nature in the context of the task – respectively continuity (Dael et al., 2012). The utterance *"This is stupid now"* supports the contraction between the previously enacted stable and fluent action patterns and environment. What follows is a reflection in order to make sense of not solving the interaction problem. More precisely, it was elaborated why the original schema does not lead to a solution of the interaction problem Therefore, the changed environment led to the failure to resolve the interaction problem. In the context of continuity, the student elaborated on "What is the concept of continuity not?" and broadened the scope of the concept.

Episode 3 – undoped task

Episode 3 depicts the professor's enactment after completing IP with graphs 1–3 (see Figure 2).

Professor:	(The green feedback appears.) Ahhh. *(Laughs.)* Probably it has to do with (...) Either it has to do with the epsilon-delta definition or with continuity, because I basically have to narrow down the neighborhood on one axis so that it always remains in the neighborhood on the other axis.
Tutor:	Based on that, how would you determine this?
Professor:	The system basically tells me in which neighborhood of the function values I have to be, and I have to accordingly make the domain, i.e., the neighborhood of the point, smaller and smaller with my fingers, so that I always remain in the neighborhood of the function values. (Next graph.) [Figure 2 graph 4 – continuous locations] *(Aligns the dashed line over the point. Adjusts the width of the delta region symmetrically around the point – green feedback appears.)* And what about (...) it also works if I am smaller too? *(Decreases the size of the delta region symmetrically around the point – green feedback.)* *(Point jumps to the next location.)* *(Makes the delta region very small.)* And one must be small enough. *(Makes the delta region very small.)* *(Laughs.)* Yes, also interesting. When you look at it from this perspective, then everything makes much more sense. *(Laughs.)*

In Episode 3 the professor is making a connection between the epsilon-delta criterion and the interaction problem. The professor also provides verbal descriptions on how to solve the interaction problem *"I have to accordingly make the domain […] so that I always remain in the neighborhood of the function values.",* which corresponds to the previously enacted stable and fluent actions *(Aligns the dashed line over the point. Adjusts the width of the delta region symmetrically around the point – green feedback appears.).*

Episode 4 – doped task

Episode 4 depicts the professor's enactment of the IP at the discontinuous point (Figure 2: graph 4).

Professor:	*(Point jumps to the location of discontinuity.) [Figure 2 graph 4 – discontinuous location] (Delta region is dynamically reduced in size symmetrically around the point. Leans closer to the screen. Widens the delta region symmetrically around the point again. Holds left edge of delta region fixed – varies the right edge of the delta region.)* What's there now? *(Makes the delta region small right to the location of the point. Widens the delta region right to the location of the point. Shifts the entire delta region to the left. Shifts the delta region symmetrically around the point. Varies the size of the area symmetrically around the point.)*
Tutor:	What's different now?
Professor:	Yeah, it's not enough with the (...) It's not about a point of a function graph anymore, it's about a lower bound or something. *(Changes position and width of the yellow area frequently. Takes hands off the tablet.)*

In Episode 4 the interaction problem was to be solved at the location of discontinuity. Here, the same processes as in the student's enactment can be observed. As in the student's enactment (see Episode 2) at first, the initial stable and fluent actions (the enacted schema in Episode 3) are carried out, followed by leaning forward as indication of an increased level of interest. The exclamation *"What's there now?"* serves as an additional indication for a changed environment, which is also represented by the last statement *"It's not about a point of a function graph anymore"*. What follows is again an enactment of previous stable and fluent actions, which made up the original *ready-to-act* artifact and serve here as a reflective tool in order to find the boundaries of the concept of continuity.

Discussion

The proposed methodology of *doping* – the deliberate introduction of impurities – an embodied design was the result of the aim to address the *concept scope*, which is theoretically explained in the German subject-matter didactics, within radical embodiment. *Doping* an embodied design has been shown to initiate a process of reflection regardless of the level of expertise (see Episodes 1-4). In this sense the enacted artifact in the form of the definition is a *ready-to-act* artifact (active use in a reflection process) as it "facilitates further actions" (Shvarts et al., 2024). In the context of the sense-making of continuity it has been shown that the role of the actions involved is dual in nature: (1) the actions – *(a) aligning the center line over the point and (b) dynamically adapting the width of the delta region* – crystallize in an artifact as *the* defining processes of continuity and (2) the actions themselves, previously crystallized in an artefact, serve as a reflective tool to expand the scope of the concept of continuity. Actions are therefore genetic (Episodes 1 and 3), in that they bring the new artefact of continuity to the fore, and extensional within a process of reflection (Episodes 2 and 4), in that they are used to set boundaries for the initial enacted concept of continuity and in this sense expand the scope of the concept of continuity.

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Potential of movement-based embodied learning in mathematics: Insights from German case studies on reactivating movements

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This paper presents findings from a recently completed dissertation project that explores the potential of "movement-based embodied learning" in mathematics education, focusing specifically on the role of "reactivating movements" in enhancing mathematical understanding. Utilizing an interdisciplinary theoretical framework, the study emphasizes the significance of whole-body movements and reveals how reactivating movements, exemplified by gestures, contribute to learning. Through qualitative analysis of ten case studies with German third-grade students using a "number carpet", the research highlights the effectiveness of both active and reactivating movements in supporting the development of key mathematical concepts, particularly place value understanding. The results underscore the need for further research into the cognitive benefits of movement in learning mathematics and advocate for a more unified conceptual framework to guide future studies.

Keywords: Place value, movement-based embodied learning, case study, learning environments.

Introduction

The idea of integrating movement and learning in subject teaching is not new and can be traced back to reform pedagogy. For over 35 years, movement-based educational concepts have been advocated, including in Germany, to incorporate movement into subject lessons, such as mathematics, to enhance learning readiness and understanding through physical activity (e.g., Laging, 2017). Over the past 20 years, advancements in research technologies have prompted studies into the embodied nature of mathematical activities, adopting a more cognitive and neuroscientific perspective on the link between movement and learning. The embodiment of mathematical activities varies widely, from finger-counting processes (e.g., Miklashevsky et al., 2021) to the role of gestures in developing mathematical understanding (e.g., Krause & Salle, 2019) and whole-body training concepts with task-specific movements related to mathematical content (e.g., Link et al., 2013). While there is consensus on the fundamental embodiment of mathematics (e.g., Flood et al., 2020), research on the intentional promotion of mathematical content through movement and its potential for meaningful mathematical learning remains underdeveloped. The complexity of movement, diverse embodiments, mathematical content, varying terminologies (e.g., Bailey et al., 2023), and different research perspectives (e.g., Sinclair & de Freitas, 2019) pose challenges for developing a coherent understanding and integrating perspectives. The dissertation project aimed to develop an interdisciplinary, holistic understanding of mathematics education and explore how movement-based embodied learning contributes to it. This article provides insights into the conceptual framework and focuses on the potential of reactivating movements for revealing mathematical content. The aim of this article is to integrate gestures, chosen as an exemplary form of reactivating movement, into the holistic and interdisciplinary framework. Furthermore, insights from case studies are used to explore the specific contributions of gestures to mathematics learning within the context of movement-based learning environments.

Theoretical frameworks

In many German-language research works, the term *Bewegtes Lernen* [active learning] has been used to describe the temporal and/or content connection between movement and learning (e.g., Laging et al., 2010). In international research that involves the body in the learning process, various terminologies are used (e.g., Bailey et al., 2023), or the general term *Embodiment* or *active learning* is applied. The issue with these general or inconsistent terms is the lack of an international definition, leading often to the use of the all-encompassing term *Embodiment*. Currently, there is a lack of a clear definition to differentiate and mutually reinforce diverse international research efforts. In this article, the term *movement-based embodied learning* is used, and the theoretically-analytically developed interdisciplinary and holistic conceptual understanding is applied to the context of mathematics.

Movement-based embodied learning in the context of mathematics

While active learning has primarily been influenced by sports science and embodiment has been heavily shaped by cognitive psychology, these concepts often evolve parallel to each other. There is a need to integrate these concepts into an interdisciplinary and holistic understanding regarding their potential for classroom learning: Movement-based embodied learning can first be framed within the context of movement-based school concepts like the *Bewegte Schule* [Moving School] (e.g., Laging, 2017), which emphasizes integrating more movement into everyday school life due to its central importance for learning. This approach advocates for movement throughout the school day, not just in physical education, and specifically situates movement-based embodied learning within subject lessons like mathematics. Following Lindgren and Johnson-Glenberg (2013) and the Embodied Learning concepts, movement-based embodied learning is seen as pedagogically-methodologically initiated teaching-learning processes, distinct from informal learning forms or educational concepts like Embodied Education (e.g., Francesconi & Tarozzi, 2012). Unlike other definitions (e.g., Skulmowski & Rey, 2018), this concept emphasizes whole-body movements to uncover central subject-related aspects. Thus, movement-based embodied learning is purposefully distinguished from traditional use of materials, with active whole-body movements in space becoming the core of the concept. For the purpose of this article, movement-based embodied learning can initially be defined in a simplified way as an approach that integrates full-body movements into the learning process from a pedagogical-methodological perspective. This approach allows learners to discover, experience, and understand content by engaging their whole body in spatial movements. Importantly, it is not merely about incorporating physical activities into lessons (such as short movement breaks) but explicitly about exploring and constructing mathematical concepts through movement. But key insights from cognitive research suggest that experiences through space movements can be *reactivated* through movements (e.g., Radünz, 2025), which adds a new but very central point to the definition of movement-based embodied learning in a second step:

Current theories on the *Embodied Cognition* approach assume that sensory-motor areas of the brain are activated during interaction with the environment and are reactivated when perceiving related objects or actions (e.g., Beilock, 2009). Research has explored the connection between actual actions and their mental representation (e.g., Kaltner et al., 2014), observation of movements (e.g., Calvo-Merino et al., 2005), verbalization of movement processes (e.g., Hauk et al., 2004), and self-directed

gestures with a representative function (e.g., Krause & Salle, 2019). Knowledge about objects and events includes not only their perception but also the motor information and action possibilities resulting from actual movement (e.g., Beilock, 2009). Imaging techniques have shown that (1) *gestures*, (2) *mental images*, (3) *observation*, and (4) *verbalizations* are generated by a brain system comparable to that of practical actions, are closely connected, and can thus be considered *reactivating movements*, as they reflect the reactivation of active movements (e.g., Radünz, 2025; see Figure 1). Consequently, movement-based embodied learning goes beyond the active movements of the whole body in space and also includes *reactivating movements* as the reactivation of one's own movement experiences. With its specific focus on gestures as a form of reactivating movement, this article touches on a well-established field of research in mathematics didactics and embodiment studies (e.g., Alibali & Nathan, 2012). It is important to clearly delineate that within the broad literature on gestures, this study focuses on self-directed gestures and their cognitive function as reactivating movements within movement-based embodied learning. This distinction is based on the Gesture-for-Conceptualization Hypothesis proposed by Kita et al. (2017), which posits that gestures integrate cognitive processes with movements, thereby fundamentally establishing a connection to movement-based learning. Furthermore, gestures are specifically categorized in this context as reactivations of motor-spatial information derived from one's own movement experiences and are initially distinguished from interactive and communicative gestures. Initial case studies have highlighted the educational significance of reactivating movements, which is illustrated through an exemplary case study, noting that the potential of movement-based embodied learning – beyond the context of mathematics – remains a significant research gap.

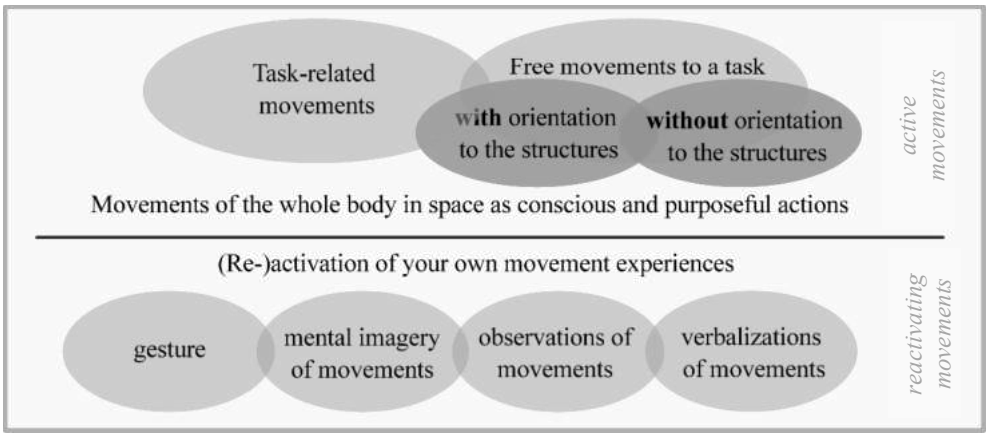

Figure 1: Forms of movement

The number carpet as systematic mathematical visualization of the place value system

Movement-based embodied learning in the context of mathematics primarily means using movements to develop key fundamental ideas in mathematics. Movements must represent the content context appropriately at a normative level and support students in building fundamental concepts at a descriptive level (e.g., vom Hofe et al., 2021). Due to the abstract nature of mathematical content, the mathematical teaching-learning process is generally mediated through suitable visual aids. To enable movements as actions using the whole body and thus leverage the full potential of bodily and spatial

perception, traditional mathematical visual aids can be adapted for whole-body use in space, while maintaining general didactic principles. Mathematics educational orientations can be realized through so-called *movement-based learning environments* (see Radünz, 2025), where movement is considered and reflected from the planning stage, integrating the full potential of the body into lesson design. However, movement-based embodied learning in mathematics is much more than mere physical movement and only unfolds its potential for supporting mathematical understanding when integrated with classical mathematics didactic concepts.

90	91	92	93	94	95	96	97	98	99
80	81	82	83	84	85	86	87	88	89
70	71	72	73	74	75	76	77	78	79
60	61	62	63	64	65	66	67	68	69
50	51	52	53	54	55	56	57	58	59
40	41	42	43	44	45	46	47	48	49
30	31	32	33	34	35	36	37	38	39
20	21	22	23	24	25	26	27	28	29
10	11	12	13	14	15	16	17	18	19
0	1	2	3	4	5	6	7	8	9

Figure 2: Number carpet as a body- and space-oriented visual aid

A fundamental idea in mathematics is the decimal place value system, and thus, developing an understanding of place value and the related bundling principle is a central goal of primary school mathematics education. Despite extensive criticisms regarding the risks of counting-based arithmetic, the hundreds board has been a present visual aid for over 150 years for the number range up to 100. However, neuroscientifically, the spatial arrangement from top to bottom, according to reading direction (e.g., Dehaene, 1997), can be questioned, as well as the fact that the hundreds board omits the zero as a central element of the decimal system. Consequently, the *number carpet* has emerged as a *body- and space-oriented visual aid* that systematically and consistently sorts numbers from zero to 99 according to place values (see Figure 2). Thus, the number carpet offers opportunities for implementing a movement-based learning environment by making the decimal place value system physically tangible and enhancing understanding through whole-body engagement. The movement-based learning environments on the number carpet integrate physical movements to develop fundamental mathematical concepts and make abstract content tangible.

Methods: Case studies as exploratory investigations

To investigate movement-based embodied learning, a qualitative and hypothesis-generating research design was employed through exploratory case studies. Given the limited implementation in practice, a suitable setting was first developed. Movement-based embodied learning environments on the number carpet were designed based on the theoretical-analytical framework, and the teaching-learning lab MATHletics was established. This lab provided an authentic setting for capturing the case studies. Over two weeks, student teachers worked with an after-school program at a primary school in Wuppertal, Germany, creating and scientifically documenting comprehensive movement-based embodied learning experiences on the number carpet.

The methodology involved ten complex case studies with third-grade students, focusing on arithmetic and place value understanding. A multi-method approach was used to holistically examine the interplay of various factors in mathematical learning through movement and to derive implications for educational practice (for more details see Radünz, 2025). This article provides partial insights into the primary data consisting of video recordings collected during the learning activities. Video recordings, observation forms, and student documents were analyzed using qualitative content analysis (Kuckartz & Rädiker, 2023). The coding system was developed both deductively, based on theoretical understanding, and inductively from the data. This analysis aimed to distill essential aspects from the data and identify key features.

Based on the coding, seven main categories were identified, each further refined through additional subcategories. In relation to the case study presented below, this section will focus on illustrating the main category 7 – reactivating movements (see Radünz, 2025). The subcategories in this instance were primarily developed deductively, grounded in the conceptual framework. For example, the subcategory *Gestures* is coded when a child uses their hands to indicate direction or position. For example, Mostafa looks at the number carpet and moves his hands diagonally (see Figure 4). Gestures involve pointing to specific positions or spatial structures, while raising hands is not considered a gesture and is coded differently. The structuring coding process identifies central scenes in which learners use gestures in the context of the moving learning unit to access mathematical content. By interpreting and combining the results from the other instruments in the sense of method triangulation, the contribution of gestures as reactivating movements of movement-based embodied learning for learning mathematics can be worked out.

Results: Insights of a case study

Figure 3: Idris's self-directed gestures while describing neighboring numbers

The case studies revealed that students used movement in various ways to comprehend mathematical content, with the contribution of movement-based embodied learning to mathematics varying significantly from an epistemological perspective. With the special focus on the contribution of gestures, two exciting functions can be highlighted: First, students employed both active and reactivating movements to accompany and support their verbal communication, indicating that movement also served a *communicative function*. Notably, students with language difficulties attributed a self-directed meaning to their movements, using gestures in various directions to guide their own speech. For example, in Figure 3, the student Idris is shown standing on the number 46 on

the number carpet, describing the neighboring numbers. The other students are seated with their backs to the mat, attempting to identify the number based solely on his descriptions. Since his peers cannot see him, Idris uses gestures in different directions while describing the neighboring numbers, which serve as a form of *self-guidance* rather than communicative gestures.

Figure 4: Mostafa's discovery and gestural representation of repdigits on the number carpet

Furthermore, the epistemological function of gestures is illustrated by a striking scene involving the student Mostafa. While two classmates are representing the various positions of repdigits on the number carpet to address the challenge of number word reversals commonly observed in German, Mostafa, simply by observing his classmates, identifies the diagonal connection between the repdigits. He examines and represents this insight through self-directed gestures (see Figure 4; left and center). He then tests this realization through active movements on the number carpet, repeatedly tracing the diagonal connection between the repdigits and communicating this discovery to his peers through movements along the diagonal and gestures (see Figure 4, right), by repeatedly stating: "Look, they are connected too". It is important to emphasize at this point that the diagonal connection of repdigits on the number carpet holds significant potential for mathematical discoveries, although this was not the focus of the intervention in the context of the support unit at that time. Thus, Mostafa independently recognized this connection by observing his classmates and using gestures, highlighting the potential of reactivating movements for the learning value of active learning.

Discussion

The case studies presented highlight the potential of movement in mathematics education, particularly the significance of reactivating movements for learning mathematical content. While the role of active movements through space remains central for understanding and discovering mathematical concepts, the observations indicate that reactivating movements – such as gestures and the mental tracking of others' movements – also play a crucial role. Mostafa's case illustrates this impressively: through observing the active movements of his classmates and subsequently representing these insights through gestures and his own active movements, he was able to gain a deeper understanding of the mathematical concept of reciprocal numbers. This shows that the various forms of movement in their active and reactivating forms are mutually dependent and closely interlinked, although the question of the concrete connection between active and reactivating forms of movement and the various forms of reactivating movement among themselves still needs to be the subject of further research. Although these results are promising, the overall research on the potential of movement for learning mathematics remains underdeveloped. The scope of the present case studies is limited as they include only a small sample, in addition to the specific German socio-cultural context, and focus on specific

scenarios. Nonetheless, the insights gained provide valuable stimuli for future research, which should further explore the interplay between active and reactivating movements in mathematics education. Particularly noteworthy is the importance of a holistic and differentiated understanding of embodied learning in mathematics education. The study has demonstrated that a clearly defined and interdisciplinary-congruent understanding of *movement-based embodied learning* is necessary to effectively advance research in this area. Such a conceptual framework can not only help relate existing studies to one another but also establish clear distinctions and connections between different approaches. Therefore, it is essential that future research pays special attention to this aspect and builds on a coherent theoretical foundation.

Thus, it is important to further explore the connection to gestures. While gestures were primarily examined here in their self-directed function as reactivations of movement experiences, the question arises as to whether gestures might hold additional significance in the context of movement-based embodied learning. Given that full-body movements can also serve communicative functions, it remains to be seen whether movement-based embodied learning could benefit more substantially from research in the specific domain of gestures, and whether the boundaries and interconnections between these areas could be more thoroughly elaborated. Overall, this study emphasizes the immense potential of integrating movement – both active and reactivating – into mathematics instruction. The insights gained lay the groundwork for further investigations aimed at understanding the conditions and mechanisms under which movement most effectively supports mathematical learning.

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Towards a theoretical framework for analysing block play constructions through the lens of crystallographic symmetry-breaking

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In this theoretical paper, I explore the possibility of using crystallographic theory as a framework for analysing young children's block play as crystallisation. Friedrich Froebel (1782-1852), the inventor of Kindergarten, made block play central to his play-based pedagogy. He was inspired by his time as an assistant to professor of crystallography Christian Samuel Weiss in Berlin, who was developing a novel system of categorising crystals according to their symmetries. While studying these crystal symmetries, Froebel experienced an epiphanic insight that the same basic symmetrical forms were key to children's understanding of Nature, and that by playing with blocks, children could imitate Nature's process of crystallisation. The crystallographic influences on Froebelian pedagogy have largely been lost in practice. As a thought experiment I attempt to analyse two early snapshots of block constructions from my doctoral study, through a crystallographic, symmetry-breaking lens.

Keywords: Kindergarten; block play; crystallography; symmetry-breaking; mathematics education.

Introduction

In my doctoral study I am analysing videos of 4-7-year-olds' free block play for evidence of proto-mathematical thinking. There is a long history of educational research into children's block play, dating back to Friedrich Froebel's invention of Kindergarten block play in the 1820s, including Froebel's own pedagogical experiments and development of blocks and block-based activities for young children and their carers, carried out by him and his colleagues at the schools he founded in Keilhau, Bad Blankenburg and elsewhere (Gura et al., 1992). Recently there has been renewed academic interest, from a historical perspective, in the profound, metaphysical inspiration Froebel drew on for his block-play pedagogy from crystallography, while working as an assistant to professor of crystallography Christian Samuel Weiss in Berlin from 1814-1816 (Friedman & Muñoz Alvis, 2023). For example, in Froebel's seminal 1826 work on Kindergarten pedagogy, *The Education of Man* (Froebel, 2005), he devotes over 20 pages to the metaphysical significance of the cubic symmetries of crystals and their relationship with child development. However, the quasi-spiritual language of Froebel's writings on the subject can be arcane, and the science of crystallography has developed significantly since – for example, embracing Dalton's then-new theory that matter consisted of combinations of atoms (which Weiss abjured) – and, despite the global popularity of his play-based Kindergarten, crystallographic aspects of Froebel's pedagogy have tended to be ignored in practice.

In this theoretical paper, as a thought experiment, I revisit Froebel's crystallographic inspirations for his block play pedagogy, alongside more recent insights from mathematical crystallography, such as group theory, Noether's theorem, entropy and symmetry-breaking, in order to explore crystallography as a potential theoretical framework for analysing patterns of block play, such as those I am observing in 4-7-year-olds. By focusing on the emergence – or crystallisation – of observable material structures when bodies interact with blocks in open-ended, free-play environments, I aim to stimulate potential

theoretical dialogue between Kindergarten pedagogy, mathematics and socio-materialist philosophies which decentre the human child by highlighting relationships with and between material objects.

Froebel's crystallographic concepts

When Froebel worked for Weiss, the field of crystallography had recently been established by René-Just Haüy in France, with his 1801 *Treatise on Minerology,* in which he showed how crystal forms corresponding to those observed in nature could be built up from unit cubes, which he could demonstrate using homemade wooden blocks (Figure 1).

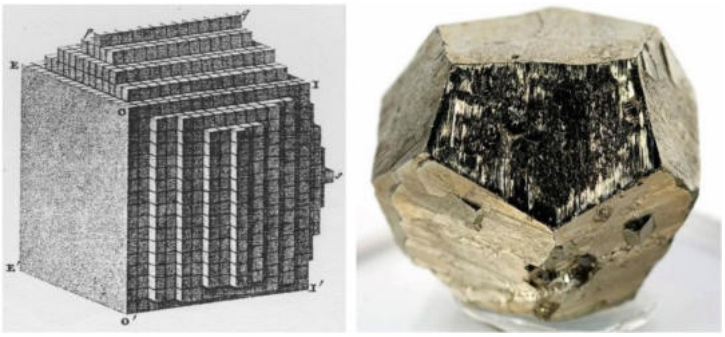

Figure 1: Haüy showed how symmetrical crystal forms, e.g., pyrite, can be built up from cubes

Weiss, rejecting the idea of atomic particles (in line with the contemporary *Naturphilosophie* idealism of his German intellectual milieu), developed a way of classifying crystals into seven 'systems' purely according to their symmetries, based on transformations of the symmetries of a cube, stretching or shearing its axes (see Figure 2).

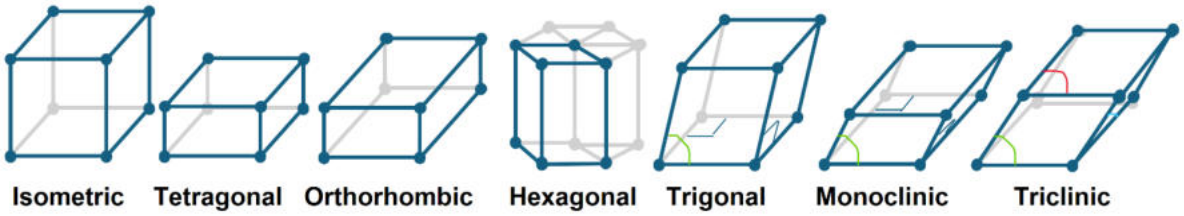

Figure 2: Weiss's seven crystal systems, each a transformation of a cube, by stretching or shearing

For example, the structure of a crystal such as diamond is based on a cubic (i.e., 'isometric') system, which has rotational symmetries. One can imagine, say, a cube of diced apple, and putting a cocktail stick in through the centre of the front face and out through the centre of the back face. If one twisted the stick a quarter turn clockwise the cube would rotate and end up looking identical. Thus, it has rotational symmetry (of order 4) about the axis of the cocktail stick. In total, the cube has 13 axes of rotational symmetry: 3 through its face-centres, 6 through its edge-centres, and 4 through its corners. Stretching a cube in one dimension to make a square-based cuboid gives the tetragonal system (e.g., apophyllite crystals), which has only 3 axes of (all face-centred) rotational symmetry. Stretching in another dimension, and shearing so that no axes are perpendicular, gives the triclinic system (e.g., turquoise crystals), which has no axes of rotational symmetry. In studying these crystal symmetries Froebel experienced an epiphanic insight, that these forms were also fundamental to our own minds:

"The simplest forms, which lie at the foundation of the fabric of the world, lay also the foundation in the minds of children for the understanding of the world" (Marenholtz-Bulow, 1876, p.1).

Turning down the offer of an academic professorship, Froebel would dedicate the rest of his life to teaching and developing his play-based Kindergarten pedagogy, centred on block play, in the belief that a child "follows the course of Nature … He likes to build and to imitate the structuring of form which we find in Nature's first activity, in the formation of crystals" (Lilley, 1967, p.38).

Froebel saw that, just as Nature generates a myriad of forms from simple basic forms such as cubes, so can a child, using simple wooden blocks. Froebel designed a sequence of manipulatives, or, 'Gifts', for children to play with as they grew: from Gift 1, a soft knitted ball meant to be reminiscent of the mother's breast, to Gift 2, a wooden ball, cylinder and cube, to Gift 3, a similar cube halved in each dimension to make eight smaller cubes, from which a vast variety of forms could be constructed in play (Figure 3). Further gifts consisted of similar cubes halved or doubled in length to make lower-symmetry bricks – e.g., tetragonal or orthorhombic cuboids – enabling more elaborate constructions.

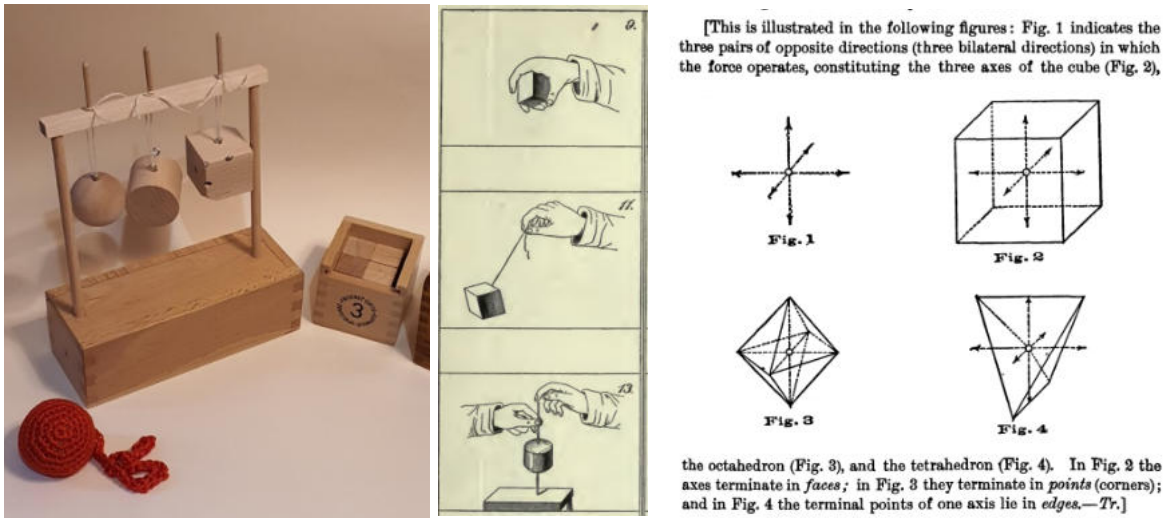

Figure 3: (Left) Froebel's Gifts 1, 2 and 3. (Centre) Froebel's illustrations of how Gift 2's cube could be spun on its axes using a string or stick. (Right) An image relating the octahedron and tetrahedron to the cube, from Froebel's *The Education of Man*.

From a crystallographic perspective Froebel added several features to highlight the symmetries of the wooden forms in Gift 2, for example drilling holes through their axes of symmetry, and fixing small eyelets to the ends of the axes. Thus, a stick could be pushed through the axis or a string through the eyelet, so the object could be spun showing the rotational symmetries. This motion added the fourth dimension of time to three dimensional shapes. For example, if the cube is spun rapidly about a face-centred axis, it appears transformed into a cylinder, revealing its invariant volume under rotation, which has its own symmetries. If the cylinder is spun rapidly about an axis perpendicular to its own axis it appears transformed into a ball. Play with Gift 2 can thus be understood as designed to support understanding of crystal symmetries and variances under rotational transformations. The smooth rolling of the spherical ball is a manifestation of infinite rotational symmetries, invariant over time.

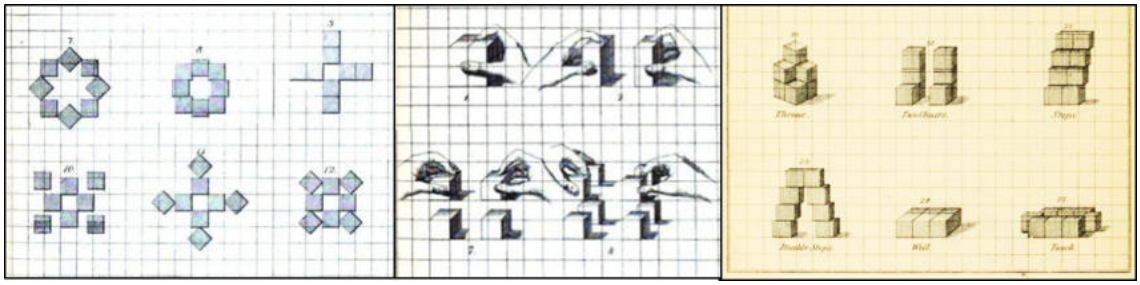

Figure 4: Gift 3, (Left) forms of beauty; (Centre) forms of knowledge; (Right) forms of life.

It is Froebel's Gift 3 of eight, one-inch wooden cubes that has become perhaps most emblematic of Kindergarten block play. For Froebel, time spent contemplating and playing with Gifts 1 and 2 was important preparation, as was storing the eight cubes together as a cube in their box, and using all eight (and no more) in constructions, maintaining a unifying, self-similar relationship with the larger cube, and via Gift 2, with the sphere. In modern Kindergarten such practices tend to be lost.

Froebel categorises block play activities into three forms: 'forms of beauty' – aesthetically pleasing symmetrical shapes, which could be made to 'dance' by transforming one into another; 'forms of knowledge' (or 'recognition') – shapes embodying abstract concepts such as numbers or fractions, as well as symbolic forms such as alphabetic letters; and 'forms of life' (or 'use') – models of people and objects familiar to the child, constructed from the eight cubes, which could be narrated and dramatised in stories, being rebuilt into new constructions as the story progressed (Figure 4).

Of these forms, the form of beauty is the most obviously crystalline in its dancing symmetries. However, with the form of knowledge, too, Froebel illustrated exploration of halving, then quartering the cube, and the new lower-symmetry cuboid forms these create, before halving quarters again into eighths returns to the cube form, pointing to an infinitely scaling crystalline symmetry of enlargement.

There are clearly many other aspects to Froebel's pedagogy, including further specific Gifts and activities (sometimes known as 'Occupations') such as weaving, painting, modelling with clay and 'peas-work' – the modelling of highly crystalline forms with toothpicks and peas – as well as singing, dancing, games, growing plants and spending time playing in Nature. Though there is not space to detail these here, I would highlight that for Froebel they were all designed to be a coherent whole, so all might in some way be related to block play and crystallisation, for example, and vice versa.

I have briefly outlined how Froebel's ideas of underlying crystallographic symmetries, gleaned from Weiss, were manifest in his block play pedagogy. Given that both block play and crystallography have evolved significantly since then I would now like to consider whether more recent, post-Weiss, crystallographic theory may also help inform crystallographic analysis of block play.

Additional crystallographic concepts: symmetry-breaking

In the decades following Froebel's time with Weiss, crystallography quickly developed from a branch of minerology to become a branch of mathematics. Galois's group theory in the 1830s enabled seeing Weiss's symmetrical analysis of crystals as more generalisable symmetry groups and began to offer ways of understanding wider physical phenomena in terms of symmetries. In 1894, Pierre Curie's dissymmetry principle theorised that any symmetry observable in an effect in a physical system can

be found in its cause. So, for example a smooth layer of iron filings scattered on a plate may have infinite rotational symmetries. That is, if one turns the plate by any angle, the iron filings look the same. But when a bar magnet is brought near the bottom of the plate, this infinite symmetry 'breaks' into a polarised pattern along the lines of the magnetic field, reflecting the field's lower, two-fold bipolar symmetry. Thus, observing and analysing the 'breaking' or lowering of the symmetry – here from infinite to two-fold symmetry – may offer a way of understanding the phenomenon:

> the occurrence of a phenomenon in a medium requires the original symmetry group of the medium to be lowered (broken) to the symmetry group of the phenomenon (or to a subgroup of the phenomenon's symmetry group). In such sense, symmetry breaking is what "creates the phenomenon" as claimed by [Pierre] Curie. (Castellani & Dardashti, 2021, p.633)

At other times the symmetry of physical systems may spontaneously break, such as when a crystal with 2- or 3-fold symmetry forms from the medium of an infinitely symmetrical gas or liquid. The final symmetry of the crystal may be highly sensitive to small variations in the medium. For example, the common crystal calcium carbonate (the main constituent of chalk and eggshells) has two symmetrically distinct forms, calcite and aragonite, both present in the shells of molluscs, which can finely control the chemical composition of the seawater their shells crystallise from. By analogy small adjustments in the block play environment may result in distinct symmetries in block constructions.

In the early twentieth century Emmy Noether's theorems made the connection between the physical laws of conservation (of mass/energy, momentum, charge etc.) and the conservation of the symmetries of a system, making symmetry – and symmetry-breaking – derived from crystallographic mathematics fundamental to understanding much of modern particle physics.

If subsequent developments in crystallography have arguably validated Froebel's focus on their importance in understanding Nature, they may also offer opportunities to reintegrate crystallographic thinking into the pedagogical analysis of Froebelian block play. For example, a tub of randomly mixed cubes may be seen as a medium with effectively infinite symmetry (i.e., by eye its symmetries are indistinguishable when the tub is rotated, mixed up etc.), analogous to scattered iron filings, or atoms in a hot gas, or a liquid solution. It may be that children's interactions with blocks break these infinite symmetries. For example, by picking up one cube, already the focus switches to a lower-symmetry object – the cube – albeit itself highly symmetrical. Connecting this cube with another to build a tower can again be seen as another symmetry-breaking – reducing the highly symmetrical cube to a lower-symmetry cuboid (or 'tetragonal' system crystal). And adding symmetrical 'arms' to either side of the tower breaks the symmetry further. While adding asymmetrical features, like a flower at the end of one arm, may break the overall symmetry to none. In theory, this offers a potential crystallographic lens for analysing block constructions observed over time as symmetry-breaking.

Experimental application

Traditionally ethnographic studies of block play have tended to categorise block constructions into types, or stages. For example two schemas introduced in the 1930s are both still commonly used (Tian, Luo & Cheung, 2020) – Johnson's seven stages (carrying blocks around, making rows or piles, bridging, enclosures, decorative patterns with symmetry, naming of block constructions, and dramatic

play with block constructions), and Guanella's five stages (nonstructural or pre-organized use of blocks, stacks or rows of blocks, bidimensional construction, tridimensional e.g., enclosures, and representational play with blocks). Though these schemas recognise the presence of symmetry occasionally, for example in repeating patterns (echoing Froebel's 'forms of beauty'), no schema I am aware of to date has used crystallographic symmetry as an organising principle for analysis.

Though any crystallographic schema would require thorough trials, and my gathering of video data of 4-7-year-olds' block play is still at an early stage, as a thought experiment I consider two snapshots of block constructions I have observed as examples for experimental crystallographic analysis.

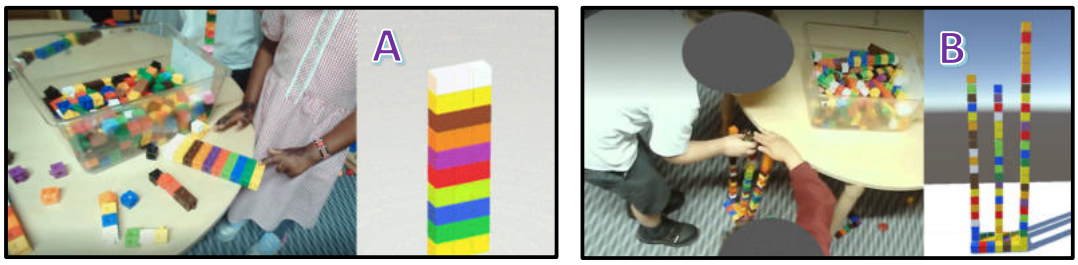

Figure 5: (Left) Model A: a rectangular form with orthorhombic symmetry. (Right) Model B: a form consisting of three tetragonal towers of different heights on a horizontal orthorhombic base.

In Figure 5 two snapshots of 5-year-olds' block constructions are presented, alongside 3D models of the constructions, re-created in 3D modelling software (with the aim of eventually being able to, e.g., view them as looping animations from various angles, and compare them side-by-side in virtual space, as well as extract coordinate data on block positions for potential symmetrical analysis). Both were taken from video clips of naturalistic, unguided free play with multilink cubes, to which the children were accustomed. Crystal system labels are applied to key phases of the construction, along with crude levelling of the symmetries – for example the cube is categorised as the isometric system (high-symmetry), and the tower, tetragonal (mid-symmetry) – to trace effects of actions on symmetry.

Model A was built up by one child, from a starting cube, made into a row of three of the same colour. This was then repeated methodically and carefully row by row upwards, with each row a different colour, with no discernible pattern to the colours.

Model B was made by two children playing together, with one creating the initial structure of three towers, and the other assisting with adding and taking away blocks over time. The towers changed rapidly, in playful conversation, both through deliberate changes to the construction, and accidents where a tower would break and fall and have to be resurrected.

Crystallographically, Model A can be seen as developing from the tub of cubes (~infinite symmetry), to a cube (isometric crystal system – high symmetry), to a row of three (tetragonal – mid symmetry), to two rows of three (orthorhombic – low symmetry), to three rows of three (tetragonal – mid symmetry), to four and then further rows (orthorhombic – low symmetry). Model B develops from the tub (~infinite symmetry), to a cube (isometric – high symmetry), to which a cube was added to start building a tower (tetragonal – mid symmetry). Three towers of different heights were added to a rectangular base (orthorhombic – low symmetry). The whole system thus became asymmetric (in crystal terms 'amorphous'), although spacing of the three tetragonal towers was approximately even.

Discussion

This theoretical paper describes a thought experiment in the possibility of developing a crystallographic theoretical framework for analysing block play constructions as symmetry-breaking. It is intended to raise questions for further theoretical development rather than draw any conclusions on limited block play data, and minimal formal method.

However, within these limited terms the thought experiment has demonstrated that block constructions may be analysed in terms of symmetries. And for constructions starting with a tub of mixed blocks (i.e., ~infinite symmetry), and then a single cube (high symmetry), constructions did tend to break these symmetries, forming lower-symmetry shapes. Thus, symmetry-breaking may offer a crystallographic basis for analysing at least one aspect of block play constructions.

As evidenced by previous studies and schemas, young children's early block constructions often tend to be 'abstract' – not recognisably models of everyday objects, which added to the fact they may be constructed with minimal if any verbal narrative, can make analysis challenging. Hence traditional schemas have often focused on observable topological features such as enclosures or bridging, or levels of dimensionality. Interpretations through a crystallographic theoretical lens may offer a supplementary framework, with the potential benefit of relating to well-defined mathematical models of symmetry, enabling, for example, computer-aided identification of symmetries across instances.

In addition, this thought experiment has highlighted some implications of the newer types of blocks, for example, interlocking cubes, or *Lego* (as well as virtual blocks such as *Minecraft, Roblox* etc.), compared with wooden blocks that rely on gravity to stay in place, on which earlier schemas are often based. Johnson's concept of balancing a 'bridge' between two stacks, for example, makes less sense if block constructions are snapped securely together. This touches on Curie's dissymmetry principle and to what extent the constructions reflect the symmetry of the force of gravity. And also suggests entropy as a phenomenon in building balanced block constructions as it is in crystallisation, – since in physics changes of state whether crystallisation or melting ultimately increase overall entropy – vividly illustrated by block towers noisily collapsing into an unordered state.

Similarly, with multi-coloured blocks now common, there may be patterns of colour, co-existing with patterns of formal construction. In virtual worlds blocks may have many other properties. This 'sticky' connectivity of multilink or virtual cubes may in fact reflect more closely the way crystals form in nature, from layers of almost weightless atoms or molecules attracted to earlier layers.

Thirdly, focusing on symmetries has opened the possibility of connection with Froebel's largely neglected original crystallographic ideas, which may help make sense of his pedagogy more generally. For example, it is hard to contemplate a rationale for the holes along the axes of symmetry of the cube in Gift 2, without reference to Froebel's knowledge of crystallographic symmetries. And the 2- and 3-symmetries of crystals may offer resonances with 2- and 3-rhythms of Kindergarten dances and songs, woven or drawn imagery, and clay models, as well as the small-number symmetries of family structures, conversational dialogue and language grammar children are learning.

In addition, the mathematical nature of symmetrical analysis may offer bridges to more formal, post-Kindergarten mathematics, where crystalline tetragonal rows of interlocking cubes are often used to

model algebra and counting, symmetrical parallel rows modelling equality, orthorhombic rectangles modelling area and multiplication, or cubes modelling volume, and so on.

In future research, these early ideas of a crystallographic theoretical framework may be developed further, iteratively, through testing on video data of children's block play, and on 3D animated models of their constructions. In addition, Froebel saw deep philosophical significance in crystals, and there may be opportunities to explore potential links with philosophers who draw on metaphysical ideas of crystallisation, immanence and virtuality, such as Simondon, Deleuze and Guattari, and newer socio-materialist philosophers such as de Freitas and Sinclair, who might assign agency to the crystal forms.

As a final reflection, one early graduate of Froebel's pedagogy, who attributed much of his creative success to his time spent at Kindergarten playing with Froebelian blocks, was renowned architect Frank Lloyd Wright, a pioneer of the modernist architecture that has influenced much of the current urban built environment. He hints at the deeper pedagogical value of Froebel's focus on the mathematical symmetries of crystals, which a crystallographic lens might elicit:

> The virtue of all this [Froebelian Kindergarten activity] lay in the awakening of the child-mind to rhythmic structure in Nature - giving the child a sense of innate cause-and-effect otherwise far beyond child-comprehension. I soon became susceptible to constructive pattern *evolving in everything I saw* [Wright's emphasis]. (Rubin, 1989, p.30)

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A plea for participatory/transformative methodologies in embodied and material studies of mathematics education

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In this paper, we reflect on methodological aspects of material and embodied studies of mathematics learning and teaching as derived from a paradigmatic shift in social science. Within (post)positivist/cognitivist and constructivist paradigms, we see a coherence across theoretical visions on cognition, main teaching practices, and core research methodologies as they all share core ontological and epistemological assumptions. Searching for a paradigmatic coherence, we resonate onto-epistemological assumptions of radical embodied and new materialist studies with those of transformative and participatory paradigms and their methodologies. An illustration from a study on embodied teaching and learning of reflection symmetry shows how the emergence of new onto-epistemological structures, iterative non-central regulation, intercorporeality, and warm participation are present in the practices of students, teachers, and also researchers.

Keywords: *radical embodiment, new materialism, participatory methodology, transformative methodology, paradigm.*

Introduction

When discussing embodied and material forms of learning mathematics, we often get a question: "Ah, it is clear that there is such a nice process going on, but how do I make a research study out of it?" The questions of assessing (Kolovou, 2022) and reporting (Sinclair, 2024) outcomes and processes of embodied learning without reducing their complexity to traditional pre-posttests and dryness of transcripts call for a methodological reflection. The situation becomes even more complicated once we bring affective processes and teachers into the research-practice landscape. If affectivity is "the fundamental 'responsivity' of life" (p. 3, Freitas et al., 2019) that flows across participants and unfolding sympathy determines the dynamics of the learning process, shall we perhaps assess affective engagement rather than cognitive outcomes? Yet, how can we do so if affect is fluid and mostly stays beneath consciousness? At the same time, even assessing the entire complexity of embodied learning might not contribute much to the future education unless teachers feel enthusiastic about new forms of learning. This is challenging as embodied approaches imply fundamental transformation of teaching habits (e.g., Macrine & Fugate, 2021). This paper aims to provide a reflection on methodological consequences of the paradigmatic shift that embodied and material focus in studying mathematics teaching and learning entails.

Classical views

In the current moment of cultural development, a traditional classroom is imagined as rows of desks oriented towards a teacher who transmits information to the students sitting still and attentive to the message. This intuitive image of students passively receiving and storing information in their heads perfectly matches cognitive science that developed in the 1950s as a response to behaviourism

(Miller, 2003). Classical cognitive science has assumed that cognition is an information-processing machine. It receives sensory signals, sophisticatedly processes them into mental representations of the outside reality, and reacts according to the received and previously stored information.

The general image of cognition—aiming to best represent reality as it is—is well aligned with the so-called positivist (or post-positivist) science paradigm. Ontologically, classical research assumes the existence of an objective universal reality that becomes an object of scientific investigation (Guba & Lincoln, 1994). Epistemologically, a subject separated from this reality conducts research aiming to cognize and represent the reality as well as possible, thus creating "knowledge of the "way things are" [that] is conventionally summarized in the form of time and context-free generalizations," (p. 109). Methodologically, this paradigm, as well as classical cognitive science, is reductionist: it tries to split the information about the world into independent variables and uncover causal or relational effects through empirical tests and interventions that do not alter the nature of the studied phenomena.

While a traditional classroom might seem outdated to a large part of the mathematics education research community, the theories and methodologies that are grounded in cognitivist assumptions are widespread. For example, Duval's popular idea of mathematical concepts existing in multiple semiotic registers builds exactly on cognitivist models of mental representations of reality (Duval, 2006). Experimental quantitative paradigms criticized in the early days of mathematics education research (e.g., Freudenthal, 1982) are now living through a partial revival due to new research technologies (e.g., Strohmaier et al., 2020). Moreover, some theories of embodied cognition (e.g., Barsalou, 2003) are in line with the information processing and representationalist paradigm.

Constructivist view

Alternative image of students' groups engaged in solving problems using manipulatives, followed by classroom negotiations on arising in problem-solving mental constructions, sounds attractive to many researchers and innovative to teachers. Such a vision of children constructing their own mathematical realities grounded many influential theories in the 1980s and 1990s, such as, for example, the theory of conceptual fields (Vergnaud, 2009). Bringing the image of students who shape their mental constructions from a mathematical classroom to broader society, we notice that this picture matches the constructivist paradigm of research in social science. This paradigm fundamentally abandons the universal ontological existence of independent reality. Instead, "Realities are apprehendable in the form of multiple, intangible mental constructions" (Guba & Lincoln, 1994, p. 110). While radical constructivism insists on the solipsistic individuality of those mental constructions and, consequently, realities (von Glasersfeld, 1991), social constructivism notices constant discursive negotiation of those intangible realities within community. Consequently, constructivist research methodologically leans on dialogic/dialectical discursive methodologies (Guba & Lincoln, 1994): researchers explore realities of the participants through interviews and negotiate the theories among each other.

Radical embodied and new materialist views and the transformative-participatory paradigm

An image of teaching/learning situation that radical embodied and new materialist studies develop is rather drastically different from the previous two images. Students are engaged in physically creating

the objects of their mathematical inquiry or discovering mathematical relations through movement (de Freitas et al., 2018; Sinclair, 2024; Shvarts & Abrahamson, 2023). Their teachers do not just discursively question students' mental constructions. Instead, they engage in co-creating new perceptual and physical objects and ways of talking about them (Flood et al., 2020).

Radical embodied and new materialist views: onto-epistemology

First of all, radical embodied and new materialist studies are *non-dualist* and, similarly to the constructivist paradigm, assert *relational* and *dynamic* ontology. This means that objects and cognizing subjects are fundamentally entangled. Ecological ontology, developed within the radical embodied approach, assumes that affordances—relational possibilities for an organism to act in the environment—are primary qualities of the world (Turvey, 2019, p. 334). It means that the environment is immediately structured for an organism along potential actions that it might wish to perform. New materialism prioritizes matter, yet not in its universal objective form, but in its performative becoming through forming new cuts and relations (assemblages) (Barad, 2007).

This entanglement defines how (material) objects and subjects are revealed to each other, thus mixing ontological and epistemological considerations. Instead of representing pre-structured reality, the world and its inhabitants constantly co-shape each other in the process of *becoming*. "There are only changing things" (Turvey, 2019, p. 329) states the nature of ecological ontology, highlighting the constant flow of transformation. The transformation includes the *emergence* of "non-decompositable" entities in complex system processes (Tompson et al., 2001, p. 420). This ability to form holistic synergetic units is, apparently, universal to quantum particles, brain networks, and bodily muscles. Karen Barad offers a notion of intra-action to capture this activity that precedes the separation of a phenomena and a cognizing subject: "It is through specific agential intra-actions that the boundaries and properties of the components of phenomena become determinate" (Barad, 2007, p.139). Both, in new materialist and in radical embodied approaches this emergence of new units is not a one-stroke but a *cyclic* and *iterative* process leading to new (relatively) stable forms: As for new materialism, "matter is a stabilizing and destabilizing process of iterative intra-activity" (p. 151), for radical embodied approaches, cognition emerges as it finds stable patterns in "cycles of operation" in interaction with environment and others (Varela et al., 1991).

In application to mathematics education, the interaction between students, teachers, and objects is seen as fundamentally embodied and pre-discursive—thus contrasting with the constructivist view. In cyclic processes of becoming, humans (and non-humans) encounter each other through *intercorporeal* coupling of sensory-motor processes across bodies (Shvarts & Abrahamson, 2023) and through *affectivity* that drives the entire interaction (Freitas et al., 2018).

Finally, the fundamental aspect of new-materialist and radical embodied onto-epistemology is *non-centralized* regulation of its future unfolding. Asserting transformations in body-brain-world system to emergent self-organization (Varela et al., 1991), the radical embodied approach to cognition rejects the central regulation of behaviour (see Shvarts & Abrahamson, 2023 for application to mathematics education). New materialism goes even further as they "decenter the human subject as the willful agent, and <…> animate the mathematical concept" (de Freitas & Sinclair, 2013, p. 460). From a

new materialist perspective, not only humans, but also non-human material entities, such as artifacts, agentically contribute to the unfolding future and possible transformation.

Transformative/participatory paradigm: onto-epistemology

In this section, we propose that the onto-epistemological assumptions of new materialism and the radical embodied approach match the onto-epistemological assumptions of the emerging transformative-participatory paradigm of research in social science. Participatory studies assume that "There is a given cosmos, a primordial reality, in which the mind actively participates. Mind and the given cosmos are engaged in a co-creative dance" (Heron & Reason, 1997, p. 279). Mind and cosmos co-shape each other, so "this ontology is thus subjective-objective" and the *emergent* reality "the fruit of an interaction of the given cosmos and the way mind engages with it." The transformative paradigm also sees reality as *emerging* and highlights the analysis of power relations in shaping the reality: "How is reality defined? By whom? Whose reality is given privilege?" (Mertens, 2007, p. 2016).

Critically, participants are engaged with the world in pre-discursive way: "Experiential knowing thus articulates reality through inner resonance with what there is and through perceptually enacting" (Heron & Reason, 1997, p. 281); only later, "this grasp is symbolized in graphic, plastic, musical, vocal, and verbal art forms." The other humans are present in this space of *affective warm participation*, forming *intercorporeality*: "when we open ourselves to meeting the given we are arrested by the presence of Other; <…> the Other declares itself to us so that we resonate with its presence in the world." (p. 281). An assumption that reality and its continuous *becoming* emerge in interaction leads "to a *cyclical* model of research that includes the establishment of partnerships between researchers and community members, including the recognition of power differences and building trust" (Mertens, 2007, p. 2018).

We claim that transformative approaches (e.g., in sustainability science) and participatory approaches (e.g., participatory action research, participatory design) form a unified participatory/transformative paradigm. They share fundamental onto-epistemological assumptions of *becoming*, *emergence*, *intercorporeality*, and *affectivity*, which are also inherent to radical embodied and new materialist studies. Those assumptions call for research methodologies that encourage *warm participation* of different stakeholders in *non-centralized iterative* processes of creating educational transformations.

Empirical illustration

We use materials from the second author's doctoral dissertation to illustrate how the qualities of *emergence*, *intercorporeality and warm participation*, and *non-central iterative regulation* are fundamental in research on embodied education at the levels of students, teachers, and researchers as the main stakeholders within such research. In a design research study of a professional development (PD) program, the teachers were invited to explore action-based embodied design activities for reflection symmetry within a low-tech environment. A fragment of an activity is as follows: There is a see-through mirror and two chess towers standing on a drawn line, orthogonal to a see-through mirror (see Figure 1). The students are invited to move two towers in a way that a mirror reflection of one tower coincides with another tower. As in all action-based embodied designs, students need to solve this motor problem and then reflect and describe the rule when the towers coincide. As

designers, we expect that students come to see that the towers need to be positioned at equal distances from the mirror to coincide. The role of a teacher is non-traditional yet critical: Teachers engage students in a reflection on their sensory-motor interaction with the environment and gradual transition from naïve to mathematical discourse in describing target relations (Flood et al., 2020).

In three iterations of designing PD program, 2 to 3 teachers were engaged in a workshop activity where we gave them a chance to experience learning environment and explained embodied teaching practices previously shown as effective in embodied teaching, such as multimodal revoicing, collaborative gesturing, and others (e.g., Flood et al., 2020). Further, each teacher went through two cycles of trying out activities with students and reflecting on them with the researcher.

Transformative aspect: Emergent structures in cyclic interactions

Level of students. After creating the target learning environment (Figure 1a), Mark moves two towers on the line from the mirror. All bodies are organized around the action field, and the perceptions of three students and the teacher are coupled to Mark's actions (Figure 1b). Moreover, Maja joins the actions by physically adjusting the position of one of the towers (Figure 1c).

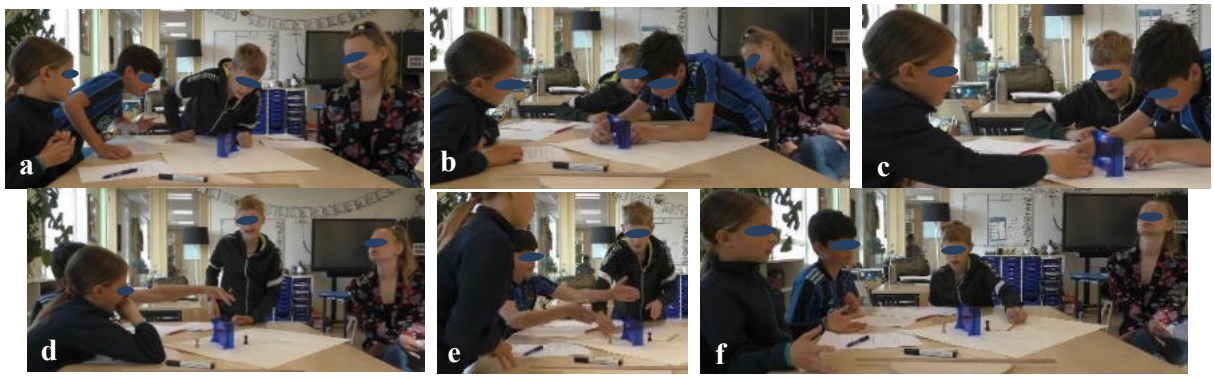

Figure l: Emergent structure of symmetry and students-teacher intercorporeal processes

Once smooth movement is achieved through *iterative attempts*, the teacher asks: "What do you see happening?" Simultaneously, very many voices and half-phrases manifest an intensive process of formulating the emerging perceptual structure: "We see that they are on each other"; "They move in the same way". Teacher asks for more specification: "When do they fall on each other?" Mark (*gesturing two sides from the mirror*, Figure 1d): "They go equally far… against the mirror". The teacher narrows the discussion even more: "And when are they precisely together"? All students together, alternating half-phrases between them, express the core idea of equidistance: "When you precisely" "same distance" "from the mirror" "the same distance" "from the mirror" (different colors mark different voices). Remarkably, their co-speech gestures gradually move away from the action space (Figure 1e,f), abstracting the *emergent perceptual structure* into a new property of symmetry.

Level of a teacher. Caty starts steering the lesson from sitting directly at the table, watching closely the children (Figure 2a); however, during the lesson, she gradually moves away from the table (Figure 2b,c), intervening only at what she calls later "critical moments". When asked at the reflection session about what she was aware of during the session, she immediately said: "I do know that what I really did consciously is to stay a little in the background (*gestures as if pushing herself back*), that I thought of 'hey, I don't want to be too much above it' (*gestures with two hands forward, above the table*), but

rather to let kids go their way and then watch also precisely what happens with the three of them." After conducting the lesson with the half-class she again recalls her emergent strategy of *sitting in the background:* "last time I was really sitting [in the background] so much (*gestures as if she is pushing herself back from the scene,* Figure 2d)." Caty did not deliberately enact the prescribed strategy in a top-down manner. Instead, the strategy *emerged* for her as an onto-epistemological entity through *iteratively* trying and adjusting her behavior to the feedback from the students, and further describing this emergent behaviour.

Figure 2: A sitting at the background strategy emerges

Level of a researcher. Equipped with the knowledge about embodied teaching practices (Flood et al., 2020), we hoped that teachers would follow our explanations in their lessons. Although exhibiting those practices from time to time, teachers barely used them as deliberate ways of acting. Instead, as we showed above, they developed their embodied ways of teaching and named them in their own ways. For us as researchers, a meta-level conceptualization *emerged*: the need for *non-linear pedagogy for teachers*. Just like we create special environments for students—*fields of promoted actions*—in which they discover in action and then describe in reflection new emergent onto-epistemological (mathematical) structures, we need to create action-reflection spaces for teachers so that embodied teaching practices emerge and reify through teachers' reflections. Although we knew principles of non-linear pedagogy ahead of the study, applying them to the case of the teacher was not straightforward. Instead, this realization *emerged in iterative attempts* to talk about embodied practices with teachers and reflect within the researchers' team on teachers' actions and reflections.

Participatory aspect: Incorporeality and affectivity in non-centralized warm participation

Level students–teacher. Fundamental co-presence of the teacher and the students is visible in our previous example (Figure 1a). Participants are coupled in an *intercorporeal functional dynamic system* (Shvarts & Abrahamson, 2023): their perception and action are directly interwoven in solving the sensory-motor problem. It is only through *warm participation* and contingent adaptivity to the bodies of the students and artifacts that Caty could distinguish "critical moments" for intervention and support. She does not just negotiate students' mental constructions, as a constructivist teacher would. Instead, she is present in the here-and-now of the students' sensory-motor discoveries.

Level teacher–researcher. Just like the co-presence of the teacher and the students is critical for the emergence of new geometrical constructions for the students, *warm participation* of the researcher is critical for the flow of the lesson and for the research discoveries. There were multiple moments when the teacher shared her *affective states* with the researcher: her laugh at students' naïve yet smart descriptions; indecisiveness in addressing unexpected situations. On Figure 1a, the students are expected to put a mirror on the line so that they "see only one line", thus orthogonally. Instead, they put the mirror directly on the line. Cathy is uncertain about how to progress and looks at a researcher

with a half-smile-half-question about what to do. The researcher makes a rotating gesture. Cathy immediately says, "Try to put the mirror in another way", which resolves the situation.

Where the direct *affective and intercorporeal presence* of the researcher and the teacher enabled enthusiastic flow of learning in a small group, in a later classroom situation, we, researchers, felt that such positive affectivity was lacking. Cathy felt the same: "in that [half-class] group <…> they were like 'how long will all of it last'". Theoretically, we could also understand what was going on: the students lacked the physical intercorporeal co-presence of a teacher in each group who would catch micro-zones of proximal development or struggles. A possible theory-based conclusion that this form of teaching is inappropriate for a classroom was interrupted by a suggestion from the next cohort of teachers, who iterated small-group lessons twice: They wanted to ask students from the small groups to become future teacher assistants, thus spreading the activity in the class. This idea would never come to us as researchers and marks *decentralization of expertise* in our research: Our researchers' theoretical expertise needs to be constantly advanced by the practical expertise of teachers.

Conclusions

In this brief overview, we show that, within each paradigm, theoretical and pedagogical visions on mathematics education are coherent with the involved research methodologies. For radical embodied and new materialist studies, we are still in the process of establishing suitable methodologies. We submit that the methodologies within the participatory/transformative paradigm might fit best: just like radical embodied and new materialist approaches, they see acquiring knowledge (in both research and education) as the process of *becoming* that leads to new onto-epistemological cuts. Research strategies of design research, lesson study, participatory action research might make a step towards new methodological standards suitable for radical embodied and new materialist studies. Such future methodologies would see (mathematics) education research as part of *transformative becoming* through *warm participatory* co-creation of *emergent* structures of future education.

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Experimental epistemology in mathematics education and biology of cognition

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We explore the intersection of "experimental epistemology" in mathematics education and theoretical biology, by highlighting the disruptive work of Guy Brousseau in France and Humberto Maturana and Francisco Varela in Chile, initiated in the 1960s and 70s. We discuss the key notions of situation, structural coupling, embodiment and metaphorising and their influence on the understanding of mathematical objects. We argue that while French mathematics education has downplayed embodiment, Brousseau's approach inherently incorporates embodied cognition. Furthermore, we show how the Chilean School of Enactivism in Mathematics Education Research is in this vein actively bridging connections between mathematics education and theoretical biology.

Keywords: Experimental epistemology, mathematics, metaphor, autopoiesis, structural coupling.

Introduction

We explore the intriguing parallel disruptive emergence of "experimental epistemology" in two seemingly disparate fields: mathematics education and theoretical biology. Focusing on the 1960s and 70s, we compare the groundbreaking work of Guy Brousseau in mathematics education with that of Humberto Maturana and Francisco Varela in theoretical biology. By tracing each approach's origins, development, and core insights, we aim to illuminate their convergences and divergences. Specifically, we examine how their perspectives address the ontological status of mathematical objects and the role of embodiment and metaphor in mathematics. Key concepts such as Brousseau's "a-didactic situation," Maturana's "structural coupling," and Varela's "enaction" provide focal points for our analysis. While French mathematics education may have temporarily downplayed the importance of embodiment, we argue that Brousseau's work implicitly incorporates it, anticipating contemporary trends in mathematics and science education[1], including the burgeoning Chilean Enactivist School (Palatnik et al., 2023, p. 161; Videla-Reyes et al., 2023). We further explore the parallels between Brousseau's a-didactic approaches and enactivist perspectives, while acknowledging key differences. These include the emphasis on creating a safe learning environment within experimental epistemology of mathematics and the distinct dynamics of object creation within each framework. Ultimately, this comparative analysis reveals valuable insights with implications for both fields, opening new avenues for mathematics education research and pedagogical innovation.

[1] See https://embodiment-steam.sciencesconf.org for a recent example.

French experimental epistemology in mathematics education

In the 1960s, after a decade teaching elementary mathematics in France, Guy Brousseau (1933-2024) became outraged by the prevailing pedagogical approach, which he found abusive, punitive and segregative towards students. He later pointed out that his concern about this critical issue, which we fully shared, fuelled his research on "experimental epistemology of mathematics" beginning in the 1960s (letter, June 20, 2023). The "cognitive abuse" inherent in such teaching practices wasn't explicitly recognized in the literature until much later, though (Watson, 2021). Brousseau, influenced by Piaget, Sartre[2] and Makarenko (1951) tried to counter this cognitive abuse by making mathematical concepts exist, before formally defining them, allowing their meaning and purpose ("raison d'être") to emerge organically (Soto-Andrade and Diaz-Rojas, 2023). He designed engaging problematic *situations,* which made sense for the students and fostered their autonomous, collaborative learning. He carefully studied the *conditions* under which such situations would trigger, or not, the construction of the mathematical concepts the teacher was aiming at. This was the start of his "Experimental Epistemology of Mathematics" (EEM), a provocative terminology, later dropped in favour of the more mainstream denomination of "Didactics of Mathematics" (Brousseau, 2002). EEM later evolved into his theory of *a-didactic situations*, where the teacher steps aside, hides her didactic intent, and lets the students explore and discover on their own. In our view, his approach foreshadows von Glasersfeld's Radical Constructivism and Varela's Enaction.

A fundamental a-didactic situation for statistical inference: "Brousseau's bottle"

Designed for 4th graders ignorant of stochastics, this situation involves an embodied, interactive "mise-en-scène" going beyond typical data-driven exercises (Warfield, 2014). It was implemented at COREM[3] in collaboration with Nadine Brousseau, and later with prospective and in-service teachers. ***The teacher fills a small opaque bottle with five marbles drawn randomly from a bag with equal numbers of blue and red marbles. A small window in the bottle's cap allows students to peek at one marble at a time. She hands such a bottle over to each small group of students (typically five) in the class and asks about the number of blue and red marbles in it.*** Students, working in small groups, explore the bottle by peeking, shaking, and recording marble colours. The teacher observes their *statistical thinking* emerging in their self-organized sampling strategies. Interestingly, French kids tend to analyse data in sets of five, favouring the most frequent composition, while older Chilean students and prospective teachers track the accumulated proportion of blue marbles with each observation. Some of our groups, working with 20 series of 5 and 100 single marble observations, were bewildered to find that the mode of the series of 5 and the accumulated frequencies contradicted each other. We prompted them to look for an embodied, enactive explanation of this phenomenon, by handling concrete objects (e.g., 100 coins on a table), since a series of observations is not so tangible and concrete after all. The learners discuss how many

[2] Particularly by his motto: "L'existence prime sur l'essence" [Existence precedes essence].

[3] Centre pour l'Observation et la Recherche sur l'Enseignement des Mathématiques, in Talence, France (1972-1999).

samplings “are enough” to be able to “guess” the composition of the bottle. Eventually, they realise that there is no magic number of observations but that this is a matter of tendency, which becomes visible when they plot, say, the accumulated frequency of blue in *n* observations as a function of *n*. Crucially, the opaque bottle cannot*, and must not,* be opened. The teacher resists pleas from students to verify their guesses. When they think they have figured out the bottle’s composition, the teacher quickly empties the bottle back into the bag, so that nobody sees the colours of the marbles inside. This mimics real-world statistical scenarios where complete observation is impossible. Students eventually grapple with the possibility of a 3-2 (blue-red) bottle masquerading as a 2-3 bottle. This sparks a crucial insight: *to create a transparent 3-2 bottle and observe its behaviour under repeated sampling*: a transition from *statistical thinking* (inferring composition) to *probabilistic thinking* (predicting behaviour)! Students are encouraged to visualize their sampling process. They eventually see it as a 2D random walk: one step right for a blue marble, one step up for red. This walk, after some wild initial fluctuations, tends to settle within a wedge-shaped area, with the slope of the line from the origin to the walker's position indicating the current red-to-blue ratio.

Experimental Epistemology and Theoretical Biology from the South

Humberto Maturana (1928-2021) left Chile in 1953 to study at University College London, before completing his M.D. Lacking a formal degree, he faced obstacles enrolling in a doctoral program due to UK university regulations. He eventually found his way to Harvard, where he completed a Ph.D. in neuroanatomy and later a postdoc at MIT under Warren McCulloch, a pioneer of cybernetics. Through experiments on the visual system of frogs, Maturana proved wrong the prevailing view that vision was simply a pixel-based light detection mechanism. His research demonstrated that neurons respond to specific features of the visual world (like edges, movement, bugs, ...) Thus, the brain, instead of analysing a set of values in a matrix of intensities, does something radically different as it filters the visual stream according to predetermined patterns generated by evolution and ecology. Thus “to view” is isomorphic to “existing in a concrete manner” (Uexküll, 1926). This contradicted the prevailing "one-way street" metaphor of perception, where the brain computes an objective reality from sensory data. In 1963, Maturana met Heinz von Foerster, who invited him to his famous *Biological Computing Laboratory* (BCL), where he wrote his seminal report “*Biology of Cognition”* (BC) (Maturana, 1969). Maturana claimed that the brain is *not* a computer decoding “reality”, instead it constructs it, in an endless helix of sensory-motor *correlations* that *become* the objects populating the Umwelt of each organism. A still novel idea today, but nonsensical in 1969. After returning to Chile in 1969, Maturana and his former student, Francisco Varela (1946-2001), created the theory of *Autopoietic Systems* by translating the idea of sensory-motor correlations to metabolic networks. This theory (Maturana & Varela, 1973/1980) posits that the main property of living systems is that they continuously *self-fabricate* themselves and that this *self-fabrication entails autonomy*. The impact of these ideas is still swelling: more than half a million web references and over 5,000 technical articles use the notion of autopoiesis, which is central to many theories about the origin and nature of living systems (Cornish-Bowden & Cárdenas, 2020). Furthermore, by blending autonomy and the creation of sensory-motor correlations, Maturana and Varela created competing visions on how this construction happens: *structural coupling* and *enaction*. In retrospect, BC opened avenues for ideas

like sensory-motor correlations, metabolic closure, enaction, structural coupling, epigenetic niche, and language as a tool to recursively generate coupled behaviour (Meyer & Brancazio, 2023).

How Biology of Cognition informs Mathematics and Mathematics Education

The core idea of Biology of Cognition (BC), sketched in Maturana (1969) and further developed in Valdes-Zorrilla et al. (2023), is that all "objects" (a cat, Pythagoras theorem, HIV…) arise from our coupling with an external medium that continuously shapes our metabolism. Objects are not properties of the medium but properties of the Perception-Action trajectories every living being engages in with respect to its medium. Mathematical structures are not ideal entities independent of human observers, they arise as *invariants* of some particular flow of interactions with our medium**,** which *is not* a property of the world. Every object is a sensorimotor correlation constructed as we interact with our medium. Mathematical learning should prioritise human experience. Instead of watching properties of pre-defined entities, learners should experience how those properties arise from basic distinctions that they spontaneously make all the time. This echoes Sartre's "existence precedes essence, " which inspired Brousseau. According to BC, every distinction is *relational* and based on the complex properties of self-fabricating metabolisms, echoing Barad (2007): "There are no pre-existing individual entities at all, including us; the world is composed of *relations* between beings." We tend, though, to focus on *binary* relations and operations, very likely because we have two hands[4]. We may see binary relations as connecting edges between not yet existing points (Paton & Sinclair, 2024). But many meaningful relations are not *binary*, but *n-ary,* so a broader visualisation emerges: *higher-dimensional relational domains*, whose *boundaries* consist of points, segments, and so on. The familiar individual objects in our Umwelt (Uexküll, 1926) are then just the *0-dimensional skeleton* (vertices) of the boundary of those higher-dimensional relational domains we are ill-equipped to fathom. BC reminds us that the way we relate with our medium is shaped by our embodiment as vertebrates, limited by our relatively small nervous systems. Other life forms, like Hoyle's Black Cloud, would construct their Umwelten in ways we can scarcely imagine. Ultimately, BC emphasizes that two fundamental properties of living systems (metabolic closure and their existence as cognitive machines) are fundamentally linked. *It is thus impossible to disentangle the biological from the cognitive aspects, as cognition and living are two inseparable facets of organisms.* We develop theories by using *experience* to explain *experience*. For instance, we use our experience of unsupported objects falling to the ground to explain why some points of light in the night sky change position. Ptolemaic astronomy, based on perfect circular motion, even with adjustments couldn't explain the observed orbits. Galileo, Kepler, and Newton introduced a new metaphor: undisturbed masses move in straight lines, and the force ruling planetary motion is the same force that causes an apple to fall. This idea, that *experience explains experience*, can be extended from physics to mathematics by broadening the notion of experience to include theoretical experiences. Non-Euclidean geometries, like Lobachevsky's, redefine lines as the shortest paths (*geodesics*) between two points. In the Euclidean plane, these are just straight lines, but in the hyperbolic plane, modelled by the unit disk with a specific metric, "lines" are arcs of circles perpendicular to the disk's

[4] Indeed our bodies shape our cognition. Interviewing octopii or slime moulds would be most commendable…

boundary. The concept of a line as a *geodesic* is experiential and depends on the ambient metric. This experiential approach allows us to understand lines in other spaces, starting from lines in the Euclidean plane. For example, on a sphere, “lines” are maximal circles, like the Equator.

A-didactic situations from the viewpoint of Biology of Cognition.

We set up a fundamental a-didactic situation based on a *random walk* (a baby avatar of *Brownian motion*), where probabilities may emerge idiosyncratically metaphorised as *portions* of the walker, enabling an embodied approach to probability. Specifically, we consider the following 2D version of the 1D random walk of a frog in Soto-Andrade and Diaz-Rojas (2023), which has been worked out with various learners at the University of Chile (first-year humanistic students, prospective mathematicians, prospective mathematics-physics teachers, in-service primary school teachers). ***A puppy, fittingly called Brownie, with a nose for trouble, escapes from home to avoid a bath. Disoriented by the city traffic’s cacophony and smells, she randomly chooses one of the four cardinal directions at each intersection, sprinting for four blocks before collapsing in exhaustion. Her worried owner needs help finding Brownie and estimating how far she might have travelled.*** Students, working in small random groups, discover that Brownie can reach some corners but not others. Roughly half of them initially believe all reachable corners are equally likely; others disagree, noting that some corners are reached by multiple paths. Statistical simulations help most students rank corners by likelihood. To quantify these likelihoods, some students analyse simulation frequencies, while others try counting paths à la Laplace. The latter notice they would have $4^4 = 256$ possible paths, from which the favourable ones need to be extracted! There are, however, more intuitive and efficient *metaphorical approaches* to this quantifying. Under some minimal prompting like: *What else, more concretely, could you imagine instead of this (rather abstract) fair random choice between the 4 cardinal directions?* Some metaphors emerge idiosyncratically: The *Solomonic Metaphor* (Brownie splits into four at each corner, with each part taking a different direction); the *Hydraulic Metaphor* (Brownie flows like water, dividing itself equally among the four streets at each intersection); the *Splitting Pack Metaphor* (unleash a pack of puppies from home, fittingly 256 in this case, which split evenly into fourths at each corner). After prompting, typically, one or two out of thirty “see” Brownie splitting into fourths. In-service teachers are not so prone to metaphorising - didactical contract obliges (Brousseau, 2012). These metaphors allow quantifying corner likelihoods.

Embodiment in Experimental Epistemology

While embodied cognition isn't as explicit in EEM as in BC, Brousseau's approach is far more embodied than traditional Cartesian French mathematics education (Soto-Andrade & Diaz-Rojas, 2023). It rejects the "conduit metaphor" of teaching as transmitting knowledge and challenges the notion of mathematical objects as revealed pre-existing truths, “parachuted from Olympus”, whose definitions are to be recited, transmitted and applied to execute tasks. This parallels Maturana's simultaneous rejection of the well-entrenched metaphor of perception as a “one-way street” in cognitive science. Brousseau saw instead mathematical objects and processes as emerging from the embodied actions of learners in *situations*. EEM in our research even connects with embodied *non-human* cognition. Indeed, in Videla-Reyes et al. (2023, pp. 259–276) we showed how in *stochastic*

dance dancers simultaneously enact all possible unfoldings of a random walk, mirroring the maze-solving behaviour of slime moulds (Nakagaki et al., 2000) and in *flow of motion dance*, all possible orbits of a *deterministic* dynamical system in a finite cyclic universe. Structural Coupling sheds new light on "Brousseau's bottle", where both sampling strategies embody the *ergodic principle:* the "universe average" equals the "long time average". Ergodicity is indeed a powerful metaphor relating space (universe) to time, which needs to be enacted to have meaning. The amazing equivalence between *ensemble sampling* and *time sampling* and the equality "ensemble mean = time series mean" need a continuous confrontation between perception (experience) and the result of some actions promoted by this experience, i. e. engaging in a perception-action loop. Lived experience allows us to check the proportion of blue balls without "opening the box": students could engage in a betting game, trying to predict the percentage of blue balls in the next 100 observations, plus minus 5%, say.

Metaphorising in Experimental Epistemology

We prefer "metaphorising" (an active process) to "metaphor" (an object "out there"), to avoid abusive *reification* (Sfard, 1997). Rather than Lakoff and Nuñez's (2000) *metaphorising* of metaphor as an inference-preserving mapping from a concrete source domain into a more abstract target domain, we favour Sfard's (1997, p. 355) view of metaphor as a dynamic, back-and-forth exchange between source and target in a circular, *autopoietic* way. In enactivist perspectives, metaphorising is just "the tip of the iceberg" of a fundamental embodied *biological process* enabled by *structural coupling* (Maturana & Varela, 1980). Aristotle, limited by his time, only partially grasped metaphorising as renaming. Nowadays, we metaphorically say that metaphorising is *looking* at something and *seeing* something else, a sort of controlled *hallucinating* in the sense of Seth (2021), as exemplified in Brownie's random walk. Metaphor is not just a rhetorical device, but a powerful *poietic* cognitive mechanism, enabling us to *construct* new concepts and theories (recall Darwin's "tree of life" metaphor). See Abrahamson et al. (2012); Lakoff and Nuñez (2000), Sfard (1997, 2009), Soto-Andrade (2020). We promote learners' spontaneous, bottom-up generation of metaphors over prescriptive, top-down uses by teachers. Metaphors emerge from a learner's active engagement with a situation, not in the void. This is mirrored in Chaplin's "The Circus" (used by Brousseau as a start-up for his didactics courses), where Charlot struggles to be funny on command, yet effortlessly elicits laughter through spontaneous action. Moreover, "the fish is the last one to see the water"… or "the metaphor", as in the case of the second nature metaphor "teaching is transmitting knowledge."

Conclusion

We have explored the convergence of disruptive parallel "experimental epistemologies" in mathematics education and theoretical biology. We highlight how both emphasise a relational and metaphorical construction of knowledge, exemplified by our stochastic examples. Connecting Biology of Cognition (BC) and Experimental Epistemology in Mathematics (EEM), we see that BC focuses on how a living being constructs the objects within its Umwelt, while EEM aims to trigger a similar object construction in a learner's Umwelt through "a-didactic situations", which challenge them to discover mathematical concepts as necessary tools for navigating problems, much like organisms adapt for survival. These situations amount to "pushing a non-swimmer into a pool with a

safety pole nearby", since they should be genuinely problematic, requiring exploration and experimentation. We should not believe, though, that learners construct in their Umwelten the object we have in mind. We glimpse the variety of constructed objects when we let the learners "do anything they like" with a bunch of sticks, for example. Some build structures, others throw them in the air… Indeed, traditional math education ignores the notion of Umwelt. Now, while both approaches share core principles, EEM operates within the constraints of a classroom, a unique environment that must balance challenge with safety, needing careful consideration of time, space, and the emotional needs of learners, to provide a safe haven for them. Mathematical education, like life itself, is less about solving problems and more about experiencing reality. By blending insights from mathematics and biology, we explore how new ideas emerge—through confronting our existence or translating concepts across contexts. This challenges traditional mathematical thinking by proposing that understanding the creation of mathematical ideas requires considering mathematics and biology as equal partners. However, this approach's future remains uncertain, as our understanding of structural coupling is still limited, and preconceptions about the *a priori* existence of concrete or ideal objects must be replaced by intuitions that need to mature for a while before becoming helpful techniques.

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Spatial perspective switching in group embodied activities

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Although evidence for the value of embodied approaches to representing mathematics concepts continues to accumulate, limited attention has been given to the complexities of children's engagement in group embodied representational tasks. This paper explores the observable responses of 5-to-7-year-old Australian children to group tasks requiring cooperation in the creation of large-scale mathematical representations using their bodies. It is argued that this type of task requires perspective-switching by at least some of the participants and involves the emergence of shared coordinated action. Given the rarity of research that draws together these task characteristics, this study provides a starting point for further investigation of the significance of spatial perspective switching in group embodied activities for children's learning.

Keywords: Mathematics, embodied representation, spatial perspectives.

Introduction

The current study focuses on the type of teaching/learning task that engages each student in using their body as a component of a large-scale mathematical representation, thereby evoking spatial awareness and reasoning, including perspective-taking. A social dimension is added to such tasks when forming the representation relies on the interaction and cooperation of more than one child. The aim of this paper is to provide an initial exploration of the role that spatial perspective switching might play in cooperative group activities involving the body movement of children around age 6 years. The study itself is focused by the question, *what can be inferred about the role of 'spatial perspective switching' from observing children's actions during the creation of group embodied representations?*

Spatial perspectives

Drawing on embodied cognition theory, the learning process can be described as involving multidirectional loops of perception, action and cognitive processing, with emphasis on the essential role of 'action' (Kiefer & Trumpp, 2012; Kim & Sankey, 2022). Integral to many aspects of learning is the development of spatial awareness, flexible spatial reasoning, spatial memory and associated spatial language (Bruce et al., 2017). It has been well argued that embodied experiences are fundamental to learning and memory, including the development of spatial reasoning (de Freitas & Sinclair, 2013; Kiefer & Trumpp, 2012). The term 'spatial reasoning' refers to "the ability to recognize and (mentally) manipulate the spatial properties of objects and the spatial relations among objects" (Bruce et al., 2017, p. 147). An aspect of 'spatial relations' that is of particular interest in this study is the phenomenon of switching spatial perspectives. In this paper, the term 'switching spatial perspectives' is used to encompass a range of changes in position, orientation and viewpoint by the learner (physically or through visualisation), in the context of physical engagement with an environment.

One way that switching spatial perspectives occurs is shifting between an egocentric frame of reference and an allocentric frame of reference (Avraamides & Kelly, 2008; Dackermann et. al., 2017). In an egocentric spatial perspective, a person thinks and acts in relation to the current positioning of their own body, giving particular meaning to concepts like forward, right, and up. An allocentric spatial frame of reference requires perceiving objects in the environment in relation to other objects and allows a person to create mental maps to navigate the three-dimensional environment, and also to create and interpret two-dimensional representations, such as maps (Shusterman, & Li, 2016). Different parts of the brain are activated during egocentric processing and allocentric processing (Galati et al., 2010), and spatial information is stored in memory according to the spatial frame of reference through which the spatial representation was formed (Galati et al., 2010; Ruggiero et al. 2016). Egocentric-based spatial representations are easier to retrieve from memory than allocentric-based representations (Ruggiero et al. 2016).

From a child developmental stance, the classic view is that egocentric spatial memory precedes allocentric representations (Acredolo, 1978; Piaget & Inhelder, 1967), although evidence of the ability of 3-to-5-year-olds to work with allocentric frames reference has been found (Vasilyeva & Lourenco, 2012). The key transition point appears to be development of a child's capacity to efficiently integrate egocentric and allocentric perceptions and experiences of an environment, which seems to emerge around age 6 years (Nardini et al., 2006; Vasilyeva & Lourenco, 2012).

Triggering spatial perspective switching

Linked to ego/allocentric switching is the ability to move between an on-ground visual-spatial perspective (from one's own perspective) and a from-above visual-spatial perspective (looking down on the scene), either through actual physical reorientation or through visualisation and mental imagery. For example, Ferrari and Soldano (2023) noted that when third-grade students were explaining, using words and gestures, the movements that two small robots (bee-bots) must perform to create symmetrical paths, one child seemed to switch between a bee-bot's view and bird's eye view, and at one point perhaps integrated the two perspectives to conceptualise the mirrored paths in a holistic way.

Dackermann et al. (2017) identified the role of interactions between different regions of personal space, and the integration of egocentric and allocentric frames of reference as mechanisms in achieving positive learning outcomes across several studies with 4-to-8-year-olds engaged in embodied spatial-numerical representations. For example, when interacting with a large-scale number line on the floor, a child might stand on a starting point and step forward to reach the next number. This is essentially an egocentric experience with the child's actions within their near space (one step). However, a task requiring the enactment of an addition or subtraction would require consideration of a greater portion of the number line, thereby triggering an allocentric perspective and an extension of the action space beyond their near space. More complex switching back and forth between (or integration of) frames of reference would be required by a task such as estimating the location of 63 between the marked locations of 0 and 100 (with no other markings present). Dackermann et al. (2017) advocate that participation in embodied tasks such as these enable the mapping of number concepts to spatial representations and build relational knowledge about

numbers. These examples of tasks with robots and floor number lines refer to perspective switching in individual actions, rather than in a group situation with social interaction.

Socio-kinesthetic interaction

It is generally assumed that the egocentric perspective comes naturally to humans and would have primacy in most situations. This notion is challenged by Tversky and Hard (2009) who found that the "mere presence of another person in a position to act on the objects" induced adult participants in their study to "describe the spatial relations from that person's point of view" without prompting (p. 124). Furthermore, "When the query about the spatial relations was phrased in terms of action, more respondents took the other's perspective than their own" (Tversky & Hard, 2009, p.124). Much of the embodied learning research with children has focused on the individual, but perhaps more attention should be given to the influence of the presence of other children in the embodied learning situations if we are to understand social contexts of classrooms.

Cooperative group tasks impose different demands on the creation of embodied representation of mathematical concepts than individual tasks. Shoval (2011) studied the behaviours and achievement of 158 second and third-grade children who engaged in learning about angles through body movement in cooperative groups using a length of rope as a representational object. Two of the five types of learning activities were "socio-kinesthetic interaction" and "sustaining the movement-aided learning activity" (Shoval, 2011 p.453). The most significant predictor of improvement in achievement was the *sustained movement-aided learning* activity, which is a combination of all the studied behaviours, including individual movement, cooperative movement, contact with an object, observing, speaking and listening (Shoval, 2011). Of particular interest in the present study is the activity of *socio-kinesthetic interaction*, which is characterised by two behaviours, cooperative movement with others and cooperative contact with physical objects pertinent to the learning activity (Shoval, 2011). *Socio-kinesthetic interaction* was positively correlated with learning, but with lower significance in comparison to other activity-types in Shoval's study. It is possible that the impact may have been moderated by the fact that social skills are still developing in young children (Slavin et al., 2003). It is also possible that attending to their own individual movement as well as coordinating with movement of other group members, challenged the capacity of some children to switch between or integrate egocentric and allocentric frames of reference. De Freitas et al. (2019) discuss the emergence of "embodied shared response-ability" in 15/16-year-old mathematics students collaborating to produce a graph via movements using WiiGraph software. The successful creation of the complex representation necessitated the development of "transindividual sympathy" that reflected the dynamic nature of the mathematical concepts (de Freitas et al., 2019). Perhaps younger students can also achieve the social connections needed for cooperative action.

Method

The context for this study is the *Embodied Learning in Early Mathematics and Science* project (2022-2024). The goal of Phase 1 (held in 2022) of the project was to investigate ways in which teachers could translate the findings of research on embodied learning from the fields of psychology, neuroscience and education (see Way & Ginns, 2024) into their mathematics teaching practice with 4-to-8-year-old children. The researchers worked collaboratively with teachers in one school in a

socio-educationally disadvantaged suburb of a major city. The school had just reopened to external visitors after the COVID-19 'lock downs', and the participating children had experienced limited social interaction with their peers for about a year. As part of the investigation, videos were made to record some of the embodied activities that were perceived by the teachers to be productive learning experiences. The current study returns to these videos as a data source for exploring a new question about the role of spatial perspective switching. Two video-recorded episodes sourced from two lessons were selected for this paper using the following criteria.

a) The focus of the tasks is the creation of a mathematical representation involving whole-body movement with the intention of promoting conceptual understanding.
b) The representational space has been 'scaled up' so that students can step into a frame of reference and become movable parts within the emerging representation.
c) Social interaction and some level of co-operation and/or collaboration to align the parts of the representation in an appropriate configuration is required for successful completion.

The data are presented as images from the videos and descriptions of the main actions of both individuals and the group as the task evolves. In all the activities, the teachers gave minimal instructions and allowed the children time to make their own decisions before deciding to intervene with suggestions for actions.

Analysis and interpretation of episodes, through repeated viewing of the videos, was guided by the following key ideas identified by previous research:

a) Embodied tasks may involve egocentric and/or allocentric spatial perspectives (e.g., Avraamides & Kelly, 2008; Dackermann et al., 2017)
b) Flexible switching between spatial perspectives may aid conceptual development and successful task completion (e.g., Dackermann et al., 2017; Ferrari & Soldano, 2023)
c) Thinking about one's 'far space' rather than only 'near space' may be a trigger for shifting from ego- to allo-centric spatial referencing (Dackermann et al., 2017)
d) The presence of others in a task may trigger perspective switching (Tversky & Hard, 2009)
e) Cooperative movement with others and cooperative contact with physical objects are pertinent to learning (Shoval, 2011)
f) Some 6-year-olds may not have sufficiently developed social/collaborative skills to co-ordinate their movement and positioning in group representation tasks. (e.g., Nardini et al., 2006; Vasilyeva & Lourenco, 2012)

These key ideas provided the basis for interpreting the actions of the children during their shared creation of a mathematical representation and for inferring the meaning of those actions.

Episode 1: Be a dot

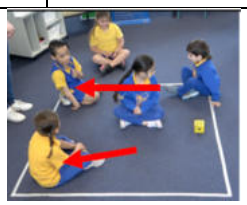
In this task 5/6-year-olds were expected to cooperate to recreate the dot pattern on a dice by positioning their bodies. When this task was first attempted, there was no square marked on the floor. When the children were asked to arrange themselves to 'be the dots on the dice' they were confused. The researcher and teacher quickly used masking tape to mark a large square on the floor to represent the face of a dice and restarted the activity.

Table 1: Episode 1 data, incorporating Figures 1A, B, C, D, E

A. The teacher rolls a 5 on the dice and asks the children to look at the number and pattern of dots.

B. Teacher invites the children to each become a dot on the dice. Three children sit inside the square but do not appear to be representing a particular dot.

C. One child leaves the square and another comes in. The teacher suggests they look at the pattern on the dice and they turn to look at the dice then glance around the square frame.



D. Without discussion about exactly which dice-dot they will 'be', two children move decisively from their original positions into two corners of the square, and a fourth child enters the square to sit in a third corner. The child who sat near the centre at the beginning remains there, turning her head and watching the other children moving around her.

E. The girl who previously left the square, returns to sit in the fourth corner to complete the pattern of five dots. The task was completed in under one minute.

Interpretation

The square marked on the floor provided a necessary environmental 'object' with which the children could interact. At first the children acted individually and presumably were perceiving an egocentric spatial frame. The prompt from the teacher appeared to help the children connect self to a dot and a significant shift to an allocentric (or 'from above') viewpoint seems to have occurred for the two children who relocated themselves to two corners of the square. This action seemed to trigger action in the remaining two children who positioned themselves appropriately in relation to others and to the square. Notably, all this occurred without the children giving verbal instructions to each other.

Episode 2: Numeral sequencing

This task captured the 6/7-year-old children's first attempt at sequencing consecutive counting numbers by physically forming a line. The teacher handed each child a numeral card (1 to 15) and asked them to organise themselves in order from smallest (teacher points near the wall) to largest (teacher gestures along a line painted on the playground). See Table 2.

Table 2: Episode 2 data, incorporating Figures 2A, B, C, D, E

A. The children mill about in a cluster next to the wall, looking at their own numeral card and not speaking much. The children holding 1, 2 and 3 position themselves next to the wall and do not move thereafter.

B. The teacher and two of the girls (holding 4 and 15) begin urging everyone to spread out. The children begin looking at each other's numerals and some children begin moving away from the cluster to stand beside a numeral consecutive to their own.

C. A few children begin pointing and giving instructions, "You have to go down there", "Find number 9"

D

E

D. The girl holding 10, moves further along the line on the ground and stands alone. Many of the children now stand facing the same direction, shoulder to shoulder along the line on the ground and turn their heads to the left and right looking up and down the line. Three children have stepped out of the line and face the other children while checking the sequence of numerals. The girl (holding 15) begins counting out loud, pointing to each child along the line to check the sequence and pauses while 10,11 and 12 get into position.
E. Everyone seems satisfied the task is complete.

Interpretation

At first the children lacked a strategy as they only had control of one numeral and perhaps could not visualise what the line would look like, suggesting egocentric operation. A few individuals began to influence the others through their own relative positioning or verbal and gestural instructions. A shared understanding of which direction to face appeared quite suddenly. Number-10-girl seems to have envisaged the whole length of the 'line' to estimate a proportional distance of 10 from 1 – thus thinking beyond 'near space'. The encouragement to spread out and find numbers before or after their own, appeared to trigger comprehension of the configuration required and the children moved into position, though children focusing only on the number before/after their own may have been thinking about 'near space', hence taking an egocentric perspective. The children providing some leadership seemed to be the ones who could take an allocentric perspective quickly, to imagine the entire number sequence along the line.

Discussion

Analysis of the two episodes indicates that 'spatial perspective switching' was integral to the creation of group embodied representations. The socio-kinesthetic interaction (Shoval, 2011) necessitated by the cooperative group task was anchored in spatial relationships between individuals, objects and the whole group. In Episode 1, the children needed the square on the floor to anchor their spatial representation and had to coordinate the arrangement of the dots on the dice with both their own position in the square and their position relative to each other. In Episode 2, the floor-line anchored the representation, and the children had to co-ordinate their position on the line in relation to the number before/after their own (egocentric perspective, near space) (Dackermann et al., 2017). Some

of the children also needed to take an allocentric perspective in the creation of the representation. This enabled "sustaining the movement-aided learning activity", which Shoval (2011) found to be a significant predictor of learning. Interestingly, children's co-operative actions were mostly triggered by observing the actions of others, rather than verbal communication. Both tasks challenged the children's capacity to translate their number knowledge into a spatial representational, yet, collectively, they succeeded with minimal teacher direction.

This study appears to be unique in that it draws together embodied mathematical representation, scaled-up representational space and social interaction with young children, to raise the idea that shifting between egocentric and allocentric spatial perspectives may play an important role in the dynamics of embodied learning and the development of mathematics concepts. The argument is currently tentative, but this study provides a starting point for further research into the role of spatial-perspective-switching in group embodied activities.

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Developing functional thinking through bimanual movements

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This study investigates the role of bimanual movements in developing functional thinking (FT) through digital-embodied nomogram tasks. Two teaching experiments, conducted in the Netherlands and China, involved a total of 76 ninth-grade students. Using a mixed-methods approach, the research combines quantitative correlation analyses with qualitative examples and mini-interviews. The findings reveal that initial FT levels, as measured by pretest scores, are significant predictors of student performance in bimanual tasks. Higher pretest scores were associated with shorter times to achieve fluency and less time spent in the fluency phase, whereas students with lower pretest scores required longer time in the fluency phase for extended exploration. Mini-interviews further showed that students with different FT levels used different strategies. Overall, our results suggest that bimanual tasks effectively support the development of FT across different student groups.

Keywords: Bimanual movements, functional thinking, digital-embodied tasks, mathematics education, nomogram.

Bimanual movement in functional thinking

The development of functional thinking (FT) is an important learning goal in mathematics education, involving a fundamental understanding of functions and continuous variation and covariation of quantities. Developing FT poses a significant challenge due to its abstract nature, especially the covariation aspect. As noted by Thompson and Carlson (2017), fostering covariational reasoning is crucial not only for FT but also for broader mathematical development. Studies have shown that students often struggle to grasp the dynamic nature of covariation, especially when required to visualize or mentally manipulate relationships between quantities (Castillo-Garsow et al., 2013; Thompson & Carlson, 2017). A taxonomy of covariational reasoning levels proposed by Thompson and Carlson (2017) provides a framework to understand these challenges, describing a progression in reasoning about covariation. At the lowest levels, students either fail to coordinate quantities (L0) or simply observe that changes in one quantity correspond to changes in another (L1). Intermediate levels involve describing general patterns of change (L2) or coordinating specific increments between quantities (L3). Higher levels require envisioning simultaneous changes over fixed intervals (L4) or, at the highest level, perceiving smooth, continuous covariation (L5). Many students struggle with the cognitive demands of transitioning from discrete reasoning to continuous, dynamic thinking, particularly in reaching the advanced understanding required at L4 and L5.

To support students' development of covariational reasoning within FT, researchers and educators have turned to embodied learning, an approach that emphasizes the connection between physical experiences and cognition. This approach is based on the idea that learning is deeply rooted in the body's interactions with the environment. According to Lakoff and Núñez (2000), mathematical thinking itself is inherently embodied, meaning that our understanding of abstract concepts is grounded in our sensorimotor experiences. Bimanual movements—coordinated actions involving both hands—can offer promising opportunities in this context. Research has shown that such

movements can help students grasp mathematical concepts by providing a concrete, physical experience of abstract ideas (Abrahamson et al., 2014; Jaber et al., 2024; Walkington et al., 2014). For instance, Abrahamson et al. (2016) demonstrated how primary school students' understanding of proportions improved through bimanual tasks that involved physically manipulating objects to represent proportional relationships. This finding aligns with broader research on embodied cognition, which suggests that engaging the body in learning activities can enhance understanding and retention of abstract concepts (Alibali & Nathan, 2012).

The potential of bimanual movements in supporting covariational aspects of FT has also been explored in multi-representational learning environments (Pittalis et al., 2024). These environments integrate different forms of function representation, such as graphs and tables, to help students develop qualitative FT, which includes interpreting covariation through visual elements like the shapes of graphs. Rolfes et al. (2022) found that learning with graphs provided richer insights into covariation, supporting the idea that graphs, by visually presenting how quantities vary simultaneously, facilitate a deeper understanding of covariation (Carlson et al., 2002).

Building on this, the purpose of this study is to explore how bimanual movements on a nomogram—a representation consisting of two parallel number lines connected by arrows—can foster FT by providing a tactile, kinesthetic learning experience. Specifically, we address the research question of how students' bimanual activity in digital-embodied tasks relates to the development of their FT.

Method

Participants

This study involved 76 grade-nine students with comparable prior education from two international schools in the Netherlands (37) and China (39). These students were assessed through a pre and posttest and did tasks from three learning modules. To ensure uniformity in the research conditions, all participants received the same instruction and engaged in identical tasks.

Instrument

This study focused on the second of a series of three learning modules designed within the Numworx platform (https://bit.ly/bimanual2). Numworx is equipped with multi-touch and real-time feedback capabilities, making it an ideal tool for supporting the understanding of functional relationships through bodily interactions. Students were invited to work on tablets, using bimanual dragging motions to explore seven different function nomograms.

Data processing

Bimanual data

The bimanual movement data was collected from the Numworx log files, which captured every millisecond location of the two draggable points (representing two hand positions) on the nomograms. This data enabled the classification of student bimanual movements into two phases: the *Exploration* phase and the *Fluency* phase (Tancredi et al., 2021). In the *Exploration* phase, students initially explored the movement rule and made trial attempts to understand it. This phase reflects their early efforts to grasp the task and experiment with the hand movements. In the *Fluency* phase, students'

movements became more consistent and smoother, which could indicate that they had developed a better practice of the bimanual coordination and better understanding of covariation aspect of FT.

To distinguish between these two phases, we first calculated the moving average of the differences between the actual and target positions of the right hand, with the target position determined by the left hand's location. The target position varied depending on the specific task, reflecting the different functional relationships. For each task, we set a task-specific threshold to differentiate the phases based on the difficulty of each task. Movements where the difference fell below the threshold were classified as part of the *Fluency* phase, indicating that the student had achieved accuracy in coordinating the two variables.

However, to prevent mislabeling brief moments of fluency, any fluency period lasting less than three seconds was reassigned to the *Exploration* phase, differing from the method used by Tancredi et al. (2021). The variable Ly represents the left hand's movement, while Ry represents the right hand's movement. By plotting these hand positions over time, we were able to identify detailed trajectories of the student's bimanual movement process, specifically differentiating between the *Exploration* and *Fluency* phases.

Pretest and posttest data

The pretest and posttest were designed to measure student's FT level using a paper-pencil format. The validity of the assessment tool was established through a multifaceted approach, incorporating expert reviews from both experienced mathematics teachers and researchers. Each test comprised 15 items, scored on a 5-point scale, with a maximum total score of 75 points. The posttest mirrored the pretest in item structure and content focus but featured variations in numbers or data. The pilot of the pretest to 51 students allowed for a classical test analysis, including calculations of p-value, variance, item reliability index (Rir), and item-total correlation (Rit), culminating in a Cronbach's alpha (α) of 0.71. To ensure the reliability of scoring the pretest and posttest, two independent coders evaluated the work. Following their independent assessments, a consensus discussion was conducted to reconcile any differences. The inter-rater reliability for these assessments was quantified using Cohen's kappa, which yielded coefficients of 0.91 for the pretest-posttest comparisons.

Correlation test

We conducted a Spearman's Rank Correlation test to explore the relationship between bimanual task performance indicators—time to fluency (TTF) and fluency time sum (FTS)—and FT improvement (pretest to posttest). This non-parametric test was used due to the non-normal distribution of bimanual data, guided by two hypotheses: ***H*1**: advanced FT levels relate to shorter TTF; ***H*2**: advanced FT levels relate to a longer FTS.

For the correlation analysis, we excluded datasets from students who did not reach a fluency phase, as their TTF could not be measured. Additionally, we removed outliers to ensure the accuracy and reliability of the results. For TTF, we excluded data points greater than 150 seconds, and for FTS, we excluded data points shorter than 3 seconds and longer than 400 seconds. After these exclusions, approximately 250 valid samples across seven tasks remained for analysis.

Mini interviews

To capture students' learning progression along the HLT (Wei et al., 2024), we conducted brief, one-to-two-minute mini interviews during the teaching experiments. The aim was to encourage students to reflect on and expand upon their written responses in the answer boxes within the learning environment through open-ended questions and follow-up prompts, promoting deeper engagement with their learning processes (Drijvers, 2003). Conducted by the research team, these interviews were held individually or in pairs, allowing insight into students' problem-solving processes and reasoning.

Results

In this section, we present two cases to explain the bimanual task and to clarify the interpretation of the exploration and fluency phases. Next, we present the results of correlation analyses between bimanual movement indicators and pre and posttest outcomes.

Case study

Figure 1 presents a student's hands movement trace for function $f(x) = x - 3$. He moved the right point upward significantly without getting green feedback, causing a decrease in movement speed and a slight repositioning of the right point. Subsequently, he realized the necessity of moving both points in parallel, which led to the achievement of the *Fluency* phase. As the student approached the top of the screen, he paused and began exploring different directions, entering a second exploration phase. Finally, both points were moved downward simultaneously with green feedback.

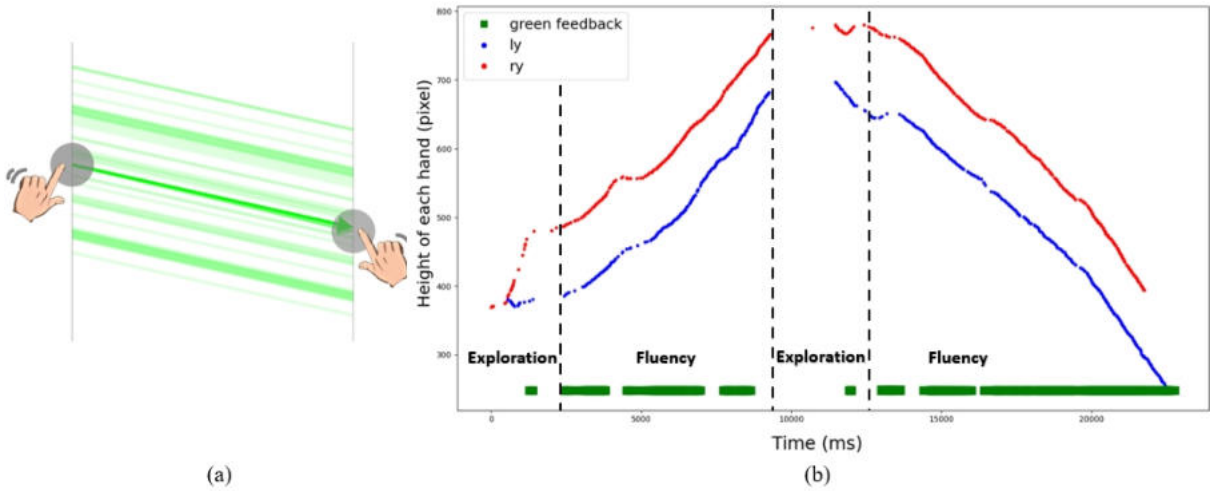

Figure 1 (a) Bimanual movement of $f(x) = x - 3$; (b) A student's hand movements across phases

For the task in Figure 2, students were assigned to first complete the nomogram of the function $f(x) = -x$. This student also went through multiple learning phases. Following the initial *Exploration* phase, his hand movements displayed a pattern: his right hand moved upwards while his left hand moved downward simultaneously. In the subsequent *Exploration* phase, he adopted an inverse approach, moving his right hand downward while moving his left hand upward, showing a geometrically symmetrical movement according to the properties of the target nomogram. After

navigating the entire screen, he described his findings in the answer box: 'To get the green line, I moved the two dots in opposite ways which created this pattern'. Many students also noted the intersection point on this nomogram and used it as a reference to adjust their movements.

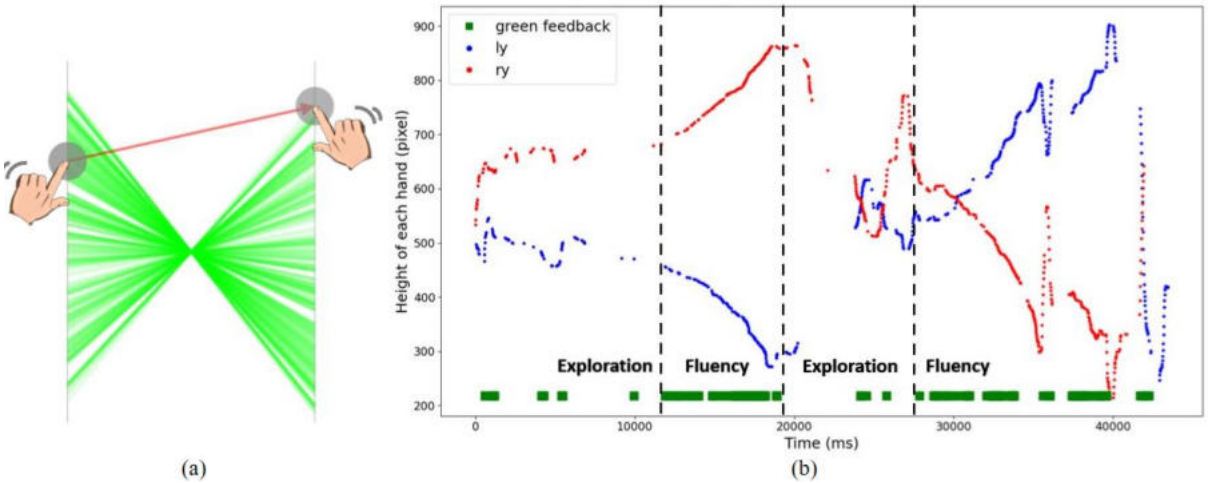

Figure 2 (a) Bimanual movement of $f(x) = -x$; (b) A student's hand movements across phases

Correlation analysis

This section presents the results of the correlation analysis between bimanual movement data and the pretest and posttest scores, which reveals students' FT levels.

Table 1: Correlation test between bimanual indicators and test scores

	Pretest	Posttest	Overall Improvement
Time to Fluency (TTF)	r = -.28, p= .02	r = -.25, p= .04	r = .11, p= .37
Fluency Time Sum (FTS)	r = -.45, p< .05	r = -.33, p< .05	r = .28, p= .05

Concerning TTF aspect, both pretest and posttest scores are negatively correlated with TTF. The stronger correlation with pretest scores indicates that initial FT levels are a more robust predictor of how quickly students reach fluency, supporting ***H1***. The positive but non-significant correlation between the overall improvement and TTF suggests that improvements in FT levels are not strongly related to how quickly students transition to fluency. This may indicate that the transition speed is more closely tied to students' initial FT levels rather than how much they improve.

For FTS aspect, a strong, significant negative correlation between FTS and pretest suggests that students who spend less time in the fluency phase have higher pretest scores. This could imply that students with better initial FT skills maintain fluency more efficiently, needing less time to stabilize their movements. The significant negative correlation between FTS and posttest indicates that higher posttest scores are associated with shorter fluency times. The positive correlation between FTS and improvements suggests that students who demonstrate greater improvements in their FT levels tend to spend more time in the fluency phase.

Discussion and conclusion

In this study, we addressed the research question of how students' bimanual activity in digital-embodied tasks relates to the development of their FT, with a focus on the covariation aspect. By analyzing movement patterns during the exploration and fluency phases and correlating these with pretest and posttest scores, we gained insights into how bimanual tasks help students understand and apply covariational relationships in the field of functions. The results from the case studies suggest that bimanual tasks support covariational reasoning by encouraging students to engage in hands-on activities that emphasize the dynamic and smooth continuous relationships between covaried quantities. The results also highlight how the transition from the initial exploratory phase to fluency enables students to progressively build a deeper and more nuanced understanding of covariation.

In the first case study, the student's movement patterns initially lacked coordination, reflecting an exploratory approach as they attempted to understand the relationship. Over time, the student learned to move both hands in parallel, indicating an emerging understanding of the relationship between the variables based on the bimanual experience. This shift from uncoordinated to coordinated movement marks a transition from exploration to fluency, demonstrating how bimanual tasks can potentially support the development of FT by allowing students to physically manipulate and observe the effects of changing variables on nomograms.

The second case study further supports this conclusion, showing how the student's movements mirrored the geometric properties of the function $f(x) = -x$. By adopting symmetrical bimanual movements that reflected the inverse relationship between the variables, the student demonstrated a deepening understanding of covariation. For example, the student connected the geometric properties of the nomogram with the type of function, stating, "I didn't mean that they (arrows in the nomogram) all pass through a single point (intersection point), then it must be a linear function. But if they pass through the point (0,0), then it's definitely a direct proportional relationship." The ability to visualize and manipulate these relationships through bimanual tasks suggests that such tasks can be potential tools for improving students' conceptual grasp of functions.

Correlations between movement indicators and FT

The correlation analyses on the whole group data provide additional evidence of the relationship between initial FT levels and performance in bimanual tasks. Higher pretest scores, indicative of stronger initial FT, were associated with shorter times to achieve fluency and less time spent in the fluency phase. Conversely, students with lower pretest scores spent more time in the fluency phase. This may suggest that students with lower initial FT need longer time to master the embodied task and mathematize their bimanual movements within the task's context: the two variables covary according to a functional relationship, which is mirrored by their bimanual movements.

The mini-interview data provide further insights into these correlations by showing the different strategies used by students based on their initial FT levels. Some students with lower pretest scores mentioned using a strategy that involved trying out multiple pairs of numbers on the nomogram and searching for patterns. This method, which likely contributed to their extended time in the fluency phase, aligns with the correlation results showing longer engagement times for these students. Their

need for more thorough exploration suggests that they require additional time to develop a robust understanding of covariational relationships. In contrast, students with higher pretest scores used a more direct approach, such as substituting values into $y = ax + b$ after identifying a few number pairs (normally less than three). This strategy facilitated a quicker transition to fluency, as these students could leverage their existing understanding of functional relationships or equations to efficiently solve the task. These strategies also suggest that prior knowledge of linear functions allows students to deliberately search for specific features during the learning process. This appears to diverge from the process often described in embodied design, where such functions are expected to emerge as mathematizations of students' verbal describing their solution of the embodied task.

Implications for educational practice

The findings have significant implications for educational practice in mathematics education. Bimanual tasks provide a dynamic, interactive means of developing FT by engaging students in hands-on exploration within a digital-embodied learning environment. Educators can use or redesign these bimanual tasks that cater to different levels of student. For students with lower initial FT levels, allowing enough time for exploration and providing structured opportunities for pattern recognition and hypothesis testing can help build a strong foundation for function learning. For students with higher initial FT levels, encouraging more targeted and efficient strategies can facilitate the transition to fluency and reinforce their understanding of covariational relationships.

In conclusion, this study investigates how bimanual task could foster FT within a digital-embodied learning environment. The examples and correlation analyses indicate that students' initial levels of FT significantly influence their learning processes and outcomes when engaging with bimanual tasks. Additionally, the strategies identified through mini interviews show that individual differences in FT shape how students approach problem-solving and the time they spend in different phases of the bimanual tasks. These findings suggest that bimanual tasks can be effective tools for fostering FT. Future research could explore the long-term impact of bimanual tasks on FT and investigate their adaptability across different educational settings and subjects.

Acknowledgment

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Body and materiality from a perspective of visuality in mathematics education

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Keywords: Embodied learning, visuality, eye movements.

Introduction

In Flores at al. (2023) we showed how mathematics and the body intertwine in encounters with arts-based activities in classrooms, providing a lived mathematical experience. This time, we want to highlight the notion of "visuality", which interrelates visual perception with sociocultural aspects, drawing attention to the active role of the gaze in the creation and invention of mathematical knowledge in the classroom. Through this poster[1], we aim to discuss how "visuality" can be a valuable concept, offering theoretical insights for research on the embodied nature of mathematical activity and learning[2].

The perspective of visuality

The term visuality, initially introduced in the book titled *Vision and Visuality* by Hall Foster (1988), was borrowed by us from American studies on visual culture to create a 'perspective of visuality in mathematics education.' (Flores, 2024). Visuality refers not only to the biological function of the eye but also, and more importantly, to the socio-cultural practices that shape the act of seeing.

Therefore, we assume that visuality operates as one of the principles underpinning our visual activities in mathematics education. This means that visualization in mathematics is not based solely on the activity of reasoning to interpret and understand visual information. It means that the concept itself is implicated not only in the eye of the mind that sees cognitively but also in the eye that resonates with affection and affect, seeing in the image that which, socially, constitutes it. This involves the eye movement of the multiplicity that encompasses gesturing hands, the mouth that speaks and the emotions that circulate in the interaction between body, mathematics, and the arts.

As a method, visuality operates as both an analytical and experimental tool, implying a more relational and dynamic view of mathematics, in which mathematical abstraction and matter are not separated but rather interpenetrate, creating a living practice where the mind, body, objects, and mathematical tools are intertwined. Mathematical knowledge is not merely something mental or formal, but a material practice that involves the body and the objects of the physical world, both in teaching and in the practice of mathematics itself.

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[1] The poster presented at the conference can be accessed at: https://gecem.ufsc.br/imagens-do-labirinto/poster/

[2] This aim is arised from the research project 'Powers of visuality in workshops with visual arts: About mathematical aesthetic practices', conducted by the first author and supported by CNPq-Brazil.

Visuality and embodied learning in experiments with art

In the last 10 years, our research group has been conducting studies[3] based on this perspective, mainly considering the arts, such as paintings, sculptures, and art installations. This involves putting the movement of looking into practice, operating different ways of producing, imagining, and inventing mathematical knowledge in the classroom. By doing this, we have focused on imagination and mathematical abstraction as socio-material practices, through sensorimotor processes, including gestures, drawings, touch, use of language, etc., within and between bodies. Therefore, these are research studies that engage in aesthetic experiences, providing access to embodied mathematical content: the body in its pulsation, in other words, in its immanent flow that characterizes life. In this direction, the body not only produces political relations but also constitutes potentials for learning.

Inviting questions

How can visuality, as a concept, change the way we understand mathematics learning and teaching? How does the idea of visuality challenge traditional approaches to mathematics, which often consider it an abstract and disembodied practice? How do emotions and affectivity influence the way we perceive and engage with mathematics visuality? What happens when the "vision" of mathematics involves the body, gestures and emotional expressions? How can this enrich or complicate mathematical learning? Can visuality be applied equally to all students, considering different learning ways? These questions can help to further deepen the debate on the implications of visuality in mathematics education, encouraging reflections on pedagogical practices, theory of knowledge and the relationship between body, affectivity and cognition in mathematical learning.

Acknowledgment

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[3] Let's have a look at: "The case of experiencing mathematics with cubist art"; "The case of experiencing mathematics with surreal art "; and The case of experiencing mathematics with abstract art", in Flores (2014) (https://www.worldscientific.com/worldscibooks/10.1142/13700-vol2)

Re-presenting abstract and concrete in the diagrams of Al-Khwarizmi

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Keywords: Al-Khwarizmi, algebra, area, diagrams, quadratic.

Introduction

This poster builds on a recent translation (Maknaoui et al., 2024) of sections on quadratic equations from Al-Khwarizmi's seminal algebra text (Al-Khwarizmi, 813/846). Using print, Al-Khwarizmi and other 8th–13th century Islamic scholars *re-presented* and clarified transient 'dust-board' based Indian mathematical algorithms (Al-Uqlidisi & Saidan, 1978). We read Al-Khwarizmi's original work to explore how his representations could inform working with present-day digital and material media, which allow for transient and rigid representations via reconfigurable visual and tactile interfaces. In exploring the diagrammatic (rectangular, area-based) representations used by Al-Khwarizmi to support his algorithms, we noticed that they disrupt the abstract-concrete dichotomy in mathematics, in ways similar to those advanced by Coles and Sinclair (2019) and Nemirovsky et al. (2025). Here we outline some of Al-Khwarizmi's approach, and how it provides an instance of a more unified and inclusive approach to abstract and concrete in the learning and teaching of mathematics.

Translating Al-Khwarizmi – roots, wealth, numbers, and areal representations

> And I have found that these three types, which are the roots and the wealth and the numbers, associate and from it there are three coupled associations; they are: wealth and roots equal a number, wealth and a number equal roots, and roots and a number equal wealth. (Al-Khwarizmi, as translated in Maknaoui et al., 2024, p. 3)

In Al-Khwarizmi's diagrams, unknown quadratic quantities (wealth) are represented as square areas, unknown roots are side lengths, and known numbers are given as money (dirhams). In associating (adding) them, Al-Khwarizmi admits an abstract notion of quantity encompassing wealth, roots and numbers, also represented as areas, while the three types of associations (equations) reflect the materiality of the different ways they come together (in the absence of negative amounts). This mixed use permeates his work, where the end goal of his 'doors' (algorithms) is not simply the determination of one unknown but engagement with the connected set of wealth, roots and numbers. The doors are presented in prose, and seem quite magical, with diagrams to make sense of and justify them.

> … like when you say, wealth and ten of its roots equal thirty-nine dirhams, … Its door is to halve the roots, and in this matter, it is five, so you multiply it by what is like it, which results in twenty-five. Then you add it to thirty-nine, resulting in sixty-four. Then take its root, which is eight, subtract from it half of the root, which is five, leaving three. This is the root of the wealth that you want, and the wealth is nine. (Al-Khwarizmi, as translated in Maknaoui et al., 2024, p. 3)

While diagrams in the text (rightmost diagram in Figure 1) are static representations of their final state, the text details the steps taken to create these diagrams, illuminating the algorithms (shown using present-day symbols, and with corresponding construction diagrams, in Figure 1).

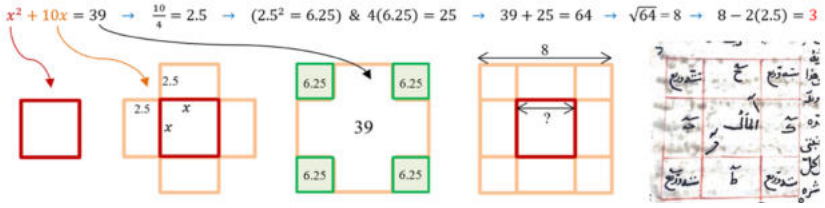

Figure 1: Algorithm and diagram for 'wealth and ten roots equal thirty-nine dirhams'

Discussion – doors to mathematics

Both Coles and Sinclair (2019) and Nemirovsky et al. (2025) counter the 'material equals concrete' and 'symbolic equals abstract' reductive approach to abstraction, and, quite differently, see concretising and abstracting as co-existing accessible experiences, offering a more inclusive approach to mathematics. Coles and Sinclair (2019, p. 467) see concretising as "productive shifts in attention … from relations to objects" while abstracting "moves from objects to relations". In Al-Khwarizmi's setting, these could correspond to solving and setting up equations respectively, with his diagrammatic environment corresponding to their 'symbolically structured environments', where learners can represent actions and be supported in making their own novel moves. Nemirovsky et al. (2025) on the other hand, see abstracting in terms of immersive "acts of as-if world-making", and concretising as being better oriented in, and having a more connected sense of, a mathematical domain. In this framing also, the setting up of equations can be seen as abstracting, playfully conjuring up worlds where wealth, lengths and money can be pooled and equated. Working with the areal diagrams can be concretising, connecting area with quadratic quantities and enabling learners to make sense of adding, multiplying and equating quantities, and clarifying algorithms.

Acknowledgment

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Collaborative reflection on embodied abstraction for geometry: questioning the research-teaching interaction

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Keywords: Embodied abstraction, teacher-researcher collaboration, geometry, primary school.

Introduction

A growing body of research supports the recognition of embodied processes in the learning and teaching of mathematics (Abrahamson et al., 2020). However, as stated in the call for contributions, sharing these insights with a broader community remains challenging. We aim here to investigate this methodological issue by problematising the interaction between research on embodied processes and teaching practices. Considering the development of embodied designs, Abrahamson et al. (2020) frame the relationship between research and teaching as a pedagogical one: "the time is ripe to educate prospective teachers, in-service teachers, and collegiate instructors regarding embodiment" (p. 17). While we recognise the importance of understanding how research can *inform* teaching, we aim to investigate how teaching practices can also *inform* research. Carrying the call to "cross traditional boundaries" (ibid., p. 20), we therefore consider in-service teachers' perspective on embodiment in teaching and learning mathematics and focus on how researchers' and teachers' perspectives *interact*. Thus, our research question reads: what knowledge arises when researchers and teachers commonly investigate the role of embodiment in teaching and learning mathematics?

Context and methods

We investigate the research-teaching interaction by focusing on a specific mathematical content and a localised teaching context. To do so, we build on research conducted by Boonstra et al. (2023) on the fostering of *embodied abstraction* in primary school geometry. This work recognized a tension between the embodied abstraction approach on geometry and the teaching of geometry at the primary level in the Netherlands. Then, a series of tasks was developed for teachers to experience and reflect on emerging embodied processes, focusing on reflectional symmetry. In the Vaud canton of Switzerland, recommendations for primary teachers also contrast with the approach of embodied abstraction. They focus on using manipulatives for building "mental images", on sensitive experiences of physical quantities (weight, length, area…) and on changes of spatial scales (the sheet of paper, the classroom or the building). Thus, the tasks developed by Boonstra et al. (2023) are likely to come into *tension* with the practices of local primary teachers, bringing the perspectives of research and teaching practices to interact. Using tasks on equidistance between reflective symmetric points, we investigate with Swiss primary teachers (also working as teacher educators) how the action-based design and its underlying principles (see Abrahamson, 2014) could relate to their teaching practices.

The methods (and, admittedly, the reasons) for conducting this investigation ground in the principles of *collaborative research* (Savoie-Zajc & Descamps-Bednarz, 2007). This approach frames the teacher-researcher relationship as *co-constructive* for knowledge about teaching practices. To co-

construct such knowledge, the setting of an activity that *reflects* teachers' practices and allows to reflect on them is required. Thus, we investigated the selected tasks in relation to the rationale of action-based design for mathematics (Abrahamson, 2014); then, we experienced the tasks and discussed their use in their respective classes; finally, teachers could try the tasks in class and reflect on the experience with the first author. Results from the first two phases are here presented, providing methodological insights on possible interactions of teachers' and researchers' perspectives.

Insights and opening question

Discussions display a research-teaching tension (resonances and divergences) on epistemological aspects of bodily experience in mathematics. In resonance with the tasks' rationale, teachers valued tasks which involve the body as a resource for solving a mathematical problem (Boonstra et al., 2023, p. 7), rather than as a medium for expressing the solution of a mathematical task. However, diverging with the tasks' rationale, teachers differentiated tasks depending on the "scale" of embodiment: they deemed important to involve the body beyond the hands and arms, asking pupils to reflect the displacements of a classmate standing on the other side of a line. This divergence can be related to their context of practice, which emphasises the variation of spatial scales in mathematics rather than sensorimotor challenges. However, varying spatial scales in geometry is also addressed in research on embodiment (Marchand et al., 2024). Thus, how to build on the epistemological diversity of "embodied research strands" (Gerofsky, 2014) to pursue this fragmented and nuanced interaction?

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Body in between multinaturalism and mathematics education

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Introduction

The aim of this text is to problematize the concept of the body from the perspective of Amerindian Multinaturalism and Perspectivism (Viveiros de Castro, 2018) as a potential framework for research and classroom practices in mathematics education. In our field, it is common to find investigations in which the body is seen merely as a biological delimitation. However, the body plays an active role in the production of mathematical knowledge, whether in its inclusion or exclusion, as well as in the desirable modulations within and arising from the project of modern schooling. Investigations such as those by Efron (2023) have explored the body's participation in acquiring mathematical knowledge, while Flores and colleagues (2023) highlight the interplay between seeing and speaking in observing the world, space, and works of art. In this poster, we introduce another point of view, Multinaturalism, in which the nature of bodies is entirely relative to perspective. We attempt to propose a different understanding of body concept in mathematics education research and classroom practices, emphasizing the contingent and relational aspects of nature, where bodies and knowledge emerge through interactions shaped by diverse perspectives and agencies.

Fixed epistemology and variable nature

In Amerindian cosmologies, the world is populated by agents: gods, spirits, animals, plants, rivers, the deads. For them, in their cosmologies, humans (in all their corporeal variations) inhabit and are constituted through exchanges, predations, rituals, among other relational dynamics. In the framework of our limited grammar and lexicon, all species (agents) are conceptualized as human, though understood through diverse mutations and corporealities, emphasizing relational perspectives rather than fixed identities. A subject is always constituted from a human perspective: the jaguar sees itself as human and sees us as non-human. Thus, unlike a Western cosmopolitics, agents have variable natures, that is, different ontologies, yet they operate and relate in the same manner, that is, within the same (human) epistemology. Particularly, regarding the concept of the body (whether animal, plant, or indigenous), its identity is established in terms of a perspective, a point of view, in relation to the one who perceives it. The body is not stable, fixed, nor does it occupy a singular position; its form is relative to the perspective of the other. As Taylor and Viveiros de Castro (2019) state:

> [...] the identity of the perceived body depends on the nature of the body of the 'person' from whom the gaze originates. The body, therefore, cannot be considered outside of its necessary relationship with an observer acting as a subject." (Taylor and Viveiros de Castro, 2019, p. 770)

If we operate with this concept of the body, perhaps a different mathematics classroom and new ways to research in mathematics education could emerge. In bodies that are always invented from perspectives, and in relationalities with other bodies, in a school territory inhabited by different agents (both human and non-human), discussions should not only focus on mathematics learning processes (in their cognitive aspects); it seems necessary to explore what kinds of belonging and commitments

could be entangled. A body only happens in the possibilities of encounters with another. A body changes (mutates) from the perspective of the other, from the one who sees it, and as a result, the effects and political project of mathematics education can also change. A body unfolds in dissonances, pauses, fissures, sparks, and suspensions, weaving possibilities in formal and informal contexts that are unlikely to be anticipated. Contingency emerges as a central point. These discussions can also be applied to research.

Some considerations (or some questions for future discussions)

To produce a classroom or research as a territory of suspensions, equivocations, and problematizations, where bodies affect one another, may initiate a political process of deconstructing (or decolonizing) discourses that imprison and inflict violence upon certain bodies, especially those that perceive the body as something fixed, possessing only a mind. If the body is not something given, but intrinsically depends on the gaze of the other and can be altered (modified) by agents (human and non-human), establishing itself in relation to the one who sees it, how can mathematics classrooms be spaces that not only produce mathematics but also create bodies, continuously nourished by others? What dynamics can be established in the classroom to enable the reading of this gaze of the other upon "my body" so that I may see myself through their eyes? If a body is constituted in dependence on the gaze of the other, what is the function of this other in a mathematics classroom? What kinds of agency would be possible with mathematics, instruments, and bodies in a mathematics classroom? How can we navigate the spaces between multinaturalism and mathematics education, particularly with this concept of variable body?

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